A GENERALIZATION OF THE DISTORTION FUNCTION AND THE ASYMPTOTIC GEOMETRY OF SUBGROUPS

By

Andy Jarnevic

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Approved:

Denis Osin
Spencer Dowdall
Mike Mihalik
Alexander Olshanskii
Paul Sheldon

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TABLE OF CONTENTS

Chapter I

Introduction

The distortion function was introduced by Gromov in [9] and given a finitely generated subgroup H of a finitely generated group G, essentially detects how much the intrinsic geometry of H differs from the geometry induced on H by G. One motivation for studying the distortion function is that it is closely related to the membership problem. Specifically, given a finitely generated subgroup H of a finitely generated group G, the membership problem of H in G is solvable if and only if the distortion function is bounded by a computable function.

The distortion function is not well-defined when the subgroup H is not finitely generated. In this thesis, we define a variation of the distortion function that can be computed in the case that H is not finitely generated. We name this function the generalized distortion function, and denote it by δ. We calculate this function in a few natural cases, and demonstrate a connection between this function and the isoperimetric spectrum defined by Osin and Rybek in [17]. One consequence of this connection is that the group $F_2 \times F_2$ contains an uncountable collection of subgroups with mutually incomparable generalized distortion functions.

We then turn our attention to an application of this function to asymptotic cones. The asymptotic cone of a group G is a metric space which captures certain aspects of the coarse geometry of G. Roughly speaking, the asymptotic cone is how the group looks from infinitely far away, and is constructed by taking a certain limit of scaled down copies of the group viewed as a metric space. The roots of asymptotic cones come from a paper of Gromov proving that finitely generated groups of polynomial growth are nilpotent [8]. Van den Dries and Wilkie added non-standard analysis to the construction in this paper, formally introducing asymptotic cones [24]. Since then, several other standard algebraic and geometric properties of groups have been shown to have natural parallels in their asymptotic cones. For instance, a finitely generated group is virtually abelian if and only if all of its asymptotic cones are quasi-isometric to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ [9], and a finitely-generated group is hyperbolic if and only if all of its asymptotic cones are $\mathbb{R}$-trees [9].

Given a finitely generated subgroup H of a group G, we define a subspace of the asymptotic cone of G which corresponds to H, and denote this subspace by $Cone_G^\omega(H)$. We show that the function δ detects whether this subspace is connected. We also show that in this case, the ordinary distortion function is not sufficient to decide whether or not $Cone_G^\omega(H)$ is connected.

We then turn our attention to convexity properties of subgroups of groups. The notion of a subgroup being strongly-quasi convex was introduced independently by Tran and Genevois [7, 23], and is a generalization of the notion of quasi-convex subgroups of hyperbolic groups that works nicely outside of the hyperbolic setting. We show that whether a subgroup is strongly quasi-convex can be detected by a natural convexity property of the embedding of $Cone^{\omega}_{G}(H)$ in $Cone^{\omega}(G)$. We use this characterization to show that any group satisfying a law cannot have an infinite, infinite index strongly quasi-convex subgroup. We also show that if H is a strongly quasi-convex subgroup of a group G, then the stabilizer of $Cone^{\omega}_{G}(H)$ in $Cone^{\omega}(G)$ under the natural action of G is the same as the commensurator of H in G.

Finally, we provide applications of these results to groups with Morse elements. Any group with a Morse element has cut points in its asymptotic cone, and it was shown in [21] that any loxodromic element of an acylindrically hyperbolic group is Morse. Thus, groups with Morse elements form a natural intermediary class between acylindrically hyperbolic groups and groups with cut points in their asymptotic cones. It is therefore natural to ask which properties of acylindrically hyperbolic groups can be extended to the class of groups with Morse elements. One possible candidate for such a property is the fact that any acylindrically hyperbolic group where all non-trivial conjugacy classes are infinite is mixed identity free. We show that with certain additional assumptions, groups with Morse elements are also mixed identity free.

The thesis is structured as follows. We begin in Chapter 2 with a more detailed overview of our results. We then provide the necessary background on a number of topics relevant to this thesis in Chapter 3. The next chapter proves some basic properties and computes some basic examples of the generalized distortion function. We also discuss the relationship between the generalized distortion function and the isoperimetric spectrum. Chapter 5 discusses the case where the subgroup H is finitely generated. We compute some examples in this case and prove that the generalized distortion function detects the connectedness of $Cone^{\omega}_{G}(H)$. Chapter 6 deals with convexity properties of $Cone^{\omega}_{G}(H)$ and their applications. We conclude by discussing a few open problems related to the results of this thesis.

Chapter II

Overview of main results

II.1 Generalized distortion

We begin by recalling the definition of the distortion function.

Definition II.1.1. Let H be a subgroup of a group G, with $G = \langle X \rangle$ and $H = \langle Y \rangle$ where X and Y are finite sets. The *distortion function* of H in G with respect to X and Y is defined by the formula

$$\Delta_{H,Y}^{G,X}(n) = \max\{|h|_Y \mid h \in H,\ |h|_X \leq n\},$$

where $|h|_Y$ denotes the word length of h with respect to the generating set Y. A subgroup H of a group G is called *undistorted* if $\Delta_{H,Y}^{G,X}$ is bounded from above by a linear function.

We consider distortion up to the following equivalence relation.

Definition II.1.2. For non-decreasing functions $f, g\colon \mathbb{N} \to \mathbb{N}$, we write that $f \preceq_\Delta g$ if there exists a constant C such that $f(n) \leq Cg(Cn)$ for all $n \in \mathbb{N}$. We write $f \sim_\Delta g$ if $f \preceq_\Delta g$ and $g \preceq_\Delta f$.

Up to this equivalence relation, the distortion function does not depend on the choice of the finite generating sets for G and H.

For the distortion function to be well-defined up to equivalence, it is necessary for both the group G and the subgroup H to be finitely-generated. Here, we provide a variation of the distortion function which can be computed even when the subgroup H is not finitely generated.

Definition II.1.3. Let H be a subgroup of a group G with finite generating set X. We will denote by $Y_{H,m}$ the set $\{h \in H \mid |h|_X \leq m\}$. When the subgroup H is clear from context, we will simply denote $Y_{H,m}$ by Y_m. We will denote by H_m the subgroup of H generated by $Y_{H,m}$.

Definition II.1.4. We will denote by T the set $\{(k,m,n) \in \mathbb{N}^3 \mid k \leq m\}$.

Definition II.1.5. Let H be a subgroup of a group G with finite generating set X. We define $\delta_H^{G,X} : T \to \mathbb{N}$ to be the function defined by $\delta_H^{G,X}(k,m,n) = \max\{|h|_{Y_m} \mid h \in H_k, |h|_X \leq n\} = \Delta_{H_k,Y_m}^{G,X}(n)$. We refer to this function as the *generalized distortion function*.

Note that the domain of δ is T, as if $k > m$, then it is possible that Y_m is not a generating set for H_k.

We consider generalized distortion up to the following equivalence.

Definition II.1.6. Given two functions $f,g : T \to \mathbb{N}$ which are non-decreasing in the first and third variable and non-increasing in the second variable, we write $f \preceq_\delta g$ if there exists a natural number C such that for all $(k,m,n) \in T$, we have that $f(k,Cm,n) \leq Cg(Ck,m,Cn)+C$, whenever $(Ck,m,Cn) \in T$. We say that f and g are equivalent, denoted $f \sim_\delta g$, if $f \preceq_\delta g$ and $g \preceq_\delta f$. We write that $f \prec_\delta g$ if $f \preceq_\delta g$, but f is not equivalent to g.

Up to this equivalence, the generalized distortion function does not depend on the choice of the finite generating set of G. We now have two equivalence relations $\sim_\delta$ and $\sim_\Delta$ for the two functions δ and Δ. In most cases, it will be clear from context which equivalence relation is relevant, and in these cases we will supress the subscripts.

Finitely generated free groups and wreath products of finitely generated groups are important examples of finitely generated groups with non-finitely generated subgroups. We calculate δ in both of these cases. For background on wreath products see section III.3.

Theorem II.1.7. *(Theorem IV.1.3) If G is a finitely generated free group, and H is any subgroup of G, then $\delta_H^G(k,m,n) \preceq n/m$.*

Theorem II.1.8. *(Theorem IV.1.7) Let A,B be finitely generated groups, and let $G = A \wr B$. If $H = A^{(B)}$ is the base group of the wreath product, then $\delta_H^G(k,m,n) \preceq n/m$.*

Next we turn our attention to a connection between the generalized distortion function and the isoperimetric spectrum defined in [17]. This function, denoted by f is a generalization of the Dehn function to finitely generated but not necessarily finitely presented groups analogous to the generalized distortion function δ. These functions are considered up to a natural equivalence relations. For details about these functions and this equivalence relation, see section III.1.

In [16], an interesting connection between the distortion function and the Dehn function is presented. Specifically, the authors prove the following.

Theorem II.1.9. *Up to equivalence, the set of Dehn functions of finitely presented groups is the same as the set of distortion functions of subgroups of $F_2 \times F_2$.*

We prove the following related theorem.

Theorem II.1.10. *Theorem (IV.2.2) If f is a function which can be realized as the isoperimetric spectrum of a finitely generated group, then f can also be realized as the generalized distortion function of a subgroup of $F_2 \times F_2$.*

Note that this theorem is weaker than the obvious analogy with II.1.9. In particular, the theorem does not claim that any function which can be obtained as the generalized distortion of a subgroup of $F_2 \times F_2$ can be obtained as the isoperimetric spectrum of a finitely generated group.

It is proven in [17] that there exists an uncountable set of groups with pairwise incomparable isoperimetric spectra. Therefore we have the following corollary of II.1.10.

Corollary II.1.11. *There exists a set of uncountably many subgroups of $F_2 \times F_2$ with mutually incomparable distortion functions.*

Note that a finitely generated group can have only countably many finitely generated subgroups. Thus, a similar result cannot be obtained for the ordinary distortion function and in fact represents a genuinely new phenomenon for the generalized distortion function.

Another result from [17] is that there exist finitely generated groups whose isoperimetric spectrum is not fundamentally independent of the variable k, i.e. f is not equivalent to any function of the two variables m, m. Hence we also obtain the following corollary of II.1.10.

Corollary II.1.12. *There exists a subgroup H of $F_2 \times F_2$ such that $\delta_H^{F_2 \times F_2}$ is essentially dependent on the variable k.*

We then turn our attention to the case where H is finitely generated. Note that in this case for sufficiently large k, the subgroup H_k will be the entire subgroup H. For this reason, the function δ is essentially independent of the variable k.

If G is a finitely generated group with finitely generated subgroup H, and X is a finite generating set for G, then $\Delta_{H,X\cap H}^{G,X}(n) = \delta_H^{G,X}(k,1,n)$ for all natural numbers n and all sufficiently large natural numbers k. It is in this sense that δ is a generalization of the distortion function.

The function δ turns out to be useful in particular to discern the connectedness of a certain subspace of the asymptotic cone of G.

Given a group G and an ultrafilter ω, we will denote the asymptotic cone of G with respect to ω by $Cone^{\omega}(G)$. We will study a natural subspace of $Cone^{\omega}(G)$ corresponding to a subgroup H. Essentially, points in the asymptotic cone of a group G can be represented by certain elements of the ultrapower G^{ω}. We denote by $Cone_G^{\omega}(H)$ the subspace of $Cone^{\omega}(G)$ consisting of points with a representative from H^{ω}. For the formal definition of this subspace, see section III.2.

We show that the function δ determines whether $Cone_G^{\omega}(H)$ is connected. Specifically, we prove the following result, which also shows that for such a subspace connectedness is equivalent to path-connectedness.

Theorem II.1.13. *Theorem V.2.13 For any finitely generated group G and any subgroup H, the following conditions are equivalent.*

1. *H is finitely generated and $\delta_H^G(k,m,n) \leq f(n/m)$ for some function $f : \mathbb{R}^{\geq 0} \to \mathbb{N}$.*

2. *$Cone_G^\omega(H)$ is path connected for all non-principal ultrafilters ω.*

3. *$Cone_G^\omega(H)$ is connected for all non-principal-ultrafilters ω.*

This theorem enables us to relate the ordinary distortion function to the connectedness of $Cone_G^\omega(H)$, and to construct pairs $H \leq G$ such that $Cone_G^\omega(H)$ is disconnected, but the distortion of H in G is small. Consider the following properties of a finitely generated subgroup H of a finitely generated group G:

1. H is undistorted in G,

2. $Cone_G^\omega(H)$ is connected for all non-principal ultrafilters ω,

3. Δ_H^G is bounded by a polynomial function.

The following theorem collects the relationship between these three properties.

Theorem II.1.14. *(V.2.18) For any finitely generated subgroup H of a finitely generated group G, the following implications hold:*

$$1 \Rightarrow 2 \Rightarrow 3$$

Further, the missing implications do not hold. Specifically, we have the following.

1. *For any $k \in \mathbb{N}$, there exists a finitely generated group G and a finitely generated subgroup H of G such that $\Delta_H^G(n) \sim n^k$ and $Cone_G^\omega(H)$ is connected for any non-principal ultrafilter ω.*

2. *For any real number $\varepsilon > 0$, there exists a finitely generated group G with a finitely generated subgroup H such that $\Delta_H^G(n) \preceq n^{1+\varepsilon}$ but $Cone_G^\omega(H)$ is disconnected for some non-principal ultrafilter ω.*

II.2 Strongly quasi-convex subgroups

We now provide the definition of a strongly quasi-convex subgroup of a finitely generated group.

Definition II.2.1. A subgroup H of a group G with finite generating set X is said to be *quasi-convex* if there exists a number M such that any geodesic in the Cayley graph $\Gamma(G,X)$ connecting two points in H is contained in the M neighborhood of H. H is said to be *strongly quasi-convex* if for all real numbers $\lambda \geq 1, C \geq 0$ there exists a constant $N(\lambda,C)$ such that any (λ,C)-quasi-geodesic in $\Gamma(G,X)$ connecting two points in H is entirely contained in the N neighborhood of H.

In general, quasi-convexity is not independent of the choice of the finite generating set of G. For instance, in the group $\mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle$, the subgroup $\langle ab \rangle$ is not quasi-convex with respect to the generating set

$\langle a,b\rangle$, but is quasi-convex with respect to the generating set $\langle ab,a\rangle$. In the case where G is hyperbolic, quasi-convexity is independent of the choice of the finite generating set.

We have the following relationship between these properties of a subgroup H of a finitely generated group G:

$$\text{strongly quasi-convex} \Rightarrow \text{quasi-convex} \Rightarrow \text{finitely generated and undistorted.}$$

None of the reverse implications hold. To see this again consider $G = \mathbb{Z}\times\mathbb{Z} = \langle a\rangle\times\langle b\rangle$. The subgroup $\langle ab\rangle$ is undistorted but not quasi-convex, and the subgroup $\langle a\rangle$ is quasi-convex but not strongly quasi-convex. However, in the case when G is hyperbolic, all of these properties are in fact equivalent.

Strong quasi-convexity is a generalization of quasi-convexity that is preserved under quasi-isometry in general. Tran [23] characterized strongly quasi-convex subgroups based on a certain divergence function, and showed that they satisfy many properties of quasi-convex sugroups of hyperbolic groups. Specifically, any strongly quasi-convex subgroup is undistorted, has finite index in its commensurator, and the intersection of any two strongly quasi-convex subgroups is strongly quasi-convex. Examples of strongly quasi-convex subgroups include peripheral subgroups of relatively hyperbolic groups and hyperbolically embedded subgroups of finitely generated groups.

We show that the property of being strongly quasi-convex is equivalent to a natural property of the embedding of $Cone^{\omega}_{G}(H)$ in $Cone^{\omega}(G)$.

Definition II.2.2. We say that a subspace T of a metric space S is *strongly convex* if any simple path in S starting and ending in T is entirely contained in T.

Theorem II.2.3. *(VI.1.12) Let H be a finitely generated subgroup of a finitely generated group G. H is strongly quasi-convex in G if and only if $Cone^{\omega}_{G}(H)$ is strongly convex in $Cone^{\omega}(G)$ for all non-principal ultrafilters ω.*

This characterization gives useful information about the structure of the asymptotic cones of groups with strongly quasi-convex subgroups. For instance, we obtain the following result.

Theorem II.2.4. *(Theorem VI.2.1) If G is a finitely generated group containing an infinite, infinite index strongly quasi-convex subgroup H, then all asymptotic cones of G contain a cut point.*

Next, we provide background and definitions for identities and mixed identities in groups.

