THE SAUER-SHELAH-PERLES INEQUALITY AND RELATIVELY COMPLEMENTED LATTICES:

THE SSP=RC CONJECTURE

By

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Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

December 17th, 2022

Nashville, Tennessee

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ACKNOWLEDGMENTS

I thank the department of mathematics at Vanderbilt for giving me an opportunity to do the math and, heavens, to be paid for doing it. It is very fulfilling, professionally, to be in a strong and active environment, and I appreciate the efforts of numerous people who keep it at level. The time I spent here was mostly good and certainly memorable. I harbor the hope that I've learned something in the process and, if so, then it is undoubtedly due to excellent courses taught by, in the order of appearance, Mike Mihalik, Costas Tsinakis, Denis Osin, and Mark Sapir.

Among all my fellow grad students and postdocs, who created this wild mathematical rumpus I enjoyed so much during my years here, I want to specifically thank Frank Wagner, the master of trivia and ridiculous jokes, who, in a well-timed manner, has thrown me a totally different problem to think about, thus diverting my efforts from writing this thesis, which helped in making it shorter and more readable. And also Andrew Moorhead, a young but seasoned captain, braving the troubled waters of universal algebra, for being a perfect guide in my early years, and for lending me his bike, which I still use.

Looking back, I want to thank several people who helped me in the early stages of my career and made it possible for me to come to Vanderbilt in the first place. I thank Grygoriy Zholtkevych, the dean of the math department at Kharkiv National University, for doing the impossible in keeping the department afloat, under impossible circumstances, and for helping me to avoid rash decisions at a point in life when I was considering leaving mathematics for good. I also thank Friedrich Wehrung, whom I never met personally, and Bernhard Ganter for being my first windows into the world of true science, as well as awfully kind and supportive people in general.

I am extremely grateful to the members of my committee, who, by being on it, have thus, perhaps unknowingly, agreed to take a look at the proof I'm about to unleash on them, which, in my view, is at times humanly unreadable.

I thank my wife Ira, who left a big part of her life behind to come and join me in the US. For a person like myself, who loves to be alone, sharing my time is not easy. But the days shared with you give my life the depth and the colors I didn't ever have before. I also thank another family member, Mr. Cat, who, among other things, has made numerous contributions to this text; sadly, all of them had to be removed during the proofreading stage.

My deep and separate gratitude is to my advisor, the great Ralph McKenzie, who was the reason I came to Vanderbilt, and turned out to be everything I was expecting and more. Although I ended up not doing universal algebra, you, Ralph, have certainly shown me the depth and beauty of this area. But, having a taste for cherry-picking tasty problems yourself, you didn't mind me finding something interesting to bite outside of it. Thank you, Ralph, for showing me not only how to enjoy math, but also how to be a mathematician. Get well, your biggest fish is yet to be caught.

Finally, I want to thank the armed forces of Ukraine for protecting my motherland and ensuring I'll have a home to return to.

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CHAPTER 1

Introduction

In this dissertation, we discuss a conjecture that a finite lattice satisfies the Sauer-Shelah-Perles inequality (SSP) if and only if it is relatively complemented (RC). It is straightforward to prove that SSP implies RC, and it is the other direction that is problematic. Our main advance in this direction is that a subset in an RC lattice, whose order-ideal of non-shattered elements has at most three minimal elements, satisfies the SSP inequality, that is, shatters at least as many elements as it has. Additionally, we show that our proof strategy does not work for five minimal elements and construct some tools that aim at disproving the conjecture.

1.1 Motivation

The Sauer–Shelah–Perles (SSP) lemma [31, 32] was developed in the context of the Vapnik-Chervonenkis (VC) dimension. Recalling the corresponding terminology, let X be a finite set (we only consider finite sets, finite lattices, and so on) and let $\mathscr{F} \subseteq 2^X$. We say that $\mathscr{F}$ *shatters* $Y \subseteq X$ if and only if for all $Z \subseteq Y$ there is $F \in \mathscr{F}$ such that $Y \cap F = Z$. The family of sets shattered by $\mathscr{F}$ is denoted by $\mathrm{Str}(\mathscr{F})$, and the size of the maximal set in $\mathrm{Str}(\mathscr{F})$ is called the *VC dimension* of $\mathscr{F}$. The original statement of the SSP lemma is then as follows

Lemma 1 (SSP lemma, originally)**.** *If $\mathscr{F} \subseteq 2^X$, for $|X| = n$, has VC dimension at most d, then*

$$|\mathscr{F}| \leq \binom{n}{0} + \cdots + \binom{n}{d}.$$

The form of the lemma that we will be dealing with is a generalization due to Pajor [29], from which Lemma 1 easily follows.

Lemma 2 (SSP)**.** *For any $\mathscr{F} \subseteq 2^X$ it holds $|\mathscr{F}| \leq |\mathrm{Str}(\mathscr{F})|$.*

Unsurprisingly, most applications of the lemma are in connection with VC dimension or its generalizations. The work of Vapnik and Chervonenkis [33] lies in the area of probability and statistics, and this direction remains active through, for example, applications to machine learning[8]. Shelah [32] used a similar concept in model theory in the definition of theories with NIP (No Independence Property), and Sauer [31] used it in addressing a combinatorial problem by Erdős. Apart from those areas of application, VC dimension has been used in convex geometry [34], computational complexity [27, 3, 24], graph theory [12], matrices with forbidden configurations [4, 5], and formal context analysis [1].

A separate line of inquiry that deals with the SSP lemma itself rather than with VC dimension is the study of *shattering-extremal* set families, that is, families shattering the same number of sets as they have [10, 9]. This approach has been applied to studying sample compression schemes [28], isometric embeddings of graphs [23], convex geometries and meet-distributive lattices [13], hyperplane arrangements [21], and graph orientations [26]. We also mention the use of Gröbner bases in a fast algorithm for recognizing shattering-extremal classes [30].

The definition of shattering can be easily reformulated in terms of lattices, namely, for a lattice L and $F \subseteq L$, F *shatters* $y \in L$ if and only if for all $z \leq y$ there is $f \in F$ such that $y \wedge f = z$. Again, $\mathrm{Str}(F) \subseteq L$ is the set of all elements shattered by F. It is trivial to show that $\mathrm{Str}(F)$ is downward closed. We say that a lattice L satisfies the SSP property if the conclusion of Lemma 2 holds for it. That is, if $|F| \leq |\mathrm{Str}(F)|$ for every $F \subseteq L$.

Lemma 2 thus states that every Boolean lattice is SSP. Apart from those, all lattices L with nonvanishing Möbius function satisfy SSP (we say that L has NMF). This includes, for example, all geometric (atomic and semimodular) lattices. The NMF condition is due to Babai and Frankl [19] and, although neat, it will not play a role in this dissertation. We refer the reader to either [11] or [19] for the corresponding definition and result. On the other hand, the SSP property implies that the lattice is relatively complemented (we say that L is RC). The proof of this was given in [11]. It is rather trivial and thus is omitted.

Lemma 3 (SSP $\Rightarrow$RC)**.** *If a lattice satisfies* SSP *then it is RC.*

Let us now review the definition of an RC lattice. The smallest and the largest element in a lattice L are denoted by 0_L and 1_L respectively. If the lattice in question is clear, we will drop the underscripts and write simply 0 and 1. We say that $x \in L$ is a *complement* of $y \in L$ if $x \wedge y = 0$ and $x \vee y = 1$. L is *complemented* if and only if every element in L has a complement, and it is *relatively complemented* if and only if every interval in L is a complemented lattice. That is, L is relatively complemented if and only if for all $x \leq y \leq z$ there is w in $[x,z] = \{u \in L \mid x \leq u \leq z\}$ such that $y \wedge w = x$ and $y \vee w = z$. An alternative structural characterization of RC lattices by Björner [6] is immensely helpful in dealing with them.

Lemma 4 (Björner)**.** *A finite lattice is RC if and only if it does not contain a* 3*-element interval, that is, there are no two elements $x < y$ such that there is a unique z satisfying $x < z < y$.*

Here are some related results; again, we refer to [11]:

- NMF condition is not necessary for SSP, that is, there are lattices whose Möbius function is zero on some intervals, which satisfy the SSP property;

- A lattice L, which is not a 3-element interval itself, whose Möbius function is nonvanishing except for on $[0,1]$ interval, satisfies SSP;
- NMF, SSP, and RC classes are closed under direct products. In particular, this implies that there are SSP lattices with Möbius function vanishing almost everywhere;
- NMF and RC classes are closed under taking duals, we do not know if that is the case for SSP lattices.

We were not able so far to construct an RC lattice that fails to be SSP, hence the conjecture:

Conjecture 1 (SSP=RC)**.** *A lattice is SSP if and only if it is RC.*

In view of Lemma 3, to prove Conjecture 1 it is sufficient to show that SSP inequality holds in any RC lattice.

The original proof of the SSP lemma is by inductive argument [31]. A related but slightly different approach is by shifting [17, 10], which turned out to be useful, for example, in dealing with Erdős matching conjecture [18, 15, 16]. A completely different proof is by using the dimensionality argument in linear algebra [20], the aforementioned NMF condition is one step away from it. Apart from generalizing the SSP condition to certain lattices, similar methods were used to generalize Frankl–Wilson and Ray-Chaudhuri–Wilson theorems about uniform set families with a small number of possible intersections to the lattice setup [2].

