VON NEUMANN EQUIVALENCE IN DEFORMATION RIGIDITY THEORY

By

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To my parents, my sister
and my wife

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CHAPTER 1

Introduction

In the series of papers [1] [2], [3], [4] written by Murray and von Neumann, they introduce the notion of Rings of Operators (now called von Neumann algebras) and their basic properties. One of the most studied classes of von Neumann algebras is the class of group von Neumann algebras [4], which consists of a von Neumann algebra $L\Gamma$ associated to every group Γ. One of the main questions in the theory of von Neumann algebras is to identify properties of a group that are "remembered" by the respective group von Neumann algebra.

In [5], Gromov introduced the notion of *measure equivalence* for groups as a measurable analogue of quasi-isometry. This notion has been studied by many [6], especially, the invariant properties of groups under measure equivalence. For example, in [7], Ornstein and Weiss showed that all amenable groups are measure equivalent. Also, from work by Furman [8] and Popa [9], we know that property (T) is invariant under measure equivalence.

Recently, Ishan, Peterson and Ruth introduced the notion of von Neumann equivalence for groups and von Neumann algebras [10]. See section 5 for the definition of von Neumann equivalence. This notion is a generalization of measure equivalence for groups, which they also extended to the von Neumann algebra setting. One of the interesting properties of von Neumann equivalence for groups is that many analytic properties such as amenability, property (T) and Haagerup property are invariant under this equivalence relation. All these properties have a characterization that uses unitary representations, and this proof relies on inducing representations via the equivalence. We use a similar technique in the von Neumann algebra setting using bimodules, to show that all the above properties for von Neumann algebras are stable under von Neumann equivalence.

In Chapter 2, we present some basic definitions and results about von Neumann algebras. We also discuss some basic properties about bimodules, such as weak-containment, mixingness etc. since we need those to define approximation properties such as property (T), amenability and Haagerup property. We give definitions of those properties using bimodules which we will later use in Chapter 4.

In chapter 3, we give the definition of von Neumann equivalence for groups and von Neumann algebras from [10]. We also present some basic properties of these notions and present the proof of the theorem from [10] that states two groups are von Neumann equivalent if and only if their group von Neumann algebras are von Neumann equivalent.

Theorem 1. *Property (T), amenability and Haagerup property for von Neumann algebras are von Neumann equivalent invariant.*

If we have an N-N-bimodule $\mathcal{H}$, we present a way to construct a new M-M-bimodule through the von Neumann equivalence. Since all the above mentioned properties have characterizations using bimodules, we use this induction of bimodules method to show the stability of these properties under von Neumann equivalence.

We also show that L^2 rigidity is a von Neumann equivalence invariant:

Theorem 2. *The property L^2-rigidity is von Neumann equivalent invariant.*

The notion of L^2-rigidity is due to Peterson [11], as an analog of vanishing first ℓ^2-Betti number, defined using derivations and their induced Markov deformations. We first introduce a way to induce malleable deformations and we show that the induced deformations uniformly converge if and only if the original deformations uniformly converge. This proof also extends to L^2-deformations. And since L^2-rigidity is defined by the uniform convergence of L^2-deformations, this shows us that the L^2-rigidity is von Neumann equivalence invariant.

The notion of proper proximality for groups is due to Rémi Boutonnet, Adrian Ioana and Jesse Peterson and first appeared in [12]. This class of countable groups contain all non-amenable bi-exact groups and non-elementary convergence groups but they don't contain any inner-amenable groups. In [10], Ishan, Peterson and Ruth showed that proper proximality is invariant under von Neumann equivalence. At the time, a similar notion for von Neumann algebras did not exist. But Ding, Kunnawalkam Elayavalli, and Peterson in [13] extended this notion to von Neumann algebras. We give the definition of proper proximality for von Neumann algebras that uses strong bimodules developed by Magajna in [14], [15], [16]. We show that this notion is von Nuemann equivalent invariant.

Theorem 3. *Proper proximality for von Neumann algebras is von Neumann equivalence invariant.*

This thesis is based on the joint work with Dr. Peterson and Dr. Ding.

CHAPTER 2

Preliminaries

We first introduce some definitions and basic properties of von Neumann algebras. Proofs of the propositions in this chapter can be found in [17] and [18].

2.1 Von Neumann Algebras

Let $\mathcal{H}$ be a Hilbert space. We define the *weak operator topology* (WOT) and the *strong operator topology* (SOT) on $\mathbb{B}(\mathcal{H})$ as the topologies induced by the seminorms $\{x \mapsto |\langle x\xi, \eta\rangle| : \xi, \eta \in \mathcal{H}\}$ and $\{x \mapsto \|x\xi\| : \xi \in \mathcal{H}\}$ respectively.

A $*$-subalgebra $M \subset \mathbb{B}(\mathcal{H})$ with $1 \in M$ is called a *von Neumann algebra* if $M = \overline{M}^{\text{WOT}}$. We can equivalently define a von Neumann algebra by considering the strong operator topology instead of the weak operator topology. This is an immediate result of the von Neumann bicommutant theorem, which we state below.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S} \subset \mathbb{B}(\mathcal{H})$ be a set. The *commutant* of $\mathcal{S}$, is the set

$$\mathcal{S}' = \{x \in \mathbb{B}(\mathcal{H}) : Sx = xS, \ \forall S \in \mathcal{S}\}.$$

Now let $\mathcal{K} \subset \mathcal{H}$ be a subspace. For $x \in \mathbb{B}(\mathcal{H})$, $\mathcal{K}$ is said to be *invariant* for x (or $\mathcal{K}$ is *x-invariant*) if $x\mathcal{K} \subset \mathcal{K}$. $\mathcal{K}$ is said to be *reducing* for x if it is invariant for x and x^*. For a $*$-subalgebra $M \subset \mathbb{B}(\mathcal{H})$, $\mathcal{K}$ is said to be *reducing* for M (or $\mathcal{K}$ is *M-invariant*) if $M\mathcal{K} \subset \mathcal{K}$.

Theorem 2.1.1 (von Neumann). *Let $M \subset \mathbb{B}(\mathcal{H})$ be a $*$-algebra with $1 \in M$. Then we have $\overline{M}^{\text{SOT}} = \overline{M}^{\text{WOT}} = M''$.*

We define two new topologies on $\mathbb{B}(\mathcal{H})$ called *σ-strong* and *σ-weak* topology, or

σ-SOT and σ-WOT. They are induced by the collections of seminorms

$$\Big\{x \mapsto \Big(\sum_{n=1}^{\infty} \|x\xi_n\|^2\Big)^{1/2} : \{\xi_n\} \in \ell^2(\mathbb{N}, \mathcal{H})\Big\}$$

and

$$\Big\{x \mapsto \Big|\sum_{n=1}^{\infty} \langle x\xi_n, \eta_n\rangle\Big| : \{\xi_n\}, \{\eta_n\} \in \ell^2(\mathbb{N}, \mathcal{H})\Big\}$$

respectively. If we embed $\mathbb{B}(\mathcal{H})$ in $\mathbb{B}(\mathcal{H}\bar{\otimes}\ell^2\mathbb{N})$ by $x \mapsto x \otimes \mathrm{id}$, then the σ-WOT (resp. σ-SOT) is the subspace topology induced by the WOT (resp. SOT) on $\mathbb{B}(\mathcal{H}\bar{\otimes}\ell^2\mathbb{N})$. Also on (norm) bounded sets, WOT (resp. SOT) and σ-WOT (resp. σ-WOT) coincide with each other.

A linear functional $\phi : M \to \mathbb{C}$ on a von Neumann algebra M is called *normal* if it is continuous with respect to the σ-WOT. Also ϕ is called *positive* if $\phi(x^*x) \geq 0$ for each $x \in M$. It can be seen that if ϕ is positive, then it is bounded. We also see that $\phi : M \to \mathbb{C}$ is positive if and only if $\|\phi\| = \phi(1)$. A *tracial state* (or simply, a *trace*) on a von Neumann algebra M is a positive linear functional τ, with $\|\tau\| = 1$ and $\tau(xy) = \tau(yx)$ for $x, y \in M$.

A von Neumann algebra M is said to be *tracial* if there is a normal trace τ on M such that τ is faithful: $x = 0$ whenever $\tau(x^*x) = 0$. A von Neumann algebra M is said to be *finite* if there is a normal faithful center-valued trace on M, i.e., there exists a unital linear map $\tau : M \to \mathcal{Z}(M)$ such that

(i) $\tau(x^*x) \geq 0$ for each $x \in M$, and $\tau(x^*x) = 0$ implies $x = 0$,

(ii) $\tau(zx) = z\tau(x)$ for each $z \in \mathcal{Z}(M)$ and $x \in M$,

(iii) for any increasing bounded net $x_i \in M_+$, $\tau(\sup x_i) = \sup \tau(x_i)$,

(iv) $\tau(xy) = \tau(yx)$ for $x, y \in M$.

Also a separable von Neumann algebra $\mathcal{M}$ is said to be *semi-finite* if there is a faithful, normal, semi-finite tracial weight on $\mathcal{M}_+$, i.e., there is a map $\mathrm{Tr} : \mathcal{M}_+ \to [0, \infty]$ such that

(i) $\mathrm{Tr}(\lambda x + y) = \lambda \mathrm{Tr}(x) + \mathrm{Tr}(y)$, for $x, y \in \mathcal{M}_+$, $\lambda > 0$,

(ii) $\mathrm{Tr}(x^* x) = \mathrm{Tr}(x x^*)$ for $x \in \mathcal{M}$,

(iii) for every $x \in \mathcal{M}_+ \setminus \{0\}$, there exist $y \in \mathcal{M}_+ \setminus \{0\}$ with $y \leq x$ and $\mathrm{Tr}(y) < \infty$,

(iv) $\mathrm{Tr}(\sup x_i) = \sup \mathrm{Tr}(x_i)$ for every bounded increasing net $x_i \in \mathcal{M}_+$,

(v) $\mathrm{Tr}(x) = 0$ if and only if $x = 0$.

Define $\mathfrak{n}_{\mathrm{Tr}} = \{x \in A \mid \mathrm{Tr}(x^* x) < \infty\}$. The definition domain of Tr is defined as

$$\mathfrak{m}_{\mathrm{Tr}} = \left\{ \sum_{i=1}^{n} y_i^* x_i \mid x_i, y_i \in \mathfrak{n}_{\mathrm{Tr}} \right\}.$$

Tr is extends to $\mathfrak{m}_{\mathrm{Tr}}$ as a linear map.

