Diagrams Benefit Symbolic Problem Solving

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Abstract

How a problem is presented can influence students' problem-solving performance. For example, including diagrams can alter students' understanding, choice of strategy and accuracy on word problems. In this study, we examined the effect of diagrams on students' performance in a symbolic problem domain. Sixty-one seventh-grade students solved algebraic equations presented in two formats: with or without an accompanying diagram. The presence of diagrams increased equation-solving accuracy and use of informal strategies. Overall, the benefits of diagrams found previously for word problems generalized to symbolic problems.

Introduction

If you had to teach children basic addition, what would be more helpful – a set of blocks they could touch and count or a list of common arithmetic equations such as "2+3=5"? External representations such as blocks and equations are commonly used tools that can impact student outcomes, such as their learning of concepts and problem solving performance (Belenky & Schalk, 2014). One way that external representations may influence student outcomes is by impacting how people internally represent problems, which in turn influences how they solve the problems (Koedinger, Alibali, & Nathan, 2008). For instance, to solve the equation "2+3= ? ", students need to correctly interpret the string of symbols, form accurate internal representations of the quantities involved, and select and execute a relevant strategy.

Further, the type of external representations may affect problem solving. One way to classify external representations is to place them on a continuum from concrete to symbolic. Concrete representations such as pictures, diagrams, and physical models are grounded in familiar experiences, connect with learners' prior knowledge, and have an identifiable perceptual correspondence with their referents (Fyfe, McNeil, Son, & Goldstone, 2014). However, they may contain extraneous perceptual details that distract learners from relevant information or inhibit transfer of knowledge to novel situations (Harp & Mayer, 1997; Kaminsky, Sloutsky, & Heckler, 2008). In contrast, symbolic representations such as formal equations and line graphs eliminate extraneous surface details, are more arbitrarily related to their referents, and represent the underlying structure of the referent more efficiently. Thus, they allow greater flexibility and generalizability to multiple contexts, but may appear as meaningless symbols to learners who lack understanding of the symbols (Nathan, 2012).

The purpose of the current study was to investigate the impact of using a concrete external representation – diagrams – to represent algebra equation problems. Algebra is an important topic for students in secondary education because it acts as a gatekeeper subject that influences later academic and career success (U.S. Department of Education, 1997). One longitudinal study found that students who enrolled in an algebra course in 8th grade were more likely to take advanced math courses in high school (Atanda, 1999). Other research suggests that students who complete Algebra II in high school are more likely to persist in and complete a four year degree than their peers (Horn, Kojaku, & Carroll, 2001). However, it can be difficult to reason about abstract and unknown quantities in algebra. According to Koedinger et al (2008), algebra is the "first abstract symbolic language" (p 367) that most people encounter in school after learning natural language. Students often have difficulty comprehending and producing algebraic equations (Payne & Squibb, 1990). Thus, this study explored if using a concrete tool such as diagrams could help students to better understand and solve symbolic algebra equations.

Potential Benefits of Diagrams

Although diagrams are a type of concrete external representation, they have important features that distinguish them from other types of illustrations. Specifically, I define diagrams to be visual representations that express information via spatial relationships. For instance, a floor plan is a diagram that uses bare shapes to represent the floor space occupied by furniture. Spatial features such as the size and relative positions of the shapes correspond to real properties of furniture. In addition, irrelevant concrete details of the referent, such as furniture height or material, can be disregarded so that only the relevant problem features and quantitative relations are depicted. In contrast, a pictorial representation would express the specific objects of the problem situation with more surface-level details, such as color. Figure 1 shows the difference

between a pictorial and a diagrammatic representation of a word problem about using a measuring stick to estimate the depth of the sea. In this example, the drawing on the left is a diagram that spatially represents quantitative information about length and depth as physical distances, but the drawing on the right is a picture that includes irrelevant concrete objects such as boats and fish. Previous research on the distinction between pictorial and diagrammatic representations has found that diagrams, but not pictures, are beneficial to problem solving (e.g. Hegarty & Kozhevnikov, 1999). Thus, we focus on diagrams in this study.

There are at least three reasons why diagrams might be helpful for solving symbolic problems such as algebraic equations. First, diagrams may help highlight relevant information that solvers should attend to. For instance, including diagrams with equations may help students extract relevant information needed to solve the problem more quickly and accurately (Larkin & Simon, 1987). This could improve problem-solving speed and also help students to check their work. For instance, students could verify that they are using the correct information to solve the problem, or check their solutions against the problem constraints more directly.

Second, diagrams may decrease working memory load and support quantitative reasoning (Munez, Orrantia, & Rosales, 2013; Murata, 2008). In one study with adults, functional magnetic resonance imaging revealed that solving a word problem by constructing a mental diagram required fewer resources for controlling attention or retrieving procedural knowledge than solving the problem by constructing a mental equation (Lee et al., 2007). Thus, at least for people familiar with diagrams, presenting diagrams may free up cognitive resources that are important for accurate problem solving. This could allow diagrams to be particularly facilitative for students with low cognitive or arithmetic abilities, or on more complex and cognitively demanding problems.

Third, diagrams may scaffold algebraic reasoning by facilitating connections between concrete and symbolic representations (Koedinger & Terao, 2002; Lee, Khng, Ng, & Ng, 2013). Specifically, diagrams may elicit students' intuitive, informal knowledge and strategies. Presenting diagrams with equations may allow students to connect this knowledge to formal, symbolic problem formats. Thus, a diagram benefit may be particularly apparent for students who are still developing familiarity with manipulating abstract symbols, such as beginning algebra students.

Evidence for a Diagram Benefit

In addition to theoretical reasons for a diagram benefit, there are several lines of work suggesting that diagrams can aid mathematical problem solving. Past research has focused on the benefits of diagrams for solving word problems.

The first line of evidence comes from research on individual differences in the spontaneous use of diagrams during word problem solving. More accurate solvers tend to use diagrams, whether by mental visualization or by drawing diagrams on paper (Edens & Potter, 2008; Hegarty & Kozhevnikov, 1999). For example, Edens and Potter (2008) found in a sample of 4th and 5th graders that producing more diagram drawings was correlated with greater accuracy. This result has been replicated in a sample of 6th grade students with and without learning disabilities (Van Garderen & Montague, 2003). Thus, diagrams can facilitate problem-solving success by 4th grade. However, generating diagrams may be an indicator of better problem understanding, rather than a technique that improves problem solving.

A second line of evidence for a diagram benefit comes from instructional practice. Countries such as Japan and Singapore have long incorporated diagrams into math instruction on a national level, and these countries typically perform at the top in international tests of

mathematics achievement (Murata, 2008; Ng & Lee, 2009). For instance, 1st and 2nd graders in Singapore are introduced to a heuristic using horizontal bar diagrams to solve word problems (Ng & Lee, 2009). Students who construct a diagram representation are able to use informal arithmetic strategies to solve algebraic problems, thus making algebraic problems accessible earlier – beginning in 3rd grade, as opposed to the 7th grade in most US classrooms (Lee et al, 2013). In fact, teaching 6th grade pre-algebra students in the US to use this diagram heuristic enabled them to solve algebra word problems as well (Koedinger & Terao, 2002). Additionally, intensive word-problem solving interventions that include practice generating diagrams have helped U.S. elementary school children improve their word problem solving success (Jitendra et al., 2007). However, it is unclear if benefits of learning to use diagrams are due to the presence of diagrams or to more general problem-solving strategy instruction.