Definition II.2.5. Given a group G, an element $g\in F_2$ is called an *identity* in G if for any homomorphism $\phi : F_2 \to G$, $\phi(g) = e$. An element $g \in F_2 * G$ is called a *mixed identity* if for all homomorphisms $\phi : F_2 * G \to G$ which are identity when restricted to G, $\phi(g) = e$. If no nontrivial elements of F_2 is an identity of G, then G is said to be *identity free*. If no non-trivial element of $F_2 * G$ is a mixed identity for G, then G is said to be *mixed identity free*, abbreviated *MIF*. Any mixed identity can be conjugated to be in the form $g_1 x^{n_1} g_2 x^{n_2} \dots g_k x^{n_k}$ where each $g_i \neq e$ and $n_i \neq 0$ for all $1 \leq i \leq k$. We call the set $\{g_1, g_2, \dots g_k\}$ the *coefficients* of the mixed identity.

In general, being mixed identity free is a much stronger property than being identity free. For example if g is a non-trivial element of G the conjugacy class of which has n elements, then G satisfies the non-trivial mixed identity $[x^{n!}, g]$.

Some consequences of a group being mixed identity free can be found in [11]. For instance, it is shown that if G is an ICC mixed identity free countable group, then a generic length function on G (in a certain technical sense) corresponds to a word length giving a Cayley graph isomorphic to a specific universal graph.

Combining Theorem II.2.4 with a result of Drutu and Sapir [6] gives the following result.

Corollary II.2.6. *(Theorem VI.2.3) If G is a finitely-generated group containing an infinite, infinite index strongly quasi-convex subgroup, then G does not satisfy a non-trivial identity.*

This result can be applied to show for instance that solvable groups and groups satisfying the law $x^n = 1$ for some $n \in \mathbb{N}$ cannot have infinite, infinite index strongly quasi-convex subgroups.

If G is a finitely generated group, then G acts naturally on its asymptotic cone. We call the stabilizer of $Cone_G^{\omega}(H)$ under this action the asymptotic stabilizer of H in G. We relate this subgroup to the commensurator of H in G.

Definition II.2.7. Let H be a subgroup of a group G. The commensurator of H in G, denoted $Comm_G(H)$ is the set of elements $g \in G$ such that $g^{-1}Hg \cap H$ is a finite index subgroup of both H and $g^{-1}Hg$.

An alternate characterization for the commensurator in the case that G is finitely generated can be found in [2].

Proposition II.2.8. *Let H be a subgroup of a group G with finite generating set X. Then the commensurator of H in G is the set of elements* $g \in G$ *such that* $d_{hausdorff}(gH, H) < \infty$.

It is useful to contrast this definition with that of the asymptotic stabilizer $Astab_G(H)$. Essentially, the commensurator is stricter in that for an element $g \in G$ to be in the commensurator it can only move elements of H a fixed finite distance. On the other hand, for g to be in the asymptotic stabilizer of H, it can move elements of H unbounded distances, as long as the distance it moves elements is sublinear in terms of the length of the elements of H. An example where these two notions differ is presented in section IV.1.

Theorem II.2.9. *(Theorem VI.2.12) If H is a strongly quasi-convex subgroup of a finitely generated group G, then the commensurator of H in G is the stabilizer of $Cone^{\omega}_{G}(H)$ in $Cone^{\omega}(G)$ under the natural action of G.*

Finally we turn our attention to applications of this result to groups with Morse elements.

Definition II.2.10. An element g of a finitely generate group G is called *Morse* if the subgroup $\langle g\rangle$ is strongly quasi-convex in G.

Any group with a Morse element has cut points in its asymptotic cone, and it was shown in [21] that any loxodromic element of an acylindrically hyperbolic group is Morse. Thus, groups with Morse elements form a natural intermediary class between acylindrically hyperbolic groups and groups with cut points in their asymptotic cones. It is therefore natural to ask which properties of acylindrically hyperbolic groups can be extended to the class of groups with Morse elements. One possible candidate for such a property is the following fact about acylindrically hyperbolic groups.

Definition II.2.11. If G is a group such that every non-trivial conjugacy class is infinite, then G is said to be an *ICC (infinite conjugacy class)* group.

Theorem II.2.12. *[10] If G is an ICC acylindrically hyperbolic group, then G does not satisfy any non-trivial mixed identities.*

We prove variants of this theorem with additional assumptions. The first involves the notion of a Morse element being ”orientable”. This notion is defined in Section VI.2.

Theorem II.2.13. *(Theorem VI.2.26) If G is an ICC group with an orientable Morse element then G is mixed identity free.*

Next we show that in the context of not necessarily orientable Morse elements, we can obtain a partial result in the direction of being mixed identity free.

Theorem II.2.14. *(Theorem VI.2.26) If G is an ICC group with a Morse element, then G does not satisfy any mixed identity with one coefficient.*

Finally, we show the following.

Theorem II.2.15. *(Theorem VI.2.26) If G is an ICC amenable group with a Morse element, then G is mixed identity free.*

Unfortunately, there are no known examples of ICC amenable groups with Morse elements so it is not clear at the moment whether Theorem II.2.15 has any applications.

Chapter III

Background

III.1 Asymptotic invariants of finitely generated groups

In this section, we recall some important definitions and facts about the distortion function and the Dehn function.

Definition III.1.1. Given a group G given by the finite presentation $P = \langle X \mid R \rangle$, and a word w in $F(X)$ which evaluates to identity in G, the area of w with respect to the presentation P, denoted $Area_P(w)$ is the least natural number n such that there exists a decomposition of w in $F(X)$ of the form $\Pi_{i=1}^{n} u_i^{-1} r_i u_i$ where each $u_i \in F(X)$ and each $r_i \in R$. If w is a word in $F(X)$ which evaluates to 1 in G, we write $w =_G 1$.

Definition III.1.2. Given a group G and a finite presentation for G, $P = \langle X \mid R \rangle$, the Dehn function $f : \mathbb{N} \to \mathbb{N}$ of G with respect to P is given by $f_P(n) = \max\{Area_P(w) \mid w \in F(X), w =_G 1, |w|_X \leq n\}$.

We consider Dehn functions up to the following equivalence.

Definition III.1.3. Given two non-decreasing functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$, we write that $f_1 \preceq f_2$ if there exists a constant $C \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f_1(n) \leq C f_2(Cn) + Cn + C$. f_1 and f_2 are said to be *equivalent*, written $f_1 \sim f_2$ if $f_1 \preceq f_2$ and $f_2 \preceq f_1$.

Up to this equivalence, the Dehn function of a finitely presented group does not depend on the choice of the finite generating set P.

Definition III.1.4. Given a group G given by finite presentation $\langle X \mid R \rangle$, a Van Kampen diagram for w over P is a directed labeled planar graph where each edge is labeled by an element in X, the boundary of each cell is a word in R, and the boundary of the entire diagram is w.

A word w in $F(X)$ evaluates to 1 in G if and only if there is a Van Kampen diagram for w. The area of a word w which evaluates to identity in G is the minimal number of cells in a Van Kampen diagram for w.

In [17], Osin and Rybek defined a generalization of the Dehn function for finitely generated but not necessarily finitely presented groups. We recall that definition here.

Definition III.1.5. Let G be the group with finite generating set X, given by the presentation $\langle X \mid R \rangle$, i.e. $G \cong F(X)/\langle\langle R \rangle\rangle$. Let $S_m = \{w \in \langle\langle R \rangle\rangle \mid |w|_X \leq m\}$. The *isoperimetric spectrum* of G is the function $f : T \to \mathbb{N}$ defined by $f(k,m,n) = \max\{Area_{S_m}(w) \mid w \in \langle\langle S_k \rangle\rangle, |w|_X \leq n\}$.

We consider this function up to the following equivalence.

Definition III.1.6. Given two functions $f,g : T \to \mathbb{N}$ which are non-decreasing in the first and third variables and non-increasing in the second variable, we write that $f \preceq_f g$ if there exists a constant C such that $f(k,Cm,n) \leq Cg(Ck,m,Cn)+C(n/m)+C$. We say that $f \sim_f g$ if $f \preceq_f g$ and $g \preceq_f f$.

Up to this equivalence, the isoperimetric spectrum does not depend on the choice of the finite generating set X. Note that the only difference between this equivalence relation and the one in Definition II.1.6 is the presence of the linear term $C(n/m)$. Again, when the relevant equivalence relation is clear from context, we will omit the subscript.

We will also need the following theorem of Olshanskii, which is useful for constructing subgroups with particular distortion functions.

Theorem III.1.7. *[15] For any group H, and any function $\ell\colon H \to \mathbb{N}$ satisfying the following conditions:*

1. *for all $h \in H$, $\ell(h) = 0$ if and only if $h = 1$,*

2. *$\ell(h) = \ell(h^{-1})$ for all $h \in H$,*

3. *$\ell(gh) \leq \ell(g)+\ell(h)$ for all $g,h \in H$,*

4. *there exists a constant a such that $|\{h \in H \mid \ell(h) \leq n\}| \leq a^n$,*

there exists a group $G = \langle X\rangle$ with $|X| < \infty$, an embedding ϕ of H in G, and a constant C such that for all $h \in H$,

$$\frac{|\phi(h)|_X}{C} \leq \ell(h) \leq C|\phi(h)|_X.$$

III.2 Asymptotic cones

We now turn to defining asymptotic cones. We begin by recalling the definition of an ultrafilter.

Definition III.2.1. A subset of the power set of the natural numbers, $\mathscr{P}(\mathbb{N})$, is called an *ultrafilter* if it satisfies the following properties.

1. For all $S,T \in \mathscr{P}(\mathbb{N})$, if $S \in \omega$ and $S \subset T$, then $T \in \omega$.

2. For all $S,T \in \omega$, $S \cap T \in \omega$.

3. For all $S \in \mathscr{P}(\mathbb{N})$, exactly one of S and $\mathbb{N} \setminus S$ is in ω.

4. $\mathbb{N} \in \omega$.

Further, if ω does not contain any finite sets, then ω is called a *non-principal* ultrafilter.

Definition III.2.2. Given an ultrafilter ω, and a sequence of real numbers (r_i), a real number L is called the *ultralimit* of (r_i) with respect to ω, denoted $\lim^{\omega} r_i$, if for all $\varepsilon > 0$ we have that $\{i \in \mathbb{N} \mid |r_i - L| < \varepsilon\} \in \omega$.

Recall that given an ultrafilter ω and any bounded sequence of real numbers, (r_i), $\lim^{\omega}(r_i)$ exists and is unique.

Now let (S,d) be a metric space, and let c_i be an unbounded, strictly increasing sequence of positive real numbers. Denote by d_i the metric on S defined by $d_i(x,y) = d(x,y)/c_i$. We call the sequence (c_i) the *scaling sequence*.

Definition III.2.3. Given a metric space (S,d), a scaling sequence (c_i), and an infinite sequence of points $z = (s_i)$ in S, denote by $S_z^{\mathbb{N}}$ the set of infinite sequences (t_i) in S such that $d_i(s_i,t_i)$ is bounded. The sequence (s_i) is called the *observation point*.

Definition III.2.4. Given $(x_i),(y_i) \in S_z^{\mathbb{N}}$, let $d^*((x_i),(y_i)) = \lim^{\omega} d_i(x_i,y_i)$.

Note that this is a bounded sequence so the limit exists. However, in general d^* will not be a metric, as there can be different sequences $(x_i),(y_i)$ such that $d^*((x_i),(y_i)) = 0$.

Definition III.2.5. We will denote by $Cone_z^{\omega}((d_i),S)$ the metric space that results from quotienting the pseudo-metric d^* by the equivalence relation $(x_i) \sim (y_i)$ if $d^*((x_i),(y_i)) = 0$. We will denote the resultant metric by d_S^{ω}. When the choice of the base point or the scaling sequence is clear, we will simply write $Cone^{\omega}(S)$. We will denote the equivalence class of (x_i) by $(x_i)^{\omega}$, so $d_S^{\omega}((x_i)^{\omega},(y_i)^{\omega}) = d^*((x_i),(y_i))$.

Definition III.2.6. A map f between two metric spaces (S,d_S) and (T,d_T) is called a (λ,C)-*quasi-isometric embedding* if for all $s,t \in S$

$$\frac{d_S(s,t)}{\lambda} - C \leq d_T(f(s),f(t)) \leq \lambda d_S(s,t) + C.$$

f is called ε-*quasi-surjective* if for all $t \in T$, there exists an $s \in S$ such that $d_T(f(s),t) \leq \varepsilon$. A map f is called a (λ,C,ε)-*quasi-isometry* if f is a (λ,C)-quasi-isometric embedding, and is ε-quasi-surjective. When we don't care about the quasi-isometry constants, we will simply call f a quasi-isometry and say that S and T are quasi-isometric.

Definition III.2.7. Let S be a metric space. A path $p\colon [0,\ell] \to S$ is called a (λ,C)-*quasi-geodesic* if p is a (λ,C)-quasi-isometric embedding.

Definition III.2.8. Given a pointed metric space (S,x) and (λ,C)-quasi-geodesic paths $p_i\colon [0,\ell_i] \to S$ such that the sequence ℓ_i/c_i is bounded and $(p_i(0)) \in S_z^{\mathbb{N}}$, let $L = \lim^{\omega} \ell_i/c_i$. If $L \neq 0$, define the ω-*limit* of the paths p_i, denoted

$$p = \lim^{\omega}(p_i)\colon [0,L] \to Cone^{\omega}(S),$$

by the following formula: $p(x) = \left(p_i\left(x\frac{\ell_i}{L}\right)\right)^{\omega}$. If $L = 0$, define $p = \lim^{\omega}(p_i)\colon \{0\} \to Cone^{\omega}(S)$ by the formula $p(0) = (p_i(0))^{\omega}$.

Definition III.2.9. A geodesic in $Cone^{\omega}(S)$ is called a *limit geodesic* if it is an ω-limit of geodesic paths.

Note that the limit of geodesics is a geodesic in the asymptotic cone. Thus, if S is a geodesic metric space, then so is $Cone^{\omega}(S)$.

A finitely generated group G can be considered as a metric space using the word metric arising from any finite generating set X. Given an ultrafilter ω, we will denote the asymptotic cone of G with respect to ω by $Cone^{\omega}(G)$ where we assume all scaling sequences are $c_i = i$ unless otherwise specified, and the observation point will always be $(e)^{\omega}$. Note that G is $(1,0,\frac{1}{2})$ quasi-isometric to its Cayley graph $\Gamma(G,X)$, and so its asymptotic cone is isometric to the asymptotic cone of $\Gamma(G,X)$. This is a geodesic space, and so we have that $Cone^{\omega}(G)$ is a geodesic space.

The asymptotic cone of G depends on the choice of a finite generating set X, an ultrafilter ω, and the choice of a scaling sequence (d_i). Note that changing the generating set of a group gives a quasi-isometric Cayley graph, and so will give a bi-Lipschitz asymptotic cone. In general, however, the other choices can matter, and a group can have many different asymptotic cones. For instance, Thomas and Velickovic exhibited a group such that one of its asymptotic cones is an $\mathbb{R}$-tree, and another is not simply connected [22]. These two choices turn out to be closely related. Specifically, given any scaling sequence (c_i) such that the sizes of the sets $S_r = \{i | c_i \in [r, r+1)\}$ are bounded, and any ultrafilter ω, there exists an ultrafilter ω' such that $Cone^{\omega}((c_i), G) = Cone^{\omega'}((i), G)$ [19]. This justifies our choice to take all scaling sequences as $c_i = i$ unless otherwise specified. In the case of groups, we will also always take the observation point to be (e).

Definition III.2.10. We say that a metric space S is transitive if for any two points $s,t \in S$ there exists an isometry $\phi\colon S \to S$ such that $\phi(s) = t$.

Recall that for any group G, $Cone^{\omega}(G)$ is a transitive space, and that any asymptotic cone is complete.

If G is a finitely generated group, then G acts on its asymptotic cone by the formula $g(g_i)^{\omega} = (gg_i)^{\omega}$. Given a subgroup H of a finitely generated group G, we denote by $Cone^{\omega}_G(H)$ the set of points in $Cone^{\omega}(G)$ with a representative of the form $(h_i)^{\omega}$ where all h_i are in H.

III.3 Wreath products

We finish with a short section recalling the definition and setting our notation for wreath products.

Definition. Given finitely generated groups A and B, we denote by $A^{(B)}$ the set of functions $f : B \to A$ with finite support.

Definition. Given two finitely generated groups A and B, B acts on $A^{(B)}$ by the following formula. If b_1, b_2 are elements of B, and f is an element of $A^{(B)}$, then $(b_1 f)(b_2) = f(b_2 b_1)$. The semidirect product of B and $A^{(B)}$ with respect to this action is denoted by $A \wr B$ and is called the *wreath product* of A and B. We refer to the subgroup $A^{(B)}$ as the *base group* of the wreath product.

Wreath products of particular interest for our purposes as they provide a natural example of a finitely generated group with non-finitely generated subgroups. In particular, the base group will not be finitely generated whenever A is infinite.

Chapter IV

The generalized distortion function

IV.1 Basic properties and examples

We begin by showing that the generalized distortion function is essentially independent of the choice of the finite generating set.

