PARADOXES OF FAIR DIVISION*

Paradoxes, if they do not define a field, render its problems intriguing and often perplexing, especially insofar as the paradoxes remain unresolved. Voting theory, for example, has been greatly stimulated by the Condorcet paradox, which is the discovery by the Marquis de Condorcet[1] that there may be no alternative that is preferred by a majority to every other alternative, producing so-called cyclical majorities. Its modern extension and generalization is Arrow's[2] theorem, which says, roughly speaking, that a certain set of reasonable conditions for aggregating individuals' preferences into some social choice are inconsistent.

In the last fifty years, hundreds of books and thousands of articles have been written about these and related social-choice paradoxes and theorems, as well as their ramifications for voting and democracy. Hannu Nurmi[3] provides a good survey and classification of voting paradoxes and also offers advice on "how to deal with them."

There is also an enormous literature on fairness, justice, and equality, and numerous suggestions on how to rectify the absence of these properties or attenuate their erosion. But paradoxes do not frame the study of fairness in the same way they have inspired social-choice theory.

To be sure, the notion that justice and order may be incompatible, or that maximin justice in John Rawls's[4] sense undercuts the motivation of individuals to strive to do their best, underscores the possible trade-offs in making societies more just or egalitarian. For example, an egalitarian society may require strictures on free choice to ward off anarchy; rewarding the worst-off members of a society may deaden competition among the most able if their added value is siphoned off to others.

Obstacles like these which stand in the way of creating a just society are hardly surprising. They are not paradoxes in the strong sense of

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[1] *Essai sur l'application de l'analyse à la probabilité dés decisions rendues à la pluralité des voix* (Paris: De L'Imprimerie royale,1785).

[2] Kenneth J. Arrow, *Social Choice and Individual Values* (New Haven: Yale, 1951; 2nd ed., 1963).

[3] *Voting Paradoxes and How to Deal with Them* (Berlin: Springer, 1999).

[4] *A Theory of Justice* (Cambridge: Harvard, 1971).

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constituting a logical contradiction between equally valid principles. Here we use paradox in a weaker sense—as a conflict among fairness conditions that one might expect to be compatible. Because we are surprised to discover this conflict, it is "nonobvious," as one of us labeled a collection of paradoxes he assembled about politics.[5]

The fair-division paradoxes we present here all concern how to divide up a set of indivisible items among two or more players. In some paradoxes, we assume the players can do no more than rank the items from best to worst; in others, we assume they can, in addition, indicate preferences over subsets, or packages, of items. While our framework is generally an ordinalist one, we do admit one cardinalization of ranks, based on the Borda count used in voting, to facilitate certain comparisons, particularly those involving allocations with different numbers of items.

The main criteria we invoke are *efficiency* (there is no other division better for everybody, or better for some players and not worse for the others) and *envy-freeness* (each player likes her allocation at least as much as those which the other players receive, so she does not envy anybody else). But because efficiency, by itself, is not a criterion of fairness (an efficient allocation could be one in which one player gets everything and the others nothing), we also consider other criteria of fairness besides envy-freeness, including two different measures of how a worst-off player fares (maximin and Borda maximin), which we contrast with a utilitarian notion of overall welfare (Borda total score). What we rule out, besides the splitting of items, is the possibility of randomizing among different allocations, which is another way that has been proposed for "smoothing out" inequalities caused by the indivisibility of items.[6]

[5] Brams, *Paradoxes in Politics: An Introduction to the Nonobvious in Political Science* (New York: Free Press, 1976); see also Fishburn, "Paradoxes of Voting," *American Political Science Review*, LXVIII, 2 (June 1974): 537-46, and Fishburn and Brams, "Paradoxes of Preferential Voting," *Mathematics Magazine*, LVI, 4 (September 1983): 207-14.