Despite our best efforts, neither of these three approaches seems to be applicable to RC lattices. Similarly, the available information about RC lattices, except for the characterization of Björner, shed little light on the SSP=RC conjecture. Of a tangential interest is a result of Dilworth [14], which enables us to restrict our attention to simple RC lattices. We thus needed to develop an original direction of attack on the conjecture.

Let us finish this introduction by outlining a potentially interesting application of the SSP property for lattices. It is known that greedoids are a natural extension of both matroids and antimatroids [7, 25]. Both these classes are related, although in different ways, to shattering-extremal families in SSP lattices. The lattices of flats of matroids are precisely geometric lattices, which are NMF, and hence SSP. At the same time, antimatroids can be characterized as precisely union-closed shattering-extremal subfamilies of sets [13]. Thus, both matroids and antimatroids can be considered subclasses of join-subsemilattices of SSP lattices, hinting at a possible connection between SSP lattices, shattering-extremality, and greedoids. Some sketchy results about RC lattices, submodular functions, and greedoids can be found in [35].

1.2 Main results and summary of contents

The following theorem is the main result of this dissertation, its statement is indicative of our approach.

Theorem 1. *In an RC lattice L every set F, whose antichain of minimal non-shattered elements has at most three elements, satisfies SSP inequality, that is,* $|F| \leq |\operatorname{Str}(F)|$.

We never prove Theorem 1 explicitly. Instead, it is a direct consequence of the following results, whose order also outlines our plan of attack on the SSP=RC conjecture in general:

- Lemma 5 in Section 2.1 shows that $\mathrm{SSP}_\omega \Rightarrow \mathrm{SSP}$, where SSP_ω is a certain relaxation of SSP that has the advantage of being iterative. That is, SSP_ω is a union of conditions SSP_k, for $k \in \mathbb{N}$. The somewhat specific statement of Theorem 1 comes from the fact that it effectively translates $\mathrm{RC} \Rightarrow \mathrm{SSP}_3$ statement to a "normal" language;

- Lemma 7 in Section 2.2 proves Theorem 1 in case the antichain of minimal non-shattered elements has at most two elements;

- Theorem 2 in Section 3.1 gives a sufficient condition for the SSP_k condition in terms of RC graphs. We still call the corresponding condition SSP_k;

- Finally, Theorem 4, stated in Section 4.5 and proven in Sections 4.5 and 4.6, proves SSP_3 for RC graphs.

This also outlines the structure of the dissertation, on which we will now elaborate. Our approach involves two relaxations of the SSP property. In Chapter 2 we describe the first one, SSP_ω, which is a union of SSP_k conditions for $k \in \mathbb{N}$. This relaxation is in terms of antichain systems, described in detail in Section 2.1. This is a proper relaxation, that is, although SSP_ω implies SSP, we have not been able to prove the implication in the opposite direction. Despite that, it feels that the gap between SSP and SSP_ω is relatively narrow, and we would expect that they either hold or fail for RC lattices together. This relaxation proves to be helpful, as, in Theorem 1, we show that RC implies SSP_1 and SSP_2: The first one is straight from the definition, and the second involves a simple but handy property of RC lattices (Lemma 6), whose formulation is motivated by SSP_k setup. We also note that SSP_1 was already proven in [11].

The second relaxation in Chapter 3 is in terms of RC graphs. It relaxes each of SSP_k, and the corresponding property is still called SSP_k, but for graphs. Again, SSP_k for graphs implies SSP_k for RC lattices. Although we do not consider this relaxation to be tight, it turns out to be useful. Namely, we can use it to prove the graph version of SSP_3. This turns out to be rather complicated. The main part of the dissertation, that is, the remaining part of Chapter 3 and Chapter4, is dedicated to this goal; in a way, the RC graphs reformulation itself was designed to address SSP_3. Out of these sections, let us single out Sections 4.2 and 4.3: They deal with a simple case of RC graphs that are called RC graphs with trivial closures. Although the latter are not important for the main proof, and in principal these two sections can be omitted, we chose to include them for illustrative purposes. In particular, Theorem 3 in Section 4.3, which proves SSP_3 for RC graphs with trivial closures, serves as a template for Theorem 4, which proves SSP_3 in full generality.

Finally, in the concluding Chapter 5, we elaborate on the possibility of disproving SSP=RC conjecture. In Section 5.1, we give a promising result in this direction: We construct a rather simple RC_5-graph that fails to be SSP. However, the relaxation from RC lattices to RC graphs is far from being tight. In the remaining part of the section, we are developing some instruments that might help with constructing an RC lattice that, in some sense, corresponds to a given RC graph. The main tool for that is RC-pumping, described in Section 5.2. Then, in Section 5.3, we show how RC-pumping can be used to disprove, in the RC lattice setup, some SSP-like properties that hold for Boolean lattices. Finally, in Section 5.4, we show that using these constructions can be, in turn, problematic. In particular, we show that a pumping of an NMF lattice cannot provide a counterexample to SSP$\Rightarrow$RC conjecture.

CHAPTER 2

Preliminary results

2.1 Antichain reformulation

A *system* $\mathscr{E}$ over a finite *index set* $\mathscr{I}$ in a lattice L is an $\mathscr{I}$-set (that is, a set indexed by $\mathscr{I}$) of pairs in $L \times L$, $\mathscr{E} = \{(x_i, y_i) \mid i \in \mathscr{I}\}$, where all x_i are distinct and form an antichain, and $y_i \leq x_i$, for all $i \in \mathscr{I}$. Note that we do not require that the elements y_i are noncomparable, or even distinct. The *dimension* of $\mathscr{E}$ is the size of $\mathscr{I}$, $\dim(\mathscr{E}) = |\mathscr{I}|$. Additionally, for $\mathscr{E}$ thus defined, for every $i \in \mathscr{I}$ we denote $C_i = \{u \in L \mid u \wedge x_i = y_i\}$ and $S_i = [x_i) = \{u \in L \mid u \geq x_i\}$, and $C(\mathscr{E}) = \bigcup_{i \in \mathscr{I}} C_x$ and $S(\mathscr{E}) = \bigcup_{i \in \mathscr{I}} S_i$.

We say that a system $\mathscr{E}$ is SSP if $|S(\mathscr{E})| \leq |C(\mathscr{E})|$. For an integer k, we say that L satisfies SSP_k if any system $\mathscr{E}$ in L of dimension at most k is SSP, and we say that L satisfies SSP_ω if L satisfies SSP_k for all k. As the name suggests, there is a connection between SSP and SSP_ω, namely

Lemma 5 ($\mathrm{SSP}_\omega \Rightarrow \mathrm{SSP}$). *If a lattice satisfies* SSP_k*, then every set in it whose antichain of minimal non-shattered elements has at most k elements shatters at least as many elements as it has.*

In particular, if a lattice satisfies SSP_ω *then it satisfies* SSP.

Proof. Let L be a finite lattice satisfying SSP_k; let us take $F \subseteq L$ and let $S_F = L - \mathrm{Str}(L)$ be a set of elements non-shattered by F. As $\mathrm{Str}(L)$ is downward closed, S_F is upward closed, and hence $S_F = [X_F) = \{u \in L \mid u \geq x \text{ for some } x \in X_F\}$, where X_F is an antichain of minimal elements of S_F. For every $x \in X_F$ there is $y_x \leq x$ such that x is non-shattered by F *through* y_x, that is, there is no $u \in F$ such that $x \wedge u = y_x$. Fixing y_x for every $x \in X_F$ defines a system $\mathscr{E} = \{(x, y_x) \mid x \in X_F\}\}$ over X_F.

Now, assuming $|X_F| \leq k$, SSP_k implies $|S(\mathscr{E})| \leq |C(\mathscr{E})|$. Note that $S(\mathscr{E}) = [X_F) = L - \mathrm{Str}(F)$, and, consequently, $\mathrm{Str}(F) = L - S(\mathscr{E})$. Also, as for no $u \in F$ and $x \in X_F$ it can happen that $x \wedge u = y_x$, we have $F \subseteq L - C(\mathscr{E})$, and, consequently, $L - F \supseteq C(\mathscr{E})$. Then

$$|F| = |L| - |L - F| \leq |L| - |C(\mathscr{E})| \leq |L| - |S(\mathscr{E})| = |L - S(\mathscr{E})| = |\,\mathrm{Str}(F)|.$$

$\square$

In view of Lemma 3 and Lemma 5, in order to establish that SSP=RC, it is sufficient to prove that $\mathrm{RC} \Rightarrow \mathrm{SSP}_\omega$.

2.2 Systems of dimension 1 and 2

Let L be an RC lattice, for $x \leq y \leq z \in L$ we define $C(x,y,z) \subseteq L$ as a (nonempty) set of complements of y in $[x,z]$. We can then fix, in an arbitrary way, a *complementation function* $c\colon \{x,y,z \in L^3 \mid x \leq y \leq z\} \to L$, such that $c(x,y,z) \in C(x,y,z)$. Note that, for fixed x and y, the function $c(x,y,\cdot)$ is one-to-one. Indeed, if $c(x,y,z_1) = c(x,y,z_2) = u$, then $z_1 = z_2 = u \vee y$. Similarly, $c(\cdot,y,z)$ is one-to-one. We now prove an intermediate structural lemma about RC lattices.

