CLASSROOM GROUPS LEARNING MATHEMATICS TOGETHER

By

Isaac Nichols-Paez

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Approved:

Rogers Hall, Ph.D.

Anita Wager, Ph.D.

Corey Brady, Ph.D.

Rich Lehrer, Ph.D.

I dedicate this dissertation to my perfectly imperfect family: Angela, Jacobo, Annabel, and Astro.

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CHAPTER 1

Introduction

1.1 Phenomenon of Interest and Its Importance

Within this dissertation, I use design-based research along with discourse and interaction analysis to study the dialectic relationship between conceptual neighborhoods (Nichols-Paez, 2018, 2019) and the forming tiny disciplinary public (Fine, 2012). Specifically, I focus on the phenomenon of classroom groups building conceptual and epistemic communities arguing that group learning is fundamentally different than individual learning, requiring new metaphors and conceptualizations of learning. Additionally, understanding classrooms as a unit is important because groups are primary for identity formation and learning (Lave and Wenger, 1991), essential for the production of cultural practices (Fine, 2012), and vital for rehumanizing mathematics (Gutiérrez, 2018).

1.2 Literature

For the last 30 years, educational researchers have been trying to better understand how to think about and support classrooms learning mathematics. Since the field's turn towards the social aspects of learning of mathematics, researchers have been working to better understand how the classroom work as a unit to create a context for students to learn. Yackel and Cobb (1996) made a major contribution to this work by conceptualizing sociomathematical norms, specific social norms which support classroom groups to engage in mathematics together. They describe classroom's development of sociomathematical norms as similar to individuals' mathematical beliefs through negotiation of taken-as-shared activity. In this way, they perpetuate using metaphors of individual learning to understand collective learning. Instead, I propose looking into metaphors of collective activity to describe collective learning. From a differing approach, Stroup et al. (2005) considered the dialectic relationship of the social and mathematical describing the social sphere as structuring the mathematics and the mathematical sphere structuring the social dynamics. Yet for the mathematical sphere structuring the social, they focus on activity structure and technological design and the primary driver, and they under theorize how this structuring plays out in more general classrooms. Ma and Hall (2018) conceptualize a type of learning unique to groups as "ensemble learning" which must satisfy three characteristics: learning must happen in group performance, performance requires a group, and participants understand performance is done as a group. Satisfying this last characteristic is especially hard in classroom because of the modularized forms of engagement and the lack of a shared enterprise. Yet, we can see classrooms as performing collectives, learning through performance of being a 'class' which requires a group to do. To add to this literature, I use three major systems which operate within classrooms as lenses into the dialectic of classroom neighborhoods and tiny publics: social systems, representational systems, and technological systems. I devote a paper to each of these systems and below is a visual representation of the systems working together and contributions from paper.

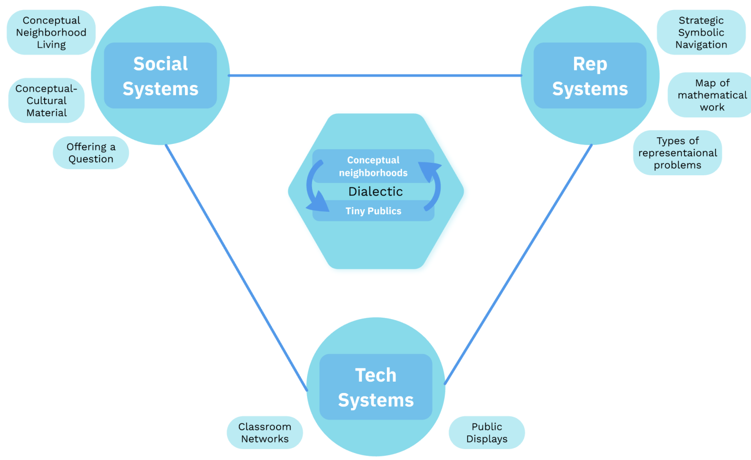

Figure 1.1: Contribution Diagram

1.3 Contribution of Paper 1

The first paper focuses on social structures and processes within classroom groups learning mathematics together. It make three main contributions. First, I develop the idea of *conceptual neighborhood living*, the aspect of classroom idioculture (Fine, 2012) specific to students building conceptual neighborhoods (Nichols-Paez, 2019). Through classrooms conceptual neighborhood living, they produce cultural items, what I call *conceptual-cultural material*. Finally, I analyze the development of one cultural item from a 2nd grade classroom which I call 'offering a question' as a rare case of convivial (Illich, 1973) building-out. The second graders learn how to take up and refine ill-structured mathematical questions and turn them into group inquiry. This fried chicken provides a new way of seeing classroom life and pathways to a more equitable, just, and rehumanizing mathematics (Gutiérrez, 2018).

1.4 Contribution of Paper 2

The second paper focuses on a way to see and understand a classroom group's representational practice. It contributes a new way to map and describe students' mathematical work with representations highlighting when students strategically use representational systems to navigate around obstacles. I call this practice *strategic symbolic navigation*. I analyze three classroom episodes of students wielding representational systems strategically, first lead by the teacher, then collectively lead, and finally lead by a student. This process demonstrates how the class is appropriating strategic symbolic navigation into their ways of symbolizing (Nemirovsky, 1994) repertoire. Finally, this framework provides a way to categorize types of obstacles which students can navigate.

1.5 Contribution of Paper 3

The third paper focuses on the needs that technology fills for classroom engaged in activity structures which rely on collective pursuit of mathematical ideas, specifically generative activities (Stroup et al., 2002). By comparing the same classroom working on very similar activities with and without technology, I am able to determine the specific strains they activity structure put on the system and the specific solutions the technology provides. I determined the activity structure put strains on the natural communication structure of the classroom, namely turn-based talk, because multiple students wanted to read out a list of up to thirty fractions. The technology allowed for students to see an aggregated version of their work in real time. This public and dynamic display is a key part of classroom networks (Brady et al., 2013) as a representational and communications infrastructure (Hegedus and Moreno-Armella, 2009), and they alleviated the strains put on the classroom system by the generative activities. This finding further supports the use and design of classroom networks and poses the question of how such technology might be used in support of social and representational systems.

1.6 Building a Foundation for Future Work

Collectively, these three papers create a map of the different systems within classroom as they learn mathematics together. Each paper focuses on a specific system while connecting it to the others. Taken together, we begin to find some principles which we can use to tweak and iterate classroom design. Classroom networks and technology can jumpstart ways of learning mathematics together and the social systems needed in its pursuit. Representational systems only solve locally constituted problems when classroom social systems support students to constitute them. Integrating these three systems within classroom design is the job of both researchers and teachers when setting up learning environments for students.

These three papers form a foundation for my research trajectory to better understand the dialectical relationship between conceptual neighborhoods and tiny publics. In future studies, I will further explore how we can design and support classroom groups to engage in conceptual neighborhood living, strategic symbolic navigation, and conceptual agency. Through long-term classroom studies and design-based research methods, I will further understand how classrooms create conceptual-cultural materials within classrooms and how this material supports their engagement in mathematics. My goal is to build a theory of how to support classroom to become tiny disciplinary publics embodying their local version of an epistemic community.

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CHAPTER 2

Building-Out by 'Offering a Question': Conceptual Neighborhood Living and Tiny Disciplinary Publics

2.1 Introduction

For me, getting to know a domain of knowledge (say, Newtonian mechanics or Hegelian philosophy) is much like coming into a new community of people.... [W]hen one enters a new domain of knowledge, one initially encounters a crowd of new ideas. Good learners are able to pick out those who are powerful and congenial. Others who are less skillful need help from teachers and friends. But we must not forget that while good teachers play the role of mutual friends who can provide introductions, the actual job of getting to know an idea or person cannot be done by a third party. Everyone must acquire skill at getting to know and a personal style for doing it.
- Seymour Papert, "Midstorms" (1980, p. 137)

The variation among theories of learning reflects the differing ways individuals engage in different disciplines and ideas. The current work strives to contribute to our understanding of how groups of learners come together to understand and engage with a discipline. I argue for a radically local perspective of this engagement where classrooms are seen first as what Fine (2012) calls "tiny publics," small groups within which culture is produced, where their epistemic culture is not measured against a discipline but locally constituted and defined by the student group. To understand classrooms in this way, I build on prior work where concepts are seen as a terrain or environment within which learners explore and build (Greeno, 1991) considering this perspective at the group level, what I call *conceptual neighborhoods*. To ground this work, I investigate one class's development of 'offering a question,' an evolving social infrastructure of the class where the group clarifies and refines an ill-formed and incomplete question or idea together to further build-out their conceptual neighborhood. Within this context, I am investigating how this practice offers the class one way of 'getting a sense of' a domain of knowledge, and specifically the implications of this one aspect of their budding culture.

2.2 Literature Review

Papert (1980) discussed a type of learning characterized as akin to getting to know a person, community, or landscape. I call this learning 'getting a sense of.' Greeno (1991) built on this idea of learning by comparing a domain's content to a conceptual *environment* in which students become familiar with traveling from idea to idea and performing specific disciplinary actions on the objects of the environment. He outlined how learning a concept like "number sense" can best be understood as 'getting a sense of' the environment of number, knowing specific landmarks, having specific tools, and being able to flexibly use these to get from point A to point B.

This type of students' learning as 'getting a sense of' an idea space has largely been left behind (see Stroup et al., 2005; Hawkins, 2007, for meaningful exceptions). One possible reason for its abandonment is that its image of a single individual within a static idea-space leaves little room for groups (like classrooms) and the effects of collective inquiry on the conceptual environment. I extend the metaphor in two ways: first by considering the environment as both changing and changeable where individuals are finding their *niche*, and second by considering how human groups operate within environments.

By extending the conceptual environment metaphor as being more than simply a terrain but an ecology of changing ideas and experiences, I am proposing a major change in the role and experience of the student and the importance of teaching and facilitation. When establishing one's niche, students pursue both problem solving (e.g. getting from point A to point B) and place/home finding (e.g. developing favorite numbers). Home finding involves a feeling of safety within the conceptual environment, heightening the importance of teachers, yet ecological niches are not always comfortable. Thus, some students create a niche with emotional, conceptual predators. In both these cases, students are developing a point-of-view of the conceptual domain and deciding where they fit in.

Next by extending the conceptual environment metaphor to the group level, I propose considering how human groups approach environments, specifically how humans build neighborhoods. From a neighborhood perspective, students are positioned to build on and with the environment expanding their niche through incorporating and creating new resources for the collective from the environment. Thus, I use the term *conceptual neighborhoods* (CN) (Nichols-Paez, 2018, 2019) to describe the collective work of groups (usually classrooms but not exclusively) living together in a conceptual environment and building their collective neighborhood resources to do and make things together. Within this conception of classroom's work, the (outside) discipline takes the role of a "found" conceptual environment determining the conceptual context, e.g. systems of thinking focused on 3D and 2D structures or focused on causes and impacts of World War II. Within this conceptual context, groups build a conceptual niche for shared experiences. Not all niches or neighborhoods are safe or welcoming, and not all classrooms build conceptual neighborhoods members want to return to. Yet when a classroom does build a space to collectively participate in the conceptual ecology of ideas, they become an epistemological community (the group-level version of Papert's epistemological agent), and their conceptual environment epitomizes a *convivial*[1] (Illich, 1973) home.

2.2.1 Three Concept Building Moves

In a prior study, I (2018) have used this metaphor to understand how groups construct concepts together by progressively building on individuals' ideas in the collective space. I (2019) have also used it to compare different classrooms and how they foster characteristically different types of conceptual neighborhoods by supporting or disallowing specific moves. In these works, I conceptualize three types of concept-constructing types: building-out, building-within (formerly filling-in), and bounding-out. *Building-out* is when the group introduces a new conceptual contribution (idea or question) as a part of their conceptual terrain worthy of co-constructing and understanding. Through building-out, students 'get a sense of' how to introduce ideas, which types of ideas are considered disciplinary, and how to assess ideas' worthiness in relation to the current work. Notice that through building-out, students are 'getting a sense' of "how;" that is, students are 'getting a sense' of a practice of 'getting a sense.' This meta level reflects a group-level version of what Papert (1980) describes as placing the child in the role of epistemologist. When the group dives into and further elaborates already-known conceptual resources and connections within a defined conceptual space, they are *building-within* the conceptual neighborhood. Through building-within, students 'get a sense of' how to fit ideas together and how the fitting of ideas solves problems, creates questions, and creates a richer conceptual space. Finally, *bounding-out* is characterized by the group's excluding a new conceptual contribution as not part of their current conceptual terrain, either forever or just for now. Through bounding-out, students 'get a sense of' how and when to keep a narrow focus, which ideas to abandon, and possibly, which ideas are worth returning to. All three of these concept-constructing types operate at this meta level in that each of these actions is done by the group, not by individuals. Individuals initiate and negotiate these group actions. I call it a "move" (e.g. a building-out move) when an individual makes a bid for the group to perform that action. However, without the community backing, their efforts are significantly hindered and ownership of the neighborhood is not collective. I (2019) have discussed the importance of building-out within classrooms and the rarity of students' having the power, both socially and mathematically, to make building-out moves. Thus, understanding how a class came to consider and take up student contributions as building-out moves, such as "offering a question," is a considerable contribution. To better analyze this process, I turn to a sociological approach to group learning.

2.2.2 Fine's Idioculture

To understand the development and functions of social order, cohesion, and cultural development, Fine (2012) argues for looking at how small groups function and how individuals operate within small groups to produce cultural material and to embody, perpetuate, and reinvent cultural narratives. Fine calls these small groups "tiny publics," referring to their ability to create culture, and he calls the local, small-group culture they produce "idioculture." Fine defines an idioculture as group members' shared "system of knowledge, beliefs, behaviors, and customs" to which members refer, and which they "employ as the basis of further interaction" (2012, p. 36). As educators, we might connect idioculture to classroom culture, and this connection is quite meaningful, because each classroom is a local version of a learning community. Yet, my focus is not on the entire idioculture of a classroom; instead, I use Fine's constructs to foreground the nature of the cultural material which classrooms are creating, specifically as it relates to the discipline. We must ask ourselves what types of *tiny disciplinary publics* we are supporting in classrooms. This push is especially important in disciplines which have cultural histories of dehumanizing specific populations (c.f. Gutiérrez, 2018). Considering such cultural histories calls into question how many 'ambitious' teaching practices which advocate for a closer approximation of the discipline (c.f. Lampert, 1990) may be perpetuating whiteness (see Barajas-Lopez and Bang, 2018; Lee, 2001, for works which question such approximations). Yet, the power of the local, of idioculture, highlights the potential of classrooms to produce new culture (knowledge, beliefs, behaviors, and customs) which might rehumanize these populations. Instead of focusing on the classroom as progressively conforming to the discipline, I focus on the local disciplinary practices of the class as potentially re-writing the discipline, and over time emancipating both the students and the discipline from the histories of dehumanization.

To further contribute to this work, Fine provides a set of tools to understand how pieces of the discipline are (or not) incorporated into the class's idioculture. In the creation of idioculture, Fine (1979, 2012) asserts five specific local components which act as criteria for statements and events produced by individuals to become part of the idioculture. "The item must be perceived as known (K), usable (U), functional (F), appropriate (A) in light of the group's status system, and triggered (T) by experience" (2012, p. 42). Fine discusses these characteristics as filters and notes they can also be viewed as seeds, i.e. the group's idioculture grows from specific seeds in the collective known culture through becoming usable, functional, and eventually appropriate (see Figure 2.1). This process is significant for classrooms when we as designers have specific cultural items in mind, and we would expect to see the items move through these stages as they become part of the groups idioculture.

Fine points out that group members must "sponsor" cultural material (what Fine calls "cultural items") by words or action, thus creating a 'triggering' event. Triggering events are of particular interest to classrooms because these events are on what the teacher (or researcher) has the most and immediate influence. Furthermore, we can characterize triggering events in the context of conceptual neighborhoods. We could see acts of building-out, building-within, or bounding-out as triggering events, and either their repetition or their novelty may mean the class builds conceptual material around any of these. For example, in my previous

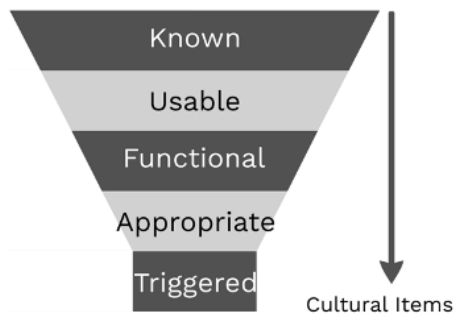


Figure 2.1: Cultural Production

work (2019), the two groups I analyzed developed very different norms around bounding-out events because different group members perceived these triggering events differently, namely as either functional and appropriate or not. Thus, in the present work, these 'forces' provide an explanatory framework to understand how "offering a question" is incorporated into the idioculture. Specifically, I investigate how students' perception of triggering events of "offering a question" change over time, and how this "becomes an item" –something that is known, usable, functional, and appropriate. Through this formation of a cultural item, the group is 'getting a sense' of building-out. More broadly speaking, I target those aspects of its idioculture connected to the three concept building moves, what I call the conceptual neighborhood living of a classroom (see Figure 2.2).

2.2.3 Conceptual Neighborhood Living

A key contribution of Fine's work is to consider how a group enacts what it means (to them) to *be* a group, e.g. a little league team enacts what it means to be a team, a baseball team, a group of mushroomers enact what it means to be an environmental group, and a class enacts what it means to be a group of learners. Thus, *conceptual neighborhood living* considers the conceptual neighborhood as the niche for a class to live and enact what it means to be a tiny disciplinary public, not merely a landscape to explore or even a sandbox within which to build. Thus while each students acts as an epistemologist, I position the entire classroom as a growing epistemic community (growing in terms of their epistemological repertoire). Via participating in the three concept-construction moves, the group develops conceptual neighborhood living and cultural items specific to these moves, what I call *conceptual-cultural material*. Conceptual neighborhood living is a subset of the classroom's idioculture specific to their enactment of the discipline, and similarly, conceptual-cultural material is a subset of the classroom's cultural items. To better understand this idea, I connect it to two pieces of well-known mathematics educational literature.

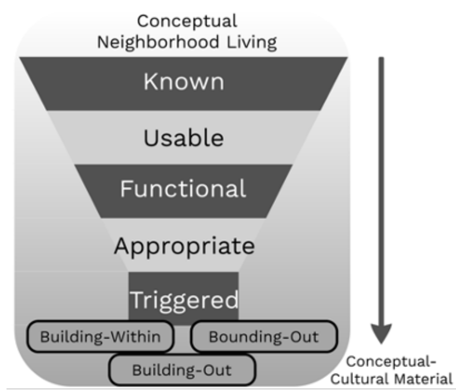


Figure 2.2: Cultural Production

Cobb, Yackel, and Wood discuss how individual learning leads to group development through the negotiation of the taken-as-shared basis of collective activity (Yackel et al., 1993); how the teacher played an authoritative role in setting up the social structures of talking about mathematics (Cobb et al., 1991) but a less-authoritative, participatory role in setting up the social-mathematical norms of the class (Yackel and Cobb, 1996). Their point of negotiation of collective activity aligns well with what Fine has found in groups' development of idiocultures, and their point of the role of the teacher describes how the teacher supported the class in their conceptual neighborhood living, specifically as it pertained to solutions to problems posed by the teacher (and research team) and determined by the school institution. In this way, much of their conceptual neighborhood living revolved around building-within an institutionalized version of second grade mathematics to develop their own local mathematical criteria (i.e. socio-mathematical norms). Thus, the conceptual-cultural material they produced had little that pertained to building-out. I contrast this discussion with a famous episode of a class engaging explicitly in building-out.