[6] Fair-division procedures that allow for the splitting of (divisible) goods or the sharing of (indivisible) goods—possibly based on a randomization process that determines time shares—are discussed in, among other places, John Broome, *Weighing Goods: Equality, Uncertainty and Time* (Cambridge: Blackwell, 1991); H. Peyton Young, *Equity in Theory and Practice* (Princeton: University Press, 1994); Hervé Moulin, *Cooperative Economics: A Game-Theoretic Introduction* (Princeton: University Press, 1995); Brams and Alan D. Taylor, *Fair Division: From Cake-Cutting to Dispute Resolution* (New York: Cambridge, 1996); Jack Robertson and William Webb, *Cake-Cutting Algorithms: Be Fair If You Can* (Natick, MA: A. K. Peters, 1998); and Brams and Taylor, *The Win-Win Solution: Guaranteeing Fair Shares to Everybody* (New York: Norton, 1999).

Our paradoxes demonstrate the opportunities as well as the limitations of fair division. Thus, for example, while the only division of items in which one player never envies the allocation of another may be nonexistent or inefficient, we note that there is always an efficient and envy-free division for two players—even when they rank all items the same—as long as they do not rank all subsets of items the same. We also show that fair division may entail an unequal division of the items.

We divide the paradoxes into three categories:

(1) The conflict between efficiency and envy-freeness (paradoxes 1 and 2).
(2) The failure of a unique efficient and envy-free division to satisfy other criteria (paradoxes 3 and 4).
(3) The desirability, on occasion, of dividing items unequally (paradoxes 5, 6, 7, 8).

While the paradoxes highlight difficulties in creating "fair shares" for everybody, they by no means render the task impossible. Rather, they show how dependent fair division is on the fairness criteria one deems important and the trade-offs one considers acceptable. Put another way, achieving fairness requires some consensus on the ground rules (that is, criteria) and some delicacy in applying them (to facilitate trade-offs when the criteria conflict).

We mention three technical points before we proceed to specific examples. First, we assume that players cannot compensate each other with side payments—the division is only of the indivisible items. Second, all players have positive values for every item. Third, a player *prefers* one set *S* of items to a different set *T* if (i) *S* has as many items as *T* and (ii) for every item *t* in *T* and not in *S*, there is a distinct item *s* in *S* and not *T* that the player prefers to *t*. For example, if a player ranks items 1 through 4 in order of decreasing preference 1234, we assume that she prefers

- the set {1,2} to {2,3}, because {1} is preferred to {3}, and
- the set {1,3} to {2,4}, because {1} is preferred to {2} and {3} is preferred to {4}

whereas the comparison between sets {1,4} and {2,3} could go either way.

I. EFFICIENCY AND ENVY-FREENESS: THEY MAY BE INCOMPATIBLE

Paradox 1: a unique envy-free division may be inefficient. Suppose there is a set of three players, {*A*, *B*, *C*}, that must divide a set of six indivisible items, {1, 2, 3, 4, 5, 6}. Assume the players strictly rank the items from best to worst as follows:

Example I:
A: 1 2 3 4 5 6
B: 4 3 2 1 5 6
C: 5 1 2 6 3 4

The *unique* envy-free allocation to (*A*, *B*, *C*) is ({1,3}, {2,4}, {5,6}), or for simplicity (13, 24, 56), whereby *A* and *B* get their best and third-best items, and *C* gets her best and fourth-best items. Clearly, *A* prefers her allocation to that of *B* (which are *A*'s second-best and fourth-best items) and that of *C* (which are *A*'s two worst items). Likewise, *B* and *C* prefer their allocations to those of the other two players. Consequently, the division (13, 24, 56) is envy-free: all players prefer their allocations to those of the other two players, so no player is envious of any other.

Compare this division with (12, 34, 56), whereby *A* and *B* receive their two best items, and *C* receives, as before, her best and fourth-best items. This division *Pareto-dominates* (13, 24, 56), because two of the three players (*A* and *B*) prefer the former allocation, whereas both allocations give player *C* the same two items (56).

It is easy to see that (12, 34, 56) is Pareto-optimal, or efficient: no player can do better with some other division without some other player or players doing worse. This is apparent from the fact that the only way *A* or *B*, who get their two best items, can do better is to receive an additional item from one of the two other players—assuming all items have some positive value for the players—but this will necessarily hurt the player who then receives fewer than her present two items. Whereas *C* can do better without receiving a third item if she receives item 1 or 2 in place of item 6, this substitution would necessarily hurt *A*, which will do worse if she receives item 6 for item 1 or 2.