A third, more direct line of research clarified this issue by experimentally manipulating whether students were given diagrams in conjunction with word problems. Munez, Orrantia, and Rosales (2013) examined how presenting novel diagrams could enhance 9th graders' accuracy on arithmetic word problems. The problems were comparison problems involving two sets of quantities and their relationship, with keywords such as "more than". The diagrams used were novel to students and consisted of vertical rectangular bars. Bar height represented magnitude and the diagram was clearly labeled with relevant variable names and numerical values from the problem statement. Using a within subjects design, the researchers found that students' accuracy and response times on isomorphic problems improved markedly when problems were presented together with a diagram than not. Further, the improvement was greatest on more difficult problems. These findings are consistent with similar research on undergraduates (Lewis, 1989).

Thus, diagrams on their own appear to aid in problem comprehension and solution, at least for older students.

However, the benefits of provided diagrams may be less robust in middle-school students. Booth and Koedinger (2012) assessed 6th to 8th grade students on three algebraic problems differing in complexity. Each problem was presented in one of three formats: equation only, word, or word-with-diagram. The diagrams used were novel to students and tailored to each problem. While high-ability students of all grades performed equally with or without diagrams, low-ability students in the 7th and 8th grades were more accurate and made fewer conceptual errors on word problems with diagrams. However, these students benefited from diagrams only on the more complex double-reference problem where the unknown variable appeared twice (e.g., N - 1 /5 * N = 30). No diagram benefit was found on the simpler single-reference problem where the variable appeared only once. This suggests that grade, ability level and problem complexity may be key moderators of a potential diagram benefit.

Although informative, some limitations of this study motivate the current study design. First, this study did not include an equation-with-diagram condition, which would have revealed whether the diagram benefit was consistent across problem types or only for word problems. Second, despite a large sample size with over 100 students in each grade, only three problems were used in the assessment. It is unclear if the results would generalize to other problems. Third, the diagrams used were inconsistent in design. Two of the diagrams used rectangular bars to represent quantity, but one image used did not meet my definition of a diagram. Specifically, that image was a pictorial representation of information that did not spatially represent any quantitative relationships. As a result, a diagram benefit was found for the first two diagrams, but not for the third image. While this agrees with previous research demonstrating the benefits of

diagrams but not pictures (e.g. Edens & Potter, 2008), there is a potential confound between problem difficulty and diagram effect because the problem using a picture was the hardest problem in all formats. Hence, these results have limited generalizability because they may be a consequence of the specific problems posed.

Further, Booth and Koedinger with previous research the question of to what extent unfamiliar diagrams can be helpful. In this study, low-ability 6th graders did not benefit from diagrams, but low-ability 7th and 8th graders did. Was the improved ability to benefit from diagrams due to increased exposure and practice with diagrams, which was part of the middle school curriculum? Or was it due to students' improvement over time in other areas, such as general problem-solving skills? Thus, it is still unclear how much familiarity with diagrams is required before they become helpful.

In summary, diagrams generally aid in problem comprehension and solution of word problems, although results are less consistent with middle-school students than with high-school and college students. Little is known about the use of diagrams with more symbolic tasks such as equation solving. Extending the diagram research to a symbolic domain will test the generalizability of the diagram effect. It should also provide insight into how children interpret and solve symbolic problems.

Current Study

The present study investigated the effectiveness of presenting diagrams alongside algebraic equations. Previous research suggests that diagrams can help students make sense of word problems, including those with an algebraic structure. However, algebra equations are more abstract than word problems. Algebra equations not only require students to understand complex mathematical structure, but also require students to decode the symbolic language of

algebra (Payne & Squibb, 1990). By investigating how students relate provided diagrams and equations, this study can also provide insight into how children interpret and solve symbolic problems.

Our primary research question was whether the presence of diagrams would influence algebraic equation-solving performance, including accuracy, strategy use, and types of errors made. We predicted that problem-solving accuracy would be higher if a diagram was provided than if it was not – a "diagram benefit". We also hypothesized that diagrams could elicit students' intuitive knowledge of quantitative relations in the problem. Similar to findings in word problems (e.g. Koedinger & Nathan, 2004), this should increase both the usage of non-algebraic strategies and the accuracy of algebraic strategies, while reducing the frequency of conceptual errors.

Our secondary research question was whether the effect of diagrams would depend on problem or student characteristics, as a test of the generalizability of the diagram benefit. We explored problem complexity, students' representation translation ability, and students' general math ability as three factors that could influence the benefits of diagrams. We varied problem complexity by including both single-reference and more difficult double-reference problems, which differed in whether the variable appeared once or twice in the equation (Table 1). We measured students' representation translation ability by asking them to convert diagrams into equations, and vice versa. We explored the importance of general math ability by working with students drawn from advanced and regular mathematics classes.

Method

Middle-school students participated in an experimenter-led classroom session. Using a within-subjects design, we manipulated the presence of diagrams during the equation solving assessment.

Participants

Participants were 62 seventh-grade students from four classes attending an independent private school in Nashville, TN. Students were tested in late October, about 2 months into their first pre-algebra course. Two classes (34 students) were in an advanced math class, which covered the same breadth of content but in greater depth. Students had experience with reading algebraic expressions and solving simple one step equations, but had not studied the equation forms used in the experiment. They did not have prior experience with the type of diagrams used in this study. Prior to the experiment, their mathematics teacher explained that they would be learning a novel way of solving algebraic equations, and that they could potentially apply the new knowledge to future classroom work. We dropped the data of one student who showed visible frustration during the experiment, had difficulty understanding the task, and did not attempt any of the assessment items. The final sample contained 61 students (33 male, mean age = 12.7 years).

Procedure

Students completed the experiment in their classrooms during their regular 50-minute math period. All students completed the tasks in the same order: introduction, diagram practice, representation translation, and equation solving. The within-subjects manipulation occurred only during equation solving. There were no time limits for any of the tasks. Calculators were not permitted on any task.

Measures

Appendix 1 provides a sample assessment booklet. On all problem-solving tasks, two types of problems were used: (a) single-reference problems, which involved only one instance of the variable, and (b) double-reference problems, in which the unknown variable appeared twice. Table 1 shows examples of equations and corresponding diagrams for each type of problem complexity.

Equations were adapted from Booth & Koedinger (2012)'s choice of single- and double-reference problems. The math teacher of the participating students checked the final set of 16 problems and verified that they were of suitable difficulty to provide us with a range of responses.

Based on the diagram approach used in Singapore, diagrams were constructed according to these guidelines: (1) Quantities are represented by rectangular bars using solid borders; (2) Dotted vertical lines divide bars into equal portions; (3) Rectangular bars were labeled internally with variables (4) Horizontal arrows over the length of a bar indicated the quantity's magnitude and were labeled with known values or '?'. Diagrams were drawn to approximate the relative quantities in each problem, but were not of the same scale across problems

Diagram Introduction. The experimenter spent about 8 minutes with the entire class to describe diagrams as a special kind of picture that represents information about numbers and quantities. She used four examples to explain the guidelines used to construct diagrams, highlighting important features (e.g. arrows, labels, dotted lines). She described how diagrams could represent each of the basic operations: addition, subtraction, multiplication, and division. For each operation, she presented and described an example diagram and asked students to copy her drawings on a worksheet. These example diagrams were much simpler that the diagrams

students would encounter in the rest of the experiment. The intent was to develop basic

knowledge of interpreting the diagrams as they were unfamiliar to the students. No references to

equations were made. Appendix 2 provides the script used.

Diagram practice. To provide brief exposure to problem-solving using diagrams in

isolation, we asked students to work individually on four problems. Students had to solve for an

unknown variable from a given diagram. The corresponding equation was not included. The first

three problems were single-reference problems and the fourth was a double-reference problem.