Lemma 6. *Let L be an RC lattice. Then for arbitrary x_a, u, $x_b \in L$ there are elements u^- and u^+, $u^- \leq u \leq u^+$ such that*

$$u^- \vee x_a = u \vee x_a = u^+ \vee x_a, \qquad\qquad u^- \wedge x_b = u \wedge x_b = u^+ \wedge x_b,$$

and

$$u^+ \vee x_b \geq x_a, \qquad\qquad u^- \wedge x_a \leq x_b.$$

The statement is illustrated in Figure 2.1 below.

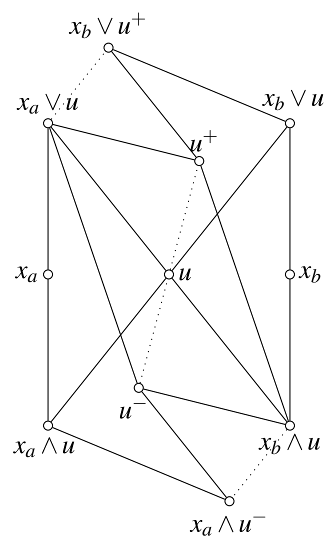


Figure 2.1: Illustration for Lemma 6.

Proof. The statement about u^- is dual to the one about u^+, so we only need to prove the latter. Now, let u^+ be a maximal element satisfying $u^+ \vee x_a = u \vee x_a$, $u^+ \wedge x_b = u \wedge x_b$ and $u^+ \geq u$. As u itself satisfies these conditions, such u^+ can be picked. We claim that $u^+ \vee x_b \geq x_a$. Indeed, let us take $w = c(u^+, (u^+ \vee x_a) \wedge (u^+ \vee x_b), u^+ \vee x_a)$; trivially, $u^+ \leq w$. As $u^+ \leq w \leq u^+ \vee x_a$, by taking a join with x_a, we get $w \vee x_a = u^+ \vee x_a = u \vee x_a$. Also,

$$\begin{aligned} w \wedge x_b &= \big(w \wedge (u^+ \vee x_a)\big) \wedge \big(x_b \wedge (u^+ \vee x_b)\big) \\ &= \big((u^+ \vee x_a) \wedge (u^+ \vee x_b)\big) \wedge w \wedge x_b = u^+ \wedge x_b = u \wedge x_b. \end{aligned}$$

So, w satisfies the conditions on u^+, and, by maximality of u^+, $w = u^+$. By the definition of w, this implies $(u^+ \vee x_a) \wedge (u^+ \vee x_b) = u^+ \vee x_a$, which is equivalent to $u^+ \vee x_b \geq u^+ \vee x_a$, implying $u^+ \vee x_b \geq x_a$. □

Lemma 7. *If $\mathscr{E}$ is a system of dimension* 1 *or* 2*, then* $|S(\mathscr{E})| \leq |C(\mathscr{E})|$.

Proof. First, let $\dim(\mathscr{E}) = 1$, that is, $\mathscr{E} = \{(x,y)\}$ for some $y \leq x$. Then, $S_{\mathscr{E}} = S_x = [x)$, and, as $x \wedge c(y,x,z) = y$ for all $z \geq x$, we get $\{c(y,x,z) \mid z \in S_x\} \subseteq C_x$. Thus, $c(y,x,\cdot)$ is an injective mapping from $S_{\mathscr{E}} = S_x$ to $C_{\mathscr{E}} = C_x$, proving $|S(\mathscr{E})| \leq |C(\mathscr{E})|$.

Now, let $\dim(\mathscr{E}) = 2$, and let $\mathscr{I} = \{a,b\}$, that is, $\mathscr{E} = \{(x_a,y_a),(x_b,y_b)\}$, for a pair of incomparable elements x_a, x_b, and for some $y_a \leq x_a$, $y_b \leq x_b$. Let $\alpha \colon S_a \to L$, $\beta \colon S_b - S_a \to C_b$ be defined as

$$\begin{aligned} &\beta(z) = c(y_b,x_b,z), \\ &\alpha(z) = \begin{cases} u = c(y_a,x_a,z), & \text{if } u \notin \beta[S_b - S_a]; \\ u^+, & \text{otherwise ,} \end{cases} \end{aligned}$$

where $u^+ = u^+(u,x_a,x_b)$ is an element provided by Lemma 6, for which $u^+ \geq u$, $u^+ \vee x_a = u \vee x_a = z$, $u^+ \wedge x_b = u \wedge x_b$ and $u^+ \vee x_b \geq x_a$. Note that we use u^+ in case $u \in \beta[S_b - S_a]$, that is, when $u = c(y_b,x_b,z)$ for some $z \in S_b - S_a$. But then $u^+ \wedge x_b = u \wedge x_b = y_b$, in particular, $u^+ \in C_b$.

As in the $\dim(\mathscr{E}) = 1$ case, β is injective to C_b. By the previous paragraph, $\alpha(z)$ also maps to $C = C_a \cup C_b$, and $x_a \vee \alpha(z) = z$, which proves that α is injective. We claim that $\alpha[S_a]$ is disjoint from $\beta[S_b - S_a]$. Indeed, if $w = \alpha(z_1) = \beta(z_2)$, then $w = u^+$ for some $u \in \beta[S_b - S_a]$. But then $z_2 = w \vee x_b = u^+ \vee x_b \geq x_a$, which is impossible, as $z_2 \in S_b - S_a$. □

We note that the construction of α and β has some level of arbitrariness, we could have just as well taken $\alpha \colon S_a - S_b \to C_a$, $\beta \colon S_b \to L$, and fix the values of β on elements mapped to $\alpha[S_a - S_b]$. An informal reason for that is that Lemma 6 allows a fair amount of symmetry; for $\mathscr{I} = \{a,b\}$, it can be equally applied to either x_a, v, x_b or to x_b, v, x_a, producing different modifications of the mappings. This phenomenon only intensifies for bigger systems: for $\dim(\mathscr{E}) = 3$, the number of potential applications is 6, and so on. We will try to make the best use of it with graph constructions in Section 3.1.

Before that, let us formulate another simple structural result that we are going to use.

Lemma 8. *Let L be an RC lattice, and suppose x_a, x_b, u, z, and $z^+ \in L$ are such that $u \vee x_b = z \geq x_a$, and $z^+ \geq z$. Then there is $u^+ \geq u$ such that $u^+ \wedge x_a = u \wedge x_a$ and $u^+ \vee x_b = z^+$.*

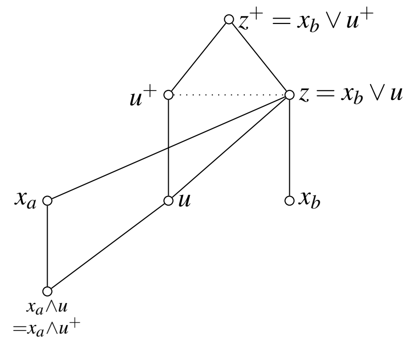


Figure 2.2: Illustration for Lemma 8.

Proof. Let $u^+ = c(u, z, z^+)$. Then $u^+ \geq u$. Also, as $z \geq x_a$, $u^+ \wedge x_a = u^+ \wedge z \wedge x_a = u \wedge x_a$. And $u^+ \vee x_b = u^+ \vee u \vee x_b = u^+ \vee z = z^+$. The argument is illustrated in Figure 2.2. $\square$

CHAPTER 3

Graph constructions

3.1 RC graphs

The idea of this section is to forget almost everything about an RC lattice L, including the fact that it is RC and that it is a lattice, and leave only the properties established in Lemma 6 and Lemma 8, but then milk them for all their worth. For this, we will need to establish a lot of terminology, so, before that, let us give a brief overview of our approach. We are going to represent the information about a system $\mathscr{E}$ over $\mathscr{I}$ in an RC lattice L as a bipartite graph G_L with black and white vertices, representing the elements of $S = S(\mathscr{E})$ and $C = C(\mathscr{E})$ respectively; moreover, if an element in L belongs to both S and C, it will produce two vertices, one black and one white, in G_L. The edges, colored with $i \in \mathscr{I}$, correspond to joins of the elements from C with x_i. Moreover, we will introduce black and white types, which we will use to track, first, whether for an element $u \in S \cup C$, it holds $u \vee x_a \geq x_b$, for $a, b \in \mathscr{I}$, and second, whether for $u \in C$ it holds $u \wedge x_a = y_a$, that is, whether $u \in C_a$, for $a \in \mathscr{I}$. Then, as it turns out, many valuable properties of systems over RC lattices, in particular the aforementioned lemmas, can be effectively reformulated and dealt with in terms of such graphs.

In the context of graph constructions, we will typically call the elements of the index set $\mathscr{I}$ *letters*. Also, we will drop the curly brackets when referring to the subsets of $\mathscr{I}$, that is, for $\mathscr{I} = \{a, b, c\}$, we write bc instead of $\{b, c\}$. Most of the upcoming definitions are over $\mathscr{I}$, that is, imply a fixed index set.