Proposition IV.1.1. *Let ϕ be a quasi-isometry between finitely-generated groups G_1 and G_2. Let X_1, X_2 be finite generating sets for G_1, G_2 respectively. Let H^1 be a subgroup of G and let $H^2 = \phi(H^1)$. Then $\delta_{H^1}^{G_1,X_1} \sim \delta_{H^2}^{G_2,X_2}$. In particular if X, Y are finite generating sets for a group G with subgroup H, then $\delta_H^{G,X}(k,m,n) \sim \delta_H^{G,Y}(k,m,n)$.*

Proof. Let ϕ be a (λ, C) quasi-isometry from G_1 to G_2 such that $\phi(H^1) = H^2$. Let $D = \lambda + C$. Fix $(k,m,n) \in T$ with $m > Dk$, and let $h \in H_k^1$ be such that $|h|_{X_1} \leq n$. Note that this means we can find a natural number p and a sequence $1 = h_1, h_2, \dots h_p = h$ such that $d_{X_1}(h_i, h_{i+1}) \leq k$ for all $1 \leq i < p$. This implies that $d_{X_2}(\phi(h_i), \phi(h_{i+1})) \leq Dk$, for all $1 \leq i < p$, so $\phi(h) \in H_{Dk}^2$. Further, we have that $|\phi(h)|_X \leq Dn$. Thus, $|\phi(h)|_{Y_{H^2},m} \leq \delta_{H_2}^{G_2,X_2}(Dk,m,Dn)$. This implies that we can find a natural number $q \leq \delta_{H_2,X}(Dk,m,Dn)$ and a sequence $1 = h_1, h_2, \dots h_q = h$ such that $d_{X_2}(\phi(h_i), \phi(h_{i+1})) \leq m$ for all $1 \leq i < q$. This means that $d_{X_1}(h_i, h_{i+1}) \leq Dm$ for all $1 \leq i < q$, and so $|h|_{Y_{H^1_{Dm}}} \leq q \leq \delta_{H^2,X}(Dk,m,Dn)$. Thus, $\delta_{H^1}^{G_1,X_1}(k,Dm,n) \leq \delta_{H^2}^{G_2,X_2}(Dk,m,Dn)$. By symmetry, $\delta_{H^1}^{G_1,X_1} \sim \delta_{H^2}^{G_2,X_2}$. □

Lemma IV.1.2. *let H be a subgroup of a group G with finite generating set X. If H' is a finite index subgroup of H, then $\delta_{H'}^G \sim \delta_H^G$.*

Proof. As H' is finite index in H, there exists a constant C such that for all $h \in H$, there exists an $h' \in H'$ such that $d_X(h,h') \leq C$. Fix $(k,m,n) \in T$. Let $h \in H'$ be such that $h \in H'_k$ and $|h|_X \leq n$. Let $p = \delta_H^G(k,m,n)$. there exists a natural number $q \leq p$ and q elements of H, $e = h_1, h_2, \dots h_q = h$, with $d_X(h_i, h_{i+1}) \leq m$. For each h_i with $1 < i < q$, let h'_i be an element of H' with $d_X(h_i, h'_i) \leq C$. As $d_X(h'_i, h_i) \leq m + 2C$, we have that $\delta_{H'}^{G,X}(k, m+2C, n) \leq \delta_H^G(k,m,n)$ and so $\delta_{H'}^G \preceq \delta_H^G$. On the other hand, let h be an element of H such that $|h|_X \leq n$ and $h \in H_k$. Let $e = h_1, h_2 \dots h_p = h$ be such that for all $1 \leq i \leq p-1$, $d_X(h_i, h_{i+1}) \leq k$ and for all $1 \leq i \leq p$, $h_i \in H$. For each $1 \leq i \leq p$ there exists an $h'_i \in H'$ such that $d_X(h_i, h'_i) \leq C$. Therefore $d_X(h'_i, h'_{i+1}) \leq 2C + k$ for all $1 \leq i \leq p-1$. Therefore $h_p \in H_{2C+k}$. There exists a natural number $q \leq \delta_{H'}^G(2C+k, m, n+C)$ and q elements of H', $e = h''_1, h''_2, \dots h''_q = h'_p$, with $d_X(h''_i, h''_{i+1}) \leq m$ for all $1 \leq i \leq p-1$. As $d_X(h'_p, h) \leq C$, we have that $|h_p|_{Y_{m+C}} \leq q+1$, and so $\delta_H^{G,X}(k, C+m, n) + 1 \leq \delta_{H'}^{G,X}(k+2C, m, n+C)$. Therefore, we have that $\delta_H^G \sim \delta_{H'}^G$. □

Theorem IV.1.3. *Let $F(X)$ be the free group on the finite generating set X, and let H be a subgroup of $F(X)$. Then $\delta_H^{F(X)}(k,m,n) \preceq n/m$.*

Proof. Fix $(k,m,n) \in T$. Let $h \in H_k$, with $|h|_X \leq n$. As $h \in H_k$, we can find $p \in \mathbb{N}$, and $w_i \in F(X)$ for $1 \leq i \leq p$ such that h is freely equal to the concatenation $w_1 w_2 \dots w_p$ and for all $1 \leq i \leq p$, $|w_i|_X \leq k$. Denote by w the concatenation $w_1 w_2 \dots w_p$. Let $\ell_1 \ell_2 \dots \ell_q$ be a reduced representation of h with each $\ell_i \in X$. Fix a reduction of w in $F(X)$. Let $f(i)$ be the preimage of ℓ_i under this reduction in $w_1 w_2 \dots w_p$. Let w_{i_j} be such that $f(j) \in w_{i_j}$. Observe that for all $1 \leq i < j \leq q$, if $w_{i,j}$ is the portion of w beginning at $f(i)$ and ending at $f(j)$, then $w_{i,j}$ is freely equal to $\ell_i \ell_{i+1} \dots \ell_j$, and hence $|w_{i,j}|_X = j - i + 1$. This implies that for all $1 \leq mj < m(j+1) \leq q$, $|w_{i_{mj}} w_{i_{mj}+1} \dots w_{i_{m(j+1)-1}}|_X \leq m + 2k$, as $f(mj)$ is within k of the start of w_{mj}, and $f(m(j+1))$ is within k of the end of $w_{m(j+1)-1}$. Finally, as

$$h = (w_1 w_2 \dots w_{i_m - 1})(w_{i_m} \dots w_{i_{2m}-1}) \dots (w_{i_{\lfloor (n/m) \rfloor - 1}} \dots w_{i_{\lfloor (n/m) \rfloor} - 1})(w_{i_{\lfloor (n/m) \rfloor}} \dots w_p)$$

we have that $|h|_{Y_{m+2k}} \leq \lfloor (n/m) \rfloor + 1$. We conclude that $\delta_H^G \preceq n/m$. $\square$

Definition IV.1.4. Given a function f from a group B to a group A, the *support* of f, denoted $supp(f)$ is the set $\{b \in B \mid f(b) \neq 1\}$.

Definition IV.1.5. Given a group B with finite generating set X, and a subset $S \subset B$, the *reach* of B, denoted *reach*(B), is the length of the shortest loop in $\Gamma(B,X)$ which starts and ends at 1, and goes through every point in S.

In [4], Olshanskii and Davis present a formula for computing the word length of an element of wreath products of finitely generated groups A and B. Here we present a version of that formula for elements in the natural copy of $A^{(B)}$ in the wreath product.

Lemma IV.1.6. *[4] Let A and B be groups with finite generating sets X and Y respectively, and let f be an element in the natural copy of $B^{(A)}$ in $A \wr B$. Then $|f|_{X \cup Y} = reach(supp(f)) + \sum_{b \in supp(f)} |f(b)|_X$.*

Theorem IV.1.7. *Let A and B be finitely generated groups. Let $G = A \wr B$, and let H be the natural copy of $A^{(B)}$ in G. Then, $\delta_H^G(m,n,k) \preceq n/m$.*

Proof. Fix $(k,m,n) \in T$. Let $f \in H_k$ with $|f|_{X \cup Y} \leq n$. By Lemma IV.1.6, *reach*$(f) \leq n$ and

$$\sum_{b \in supp(f)} |f(b)|_X \leq n.$$

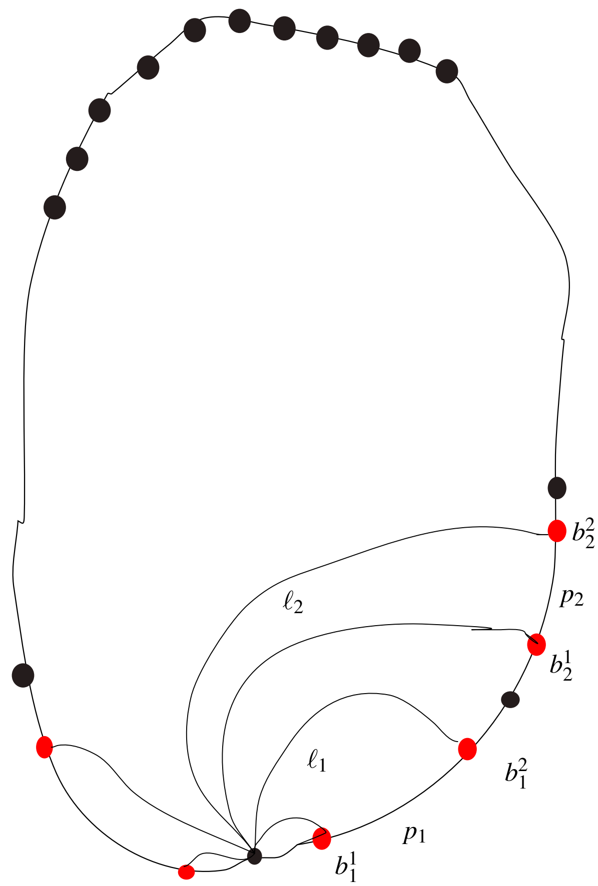

Figure IV.1: Theorem IV.1.6

Let ℓ be a loop in $\Gamma(B,Y)$ starting and ending at e and containing all elements in $supp(f)$ with length equal to $reach(f)$. As $f \in H_k$, if $b \in supp(f)$, then $|b|_Y \leq k$. For $1 \leq i \leq \lceil n/m \rceil$, let p_i be a subpath of ℓ of length less than or equal to m such that each p_i is disjoint and the p_is cover ℓ. For each p_i let b_i^1 be the first point on p_i in $supp(f)$, and let b_i^2 be the last point on p_i in $supp(f)$. Let p_i^1 be a geodesic connecting e to b_i^1, and let p_i^2 be a geodesic connecting b_i^2 to e. Let ℓ_i be the the loop obtained by concatenating p_i^1, the portion of p_i between b_i^1 and b_i^2 and p_i^2. The length of ℓ_i is at most $2k+m \leq 3m$. Let f_i be the restriction of f to p_i, and let

$$n_i = \sum_{b \in p_i} |f(b)|_X.$$

As the length of $\ell_i \leq 3m$, $|f_i|_{Y_{4m}} \leq \lceil n_i/m \rceil$. Thus as $f = f_1 f_2 \dots f_{\lceil n/m \rceil}$, we have that

$$|f|_{Y_{4m}} \leq \sum_{i=1}^{\lceil n/m \rceil} \left\lceil \frac{n_i}{m} \right\rceil \leq \sum_{i=1}^{\lceil n/m \rceil} \left(\frac{n_i}{m} + 1\right) \leq 2\frac{n}{m} + 1.$$

Thus, $\delta_{H,X\cup Y}(k,4m,n) \leq 2n/m + 1$, so $\delta_H \preceq n/m$. □

IV.2 A group with uncountably many incomparably distorted subgroups

We will show that the set of all isoperimetric spectra of 2 generated groups is contained in the set of all generalized distortion functions of subgroups of $F_2 \times F_2$. In order to prove this, we need the following result from [16].

Lemma IV.2.1. *Let Δ be a Van Kampen diagram over a presentation $\langle X \mid R \rangle$, where X is a symemtric generating set, and R is closed under cyclic shifts. Let w be the boundary word of Δ. Then w is equal in $F(X)$ to a word of the form $u_1 r_1 u_2 r_2 \dots r_n u_{n+1}$, where:*

1. *For all $1 \leq i \leq n$, $R_i \in R$;*
2. *$n \leq Area(\Delta)$;*
3. *$u_1 u_2 \dots u_{n+1} = 1$;*
4. *$\sum_{i=1}^{n+1} |u_i|_X \leq 4e$ where e is the number of edges in Δ.*

Theorem IV.2.2. *Let G be the group given by the presentation $\langle X \mid R \rangle$, and let H be the subgroup of $F_2 \times F_2 = \langle x,y \rangle \times \langle a,b \rangle$ generated by elements of the form (f,f) for $f \in F_2$ and $(1,r)$ for $r \in R$. Then, $f_G \sim_f \delta_H$, where f_G is the isoperimetric spectrum of G.*

Proof. Fix $(k,m,n) \in T$ with $k \geq 2$. Let $h = (u,v) \in H_k$ such that $|h|_X = |u|_X + |v|_X \leq n$. We can write $h = (h_1,h_1)(h_2,h_2)\dots(h_p,h_p)(1,w)$, with each $|h_i|_X \leq m$, and $p \leq \lceil |u|_X/m \rceil \leq \lceil n/m \rceil$. Note that $(1,w) \in H_k$, as $h \in H_k$, and as $k \geq 2$, each $(h_i,h_i) \in H_k$. Further, $|w|_X \leq |u|_X + |v|_X \leq n$. As $w \in H_k$, w evaluates to 1 in the group given by the presentation $\langle a,b \mid S_k \rangle$ where $S_k = \{w \in \langle\langle R \rangle\rangle \mid |w|_X \leq k\}$. Let Δ be a minimal Van Kampen diagram for w over the presentation $\langle a,b \mid S_m \rangle$. This diagram will have less than $f_{G,X}(k,m,n) = q$ cells. Thus, by Lemma IV.2.1, we can write w as $u_1 r_1 u_2 r_2 \dots r_q u_{q+1}$, with each $r_i \in S_m$, $u_1 u_2 \dots u_{q+1} = 1$, and $\sum_{i=1}^{n+1} |u_i|_X \leq 4e$, where e is the number of edges in Δ. As each $r_i \in S_m$, $|r_i| \leq m$. The number of edges in Δ will be less than or equal to $mq + |w|_X$, as each edge of Δ either belongs to a face of Δ, or is on the boundary of Δ. We can also write each u_i as a product of t elements of F_2 whose length is less than or equal to m, with

$t \leq \lceil |u_i|_X/m \rceil$. Thus, we can write w as a product of no more than

$$f(k,m,n) + \sum_{i=1}^{q+1} \lceil |q_i|/m \rceil \leq f(k,m,n) + \sum_{i=1}^{q+1} |u_i|/m + 1$$

$$\leq q + 4e/m + q + 1 \leq q + 4(mq + |w|)/m + q + 1 \leq 6q + 4n/m + 1$$

elements of length less than or equal to m. This in turn means that $|w|_{Y_{2m}} \leq 6f(k,m,n) + 5n/m + 1$, so $\delta_H^G \preceq_f f_G$.

Now fix $(k,m,n) \in T$, and let w be a word equal to 1 in $\langle a,b | S_k \rangle$ of length less than or equal to n whose area with respect to the presentation $\langle a,b \mid S_m \rangle$ is equal to $f(k,m,n)$. Note that if $(u,v) \in H$, then $u =_G v$. Let $(u_1,v_1)\ldots(u_p,v_p) = (1,w)$ with $|(u_i,v_i)|_X \leq m$ for all $1 \leq i \leq p$. We can then write that

$$w = \Pi_1^p u_1 u_2 \ldots u_i (u_i^{-1} v_i)(u_1 u_2 \ldots u_i)^{-1},$$

where each $u_i^{-1} v_i \in S_m$. Thus, $p \geq Area_{S_m}(w) = f(k,m,n)$, so $\delta_H^G \succeq_f f_G$. □

In [17], it is shown that there is a collection of uncountably many 2-generated groups with pairwise incomparable isoperimetric spectra. It therefore follows from IV.2.2 that $F_2 \times F_2$ contains an uncountable collection of subgroups with incomparable distortion. Further, [17] shows that there exist groups whose isoperimetric spectrum depends essentially on k, and so the same result follows for generalized distortion functions of subgroups of groups.

Chapter V

Generalized distortion and asymptotic cones

V.1 Computing the generalized distortion function in the case where H is finitely generated

We begin by defining a variant of distortion that will help us calculate generalized distortion for a variety of finitely generated subgroups of finitely generated groups.

Definition V.1.1. Let H be an infinite subgroup of a group G and let Y,X be finite generating sets of H and G respectively. Define the *lower distortion function* of H in G, denoted $\nabla_{H,Y}^{G,X}(n)$, by the formula

$$\nabla_{H,Y}^{G,X}(n) = \min\{|h|_Y \mid |h|_X > n, h \in H\}.$$

We consider lower distortion up to the same equivalence as distortion, and denote by ∇_H^G the function $\nabla_{H,Y}^{G,X}$ for some choices of the finite generating sets X,Y.

