Monoidal Structures in Equivariant Algebra

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Introduction

1.1 Background

Equivariant homotopy theory is the study of homotopical invariants, like cohomology, of spaces with an action by a fixed group G. Since not every space admits a non-trivial action by G, we expect the class of spaces with G-action to systematically admit interesting structures not present for all spaces. This extra structure endows invariants of spaces with G-action with additional data and the study of the resulting algebraic structures is known as *equivariant algebra*. Equivariant algebra occupies an important place in algebraic topology and played a crucial role in Hill, Hopkins, and Ravenel's solution to the Kervaire invariant one problem [HHR16].

The most fundamental object in equivariant algebra is the *Mackey functor*. Roughly, a Mackey functor M for a finite group G consists of a collection of abelian groups $M(H)$ indexed on the subgroups $H \leq G$. These groups are connected by a system of additive operations, called transfer and restriction, which provide computational power. Mackey functors were originally defined by Dress to axiomatize various structures that arise in representation theory [Dre73]. They provide a useful framework for approaching problems in both topology and algebra when a finite group is acting on objects of interest. The connection to equivariant homotopy theory comes from the following example.

Example 1.1.1. Let X be a space with action by a finite group G and let E be a *genuine G-spectrum*, i.e. a cohomology theory for spaces with G-action. For every n, there is a Mackey functor $E^n(X)$.

Mackey functors play a role in equivariant homotopy theory analogous to the role of abelian groups in ordinary algebraic topology. The use of Mackey functors in this way has its roots in work of Bredon, and was fully realized by a program of work due to Peter May and collaborators [Bre67, LMM81, May96]. Genuine equivariant cohomology theories are an alternative to the older Borel cohomology theories and have certain technical advantages. For example, the genuine theories satisfy a version of Poincaré duality for all compact G-CW complexes. By contrast, the Borel cohomology theory only satisfies Poincaré duality for G-CW complexes with free G-action. The duality of genuine theories was utilized, for example, in Manolescu's disproof of the triangulatation conjecture in high dimensions [Man16].

While Mackey functors are abelian groups in the equivariant setting, there are several generalizations of rings. A *Green functor* is (essentially) a Mackey functor R such that $R(H)$ is a ring for all $H \leq G$. A *Tambara functor* is a commutative Green functor with additional multiplicative operations known as *norm maps*. Norm structures were first studied in connection to equivariant stable homotopy theory by Greenlees and May, and then more systematically by Hill, Hopkins, and Ravenel [GM97, HHR16]. The norm operations of a Tambara functor provide significant advantages when performing computations and were a key ingredient in Hill, Hopkins, and Ravenel's seminal work on the Kervaire invariant one problem.

If E is a genuine G-ring spectrum (i.e. a multiplicative equivariant cohomology theory), and X is any G-space then $E^0(X)$ is naturally a Tambara functor, and hence also a Green functor. More generally, Angeltveit and Bohmann show the collection of Mackey functors $E^\star(X)$ fit into a graded Tambara functor [AB18]. Other examples of Green and Tambara functors arise naturally from group representation rings, Grothendieck–Witt rings, and Galois field extensions.

Hill, Hopkins, and Ravenel's solution to the Kervaire invariant problem motivated a tremendous amount of recent work in developing a deeper understanding of equivariant algebra, particularly the ways it relates to equivariant stable homotopy theory [Boh14, BGHL19, BH22, BH18, Hil17, HH14, HH16, Hoy14, Maz13]. Recent work of Blumberg and Hill provides clarity on the sorts of additive and multiplicative structures which can emerge in equivariant homotopy theory indexed on an *incomplete G-universe* and concludes with a conjecture regarding the ways these additive and multiplicative structures interact [BH22]. In the next section we describe this conjecture in more detail. The main result of this thesis is a proof of this conjecture.

1.2 Summary of results

We begin by describing two variations on Mackey and Tambara functors. The first variant is the *incomplete Tambara functors* of Blumberg and Hill [BH18], which are like Tambara functors but they are missing some norms maps. These structures arise naturally when studying localizations of equivariant ring spectra. The reader unfamiliar with the theory of spectra in stable homotopy theory should think of localization as the analogue of classical localization of rings and modules at multiplicative subsets in commutative algebra. In contrast to the non-equivariant setting, if E is a commutative ring in G-spectra there can exist localizations $\tilde{E}$ of E that are *not* commutative rings. In particular, $\tilde{E}^0$ can fail to be a Tambara functor because the process of localization makes some of the norms impossible to define [HH14]. Blumberg and Hill observed that while some of the norms are lost, some of the norms remain, implying $\tilde{E}^0$ is an incomplete Tambara functor [BH15, BH18].

The second variant we need are the *incomplete Mackey functors*. In analogy with incomplete Tambara functors, an incomplete Mackey functor is like a Mackey functor which only has some of the transfer maps. Such structures were first studied by Lewis as a natural home for the unstable equivariant homotopy groups of G-spaces, as well as homotopy groups of G-spectra indexed on incomplete G-universes [Lew92]. Recently, these structures have been studied by Blumberg and Hill in connection to novel incomplete models for equivariant stable homotopy categories [BH21].

Both "incomplete" settings arise from considering only restricted collections of operations, norms or transfer, allowed in our algebraic structures. A bi-incomplete Tambara functor is a Tambara functor in which we are only allowed some of the norms and some of the transfers. These objects were introduced by Blumberg and Hill in [BH22] with examples coming from ring spectra indexed on incomplete G-universes. In the same paper, Blumberg and Hill conjectured a generalization of the Hoyer–Mazur theorem [Hoy14, Maz13], predicting that the bi-incomplete Tambara functors should serve as a form of equivariant ring objects for incomplete Mackey functors. Our first main theorem, stated imprecisely here and more precisely in the main text, answers this conjecture in the affirmative.

Theorem A (Theorem 3.4.4)**.** *Any category of (additively) incomplete Mackey functors admits an equivariant symmetric monoidal structure for every compatible collection of multiplicative norm maps. In each of these structures, the equivariant commutative monoids are exactly the bi-incomplete Tambara functors.*

The proof of Theorem A requires understanding the interplay of various incomplete systems of norms and transfers. Because the norms and transfers of a bi-incomplete Tambara functor must interact in non-trivial ways, we cannot pick arbitrary collections of available norms and transfers and expect to get a coherent and useful algebraic object. We keep track of available norms and transfers of a bi-incomplete Tambara functor by using *indexing categories*, written $\mathcal{O}_{\mathrm{nm}}$ and $\mathcal{O}_{\mathrm{tr}}$, which are certain subcategories of the category of finite G-sets. A pair of indexing categories $(\mathcal{O}_{\mathrm{nm}}, \mathcal{O}_{\mathrm{tr}})$ are called *compatible* if they are able to index a bi-incomplete Tambara functor.

The first step toward proving Theorem A is to show that every compatible pair of indexing categories determines an equivariant symmetric monoidal structure on the category of incomplete Mackey functors with transfers indexed by $\mathcal{O}_{\mathrm{tr}}$. The construction requires us to define *norm functors* which allow us to move between incomplete K-Mackey functors and incomplete H-Mackey functors for subgroups $K \leq H$. These norm functors are an algebraic analogue of the famous Hill–Hopkins–Ravenel norm on equivariant spectra. In the setting of ordinary Mackey functors, the algebraic norms were first constructed by Mazur in the case $G = \mathbb{Z}/p^n$, and later for all finite groups by Hoyer. These norms play an important role in other constructions in equivariant algebra such as generalizations of topological Hochschild homology for Green functors [BGHL19].

Constructing the algebraic norm functors requires understanding exactly the conditions under which two pairs of indexing categories can be compatible. Blumberg and Hill provide one characterization which depends heavily on computation of the coinduction functors

$$\mathrm{Map}_K(H, -)\colon\ \mathrm{Set}^K \to \mathrm{Set}^H$$

for subgroups $K \leq H \leq G$. Unfortunately it is, in practice, not a simple task to determine the orbit decomposition of coinduced sets.

Our second main theorem provides a way to check that two indexing categories are compatible without needing to compute any coinduction functors. We achieve this by reframing the problem in terms of *transfer systems*, a tool developed independently by Rubin and Balchin–Barnes–Roitzheim to give an alternate description of the combinatorics of indexing categories [BBR21, Rub21]. In Definition 2.5.6 we give a notion of a compatible pair of transfer systems which can be checked without any difficult computations. Our second main theorem is:

Theorem B (Theorem 2.5.10)**.** *Compatibility of a pair of indexing categories is equivalent to a compatibility condition (Definition 2.5.6) on the associated pair of transfer systems.*

Theorem B provides enough understanding of compatible pairs to define the norm functors, but it is also of independent interest. Important examples of compatible pairs of indexing categories arise naturally from equivariant Steiner and linear isometries operads indexed on G-universes. Rubin has, in some cases, classified the kinds of indexing categories that can arise from the equivariant Steiner operads, but a classification of those determined by linear isometries operads remains unknown [Rub21]. Theorem B could provide an avenue for understanding the linear isometries operads via their compatibility with the Steiner operads.

The models of equivariant symmetric monoidal structures used in Theorem A are the symmetric monoidal Mackey functors defined by Hill and Hopkins [HH16, HM19]. These can be thought of as Mackey functors in symmetric monoidal categories: a symmetric monoidal Mackey functor $\mathcal{C}$ consists of symmetric monoidal categories $\mathcal{C}(H)$ for every subgroup $H \leq G$. The categories $\mathcal{C}(H)$ are connected by strong monoidal functors that mirror data of ordinary Mackey functors. These functors are subject to a rather large amount of coherence data. We prove this coherence can be repackaged as a pseudo-functor which takes values in the 2-category of categories. We call these objects *categorical Mackey functors* and show that they can recover most of the examples of Hill and Hopkins.

Theorem C (Theorem 4.2.6)**.** *Every categorical Mackey functor determines a symmetric monoidal Mackey functor in the sense of [HH16].*

Packaging the coherence this way allows for efficient construction of examples, such as that of Theorem A. A more subtle advantage of Theorem C is that our pseudo-functors take values in the 2-category of categories, instead of symmetric monoidal categories. Valuing our pseudo-functors in a less structured 2-category should allow one to define a notion of symmetric monoidal Tambara functors, though we do not pursue this here. We expect that such structures play an important role in studying the ring structures on equivariant spectra realized as *spectral Mackey functors* in the sense of Guillou–May [GM11, BO15]. Finally, organizing symmetric monoidal Mackey functors in this way sheds some light on the notion of equivariant commutative monoids.

Theorem D (Theorem 4.5.3)**.** *For a symmetric monoidal Mackey functor $\mathcal{M}$ coming from a categorical Mackey functor, its category of equivariant commutative monoids is the universal extension of $\mathcal{M}$ to a product preserving pseudo-functor $\mathcal{A}_2^G \to \mathrm{Cat}$, where $\mathcal{A}_2^G$ is a 2-categorical version of the Burnside category.*

After a preprint [Cha22] of some of this work was made available online, it was brought to my attention that a version of Theorem A was proved independently in the Ph.D. Thesis of Ivo Vekemans. Our work differs in several ways. In this thesis we approach the problem by first studying the ways that indexing categories can interact by proving Theorem B. Beyond its independent interest, this theorem has the virtue of also providing an affirmative answer to Conjecture 7.90 of [BH22]. From here, we work in the framework of the symmetric monoidal Mackey functors of Hill and Hopkins [HH16] to prove Theorem A.

By contrast, Vekemans' work provides an interesting and more general description of categories of incomplete Mackey and Tambara functors as modules over the Burnside bicategories in a particular sense. This approach is similar, though still distinct, from the work of this thesis in Chapter 4. We explain the differences in more detail in Remark 4.2.13.

1.3 Notations and conventions

In this thesis the letter G is reserved for a finite group. The groups H and K will always be subgroups of G. If $K \leq H$, the functors Res_K^H, Ind_K^H, and $\mathrm{Map}_K(H, -)$ denote the restriction, induction, and coinduction functors of G-sets, respectively. In Chapter 3 we make use of extensions of these functors to various Burnside categories, we denote these extensions by ρ_K^H, I_K^H, and C_K^H respectively.

If X is a G-set we write X^H for the set of H-fixed points of X. For any $x \in X$ we write G_x or $\mathrm{Stab}_G(x)$ for the stabilizer of x in G. We write $H_x = H \cap G_x$. If $g \in G$ we write $H^g = g^{-1}Hg$ for the conjugate subgroup.

The category of sets will be denoted by Set. The category of finite G-sets will be written Set^G and similar notation will be used for finite H-sets and finite K-sets. When using large categories, we assume a choice of Grothendieck universe of small sets and all large categories, in particular Set, are assumed to consist of only objects which exist in this universe. We will make no further comment on the size of our categories; the interested reader should look at [Bor94, Section 1.1] or [Shu08].

If $\mathcal{C}$ is any category we will write $\mathrm{Set}^{\mathcal{C}}$ for the category of functors from $\mathcal{C}$ in Set. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, will we use $F^* \colon \mathrm{Set}^{\mathcal{D}} \to \mathrm{Set}^{\mathcal{C}}$ for the functor given by precomposition with F. This functor has a left adjoint and right adjoint which we denote by $F_!$ and F_* respectively. If not otherwise specified, we assume these adjoint are given by the pointwise Kan extension formulae.

Chapter 2

Mackey and Tambara functors

The starting point of equivariant algebra is to replace the category of abelian groups with the category of Mackey functors. These objects are systems of abelian groups, indexed on the subgroups of a finite group G, which are connected by collections of group homomorphisms called *transfers* and *restrictions*. The operations of transfer and restriction give computational power to examples coming from equivariant topology and are an essential part of understanding the structure of equivariant homotopical invariants.

Unfortunately, requiring that our invariants have this additional structure can be fairly restrictive. For example, if X is a connected, finite G-CW complex then it is *not* true that the homotopy groups $\pi_n(X)$ admit the structure of a Mackey functor. The issue is that defining the transfer homomorphisms requires additional structure on the space X. To alleviate this, one can instead consider "incomplete" versions of Mackey functors which have only some of their transfers. We keep track of which transfers are preserved using the data of *indexing categories*.

The story of commutative rings in equivariant algebra is similar. The most powerful equivariant version of a commutative ring is known as a Tambara functor, which is a system of commutative rings which are connected by transfers, restrictions, and additional multiplicative operations called *norms*. In topology, examples of Tambara functors arise from genuine equivariant ring spectra which represent multiplicative cohomology theories for G-spaces. An observation of Blumberg and Hill is that if we allow a weaker notion of equivariant ring spectrum then we recover algebraic data which is just like a Tambara functor, except that it is missing some of its norms. Such algebraic objects are known as *incomplete Tambara functors*.

In this chapter we begin with background on indexing categories, Mackey functors, and Tambara functors. We then recall Blumberg and Hill's definition of bi-incomplete Tambara functors which serve the role of commutative rings for incomplete Mackey functors. Keeping track of bi-incomplete Tambara functors requires that we use two different indexing categories which interact in non-trivial ways. We prove a characterization of which pairs of indexing categories are sufficiently compatible in Theorem 2.5.10. As a corollary, we answer a conjecture of Blumberg and Hill in the affirmative; this is stated as Corollary 2.5.11.

2.1 Indexing Categories

Throughout this paper, systems of transfers and norms for Mackey and Tambara functors for a finite group G are indexed by *indexing categories*. We begin this section by recalling the definitions and some basic examples. At the end of the section we prove some new results regarding the interaction of indexing categories with the normal cores of subgroups H of a group G.

An indexing category $\mathcal{O}$ is a particularly nice subcategory of the category of finite G-sets. Let us first recall some definitions which make precise what we mean.

Definition 2.1.1. Let $\mathcal{C}$ be a category with finite coproducts. A subcategory $\mathcal{O} \subset \mathcal{C}$ is

1. *wide* if it contains all the objects,

2. *finite coproduct complete* if it has finite coproducts, and they agree with the coproduct in $\mathcal{C}$,

3. *pullback stable* if for any pullback square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{C}$ with f in $\mathcal{O}$ we also have g in $\mathcal{O}$.

Definition 2.1.2. An *indexing category* $\mathcal{O}$ is a wide, pullback stable, finite coproduct complete subcategory of Set^G. For any $H \leq G$, an H-set X is called *$\mathcal{O}$-admissible* if there is a morphism $G \times_H X \to G/H$ in $\mathcal{O}$.

Example 2.1.3. For any group G we have the complete indexing category $\mathcal{O}^{gen} = \mathrm{Set}^G$.

Example 2.1.4. For any group G we have the trivial indexing category $\mathcal{O}^{tr}$. A map $f\colon X \to Y$ is in $\mathcal{O}^{tr}$ if and only if for any orbit $X_0 \subset X$, the restriction $f\colon X_0 \to Y$ is an isomorphism onto its image. Essentially, the only maps in $\mathcal{O}^{tr}$ are fold maps and inclusions.

Non-trivial examples come from representation theory.

Example 2.1.5. Let V be a finite dimensional real representation of the group G with $V^G \neq 0$. For $H \leq G$ a subgroup, we say that a finite H-set X is V-admissible if there is an H-equivariant embedding $X \to V$. There is an indexing category $\mathcal{O}_V$ which is generated by the maps $f\colon G/K \to G/H$ for subgroups $K \leq H$ such that H/K is V-admissible.

When V is a finite dimensional trivial representation we have $\mathcal{O}_V = \mathcal{O}^{tr}$. When V is the regular representation of G then $\mathcal{O}_V = \mathcal{O}^{gen}$. In general, $\mathcal{O}_V$ will be something in between. These indexing categories are called the Steiner indexing categories and are studied in [BH15] and [Rub21].

Example 2.1.6. Let $H \leq G$ be a subgroup. Every G-indexing category $\mathcal{O}$ determines an H-indexing category $i_H^*\mathcal{O}$. A map of H-sets $f\colon S \to T$ is in $i_H^*\mathcal{O}$ if the induced G-map

$$G \times_H f\colon G \times_H S \to G \times_H T$$

is in $\mathcal{O}$. It follows from the definitions that for any $K \leq H$, the $i_H^*\mathcal{O}$-admissible K-sets are the same at the $\mathcal{O}$-admissible K-sets. Note that for any $K \leq H$, the induction functor

$$H \times_K (-)\colon\ \mathrm{Set}^K \to \mathrm{Set}^H$$

restricts to a functor

$$H \times_K (-)\colon i_K^*\mathcal{O} \to i_H^*\mathcal{O}.$$

Our definition of an $\mathcal{O}$-admissible set is not exactly the same as one given in [BH22], though it is equivalent. The difference is that Blumberg and Hill define admissibility for *indexing systems*, which are structures that carry data equivalent to that of indexing categories. Indexing systems play no explicit role in this paper, and so we have worded our statements in terms of indexing categories. In keeping with this, the next three lemmas are not new, although we include proofs because we do not know a reference in which they are stated in this language.

Lemma 2.1.7 ([BH18, Proposition 3.1])**.** *Suppose $\mathcal{O}$ is an indexing category and $H \leq G$ is a subgroup. If X is an $\mathcal{O}$-admissible H-set then every orbit of $X_0 \subset X$ is also $\mathcal{O}$-admissible.*

Proof. By assumption, we have a map $f\colon G \times_H X \to G/H$ in $\mathcal{O}$ which allows us to construct the composite

$$G \times_H X_0 \to G \times_H X \xrightarrow{f} G/H$$

where the first map is the inclusion. By Proposition 3.1 of [BH18], all monomorphisms of Set^G are in $\mathcal{O}$ and thus we have constructed a map $G \times_H X_0 \to G/H$ that is in $\mathcal{O}$. □

Lemma 2.1.8 ([Rub21, Proposition 3.3])**.** *Let $\mathcal{O}$ be an indexing category. If H/K is $\mathcal{O}$-admissible, then for all subgroups $L \leq G$ we have that $(H \cap L)/(K \cap L)$ is also $\mathcal{O}$-admissible.*

Proof. Since $H \cap L = H \cap (H \cap L)$ we can replace L by $H \cap L$ and reduce to the case where $L \leq H$. Unwinding the definitions, it suffices to construct a map $G/(L \cap K) \to G/L$ in $\mathcal{O}$. By assumption, we have a map $f\colon G/K \to G/H$ in $\mathcal{O}$. Consider the following pullback diagram:

$$\begin{array}{ccc} G/L \times_{G/H} G/K & \longrightarrow & G/K \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ G/L & \longrightarrow & G/H \end{array}$$

The map p must be in $\mathcal{O}$ by pullback stability and the fact that f is in $\mathcal{O}$. By Lemma 2.1.7, we are done if we can show the pullback $G/L \times_{G/H} G/K$ has an orbit isomorphic to $G/(L \cap K)$. This is immediate since the element (eK, eL) in the pullback has stabilizer exactly $L \cap K$. □

When applying Lemma 2.1.8 we say $(H \cap L)/(K \cap L)$ is obtained from H/K by intersection with L. In this language, the lemma can be summarized by saying the collection of $\mathcal{O}$-admissible sets is closed under intersection with subgroups.

Our next lemma states that the induction functors associated to subgroups $K \leq H$, with H/K an $\mathcal{O}$-admissible H-set, give a function from admissible K-sets to admissible H-sets. Blumberg and Hill refer to this property as closure under self-induction.

Lemma 2.1.9 (Closure under self-induction)**.** *Suppose $K \leq H$ are subgroups and H/K is an $\mathcal{O}$-admissible H-set. If T is an $\mathcal{O}$-admissible K-set, then $H \times_K T$ is an $\mathcal{O}$-admissible H-set.*

Proof. The assumptions guarantee maps $f\colon G \times_K T \to G/K$ and $g\colon G \times_H (H/K) \to G/H$ in $\mathcal{O}$. Since there is an isomorphism $i\colon G/K \cong G \times_H (H/K)$, which must be in $\mathcal{O}$, we see the composite $g \circ i \circ f\colon G \times_K T \to G/H$ is a morphism in $\mathcal{O}$. The result now follows from the isomorphism $G \times_K T \cong G \times_H (H \times_K T)$. □

If S is a G-set, we write $G_s = \{g \in G \mid g \cdot s = s\}$ for the stabilizer of s in G.

Lemma 2.1.10 ([BH18, Section 3])**.** *A map $f\colon S \to T$ is in $\mathcal{O}$ if and only if for any $s \in S$ we have that $G_{f(s)}/G_S$ is an $\mathcal{O}$-admissible $G_{f(s)}$-set.*

In the remainder of this section we make a few new observations on the structure of indexing categories. Of particular interest is the interaction between indexing categories and the *cores* (Definition 2.1.12 below) of various subgroups $H \leq G$. We begin with a modest generalization of Lemma 2.1.8.

Corollary 2.1.11. *If H_1/K_1 and H_2/K_2 are two $\mathcal{O}$-admissible sets, then $(H_1 \cap H_2)/(K_1 \cap K_2)$ is also $\mathcal{O}$-admissible.*

Proof. Intersecting H_1/K_1 with H_2 yields that $(H_1 \cap H_2)/(K_1 \cap H_2)$ is $\mathcal{O}$-admissible. Similarly, intersecting H_2/K_2 with K_1 gives us that $(K_1 \cap H_2)/(K_1 \cap K_2)$ is $\mathcal{O}$-admissible. We are now done, as

$$(H_1 \cap H_2)/(K_1 \cap K_2) \cong (H_1 \cap H_2) \times_{K_1 \cap H_2} (K_1 \cap H_2)/(K_1 \cap K_2)$$

with the right hand side $\mathcal{O}$-admissible by Lemma 2.1.9. □

Definition 2.1.12. For a subgroup $H \leq G$ the *core* of H in G, denoted $\mathrm{Core}_G(H)$, is the intersection of all conjugates of H. That is,

$$\mathrm{Core}_G(H) = \bigcap_{g \in G} H^g$$

where $H^g = gHg^{-1}$.

Remark 2.1.13. The core of a subgroup admits several equivalent definitions. In particular, it can be described as the kernel of the group homomorphism $G \to \Sigma_{|G/H|}$ which realizes the left G-action on the set of H-cosets. While the definition above is convenient for indexing categories, especially in light of Corollary 2.1.11, this second description can be helpful to have in mind later when considering the coinduction functors $\mathrm{Map}_H(G, -)\colon \mathrm{Set}^H \to \mathrm{Set}^G$. It also makes evident that the core is always a normal subgroup. Indeed, another equivalent characterization of the $\mathrm{Core}_G(H)$ is the largest normal subgroup of G contained in H.

Lemma 2.1.14. *Suppose H/K is an $\mathcal{O}$-admissible H-set. Then for all $g \in G$ the sets H^g/K^g are $\mathcal{O}$-admissible.*

Proof. The result follows from noticing that

$$G \times_H H/K \cong G/K \cong G/K^g \cong G \times_{H^g} H^g/K^g$$

and the fact that indexing categories contain all isomorphisms. □

Proposition 2.1.15. *Suppose $\mathcal{O}$ is an G-indexing category and $K \leq H \leq G$ is a chain of subgroups. If H/K is $\mathcal{O}$-admissible then so is* $\mathrm{Core}_G(H)/\mathrm{Core}_G(K)$. *In particular, if G/H is admissible then so is $G/\mathrm{Core}_G(H)$.*

Proof. The result follows from repeated applications of Lemma 2.1.14 and Corollary 2.1.11. □

Proposition 2.1.15 is useful when working with indexing categories because it allows us to effectively replace an arbitrary subgroup $H \leq G$ with the $\mathrm{Core}_G(H)$, which is a normal subgroup.

This is especially helpful when dealing with coinduced G-sets. For now, we content ourselves with the following example.

Example 2.1.16. Suppose $G = \Sigma_n$ is the symmetric group on n elements for $n \geq 5$. Let H be any proper subgroup of Σ_n, other than the alternating group. Since $\mathrm{Core}_{\Sigma_n}(H)$ is a normal subgroup of Σ_n that is contained in H it must be the trivial group. By Proposition 2.1.15, if $\mathcal{O}$ is an indexing category with Σ_n/H admissible then Σ_n/e is also admissible.

The previous example illustrates the surprising ways the core can affect the admissible sets of an indexing category. In Section 2.5, specifically the proof of Proposition 2.5.7, the interaction between the cores of subgroups and indexing categories plays a central technical role in our characterization of *compatible* indexing categories. We will define compatibility of indexing categories precisely in Definition 2.4.2 below. Presently, it is sufficient to think about a pair of indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$ as being compatible if $\mathcal{O}_m$ acts on $\mathcal{O}_a$ in some algebraic sense. We end this section by collecting a few corollaries illustrating the ways that the core sheds light on the rigid structure of compatible pairs of indexing categories.

Corollary 2.1.17. *Suppose H is a subgroup of G such that $\mathrm{Core}_G(H) = e$ is the trivial group and suppose $\mathcal{O}_m$ is an indexing category with G/H an $\mathcal{O}_m$-admissible G-set. If $(\mathcal{O}_m, \mathcal{O}_a)$ is a compatible pair of indexing categories, then $\mathcal{O}_a$ is the complete indexing category.*

Proof. By Proposition 2.1.15, we have that G/e is $\mathcal{O}$-admissible. The result is now the conclusion of [BH22, Proposition 7.69]. □

Corollary 2.1.18. *Suppose G is a simple group, $H \leq G$ a proper subgroup, and $\mathcal{O}_m$ is an indexing category. If G/H is $\mathcal{O}_m$-admissible then the only indexing category $\mathcal{O}_a$ which is compatible with $\mathcal{O}_m$ is the complete indexing category.*

2.2 Incomplete Mackey functors

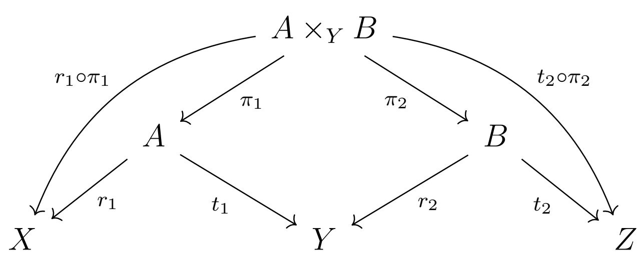
In this section we review the notion of incomplete Mackey functors. Most of the material from this section can be found in [BH22].

If X and Y are two G-sets, then a *span* between X and Y is a pair of maps

$$[X \xleftarrow{r} A \xrightarrow{t} Y] \tag{2.1}$$

where r, t are equivariant maps of G-sets. We refer to the G-set A in the span as the middle G-set and the morphisms r and t as the legs of the span. Two spans are *isomorphic* if the there is an isomorphism of their middle G-sets which is compatible with the legs.

Definition 2.2.1. Let $\mathcal{O}$ be an indexing category for a finite group G. The *$\mathcal{O}$-Burnside Category* $\mathcal{A}^G_{\mathcal{O}}$ is the category whose objects are finite G-sets and morphisms $\mathcal{A}^G_{\mathcal{O}}(X, Y)$ are isomorphism classes of spans (2.1) with $t \in \mathcal{O}$. Composition is given by pullback, i.e. the composition of $[X \xleftarrow{r_1} A \xrightarrow{t_1} Y]$ and $[Y \xleftarrow{r_2} B \xrightarrow{t_2} Z]$ is the class of the span along the top of



where the middle square is a pullback.

Remark 2.2.2. Composition in the category $\mathcal{A}^G_{\mathcal{O}}$ is well defined because the indexing category $\mathcal{O}$ is pullback stable. In many ways this is the point of indexing categories, although they were invented in [BH18] to manage incomplete systems of norms, not transfers.