After 10 minutes of individual work, the experimenter announced the correct answer for each

problem. Students checked their own work. The intent was to increase familiarity with the

diagrams.

Representation-translation task. We constructed four items to measure how well students

could translate between diagrams and equations. The first two problems required students to

choose between two diagrams that described a given equation. Students received one point for

circling the correct diagram. The next two problems required students to write an equation

describing a given diagram. Students received one point for writing a valid equation; expressions

(e.g. 2x) were not valid. For both pairs of problems, a single-reference problem was presented

first, followed by a double-reference problem. Students first read directions and a completed

example before attempting each pair of problems. No feedback was provided. We included this

task as a measure of how well students were able to make the connection between diagrams and

equations.

Equation-solving assessment. To evaluate our primary research question, we designed

eight algebra problems contrasting two factors, presentation format and problem complexity.

Students saw four problems as equations and an isomorphic set of four problems as equations

with accompanying diagrams, using different variable letters and constant values. For each

presentation format, the first two problems were single-reference and the other two problems

were double-reference. We designed four counterbalanced forms of the equation-solving

assessment to control for order of presentation (equation-first or equation-with-diagrams first),

and version (which problems had diagrams). Problems were blocked by presentation format: for

example, the first four problems would be in equation format, and the next four problems would

be in equation-with-diagram format. On the four equation-with-diagram problems, students also

indicated if they had used the diagram to solve each problem by circling "yes" or "no" inside a

small box below the diagram. This provided a measure of diagram use frequency.

In addition, a digital clock was shown on a projector screen for this section of the

experiment. Students were prompted to write down the time that they had completed each page

of the assessment. This was used as a measure of problem solving speed.

Coding

Each problem was scored as correct or incorrect. Students received one point for each

correct answer. We coded students' strategy use and types of errors committed based on their

written work, using the scheme outlined in Tables 2 and 3. These schemes were adapted from

previous research on students' solution of algebra equations and word problems (Koedinger,

Alibali, and Nathan, 2008).

Apart from cases where students did not attempt a problem or did not show written work,

three main strategy codes were given: algebra, unwind, and guess and check. An "algebra" code

was given to students who demonstrated use of symbolic manipulations by writing an algebraic

expression or equation. An "unwind" code was given to students who used informal methods to

logically derive a solution, such as performing arithmetic operations on known values without

working with the variable. A "guess and check" code was given to students who tried different values of the unknown variable in order to find a solution which fit the problem. An "other" code was given to responses with ambiguous work or other approaches, such as when the student wrote down some numbers or equations that did not follow from the problem. Strategy codes were given based on the general approach taken, and incomplete or inaccurate implementations were still coded if it was clear which strategy was employed. We were primarily interested in the unwind and algebra strategies, which are essentially formal and informal versions of logical problem-solving.

Similarly, two main error codes were given for inaccurate or incomplete attempts with written work. An "arithmetic" code was given to students whose answers were wrong only because of computational errors, but who otherwise understood the problem and made a valid attempt at solving the equation, regardless of the specific strategy used. A "conceptual" error was given to students who used invalid approaches. For instance, they might have ignored parentheses in an algebraic strategy or used the wrong operations in an unwind strategy. Incomplete approaches were also considered conceptual errors because these suggest that students had partial but incomplete understandings. In addition, students who received a "no attempt" strategy code received the same "no attempt" error code; this suggested that the student had such a low comprehension of the problem that they could not even begin. A "copy slip" code was given to solutions that would have been correct, if not for errors due to miscopying a value between steps.

To establish inter-rater reliability, a second rater coded the written responses of 25% of the children. Inter-rater agreement was high as indicated by Cohen's kappa ($\kappa = .85$ for strategies, $\kappa = .92$ for errors).

Results

To evaluate the effect of counterbalanced forms, a 2 (order) x 2 (version) ANOVA on equation-solving assessment scores was done. This analysis revealed no significant effects of either factor or their interaction, F's < 1. Thus the subsequent analyses treated the four forms as equivalent.

Recall our primary research question of whether the presence of diagrams would influence student performance on algebraic equation problems, and secondary research question of how problem and student characteristics would affect these patterns. Thus our main independent variable was presentation format. Our dependent variables were students' accuracy, strategy use, errors made, and solution times. Where relevant, we also assessed the interaction of these variables with problem complexity, student's general math ability and students' representation translation ability.

Accuracy

We evaluated students' accuracy on the equation-solving assessment using a repeated-measures ANOVA with presentation format (equation or equation-with diagram) and problem complexity (single- or double- reference) as within-subjects variables and math ability (regular or advanced class) as the between-subjects variable. As shown in Figure 1, students provided more correct answers when solving equations with provided diagrams (48%) than without (36%), $F(1,59) = 11.6$, $p = .001$, $\eta_p^2 = .16$. Students also solved fewer double-reference equations correctly (32%) than single-reference equations (50%), $F(1,59) = 17.6$, $p < .001$, $\eta_p^2 = .23$. Students in advanced classes solved more problems correctly (56%) than students in regular classes (22%), $F(1,60) = 24.28$, $p < .001$, $\eta_p^2 = .29$. None of the two-way or three-way interactions between presentation format, problem complexity and math ability were significant

(F's < 1). Thus, the positive effect of diagrams was consistent across simple and complex problems and across students with low and high ability, with a moderate to large effect size.

Students were moderately successful on the representation translation task (M = 2.44 out of 4, SD =1.35). Students were equally successful when the task involved single-reference problems (62% accurate) and double-reference problems (61% accurate), $t(60)$ < 1. Students in the advanced math classes also scored higher on this task than students in the regular math classes (M = 1.70 vs. 3.03), $t(59)$ = 4.35, p < .001, d = .55. To assess the impact of representation translation ability to equation-solving performance, we conducted correlations between students' representation translation scores with students' accuracy on the equation-solving assessment. Student's representation translation scores were positively correlated with their total equation-solving accuracy (r = .492, p < 0.001) and with the equation-solving accuracy on the four problems with diagrams (r = .504, p < 0.001). Further, to test if representation ability contributed to a diagram benefit over and beyond students' general equation-solving ability, we calculated a diagram effect score for each student (equation-with-diagram accuracy – equation-only accuracy) and found a small but insignificant correlation between representation translation scores and diagram effect scores (r = .21, p > .1).

Diagram Use

Eleven students failed to indicate whether they used a diagram for a specific problem at least once. The remaining 50 students reported using the diagram on a majority of the equation-with-diagram problems (60%). Students reported using diagrams more often on double-reference problems (72%) than on single-reference problems (49%), $t(49)$ = 3.34, p < .01. However, reported diagram use was not correlated with accuracy on equation-with-diagram problems (r = -.045, p > .75).

Strategy

Next consider the strategies students used to solve equations. Table 4 shows the frequency of strategy use and the average success of using each strategy, divided by presentation format and problem complexity. Students did not attempt 12% of all questions and did not show written work on 21% of all questions. Overall, students used an unwind strategy twice as often as an algebra strategy (36% vs. 16% of problems), $t(60) = 4.31$, $p < .001$.

We conducted two separate repeated-measures ANOVAs to evaluate how presentation format affected students' frequency of using the unwind and algebra strategies. For both tests, we used presentation format (equation or equation-with diagram) and problem complexity (single- or double- reference) as within-subjects variables. Though not reliable, the presence of diagrams marginally increased use of the unwind strategy from 32% to 38%, $F(1, 59) = 2.48$, $p = .12$, $\eta_p^2 = .04$. Students also used fewer unwind strategies on double-reference problems (24%) than on single-reference problems (46%), $F(1, 59) = 20.47$, $p < .001$, $\eta_p^2 = .25$. Results were different for the frequency of algebra strategy use. Neither presentation format nor problem complexity affected students' use of the algebra strategy, F's < 1. There were no significant interactions between presentation format and problem complexity for either strategy use, F's < 1.