Ball (1993) introduces an example of a class engaged in creating conceptual-cultural material by building-out, when Sean, a student, makes a claim about even and odd numbers. Horn (2008) closely analyzes when Sean first made this claim, studying the reaction of the class and the teacher and describing the interaction as "accountable argumentation." I want to highlight two important aspects of this episode. First, Sean's idea started as ill-formed, with his sense that the numbers 0 and 6 were in a space of "could be" even or odd but are not either. Another student, Mei, clarifies and generalizes Sean's notion to the number 10, and through this exchange, the class creates a new category of numbers, eventually known as 'Sean numbers." In my terms, the class built-out Sean's idea through this interaction. The conceptual-cultural material produced within this interaction mimics much of the epistemological traditions seen in academics (Perry et al., 1999), especially in mathematics (Lakatos, 2015). Yet, this form of knowledge production is not the only one available to classrooms (Belenky, 1986). Different from making a claim, "offering a question" invites the class to collaborate and wonder.

Like what Mei did in Ball's class, responding to an "offering a question" move demands a high level of compassion, as the responder must come to understand what the offeror is trying to say. This paper aims to illuminate how students can come to view questions and contributions from this place of empathy through considering a classroom's conceptual neighborhood living and their development of "offering a question". Conceptual neighborhood living provides a way of talking about the disciplinary aspects of the class's idioculture, and Fine (2012) provides additional analytical mechanisms to unpack the students' performance of that idioculture. In this way, students are 'getting a sense' of playing their part within their own tiny disciplinary public. Thus, this paper contributes to the small set of analyses that foregrounds part-whole relations in collective views of learning, moving beyond "modularized" visions of the classroom (see Ma and Hall, 2018; Scardamalia and Bereiter, 1994, for other examples).

2.2.4 Research Questions

When considering the conceptual neighborhood and conceptual neighborhood living of a classroom, our main question is: What conceptual-cultural material is the class creating around building-out, building-within, and bounding-out? The norms, nicknames, and patterns of these three types of conceptual building provide a lens into the classroom as a tiny disciplinary public. For the current work, I explore a class developing "offering a question" as a piece of conceptual-cultural material of building-out. Specifically, I investigate how this activity became known, usable, functional, and appropriate to the class, and then the implications this has on the class's ability to build conceptual neighborhoods.

2.3 Data and Methods

This paper's argument gains force when it moves beyond description and provides answers to key questions around the design of learning environments. To demonstrate this, I present an analysis of a classroom where students' ideas are prioritized, and where we can see significant conceptual development happening with students at a young age (see Chapter 3 of this volume). The analysis describes how 'offering a question' became an item of the class's idioculture through three triggering events. By analyzing this process, we can understand both the conditions within which a building-out item of classroom idioculture was created and what the class as a unit had to 'get a sense of' in order to create that item.

2.3.1 Data Set

The overarching research project from which this analysis comes spanned multiple years and aimed to better understand how to support elementary students' understanding of Geometry. It took place in a Midwest elementary school from 1992 to 1996. The research team worked with a small group of second grade teachers to develop instructional designs and design principles of teaching and learning geometry (Lehrer et al., 1998). One of the content focuses of the researcher-teacher partnership was students drawing nets, and Ms. C was one of the second-grade teachers creating and implementing this design. Ms. C first implemented nets during the first few months of the 1994-5 academic year (Sept. 1994 – Nov. 1994). The research team co-designed curricular activities while observing Ms. C's class and impromptu interviews with students. The class contained a diverse group of students with respect to ability, race, and socioeconomic status. I focused on this component of the project for two reasons. First, during this time, the first major geometry unit of the school year was completed, and Ms. C's classroom constructed many of its cultural norms around doing mathematics, and specifically doing geometry. Second, the time span of the unit was long enough that I conjectured I would see products of the class's developing idioculture while working on the same mathematical idea, namely nets.

Data supporting the current analysis were videos of classroom sessions and post-session interviews with Ms. C, transcripts of videos, and field notes from the research team with embedded pictures of students' work. Within the timeframe Sept. 1994 – Nov. 1994, seven classroom sessions were documented, each with a post-session interview with Ms. C and the research team. The first classroom session (9/12/94) was not recorded but documented via detailed field notes. The second classroom session (9/27/94) documented students working at one of a few math centers with brief field notes taken and indications of a video recording being lost. The next three class sessions (9/28/94, 10/13/94, and 10/18/94) were recorded with brief field notes. The final two class sessions (11/09/94, 11/16/94) were recorded with two cameras, to capture more of the simultaneous small group work, and brief field notes were taken. For close analyses to understand the interactions that occasioned the classroom's idioculture development, I primarily used classroom video.

2.3.2 Methods

While reviewing the field notes from the first two days and watching video recordings of the next five, I mapped out the curricular and idiocultural progression of the seven periods, taking note of any episodes of particular interest. I identified whole-class discussions as the primary source of classroom idiocultural development, and I began to look more closely at these sections of the video. Specifically, I laid out the arcs and themes of each class period, how I thought those arcs and themes built on each other, and how they resulted in any observed mathematical learning. This process left me with a "wide-angle lens" understanding of the classroom's development during this time period, but more specific analyses would be needed to understand specific steps within that development.

For the second phase, I stitched together all the recorded whole-class discussions from the Fall 1994 data into a single video "reel." Much of the student work was done in small groups, so by focusing on whole-class discussions, I reduced the length of video data to two hours. While much idiocultural develop happened within the small groups, I observed whole-class discussions were unique in the fact that they were the most salient shared experiences of the class (i.e. the 'whole' aspect of 'whole-group')[2]. Thus, I predicted these times would be both the most impactful for idiocultural development. Additionally, by viewing the whole class discussions in chronological order, I was better able to observe the changes in whole-group dynamics, honing in on changes in how the classroom group learned mathematics together. While changes from class to class were subtle, I observed some distinctive changes from the first whole-class discussion (9/28/94) to the final whole-class discussion (11/16/94). While I observed sophisticated mathematical development across the classes as well, this analysis would be out of the scope of this paper (see chapter 3 of this volume). Instead, I focused on the changes in the whole-group dynamics between the beginning and ending class sessions. The changes in group dynamics around conceptual building were the group creating and refining conceptual-cultural material and evidence of the group 'getting a sense of' what it means to be an epistemological collective. Thus, I captured key elements of the classroom's conceptual neighborhood living.

To better understand socio-genetic development of this classroom, its processes, and its idioculture, I decided to compare the first whole-class discussion (9/28/94) with the final whole-class discussion (11/16/94). Within this phase, I conducted three stages of analysis, in two of which I used discourse analysis (Cazden, 1988; Gee, 2004), and in one of which I used interaction analysis (Goodwin, 2017; Hall and Stevens, 2015). In the first stage, I performed bottom-up code generation, looking into two types of units of analysis. First, I looked to code the classroom's "conceptual resources," the ideas and mathematical concepts individuals were drawing on to communicate and solve problems together. Second, I coded chunks of interaction where participants engaged in mathematics. These two types of units are usually characterized as 'content' and 'practices,' but since my goal was to understand the mathematical idioculture, i.e. the local mathematics culture, I attempted to curtail my coding bias towards a canonical division between content and practices. In fact, in this classroom, 'content' and 'practices' were interactively constituted to a high degree. The nature of interactions contributed to features of the conceptual resources, and the conceptual resources at students' disposal impacted the ways they interacted with each other (see chapter 3). Because of these interactions, I developed codes specific to the first whole-class discussion and then codes specific to the last whole-class discussion separately. By approaching the two clips as almost two different classrooms, I created codes that were very locally situated in students' present work. Then by comparing the codes, I could connect the different types of resources and interactions across the two class sessions.

After coding both whole-class discussions, I compared the codes looking for both commonalities and developmental changes from the first to the last whole-class discussions. First, I looked for commonalities across the codes, grouping them into quite specific category types. It was at this point that I identified Ms. C's "offering a question" as an interesting and impactful part of the classroom development. Then by looking within these types of conceptual resources and chunks of interactions, I was able to describe how they changed over time. Throughout both steps, I continually returned to the video record to better understand and compare codes. Finally, to develop this comparison further, I looked to reduce my data one more time in order to perform a detailed Interaction Analysis (Goodwin, 2017; Hall and Stevens, 2015) to better understand the details of the types of codes and their changes from one class to another. To do this, I looked for a 'spanning set,' a collection of the fewest moments that together contained all the codes for each class. I covered all codes with two moments in the first whole-class discussion and then again with three moments from the final whole-class discussion. I then analyzed the video record of each of these moments in preparation for an interaction analysis session, creating better transcripts, analyzing the time sequential nature of each moment, and looking into how the different conceptual resources connected with the chunks of interaction. By better understanding and thus comparing these moments, I brought the picture of this tiny disciplinary public into focus, investigating the triggers codes and the participatory agency of difference students with the conceptual resources and within the interactional chunks. This later point became especially important in unpacking the class's known, usable, functional, and appropriate culture. At this point, I brought in Dr. Corey Brady as a conversational partner to clarify and refine my thinking in an IA session. We watched the moments together, stopping as needed, and discussing the implications I was drawing based on the record. After the IA session, I further unpacked the observations we collectively made about the video record and enhanced the narrative of the developing idioculture. Finally, I triangulated this narrative with the video record as a whole, to ensure that the interpretations of change from first to last were consistent with the view offered by the video record as a whole. This final look through concluded the analysis.

2.4 Analysis

While the entire analysis of the classroom's conceptual neighborhood living is out of the scope of this paper, I focus on one type of chunk of interaction, namely "offering a question." Within the spanning set (and the entire first and last whole-class discussions) 'offering a question' was triggered only thrice, but I observed a notable change in how students participated within the interaction across the three triggering events. This change illustrates how 'offering a question' became known and usable (triggering event 1 in first whole-class) and then functional and appropriate (triggering event 2 in last whole-class) and then triggered by students, in opposition to the teacher (triggering event 3). In each of the first two events, the teacher asks a seemingly spontaneous question, which is based on the topics and concepts at hand but not in line with their current goal, yet she manages to skillfully connect this spontaneous work to other mathematical ideas, showing how ill-formed questions lead to rich mathematics. While these two events are separated across time, they establish seeds of known, useful, functional, and appropriate culture of 'offering a question' with which students grow the final triggering event.

2.4.1 Triggering Event 1

The first triggering event occurred as the class was assembling different ways to make a net of a cube. In an earlier session, Ms. C had introduced nets (2D representations that can be folded into their 3D referents), through an activity where students deconstructed cereal boxes at an "open explore" station. On the day of Triggering Event 1 (9/28/94), the class was divided into pairs of students. Each pair was given six square polydrons (plastic construction materials consisting of shapes that can snap together and hinge at the seams) and invited to create as many different nets of a cube as they could. After working in pairs, Ms. C gathered the class together to make a shared list of all the different ways to net a cube, with the pairs taking turns, each donating one new net to the classroom collection. Ms. C took the role of scribe, re-presenting the net contributions as drawings on a large piece of graph-ruled poster paper. As Ms. C began to inscribe the sixth net onto the paper, she asked a seemingly spontaneous (and confusing) question.

23:22 Ms. C: ... All right. Now, I just have to um draw that (draws net on large paper). I'm glad I'm using um graph paper that already has the squares on for me. It makes my job a lot easier. Next time we do nets um or projects with cubes, I'll give you graph paper also if that will help you to record your ideas (finished drawing, sits up, puts cap on marker) about how to put the square polydrons together. The reason I didn't today is because I wanted you to find out for yourself that really... what parts do you need to make every one of these nets for a cube.
23:56 S: Squares.
23:57 Ms. C: Squares. And how many do you need for each net?
23:59 S: Six.
24:00 Ms. C: Six. Is there any other way to net a cube? (makes facial expression with head tilted and mouth slightly open, smiles slightly as students answer)

In the first utterance, Ms. C explains her choice of materials and justifies her learning goal. This type of description was common for her, but it leads to her question "Is there any other way to net a cube?" On the surface, this question does not make much sense as anything new because this is what students were contributing, different ways to net a cube. Ms. C's line of question implies her meaning, which is, is there another way to make a cube, other than by using six squares, but her initial phrasing does not make this explicit. Additionally, Ms. C's facial expression seems to highlight for the class that her question is of a different kind, what I call "an offering question," where the ideas are not yet fully formed and where there might possibly be a surprising answer. Her tone of voice also seems to imply an answer in the negative, but the phrasing of the question implies positive because they had been making ways to net a cube all class period. Through this, Ms. C invites students to suspend the way of thinking within the assignment bounds and explore, but she also safeguards herself in case the class gives a resounding negative. In response, the class gives a mixed answer, unsure of what she is asking.

24:02 S: No. [others- yes]
24:07 Ms. C: How could you do it? (moves recorded net back to carpet)
24:08 S: (Inaudible).
24:10 Ms. C: Anybody. Wait. Alex thinks, how could you do it?
24:14 Justin: You could take four.
24:15 Ms. C: Wait. Alex's idea first and then yours Justin. Go ahead.

At first, the class reacts in the negative, catching onto Ms. C's tone of voice, but as Ms. C continues to smile, some students change their answer. This process implies not just that students are not sure of the answer, but they are not sure of how to answer or what kind of answer is appropriate. Thus, we see a lack of known culture on which to respond to this type of activity. Yet, two students, Alex and Justin, feel comfortable enough to hazard a try to respond to Ms. C's 'offering.' First, Alex responds by attempting to draw Ms. C back into the activity of nets of a cube with six squares.

24:18 Alex: Well on my paper, maybe I was just thinking of this way. (turns paper towards T, holds tilted up, points to spot on paper)
24:28 Ms. C: Is it a way they don't have to use six squares? That's what I'm asking. Is there a way to net a cube – to draw a net for a cube someway that won't use six squares?
24:40 Alex: Well… I thought [it was
24:43 Ms. C: [You thought I was asking something else? What do you think, Justin?

Alex's reference to his paper is a direct referral back to the activity at hand prior to Ms. C's question, referring to the known, usable, functional, and appropriate culture of the class. Alex demonstrates the class's momentum to stay within the bounds of class activity, but Ms. C refuses to return to that space. Yet the refusal provides an opportunity for her to clarify her question for the entire class, and next Justin offers a way to follow Ms. C on her 'offered' tangent by using 24 squares instead of 6.

24:46 JUSTIN: Well you can take, you'd can take like…

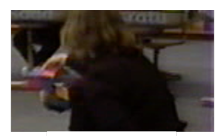
24:58 Ms. C: Hold that idea. Do you understand what Justin is saying?

25:04 Class: “yea:” and “No::”
25:05 Ms. C: Ok. [I need to show you.
25:05 JUSTIN: [(moves out of desk to box behind him, apparently to get more polydrons)
25:06 Rich: Justin’s gonna
25:09 Ms. C: Justin, can you [come up here and help me show them?
25:11 JUSTIN: [(moves to front of room, starts putting polydrons together close to T.)

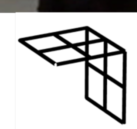
First, Justin builds his idea in his gestural field, directed towards Ms. C. This excludes many class members. Ms. C then tells him to “hold that idea” and asks the class if they understand what Justin is saying, reintroducing them into the conversation and ensuring the group participate both in processing the idea and engaging in the cultural item of building the idea into their repertoire. When some students in the class respond with “no,” Ms. C takes time and floor space at the front of the room to make sure the whole class can see Justin’s idea and (eventually) can participate in its creation. By doing this, Ms. C supports all class members to begin to understand what is involved during ‘offering a question,” making it as a cultural item much more known to the whole group (rather than just to Justin and Ms. C).

25:16 Ms. C: Okay. All right. He’s going to work on this for a while (brings extra polydron squares for Justin) and we’ll get it together. (begins putting polydrons together) This is what Justin is proposing. He’s saying to make a cube from the polydrons you cannot always six squares. He’s saying if you took four of the polydron squares and connected them like this (sits up, holds up four squares link to create a larger square, shows to class, opens up towards Justin) – am I right, Justin?

25:38 JUSTIN: Right.
25:38 Ms. C: Use this (puts 4-connected squares on her palm facing up, places other hand on top) [for the top of a cube,
25:39 JUSTIN: [(holds another 2x2-connected squares up towards class while watching Ms. C)



25:39 Ms. C: [[that for a side of the cube.
25:39 Ms. C and Justin: [[(bring the two 2x2-connected squares together along an edge at a 90-degree angle 25:42 Ms. C: [[[Do you see?...



25:42 JUSTIN: [[[A::nd another one here.
25:44 Ms. C: Okay. Keep making these and we’ll put them together. Okay. He’s going to work on that while we discuss the other nets that we’ve seen so far, which shows Jillian and Ashley. Okay. Is this net different from the others you see laying here?

Ms. C works with Justin to demonstrate his idea to the class, checking her understanding with him in the first image and collaboratively showing how two 2x2 sides of the large cube will sit together once made (1/3 of the entire cube). Justin stays cued into the conversation, keeping an eye on Ms. C while she focuses on ensuring collective attention and participation. Further, Ms. C gives Justin the task “to work on that” communicating to the class how her question and his idea have led to productive and valuable work, a core possibility of ‘offering a question.’ A few minutes later, Ms. C uses Justin’s work to launch a visualizing-counting problem.

27:31 Ms. C: Think about this for a second. If, do you understand the way that Justin is thinking about a cube?

27:38 Class: “Yeah” and “yes”
27:38 Ms. C: How many square polydron pieces is it going to take to make that cube that Justin is working on? How many squares will we need to make the cube he is thinking about?

While the class is easily convinced Justin’s idea would be different from six squares, Ms. C follows up with questioning how different from six squares, launching a visualization-counting question. This question is key on two levels. First, the question of ‘how many squares’ is deeply connected to Ms. C’s clarification of her idea as tangent to the bounds of the activity which was to make nets of a cube with 6 squares. Second, this question makes Justin’s idea quantitatively present for students because the answer involves 6 of something, specifically the 2x2 squares. In this way, Justin’s cube is going to highlight the six-ness of cubes, while violating the literal 6 polydron squares rule. This new question is how Ms. C continues to show the usefulness of the ‘offering a question’ by supporting the class to see how mathematically rich a space inspired by it can become.

In this first triggering event, Ms. C supports the class to experience and participate in a new type of activity, namely what I call “offering a question,” and in doing so, she supports this activity to become both known and usable to the group, two of the criteria that Fine discusses. Within the activity, Ms. C offers a question which is spontaneous, ill-formed, and tangential; and together, first with Alex and then with Justin, the group clarifies the question, generates a possible answer, and then creates more mathematically rich questions from the answer. Through this experience, the group is ‘getting a sense of’ how to respond to such spontaneous, ill-formed, and tangential questions and its place within the classroom’s niche to inspire new ideas, questions, and lines of mathematical investigation.