If M is a von Neumann algebra, then we denote by M^{op} the opposite algebra, which we will identify as a von Neumann subalgebra of $\mathcal{B}(L^2(M))$ via the formula $x^{\mathrm{op}} = J x^* J$, where J is Tomita conjugation. On occasion we will have to consider von Neumann algebras $\mathcal{M}$ that contain isomorphic copies of both M and M^{op} for a finite von Neumann algebra M. In this situation if $a \in M$, then we will write a^o for the corresponding element in $M^{\mathrm{op}} \subset \mathcal{M}$, and we will write a^{op} for the corresponding element in $\mathcal{M}^{\mathrm{op}} \subset \mathcal{B}(L^2(\mathcal{M}))$. In doing so we we hope to avoid cumbersome notation and confusion as much as possible.

The *left-regular* and *right-regular* representations of a group Γ are the representations $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ and $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ given by

$$(\lambda_g \xi)(\cdot) = \xi(g^{-1} \cdot)$$

and

$$(\rho_g \xi)(\cdot) = \xi(\cdot g)$$

respectively. The (left) *group von Neumann algebra* is defined as $\mathrm{L}\Gamma = \lambda(\Gamma)''$. Similarly, we define $\mathrm{R}\Gamma = \rho(\Gamma)''$.

Proposition 2.1.2. *Let Γ be a discrete group. Then $\tau(x) = \langle x\delta_e, \delta_e\rangle$ defines a normal faithful trace on $\mathrm{L}\Gamma$. In particular, $\mathrm{L}\Gamma$ is a finite von Neumann algebra.*

Let M be a tracial von Neumann algebra, $B \subset M$ a von Neumann subalgebra and let $e_B \in \mathbb{B}(\mathrm{L}^2 M)$ be the projection onto $\mathrm{L}^2 B \subset \mathrm{L}^2 M$. The von Neumann algebra $\langle M, e_B\rangle$ generated by M and by the projection e_B is called the *extension of M by B*, or the *Jone's basic construction* for $B \subset M$.

2.2 Hilbert Modules

The notion of Hilbert modules were first developed by Paschke [19] and Rieffel [20] in order to develop the non-commutative analogue of fiber bundles over topological spaces.

Let A be a $\mathbb{C}^*$-algebra. A (*right*) *pre-Hilbert module over A* consists of a complex vector space $\mathcal{X}$ which is an algebraic right A-module, equipped with an A-valued inner-product $\langle\cdot,\cdot\rangle$ such that for all $\xi, \eta, \zeta \in \mathcal{X}$, $\lambda \in \mathbb{C}$, and $a \in A$ we have:

(i) $\langle\xi,\xi\rangle \geq 0$ and $\langle\xi,\xi\rangle = 0$ if and only if $\xi = 0$,

(ii) $\langle\xi, \eta + \lambda\zeta\rangle = \langle\xi,\eta\rangle + \lambda\langle\xi,\zeta\rangle$,

(iii) $\langle\xi, \eta a\rangle = \langle\xi,\eta\rangle a$,

(iv) $\langle\xi,\eta\rangle^* = \langle\eta,\xi\rangle$.

Lemma 2.2.1. *For all $\xi, \eta \in \mathcal{X}$, we have $\langle\xi,\eta\rangle\langle\eta,\xi\rangle \leq \|\langle\eta,\eta\rangle\|\langle\xi,\xi\rangle$.*

Now set $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$. Then from the previous lemma $\|\langle \xi, \eta \rangle\| \leq \|\xi\| \|\eta\|$, and therefore it follows that $\| \cdot \|$ gives a norm on $\mathcal{X}$. Moreover, for $\xi \in \mathcal{X}$ and $a \in A$ we have $\|\xi a\| = \|a^* \langle \xi, \xi \rangle a\|^{1/2} \leq \|\xi\| \|a\|$. Thus $\mathcal{X}$ is a normed A-module.

A pre-Hilbert module $\mathcal{X}$ over A is a *Hilbert A-module* if it is complete with respect to the norm $\|\xi\| = \|\langle \xi, \xi \rangle_A\|^{1/2}$.

Example. If A is a $\mathbb{C}^*$-algebra, choose $\mathcal{X} = A$ with the inner product $\langle \xi, \eta \rangle = \xi^* \eta$. Then this is a Hilbert A-module.

Let $E \subset \mathbb{B}(\mathcal{K}_1)$ and $F \subset \mathbb{B}(\mathcal{K}_2)$ be two operator spaces, i.e., closed subspaces in $\mathbb{B}(\mathcal{K}_i)$. Now embed $E \otimes F \subset \mathbb{B}(\mathcal{K}_1 \overline{\otimes} \mathcal{K}_2)$. We denote by $E \otimes_{\min} F$, the $\mathbb{C}^*$-closure of $E \otimes F$ and in the case where both E and F are weakly-closed, $E \overline{\otimes} F$, the σ-WOT closure of $E \otimes F$. Also if we have two operator spaces E and F, then we can show that the algebra of completely bounded maps from E to F, denoted $\mathrm{CB}(E, F)$, is again an operator space using Ruan's axioms ([21]). So it is now easy to see that any Hilbert space $\mathcal{H} \cong \mathrm{CB}(\mathbb{C}, \mathcal{H})$ has an operator space structure.

Suppose M is a von Neumann algebra and $\mathcal{H}$ is a Hilbert space. Then on the algebraic tensor product $M \otimes \mathcal{H}$ we can define an M-valued inner product given by $\langle x \otimes \xi, y \otimes \eta \rangle_M = \langle \eta, \xi \rangle y^* x$. This inner product extends continuously to $M \otimes_{\min} \mathcal{H}$. Then the Hilbert M-module structure on $M \otimes_{\min} \mathcal{H}$ extends to $M \overline{\otimes} \mathcal{H}$ such that $\langle \cdot, \cdot \rangle_M$ is separately σ-weakly continuous.

2.3 Bimodules

Let M and N be tracial von Neumann algebras. A *(left) M-module* is a Hilbert space $\mathcal{H}$ together with a normal unital $*$-homomorphism (not necessarily injective) $\pi : M \to \mathbb{B}(\mathcal{H})$. If π is faithful, we say the M-module is *faithful*. A *right M-module* is a Hilbert space $\mathcal{H}$ together with a normal unital $*$-homorphism $\rho : M^{\mathrm{op}} \to \mathbb{B}(\mathcal{H})$.

An *M-N-bimodule* (also called a *correspondence* from M to N) is a Hilbert space $\mathcal{H}$ together with normal unital $*$-homomorphisms $\pi : M \to \mathbb{B}(\mathcal{H})$ and $\rho : N^{\mathrm{op}} \to$

$\mathbb{B}(\mathcal{H})$ such that $\pi(M)$ and $\rho(N^{\mathrm{op}})$ commutes. We refer to π as the left action and to ρ as the right action. Also we use the notation $a\xi b := \pi(a)\rho(b^{\mathrm{op}})\xi$, for $a \in M$, $b \in N$ and $\xi \in \mathcal{H}$. An M-N-bimodule $\mathcal{H}$ is sometimes denoted by ${}_M\mathcal{H}_N$. Also we denote the collection of all the M-N-bimodules by $\mathcal{B}\mathrm{imod}(M, N)$.

For a von Neumann algebra M the *trivial M-M-bimodule* is the bimodule L^2M with the left and right actions given by $x \cdot \hat{z} \cdot y = \widehat{xzy}$ for $x, y \in M$ and $\hat{z} \in L^2M$. The *coarse M-M-bimodule* is the bimodule defined by $L^2M \bar{\otimes} L^2M$ with actions $x \cdot (\xi \otimes \eta) \cdot y = x\xi \otimes \eta y$ where $x, y \in M$ and $\xi, \eta \in L^2M$.

Suppose ${}_M\mathcal{H}_M$ is an M-M-bimodule. We say a non-zero vector $\xi \in \mathcal{H}$ is a *central vector* (or *M-central vector*) if $x\xi = \xi x$ for all $x \in M$. A sequence of non-zero vectors $\{\xi_n\}$ in $\mathcal{H}$ is said to be *almost central* (or *almost M-central*) if $\|x\xi_n - \xi_n x\| \to 0$ for all $x \in M$. These are the equivalent notions to invariant and almost invariant vectors in the group case.

Let M and N be von Neumann algebras and let $\mathcal{H}$ be an M-N bimodule, i.e., $\mathcal{H}$ is a Hilbert space endowed with commuting normal $*$-representations of M and N^{op} on $\mathcal{H}$.

We fix a normal faithful state φ on N, and let $L^2(N, \varphi)$ be the corresponding GNS-construction with canonical cyclic vector 1_φ. We let $\mathcal{H}^L_\varphi$ denote the space of left bounded vectors (with respect to φ), i.e., $\xi \in \mathcal{H}^L_\varphi$ if and only if there is $K > 0$ such that $\|\xi x\| \le K\|x1_\varphi\|$, for all $x \in N$. In this case we may define the bounded operator $L_\xi : L^2(N, \varphi) \to \mathcal{H}$ by $L_\xi(x1_\varphi) = \xi x$. We also set

$$\mathcal{H}^L = \left\{ L_\xi \mid \xi \in \mathcal{H}^L_\varphi \right\} = \mathbb{B}_{N^{\mathrm{op}}}(L^2(N, \varphi), \mathcal{H}),$$

the space of N^{op}-modular operators in $\mathbb{B}(L^2(N, \varphi), \mathcal{H})$. The bimodule structure on $\mathcal{H}^L$ is then given by pre and post composition and hence this is a dual normal M-N operator bimodule.

We may check that $L_\xi^* L_\eta \in JNJ' \cap \mathbb{B}(L^2(N,\varphi)) = N$ and so we endow an (linear in the second variable) N-valued inner-product on $\mathcal{H}^L$ by

$$\langle \xi, \eta \rangle_N = L_\xi^* L_\eta.$$

In this way we may view $\mathcal{H}^L$ as a dual Hilbert N-module that is a normal left M-module.

Conversely, if $\mathcal{K}$ is a normal Hilbert N-module and we have a normal representation of M in $\mathbb{B}_N(\mathcal{K})$ then we may consider the Hilbert space $\mathcal{K}\overline{\otimes}_N L^2(N,\varphi)$, which is a normal Hilbert M-N bimodule.

Let M and N be two tracial von Neumann algebras, $\mathcal{H} \in \mathcal{B}\text{imod}(M,N)$, $\varepsilon > 0$, $E \subset M$ and $F \subset N$ be finite sets and $S = \{\xi_1, \ldots, \xi_n\} \subset \mathcal{H}$. Then define $V(\mathcal{H}; \varepsilon, E, F, S)$ as the set of all $\mathcal{K} \in \mathcal{B}\text{imod}(M,N)$ where $\exists \eta_1, \ldots, \eta_n \in \mathcal{K}$ such that $\left|\langle x\xi_i y, \xi_j\rangle - \langle x\eta_i y, \eta_j\rangle\right| < \varepsilon$, $\forall x \in E, y \in F, \forall i, j$. The *Fell topology* on $\mathcal{B}\text{imod}(M,N)$ is induced by the basic sets of the form $V(\mathcal{H}; \varepsilon, E, F, S)$.