Example 2.2.3. When $\mathcal{O} = \mathrm{Set}^G$ is the complete indexing category we refer to $\mathcal{A}^G_{\mathcal{O}}$ as simply the Burnside category of G and denote it by $\mathcal{A}^G$.

Morphisms in the $\mathcal{A}^G_{\mathcal{O}}$ factor as composites of two nice families of maps called the restrictions and transfers. For a map $f\colon X \to Y$ in Set^G, we define the restriction of f by

$$R_f = [Y \xleftarrow{f} X = X].$$

Similarly, if $f \in \mathcal{O}$ we can define the transfer of f by

$$T_f = [X = X \xrightarrow{f} Y].$$

Using these, we can decompose an arbitrary span as

$$[X \xleftarrow{f} A \xrightarrow{g} Y] = T_g \circ R_f. \tag{2.2}$$

When $f\colon G/K \to G/H$ for $K \leq H$ is the canonical quotient we denote R_f and T_f by R_K^H and T_K^H respectively. Finally, if $f\colon G/H \to G/H^g$ is a conjugation isomorphism, we denote T_f by c_g, the conjugation by g. Choosing $c_{g^{-1}} = R_f$ leads to the same conjugation maps.

The $\mathcal{O}$-Burnside category of a group G is a semi-additive category with finite products given by disjoint union of G-sets. It follows that a product preserving functor $M\colon \mathcal{A}_{\mathcal{O}}^G \to \mathrm{Set}$ naturally takes values in commutative monoids. For any finite G-set X, the addition on $M(X)$ is given by

$$M(X) \times M(X) \cong M\left(X \amalg X\right) \xrightarrow{M(T_\nabla)} M(X)$$

where $\nabla\colon X \amalg X \to X$ is the fold map. The unit is given by the image of

$$M(\emptyset) \xrightarrow{M(T_{\emptyset \to X})} M(X)$$

Note $M(\emptyset)$ must be a singleton set as M is product preserving.

Definition 2.2.4. A *semi $\mathcal{O}$-Mackey functor* is a product preserving functor $M\colon \mathcal{A}_{\mathcal{O}}^G \to \mathrm{Set}$. An *$\mathcal{O}$-Mackey functor* is a semi $\mathcal{O}$-Mackey functor that is group complete, in the sense that for all X the commutative monoid $M(X)$ is actually an abelian group. Any semi $\mathcal{O}$-Mackey functor M determines a $\mathcal{O}$-Mackey functor M^+ defined by letting $M^+(X)$ be the group completion of the commutative monoid $M(X)$ for every finite G-set X.

A morphism of (semi) $\mathcal{O}$-Mackey functors is a natural transformation of product preserving functors. We denote the category of $\mathcal{O}$-Mackey functors for a group G by $\mathrm{Mack}^G_{\mathcal{O}}$. When $\mathcal{O} = \mathrm{Set}^G$ is the complete indexing category we recover the category of Mackey functors and denote this category by Mack^G.

Remark 2.2.5. It is common to define Mackey functors to be product preserving functors from the Burnside category into the category of abelian groups. While this definition is equivalent, we have elected to value our Mackey functors in the category of sets for two reasons. First, this definition lines up more readily with the view that the Burnside category is the multisorted Lawvere theory whose models are Mackey functors. Second, and more importantly, we wish to have a better analogy with Tambara functors which are also given as product preserving functors from a specified category $\mathcal{P}^G$ into sets. Unlike Mackey functors, however, Tambara functors cannot be defined as functors that land in abelian groups. The essential difference is that Tambara functors are a model for equivariant ring structures and the bilinear multiplication map $R \times R \to R$ of a ring R is not a map of abelian groups.

While there are many objects in $\mathcal{A}^G_{\mathcal{O}}$, it suffices to specify the value of a Mackey functor M on only finitely many objects. Since every finite G-set is the union of its orbits, and all orbits are isomorphic to G/H for some subgroup H, an $\mathcal{O}$-Mackey functor M is determined by its value on the objects G/H as well as on spans $[G/H \xleftarrow{r} G/K \xrightarrow{t} G/L] = T^L_K R^H_K$ and conjugations c_g. Abusing notation, we use R^H_K (resp. T^H_K) to denote both the restriction (resp. transfer) in $\mathcal{A}^G_{\mathcal{O}}$ as well as the map induced by applying M.

The following lemma gathers together several standard and useful facts about $\mathcal{O}$-Mackey functors. The proofs are all standard and, with the exception of the double coset formula, straightforward. The double coset formula follows from a computation of the orbit decomposition for the product of transitive H-sets: $H/L \times H/K$.

Lemma 2.2.6. *Let H be a subgroup of G. The restriction and transfer maps of a semi-$\mathcal{O}$-Mackey functor M satisfy the following properties, assuming the transfers exist.*

1. *(Additivity) All transfers, restrictions, and conjugations are additive maps of commutative monoids.*

2. *(Composition) For $K \leq L \leq H$ we have $R^L_K R^H_L = R^H_K$ and $T^H_L T^L_K = T^H_K$.*

3. *(Double Coset Formula) Let K and L be subgroups of H and $\gamma_1, \ldots, \gamma_n$ be a choice of representatives for the double cosets $K\backslash H/L$. If H/L is $\mathcal{O}$-admissible, we have the following equality of maps:*

$$R^H_K T^H_L = \sum_{\gamma_i} T^K_{K\cap L^{\gamma_i}} R^{L^{\gamma_i}}_{K\cap L^{\gamma_i}} c_{\gamma_i}$$

4. *(Weyl Group Action) The Weyl group $W_G(H) = N_G(H)/H$ acts naturally on the G-set G/H. Applying M to this action gives $M(G/H)$ a canonical $W_G(H)$ action.*

Remark 2.2.7. In the statement of the double coset formula we note that $\mathcal{O}$-admissibility of H/L is sufficient to imply both sides of the equation are well-defined. The left hand side is clear. To see the right hand side is well defined, note that $K/(K \cap L^{\gamma_i})$ is admissible for all i by Lemma 2.1.8. As a warning, it seems it is possible for all the transfer on the right hand side to be well defined but for the transfer on the left to still be undefined.

Examples of Mackey functors include the Burnside Mackey functor, representation rings, and Galois field extensions discussed below. For simplicity, we write the examples as complete Mackey functors, i.e. with $\mathcal{O} = \mathrm{Set}^G$. Incomplete versions can be obtained for any $\mathcal{O}$ by restricting our functors along the inclusion $\mathcal{A}^G_{\mathcal{O}} \to \mathcal{A}^G$.

Example 2.2.8. For any object $X \in \mathcal{A}^G$ there is a represented semi-Mackey functor $A_X = \mathcal{A}^G(X, -)$. The group completion $A^+_{G/G}$ is called the Burnside Mackey functor. One can check that $A_{G/G}(G/H)$ is isomorphic to the set of isomorphism classes of finite H-sets with addition given by disjoint union. It follows that $A^+_{G/G}(G/H)$ is the free abelian group on the set of transitive H-sets. The restriction R^H_K is given by the restriction of H-sets to K-sets while the transfer T^H_K is given by induction.

Example 2.2.9. For every $H \leq G$, let $\mathrm{Rep}(H)$ denote the abelian group of virtual complex H-representations. For subgroups $K \leq H$, these groups are related via the classical induction and restriction homomorphisms $\mathrm{Ind}^H_K\colon \mathrm{Rep}(K) \to \mathrm{Rep}(H)$ and $\mathrm{Res}^H_K\colon \mathrm{Rep}(H) \to \mathrm{Rep}(K)$. This produces the complex representation ring Mackey functor R_G, which is given by $R_G(G/H) = \mathrm{Rep}(H)$ with transfers and restrictions are given by induction and restriction homomorphisms. Varying the kinds of representations considered leads to other examples, such as the real orthogonal representation ring RO_G, the unitary representation ring RU_G, etc.

Example 2.2.10. Let $F \subset L$ be a a finite Galois field extension with Galois group G. We obtain a Mackey functor $\mathbb{L}$ given by $\mathbb{L}(G/H) = L^H$, the H-fixed points of L. The restriction R^H_K associated to $K \leq H$ is given by inclusion of fixed points. The transfers T^H_K are given by the classical field trace: picking representatives $h_1K, \ldots, h_nK$ for the left cosets H/K we define tr: $L^K \to L^H$ by

$$\mathrm{tr}(x) = \sum_{i=1}^{n} h_i x$$

which does not depend on the choice of representatives as $x \in L^K$. In fact, the same transfers and restrictions give a Mackey functor for any ring R with G-action.

Examples of incomplete Mackey functors come from unstable equivariant homotopy theory.

Example 2.2.11. Recall the trivial indexing category $\mathcal{O}^{tr}$ of Example 2.1.4. An $\mathcal{O}^{tr}$-Mackey functor is more commonly known as a coefficient system. Such an object consists of a system of abelian groups, indexed by the subgroups of G, and connected by only the restriction maps. Given a based G-space X and $n \geq 2$, we have the n-th homotopy coefficient system of X given by $G/H \mapsto \pi_n(X^H)$ where X^H denotes the H-fixed points of X. For subgroups $K \leq H$, the restriction map $R^H_K \colon \pi_n(X^H) \to \pi_n(X^K)$ is induced by the inclusion of fixed points $X^H \to X^K$.

Example 2.2.12. Suppose X is a (based) G-space which is also an infinite loop space. In the non-equivariant setting, such a space is equivalent to a connective spectrum; the failure of X to be a genuine G-spectrum is measured by the non-existence of transfer maps in the homotopy coefficient systems. More precisely, for a genuine Ω G-spectrum Y, the homotopy groups $\pi_n(Y)$ fit into a Mackey functor with the transfers determined by the delooping isomorphisms $\Omega^V(Y_{n+V}) \cong Y_n$. To build in transfer maps to the homotopy coefficient system of X is equivalent to making choices for equivariant deloopings $X \cong \Omega^V(X_V)$ for various non-trivial G-representations V.

Such deloopings can be difficult to construct, although models in the case $G = C_2$ have been constructed by Liu, and used in computations of equivariant cohomology by Petersen [Liu20, Pet22]. For more on the connections between incomplete Mackey functors and stability in topology we refer the reader to the work of Lewis and Blumberg–Hill [Lew92, BH21].

2.3 Tambara functors

Examples 2.2.8, 2.2.9, and 2.2.10 above are all classical examples of Mackey functors. Notice, though, that all three examples actually have more structure than simply an abelian group for all subgroups $H \leq G$. If M is any Mackey functor from these examples, $M(G/H)$ is actually a commutative ring for all H. This ring structure and the norm maps described below lead to the notion of Tambara functors, originally defined in [Tam93]. Tambara functors, similar to Mackey functors, are defined by first giving an additive category $\mathcal{P}^G$ and letting (semi) Tambara functors be the product preserving functors from this category into sets.

Eventually we need to define bi-incomplete Tambara functors, which depend on a pair of indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$, one for the multiplicative norms and one for the additive transfers. Since the composition laws in the category $\mathcal{P}^G$ are a bit elaborate, we first give the definition in the case of complete indexing categories. We then proceed to explain the necessary modifications to create a category we denote $\mathcal{P}^G_{\mathcal{O}_m,\mathcal{O}_a}$ which is the domain of a bi-incomplete Tambara functor.

For a group G, the *polynomial category* $\mathcal{P}^G$ is the category whose objects are finite G-sets and with morphism sets $\mathcal{P}^G(X, Y)$ consisting of isomorphism classes of *bispans*

$$[X \xleftarrow{r} A \xrightarrow{n} B \xrightarrow{t} Y]$$

where r, n, and t are equivariant maps of G-sets. The composition of morphisms is described below. This category, like the Burnside category, is semi-additive and thus for any product preserving functor $S\colon \mathcal{P}^G \to \mathrm{Set}$ and any $X \in \mathcal{P}^G$, the set $S(X)$ is naturally a commutative monoid with addition given by

$$S(X) \times S(X) \cong S(X \amalg X) \xrightarrow{S([X \amalg X \xleftarrow{=} X \amalg X \xrightarrow{=} X \amalg X \xrightarrow{\nabla} X])} S(S).$$

Putting the fold map ∇ in the middle of the bispan, instead of on the right, leads to a second map $S(X) \times S(X) \to S(X)$ which we call the multiplication. The composition laws of $\mathcal{P}^G$ are such that the addition and multiplication make $S(X)$ into a commutative semiring (a ring possibly without additive inverses).

Definition 2.3.1. A *semi-Tambara functor* is a a product preserving functor $\mathcal{P}^G \to \mathrm{Set}$. A *Tambara functor* is a semi-Tambara functor S such that $S(X)$ is a ring for all X. A morphism of Tambara functors is a natural transformation of product preserving functors and the category of Tambara functors for a group G is denoted by Tamb^G.

Before describing the composition laws for $\mathcal{P}^G$, we highlight the key difference between the categories $\mathcal{A}^G$ and $\mathcal{P}^G$, namely the extra map $A \xrightarrow{n} B$ in the bispan. The map n in this category exists to parameterize the norm maps $N_K^H \colon S(G/K) \to S(G/H)$ for Tambara functors S. These norm maps are multiplicative maps between the various rings comprising a Tambara functor. Between the transfers, norms, and restrictions the category $\mathcal{P}^G$ has three distinguished kinds of morphisms associated to any map $f \colon X \to Y$ in Set^G. We denote these by

$$R_f = [Y \xleftarrow{f} X = X = X],$$

$$N_f = [X = X \xrightarrow{f} Y = Y],$$

$$T_f = [X = X = Y \xrightarrow{f} Y],$$

and refer to these maps as the restriction, norm, and transfer associated to the map f. Just as we did for Mackey functors, we abuse notation and sometimes use R_f, N_f, and T_f for both the maps in $\mathcal{P}^G$ as well as for the maps induced by a Tambara functor S. Further, when $f \colon G/K \to G/H$ is a canonical quotient for $K \leq H$, we write R_K^H, T_K^H and N_K^H.

To describe the composition laws in $\mathcal{P}^G$, we first note than an arbitrary bispan

$$\omega = [X \xleftarrow{r} A \xrightarrow{n} B \xrightarrow{t} Y]$$

can be written as the composition $T_t \circ N_n \circ R_r$. Given another bispan

$$\omega' = [Y \xleftarrow{r'} C \xrightarrow{n'} D \xrightarrow{t'} Z],$$

we need to know how to write $\omega' \circ \omega = T_{t'} N_{n'} R_{r'} T_t N_n R_r$ as a composite of the form $T_{t''} N_{n''} R_{r''}$. To do this, we need to describe the interchange rules relating the various T's, N's and R's; these are summarized over the course of a few lemmas. The proof that the definitions of compositions given in the following lemmas defines a category can be found in the original paper of Tambara [Tam93].

Lemma 2.3.2. *For G-equivariant maps $f\colon X \to Y$ and $g\colon Y \to Z$ we have*

1. $T_{g \circ f} = T_g \circ T_f$,
2. $N_{g \circ f} = N_g \circ N_f$,
3. $R_{g \circ f} = R_f \circ R_g$.

Lemma 2.3.3. *For any pullback square*

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$$

in Set^G, we have both $R_f \circ N_g = N_k \circ R_h$ and $R_f \circ T_g = T_k \circ R_h$.

Remark 2.3.4. The composition law $R_f \circ T_g = T_k \circ R_h$ in Lemma 2.3.3 is exactly the same composition law satisfied by restrictions and transfers in the Burnside category. Indeed, there is a functor $i_{add}\colon \mathcal{A}^G \to \mathcal{P}^G$ that is the identity on objects and acts on morphisms by

$$[X \xleftarrow{r} A \xrightarrow{t} Y] \mapsto [X \xleftarrow{r} A = A \xrightarrow{t} Y].$$

Restricting along i_{add} gives a forgetful functor $i_{add}^*\colon \mathrm{Tamb}^G \to \mathrm{Mack}^G$ analogous to sending a commutative ring to its underlying abelian group.

Since the norms and restrictions satisfy the same interchange law as transfers and restrictions there is a second functor $i_{mult}\colon \mathcal{A}^G \to \mathcal{P}^G$ which is the identity on objects and acts on morphisms by

$$[X \xleftarrow{r} A \xrightarrow{t} Y] \mapsto [X \xleftarrow{r} A \xrightarrow{t} Y = Y].$$

Restricting along this functor gives a functor from the category of Tambara functors to the category of semi-Mackey functors analogous to sending a commutative ring to its underlying multiplicative monoid.

It remains to explain how the transfers and norm maps interchange. To do so, we first need to describe the *exponential diagrams*. For $X \in \mathrm{Set}^G$ we denote the category of G-sets over X by $\mathrm{Set}^G_{/X}$. Associated to an equivariant map $f\colon X \to Y$, we have a functor $f^*\colon \mathrm{Set}^G_{/Y} \to \mathrm{Set}^G_{/X}$ given by pullback. By the adjoint functor theorem f^* has a right adjoint, called the dependent product, which we denote by $f_*\colon \mathrm{Set}^G_{/X} \to \mathrm{Set}^G_{/Y}$.

Example 2.3.5. Let $n\colon G/H \to G/G$ be the collapse map. There are equivalences of categories $\mathrm{Set}^G_{/(G/H)} \cong \mathrm{Set}^H$ and $\mathrm{Set}^G_{/(G/G)} \cong \mathrm{Set}^G$ and, passing through these equivalences, one can show that $n^*\colon \mathrm{Set}^G \to \mathrm{Set}^H$ is the usual restriction functor. By the uniqueness of adjoints, it follows that $n_*\colon \mathrm{Set}^H \to \mathrm{Set}^G$ is naturally isomorphic to the coinduction functor.

Given two composable maps $A \xrightarrow{t} B \xrightarrow{n} C$ in Set^G, we can form the associated *exponential diagram*

$$\begin{array}{ccccc} B & \xleftarrow{t} & A & \xleftarrow{r'} & B \times_C n_*(A) \\ \downarrow n & & & & \downarrow n' \\ C & & \xleftarrow{t'} & & n_*(A) \end{array} \tag{2.3}$$

in which n' is the projection and t' is the map realizing $n_*(A)$ as an object in $\mathrm{Set}^G_{/C}$. Noting that $B \times_C n_*(A) = n^* n_*(A)$, the map $r' \colon n^* n_*(A) \to A$ is the counit of the adjunction.

Proposition 2.3.6. *Given composable maps $A \xrightarrow{t} B \xrightarrow{n} C$ in Set^G, the norm N_n and transfer T_t in $\mathcal{P}^G$ interchange as $N_n T_t = T_{t'} N_{n'} R_{r'}$ where t', n', and r' are the maps in the exponential diagram* (2.3).

Example 2.3.7. Let $G = C_2 = \{e, \sigma\}$ be the cyclic group of order 2, let $S \in \mathrm{Tamb}^{C_2}$ and let $a, b \in S(C_2/e)$. Since addition is the transfer associated to the fold maps, the exponential diagram can be used to compute that

$$N_e^{C_2}(a+b) = N_e^{C_2}(a) + N_e^{C_2}(b) + T_e^{C_2}(a \cdot c_\sigma(b)).$$

For a derivation of the above formula, see Section 7 of [Str12].

2.4 Bi-incomplete Tambara functors

We now turn to the problem of defining bi-incomplete Tambara functors. Suppose we are given a pair of indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$. The idea is to take a wide subcategory $\mathcal{P}^G_{\mathcal{O}_m, \mathcal{O}_a} \subset \mathcal{P}^G$ with the transfers restricted to maps in $\mathcal{O}_a$ and the norms restricted to $\mathcal{O}_m$. Due to the complicated composition laws of $\mathcal{P}^G$, some care has to be taken in which indexing categories we can choose and still have $\mathcal{P}^G_{\mathcal{O}_m, \mathcal{O}_a}$ be a category.

Example 2.4.1. Let $G = C_2$ be the group with two elements and suppose $(\mathcal{O}_m, \mathcal{O}_a) = (\mathcal{O}^{gen}, \mathcal{O}^{tr})$. It follows from Example 2.3.7 that $\mathcal{P}^G_{\mathcal{O}_m, \mathcal{O}_a}$ as described in the above paragraph is *not* a category. Indeed, if $\nabla \colon C_2/e \amalg C_2/e \to C_2/e$ is the fold map, the example shows that defining the composition $N_e^{C_2} \circ T_\nabla$ in $\mathcal{P}^G_{\mathcal{O}_m, \mathcal{O}_a}$ requires $T_e^{C_2}$. Since $C_2/e \to C_2/C_2$ is not a morphism in $\mathcal{O}^{tr}$ this means the composition in $\mathcal{P}_{(\mathcal{O}_m, \mathcal{O}_a)}$ would not be defined.

The necessary condition to put on a pair of indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$ is studied in [BH22] and leads to the following definition which originates in [BH22, Definition 7.29].

Definition 2.4.2. A pair of indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$ is *compatible* if for all maps $n\colon S \to T$ in $\mathcal{O}_m$, we have

$$n_*((\mathcal{O}_a)_{/S}) \subset (\mathcal{O}_a)_{/T}$$

where n_* is the dependent product functor.

Theorem 2.4.3 (Theorem 7.30 of [BH22])**.** *If $(\mathcal{O}_m, \mathcal{O}_a)$ is a compatible pair of indexing categories then $\mathcal{P}^G_{\mathcal{O}_m,\mathcal{O}_a} \subset \mathcal{P}$ is a wide subcategory with morphisms $\mathcal{P}^G_{\mathcal{O}_m,\mathcal{O}_a}(X, Y)$ given by isomorphism classes of bispans*

$$[X \xleftarrow{r} A \xrightarrow{n} B \xrightarrow{t} Y]$$

with $n \in \mathcal{O}_m$ and $t \in \mathcal{O}_a$.

Definition 2.4.4. A semi $(\mathcal{O}_m, \mathcal{O}_a)$-Tambara functor is a product preserving functor

$$S\colon \mathcal{P}^G_{\mathcal{O}_m,\mathcal{O}_a} \to \mathrm{Set}\,.$$

A $(\mathcal{O}_m, \mathcal{O}_a)$-Tambara functor is a semi $(\mathcal{O}_m, \mathcal{O}_a)$-Tambara functor S such that $S(X)$ is additively group complete for all G-sets X. When the specific indexing categories are implicit, we refer to such objects as (semi) bi-incomplete Tambara functors. The category of $(\mathcal{O}_m, \mathcal{O}_a)$-Tambara functors is denoted $\mathrm{Tamb}^G_{(\mathcal{O}_m,\mathcal{O}_a)}$.

The main example of interest comes to us from topology.

Definition 2.4.5. A *G-universe* $\mathcal{U}$ for a finite group G is a real orthogonal representation of G such that $\mathcal{U}^G \neq 0$ and for any finite dimensional subrepresentation $V \subset \mathcal{U}$ there exist G-equivariant embeddings $V^n \to \mathcal{U}$ for all n.

Example 2.4.6. A complete G-universe $\mathcal{U}$ is one such that every real orthogonal G-representation embeds into $\mathcal{U}$ infinitely many times. A model for $\mathcal{U}$ is given by taking the infinite direct sum of the real regular representation of G.

Example 2.4.7. A trivial G-universe $\mathcal{U}_{tr}$ is any G-universe with trivial G-action.

Universes index different categories of genuine G-spectra $Sp^G_{\mathcal{U}}$. Different universes change the associated categories of spectra in a subtle, but important, way related to the dualizability of the the orbits G/H. Briefly, the suspension spectrum $G/H_+ \in Sp^G_{\mathcal{U}}$ is dualizable if and only if there is a G-equivariant embedding $G/H \to \mathcal{U}$. For full details see [LMSM86, Section II.6].

If $\mathcal{U}$ is a complete G-universe, then every G-orbit has an embedding into $\mathcal{U}$ and so all orbits are self-dual. As a result, the canonical quotient map $G/K \to G/H$ associated to subgroups $K \leq H$ induces two maps of G-spectra:

$$r^H_K : \Sigma^\infty_G G/K_+ \to \Sigma^\infty_G G/H_+$$
$$t^H_K : \Sigma^\infty_G G/H_+ \to \Sigma^\infty_G G/K_+$$

For any G-spectrum X indexed on $\mathcal{U}$, the assignment

$$G/H \mapsto [\Sigma^\infty_G G/H_+, X] \tag{2.4}$$

is a Mackey functor, where restriction and transfer are given by $(r^H_K)^*$ and $(t^H_K)^*$ respectively. Here, we have used $[-,-]$ to denote the homotopy classes of maps in the equivariant stable homotopy category. If X is a G-E_∞ ring spectrum then the assignment (2.4) yields a Tambara functor.

When the universe $\mathcal{U}$ is not complete the situation is a bit more subtle. The universe $\mathcal{U}$ determines two operads, the Steiner operad $S_{\mathcal{U}}$ and linear isometries operad $L_{\mathcal{U}}$, which in turn determine indexing categories we denote $\mathcal{O}_S$ and $\mathcal{O}_L$. The types of indexing categories arising this way are studied in [Rub21].

Proposition 2.4.8 ([BH22, Proposition 7.82])**.** *The pair $(\mathcal{O}_L, \mathcal{O}_S)$ is compatible.*

Correspondingly, in models of the equivariant stable homotopy category indexed on incomplete universes, such as those studied in [BH21], the assignment (2.4) naturally takes values in $\mathcal{O}_S$-Mackey functors. If X is an equivariant ring spectrum in this category, we obtain a $(\mathcal{O}_L, \mathcal{O}_S)$-Tambara functor.

2.5 Compatibility and Transfer Systems

Given a pair of indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$, it is not straightforward from the definitions to check whether or not the pair is compatible. Most of the difficulty stems from problems computing the image of the dependent product functor $n_*\colon \mathrm{Set}^G_{/S} \to \mathrm{Set}^G_{/T}$ associated to a map $n\colon S \to T$ in Set^G. The goal of this section is to alleviate this by providing a combinatorial characterization of when two indexing categories are compatible that can be checked without computing any dependent products.

In [BH22], Blumberg and Hill provide an alternative characterization of compatibility which, in light of Example 2.3.5, amounts to only checking the dependent products along the canonical projections $n\colon G/K \to G/H$ in $\mathcal{O}_m$. As remarked in that example, such functors are given by the coinduction functor which sends an K-set X to the H-set $\mathrm{Map}_K(H, X)$.

Example 2.5.1. Let $H = e$ and $G = C_2$, the group with two elements. Write $*$ for both the trivial G-set and trivial H-set. There are four elements of $\mathrm{Map}_e(C_2, * \amalg *)$ and a straightforward computation yields

$$\mathrm{Map}_e(C_2, * \amalg *) \cong * \amalg * \amalg C_2/e$$

which (essentially) gives the formula from Example 2.4.1. More generally, we have an isomorphism:

$$\mathrm{Map}_e(C_2, n \cdot [*]) \cong n \cdot [*] \amalg \left(\frac{n(n-1)}{2}\right) \cdot [C_2/e]$$

Readers who are more familiar with computations in Tambara functors may recognize that this equation arises in the norm in the C_2-Burnside Tambara functor.

Theorem 2.5.2 ([BH22, Theorem 7.65])**.** *A pair of indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$ is compatible if and only if for every pair of subgroups $K \leq H$ such that H/K is $\mathcal{O}_m$-admissible and every $\mathcal{O}_a$-admissible K-set T, the coinduced H-set $\mathrm{Map}_K(H, T)$ is also $\mathcal{O}_a$-admissible.*

Unfortunately, this characterization of compatibility can still be difficult to check in practice due to the somewhat unwieldy nature of the coinduction functors. In particular, Example 2.5.1

shows the coinduction functor does not preserve coproducts. Because of this, one of the standard techniques of equivariant algebra, separating a G-set into its orbits, often does not work when considering coinduction. In this section we provide a characterization of compatibility, Definition 2.5.6 below, that is more easily checked and does not make any reference to the coinduction functors.

The main result of this section is stated in terms of transfer systems, a notion we now recall.

Definition 2.5.3. For a finite group G, a G-transfer system $\mathcal{T}$ is a partial order $\leq_{\mathcal{T}}$ on the set of subgroups of G such that

1. (refines subset relation) if $K \leq_{\mathcal{T}} H$ then $K \leq H$,
2. (closure under conjugation) if $K \leq_{\mathcal{T}} H$, then for all $g \in G$ we have $K^g \leq_{\mathcal{T}} H^g$,
3. (closure under intersection) If $K \leq_{\mathcal{T}} H$ then for all $L \leq G$ we have $(K \cap L) \leq_{\mathcal{T}} (H \cap L)$.

Notation 2.5.4. If $\mathcal{T}$ is a G-transfer system it is convenient to represent $\mathcal{T}$ as a directed graph with a node for each subgroup $H \leq G$ and an arrow $K \to H$ if and only if $K \leq_{\mathcal{T}} H$. Figure 2.1 gives one example and one non-example of transfer systems for the cyclic group C_4 drawn in this notation.

Later, we will have two transfer systems $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T}_1$ refines in $\mathcal{T}_2$ in the sense that if $K \leq_{\mathcal{T}_1} H$ then $K \leq_{\mathcal{T}_2} H$. In such cases, we write $\mathcal{T}_1 \leq \mathcal{T}_2$ and represent both transfer systems as a single directed graph with two sets of arrows: a solid arrow $K \to H$ if $K \leq_{\mathcal{T}_1} H$ and a dashed arrow $K \dashrightarrow H$ if $K \leq_{\mathcal{T}_2} H$ but not necessarily $K \leq_{\mathcal{T}_1} H$. When we need to be clear about whether or not $K \leq_{\mathcal{T}_1} H$ we write it explicitly. An example is given in Figure 2.2.