We define a *closure type* K over $\mathscr{I}$ as an intersection closed family of subsets of $\mathscr{I}$, in particular, we consider $\mathscr{I}$ to be an intersection of an empty set of subsets, and thus $\mathscr{I} \in K$. It is well known that any such family uniquely corresponds to a *closure operator*, that is, to an increasing, monotonous, and idempotent operator $\kappa \colon 2^{\mathscr{I}} \to 2^{\mathscr{I}}$. Given K, the corresponding operator κ is constructed by putting $\kappa(I)$ to be a unique minimal set in K above I, and, in the other direction, given κ, K is just the image, or, alternatively, the set of the fixed points of κ.

Yet another way of defining an intersection closed family is by the set of *implications*. Formally, an implication is an ordered pair (P, Q), denoted $P \to Q$, for $P, Q \subseteq \mathscr{I}$, where P is called the *premise*, and Q the *consequence* of the implication. Then $I \subseteq \mathscr{I}$ *admits* an implication $P \to Q$ if either $P \nsubseteq I$ or $Q \subseteq I$. A family $K \subseteq 2^{\mathscr{I}}$ *admits* an implication if every set in K admits it. Then, for a family $\mathscr{F}$ of implications, the family $K_{\mathscr{F}}$ of all sets admitting $\mathscr{F}$, that is, admitting every implication in $\mathscr{F}$, is intersection closed. For all practical purposes, it is typically enough to consider only certain classes of implications, in particular, we will only use the implications $P \to Q$ with $|Q| = 1$. Then, if $\mathscr{F} = \mathscr{F}(K)$ is a family of all implications admitted by K, then $K = K(\mathscr{F}(K))$.

For a closure type K we will write both $I \in K$, for $I \subseteq \mathscr{I}$, and $P \to Q \in K$, where the latter denotes the fact that K admits $P \to Q$. Note that admitting certain implications prohibits K from containing certain sets, for example, $a \to b \in K$ and $a \in K$ are incompatible. Note that the *top*, that is, the largest element in K, is always $\mathscr{I}$. However, K also has a unique *bottom*, that is, the smallest element, which can be different for different closure types. We consider the set of all closure types to be partially ordered by inverse inclusion, that is, $K_1 \leq K_2$ if $K_1 \supseteq K_2$. This partial order on closure types is a lattice, with intersection as join and union followed by adding all possible intersections as meet. Thus, the maximal closure type is $\{\mathscr{I}\}$, and the minimal is $2^{\mathscr{I}}$. Let us also note that $K_1 \leq K_2$ if and only if K_2 admits all implications that K_1 does, that is, if and only if $\mathscr{F}(K_1) \subseteq \mathscr{F}(K_2)$.

We define an *extended black type T'* simply as a closure-type such that $\emptyset \notin T'$. The latter is equivalent to saying that the bottom of T' is nonempty. And we define an *extended white type T'* as a closure-type $K_{T'}$, such that $\emptyset \in K_{T'}$, together with a nonempty subset $L_{T'} \subseteq \mathscr{I}$ of *capital letters* of T'.

As the names suggest, together with extended black and white types, we will have black and white types, which only carry partial information about their counterparts, but are easier to deal with. So, a *black type T* is a nonempty subset of $\mathscr{I}$. For an extended black type T', its black type $T = T(T')$ is the bottom of T'. And a *white type T* is a preorder A_T (that is, a reflexive and transitive relation) on $\mathscr{I}$, called the set of *arrows* of T, together with a nonempty subset $L_T \subseteq \mathscr{I}$ of *capital letters* of T. For an extended white type T', its white type $T = T(T')$ has the same set of capital letters, and A_T is the set of the implications with one-element premise and consequence admitted by $K_{T'}$, treated as a relation in $\mathscr{I} \times \mathscr{I}$, formally, $A_T = \{(a,b) \mid a \to b \in K_{T'}\}$. Complying with the notation for implications, we denote the elements of A_T by $a \to b$ instead of (a,b). We need to elaborate that, thus defined, A_T is reflexive and transitive. Indeed, for any $a \in \mathscr{I}$ and $I \in K_{T'}$, $a \in I$ implies $a \in I$, hence $K_{T'}$ admits $a \to a$, and hence $a \to a \in A_T$ for all $a \in \mathscr{I}$. Thus, A_T is reflexive. For the transitivity, suppose $a \to b, b \to c \in A_T$, that is, any $I \in K_{T'}$ admits $a \to b$ and $b \to c$. But then either $a \notin I$, or $a \in I$, which implies $b \in I$, which implies $c \in I$. In both cases, I admits $a \to c$, and so $K_{T'}$ admits $a \to c$, proving the transitivity of A_T.

Let us separately note that neither black nor white types carry all the information about their extended types. This is rather obvious for black types, as, say, a black type a can correspond to any extended black type (which is just a closure type) with bottom a. For example, the extended type here can be $\{a, abc\}$, or $\{a, ab, ac, abc\}$, or something in between (here $\mathscr{I}$ is implied to be abc). This is, however, also true for white types. Clearly, a white type T shares with its extended type T' the set of capital letters, so the information is lost when going from $K_{T'}$ to A_T. For example, $A_T = \{a \to a, b \to b, c \to c\}$ might be obtained from $K_{T'} = 2^{abc}$, or from $K_{T'} = \{\emptyset, a, b, c, abc\}$. While in the first case $K_{T'}$ admits only trivial implications, that is, only implications of the form $I \to I$, in the second case $K_{T'}$ also admits $ab \to c$, $ac \to b$, and $bc \to a$. These, however, have premises of size two, which are not captured by A_T.

Let us now introduce the order on black and white types, both extended and not, and show how they are related to each other. The extended black types are closure types, and the order on them remains the same. Black types are simply subsets of $\mathscr{I}$ and we order them by inclusion. Now, we claim that for extended black types T_1' and T_2', with black types T_1 and T_2 respectively, $T_1' \leq T_2'$ implies $T_1 \leq T_2$. Indeed, $T_1' \leq T_2'$ implies $T_1' \supseteq T_2'$. Recall that T_2 is the bottom of T_2', hence $T_2 \in T_2' \subseteq T_1'$, and T_1, which is the bottom of T_1', is a subset (not necessarily proper) of T_2.

For extended white types T_1' and T_2' we say that $T_1' \leq T_2'$ if $K_{T_1'} \leq K_{T_2'}$ and $L_{T_1'} \subseteq L_{T_2'}$. The fact that this is a partial order is obvious. And for white types T_1 and T_2 we say that $T_1 \leq T_2$ if $A_{T_1} \subseteq A_{T_2}$ and $L_{T_1} \subseteq L_{T_2}$. Then again, for extended white types T_1' and T_2' with white types T_1 and T_2 respectively, $T_1' \leq T_2'$ implies $T_1 \leq T_2$. Indeed, recall that $K_{T_1'} \leq K_{T_2'}$ is equivalent to $\mathscr{F}(K_{T_1'}) \subseteq \mathscr{F}(K_{T_2'})$. So if $T_1' \leq T_2'$ then $A_{T_1} = \mathscr{F}(K_{T_1'}) \cap \{a \to b \mid a,b \in \mathscr{I}\} \subseteq \mathscr{F}(K_{T_2'}) \cap \{a \to b \mid a,b \in \mathscr{I}\} = A_{T_2}$. And $L_{T_1} = L_{T_1'} \subseteq L_{T_2'} = L_{T_2}$. We also note that it will make sense for us to consider the white types (extended or not), to be preordered simply by their closure type (the set of arrows); we will not introduce a separate notation for that, as it can be indicated by saying that $K_{T_1'} \leq K_{T_2'}$ ($A_{T_1} \subseteq A_{T_2}$).

We will be dealing quite a lot with white types, so we are going to introduce additional notation for them, which we explain by example. In the list below, A_T^* stands for the set of *nontrivial* arrows of A_T, that is, $A_T^* = A_T - \{a \to a \mid a \in \mathscr{I}\}$. So let the white type T be defined as:

$A \to b$	$L_T = A$ and $A_T^* = \{a \to b\}$;
$a \to B \to c$	$L_T = B$ and $A_T^* = \{a \to b, a \to c, b \to c\}$. Here A_T is obtained as a reflexive transitive closure of the arrows $a \to b$ and $b \to c$, indicated in the notation;
$[Abc] \to d$	$L_T = A$ and $A_T^* = \{a \to b, b \to a, a \to c, c \to a, b \to c, c \to b, a \to d, b \to d, c \to d\}$. The $[Abc]$ part of the notation indicates, apart from the fact that $A \in L_T$, that a, b, and c are in the same equivalence class induced by the preorder A_T;
$a \to BC \to d$	$L_T = BC$ and $A_T^* = \{a \to b, a \to c, a \to d, b \to d, c \to d\}$. Arrows from a and to d are applicable to both b and c, but the fact that b and c are not in square brackets indicate that they are not in the equivalence class induced by A_T, and hence there are no arrows between them;
$A \to b, C$	$L_T = AC$ and $A_T^* = \{a \to b\}$. Comma indicates that this type is obtained as a join (in the poset of white types, which turns out to be a lattice) of $A \to b$ and C;

$ab \to c$ $\quad$ $L_T = \emptyset$ and $A_T^* = \{a \to b, b \to c\}$. The arrows are inferred similarly to the previous example, and T has no capital letters, which is allowed. This notation is not to be confused with an implication $ab \to c$, which only makes sense for a closure type, but cannot be captured by the arrows of a white type.