Now we define weak-containment for bimodules as follows:

Let $\mathcal{H}$ and $\mathcal{K}$ be two M-M-bimodules. We say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$, denoted by $\mathcal{H} \prec \mathcal{K}$, if $\mathcal{H} \in \overline{\{\mathcal{K}^{\oplus\infty}\}}$ in the Fell topology on $\mathcal{B}\text{imod}(M,N)$. Notice that this is equivalent to the statement: for any $\xi, \eta \in \mathcal{H}$ and $F \subset M$ where $|F| < \infty$, there exists $\xi_n, \eta_n \in \mathcal{K}$ such that $\langle x\xi y, \eta\rangle = \sum \langle x\xi_n y, \eta_n\rangle$ for every $x, y \in M$.

And we also give the definition for mixingness for bimodules, which is due to Peterson and Sinclair [22]. This definition is inspired by the mixing representations in the group setting. We say that an M-M-bimodule $\mathcal{H}$ is *mixing* if for any net $u_i \in \mathcal{U}(M)$ with $u_i \to 0$ weakly, we have

$$\lim_{i\to\infty} \sup_{\|x\|\leq 1} \left|\langle u_i \xi x, \eta\rangle\right| = \lim_{i\to\infty} \sup_{\|x\|\leq 1} \left|\langle x\xi u_i, \eta\rangle\right| = 0$$

for all $\xi, \eta \in \mathcal{H}$. If $B \subset M$ is a von Neumann subalgebra we say $\mathcal{H}$ is *mixing relative* *to* B if for any net $u_i \in \mathcal{U}(M)$ with $\|\mathbb{E}_B(au_ib)\|_2 \to 0$ for every $a, b \in M$, we have

$$\lim_{i\to\infty} \sup_{\|x\|\leq 1} \left|\langle u_i\xi x, \eta\rangle\right| = \lim_{i\to\infty} \sup_{\|x\|\leq 1} \left|\langle x\xi u_i, \eta\rangle\right| = 0$$

for every $\xi, \eta \in \mathcal{H}$.

2.4 Approximation Properties

Property (T)

Let $N \subset M$ be a trace preserving inclusion of tracial von Neumann algebras M and N with trace τ. The inclusion $N \subset M$ said to have the *relative property* (T) (or that $N \subset M$ is a *rigid embedding*, or N is a *relatively rigid von Neumann subalgebra of* M) if for any $\varepsilon > 0$, there exists a finite subset $F \subset M$ and $\delta > 0$ with the following property: If $\phi : M \to M$ is a trace preserving u.c.p. map satisfying $\max_{x\in F} \|\phi(x) - x\|_2 \leq \delta$, then $\|\phi(a) - a\|_2 < \varepsilon\|a\|$ for all $a \in N$. We say M has *property* (T) if the identity inclusion $M \subset M$ has property (T). This definition is independent of the choice of trace on M.

The condition that ϕ be unital and τ-preserving can be relaxed in the following way: If $\phi : M \to M$ is a c.p. map such that $\phi(1) \leq 1$ and $\tau \circ \phi \leq \tau$, then the map $\tilde{\phi} : M \to M$ defined by

$$\tilde{\phi}(x) = \phi(x) + \frac{(\tau - \tau \circ \phi)(x)}{1 - \tau \circ \phi(1)} \big(1 - \phi(1)\big)$$

($\tilde{\phi} = \phi + \tau - \tau \circ \phi$ if $\phi(1) = 1$) is a u.c.p. map such that $\tau \circ \tilde{\phi} = \tau$. Also note that by Kadison's inequality, $\tau \circ \phi \leq \tau$ implies

$$\|\phi(a)\|_2 = \tau\big(\phi(a)^*\phi(a)\big)^{1/2} \leq \tau\big(\phi(a^*a)\big)^{1/2} \leq \tau(a^*a)^{1/2} = \|a\|_2$$

for each $a \in M$. In particular, ϕ is normal.

The definition of property (T) for von Neumann algebras is due to Connes and Jones [23], and the relative property (T) is due to Popa [24], see also [9]. This notion is consistent with property (T) for groups, i.e, a group Γ has property (T) if and only if $L\Gamma$ has property (T).

There is another characterization of property (T) for von Neumann algebras using bimodules. This characterization aligns well with the way we defined property (T) for groups using representations since bimodules are the direct counterpart for unitary representations in the world of von Neumann algebras. The following theorem gives us another way to define property (T):

Theorem 2.4.1. *Suppose M is a II_1 factor. The following are equivalent:*

(i) *M has property (T);*

(ii) *If $\mathcal{H}$ is an M-M-bimodule with almost M-central vectors, then $\mathcal{H}$ contains an M-central vector;*

(iii) *If $\mathcal{H}$ is an M-M-bimodule with almost M-central vectors $\{\xi_n\}$ and $\langle x\xi_n, \xi_n\rangle = \langle \xi_n x, \xi_n\rangle = \tau(x)$ for all $x \in M$, then there exists $\{\eta_n\} \subset \mathcal{H}$ such that each η_n is M-central and $\|\xi_n - \eta_n\| \to 0$.*

Amenability

We say a von Neumann algebra M is *amenable* if it has a concrete representation as a von Neumann subalgebra of some $\mathbb{B}(\mathcal{H})$ such that there exists a conditional expectation $E : \mathbb{B}(\mathcal{H}) \to M$. The existence of E does not depend on the representation $M \subset \mathbb{B}(\mathcal{H})$. We also have that a countable group Γ is amenable if and only if $L\Gamma$ is amenable.

Amenability for groups can be defined in terms of representations: we say a group Γ is amenable if the trivial representation 1_Γ is weakly contained in the left regular representation, i.e., $1_\Gamma \prec \lambda_\Gamma$. There is an analogue of this characterization of amenability for von Neumann algebras given by the following theorem:

Theorem 2.4.2. *Suppose M is a finite von Neumann algebra. Then we have M is amenable if and only if the trivial bimodule is weakly contained in the coarse bimodule, i.e.,* $L^2M \prec L^2M \otimes L^2M$.

Haagerup Property

For groups, Haagerup property is defined as follows: A group Γ has the *Haagerup property* if Γ has a mixing representation which has almost invariant vectors. Notice that this notion is, in a way, at the 'opposite' end of the spectrum to property (T) since mixing representations don't contain invariant vectors. And we define Haagerup property for von Neumann algebras simlarly: We say a finite von Neumann algebra has the *Haagerup property* if M has a mixing bimodule with almost M-central vectors. Of course these two definitions are compatible with each other, i.e., a group Γ has the Haagerup property if and only if $L\Gamma$ has Haagerup property.

CHAPTER 3

Von Neumann Equivalence

In this section we present some basic results about von Neumann equivalence for groups and von Neumann algebras. The notion of von Neumann equivalence is first introduced by Ishan Ishan, Jesse Peterson and Lauren Ruth in [10].

3.1 For Groups

Let $\sigma : \Gamma \curvearrowright \mathcal{M}$ be an action of a discrete group Γ on a von Neumann algebra $\mathcal{M}$. A *fundamental domain* for the action is a projection $p \in \mathcal{M}$, such that $\{\sigma_g(p)\}_{g\in\Gamma}$ gives a partition of unity, i.e., $\{\sigma_g(p)\}_{g\in\Gamma}$ is a collection of orthogonal projections which add up to 1.

A *von Neumann coupling* between two countable groups Γ and Λ consists of a semi-finite von Neumann algebra $(\mathcal{M}, \mathrm{Tr})$ and a trace preserving action $\Gamma \times \Lambda \curvearrowright \mathcal{M}$ such that there are finite trace fundamental domains p and q, for Γ and Λ-actions respectively. And we say Γ and Λ are *von Neumann equivalent* if there is a von Neumann coupling between them. This is denoted by $\Gamma \sim_{\mathrm{vNE}} \Lambda$.

This is in fact an equivalence relation. Reflexivity can be seen by looking at the Γ-coupling $\ell^\infty\Gamma$ and symmetry is obvious. The following proposition gives us the transitivity:

Proposition 3.1.1. *Let Γ, Λ and Σ be three countable groups and assume the semifinite von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ are (Γ, Λ) and (Λ, Σ) couplings respectively. Consider the natural actions of Γ and Σ on $\mathcal{M}\overline{\otimes}\mathcal{N}$ and the diagonal action of Λ on $\mathcal{M}\overline{\otimes}\mathcal{N}$. Then $\mathcal{M}\overline{\otimes}\mathcal{N}$ has a Λ fundamental domain and the induced semi-finite trace on $(\mathcal{M}\overline{\otimes}\mathcal{N})^\Lambda$ gives a (Γ, Σ) von Neumann coupling.*

Suppose $\sigma : \Lambda \curvearrowright \mathcal{M}$ is a trace preserving action of a discrete group Λ on a semi-finite von Neumann algebra $\mathcal{M}$ which has a finite trace fundamental domain p and $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a unitary representation. We can give a dual $\mathcal{M}$-structure for $\mathcal{M}\overline{\otimes}\mathcal{H}$ by defining the $\mathcal{M}$-valued inner product,

$$\langle a \otimes \xi, b \otimes \eta \rangle_{\mathcal{M}} = \langle \eta, \xi \rangle a^* b; \quad \text{for } a, b \in \mathcal{M} \text{ and } \xi, \eta \in \mathcal{H}.$$

So for $s \in \Lambda$ and $x, y \in \mathcal{M}\overline{\otimes}\mathcal{H}$, $\langle (\sigma_s \otimes \pi_s)x, (\sigma_s \otimes \pi_s)y \rangle_{\mathcal{M}} = \sigma_s(\langle x, y \rangle_{\mathcal{M}})$. Thus the space of fixed points $(\mathcal{M}\overline{\otimes}\mathcal{H})^{\Lambda}$ is a dual Hilbert $\mathcal{M}^{\Lambda}$-module. So there is a normal representation of $\mathcal{M}$ on $\mathcal{M}\overline{\otimes}\mathcal{H}$ given by the left multiplication and thus we can represent $\mathcal{M}^{\Lambda}$ on $(\mathcal{M}\overline{\otimes}\mathcal{H})^{\Lambda}$.

If $\Gamma \curvearrowright \mathcal{M}$ is another trace-preserving action of a discrete group Γ on $\mathcal{M}$ which commutes with the action of Λ, there is a Γ-invariant trace τ on $\mathcal{M}^{\Lambda}$ (Proposition 4.2 in [10]). Define a scalar-valued inner product on $(\mathcal{M}\overline{\otimes}\mathcal{H})^{\Lambda}$ by $\langle y, x \rangle = \tau(\langle x, y \rangle_{\mathcal{M}})$. Denote the corresponding Hilbert space completion as $(\mathcal{M}\overline{\otimes}\mathcal{H})^{\Lambda}_{\tau}$. Now we denote by $\pi_{\mathcal{M}}$ the unitary representation $\Gamma \curvearrowright (\mathcal{M}\overline{\otimes}\mathcal{H})^{\Lambda}_{\tau}$. In this way we can induce unitary representations through von Neumann equivalence.