2.4.2 Triggering Event 2

The second and third Triggering Events come from a session a month and a half later (11/16/94), when I argue that the classroom group had further appropriated the item of “offering a question” into their classroom idioculture. Evidence for this comes in the way students participate in the different activities involved in producing, stabilizing, responding to, and analyzing an ill-formed and tangential question. At this point, the class had mostly finished with nets having iteratively made first small and then large paper nets. Their next unit of focus was volumetric measure, and prior to the next triggering event, they had worked with multi-link cubes to discover different ways to make “apartment buildings,” right rectangular prisms, out of a certain number of “apartments,” unit cubes. Their work on the current day would focus on connecting this new type of way to see 3D objects with students’ understanding of nets by working across representations, to connect 5 different paper volume buildings (pre-built with dotted lines to mimic cubes) with their net representations (same dotted lines but representing squares of surface area). Ms. C was laying out the 5 types of volume buildings on a stool towards the front of the class and the 5 nets could be seen on the board. Reflecting this physical space, Ms. C was laying out the conceptual resources of the two domains which students had previously worked separately, volume and nets. As she counted out the different type of volume buildings and placed them in the public arena (thus pointing to the conceptual landmarks the physical objects represent), she observed a resource possibility missing and asked the class another seemingly spontaneous question.

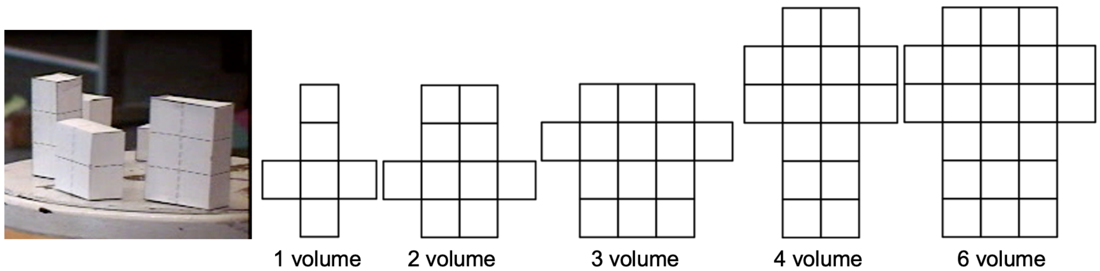


Figure 2.3: Visible resources for activity

12:03 Ms. C: Ok so we have 1 2 3 4, and 6 (moves paper shapes so students can see all 5) why do you think there, why isn't there one for 5?
12:11 Student: Cause um [because
12:12 Student: [It's an odd number
12:13 Student: You can't make it- well [if you could 5
12:15 Student: [You can't make an odd number

Again, this event opens with Ms. C asking a seemingly spontaneous question "why isn't there one for 5?" She is making space within the activity structure and schedule to offer a not-so relevant question communicating that these types of questions are worthy of the class's consideration. Students' quick responses indicate this type of question has by now become known, usable, functional, and appropriate. Yet, the question is still of the ill-formed type like trigger event 1 because it is asking about the negation ("why isn't") of an example within a series of objects, a series which she, as the authority, introduced. A reasonable implication would be the impossibility of such an object, a bounding-out of such an object rather than a building-out to that object. Students follow this negation to look for reasons why "you can't make it"—why we shouldn't allow 5's in our neighborhood—and they identify 5's odd quality as the main factor.

12:17 Ms. C: (holds up 3-volume building off camera)
12:17 Student: (smiles and giggles)
12:19 Ms. C: How many are how many made this one? How [many cubes
12:20 Student: [3
12:21 Students: [3
12:21 Ms. C: [3 is [[3 odd?

Ms. C challenges the claim that odd-numbered volume shapes are impossible simply by holding up the

3-volume shape demonstrating how the class already has odd quality buildings in the scene. She also makes sure the whole class understands the impact of her example by verbalizing what the shape represents. Yet, the students' thinking was based on logic and their work with multi-link cubes previously, and a student offers the following justification.

12:22 Student: [[But if you have if you have 4 and then you added a 5 on it, then you would have like one cube over here (gestures holding imaginary 4-volume building with left hand and adding cube with right hand)
12:30 Ms. C: ((Off camera)) (Places 1 volume cube on top of 4 volume apartment building)
12:30 Student: Ye[ah like that
12:30 Students: [Yeah

Ms. C again provides space for students to share their thinking and perspective, and she even 'revoices' the student's gestural communication by performing a similar action with the actual shapes at the front. None of this work relates (directly) to the activity Ms. C is setting up, namely matching the five shapes with their nets), yet their work is relevant to the conceptual space within which they are working and to Ms. C's question; this furthers the class's cultural association with responding to spontaneous and tangential questions, but the process next cements the activity as 'offering a question' by building-out to the 5-volume building.

12:31 Ms. C: Hmm. Could we have made one that looked like, an apartment building with 5 [apartments?
12:37 Students: [Yeah
12:39 Ms. C: What would it look like?
12:41 Students: (shout out ideas)

Ms. C rephrases the question without a negative and without the 'why' using "could," which positively frames the question aligning it to building-out into a possibility space. As with triggering event 1, the students are experiencing a clarification of the original question and then an invitation to connect the newly clarified question to their current work, which a student does next (literally and metaphorically). Through this process, the group continues to incorporate 'offering a question' into their idioculture.

To describe the imagined 5-volume building, Katie makes a building-within motion to connect it to the 3-volume building by leaning up to the front space and (without permission) moving the paper buildings. Ms. C's permission and then active support of her actions further solidifies the group's sense that this activity is appropriate, the final filter of cultural items. Mathematically, Katie's first proposal extends the 3-volume building with imaginary blocks, and her next proposal draws on connecting the 3-volume building with the 2-volume building end to end. Ms. C responds to this linking with an audible gasp, marking the novelty and creativity of this move. Ms. C follows this by holding the linked pair up for the class to see and describe to make sure this new conception of a 5-volume shape is available for the class.

This second triggering event occurred after Ms. C offered a spontaneous question to the class. Similar to

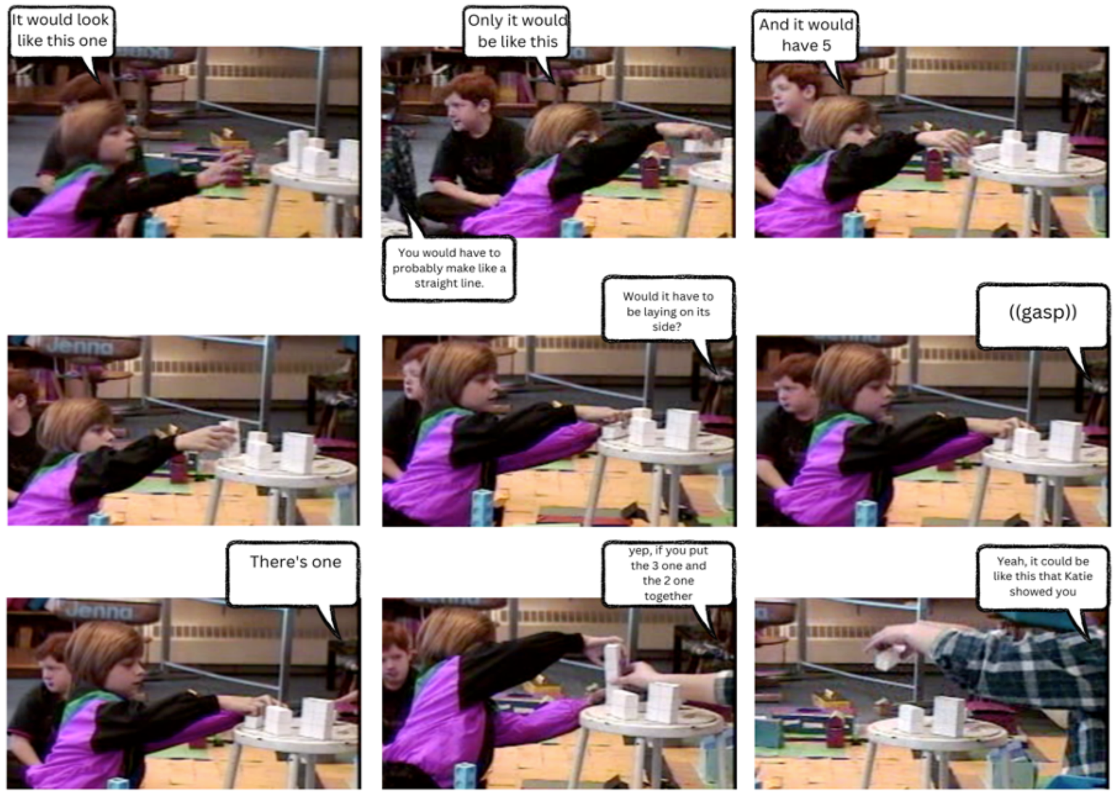


Figure 2.4: Toon-strip of classroom dialogue in Triggering Event 2

trigger event 1, the class clarified the question together, but this time they used specific building-within actions to incorporate this new concept into their current work, solidifying the functionality and appropriateness of this process. Ms. C provided the class with the time and space to do this through access to materials at the front, which further communicated to the students the worthiness of this type of task. Specifically, this interaction sequence reinforced the message to students that ill-formed, tangential questions could contribute in powerful ways to their mathematical investigations.

2.4.3 Triggering Event 3

The final triggering event occurred immediately after the previous one, as Ms. C was still ensuring all students had access to the new conception of a 5-volume shape. Unlike the previous two, this event was not triggered by Ms. C, the teacher, but instead by a student's suggestion. The transfer of agency at the point of initiation indicates a further level of incorporation of "offering a question" into the classroom's idioculture.

As Ms. C is holding up the 5-volume building for the class to see, a student suggests "and the 1 one" meaning to add the 1-volume building to the structure. Two interpretations of this suggestion are possible. First, the student may be still developing the notion of volume and took the 2-volume building as a single

Figure 2.5: Toon-strip of classroom dialogue in Triggering Event 3

unit of volume, thus not satisfying the class's goal and necessitating the addition of the 1-volume building. Or the student may have been extending to include all the possible shapes in the set. The other two shapes (4-volume and 6-volume) would not be possible to add. Ms. C operates on the first interpretation, and she moves to bound-out the contribution by further clarifying the class's goal stating in the second box "we were looking for 5." In this way, Ms. C is operating as Alex operated in Triggering Event 1, aiming to continue the activity in the direction it was heading before the question was offered. However, the classroom group recognizes the opportunity to pursue the new suggestion. The "no" in the fourth box and the "it's an odd number" in the sixth box indicate this clarification supported the whole class to keep up with the work of their peers. Yet, this bounding-out was temporary.

In the sixth and seventh boxes, three students reframe the contribution as "offering a question" and exploring what is possible using "if" statements. In this way, the class implicitly clarifies the student's suggestion as something like "what if you added the 1 one?" Importantly, more than a single student participates in this implicit clarification indicating that the class has identified when and how an idea has been presented to them through 'offering a question,' even if not done by the teacher or as an explicit question. Additionally, the clarification represents a continuation of triggering events 1 and 2 demonstrating how 'offering a question' has become a part of the group's idioculture, even when the main authority of the class, Ms. C, frames the trigger as (somewhat) inappropriate. Now that the suggestion has been implicitly clarified, Ms. C adds the 1-volume building to their collective shape, and next she moves to connect and locate this new building concept within the class's previous conceptual neighborhood.

13:07 Student: Six
13:07 Ms. C: So:: (spreads out 4 vol and 6 vol buildings) which of these buildings has exactly the same amount of space inside it as this (gestures with finger the length of 3+2+1 vol building) big [tall one we just made.
13:13 Student: [The taller [[one
13:14 Student: [[The tallest one
13:15 Ms. C: (points to 6 vol building) yeah, so two (picks up 6 vol building) different ways to show-
13:18 Student: -six-
13:19 Ms. C: -six apartments worth of space or six rooms of space, ok what I want to do now is actually look at those shapes a little but I want to see if you can figure out which net is going to become which of these shapes. So...

Again, Ms. C shows the richness of the mathematics that come from investigating these types of questions, even if the question is from a student. This connection is made even more meaningful for it is deeply connected to the generative activity 'find several ways of making [volume amount]' which students will do in small groups in a few minutes. The repetition connecting spontaneous questions to rich mathematical pursuits creates the feedback loop which encourages students to both offer such ill-formed questions or ideas trusting on the group to refine them and maintain an open mind to others' ideas even if they are not fully formed. Additionally, this event shows how students have fully incorporated 'offering a question' into their idioculture at this point. Important to note is the time of over a month that it took for this conceptual-cultural material to form with these students across an entire unit of mathematics. 'Offering a question' did not occur meaningfully in any of the other whole-class discussions or small-group discussions within this period. Thus, 'offering a question' has not yet become a driver in most of the conceptual construction of the class. Rather this final triggering event implies the class has produced a specific (and novel) piece of conceptual-cultural material around building-out. Investigation of additional data after this is needed to see how it can/did drive conceptual construction.

2.4.4 Findings

Through these three events, I am able to show how the class is developing a piece of conceptual-cultural material around building-out which is convivial in nature. In each of the three events, the class incorporates a new conceptual resource into their repertoire and niche by refining an ill-formed question or idea. Through

Ms. C's authority to create and make public the idea space and her skill to show the richness of those ideas, the class comes to see these ill-formed questions and ideas as valuable and worthy of refinement and investigation. Because of this fact, the students are able to lead a micro conceptual revolution in the final triggering event to reinterpret a contribution from counter to their collective goal into a new place to explore. Students are taking on more conceptual agency and are becoming more comfortable with 'getting a sense' of an idea even when it is murky and not fully understood by the rest of the class, the teacher included. Even greater though, the class is distributing epistemic agency and growing as an epistemic body because the class has become more comfortable with unexpected building-out.

Considering the implication of this piece of conceptual-cultural material to the classroom's conceptual neighborhood living, the class is developing a skill of warmly taking up ill-formed questions and ideas, refining them until they are well understood, and finally finding math within this new space. This component is only one element of how the class as a tiny disciplinary public is creating knowledge, but it is a very impactful component because it broadens the conceptual agency of students and the epistemic distribution of the class. The three episodes of conceptual building from these triggering events were relatively short, but they show a possibility of students 'high jacking' the curriculum within the discipline's domains (the conceptual terrain). Through this facilitated revolution, the class (locally) lays claim to that conceptual terrain and establishes a niche dominance of sorts. If we were to only view the rich, high-level mathematical practices as measured against the discipline, we would risk missing the epistemic agency students establish and may attribute much of the class's progress to the conceptual terrain, a serious injustice to both Ms. C and her students. This level of disciplinary autonomy for the whole class group is extremely rare within the literature (see notable example Ball 2013) and additionally notable for its convivial nature. Additionally, this form of conceptual neighborhood living is important if we hope to support students' conceptual and epistemic agency in a way that rehumanizes classrooms.

2.5 Discussion

By analyzing one piece of this classroom's conceptual-cultural material, I am able to show some key aspects of the tiny disciplinary public they are forming. Specifically, the group is learning how to build-out their conceptual repertoire through taking up ill-formed ideas, even when those ideas are not directly connected to the specific task they were set. In their tiny public the classroom group has collective mathematical agency akin to Yackel and Cobb's (1996) notion of autonomy, except at the group level. The group can perform tasks normally reserved for the teacher, namely posing new questions and problems.

Providing students access to the role of creating questions for each other has a growing body of research around it in math education (Singer et al., 2013) known as 'problem posing.' Yet, the types of 'problems' which students from the problem posing literature are supported to pose often remain strictly within what is known as "school mathematics," i.e. closed form questions either within a very defined context or within a very defined mathematical process. Rarely are students supported to learn how to take ill-formed notions and questions and turn them into what is known as "inquiry mathematics," which is what we see in this classroom. With this analysis, I show how these students are learning and growing their part within the learning community through the performance of that community in 'offering a question' and a critical mass of students know when and how to activate it. Yet, students' understanding of the kind of activity with which they are engaging remained implicit, i.e. we did not find evidence that students come to understand the necessity of the group to do mathematics. Satisfying such a criterion would mean the class is engaged in "ensemble learning" (Ma & Hall, 2018). This type of learning would involve the development conceptual-cultural material where members hold each other accountable to the group and hold the group accountable to members, such as students asking the group to refine their ill-formed idea or calling out students for automatically moving to bound-out such ideas. Bielaczyc (2013) finds that such part-whole perspectives were intractable to maintain within conventional K-12 classrooms due to how teachers framed the motivation for collective work as only improvement of individual work. In contrast, we have evidence that students' motivation for collective work was convivial in nature. The differences between these motivations reflects the added difficulty of ensemble learning in classrooms because of the unclear (and negotiable) shared enterprise of the class. Students form a marching band with a (somewhat) shared understanding of what a marching band does and is for. Students form a classroom without such an understanding and, I argue, should be encouraged to negotiate what their classroom does and is for. Triggering event 3 is an example of this, and it demonstrates how the class is forming a collective but may have not yet made it to an ensemble.

Conceptual neighborhoods and conceptual neighborhood living are two other significant contributions of this work bringing 'getting a sense of' to the group level. The construct of the conceptual neighborhood provides a way to understand how classrooms 'get a sense of' conceptual ideas, and conceptual neighborhood living provides a way to understand how classrooms 'get a sense of' 'getting a sense of' within this discipline, i.e. positioning the classroom group as an growing epistemic community, and thus they 'get a sense of' the discipline as a whole. Furthermore, I claim conceptual neighborhoods and conceptual neighborhood living occurs in all kinds of classrooms. Some classrooms create convivial conceptual neighborhoods where plural ways of working and knowing thrive, where students want to live, play, and work. Other classrooms create more dangerous conceptual neighborhoods where knowing takes on singular form and social and emotional threat is high. Yet, we can use the Fine's tools to unpack and understand these classrooms and how the assembly of these latter classrooms form counterproductive cultural narratives such as 'math has only one right answer' and 'drill and kill is best for struggling math students.' On the other hand, as we think about how to support classrooms to develop as tiny disciplinary publics counter to these narratives creating a convivial relationship with mathematical ideas, conceptual neighborhood and conceptual neighborhood living provide language and methods to analyze classroom's developing cultural forms and their sociogenetic development.

Notes

1. "I intend it to mean autonomous and creative intercourse among persons, and the intercourse of persons with their environment; and this in contrast with the conditioned response of persons to the demands made upon them by others, and by a man-made environment." Illich, 1973 p. 7

2. Note that at this point moving forward, I will use whole group and group interchangeably and always specific if discussing a sub-component of the class.