We also compared how math ability influenced students' strategy use, both with and without diagrams. Using independent sample t-tests, we found that students in advanced classes used the algebra strategy more frequently (25%) than students in regular classes (5%), $t(59) = 3.62$, $p < .001$. Use of the unwind strategy was similar for students in the advanced (39%) and regular classes (30%), $t(59) = 1.24$, $p > .1$. Next, we used paired t-tests to separate assess the effect of diagrams for advanced and regular students. For advanced students, diagrams marginally increased use of the unwind strategy from 34% to 44%, $t(33) = 1.91$, $p = .065$, but did

not affect their use of the algebra strategy. Presenting diagrams did not affect strategy choice for students in regular classes (t's < 1).

In addition, we evaluated the effectiveness of each strategy by considering the percentage of problems where the specific strategy resulted in a correct answer. Overall, students were most successful at implementing the algebra (56% accuracy), unwind (61%), and guess-and-check strategies (55%). The presence of diagrams appeared to increase the success of these strategies, although the overall frequencies are too small to conduct meaningful statistical analyses on.

Errors

Recall that students' strategies were coded for the general approach taken, and allowed room for error. Thus, students' written work was given a separate code for the type of error committed. Figure 4 shows the percentage of all problems where students committed each error. Overall, conceptual errors were the most frequent (28% of all problems), followed by indeterminate errors ("No work", 14%), no attempt errors (12%), and arithmetic errors (4%).

We conducted paired t-tests to evaluate the effects of presentation type on frequency of different errors. Specifically, students made fewer conceptual errors on equation-with diagram problems (21%) compared to equation-only problems (34%), $t(33) = 3.51$, $p < .001$.

We also evaluated the effect of problem complexity on student errors. Figure 5 presents the same data on students' errors, divided by problem complexity. Given this limited sample, we restrict our analyses to descriptive comparisons. With the exception of arithmetic errors, students make more errors on double-reference problems than single-reference problems. For instance, greater problem complexity increased the frequency of conceptual errors from 25% to 31% and tripled the frequency of no attempt errors from 6% to 18%, but reduced the frequency of arithmetic errors from 6% to 2%.

Solution times

Finally, consider how long students took to solve equations. 16 students (26% of the sample) failed to write down at least one time-point information. The remaining 45 students took about 14 minutes (M = 834 seconds, SD = 303 seconds) to complete the 16-item assessment. Figure 3 shows the average time (in seconds) that students spent on each two-problem section of the assessment, divided by presentation format and problem complexity. Students were slightly slower to solve the four equation-only problems (M = 431 seconds) than the equation-with-diagram problems (M = 407 seconds). This slight difference was due to the longer average time spent on double-reference equation-only problems; all other solution times were similar. No reliable differences were found.

Discussion

External representations such as diagrams generally support learning and problem solving. However, incorporating diagrams with symbolic problems has largely gone unstudied. Diagrams similar to the ones used in this study have enhanced students' understanding and performance on algebraic word problems (e.g. Booth & Koedinger, 2012; Koedinger & Terao, 2002), so we conducted this experiment to test whether diagrams might benefit students on a more difficult problem type, namely, algebraic equation problems.

We found a clear diagram benefit for students' equation-solving accuracy. Presenting diagrams alongside algebra equations enhanced students' accuracy, and this diagram benefit was independent of problem complexity and students' math proficiency. This contrasts with a previous finding in the word problem literature, where diagrams were most helpful for more difficult word problems and for students with lower ability (Booth & Koedinger, 2012; Lewis,

1989). Other research has found that concrete representations might even reduce performance on complex problems. For instance, word problems can be seen as a concrete representation of algebraic problems, because they help solvers link abstract information about quantities to real-world situations. However, one study found that college students were more accurate solving double-reference problems in equation format than in word problem format, even though they benefit from the concreteness of word problems on simpler single-reference problems (Koedinger et al, 2008). Why did diagrams provide a clearer benefit in this study than in previous work with word problems?

Students may have been able to better understand the diagrams used in the current study than in previous studies. Although students reported never seeing the diagrams before, we did provide an introductory lesson to familiarize students with diagrams before the assessment. Unlike in Booth and Koedinger's (2012) study, we also used a consistent diagram type for all problems. However, although representation translation ability correlated with accuracy on problems containing diagrams, it also correlated with accuracy on problems without diagrams. Further, although students in more advanced math classes were better at translating between diagrams and equations, they did not benefit more from diagrams than students in the regular math classes. Taken together, the evidence suggests that better understanding of diagrams only played a minor role for helping students to benefit from diagrams.

The fact that algebra equations are generally more difficult than equivalent word problems might explain why a diagram benefit was independent of problem complexity and math ability in this study. When all problems are difficult for most students, diagrams can aid performance across problem complexity and students' math proficiency. Even high school students may make persistent errors in understanding and solving algebra equations, although

they are more accurate on word problems (Koedinger & Nathan, 2004). Thus, the simpler single-reference equation problems used in this study were probably challenging enough that students benefited from an alternate concrete representation. Similarly, even students of higher math ability found the problems challenging enough to benefit from including a diagram. Students also indicated more frequent use of diagrams on more complex double-reference problems. This suggests that problem difficulty is a relevant factor when students consider what representations, tools, or strategies to use in problem-solving.

Our results on students' strategy use and errors made suggest several additional explanations. One possible explanation is that diagrams may influence internal representation. Our finding of a diagram benefit is consistent with various cognitive models of problem solving. These models generally posit that constructing appropriate mental models of a problem is key to successful problem solving (Johnson-Laird, 1983; Koedinger & Nathan, 2004). For example, compared to problems posed in difficult formal terms (such as an algebra equation), students who have the support of concrete representations can conceive of the problem in more intuitive terms, and make fewer conceptual errors when solving the problem. Our results match their predictions of increased accuracy and fewer conceptual errors. By providing students with a pre-constructed diagram, we may have removed some of the difficulty of constructing a usable internal representation of the problem. By providing an extra external representation on paper, we may also have reduced students' working memory and attention demands by offloading some cognitive processing onto perceptual processing (e.g. Larkin & Simon, 1987).

Second, diagrams may facilitate informal reasoning. Consistent with previous research on word problems (Koedinger & Nathan, 2004), students in the current study tended to use more informal, non-algebraic strategies when concrete diagrams were present. Similar to how adding a

concrete story context can improve performance on arithmetic problems by activating real-world knowledge of common operations and quantitative relations (Carraher, Carraher, & Schliemann, 1985), adding a concrete diagram might improve children's performance on algebra equations by activating informal strategies that do not rely on newer algebraic strategies that students are still acquiring. Furthermore, the presence of diagrams increased the general effectiveness of each strategy, perhaps by allowing students to check their conceptual understanding of the problem and the accuracy of their procedures.