Proposition 3.1.2 (Prop 6.15, [10])**.** *Let $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ and $\rho : \Lambda \to \mathcal{U}(\mathcal{K})$ be two unitary representations of the group Λ and let $\Gamma \times \Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ be a von Neumann coupling. If $\pi \prec \rho$, then we have $\pi_{\mathcal{M}} \prec \rho_{\mathcal{M}}$, and if π is weak-mixing, then $\pi_{\mathcal{M}}$ has no non-zero invariant vectors. Moreover, $\lambda_{\mathcal{M}}$ is a multiple of the left-regular representation of Γ.*

Proposition 3.1.3. *Let $\mathcal{M}$ be a von Neumann coupling for Γ and Λ. If Γ has property* (T)*, then so is Λ.*

Proof. Let π be a unitary representation of Λ which has almost invariant vectors, or equivalently, $1_{\Lambda} \prec \pi$. Then $1_{\Gamma} \subset (1_{\Lambda})_{\mathcal{M}} \prec \pi_{\mathcal{M}}$. But since Γ has property (T), $\pi_{\mathcal{M}}$ contains non zero Γ-invariant vectors. So π is not weak-mixing implying π has a finite dimensional Λ-invariant sub representation. So Λ has property (T). $\square$

Proposition 3.1.4. *Let $\mathcal{M}$ be a von Neumann coupling for Γ and Λ. If Γ is amenable, then so is Λ.*

Proof. Notice that if Λ is amenable, then $1_\Lambda \prec \lambda_\Lambda$. Then $1_\Gamma \subset (1_\Lambda)_{\mathcal{M}} \prec (\lambda_\Lambda)_{\mathcal{M}}$. But since $(\lambda_\Lambda)_{\mathcal{M}}$ is a multiple of the left regular representation for Γ, we have that Γ is amenable. □

3.2 For von Neumann Algebras

Let $M \subset \mathcal{M}$ be an inclusion of semi-finite von Neumann algebras where $\mathcal{M}$ is a factor. A *fundamental domain for M inside* $\mathcal{M}$ is a realization of the standard representation $M \subset \mathbb{B}(\mathrm{L}^2M)$ as an intermediate von Neumann subalgebra $M \subset \mathbb{B}(\mathrm{L}^2M) \subset \mathcal{M}$, i.e., an embedding $\theta : \mathbb{B}(\mathrm{L}^2M) \hookrightarrow \mathcal{M}$ such that $\theta|_M = \mathrm{id}$. The fundamental domain is *finite* if finite-rank projections in $\mathbb{B}(\mathrm{L}^2M)$ are mapped to finite projections in $\mathcal{M}$.

Notice that if $P = \mathbb{B}(\mathrm{L}^2M)' \cap \mathcal{M}$, then we have $\mathcal{M} \cong \mathbb{B}(\mathrm{L}^2M)\overline{\otimes}P$. In fact this is given by the map $\mathbb{B}(L^2M)\overline{\otimes}P \ni T \otimes x \mapsto Tx \in \mathcal{M}$.

Definition. A *von Neumann coupling between two finite, countably decomposable von Neumann algebras M and N* consists of a semi-finite von Neumann algebra $\mathcal{M}$ together with embeddings of M and N^{op} into $\mathcal{M}$ such that $N^{\mathrm{op}} \subset M' \cap \mathcal{M}$ and such that each inclusion $M \subset \mathcal{M}$ and $N^{\mathrm{op}} \subset \mathcal{M}$ has finite fundamental domains. A von Neumann coupling $\mathcal{M}$ between M and N is denoted by ${}_M\mathcal{M}_N$.

Two von Neumann couplings ${}_M\mathcal{M}_N$ and ${}_M\mathcal{N}_N$ are isomorphic if there is an isomorphism Φ between $\mathcal{M}$ and $\mathcal{N}$ such that $\Phi|_M = \Phi|_{N^{\mathrm{op}}} = \mathrm{id}$.

For simplicity of the proofs, we assume the semi-finite von Neumann algebra $\mathcal{M}$ in the above definition is in fact a factor. But most of the properties work even when $\mathcal{M}$ is not a factor.

Definition. Two finite, countably decomposable von Neumann algebras M and N are said to be *von Neumann equivalent*, denoted by $M \sim_{\mathrm{vNE}} N$ if there exists a von Neumann coupling between them.

Suppose ${}_M\mathcal{M}_N$ and ${}_N\mathcal{N}_Q$ are M-N and N-Q von Neumann couplings respectively. Let $\theta_{\mathcal{M}} : \mathbb{B}(\mathrm{L}^2N) \to \mathcal{M}$ and $\theta_{\mathcal{N}} : \mathbb{B}(\mathrm{L}^2N) \to \mathcal{N}$ be fundamental domains for $N^{\mathrm{op}} \subset \mathcal{M}$ and $N \subset \mathcal{N}$ respectively, and set $P_1 = \theta_{\mathcal{M}}\big(\mathbb{B}(\mathrm{L}^2N)\big)' \cap \mathcal{M}$ and $P_2 = \theta_{\mathcal{N}}\big(\mathbb{B}(\mathrm{L}^2N)\big)' \cap \mathcal{N}$. Then we have isomorphisms $\tilde{\theta}_{\mathcal{M}} : P_1 \overline{\otimes} \mathbb{B}(\mathrm{L}^2N) \to \mathcal{M}$ and $\tilde{\theta}_{\mathcal{N}} : \mathbb{B}(\mathrm{L}^2N) \overline{\otimes} P_2 \to \mathcal{N}$ with $\tilde{\theta}_{\mathcal{M}}(a \otimes x) = a\theta_{\mathcal{M}}(x)$ and $\tilde{\theta}_{\mathcal{N}}(x \otimes b) = \theta_{\mathcal{N}}(x)b$ for $a \in P_1$, $b \in P_2$ and $x \in \mathbb{B}(\mathrm{L}^2N)$. Now define the composition of the couplings ${}_M\mathcal{M}_N$ and ${}_N\mathcal{N}_Q$ to consist of the von Neumann algebra

$$\mathcal{M}\overline{\otimes}_N\mathcal{N} = P_1 \overline{\otimes} \mathbb{B}(\mathrm{L}^2N) \overline{\otimes} P_2.$$

Notice that $\mathcal{M} \hookrightarrow \mathcal{M}\overline{\otimes}_N\mathcal{N}$ and $\mathcal{N} \hookrightarrow \mathcal{M}\overline{\otimes}_N\mathcal{N}$ respectively by $\tilde{\theta}_{\mathcal{M}}^{-1} \times 1$ and $1 \times \tilde{\theta}_{\mathcal{N}}^{-1}$. Since $\mathcal{Z}(\mathcal{M}) \subset P_1$ and $\mathcal{Z}(\mathcal{N}) \subset P_2$, we have $\mathcal{Z}(\mathcal{M})\overline{\otimes}\mathcal{Z}(\mathcal{N}) \subset \mathcal{M}\overline{\otimes}_N\mathcal{N}$.

From the above construction we can show that von Neumann equivalence for von Neumann algebras is in fact an equivalence relation, which we give the proof below:

Proof. For any finite, countably decomposable M and N, notice $M \sim_{\mathrm{vNE}} M$ by considering the trivial von Neumann coupling $\mathbb{B}(\mathrm{L}^2M)$ with standard embeddings of M and M^{op}. This gives us the reflexivity. If ${}_M\mathcal{M}_N$ is a von Neumann a coupling between M and N, then $\mathcal{M}^{\mathrm{op}}$ gives us a von Neumann coupling between N and M. Thus the relation is symmetric.

Now suppose $M \sim_{\mathrm{vNE}} N$ and $N \sim_{\mathrm{vNE}} Q$ realized by the couplings ${}_M\mathcal{M}_N$ and ${}_N\mathcal{N}_Q$ respectively. Then consider the composition coupling $\mathcal{M}\overline{\otimes}_N\mathcal{N}$ constructed above. Using the same notation in the construction, notice that we have isomorphisms $\mathcal{M}\overline{\otimes}_N\mathcal{N} \cong \mathcal{M}\overline{\otimes}P_2 \cong P_1\overline{\otimes}\mathcal{N}$. Since P_1 and P_2 are finite, $\mathcal{M}\overline{\otimes}_N\mathcal{N}$ with the embeddings given in the construction gives us a finite fundamental domains for M and Q^{op}. Also notice

$$(\tilde{\theta}_{\mathcal{M}}^{-1} \otimes 1)(M) \subset (\tilde{\theta}_{\mathcal{M}}^{-1} \otimes 1)((N^{\mathrm{op}})' \cap \mathcal{M}) = P_1 \overline{\otimes} N \overline{\otimes} 1$$

and

$$(1 \otimes \tilde{\theta}_{\mathcal{N}}^{-1})(Q^{\mathrm{op}}) \subset (1 \otimes \tilde{\theta}_{\mathcal{M}}^{-1})(N' \cap \mathcal{N}) = 1 \overline{\otimes} N^{\mathrm{op}} \overline{\otimes} P_2.$$

Therefore, the copies of M and Q commute with each other. So $\mathcal{M}\overline{\otimes}_N\mathcal{N}$ is a von Neumann coupling between M and Q and, hence $M \sim_{\mathrm{vNE}} Q$. □

These definitions for groups and von Neumann algebras behave well with each other:

Theorem 3.2.1 (1.5 in [10]). *Let Γ and Λ be two countable groups. Then $\Gamma \sim_{\mathrm{vNE}} \Lambda$ if and only if $L\Gamma \sim_{\mathrm{vNE}} L\Lambda$.*

Proof. Suppose $\mathcal{M}$ is an $L\Gamma$-$L\Lambda$ von Neumann coupling. Identify Γ and Λ as subgroups of $\mathcal{U}(L\Gamma)$ and $\mathcal{U}(L\Lambda)$ respectively, and consider the conjugation actions of Γ and Λ on $\mathcal{M}$. Since we have a fundamental domain $L\Gamma \subset \mathbb{B}(\ell^2\Gamma) \subset \mathcal{M}$, the rank-one projection $[\delta_e]$ spanned by $\delta_e \in \ell^2\Gamma$ gives us a finite trace fundamental domain for the conjugation action of Γ. Similarly, we get a finite trace fundamental domain for the action of Λ on $\mathcal{M}$.