Independent work of Rubin [Rub21] and Balchin–Barnes–Roitzheim [BBR21] has shown that, given a transfer system $\mathcal{T}$, one can build an indexing category $\mathcal{O}$ generated by maps $G/K \to G/H$ such that $K \leq_{\mathcal{T}} H$. This process can also be turned around, producing a transfer system from every indexing category, and these two constructions are mutual inverses.

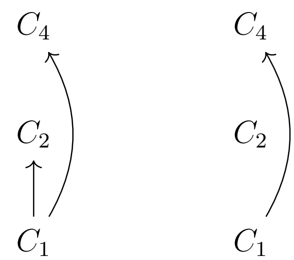


Figure 2.1: Two poset structures on the set of subsets of the cyclic group C_4. The left graph is a transfer system. The right graph is not a transfer system as it fails to satisfy closure under intersection.

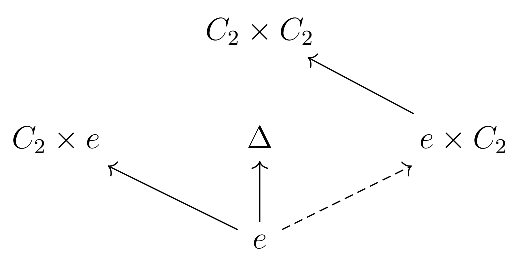


Figure 2.2: Two transfer systems $\mathcal{T}_1 \leq \mathcal{T}_2$ on the group $C_2 \times C_2$. The group Δ is the diagonal subgroup given by the image of the diagonal map $C_2 \to C_2 \times C_2$. This pair of transfer systems is not compatible.

Theorem 2.5.5 ([Rub21, Theorem 3.7]). *The posets of indexing categories and transfer systems for a finite group G are equivalent.*

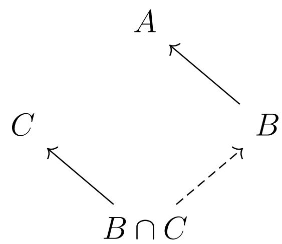
The theorem above implies that we could just as easily have defined bi-incomplete Tambara functors in terms of the pair of transfer systems $(\mathcal{T}_m, \mathcal{T}_a)$ that are equivalent to $(\mathcal{O}_m, \mathcal{O}_a)$. Unwinding the definitions, we see that the category $\mathcal{P}^G_{\mathcal{O}_m,\mathcal{O}_a}$ has a transfer T^H_K if and only if $K \leq_{\mathcal{T}_a} H$ and a norm N^H_K if and only if $K \leq_{\mathcal{T}_m} H$. Of course, describing $\mathcal{P}^G_{\mathcal{O}_m,\mathcal{O}_a}$ in terms of transfer systems requires that we have a notion of compatible transfer systems.

Definition 2.5.6. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two G-transfer systems. We say $(\mathcal{T}_1, \mathcal{T}_2)$ is a *compatible* pair if

1. $\mathcal{T}_1 \leq \mathcal{T}_2$,

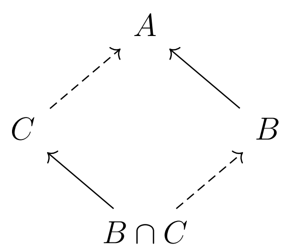
2. whenever A is a subgroup of G and $B, C \leq A$ are subgroups such that $B \leq_{\mathcal{T}_1} A$ and $(B \cap C) \leq_{\mathcal{T}_2} B$ then $C \leq_{\mathcal{T}_2} A$.

We pause to explain the second requirement of compatibility in terms of our graphical representations of transfer systems. The conditions that $B \leq_{\mathcal{T}_1} A$ and $(B \cap C) \leq_{\mathcal{T}_2} B$ can be represented graphically by saying the following subgraph appears in the directed graph representing these transfer systems:



(2.5)

Here the solid arrow $B \cap C \to C$ exists by intersection of the solid arrow $B \to A$ with C. Two transfer systems $\mathcal{T}_1 \leq \mathcal{T}_2$ are compatible if any subgraph of the form (2.5) is actually a part of a diamond:



(2.6)

We stress that once such a diagram is drawn it is an easy matter to check that the two transfer systems are compatible, as the graphs of the transfer systems can simply be checked visually. An example of a pair of transfer systems that are not compatible is found in Figure 2.2. To make this pair compatible, we would need to add the relations $\Delta \leq_{\mathcal{T}_2} C_2 \times C_2$ and $C_2 \times e \leq_{\mathcal{T}_2} C_2 \times C_2$.

Our definition of compatible transfer systems will only have value if we can show it is related to the notion of compatible indexing categories. In the remainder of this section we show they are in fact equivalent. As a first step, we have:

Proposition 2.5.7. *Suppose* $(\mathcal{O}_m, \mathcal{O}_a)$ *is a pair of indexing categories with corresponding pair of transfer systems* $(\mathcal{T}_m, \mathcal{T}_a)$*. If* $(\mathcal{O}_m, \mathcal{O}_a)$ *is compatible then* $(\mathcal{T}_m, \mathcal{T}_a)$ *is as well.*

Before proceeding with the proof, we note that the condition $\mathcal{T}_m \leq \mathcal{T}_a$ follows from Corollary 7.70 of [BH22] which says we have $\mathcal{O}_m \leq \mathcal{O}_a$. It suffices then to show the pair $(\mathcal{T}_m, \mathcal{T}_a)$ meets condition (2) of Definition 2.5.6.

Lemma 2.5.8. *If $(\mathcal{O}_m, \mathcal{O}_a)$ is compatible then $(\mathcal{T}_m, \mathcal{T}_a)$ meets condition (2) of Definition 2.5.6 in the special case where $A = BC$ is the product of the subgroups B and C. That is, if the product $BC = \{bc : b \in B, c \in C\}$ is a subgroup of G such that $B \leq_{\mathcal{T}_m} BC$ and $(B \cap C) \leq_{\mathcal{T}_a} B$, then $C \leq_{\mathcal{T}_a} BC$.*

Proof. Let $A = BC$. Our goal is to show A/C is $\mathcal{O}_a$-admissible. Writing $D = B \cap C$ we have, by Theorem 2.5.2, that the set $\mathrm{Map}_B(A, B/D)$ is $\mathcal{O}_a$-admissible. Since indexing categories are closed under subobjects (Lemma 2.1.7), we are done if we show there is an element $\mu \in \mathrm{Map}_B(A, B/D)$ whose stabilizer $A_\mu \subset A$ is exactly C as this implies

$$A/A_\mu = (BC)/C \subset \mathrm{Map}_B(A, B/D)$$

is $\mathcal{O}_a$ admissible.

Given any element $x \in A$ we can write $x = bc$ for some $c \in C$ and $b \in B$. We define the map $\mu\colon A \to B/D$ by

$$\mu(x) = \mu(bc) = bD$$

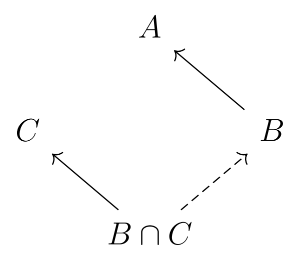
Since $B \cap C = D$, the element b is unique up to right multiplication by an element of D and so this map does not depend on the choice of b. Since the element c has no bearing on the value of μ we have $\mu(xc') = \mu(x)$ for any $c' \in C$ and thus $C \subset A_\mu$.

To show the reverse inclusion, take any $x \notin C$ (if $A = BC = C$ the result is trivial) and, as before, write $x = bc$. Since $x \notin C$, we have $b \notin D$ and thus

$$(x \cdot \mu)(e) = \mu(x) = bD \neq eD = \mu(e)$$

and so $x \notin A_\mu$. It follows that $A_\mu \subset C$ proving that $A_\mu = C$. $\square$

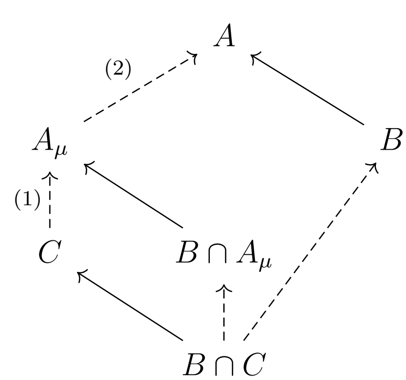
Proof of Proposition 2.5.7. By hypothesis we have a subgraph



in our transfer systems $(\mathcal{T}_m, \mathcal{T}_a)$ and our goal is to show there is a dashed arrow $C \dashrightarrow A$.

We proceed by induction on the index $k = [A : C]$, where the base case is trivial. Supposing that $k > 1$, let n be the number of double cosets $B\backslash A/C$ and write $\underline{n} = \{1, 2, \ldots, n\}$ for the trivial B-set with n elements. Note that $\underline{n}$ is admissible in all indexing categories by closure under finite coproducts. We may assume $n > 1$ as $n = 1$ occurs only if $A = BC$, which was covered in Lemma 2.5.8.

By Theorem 2.5.2, the set $\mathrm{Map}_B(A, \underline{n})$ is $\mathcal{O}_a$-admissible. Our goal is produce an element $\mu \in \mathrm{Map}_B(A, \underline{n})$ so that $C \subset A_\mu \subsetneq A$. If we can produce such a μ, we have the following subgraph in our transfer systems



where the arrow labeled (1) exists by induction hypothesis as the index $[A_\mu : C]$ is less than k, the arrow labeled (2) exists since A_μ is the stabilizer of an object in $\mathrm{Map}_B(A, \underline{n})$, which is an $\mathcal{O}_a$-admissible A-set, and all other arrows exist either by hypothesis or by the closure under intersection property of transfer systems. Composing arrows (1) and (2) yields the result.

To define μ, let $a_1, a_2, \ldots, a_n$ be representatives for the double cosets $B\backslash A/C$. For simplicity, take $a_1 = e$ to be the unit. For any $x \in A$, there is a unique $i_x \in \underline{n}$ so that $x \in Ba_{i_x}C$ and we define $\mu(x) = i_x$. Since the double cosets are fixed under left multiplication by B we see that μ is B-equivariant. Similarly, as the double cosets are fixed under right multiplication by C, we have $C \subset A_\mu$. Finally, we note the containment $A_\mu \subset A$ is strict as

$$(a_2 \cdot \mu)(e) = \mu(a_2) = 2 \neq 1 = \mu(e)$$

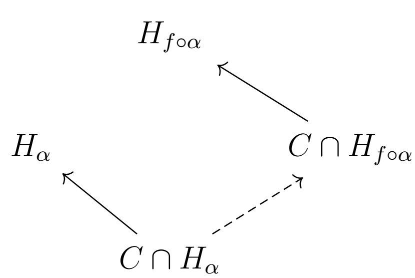
so $a_2 \cdot \mu \neq \mu$. We have used here the assumption that $n > 1$. □

To show the reverse direction, that compatible transfer systems yield compatible indexing categories, we prove a stronger result. This leads to an affirmative answer to Conjecture 7.90 of [BH22].

Proposition 2.5.9. *Suppose that $(\mathcal{T}_m, \mathcal{T}_a)$ is a compatible pair of transfer systems with corresponding indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$. If H/K is an $\mathcal{O}_m$-admissible H-set and $f\colon S \to T$ is a map in $i_K^*\mathcal{O}_a$ then $\mathrm{Map}_K(H, f)$ is a map in $i_H^*\mathcal{O}_a$. Taking $T = K/K$, we see that $\mathrm{Map}_K(H, S)$ is an $\mathcal{O}_a$-admissible H-set for all $\mathcal{O}_a$-admissible K-sets S. In particular, $(\mathcal{O}_m, \mathcal{O}_a)$ is compatible.*

Proof. To show $\mathrm{Map}_K(H, f)$ is a map in $i_H^*\mathcal{O}_a$, we must show for all $\alpha \in \mathrm{Map}_K(H, S)$ that $H_{f\circ\alpha}/H_\alpha$ is an $\mathcal{O}_a$-admissible set. Shifting to transfer systems, we must show $H_\alpha \leq_{\mathcal{T}_a} H_{f\circ\alpha}$.

Writing $C = \mathrm{Core}_H(K)$, we claim that we have the following subgraph in the directed graph for the pair $(\mathcal{T}_m, \mathcal{T}_a)$.



The solid arrows exist by Lemma 2.1.15 and closure under intersection. If we can establish the existence of the dashed arrow, we can use compatibility of our transfer systems to complete the diagram by filling in the last side of the diamond with a dashed arrow, completing the proof.

We claim that

$$C \cap H_\alpha = C \cap \left(\bigcap_{h \in H} h^{-1} K_{\alpha(h)} h \right)$$

where $K_{\alpha(h)}$ is the stabilizer of $\alpha(h)$ in K. To prove the claim, we first note that for all $x \in C$ we have

$$(x \cdot \alpha)(h) = \alpha(hx) = \alpha(hxh^{-1}h) = (hxh^{-1})\alpha(h)$$

where the last equality follows from the fact that C is a normal subgroup of H. Thus $x \in C \cap H_\alpha$ if and only if for all $h \in H$ we have $\alpha(h) = (hxh^{-1})\alpha(h)$ if and only if $x \in h^{-1}K_{\alpha(h)}h$ for all h, establishing the claim.

The same argument shows

$$C \cap H_{f\circ\alpha} = C \cap \left(\bigcap_{h \in H} h^{-1} K_{(f\circ\alpha)(h)} h \right).$$

Since $f\colon S \to T$ is in $i_K^* \mathcal{O}_a$ we have, by Lemma 2.1.10, that $K_{(f\circ\alpha)(h)}/K_{\alpha(h)}$ is $\mathcal{O}_a$-admissible for any h. The $\mathcal{O}_a$-admissibility of $(C \cap H_{f\circ\alpha})/(C \cap H_\alpha)$ now follows from Lemma 2.1.8. □

Together, Propositions 2.5.7 and 2.5.9 justify the claim that compatibility for indexing categories and transfer systems are the same.

Theorem 2.5.10. *Suppose $(\mathcal{O}_m, \mathcal{O}_a)$ is a pair of indexing categories. If $(\mathcal{T}_m, \mathcal{T}_a)$ is the corresponding pair of transfer systems then $(\mathcal{O}_m, \mathcal{O}_a)$ is compatible, in the sense of Definition 2.4.2, if and only if $(\mathcal{T}_m, \mathcal{T}_a)$ is compatible, in the sense of Definition 2.5.6.*

Finally, we end this section with the following corollary, affirmatively answering a conjecture of Blumberg and Hill. This corollary overcomes a technical obstacle in defining norms of incomplete Mackey functors. After unwinding definitions, the proof is immediate from Proposition 2.5.9.

Corollary 2.5.11 ([BH22, Conjecture 7.90])**.** *If $(\mathcal{O}_m, \mathcal{O}_a)$ is a compatible pair of indexing cate-*

gories then for every $\mathcal{O}_m$-admissible H-orbit H/K, coinduction restricts to a functor

$$\mathrm{Map}_K(H, -)\colon i_K^*\mathcal{O}_a \to i_H^*\mathcal{O}_a,$$

where $i_K^*\mathcal{O}$ and $i_H^*\mathcal{O}$ are the indexing categories of Example 2.1.6.

Chapter 3

Equivariant symmetric monoidal structures

A symmetric monoidal category is a category together with an operation, denoted $\otimes$, which endows the category with a way to combine two elements into a product. As the notation for the product suggests, the primordial example is the category Ab of abelian groups equipped with the tensor product. Abstracting the idea of the tensor product to a general category allows us to make sense of rings in new settings.

A ring R is an abelian group equipped with group homomorphisms

$$\mu\colon R \otimes R \to R \quad \text{and} \quad \eta\colon \mathbb{Z} \to R$$

which realize the multiplication and choice of unit, respectively. The maps μ and η are subject to some conditions which describe the associativity and unitality of multiplication. The important take away is that the maps μ and η, and the relations between them, can be written down in any symmetric monoidal category and give rise to a definition of a *monoid* in these categories. The name comes from the observation that monoids (in the usual sense) are exactly the monoids of the category of Sets equipped with the cartesian product as its monoidal product.

Pushing forward these ideas into the setting of equivariant algebra, the category of Mackey functors admits a symmetric monoidal product $\square$ called the box product. One might hope to recover Tambara functors as the (commutative) monoids in this category, but this is not the case. To get Tambara functors we need to consider an extension of symmetric monoidal categories, due to [HH16], called *symmetric monoidal Mackey functors*. We recall this construction in Section 3.1. Symmetric monoidal Mackey functors admit a notation of equivariant commutative monoids and the Hoyer–Mazur theorem asserts that these are precisely the Tambara functors. A conjecture of Blumberg and Hill posits that a similar theorem should hold relating bi-incomplete Tambara functors and incomplete Mackey functors [BH22, Conjecture 7.94]. We prove this conjecture in Theorem 3.4.4.

3.1 Symmetric Monoidal Mackey Functors

In this section we recall the notion of G-symmetric monoidal Mackey functors in the sense of Hill and Hopkins [HH16] and lay the groundwork for showing that bi-incomplete Tambara functors are precisely the $\mathcal{O}_m$-commutative monoids in $\mathcal{O}_a$-Mackey functors. This result is analogous to the fact that commutative rings are the commutative monoids in the category of abelian groups. While this is a helpful analogy to have in mind, we stress that Tambara functors are *not* the commutative monoids in the monoidal category of Mackey functors. Although Mackey functors do have a symmetric monoidal product, called the box product, the commutative monoids are the commutative Green functors.

A crucial insight of Hill and Hopkins is that recovering Tambara functors from the category of Mackey functors requires considering a form of equivariant symmetric monoidal structure on the category of Mackey functors. Just as a symmetric monoidal category is a form of monoid object in categories, an equivariant symmetric monoidal structure should be a form of equivariant monoid object, i.e. a semi-Mackey functor, in categories. To keep our presentation brief, we limit our presentation to only what we need, referring the interested reader to [HH16] for further discussion.

Definition 3.1.1. A *symmetric monoidal $\mathcal{O}$-Mackey functor $\mathcal{C}$* consists of symmetric monoidal categories $\mathcal{C}(H)$ for every subgroup $H \leq G$, together with strong monoidal functors

$$R^H_K \colon \mathcal{C}(H) \to \mathcal{C}(K), \quad c_g \colon \mathcal{C}(H) \to \mathcal{C}(H^g)$$

for any pair $K \leq H$, and for any $g \in G$. If H/K is an $\mathcal{O}$-admissible H-set, we also have a strong monoidal functor $N^H_K \colon \mathcal{C}(K) \to \mathcal{C}(H)$. The functors N^H_K, R^H_K and c_g are called the norm, restriction, and conjugation respectively. We require further the following coherence data:

1. isomorphisms $N^H_K N^K_L \cong N^H_L$, $R^K_L R^H_K \cong R^H_L$, and $c_g c_h \cong c_{gh}$ whenever these makes sense,

2. For any $g \in G$, isomorphisms $c_g N_K^H \cong N_{K^g}^{H^g} c_g$ and $c_g R_K^H \cong R_{K^g}^{H^g}$

3. For subgroups $K, L \leq H$, with H/L $\mathcal{O}$-admissible, the norms and restrictions are required to have a natural isomorphism

$$R_K^H N_L^H \cong \bigotimes_{\gamma_i} N_{K \cap L^{\gamma_i}}^K R_{K \cap L^{\gamma_i}}^{L^{\gamma_i}} c_{\gamma_i} \tag{3.1}$$

where the product on the right is the symmetric monoidal product in $\mathcal{C}(K)$ indexed over a collection of double coset representatives for $L \backslash H/K$. We refer to this isomorphism as the double coset formula.

Remark 3.1.2. Just as in Remark 2.2.7, the double coset formula makes sense as along as H/L is $\mathcal{O}$-admissible as this implies that $K/K \cap L^{\gamma_i}$ is $\mathcal{O}$-admissible for all γ_i.

Example 3.1.3. For any G-indexing category $\mathcal{O}$, there is a symmetric monoidal $\mathcal{O}$-Mackey functor $\mathrm{Set}^{\mathcal{O}}$ given by $\mathrm{Set}^{\mathcal{O}}(H) = \mathrm{Set}^H$, where Set^H is the category of H-sets with the monoidal product given by disjoint union. The norm maps N_K^H are given by the induction functors

$$H \times_K (-) \colon \mathrm{Set}^K \to \mathrm{Set}^H .$$

The restrictions are the usual forgetful functors, and the conjugations are given by the usual isomorphism of categories $c_g \colon \mathrm{Set}^H \cong \mathrm{Set}^{H^g}$.

If we instead use the cartesian product as the monoidal product of Set^H we can construct a different symmetric monoidal $\mathcal{O}$-Mackey functor by taking the same restrictions and conjugations, and taking the norms to be coinduction functors. The change in norms is necessitated by the fact that the norms must be strong monoidal.

Remark 3.1.4. The double coset formula is an exact analogue the formula for ordinary Mackey functors from Lemma 2.2.6 (3). For Mackey functors, the double coset formula comes from the composition laws for the Burnside category. A symmetric monoidal Mackey functor could be defined as a 2-product preserving pseudo-functor out of an appropriate bicategorical Burnside category. We return to this perspective in Chapter 4. A similar construction is studied by Balmer and Dell'Ambrogio in what they call Mackey 2-functors [BD20], though the structures they consider are not general enough to include all symmetric monoidal Mackey functors. In particular, they consider only examples in which the norm N_K^H is both a left and right adjoint of R_K^H.

With symmetric monoidal Mackey functors as our model for equivariant symmetric monoidal categories, we turn to task of defining equivariant monoids in these categories. To motivate the definitions, we consider the example of equivariant stable homotopy in which our equivariant monoids need to correspond to ring spectra.

For $\mathcal{O} = \mathrm{Set}^G$, there is a symmetric monoidal Mackey functor $G/H \mapsto Sp^H$, the category of genuine H-spectra. The restriction functors are the usual restrictions $\mathrm{Res}_K^H \colon Sp^H \to Sp^K$ and the norms $N_K^H \colon Sp^K \to Sp^H$ are given by the Hill–Hopkins–Ravenel norm [HHR16]. For any $H \leq G$, denote the category of commutative ring spectra in Sp^H by Comm^H.

Proposition 3.1.5 ([HHR16, Proposition 2.27])**.** *The norm and restriction functors restrict to an adjunction*

$$\mathrm{Comm}^K \underset{R_K^H}{\overset{N_K^H}{\rightleftarrows}} \mathrm{Comm}^H$$

with N_K^H as the left adjoint.

Proposition 3.1.5 says that a G-ring spectrum E, in addition to being a commutative monoid in Sp^G, comes equipped with counit maps $\mu_H^G \colon N_H^G R_H^G E \to E$, called norm multiplications by [AB18], for any subgroup $H \leq G$. Similarly, every H-ring spectrum F comes equipped with unit maps $\eta_H^G \colon F \to R_H^G N_H^G F$. The Hill–Hopkins model for equivariant commutative monoids is essentially just ordinary commutative monoids, together with coherent collections of unit and counit maps μ_H^G and η_H^G. We begin by discussing the unit maps, which it turns out exist for all ordinary commutative monoid in $\mathcal{C}(G)$ for any symmetric monoidal Mackey functor $\mathcal{C}$.

For any subgroup $H \leq G$, the double coset representatives $\gamma_1, \ldots, \gamma_n$ for $H\backslash G/H$ can be picked so that $\gamma_1 = e$ is the identity of G. This leads to a decomposition in the double coset formula:

$$R^G_H N^G_H \cong \mathrm{id}_{\mathcal{C}(H)} \otimes \left(\bigotimes_{i>1} N^H_{H\cap H^{\gamma_i}} R^{H^{\gamma_i}}_{H\cap H^{\gamma_i}} c_{\gamma_i} \right) \tag{3.2}$$

Suppose x is a monoid in the symmetric monoidal category $\mathcal{C}(H)$ and write $\eta_x \colon 1_H \to x$ for the unit. Since the restriction, norm, and conjugation functors are all strong monoidal, they all preserve monoids and so there are unit maps $\eta_i \colon 1_H \to N^H_{H\cap H^{\gamma_i}} R^{H^{\gamma_i}}_{H\cap H^{\gamma_i}} c_{\gamma_i}(x)$ for every i. In light of (3.2), we can define a map $\eta^G_H \colon x \to R^G_H N^G_H(x)$ by

$$x \cong x \otimes \left(\bigotimes_{i>1} 1_H \right) \xrightarrow{1\otimes(\bigotimes \eta_i)} x \otimes \left(\bigotimes_{i>1} N^H_{H\cap H^{\gamma_i}} R^{H^{\gamma_i}}_{H\cap H^{\gamma_i}} c_{\gamma_i}(x) \right) \cong R^G_H N^G_H(x) \tag{3.3}$$

The maps η^G_H function as the unit maps for our equivariant commutative monoids. While these maps exist for any commutative monoid, the counits represent non-trivial data which differentiates equivariant commutative monoids from the ordinary variety.

Definition 3.1.6 (See [Hoy14, Lemma 2.7.3])**.** For $\mathcal{C}$ a symmetric monoidal $\mathcal{O}$-Mackey functor, an *$\mathcal{O}$-commutative monoid* is a monoid (in the usual sense) $x \in \mathcal{C}(G)$ together with *norm multiplications* $\mu^H_K \colon N^H_K R^G_K(x) \to R^G_H(x)$ whenever H/K is $\mathcal{O}$-admissible. When the groups are clear from context, we may simply write μ for instead of μ^H_K. The norm multiplications are subject to the following coherence data:

1. For any $H \le G$ we have the triangle identity

$$\begin{array}{ccc} R^G_H(x) & \xrightarrow{\eta^G_H} & R^G_H N^G_H R^G_H(x) \\ & \searrow\!\!\!\!\searrow & \Big\downarrow R^G_H(\mu^G_H) \\ & & R^G_H(x) \end{array}$$

   where η^G_H is the map (3.3).

2. For all $L \le K \le H$, the map μ^H_L is equal to

$$N^H_L R^G_L(x) \cong N^H_K N^K_L R^G_L(x) \xrightarrow{N^H_K(\mu^K_L)} N^H_K R^G_K(x) \xrightarrow{\mu^H_K} R^G_H(x).$$

3. For any $K \leq H$ and $g \in G$ we have a commuting square

$$\begin{array}{ccc}
c_g N_K^H R_K^G(x) & \xrightarrow{c_g \mu_K^H} & c_g R_H^G(x) \\
\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \cong} \\
N_{K^g}^{H^g} R_{K^g}^G(x) & \xrightarrow{\mu_{K^g}^{H^g}} & R_{H^g}^G(x)
\end{array}$$

where the vertical isomorphisms come from the structural isomorphisms (2) of Definition 3.1.1.

4. (compatibility with the double coset formula). Let L, K, and H be subgroups of G with $L \leq K$. Fix representatives $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ for the double cosets $H\backslash G/L$ and $H\backslash G/K$ respectively. The following diagram must commute:

$$\begin{array}{ccc}
\bigotimes_{\alpha_i} N^K_{K\cap H^{\alpha_i}} N^{K\cap H^{\alpha_i}}_{L\cap H^{\alpha_i}} R^G_{L\cap H^{\alpha_i}}(x) & \xrightarrow{\otimes N^K_{K\cap H^{\alpha_i}}(\mu^{K\cap H^{\alpha_i}}_{L\cap H^{\alpha_i}})} & \bigotimes_{\alpha_i} N^K_{K\cap H^{\alpha_i}} R^G_{K\cap H^{\alpha_i}}(x) \\
\big\uparrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle F} \\
N_L^K R_L^G N_H^G R_H^G(x) & \overset{\alpha_L^K}{\dashrightarrow} & \bigotimes_{\beta_i} N^K_{K\cap H^{\beta_i}} R^G_{K\cap H^{\beta_i}}(x) \\
\big\downarrow{\scriptstyle N_L^K R_L^G(\mu_H^G)} & & \big\downarrow{\scriptstyle \otimes \mu^K_{K\cap H^{\beta_i}}} \\
N_L^K R_L^G(x) & \xrightarrow{\mu_L^K} & R_K^G(x)
\end{array}$$

The isomorphism comes from the double coset formula applied to $R_L^G N_H^G$. To define the map F, note that if $\alpha_i \in H\beta_j K$, then $K \cap H^{\alpha_i} \cong K \cap H^{\beta_j}$ and so we have well defined multiplications

$$\bigotimes_{\alpha_i \in K\beta_j H} N^K_{K\cap H^{\alpha_i}} R^G_{K\cap H^{\alpha_i}}(x) \to N^K_{K\cap H^{\beta_j}} R^G_{K\cap H^{\beta_j}}(x)$$

and F is the product of all these maps. The dotted arrow α_L^K is defined to be the composite along the top of the diagram and is included for future reference.