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CHAPTER 3

Strategic Symbolic Navigation

I propose to retain the time-honoured technical term 'proof' for a thought-experiment – or 'quasi-experiment' – which suggests a decomposition of the original conjecture into subconjectures or lemmas, thus embedding it in a possibly quite distant body of knowledge.
- Imre Lakatos, "Proofs and Refutations" (2015, p. 10)

3.1 Introduction

In the above working definition of proof, Lakatos presents a subtly simple, yet deep interpretation of mathematical discovery. Specifically, he identifies a kind of authentic mathematical work that does not consist of a logical sequencing of statements to verify a claim, but rather is an exploratory process of speculation and conceptualization where the clever mathematician decomposes a mere statement from one mathematical realm into pieces of which can be inculcated to a more tamed universe. We take inspiration from this definition, seeing two key actions within its depths: converting and operating within. Converting is the act of mapping aspects from one realm to counterparts in another (possibly quite distant) realm. Operating within is the act of manipulating concepts and constructs within a specific realm, possibly to make some sort of progress or to explore the realm's possibilities, decomposing them or formulating something new. These two actions are central to specific type of authentic mathematical work which wields the generative use of representational systems, a core of mathematics' power.

Educational researchers investigating students' authentic work with representations have investigated these two different "ways of symbolizing" (Nemirovsky, 1994) but rarely are they put together in the coordinated way Lakatos is suggesting. DiSessa and colleagues (cf. diSessa et al., 1991; Enyedy, 2005) tap into converting when they support students to create and analyze different representational rules to capture aspects of the world. Stroup and colleagues (Ares et al., 2009; Stroup et al., 2005) tap into operating within when they support students to playfully explore the bounds of a representational system. While these 'component' activities are all deep in their respective ways, they frame students' work as becoming familiar with a mathematical terrain on a local scale. To support students in experiencing a view of the mathematical practices that Lakatos describes, students need to see how these two key actions can be coordinated to resolve conceptual difficulties in their own mathematical investigations. Thus, our focus goes beyond students' activity in merely practicing with a given representation system or solving routine problems designed to promote or assess procedural skills or fluency. To better understand this, we propose a framework to characterize students' symbol use supporting the identification and investigation of moments when students experience the power of mathematics.

3.2 Strategic Symbolic Navigation

3.2.1 Framework Introduction

Building upon Lakatos's conceptualization of proof, we present the following framework to better understand how mathematicians and students can work both within and across representation systems, specifically to resolve conceptual problems or circumvent obstacles they encounter in authentic mathematical work. To illustrate the relationship between the two actions, converting and operating within, described above, we depict them in the single (visual) framework of a commutative diagram. Though the two actions we are describing differ in kind, in the visual display, they are represented by complementary movements in orthogonal directions. Horizontal movement indicates converting actions as learners pass from one realm to another. Vertical movement indicates operating within actions where students are working within the confines of a given realm or representation system. We use a dotted, vertical line to separate realms. We use arrows to indicate specific work that is being done, referenced, or attempted. Depending on the nature of the work, these arrows can be solid, dashed, dash-dotted, or dotted (see key below). We maintain that moving across Realms (converting) and working inside one Realm (operating within) are fundamentally different actions, either of which can 'do work', but framing them together enables us to capture sequences of mathematical actions and assess the strategic value of such sequences. We describe the successful chaining of 'converting' and 'operating within' actions to accomplish desired mathematical objectives as Strategic Symbolic Navigation. Below is a generic depiction, using the framework, of a sequence of mathematical activity in which a conceptual roadblock exists in the World Realm and the strategy of passing to the Realm of a representation system is used to circumvent the roadblock.

1. Bold Regions = Realms
2. Horizontal Lines:
   a. Solid = full conversion
   b. Dashed = alignment of specific pieces
   c. Dash-dotted = impossible conversions
   d. Dotted = ambiguous/unknown conversion
3. Vertical Lines:
   a Down Arrow = operation within a realm
   b Up arrow = reference

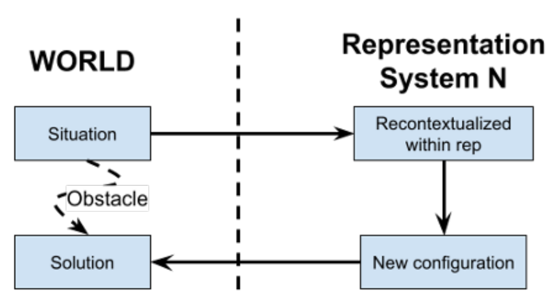

Figure 3.1: Generic version of Strategic Symbolic Navigation

Many descriptions by mathematicians of their own work involve diagrams like the one above, accompanied by the statement that “the diagram commutes.” For example, the commutative diagram below is the first illustration in Eilenberg and MacLane’s (1945) seminal paper “General theory of natural equivalences” which launched the field of Category Theory. The concepts of category and functor were introduced there, as inspired by the example of the relation between a (finite-dimensional, real) vector space (L) and its conjugate space $(T(L))$, of all real-valued linear functions on L. There is a “natural” isomorphism, tau, between the vector space L and its double-dual $T^2(L)$ (the space of real valued-linear functions on the space of real-valued linear functions on L), where, for each element x of the vector space, the function f in $T(T(L))$ is the evaluator function Ev_x. This can be shown to be a real-valued linear function on the dual space of L.

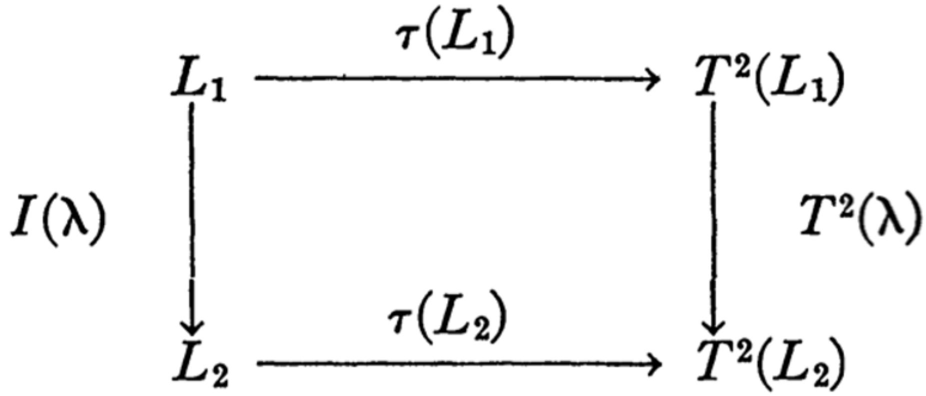

Figure 3.2: Commutative Diagram

Eilenberg and MacLane explicitly articulate how this commutative diagram works, explaining that they are using it to summarize the conceptual “connections between these isomorphisms” (i.e., tau and lambda): “[t]he statement that the two possible paths from L_1 to $T^2(L2)$ in this diagram are in effect identical is what we shall call the ‘naturality’ or ‘simultaneity’ condition. . . .” (Eilenberg and MacLane, 1945, p. 233).
Since the time of that paper, commutative diagrams have become a very common conceptual organizer for contemporary mathematicians to communicate their conceptual work as they coordinate thinking both across and within Realms, in the sense we have described, to the point that commutative diagrams are “perhaps the most ubiquitous diagrams appearing in mathematical journals nowadays” (De Toffoli, 2022, p.85).
Further, an important class of such commutative diagrams (and of the mathematical work they describe) involves mathematicians strategically navigating around obstacles they encounter in one domain and that they manage to resolve by connecting that domain with one or more other domains. For example, in the field of algebraic topology, mathematicians computing homology and cohomology groups for a new space or manifold may exploit another manifold’s known structure by finding a suitable mapping from the known space to the

unknown (or vice versa) and performing what some mathematicians affectionately call “diagram chasing” to deduce the unknown structures.

To be clear, commutative diagrams such as these may or may not act as real-time tools that aid in a discovery as it unfolds, but they do point to features of a “logic of discovery” (Lakatos, 1976), in which making connections across Realms that might initially be thought of as ‘distant’ can be a very generative mathematical ‘move.’ Instrumental uses of commutative diagrams (e.g., in the example of computing homology or cohomology groups) relate by analogy to more conceptual/heuristic, and occasionally grander, strategies—such as the Langlands Program (Langlands, 1970; Mueller and Shahidi, 2021)—of connecting “distant” fields. In the case of conceptual “Converting” and “Operating within,” the diagram itself may not give direct insights into the conceptual history of its ‘arrows’ or the intuitions behind them. Nevertheless, even in such instances, it can provide a map of the mathematical work being described. Accounting for the genesis of the conceptual pathways depicted in any given diagram may indeed be a separate and quite deep inquiry.

When considering the genesis of these moments, we must remember that strategic symbolic navigation is a complex and sophisticated practice. Lehrer and Lesh (2003) argue for distributing such complex practices across the classroom collective to leverage the group as a way for young students to gain access experiences of the power of the practice. Specifically, they build upon Krummheuer’s (1998) framework in which classroom norms are used to format conversations in mathematical ways, and in doing so distribute a target practice across both people (through dialogue) and time (through repeated experiences of similarly formatted conversations). For our purposes, we understand this formatting as a form of collective progressive acquisition of complex practices. Thus, as we look to understand the possibility of students’ fully wielding the power of representational systems, we hope to see their experiences build over time, not just in their familiarity with the representations, but also in how students participate in the practices with greater leadership and autonomy. Specifically, we would expect to see three types of classroom engagement in strategic symbolic navigation: teacher-led, whole-class, and individual student-led. In the ‘teacher-led’ category, the teacher initiates and maintains the episode of strategic symbolic navigation, and the participation structure has a strong asymmetry towards the teacher. In the ‘whole-class’ category, the participation structure exhibits a more equal balance among members, who contribute to the collective achievement of strategic symbolic navigation. The relations between these two categories have features similar to the transition, within language acquisition, from unidirectional language into actual discourse (c.f. Bruner, 1985). Finally, in the ‘student-led’ category, participation switches back to exhibit strong asymmetry towards a specific student, who is largely driving the entire practice. We may even expect to see instances of these types to appear in the order above, as the class and its members progressive engage with the practice and it becomes a part of their shared repertoire.

In this article, we apply the framework of converting and operating within, to analyze the activity of a group of very young (2nd grade) mathematicians, engaged in an extended study of the geometry of three-dimensional objects and their two-dimensional representations as nets. The research questions that guide our study are:

1. How is the collective work of the 2nd grade class described by the actions of converting and operating within and the diagrams orchestrating these actions?

2. How does the class use converting and operating within actions together and strategically to approach problems, and do they have different participatory structures?

3. For such key moments of converting and operating within, what is the genesis of the mathematical 'move' in interaction (i.e., in students' thinking and discourse, in the teacher's facilitation, and in the space of talk and interaction at the classroom level)?

In the remainder of the article, we begin with a brief look into the literature of students working with representations and then go specifically into the mathematics of nets, giving two examples of students' work with 3D objects and nets that illustrate converting and operating within, in this context. Then we outline our analytical approach to understanding not only the pathways that this classroom group traverses but also the genesis of those pathways in open investigations demonstrating this by reanalyzing part of Nemirovsky's (1994) interview. Next, we present our case study of the emergence of strategic symbolic navigation in the classroom, analyzing three key vignettes from the group's broader investigation. Finally, we close with a discussion of the significance of this approach for research on mathematical discourse in classrooms.

3.2.2 Learning and Representations

Beginning in the early 1990s, diSessa et al. (1991) introduced the concept of meta-representational competence (MRC), focusing on how students compare and refine representations of bodies in motion (diSessa and Sherin, 2000; Sherin, 2000). Later, (diSessa, 2004) synthesized this work to characterize MRC as a set of skills beyond fluency with specific representations or representational systems. MRC research focused on how students create, critique, interpret, explain, and understand representations as methods of unpacking and seeing the world. Within the space of MRC, researchers have explored graphing (diSessa et al., 1991), topography (Azevedo, 2000; Enyedy, 2005), and even Algebra (Izsák et al., 2009). For our purposes, we will focus on the work of topography for two reasons. First, the trajectory of this topic with the MRC literature has been well documented from formal interviews of students' naive conceptions (Azevedo, 2000) into a classroom study (Enyedy, 2005) to understand how a group builds on these native conceptions to formulate a collective system. Second, the classroom study was done with 2nd and 3rd grade students, a much younger student population than many other studies of MRC, and moreover, it invokes 2D representations of 3D objects as nets do.

Azevedo (2000) interviews 9th grade students with an open-ended task design to draw 2D representations of 3D terrains. Through these interviews, Azevedo proposes topography as a space where students have many ”constructive resources” on which to build towards more normative representational systems. Enyedy (2005) continues this line of MRC by conducting a study of 2nd- and 3rd-grade students' invention of topographical lines. He explores the social process of this invention, arguing the identification and solving of a collective problem is crucial to both initiate and guide the class's representational work. The power of Enyedy's argument is amplified by his showing that even young children have the capacity to develop and utilize complex representations. Students need to have the opportunity to experience how collective problems guide their work with scientific and mathematical representations, and these experiences are especially powerful for young children.

Similarly, in research on symbolizing, modeling, and problem solving, Lesh (2003) argues for starting with complex, real-world problems for students to mathematize. Through iteratively representing and testing their representations, they create and critique solution paths. He further claims that the value of this mathematizing process is fundamentally different from arguments justifying mathematics (and its representations) by demonstrating their use in the world. Cobb (2002) takes this argument a step further by considering how systems of mathematizing become taken-as-shared ways of seeing the world through the co-development of classroom practices around representations. From more developmental perspectives, van Oers (2002) and Lehrer and Pritchard (2002) consider how mathematizing and representations grow from students' experiences. Lehrer and Pritchard (2002) describe a process for building such experiences when student-generated depictions of a playground from a 3rd grade class become progressively mathematized to capture desired properties of the playground system. Like Enyedy (2005), they find that the classroom discourse around a collective problem motivated the shift.

On the other hand, Stroup et al. (2004, 2005) consider representational systems as play spaces for students to explore, discover, and create. Through generative activities and classroom networks, Stroup et al claim that activities in the mathematics classroom can stage a dialectic interplay between the social sphere of students and the mathematical sphere of representations. When a group of students operate within a representational space with different intentions, conventions, ideas, etc, the class as a whole creates the mathematical space of possibility, through what Stroup et al. (2005) terms “space-creating play.” Carmona et al. (2011) describe a case of this where three undergraduate classrooms create very different mathematical spaces while working through the same activity. Unlike MRC and mathematizing activities where a collective problem (largely provided by the teacher) does not guide students' work. Instead, student engagement and motivation rely on collective challenges emergent from the dimensions of the mathematical space (Stroup 2005) and the technology-mediated feedback (Nichols-Paez and Brady, 2019).

3.2.2.1 Nets

Nets are 2D representations of the surfaces of 3D objects. A structure of (non-overlapping) polygonal faces are connected by edges and can fold along these edges to form the 3D object. The process of representing worldly 3D objects as nets is advantageous introduction for young students in part because, as a representational system, nets afford physical manipulation. They can deconstruct some real-world 3d objects (like cereal boxes) into flat 2d objects; and conversely, when they create a net, they can assemble the corresponding 3d object through physical manipulation. In doing so, students begin to align specific aspects of a 3D object (like the top, bottom, sides, etc.) with aspects of its net and vice versa.

Piaget and Inhelder (1956) dedicated a chapter in The Child's Conception of Space to rotating and developing solids (to "develop" a solid is to create a net for it); and his description of the mental operations involved in manipulating nets offer a clue to their value in mathematics education research. Piaget argued that children's perceptual abilities with 3D objects and nets relied on the reversible mental operation of coordinated folding or unfolding operations within Euclidean space. Specifically, he noted how very young children perform "simple actions" of rotating individual components based on proximity, but "operations" as "a system of actions linked together in a reversible and transitive way" take much longer to develop (p. 292). Additionally, Piaget also notes that if children are given the chance to fold and unfold paper 3D objects, they can be "two or three years in advance of children who lack this experience" (p. 276). The potential for learning in this dimension of explicit manual experience with nets has been explored since Piaget.

Cohen (2003) found that even adults struggle with nets of curved surfaces if they have not had lack explicit experiences with such nets. This finding is especially provocative because Piaget found such curved solids to be the first ones for which students were able to draw nets (as opposed to a cube or a pyramid). On the other hand, Harris et al. (2013) found that children as young as 5 could connect pre-drawn nets to their 3D counterparts. Bourgeois (1986), working with third grade students, found that certain shapes and certain nets of shapes are much harder for students to associate with their 3D referent than others. Yet as Piaget argued, associating an already-drawn net to a 3D object can involve very different processes than that of generating a net. Diezmann and Lowrie (2009) found young students used matching and elimination techniques rather than visualization for multiple choice questions involving the correspondence between a 3D object and its net. While Piaget (1956) posited a developmental trajectory for students and Bourgeois (1986) argued specific shapes and nets determine difficulty, we agree with Wright and Smith (2017) that students' work with nets is "at the intersection" of the net complexity (e.g. number and type of coordinated folds or unfolds) and students' prior experiences with nets (p. 371). Further investigating this, Wright and Smith looked into how students associated a given 'net' as able to be folded to a pyramid (one interview) or a cube (another interview). They found a significant order effect (i.e. students did better on the second interview after working with nets in the first) regardless of which came first (i.e., pyramid then cube or cube then pyramid). Yet, they found no significant association at the individual level, meaning students' ability to anticipate cube nets seemed to have little effect on their ability to anticipate pyramid nets. This evidence further supports that while experience with nets matters, students do not spontaneously develop a generalized scheme for nets. Also, few to no recent studies have investigated students' creation of nets, even though this has been identified as an area of focus within curriculum standards.

The US Common Core (CCSSO, 2010) curriculum identifies working with nets as a goal for grade 6. They suggest that students should "[r]epresent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures" and that they should "[a]pply these techniques in the context of solving real-world and mathematical problems" (Standard 6.G.4). Thus, we see a disconnect between the research being done on nets and the role nets play in curriculum and beyond. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)

For our research, nets are particularly valuable because they can be realized both as physical objects and as drawn representational diagrams. Physical nets can be created by disassembling boxes and other familiar objects; these suggest paper or cardboard objects that children can construct and then assemble into box-like objects; plastic manipulatives like polydrons can enable rapid and repeatable creation of physical nets from polygonal units; and drawings suggesting these physical nets can be created by children as representational diagrams.

3.2.3 Actions of connecting and operating within, in the context of nets

To better understand the framework of connecting and operating within, and its potential application to the work of young mathematicians investigating nets, we next present two snippets of data that characterize these two separate actions.

The first snippet showcases a student mentally converting a physical 3D object into a 2D net in order to identify and count the parts of the net (horizontal arrows connecting Realms). Because he is still learning the representational system, his conversion is not seamless, and we can see his movement back and forth to align different pieces of the 3D object with pieces of this mental version of its net. The second snippet showcases a student operating within the 2D representational system of nets but intentionally violating rules in order to create a 'non-net' with the correct number of polydrons (vertical arrows within Realm). Together, these snippets demonstrate the difference in the two actions (converting and operating within); the accessibility of the framework; and its usefulness to better see students' work with representations. Yet, these brief analyses do not investigate the genesis of students' work, which is investigated in the following section.