The problem with efficient allocation (12, 34, 56) is that it is not *assuredly* envy-free. In particular, *C* will envy *A*'s allocation of 12 (second-best and third-best items for *C*) if she prefers these two items to her present allocation of 56 (best and fourth-best items for *C*). In the absence of information about *C*'s preferences for subsets of items, therefore, we cannot say that efficient allocation (12, 34, 56) is envy-free.[7]

[7] Henceforth we shall mean by 'envy-free' a division such that, no matter how the players value subsets of items consistent with their rankings, no player prefers any other player's allocation to her own. If a division is not envy-free, we call it *envy-possible* if a player's allocation *may* make her envious of another player, depending on how she values subsets of items, as illustrated by division (12, 34, 56) in the text. It is *envy-ensuring* if it causes envy, independent of how the players value subsets

But the real bite of paradox 1 stems from the fact that not only is inefficient division (13, 24, 56) envy-free, but it is uniquely so—there is no other division, including an efficient one, that guarantees envy-freeness. To show this in example I, note first that an envy-free division must give each player her best item; if not, then a player might prefer a division, like envy-free division (13, 24, 56) or efficient division (12, 34, 56), that does give each player her best item, rendering the division that does not envy-possible or envy-ensuring. Second, even if each player receives her best item, this allocation cannot be the only item she receives, because then the player might envy any player that receives two or more items, *whatever* these items are.

By this reasoning, then, the only possible envy-free divisions in example I are those in which each player receives two items, including her top choice. It is easy to check that no efficient division is envy-free.[8] Similarly, one can check that no inefficient division, except (13, 24, 56) that gives each player two items—including her best—is envy-free, making this division uniquely envy-free.

Paradox 2: there may be no envy-free division, even when all players have different preference rankings. While it is bad enough when the only envy-free division is inefficient (paradox 1), it seems even worse when there is no envy-free division. This is trivial to show when players rank items the same. For example, if two players both prefer item 1 to item 2, then the player that gets item 2 will envy the player that gets item 1.

In the following example, each of three players has a different ranking of three items:

Example II:
A: 1 2 3
B: 1 3 2
C: 2 1 3

There are three divisions, in which each player gets exactly one item, which are efficient—(1, 3, 2), (2, 1, 3), and (3, 1, 2)—in each of which at least one player gets her best item. It is evident that none is

of items. In effect, a division that is envy-possible has the potential to cause envy. By comparison, an envy-ensuring division always causes envy, and an envy-free division never causes envy.

[8] We previously showed that division (12, 34, 56) is not envy-free. As another example, consider efficient division (16, 34, 25). Whereas neither *B* nor *C* envies each other or *A*, *A* might prefer either *B*'s 34 or *C*'s 25 allocations, making this division envy-possible.

envy-free, because the player that gets item 1 in each (*A* or *B*) will be envied by at least one of the other two players. For instance, in the case of the division (2, 1, 3), both *A* and *C* will envy *B*.

Can an inefficient division be envy-free, as was the case in example I? It is not hard to see that this situation cannot occur in example II for the reason given above: the player that gets item 1 will be envied. But in the case of an inefficient division, "trading up to efficiency" reduces the *amount* of envy. For example, consider inefficient division (2, 3, 1), in which each player receives her second-best choice. Because *A* envies *C*, *B* envies *C*, and *C* envies *A*, a trade of items 1 and 2 between *A* and *C* is possible. It yields efficient division (1, 3, 2), in which only *B* envies *A*.

Besides (2, 3, 1), the other two inefficient divisions—(1, 2, 3) and (3, 2, 1)—also allow for trading up to efficiency. In the first, a trade of items 2 and 3 between *B* and *C* yields efficient division (1, 3, 2); in the second, a trade of items 1 and 2 between *B* and *C* yields efficient division (3, 1, 2). Three-way trades are also possible. For instance, starting from inefficient division (3, 2, 1), a three-way trade, whereby *A* sends item 3 to *B*, *B* sends item 2 to *C*, and *C* sends item 1 to *A*, yields efficient division (1, 3, 2).