Definition 3.1.7. A morphism of $\mathcal{O}$-commutative monoids x and y is a morphism $f\colon x \to y$ of monoids in $\mathcal{C}(G)$ such that for any subgroups $K \leq H$ with H/K $\mathcal{O}$-admissible, the following square commutes

$$\begin{array}{ccc} N_K^H R_K^G(x) & \xrightarrow{N_K^H R_K^G(f)} & N_K^H R_K^G(y) \\ \downarrow{\scriptstyle \mu_K^H} & & \downarrow{\scriptstyle \mu_K^H} \\ R_H^G(x) & \xrightarrow{R_H^G(f)} & R_H^G(y) \end{array}$$

We now explain the presence of the dotted arrow α_L^K in (4) above. Note the double coset formula gives an isomorphism

$$R_K^G N_H^G R_H^G(x) \cong \bigotimes_{\beta_i} N_{K\cap H^{\beta_i}}^K R_{K\cap H^{\beta_i}}^G(x).$$

which allows us to consider the codomain of α_L^K to be $R_K^G N_H^G R_H^G(x)$. Passing through this isomorphism the bottom rectangle of the diagram in (4) is

$$\begin{array}{ccc} N_L^K R_L^G(N_H^G R_H^G(x)) & \xrightarrow{\alpha_L^K} & R_K^G N_H^G R_H^G(x) \\ \downarrow{\scriptstyle N_L^K R_L^G(\mu_H^G)} & & \downarrow{\scriptstyle R_K^G(\mu_H^G)} \\ N_L^K R_H^G(x) & \xrightarrow{\mu_L^K} & R_K^G(x) \end{array}$$

which commutes by (4). In particular the α_L^K serve as norm multiplications for $N_H^G R_H^G(x)$ and the map $\mu_H^G \colon N_H^G R_H^G(x) \to x$ is a map of $\mathcal{O}$-commutative monoids.

Proposition 3.1.8. *For any $\mathcal{O}$-commutative monoid x and any $H \leq G$ with G/H an $\mathcal{O}$-admissible set, the object $N_H^G R_H^G(x)$ is also an $\mathcal{O}$-commutative monoid. Moreover, the map $\mu_H^G \colon N_H^G R_H^G(x) \to x$ is a morphism of $\mathcal{O}$-commutative monoids.*

Remark 3.1.9. Proposition 3.1.8 should be thought of as a weak form of Proposition 3.1.5 in our setting. While we do not assert (yet) that the norm and restrictions give an adjunction on categories of $\mathcal{O}$-commutative monoids, we have that the map μ_H^G lives in the correct category to be a candidate for a counit map. This provides an important technical step in our proof of the generalized Hoyer–Mazur theorem in Section 3.4.

3.2 The $\mathcal{O}_m$-symmetric monoidal Mackey functor of $\mathcal{O}_a$-Mackey functors

Let $(\mathcal{O}_m, \mathcal{O}_a)$ be a compatible pair of indexing categories. In this section we construct a symmetric monoidal $\mathcal{O}_m$-Mackey functor of $\mathcal{O}_a$-Mackey functors. This construction has previously been carried out in the complete case $(\mathcal{O}_m, \mathcal{O}_a) = (\mathcal{O}^{gen}, \mathcal{O}^{gen})$ by Mazur [Maz13] for $G = C_{p^n}$ and by Hoyer [Hoy14] for general G. This section lays the groundwork for Section 3.4 where we generalize the Hoyer–Mazur theorem by characterizing $(\mathcal{O}_m, \mathcal{O}_a)$-Tambara functors as the $\mathcal{O}_m$-commutative monoids in $\mathcal{O}_a$-Mackey functors.

Our first aim is to construct the restriction, norm, and conjugation functors which make up the structure of our $\mathcal{O}_m$-symmetric monoidal Mackey functor. We first define analogous functors on Burnside categories with the aim being to upgrade these to operations on Mackey functor categories by left Kan extending. The functor underlying the norm is an extension of the coinduction functor

$$\mathrm{Map}_K(H, -)\colon \mathrm{Set}^K \to \mathrm{Set}^H$$

to the Burnside categories $\mathcal{A}^K_{\mathcal{O}_a}$ and $\mathcal{A}^H_{\mathcal{O}_a}$. A priori, it is not clear that coinduction extends to a functor of incomplete Burnside categories. In the case of interest, our work in Section 2.5 on compatible transfer systems, specifically Corollary 2.5.11, provides exactly the justification we need to give such an extension.

A convenient consequence of defining all of our operations as left Kan extensions is that checking the coherence data of Definition 3.1.1 reduces to checking for similar coherence at the level of functors on the Burnside categories. In particular, the proof that the double coset formula holds amounts to the fact that a similar formula holds on the level of sets with group action. Since the functors on the Burnside category are very explicit, this data is easy to check by hand.

For any subgroup $H \leq G$, we have an indexing category $i_H^*\mathcal{O}_a$ as in Example 2.1.6. To clean up notation in this section, we denote the associated incomplete Burnside categories $\mathcal{A}^H_{i_H^*\mathcal{O}_a}$ by $\mathcal{A}^H_{\mathcal{O}_a}$. Similarly, we denote the category of $i_H^*\mathcal{O}_a$-H-Mackey functors by $\mathrm{Mack}^H_{\mathcal{O}_a}$.

For subgroups $K \leq H$, the categories Set^K and Set^H are connected by the functors

$$\mathrm{Res}^H_K \colon \mathrm{Set}^H \to \mathrm{Set}^K,$$
$$\mathrm{Map}_K(H, -) \colon \mathrm{Set}^K \to \mathrm{Set}^H .$$

For any choices of K and H the functor $\mathrm{Res}^{\mathrm{H}}_{\mathrm{K}}$ extends to a functor $\rho^H_K \colon \mathcal{A}^H_{\mathcal{O}_a} \to \mathcal{A}^K_{\mathcal{O}_a}$ because $\mathrm{Res}^{\mathrm{H}}_{\mathrm{K}}$, being the right adjoint to induction, preserves pullback diagrams and restricts to a functor $i^*_H\mathcal{O}_a \to i^*_K\mathcal{O}_a$. The coinduction functor extends to a functor $C^H_K \colon \mathcal{A}^K_{\mathcal{O}_a} \to \mathcal{A}^H_{\mathcal{O}_a}$ when H/K is $\mathcal{O}_m$-admissible; coinduction preserves pullbacks because it is a right adjoint and it restricts to a functor on indexing categories by Corollary 2.5.11.

Similarly, for any $g \in G$ we have a conjugation isomorphism $c_g \colon \mathrm{Set}^H \to \mathrm{Set}^{H^g}$. Abusing notation a bit, this extends to an isomorphism $c_g \colon \mathcal{A}^H_{\mathcal{O}_a} \to \mathcal{A}^{H^g}_{\mathcal{O}_a}$ of Burnside categories.

Definition 3.2.1. For any $H \leq G$ and subgroup $K \leq H$ the *restriction functor*

$$R^H_K \colon \mathrm{Mack}^H_{\mathcal{O}_a} \to \mathrm{Mack}^K_{\mathcal{O}_a}$$

and *conjugation functor*

$$c_g \colon \mathrm{Mack}^H_{\mathcal{O}_a} \to \mathrm{Mack}^{H^g}_{\mathcal{O}_a}$$

are given by left Kan extension along ρ^H_K and c_g respectively. If H/K is an $\mathcal{O}_m$-admissible H-set we define the *norm functor*

$$N^H_K \colon \mathrm{Mack}^K_{\mathcal{O}_a} \to \mathrm{Mack}^H_{\mathcal{O}_a}$$

by left Kan extension along C^H_K.

Remark 3.2.2. A priori, the left Kan extension of an $\mathcal{O}_a$-K-Mackey functor $M \colon \mathcal{A}^K_{\mathcal{O}_a} \to \mathrm{Set}$ along $C^H_K \colon \mathcal{A}^K_{\mathcal{O}_a} \to \mathcal{A}^H_{\mathcal{O}_a}$ is an object in the presheaf category $\mathrm{Fun}(\mathcal{A}^H, \mathrm{Set})$. That $N^H_K(M) = (C^H_K)_!M$ is actually a Mackey functor (i.e. preserves products) is the main result of [BD77].

Remark 3.2.3. While

$$(C_K^H)_!\colon \mathrm{Fun}(\mathcal{A}_{\mathcal{O}_a}^K, \mathrm{Set}) \to \mathrm{Fun}(\mathcal{A}_{\mathcal{O}_a}^H, \mathrm{Set})$$

is the left adjoint of $(C_K^H)^*$, it is worth noting that $N_K^H\colon \mathrm{Mack}_{\mathcal{O}_a}^K \to \mathrm{Mack}_{\mathcal{O}_a}^H$ is *not* a left adjoint. In particular, the norm N_K^H will usually fail to commute with colimits computed in the category of Mackey functors. Nevertheless, the norm does commute with colimits computed in the presheaf category $\mathrm{Fun}(\mathcal{A}_{\mathcal{O}_a}^K, \mathrm{Set})$ which is sufficient for many purposes.

Since the norm, restriction, and conjugation functors are defined via left Kan extension, we can compute their value on represented functors using the Yoneda Lemma.

Lemma 3.2.4. *Let $L \leq K \leq H$ be a chain of subgroups with H/K an $\mathcal{O}_m$-admissible H-set and let $g \in G$. For any K-set X, let $A_X = \mathcal{A}_{\mathcal{O}_a}^K(X, -)$ be the represented Mackey functor of Example 2.2.8. The norm, restriction, and conjugation of A_X are can be computed as $N_K^H(A_X) \cong A_{\mathrm{Map}_K(H,X)}$, $R_L^K(A_X) \cong A_{\mathrm{Res}_L^K(X)}$, and $c_g(A_X) = A_{c_g(X)}$*

While our definition of N_K^H is completely analogous to Hoyer's, our definition of the restrictions R_K^H needs some justification. To define the restrictions, Hoyer defines a functor $I_K^H\colon \mathcal{A}^K \to \mathcal{A}^H$, which is the extension of the induction functor $H \times_K (-)\colon \mathrm{Set}^K \to \mathrm{Set}^H$, and defines R_K^H by precomposition with I_K^H. While this definition still makes sense, it is convenient to define the restrictions as a left Kan extension because it makes them easier to compare with the norm functors. For completeness, we show our definition is equivalent to Hoyer's.

Proposition 3.2.5. *The functors ρ_K^H and I_K^H form an ambidextrous adjunction. That is, each is both a left and right adjoint of the other.*

Proof. To clean up notation, we write simply ρ and I. We first show there are natural bijections:

$$\mathcal{A}_{\mathcal{O}_a}^K(X, \rho Y) \leftrightarrow \mathcal{A}_{\mathcal{O}_a}^H(IX, Y)$$

Going from left to right, we send a morphism $[X \xleftarrow{r} A \xrightarrow{t} \rho Y]$ to $[IX \xleftarrow{Ir} IA \xrightarrow{\hat{t}} Y]$ where the map $\hat{t}\colon IA \to Y$ is the adjunct of the map t along the adjunction between induction and restriction of H-sets. To see that $\hat{t}$ is a morphism in the indexing category $i_H^*\mathcal{O}_a$, consider the decomposition of $\hat{t}$ as

$$IA \xrightarrow{It} I\rho Y \xrightarrow{\epsilon} Y$$

where ϵ is the counit of the adjunction on H-sets. The map It is in $i_H^*\mathcal{O}_a$ by closure under self-induction (Lemma 2.1.9), so it remains to show ϵ is a morphism in the indexing category. The counit is the map

$$H \times_K \mathrm{Res}_K^H Y \to Y$$

which sends a class $[h, y]$ to the element hy. It is a straightforward exercise that the stabilizer of $[h, y]$ in $H \times_K \mathrm{Res}_K^H Y$ is exactly $h\mathrm{Stab}_H(y)h^{-1}$, which is also the stabilizer of hy. It follows that the counit is the coproduct of many fold maps, and thus is in $i_H^*\mathcal{O}_a$ by finite coproduct completeness.

To build the inverse, suppose we are given a morphism $[IX \xleftarrow{p} B \xrightarrow{s} Y]$. By Proposition 2.16 of [BH18], the functor I is an *essential sieve* meaning there is a K-set B', unique up to isomorphism, and a K-equivariant map $p' \colon B' \to X$ so that $B \cong IB'$ and p factors as $B \cong IB' \xrightarrow{Ip'} IX$. It follows that any morphism $[IX \xleftarrow{p} B \xrightarrow{s} Y]$ in $\mathcal{A}^H(IX, Y)$ is equal to one of the form $[IX \xleftarrow{Ip'} IB' \xrightarrow{s} Y]$. We send such a morphism to $[X \xleftarrow{p'} B' \xrightarrow{\hat{s}} \rho Y]$, where again $\hat{s}$ is coming from the adjunction between restriction and induction. Showing $\hat{s}$ is a morphism in $i_K^*\mathcal{O}_a$ is similar to the above argument for the adjunct $\hat{t}$.

That the two constructions described above are inverse to one another can be understood by considering what happens to the restriction and transfer maps of $[X \xleftarrow{r} A \xrightarrow{t} \rho Y]$ separately. For the transfer, both constructions simply replace the map by its adjunct across the $\mathrm{Ind}_K^H \dashv \mathrm{Res}_K^H$ adjunction. For the restriction, the first construction applies I, and the second uses the fact that I is an essential sieve to undo this. It follows that the two maps described above are mutually inverse. Moreover, since both the $\mathrm{Ind}_K^H \dashv \mathrm{Res}_K^H$ adjunction and the essential sieve property of I are natural, this bijection is also natural in either argument establishing that ρ is the right adjoint of I. That this adjunction is ambidextrous is established by simply turning all the spans around and repeating the construction. □

Corollary 3.2.6. *The functor* $(I_K^H)^*\colon \operatorname{Mack}^H \to \operatorname{Mack}^K$ *is a model for left Kan extension along* ρ_K^H *and thus* $(I_K^H)^* \cong R_K^H$.

We need to show that R_K^H, N_K^H, and c_g are all strong monoidal functors. The monoidal product on the categories Mack^H and Mack^K is the box product, defined by Day convolution [Day70]. Briefly, if M and N are H-Mackey functors we define $M \square N$ by the left Kan extension diagram

$$\begin{array}{ccc} \mathcal{A}^H_{\mathcal{O}_a} \times \mathcal{A}^H_{\mathcal{O}_a} & \xrightarrow{M\times N} & \mathrm{Set} \\ {\scriptstyle\times}\big\downarrow & \nearrow_{M\square N = \mathrm{Lan}_\times (M\times N)} & \\ \mathcal{A}^H_{\mathcal{O}_a} & & \end{array}$$

where $M \times N$ sends (S,T) to $M(S) \times N(T)$ and the vertical map $\times$ is given by cartesian product on H-sets.

Lemma 3.2.7. *For any* $K \le H$*, the functors* N_K^H *(assuming* H/K *is* $\mathcal{O}_m$*-admissible),* R_K^H*, and* c_g *are all strong monoidal.*

Proof. Because the functors ρ_K^H, C_K^H, and c_g all preserve cartesian products, they are all strong monoidal functors. The result follows from the general fact (see [DS95, Propsition 1]) that left Kan extension along strong monoidal functors is a strong monoidal. □

Proposition 3.2.8. *With the choices of norms, restrictions, and conjugations from Definition 3.2.1, the assignment* $G/H \mapsto \operatorname{Mack}^H_{\mathcal{O}_a}$ *forms a symmetric monoidal* $\mathcal{O}_m$*-Mackey functor.*

Proof. The restrictions norms, and conjugations are all strong monoidal by Lemma 3.2.7 so it remain to establish (1)–(3) of Definition 3.1.1. For subgroups $L \le K \le H$, there are canonical natural isomorphisms $R_L^K R_K^H \cong R_L^H$, $N_K^H N_L^K \cong N_L^H$, and $c_g c_h \cong c_{gh}$ coming from the fact that left Kan extension along a composite is isomorphic to the composite of left Kan extensions. This establishes (1), and (2) follows similarly, so it remains to establish the double coset formula.

For any $L, K \le H$ and any L-set X there is an isomorphism of H-sets

$$\mathrm{Res}^H_K \operatorname{Map}_L(H, X) \cong \prod_{\gamma_i} \operatorname{Map}_{K\cap L^{\gamma_i}}(K, \mathrm{Res}^{L^{\gamma_i}}_{K\cap L^{\gamma_i}}(c_{\gamma_i} X))$$

where the γ_i run over a transversal of the double cosets $K\backslash H/L$. We defer the proof of this isomorphism to Lemma 3.5.1 below. For any $L \leq G$ and L-set T write A_T for the represented Mackey functor $\mathcal{A}^L_{\mathcal{O}_a}(T, -)$. It is a property of Day convolution that there are natural isomorphisms

$$A_{S\times T} \cong A_S \,\square\, A_T$$

for any pair of L-sets S and T. Using this, Lemma 3.2.4 and the set-level isomorphism above it follows that the double coset formula holds for all represented Mackey functors A_T.

To prove the double coset formula for an arbitrary Mackey functor M, we consider M as an object in the presheaf category Fun(A^H, Set) of functors from A^H to Set. This category, like all presheaf categories, is generated under colimits by the representable functors A_T and so we may write

$$M \cong \varinjlim_I A_{T_i}$$

for some index category I. By Remark 3.2.3, the norm and restriction commute with colimits in Fun(A^H, Set) so we have

$$\begin{aligned}
R^H_K N^H_L(M) &\cong R^H_K N^H_L \left(\varinjlim_I A_{T_i}\right) \\
&\cong \varinjlim_I R^H_K N^H_K(A_{T_i}) \\
&\cong \varinjlim_I \bigotimes_{\gamma_i} N^K_{K\cap L^{\gamma_i}} R^{L^{\gamma_i}}_{K\cap L^{\gamma_i}} c_{\gamma_i}(A_{T_i}) \\
&\cong \bigotimes_{\gamma_i} N^K_{K\cap L^{\gamma_i}} R^{L^{\gamma_i}}_{K\cap L^{\gamma_i}} c_{\gamma_i}\left(\varinjlim_I A_{T_i}\right) \\
&\cong \bigotimes_{\gamma_i} N^K_{K\cap L^{\gamma_i}} R^{L^{\gamma_i}}_{K\cap L^{\gamma_i}} c_{\gamma_i}(M)
\end{aligned}$$

where the third isomorphism uses the fact that A_{T_i} is represented, and the fourth isomorphism uses the fact that Day convolution, as a left Kan extension, commutes with colimits in the presheaf category. $\square$

3.3 Norms and restrictions on categories of Tambara functors

In the last section, we endowed the categories of $\mathcal{O}_a$-Mackey functors with the structure of a symmetric monoidal $\mathcal{O}_m$-Mackey functor. We now turn our attention to characterizing the $\mathcal{O}_m$-commutative monoids in $\mathcal{O}_a$-Mackey functors. In [BH22], it is conjectured that the $\mathcal{O}_m$-commutative monoids are exactly the bi-incomplete Tambara functors. In this section we lay the groundwork for proving this conjecture by studying how Tambara functors interact with the norm and restriction functors for Mackey functors. We will show, in particular, that the norm or restriction of any Tambara functor is again a Tambara functor.

It is convenient to phrase the main result of this section in slightly different language. We construct *Tambara norm functors*

$$\mathcal{N}_K^H\colon \operatorname{Tamb}^K_{(\mathcal{O}_m,\mathcal{O}_a)} \to \operatorname{Tamb}^H_{(\mathcal{O}_m,\mathcal{O}_a)}$$

for every $\mathcal{O}_m$-admissible H/K analogous to the Mackey norm functors from Definition 3.2.1. In Theorem 3.3.7 we show these two constructions agree after applying the forgetful functors from Tambara functors to Mackey functors. Similar results when the indexing categories are complete are due to Hoyer and Mazur [Hoy14, Maz13] and we adapt the proof of Theorem 2.3.3 in [Hoy14], and correct a small oversight. Similarly, we construct *Tambara restriction functors*, and also show these are compatible with the forgetful functors from Tambara functors to Mackey functors.

For any $K \leq H$, the restriction $\operatorname{Res}^H_K\colon \operatorname{Set}^H \to \operatorname{Set}^K$ and induction $H \times_K (-)\colon \operatorname{Set}^K \to \operatorname{Set}^H$ extend to functors $\rho^H_K\colon \mathcal{P}^H_{(\mathcal{O}_m,\mathcal{O}_a)} \to \mathcal{P}^K_{(\mathcal{O}_m,\mathcal{O}_a)}$ and $I^H_K\colon \mathcal{P}^K_{(\mathcal{O}_m,\mathcal{O}_a)} \to \mathcal{P}^H_{(\mathcal{O}_m,\mathcal{O}_a)}$ on polynomial categories because both preserve pullbacks and exponential diagrams.

Definition 3.3.1. The Tambara norm functor $\mathcal{N}_K^H\colon \operatorname{Tamb}^K_{(\mathcal{O}_m,\mathcal{O}_a)} \to \operatorname{Tamb}^H_{(\mathcal{O}_m,\mathcal{O}_a)}$ is given by left Kan extension along the functor I^H_K. The Tambara restriction functor $\mathcal{R}^H_K$ is defined by left Kan extension along ρ^K_H.

Remark 3.3.2. One can show that I^H_K is the right adjoint of ρ^H_K and it follows formally that, as in the case of Mackey functors, $\mathcal{R}_K^H$ is naturally isomorphic to the precomposition $(I_K^H)^*$. It follows there is an adjunction $\mathcal{N}_K^H \dashv \mathcal{R}_K^H$.

Just as Lemma 3.2.4 computes the norms of represented Mackey functors, we can compute the norms of represented Tambara functors. For any $H \leq G$ and H-set T, write P_T for the represented Tambara functor $\mathcal{P}^H_{(\mathcal{O}_m,\mathcal{O}_a)}(T,-)$.

Lemma 3.3.3. *For any $K \leq H$ and any H-set T there is an isomorphism of Tambara functors $\mathcal{R}_K^H(P_T) \cong P_{\mathrm{Res}_K^H(T)}$. Similarly, if S is any K-set then $\mathcal{N}_K^H(P_S) \cong P_{H\times_K S}$.*

If $i_H = i_{add}\colon \mathcal{A}^H_{\mathcal{O}_a} \to \mathcal{P}^H_{(\mathcal{O}_m,\mathcal{O}_a)}$ is the inclusion functor of Remark 2.3.4, then the forgetful functor which sends a Tambara functors to its underlying additive Mackey functor is $U_H = i_H^*$. The main result of this section says that the Tambara norm functors "commute" with the forgetful functors U_H in the sense that there are isomorphisms $U_H\mathcal{N}_K^H \cong N_K^H U_K$ where N_K^H is the Mackey norm functor. Before proving this result, we quickly prove the analogous result for the Tambara restriction functors.

Lemma 3.3.4. *For any pair $K \leq H$ of subgroups of G, there is a natural isomorphism of restriction functors $U_K\mathcal{R}_K^H \cong R_K^H U_H$.*

Proof. Consider the following commutative diagram of functors:

$$\begin{array}{ccc} \mathcal{A}^K_{\mathcal{O}_a} & \xrightarrow{i_K} & \mathcal{P}^K_{(\mathcal{O}_m,\mathcal{O}_a)} \\ \downarrow{\scriptstyle I_K^H} & & \downarrow{\scriptstyle I_K^H} \\ \mathcal{A}^H_{\mathcal{O}_a} & \xrightarrow{i_H} & \mathcal{P}^H_{(\mathcal{O}_m,\mathcal{O}_a)} \end{array}$$

Since this diagram commutes, there is a natural isomorphism of functors

$$(i_H \circ I_K^H)^* \cong (I_K^H \circ i_K)^*.$$

The result now follows from Corollary 3.2.6 and Remark 3.3.2 which identify both the Mackey and Tambara restriction functors precomposition with I_K^H. □

Comparing the Tambara and Mackey restriction functors with the forgetful functors U_K and U_H is easy because all functors involved are precomposition functors. Comparing the Tambara and Mackey norms requires more care because it compares precomposition functors with left Kan extensions. The correct categorical framework in which to approach such comparisons is the calculus of mate diagrams, which we now recall; for more discussion see [Shu11, Part 1].

Consider the following square of functors, inhabited by a natural transformation α.

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle h}\big\downarrow & \overset{\alpha}{\Swarrow} & \big\downarrow{\scriptstyle k} \\ C & \xrightarrow{\;g\;} & D \end{array} \tag{3.4}$$

For any category $\mathcal{C}$, we denote the category of functors and natural transformations from $\mathcal{C}$ to Set by $\mathrm{Set}^{\mathcal{C}}$. The square (3.4) determines another square of functors:

$$\begin{array}{ccc} \mathrm{Set}^A & \xleftarrow{\;f^*\;} & \mathrm{Set}^B \\ {\scriptstyle h^*}\big\uparrow & \overset{\alpha^*}{\Swarrow} & \big\uparrow{\scriptstyle k^*} \\ \mathrm{Set}^C & \xleftarrow{\;g^*\;} & \mathrm{Set}^D \end{array} \tag{3.5}$$

Because all Set valued functors admit left Kan extensions, the functors h^* and k^* in (3.5) admit left adjoints we denote by $h_!$ and $k_!$ respectively. We denote the units and counits of these adjunctions by η_h, η_k, ϵ_h and ϵ_k respectively. Using the units and counits, we can define a natural transformation $\beta\colon h_! f^* \Rightarrow g^* k_!$ as the composite

$$h_! f^* \xRightarrow{h_! f^* \cdot \eta_k} h_! f^* k^* k_! \xRightarrow{h_! \cdot \alpha^* \cdot k_!} h_! h^* g^* k_! \xRightarrow{\epsilon_h \cdot g^* k_!} g^* k_!$$

which fills the square

$$\begin{array}{ccc} \mathrm{Set}^A & \xleftarrow{\;f^*\;} & \mathrm{Set}^B \\ {\scriptstyle h_!}\big\downarrow & \overset{\beta}{\Searrow} & \big\downarrow{\scriptstyle k_!} \\ \mathrm{Set}^C & \xleftarrow{\;g^*\;} & \mathrm{Set}^D \end{array} \tag{3.6}$$

Definition 3.3.5. The square (3.6) is called the *mate* of the square (3.5). We say the square (3.4) is *exact*, or satisfies the *Beck–Chevalley condition*, if β is a natural isomorphism.

In this language, we construct a natural transformation $\alpha\colon i_H C_K^H \Rightarrow I_K^H i_K$ so that the square

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{O}_a}^K & \xrightarrow{i_K} & \mathcal{P}_{(\mathcal{O}_m,\mathcal{O}_a)}^K \\ \downarrow{\scriptstyle C_K^H} & \overset{\alpha}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle I_K^H} \\ \mathcal{A}_{\mathcal{O}_a}^H & \xrightarrow{i_H} & \mathcal{P}_{(\mathcal{O}_m,\mathcal{O}_a)}^H \end{array} \tag{3.7}$$

is exact. For a K-set T, the component $\alpha_T\colon I_K^H i_K(T) \to i_H C_K^H(T)$ is represented by the bispan

$$H \times_K T \xleftarrow{H\times_K(\epsilon_T^C)} H \times_K \operatorname{Res}_K^H \operatorname{Map}_K(H,T) \xrightarrow{\epsilon^I_{\operatorname{Map}_K(H,T)}} \operatorname{Map}_K(H,T) \xrightarrow{=} \operatorname{Map}_K(H,T)$$

where ϵ^C and ϵ^I are the counits of the coinduction-restriction and induction-restriction adjunctions respectively. This is indeed a natural transformation, although the proof is rather involved and we defer it to Section 3.5.

Lemma 3.3.6. *The maps $\alpha_T\colon I_K^H i_K(T) \to i_H C_K^H(T)$ assemble into a natural transformation $\alpha\colon i_H C_K^H \Rightarrow I_K^H i_K$.*

Equipped with the natural transformation $\alpha\colon I_K^H i_K \Rightarrow i_H C_K^H$, we obtain the mate transformation

$$N_K^H U_K = (C_K^H)_! I_K^* \overset{\beta}{\Rightarrow} i_H^* (I_K^H)_! = U_H \mathcal{N}_K^H$$

and the main result of this section is that β is a natural isomorphism.

For any bi-incomplete Tambara functor $S\colon \mathcal{P}_{(\mathcal{O}_m,\mathcal{O}_a)}^K \to \mathrm{Set}$ and H-set Y the pointwise Kan extension formula allows us to write the elements of $(C_K^H)_! U_K(S)(Y)$ as equivalence classes of pairs $(\omega\colon C_K^H B \to Y, x \in R_K^H(B))$ where ω is a morphism in $\mathcal{A}_{\mathcal{O}_a}^H$. The equivalence classes are generated by the relations

$$(\omega \circ C_K^H(\omega'), x) \sim (\omega, S(i_K(\omega'))(x))$$

for any maps ω' in $\mathcal{A}^K_{\mathcal{O}_a}$. The component $\beta_{S,Y}\colon (C^H_K)_! i^*_K(S)(Y) \to i^*_H(I^H_K)_!(S)(Y)$ of β sends the class represented by $(\omega\colon C^H_K B \to Y, x)$ to the class of $(i_H(\omega)\circ \alpha_B\colon I^H_K B \to Y, x)$.

Theorem 3.3.7 ([Hoy14, Theorem 2.3.3])**.** *For any $\mathcal{O}_m$-admissible H/K, the square (3.7) is exact. That is, the natural transformation β is a natural isomorphism of functors $U_H \mathcal{N}^H_K \cong N^H_K U_K$.*

Remark 3.3.8. In proving a version of Theorem 3.3.7, Hoyer defines a map, which in our notation is β^{-1}, and shows it is well defined. It appears that Hoyer's proof does not show that the inverse of his map, which we call β, is well defined. The advantage of first defining the natural transformation α is that the well-definedness of β is immediate from the fact that it is the mate of the natural transformation α. Moreover, it is easier to check that α is a well defined natural transformation because it is relating the functors I^H_K and C^H_K instead of their respective left Kan extensions.