Finally, for a white type T_1, defined as above, and a white type T_2, we often write $T_1 \in T_2$ to denote $T_1 \leq T_2$. For example, $A, B \to c \in T$ indicate that $L_T \supseteq \{A, B\}$ and $A_T \supseteq \{b \to c\}$. Typically, we do it when T_1 is very simple.

We are now going to define an *RC graph*. The definition is long and will be interrupted for some comments and explanations. It contains points (G1)–(G11) and (E1)–(E3). So, an *RC graph* Γ over $\mathscr{I}$ is

(G1) Γ is a bipartite graph with partitions C_Γ and S_Γ. We call vertices in C_Γ *white* and in S_Γ *black*, and use similar rule when drawing these graphs. The set of all vertices of Γ is denoted by $V_\Gamma = C_\Gamma \sqcup S_\Gamma$;

Note. When working with RC graphs, we will omit the Γ underscript whenever the graph is clear from context, that is, write C, S, and V, instead of C_Γ, S_Γ, and V_Γ; same is true for other notation accociated with RC graphs. Typically, we use variables u, v, w for white vertices, x, y, z for black vertices, and a, b, c for letters of $\mathscr{I}$.

(G2) All edges are unoriented and colored with letters from $\mathscr{I}$; for $a \in \mathscr{I}$, we refer to an edge colored with a as to a-edge;

(G3) Every white (black) vertex u has a white (black) extended type $\mathscr{T}(u)$, associated with it. A white(black) type of u is the (non-extended) type corresponding to $\mathscr{T}(u)$, and it is denoted by $\tau(u)$. If the color of u is clear, we will refer to $\mathscr{T}(u)$ and $\tau(u)$ simply as to the extended type and the type of u respectively;

For white vertices we will sometimes need to address particular elements of their types. Thus, for $u \in C$, we write $L(u)$ instead of $L_{\tau(u)}$ or $L_{\mathscr{T}(u)}$, $A(u)$ instead of $A_{\tau(u)}$, and $K(u)$ instead of $K_{\mathscr{T}(u)}$. As for the latter, we also sometimes abuse the fact that extended black types are simply closure types, and write $K(x)$ as a synonym for $\mathscr{T}(x)$ for black vertices. Note that both $K(u)$ and $K(x)$ are closure-types, and so we call them *closure-types* of u and x respectively.

(G4) For every white vertex u and $a \in \mathscr{I}$, there is exactly one a-edge, adjacent to u, edges with different colors can be parallel. We denote by $\eta_a(u)$ a unique black vertex connected to u by an a-edge;

Sometimes we write $\eta_u(a)$ instead of $\eta_a(u)$, which is a matter of convenience. However, using η_u has an advantage: Note that η_u is a function from $\mathscr{I}$ to S, and we call a function like this a *neighborhood*. By (G4), setting the edges of Γ is equivalent to assigning neighborhoods to all its white vertices. This approach will turn out to be useful later.

(G5) The color of an edge adjacent to a black vertex x is contained in $\tau(x)$;

(G6) For a white vertex u and a, I such that $a \in I \subseteq \mathscr{I}$, $I \in K(u)$ if and only if $I \in K(\eta_a(u))$. In particular, for $a, b \in \mathscr{I}$, it holds $a \to b \in \tau(u)$ if and only if $b \in \tau(\eta_a(u))$;

Let us elaborate on how the first part of (G6) implies the second. Suppose $a \to b \in \tau(u)$, and let $I = \tau(\eta_a(u))$. Then I is the bottom of $K(\eta_a(u))$, in particular, $I \in K(\eta_a(u))$. By (G5), $a \in \tau(\eta_a(u)) = I$, and so $I \in K(u)$. But then $a \to b$ implies $b \in I$. In the other direction, suppose $b \in \tau(\eta_a(u))$. By (G5), $a \in \tau(\eta_a(u))$, and so $ab \subseteq I$, for any $I \in K(\eta_a(u))$. But then for any $I \in K(u)$ such that $a \in I$ it holds $b \in I$, and so $a \to b \in K(u)$, and, consequently, $a \to b \in \tau(u)$.

Let us note that (G6) means that for a white vertex u, its closure type is defined by the closure types of its *neighbors*, that is, black vertices $\{\eta_i(u) \mid i \in \mathscr{I}\}$.

(G7) Γ is equipped with a partial order $\trianglelefteq$ on its vertices, such that $u \trianglelefteq \eta_a(u)$ and $u \trianglelefteq v$ implies $\eta_a(u) \trianglelefteq \eta_a(v)$, for all white vertices u and v and all $a \in \mathscr{I}$. Moreover, $x \trianglelefteq u$ for no black vertex x and white vertex u;

(G8) For all (black or white) vertices u and v, $u \trianglelefteq v$ implies $K(u) \leq K(v)$;

We emphasize that (G8) is about closure types, not extended types. Practically, it means that for white vertices u and v, $u \trianglelefteq v$ does not imply $L(u) \subseteq L(v)$.

(G9) Γ is equipped with a *closure structure* $\mathscr{C}$, defined below;

A *closure structure* $\mathscr{C}$ over the set S of black vertices of Γ is a family of functions $\{c_I \colon S \to S \mid I \subseteq \mathscr{I}\}$, such that

(C1) For $x \in S$ and $I \subseteq \mathscr{I}$, $x \trianglelefteq c_I(x)$ and $I \cup \tau(x) \subseteq \tau(c_I(x))$;

(C2) For $I \subseteq \tau(x)$, $c_I(x) = x$, in particular, $c_I \circ c_I(x) = c_I(x)$ and $c_\emptyset(x) = x$, for all $x \in S$;

(C3) $c_I \circ c_J(x) = c_J \circ c_I(x) = c_{I \cup J}(x)$, for all $I, J \subseteq \mathscr{I}$ and $x \in S$.

Let us note that the definition of the closure structure only uses S, types of the vertices of S, and the order on S. Later we are going to use it in context of these relaxed conditions.

(G10) For a black vertex x, $\mathscr{T}(x) = \{\tau(c_I(x)) \mid I \subseteq \mathscr{I}\}$;

For a set $I \subseteq \mathscr{I}$, an I-path is a path such that all its edges have colors in I. In particular, we consider an empty path, that is, a path consisting of a single vertex, to be a $\emptyset$-path. For a finite sequence $R = (r_i)_{i=1}^n$, $r_i \in \mathscr{I}$, an R-path is a path $(e_i)_{i=1}^n$ such that the color of e_i is r_i. In both definitions, neither edges nor vertices are assumed to be distinct.

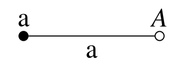
(G11) For black vertices x and y, if there is an I-path between x and y, then $c_I(x) = c_I(y)$. In particular, if in this case $I \subseteq \tau(x)$, then $\tau(x) \supseteq \tau(y)$, and if $I \subseteq \tau(x), \tau(y)$, then $x = y$.

Indeed, if $I \subseteq \tau(x)$, then $c_I(y) = c_I(x) = x$, and then, by (C1), $I \cup \tau(y) \subseteq \tau(x)$, which implies $\tau(y) \subseteq \tau(x)$. And if $I \subseteq \tau(x), \tau(y)$, then $x = c_I(x) = c_I(y) = y$.

While (G1)–(G11) outline the general structure of Γ and can be derived from a system in a lattice without the RC requirement, we will now add the *extension properties* (E1)–(E3), capturing the properties peculiar to RC case, in particular, Lemma 6 and Lemma 8.

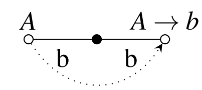
For black or white vertices u and v, and $a \in \mathscr{I}$, we say that u and v are a-neigbors if there is an a-edge between u and v. Note that if u is white, then its only a-neighbor is $\eta_a(u)$.

(E1) For a black vertex x and $a \in \tau(x)$ there is an a-neighbor u of x such that $A \in \tau(u)$;

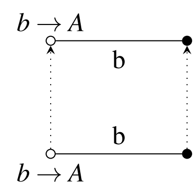


Note that (G5) and (E1) imply that the type of a black vertex is the set of colors of its adjacent edges. For a black vertex x and $a \in \tau(x)$, we denote by $\eta_a(x)$ the nonempty set of white vertices connected to x by an a-edge;

(E2) For a white vertex u and $a, b \in \mathscr{I}$, if $A \in \tau(u)$, then there is a b-b-path from u to (a white vertex) $u^+ \trianglerighteq u$ such that $A \to b \in \tau(u^+)$;



(E3) Let a white vertex u be such that $b \to A \in \tau(u)$, $x = \eta_b(u)$, and x^+ be a black vertex such that $x^+ \trianglerighteq x$. Then there is a white vertex $u^+ \trianglerighteq u$ such that $b \to A \in \tau(u^+)$ and $x^+ = \eta_b(u^+)$.