Conversely suppose $\mathcal{M}$ is a Γ-Λ von Neumann coupling. Set $\mathcal{N} = \mathcal{M} \rtimes (\Gamma \times \Lambda)$, and then we have embeddings $L\Gamma$, $L\Lambda \subset \mathcal{N}$. Suppose p is a fundamental domain for the action $\sigma : \Gamma \curvearrowright \mathcal{M}$, then the embedding $\theta : \ell^\infty\Gamma \hookrightarrow \mathcal{M}$ given by $\theta(f) = \sum_{g\in\Gamma} f(g)\sigma_{g^{-1}}(p)$ is Γ-equivariant, and now consider the embedding,

$$\mathbb{B}(\ell^2\Gamma) \cong \ell^\infty\Gamma \rtimes \Gamma \subset \mathcal{M} \rtimes \Gamma \subset \mathcal{N}.$$

Thus this is a fundamental domain for $L\Gamma$ inside $\mathcal{N}$. Also notice that by looking at the isomorphism $\mathbb{B}(\ell^2\Gamma) \cong \ell^\infty\Gamma \rtimes \Gamma$ and θ, all finite-rank projections in $\mathbb{B}(\ell^2\Gamma)$ have finite trace in $\mathcal{M}$, and hence they are finite in $\mathcal{N}$. Therefore this fundamental domain is finite. Similarly, we have a finite fundamental domain for $L\Lambda$ in $\mathcal{N}$ and therefore, $\mathcal{N}$ is an $L\Gamma$-$L\Lambda$ von Neumann coupling. □

CHAPTER 4

Invariant Properties

4.1 Inducing bimodules via von Neumann equivalence

A general abstract approach

Suppose $\mathcal{M}$ is a von Neumann algebra and we have an embedding $N^{\mathrm{op}} \subset \mathcal{M}$. Suppose also that $\mathcal{H}$ is an N-correspondence. We let $\mathcal{H}_0^R$ denote the space of right bounded vectors, i.e., $\xi \in \mathcal{H}_0^R$ if there exists $C > 0$ such that $\|x\xi\| \leq C\|x\|_2$. We let $\mathcal{H}^R = \mathcal{B}_N(L^2N, \mathcal{H})$ denote the space of left N-modular bounded operators. Each $\xi \in \mathcal{H}_0^R$ gives rise to a corresponding operator $R_\xi \in \mathcal{H}^R$ given by $R_\xi(x) = x\xi$, and it is well known that every operator in $\mathcal{H}^R$ is of this form.

The operator space $\mathcal{H}^R$ is then a left N-Hilbert module, which we may view as an N^{op}-Hilbert module, and $\mathcal{M}$ is a Hilbert $\mathcal{M}$-module with a normal representation of N^{op} on $\mathcal{M}$ given by left multiplication. We may therefore construct the normal Hilbert $\mathcal{M}$-module as the relative tensor product

$$\mathcal{H}^R \,\overline{\otimes}_{N^{\mathrm{op}}}\, \mathcal{M} = \mathcal{B}_{\mathcal{M}^{\mathrm{op}}}(L^2(\mathcal{M}, \mathrm{Tr}), \mathcal{H} \,\overline{\otimes}_{N^{\mathrm{op}}}\, L^2(\mathcal{M}, \mathrm{Tr})).$$

If $\xi \in \mathcal{H}_0^R$ and $T \in \mathcal{M}$ then the operator $R_\xi \otimes_{N^{\mathrm{op}}} T \in \mathcal{H}^R \,\overline{\otimes}_{N^{\mathrm{op}}}\, \mathcal{M}$ is given by $(R_\xi \otimes_{N^{\mathrm{op}}} T)(\hat{Z}) = \xi \otimes_{N^{\mathrm{op}}} \widehat{TZ}$. The span of such operators is weakly dense in $\mathcal{H}^R \,\overline{\otimes}_{N^{\mathrm{op}}}\, \mathcal{M}$. Note that since we are considering the Connes-fusion with respect to N^{op} it follows that if $\xi \in \mathcal{H}_0^R$, $T \in \mathcal{M}$ and $x \in N$ then we have

$$R_{x\xi} \otimes_{N^{\mathrm{op}}} T = R_\xi \otimes_{N^{\mathrm{op}}} x^{\mathrm{op}} T.$$

Note that N^{op} acts on $L^2(\mathcal{M}, \mathrm{Tr})$ via left-multiplication and that the right action of N on $\mathcal{H}$ gives a left action of N^{op} on $\mathcal{H}^R \,\overline{\otimes}_{N^{\mathrm{op}}}\, L^2(\mathcal{M}, \mathrm{Tr})$.

We let

$$\mathrm{Ind}^{\infty}_{\mathcal{M}}(\mathcal{H}) = \mathcal{B}_{N^{\mathrm{op}}-\mathcal{M}^{\mathrm{op}}}(L^2(\mathcal{M}, \mathrm{Tr}), \mathcal{H} \,\overline{\otimes}_{N^{\mathrm{op}}}\, L^2(\mathcal{M}, \mathrm{Tr})) \tag{4.1}$$

denote the space of N^{op}-intertwiners in $\mathcal{H}^R \,\overline{\otimes}_{N^{\mathrm{op}}}\, \mathcal{M}$, i.e., $T \in \mathrm{Ind}^{\infty}_{\mathcal{M}}(\mathcal{H})$ if

$$(x^{\mathrm{op}} \otimes 1)T = Tx^{\mathrm{op}} \tag{4.2}$$

for all $x \in N$. If $S, T \in \mathrm{Ind}^{\infty}_{\mathcal{M}}(\mathcal{H})$ we then have

$$\langle S, T\rangle_{\mathcal{M}} := S^*T \in (N^{\mathrm{op}})' \cap \mathcal{M}.$$

It therefore follows that $\mathrm{Ind}^{\infty}_{\mathcal{M}}(\mathcal{H})$ is a dual normal Hilbert $(N^{\mathrm{op}})' \cap \mathcal{M}$-module. Note that we also have a normal left $(N^{\mathrm{op}})' \cap \mathcal{M}$ action on $\mathrm{Ind}^{\infty}_{\mathcal{M}}(\mathcal{H})$ via the maps $(1 \otimes x)$.

A concrete approach

We now suppose that we have a fundamental domain $N^{\mathrm{op}} \subset \mathcal{B}(L^2(N)) \subset \mathcal{M}$. We then also have a copy of $N \subset \mathcal{B}(L^2(N)) \subset \mathcal{M}$. Recall that in this setting if we are given $a \in M$, then we write a^{op} to denote the corresponding element in $N^{\mathrm{op}} \subset \mathcal{M}$, while we will use a^o if we view a as an element in $\mathcal{M}$ and consider the corresponding element in $\mathcal{M}^{\mathrm{op}}$.

The fundamental domain then gives a decomposition $\mathcal{M} = \mathcal{B}(L^2(N, \varphi))\overline{\otimes} P$ where $P = \mathcal{B}(L^2(N, \varphi))' \cap \mathcal{M}$. We then have a unitary

$$V : L^2(N) \,\overline{\otimes}\, L^2(P) \,\overline{\otimes}\, L^2(N) \to L^2(\mathcal{M}, \mathrm{Tr})$$

taking $\hat{x} \otimes \hat{z} \otimes \hat{y}$ to $(xP_{\hat{1}}y) \otimes \hat{z}$, for $x, y \in N$ and $z \in P$. More generally, given a N-correspondence $\mathcal{H}$ we have the following.

Lemma 4.1.1. *There exists a unitary*

$$\mathcal{V} : \mathcal{H} \mathbin{\overline{\otimes}} L^2(P) \mathbin{\overline{\otimes}} L^2(N) \to \mathcal{H} \mathbin{\overline{\otimes}}_{N^{\mathrm{op}}} L^2(\mathcal{M}, \mathrm{Tr})$$

taking $\xi \otimes \hat{z} \otimes \hat{y}$ *to* $\xi \otimes_{N^{\mathrm{op}}} ((P_{\hat{1}} y) \otimes \hat{z})$, *for* $y \in N$, $z \in P$, *and* $\xi \in \mathcal{H}$.

Proof. Notice that for $\xi, \eta \in \mathcal{H}$, $z_1, z_2 \in P$ and $y_1, y_2 \in N$,

$$\begin{aligned} \langle \mathcal{V}(\xi \otimes z_1 \otimes y_1), \mathcal{V}(\eta \otimes z_2 \otimes y_2) \rangle &= \langle \xi \otimes_{N^{\mathrm{op}}} (P_{\hat{1}} y_1 \otimes z_1), \eta \otimes_{N^{\mathrm{op}}} (P_{\hat{1}} y_2 \otimes z_2) \rangle \\ &= \left\langle P_{\hat{1}} y_1, (L_\xi^* L_\eta) P_{\hat{1}} y_2 \right\rangle_{L^2 N \overline{\otimes} L^2 N} \langle z_1, z_2 \rangle_{L^2 P} \\ &= \langle \xi, \eta \rangle_{\mathcal{H}} \langle y_1, y_2 \rangle_{L^2 N} \langle z_1, z_2 \rangle_{L^2 P}. \end{aligned}$$

And we can check that the above equality holds for any finite linear combination of such elements and then we can extend the map to the whole space. □

Proposition 4.1.2. *Using the notation above, we have an isomorphism of* $((N^{\mathrm{op}})' \cap \mathcal{M})$*-*$((N^{\mathrm{op}})' \cap \mathcal{M})$ *bimodules* $\phi : \mathcal{H}^L \mathbin{\overline{\otimes}} P \to \mathrm{Ind}_{\mathcal{M}}^{\infty}(\mathcal{H})$ *that satisfies*

$$\phi(L_\xi \otimes x) = \mathcal{V}(L_\xi \otimes x \otimes 1) V^*,$$

for $\xi \in \mathcal{H}^L$ *and* $x \in P$.

Proof. Note that by tensoring with the identity we may view $\mathcal{H}^L \mathbin{\overline{\otimes}} P$ as the space of operators in $\mathcal{B}(L^2(N) \mathbin{\overline{\otimes}} L^2(P) \mathbin{\overline{\otimes}} L^2(N), \mathcal{H} \mathbin{\overline{\otimes}} L^2(P) \mathbin{\overline{\otimes}} L^2(N))$ that intertwine the operators $a^o \otimes c^{\mathrm{op}} \otimes S$, where $a \in N$, $c \in P$, and $S \in \mathcal{B}(L^2 N)$. We may then define the isometric map $\phi : \mathcal{H}^L \mathbin{\overline{\otimes}} P \to \mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}), \mathcal{H} \mathbin{\overline{\otimes}}_{N^{\mathrm{op}}} L^2(\mathcal{M}, \mathrm{Tr}))$ by setting $\phi(T) = \mathcal{V} T V^*$.