3.2.3.1 Operating within the Net Realm

We see in the transcript below two students working with 6 polydron squares. (This episode occurred on Day 3 - 9/28/94.) Within the clip two students respond to a researcher's (Rich Lehrer) question to make an arrangement of squares that would not make a cube by operating within the 2D space by starting with a known net and moving key squares to create a non-net. Below is the diagram that describes their work:

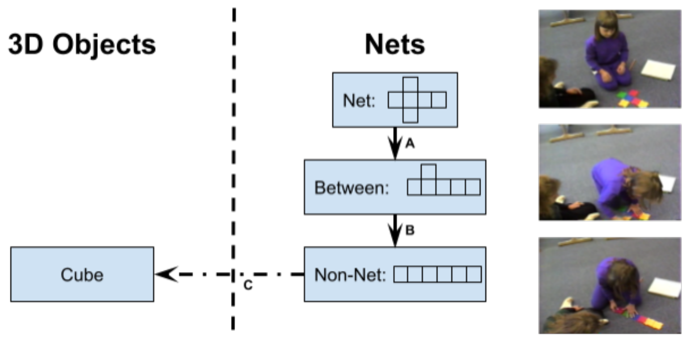

Figure 3.3: Operating within the Net Realm

01:20 Rich: Do you think there's any way at all you could put them together so they wouldn't fold up to make a cube?
01:24 S: Yeah
01:25 Rich: Yeah? Could you, could you show me one of those? [That definitely wouldn't work.
01:29 Jillian: [(reaches for polydrons, pulling off off-set square - J's right - and connecting it to the 4-in-a-row - J's top - to make 5-in-a-row.)

Here we see the first instance of operation within the representational realm of nets (arrow A). Jillian moves one of the squares from their a starting net into a place that seems to violate some unspoken rule, and while this is enough to create a non-net mathematically, she continues to operate within the net realm.

01:35 Rich: [It's not going to work? How come?
01:35 Jillian: [(attempts to remove off-set square - J's left - but breaks the 5-in-a-row. pulls off original target square and then reattaches to maintain 5-in-a-row. attaches removed square to the 5-in-a-row - J's bottom - to make 6-in-a-row.

In arrow B described here, Jillian does another operation within nets, moving the second square. The motivation of this second operation is somewhat unclear, but the move shows that Jillian is unsatisfied with moving the first square either because she is not sure if it is enough to create a non-net or because she specifically wants to show the 6-in-a-row as a non-net. Lastly, Rich asks the students to justify the non-net prompting arrow C.

01:43 Rich: Oh. Why won't that work?
01:48 S: [(reaches for 6-in-a-row picking up - S's left and J's bottom - lifting the 'chain' of squares.
01:50 Jillian: [(reaches around the pick up other side - S's right and Jillian's top - but only lifts a single square bending it orthogonal to ground and the square connected to it. disconnects orthogonal square accidentally

from 6-in-a-row)
01:50 S or Jillian: [Because there wouldn't be anything like
01:52 Jillian or S: On the side
01:53 S or Jillian: On the side

For these students, the lack of 'side' squares are the key reason that the 6-in-a-row does not create a cube. This statement highlights that these students seem to be thinking about covering the cube rather than a positional version of nets. Additionally, they couple their synchronous utterance of "there won't be anything like ... on the side," with the collaborative, physical motion (connecting) of trying to turn the 6-in-a-row into a cube.

3.2.3.2 Connecting from the 3D Objects realm into the Nets realm

We can see in the transcript below a student building on their prior experiences of seeing worldly objects physically turn into nets. Now, he is starting to align the two Realms (i.e., identify correspondences between an object and its net) in the planning stage of drawing a net that can be cut out and folded into the 3D object. (This episode occurred on Day 4 - 10/13/94.)

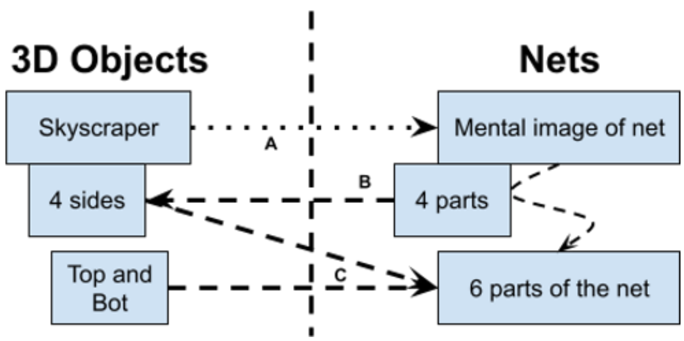

Figure 3.4: Connecting from the 3D Objects to Nets

16:42 Ms. C: ... Let's keep thinking. How many parts (points to paper example) will my net for the skyscraper have? Paul, how many nets (points to paper example) do you think a net for the skyscraper will have? He's thinking. I like the way his eyes are taking that apart in his mind.
17:04 PAUL: Four.
17:05 Ms. C: Okay. What four parts are you thinking about?
17:07 Paul: (gets up. goes to paper example. points at the 4 sides, turning the skyscraper)

Paul first moves mentally from the realm of 3D objects (a rectangular prism) into the representational realm of nets (arrow A), counting the parts needed (he comes up with 4 parts). After an invitation from the teacher, Paul aligns the 4 parts he counted with the physical object by pointing (arrow B).

17:14 Ms. C: And then you'd have the whole skyscraper?
17:16 Paul: (shakes head slightly, points to top with right hand and bottom with left) [Six
17:17 BRITTANY: [You need the top and the bottom?

When the teacher prompts him to consider if he would "have the whole skyscraper," he (and Brittany) realize he isn't quite right. By holding the object and counting off the parts of the object which would connect to the net, Paul recognizes he would still need the top and the bottom, to which he points, and then adds these to his mental net to get a total of 6 parts. In this short interaction, we see Paul (along with another student) fluidly moving between the 3D objects realm and the representational system of nets and align faces of the skyscraper with parts of the net.

3.2.4 A Case of Symbol-Use: Strategic Symbolic Navigation

While the framework provides us a way to see mathematical actions within these short clips, its value lies in supporting analyses of extended classroom investigations with representations. In applying it at this level, we take inspiration from a piece by Ricardo Nemirovsky written 30 years ago (Nemirovsky, 1994). There, Nemirovsky makes the argument that describing students' use of symbol systems is very separate from describing the symbol systems themselves, and that we must look at this 'symbol-use' and their 'ways of symbolizing' rather than just the symbol system. Adopting this language, we can recognize that both converting and operating within are different ways of symbolizing, but more than this, we can view strategic symbolic navigation as a special type of symbol use that supports students to understand the power of representations. A given representation of a problem or concept is not maintained merely for reasons of tradition or convention, but because that representation continues to do work for the people wielding it. If students never have the experience of identifying a (possibly distant) representation system and converting to it because it will solve a problem, which they deem important, they are much less likely either to be invested in learning that representational system or in recognizing how their use of it can be meaningful and generative. Our framework provides a qualitative lens into the classroom system and highlights how students are wielding representations and when they access this navigation power.

To better understand how our framework can support data analysis of students' work with representations, we offer a reinterpretation of a section of Nemirovsky's original analysis, specifically his analysis of Passage 1 and Vignette 1 of Passage 2 (p. 398 - 403). While we agree with most of Nemirovsky's conclusions, we hope to show how diagramming and analyzing the student's (Laura's) work in terms of actions of converting and operating within can further illuminate her interactions with representations and the conceptual resources which led to those interactions. First, we summarize what Laura did in the interview and Nemirovsky's conclusions. Next, we reanalyze the two parts, creating a commutative diagram for each. Finally, we describe what exactly the framework provides for analyzing a student in an interview setting as a precursor to our analyses of collective work.

Passage 1 involves Laura watching two videos of trains moving in different ways and then creating two graphs to describe the motions. After watching the first video of a train moving from right to left slowing down, she creates a speed vs time graph, sloped down. Next, Laura watches a train moving from left to right and creates a speed vs distance graph; when prompted, she says "time didn't even faze me" because the train was "going backwards." Then, in Vignette 1 of Passage 2, Laura begins to interact with a motion sensor, a car, and a data display starting with a distance vs time graph. After some playful exploration, Laura creates a distance vs time graph where the distance increases steadily, tapers off, and then decreases steadily. After this, Nemirovsky (the interviewer) asks Laura to create a velocity vs time graph. Laura struggles when she is looking at the distance vs time graph, but she is able to create the graph by remembering how she moved the car. Finally, she compares her graph with a computer-generated velocity vs time graph saying they are different because she "didn't know there was negatives."

In Passage 1, Nemirovsky concludes that speed vs time and speed vs distance graphs are more readily available descriptions of the world for Laura than distance vs time graphs. Additionally, the directionality of the train's movement had a major explicit impact on Laura's thinking only after seeing two videos in which the train was moving in different directions. In Vignette 1 of Passage 2, Nemirovsky concludes that Laura identifies "negatives" as the main difference between her graph and the computer-generated graph, rather than the slope of the two graphs. He also discusses Laura's constant use of the car context in her descriptions of the graphs and how Laura's embodied memory of the movement of the car is more useful than the distance vs time graph in creating her speed vs time graph.

Moving to our framework, the first step is to identify the different realms across which Laura is operating, then to characterize her speech and gestures in terms of conceptual movement converting across these realms or conceptual movement operating within them. In Passage 1, Laura interacts with 3 realms: the Video on the screen, a Discrete, qualitative velocity-time graph, and a Continuous, qualitative velocity-position graph. Next, we characterized Laura's actions with the diagram below. A key insight from this process is the question of Laura's relationship with the quantification of the video (in which the train's motion is "divided into units and time") and her choice of graph (both qualitative on both axes). While the difference between these two seems striking, even though Laura did not fully represent quantitative measures in her graphs, they may have been a resource to her as she created the graphs. This possibility seems especially plausible in the discrete graph where the video is chunked into five parts to be represented.

In Vignette 1 of Passage 2, Laura begins to work with a Quantitative position-time graph through moving a car in a restricted (1D) path. After prompting from Nemirovsky, she creates a Qualitative velocity-time graph and then compares it to the (computer-generated) Quantitative velocity-time graph. Thus within this clip, Laura works with four different realms (as seen in Passage 2 Analysis). We can see Nemirovsky's conclusions in the diagram through Laura's alignment of both the Position-time graph realm and the Qualitative

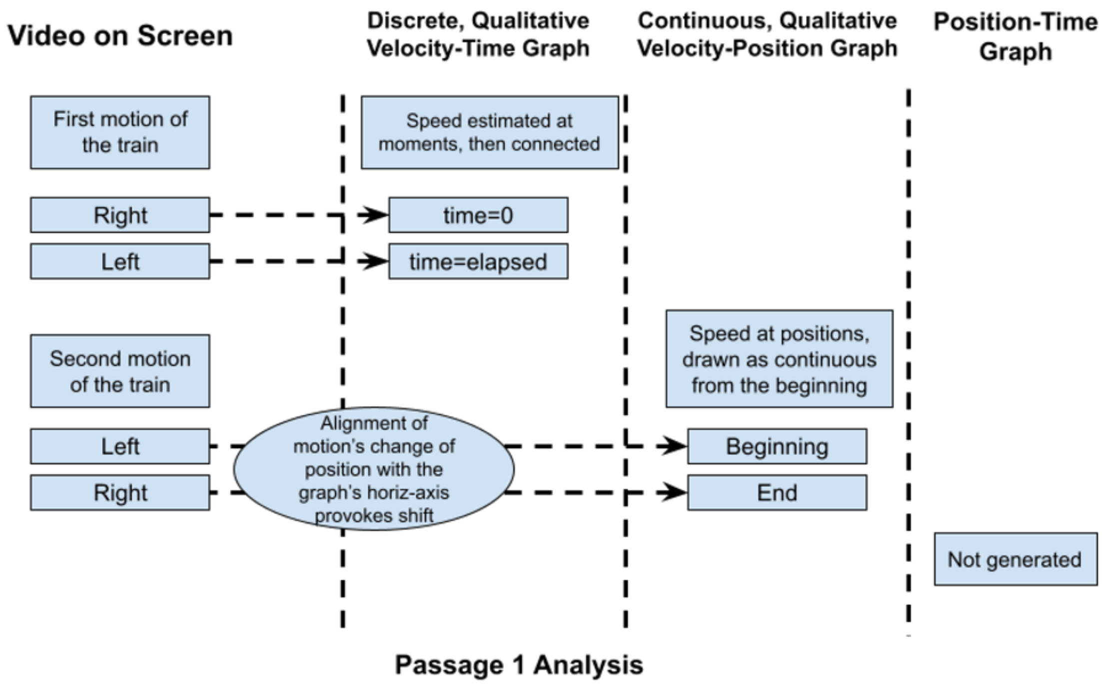

Figure 3.5: Mathematical work of Passage 1

velocity-time graph realm with the physical movement of the car realm. We also see Laura's obstacle related to "negatives" and how "theirs was numbered different." An advantage of our framework is the ease of making such conclusions based on the process of creating and seeing the diagram. Additionally, we see how "numbers" (not just negatives) and quantification play a large role in Laura's thinking. Her main question when she is stuck is "Should I number it? Should I number the speed?" Nemirovsky directs her to consider just the shape of the graph, but this question seems to come back to haunt Laura when she looks at the Quantitative velocity-time graph as being "numbered different."

Our framework contributes two main ideas to Nemirovsky's analysis. First, we are able to highlight the conceptual resources available to Laura through analyzing the specific realms she is creating and interacting with. By highlighting such resources, we can reflect on how Nemirovsky and Laura might have made different uses of the resources in the realm in the creation of the Qualitative velocity-time graph. Secondly, the visual diagram provides a space to easily see the sophisticated actions Laura is doing to align one realm with another and how this alignment interacts with the conceptual resources of that realm. Specifically, we have a language to talk about the 'obstacle' that Laura identifies when she tries to align her Qualitative velocity-time graph with the computer-generated Quantitative velocity-time graph.

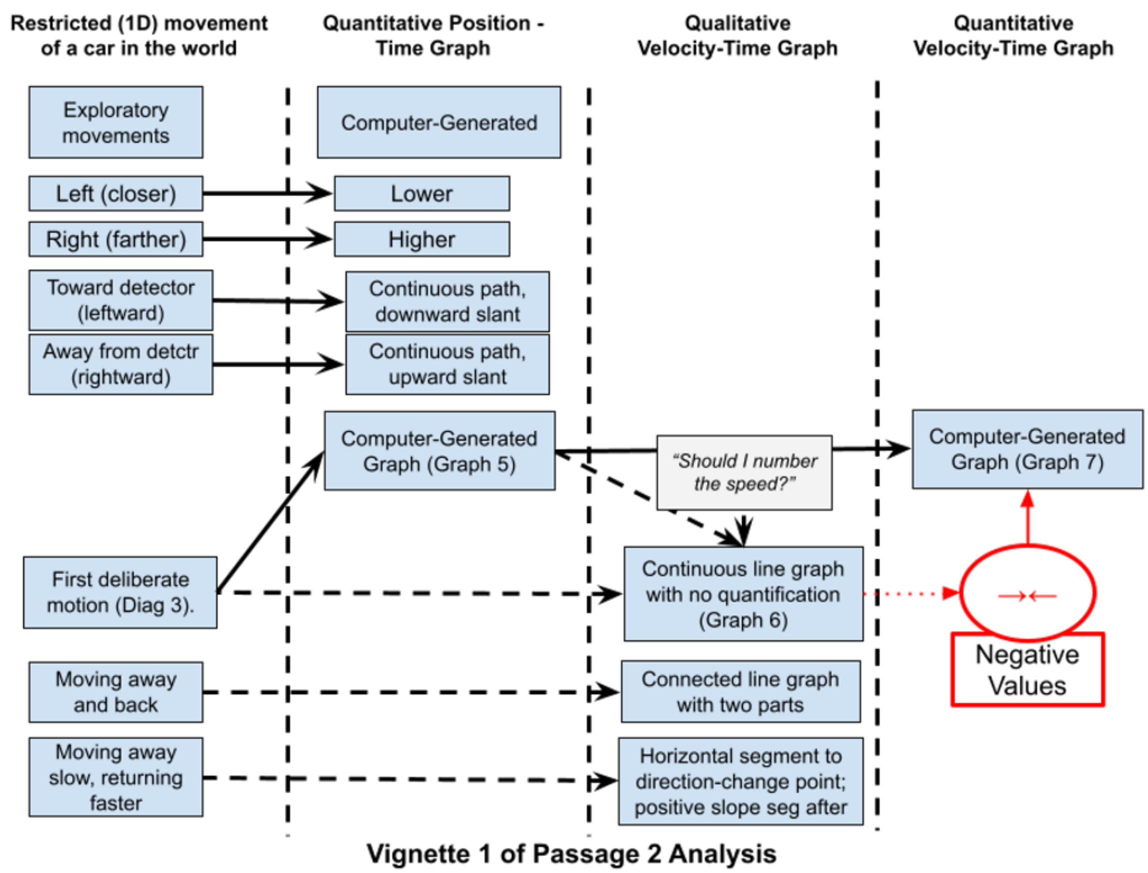

Figure 3.6: Mathematical work of Vignette of Passage 2

3.3 Data and Methods

3.3.1 The Case Study

3.3.1.1 Mathematical and Representational Context

Before launching into the nets-work of the 2nd grade students, we must understand the mathematical and representational context within which they are working. Within our data set, these second graders work with two main representational systems. First, they work with nets, seeing them connected to their real world and then creating them in various modalities. Nets require two main features: all polygonal faces are present connected and make up a single network and no polygonal faces may overlap. These characteristics make nets ideal for turning a diagram on a 2D material (like paper or cardboard) into a hollow 3D object (like a cube or a cereal box). In our data, students have multiple experiences of seeing nets in the world (unmaking a cereal box) and of making nets. See Learning Path below for further details. Second, students work with a volumetric representation by operating with multi-link cubes to create "apartment buildings" with different (or the same) number of apartments. "Apartment" acts as a metaphor for unit cubic-volume, and students experience putting together different builds with similar numbers of apartments and looking at buildings to determine the number of apartments within it. These two representational realms along with the world realm create a triangle of reference where students cross each realm to another and operate within each realm. Below, we have a series of tables to understand these spaces more fully where we describe an (incomplete) inventory of actions students take along the different horizontal and vertical lines.