Trading up to efficiency is also possible in example I: by exchanging items 2 and 3, *A* and *B* can turn inefficient division (13, 24, 56) into efficient division (12, 34, 56). As in example II, however, no efficient division is envy-free in example I. The difference between examples I and II is that example II does not admit even an inefficient envy-free division.

II. UNIQUE EFFICIENT AND ENVY-FREE DIVISIONS: THEIR INCOMPATIBILITY WITH OTHER CRITIERIA

Paradox 3: a unique efficient and envy-free division may lose in voting to an efficient and envy-possible division. So far we have shown that efficiency and envy-freeness may part company either by there being no envy-free division that is also efficient (example I), or no envy-free division at all (example II). But when these properties coincide, and there is both an efficient and an envy-free division, it may not be the choice of a majority of players, as illustrated by the following example:

Example III:
A: 1 2 3 4 5 6
B: 5 6 2 1 4 3
C: 3 6 5 4 1 2

There are three efficient divisions in which (*A*, *B*, *C*) each get two items: (12, 56, 34); (12, 45, 36); and (14, 25, 36). But only the third division, (14, 25, 36), is envy-free. Whereas *C* might prefer *B*'s 56 allocation in the first division, and *B* might prefer *A*'s 12 allocation in the second division, no player prefers another player's allocation in (14, 25, 36).

But observe that both *A* and *B* prefer the first division, (12, 56, 34), to the envy-free third division, (14, 25, 36), because they get their top two items in the first division; only *C* gets her top two items in (14, 25, 36). Hence, the first division would defeat the envy-free third division, (14, 25, 36), by simple majority rule.

The situation is not so clear-cut when we compare the second division, (12, 45, 36), with the envy-free (14, 25, 36). In fact, there would be a tie vote: *C* would be indifferent, because she gets her top two items, 36, in each division; *A* would prefer the second division (top two items versus best and fourth-best items); and *B* would prefer the envy-free division, (14, 25, 36) (best and third-best items versus best and fifth-best items).

Thus, if there were a vote, the unique envy-free division, (14, 25, 36), would lose to the envy-possible division, (12, 56, 34), and it would tie with the other envy-possible division, (12, 45, 36). If there were "approval voting,"[9] and *A*, *B*, and *C* voted only for the divisions that give each player her two best items, then the envy-free division, (14, 25, 36), would get 1 vote, compared to 2 votes each for both of the envy-possible divisions, (12, 56, 34) and (12, 45, 36). In sum, players will choose an envy-possible over the unique envy-free division, (14, 25, 36), in either pairwise comparisons or approval voting.

Paradox 4: neither the Rawlsian maximin criterion nor the Borda total-score criterion may choose a unique efficient and envy-free division. Besides using voting to select an efficient division, consider the following Rawlsian maximin criterion to distinguish among efficient divisions: choose the division that maximizes the minimum rank of items that players receive, making a worst-off player as well off as possible. To illustrate in example III, envy-possible division (12, 45, 36) gives a fifth-best item to *B*, whereas each of the two other efficient divisions gives a player, at worst, a fourth-best item. Between the latter two divisions, the envy-possible division, (12, 56, 34) is, arguably, better than the envy-free division, (14, 25, 36), because it gives the other two players—those which do not get a fourth-best item—their two best

[9] Under approval voting, voters can vote for as many alternatives as they like; each alternative approved of receives one vote, and the alternative with the most votes wins. See Brams and Fishburn, *Approval Voting* (Boston: Birkhäuser, 1983).

items, whereas envy-free division (14, 25, 36) does not give *B* her two best items.[10]

A modified Borda count would also give the nod to the envy-possible division, (12, 56, 34), compared not only with the envy-free division, (14, 25, 36), but also with the other envy-possible division, (12, 45, 36). Awarding 6 points for obtaining a best item, 5 points for obtaining a second-best item,..., 1 point for obtaining a worst item in example III, the latter two divisions give the players a total of 30 points, whereas envy-possible division (12, 56, 34) gives the players a total of 31 points,[11] which we call their *Borda total scores* and use as a measure of the *overall* utility or welfare of the players. Hence an envy-possible division beats the unique envy-free division, based on both the maximin criterion and the Borda total-score criterion. (We shall later apply the Borda count to individual players, asking what division maximizes the minimum Borda score that any player receives.)