Note that if $a, b, c, y \in N$, $d, z \in P$, and $\xi \in \mathcal{H}$, then we have

$$\begin{aligned} a^o \otimes ((bP_{\hat{1}}c) \otimes d)^{\text{op}} \mathcal{V}(\xi \otimes \hat{z} \otimes \hat{y}) &= a^o \otimes ((bP_{\hat{1}}c) \otimes d)^{\text{op}} (\xi \otimes_{N^{\text{op}}} ((P_{\hat{1}}y) \otimes z)) \\ &= \xi a \otimes_{N^{\text{op}}} (\tau(yb) P_{\hat{1}} c \otimes zd) \\ &= \mathcal{V}(a^o \otimes d^{\text{op}} \otimes cP_{\hat{1}}b)(\xi \otimes \hat{z} \otimes \hat{y}). \end{aligned}$$

Hence we have

$$a^o \otimes ((bP_{\hat{1}}c) \otimes d)^{\text{op}} \mathcal{V} = \mathcal{V}(a^o \otimes d^{\text{op}} \otimes cP_{\hat{1}}b). \tag{4.3}$$

We similarly have

$$(a^o \otimes 1)(bP_{\hat{1}}c) \otimes d)^{\text{op}} V = V(a^o \otimes d^{\text{op}} \otimes cP_{\hat{1}}b) \tag{4.4}$$

From (4.3) and (4.4) it then follows that an operator

$$T \in \mathcal{B}(L^2(N) \overline{\otimes} L^2(P) \overline{\otimes} L^2(N), \mathcal{H} \overline{\otimes} L^2(P) \overline{\otimes} L^2(N))$$

intertwines the actions of $N^{\text{op}} \subset \mathcal{M}$ and $\mathcal{M}^{\text{op}}$ if and only if $\mathcal{V}TV$ intertwines the actions of N^{op} and $P^{\text{op}} \overline{\otimes} \mathcal{B}(L^2 N)$, i.e., $T \in \mathcal{H}^L \overline{\otimes} P$ if and only if $\mathcal{V}TV \in \text{Ind}_{\mathcal{M}}^{\infty}(\mathcal{H})$. Hence $\phi : \mathcal{H}^L \overline{\otimes} P \to \text{Ind}_{\mathcal{M}}^{\infty}(\mathcal{H})$ is a well-defined isometric bijection.

Moreover, as $\mathcal{V}$ and V both intertwine $(N^{\text{op}})' \cap \mathcal{M}$ it follows that the map ϕ is $(N^{\text{op}})' \cap \mathcal{M}$-bimodular. □

We now suppose that we have a finite von Neumann algebra $M \subset (N^{\text{op}})' \cap \mathcal{M}$. We then let $\text{Ind}_{\mathcal{M}}(\mathcal{H})$ denote the M-M correspondence $\text{Ind}_{\mathcal{M}}^{\infty}(\mathcal{H}) \overline{\otimes}_M L^2(M)$. In other words, take the closure of $\text{Ind}_{\mathcal{M}}^{\infty}(\mathcal{H})$ with respect to the norm $\| \cdot \|_\tau$ which is induced by the inner product given by $\langle S, T \rangle = \tau(S^*T)$ where τ is the trace on $(N^{\text{op}})' \cap \mathcal{M}$. Then by the above proposition we have $\text{Ind}_{\mathcal{M}}(\mathcal{H}) \cong \mathcal{H} \overline{\otimes} L^2 P$ as M-M-bimodules.

4.2 Property (T), Amenability and Haagerup Property

Using the previous proposition, we get the following straightforward result about almost central vectors:

Proposition 4.2.1. *If* $\mathcal{H}$ *has bi-tracial almost central vectors then so does* $\mathrm{Ind}_{\mathcal{M}}(\mathcal{H})$.

Proof. Indeed, if we take the second perspective above then if ξ_n are almost N-central we have $\xi_n \otimes 1$ are almost $N\overline{\otimes}P$-central and hence are almost M-central. $\square$

The following proposition gives us a way to show obtain central vectors in our original bimodule, given that the induced bimodule contains a central vector.

Proposition 4.2.2. *Suppose* $\eta_0 \in \mathcal{H}$ *and* $\xi_0 = \phi(\eta_0 \otimes 1)$. *If there is a* M*-central vector* $\xi \in \mathrm{Ind}_{\mathcal{M}}(\mathcal{H})$, *then there is a* N^{op}*-central vector* $\eta \in \mathcal{H}$ *such that* $\|\eta - \eta_0\| \leq \|\xi - \xi_0\|$.

Proof. Suppose $\xi \in \mathrm{Ind}_{\mathcal{M}}(\mathcal{H})$ is a non-trivial M-central vector, we can choose this ξ to be in $\mathrm{Ind}_{\mathcal{M}}^{\infty}(\mathcal{H})$. Indeed, choose $\xi_0 \in \mathrm{Ind}_{\mathcal{M}}^{\infty}(\mathcal{H})$ such that $\|\xi - \xi_0\|_\tau < \frac{1}{2}\|\xi\|_\tau$ and let η be the element with minimal $\|\cdot\|_\tau$-norm in the $\|\cdot\|_\tau$-closure of the convex hull of $\{u\xi_0 u^* : u \in \mathcal{U}(M)\}$. Then η is M-central and is non-zero since $\|\xi - \eta\|_\tau < \frac{1}{2}\|\xi\|_\tau$. Thus we can assume $\xi \in \mathrm{Ind}_{\mathcal{M}}^{\infty}(\mathcal{H})$.

So

$$\xi \in \mathcal{B}_{N^{\mathrm{op}}-(M'\cap\mathcal{M})}(L^2(M'\cap\mathcal{M}), \mathcal{H} \otimes_{N^{\mathrm{op}}} L^2(M'\cap\mathcal{M})).$$

Considering the decomposition $L^2(\mathcal{M}, \mathrm{Tr}) \simeq L^2(M'\cap\mathcal{M})\overline{\otimes}L^2M$ and the fact that $N^{\mathrm{op}} \subset M'\cap\mathcal{M}$ we see that $VP_{L^2N\otimes\hat{1}\otimes\hat{1}}$ maps into $L^2(M'\cap\mathcal{M})$. Define

$$\eta := P_{\mathcal{H}\otimes\hat{1}\otimes\hat{1}}\mathcal{V}^*\xi VP_{L^2N\otimes\hat{1}\otimes\hat{1}} \in \mathcal{B}_{N^{\mathrm{op}}-N^{\mathrm{op}}}(L^2N, \mathcal{H}) \simeq \mathcal{H}^{N^{\mathrm{op}}}$$

and also notice that $\eta_0 = P_{\mathcal{H}\otimes\hat{1}\otimes\hat{1}}\mathcal{V}^*\xi_0 VP_{L^2N\otimes\hat{1}\otimes\hat{1}}$. Thus we have

$$\|\eta - \eta_0\|^2 \leq \|\mathcal{V}^*(\xi - \xi_0)V\|^2 = \|V^*(\xi - \xi_0)^*(\xi - \xi_0)V\| = \|\xi - \xi_0\|^2. \qquad \square$$

Now we can state the following theorem:

Theorem 4.2.3. *Property (T) for von Neumann algebras is a von Neumann equivalent invariant.*

Proof. Let M, N be two finite von Neumann algebras that are von Neumann equivalent through $\mathcal{M}$ and suppose M has property (T). Let $\mathcal{H}$ be an N-bimodule with almost central bi-tracial vectors η_n, then $\mathrm{Ind}_{\mathcal{M}}(\mathcal{H})$ is an M-bimodule with almost central bi-tracial vectors $\xi_n := \eta_n \otimes 1$ by Proposition 4.2.1. By property (T) of M, there exists M-central vectors $\tilde{\xi}_n$ such that $\lim_{n\to\infty} \|\xi_n - \tilde{\xi}_n\| = 0$. It follows from Proposition 4.2.2 that there exists N-central vectors $\tilde{\eta}_n$ with $\lim_{n\to\infty} \|\eta_n - \tilde{\eta}_n\| = 0$. □

The following lemma will help us prove the mixingness of the induced bimodule whenever the original bimodule is mixing, which in turn will give us the stability of the Haagerup property under von Neumann equivalence.

Lemma 4.2.4. *Suppose A and B are von Neumann subalgebras of $\tilde{N}$ and let $\mathcal{K}$ be an $\tilde{N}$-$\tilde{N}$ bimodule which is mixing relative to B. If the A-A bimodule $L^2\langle \tilde{N}, e_B\rangle$ is mixing, then so is ${}_A\mathcal{K}_A$.*

Proof. Let $u_n \in \mathcal{U}(A)$ be a sequence that converges to 0 weakly, $\xi, \eta \in \mathcal{K}$. Since $L^2\langle \tilde{N}, e_B\rangle$ is mixing we have for all $a, b \in \tilde{N}$,

$$\begin{aligned}
\|E_B(au_nb)\|_2^2 &= \tau\big(E_B(au_nb)E_B(b^*u_n^*a^*)\big) = \tau\big(au_nbE_B(b^*u_n^*a^*)\big) \\
&= \mathrm{Tr}\left(au_nbe_BE_B(b^*u_n^*a^*)\right) = \mathrm{Tr}(u_nbe_Bb^*u_n^*a^*e_Ba) \\
&= \big\langle u_n\widehat{be_Bb^*}u_n^*, \widehat{ae_Ba^*}\big\rangle_{\mathrm{Tr}} \leq \sup_{x\in(A)_1} \left|\langle u_n\widehat{be_Bb^*}x, \widehat{ae_Ba^*}\rangle_{\mathrm{Tr}}\right| \to 0.
\end{aligned}$$

Also because $\mathcal{K}$ is mixing relative to B as an $\tilde{N}$-$\tilde{N}$ bimodule, $\sup_{y\in(\tilde{N})_1} \left|\langle u_n\xi y, \eta\rangle\right| \to 0$. Therefore $\sup_{y\in(A)_1} \left|\langle u_n\xi y, \eta\rangle\right| \leq \sup_{y\in(\tilde{N})_1} \left|\langle u_n\xi y, \eta\rangle\right| \to 0$. □

Note the above computation shows that if $M \sim_{vNE} N$ through $\mathcal{M}$, with N having finite fundamental domain Q, then for any $u_n \in \mathcal{U}(M)$ converging to 0 weakly, we have $\|E_Q(xu_ny)\|_2 \to 0$ for any x, $y \in N\overline{\otimes}Q$.