In particular, by putting $a = b$ in (E3), we get a statement resembling (E1): For black verices $x \trianglelefteq x^+$ such that $a \in \tau(x), \tau(x^+)$, for any a-neighbor u of x such that $A \in \tau(u)$, there is an a-neighbor u^+ of x^+ such that $u^+ \trianglerighteq u$ and $A \in \tau(u^+)$.

This concludes the definition of an RC graph.

Prior to giving an example of an RC graph and showing how RC graphs can be constructed from systems over RC lattices, let us bring to notice the apparent redundancy in this definition. First of all, all the information about the types (extended and not), except for capital letters of the white types, as well as the closure system, can be recovered solely from the edges of the graph. Indeed, (G6) enables us to restore the (extended) white types from the (extended) black types. As noted, (G5) and (E1) restore the black types from the edges, and (G10) restores the extended black types from the black types and the closure structure. Finally, the following proposition enables us to restore the closure structure from the edges and the black types.

For an RC graph Γ and $I \subseteq \mathscr{I}$, we define an *I-component* of Γ as a connected component of Γ after removing all edges with colors not in I.

Proposition 1. *For an RC graph Γ over $\mathscr{I}$, any $x \in S$ and $I \subseteq \mathscr{I}$, $y = c_I(x)$ is a unique black vertex in the same $(I \cup \tau(x))$-component of Γ as x, for which $I \cup \tau(x) \subseteq \tau(y)$. Moreover, for any $a \in \tau(x)$, there is an $(I \cup a)$-path from x to $c_I(x)$.*

Proof. Let us take $x \in S$, $a \in \tau(x)$, and $b \in I - \tau(x)$. By (E1), there is an a-neighbor $u \in C$ of x such that $A \in \tau(u)$. By (E2), there is $u^+ \trianglerighteq u$ such that $A \to b \in \tau(u^+)$ and there is a b-b-path between u and u^+. Then, by (G7), $x^+ = \eta_a(u^+) \trianglerighteq \eta_a(u) = x$. So, $\tau(x^+) \supseteq \tau(x)$. Also, by (G6), $A \to b \in \tau(u^+)$ implies $b \in \tau(x^+)$. Note also that there is an ab-path between x and x^+. Repeating this process for every letter from $I - \tau(x)$, we get a vertex $y \trianglerighteq x$ such that $\tau(y) \supseteq \tau(x) \cup I$ and there is an $(a \cup I)$-path between x and y. Note that if $I \subseteq \tau(x)$ then we can simply take $y = x$.

Clearly then, y is in the same $(I \cup \tau(x))$-component as x, and, by (G11), $c_{I \cup \tau(x)}(x) = c_{I \cup \tau(x)}(y) = y$, where the second equation is by (C2). Using (C3) and (C2), the latter can be rewritten as $y = c_{I \cup \tau(x)}(x) = c_I \circ c_{\tau(x)-I}(x) = c_I(x)$. Finally, if there are black vertices y_1 and y_2 in the same $(I \cup \tau(x))$-component as x, such that $I \cup \tau(x) \subseteq \tau(y_1), \tau(y_2)$, then $y_1 = c_{I \cup \tau(x)}(x) = y_2$, and thus such y is unique. □

Corollary 1. *Any connected component of Γ (which is the same as $\mathscr{I}$-component), has a unique vertex t, called the* top *of this component, such that $\tau(t) = \mathscr{I}$. Additionally, $t \trianglerighteq x$ for any black or white vertex x from this connected component. In case Γ is connected, we call t simply the top of Γ.*

If the types and closures can be restored from the basic structure of an RC graph, why then put them into the definition? The first answer is that they directly correspond to the basic properties of the elements of an RC lattice from which the graph is abstracted. Moreover, the extension conditions, abstracting the properties essential to proving SSP, are much easier to formulate in terms of types. Another reason is that a little later we are going to relax the definition of an RC graph (thus defining a *relaxed* RC graph, or an RRC graph), and, for RRC graphs, types and closures will no longer be derivable from edges and have to be imposed externally. Let us now proceed with an example of an RC graph.

We introduce the following notation convention. For a black or white type T, whenever we name a, respectively, black or white vertex by $\underline{T}_i$, we assume that $\tau(\underline{T}_i) \geq T$; here i is an optional index used to further distinguish the vertices. If in this case $\tau(\underline{T}_i) = T$, we say that this naming is *exact*. By default, all names of this kind used in the examples are exact.

Example 1. Figure 3.1 shows an example of an RC-graph Γ over $\mathscr{I} = abc$. All names of the vertices are exact. Dotted arrows indicate the *basis* of the order on the white vertices of Γ, where by basis we understand a relation whose reflexive transitive closure is the corresponding order.

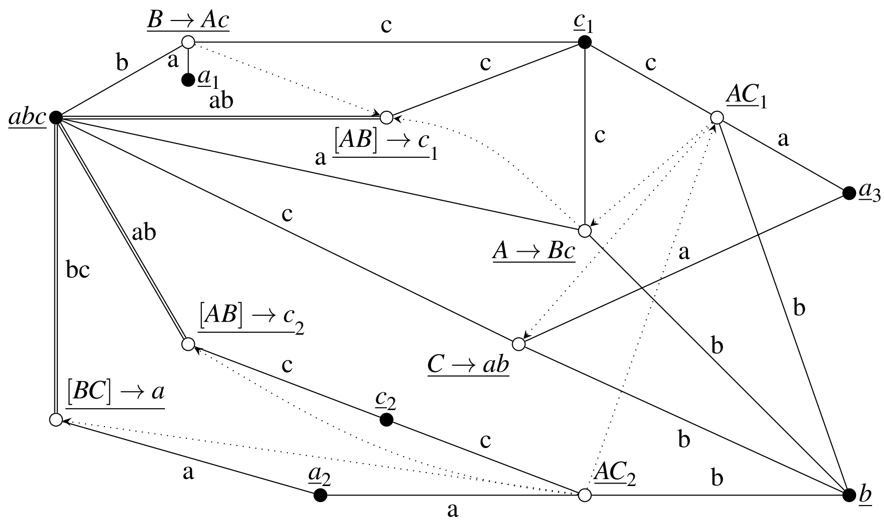


Figure 3.1: RC graph Γ.

Let us elaborate on this example. The properties (G1)–(G6) are either by construction or can be easily checked. In particular, the alignment of types, established by (G6), can be checked for types and can be used as a definition for the extended white types based on the extended black types.

Proposition 1 can be used to define the closure structure and thus to satisfy (G9). For Γ from Example 1, the closure structure turns out to be extremely simple. Let us formulate this as separate property. We say that an RC graph Γ has *trivial closures* if the types of all its black vertices are either single-element or equal to $\mathscr{I}$; note that the latter case, by Corollary 1, implies that the corresponding vertex is the top of its component. The reason for the name is that if Γ has trivial closures, then for every $x \in S$ and any $a \notin \tau(x)$, $c_a(x) = t$, where t is the top vertex in the same connected component as x. In particular, the functions c_I are completely defined as $c_I(x) = x$ if $I \subseteq \tau(x)$ and $c_I(x) = t$ otherwise. Similarly, by (G10), an extended type of any black vertex x is $\{\tau(x), \mathscr{I}\}$, which is just $\{\mathscr{I}\}$ whenever x is a top vertex.

As Γ is connected, it has a unique top vertex, which is $\underline{abc}$; recall that by Corollary 1 this implies $\underline{abc} \geq x$ for any black vertex x. Then the above argument uniquely defines the closure structure of Γ, and it is easy to check that the properties (G9)–(G11) are satisfied.

The order on the white vertices is given by its basis, for example, $\underline{[AB] \to c}_1 \trianglerighteq \underline{AC}_1$, as $\underline{[AB] \to c}_1 \trianglerighteq \underline{A \to Bc}$ and $\underline{A \to Bc} \trianglerighteq \underline{AC}_1$ are presented in the picture. If, like in this case, the order on the black vertices is not given, we asume that it is the minimal order implied by (C1) part of (G9), and by (G7), that is, it is a reflexive transitive closure of $\{\eta_a(u) \trianglelefteq \eta_a(v) \mid u \trianglelefteq v \in C, a \in \mathscr{I}\} \cup \{x \trianglelefteq c_I(x) \mid x \in S, I \subseteq \mathscr{I}\}$. Note that the second part does not tell anything other than $x \trianglelefteq \underline{abc}$ for any black vertex x. The first part, however, gives some nontrivial information that has to be carefully collected. In this example, the nonobvious ordered pairs produced by this rule, are $\underline{a}_2 = \eta_a(\underline{AC}_2) \trianglelefteq \eta_a(\underline{AC}_1) = \underline{a}_3$ and $\underline{c}_2 = \eta_c(\underline{AC}_2) \trianglelefteq \eta_c(\underline{AC}_1) = \underline{c}_1$. Finally, as the order between the black and the white vertices is not given explicitly, we assume that $u \trianglelefteq x$ whenever there are y, v, and a such that $\eta_a(v) = y \trianglelefteq x$, for $u \trianglelefteq v$. Then the nonobvios ordered pairs of this kind, that is, ordered pairs other than $u \trianglelefteq \underline{abc}$ and $u \trianglelefteq \eta_a(u)$, are $\underline{AC}_2 \trianglelefteq \underline{a}_3, \underline{c}_1$. It can be now easily checked that (G7) and (G8) are indeed satisfied.