III. THE DESIRABILITY OF UNEQUAL DIVISIONS (SOMETIMES)

Paradox 5: an unequal division of items may be preferred by all players to an equal division. In section II, we showed that neither (i) pairwise comparison voting or approval voting (paradox 3), nor (ii) the maximin criterion or the Borda total-score criterion (paradox 4), always selects a unique efficient and envy-free division. In the following example, there is also a unique efficient and envy-free division—in which all players receive the same number of items (henceforth called an *equal division*)—but there may be grounds for choosing an efficient but unequal envy-possible division:

Example IV:
A: 1 2 3 4
B: 2 3 4 1

[10] This might be considered a second-order application of the maximin criterion: if, for two divisions, players rank the worst item any player receives the same, consider the player that receives a next-worst item in each, and choose the division in which this item is ranked higher. This is an example of a *lexicographic decision rule*, whereby alternatives are ordered on the basis of a most important criterion; if that is not determinative, a next-most important criterion is invoked, and so on, to narrow down the set of feasible alternatives.

[11] The standard scoring rules for the Borda count in this 6-item example would give 5 points to a best item, 4 points to a second-best item,..., 0 points to a worst item. We depart slightly from this standard scoring rule to ensure that each player obtains some positive value for all items, including her worst choice, as assumed earlier.

It is not difficult to show that (13, 24) is the only efficient and envy-free division. Two other equal divisions, (12, 34) and (14, 23), while better for one player and worse for the other, are envy-possible.

The above three equal divisions all give Borda total scores of 12 to their players. If we eliminate the envy-possible division, (14, 23), on the grounds that it fails the maximin criterion by giving *A* her worst item (item 4), then the comparison reduces to that between envy-free division, (13, 24), and envy-possible division, (12, 34).

Curiously, it is possible that *both* *A* and *B* prefer the unequal envy-possible division, (134, 2), to the equal envy-possible division, (12, 34).[12] Thus, unequal divisions might actually be better for all players than equal divisions.

Ruling out equal division (12, 34) in such a situation, let us compare (134, 2) with the envy-free (equal) division (13, 24). Clearly, (134, 2) is better than (13, 24) for *A*, but it is worse for *B*.

This leaves open the question of which of these two divisions, involving an equal and an unequal division of the items, comes closer to giving the two players "fair shares." As the next paradox shows, an unequal division may actually be more egalitarian—as measured by Borda scores for individual players—than an equal division.

Paradox 6: an unequal division of items may (i) maximize the minimum Borda scores of players (Borda maximin) and (ii) maximize the sum of Borda scores (Borda maxsum). In paradox 5, we showed that an unequal but envy-possible division of items may compare favorably with an equal and envy-free division. To make this kind of comparison more precise, consider the following example:

Example V:
A: 1 2 3 4 5 6 7 8 9
B: 3 1 2 4 5 6 7 8 9
C: 4 1 2 3 6 5 7 8 9

There are exactly two unequal divisions, (12, 357, 4689) and (12, 3589, 467), that maximize the minimum Borda scores of players, which are [17, 17, 17] for both divisions.[13] On the other hand, there are two equal divisions, (129, 357, 468) and (129, 358, 467), that maximize the minimum Borda scores of players, which are [18, 17, 16] for the first division and [18, 16, 17] for the second division. These are all the divisions whose Borda total scores are 51, which, it

[12] This is true if *A* prefers 34 to 2, and *B* prefers 2 to 34.

[13] Henceforth we shall indicate the Borda scores of players [in brackets] to distinguish them from item allocations (in parentheses).

can be shown, is the maximal sum, or *Borda maxsum*, among all possible divisions (equal or unequal).