According to the National Council of Teachers of Mathematics' (NCTM), developing facility with multiple representations is an important outcome of elementary math education (NCTM, 2000). This study suggests that combining concrete and symbolic representations is feasible and beneficial for students. It would be worthwhile to see if teaching children how to generate their own diagrams is beneficial for equation solving. Would learning to construct algebraic equations in a diagrammatic way help students to grasp the underlying mathematical structures involved? An emerging line of research suggests that combining concrete and symbolic representations in an instructional sequence can integrate their advantages and mitigate their disadvantages (Fyfe et al, 2014). This technique, known as "concreteness fading", uses concrete materials to introduce learners to new concepts, before gradually removing perceptual details so as to encourage learners to generalize their understandings. Diagrams are a useful concrete representation of quantitative relations in equations. Using diagrams in combination with equations, and gradually fading the inclusion of diagrams may facilitate learning and performance in algebra. Future research needs to evaluate whether such an instructional sequence indeed benefits learning and problem solving.

Despite the positive contributions of the current study, the scope of this study is still very limited. The present study is unable to provide detailed accounts of how students used the diagram. Simply coding errors from students' written work provides a limited perspective into students' problem-solving process. Students may not have written down all their work – in fact, more than 20% of all problems contained no written work.

Also, using math class assignment as a proxy for students' general math ability may have introduced confounding factors that affect how students reason using external representations, such as the amount of experience with visual representations, type of problem-solving strategies used in the classroom, and interest in using diagrams. These are real concerns because students with higher math ability are more likely to spontaneously generate useful diagrams when solving problems (Edens & Potter, 2008). Thus, students of different math ability may have benefited from diagrams due to different reasons. For example, higher-ability students may have benefited because the diagrams more closely matched their own internal representation of the problem, whereas low-ability students may have benefited because the diagrams highlighted important information that they might otherwise have ignored or misinterpreted.

Thus, a mechanistic account of diagrammatic reasoning is still lacking. Future research should investigate specific processes that problem-solvers engage in when using a diagram. Do students iterate between representations and try to process both to understand the problem? Or do students fixate on the more concrete or familiar representation, effectively treating that as a self-contained problem? The kind of processes that students engage in can inform how teachers try to incorporate such diagrams into their instruction. Tracking participants' eye movements may also reveal diagram elements that are particularly helpful, distracting, or ignored, which can inform the design of effective diagrams and other visual aids.

Another interesting direction for further research is on the role of representation translation skill for benefiting from multiple representations. Research on perceptual learning has demonstrated how extensive practice with translating between representations, such as graphs, word problems, and equations, can improve students' conceptual knowledge and problem-solving speed (Kellman et. al, 2008). The idea is that students can use perceptual processes to recognize a common underlying mathematical structure across different modes of representation. In the present study, students with greater general math ability scored higher on the representation translation test, supporting the same relationship between general mathematical ability and translation skill. More generally, Ainsworth (2006) has proposed that learners' ability to integrate multiple representations is a crucial component of learning and benefiting from multiple representations. For instance, domain experts are more adept at interpreting diagrams within their domains because they can easily make connections between the diagrams and the problem situation, which in turn supports their learning from those diagrams. It would be worthwhile to investigate how representation translation ability influences the acquisition of conceptual and procedural knowledge, and how this ability should best be supported.

In summary, the current study extends previous research of a diagram benefit in problem solving to a symbolic domain. Providing novel diagrams enhanced students' accuracy on difficult algebra equation problems independent of the problem and student characteristics studied. Concrete external representations may be more powerful than previously leveraged, especially when combined with symbolic problems.

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Table 1

Example equations and diagrams from the equation-solving assessment

Problem complexity	Presentation Format	
	Equation	Diagram
Single-reference	$(x - 45) \div 3 = 20.5$	
Double-reference	$N - \dfrac{1}{5}N = 30$	

Table 2

Children's strategies on equation solving assessment

Strategy	Definition	Example responses $(x-45) / 3 = 20.5$
Algebra	Student uses algebraic manipulations to derive solution. A partially-solved equation is written.	$x - 45 = 3 * 20.5$ $x = 61.5 + 45$ …
Unwind	Student works backward using arithmetic strategies to derive solution.	$20.5 * 3 = 61.5$ $61.5 + 45 = 106.5$
Guess and Check	Student substitutes different values of the variable into the provided equation.	$(90-45) / 3 = 15$ $(105-45) / 3 = 20$ $(108-45) / 3 = 21$
Other	Student uses other non-algebraic strategies, or strategy is ambiguous	Student draws a diagram
Answer Only	Answer is provided without any working.	$x = 106.5$
No Attempt	Student leaves problem blank.	Student writes "I don't know" or "Skip"

Note: Codes are assigned based on students' overall approach, even if errors are made in the process.

Table 3

Children's errors on equation-solving assessment

Error	Definition	Example $(x-45) / 3 = 20.5$
Arithmetic	Student makes a computational error, but solution is otherwise correct	$3 * 20.5 = 60.15$ $x = 45 + 60.15$ …
Conceptual	Student employs invalid or incomplete strategies	$x - 45 = 3 + 20.5$ …
Copy slip	Student miscopies a value from the problem or from own work, but solution is otherwise correct	$(x-45) / 3 = 205$ $x - 45 = 615$ …
No Work	Student writes an incorrect answer without any work shown.	$x = 50$
No Attempt	Student leaves problem blank.	

Table 4

Percentage of problems attempted and solved correctly with different strategies for equation-only and equation-with-diagram equations

	Single-reference		Double-reference		Total	
	% Used	% Correct	% Used	% Correct	% Used	% Correct
Equation-only						
Algebra	17	57	16	45	17	51
Unwind	44	57	20	38	32	51
Guess and Check	7	44	3	50	5	46
Other	10	8	20	25	15	19
Answer only	16	36	21	15	19	24
No Attempt	6	-	20	-	13	
Equation-with-Diagram						
Algebra	16	65	14	59	15	62
Unwind	48	74	29	60	39	69
Guess and Check	3	50	4	80	4	67
Other	7	38	9	27	8	32
Answer only	20	25	28	29	24	27
No Attempt	7	-	16	-	12	

Note: % Used may not add up to 100 because of rounding error.

Figure 1

Example of schematic and pictorial drawing. Adapted from Edens & Potter, 2008, p.186
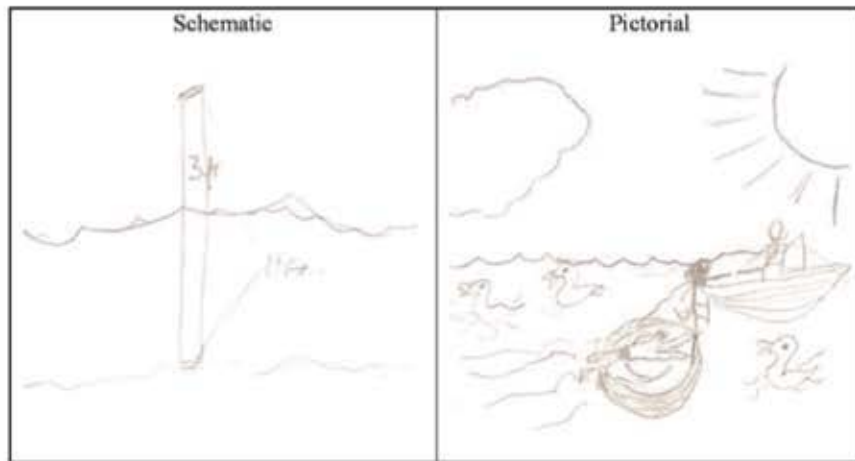
Figure 2

Accuracy on equation-solving assessment by problem complexity and presentation format