Example V.1.2. For $p \in \mathbb{N}, p \geq 2$, let $G = BS(1,p) = \langle a,b | b^{-1}ab = a^p \rangle$, and let $H = \langle a \rangle$. Note that $a^{p^n} = b^{-n}ab^n$, and so $\Delta_H^G(n) \succeq p^n$. In fact, $\Delta_H^G \sim p^n$ [9]. Next, note that if $k < p^n$, then we can write $k = \sum_{i=0}^{n-1} c_i p^i$, with $0 \leq c_i < p$. This in turn means that we can write $a^k = \prod_{i=0}^{n-1} b^{-i} a^{c_i} b^i = (\prod_{i=0}^{n-1} a^{c_i} b^{-1}) b^{n-1}$. This implies that $|a^k|_X \leq n + n(p) = n(p+1)$. Thus, $\nabla_H^G(n) \succeq p^n$.

Example V.1.3. Let G be the discrete Heisenberg group, i.e. the group of all upper triangular integer matrices with ones along the diagonal, and let H be the center of this group, i.e. the subgroup of all matrices of the form $\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $c \in \mathbb{Z}$. Let X be the generating set for the group G given by $G = \langle x,y,z \rangle$ where $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and $z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and let $Y = \{z\}$, a generating set for H. Note that $x^n y^n x^{-n} y^{-n} = z^{n^2}$. Now let m be a natural number such that $(n-1)^2 < m < n^2$. We know that $|z^{n^2}|_X \leq 4n$. Thus,

$$|z^m|_X \leq 4n + (n^2 - (n-1)^2) = 4n + 2n - 1 \leq 6n.$$

Thus, if $m \leq n^2$, then $|z^m|_X \leq 6n$, and so $\nabla_H^G(n) \succeq n^2$.

Now we will show that if $|h|_X \leq n$, then $|h|_Y \leq n^2$. Let $f\colon G \to \mathbb{N}$ be the function given by $f\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = |a|$, and let $k\colon G \to \mathbb{N}$ be the function given by $k\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = |b|$. We have that

$$f(gx) \leq f(g)+1,\, f(gy) = f(g),\, f(gz) = f(g),$$

and thus if $|g|_X \leq n$, then $f(g) \leq n$. Similarly,

$$k(gx) = k(g),\, k(gy) \leq f(g)+k(g),\, k(gz) \leq k(g)+1.$$

Thus if $|g|_X \leq n$, then $k(g) \leq n^2$. If $h \in H$, then $|h|_Y = k(h)$, and so if $|h|_X \leq n$, then $|h|_Y \leq n^2$. Thus, $\Delta_H^G(n) \preceq n^2$.

Example V.1.4. Let $G = \langle a,b,c \mid [a,b]=1, [a,c]=1, c^{-1}bc = b^2\rangle \cong \mathbb{Z} \times BS(1,2)$, and let $H = \langle a,b\rangle \cong \mathbb{Z}\times\mathbb{Z}$. Let $X = \{a,b,c\}$. Note that $|b^{2^n}|_X \leq 2n+1$, so $\Delta_H^G(n) \succeq 2^n$, but $|a^n|_X = n$, and so $\nabla_H^G(n) \preceq n$. Thus, we have that $\Delta_H^G \not\sim \nabla_H^G$.

Recall that if H is a finitely generated subgroup of the finitely generated group G, then the function δ is essentially independent of the variable k, as for sufficiently large k, $H_k = H$.

Note that if f_1, f_2, g_1 and g_2 are strictly increasing functions such that $f_1(n) \sim f_2(n)$ and $g_1(n) \sim g_2(n)$ then $f_1(n)/g_1(m) \cong f_2(n)/g_2(m)$. Thus, we can state the following proposition.

Proposition V.1.5. *For a finitely generated, infinite subgroup H of a finitely generated group group G, the following inequalities hold*

$$\frac{\Delta_H^G(n)}{\Delta_H^G(m)} \preceq \delta_H^G(k,m,n) \preceq \frac{\Delta_H^G(n)}{\nabla_H^G(m)}. \tag{1}$$

Proof. First, choose a finite generating set X for G containing a generating set Y for H. Fix $n \in \mathbb{N}$ and let h be an element of H such that $|h|_X \leq n$, and $|h|_Y = \Delta_{H,Y}^{G,X}(n)$. By definition, if $k \in Y_m$ then $|k|_X \leq m$, and so $|k|_Y \leq \Delta_{H,Y}^{G,X}(m)$. Thus, $|h|_{Y_m} \geq \left\lceil \Delta_{H,Y}^{G,X}(n)/\Delta_{H,Y}^{G,X}(m) \right\rceil$, and we obtain the first inequality in (1). For the next inequality, note that if $|h|_X \leq n$, then $|h|_Y \leq \Delta_{H,Y}^{G,X}(n)$. Thus, we can write h as a product of at most $\left\lceil \Delta_{H,Y}^{G,X}(n)/(\nabla_{H,Y}^{G,X}(m)-1) \right\rceil$ elements of length less than or equal to $\nabla_{H,Y}^{G,X}(m)-1$ with respect to Y. Note that if h is an element of H such that $|h|_Y < \nabla_{H,Y}^{G,X}(m)$, then by the definition of $\nabla_{H,Y}^{G,X}$, $|h|_X \leq m$, and $h \in Y_m$. This gives the second inequality in (1). $\square$

Definition V.1.6. We call a subgroup H of a group G *uniformly distorted* if $\Delta_H^G \sim \nabla_H^G$.

Combining the previous observations gives the following corollary.

Corollary V.1.7. *If H is a uniformly distorted finitely generated subgroup of a finitely generated group G, then $\delta_H^G(k,m,n) \sim \frac{\Delta_H^G(n)}{\Delta_H^G(m)} \sim \frac{\Delta_H^G(n)}{\nabla_H^G(m)}$.*

Example V.1.8. Example V.1.2 showed that if $G = BS(1,p) = \langle a,b \mid b^{-1}ab = a^p \rangle$ and $H = \langle a \rangle$, then H is uniformly distorted in G, so we can apply Corollary 3.7 to get that $\delta_H^G(k,m,n) \sim p^{n-m}$.

Example V.1.9. Example V.1.3 showed that if G is the discrete Heisenberg group, and H is the center of G then H is uniformly distorted in G and we have from Corollary 3.7 that $\delta_H^G(k,m,n) \sim (n/m)^2$.

We conclude this section with an example demonstrating that for a group G with finitely generating set X, $\delta_H^{G,X}(k,n-1,n)$ can grow very quickly.

Example V.1.10. Let H be a finitely generated subgroup of a finitely generated group G such that the membership problem is undecidable, and let X be a finite generating set for G containing a generating set of H. The existence of such subgroups was demonstrated independently by Mihailova and Rips [20] [13]. Gromov [9] showed that the distortion function of H in G is bounded by a computable function if and only if the membership problem is solvable. Fix k such that $H_k = H$, and note that

$$\Delta_{H,Y}^{G,X}(n) = \delta_H^{G,X}(k,1,n) \leq \delta_H^{G,X}(k,1,2)\delta_H^{G,X}(k,2,3)\ldots\delta_H^{G,X}(k,n-1,n).$$

Thus, if $\delta_H^{G,X}(k,n-1,n)$ is bounded by a computable function, then so is $\Delta_{H,Y}^{G,X}(n)$, a contradiction. Thus, $\delta_H^{G,X}(k,n-1,n)$ is not bounded by any computable function.

V.2 Connectedness in asymptotic cones

We begin by defining an analog of the generalized distortion function for the case of a metric space S.

Definition V.2.1. Given a metric space S, a real number $r > 0$, and two points $s,t \in S$, an *r-path* connecting s and t is a sequence of points $s = s_0, s_1, \ldots, s_k = t$ with $d_S(s_i, s_{i+1}) \leq r$ for all $0 \leq i < k$. We call k the *length* of the r-path. We say a metric space S is *r-connected* if for any two points $s,t \in S$ there exists an r-path connecting s and t. If (S,s) is a pointed r-connected metric space, and t is in S, let $|t|_r$ be the length of the shortest r-path connecting s and t.

Definition V.2.2. Let (S,s) be a proper r-connected pointed metric space. Define $\nu_S(m,n) \colon \mathbb{R}^{\geq r} \times \mathbb{R}^{\geq 0} \to \mathbb{N}$ to be $\max\{|t|_m | \, d_S(s,t) \leq n\}$.

Lemma V.2.3. *ν_S is well-defined, i.e. for all real numbers $m \geq r, n$ there exists a constant $K \in \mathbb{R}$ such that for any point $t \in S$ with $d(s,t) \leq n$, $|t|_m \leq K$.*

Proof. Fix $n \in \mathbb{R}^{\geq 0}$, and let B be the closed ball centered at s of radius n. As B is compact, it can be covered by some finite number p of open balls of radius m. Let $s_1, \ldots s_p$ be the centers of these balls. As S is r-connected for each s_i there exists a sequence of points

$$s = s_{0,i}, s_{1,i}, \ldots, s_{K_i,i} = s_i$$

with $d_S(s_{j,i}, s_{j+1,i}) \leq m$ for all $0 \leq i < K_i$. Let $K = \max\{K_i \mid 1 \leq i \leq p\}$. Any point in B is within m of some s_i, and so $\nu_S(m,n) \leq K+1$. □

If H is a finitely-generated subgroup of a finitely generated group G, and X is a finite generating set of G, then H is n-connected and proper for some $n \in \mathbb{N}$ with respect to the word metric induced by X. It is clear in this case that δ_H^G is the restriction of ν_H to $\mathbb{N} \times \mathbb{N}$, where we consider H with the word metric induced from G, and k is large enough that Y_k generates H.

Definition V.2.4. Given two functions $f, g\colon \mathbb{R}^{\geq r} \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ which are non-increasing in the first variable, and non-decreasing in the second variable, we write $f \preceq_\nu g$ if there exists a constant $C \in \mathbb{R}$ such that $f(Cm,n) \leq Cg(m,Cn)$ for all $m,n \in \mathbb{R}^{\geq 0}, m \geq r$ and we say that $f \sim_\nu g$ if $f \preceq_\nu g$ and $g \preceq_\nu f$.

Again, we will repress the subscript ν whenever the relevant equivalence relation is clear from context.

Essentially, ν measures how far away S is from being a geodesic metric space. For instance, if S is geodesic, then $\nu_S(m,n) = \lceil n/m \rceil$.

Lemma V.2.5. *If $(S,s),(T,t)$ are proper, r-connected pointed metric spaces, and f is a $(\lambda, C, \varepsilon)$-quasi-isometry between S and T such that $f(s) = t$, then, $\nu_S \sim \nu_T$.*

Proof. First, fix $n \in \mathbb{R}^{\geq 0}, m \in \mathbb{R}^{\geq r}$, and let $y \in S$ with $d_S(s,y) \leq n$. This implies that $d_T(t, f(y)) \leq \lambda n + C$. Let $K = \nu_T(m, \lambda n + C)$. There exist $K+1$ points $y_0, y_1 \ldots y_K$ such that

$$t = y_0, y_1, \ldots, y_K = f(y)$$

with $d_T(y_i, y_{i+1}) \leq m$. By quasi-surjectivity, for each i there exists an $y'_i \in S$ such that $d_T(f(y'_i), y_i) \leq \varepsilon$. Thus, $d_T(f(y'_i), f(y'_{i+1})) \leq m + 2\varepsilon$, and so

$$d_S(y'_i, y'_{i+1}) \leq \lambda(m + 2\varepsilon) + C \leq \lambda' m$$

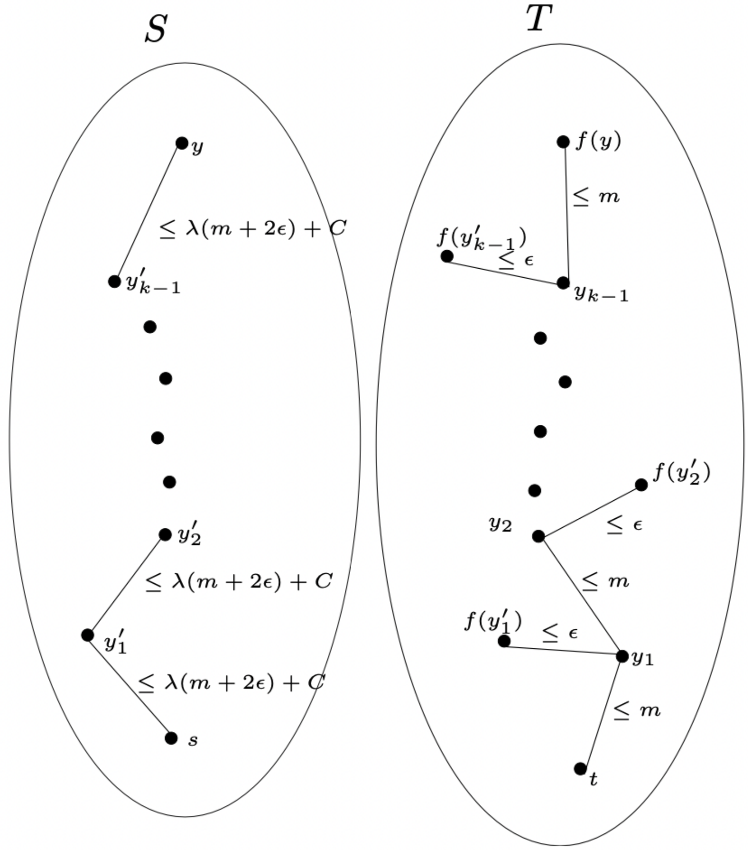

Figure V.1: Lemma V.2.5

for some fixed λ' as $m \geq r$. Note that we can choose y'_0 to be s, and y'_K to be y. Thus $\nu_S(\lambda' m, n) \leq \nu_T(m, \lambda n + C)$. If $\lambda n + C \leq m$, we have that $\nu_T(m, \lambda n + C) = 1$, so we can assume that $\lambda n + C$ is greater than r as well, and we have that $\nu_S(\lambda' m, n) \leq \nu_T(m, \lambda'' n)$ for some fixed λ''. By symmetry, $\nu_T \preceq \nu_S$, and so $\nu_T \cong \nu_S$. $\square$

Definition V.2.6. Call a metric space S *asymptotically transitive* if $Cone^{\omega}(S)$ is transitive for all ultrafilters ω.

Theorem V.2.7. *Let r be a positive number and let (S, s) be an asymptotically transitive proper r-connected pointed metric space. The following are equivalent:*

1. *there exists a function $f\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that for all $m \geq r, n \geq 0$,*
   $\nu_S(m, n) \leq f(n/m)$,
2. *there exists a constant K such that $\nu_S(i, 4i) \leq K$ for all real numbers $i \geq r$,*
3. *$Cone^{\omega}(S)$ is path connected for all non-principal ultrafilters ω,*

4. *$Cone^{\omega}(S)$ is connected for all non-principal ultrafilters ω.*

Note that the implication 1) $\Rightarrow$ 2) is clear, simply by letting $K = f(4)$. The implication 3) $\Rightarrow$ 4) is also immediate.

To show that 2) implies 3) we will need the following lemma.

Lemma V.2.8. *Let $r \in \mathbb{R}^{\geq 0}$. If (S,s) is an asymptotically transitive, proper, r-connected, pointed metric space and there exists a constant K such that $\nu_S(i,4i) \leq K$ for all real numbers $i \geq r$, then for any points $p = (y_i)^{\omega}, q = (z_i)^{\omega} \in Cone^{\omega}(S)$, there exist $K+1$ points*

$$p = p_0, p_1, p_2, ..., p_K = q$$

in $Cone^{\omega}(S)$ such that $d_S^{\omega}(p_i, p_{i+1}) \leq d_S^{\omega}(p,q)/2$.

This lemma is reminiscent of a lemma in [18] used to prove that any group satisfying a quadratic isoperimetric inequality has a simply connected asymptotic cone. In that paper, Papasoglu used the isoperimetric inequality to build sequences of loops to fill a loop in the asymptotic cone. This is very similar to the approach we will use to prove that 2) implies 3). Similar ideas can also be found in [12], [3], and [19]

Proof. If $(y_i)^{\omega} = (z_i)^{\omega}$, the result is trivial, so let $(y_i)^{\omega}$ and $(z_i)^{\omega}$ be points in $Cone^{\omega}(S)$ such that

$$d_S^{\omega}((y_i)^{\omega}, (z_i)^{\omega}) = C > 0.$$

Note that by the transitivity of $Cone^{\omega}(S)$, we can assume that $(y_i)^{\omega} = (s)^{\omega}$. This means in particular that $d_S(s, z_i) \leq 2Ci$ ω-almost surely. Note that $Ci/2 \geq r$ ω-almost surely, and hence $\nu_S(Ci/2, 2Ci) \leq K$ ω-almost surely. It follows that there exist points $s = y_{i,0}, y_{i,1}, ..., y_{i,K} = z_i$ with $d_S(y_{i,j}, y_{i,j+1}) \leq Ci/2$ for all $0 \leq j \leq K-1$ ω-almost surely. Now define $p_j = (y_{i,j})^{\omega}$. Note that

$$d_S^{\omega}(p_j, p_{j+1}) = \lim{}^{\omega} d_S(y_{i,j}, y_{i,j+1})/i \leq C/2,$$

and so we have our desired $p_0, ..., p_K$. $\square$

We will also need the following Lemma in order to prove that 4) implies 1).