Finaly, we need to explicitly check the extension conditions. (E1) is obvious, moreover, for all $x \in S$, except for the top vertex, and any $a \in \tau(x)$, the choice of the corresponding neighbor is unique, and checking (E2) is also straightforward. Perhaps the least obvious condition is (E3), but it also might be checked explicitly. For example, for $x = \underline{c}_2$, $u = \underline{AC}_2$, and $x^+ = \underline{c}_1$, the corresponding u^+ is $\underline{AC}_1$. Note that if we relax the order on the white vertices, and, instead of $\underline{AC}_2 \trianglelefteq \underline{AC}_1$, include into the basis $\underline{AC}_2 \trianglelefteq \underline{A \to Bc}$ and $\underline{AC}_2 \trianglelefteq \underline{C \to ab}$, then (E3) would break, but all other conditions would still hold.

We say that Γ is an RC$_k$-graph, or an RC graph of dimension k, for $k \geq 1$, if Γ is an RC graph over $\mathscr{I}$ with $|\mathscr{I}| = k$, and we say that Γ is SSP if $|C_\Gamma| \geq |S_\Gamma|$. Then Γ from Example 1 is an RC$_3$-graph, and, as $|C_\Gamma| = 8$ and $|S_\Gamma| = 7$, it is SSP. Let us now show how RC graphs are constructed from the systems in RC lattices.

Theorem 2. *Let L be an RC lattice and $\mathscr{E} = \{(x_i, y_i) \mid i \in \mathscr{I}\}$ be a system in L over $\mathscr{I}$. Let us define $\Gamma = \Gamma(L, \mathscr{E})$ as:*

- *S_Γ and C_Γ are disjoint copies of $S_\mathscr{E}$ and $C_\mathscr{E}$. For $z \in S_\mathscr{E}$, we denote by z_s its copy in S_Γ, and similarly, for $u \in C_\mathscr{E}$, u_c is its copy in C_Γ. Note that, for $v \in S_\mathscr{E} \cap C_\mathscr{E}$, the vertices v_s and v_c are different;*
- *For $a \in \mathscr{I}$ and $u_c \in C_\Gamma$, the a-neighbor of u_c is $z_s \in S_\Gamma$, for $z = u \vee x_a$;*
- *For $u1_c, u2_c \in C_\Gamma$, $u1_c \trianglelefteq u2_c$ if and only if $u1 \leq_L u2$, the order between $u_c \in C_\Gamma$ and $z_s \in S_\Gamma$, and between $z1_s, z2_s \in S_\Gamma$, is defined similarly. Note that for $z_s \in S_\Gamma$ and $u_c \in C_\Gamma$, $z_s \ntrianglelefteq u_c$, even if $z \leq_L u$;*
- *For $z_s \in S_\Gamma$ and $I \subseteq \mathscr{I}$, $c_I(z_s) = \left(z \vee \bigvee\{x_i \mid i \in I\}\right)_s$;*
- *For $u_c \in C_\Gamma$ and $a \in \mathscr{I}$, $A \in \tau(u_c)$ if and only if $u \wedge x_a = y_a$;*
- *For a black vertex z_s, its closure type (which is the same as its extended type), defined by implications, is $K(z_s) = K(\mathscr{F}(z_s))$, for*

$$\mathscr{F}(z_s) = \left\{ I \to a \mid I \subseteq \mathscr{I}, a \in \mathscr{I}, z \vee \bigvee\{x_i \mid i \in I\} \geq_L x_a \right\},$$

  and its type is

$$\tau(z_s) = \{a \in \mathscr{I} \mid z \geq_L x_a\}.$$

Similarly, for a white vertex u_c, its closure type is $K(u_c) = K(\mathscr{F}(u_c))$ for

$$\mathscr{F}(u_c) = \left\{ I \to a \mid I \subseteq \mathscr{I} - \emptyset, a \in \mathscr{I}, u \vee \bigvee \{x_i \mid i \in I\} \geq_L x_a \right\},$$

and the set of arrows $A(u_c)$ of its type is

$$A(u_c) = \{a \to b \mid a, b \in \mathscr{I}, u \vee x_a \geq_L x_b\}.$$

Then thus defined Γ is an RC graph over $\mathscr{I}$. Moreover, $|S_\Gamma| = |S_\mathscr{E}|$ and $|C_\Gamma| = |C_\mathscr{E}|$. In particular, Γ is SSP if and only if $\mathscr{E}$ is.

Proof. A big part of the proof deals with the closure types and their representation by implications, so prior to checking the conditions for Γ, let us prove the following: For $z_s \in S_\Gamma$, $\mathscr{F}(K(z_s)) = \mathscr{F}(z_s)$, and similarly, for $u_c \in C_\Gamma$, $\mathscr{F}(K(u_c)) = \mathscr{F}(u_c)$.

We are only going to prove the first one, as the proof of the second one is similar. Let us, however, first elaborate on what is that statement that we are proving. So, $\mathscr{F}(z_s)$ is a set of implications, explicitly defined in the statement of the theorem, based on how z joins with x_i's in L. Then $K(\mathscr{F}(z_s))$ is a closure type, that is, an intersection-closed family; Note that whatever the set $\mathscr{F}$ of implication is, $K(\mathscr{F})$ is always intersection closed. Then $K(z_s)$, that is, the closure type of z_s, is defined to be $K(\mathscr{F}(z_s))$. Now, we are interested in the set $\mathscr{F}(K(z_s)) = \mathscr{F}(K(\mathscr{F}(z_s)))$ of all implications, satisfied by $K(z_s)$. In general, for an arbitrary $\mathscr{F}$, $\mathscr{F}(K(\mathscr{F})) \supseteq \mathscr{F}$, and the containment can be proper. For example, if $\mathscr{F} = \{a \to b, b \to c\}$, then $\mathscr{F}(K(\mathscr{F})) = \{a \to b, b \to c, a \to c\} \cup \{I \to a \mid I \subseteq \mathscr{I}, a \in I\}$. Our goal is then to prove that, for thus defined $\mathscr{F}(z_s)$, this containment becomes an equality.

As noted, $\mathscr{F}(K(z_s)) \supseteq \mathscr{F}(z_s)$. In the other direction, let us take an arbitrary implication $I \to a \in \mathscr{F}(K(z_s))$. It is well known and easy to check that the latter holds if and only if $a \in I'$, where I' is the minimal set in $K(z_s)$ above I. We claim that $I' = I'' = \{b \in \mathscr{I} \mid z \vee \bigvee \{x_i \mid i \in I\} \geq_L x_b\}$. Note that, by construction, $I \to b \in \mathscr{F}(z_s)$, for all $b \in I''$, and so proving $I' = I''$ would imply $I \to a \in \mathscr{F}(z_s)$.

Trivially, $I'' \supseteq I$. To prove that $I'' \in K(z_s)$, let us take an arbitrary $J \to b \in \mathscr{F}(z_s)$ and show that I'' admits it. If $J \not\subseteq I''$, this holds automatically, so let $J \subseteq I''$. As $J \to b \in \mathscr{F}(z_s)$, then

$$x_b \leq_L z \vee \bigvee_{i \in I} x_i \leq_L z \vee \bigvee_{i \in I''} x_i \leq_L z \vee \bigvee_{i \in I''} \left(\bigvee_{j \in I} x_j \right) = z \vee \bigvee_{j \in I} x_j,$$

and so $b \in I''$, and thus I'' admits $J \to b$. Finally, suppose there is $J \in K(z_s)$ such that $I \subseteq J \subsetneq I''$, and let $b \in I'' - J$. But then J does not admit an implication $I \to b \in \mathscr{F}(z_s)$. This proves that $I' = I''$, and that

$I \to a \in \mathscr{F}(z_s)$, as needed.

Let us now proceed with checking that Γ is indeed an RC graph. The conditions (G1), (G2), and (G4) are by construction; (G3) is also by construction, but we need to separately check that the bottom of a closure type $K(z)$ of z is nonempty whenever z is black, and empty whenever z is white; Recall that this restriction is imposed by the definition of extended black and white types respectively. Moreover, as we define both types and extended types explicitly, we need to check that they agree with each other. Indeed, for a black vertex z_s, let $J = \tau(z_s) = \{a \in \mathscr{I} \mid z \geq_L x_a\}$. Then $\emptyset \to j \in K(z_s)$, for all $j \in J$, and also $I \to a \notin K(z_s)$, for all $I \subseteq J$ and $a \notin J$. This implies that J is indeed the bottom of $K(z_s)$, moreover, $z_s \in S_{\mathscr{E}}$ implies $J \neq \emptyset$, as required by the definition of the black type. And, for a white vertex u_c, notice that, by construction, $K(u_c)$ contains no implications of the form $\emptyset \to a$, and so $\emptyset \in K(u_c)$. Also,

$$
\begin{aligned}
A(u_c) &= \{a \to b \mid a, b \in \mathscr{I}, u \vee x_a \geq_L x_b\} \\
&= \{a \to b \mid a \to b \in \mathscr{F}(u_c)\} = \{a \to b \mid a \to b \in \mathscr{F}(K(u_c))\},
\end{aligned}
$$

and so the type of u_c corresponds to its extended type, as needed.