Average accuracy on equation-solving assessment
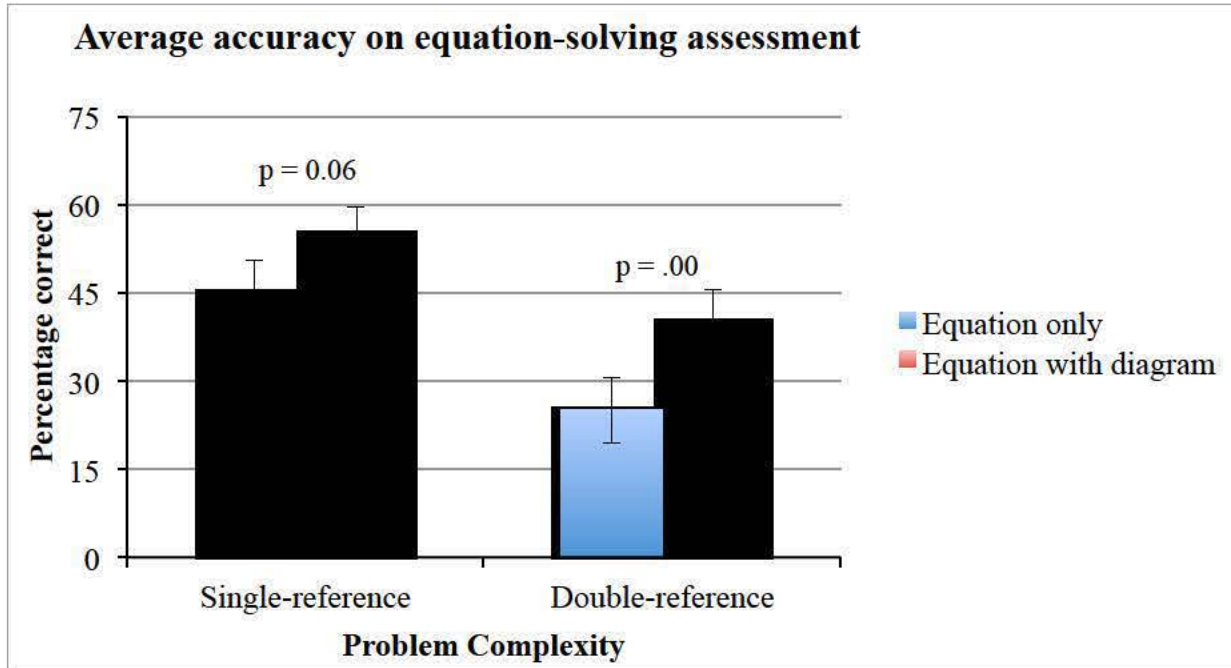
Note. Error bars represent standard errors.

** Data used to create the graph is below:

Average accuracy by problem complexity and presentation format

	Presentation Format	
Problem Complexity	Equation Only	Equation with Diagram
Single-reference	45 (5.5)	55 (5.5)
Double-reference	25 (4.5)	40 (5.5)

Note. N = 61. Scores are mean(SE)

Figure 3

Response times on equation-solving assessment by problem complexity and presentation format
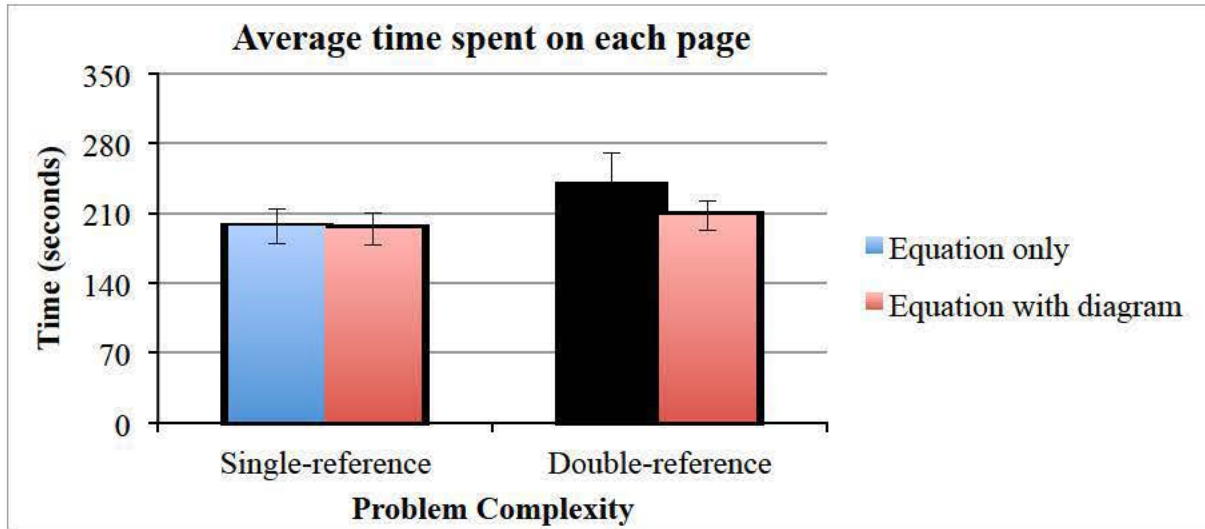
Note. N = 45. Scores are time in seconds taken to solve two problems. Error bars represent standard errors.

Data used to create the graph is below:

Average response time by problem complexity and presentation format

	Presentation Format	
Problem Complexity	Equation Only	Equation with Diagram
Single-reference	197 (17)	195 (16)
Double-reference	238 (33)	208 (15)

Note: N = 45. Scores are mean times with standard errors in parentheses

Figure 4.

Percentage of problems with different errors by presentation format

Frequency of errors
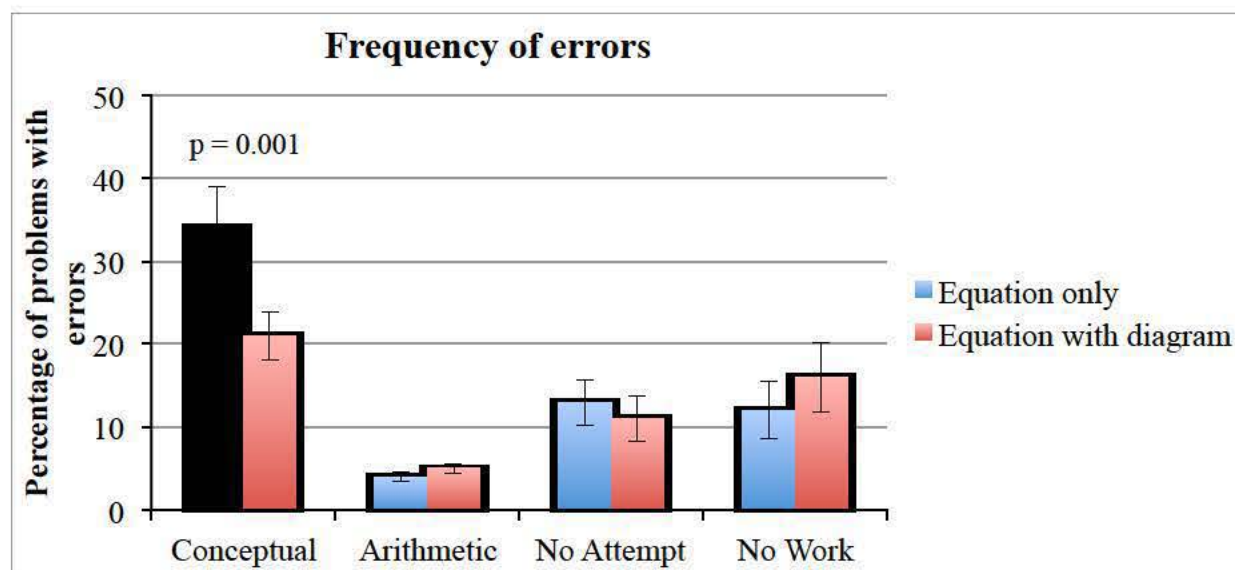
Note. Error bars represent standard errors.