Proof. For notational brevity, we fix the subgroups K and H and suppress them from the notation when clear, writing C instead of C^H_K, ρ instead of ρ^H_K and so on. Fixing a bi-incomplete K-Tambara functor S and an H-set Y, we need to show the components $\beta_{S,Y}\colon C_! i^*_K(S)(Y) \to i^*_H I_!(S)(Y)$ are bijections. To show surjectivity, note that by Lemma 2.3.4 of [Hoy14], and the discussion that follows, an arbitrary element in the codomain of $\beta_{S,Y}$ is a class represented by a pair (λ, x) where λ is a bispan of the form

$$\lambda = [I\rho B \xleftarrow{=} I\rho B \xrightarrow{\epsilon^I} B \xrightarrow{h} Y].$$

One can check directly that $\beta_{S,Y}([C\rho B \xleftarrow{\eta^C} B \xrightarrow{h} Y, x]) = [\omega, x]$ and thus our map is surjective. Since the element $[\lambda, x]$ is arbitrary, the assignment

$$[\lambda, x] \mapsto [C\rho B \xleftarrow{\eta^C} B \xrightarrow{h} Y, x] \tag{3.8}$$

defines a section of β which Hoyer shows is well defined.

It remains to show our section (3.8) is surjective. To see this, suppose we are given an arbitrary element $[\sigma, x] \in (C^H_K)_! i^*_K(S)(Y)$ where σ is represented by the span $CX \xleftarrow{r} A \xrightarrow{t} Y$ and $x \in S(X)$. We can factor the map r as the composite $A \xrightarrow{\eta^C} C\rho(A) \xrightarrow{C(\hat{r})} CX$ where $\widehat{r}$ is the adjunct of r. Using the defining relation of the pointwise Kan extension formula, we see that $[\sigma, x] = [\sigma', S(\hat{r})(x)]$ where σ' is represented by the span $C\rho A \xleftarrow{\eta^C} A \xrightarrow{t} Y$. Since $[\sigma', S(\hat{r})(x)]$ is in the image of (3.8), we are done. □

3.4 Bi-incomplete Tambara Functors are $\mathcal{O}_m$-Commutative Monoids

In this section we prove the $\mathcal{O}_m$-commutative monoids in $\mathcal{O}_a$-Mackey functors are exactly the $(\mathcal{O}_m, \mathcal{O}_a)$-Tambara functors. We begin by showing that for any G-Tambara functor S, the underlying Mackey functor $U_G(S)$ is always an $\mathcal{O}_m$-commutative monoid by constructing norm multiplications

$$\mu_K^H \colon N_K^H R_K^G U_G(S) \to R_H^G U_G(S)$$

for all pairs $K \leq H$ of subgroups of G. The construction of the μ_K^H rely heavily on our work from Section 3.3 comparing the norm and restriction functors with the forgetful functor U_G.

After establishing that Tambara functors give $\mathcal{O}_m$-commutative monoids we turn our attention to the inverse construction of constructing a Tambara functor from an $\mathcal{O}_m$-commutative monoid M. The essential step is to use the norm multiplications to build operations

$$\nu_K^H \colon M(G/K) \to M(G/H)$$

which we call the *external norms* of M. We show that if M is the underlying monoid of a Tambara functor then the external norms agree with the usual internal norms of the Tambara functor. Not only does this show that two constructions are mutually inverse, but is also implies that the external norms are compatible with the transfers and restrictions in exactly the same way as the norms of a Tambara functor, allowing us to conclude that every $\mathcal{O}_m$-commutative monoid equipped with external norms is a Tambara functor.

Proposition 3.4.1. *For any Tambara functor $S \in \mathrm{Tamb}^G_{(\mathcal{O}_m, \mathcal{O}_a)}$, the Mackey functor $U_G(S)$ is an*

$\mathcal{O}_m$*-commutative monoid. Moreover, this gives a functor*

$$U_G\colon \operatorname{Tamb}^G_{(\mathcal{O}_m,\mathcal{O}_a)} \to \operatorname{Comm}_{\mathcal{O}_m}(\operatorname{Mack}_{\mathcal{O}_a}). \tag{3.9}$$

Proof. For any map $h\colon G/K \to G/H$ in $\mathcal{O}_m$ we define the norm multiplication

$$\mu_K^H\colon N_K^H R_K^G U_G(S) \to R_H^G U_G(S)$$

to be the unique map so that the following diagram commutes:

$$\begin{array}{ccccc} N_K^H R_K^G U_G(S) & & \xrightarrow{\mu_K^H} & & R_H^G U_G(S) \\ \downarrow\cong & & & & \downarrow\cong \\ U_G\mathcal{N}_K^H\mathcal{R}_K^G S & \xrightarrow{\cong} & U_G\mathcal{N}_K^H\mathcal{R}_K^H\mathcal{R}_H^G S & \xrightarrow{U_G\cdot\epsilon_K^H} & U_G\mathcal{R}_H^G S \end{array} \tag{3.10}$$

where $\epsilon_K^H\colon \mathcal{N}_K^H\mathcal{R}_K^H\mathcal{R}_H^G S \to \mathcal{R}_H^G S$ is the counit of the $\mathcal{N}_K^H \dashv \mathcal{R}_K^H$ adjunction and the vertical maps are the isomorphisms of Theorem 3.3.7. The coherence data of Definition 3.1.6 is checked directly using the explicit form of the isomorphism of Theorem 3.3.7 and the fact the counits ϵ_K^H can be explicitly computed using the pointwise Kan extension formula.

To see U_G is a functor we must show that for any map $F\colon S \to T$ of Tambara functors that the underlying map $f = U_G(F)\colon U_G(S) \to U_G(T)$ is a morphism of $\mathcal{O}_m$-commutative monoids. This amounts to showing that the square

$$\begin{array}{ccc} N_K^H R_K^G U_G(S) & \xrightarrow{N_K^H R_K^G f} & N_K^H R_K^G U_G(T) \\ \downarrow \mu_K^H & & \downarrow \mu_K^H \\ R_H^G U_G(S) & \xrightarrow{R_H^G f} & R_H^G U_G(T) \end{array}$$

commutes for any choices of $K \le H$ but this is immediate since every map in the diagram (3.10) is natural in S. $\square$

The remainder of this section is devoted to constructing an inverse to the functor U_G. Since every $\mathcal{O}_m$-commutative monoid has the structure of a Green functor, and Tambara functors are essentially Green functors with norm maps, it suffices to construct functorial norm operations on every $\mathcal{O}_m$-commutative monoid.

Let M be an $\mathcal{O}_m$-commutative monoid in $\mathcal{O}_a$-Mackey functors. Any element $x \in M(G/K)$ determines a map $\hat{x}\colon A_{K/K} \to R^G_K M$. If H/K is $\mathcal{O}_m$-admissible, we can form the following composite

$$A_{H/H} \cong N^H_K A_{K/K} \xrightarrow{N^H_K(\hat{x})} N^H_K R^G_K M \xrightarrow{\mu} R^G_H M. \tag{3.11}$$

where $\mu\colon N^H_K R^G_K M \to R^G_H M$ is the norm multiplication. By the Yoneda lemma, the map (3.11) corresponds uniquely to an element $\nu^H_K(x) \in R^G_H M(H/H) \cong M(G/H)$ which we call the *external norm* of x. When $M = U_G(S)$ comes from a Tambara functor, the external norms recover the usual internal norms.

Proposition 3.4.2. *For any Tambara functor S and element $x \in S(G/K)$ the external norm $\nu^H_K(x)$ is equal to the internal norm $N^H_K(x)$.*

Proof. Using the definition of the norm multiplications on a Tambara functor, the composite (3.11) defining $\nu^H_K(x)$ becomes

$$A_{H/H} \cong N^H_K A_{K/K} \xrightarrow{N^H_K(\hat{x})} N^H_K U_K \mathcal{R}^G_K S \xrightarrow{\beta\cdot\mathcal{R}^G_K} U_H \mathcal{N}^H_K \mathcal{R}^G_K S \xrightarrow{U_H\cdot\epsilon_{R^G_H S}} U_H \mathcal{R}^G_H S \tag{3.12}$$

where $\epsilon_{R^G_H S}\colon \mathcal{N}^H_K \mathcal{R}^G_K S \to \mathcal{R}^G_H S$ is the counit of the $\mathcal{N}^H_K \dashv \mathcal{R}^H_K$ adjunction.

The external norm $\nu^H_K(x)$ is equal to the image of the element $\mathrm{id}_{H/H} \in A_{H/H}(H/H)$ under the map (3.12). Evaluating all the functors in (3.12) at the object H/H, and using the pointwise Kan extension formula, we can can compute where $\mathrm{id}_{H/H}$ is sent at every step in the composition:

$$\begin{aligned} \mathrm{id}_{H/H} &\mapsto [C(K/K) \xleftarrow{\cong} H/H \xrightarrow{=} H/H, \mathrm{id}_{K/K} \in A_{K/K}(K/K)] \\ &\mapsto [C(K/K) \xleftarrow{\cong} H/H \xrightarrow{=} H/H, x \in U_K \mathcal{R}^G_K S(K/K)] \end{aligned}$$

$$\mapsto [I\rho(H/H) \xleftarrow{=} I\rho(H/H) \xrightarrow{\epsilon^I_{H/H}} H/H \xrightarrow{=} H/H, x \in \mathcal{R}^G_K S(K/K)]$$
$$\mapsto \mathcal{R}^G_H(S)(N_{\epsilon^I_{H/H}})(x).$$

In the last line, we use the Yoneda lemma to identify $\mathcal{R}^G_K S(K/K)$ with $\mathcal{R}^G_H S(H/K)$ and call the element x the same thing in both sets. Under the identification $\mathcal{R}^G_H(S)(H/H) \leftrightarrow S(G/H)$, this element corresponds to $S(N_{I^G_H(\epsilon^I_{H/H})})(x)$. There is an isomorphism

$$I\rho(H/H) = H \times_K (\mathrm{Res}^H_K(H/H)) \cong H/K$$

and after this identification the map $\epsilon^I_{H/H} \colon H/K \to H/H$ is the canonical quotient and so we have $\nu^H_K(x) = S(N_{G/K \to G/H})(x) = N^H_K(x)$. □

We can define an external norm $\nu_p \colon M(S) \to M(T)$ for any map $p \colon S \to T$ in Set^G. First, if $p \colon G/K \to G/H$ is the canonical quotient we define $\nu_p = \nu^H_K$. Any other map p in Set^G is isomorphic to a disjoint union of canonical quotients

$$p \colon \coprod_{i=1}^{n} G/K_i \to G/H$$

for some subgroups $K_i \leq H$. For such p, we define ν_p to be the composition

$$\prod_{i=1}^{n} M(G/K_i) \xrightarrow{\prod \nu^H_{K_i}} \prod_{i=1}^{n} M(G/H) \xrightarrow{\mu} M(G/H)$$

where μ is the multiplication map that exists because M is a Green functor.

Proposition 3.4.3. *Suppose that $M \in \mathrm{Mack}^G_{\mathcal{O}_a}$ is an $\mathcal{O}_m$-commutative monoid. Then there exists a $(\mathcal{O}_m, \mathcal{O}_a)$-Tambara functor $S \colon \mathcal{P}^G_{(\mathcal{O}_m, \mathcal{O}_a)} \to \mathrm{Set}$ whose underlying Green functor is M. The internal norms of S are given by the external norms ν^H_K.*

Proof. We proceed by induction on the size of the group G. The base case of the trivial group is immediate since both Tambara functors and $i_e^*\mathcal{O}_m$-commutative monoids are just commutative rings. Suppose then that the result is true for all groups H with $|H| < |G|$.

By definition, the Mackey functor M has the structure of a commutative monoid in Mack^G, i.e. a Green functor. Since a Tambara functor is just a Green functor with additional norm maps, it suffices to show how the $\mathcal{O}_m$-commutative monoid structure determines norms $M(G/K) \to M(G/H)$ for each $K \leq H$ with H/K an $\mathcal{O}_m$-admissible H-set. As indicated in the statement of the proposition, the norm maps are given by the external norm maps $\nu_K^H \colon M(G/K) \to M(G/H)$. All that remains is to check the external norms satisfy the necessary compatibility with the transfers and restriction maps of our Green functor M.

We first handle compatibility with transfers. Suppose we have $L \leq K \leq H$, and suppose we have picked an exponential diagram in Set^G

$$\begin{array}{ccccc} E & & \xrightarrow{\quad\beta\quad} & & F \\ \downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\gamma} \\ G/L & \xrightarrow{p} & G/K & \xrightarrow{q} & G/H \end{array}$$

where p and q are the canonical quotient maps. We need to show that for any $x \in M(G/L)$ that $\nu_K^H(T_L^K(x)) = T_\gamma \nu_\beta R_\alpha(x)$. We assume that L is a proper subgroup of H, as otherwise there is nothing to show.

We claim there is a Tambara functor S and a map $f \colon S \to M$ of $\mathcal{O}_m$-commutative monoids so that $x = f(y)$ for some $y \in S(G/L)$. Granting this, the naturality of external norms, transfers and restrictions implies

$$\nu_K^H(T_L^K(x)) = \nu_K^H(T_L^K(f(y))) = f(\nu_K^H(T_L^K(y))) = f(T_\gamma \nu_\beta R_\alpha(y)) = T_\gamma \nu_\beta R_\alpha(x)$$

where the third equality uses the fact that, by Proposition 3.4.2, the external norms in $U_G S$ must be equal to the internal norms of S and thus are sufficiently compatible with the transfers. The same argument proves compatibility between the external norms and the restrictions and so it remains to prove the claim.

By the induction hypothesis, and the assumption that $L < H$ is proper, the $i_L^* \mathcal{O}_m$-commutative monoid $R_L^G(M)$ is isomorphic to $U_L(S')$ for some bi-incomplete L-Tambara functor S'. Applying the Tambara norm, and using Theorem 3.3.7, we have an equivalence of $\mathcal{O}_m$-commutative monoids $U_G\mathcal{N}_L^G S' \cong N_L^G R_L^G M$. In the commutative square

$$\begin{array}{ccc} R_L^G N_L^G R_L^G(M)(L/L) & \xrightarrow{\cong} & N_L^G R_L^G(M)(G/L) \\ \big\downarrow{\scriptstyle R_L^G(\mu_L^G)} & & \big\downarrow{\scriptstyle \mu_L^G} \\ R_L^G(M)(L/L) & \xrightarrow{\quad\cong\quad} & M(G/L) \end{array}$$

the left vertical arrow is surjective by (1) of Definition 3.1.6. It follows that the right arrow is as well and so x is in the image of the map $U_G\mathcal{N}_L^G(S') \cong N_L^G R_L^G(M) \xrightarrow{\mu_L^G} M$. This map is a morphism of $\mathcal{O}_m$-commutative monoids by Proposition 3.1.8 so we have proven the claim. □

Proposition 3.4.3 gives the object function of a functor

$$\Phi\colon \mathrm{Comm}_{\mathcal{O}_m}(\mathrm{Mack}_{\mathcal{O}_a}) \to \mathrm{Tamb}^G_{(\mathcal{O}_m,\mathcal{O}_a)} \tag{3.13}$$

from the category of $\mathcal{O}_m$-commutative monoids in $\mathcal{O}_a$-Mackey functors to the category of Tambara functors. On morphisms, this functor sends a map of $\mathcal{O}_m$-commutative monoids to the map of underlying commutative monoids i.e. to the map of Green functors. The compatibility conditions on morphisms of $\mathcal{O}_m$-commutative monoids imply that such maps of Green functors commute with the external norms and hence are maps of Tambara functors. Since two maps of Tambara functors are the same if and only if the underlying maps of Mackey functors are the same, this functor is faithful. On the other hand, by Theorem 3.4.2, the composite $\Phi \circ U_G$ is actually the identity functor, and thus Φ is also full and surjective on objects.

Theorem 3.4.4 ([BH22, Conjecture 7.94])**.** *For any compatible indexing categories $(\mathcal{O}_m, \mathcal{O}_a)$, the functor*

$$\Phi\colon \mathrm{Comm}_{\mathcal{O}_m}(\mathrm{Mack}_{\mathcal{O}_a}) \to \mathrm{Tamb}^G_{(\mathcal{O}_m,\mathcal{O}_a)}$$

is an equivalence of categories.

3.5 Proofs of Technical Lemmas

This section contains the proofs of two lemmas used earlier in this chapter. We have deferred the proofs of these lemmas as the details are non-essential to understanding the goals of the paper and consist mostly of diagram chases and formal category theory. The first lemma is a double coset formula for the coinduction functor, used in the proof of Proposition 3.2.8. This formula is surely well-known, but we could not find a reference. The second is the proof of Lemma 3.3.6 that the maps $\alpha\colon i_H C_K^H \Rightarrow I_K^H i_K$ actually assemble into a natural transformation. We first give the proof of the double coset formula.

Lemma 3.5.1. *Let H be a finite group and suppose L and K are two subgroups of H. Let $\gamma_1, \ldots, \gamma_n$ be a collection of representatives for the double cosets $L\backslash H/K$. For any L-set X, there is an isomorphism of K-sets*

$$\operatorname{Res}_K^H \operatorname{Map}_L(H, X) \cong \prod_{\gamma_i} \operatorname{Map}_{K\cap L^{\gamma_i}}(K, \operatorname{Res}_{K\cap L^{\gamma_i}}^{L^{\gamma_i}}(c_{\gamma_i} X))$$

where $c_{\gamma_i}(X)$ is the L^{γ_i}-set with the same objects as X but and action defined by $(\gamma_i l \gamma_i^{-1}) \cdot x = lx$.

Proof. For any group G, let BG denote the category with one element and morphism set G. We adopt the convention that the composite

$$\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet$$

is $g_1 g_2$, instead of $g_2 g_1$.

The category Set^G is equivalent to the category of functors $\operatorname{Fun}(BG, \operatorname{Set})$. For any $J \leq G$, the inclusion of subcategories $i_J\colon BJ \to BG$ determines the restriction $\operatorname{Res}_J^G = i_J^*\colon \operatorname{Set}^G \to \operatorname{Set}^J$. By uniqueness of adjoints, the coinduction functor $\operatorname{Map}_J(G, -)\colon \operatorname{Set}^J \to \operatorname{Set}^G$ is isomorphic to right Kan extension functor $(i_J)_*$.

Consider the following square of groupoids

$$\begin{array}{ccc} \mathrm{Comma}(i_K, i_L) & \xrightarrow{u} & BL \\ {\scriptstyle v}\downarrow & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle i_L} \\ BK & \xrightarrow{i_K} & BH \end{array}$$

where $\mathrm{Comma}(i_K, i_L)$ is the comma category i_K/i_L and ϕ is the canonical comma natural transformation. The objects of $\mathrm{Comma}(i_K, i_L)$ are all the elements $h \in H$ and an arrow $h \to h'$ is a pair $(k, l) \in K \times L$ such that $khl^{-1} = h'$.

By Proposition 1.26 of [Gro13], the comma square is exact, meaning there is a natural isomorphism $v_*u^* \cong i_K^*(i_L)_*$. For any L-set X, we have observed that $i_K^*(i_L)_*(X) \cong \mathrm{Res}_K^H \mathrm{Map}_L(H, X)$ is the left hand side of our claimed isomorphism. It remains to identify $v_*u^*(X)$.

From the description of morphisms in $\mathrm{Comma}(i_K, i_L)$ we see that $\pi_0\mathrm{Comma}(i_k, i_L)$ is in bijection with the double cosets $K\backslash H/L$. For any γ_i, we have $\pi_1(\mathrm{Comma}(i_K, i_L), \gamma_i)$ is equal to the set of pairs (k, l) such that $k = \gamma_i l \gamma_i^{-1}$, which is naturally isomorphic to $K \cap L^{\gamma_i}$. Since every groupoid is equivalent to the disjoint union of the fundamental groups of its components, we have an isomorphism of groupoids

$$\mathrm{Comma}(i_K, i_L) \cong \coprod_{\gamma_i} B(K \cap L^{\gamma_i})$$

which gives a natural isomorphism

$$\mathrm{Fun}(\mathrm{Comma}(i_K, i_L), \mathrm{Set}) \cong \prod_{\gamma_i} \mathrm{Fun}(B(K \cap L^{\gamma_i}), \mathrm{Set}) \cong \prod_{\gamma_i} \mathrm{Set}^{B(K \cap L^{\gamma_i})} \tag{3.14}$$

After identification (3.14), the map

$$v^* \colon \mathrm{Set}^K \to \prod_{\gamma_i} \mathrm{Set}^{B(K \cap L^{\gamma_i})}$$

is the product of the restriction functors $\mathrm{Res}^K_{K \cap L^{\gamma_i}}$ and the map u^* is the product of the conjugations isomorphic $BL \cong B(L^{\gamma_i})$ followed by the restrictions $\mathrm{Res}^{L^{\gamma_i}}_{K\cap L^{\gamma_i}}$. Given any L-set X, we have computed

$$v_* u^*(X) \cong \prod_{\gamma_i} \mathrm{Map}_{K\cap L^{\gamma_i}}(K, \mathrm{Res}^{L^{\gamma_i}}_{K\cap L^{\gamma_i}}(c_{\gamma_i}X))$$

completing the proof. □

Remark 3.5.2. One can surely write down an actual K-equivariant bijection realizing the double coset formula we just proved. We have included a categorical proof of the double coset formula for two reasons. First, it offers some clarity as to why the statement is true. Secondly, and more importantly, it is thematically connected to Lemma 3.3.6, which proves the existence of a natural transformation filling an exact square.

The rest of this section is devoted to the proof of Lemma 3.3.6. Recall we have a square

$$\begin{array}{ccc} \mathcal{A}^K_{\mathcal{O}_a} & \xrightarrow{i_K} & \mathcal{P}^K_{(\mathcal{O}_m,\mathcal{O}_a)} \\ \downarrow C^H_K & \overset{\alpha}{\swarrow} & \downarrow I^H_K \\ \mathcal{A}^H_{\mathcal{O}_a} & \xrightarrow{i_H} & \mathcal{P}^H_{(\mathcal{O}_m,\mathcal{O}_a)} \end{array} \tag{3.15}$$

of functors that we are trying to show is exact. For any K-set T, the component

$$\alpha_T \colon I^H_K i_K(T) \to i_H C^H_K(T)$$

is represented by the bispan

$$H \times_K T \xleftarrow{H\times_K(\epsilon^C_T)} H \times_K \mathrm{Res}^H_K \mathrm{Map}_K(H,T) \xrightarrow{\epsilon^I_{\mathrm{Map}_K(H,T)}} \mathrm{Map}_K(H,T) \xrightarrow{=} \mathrm{Map}_K(H,T)$$

where ϵ^C and ϵ^I are the counits of the coinduction-restriction and induction-restriction adjunctions respectively. In order to keep the presentation a bit more organized, we abuse notation slightly and replace, whenever it won't cause confusion, all instances of induction, restriction, and coinduction functors by the symbols I, ρ, and C respectively so that the bispan α becomes simply

$$IT \xleftarrow{I\epsilon_T^C} I\rho CT \xrightarrow{\epsilon_{CT}^I} CT \xrightarrow{=} CT. \tag{3.16}$$

Lemma 3.5.3 (Lemma 3.3.6)**.** *The maps* $\alpha_T\colon I_K^H i_K(T) \to i_H C_K^H(T)$ *defined by* (3.16) *assemble into a natural transformation* $\alpha\colon i_H C_K^H \Rightarrow I_K^H i^K$.

Proof. We need to show that the maps α_T are sufficiently natural. Suppose we are given a morphism $\omega\colon T \to T'$ in $\mathcal{A}_{\mathcal{O}_a}^K$ which is represented by the span $T \xleftarrow{f} A \xrightarrow{g} T'$. We need to show that the square

$$\begin{array}{ccc} IT & \xrightarrow{Ii_K(\omega)} & IT' \\ \downarrow{\scriptstyle \alpha_T} & & \downarrow{\scriptstyle \alpha_{T'}} \\ CT & \xrightarrow{i_H C(\omega)} & CT' \end{array} \tag{3.17}$$

commutes in $\mathcal{P}_{(\mathcal{O}_m,\mathcal{O}_a)}^H$.

To evaluate the top right composite, fix a choice of pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{u} & \rho CT' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \epsilon^C} \\ A & \xrightarrow{g} & T' \end{array} \tag{3.18}$$

and consider the diagram

$$\begin{array}{ccccccc} IT & \xleftarrow{I(f)} & IA & \xrightarrow{=} & IA & \xrightarrow{I(g)} & IT' \\ & & & & \uparrow{\scriptstyle I(v)} & (1) & \uparrow{\scriptstyle I(\epsilon^C)} \\ & & & & IP & \xrightarrow{I(u)} & I\rho CT' \\ & & & \nearrow{\scriptstyle F} & & (2) & \downarrow{\scriptstyle \epsilon^I} \\ & & I\rho CA & \xrightarrow{\epsilon^I} & CA & \xrightarrow{Cg} & CT' \\ & & & & & & \downarrow \\ & & & & & & CT' \end{array} \tag{3.19}$$

where the square (1) is the result of applying $H \times_K (-)$ to the pullback (3.18) and is thus a pullback. We claim the trapezoid (2) is an exponential diagram for the composable arrows

$$IP \xrightarrow{I(u)} I\rho CT' \xrightarrow{\epsilon^I} CT'$$

and that the map F is such that $I(v) \circ F = I(\epsilon^C_A)$. We defer the proof of both claims to Lemma 3.5.4 below.

The diagram (3.19), and the composition laws in the category $\mathcal{P}^H_{(\mathcal{O}_m, \mathcal{O}_a)}$ tells us the composite $\alpha_{T'} \circ I^H_K i_K(\omega)$ is the bispan

$$IT \xleftarrow{I(f \circ \epsilon^C)} I\rho CA \xrightarrow{\epsilon^I} CA \xrightarrow{C(g)} CT' \tag{3.20}$$

We compute left-bottom composite of (3.17) using the diagram

$$\begin{array}{ccccccc}
IT & & & & & & \\
\uparrow{\scriptstyle I(\epsilon^C)} & & & & & & \\
I\rho CT & \xleftarrow{I\rho C(f)} & I\rho CA & & & & \\
\downarrow{\scriptstyle \epsilon^I} & (4) & \downarrow{\scriptstyle \epsilon^I} & & & & \\
CT & \xleftarrow{C(f)} & CA & & & & \\
\downarrow{\scriptstyle =} & (3) & \downarrow{\scriptstyle =} & & & & \\
CT & \xleftarrow{C(f)} & CA & \xrightarrow{=} & CA & \xrightarrow{C(g)} & CT'
\end{array} \tag{3.21}$$

in which squares (3) and (4) are both pullbacks. To see that (4) is a pullback, note that for any $X \in \mathrm{Set}^H$, we have an isomorphism $I\rho(X) \cong H/K \times X$ and the morphism $\epsilon^I \colon I\rho(X) \to X$ is the projection map.

The diagram (3.21), and the composition laws in $\mathcal{P}^H_{(\mathcal{O}_m, \mathcal{O}_a)}$ tells us the composite $i_H C(\omega) \circ \alpha_T$ is the bispan represented by

$$IT \xleftarrow{I(\epsilon^C \circ \rho C(f))} I\rho CA \xrightarrow{\epsilon^I} CA \xrightarrow{C(g)} CT'$$

which, by the naturality of ϵ^C, is equal to (3.20). □

Lemma 3.5.4. *The trapezoid (2) from the proof of Lemma 3.5.3 is an exponential diagram. The composite $I(v) \circ F$ is equal to $I(\epsilon_A^C)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} Q & \xrightarrow{\gamma} & CP & \xrightarrow{C(v)} & CA \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle C(u)} & & \downarrow{\scriptstyle C(g)} \\ CT' & \xrightarrow{\eta^C} & C\rho CT' & \xrightarrow{C(\epsilon^C)} & CT' \end{array}$$

in which Q is chosen so that left square is a pullback. Since the right square is C applied to the pullback square (3.18), and coinduction preserves pullbacks, the outside rectangle is a pullback. Moreover, since the composite along the bottom is the identity by a triangle identity, we could have chosen Q so that $Q = CA$, $\delta = C(g)$, and $C(v) \circ \gamma = \mathrm{id}_{CA}$.

Lemma 2.3.5 of [Hoy14] says precisely that there is an exponential diagram

$$\begin{array}{ccccc} & & I\rho Q & \xrightarrow{\epsilon^I} & Q \\ & \swarrow{\scriptstyle I(\widehat{\gamma})} & & & \downarrow{\scriptstyle \delta} \\ IP & \xrightarrow[I(u)]{} & I\rho CT' & \xrightarrow[\epsilon^I]{} & CT' \end{array}$$

where $\widehat{\gamma}$ is the adjunct of γ along the restriction-coinduction adjunction. Taking $F = I(\widehat{\gamma})$ proves the first claim.