Notice that the worst-off player in the two unequal divisions garners 17 points (so does the best-off player, because the Borda scores of all players are the same), whereas the worst-off player in the two equal divisions receives fewer points (16). By the maximin criterion, but now based on Borda scores, the unequal divisions are more egalitarian. We call this the *Borda maximin* criterion, which is especially useful in comparing equal and unequal divisions.[14]

None of the four equal or unequal divisions is envy-free—all are envy-possible or envy-ensuring. Likewise, all four divisions are "efficient-possible" in the sense that there may be a more efficient division, but this is not guaranteed. Take, for example, the unequal division (12, 357, 4689). *B* or *C* might prefer *A*'s 12 allocation, just as *A* might prefer *B*'s or *C*'s allocation, so a trade could make two, or even all three, players better off. Unlike our previous examples, in which divisions called "efficient" were all "efficient-ensuring" (that is, there were no trades that could improve the lot of all traders, however players valued subsets of items), this is not the case in example V.

The Borda maximin criterion seems a reasonable one to distinguish among all efficient-possible and envy-possible divisions. In example V, it is not only unequal divisions that do best on this criterion, but these divisions are also Borda maxsum, making them both fair and utility-maximizing (according to the Borda cardinalization of utility—more on its limitations, illustrated in example VII, later).

Paradox 7: an unequal division of items may be Borda maxsum but not Borda maximin. There was no conflict between Borda maxsum and Borda maximin in example V—two unequal divisions satisfied both these properties. But as the next example illustrates, this need not be the case:

Example VI:
A: 1 2 3
B: 1 3 2

[14] To see why, consider the two unequal divisions in example IV in which no player receives a fourth-best item: (1, 234) and (123, 4). To call these divisions maximin—like equal division (13, 24), in which no player receives a fourth-best item as well—seems highly questionable, because the player receiving her top three items in these two divisions can hardly be considered worse off (because she receives a third-best item) than the player receiving only her top item. Indeed, the Borda scores of the players in the two unequal divisions, [4, 9] and [9, 2], reveal how inegalitarian these divisions are, particularly when compared with equal Borda maximin division (13, 24) with Borda scores of [6, 6].

There are two unequal maxsum divisions, (12, 3) and (2, 13), whose Borda scores are, respectively, [5, 2] and [2, 5]. Each gives a Borda total score of 7, and a minimum Borda score of 2 for a player.

By contrast, there are two Borda maximin divisions, (1, 23) and (23, 1), both of which give Borda scores of [3, 3]. While they give the players a lower Borda total score (6) than the Borda maxsum divisions, they give the players a higher minimum score of 3.

Presumably, the egalitarian would choose one of the two Borda maximin divisions, whereas the utilitarian would choose one of the two Borda maxsum divisions. Because there are an odd number of items to divide in example VI, all the divisions between *A* and *B* are necessarily unequal. But both Borda maximin divisions and Borda maxsum divisions can be the either equal (unlike example VI, the Borda maximin division in example IV is equal, as discussed in footnote 14) or unequal (two of the Borda maxsum divisions in example V are equal and two are unequal). It also turns out that Borda maxsum and Borda maximin scores can be arbitrarily far apart.[15]

We believe that when Borda maximin and Borda total scores choose different divisions, Borda maximin generally gives the fairer division by guaranteeing that the Borda score of the worst-off player is as great as possible.[16] As we shall show in our final paradox, however, a Borda maximin division may be quite implausible, depending on how players value subsets; or it may not be envy-free when, at the same time, there exists an envy-free division that is neither maximin nor Borda maximin.

Paradox 8: if there are envy-free divisions, none may be maximin or Borda maximin. In the following example, there are two players but an odd number of items, so no equal division of the items is possible:

Example VII:
A: 1 2 3 4 5
B: 1 2 3 4 5

[15] Brams, Edelman, and Fishburn, "Fair Division of Indivisible Items" (unpublished).