Types of Connecting: 3D Objects-Net	Types of Operating within: Nets Actions
Cutting and pasting/hooking edges	Counting faces
Folding and unfolding edges	Comparing size (area) of faces
(Deconstructing and reconstructing)	Adding/moving faces
	Changing the shape/size of specific faces
	Rotate and mirror entire net

Table 3.1: Nets Table

Types of Connecting: 3D Objects-Volume	Types of Operating within: Volume Actions
"Think of as" apartments or cubes"	Linking or unlinking volumetric shapes
"One [apartment's or cube's] worth of space"	Adding or removing 1 volume
Imagined linked pieces as a whole	Rearranging same volume

Table 3.2: Volume Table

Types of Connecting: Nets-Volume
Quantifying (Gridlines)

Table 3.3: Nets-Volume Table

3.3.1.2 Learners' Context and Data sources

The overarching research project from which this analysis comes spanned multiple years and aimed to better understand how to support elementary students' understanding of Geometry. It took place in a Midwest elementary school from 1992 to 1996. The research team worked with a small group of second grade teachers to develop instructional designs and design principles of teaching and learning geometry (Lehrer et al., 1998). One of the content focuses of the researcher-teacher partnership was students drawing nets, and Ms. C was one of the second-grade teachers creating and implementing this design. Ms. C first implemented nets during the first few months of the 1994-5 academic year (Sept. 1994 – Nov. 1994). The research team co-designed curricular activities while observing Ms. C's class and impromptu interviews with students. The class contained a diverse group of students with respect to ability, race, and socioeconomic status. During this period the research team acted in an observational capacity, offering resources and a sounding board for the teacher's reflection but taking little to no role in design. I focused on this component of the project for two reasons. First, during this time, the first major geometry unit of the school year was completed, and Ms. C's classroom constructed many of its cultural norms around doing mathematics, and specifically doing geometry. Second, the time span of the unit was long enough that I conjectured I would see products of the class's developing idioculture while working on the same mathematical idea, namely nets.

Within the timeframe Sept. 1994 – Nov. 1994, seven classroom sessions were documented each with a post-session interview with Ms. C and the research team. The first classroom session (9/12/94) was not recorded but documented via field notes. The second classroom session (9/27/94) documented students working at one of a few math centers with field notes taken and indications of a video recording being lost. The next three class sessions (9/28/94, 10/13/94, and 10/18/94) were recorded but did not have field notes. (sorry, not all the paper capture made it to digital form) The final two class sessions (11/09/94, 11/16/94) were recorded with two cameras, to capture more of the simultaneous small group work. I primarily used classroom video to understand student thinking and classroom development.

3.3.1.3 Learning Path

The 2nd grade class convened in the fall of 1994. For the teacher (called "Ms. C" in the transcripts), the first mathematical focus was geometry. Specifically, she wanted to support students to visualize and get to know 3D objects, and she used nets as a first step. The first activity (Day 1 - 9/12/94) was a collective visualization activity where Ms. C led the students in visualizing a specific 3D object from their life, a cereal box, and asked them how they could flatten the box. Next, she brought an actual cereal box into the class and took it apart where the manufacturer glued the sides together to show students how it had started as a flat piece of cardboard with specific parts (Day 2 - 9/27/94). This activity was when she introduced the term "net." From this launching point, the class explored nets in multiple ways. They built nets with premade polygons ("polydrons") which clip together with a hinge to fold into 3D objects, exploring how to draw nets and then how to use the drawings as documentation of different ways to net a cube (Day 3 - 9/28/94). They looked at 3 different paper 3D objects (made by the teacher), each more difficult to net than the last, and they (iteratively) hand drew nets for them (Day 4, 5, and 6 - 10/13/94, 10/18/94, and 11/09/94). They started by creating nets for smaller versions of the 3D objects so they could more quickly create multiple versions and adjust. After they were successful in the small version, they moved to working on cardstock to make a larger version as a final product. Ms. C then moved the class into cubic volume (Day 7 – class not documented). One class session combined these two topics, nets and cubic volume, creating very fruitful connections between them (Day 8 - 11/16/94).

3.3.1.4 Data Selection

We selected specific episodes for the current paper, both to demonstrate the framework and as an existence proof that 2nd graders do have the ability to engage in Strategic Symbolic Navigation. We began by review the Day 1 and 2 field notes and fully watching the videos of Day 3, 4, 5, 6, and 8 class sessions. We decided to focus on whole-class discussions based on the sophistication of the practice and how we expect students to progressively engage from teacher-lead to eventually student-lead. After we familiarized ourselves with the entire arc of the class's mathematical development, we decided to first focus on the final whole-class discussion (Day 8 - 11/16/94) because we noted students were strategically navigating both nets and cubic volume representational systems. We chunked the video into different episodes of the class (or a student) working with representations and coded each episode as converting, operating-within, or strategic use of both, along with identifying the active representational realms, depending on the episode's content. These codes were purposefully broad capturing the action of entire episode, and this labeling provided us with a map to decide which clips to further investigate and which types of actions were not present in each whole-class discussion. Based on the literature, we expected to observe teacher-lead, whole-class, and student-lead, but not necessarily within the same class session. We found that this final class session did not contain a teacher-lead episode of strategic symbolic navigation. We then returned to the prior class sessions and found such an episode on Day 4. We analyzed these moments by first generating a visual commutative diagram and then digging into the genesis and meaning of each major representational move within the video record. At times, this understanding lead to a revision of the visual diagram to better describe the work of the class or a student.

3.4 Findings

3.4.1 Episode 1: Teacher-led Strategic symbolic navigation

The below episode occurs immediately before the connecting episode presented earlier (Day 4). Students are just beginning to work with nets, and the teacher is supporting them to align both faces with the shapes of the net and properties of those faces with characteristics of the net. Prior to this episode, some students had created nets for various shapes on their own, but as a whole class, they had worked exclusively with creating nets of a cube. The 3D shape they work with in this episode is a rectangular prism with a square base (the skyscraper?). A large paper example is available to the class to reference, and the episode starts with an epistemic question (how do you know?) posed by the teacher which pushes students to operate within the world of 3D objects and relate this operation to the world of nets. The teacher continues to ask specific questions to support students to think about size relations among the different parts of the (to-be-created) net.

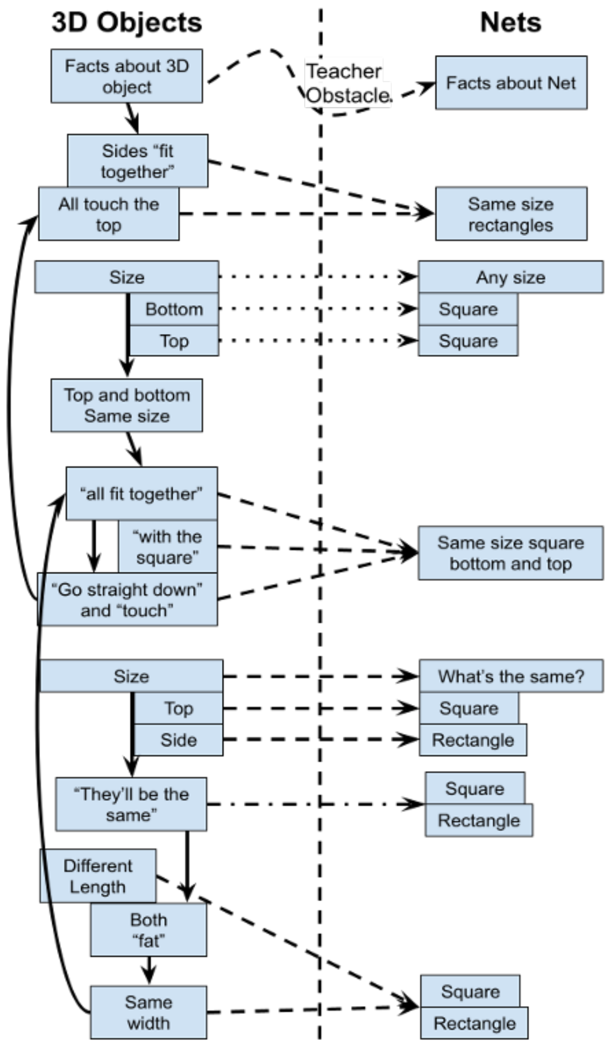

Figure 3.7: Episode 1: Map of Mathematical Work

15:14 Ms. C: But I mean, none of you came up and measured the skyscraper, did you? I didn't see anyone with a ruler or using their hands or even their feet — how do you know that these are the same size? You didn't measure them.

The teacher poses an epistemic question, a "how do you know" question (arrow A), in response to students' ideas of the size relations within a 3D figure and their corresponding relations within the net. Through this question, Ms. C questions the connecting of the students and creates an obstacle for the class to circumvent. Additionally, she contrasts the students' logical thinking with the act of measuring, which would be another way to ‘know’ that the rectangles are all the same size.

15:23 JUSTIN: Because they all fit together [stand vertical like a (motions hands from high to low).
15:24 STUDENT: [They all touch the top.
15:29 STUDENT: None of them is smaller than the others.

Next, students describe the operations within 3D objects to justify the size relations using words like “fit together” and “touch the top” (arrows B). One interpretation of “fit together” would be that the students are imagining putting the net together and the pieces fitting together as they are folded, but as we can see from Justin’s gestures, the fitting together (from his point of view) is from orthogonal movement and symmetry, not a rotary motion which would imply folding.

15:31 Ms. C: Can I use any size square for the bottom and any size square for the top?
15:34 STUDENTs: “No::”
15:34 STUDENT: Yeah. They [have to be the same size.
15:35 STUDENT: [They have to be the same size
15:37 Ms. C: [Why?
15:37 STUDENT: [They all have to fit together
15:38 STUDENT: [[They have to fit together.
15:38 STUDENT: [[They have to fit together.

Next, Ms. C turns the class’s focus to a different shape of the net, the squares (arrow C). Earlier in the discussion, the class determined that two squares were needed for the net, one for the top and one for the bottom. Ms. C asks about the size of these two squares, and students articulate they have to “fit together” (arrow D). Yet, the specific size of the square does not seem to be necessary, at least not for all students, which is an interesting contrast to a thinking that would have happened if their focus had been on measuring. Justin articulates what he means by fitting together.

15:38 Justin: [[Because they all fit together. Because rectangles have to fit together with ((brings hands towards each other palms facing each other punctuating “with” like tapping the sides of the building)) the squares ((rotates hands bringing palms together so that right hand is on bottom of building and left is on the side at a right angle)) [the other two are on ((moves left hand to top of imagined building, brings hands towards each)).
15:45 Sara: [They have to go straight down and they have to touch the square.

Justin uses his gestures to describe how the components of the 3D object “fit together” adding a new idea of “with the squares.” He shows this by bringing the edge of his palms together as if they are an edge of the 3D object (see picture). Sara uses the idea of “straight down” of the rectangles and touching the squares (arrow E). A core element of both of these ideas is the relationship between the square and the rectangles, and Ms. C pulls this element out more explicitly next.

15:49 Ms. C: Will anything be the same about the square that's on top and the rectangle that's on the side? Will anything be the same about those two shapes? I know one will be a square and one will be a rectangle.
15:59 STUDENT: [They'll both be
15:59 STUDENT: [They'll both be the same
16:00 Katie: They won't be the length but they will both be the same, they'll both be fat. Both both (inaudible) (gestures with left hand to indicate same width).

Ms. C next asks students to go deeper within the size relations of the faces (arrow F). Prior to this, the students had been talking about the relations between whole faces, but now she is asking if there is a relationship between components of the square and components of the rectangle. Multiple students have a sense that something about them is "the same" (arrow G), but Katie articulates that one component (what she calls "length") won't be the same but some other component (what she calls "fat" and gestures width) will be the same (arrow H).

16:07 Ms. C: [The same. The same
16:07 JUSTIN: [They'll both have to be the same size
16:09 Ms. C: Oh, the same size? Is this shape (off camera points to square) the same size as this shape(off camera points to rectangle)?
16:13 STUDENTs: "No"
16:13 Justin: I mean the um the (inaudible) have to match each other, a- across from each other (gestures top and bottom multiple times switching which hand is on top and which is on bottom).

Ms. C makes a point to highlight how we can not say the whole face of the square and rectangle are the same (arrow I) but only that components are the same. Justin either revises his thinking or explains the miscommunication that the whole faces we can say "have to be the same size" are the squares on top and bottom.

16:18 Ms. C: Okay. What Katie said is they won't be the same length, because she's saying the square is only this long whatever and the rectangle is this long, but they will be the same — she was talking about how fat they are ((motions hands across from each other)) but the same width ((picks up paper example)). How wide ((turns example so top faces the class. measures width of top with thumb and pointer finger)) the square is is ((turns example right side up as motioning down the side keeping thumb and pointer same width)) how wide the rectangle is. But the rectangle is ((mimics the same thumb and pointer down as before but no longer next to the example)) much longer. Because ((creates square with two hands)) if the rectangle was ((mimics hands across from each other for fat as before. brings hands down with palms up. motions to Katie)) fatter than the square, would they fit together ((holds left hand palm up fingers pointed to right and right hand at a right angle fingers pointed forward))?
16:41 STUDENTs: "No"

Ms. C revoices Katie's idea making sure to use both her language and more mathematical language of "width" (arrow J) and further highlight the component relation and difference between the square and rectangle (arrow K). Furthermore, she returns to the students' argument that if they were not related in this way,

they wouldn't "fit together" (arrow L). With this reference, Ms. C validates the students' logical reasoning of size through 'fit together' in contrast to her measuring notion posed at the beginning of the episode.
Within this episode, we see the class manipulate the 3D object mentally and gesturally, align specific parts of the 3D paper shape with their to-be-drawn net, and use these to make logical assertions of the size relations of shapes within the net. At this point, students are more comfortable operating within the realm of 3D objects than operating within the realm of nets. Yet as the class continues to make such connections between the two realms, we would expect students fluency both with connecting the two and with operating within nets to increase. In fact, we would expect students operating within the 3D objects realm to increase as well because of the feedback loop between them.

3.4.2 Episode 2: Collective Strategic Symbolic Navigation

Our next episode shows the class as a collective strategically using volume to navigate and explore what is possible in the world and in volume (Day 8). In this previous lesson, Ms. C introduced apartments and apartment buildings as a metaphor for cubic volume where they ask how many (cubic) apartments fit into different (rectangular prisms) apartment buildings or what different apartment buildings have the same number of apartments. Multi-link cubes acted as individual apartments with which students link to assemble an apartment building. In the current episode, the teacher still leads the class through asking questions, especially at the beginning, but those questions play a very different role in this episode than in Episode 1. In Episode 1, Ms. C's questions directed students to make specific connections between the 3D objects realm and the representational realm of nets; in this Episode, Ms. C's questions open a space of possibility and keep it open. Additionally when she shuts down a suggestion, multiple students work to position the contribution as another possibility.

12:03 Ms. C: Ok so we have 1 2 3 4, and 6 ((moves paper shapes so students can see all 5) why do you think there, why isn't there one for 5?
12:11 Student: Cause um [because
12:12 Student: [It's an odd number
12:13 Student: You can't make it- well [if you could 5
12:15 Student: [You can't make an odd number

In this opening question, Ms. C asks a question that flows from talk. Whether or not she had planned to ask the question, it gives the impression of being spontaneous. The paper shapes are 3D objects that came from nets (which are displayed on the board behind Ms. C) meaning the correct representational realm to think of these shapes is somewhat ambiguous. These shapes sit in-between the nets realm which has held most of the class's attention (Day 1-7) vs. volume which students have just started with multi-link cubes (Day 8). Using the representational realm of volume, students very quickly identify the key feature of "5" as

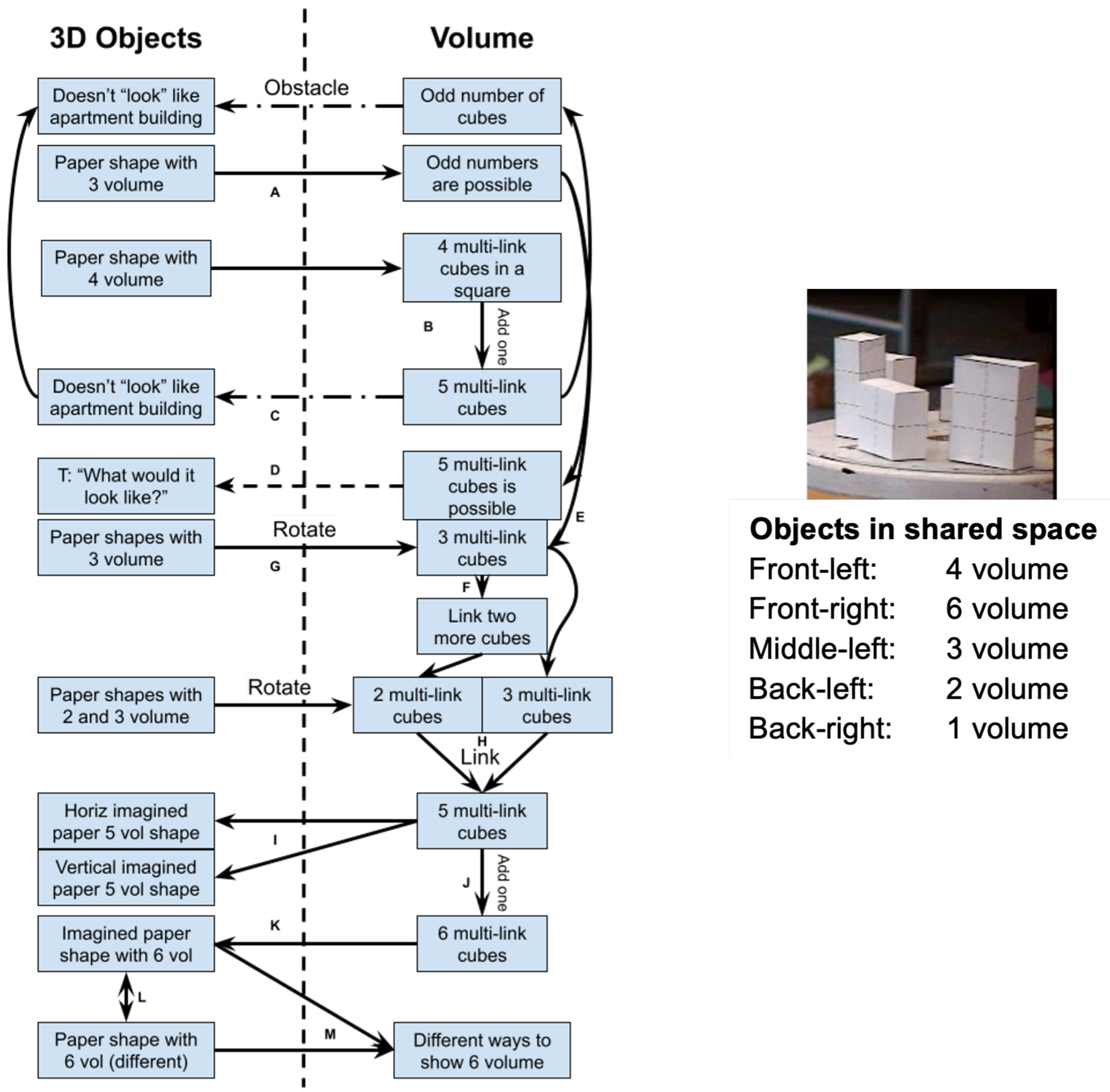

Figure 3.8: Episode 2: Mathematical Work

odd (Obstacle in diagram), different from 4 and 6. Further, they generalize their claim to "you can't make an odd." Ms. C quickly challenges this claim next:

12:17 Ms. C: ((holds up 3-volume building off camera))
12:17 Student: ((smiles and giggles))
12:19 Ms. C: How many are how many made this one? How [many cubes
12:20 Student: [3
12:21 Students: [3
12:21 Ms. C: [3 is [[3 odd?