To prove $I(v) \circ F = I(\epsilon_A^C)$, it suffices to show that $v \circ \widehat{\gamma} = \epsilon_A^C$. By definition, $\widehat{\gamma}$ is equal to the composite $\epsilon_P^C \circ \rho(\gamma)$ and we have

$$v \circ \widehat{\gamma} = v \circ \epsilon_P^C \circ \rho(\gamma) = \epsilon_A^C \circ \rho C(v) \circ \rho(y) = \epsilon_A^C \circ \mathrm{id}_{\rho CA} = \epsilon_A^C$$

where the second equality uses naturality of ϵ^C and the third follows from our choice of γ. □

Chapter 4

Categorical Mackey functors

In Definition 3.1.1, we defined symmetric monoidal Mackey functors as collections of symmetric monoidal categories which are linked by strong monoidal functors. These functors satisfy some relations, though the exact nature of where these relations come from and exactly how natural they should be is a bit opaque. In this chapter we revisit the definition of symmetric monoidal Mackey functors and show how all of our examples of interest arise from a variant which we call a *categorical Mackey functor*. We should note that the categorical Mackey functors we describe here are different from those considered in [BO15], though those examples are special cases of ours.

There are several advantages to using categorical Mackey functors. First, the coherence data of the strong monoidal functors becomes apparent. Second, we show in Section 4.3 how to efficiently construct examples. Finally, in Theorem 4.5.3, we use this new perspective to shed some light on the definition of G-commutative monoids, proving a new characterization of these objects via a universal property.

The rest of this chapter is organized as follows. Section 4.1 contains a review of the theory of bicategories which will be essential in the remainder of the chapter. In Section 4.2 we give the definition of categorical Mackey functors and observe that every categorical Mackey functor determines a symmetric monoidal Mackey functor. In Section 4.3 we provide tools for constructing new categorical Mackey functors. We conclude by studying a universal property for G-commutative monoids in Section 4.5.

4.1 A crash course in bicategories

In this section we review the notion of a bicategory and discuss the examples of the bicategorical Burnside category and Cat, the bicategory of small 1-categories.

Definition 4.1.1. A bicategory $\mathcal{B}$ consists of a collection of objects $\mathrm{ob}(\mathcal{B})$ together with categories $\mathcal{B}(x, y)$ for each pair of objects $x, y \in \mathrm{ob}(\mathcal{B})$. The objects of the categories $\mathcal{B}(x, y)$ are called 1-morphisms and are denoted by single stemmed arrows $f\colon x \to y$. Morphisms between f and g in $\mathcal{B}(x, y)$ are denoted by double stemmed arrows $\alpha\colon f \Rightarrow g$. We refer to the morphisms in $\mathcal{B}(x, y)$ as 2-morphisms and their composition as vertical composition.

In addition to the above, we have the following:

1. (Horizontal Composition) For $x, y, z \in \mathrm{ob}\mathcal{B}$, a horizontal composition bi-functor

$$c_{x,y,z}\colon \mathcal{B}(y, z) \times \mathcal{B}(x, y) \to \mathcal{B}(x, z)$$

   Rather than write $c_{x,y,z}$, we often denote horizontal composition by concatenation of 1-morphisms.

2. (Unitors) For all $b \in \mathcal{B}$, an object $1_b \in \mathcal{B}(b, b)$ and specified natural isomorphisms

$$c_{a,b,b}(1_b, -) \cong \mathrm{id}_{\mathcal{C}_{a,b}} \cong c_{a,a,b}(-, 1_a)$$

3. (Associators) Natural isomorphisms

$$\alpha_{w,x,y,z}\colon c_{w,y,z} \circ (\mathrm{id}_{\mathcal{B}(y,z)} \times c_{w,x,y}) \Rightarrow c_{w,x,z} \circ (c_{x,y,z} \times \mathrm{id}_{\mathcal{B}(w,x)})$$

The associators and unitors are required to satisfy some coherence axioms that we do not include here. The interested reader can find these diagrams in [JY21, Section 2.1].

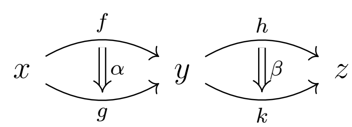
Sometimes we will refer to the objects of a bicategory as 0-cells, the 1-morphisms as 1-cells and the 2-morphisms as 2-cells. This terminology is especially helpful when visualizing diagrams.

Remark 4.1.2. The term bifunctor in (1) of the definition above means if we fix a particular $f \in \mathcal{B}(y, z)$ then

$$c_{x,y,z}(f, -)\colon \mathcal{B}(x, y) \to \mathcal{B}(x, z)$$

is a functor. For α a 2-cell in $\mathcal{B}(x, y)$ we write $c_{x,y,z}(f, \alpha)$ as $f \cdot \alpha$ and this element is called the *whiskering* of α with f. Similarly, for $g \in \mathcal{B}(x, y)$ and β a 2-morphism of $\mathcal{B}(y, z)$ we have the whiskering $\beta \cdot g$.

Whiskering satisfies certain coherence laws with respect to the vertical composition. Rather than write down the coherence explicitly, let us just say any sufficiently simple planar diagram of 0, 1 and 2 cells has exactly one vertical composition. For instance, in the diagram



we have $(\beta \cdot g) \circ (h \cdot \alpha) = (k \cdot \alpha) \circ (\beta \cdot f)$. Full details on the coherence of pasting diagrams in bicategories can be found in [JY21, Theorem 3.6.6].

Bicategories are just like regular categories, except that almost every statement only holds up to an invertible 2-cell. For example, two objects $x, y \in \mathcal{B}$ are *equivalent* if there are 1-morphisms $f \colon x \to y$ and $g \colon y \to x$ so that $fg \cong 1_y$ and $gf \cong 1_x$. Note that g is not the unique "inverse" to f, although any two inverses are equivalent.

Example 4.1.3. We write Cat for the bicategory of small categories. The objects are small 1-categories and for any categories x and y, the morphism category $\mathrm{Cat}(x, y)$ is the category of functors and natural transformations from x to y. Horizontal and vertical composition are given by composition of functors and natural transformations. The associators and unitors are identities.

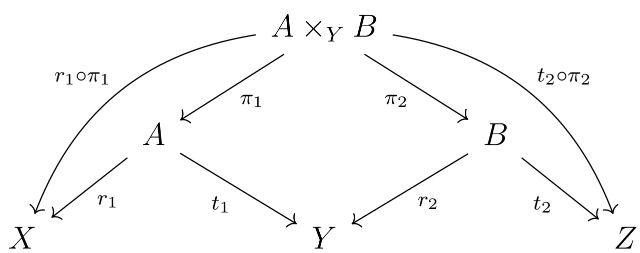
Example 4.1.4. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. Then there is a bicategory $\mathcal{BC}$ with a single 0-cell $*$ and $\mathcal{BC}(*) = \mathcal{C}$. The horizontal composition in $\mathcal{BC}$ is the monoidal product $\otimes$ and the associators are the associativity data of the monoidal category. The vertical composition is the usual composition in $\mathcal{C}$. One of the motivating reasons for studying bicategories is that in fact monoidal categories are equivalent to bicategories with a single object. Thus bicategories can be thought of as monoidal categories (or rings) with many objects.

Example 4.1.5. The Burnside bicategory $\mathcal{A}^G$ of a finite group G is the bicategory whose objects are finite G-sets and morphisms $\mathcal{A}^G(X,Y)$ are spans – not isomorphism classes of spans –

$$X \xleftarrow{r} A \xrightarrow{t} Y$$

where r, t are equivariant maps of G-sets. A 2-morphism between spans $X \xleftarrow{r} A \xrightarrow{t} Y$ and $X \xleftarrow{r'} A' \xrightarrow{t'} Y$ is a G-equivariant map $f\colon A \to A'$ so that the obvious diagram commutes. Vertical composition is by composition of G-equivariant maps.

Horizontal composition is given by pullback of G-sets, i.e. the composition of $X \xleftarrow{r_1} A \xrightarrow{t_1} Y$ and $Y \xleftarrow{r_2} B \xrightarrow{t_2} Z$ is the span along the top of



where the middle square is the pullback. It is important we make a specific choice of pullback so this is well defined. We make the choice:

$$A \times_Y B = \{(a,b) \in A \times B \mid t_1(a) = r_2(b)\}.$$

Note this pullback construction is not strictly associative, and so we need associators which are given by the obvious isomorphism $(A \times_Y B) \times_Z C \cong A \times_Y (B \times_Z C)$ whenever necessary.

Finally, given three spans

$$\omega_1 = [X \xleftarrow{r} A \xrightarrow{t} Y]$$
$$\omega_2 = [X \xleftarrow{r'} B \xrightarrow{t'} Y]$$
$$\omega_3 = [Y \xleftarrow{r''} C \xrightarrow{t''} Z]$$

and a 2-cell $\gamma\colon \omega_1 \Rightarrow \omega_2$ realized by a map $\gamma\colon A \to B$, the whiskering is $\omega_3 \cdot \gamma\colon \omega_3\omega_1 \Rightarrow \omega_3\omega_2$ is realized by the map $\gamma \times_Y \mathrm{id}_C \colon A \times_Y C \to B \times_Y C$.

4.1.1 Pseudo-functors

Just as we do for ordinary categories, we often want a way of comparing different bicategories through some kind of functor. Since the composition of 1-cells in a bicategory is not actually associative, our functors will need to take this into account as a part of their data. This leads naturally to the notion of a *pseudo-functor*.

Definition 4.1.6. A *pseudo-functor* $F \colon \mathcal{B} \to \mathcal{C}$ between two bicategories consists of

1. an object function, $F \colon \mathrm{ob}(\mathcal{B}) \to \mathrm{ob}(\mathcal{C})$,
2. functors $F_{x,y} \colon \mathcal{B}(x, y) \to \mathcal{C}(Fx, Fy)$ for any $x, y \in \mathcal{B}$,
3. specified isomorphisms $\mu_{f,g} \colon F(g)F(f) \overset{\simeq}{\Rightarrow} F(gf)$ for any composable 1-cells f and g,
4. and isomorphisms $1_{F(x)} \overset{\simeq}{\Rightarrow} F_{x,x}(1_x)$.

The composition and unit isomorphisms must satisfy certain compatibility with the associators and unitors of the bicategories $\mathcal{B}$ and $\mathcal{C}$. Full details can be found in [JY21, Definition 4.1.2]

Pseudo-functors describe situations in which composition of morphisms is not strictly associative. Such constructions are particularly common when morphisms in either the target or source category are not described by functions on sets, such as the Burnside category. They are also fairly common even when working with concrete and elementary constructions.

Example 4.1.7. Let G be a finite group and let $\mathcal{C}$ denote the subgroup poset of G, considered as a 1-category in the usual way. We would like to define a functor $\mathcal{C} \to \mathrm{Cat}_1$, the category of 1-categories and functors, that sends $H \leq G$ to the category Set^H of finite H-sets. Our functor should send a morphism $K \leq H$ in $\mathcal{C}$ to the induction functor $H \times_K (-) \colon \mathrm{Set}^K \to \mathrm{Set}^H$. However, this is not a functor, because for any chain $L \leq K \leq H$ and any L-set S

$$H \times_K (K \times_L S) \neq H \times_L S.$$

While the equation above does not hold as a strict equality, it does hold up to natural isomorphism and one can realize the desired functor as a pseudo-functor between bicategories.

Example 4.1.8. If X is a topological space, let $\mathrm{Vect}(\mathrm{X})$ denote the category of real, finite dimensional vector bundles over X. If $f\colon X \to Y$ is continuous and $E \to Y$ is a vector bundle, then we can construct a pullback bundle $f^*(E) \to X$. This describes a functor $f^*\colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ and one can ask if this gives a contravariant functor from topological spaces into the category of categories. The answer to this question depends on the choice of pullbacks, but the usual choice of pullbacks does not work since it is not associative. It is reasonably straightforward, however, to turn this construction into a contravariant pseudo-functor.

Remark 4.1.9. Every category $\mathcal{C}$ determines a bicategory $\widehat{\mathcal{C}}$ with the same objects and 1-morphisms as $\mathcal{C}$, and only identity 2-cells. This allows us to consider 1-categories as particular examples of bicategories and thus to make sense of pseudo-functors between 1-categories and arbitrary bicategories. Of course, this is of the most interest when the target is not a 1-category.

4.1.2 Products in bicategories

As with ordinary functors between 1-categories, we can make sense of pseudo-limits of pseudo-functors $F\colon \mathcal{B} \to \mathcal{C}$. The whole definition (see [Bor94, Chatper 7]) is a bit unwieldy, but in the case of products we can be quite explicit. The interested reader can check the definition we give below is equivalent to the notion of a pseudo-limit of the pseudo-functor $F\colon \underline{2} \to \mathcal{B}$ where $\underline{2}$ is the bicategory with 2 objects and only identity morphisms.

Definition 4.1.10. Let $\mathcal{B}$ be a bicategory and $A, B \in \mathcal{B}$. A *product* of A and B in $\mathcal{B}$, if it exists, is an object P together with maps $\pi_A\colon P \to A$ and $\pi_B\colon P \to B$ such that if P' is some other object of $\mathcal{B}$, equipped with 1-morphisms $\pi'_A\colon P' \to A$ and $\pi'_B\colon P' \to B$, then there is a 1-morphism $u\colon P' \to P$ such that $\pi'_A \cong \pi_A \circ u$ and $\pi'_B \cong \pi_B \circ u$. Moreover, u is unique up to a unique 2-isomorphism, in the sense that for any other choice $v\colon P' \to P$ and 2-isomorphisms $\alpha\colon \pi_A u \Rightarrow \pi_A v$ and $\beta\colon \pi_B u \Rightarrow \pi_B v$, there is a unique 2-isomorphism $\gamma\colon u \Rightarrow v$ so that $\pi_A \cdot \gamma = \alpha$ and $\pi_B \cdot \gamma = \beta$.

Example 4.1.11. We consider the bicategory Cat of 1-categories, functors, and natural transformations. A product here is given by the usual product of 1-categories. Explicitly, the product of categories A and B is the category $A \times B$ with objects pairs (a, b) of objects in A and B. A morphism $(f, g)\colon (a, b) \to (a', b')$ is a pair of morphisms $f\colon a \to a'$ and $g\colon b \to b'$ in A and B.

Lemma 4.1.12. *Let $\mathcal{A}^G$ be the Burnside bicategory of a finite group G and let A, B be finite G-sets. Then the disjoint union $A \amalg B$ is a product of A and B in $\mathcal{A}^G$.*

Proof. The projection spans $\pi_A\colon A \amalg B \to A$ and $\pi_B\colon A \amalg B \to B$ are given by

$$\pi_A = [A \amalg B \xleftarrow{i_A} A \xrightarrow{=} A]$$
$$\pi_B = [A \amalg B \xleftarrow{i_B} B \xrightarrow{=} B]$$

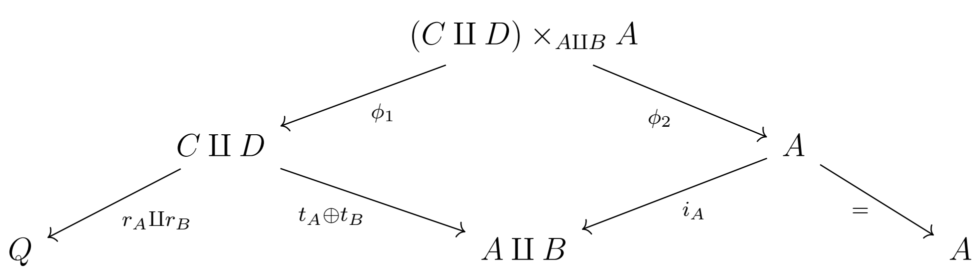
where the left arrow in each span is the obvious inclusion. Given any other element $P \in \mathcal{A}^G$ and spans

$$p_A = [Q \xleftarrow{r_A} C \xrightarrow{t_A} A]$$
$$p_B = [Q \xleftarrow{r_B} D \xrightarrow{t_B} B]$$

we can define a span $u\colon Q \to A \amalg B$ by

$$u = [Q \xleftarrow{r_A \amalg r_B} C \amalg D \xrightarrow{t_A \oplus t_B} A \amalg B]$$

The composite $\pi_A u$ is computed as



where the inner diamond is a pullback in G-sets. We define an isomorphism

$$f_A\colon C \to (C \amalg D) \times_{A\amalg B} A$$

by $f_A(c) = (c, t_A(c))$ for $c \in C$. One readily checks that $(r_A \amalg r_B) \circ \phi_1 \circ f_A = r_A$ and $\phi_2 \circ f_A = t_A$ so that $\pi_A \circ u$ is isomorphic, via the 2-cell f_A, to p_A as desired. A similarly defined f_B will show that $\pi_B \circ u$ is isomorphic to p_B.

It remains to show the 2-uniqueness of $u\colon Q \to A \amalg B$. Suppose

$$v = [Q \xleftarrow{r} E \xrightarrow{t} A \amalg B]$$

is a span such that there are 2-isomorphisms $\alpha\colon \pi_A u \Rightarrow \pi_A v$ and $\beta\colon \pi_B u \Rightarrow \pi_B v$. We need to construct a 2-cell $\gamma\colon u \Rightarrow v$ so that $\pi_A \cdot \gamma = \alpha$ and $\pi_B \cdot \gamma = \beta$. The 2-cell α is realized by an isomorphism of G-sets $\overline{\alpha}\colon (C \amalg D) \times_{A\amalg B} A \to E \times_A A$. Since every element of the domain is of the form $(c, t_A(c))$ for some $c \in C$, we see that $\overline{\alpha}$ is given $\alpha(c, t_A(c)) = (g_A(c), t_A(c))$ for some equivariant function $g_A\colon C \to E$. Similarly, there is a G-map $g_B\colon D \to E$ that determines β.

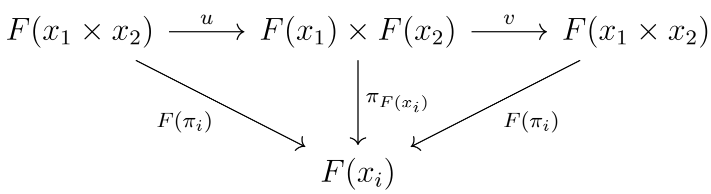
We define $\gamma\colon C \amalg D \to E$ to be the coproduct $g_A \amalg g_B$. One can easily check that $\gamma \times_{A\amalg B} A = \alpha$ and $\gamma \amalg_{A\amalg B} B = \beta$. Moreover, γ is the unique map with this property, by the universal property of the coproduct. □

Corollary 4.1.13. *The Burnside bicategory is generated under products by the transitive G-sets G/H, in the sense that every object in $\mathcal{A}^G$ is isomorphic to a finite product of elements of this form.*

Going forward, we will interested in pseudo-functors which preserve products.

Definition 4.1.14. A pseudo-functor $F\colon \mathcal{B} \to \mathcal{C}$ preserves products if for any product $x_1 \times x_2$ in $\mathcal{B}$, with projections $\pi_i\colon x_1 \times x_2 \to x_i$, the image $F(x_1 \times x_2)$ is a product with projections $F(\pi_i)$.

Suppose $F\colon \mathcal{B} \to \mathcal{C}$ preserves products. It follows from the definition that there must be morphisms $u\colon F(x_1 \times x_2) \to F(x_1) \times F(x_2)$ and $v\colon F(x_1) \times F(x_2) \to F(x_1 \times x_2)$ so that the following diagram commutes up to some specified invertible 2-cells for $i = 1, 2$.

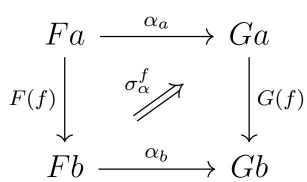


It follows from Definition 4.1.10 that vu and uv are uniquely isomorphic to the identities. In the case where $\mathcal{C} = \mathrm{Cat}$, u and v form an adjoint equivalence of categories in the usual sense.

Definition 4.1.15. If $F, G\colon \mathcal{B} \to \mathcal{C}$ are two pseudo-functors between bicategories, a pseudo-natural transformation $\alpha\colon F \Rightarrow G$ consists of:

1. 1-cells $\alpha_b\colon F(b) \to G(b)$ for all $b \in \mathcal{B}$,

2. For all 1-cells $f\colon a, b$ in $\mathcal{B}$, an invertible 2-cell σ_α^f in $\mathcal{C}$ filling the following square:



We also require that all reasonable diagrams involving the σ_α^f and the associators and unitors for F and G commute. Again, we refer the reader to Johnson–Yau for the full details [JY21, Section 4.2].

Lemma 4.1.16. *The collection of bicategories, pseudo-functors, and pseudo-natural transformations assembles into a bicategory, denoted* BiCat.

4.1.3 (2,1)-bicategories

In this subsection we highlight a particularly nice class of bicategories.

Definition 4.1.17. A $(2, 1)$-bicategory is a bicategory in which every 2-cell is an isomorphism.

Remark 4.1.18. The choice of notation $(2,1)$-bicategory is probably not illuminating to those encountering higher categories for the first time. A bicategory is a particular version of a 2-category, meaning that it is like a category except that morphisms between two objects are also a category. One could inductively define an n-category which is like a category except that the morphisms between any two objects form an $(n-1)$-category. In this context, an (n,k)-category is an n-category such that every morphism from dimension $k+1$ to n is invertible.

Example 4.1.19. Every bicategory $\mathcal{B}$ admits a sub-bicategory $\mathcal{B}_{(2,1)}$ which is a $(2,1)$-bicategory. $\mathcal{B}_{(2,1)}$ has the same objects and 1-cells as $\mathcal{C}$, but only the isomorphism 2-cells. An important example going forward will be $\mathcal{A}^G_{(2,1)}$, which we refer to as the *truncated Burnside bicategory*.

Example 4.1.20. Recall from Remark 4.1.9 that every category $\mathcal{C}$ determines a bicategory $\widehat{\mathcal{C}}$ with only identity 2-cells. Evidently $\widehat{\mathcal{C}}$ is always a $(2,1)$-bicategory.

Example 4.1.19 provides a nice example of a pseudo-functor which preserves products.

Lemma 4.1.21. *A bicategory $\mathcal{B}$ admits products if and only if $\mathcal{B}_{(2,1)}$ admits products. Moreover, the inclusion pseudo-functor $\mathcal{B}_{(2,1)} \to \mathcal{B}$ is product preserving.*

Proof. The only 2-cells appearing in the definition of a product are isomorphisms and thus all the data of a product in $\mathcal{B}$ determines a product in $\mathcal{B}_{(2,1)}$ and vice versa. □

We end this section with a discussion of opposite categories. Since there are two kinds of morphisms in a bicategory there are several things one might mean by the opposite category depending on whether one wishes to invert 1-cells, 2-cells, or both. The next definition sets notation for these choices.

Definition 4.1.22. Let $\mathcal{B}$ be a bicategory. We define the following dual categories:

1. the op-dual $\mathcal{B}^{op}$ has the same 0-cells and 2-cells as $\mathcal{B}$ but $\mathcal{B}^{op}(x,y) = \mathcal{B}(y,x)$.

2. the co-dual $\mathcal{B}^{co}$ has the same 0-cells and 1-cells as $\mathcal{B}$ but $\mathcal{B}^{co}(x,y) = \mathcal{B}(x,y)^{op}$.

3. the coop-dual $\mathcal{B}^{coop}$ has the same 0-cells as $\mathcal{B}$ but $\mathcal{B}^{coop}(x,y) = \mathcal{B}(y,x)^{op}$.

We make further use of the co-dual of a bicategory in Section 4.3. For now though, we finish this section with the following observation.

Lemma 4.1.23. *Let $\mathcal{B}$ be a bicategory. If $\mathcal{B}$ is a $(2,1)$-bicategory then $\mathcal{B}$ is isomorphic to $\mathcal{B}^{co}$.*

Proof. The functor $\mathcal{B} \to \mathcal{B}^{co}$ which is the identity on 0-cells and 1-cells but sends every two cell to its inverse is an isomorphism of bicategories. □

4.1.4 Adjoints in bicategories

The data of an adjunction between functors can be phrased, using units and counits, in terms of just functors and natural transformations. That is, we can think about adjunctions as particular relations which exist between 1-cells and 2-cells in the bicategory Cat of small 1-categories. This allows us to make sense of adjunctions in arbitrary bicategories.

Definition 4.1.24. Suppose $f\colon b \to c$ is a 1-cell in a bicategory $\mathcal{B}$. A 1-cell $g\colon c \to b$ is a *left adjoint* of f if there exist 2-cells $\epsilon\colon gf \to 1_b$ and $\eta\colon 1_c \to fg$ such the following diagrams commute:

$$\begin{array}{ccccc} g \circ 1_c & \xrightarrow{g\cdot\eta} & g \circ (f \circ g) & \longrightarrow & (g \circ f) \circ g \\ \downarrow & & & & \downarrow{\scriptstyle \epsilon\cdot g} \\ g & & \longrightarrow & & 1_b \circ g \end{array}$$

$$\begin{array}{ccccc} 1_c \circ f & \xrightarrow{\eta\cdot f} & (f \circ g) \circ f & \longrightarrow & f \circ (g \circ f) \\ \downarrow & & & & \downarrow{\scriptstyle f\cdot\epsilon} \\ f & & \longrightarrow & & f \circ 1_b \end{array}$$

where the unlabeled morphisms are unitors and associators.

Example 4.1.25. In the bicategory Cat, an adjoint pair is simply an adjoint pair of functors. After replacing the unitors and associators with the appropriate identities, the coherence diagrams become the usual triangle identities.

Lemma 4.1.26 ([JY21, Proposition 6.1.7])**.** *Suppose $f\colon b \to c$ is a 1-cell in a bicategory $\mathcal{B}$ with left adjoint $g\colon c \to b$. For any pseudo-functor $F\colon \mathcal{B} \to \mathcal{C}$, the 1-cell $F(g)$ is a left adjoint of $F(f)$.*

Lemma 4.1.27. *For any map of finite G-sets $f\colon S \to T$, the transfer T_f is a left adjoint of R_f, in the bicategory $\mathcal{A}^G$.*

Proof. The spans $R_f \circ T_f$ and $T_f \circ R_f$ are given by

$$R_f \circ T_f = [S \xleftarrow{\pi_1} S \times_T S \xrightarrow{\pi_2} S]$$
$$T_f \circ R_f = [T \xleftarrow{f} S \xrightarrow{f} T]$$

The unit 2-cell $\eta\colon 1_S \Rightarrow R_f T_f$ is given by the diagonal $S \to S \times_T S$, while the counit $\epsilon\colon T_f R_f \Rightarrow 1_T$ is given by $f\colon S \to T$. It is straightforward to check the necessary coherence. □

Corollary 4.1.28. *If $\mathcal{C}\colon \mathcal{A}^G \to \mathrm{Cat}$ is a pseudo-functor then there is an adjunction $\mathcal{C}(T_f) \dashv \mathcal{C}(R_f)$ for any morphism f of finite G-sets.*

4.2 Categorical Mackey functors

In this section we lay out the definition of categorical Mackey functors and discuss their basic properties.

Definition 4.2.1. A *categorical Mackey functor* is a product preserving pseudo-functor

$$M\colon \mathcal{A}^G_{(2,1)} \to \mathrm{Cat}\,.$$

A morphism $\alpha\colon M \to N$ of categorical Mackey functors is a pseudo-natural transformation α which respects products in the sense that the following diagram commutes for any $A, B \in \mathcal{A}^G_{(2,1)}$

$$\begin{array}{ccc} M(A \amalg B) & \xrightarrow{\cong} & M(A) \times M(B) \\ \downarrow{\scriptstyle \alpha_{A\amalg B}} & & \downarrow{\scriptstyle \alpha_A \times \alpha_B} \\ N(A \amalg B) & \xrightarrow{\cong} & N(A) \times N(B) \end{array}\,.$$

Before giving any examples, we clarify the data that must be given to determine such an object.

Given any $X \in \mathcal{A}^G$, Corollary 4.1.13 says there exists subgroups $H_i \le G$ so that

$$X \cong \coprod_{i=1}^{n} G/H_i \tag{4.1}$$

It follows that given a categorical Mackey functor M, we must have isomorphisms of categories:

$$M(X) \cong \prod_{i=1}^{n} M(G/H_i). \tag{4.2}$$

In particular, up to categorical equivalences, M is determined by its values on the G/H. In principle one can use this to define our Mackey functor by specifying its values on the orbits G/H, fixing isomorphisms of the form of (4.1) and defining $M(X)$ by taking the equivalence (4.2) to be the identity. Of course, to make this precise we need to specify some coherence data. The next proposition enumerates how much data we need.