Lemma V.2.9. *If S is a connected metric space, then for any real number $r > 0$, S is r-connected.*

Proof. For a fixed $r > 0$, and fixed $p \in S$, consider the set C of points q such that there exists a finite sequence of points $p = p_0, p_1 \dots p_K = q$ with $d(p_i, p_{i+1}) \leq r$. If $x \in C$, then clearly $B_r(x) \subset C$, and so C is open.

Similarly, if $x \notin C$, then $B_r(x) \subset S \setminus C$, so C is closed. Hence, C is open, closed and non-empty, so $C = S$, as desired. □

We are now ready to prove the theorem.

Proof. We begin by proving 2) implies 3).

Let $p, q \in Cone^{\omega}(S)$, and let $C = d_S^{\omega}(p,q)$. We will define a uniformly continuous function f from numbers of the form a/K^n with $a, n \in \mathbb{N}$ $a \leq K^n$ to the asymptotic cone such that $f(0) = p$ and $f(1) = q$. Note that this is sufficient, since asymptotic cones are complete, and these numbers are dense in the interval $[0,1]$.

We will define the function inductively as follows. First, define $f(0) = p$ and $f(1) = q$. Then, fix $n \in \mathbb{N}$, and assume we've defined f on all numbers of the form a/K^n in such a way that for all $s \in \mathbb{N} \cup \{0\}$ with $s < K^n$

$$d_S^{\omega}\left(f\left(\frac{s}{K^n}\right), f\left(\frac{s+1}{K^n}\right)\right) \leq \frac{C}{2^n}.$$

Now let $t = (K\ell + b)/K^{n+1}$ where $1 \leq b < K$ and $\ell \in \mathbb{N} \cup \{0\}$, $\ell \leq K^{n-1}$ According to Lemma V.2.8, there exist points $p_0, p_1, \ldots, p_K$ such that

$$f\left(\frac{\ell}{K^n}\right) = p_0, p_1, \ldots, p_K = f\left(\frac{\ell+1}{K^n}\right),$$

and

$$d_S^{\omega}(p_i, p_{i+1}) \leq \frac{d_S^{\omega}(f(\frac{\ell}{K^n}), f(\frac{\ell+1}{K^n}))}{2} \leq \frac{C}{2^{n+1}}.$$

Let $f(t) = p_b$. It is straightforward to verify that f is uniformly continuous.

We will now show that 4) implies 1) by contradiction. Assume that $Cone^{\omega}(S)$ is connected, and that $\nu_S(m,n)$ is not bounded by any homogeneous function. Hence there exists a $c \in \mathbb{R}^{>0}$ such that $\nu_S(n, cn)$ is not bounded. Let n_i be a sequence of natural numbers such that $\nu_S(n_i, cn_i) \geq i$. Let ω be an ultrafilter containing $\{n_i | i \in \mathbb{N}\}$. Consider a sequence of points $t_i \in S$ such that $d_S(s, t_i) \leq ci$, and $|t_i|_i = \nu_S(i, ci)$. According to Lemma V.2.9, we can pick points $(s)^{\omega} = p_0, p_1, \ldots, p_k = (t_i)^{\omega}$ in $Cone^{\omega}(S)$ such that $d_S^{\omega}(p_i, p_{i+1}) \leq \frac{1}{2}$. Let $p_j = (t_{i,j})^{\omega}$. We have that $d_S(t_{i,j}, t_{i,j+1}) \leq i$ ω-almost surely, so $\nu_S(i, ci) = |t_i|_i \leq k$ ω-almost surely. On the other hand if $j > k$, then $\nu_S(n_j, cn_j) > k$. However,

$$\{n_j | j > k\} = \{n_j | j \in \mathbb{N}\} \cap \{n | n > n_k\} \in \omega,$$

a contradiction. □

We now want to study how distortion of groups relates to connectedness in asymptotic cones. We begin by defining a natural subspace of the asymptotic cone of G corresponding to H.

Definition V.2.10. Let T be a subspace of a metric space S. Denote by $Cone^{\omega}_{S}(T)$ the set of all points in $Cone^{\omega}(S)$ with a representative $(t_i)^{\omega}$ with each component in T.

Lemma V.2.11. *For all subspaces $T \subset S$, $Cone^{\omega}_{S}(T)$ is closed in $Cone^{\omega}(S)$.*

Proof. Note that $Cone^{\omega}_{S}(T) = Cone^{\omega}(T)$ where we consider T under the induced metric from S. Since asymptotic cones are complete, this is a complete metric space. A complete subspace of a complete metric space is closed and so we have that $Cone^{\omega}_{S}(T)$ is closed in $Cone^{\omega}(S)$. $\square$

Note that we can think about a subgroup H of a group G as a subspace of the metric space we get by considering the word metric on G.

Lemma V.2.12. *If H is a subgroup of a finitely generated group G such that $Cone^{\omega}_{G}(H)$ is connected for all ultrafilters ω, then H is finitely generated.*

Proof. Let H be a subgroup of a finitely generated group G, and let X be a finite generating set for G. We call an element h of H *reducible* if there exists a constant $k \in \mathbb{N}$ and k elements of H, $h_1, h_2 \dots h_k$, with $|h_i|_X < |h|_X$ for all $0 \leq i \leq k$ such that $h = h_1 h_2 \dots h_k$. We call an element $h \in H$ *irreducible* if it is not reducible. We can assume that there exists no i such that all elements $h \in H$ with $|h|_X \geq i$ are reducible, as this would imply that H is finitely generated. Thus we can find a sequence (h_i) of irreducible elements of H such that $|h_i|_X > |h_{i-1}|_X$ for all i. Fix an ultrafilter ω and consider the asymptotic cone $Cone^{\omega}_{G}(H)$ with respect to ω and the scaling sequence $(|h_i|_X)$. Assume this asymptotic cone is connected. As $(h_i)^{\omega} \in Cone^{\omega}_{G}(H)$, there exist points $(e)^{\omega} = p_0, p_1, \dots, p_k = (h_i)^{\omega}$ with $d(p_i, p_{i+1}) \leq 1/4$ for all $0 \leq i < k$. Let $p_j = (h_{i,j})^{\omega}$. We have that $|h_{i,j}^{-1} h_{i,j+1}|_X \leq |h_i|_X/2$ ω-almost surely. Finally, note that $h_i = h_{i,k} = h_{1,i}(h_{i,1}^{-1} h_{i,2}) \dots (h_{i,k-1}^{-1} h_{i,k})$. This, however, implies that h_i is ω-almost surely reducible, a contradiction. $\square$

We can apply Theorem V.2.7 to a subgroup H of a finitely generated group G, where H is given the word metric induced from G. In this case, the relationship between ν_H and δ^G_H combined with Lemma V.2.12 gives the following theorem.

Theorem V.2.13. *The following are equivalent for a subgroup H of a finitely generated group G:*

1. *H is finitely generated and there exists a constant K such that $\delta^G_H(k, i, 4i) \leq K$ for all i and for all sufficiently large k.*

2. *H is finitely generated and there exists a function f such that $\delta^G_H(k, m, n) \leq f(\frac{n}{m})$ for all sufficiently large k.*

3. $Cone^{\omega}_{G}(H)$ *is path connected for all ultrafilters* ω.

4. $Cone^{\omega}_{G}(H)$ *is connected for all ultrafilters* ω.

Example V.2.14. We have previously seen that if $G = BS(1,p) = \langle a,b \mid b^{-1}ab = a^{p}\rangle$, and $H = \langle a\rangle$ then $\delta^{G}_{H}(k,m,n) \sim p^{n-m}$. Thus, $\delta^{G}_{H}(k,i,2i)$ is unbounded, and we can conclude from Theorem V.2.7 that there exists an ultrafilter ω such that $Cone^{\omega}_{G}(H)$ is disconnected.

Example V.2.15. If G is the discrete Heisenberg group, and H is the center of G, then we have seen in a previous example that $\delta^{G}_{H}(k,m,n) \sim n^2/m^2$, and so $\delta^{G}_{H}(k,i,4i)$ is bounded, and $Cone^{\omega}_{G}(H)$ is connected for all ultrafilters ω.

We now want to relate the connectedness of $Cone^{\omega}_{G}(H)$ to the distortion of H in G. In order to do this, we need a couple preliminary results.

Definition V.2.16. A function $f\colon\mathbb{R}^{\geq 1} \to \mathbb{R}$ is called *superlinear* if for all $k \in \mathbb{R}$ the set $\{x \mid f(x) \leq kx\}$ is bounded. f is called *sublinear* if for all $k \in \mathbb{R}$ the set $\{x \mid f(x) \geq kx\}$ is bounded.

Lemma V.2.17. *Let* $f\colon\mathbb{R}^{\geq 1} \to \mathbb{R}$ *be an increasing, sublinear function with* $f(r) \leq r$ *for all real numbers* $r \geq 1$. *There exists a function* $\ell\colon\mathbb{R}^{\geq 1} \to \mathbb{R}^{\geq 1}$ *satisfying the following properties:*

1. *for all* $m,n \in \mathbb{N}$, $\ell(m)+\ell(n) \geq \ell(m+n)$.

2. *for all* $n \in \mathbb{N}$, $\ell(n) \geq f(n)$.

3. *for all* $k \in \mathbb{N}$, *there exists a* $p_k \in \mathbb{R}$ *such that* ℓ *is constant on the interval* $[p_k, kp_k]$.

Proof. We will define p_k and ℓ by induction on k. First let $p_1 = 1$ and let $\ell(1) = 1$. Assume we have defined p_k and $\ell(n)$ for $n \leq kp_k$ in a way that satisfies properties 1-3. Let p_{k+1} be the least real number such that for all $r \in \mathbb{R}$, if $r \geq (k+1)p_{k+1}$, then $f(r) \leq r/(k+1)!$. For $s \in \mathbb{R}$, if $kp_k < s \leq p_{k+1}$ define $\ell(s) = s/k!$. For $s \in \mathbb{R}$, $p_{k+1} \leq s \leq (k+1)p_{k+1}$, define $\ell(s) = p_{k+1}/k!$. By definition,

$$\ell((k+1)p_{k+1}) = p_{k+1}/k! = (k+1)p_{k+1}/(k+1)!\,.$$

We will now show that ℓ satisfies properties 1-3. First, fix $r \in \mathbb{R}^{\geq 1}$, and let $k \in \mathbb{N}$ such that $kp_k \leq r \leq (k+1)p_{k+1}$. If $kp_k < r < p_{k+1}$, then $\ell(r) = r/k!$, and if $s < r$, then $\ell(s) \geq s/k!$. Thus, if $p+q = r$, then

$$\ell(p)+\ell(q) \geq p/k! + q/k! = r/k! = \ell(r).$$

If $p_{k+1} < r \leq (k+1)p_{k+1}$, then $\ell(r) = \ell(p_{k+1})$, and property 1 follows immediately as ℓ is increasing. For $s \in \mathbb{R}$, if $kp_k \leq s \leq p_{k+1}$, then $\ell(s) = s/k! > f(s)$ by definition. If $p_{k+1} \leq s \leq (k+1)p_{k+1}$, then

$$\ell(s) = \ell((k+1)p_{k+1}) = (k+1)p_{k+1}/(k+1)! \geq f((k+1)p_{k+1}) \geq f(s),$$

so ℓ satisfies property 2. It is clear that this definition of ℓ satisfies property 3. $\square$

We are now ready to relate the connectedness of $Cone^{\omega}_{G}(H)$ to the distortion of H in G.

Theorem V.2.18. *If H is a finitely generated subgroup of a finitely generated group G, then the following implications hold.*

1. *If $\Delta^G_H(n)$ is linear, then $Cone^{\omega}_{G}(H)$ is connected for all ultrafilters ω.*
2. *If $Cone^{\omega}_{G}(H)$ is connected for all ultrafilters ω, then $\Delta^G_H(n) \preceq f$ for some polynomial f.*
3. *For every increasing, superlinear function $\phi : \mathbb{N} \to \mathbb{N}$ there exists a group G with a subgroup H such that $Cone^{\omega}_{G}(H)$ is disconnected for some ultrafilter ω, but $\Delta^G_H(n) \preceq \phi$.*
4. *For all $k \in \mathbb{N}$, there exists a group G with a subgroup H such that $Cone^{\omega}_{G}(H)$ is connected for all ultrafilters ω, and $\Delta^G_H \sim n^k$.*

Proof. We will begin by proving claim 1.

If H is a subgroup of G, then we can define a continuous function ρ from $Cone^{\omega}(H)$ to $Cone^{\omega}_{G}(H)$ by $\rho((h_i)^{\omega}) = (h_i)^{\omega}$. For all $h \in H$, $|h|_X \leq C|h|_Y$ for some fixed constant C, so ρ is well-defined. Assume $(h_i)^{\omega} \in Cone^{\omega}_{G}(H)$. This means that there exists B such that for all $i \in \mathbb{N}$, $|h_i|_X/i \leq B$. Distortion is linear means that there exists D such that

$$\frac{|h_i|_Y}{i} \leq D\frac{|h_i|_X}{i} \leq DB.$$

Thus, ρ is surjective, and $Cone^{\omega}_{G}(H)$ is connected, as $Cone^{\omega}_{G}(H)$ is connected.

Now we prove the second claim in Theorem V.2.18.

Assume that $Cone^{\omega}_{G}(H)$ is connected in $Cone^{\omega}(G)$, and hence that $\delta^G_H(k,i,2i)$ is bounded by some constant K for all i for sufficiently large k. By induction we have that $\Delta^G_H(2^n) = \delta^G_H(k,1,2^n) \leq K^n$ for all $n \in \mathbb{N}$.

Now let $n \in \mathbb{N}$, and let $m \in \mathbb{R}$ such that $2^{m-1} \leq n < 2^m$. We have that

$$\Delta^G_H(n) \leq \Delta^G_H(2^m) \leq K^m = (2^m)^{\log_2 K} \leq (2n)^{\log_2 K}.$$

Thus, $\Delta^G_H(n) \preceq n^{\log_2 K}$.

We will now prove the third claim of the theorem. Let ϕ be a superlinear, increasing function $\mathbb{N} \to \mathbb{N}$. ϕ can be extended to an invertible, increasing, superlinear function from $\mathbb{R}^{\geq 1}$ to $\mathbb{R}$. We can now apply Lemma V.2.17 to ϕ^{-1} to get a function ℓ which is always larger than ϕ^{-1}. We can then restrict ℓ to the natural numbers and take ceilings to get a function from $\mathbb{N}$ to $\mathbb{N}$. We can extend this to a function from $\mathbb{Z}$ to $\mathbb{Z}$ by defining $\ell(0) = 0$ and $\ell(-z) = \ell(z)$ for $z < 0$. As $\ell \geq \phi^{-1}$, we have that $\phi(\ell(n)) \geq n$. If ϕ is subexponential, then this ℓ now satisfies all of the conditions of Theorem III.1.7, and hence there exists a group $G = \langle X \rangle$, a constant C and an embedding $\psi \colon \mathbb{Z} \to G$ such that

$$\frac{\ell(n)}{C} \leq |\psi(n)|_X \leq C\ell(n).$$

Now note that if $|\psi(n)|_X \leq m$, then $\ell(n) \leq C|\psi(n)|_X \leq Cm$, and so $n < \phi(\ell(n)) \leq \phi(Cm)$. Hence, distortion is bounded by ϕ. On the other hand, $\ell(p_k) = \ell(p_k + 1) = \cdots = \ell(kp_k)$ implies that $C|\psi(q)|_X > \ell(p_k)$ for all $p_k \leq q \leq kp_k$ while $|\psi(kp_k)|_X \leq C\ell(p_k)$, and so $\delta_H^G(k, \ell(p_k)/C, C\ell(p_k)) \geq k$. By Theorem V.2.7, $Cone_G^{\omega}(H)$ is disconnected for some ultrafilter ω.

Note that if ϕ is superexponential, then claim 2 of Theorem V.2.18 shows that $Cone_G^{\omega}(H)$ is not connected for all ultrafilters ω.

Part 4 of the theorem can also be proven using this method.