Data used to create the graph is below:

Average frequency of different errors

	Presentation Format	
	Equation Only	Equation with Diagram
Conceptual	34 (4.0)	21 (3.0) *
Arithmetic	4 (1.3)	5 (1.4)
No Attempt	13 (3.0)	12 (3.0)
No Work	12 (3.3)	15 (3.7)
Copy	2 (0.9)	0 (-)

Note: Scores are percentages of all problems with that error, presented as means with standard errors in parentheses. * denotes differences at $p = 0.05$ level of significance.

Figure 5.
Percentage of problems with different errors by presentation format

Frequency of errors by problem complexity
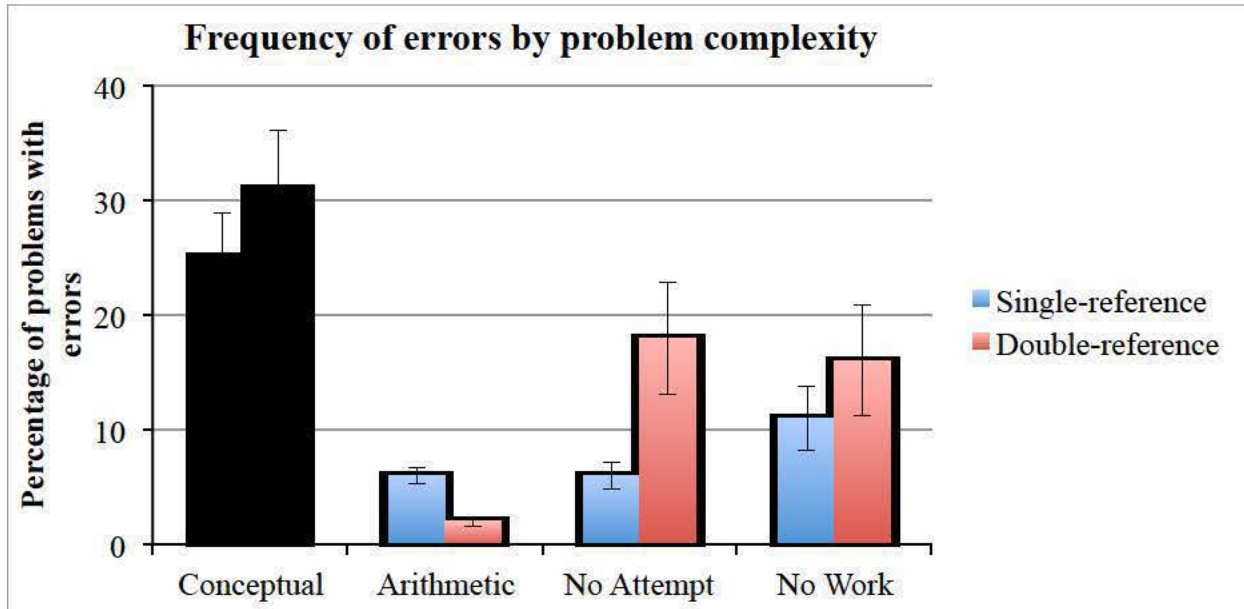
Note. Error bars represent standard errors.

Data used to create the graph is below:

Percentage of problems with different errors

	Problem Complexity	
	Single-reference	Double-reference
Conceptual	25 (3.9)	31 (5.1)
Arithmetic	6 (0.7)	2 (0.4)
No Attempt	6 (1.2)	18 (4.9) *
No Work	11 (2.8)	16 (4.8) *
Copy slip	2 (.01)	0 (-)

Note: Scores are percentages of all problems with that error, presented as means with standard errors in parentheses. * denotes differences at $p = 0.05$ level of significance.

Appendix 1.
Assessment, Form A.

Directions: For each diagram, find the unknown value.
Show all your work. Write your answer in the answer blank.

1.

$$13 \qquad 3$$

$$x = ?$$

Answer: $x =$ _____

2.

$$h = ? \quad 5$$

| h | | h | |
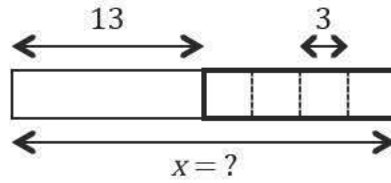
$$32$$
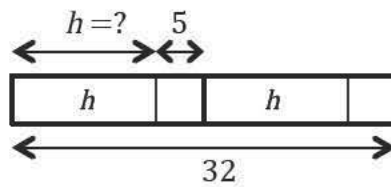
Answer: $h =$ _____

3.

Answer: $B =$ _____

4.

Answer: $Y =$ _____

STOP
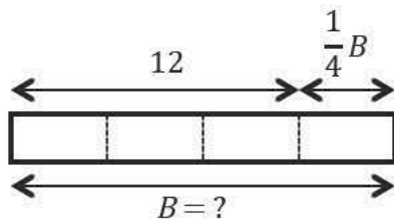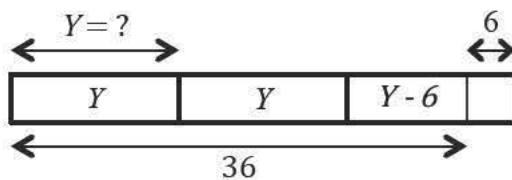
Wait for instructions from the teacher.
Do not turn the page until told to do so.

Directions: For each equation, <u>circle</u> the diagram that represents the same problem. Make sure the diagram captures all the information in the equation.

Practice.

$$x + 20 = 45$$
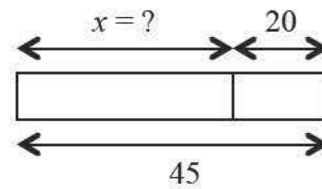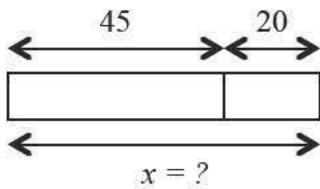
1.

$$(x - 13) \div 4 = 2.5$$
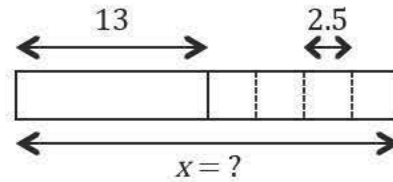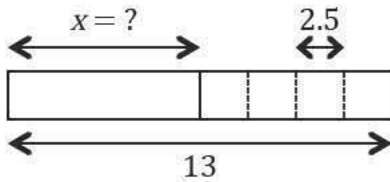
2.

$$x - \frac{2}{5}x = 21$$
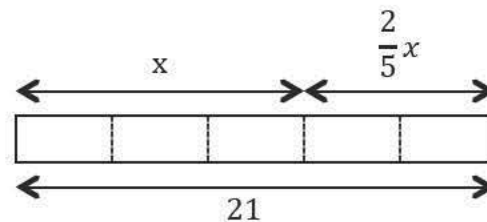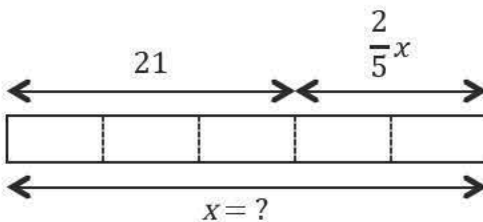
Write the time now before turning the page:

Directions: For each diagram, <u>write an equation</u> that represents the same problem. Do not solve it.