Proposition 4.2.2. *The following data determines a categorical Mackey functor:*

1. *symmetric monoidal categories $M(G/H)$ for each $H \le G$,*
2. *strong monoidal restriction functors $R^H_K \colon M(G/H) \to M(G/K)$ for each $K \le H \le G$,*
3. *strong monoidal transfer functors $T^H_K \colon M(G/K) \to M(G/H)$ for each $K \le H \le G$,*
4. *strong monoidal conjugation functors $c_g \colon M(G/H) \to M(G/H^g)$ for all $g \in G$,*
5. *natural isomorphisms $R^K_L R^H_K \cong R^H_K$ and $T^H_K T^K_L \cong T^H_L$ for any chain $L \le K \le H$,*
6. *equalities $c_{g_1} c_{g_2} = c_{g_1 g_2}$,*
7. *The following double coset isomorphism for $J, K \le H$:*

$$R^H_J T^H_K \cong \bigoplus_{\gamma \in K\backslash H/J} T^J_{J\cap K^\gamma} R^{K^\gamma}_{J\cap K^\gamma} c_\gamma$$

where $\oplus$ is the monoidal product of $M(J)$,

8. *T^H_H, R^H_H are both the identity,*

9. *c_g is the identity whenever $g \in H$,*

10. *for any $J \leq L \leq K \leq H$, the two ways of applying (5) to get from $T^H_K T^K_L T^L_J$ to T^H_J are the same. A similar statement holds for restrictions,*

11. *equalities $c_g T^H_K = T^{H^g}_{K^g} c_g$ and $c_g R^H_K = R^{H^g}_{K^g} c_g$. That is, the c_g are natural transformations between transfers and restrictions.*

Proof. Consider pairs (n, α) where n is a non-negative integer and $\alpha\colon G \to \Sigma_n$. Such a pair determines a G-action on the ordered set $\{1, \ldots, n\}$ and we identify the pair (n, α) with the resulting G-set. Without loss of generality we may assume that every object of $\mathcal{A}^G$ is of this form. Let $X \subset (n, \alpha)$ be an orbit. We write $X^{\min}$ for the minimum element of X in the ordering on $\underline{n}$. For any G-set (n, α), we define a total ordering on collection of orbits $X_1, \ldots, X_k$ of (n, α) by declaring $X_i \leq X_j$ whenever $X_i^{\min} < X_j^{\min}$.

We fix a G-set (n, α) and let $X_1, \ldots, X_k$ be the orbits, ordered as above. Given the data from the statement, we define a categorical Mackey functor $\mathcal{M}$ on the object (n, α) by

$$\mathcal{M}(n, \alpha) = \prod_{i=1}^{k} M(G/H_i)$$

where $H_i = \mathrm{Stab}(X_i^{\min})$. It remains to check that this extends to an actual pseudo-functor.

We first define the transfers. Let (m, β) be another G-set and let $Y_1, \ldots Y_j$ be its orbits, ordered as above, and write K_i for the stabilizer of $Y_i^{\min}$. Any G-map $f\colon (n, \alpha) \to (m, \beta)$ determines a map $\phi_f\colon \underline{k} \to \underline{j}$ defined by $f(X_i) = Y_{\phi(i)}$. Pick a $g_i \in G$ such that $g_i Y^{\min}_{\phi(i)} = f(X_i^{\min})$ so that $H_i \subset K^{g_i}_{\phi(i)}$ and define

$$\mathcal{M}(T_f)\colon \prod_{i=1}^{k} M(G/H_i) \to \prod_{\ell=1}^{j} M(G/K_i)$$

to be the "matrix" whose $(i, \phi(i))$ entry is $c_{g_i^{-1}} T_{H_i}^{K^{g_i}_{\phi(i)}}$ with all other entries 0. Since any two choices of g_i differ by an element in $K_{\phi(i)}$, this is well defined by (6) and (9).

For restrictions along f, we let

$$\mathcal{M}(R_f)\colon \prod_{i=1}^{j} M(G/K_i) \to \prod_{i=1}^{k} M(G/H_i)$$

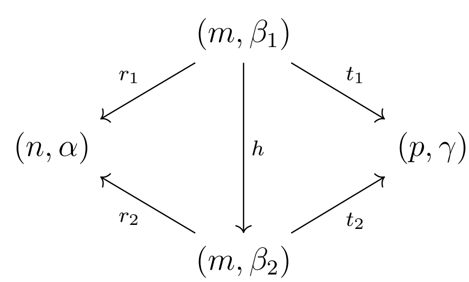
be the matrix whose $(\phi(i), i)$ entry is $R_{H_i}^{K_{\phi(i)}^{g_i}} c_{g_i}$ and all other entries are zero.

On any arbitrary span $\omega = [(n,\alpha) \xleftarrow{r} (m,\beta) \xrightarrow{t} (p,\gamma)]$ define

$$\mathcal{M}(\omega) = \mathcal{M}(R_r)\mathcal{M}(T_t)$$

and let the associators be given by the data of (6), (6), and (7). Using (8) and (9) we can choose the unitors to be identities. The fact that these associators and unitors form a pseudo-functor is the data of (10) and (11). All that remains is to define our functor on 2-cells.

Let $\omega_i = [(n,\alpha) \xleftarrow{r_i} (m_i,\beta_i) \xrightarrow{t_i} (p,\gamma)]$ be any two spans in $\mathcal{A}^G_{(2,1)}$. Since all 2-cells of $\mathcal{A}^G_{(2,1)}$ are invertible, there is a 2-cell $h\colon \omega_1 \Rightarrow \omega_2$ if and only if $m_1 = m_2 = m$ and we have a commutative diagram of G-sets:



Since $h\colon \underline{m} \to \underline{m}$ is an isomorphism, $\mathcal{M}(T_f)$ consists only of index shuffling on the product of categories and various conjugations functors c_g. By the definition of $\mathcal{M}$ on transfer and restriction morphisms, and using the fact that conjugations are assumed to commute with the other data strictly by (6) and (11), we have

$$\mathcal{M}(T_h)\mathcal{M}(R_{r_1}) = \mathcal{M}(R_{r_2}) \quad \text{and} \quad \mathcal{M}(T_{t_2})\mathcal{M}(T_h) = \mathcal{M}(T_{t_1}).$$

By (6) we have that $\mathcal{M}(T_h) \circ \mathcal{M}(T_{h^{-1}})$ is the identity functor, which implies

$$\mathcal{M}(\omega_1) = \mathcal{M}(T_{g_1})\mathcal{M}(R_{f_1}) = \mathcal{M}(T_{g_1})\mathcal{M}(T_h)\mathcal{M}(T_{h^{-1}})\mathcal{M}(R_{f_1}) = \mathcal{M}(T_{g_2})\mathcal{M}(R_{f_2}) = \mathcal{M}(\omega_2)$$

and so for any two cell $h\colon \omega_1 \Rightarrow \omega_2$ we have $\mathcal{M}(\omega_1) = \mathcal{M}(\omega_2)$ so we can just take $\mathcal{M}(h)$ to be the identity two cell. □

Remark 4.2.3. The proof of the preceding proposition relies heavily on the fact that the conjugation functors c_g are assumed to strictly commute with everything in sight. While this is a bit "evil," it seems to be a reasonable assumption in examples.

Remark 4.2.4. The data of Proposition 4.2.2 is quite bulky. In many cases, it is infeasible, or at least undesirable, to check all this data. In Section 4.3 we provide a different method of producing new, more elaborate examples from existing examples, allowing us to limit the number of times we must invoke Proposition 4.2.2.

An ordinary Mackey functor is usually given as a product preserving functor from the ordinary Burnside 1-category to the category of abelian groups. Similarly, a semi-Mackey functor is a product preserving functor from the Burnside 1-category to commutative monoids. Interestingly, the data of a semi-Mackey functor is equivalent to a functor that takes values in the category of sets. Essentially, this works because a product preserving functor from the Burnside 1-category is a monoidal functor into Set, where both categories have the categorical product as the monoidal product. Since each object of the Burnside 1-category is a commutative monoid, it follows that $M(X)$ is also a commutative monoid.

The next proposition categorifies the argument of the previous paragraph to the present setting, where the role of commutative monoids is played by symmetric monoidal categories. The proof makes heavy use of the technical results of a paper of Day and Street [DS97].

Proposition 4.2.5. *Let $M\colon \mathcal{A}^G_{(2,1)} \to \mathrm{Cat}$ be a categorical Mackey functor. For every $b \in \mathcal{A}^G_{(2,1)}$, the category $M(b)$ comes equipped with a symmetric monoidal structure.*

Proof. For this proof, we abbreviate $M(b)$ by M_b. Similarly, if $f\colon x \to y$ is a 1-cell in $\mathcal{A}^G_{(2,1)}$ then we abbreviate $M(f)$ by M_f.

By [CKWW07, Theorem 2.15], any bicategory with binary products underlies a symmetric monoidal bicategory with monoidal product given by the binary product. Following [DS97], any object $b \in \mathcal{A}^G_{(2,1)}$ is a *pseudo-monoid*, meaning it admits a coherently associative and unital multiplication. In our case, the multiplication $\mu\colon b \amalg b \to b$ is given by the span

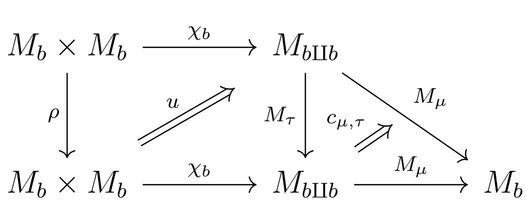
$$b \amalg b = b \amalg b \xrightarrow{\nabla} b$$

where ∇ is the fold map. Our explicit choice of pullbacks along identities implies this multiplication is strictly unital. The unit $\emptyset \to b$ is given by the transfer along the unique map from the empty set to B in G-sets. The diagrams of [DS97, Section 3] are easily checked. By [DS97, Proposition 5], we immediately get that M_b is a monoidal category with product given by the composite

$$M_b \times M_b \xrightarrow{\chi_b} M_{b\amalg b} \xrightarrow{M_\mu} M_b$$

where χ_b is a categorical equivalence, unique up to unique natural isomorphism, which witnesses the fact that M is product preserving. It remains to check that this product is symmetric.

Let $\rho\colon M_b \times M_b \to M_b \times M_b$ be the switch map of categories and let $\tau\colon b \amalg b \to b \amalg b$ be the switch map in $\mathcal{A}^G_{(2,1)}$. We have a pasting diagram

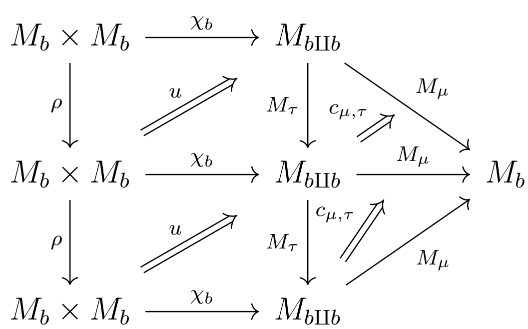


in Cat. The 2-cell u is the unique 2-cell which witnesses the fact that the 1-cells $M_\tau\chi_b$ and $\chi_b\rho$ agree, up to natural equivalence, after either projection $M_{b\amalg b} \to M_b$. The transformation $c_{\nabla,\tau}$ is the natural transformation which realizes the fact that M preserves composition up to natural isomorphism. The resulting 2-cell

$$c_{\mu,\tau} \circ u \colon M_\mu \chi_b \rho \Rightarrow M_\mu \chi_b$$

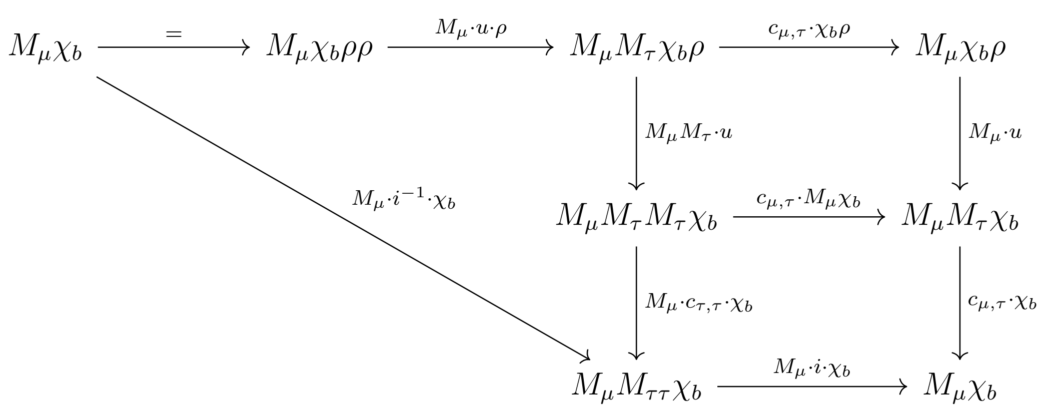
gives a braiding which we need to check is symmetric; that is, we need to check that doing the braiding twice is the identity.

The relevant diagram is



(4.3)

which realizes a 2-cell $M_\mu \chi_b \Rightarrow M_\mu \chi_b$ since ρ^2 is the identity. Consider the diagram



where i is the unitor for M. If we show this diagram commutes we are done as the top-right composite is a 2-cell represented by the pasting diagram (4.3) while the bottom composite of the diagram is the identity. The top-right square represents the two ways of evaluating the sub pasting diagram of (4.3) consisting of the top-left square and the bottom right triangle; this commutes because Cat is a strict 2-category. The bottom-right square commutes because M is a pseudo-functor and this square is a whiskering of χ_b with two ways of resolving $M_\mu M_\tau M_\tau$ to $M_{\mu\tau\tau} = M_\mu$.

To see the left triangle commutes, note that it is the whiskering of M_μ with the composite

$$(c_{\tau,\tau} \cdot \chi_b)(M_\tau \cdot u)(u \cdot \tau)\colon \chi_b \to M_{\tau\tau}\chi_b.$$

Since $M_{\tau\tau}\chi_b$ and χ_b are maps $M_b \times M_b \to M_{b\amalg b}$ which respect the projections up to isomorphism, they must be uniquely isomorphic by the 2-uniqueness of bicategorical products and so both ways around the triangle must be the same.

□

The symmetric monoidal structures on $M(b)$ specified by the last proposition interact nicely with the categorical transfer, restriction, and conjugation functors. In particular, all of these functors are strong monoidal, providing the link between our categorical Mackey functors and the symmetric monoidal Mackey functors of Definition 3.1.1.

Theorem 4.2.6. *Every categorical Mackey functor M determines a symmetric monoidal Mackey functor $\widetilde{M}$ with $\widetilde{M}(H) = M(G/H)$, where $M(G/H)$ is endowed with the symmetric monoidal structure of Proposition 4.2.5.*

In certain cases, we can identify the symmetric monoidal structure of Proposition 4.2.5 as a cocartesian monoidal structure.

Lemma 4.2.7. *Let $M\colon \mathcal{A}^G_{(2,1)} \to \mathrm{Cat}$ be a categorical Mackey functor. If M is the restriction of a pseudo-functor $M\colon \mathcal{A}^G \to \mathrm{Cat}$, then for any object $b \in \mathcal{A}^G_{(2,1)}$, the symmetric monoidal product on $M(b)$, identified in Proposition 4.2.5, is a categorical coproduct.*

Proof. Recall that for any $b \in \mathcal{A}^G_{(2,1)}$, the symmetric monoidal product on $M(b)$ is

$$\mu_b\colon M(b) \times M(b) \cong M(b \amalg b) \xrightarrow{M(T_\nabla)} M(b)$$

By Lemma 4.1.27, the map T_∇ is the left adjoint of R_∇ and so, by Lemma 4.1.26, the functor μ_b

must be a left adjoint of

$$D_b \colon M(b) \xrightarrow{M(R_\nabla)} M(b \amalg b) \cong M(b) \times M(b)$$

Since the composite of R_∇ with either projection map is the identity, we see that D_b must be equivalent to the diagonal functor. Since the coproduct functor is characterized as the left adjoint to the diagonal functor, this concludes the proof. □

We end this section with several examples. With the exception of Examples 4.2.8 and 4.2.9 all of these examples can be checked to be actual categorical Mackey functors using Proposition 4.2.2.

Example 4.2.8. The categorical Mackey functor $\mathrm{Set}_{\amalg,iso}$ is defined by $\mathrm{Set}_{\amalg,iso}(G/H) = \mathrm{Set}^H_{iso}$, the category of finite G-sets and isomorphisms. The symmetric monoidal product is the disjoint union of H-sets. The restrictions and transfers are given by restriction and induction of sets with group actions. This example is representable in the sense that

$$\mathrm{Set}_{\amalg,iso}(G/H) \cong \mathcal{A}^G_{(2,1)}(G/G, G/H)$$

and so the fact that this is actually a product preserving pseudo-functor is entirely formal.

Example 4.2.9. The categorical Mackey functor $\mathrm{Set}_\amalg$ is given by $\mathrm{Set}_\amalg(G/H) = \mathrm{Set}^H$. The symmetric monoidal product is the disjoint union of H-sets. The restrictions and transfers are given by restriction and induction of G-sets. Like Example 4.2.8, this example is is representable in the sense that for all H we have equivalences of categories

$$\mathrm{Set}_\amalg(G/H) \cong \mathcal{A}^G(G/G, G/H)$$

and the transfers and restrictions are induced by maps in $\mathcal{A}^G$. In particular, $\mathrm{Set}_\amalg$ is actually restricted from a product preserving functor $\mathcal{A}^G \to \mathrm{Cat}$ and is in fact the universal example of such a functor.

Example 4.2.10. The categorical Mackey functor $\mathrm{Set}_\times$ is given by $\mathrm{Set}_\times(G/H) = \mathrm{Set}^H$. The symmetric monoidal product is the Cartesian product of H-sets. The restrictions and transfers are given by restriction and coinduction of G-sets.

Example 4.2.11. The categorical Mackey functor $\mathcal{A}^G_\amalg$ is given by $\mathcal{A}^G_\amalg(G/H) = \mathcal{A}^H_1$, where the subscript 1 means we are considering the Burnside 1-category. The symmetric monoidal product is the disjoint union of H-sets. The restrictions and transfers are given by the extension of restriction and induction of finite sets with group action to functors on the Burnside categories.

Example 4.2.12. The categorical Mackey functor $\mathcal{A}^G_\times$ is given by $\mathcal{A}^G_\times(G/H) = \mathcal{A}^H_1$. The symmetric monoidal product is the Cartesian product of H-sets. The restrictions and transfers are given by the extension of restriction and coinduction of finite sets with group action to functors on the Burnside categories.

Remark 4.2.13. The categorical Mackey functor $\mathrm{Set}_{\amalg,iso}$ of Example 4.2.8 plays a central role in our theory of categorical Mackey functors. In [HH16], Hill and Hopkins study structures called *G-symmetric monoidal categories* which amount to objects which are, in some sense, modules over $\mathrm{Set}_{\amalg,iso}$. This idea is carried further in the work of Vekemans, who develops the notion for incomplete G-symmetric monoidal structures which are indexed by various indexing categories.

The analogy with categorical Mackey functors can loosely be described as follows: the collection of categorical Mackey functors should be thought of as a symmetric monoidal category with $\mathrm{Set}_{\amalg,iso}$ as the unit. Since the unit in a monoidal category is always a monoid, we can make sense of modules over the unit and in fact every object is a module over the unit. In particular, morally there should be no difference between considering categorical Mackey functors and considering modules over $\mathrm{Set}_{\amalg,iso}$. In classical terms, we have just described how one identifies modules over a commutative ring R with modules over the R-algebra R.

Of course, while the analogy is quite strong it is a different task entirely to make the analogy precise. We have taken the route that best suits our purposes for this thesis, but overall it seems to be a matter of taste.

4.3 Constructing categorical Mackey functors

In this section we describe a method of constructing examples of categorical Mackey functors from old ones. The goal of this method is to bypass, to a certain degree, the necessity of checking the data of Proposition 4.2.2. In particular, suppose $\mathcal{M}\colon \mathcal{A}^G_{(2,1)} \to \mathrm{Cat}$ is a rule which assigns objects to objects, 1-cells to 1-cells and 2-cells to 2-cells. We wish to show that $\mathcal{M}$ gives an actual categorical Mackey functor. If the square

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$$

is a pullback diagram in Set^G the composition law in the Burnside category says that if $\mathcal{M}$ is a pseudo-functor then there must be natural isomorphisms

$$\mathcal{M}(T_k)\mathcal{M}(R_h) \cong \mathcal{M}(R_f)\mathcal{M}(T_g)$$

which can be hard to construct ad hoc. Even worse, showing these isomorphisms are sufficiently natural presents its own challenge.

The plan is to factor the desired categorical Mackey functor as a composite

$$\mathcal{A}^G_{(2,1)} \xrightarrow{T_{\mathcal{M}}} \mathrm{Cat}^{co}_{\times} \xrightarrow{\mathrm{Mod}} \mathrm{Cat}$$

where $\mathrm{Cat}_{\times}$ is the bicategory of categories with finite products, functors, and natural transformations and the superscript "co" indicates that we have taken the co-dual in the sense of Definition 4.1.22. We emphasize that the 1-morphisms in $\mathrm{Cat}_{\times}$ do *not* need to be product preserving as this would be too restrictive for our examples.

The pseudo-functor $T_{\mathcal{M}}$ should be a categorical Mackey functor which is already constructed. For instance, if we pick $T_{\mathcal{M}}$ to be the categorical Mackey functor of Example 4.2.12, the composite

$\mathrm{Mod} \circ T_{\mathcal{M}}$ will give a categorical Mackey functor of Mackey functors. The underlying symmetric monoidal Mackey functor will be the one studied in Chapter 3.

Our pseudo-functor Mod sends a category $\mathcal{C}$ with products to

$$\mathrm{Mod}(\mathcal{C}) = \mathbf{Fun}_{\times}(\mathcal{C}, \mathrm{Set}),$$

the category of product preserving functors from $\mathcal{C}$ to the category of sets. If $f: \mathcal{C} \to \mathcal{D}$ is any functor then $\mathrm{Mod}(f)$ is given by left Kan extension along f. When we need to be explicit, we will take left Kan extensions along non-identity functors to be defined by the pointwise formula; we require left Kan extension along identity functor to be the identity functor. The fact that left Kan extensions of product preserving functors preserve products is [BD77, Theorem 1.5(iii)]. If $f, g: \mathcal{C} \to \mathcal{D}$ are two functors and $\alpha: f \Rightarrow g$ is a natural transformation (a morphism in $\mathrm{Cat}_{\times}^{co}(g, f)$), then the natural transformation given by the composite

$$\mathrm{id}_{\mathrm{Fun}_{\times}(\mathcal{C},\mathrm{Set})} \overset{\eta}{\Rightarrow} f^* f_! \overset{\alpha^*}{\Longrightarrow} g^* f_!$$

has an adjunct which is $\mathrm{Mod}(\alpha): g_! \Rightarrow f_!$. Phrased another way, for any $M: \mathcal{C} \to \mathrm{Set}$, the natural transformation $\mathrm{Mod}(\alpha)_M: g_! M \Rightarrow f_! M$ is the unique natural transformation so that

$$(\mathrm{Mod}(\alpha) \cdot g) \circ \eta_M^g = (f_! M \cdot \alpha) \circ \eta_M^f \tag{4.4}$$

where the equality is as natural transformations from M to $f_!(M) \cdot g$.

Lemma 4.3.1. Mod *is a pseudo-functor.*

Proof. Given two composable functors f and g, the left Kan extension of the composite is canonically isomorphic to the composite of the left Kan extensions, providing the necessary isomorphisms $\mathrm{Mod}(f \circ g) \cong \mathrm{Mod}(f) \circ \mathrm{Mod}(g)$. The same reasoning provides both the associators and the unitors, and the universal properties which make these choices canonical imply all the necessary coherence.

It remains to check that for any categories $\mathcal{C}$ and $\mathcal{D}$ with products that Mod gives an actual functor:

$$\mathrm{Cat}_\times(\mathcal{C},\mathcal{D})^{\mathrm{op}} \to \mathrm{Cat}(\mathrm{Mod}(\mathcal{C}),\mathrm{Mod}(\mathcal{D})).$$

For any $f\colon \mathcal{C} \to \mathcal{D}$, note that if we plug $f = g$ and $\alpha = \mathrm{id}_f$ into equation (4.4) we get the defining equation for $\mathrm{Mod}(\mathrm{id}_f)$. Since $\mathrm{id}_{f_!}$ also satisfies this equation these two natural transformations agree at all M and thus $\mathrm{Mod}(\mathrm{id}_f) = \mathrm{id}_{f_!}$. It remains to show that if $g, h\colon \mathcal{C} \to \mathcal{D}$ and $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$ then we have an equality

$$\mathrm{Mod}(\alpha) \circ \mathrm{Mod}(\beta) = \mathrm{Mod}(\beta \circ \alpha).$$

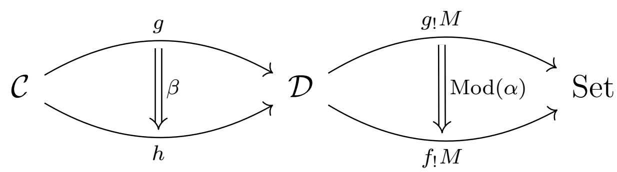
For all $M\colon \mathcal{C} \to \mathrm{Set}$, $\mathrm{Mod}(\beta \circ \alpha)_M$ is the unique 2-cell $h_!M \to f_!M$ so that

$$(\mathrm{Mod}(\beta \circ \alpha)_M \cdot h) \circ \eta^h_M = (f_!M \cdot (\beta \circ \alpha)) \circ \eta^f_M \tag{4.5}$$

Consider the following commutative diagram where all arrows are natural transformations:

$$\begin{array}{ccccc}
M & \xrightarrow{\quad = \quad} & M & \xrightarrow{\eta^h_M} & h^*h_!M \\
\Big\downarrow{\scriptstyle \eta^f_M} & & \Big\downarrow{\scriptstyle \eta^g_M} & & \Big\downarrow{\scriptstyle \mathrm{Mod}(\beta)\cdot h} \\
 & & g^*g_!M & \xrightarrow{(g_!M)\cdot\beta} & h^*g_!M \\
 & & \Big\downarrow{\scriptstyle \mathrm{Mod}(\alpha)\cdot g} & & \Big\downarrow{\scriptstyle \mathrm{Mod}(\alpha)\cdot h} \\
f^*f_!M & \xrightarrow[(f_!M)\cdot\alpha]{} & g^*f_!M & \xrightarrow[(f_!M)\cdot\beta]{} & h^*f_!M
\end{array} \tag{4.6}$$

The top right square commutes by definition of $\mathrm{Mod}(\beta)$, the left rectangle commutes by the definition of $\mathrm{Mod}(\alpha)$, and the bottom right square commutes because the two ways around represent the two possible composites of the pasting diagram



in the strict 2-category Cat.

Commutativity of the diagram (4.6) gives

$$((\mathrm{Mod}(\alpha) \circ \mathrm{Mod}(\beta)) \cdot h) \circ \eta^h_M = (f_! M \cdot (\beta \circ \alpha)) \circ \eta^f_M$$

which means that $\mathrm{Mod}(\alpha) \circ \mathrm{Mod}(\beta)$ satisfies the defining property (4.5) of $\mathrm{Mod}(\beta \circ \alpha)$ and so the two natural transformations are equal. □

Remark 4.3.2. Nowhere in this the above proof did we use anything about the category of sets other than the fact that it admits all pointwise left Kan extensions and that these Kan extensions send product preserving functors to product preserving functors as in [DS95]. Thus one could extend the above result by replacing $\mathrm{Cat}_\times$ with the bicategory of small categories with products enriched in some sufficiently nice category $\mathcal{V}$. We will not pursue this further in this thesis.

We now need to show that Mod preserves products. First, we need to identify the products in $\mathrm{Cat}_\times$.

Lemma 4.3.3. *The bicategorical product of* $\mathrm{Cat}_\times$ *is the usual product of* 1*-categories.*

Proof. The only point that really needs addressing is the fact that the product $A \times B$ of two categories A and B with finite products is itself a category with finite products. Given two objects (a_1, b_1) and (a_2, b_2) of $A \times B$, a product of these two elements is given by $(a_1 \times_A a_2, b_1 \times_B b_2)$ where $\times_A$ and $\times_B$ denote products in A and B. □

Lemma 4.3.4. *The pseudo-functor* $\mathrm{Mod}\colon\ \mathrm{Cat}^{co}_\times \to \mathrm{Cat}$ *is product preserving.*

Proof. Let A_1 and A_2 be categories with finite products and let $\pi^i\colon A_1 \times A_2 \to A_i$ be the projections. For this proof, we write $\mathcal{S}$ for the category of sets. The first step is to compute what the functor Mod does to the projection functors π^i. To this end, for any functor $F\colon A_1 \times A_2 \to \mathcal{S}$ we define a functor $F_1\colon A_1 \to \mathcal{S}$ given on objects by $F_1(x) = F(x,*)$, where $*$ is a fixed terminal object of A_2; this terminal element exists because A_2 has all finite products. F_1 does the obvious thing on morphisms. We claim that F_1 is a left Kan extension of F along π^1. That is, we there are natural bijections of sets

$$\pi^1_!(F)(x,y) \cong F(x,*) \tag{4.7}$$

for all (x,y).

To see this, define a natural transformation $\eta\colon F \Rightarrow F_1 \circ \pi^1$ with component

$$\eta_{(x,y)}\colon F(x,y) \xrightarrow{F(\mathrm{id}_x,!)} F(x,*) = (F \circ \pi^1)(x,y)$$

for any $(x,y) \in A_1 \times A_2$. Given any other functor $G\colon A_1 \to \mathcal{S}$ and natural transformation $\gamma\colon F \Rightarrow G \circ \pi^1$, we have a diagram

$$\begin{array}{ccccc} F(x,y) & \xrightarrow{\gamma_{(x,y)}} & G\pi^1(x,y) & = & G(x) \\ \downarrow{\scriptstyle \eta_{(x,y)}} & & \downarrow{\scriptstyle G\pi^1(\mathrm{id_x},!)} & & \| \\ F(x,*) & \xrightarrow{\gamma_{(\mathrm{id}_x,!)}} & G\pi^1(x,*) & = & G(x) \end{array}$$

in $\mathcal{S}$ which commutes by naturality of γ. We see that every such γ factors uniquely through η establishing the claim.