Fix $k \in \mathbb{N}$, and for $z \in \mathbb{Z}$ let $\ell(z) = \left\lceil |z|^{\frac{1}{k}} \right\rceil$. Let G be a group with finite generating set X and ψ an embedding of $\mathbb{Z}$ into G such that

$$\frac{\ell(z)}{C} \leq |\psi(z)|_X \leq C\ell(z).$$

Note that if $|\psi(z)|_X \leq m$, then

$$|z|^{1/k} \leq \left\lceil |z|^{1/k} \right\rceil = \ell(z) \leq C|\psi(z)|_X \leq Cm,$$

which implies that $|z| \leq C^k m^k$. Thus $\Delta_H^G(m) \preceq m^k$. Now note that $\ell(m^k) = m$, so $|\psi(m^k)|_X \leq Cm$, which implies $\Delta_H^G(Cm) \geq m^k$. Thus, $\Delta_H^G(m) \sim m^k$. The above calculations show that if $|\psi(z)|_X \leq 4i$, then $|z| \leq 4^k C^K i^k$. Further, if $|z| \leq (i/C)^K$ then $|\psi(z)|_X \leq C\ell(z) \leq i$. Thus, $\delta_H^G(k, i, 4i) \leq 4^k C^{2k}$, and so by Theorem V.2.7 we have that $Cone_G^{\omega}(H)$ is connected. $\square$

Chapter VI

Convexity in asymptotic cones

VI.1 Strongly quasi-convex subgroups

Definition VI.1.1. A subspace T of a metric space S is called *Morse* if for all constants λ, C there exists a constant M such that any (λ, C)-quasi-geodesic connecting points in T is contained in the M neighborhood of T.

Definition VI.1.2. We say a subset T of a metric space S is *strongly convex* if every simple path starting and ending in T is entirely contained in T.

Theorem VI.1.3. *Let T be a closed subspace of a geodesic metric space S. Assume that $Cone_S^\omega(T)$ is strongly convex in $Cone^\omega(S)$ for all ultrafilters ω and for any two points t_1, t_2 in $Cone_S^\omega(T)$ there exists an isometry ϕ of $Cone^\omega(S)$ fixing $Cone_S^\omega(T)$ such that $\phi(t_1) = t_2$. Then T is Morse.*

Proof. Assume T is not Morse. This means that there exist constants $\lambda \geq 1, C \geq 0$ such that for all $i \in \mathbb{N}$ there exists a (λ, C)-quasi-geodesic $p_i\colon [0, k_i] \to S$ parameterized by length, and $s_i \in [0, k_i]$ with $p_i(0)$ and $p_i(k_i)$ in T and $d_S(p_i(s_i), T) \geq i$. For all i let

$$d_i = \sup\{d_S(p_i(s), T) \mid s \in [0, k_i]\}. \tag{2}$$

We can choose our paths p_i to make the sequence (d_i) increasing with all $d_i > C$. For each i, let s_i be a point in $[0, k_i]$ such that $d_S(p_i(s_i), T) = d_i$ (such a point exists as paths are compact). Let $s_i^\ell = \max\{s_i - 3\lambda d_i, 0\}$, and similarly let $s_i^r = \min\{s_i + 3\lambda d_i, k_i\}$. By (2) $d_S(p_i(s_i^\ell), T)$ and $d_S(p_i(s_i^r), T)$ are less than or equal to d_i. Let $d_S(p_i(s_i^\ell), T) = k_i^\ell$, and $d_S(p_i(s_i^r), T) = k_i^r$. Let t_i^ℓ be a point in T such that $d_S(p_i(s_i^\ell), t_i^\ell) = k_i^\ell$, and let $p_i^\ell\colon [0, k_i^\ell] \to \Gamma(G)$ be a geodesic from t_i^ℓ to $p_i(s_i^\ell)$. Note that by assumption we can take $t_i^\ell = t$ where t is some fixed point in T by taking an isometry fixing T sending t_i^ℓ to t. Similarly, let $p_i^r\colon [0, k_i^r]$ be a geodesic from $p_i(s_i^r)$ to a point $t_i^r \in T$ such that $d_S(t_i^r, p_i(s_i^r)) = k_i^r$. Denote by $p_i^m\colon [s_i^\ell, s_i^r] \to S$ the segment of p_i from $p_i(s_i^\ell)$ to $p_i(s_i^r)$.

We will need the following lemma.

Lemma VI.1.4. 1. *For all $i \in \mathbb{N}$, if $s_i^\ell \neq 0$, $a \in [s_i, s_i^r]$, and $b \in [0, k_i^\ell]$, then $d_S(p_i^m(a), p_i^\ell(b)) \geq d_i$.*

2. *For all $i \in \mathbb{N}$, if $s_i^r \neq k_i$, $a \in [s_i^\ell, s_i]$, and $b \in [0, k_i^r]$, then $d_S(p_i^m(a), p_i^r(b)) \geq d_i$.*

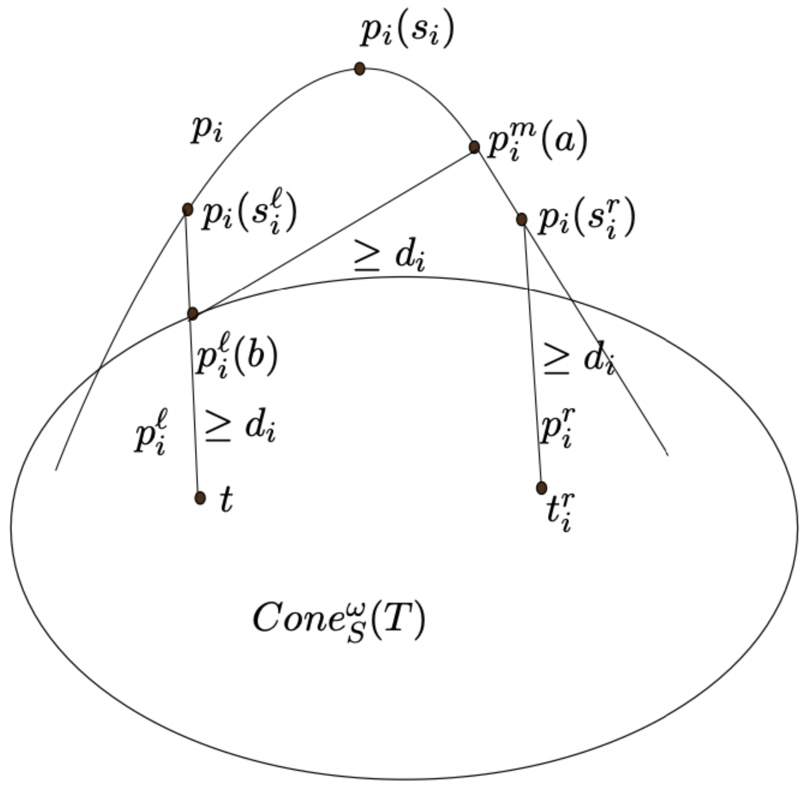

Figure VI.1: Theorem 5.3

Proof. First, if $s_i^\ell \neq 0$, then $s_i^\ell = s_i - 3\lambda d_i$. Now note that

$$d_S(p_i^m(a), p_i^m(s_i^\ell)) \geq \frac{3\lambda d_i}{\lambda} - C = 3d_i - C > 3d_i - d_i = 2d_i,$$

as p_i is a (λ, C) geodesic, and we assumed that $d_i > C$. Thus, as $d_S(p_i^\ell(b), p_i^m(x_i^\ell)) \leq d_i$, $d_S(p_i^m(a), p_i^\ell(b)) \geq d_i$. The second claim follows similarly. $\square$

We return to the proof of Theorem VI.1.3.

Fix an ultrafilter ω, and consider the asymptotic cone of S with respect to ω and the scaling sequence d_i. By construction, $d_S(t, p_i^\ell(k_i^\ell)) \leq d_i$, and so $(p_i^\ell(k_i^\ell))^\omega \in Cone^\omega(G)$. As $|s_i^\ell - s_i^r| \leq 6\lambda d_i$, we have that $d_S(p_i(s_i^\ell), p_i(s_i^r)) \leq 6\lambda^2 d_i + C$,. and so as $(p_i(s_i^\ell))^\omega \in Cone^\omega((d_i), S)$, we have that $(p_i(s_i^r))^\omega \in Cone^\omega((d_i), S)$. As $d_S(p_i(s_i^r), p_i^r(k_i^r)) = d(p_i^r(0), p_i^r(k_i^r)) \leq d_i$, we have that $(p_i^r(k_i^r))^\omega \in Cone^\omega((d_i), S)$. Thus we can define

$$k^\ell = \lim^\omega \frac{k_i^\ell}{d_i}, s^\ell = \lim^\omega \frac{s_i^\ell}{d_i}, s^r = \lim^\omega \frac{s_i^r}{d_i}, k^r = \lim^\omega \frac{k_i^r}{d_i},$$

and we can define $p^\ell\colon[0,k^\ell]\to Cone^\omega((d_i),S)$ as $\lim^\omega(p_i^\ell)$, $p^m\colon[s^\ell,s^r]\to Cone^\omega((d_i),S)$ as $\lim^\omega(p_i^m)$, and $p^r\colon[0,k^r]$ as $\lim^\omega(p_i^r)$. We have that p^ℓ and p^r are geodesics, and p^m is a $(\lambda,0)$ quasi-geodesic, and hence all are simple.

Now we have three simple paths, p^ℓ, p^m, p^r, such that $p^\ell(0)$ and $p^r(k^r)$ are in $Cone_S^\omega((d_i),T)$, and p^ℓ and p^r both intersect p^m. Unfortunately, the concatenation of these three paths may not be simple, as p^ℓ and p^r could intersect p^m in more than once. To deal with this case, we need the following lemma.

Lemma VI.1.5. *Let $s=lim^\omega s_i/d_i$.*

1. *If $a\in[0,k^\ell]$,and $b\in[s^\ell,s^r]$, with $p^\ell(a)=p^m(b)$, then $b\le s$.*
2. *if $a\in[0,k^r]$, and $b\in[s^\ell,s^r]$, with $p^r(a)=p^m(b)$, then $b\ge s$.*

Proof. Note that if $\{i|k_i^\ell=0\}\in\omega$, then p^l is a trivial path, and the result is clear. Otherwise, we have that $\{i|k_i^\ell\neq 0\}\in\omega$. In this case we can use Lemma VI.1.4 to say that if $(b_i)^\omega$ is on p^ℓ and $(a_i)^\omega$ is on p^m after s, then $d_S^\omega((b_i)^\omega,(a_i)^\omega)\ge\lim^\omega\frac{d_i}{d_i}\ge 1$. The proof of claim 2 follows similarly. □

Thus, we can form a simple path which starts and ends in $Cone_S^\omega((d_i),T)$ as follows. Let

$$p=\max\{t\in[s^\ell,s^r]\mid\exists a\in[0,k^\ell]\ p^\ell(a)=p^m(t)\},$$

and let

$$q=\min\{t\in[s^\ell,s^r]\mid\exists a\in[0,k^r]\ p^r(a)=p^m(t)\}.$$

We obtain a simple path by following p^ℓ up to $p^m(p)$, then following p^m up to $p^m(q)$, and finally following p^r back to $p^r(k^r)$. This path contains $p^m(s)$ by Lemma VI.1.5. Finally, as $p^m(s)=(p_i^m(s_i))^\omega$,

$$d_S^\omega(p^m(s),Cone_S^\omega((d_i)T))=\lim^\omega\frac{d_S(p_i^m(s_i),Cone_S^\omega((d_i),T))}{d_i}=\lim^\omega\frac{d_i}{d_i}=1.$$

Thus, we have a simple path starting and ending in $Cone_S^\omega(T)$, which is not entirely contained in $Cone_S^\omega(T)$. □

In order to prove a partial converse of this statement we will need the following results from Drutu, Mozes and Sapir [5]. Note that an error was found in this paper [1], but none of the following lemmas were affected.

Lemma VI.1.6. *([5] Lemma 2.3) Let S be a geodesic metric space, ω an ultrafilter, and B a closed subset of $Cone^\omega(S)$. If x,y are in the same connected component of $Cone^\omega(S)\setminus B$, then there exists a sequence of* *paths* $(p_i)_{i=1}^n$ *such that each path is a limit geodesic in* X*, and the concatenation of the paths* p_i *is a simple path from* x *to* y*.*

Definition VI.1.7. A path is called C bi-lipschitz if it is a $(C,0)$ quasi-geodesic.

Lemma VI.1.8. *([5] Lemma 2.5) In the same setting as Lemma 5.6, let* p *be a simple path in* $Cone^{\omega}(S)$ *which is a concatenation of limit geodesics. For all* δ *there exists a constant* C *and a* C *bi-Lipschitz path* p' *such that the Hausdorff distance between* p *and* p' *is less than* δ*, and* p' *is also a concatenation of limit geodesics connecting the same points.*

Lemma VI.1.9. *([5] Lemma 2.6) Let* p *be a C-bi-Lipschitz path in* $Cone^{\omega}(S)$ *which is a concatenation of limit geodesics. There exists a constant* C' *and a sequence of paths* (p_n) *in* S *such that each* p_n *is* C' *bi-Lipschitz, and* $lim^{\omega}(p_n) = p$*.*

Theorem VI.1.10. *If* T *is a Morse subspace of a metric space S, then* $Cone^{\omega}_S(T)$ *is strongly convex in* $Cone^{\omega}(S)$*.*

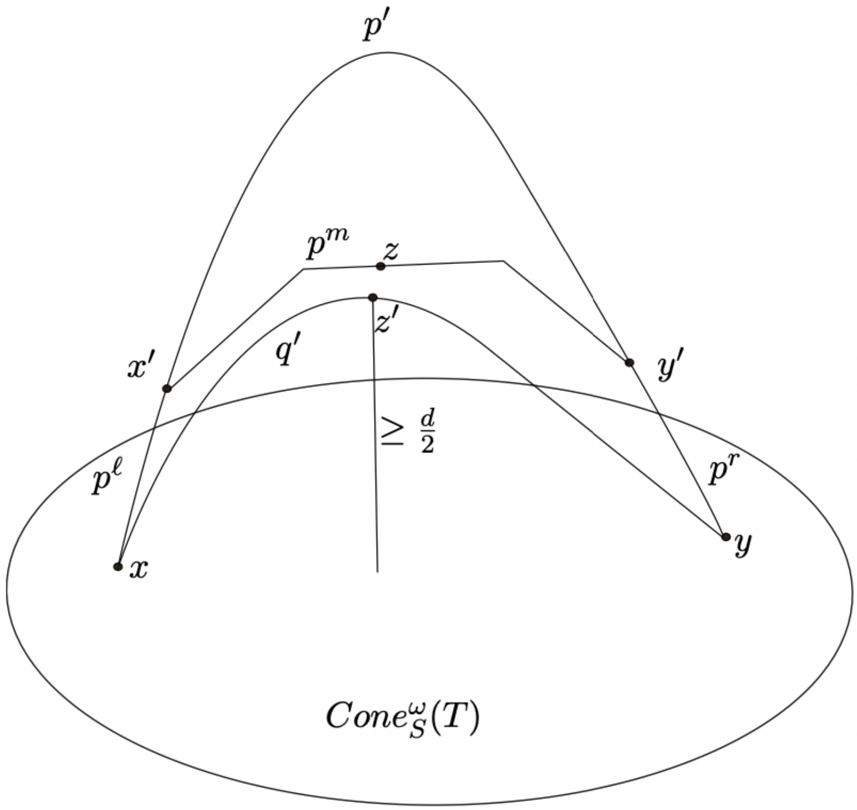

Figure VI.2: Theorem 5.9

Proof. Let p be a simple path in $Cone^{\omega}(S)$ starting and ending in $Cone^{\omega}_{S}(T)$ but not entirely contained in $Cone^{\omega}_{S}(T)$. As $Cone^{\omega}_{S}(T)$ is closed, there is a subpath p' of p which starts and ends in $Cone^{\omega}_{S}(T)$ but no interior point of p' is in $Cone^{\omega}_{S}(T)$. Let x be the initial point of p and let y be the terminal point of p. Let x', y' be points on p' such that

$$\max\{d^{\omega}_{S}(x,x'), d^{\omega}_{S}(y,y')\} < \frac{d^{\omega}_{S}(x,y)}{2},$$

and let p^l, p^r be limit geodesics from x to x' and from y' to y respectively. Let p^m be a concatenation of limit geodesics connecting x' to y' avoiding $Cone^{\omega}_{S}(T)$. Such a path exists by Lemma VI.1.6 as $Cone^{\omega}_{S}(T)$ is closed. The concatenation of p^l p^m and p^r may not be simple, so we let a be the first point of p^l on p^m, and b be the last point of p^r on p^m. By the choice of x' and y', p^{ℓ} does not intersect p^r, so we can obtain a simple path by following p^{ℓ} from x to a, p^m from a to b, and p^r from b to y. Call this concatenation q.