[16] To be sure, assuming that the differences in ranks are all equal, as Borda scoring does, is a simplification. If cardinal utilities could be elicited that reflect the players' intensities of preference, then these utilities—instead of the rank scores—could be used to equalize, insofar as possible, players' satisfaction with a division of the items. For fair-division bidding schemes that incorporate cardinal information, see Brams and Taylor, *Fair Division* and *The Win-Win Solution*; Brams and D. Marc Kilgour, "Competitive Fair Division," *Journal of Political Economy*, CIX, 2 (April 2001): 418-43; and Claus-Jochen Haake, Matthias G. Raith, and Francis E. Su, "Bidding for Envy-Freeness: A Procedural Approach to *n*-Player Fair-Division Problems," *Social Choice and Welfare* (forthcoming).

Because the players rank the items exactly the same, all divisions are efficient, making the choice of a fairest one appear difficult.

Only six divisions, however, are what Brams and Fishburn[17] call *undominated splits*:

(1, 2345); (12, 345); (13, 245); (14, 235); (15, 234); (145, 23)

These divisions are those in which, in the absence of information about preferences over subsets, either of the two allocations in each might be preferred by a player, making each undominated. All these divisions, therefore, are envy-possible.

The Borda maximin divisions are (13, 245) and (14, 235), which give Borda scores of, respectively, [8, 7] and [7, 8] to the players. But neither division might be envy-free if, say, *both* players prefer allocation 13 to 245 in the first and allocation 14 to 235 in the second—that is, both prefer the "same side" of each division. These preferences imply that both prefer allocation 12 to 345 in the second division, (12, 345), and allocation 145 to 23 in the sixth division, (145, 23), precluding these divisions, as well, from being envy-free.

Thus, the preferences of *A* and *B* assumed above would eliminate four of the undominated splits from being envy-free, allowing the two remaining divisions to be so. For example, *A* might prefer allocation 1 in the first division and allocation 15 in the fifth, whereas *B* might prefer the complements: allocation 2345 in the first, and allocation 234 in the fifth.

In none of our previous examples with envy-free divisions was such a division not Borda maximin. But as we have just illustrated, there may be several envy-free divisions, none of which is Borda maximin. This divergence points to the limitation of Borda maximin as a criterion for choosing divisions, because Borda scoring may not reflect the intensity of player preferences that can be better gleaned from player preferences over subsets.

But we do not need to know player preferences over subsets to show that an envy-free division may not be Borda maximin:

Example VIII:
A: 1 2 3 4 5 6 7 8 9
B: 5 8 1 2 6 7 3 4 9
C: 3 4 9 1 2 5 6 7 8

[17] "Fair Division of Indivisible Items between Two People with Identical Preferences: Envy-freeness, Pareto-optimality, and Equity," *Social Choice and Welfare*, XVII, 2 (February 2000): 247-67.

It is easy to see that division (127, 568, 349) is envy-free, but it gives a seventh-best item to *A*. By contrast, division (126, 587, 349) gives sixth-best items to *A* and *B* and the same 349 allocation to *C*.

Because no other divisions (equal or unequal) give players lowest-ranked items that are as high as sixth-best, division (126, 578, 349) is maximin. But it is not envy-free: *B* may envy *A*, because she may prefer allocation 126 to 578, making this division envy-possible.

The Borda scores of the envy-free division are [20, 22, 24], whereas those of the maximin division are [21, 21, 24], so the maximin division is also Borda maximin. Both the envy-free and maximin/Borda maximin divisions have total Borda scores of 66, which is also the Borda maxsum in example VIII.

This example illustrates what we think is our most striking paradox. Specificially, without any special assumptions about the preferences of the players for subsets of items, it shows the clash between envy-freeness and both maximin and Borda maximin. Furthermore, because there are no unequal divisions in example VIII that satisfy any of our fairness criteria—or, for that matter, the Borda maxsum criterion—it highlights the difficulty of choosing a fairest allocation, even in the equal-division case: Should one help the worst-off, or avoid envy, when one cannot do both?[18]

[18] This question is more fully explored in Brams and Daniel King, "Efficient Fair Division: Help the Worst Off or Avoid Envy?" (unpublished), in which it is shown not only that maximin and Borda maximin divisions may not be envy-free but also that *all* such divisions may actually ensure envy. To illustrate this conflict, consider the following example:

A: 1 2 3 4 5 6
B: 1 2 3 4 5 6
C: 1 5 4 6 2 3

There are four Borda maximin divisions—(14, 23, 56), (23, 14, 56), (13, 24, 56), (24, 13, 56), each giving a minimum Borda score of 8 to a player—which are also maximin divisions (a worst-off player receives a fourth-best item). In addition, there are two maximin divisions which are not Borda maximin divisions—(12, 34, 56), (34, 12, 56)—which also give a worst-off player a fourth-best item. All six divisions ensure envy: in each, one player prefers another player's two items to her own. This example demonstrates that maximin and Borda maximin divisions, rather than just preclude envy-freeness, may *guarantee* envy (that is, be envy-ensuring rather than just envy-possible, as in example VIII). Furthermore, unlike example II, in which the unique efficient maximin and Borda maximin division, (1, 3, 2), is also envy-ensuring, the present example involves each player's receiving two items, which one might think would be sufficient to allow a maximin or Borda maximin division to be envy-possible, if not envy-free. This, however, is not the case, underscoring the seriousness of the conflict among our fairness criteria.

IV. CONCLUSIONS

The eight paradoxes pinpoint difficulties in dividing up indivisible items so that each player feels satisfied, in some sense, with her allocation. The first two paradoxes show that efficient and envy-free divisions may be incompatible because the only envy-free division may be inefficient, or there may be no envy-free division at all.

Both of these paradoxes require at least three players.[19] When there are only two players, even when they rank items exactly the same, it turns out that efficient and envy-free divisions can always be found, except when the players have the same preferences over all subsets of items.[20]

But the existence of even a unique efficient and envy-free division may not be chosen by the players for other reasons. In particular, such a division will not necessarily be selected when players vote for the division or divisions that they prefer. Also, a unique efficient and envy-free division will not necessarily be the division that maximizes the minimum rank of items that players receive, so the Rawlsian maximin criterion of making the worst-off player as well off as possible may not single it out.

As a way of measuring the value of allocations to find those divisions which are most egalitarian, especially in comparing equal and unequal divisions, we used Borda scoring based on player rankings of the items. We showed that a Borda maximin division may not be a Borda maxsum division, indicating the possible conflict between egalitarian and utilitarian outcomes.

This difference may show up when there are as few as two players dividing up three items, making it impossible to divide the items equally between the players. But even when this is possible, unequal divisions of items may be the only ones that satisfy the Borda maximin criterion. While indicating a preference for this criterion over the Borda maxsum criterion when the two clash, we illustrated how Borda maximin divisions may fail badly in finding envy-free divisions. Indeed, there may be no overlap between Borda maximin and envy-free divisions.

Our purpose is not just to indicate the pitfalls of fair division by exhibiting paradoxes that can occur. There are also opportunities, but these depend on the judicious application of selection criteria when not all criteria can be satisfied simultaneously.

[19] Edelman and Fishburn, "Fair Division of Indivisible Items among People with Similar Preferences," *Mathematical Social Sciences*, XLI, 3 (May 2001): 327-47.

[20] Brams and Fishburn, "Fair Division of Indivisible Items between Two People with Identical Preferences: Envy-freeness, Pareto-Optimality, and Equity."

Several recent papers have suggested constructive procedures for finding the most plausible candidates for fair division of a set of indivisible items.[21] We find this direction promising, because it is potentially applicable to ameliorating, if not solving, practical problems of fair division—ranging from the splitting of the marital property in a divorce to determining who gets what in an international dispute. While some conflicts are ineradicable, as the paradoxes demonstrate, the trade-offs that best resolve these conflicts are by no means evident.

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[21] Brams and Fishburn, "Fair Division of Indivisible Items between Two People with Identical Preferences: Envy-freeness, Pareto-Optimality, and Equity"; Edelman and Fishburn; Brams, Edelman, and Fishburn; Brams and King; and Dorothea Herreiner and Clemens Puppe, "A Simple Procedure for Finding Equitable Allocations of Indivisible Goods," *Social Choice and Welfare* (forthcoming).