Define a functor $\Phi\colon \operatorname{Mod}(A_1)\times\operatorname{Mod}(A_2) \to \operatorname{Mod}(A_1\times A_2)$ by $\Phi(F,G)(x,y) = F(x)\times G(y)$. It is not hard to see this is an equivalence of categories and, moreover, we have

$$(\pi^1_!\Phi(F,G))(a) \cong F(a) \times G(*) \cong F(a)$$

where the first isomorphism uses (4.7) and the second usses the fact that $G(*)$ must be a singleton because G is product presrving. It follows that $\mathrm{Mod}(A \times B)$ satisfies the property of the bicategorical product and carries projections to projections. □

Corollary 4.3.5. *If* $T\colon \mathcal{A}^G_{(2,1)} \to \mathrm{Cat}^{co}_{\times}$ *is a product preserving pseudo-functor then* $\mathrm{Mod}_T = \mathrm{Mod} \circ T$ *is a categorical Mackey functor.*

Remark 4.3.6. A category T with finite products such that every object is isomorphic to a finite product of some fixed object $x \in T$ is called a *Lawvere Theory*. Such categories, first studied systematically by Bill Lawvere, give a categorical approach to constructions in universal algebra [Law63]. Briefly, the morphisms of the theory T encode the various structure maps present in the algebraic theory of interest. The *models* of theory T are the product preserving functors $T \to \mathrm{Set}$ and encode the collection of all sets with the desired structure maps.

Generalizing, one considers *multisorted* Lawvere theories T in which we require that our objects be generated under products by a collection of objects $\{x_i\}_{i \in I}$. The main example of interest in this paper are the Burnside 1-categories $\mathcal{A}^H$, which are the multisorted Lawvere theories whose models are exactly the H-semi-Mackey functors. With this lens, one can interpret Corollary 4.3.5 as saying that if we can construct a "Mackey functor of Lawvere theories," then the associated collection of models is necessarily a categorical Mackey functor. The usefulness of the corollary comes from the fact that the construction of the theories as a Mackey functor is, in many cases, easier than the construction of the Mackey functor of models.

All of the examples at the end of section 4.2 lift to functors $\mathcal{A}^G_{(2,1)} \to \mathrm{Cat}^{co}_{\times}$ and so post composition with Mod yields new categorical Mackey functors. We focus on Example 4.2.12 which is the categorical Mackey functor determined by $G/H \mapsto \mathcal{A}^H$ with restriction and transfer given by restriction and coinduction of H-sets. Composition with Mod yields a categorical Mackey functor whose value on G/H is exactly the category of H-semi-Mackey functors. This example is exactly the one we studied in Chapter 3.

Proposition 4.3.7. *The categorical Mackey functor* $\mathrm{Mod}_{\mathcal{A}^G_{\times}}$ *agrees with the G-symmetric monoidal Mackey functor of Proposition 3.2.8 in the case where the indexing category* $\mathcal{O}$ *is* Set^G.

4.4 G-commutative monoids

Throughout this section we fix a categorical Mackey functor $\mathcal{C}\colon \mathcal{A}^G_{(2,1)} \to \mathrm{Cat}$. By Theorem 4.2.6, the categories $M(H) = \mathcal{C}(G/H)$ form a symmetric monoidal Mackey functor in the sense of Definition 3.1.1. The goal of this section and the next is to consider how the G-commutative monoids fit into the framework of categorical Mackey functors. In this section we lay the groundwork by showing how to recover the G-commutative monoids from constructions involving only categorical Mackey functors. In the next section we prove that G-commutative monoids of a categorical Mackey functor $\mathcal{C}$ satisfy the universal property of being a kind of right Kan extension of $\mathcal{C}$.

Recall that for a symmetric monoidal Mackey functor M, the G-commutative monoids of M are the subcategory $\mathrm{Comm}_G(M) \subset M(G)$ consisting of objects x together with norm multiplications

$$\mu^H_K\colon N^H_K R^G_K(x) \to R^G_H(x)$$

for subgroups $K \leq H \leq G$ and subject to the coherence data of Definition 3.1.6.

The main result of this section recovers the G-commutative monoids of a symmetric monoidal Mackey functor M which arises from a categorical Mackey functor $\mathcal{C}$. Recall the categorical Mackey functor $\mathrm{Set}_{\amalg}$ of Example 4.2.9 which is given by

$$\mathrm{Set}_{\amalg}(G/H) = \mathrm{Set}^H$$

with transfer and restriction given by induction and restriction respectively.

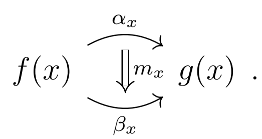
Theorem 4.4.1. *Let $\mathcal{C}$ be a categorical Mackey functor and let M be the symmetric monoidal Mackey functor determined by Theorem 4.2.6. There is an equivalence of categories*

$$\mathrm{Comm}_G(M) \cong [\mathrm{Set}_{\amalg}, \mathcal{C}]$$

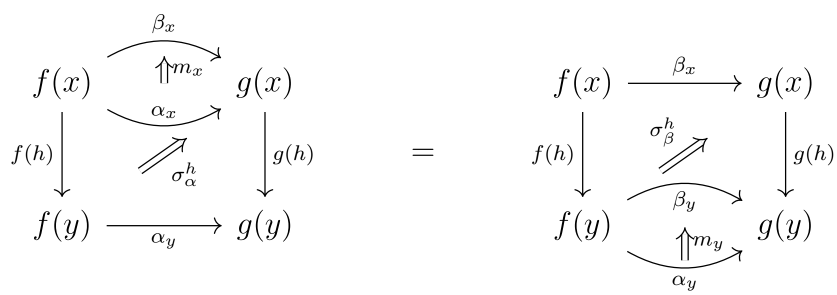
where $[-,-]$ *denotes the category of maps of categorical Mackey functors.*

Before giving the proof we need to unpack the data of the category $[\mathrm{Set}_{\mathrm{II}}, \mathcal{C}]$. The objects are product preserving natural transformations, but we need to define the morphisms of this category. Morphisms between natural transformations are handled by *modifications*.

Definition 4.4.2. Let $f, g \colon \mathcal{B} \to \mathcal{D}$ be a pair of pseudo-functors and let $\alpha, \beta \colon f \Rightarrow g$ be a pair of pseudo-natural transformations. A *modification* $m \colon \alpha \to \beta$ is the data of a 2-cell $m_x \colon \alpha_x \Rightarrow \beta_x$ in $\mathcal{D}$ for all $x \in \mathcal{B}$ filling the diagram



For every map $h \colon x \to y$ in $\mathcal{B}$ we require the that the modification 2-cells be compatible with the 2-cells that come from the pseudo-natural transformations. Diagramatically we write this coherence as an equality of pasting diagrams



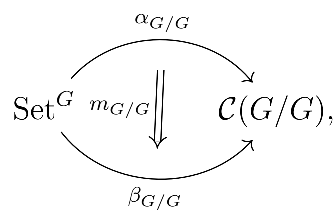
where σ_α^h and σ_β^h are the invertible 2-cells which exist as a part of the data of α and β being pseudo-natural transformations (Definition 4.1.15). In equations, the diagrams can be read as an equality $m_x \circ \sigma_\alpha^h = \sigma_\beta^h \circ m_y$.

The collection of bicategories, pseudo-functors, pseudo-natural transformations, and modifications form a weak version of a 3-category called a tricategory [GPS95]. In particular, the pseudo-natural transformations and modifications form a one category which is what we consider in the above theorem. In our case, the morphisms in $[\mathrm{Set}_{\mathrm{II}}, \mathcal{C}]$ are determined by much less data than an arbitrary modification.

Let $\alpha, \beta \colon \mathrm{Set}_{\amalg} \Rightarrow \mathcal{C}$ be two product preserving natural transformations. Since $\mathrm{Set}_{\amalg}$ and $\mathcal{C}$ are pseudo-functors which take values in Cat, a modification m between $\mathrm{Set}_{\amalg}$ and $\mathcal{C}$ has the data of actual natural transformations m_x for all $x \in \mathcal{A}^G_{(2,1)}$. We will show that really only one of these natural transformation, namely

$$m_{G/G} \colon \alpha_{G/G} \Rightarrow \beta_{G/G}$$

matters. That is, the data of a modification is determined by the relatively manageable diagram



though we can do even better. It turns out that the data of the natural transformation $m_{G/G}$ is itself entirely determined by a single morphism in $\mathcal{C}(G/G)$, namely the component $m_{G/G,G/G}$ of $m_{G/G}$ at the object G/G. The key to all of this is that the single object $G/G \in \mathrm{Set}_{\amalg}(G/G)$ determines every object in $\mathrm{Set}_{\amalg}(X)$ for all X. We make this argument precise in the next lemma.

Lemma 4.4.3. *Let $\alpha, \beta \colon \mathrm{Set}_{\amalg} \Rightarrow \mathcal{C}$ be two product preserving natural transformations. Then two modifications m and n from α to β are the same if and only if the component natural transformations m_x and n_x are the same for the object $x = G/G$. Moreover, the natural transformations $m_{G/G}$ and $n_{G/G}$ agree if and only if the components*

$$m_{G/G,G/G},\ n_{G/G,G/G} \colon \alpha_{G/G}(G/G) \to \beta_{G/G}(G/G)$$

are equal.

Proof. If m and n are the same then certainly all their components agree. Conversely, suppose that $m_{G/G} = n_{G/G}$. For X a finite G-set we write $f \colon X \to G/G$ for the collapse map. Let σ_α be the invertible 2-cell which fills the diagram

$$\begin{array}{ccc} \mathrm{Set}_{\amalg}(G/G) & \xrightarrow{\alpha_{G/G}} & \mathcal{C}(G/G) \\ {\scriptstyle T_f}\uparrow & \overset{\sigma_\alpha}{\nearrow\!\!\!\nearrow} & {\scriptstyle T_f}\uparrow \\ \mathrm{Set}_{\amalg}(X) & \xrightarrow{\alpha_X} & \mathcal{C}(X) \end{array}$$

which exists by the definition of a pseudo-natural transformation. Define σ_β similarly. The definition of a modification gives equalities

$$\sigma_\beta \circ m_X = m_{G/G} \circ \sigma_\alpha \quad \text{and} \quad n_{G/G} \circ \sigma_\alpha = \sigma_\beta \circ n_X.$$

Putting these together with the assumption $m_{G/G} = n_{G/G}$ and the invertibility of σ_β gives us that $m_X = n_X$ for all X proving the first claim.

To prove the second claim we need to check that the natural transformation $\mu = m_{G/G}$ depends only on the G/G component. Note that the domain category $\mathrm{Set}_{\amalg}(G/G) = \mathrm{Set}^G$ is generated under disjoint union by the orbits G/H. Since the functors $\alpha_{G/G}$ and $\beta_{G/G}$ are strong monoidal, we see that μ is determined by its components at the orbits G/H for $H \leq G$. There is a commutative diagram

$$\begin{array}{ccccc} \alpha_{G/G}(G/H) & \xrightarrow{\cong} & T_H^G(\alpha_{G/H}(H/H)) & \xrightarrow{\cong} & T_H^G R_H^G(\alpha_{G/G}(G/G)) \\ \downarrow{\scriptstyle \mu_{G/H}} & & \downarrow{\scriptstyle T_H^G(m_{G/H,H/H})} & & \downarrow{\scriptstyle T_H^G R_H^G(\mu_{G/G})} \\ \beta_{G/G}(G/H) & \xrightarrow{\cong} & T_H^G(\beta_{G/H}(H/H)) & \xrightarrow{\cong} & T_H^G R_H^G(\beta_{G/G}(G/G)) \end{array} \tag{4.8}$$

where the unlabeled isomorphisms come from the data of α and β being pseudo-natural transformations. The squares commute because m is a modification. It follows that the components $\mu_{G/H}$ are determined entirely by the data of α, β, and $\mu_{G/G}$ and thus there is at most one modification n with $n_{G/G,G/G} = \mu_{G/G}$ □

We can now give the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. Let $\alpha\colon \mathrm{Set}_{\amalg} \Rightarrow \mathcal{C}$ be a product preserving pseudo-natural transformation of categorical Mackey functors and let $x = \alpha_{G/G}(G/G) \in \mathcal{C}(G/G)$. For any chain of subgroups

$K \leq H \leq G$ there is a unique H-equivariant map

$$\mu_K^H \colon H \times_K \operatorname{Res}_K^G(G/G) \to \operatorname{Res}_K^G(G/G)$$

which, after applying the functor $\alpha_{G/H} \colon \operatorname{Set}^H \to \mathcal{C}(G/H)$, induces a map

$$\mu_K^H \colon N_K^H R_K^G(x) \to R_H^G(x)$$

and this collection of maps define norm multiplications of x. The coherence of Definition 3.1.6 is all immediate from pseudo-naturality of α and so x is a G-commutative monoid. To see this is a functor, suppose we have a modification $m \colon \alpha \to \beta$. Then

$$m_{G/G,G/G} \colon \alpha_{G/G}(G/G) \to \beta_{G/G}(G/G)$$

is a map of G-commutative monoids which makes the assignment $\alpha \mapsto \alpha_{G/G}(G/G)$ functorial. This functor, call it Φ, is faithful by Lemma 4.4.3. To show Φ is essentially surjective and full we will produce a functor going the other way which is a section.

Suppose x is an object in $\operatorname{Comm}_G(M)$. We define a natural transformation

$$\widehat{x} \colon \operatorname{Set}_{\amalg} \to \mathcal{C}$$

as follows. For every finite G-set Y and object $f \in \operatorname{Set}_{\amalg}(Y) = \mathcal{A}^G(G/G, Y)$ we define

$$\widehat{x}_Y(f) = \mathcal{C}(f)(x) \in \mathcal{C}(Y).$$

and we must show this is a functor $\widehat{x}_Y(f) \colon \mathcal{A}^G(G/G, Y) \to \mathcal{C}(Y)$. We will show the case when $Y = G/G$, the general case being essentially the same. Thus $\mathcal{A}^G(G/G, G/G) = \operatorname{Set}^G$ is the category of finite G-sets. For every finite G-set Z pick an isomorphism

$$\eta_Z\colon \coprod_{i=1}^{N_Z} G/H_i^Z \to Z.$$

The point of this is that if we can define $\widehat{x}_{G/G}$ on maps of the form

$$v\colon \coprod_{i=1}^{N} G/H_i \to \coprod_{i=1}^{M} G/K_i$$

then we can conjugate these by the isomorphisms $\mathcal{C}(\eta_Z)$ and $\mathcal{C}(\eta_Z)^{-1}$ to extend $\widehat{x}$ to all kinds of maps. Since

$$G/H_i \cong T_{H_i}^G R_{H_i}^G(G/G) \in \mathrm{Set}_{\amalg}(G/G)$$

the value of $\widehat{x}(v)$ is determined by a coherent choice of maps of the form

$$T_{H_i}^G R_{H_i}^G(x) \to T_{K_j}^G R_{K_j}^G(x)$$

for $H_i \leq K_j$ which are provided by the norm multiplications of x. The coherence follows from the data of Definition 3.1.6. At that remains is to define the invertible 2-cells of $\widehat{x}$.

For any map $\omega\colon Y \to Z$ in $\mathcal{A}^G_{(2,1)}$ we need to define an invertible 2-cell $\sigma^{\omega}_{\widehat{x}}$

$$\begin{array}{ccc} \mathcal{A}^G(X,Y) & \xrightarrow{\widehat{x}_Y} & \mathcal{C}(Y) \\ {\scriptstyle \omega_*}\downarrow & \overset{\sigma^{\omega}_{\widehat{x}}}{\nearrow} & \downarrow{\scriptstyle \mathcal{C}(\omega)} \\ \mathcal{A}^G(X,Z) & \xrightarrow{\widehat{x}_Z} & \mathcal{C}(Z). \end{array}$$

Unwinding the definitions, for any object $f\colon X \to Y$ we need an equivalence

$$\mathcal{C}(\omega \circ f)(x) \xrightarrow{\sim} \mathcal{C}(\omega)\mathcal{C}(f)(x)$$

which we can take to be the data given to us by the fact that $\mathcal{C}$ is a pseudo-functor. All the necessary coherence follows from the assumption that $\mathcal{C}$ is actually a pseudo-functor. □

4.5 The universal property of G-commutative monoids

In this section we offer one final perspective on G-commutative monoids, characterizing these objects via a universal property. This universal property exists in the context of the following extension problem: given a categorical Mackey functor $\mathcal{C}\colon \mathcal{A}^G_{(2,1)} \to \mathrm{Cat}$, when does $\mathcal{C}$ admit an extension to $\mathcal{A}^G$? That is, when can we construct a functor $\widehat{\mathcal{C}}$ such that the diagram

$$\begin{array}{ccc} \mathcal{A}^G_{(2,1)} & \xrightarrow{\mathcal{C}} & \mathrm{Cat} \\ {\scriptstyle i}\big\downarrow & \nearrow_{\widehat{\mathcal{C}}} & \\ \mathcal{A}^G & & \end{array} \tag{4.9}$$

commutes up to pseudo-natural equivalence where i is the inclusion.

Of course, such an extension problem does not always admit a solution. Indeed, by Lemma 4.2.7 an extension $\widehat{\mathcal{C}}$ to $\mathcal{A}^G$ can only exist if the symmetric monoidal product on $\mathcal{C}(G/H)$ is the coproduct for all $H \leq G$. Symmetric monoidal categories in which the monoidal product is the categorical coproduct are called *co-cartesian*. This motivates the following definition.

Definition 4.5.1. A categorical Mackey functor $M\colon \mathcal{A}^G_{(2,1)} \to \mathrm{Cat}$ is *co-cartesian* if it is restricted from a product preserving pseudo-functor $\widehat{M}\colon \mathcal{A}^G \to \mathrm{Cat}$.

As noted in Example 4.2.9, $\mathrm{Set}_{\amalg}$ gives us a canonical example of a co-cartesian categorical Mackey functor. More generally, if X is any finite G-set we could define a co-cartesian categorical Mackey functor by the assignment

$$Y \mapsto \mathcal{A}^G(X, Y)$$

and we will call this $\mathrm{Set}^X_{\amalg}$. These representable categorical Mackey functors will offer us a way to solve a weaker version of our extension problem which we now state.

Our new extension problem begins by considering the question: if we cannot get an exact solution to (4.9), how close we can get to an solution? That is, can we always construct a diagram

$$\begin{array}{ccc} \mathcal{A}^G_{(2,1)} & \xrightarrow{\mathcal{C}} & \mathrm{Cat} \\ {\scriptstyle i}\downarrow & \overset{\alpha}{\Nearrow} \quad \nearrow_{\widehat{\mathcal{C}}} & \\ \mathcal{A}^G & & \end{array} \tag{4.10}$$

where α is some 2-cell? Of course, it is likely the case that many such α exist. More interesting is whether or not we can construct and identify a universal example of such a diagram. Before proceeding to the general construction, we consider more closely the case where $G = e$ is the trivial group.

Example 4.5.2. When $G = e$, a categorical Mackey functor $\mathcal{C}\colon \mathcal{A}^e_{(2,1)} \to \mathrm{Cat}$ is exactly the data of a symmetric monoidal category. A product preserving functor $\widehat{\mathcal{C}}\colon \mathcal{A}^e \to \mathrm{Cat}$ is exactly a co-cartesian monoidal category. Thus a solution to our weak extension problem is precisely a co-cartesian monoidal category $\widehat{\mathcal{C}}$ together with a strong monoidal functor $\alpha\colon \widehat{\mathcal{C}} \to \mathcal{C}$ such that for any co-cartesian mononidal category $\widehat{\mathcal{D}}$ and strong monoidal functor $\beta\colon \widehat{\mathcal{D}} \to \mathcal{C}$ there is a unique factorization of β as a composite

$$\widehat{\mathcal{D}} \xrightarrow{f} \widehat{\mathcal{C}} \xrightarrow{\alpha} \mathcal{C}$$

The solution to this problem is to take $\widehat{\mathcal{C}}$ to be the category of commutative monoids in $\mathcal{C}$ and take α to simply be the functor which forgets the monoid structures. Since strong monoidal functors preserve commutative monoids, and every object in a co-cartesian monoidal category has a unique structure as a commutative monoid, any functor β must factor as desired.

The observation which we use to characterize G-commutative monoids is that Theorem 4.4.1 allows us to run essentially the same argument as the last example.

Theorem 4.5.3. *For a categorical Mackey functor $\mathcal{C}$ there is always a solution $(\widehat{\mathcal{C}}, \alpha)$ to the extension problem* (4.10)*. Explicitly, given any other pair*

$$(\mathcal{D}\colon \mathcal{A}^G \to \mathrm{Cat},\ \beta\colon i^*(\mathcal{D}) \to \mathcal{C}),$$

there is a pseudo-natural transformation $\widehat{\beta}\colon \mathcal{D} \to \widehat{\mathcal{C}}$, unique up to modification equivalence, such that the pseudo-natural transformation β factors as a composite

$$i^*(\mathcal{D}) \xrightarrow{i^*(\widehat{\beta})} i^*(\widehat{\mathcal{C}}) \xrightarrow{\alpha} \mathcal{C}.$$

Finally, there is an equivalence of categories $\widehat{\mathcal{C}}(G/G) \cong Comm_G(\mathcal{C})$.

Proof. Recall that we have defined the co-cartesian categorical Mackey functors $\mathrm{Set}_{\amalg}^X$ by

$$\mathrm{Set}_{\amalg}^X(Z) = \mathcal{A}^G(X, Z)$$

for any finite G-sets X and Z. Define the functor $\widehat{\mathcal{C}}\colon \mathcal{A}^G \to \mathrm{Cat}$ by the rule

$$\widehat{\mathcal{C}}(X) = [i^* \mathrm{Set}_{\amalg}^X, \mathcal{C}]$$

where, as in Theorem 4.4.1, the brackets $[-,-]$ denote a category of pseudo-natural transformations and modifications. For any map $f\colon X \to Y$ in $\mathcal{A}^G$, we get an induced pseudo-natural transformation

$$f^*\colon\ \mathrm{Set}_{\amalg}^Y \to \mathrm{Set}_{\amalg}^X$$

and we define $\widehat{\mathcal{C}}(f)$ to be the map

$$(f^*)^*\colon [i^* \mathrm{Set}_{\amalg}^X, \mathcal{C}] \to [i^* \mathrm{Set}_{\amalg}^Y, \mathcal{C}]$$

and a straightforward unwinding of definitions shows that

$$\widehat{C}(f) \circ \widehat{C}(g) = ((f \circ g)^*)^*$$

whenever f and g are composable and so pseudo-functoriality is immediate from the fact that $\mathcal{A}^G$ is a bicategory.

If $f, g\colon X \to Y$ are 1-cells in $\mathcal{A}^G$ and $\gamma\colon f \Rightarrow g$ is a 2-cell then we get induced 2-cells $\gamma^*\colon f^* \Rightarrow g^*$ and $(\gamma^*)^*\colon (f^*)^* \Rightarrow (g^*)^*$ which serve the role of $\widehat{\mathcal{C}}(\gamma)$. Again, these 2-cells have all the necessary coherence because $\mathcal{A}^G$ is a bicategory.

Thus we have a co-cartesian categorical Mackey functor $\widehat{C}\colon \mathcal{A}^G \to \mathrm{Cat}$ and the identification

$$\widehat{\mathcal{C}}(G/G) \cong \mathrm{Comm}_G(\mathcal{C})$$

is immediate from the definition and Theorem 4.4.1. It remains to define the functor $\alpha\colon \widehat{\mathcal{C}} \to \mathcal{C}$ and show that this satisfies the claimed universal property.

To define α, let X be a finite G-set and let $\mathrm{Set}^X_{\amalg,\mathrm{iso}}$ be the categorical Mackey functor of Example 4.2.8 defined by

$$\mathrm{Set}^X_{\amalg,\mathrm{iso}}(Y) = \mathcal{A}^G_{(2,1)}(X, Y)$$

for all $Y \in \mathcal{A}^G_{iso}$. By the bicategorical Yoneda lemma [JY21, Section 8.3] there are equivalences of categories

$$e_X\colon [\mathrm{Set}^X_{\amalg,\mathrm{iso}}, \mathcal{C}] \xrightarrow{\cong} \mathcal{C}(X),$$

which are natural in X, where e_X sends a pseudo-natural transformation β to $\beta_X(\mathrm{id}_X) \in \mathcal{C}(X)$. For all X we have an obvious inclusion functor $j_X\colon \mathrm{Set}^X_{\amalg,\mathrm{iso}} \to i^*\,\mathrm{Set}^X_{\amalg}$ and these inclusion commute with the transfers and restrictions. Thus at all X we get functors

$$\widehat{\mathcal{C}}(X) = [i^*\,\mathrm{Set}^X_{\amalg}, \mathcal{C}] \xrightarrow{j_X^*} [\mathrm{Set}^X_{\amalg,\mathrm{iso}}, \mathcal{C}] \xrightarrow{e_X} \mathcal{C}(X)$$

and these give the components of $\alpha\colon i^*\widehat{C} \Rightarrow \mathcal{C}$.

Finally, to see that this construction is universal, suppose that $\mathcal{D}$ is a co-cartesian categorical Mackey functor and $\beta\colon i^*\mathcal{D} \to \mathcal{C}$ is a product preserving pseudo-natural transformation. We claim the functor

$$i^*\colon [\mathrm{Set}^X_{\amalg}, \mathcal{D}] \to [i^*\,\mathrm{Set}^X_{\amalg}, i^*(\mathcal{D})] \tag{4.11}$$

is an equivalence of categories. Granting this, we have that

$$\Phi_X \colon \mathcal{D}(X) = [\mathrm{Set}_{\mathrm{II}}^X, \mathcal{D}] \xrightarrow{i^*} [i^* \mathrm{Set}_{\mathrm{II}}^X, i^*(\mathcal{D})] = \widehat{(i^*\mathcal{D})}(X)$$

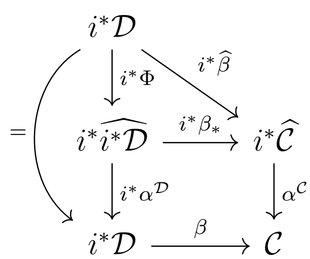
is an equivalence of categories for all X. The Φ_X assemble into a pseudo-natural equivalence $\Phi \colon \mathcal{D} \to \widehat{i^*\mathcal{D}}$ and so the functor

$$\Phi^* \colon [\widehat{i^*\mathcal{D}}, \widehat{\mathcal{C}}] \to [\mathcal{D}, \widehat{\mathcal{C}}]$$

is an equivalence of categories. Thus the functor

$$\beta_* \colon \widehat{i^*\mathcal{D}} \to \widehat{\mathcal{C}}$$

determines, up to modification isomorphism, a pseudo-natural transformation $\widehat{\beta} \colon \mathcal{D} \to \widehat{C}$. Unwinding the definitions we see that the diagram



commutes and so we are done once we show the functor (4.11) is an equivalence of categories.

The functor i^* is faithful since, by the same argument as the proof of Theorem 4.4.3, the modifications m making up the morphisms in both categories are determined entirely by the the source, the domain, and the morphism $\mu_X = m_{X,\mathrm{id_X}}$.

To see the functor is full, suppose that

$$m \in [i^* \mathrm{Set}_{\mathrm{II}}^X, i^*(\mathcal{D})]$$

is some modification. A modification $m' \in [\mathrm{Set}_{\mathrm{II}}^X, \mathcal{D}]$ will satisfy $i^*m' = m$ if and only if $m'_{X,\mathrm{id}_X} =$

m_{X,id_X}. Thus it suffices to show that there is some modification m' with this property. But m_{X,id_X} is a morphism in $(i^*\mathcal{D})(X)$ so the equivalence

$$(i^*\mathcal{D})(X) = \mathcal{D}(X) \cong [\mathrm{Set}_{\mathrm{II}}^X, \mathcal{D}]$$

proves that such an m' exists.

Finally, to see i^* is essentially surjective note that every product preserving pseudo-natural transformation $\gamma\colon i^* \mathrm{Set}_{\mathrm{II}}^X \to i^*\mathcal{D}$ is determined up to modification equivalence by the object $d = \gamma_X(\mathrm{id}_X)$. To see this, one simply repeats the proof of the ordinary Yoneda lemma, using the fact that the element $id_X \in i^* \mathrm{Set}_{\mathrm{II}}^X(X)$ generates every object in $\mathrm{Set}_{\mathrm{II}}^X(Y)$ for all Y. On the other hand, this object d determines a pesudo-natural transformation $\widehat{d}\colon \mathrm{Set}_{\mathrm{II}}^X \Rightarrow \mathcal{D}$, which is essentially unique with respect to the property that $\widehat{d}_X(\mathrm{id}_X) = d$ and $i^*(\widehat{d}) \cong \gamma$. □

We end with an application. A consequence of Theorem 4.5.3 is that the G-commutative monoids of a categorical Mackey functor fit into a categorical Mackey functor of their own, namely

$$\widehat{C}(X) = [\mathrm{Set}_{\mathrm{II}}^X, \mathcal{C}],$$

which happens to be co-cartesian. Moreover, the transfers and restrictions of $\widehat{C}$ are the same as those of $\mathcal{C}$. Because $\widehat{C}$ is co-cartesian, the transfers and restrictions are adjoint to one another, with transfers as the left adjoints. This provides, among other things, an immediate proof of Proposition 3.1.8.

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