Let z be a point on q such that $d^{\omega}_{S}(z, Cone^{\omega}_{S}(T)) = d > 0$. Using lemma VI.1.8, we can find a path q' such that q' is a C bi-Lipschitz path which is a concatenation of limit geodesics, and the Hausdorff distance between q and q' is less than $\frac{d}{2}$. Thus, there is a point z' on q' such that $d^{\omega}_{S}(z,z') \leq d/2$, so $d^{\omega}_{S}(z', Cone^{\omega}_{S}(T)) \geq d/2$.

Finally we can apply Lemma VI.1.9 to this new path q' to get that $q' = \lim^{\omega}(q_n)$ with each q_n being a C' bi-Lipschitz path starting and ending in T. Thus, as T is Morse, each path is in some fixed neighborhood of T. This implies that $q = \lim^{\omega}(q_n)$ is entirely contained in $Cone^{\omega}_{S}(T)$, a contradiction.

Thus, if T is Morse in S, then $Cone^{\omega}_{S}(T)$ is strongly convex in $Cone^{\omega}(S)$. ☐

Definition VI.1.11. A subgroup H of a group G with finite generating set X is called *strongly quasi-convex* if it is Morse as a subspace of the Cayley graph G with respect to X.

Note that if H is a subgroup of G, then for any two points $(h_i)^{\omega}, (k_i)^{\omega}$ in $Cone^{\omega}_{G}(H)$ there exists an isometry of $Cone^{\omega}(G)$ fixing $Cone^{\omega}_{G}(H)$ which sends $(h_i)^{\omega}$ to $(k_i)^{\omega}$. Thus, we can combine the previous two results to give:

Theorem VI.1.12. *A subgroup H of a group G is strongly quasi-convex if and only if $Cone^{\omega}_{G}(H)$ is strongly convex in $Cone^{\omega}(G)$ for all ultrafilters ω.*

VI.2 Applications

We begin this section by proving a large class of groups cannot contain infinite infinite index strongly quasi convex subgroups.

Theorem VI.2.1. *If a path connected metric space S contains a proper closed strongly convex subspace T consisting of more than one point, then S contains a cut point.*

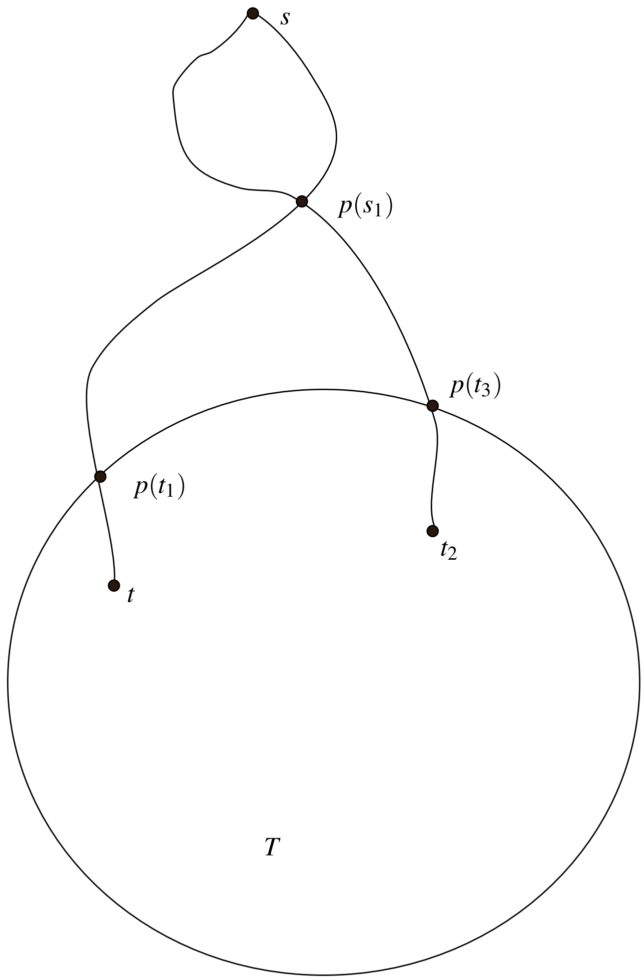

Figure VI.3: Theorem VI.2.1

Proof. Let $s \in S \setminus T$, and let $t \in T$. Let $p\colon [0,\ell] \to S$ be a simple path connecting s and t. Let $t_1 = \min\{a \in [0,\ell] \mid p(a) \in T\}$. This is well-defined as T is closed. We will show that $p(t_1)$ is a cut point. Let $t_2 \neq p(t_1)$ be a point in T. If $p(t_1)$ is not a cut point, then there exists a path $p'\colon [0,k]$ connecting s and t_2 such that $p(t_1)$ is not on p'. Let $t_3 = \min\{a \in [0,k] \mid p'(a) \in T\}$. Let $s_1 = \max\{a \in [0,t_1] \mid p(s_1) \in p'\}$ Create a simple path by following p from t_1 to s_1 and then following p' from s_1 to t_2. This is a simple path connecting two points of T that is not entirely contained in T, a contradiction. $\square$

Sapir and Drutu [6] proved the following theorem.

Theorem VI.2.2. *If G is a non-virtually cylic group satisfying a law, then no asymptotic cone of G contains a cut point.*

If H is an infinite, infinite index subgroup of a finitely-generated group G, then it is easy to see that

$Cone^{\omega}_{G}(H)$ is a proper subspace of $Cone^{\omega}(G)$ that consists of more than one point. Thus, we can combine the previous two results to get the following corollary.

Corollary VI.2.3. *If G is a finitely-generated group containing an infinite, infinite index strongly quasi-convex subgroup H, then G does not satisfy a law.*

We will now show that if H and K are strongly quasi-convex subgroups of a finitely generated group G, then $Cone^{\omega}_{G}(H \cap K) = Cone^{\omega}_{G}(H) \cap Cone^{\omega}_{G}(K)$. This is not true in general for any two subgroups of a finitely generated group.

Example VI.2.4. Let $G = \mathbb{Z} \times BS(1,2) = \langle a,b,c | [a,b] = [a,c] = 1, b^{-1}cb = c^2 \rangle$. Consider the subgroups $H = \langle (a,e) \rangle$ and $K = \langle (a,c) \rangle$ where e is the identity element in $BS(1,2)$. It is clear that $H \cap K = \{(a^0,e)\}$, but $((a^i,c^i))^{\omega} = ((a^i,e))^{\omega} \in Cone^{\omega}(H) \cap Cone^{\omega}(K)$.

Notation VI.2.5. If γ is a geodesic in a metric space S containing the points s and t, then we will denote by $[s,t]_{\gamma}$ the subsegment of γ connecting s and t.

We will need the following lemma, which is based heavily on ideas found in Lemma 3.2 of [23].

Lemma VI.2.6. *Let a,b,c be points in a metric space X, and let γ_1 and γ_2 be geodesics connecting a to b and b to c respectively. Let $K = d(b,c)$, and assume that $d(a,b) \geq 3K$. There exists a $(2,0)$ quasi-geodesic γ_3 connecting a to c such that there exists a point $x \in \gamma_1 \cap \gamma_3 \cap B_{3K}(b)$.*

Proof. Denote the concatenation of γ_1 and γ_2 by γ. If γ is a $(2,0)$ quasi-geodesic, then we are done, so assume γ is not a $(2,0)$ quasi-geodesic. As $d(a,b) \geq 3d(b,c)$, we have that $d(a,c) \geq d(a,b) - d(b,c) \geq (2/3)d(a,b)$. On the other hand $d_{\gamma}(a,c) = d(a,b) + d(b,c) \leq (4/3)d(a,b)$. Thus, $d_{\gamma}(a,c)/d(a,c) \leq 2$. Therefore, there exists a maximal subsegment $[x,y]$ of γ such that $d_{\gamma}(x,y)/d(x,y) = 2$. Let γ_3 be the concatenation of $[a,x]_{\gamma_1}$, $[x,y]$ and $[y,c]_{\gamma_2}$ where $[x,y]$ is a geodesic connecting x and y. It follows from the proof of Lemma 3.2 from [23] that γ_3 is a $(2,0)$ quasi-geodesic.

We will now show that $d(x,b) \leq 3d(b,c)$. As $d_{\gamma}(x,y) = 2d(x,y)$, we have that

$$d(x,b) + d(b,y) = d_{\gamma}(x,y) = 2d(x,y) \geq 2d(x,b) - 2d(y,b).$$

Rearranging gives that $3d(b,y) \geq d(x,b)$. As $d(b,c) \geq d(b,y)$, we have that $3d(b,c) \geq d(b,x)$. $\square$

Notation VI.2.7. Let S be a metric space and let T be a subspace of S. We denote by $\mathcal{N}_{\varepsilon}(T)$ the set of points s in S such that there exists a $t \in T$ with $d(s,t) \leq \varepsilon$.

Lemma VI.2.8. *Let (S,s) be a pointed geodesic metric space, with Morse subspaces U,V. Let $(x_i)^\omega \in Cone^\omega_S(U) \cap Cone^\omega_S(V)$. Assume $(x_i)^\omega \neq (s)^\omega$. Then there exists a real number $\varepsilon \geq 0$ such that $(x_i)^\omega \in Cone^\omega_S(U \cap \mathcal{N}_\varepsilon(V))$.*

Proof. Let $(u_i)^\omega$ be a representative of $(x_i)^\omega$ with all components in U. Let $(v_i)^\omega$ be a representative of $(x_i)^\omega$ with all components in V. For each $i \in \mathbb{N}$, let $\gamma_{1,i}$ be a geodesic connecting s to u_i, and let $\gamma_{2,i}$ be a geodesic connecting u_i to v_i. As $(x_i)^\omega \neq (s)^\omega$, we have that $d(s,u_i) \geq 3d(u_i,v_i)$ ω-almost surely. Thus, By Lemma VI.2.6 we can ω-almost surely find a $(2,0)$ quasi-geodesic $\gamma_{3,i}$ containing a point $z_i \in \gamma_1 \cap B_{3d(u_i,v_i)}(u_i)$. As z_i is on the geodesic γ_1 which connects two points of the Morse subset U, there exists a constant D such that for all $i \in \mathbb{N}$ there exists a point $y_i \in U$, with $d(y_i,z_i) \leq D$. As $(u_i)^\omega = (v_i)^\omega$, we have that $\lim^\omega d(u_i,v_i)/i = 0$. Thus

$$d^\omega_S(y_i,u_i) = \lim{}^\omega d_S(y_i,u_i)/i \leq \lim{}^\omega(d_S(y_i,z_i)+d_S(z_i,u_i))/i \leq \lim{}^\omega(D+3d_S(u_i,v_i))/i = 0.$$

Thus, $(y_i)^\omega = (u_i)^\omega$. Finally, as z_i is on the $(2,0)$ quasi-geodesic $\gamma_{3,i}$ connecting two points of the Morse subspace V, there exists a constant E such that z_i is within E of some point in V, so y_i is within $\varepsilon = D+E$ of a point in V. □

Theorem VI.2.9. *Given a finitely generated group G and strongly quasi-convex subgroups H and K, $Cone^\omega_G(H) \cap Cone^\omega_G(K) = Cone^\omega_G(H \cap K)$.*

Proof. Fix a finite generating set X for G. It is clear that $Cone^\omega(H \cap K) \subset Cone^\omega(H) \cap Cone^\omega(K)$. Assume $(h_i)^\omega \in Cone^\omega(H) \cap Cone^\omega(K)$. If $(h_i)^\omega = (e)^\omega$, then clearly $(h_i)^\omega \in Cone^\omega_G(H \cap K)$, so we can assume otherwise. According to Lemma VI.2.8, there exists a representative $(h'_i)^\omega$ of $(h_i)^\omega$ and a constant C such that for all $i \in \mathbb{N}$, $h'_i \in H$ there exists a $k_i \in K$ with $d(h'_i,k_i) \leq C$. There exist only finitely many elements $g \in G$ with $|g|_X \leq C$. Thus, there exists an element $g \in G$ such that $h'_i g = k_i$ ω-almost surely. Let j be the least i such that $h'_j g = k_j$. If $h'_i g = k_i$, let $h''_i = h'_i h_j^{-1} = k_i k_j^{-1} \in H \cap K$. Otherwise, let $h''_i = e \in H \cap K$. Clearly $(h''_i)^\omega = (h'_i)^\omega = (h_i)^\omega$. So $(h_i)^\omega \in Cone^\omega(H \cap K)$, and $Cone^\omega(H) \cap Cone^\omega(K) = Cone^\omega(H \cap K)$. □

Lemma VI.2.10. *Let G be a finitely generated group with finite generating set X, and let H,K be subgroups of G such that $H \leq K$. Then $Cone^\omega_G(H) = Cone^\omega(K)$ for all ultrafilters ω if and only if H is finite index in K.*

Proof. First, if H is finite index in K, then there exists a finite set S of elements of K such that $K = \sqcup_{s \in S} Hs$. Let $M = \max\{|s|_X \mid s \in S\}$. Any element $k \in K$ is within M of an element of H, and hence $Cone^\omega_G(H) = Cone^\omega_G(K)$.

Now assume that H is of infinite index in K. For each right coset C of H in K, let k_C be an element of C such that for any other $k \in C, |k|_X \geq |k_C|_X$. As H is infinite index in K, we have that there exist cosets C such that $|k_C|_X$ is arbitrarily large. Fix a sequence of right cosets C_i such that $|k_{C_i}|_X$ is an increasing unbounded sequence. Consider the point $(k_{C_i})^\omega \in Cone^\omega_G(K)$ where $Cone^\omega(G)$ is an asymptotic cone with respect to the scaling sequence $(|k_{C_i}|_X)$. If $d_X(k_{C_i}, H) < |k_{C_i}|_X$, then there exists an $h \in H$ such that $d_X(k_{C_i}, h) = d_X(h^{-1}k_{C_i}, e) = |h^{-1}k_{C_i}|_X$, but this contradicts the minimality of $|k_{C_i}|_X$. Thus, $d_X(H, k_{C_i}) = |k_{C_i}|_X$ and $d((k_{C_i})^\omega, Cone^\omega_G(H)) = 1$, so $Cone^\omega_G(H)$ does not equal $Cone^\omega_G(K)$. □

Definition VI.2.11. Given a finitely generated group G and a subgroup H, the *asymptotic stabilizer* of H in G, denoted $Astab_G(H)$, is defined to be $\{g \in G \mid gCone^\omega_G(H) = Cone^\omega_G(H)\}$.

Theorem VI.2.12. *If H is a strongly quasi-convex subgroup of G, then $Comm_G(H) = Astab_G(H)$.*

Proof. It is clear that $Comm_G(H) \subset Asbat_G(H)$. Let $g \in Astab_G(H)$. This means that

$$Cone^\omega_G(g^{-1}Hg) = g^{-1}Cone^\omega_G(Hg) = g^{-1}Cone^\omega_G(H) = Cone^\omega_G(H).$$

Thus, by Theorem VI.2.9, we have that $Cone^\omega_G(H \cap g^{-1}Hg) = Cone^\omega_G(H) \cap Cone^\omega_G(g^{-1}Hg) = Cone^\omega_G(H)$. By Lemma VI.2.10, we have that $g^{-1}Hg \cap H$ is finite index in H. A similar argument shows that $g^{-1}Hg \cap H$ is finite index in $g^{-1}Hg$, and thus $Comm_G(H) = Astab_G(H)$. □

This is not the case for a general subgroup of a group.

Example VI.2.13. Let G, H and K be as in VI.2.4. Consider the *HNN* extension of G given by the presentation $\langle a,b,c,t | [a,b] = [a,c] = 1, b^{-1}cb = c^2, t^{-1}at = ab\rangle$. It is clear that no power of t is in the commensurator of H, but $d(t^{-1}a^nt, a^n) = d((ab)^n, a^n) = |b^n|_{\{a,b,t\}} \preceq \log_2(n)$. On the other hand $|a^n|_{\{a,b,t\}} \sim n$, so $t \in Astab_G(H)$. In particular, $Comm_G(H)$ is an infinite index subgroup of $Astab_G(H)$.

We now recall a result about the structure of virtually cyclic groups.

Theorem VI.2.14. *If G is an infinite virtually cyclic group then exactly one of the following two statements holds:*

1. *There exists an element $a \in G$ of infinite order and a finite normal subgroup H such that for all $g \in G$ there exists an $n \in \mathbb{N}$ and an $h \in H$ such that $g = ha^n$.*
2. *There exists an element $a \in G$ of infinite order, a finite normal subgroup H, and an element $f \in G$ satisfying $f^2 = e$ and $faf = a^{-1}$, such that for all $g \in G$ there exist an $n \in \mathbb{N}$, an $h \in H$ and $\varepsilon \in \{0,1\}$ such that $g = ha^nf^\varepsilon$.*

If G satisfies 1, then we say that a is orientable in G.