Example:
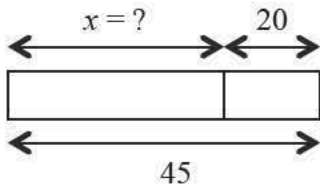
$$x + 20 = 45$$

1.
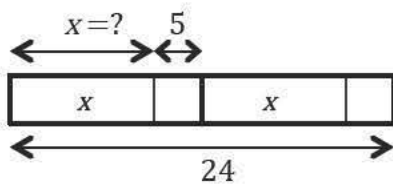
2.
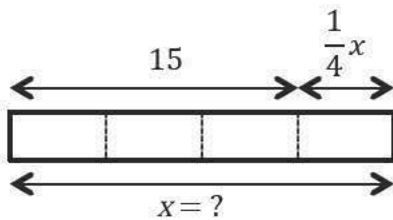
Write the time now before turning the page:

Directions for next section:
Work on each problem, show all your work, and write your answer in the answer blank.

Write the time inside the box at the bottom of each page before moving on.

A digital clock is shown on the screen.

Write the time now before turning the page:

Directions: Solve the following 4 equations. Show all your work. Write your answer in the answer blank.

1. $(x - 45) \div 3 = 20.5$

Answer: $x =$ _____

2. $3(t + 6) = 48$

Answer: $t =$ _____

Write the time now before turning the page:

3. $N - \frac{1}{5}N = 30$

Answer: $N =$ _____

4. $p + \frac{2}{3}p = 35$

Answer: $p =$ _____

⚠️ | Write the time now before turning the page:

Directions: Solve the following 4 equations. Show all your work. Write your answer in the answer blank. You may use the diagrams to help you. Circle YES or NO so we know if you used it.

5. $(y - 25.5) \div 5 = 9$

25.5   9
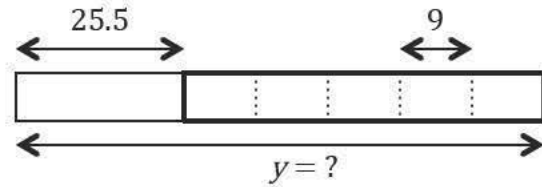
$y = ?$

Did you use the
diagram on
problem 5?

YES   NO

Answer: $y =$ _____

6. $4(q + 0.5) = 32$

$q = ?$ 0.5

| q | | q | | q | | q | |

32
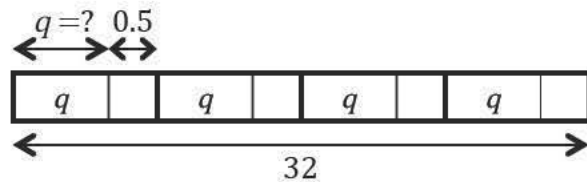
Did you use the
diagram on
problem 6?

YES   NO

Write the time now before turning the page:

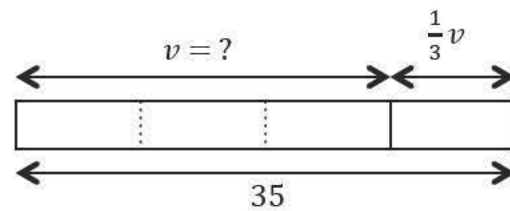7. $M - \frac{1}{3}M = 17$
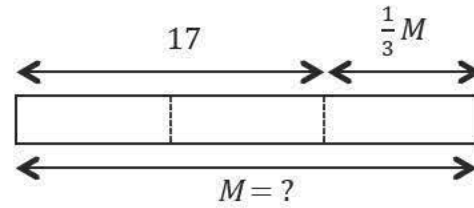
Answer: $M =$ _____

Did you use the
diagram on
problem 7?

YES   NO

8. $v + \frac{1}{3}v = 32$

Answer: $v =$ _____
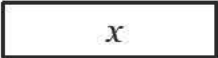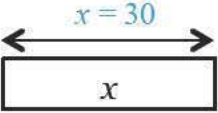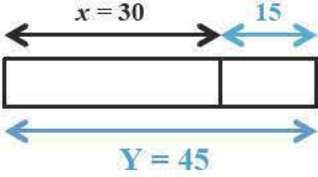
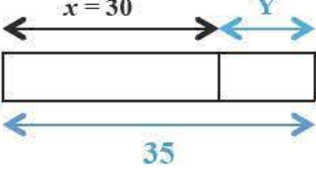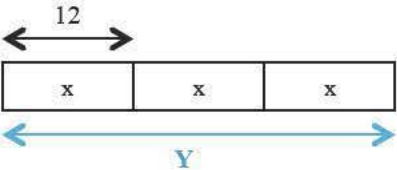Did you use the
diagram on
problem 8?

YES   NO

You have reached the end.
Write the time now:

Appendix 2.

Diagram instruction script.

SLIDE 1 – blank	Today, we will talk about some math ideas and solve some math problems using diagrams. First, you will learn about diagrams and how to read the information found inside a diagram. These diagrams are used in math lessons in other countries. You might find it new or difficult, but that's okay. This is not a test. The most important thing is to try to understand them and think about what the diagrams mean. These diagrams can help you to practice and improve your math skills.
SLIDE 2 - examples	Here are some examples of diagrams. Let's try to understand what the diagrams mean. You can think about diagrams as a special type of picture. Diagrams help you to show information about numbers and quantities.
SLIDE 3 x	In these diagrams, we use bars to represent quantities. This is similar to how we use letters to represent unknown quantities in equations. For example, this diagram (**Point to bar**) represents the unknown quantity "x". (**Point to X**). (**Click to next slide; show arrow**)
SLIDE 4 $x = 30$ x	This arrow tells us the *value* of the quantity X (**Point to arrow**). Looking at this diagram now, we don't know the value of X. However if we find the value of x is 30, we write x = 30. (**Write x = 30 above arrow**). Now, we are going to look at some examples. I want you to label your bar as we do this together.
SLIDE 5 - Addition $x = 30$ 15 $Y = 45$	Diagrams can represent addition and subtraction as part-whole relations. Look at the diagram for addition on your page. Notice that I added a smaller rectangle (**point to smaller rectangle**) so we have two rectangles (**point to both**) that combine to make a larger rectangle (**gesture over entire rectangle**). Now we have 3 quantities – two smaller parts, and one larger whole. We can use arrows to indicate each of their values (**draw arrow over smaller bar**). Let's label the smaller part as 15 (**write 15**) and the larger whole as Y (**Write "Y ="**). You do the same on your paper.

	What is the value of Y?
	   (Wait for answer. If correct, write 45).
	How did you get 45?
	   (Repeat what student says).

SLIDE 6 - Subtraction $x = 30$ Y 35	Here is a different problem on subtraction. Remember to label your diagrams too! Let's label the smaller part as Y **(write "Y=")** and the larger whole as 35. **(write 35)** Now, what is the value of Y? **(Wait for answer. If correct, write "5")**. So this is how diagrams can show you addition and subtraction problems

SLIDE 7 - multiplication
 	
12
 	
x   x   x
 	
Y
	Diagrams can also be used to show multiplication and division problems.
	
	Let's look at this diagram for multiplication.
	There are 3 equal parts, all with the value X. The value of X is shown in this diagram. What is the value of X? **(Wait for answer)**
	   (Repeat "Yes, X equals 12 because it is labeled there.)
	
	Okay. In this diagram, 3 equal parts combine to make a larger rectangle. We can draw an arrow over the entire bar and label this total as Y. **(Draw arrow, label "Y=")**.
	
	What is the value of Y?
	   (Wait for answer. If correct, label "36").
	How did you get that answer?
	   (Wait for strategy)
	"Right, Y equals 3 times X"