Ms. C challenges the claim that odd-numbered volume shapes are impossible simply by holding up the 3-volume shape (arrow A) connecting the paper shape (made from a net) with the volume realm. She makes sure the whole class understands this connecting by engaging them in the contribution. Yet, the students' thinking was based on volume logic and their work with multi-link cubes previously, and a student offers a description of this logic.

12:22 Student: [[But if you have if you have 4 and then you added a 5 on it, then you would have like one cube over here ((gestures holding imaginary 4-volume building with left hand and adding cube with right hand))
12:30 Ms. C: ((Off camera)) ((Places 1 volume cube on top of 4 volume apartment building))
12:30 Student: Ye[ah like that
12:30 Students: [Yeah

Ms. C provides space for the student to justify their thinking both verbally (with gesture) and then Ms. C 'revoices' the thinking by performing the action the student describes (arrow B). Yet, the action the student describes and Ms. C's action are slightly different. The student seems to be thinking about extending the 4-volume shape horizontally putting the imagined cube on the side whereas Ms. C performs the action vertically putting the actual cube on top. This difference becomes more meaningful later. Lastly, the class confirms that this operation creates something that is NOT a valid 5 volume shape (arrow C).

12:31 Ms. C: Hmm. Could we have made one that looked like, an apartment building with 5 [apartments?
12:37 Students: [Yeah
12:39 Ms. C: What would it look like?
12:41 Students: (shout out ideas)

Next, Ms. C offers the question again, but now it does not have a negative. Instead, she frames the question with possibility ('could') and asks "what would it look like?" (arrow D). With this, students quickly take up the positive position.

In the above strip, Katie takes the initiative to move to the front and work with the physical shapes (without explicit permission). Her idea is to use the 3-volume shape and extend it to have a volume of 5. Note the 3-volume shape was Ms. C's counter example earlier (arrow E). Katie does this first with imagined cubes comparable to the multi-link cubes to show her basic idea (arrow F). Additionally, Katie finds it important to lay the 3-volume shape down and extend it horizontally (arrow G) like the student extending the 4-volume shape. She does this again after putting the 3-volume shape back upright. She reconsiders and turns it down again to connect the 3-volume and the 2-volume shapes (arrow H). This idea is met with an audible gasp from Ms. C and "There's one." Right before this, Ms. C questions Katie about the need for it "to be laying on its side," which prompts Katie to turn the linked shapes vertical as a unit. This process is interesting because a key component of the representational realm of volume for the class group is how to make "different ways to show [x] apartments worth of space," so vertical vs horizontal 5-in-a-row cubes seem to be different ways to

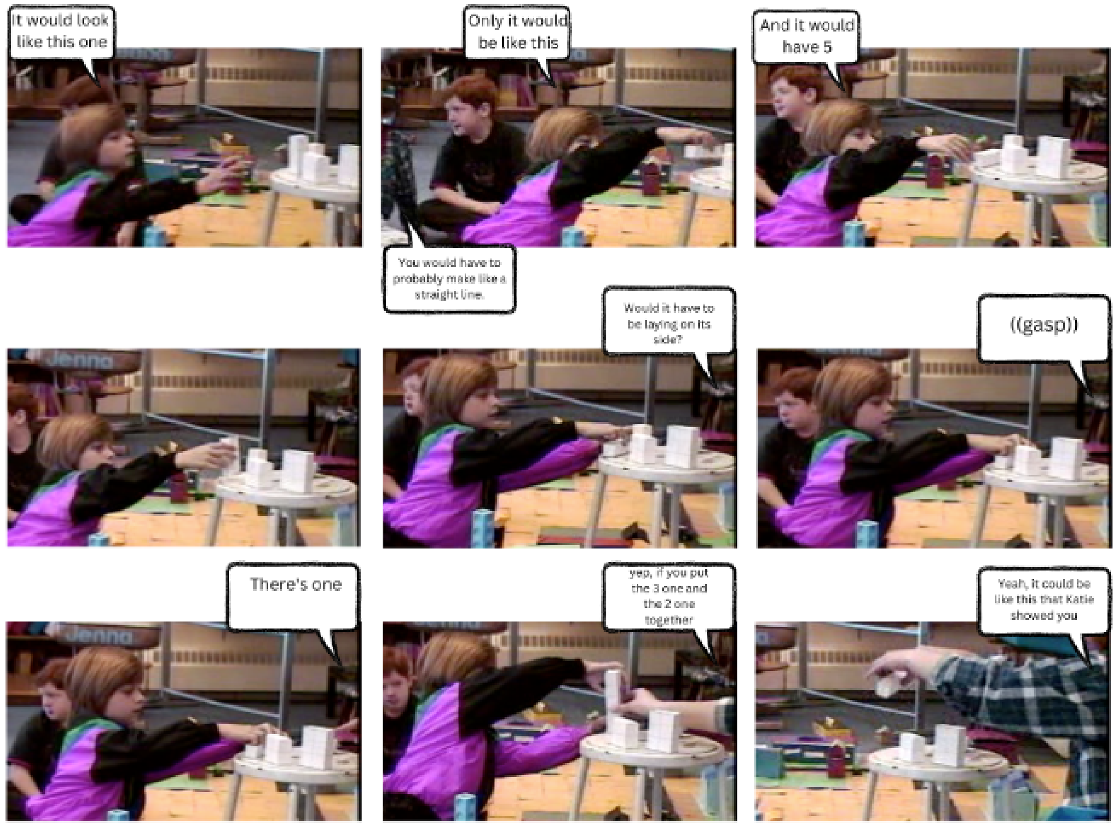

Figure 3.9: Episode 2: Toon-strip 1 of classroom dialogue

show 5 units of volume worth of space. Lastly, Ms. C holds the linked shapes up for the class to see (arrows I).

As Ms. C shows the class the vertical 5-volume shape, a student suggests "and the 1 one," i.e. adding the last single cube to the linked pair (arrow J). It is unclear the motivation behind this suggestion, but Ms. C brings the class's attention to the goal ("looking for five") and whether they have achieved it emphasizing the connecting of the linked shapes to the imagined paper shape (arrows I again). Yet after this, three different students are interested in what would happen "if" you put the single cube, so Ms. C adds it.

13:07 Student: Six
13:07 Ms. C: So:: ((spreads out 4 vol and 6 vol buildings)) which of these buildings has exactly the same amount of space inside it as this ((gestures with finger the length of 3+2+1 vol building)) big [tall one we just made.
13:13 Student: [The taller [[one
13:14 Student: [[The tallest one
13:15 Ms. C: ((points to 6 vol building)) yeah, so two ((picks up 6 vol building)) different ways to show-
13:18 Student: -six-
13:19 Ms. C: -six apartments worth of space or six rooms of space, ok what I want to do now is actually look at those shapes a little but I want to see if you can figure out which net is going to become which of these shapes. So...

Figure 3.10: Episode 2: Toon-strip 2 of classroom dialogue

In this final part, Ms. C seamlessly links the new idea and construction to the ongoing task of "different ways to show [x] apartments worth of space" by asking which of the net-make shapes has "the same amount of space" (arrow L and M). This motion is quite deft especially since Ms. C seemed very resistant to the idea of "adding the 1 one," and how their current activity would connect to the students' ongoing work is unclear. Yet with this motion, she further relates these net-made shapes to the class's volume thinking. In this episode of strategic symbolic navigation, the class finds a way to make "one for 5" by both connecting the net-made 3D objects with the representational realm of volume as linkable cubes. Different from the commutative diagrams of mathematics, the work of the class is messy and non-standard, yet within the interplay between the diagram and the genesis analysis, we see how the students are becoming fluid with the motions between (connecting) and within (operating within) the realms to solve a question of what is possible. In the next episode, a student builds on this fluency to solve a collective problem.

3.4.3 Episode 3: Student-led Strategic Symbolic Navigation

The following episode builds upon the previous one to showcase how one student (Nicole) connected 3D objects and their nets by navigating into volumetric thinking (Day 8). While Episode 2 connected net-made objects to volume, Nicole aligned specific parts of nets with the 3D object and then performed an mental operation within that is very connected to the operation of unlinking a multi-link cube from a structure. Nicole's intellectual leap solves a collective problem identified earlier in the class. Below is pictured what was in the front of the class and depictions of what Ms. C put taped on the blackboard behind them.

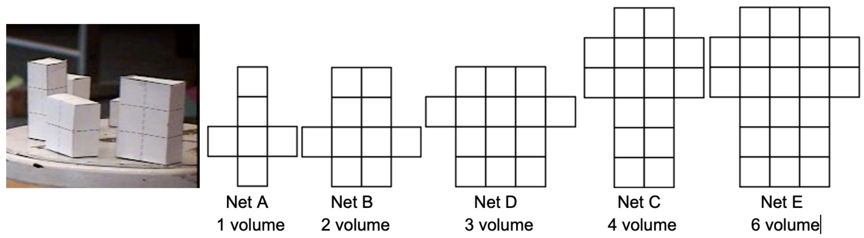

Figure 3.11: Episode 3: Visible Resources for the Activity

The collective problem was identified earlier in the class when two students (Justin and Brittany) disagree about which net goes to which world object but neither seem to be able to create an argument that is fully satisfactory. The following diagram describes the problem.

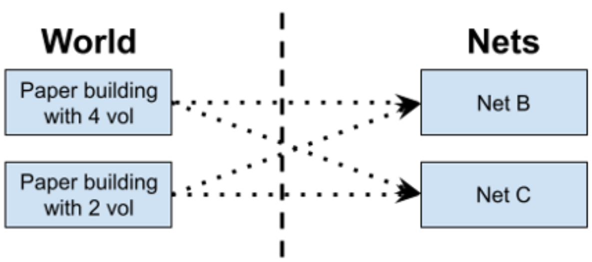

Figure 3.12: Episode 3: Collective Problem

After the disagreement and lack of conclusion on either side, the teacher moves to a process-of-elimination method guiding students to the shapes they can (more) easily agree upon. This method hits a snag when one student (Nicole) disagrees with the rest of the class. Yet after changing her mind and explaining why the class was right, Nicole solves the collective problem for Net B. Below is the diagram which documents the class's work through the episode.

17:45 Ms. C: ((holds 3 volume shape in air)) It is 'D'? What makes you think it is? What you when you look at 'D' what do you see that makes you think this shape?
17:52 NICOLE: Four sides down the block and three sides in the top and the bottom.
18:02 Ms. C: Okay, so this would be the top ((at board points to square on left)) and this would be the

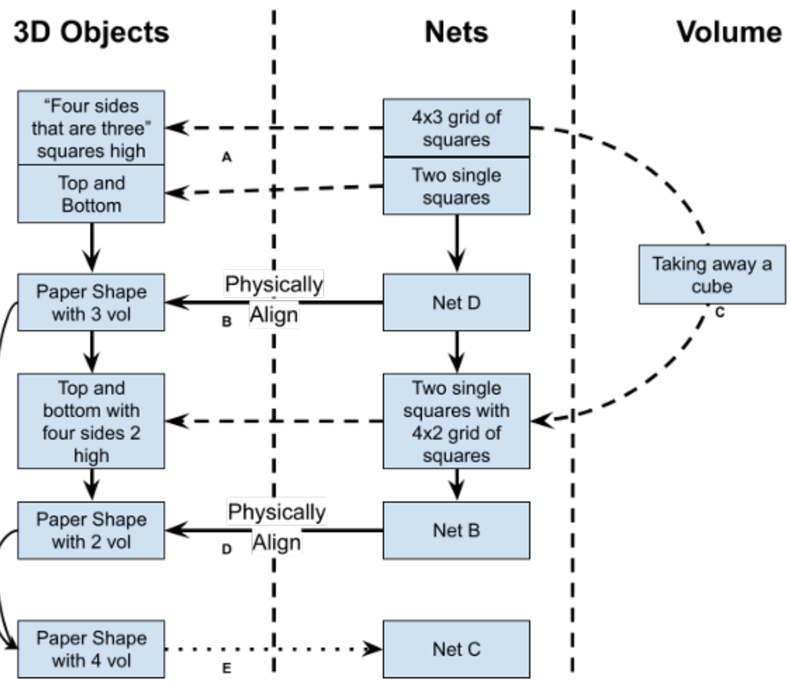

Figure 3.13: Episode 4: Mathematical work

bottom((points to square on right))? And here you would have ((runs finger down middle of net)) the four ((sits back down)), did you say four sides that are what?
18:11 NICOLE: Four sides that are three.
18:14 Ms. C: Four sides that are three, um, squares high?
18:17 NICOLE: ((nods her head yes))
18:18 Ms. C: Yep, I think. I think she's right ((holds 3 volume sideways up to net D between top and bottom squares)). And I think the rest of you who said that first are right. If I put that on there does it make it easier for you to see [how 'D' could become.

Within her explanation, Nicole aligns specific parts of net D with specific parts of the 3D object describing the center part of the net as a configuration of squares that align to the configuration of faces on the 3D object (arrow A). Nicole sweeps the alignment of these parts to form a connecting between net D and the 3-volume paper shape (arrows B within both 3D objects and Nets). Ms. C revoices this alignment adding gesture to the parts of the net and fully articulating Nicole's idea of "three" as "three squares high." Ms. C follows this up by physically aligning the 3D object up to the net (arrow C) to support students' mental connecting of net D with the 3 volume shape. Yet, Nicole takes her thinking further.

18:18 NICOLE: [B's two.
18:20 Ms. C: B is two what?
18:21 BRITTANY: Two square.
18:22 NICOLE: B is the two one.
18:24 Ms. C: Why do you think so? The [B
18:25 NICOLE: [It's not as big as C, I mean D, and it has, and it's exactly like D only it has two squares and Justin's right for C.

18:37 Ms. C: Okay, so Nicole is saying that she thinks net B is almost exactly like net D, except instead of each side of the building be three squares high it's two squares high. She thinks that B could become the two apartment building. Do you think Nicole is right?
19:05 STUDENTs: Yes!

Nicole makes a significant leap to relate her argument of net D as the 3-volume shape to net B as the 2-volume shape (arrows D). She describes a relation between net B and net D in similar terms to the volume relation of the 2-volume shape and the 3-volume shape as if taking away a cube (arrow E). When done on the net, the configuration of squares changes from 4 by 3 to 4 by 2 ("only it has two squares"). This motion is a new type of operating within the net realm, and Ms. C backs it up by physically aligning the 2-volume shape with net B same as she did with net D (arrow F). Additionally, Nicole returns to the process-of-elimination method (arrows G) to conclude the last net (net C) must go to the last paper shape (4-volume shape) (arrow H).

19:07 Ms. C: So what does that mean that C will turn into?
19:09 STUDENTs: The square one
19:12 STUDENTs: The four
19:13 Ms. C: This [one? ((holds up 4-volume shape))
19:14 Student: C
19:14 STUDENTs: Yeah
19:15 Student: No::
19:16 STUDENTs: Yeah ((louder)) (inaudible chatter)

In this final part, Ms. C attempts to follow Nicole's final argument to the conclusion that net C turns into the 4-volume shape (arrows G and H), yet this process is not satisfactory for all students, which implies some students hold a slightly different standard to connect a net with a 3D object.

In this final episode, Nicole navigates around the obstacle between net B and C and the 2- and 4-volume shapes by applying a similar action to the volume action of taking away one cube. Such a process is non-trivial in the representational realm of nets because of the non-additive relationship between surface area and volume. Instead of removing a net of an entire cube, Nicole mentally works out the configuration of squares from the cube net which are necessary to subtract (a row of four). This type of navigation shows how fluent with the representational realm of nets Nicole has become.

3.5 Discussion

Within the episodes presented in this paper, we make the argument that students have two broad ways of symbolizing at their disposal, connecting (deciding which aspects of a situation/object are represented by particular aspects of an inscription and re-envisioning that situation/object through the inscription and operating within, (treating the inscription as a notational system that allows for composition, deletion etc -take away a cube) and that these actions are qualitatively different. Furthermore, we propose chaining these two ways of symbolizing to successful circumnavigate an obstacle, strategic symbolic navigation, as a powerful way for students to learn representations. The three episodes we present in this paper are three ways a classroom group can appropriate a representational system into their class repertoire. Further analysis would be needed to understand the mechanisms which made these episodes possible, but the importance of gesture within the record and physicality of the two representational systems are two fruitful avenues. Additionally, the current analysis demonstrates two key additions of our framework: better understanding the conceptual resources at students' disposal, and a language and inventory of a class's collective problems and their work to solve them.

First, by analyzing students' work in terms of the representational realms they are connecting and operating within, we can see how they are (or are not) making use of the conceptual resources within those realms and which are most important for the local class. In Episode 1, the class uses of their experiences with 3D objects, with how closed 3D objects "fit together," with symmetry and alignment. In Episode 2, they use imagined cubes, imagined likeslinks between shapes, and the operation of linking those shapes. Finally in Episode 3, Nicole uses this conceptual resource of an operation to link net-made shapes on the nets themselves.

Second, our framework provides a language to better describe both students' work with representations and the collective problems they are trying to solve. For example, in Enyedy's (2005) paper, remember the class's collective problem is the fact that they must put away antheir ongoing build of a block city everyday, and they need a way to quickly and efficaciously rebuild where they left off. This problem is a problem of the physical classroom's space and the temporary nature of block buildings, a vertical obstacle within the realm of building block cities in classrooms. The class uses the compactness and semi-permanence of paper drawings as a representational solution and the focus of the class work is connecting the block city to drawings and back. Additionally, our framework provides an inventory of the types of obstacles students may encounter or facilitators may try to prompt. A vertical obstacle within the referent realm (like the one just described) is one. A horizontal object is another where the issues is one of connecting, like the one solved by Nicole. Finally, a vertical obstacle within a representational realm is also possible.

The final contribution of our paper is to highlight the curricular and research possibilities for the representational realm of nets in the younger grades. As we described earlier, the common core's standard (6.G.4 of CCSSO 2010) is a 6th grade standard. The current analysis demonstrates how students as young as 2nd grade can strategically explore and use nets to better understand 3D space. One possible avenue for additional research would be to consider the representational realm of nets as akin to a "micro-world" (Papert, 1980) for representation. Papert conceptualizes a "micro-world" as "…a subset of reality or a constructed reality whose structure matches that of a given cognitive mechanism so as to provide an environment where the latter can operate effectively" (p. 204). Nets provide a specific, physical action to connect representations to referents creating an environment for students to connect and operate on representations and learn. Such an environment for representations may act to help students 'get to know' systems of representation.