Definition VI.2.15. An element a of a finitely generated group G is called *Morse* if the subgroup generated by a is strongly quasi-convex.

The following theorem is proven in [23].

Theorem VI.2.16. *If H is a strongly quasi-convex subgroup of a finitely generated group G, then H is finite index in its commensurator. In particular, if a is a Morse element, then $Comm_G(\langle a\rangle)$ is virtually cyclic.*

Definition VI.2.17. If a is an infinite order element of a group G, then the *elementary subgroup* of a, denoted $E(a)$ is the largest subgroup of G in which $\langle a\rangle$ is a finite index subgroup, if such a subgroup exists.

Remark. *Theorem VI.2.16 implies that if a is a Morse element of a group G, then $E(a)$ exists and is equal to $Comm_G(H) = Astab_G(H)$.*

Definition. If a is a Morse element of a group G, then we call *a orientable* if a is orientable in its commensurator.

Lemma VI.2.18. *Let a be a Morse element of a finitely generated group G, and let H, f be as in Theorem VI.2.14 for the commensurator of $\langle a\rangle$. Then if $h_1, h_2 \in H$ and $n, m \in \mathbb{N}$ with $|n| \neq |m|$, then $h_1 a^n$ is not conjugate to $h_2 a^m$.*

Proof. Assume that there exists a $t \in G$ such that $t^{-1}h_1a^nt = h_2a^m$. This implies that $t \in Comm_G(\langle a\rangle)$. Thus, we have that $h_2a^m = t^{-1}h_1tt^{-1}a^nt$, and so, as H is normal in $Comm_G(\langle a\rangle)$, $t^{-1}a^nt = t^{-1}h_1^{-1}th_2a^m = h_3a^m$ for some $h_3 \in H$. Assume without loss of generality that $|n| < |m|$. Note that $n \neq 0$, as otherwise h_1a^n would have finite order while h_2a^m would have infinite order. We will show by induction that for all $k \in \mathbb{N}$, $t^{-k}h_1a^{n^k}t = ha^{m^k}$ for some $h \in H$. Assume that the result holds for $k-1$ and note that

$$t^{-k}h_1a^{n^k}t^k = t^{-k}h_1t^kt^{-(k-1)}t^{-1}a^{n^k}tt^{k-1} =$$
$$t^{-k}h_1t^kt^{-(k-1)}h_3a^{mn^{k-1}}t^{k-1} = t^{-k}h_1t^kt^{-(k-1)}h_3t^{k-1}a^{m^k} = ha^{m^k}$$

for some $h \in H$. But this implies that $\langle a\rangle$ is exponentially distorted in $Comm_G(\langle a\rangle)$, a contradiction as $\langle a\rangle$ is a finite index subgroup of $Comm_G(H)$. $\square$

We will need the following Lemma due to Neumann in [14].

Lemma VI.2.19. *Let $k \in \mathbb{N}$, $H_1, H_2, \ldots, H_k$ be infinite index subgroups of a group G and $g_1, g_2, \ldots g_k$ be elements of G. Then $G \neq \bigcup_{i=1}^{k} H_ig_i$.*

Notation VI.2.20. We denote the conjugacy class of an element G of a group G by $(g)^G$. We denote the centralizer of an element G of a group G by $C_G(g)$.

Lemma VI.2.21. *Let G be an ICC group, and let a be an orientable Morse element. Then for any $n \in \mathbb{N}$ and $g_1, g_2 \dots g_n \in G \setminus \{e\}$, there exists $t \in G$ such that $t^{-1} g_i t \notin Comm_G(\langle a \rangle)$ for all $i \in \{1, 2 \dots n\}$.*

Proof. First note that for any $g \in G$, $(g)^G \cap Comm_G(H)$ is finite by Lemma VI.2.18 and the fact that a is orientable. For each g_i let $k_i = |(g_i)^G \cap Comm_G(\langle a \rangle)|$ and let $(g_i)^G \cap Comm_G(\langle a \rangle) = \{h_{i,1}, h_{i,2}, \dots h_{i,k_i}\}$. For each $h_{i,j} \in (g_i)^G \cap Comm_G(\langle a \rangle)$ let $t_{i,j}$ be such that $t_{i,j}^{-1} g_i t_{i,j} = h_{i,j}$. The set of $t \in G$ such that $t^{-1} g_i t \in Comm_G(\langle a \rangle)$ can be written as $\sqcup_{j=1}^{k_i} C_G(g_i) t_{i,j}$. Thus, the set of $t \in G$ such that $t^{-1} g_i t \in Comm_G(\langle a \rangle)$ for some $i \in \{1, 2, \dots n\}$ can be written as $\cup_{i=1}^{n} \sqcup_{j=1}^{k_i} C_G(g_i) t_{i,j}$. As G is ICC, this is a finite union of cosets of subgroups of infinite index and hence by Lemma VI.2.19, does not cover G. Thus there exists $t \in G$ such that $t^{-1} g_i t \notin Comm_G(\langle a \rangle)$ for all $i \in \{1, 2, \dots n\}$. ☐

Lemma VI.2.22. *Let G be an ICC group with Morse element a. Then for any $g \in G \setminus \{e\}$, there exists an element $t \in G$ such that $t^{-1} g t \notin Comm_G(\langle a \rangle)$.*

Proof. Let H, f be as in VI.2.14.

Assume that $(g)^G \subset Comm_G(H)$. By VI.2.18, $(g)^G$ can only contain finitely many elements of the form ha^n where $h \in H$ and $n \in \mathbb{N}$. Thus as G is ICC, there must exist $h_1, h_2 \in H$ and $n, m \in \mathbb{N}$ with $n \neq m$ such that $h_1 a^n f, h_2 a^m f \in (g)^G$. Note that

$$h_1 a^n f h_2 a^m f = h_3 a^n f a^m f = h_3 a^{n-m}$$

for some $h_3 \in H$ as H is normal. For all t in G,

$$t^{-1} h_3 a^{n-m} t = t^{-1} h_1 a^n f t t^{-1} h_2 a^m f t \in Comm_G(\langle a \rangle),$$

as both $t^{-1} h_1 a^n f t$ and $t^{-1} h_2 a^m f t$ are in $(g)^G \subset Comm_G(\langle a \rangle)$, but this contradicts Lemma VI.2.18 and the fact that G is ICC. ☐

Lemma VI.2.23. *Let G be an amenable group, let a be a Morse element of G, and let $g_1, g_2, \dots g_n$ be elements of $G \setminus \{e\}$. Then there exists an element $t \in G$ such that $t^{-1} g_i t \notin Comm_G(\langle a \rangle)$ for all $i \in \{1, 2 \dots n\}$.*

Proof. If a is orientable in its commensurator, then we are done by Lemma VI.2.21, so assume a is not orientable in its commensurator. Let H and f be as in Theorem VI.2.14 for the commensurator of a, and let μ be a left invariant finitely additive measure for G. For $g \in G$ define $T_g = \{t \in G \mid t^{-1} g t \in Comm_G(\langle a \rangle)\}$. If $h = t^{-1} g t$ then $T_h = t^{-1} T_g$ so as μ is left invariant, we have that if g is conjugate to h then $\mu(T_g) = \mu(T_h)$. We will show that for $g \neq e, \mu(T_g) = 0$. Assume that there exists a $g \in G \setminus \{e\}$ with $\mu(T_G) = p > 0$. If $(g)^G \cap Comm_G(H)$ is finite, then T_g is a finite union of cosets of $C_G(g)$, and so has measure zero, so we can assume that $(g)^G \cap Comm_G(\langle a \rangle)$ is infinite. Let $M \in \mathbb{N}$ be such that $Mp > 1$. By Lemma VI.2.18 we can find M elements

$$k_1 = h_1 a^{n_1} f, k_2 = h_2 a^{n_2} f, \dots k_M = h_M a^{n_M} f \in (g)^G \cap Comm_G(\langle a \rangle)$$

with $n_i \neq n_j$ whenever $i \neq j$. Thus $k_i k_j$ is of the form ha^n for some $h \in H$ and some $n \in \mathbb{N}$. By Lemma VI.2.18, we have that $\mu(T_{k_i k_j}) = 0$ whenever $i \neq j$. $T_{k_i} \cap T_{k_j} \subset T_{k_i k_j}$, so $\mu(T_{k_i} \cap T_{k_j}) = 0$ whenever $i \neq j$. On the other hand $\mu(T_i) = p$ for all $i \in \{1, 2 \dots M\}$ so $\mu(T_i \bigcap_{i \in \{1,2,\dots i-1, i+1, \dots M\}} T_i^c) = p$. But this is a collection of M disjoint sets, contradicting that $Mp > 1$. □

Lemma VI.2.24. *let S be a metric space, and let $T_1, T_2, \dots T_n$ be strongly convex subspaces of S. For all $1 \leq i \leq n$, let p_i be a non-trivial geodesic in T_i such that the concatenation of the p_is forms an n-gon. Then either there exists an $1 \leq i \leq n-1$ such that the intersection of T_i and T_{i+1} consists of more than one point or the intersection of T_1 and T_n consists of more than one point.*

Proof. The result is trivial if $n = 2$. In the case of $n = 3$ note that the concatenation of p_2 and p_3 is a path that begins and ends in T_1. Thus the path is either entirely contained in T_1 or is not simple. In either case the result follows. Now fix n and assume that the result holds for all $m < n$. Again, note that the concatenation of p_2 through p_n begins and ends in T_1 and so is either contained in T_1 (in which case the result follows immediately) or is not simple. In the latter case, we have a point p which occurs on two different geodesics p_i and p_j. This splits the n-gon into two different polygons, at least one of which has fewer than n sides. The result now follows by induction. □

Definition VI.2.25. Given a group G with finite generating set X and an ultrafilter ω, we will denote by $\Pi G/\omega$ the group consisting of equivalence classes of sequences (g_i) with $g_i \in G$ for all $i \in \mathbb{N}$ where two sequences $(g_i), (h_i)$ are considered equivalent if $g_i = h_i$ ω-almost surely. We will denote the equivalence class of (g_i) by $(g_i)^{\omega,L}$ to distinguish elements of $\mathscr{L}(G)$ from points in $Cone^{\omega}(G)$. Finally, we will denote by $\mathscr{L}(G)$ the subgroup of $\Pi G/\omega$ where $(g_i)^{\omega,L} \in \mathscr{L}$ if and only if there exists a constant $C \in \mathbb{N}$ such that $|g_i|_X \leq Ci$ ω-almost surely.

Theorem VI.2.26. *If G is an ICC group with orientable Morse element a, and w is a nontrivial element of $F_2 * G$, then w is not a mixed identity of G if any one of the following conditions hold.*

1. *a is an orientable Morse element.*
2. *w has only one coefficient.*
3. *G is amenable.*

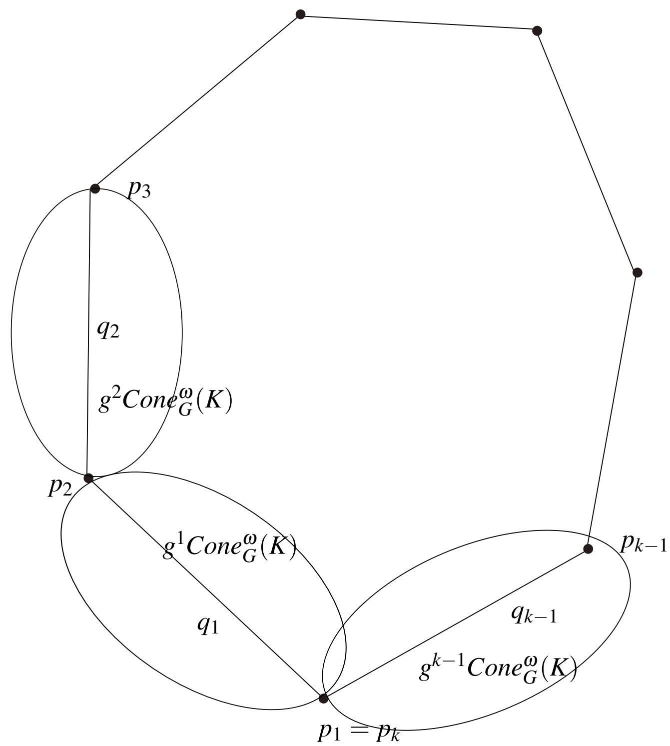

Figure VI.4: Theorem 5.13

Proof. We begin by proving part 1. Let w be a non-trivial mixed identity satisfied by G. By taking cyclic conjugates, we can assume without loss of generality that w is of the form $g_1x^{n_1}g_2x^{n_2}\dots g_kx^{n_k}$ where $g_i \in G\setminus\{e\}$ and $n_i \neq 0$ for all $1 \le i \le k$. Let a be a Morse element of G such that $g_1, g_2 \dots g_k$ do not belong to the commensurator of $\langle a\rangle$. This is possible by Lemma VI.2.21 and the fact that a conjugate of a Morse element is Morse. Let $K = Comm_G(\langle a\rangle)$. For $j \in \{1,2,\dots k\}$, let $p_j = (g_1a^{n_1i}g_2a_2^{n_2i}\dots g_ja^{n_ji})^{\omega}$. As w is a mixed identity, $p_k = (e)^{\omega}$. Let q_i be a path from p_i to p_{i+1} for all $1 \le i \le k-1$, and let q_k be a path from p_k to p_1. Let $g^i = (g_1a^{n_1i}g_2a_2^{n_2i}\dots g_ja^{n_ji}g_{i+1})^{\omega,L}$ for i between 1 and k. For $1 \le i \le k$, q_i begins and ends in $g^iCone^{\omega}_G(K)$. q_k begins and ends in $(g_1)^{\omega,L}Cone^{\omega}_G(K)$. As $\mathscr{L}(G)$ acts on $Cone^{\omega}(G)$ by isometries we can form a k-gon q where each side is entierly contained in a strongly convex set by concatenating $q_1, q_2 \dots q_k$. We can use Lemma VI.2.24 to conclude that there exists a j such that $g^{j-1}Cone^{\omega}_G(K) \cap g^jCone^{\omega}_G(K)$ consists of more than one point. Thus, we have that $(a^{n_j i} g_{j+1})^{\omega,L} Cone_G^{\omega}(K) \cap Cone_G^{\omega,L}(K)$ contains at least two points, and so

$$g_{j+1} Cone_G^{\omega}(K) \cap Cone_G^{\omega}(K) = Cone_G^{\omega}(g_{j+1} K g_{j+1}^{-1}) \cap Cone_G^{\omega}(K) = Cone_G^{\omega}(g_{j+1} K g_{j+1}^{-1} \cap K)$$

contains at least two points. But this implies that $g_{j+1} K g_{j+1}^{-1} \cap K$ is an infinite subgroup of G, which contradicts the fact g_{j+1} does not belong to the commensurator of $\langle a \rangle$.

Part 2 of the theorem can be proven using the exact same method, but replacing Lemma VI.2.21 with Lemma VI.2.22.

Part 3 can be proven the same way replacing VI.2.21 with VI.2.23. □

Chapter VII

Directions for future research

In this chapter, we present several potential directions for further research based on the results presented in this thesis.

We begin with some further questions in the direction of the connection between the generalized distortion function and the isoperimetric spectrum.

Question. *Is the set of functions that can be realized as the generalized distortion functions of subgroups of $F_2 \times F_2$ the same as the set of functions which can be realized as isoperimetric spectra of finitely generated groups?*

Note that this would be a more natural generalization of Theorem II.1.9. The difficulty is that the construction used in the proof of this result does not always return a finitely generated group given a subgroup of $F_2 \times F_2$, and so we were only able to recover one direction of this argument.

This begs another natural question.

Question. *Is the set of functions which can be realized as the generalized distortion function of a subgroup of a finitely generated group the same as the set of functions which can be realized as isoperimetric spectra of finitely generated groups?*

The answer to this question is likely no. One reason to believe this is that the corresponding question for the ordinary distortion function and the Dehn function is negative. The easiest way to see this is to note that there is a gap between linear and quadratic in the set of possible Dehn functions, but no such gap exists for distortion functions. This can be seen easily from III.1.7. It is possible that a similar trick could work in the case of generalized distortion functions and isoperimetric spectra, however no such gap is known for the isoperimetric spectrum. In any case, it is clear that any proof in the negative direction will require some form of obstruction for a function to be an isoperimetric spectrum.

Perhaps the most obvious conjecture related to this thesis is the following.

Conjecture VII.0.1. *If G is a finitely generated ICC group with a Morse element, then G does not satisfy any non-trivial mixed identity.*

The clearest evidence for this conjecture is Theorem VI.2.26. Essentially, the goal is to remove the orientability condition from the theorem. This could be accomplished by removing the orientability condition from Lemma VI.2.21. Alternatively, it could be achieved by a different argument utilizing more directly the geometry of asymptotic cones of groups with Morse elements.

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