Proposition 4.2.5. *Let M and N be finite von Neumann algebras such that $M \sim_{vNE} N$ through $\mathcal{M}$. Given normal N-N bimodules $\mathcal{K}$ and $\mathcal{H}$, the following hold:*

(i) *If $\mathcal{K} \prec \mathcal{H}$, then $\mathrm{Ind}_{\mathcal{M}}(\mathcal{K}) \prec \mathrm{Ind}_{\mathcal{M}}(\mathcal{H})$.*

(ii) *If $\mathcal{H}$ is a mixing N-bimodule, then $\mathrm{Ind}_{\mathcal{M}}(\mathcal{H})$ is a mixing M-bimodule.*

(iii) *$\mathrm{Ind}_{\mathcal{M}}(L^2(N)\overline{\otimes}L^2(N))$ is a multiple of the coarse bimodule of M.*

Proof. Suppose first that $\mathcal{K} \prec \mathcal{H}$. Replacing $\mathcal{H}$ with $\mathcal{H}^{\oplus\infty}$, we may assume that $\mathcal{H}$ has infinite multiplicity. Since $\mathcal{K} \prec \mathcal{H}$, for any ξ, $\eta \in \mathcal{K}^L$, there exist nets $\{\xi_i\}$ and $\{\eta_i\} \subset \mathcal{H}$ such that $\langle x\xi_i y, \eta_i\rangle \to \langle x\xi y, \eta\rangle$ for any $x, y \in N$. As $\mathcal{H}^L$ is dense in $\mathcal{H}$, for each i, we may choose sequences $\{\xi_{i,n}\}_n$ and $\{\eta_{i,n}\}_n$ inside $\mathcal{H}^L$ which converge to ξ_i and η_i, respectively. It's easy to see that for any p, $q \in P$ and a, $b \in N\overline{\otimes}P$, we have $\langle a(\xi_{i,n} \otimes p)b, \eta_{i,n} \otimes q\rangle \to \langle a(\xi \otimes p)b, \eta \otimes q\rangle$. As elements of the form $\xi \otimes p$ span a dense subspace of $\mathrm{Ind}_{\mathcal{M}}(\mathcal{K})$, this then shows (1).

Suppose $\mathcal{H}$ is a mixing N-bimodule. Then it's clear that $\mathcal{H}\overline{\otimes}L^2(P) = \mathrm{Ind}_{\mathcal{M}}(\mathcal{H})$ is mixing relative to P as a $N\overline{\otimes}P$-bimoudle. Notice that

$$L^2(\langle N \otimes P, e_P\rangle) = L^2(\mathcal{B}(L^2N)\overline{\otimes}P) \cong L^2(\mathcal{M})$$

and hence it follows from Lemma 4.2.4 that $\mathrm{Ind}_{\mathcal{M}}(\mathcal{H})$ is mixing as an M-bimodule.

For (3), a direct computation shows that

$$\mathrm{Ind}_{\mathcal{M}}(L^2(N)\overline{\otimes}L^2(N)) \cong L^2(N)\,\overline{\otimes}\,L^2(P)\,\overline{\otimes}\,L^2(N) \cong L^2(M)\,\overline{\otimes}\,L^2(Q)\,\overline{\otimes}\,L^2(M)$$

as M-M bimodules, where Q is given by the fundamental domain of $M \subset \mathcal{M}$. Therefore $\mathrm{Ind}_{\mathcal{M}}(L^2N\overline{\otimes}L^2N) \cong (L^2(M)\,\overline{\otimes}\,L^2(M))^{\oplus\infty}$. ∎

Theorem 4.2.6. *Amenability and Haagerup property for von Neumann algebras are von Neumann equivalent invariant.*

Proof. Amenability is characterized by having the trivial bimodule weakly contianed in the coares bimodule, thus this follows from (1) and (3) of Proposition 4.2.5.

Haagerup property is characterized by having a mixing bimodule with almost central vectors. This is a direct consequence of (1) and (2) of Proposition 4.2.5. □

4.3 Deformations and L^2-Rigidity

First, we give brief introduction to malleable deformations. Let $M \subset \tilde{M}$ be a trace preserving inclusion of finite von Neumann algebras.

- A *malleable deformation* α *of* M *inside* $\tilde{M}$ is a group homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(\tilde{M})$ such that $\|\alpha_t(x) - x\|_2 \to 0$ as $t \to 0$ for each $x \in \tilde{M}$.

- An *s-malleable deformation* (α, β) *of* M *inside* $\tilde{M}$ is α as above, together with $\beta \in \mathrm{Aut}(\tilde{M})$ satisfying $\beta|_M = \mathrm{id}$, $\beta^2 = \mathrm{id}$ and $\beta\alpha_t = \alpha_{-t}\beta$ for all $t \in \mathbb{R}$.

We say that the malleable deformation $\alpha_t \in \mathrm{Aut}(\tilde{M})$ converges uniformly on $(M)_1$ if $\sup_{x\in(M)_1} \|\alpha_t(x) - x\|_2 \to 0$ as $t \to 0$. Notice that we have the following basic result about malleable deformations:

Proposition 4.3.1. *Let* (M, τ) *be a tracial von Neumann algebra and* $\alpha_t : M \to M$ *for* $t \in \mathbb{R}$ *a one-parameter group of trace preserving automorphisms of* M*. The following are equivalent:*

(i) $t \mapsto \alpha_t$ *is pointwise sot-continuous.*

(ii) $t \mapsto \alpha_t$ *pointwise strong** *topology continuous.*

(iii) $t \mapsto \alpha_t$ *is pointwise continuous in* $\|\cdot\|_2$.

The following result allows us to induce malleable deformations using von Neumann equivalence and show that these induced deformations converge uniformly if and only if the original deformations converge uniformly.

Proposition 4.3.2. *Let M and $N \subset \tilde{N}$ be finite von Neumann algebras such that $M \sim_{vNE} N$ through $\mathcal{M}$. Let $\{\alpha_t\}_t \subset \mathrm{Aut}(\tilde{N})$ be a deformation for $\tilde{N}$ and set $\tilde{\alpha}_t := \alpha_t \otimes \mathrm{id} \in \mathrm{Aut}(\tilde{N}\overline{\otimes}P)$. If $\tilde{\alpha}_t$ converges to id uniformly on $(M)_1$ if and only if α_t converges to id uniformly on $(N)_1$.*

Proof. First note that it suffices to consider the case $\tilde{N} = N$ as we may replace α_t with $E_N \circ \alpha_t \circ E_N$. Denote by $(\mathcal{H}_t, \eta_t)$ the Stinespring dilation coming from α_t, where $\eta_t = \hat{1} \otimes \hat{1}$. Notice that

$$\begin{aligned}\|\alpha_t(x) - x\|_2^2 &= \tau(\alpha_t(x)^*\alpha_t(x)) + \tau(x^*x) - \tau(\alpha_t(x)^*x) - \tau(\alpha_t(x)x^*)\\ &\leq \tau(\alpha_t(x^*x)) + \tau(x^*x) - \tau(\alpha_t(x^*)x) - \tau(\alpha_t(x)x^*)\\ &= \|x\eta_t - \eta_t x\|_{\mathcal{H}_t}^2\end{aligned}$$

for $x \in N$. It is straightforward to check that the Stinespring dilation $(\mathcal{K}_t, \xi_t)$ of $\tilde{\alpha}_t$ coincides with $(\mathcal{H}_t\overline{\otimes}L^2P, \eta_t \otimes 1)$, which is the induced bimodule $\mathrm{Ind}_{\mathcal{M}}(\mathcal{H}_t)$ of $\mathcal{H}_t$ through the von Neumann equivalence.

For each t, let $\tilde{\xi}_t \in \overline{\mathrm{conv}}^{\|\cdot\|_2}\{u\xi_t u^* \mid u \in \mathcal{U}(M)\}$ be the vector with minimal norm. Then $\tilde{\xi}$ is a M-central vector in $\mathcal{H}_t\overline{\otimes}L^2P$. Since $\epsilon(t) := \sup_{x\in(M)_1} \|\tilde{\alpha}_t(x) - x\|_2 \to 0$ as $t \to 0$, we see that $\|\xi_t - \tilde{\xi}_t\| \leq \epsilon(t)$. Then it follows from Proposition 4.2.2 that $\|\eta_t - \tilde{\eta}_t\| \leq \epsilon(t)$, which implies that $\lim_{t\to0} \sup_{x\in(N)_1} \|\alpha_t(x) - x\|_2 = 0$. In fact we have

$$\begin{aligned}\|\alpha_t(x) - x\|_2 &\leq \|x\eta_t - \eta_t x\|_{\mathcal{H}_t}\\ &\leq \|x(\eta_t - \tilde{\eta}_t)\|_{\mathcal{H}_t} + \|(\tilde{\eta}_t - \eta_t)x\|_{\mathcal{H}_t}\\ &\leq 2\|x\|\|\eta - \tilde{\eta}\|_{\mathcal{H}_t}.\end{aligned}$$

for $x \in N$. □

Notice that we also have the converse of the above result. In fact, if α_t converges uniformly on $(N)_1$, then $\tilde{\alpha}_t = \alpha_t \otimes \mathrm{id}$ converges uniformly on $(N \overline{\otimes} P)_1$ which in turn gives us $\tilde{\alpha}_t$ converges uniformly on $(M)_1$.

The notion of L^2-rigidity for von Neumann algebras was introduced in [11] and can be thought of as the von Neumann algebra counterpart of the vanishing of 1-cohomology in the left regular representation of groups. The definition of L^2-rigidity which we introduced below is from [22], which is a slight modification from the original definition in [11].

Let M be a finite von Neumann algebra and let $\mathcal{H}$ be M-M bimodule. Then a derivation δ is an unbounded operator $\delta : L^2M \to \mathcal{H}$ such that the domain $\mathcal{D}(\delta)$ is a $\|\cdot\|_2$-dense, $*$-subalgebra of M and $\delta(xy) = x\delta(y) + \delta(x)y$ for each $x, y \in \mathcal{D}(\delta)$. A derivation δ is closable and real if δ is closable as an unbounded operator and there is an anti-linear involution $\mathcal{J}$ on $\mathcal{H}$ such that $\mathcal{J}(x\xi y) = y^*\mathcal{J}(\xi)x^*$ and $\mathcal{J}(\delta(z)) = \delta(z^*)$ for $x, y \in M$, $\xi \in \mathcal{H}$, $z \in \mathcal{D}(\delta)$.

If δ is closable, then it is well known that $\mathcal{D}(\overline{\delta}) \cap M$ is again a $*$-subalgebra and δ restricted to $\mathcal{D}(\overline{\delta}) \cap M$ is also a derivation.

Now we define the u.c.p. map $\Phi^t : M \to M$ by $\Phi^t(x) := \exp(-t\delta^*\overline{\delta})$ for $t > 0$ and $x \in \mathcal{D}(\overline{\delta}) \cap M$. The collection of these trace preserving u.c.p. maps is known as the Markov semigroup (since $\Phi^t \circ \Phi^s = \Phi^{t+s}$ for $t, s > 0$), and we have that $\|\Phi^t(x) - x\|_2 \to 0$ as $t \to 0$ for all $x \in M$.