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CHAPTER 4

INFRASTRUCTURE TO SUPPORT STUDENTS EXERCISING CONCEPTUAL AGENCY

[1]

4.1 Introduction

Generative activities are activities operating at the individual, small group, and whole class within which students are actively constructing connections and relations of mathematical ideas in both prepared and emergent participation structures that reflect and build on the mathematical ideas that the group creates (Stroup et al., 2002, 2004, 2005; Ares et al., 2009). In these types of activities, the class's social group functions to explore mathematical structures together and uses their social dynamics as a purposeful resource to support their exploration. A common means for designing and developing such activities take a standard, closed-form question as a starting point, and "inverts" it, making the answer of the standard question into the prompt for the generative activity. For example, instead of asking students to "simplify 4(x-3)+12" (a closed-form question, with correct answer "4x"), one might ask them each to create several expressions that are "the same as 4x" (Stroup et al., 2002). By inverting the traditional one-correct-answer task, generative activities provide ways for students to construct or apply mathematical principles (e.g., exploring additive inverses by repeatedly adding "+x-x" to an expression known to be equivalent to 4x. When this kind of construction is occurring in parallel across the classroom, students are able to use the diversity of their group and their ideas for experimentation to generate a mathematical space.

Stroup et al. (2002; 2005) describe the resonance of generative activities with classroom network technologies to provoke new theoretical, methodological, and design frameworks. They articulate two main principles in the flow of a generative activity: (a) space-creating play and (b) dynamic structure. Space-creating play is the idea of students generating a mathematical space via experimentation, exploration, and playfulness. Dynamic structure refers to the emergent set of connections and meanings that appear as the students produce mathematical creations and respond to each other's work, both by commenting and by imitating, expanding on, or combining work to make new creations. Dynamic structure makes use of a functional sense of activity structure that is brought into being through students' playful actions and characterizes the unfolding space students are generating. Stroup et al. use these two ideas to argue that the relationship between mathematical/scientific structures and social structures is dialectical, with each mutually building off of the other. Essential to this process is the collective, public display of students' mathematical space, either in some physical/digital inscription or through social display.

As a complementary perspective of these public displays, we can consider them a space for conocimiento (Anzaldúa, 1987 cited by Gutiérrez, 2012), or sense of becoming familiar, connecting, and receptive of others. Through students' shared solidarity in generating the mathematical space, they develop their conocimiento of both the unfolding mathematical structures and the persons engaged in the display. Additionally, public displays of their work at the whole-class level may support students' sense of *nos/otras* (Anzaldúa, 1987 cited by Gutiérrez, 2012), or the juxtaposition of the collective and the individual. Further connections of this perspective with generative activities and classroom networks is unexplored and possibly very fruitful because of their differences in framing knowledge but similarities in positioning participants as generators of that knowledge.

Though Stroup et al. further describe the *resonance* between generative activities and classroom networks, arguing that the networked classroom is particularly suited to support a dialectic relationship between space-creating play and dynamic structure, few studies have explored these constructs in mathematics classrooms without the technology[2]. Substantial research has shown the impact of these new networking technologies and their resonance with generative activities (e.g. Ares et al., 2009; Ares, 2013; Stroup et al., 2011), but these technologies are both largely unattainable for most classrooms and still going through continuous re-design. Thus, we need to understand the specific features of the classroom network critical to fostering collective mathematics inquiry through space-creating play and dynamic structure and which are optative. Furthermore, understanding which of the features should be customizable and which are fairly generic to collaboration will both support continued technology design and strengthen the underlying theory of collective mathematics. To investigate these features of classroom networks, we investigate 1) Do generative activities and collective mathematical exploration put strain on normal classroom infrastructure? (and, how?) and 2) Which aspects of classroom networks alleviate that pressure? (and, how?).

4.2 Classroom Networks and the Group-based Cloud Computing System (GbCC)

Classroom networks have been an area active, but uneven, research and development for over 20 years (or much longer, depending on one's definition (see Abrahamson, 2006; Abrahamson and Brady, 2014; Roschelle et al., 2004), with a varied history of research and commercialization efforts. For the purposes of this paper, a classroom network (c.f. Brady et al., 2013) is a representation and communications infrastructure (Hegedus and Moreno-Armella, 2009) consisting of hardware, software, and curricular/activity components. The hardware includes a set of devices (laptops, smartphones, or other custom communications- enabled "computers"), with each student (or, less commonly, each small group), having a device. These devices are networked to communicate directly or indirectly with each other and with a teacher computer, which is connected to a public display (usually a digital projector). Software, running on the classroom computers and/or on a networked server, provides aspects of communications infrastructure by routing messages among the participating devices in configurable, activity-specific ways. Software also provides a representation infrastructure, offering students and teachers views of the activity and tools to contribute that are appropriate for the discipline, the activity, and the participants' roles. Finally, at the curricular/activity level, "documents" or other specifications of roles or goals can be sent to participants to configure their devices and displays, and to facilitate the activity in real time.

GbCC (Brady et al., 2018) is a system of this kind, emphasizing flexible programmability and rich discipline-specific representations for mathematics, science, and the social sciences. It leverages browser-based open-source tools, building upon the NetLogo Web agent-based modeling environment (Wilensky, 2015), married with GeoGebra Web (https://www.geogebra.org/) as a dynamic mathematics platform for geometry and algebra in Euclidean and Cartesian representations; and several other extensions to support mapping (Leaflet, https://leafletjs.com/) and 2d physics (Box2d, https://box2d.org/). As a platform for design-based research environment, its programmability supports an open-ended array of activity structures, and it can be run on any browser-enabled device (phones, tablets, or laptops). Its flexibility, configurability, and programmability make it ideally suited to exploring our research questions.

4.3 Data and Methods

The current study was a single four-week cycle from a larger design-based research (DBR) project.[3] The 20 participants came from a 5th grade classroom at a public middle school serving a racially (39% Black, 6% Hispanic, 4% Asian) and economically (41% free or reduced lunch) diverse population within a large metropolitan district in a midsize southern city in the USA. The class period of the DBR study was not students' normal mathematics class but a time when students were tracked based on standardized tests in order to provide individualized attention (called Personal Learning Time, PLT, in the school). The participants from the current study were considered math tier 2 students (i.e., on target but needing some extra time for mathematics). Because of the nature of standardized testing and the flexibility of this class period, students moved from tier to tier or subject to subject depending on the most current testing. Thus about half of the students in the current study had participated in a prior implementation of a design cycle with generative activities without technology. The first author facilitated about 2 class sessions each week over a four week period totaling of 8 sessions, each 30-45 minutes in length, and the classroom teacher either co-facilitated or pulled specific students for individual work.

The primary data source for the current study was design and field notes taken by the first author. Audio and video recordings of each lesson were also collected and used to triangulate findings. Analysis was ongoing and continuous throughout the design where the humble theories of the class's mathematical thinking and engagement were revised after each lesson (Cobb et al., 2003), in conversations among the researchers and with the teacher. Posterior analysis took the form of reviewing the progression of the lessons contrasted with the predicted learning trajectory. We paid special attention to anticipated and unanticipated challenges and strains on the classroom system prior to introducing network technology and the nature of how those challenges and strains changed when using it.

4.4 Mathematical Context and Predicted Learning Trajectory

We chose to target 5th grade fractions standards involving equivalence, operations, and comparison for this study. Fractions have been found to be a particularly difficult concept for students, yet they can be readily used as the basis for generative activities because the mathematical space of equivalent fractions is both core to the standards and very rich. We created a sequence of generative activities, to explore equivalence for the first two weeks and then operations on fractions for the second two weeks. The activity for both topics followed a similar rough structure. The first day of each of the two weeks focused on "space-creating play" to generate the space of ways to make $\frac{1}{2}$, either with equivalent fractions or with fraction operations, depending on the topic. Students worked in small groups during these times, to foster connections in their space-creating play and reflection on the dynamic structure they were creating. Following this small group work to make $\frac{1}{2}$, a whole-class discussion explored the different kinds of objects in the space (to make $\frac{1}{2}$) and the mathematical principles students used to generate the space. Building off this the following class session (a week later), students returned to small groups to generate ways to make a fraction of their group's choice followed by another whole-class discussion of the mathematical principles. This trajectory was supported by research both on fractions (Lamon, 2012) and generative activities (Stroup et al., 2002, 2005), the key difference from the latter was the lack of networking technology. Beyond the curricular goals, we predicted the generative activities would support students to take conceptual agency (Boaler and Greeno, 2000) in the classroom to create mathematical principles of equivalence and operation and to voice their conceptual perceptions even without technology. We remained open to the question of whether these technologies would be needed, by observing the classroom system, students' engagement in the tasks, and the degree to which they exercised conceptual agency.

4.5 Results

Through our design and analysis of generative activities to support students' conceptual agency in exploring fractions *without* technology, we found that these activities put multiple strains on the classroom system for students to engage and participate. Without the technological infrastructure and additional ways to participate in the activity, the whole-class discussions led by the first author were not able to support students to have a platform to show the work they did in small groups, or to have much of a "voice" at the whole-class level. This central strain reduced students' engagement over time, and following the second whole-class discussion (week 2), the necessity of additional infrastructural support was apparent, both to the authors and to the classroom teacher. Upon the introduction of technology, students' re-engagement in the generative-activity process was visible, as usual with the introduction of any new technology. Yet more meaningfully, students' engagement was sustained through the last two weeks, and their conceptual agency increased in that time. This process contrasted significantly with the time without technology when their engagement and utilization of conceptual agency decreased over the course of the same time period. By comparing the strains on the classroom system during generative activities without technology and how the infrastructure provided by the technology relieved those strains, we can begin to identify some of the crucial features of classroom networks.

4.5.1 Generative Activities' Strains on the Classroom System

Progressively throughout the first two weeks of equivalent-fraction generative activities, we documented how students became less and less engaged and utilized their conceptual agency less and less. This process came to a climax when the classroom teacher requested a change in the activity in order to re-engage students at the end of week two. Upon analysis of the design, students' disengagement was progressive. Students engaged readily in the initial generative activity convening the space-creating play in almost all the small groups. Some groups even utilized their conceptual agency to recognize patterns and methods in their generation of equivalent fractions. Yet, during the whole-class discussion, students struggled to know how to participate in productive ways and see their hard work validated. Multiple students made various bids to read aloud their list of fractions in its entirety, but with upwards of over 30 fractions, this was not logistically possible. Moreover, without a means to organize or represent these contributions visibly, a reading would not have contributed to the dynamic structure. Instead, the first author focused on having students share out their methods of generating fractions and patterns they observed in their set of equivalences. While students did engage in the discussion and built multiplicative conceptual resources for fractions, field notes capture a number of students' feelings of discontent.

The following week, the first author launched another generative activity to build on students' work with $\frac{1}{2}$ by generating fractions the same as a fraction of their choice. Unlike the start of the previous activity, the teacher and the first author struggled to support students to begin the activity (even to choose a fraction), and to convene space-creating play in their small groups. In the students' eyes, the small group work had lost its importance and meaning after the previous week's whole-class discussion when they perceived their work was left unchecked, ungraded, and unshared with the class. While either adult was present, students would work together to generate equivalent fractions, but their motivation reflected a perceived lack of importance of their work at the whole-class level. Thus, students' patterns and methods were much less robust during the whole-class discussion the following day, and fewer students participated. Additionally, one of the students from the previous week made another bid to read all of her fractions aloud, demonstrating a continued desire to showcase her work at the whole class level, to hear her voice as part of the group, and receive validation for the effort she had put in. Because of students' steeply declining engagement, we decided to introduce technology to re- engage students and support their sustained participation in generative activities. Our prediction was that the introduction of technology would quickly re-engage students with the task of generative activities, and that comparison in students' sustained engagement would reveal the some of the crucial features of classroom networks to support students' collective mathematics in generative activities.

4.5.2 Adjusted Learning Trajectory and Use of Classroom Networks

Because of the strains of the classroom system for students to see their work as meaningful at the whole-class level, we adjusted the research plan to incorporate GbCC support for the activities in the final two weeks. Since the activities designed with the technology did not strictly align with the original learning plans, we adjusted the curricular goals to target fraction comparison instead of fraction operations. We planned to use GbCC's public display to create a joint representation for students to see a reflection of themselves and their classmates as they engage with mathematics. The classroom network assembled students' fraction input as a character moving on a vertical line between a teacher-defined maximum and minimum value, with its y-coordinate corresponding to the fraction value. The class appeared as a collection of these characters moving between the max and min values. If a student's fraction input was outside of this range, their character was shown into a gray area above or below. The goal of the first week was for students to make connections from their work with equivalence within the technology as a way to begin to understand the representational forms it used and then for the class to quickly transition into comparing 'easy' fractions. We wanted students to have the chance to explore within a technologically enhanced representational world and for the class to see each other's explorations to discuss our methods and strategies. In this way, the classroom network would provide additional communicative pathways for students to feel their work and their classmates' work were meaningful at the whole-class level. We planned to end the activity sequence with supporting students to see the *density* of fractions (i.e. that between any two fractions there is another fraction). We conceptualized this as a 'zooming in' effect with the technology where the teacher could make the range a subset of the previous defined range and fractions could still be found.

The first two days of implementing GbCC went as predicted. The technology served to revive students' engagement and enthusiasm while also providing additional tools and representations to the work they were doing as a whole class. The public, anonymous display provoked a collective responsibility to fill it, positioning students to hold each other accountable during the activity, and during whole-class discussions, this public representation was a collective object for us to reference. During this space-creating play, students exercised their conceptual agency by choosing personally relevant numbers (not something seen the previous week). For example, one student found the fraction equivalent to 1/7th where the numerator was her birthday (mmddyy). Students patterns and methods extended the ideas from previous weeks using multiplicative relationships to generate equivalent fractions.

The final week of the study focused on comparing fractions, with the goal of students' having insight into the density of fractions. We started with a whole-class discussion of the previous weeks' work and asking if students had ways to know if one fraction is bigger than another (no technology). Even without technology, the class sustained a meaningful discussion, leveraging the collective perspective provided by the classroom network activities the previous week. In the following two days of activities, students sustained engagement and motivation, unlike the second week without classroom networks. Furthermore, students' utilization of conceptual agency grew as their fluency with the technology grew, compared to declining as their engagement declined, in the first two weeks. As students interacted with and in the mathematical space, a few began to use the public display as a dynamic representation - moving their characters across the screen by manipulating their fraction input successively. This type of play showcased how the classroom network became an embedded infrastructure for students to represent movement and communicate their actions to me and to others. Additionally, while these playful actions were unexpected and in fact went against the underlying goal of the activity for students to develop insight into the density of fractions, students were developing individual and share-able fluency with manipulating and comparing fractions in service of the personally- meaningful goal to predict the movement of their character up, down, and into the middle. Such spontaneous, and unpredicted, utilization of conceptual agency was not present without the classroom network's representational and communication infrastructure.

4.5.3 Crucial Features of the Classroom Network

The above analysis explored how generative activities strained a classroom system without adequate representational and communicational infrastructure and identified features of classroom networks that were crucial to relieving those strains and supporting students in utilizing their conceptual agency. The collective, public representation of students' work with fractions was the focal point of two such crucial features that supported collective mathematics and that were very difficult to provide without technological support. First, as demonstrated in the first week and the follow-up discussion without technology, the *public display of an aggregated representation of students' contributions* provided an essential means of discussing the activity, referring to students' work in context, facilitating activity flow, and sustaining students' attention. Leveraging this feature, we were able to facilitate whole-class discussions where students engaged in illuminating the underlying multiplicative structure of equivalent fractions and continue the conversation even when the technology was temporarily removed. These types of whole-class discussions were very different prior to implementing the technology when students did not have such a collective orientation, and they made multiple bids to reorient the discussion towards what they felt was important (e.g., their personal lists of equivalent fractions).

The second crucial feature relating to the public display was *the communal, real-time dynamic nature of the public representation*. Students displayed a sense of both collective effort and individual publicity, or *nos/otras* (Gutiérrez, 2012). Simultaneously feeling both connected to the community and represented as an individual was essential for collective mathematics. The importance of this feature was demonstrated first when the classroom network was first introduced as students began to hold each other accountable to participate in the activity, and it grew further when students began utilizing their conceptual agency and publicizing their new skill of predicting the movement of their character, showing their abilities to others and sharing how they did it.

4.6 Discussion: Students Utilizing Conceptual Agency with the Technology

Understanding how the representational and communicational infrastructure of classroom networks support students' space-creating play and utilization of their conceptual agency can provide insight into these technologies' functionality and support their ongoing design. At the same time, it also can inform efforts to enact generative activities without classroom networks, identifying needs and resources for alternative supports in such classrooms. Based on our comparison here of a classroom with and without technology, two crucial features of the dynamic infrastructure emerged, in the collective orientation provided by the public representation and the simultaneous communicative avenues of collective and individual voice developing a sense of nos/otras. These aspects are vital to keep in mind as we continue to design classroom networks, infrastructure, and activities to further support students exercising their conceptual agency.

Additionally, generative activities need to be flexible enough to support students' adaptation of the task as they exercise their conceptual agency. Similar to work in microworlds (Edwards, 1998), generative activities supported by classroom networks are not capsules of disciplinary learning and conceptual agency. Rather, we need to design for and encourage students to make expressive and unpredicted conceptual moves as they interact with the representations and concepts of the activities. On the other hand when the classroom system does not have the infrastructure of classroom networks, traditional infrastructures must be adjusted to foster collective orientation and nos/otras. Specifically, students need some form of collective representation of the concept to orient their individual or small-group work towards each other. Furthermore, social infrastructure must support students as they make their work public to both hear their own voice and, metaphorically, hear the voice of the choir. Over time, classroom systems can develop these types of social infrastructures through socio-mathematical norms, but classroom networks may foster more rapid development of them or a lower threshold of effort for sustaining them over time.

4.7 Implications for Further Research

Classroom networks provide a flexible space for students to interact, both with mathematical ideas and with each other, and a dynamic, public display of their work as it unfolds. This space quickly creates infrastructure in the class to foster students' prolonged engagement and utilization of their conceptual agency. Yet pragmatically, teachers, administrators, and researchers may question the necessity of this technology when compared to its cost and disruption. By observing and documenting first how a classroom group experienced strain without the technology and then was supported by it, we understand better the value of the technology, what types of additional activities may supplement it, and ideas on how we might support the classrooms without it. Additional work should compare other types of representational and communication infrastructures (Hegedus and Moreno-Armella, 2009) and curriculum activity systems (Roschelle et al., 2004) to better understand how students participate in collective inquiry and the necessary of these infrastructures to support students in exercising their conceptual agency. Specifically, previous studies have shown collective inquiry is possible without technology (e.g. Ball, 1993; Lehrer et al., 2013; Fiori and Selling, 2016), and exploring the infrastructure imbedded in these types of classrooms will provide insight into both the dynamics of group mathematics learning and into the design of networking technology.

Notes

1. This chapter is adapted from "INFRASTRUCTURE TO SUPPORT STUDENTS EXERCISING CONCEPTUAL AGENCY" published in *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* and has been reproduced with the permission of the publisher and my co-author Dr. Corey Brady.

Nichols-Paez, I., & Brady, C. (2019). Infrastructure to support students exercising conceptual agency. *Proceedings of the Forty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

2. Stroup's introduction of the construct of generative activities clarifies that their roots lie outside of mathematics, connecting to work in reading comprehension by Wittrock and in shared identity building by Freire (the identification of a community's "generative words").

3. A disruption of losing half the participants and gaining the same number of new students caused analysis of the two design cycles to lose much of its meaning, but the class during the analyzed cycle remained intact.

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