Definition. A finite von Neumann algebra M is L^2-rigid if for any trace preserving inclusion of finite von Neumann algebras $M \subset \tilde{M}$, a $\tilde{M}$-$\tilde{M}$ bimodule $\mathcal{H}$ such that $\mathcal{H}$ embeds in $(L^2M \overline{\otimes} L^2M)^{\oplus\infty}$ when viewed as an M-M bimodule, and any real closable derivation $\delta : \tilde{M} \to \mathcal{H}$, we have that the induced Markov semigroup $\{\Phi^t\}$ converges uniformly on $(M)_1$, i.e.,

$$\lim_{t\to 0} \sup_{x\in(M)_1} \|\Phi^t(x) - x\|_2 = 0.$$

The deformations coming from the derivations are known as L^2-deformations. We say that a group Γ is L^2-rigid if its von Neumann algebra $L\Gamma$ is L^2-rigid. In [22], it is shown that L^2-rigidity is stable under orbit equivalence, i.e., if two groups Γ and Λ have free, ergodic, measure preserving actions which are orbit equivalent and if Λ is L^2-rigid, then so is Γ. But since von Neumann equivalence is closely related to measure equivalence, and hence orbit equivalence in the group setting, it is a natural question to ask whether the L^2-rigidity is stable under von Neumann equivalence. As it turns out, the answer to this question is yes.

Theorem 4.3.3. *Let M and N be two finite von Neumann algebras with $M \sim_{vNE} N$. If M is L^2-rigid, then so is N.*

Proof. Suppose $N \subset \tilde{N}$ and let $\delta : \tilde{N} \to \mathcal{H}$ be a closable real derivation where $\mathcal{H}$ embeds in the coarse bimodule of N when viewed as an N-N bimodule. Let P be the finite von Neumann algebra coming from the finite fundamental domain for N such that $M \subset N\overline{\otimes}P$. Then we define the derivation $\tilde{\delta} : \tilde{N}\overline{\otimes}P \to \mathcal{H}\overline{\otimes}L^2P$ by $\tilde{\delta}(x \otimes y) = \delta(x) \otimes y$ for $x \in \tilde{N}$, $y \in P$. Clearly we have $\mathcal{H}\overline{\otimes}L^2P \hookrightarrow (L^2M\overline{\otimes}L^2M)^{\oplus\infty}$ when viewed as an M-M bimodule. Notice that if the L^2-deformations coming from δ is $\{\eta_t\}$, then $\tilde{\delta}$ induces deformations of the form $\{\eta_t \otimes \mathrm{id}\}$. But since M is L^2-rigid, we have that $\{\eta_t \otimes \mathrm{id}\}$ converges uniformaly on $(M)_1$ and by 4.3.2 we have $\{\eta_t\}$ converges uniformly on $(N)_1$ giving us the L^2-rigidity of N. □

4.4 Proper Proximality

We first give some basic definitions. For further reading on properly proximal von Neumann algebras and their properties we refer the reader to [13]. An *operator system* is a subspace $X \subset \mathbb{B}(\mathcal{H})$ such that $1 \in A$ and $A^* = A$. Suppose A is a unital C^*-algebra and suppose $\pi : A \to \mathbb{B}(\mathcal{H})$ is a unital $*$-homomorphism. We say that X

is an *operator A-system* if X is a $\pi(A)$-$\pi(A)$-bimodule with the bimodule structure is induced by operator composition. For a von Neumann algebra M, an operator M-system $X \subset \mathbb{B}(\mathcal{H})$ is said to be *normal* if the above homomorphism π can be chosen to be normal and faithful.

If X is a normal operator M-system we define the *M-topology* on X as the topology induced by the seminorms:

$$s_{\rho,\omega}(x) = \inf_{\substack{x=a^*yb \\ a,b\in M}} \left\{ \rho(a^*a)^{1/2} \|y\| \omega(b^*b)^{1/2} \right\}$$

where ω and ρ are positive linear functionals on M. We denote by $X^\sharp$, the space of bounded linear functionals φ in X^* such that $(a,b) \mapsto \varphi(axb)$ is separately ultraweakly continuous for each $x \in X$. It is known that if X is a normal operator M-system, then $X^\sharp$ coincides with the linear functionals that are continuous with the M-topology defined above.

Lemma 4.4.1. *Let M be a von Neumann algebra, let F be a normal M-system and let $E \subset F$ be an M-subsystem. If $\eta \in E^\sharp$, then there exists an extension $\tilde{\eta} \in F^\sharp$ so that $\|\tilde{\eta}\| = \|\eta\|$.*

Proof. Take $\eta \in E^\sharp$ with $\|\eta\| = 1$. Then there exist Hilbert spaces $\mathcal{H}_i$, unit vectors $\xi_i \in \mathcal{H}_i$, normal representations $\pi_i : M \to \mathbb{B}(\mathcal{H}_i)$, and an M-bimodular complete contraction $\phi : E \to \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ so that for $a, b \in M$ and $x \in E$ we have

$$\eta(axb) = \langle \pi_1(a)\phi(x)\pi_2(b)\xi_2, \xi_1 \rangle.$$

We let $\tilde{\phi} : F \to \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ be a completely contractive M-bimodular extension (see [25]), and then define $\tilde{\eta} \in F^*$ to be an extension of η with $\|\tilde{\eta}\| \le 1$ by $\tilde{\eta}(x) = \langle \phi(x)\xi_2, \xi_1 \rangle$. Since $\tilde{\phi}$ is M-bimodular, for $a, b \in M$ and $x \in F$ we have $\tilde{\eta}(axb) = \langle \pi_1(a)\tilde{\phi}(x)\pi_2(b)\xi_2, \xi_1 \rangle$, and since π_1 and π_2 are normal we then see that $\tilde{\eta} \in F^\sharp$. □

Corollary 4.4.2. *Let M be a von Neumann algebra and let F be a normal M-system. Then $F^\sharp$ is spanned by states in $F^\sharp$.*

Proof. We let A be a normal M-C^*-algebra that contains F as an M-subsystem. By Lemma 5.3 in [13] we have that $A^\sharp$ is spanned by states in $A^\sharp$. The result then follows from Lemma 4.4.1 □

Lemma 4.4.3. *Let M and N be von Neumann algebras and let E and F be normal M and N-systems, respectively. If $\phi : E \to F$ is completely positive such that the restriction of ϕ to M defines a normal map from M to N, then ϕ is a continuous map between E with the weak M-topology, and F with the weak N-topology.*

Proof. To prove that ϕ is continuous with respect to the weak M-topology and weak N-topology we need to check that if $\eta \in F^\sharp$ then $\eta \circ \phi \in E^\sharp$. Note that if $\eta \in F^\sharp$ is a state, then $\eta_{|N}$ is normal and hence $(\eta \circ \phi)_{|M}$ is normal, from which it follows that $\eta \circ \phi \in E^\sharp$. By Corollary 4.4.2 every linear functional in $E^\sharp$ is in the span of states, and the general result then follows. □

Suppose M is a von Neumann algebra, then we define the norm $\| \cdot \|_{\infty,1}$ on M by looking at elements in M as operators going from M into L^1M and taking the operator norm. We denote by $\mathbb{K}^{\infty,1}$ the $\| \cdot \|_{\infty,1}$-closure of the compact operators $\mathbb{K}(L^2M)$ on L^2M. Now define

$$\mathbb{S}(M) = \big\{T \in \mathbb{B}(L^2M) : [T, JxJ] \in \mathbb{K}^{\infty,1},\ \forall x \in M\big\}.$$

Notice that clearly we have $M \subset \mathbb{S}(M)$. We say a von Neumann algebra M is *properly proximal* if there is no M-central state φ on $\mathbb{S}(M)$ such that $\varphi|_M$ is normal.

Theorem 4.4.4. *Let M N be two finite von Neumann algebras that are von Neumann equivalent through $\mathcal{M}$. If M is properly proximal, then so is N.*

Proof. Since $M \sim_{vNE} N$, we have $\mathcal{M} \cong \mathbb{B}(L^2M)\overline{\otimes}Q \cong \mathbb{B}(L^2N)\overline{\otimes}P$, $M \subset N\overline{\otimes}P$ and $N^{\text{op}} \subset M^{\text{op}}\overline{\otimes}Q$, where P and Q are both finite. Denote by $\mathbb{K}^0_{\text{Tr}}(\mathcal{M})$ the norm closure of the definition ideal $\mathfrak{m}_{\text{Tr}}$ of $\mathcal{M}$ and let $\mathbb{K}_{\text{Tr}}(\mathcal{M})$ be the M-M and $M^{\text{op}}\overline{\otimes}Q$-$M^{\text{op}}\overline{\otimes}Q$ closure of $\mathbb{K}^0_{\text{Tr}}(\mathcal{M})$.

Consider $\mathbb{S}(M) \hookrightarrow \mathbb{S}(M) \otimes 1_Q \subset \mathcal{M}$ and under this embedding we have

$$\mathbb{S}(M) \subset \{T \in \mathcal{M} \mid [T, x] \in \mathbb{K}_{\text{Tr}}(\mathcal{M}), \text{ for any } x \in M^{\text{op}}\overline{\otimes}Q\},$$

which lies in $\mathbb{S}(N; \mathcal{M}) := \{T \in \mathcal{M} \mid [T, x] \in \mathbb{K}_{\text{Tr}}(\mathcal{M}), \text{ for any } x \in N^{\text{op}}\}$.

We claim that $\text{id} \otimes \tau_P : \mathbb{S}(N; \mathcal{M}) \to \mathbb{S}(N)$. Indeed, it suffices to show $\text{id} \otimes \tau_P : \mathbb{K}_{\text{Tr}}(\mathcal{M}) \to \mathbb{K}^{\infty,1}(N)$. Since $\text{id} \otimes \tau_P$ is trace preserving, we have $\text{id} \otimes \tau_P : \mathbb{K}^0_{\text{Tr}}(\mathcal{M}) \to \mathbb{K}(L^2N)$, and hence it suffices to show that $\text{id} \otimes \tau_P$ is continuous between $\mathcal{M}$ with the weak M and weak $M^{\text{op}}\overline{\otimes}Q$-topologies and $\mathbb{B}(L^2N)$ with the weak N and N^{op}-topologies. This then follows from Lemma 4.4.3 once we observe that the restriction of $\text{id} \otimes \tau_P$ defines a normal map from M to N and $M^{\text{op}}\overline{\otimes}Q$ to N^{op}.

Now suppose N is not properly proximal, i.e., there exists $\varphi \in \mathbb{S}(N)^*$ that is N-central and $\varphi_{|N} = \tau_N$. Then $\psi := \varphi \circ (\text{id} \otimes \tau_P) : \mathbb{S}(N; \mathcal{M}) \to \mathbb{C}$ is $N\overline{\otimes}P$-central with $\psi_{|N\overline{\otimes}P} = \tau_{N\overline{\otimes}P}$ and hence restricting to $\mathbb{S}(M)$ gives an M-central state with $\psi_{|M} = \tau_M$. □

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