For (G5), it is enough to note that if an a-edge is adjacent to some z_s, then $z = u \vee x_a$ for some $u \in C_{\mathscr{E}}$, and hence $z \geq x_a$ and $a \in \tau(z_s)$. For (G6), let us take an arbitrary white vertex u_c and $a \in \mathscr{I}$, and let $z_s = \eta_a(u_c)$, that is, $z = u \vee x_a$. Then, straight from the definition of $\mathscr{F}(u_c)$ and $\mathscr{F}(z_s)$, for any $I \subseteq \mathscr{I}$ such that $a \in I$ and any $b \in \mathscr{I}$ it holds $I \to b \in \mathscr{F}(u_c)$ if and only if $I \to b \in \mathscr{F}(z_s)$. Recall that an intersection closed family is a set of fixed points of its coresponding closure operator, which, as we argued, is $I \mapsto I' = \{b \mid I \to b \in \mathscr{F}(u_c)\}$, and similarly for z_s. But then, for I such that $a \in I$, $I = \{b \mid I \to b \in \mathscr{F}(u_c)\}$ if and only if $I = \{b \mid I \to b \in \mathscr{F}(z_s)\}$, and hence $I \in K(u_c)$ if and only if $I \in K(z_s)$.

The conditions (G7), (G8), and (G9), establishing the order on the vertices and the closure structure, and relating the latter to the closure types, are obvious by construction. To prove (G10), let $I \in \mathscr{T}(z_s)$, and let $z' = z \vee \bigvee\{x_i \mid i \in I\}$, which means $z'_s = c_I(z_s)$. Then $\tau(z'_s) \supseteq I$. On the other hand, if $\tau(z'_s) \supsetneq I$, then there is some $b \in \tau(z') - I$ and, by the definition of z' and $\mathscr{T}(z_s)$, this implies $I \to b \in \mathscr{T}(z_s)$, and hence $I \notin \mathscr{T}(z_s)$, a contradiction. This proves $\mathscr{T}(z_s) \subseteq \{\tau(c_I(z_s)) \mid I \subseteq \mathscr{I}\}$. In the other direction, let us take an arbitrary $I \subseteq \mathscr{I}$ and let, as before, $z' = z \vee \bigvee\{x_i \mid i \in I\}$, implying $z'_s = c_I(z_s)$, and let $J = \tau(z'_s) = \{a \in \mathscr{I} \mid z' \geq_L x_a\}$; In particular, $J \supseteq I$. But then trivially $z' = z \vee \bigvee\{x_i \mid i \in J\}$ and $J \to b \notin \mathscr{T}(z_s)$ for any $b \notin J$. Thus, $J \in \mathscr{T}(z_s)$, finishing the proof of (G10).

For (G11), note that if there is an I-path between any black vertices $z1_s$ and $z2_s$, it easily implies $z1 \vee \bigvee\{x_i \mid i \in I\} \geq z2$ and, consequently, $z1' = z1 \vee \bigvee\{x_i \mid i \in I\} = z2 \vee \bigvee\{x_i \mid i \in I\} = z2'$. But, by definition, $z1'_s = c_I(z1) = z2'_s = c_I(z2)$.

Finally, let us prove the extension conditions. For (E1), let z_s be a black vertex such that $a \in \tau(z_s)$, then $x_a \leq_L z$ and for $u = c(y_a, x_a, z)$ it holds $u \vee x_a = z$, that is, $\eta_a(u) = z$, and $u \wedge x_a = y_a$, that is, $A \in \tau(u)$. For (E2), let us take a white vertex u_c such that $A \in \tau(u_c)$, meaning $u \wedge x_a = y_a$. Let us take u^+, guaranteed by Lemma 6, such that $u^+ \geq_L u$, $u^+ \wedge x_a = u \wedge x_a = y_a$, $u^+ \vee x_b = u \vee x_b$, and $u^+ \vee x_a \geq_L x_b$. Then $u_c^+ \trianglerighteq u_c$, $A \to b \in \tau(u_c^+)$, and u_c to $(u \vee x_b)_s$ to u_c^+ is a b-b-path from u_c to u_c^+. Finally, (E3) is a direct counterpart of Lemma 8. $\square$

Corollary 2. *If every* RC_k*-graph is* SSP *then every RC lattice satisfies* SSP_k.

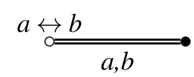
We note that Lemma 6 can be strengthened, in a straightforward way, to enable a slightly more general version of (E2)

(E2*) For a white vertex u, $I \subseteq \mathscr{I}$, and $b \in \mathscr{I}$, if $I \subseteq L(u)$, that is, if all letters in I are capital in $\mathscr{T}(u)$, then there is a b-b-path from u to (a white vertex) $u^+ \trianglerighteq u$ such that $I \subseteq L(u^+)$ and $K(u^+)$ admits an implication $I \to b$.

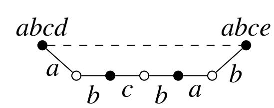
This property will not be used, so we do not include it in the definition of an RC graph and give it just for a reference. Let us also state a couple of additional conditions satisfied by RC graphs, which follow from the ones in the definition.

Lemma 9. *An RC graph satisfies (E4) and (E5), where*

(E4) For a white vertex u, $a \leftrightarrow b \in \tau(u)$ if and only if $\eta_a(u) = \eta_b(u)$, that is, if and only if a and b-neighbors of u coincide;



(E5) If black vertices x and y are connected by $\tau(x) \cap \tau(y)$-path, then $x = y$.



Proof. (E5). Let Γ be the RC graph in question, and suppose x and y are black vertices of Γ, such that there is an $(\tau(x) \cap \tau(y))$-path ρ between x and y. Then ρ is also a $\tau(x)$-path, and hence, by (G11), $\tau(x) \supseteq \tau(y)$. Similarly, ρ is a $\tau(y)$-path, so $\tau(y) \supseteq \tau(x)$, and hence $\tau(x) = \tau(y)$. But then, again by (G11), $x = y$, finishing the proof.

(E4, $\Rightarrow$). For a white vertex u with $a \leftrightarrow b \in \tau(u)$, let e_a and e_b be a and b-edges of u, and let $x_a = \eta_a(u)$ and $x_b = \eta_b(u)$. As $a \to a$ and $a \to b \in \tau(u)$ (the first one by the reflexivity of $A(u)$), by (G6), we have $a, b \in \tau(x_a)$; similarly, $a, b \in \tau(x_b)$, so $a, b \in \tau(x_a) \cap \tau(x_b)$. But then (e_a, e_b) is a $\tau(x_a) \cap \tau(x_b)$-path between x_a and x_b, so, by (E5), $x_a = x_b$.

(E4, $\Leftarrow$). If, for $a, b \in \mathscr{I}$, a and b-neighbors of u are the same black vertex x, then, by (G5), $a, b \in \tau(x)$. But then, by (G6), $a \to b, b \to a \in \tau(u)$. $\square$

3.2 Some transformations of RC graphs

We will now give three simple considerations which would enable us, in context of proving SSP_k, to restrict our attention to RC graphs satisfying some additional properties.

Lemma 10. *Let Γ be an RC graph over $\mathscr{I}$, and let Ω be a connected component of Γ (with the order on the vertices and the closure system respectively restricted to Ω). Then Ω is an RC graph over $\mathscr{I}$.*

Proof. The only nonobvious part of the lemma is that the closure system can be properly restricted to Ω, that is, that for $I \subseteq \mathscr{I}$ and a black vertex $x \in \Omega$, $c_I(x) \in \Omega$. But this is a direct consequence of Proposition 1. $\square$

Lemma 11. *Let Γ be an RC graph over $\mathscr{I}$, let η be a neighborhood, that is, $\eta : \mathscr{I} \to S$, and let $U \subseteq C = \{u \in C \mid \eta_u = \eta\}$; suppose also that U is nonempty. Let Γ' be a graph obtained from Γ by removing U from C and adding a white vertex u' in the following way:*

- $\eta_{u'} = \eta$;
- $K(u') = K(u)$, *for all* $u \in U$;
- $L(u') = \bigcup\{L(u) \mid u \in U\}$;
- $u' \trianglelefteq u'$, *and, for* $v \in C \sqcup S - U$, $v \trianglelefteq u'$ *if and only if* $v \trianglelefteq u$ *for some* $u \in U$, *and* $v \trianglerighteq u'$ *if and only if* $v \trianglerighteq u$ *for some* $u \in U$;
- *The order on the remaining vertices is modified by adding* $v \trianglelefteq x$ *whenever there are* $u_1, u_2 \in U$ *such that* $v \trianglelefteq u_1$ *and* $u_2 \trianglelefteq x$, *for* $v \in C - U$ *and* $x \in C \sqcup S - U$.

Then Γ' is an RC graph.

Proof. First of all, let us note that $K(u_1) = K(u_2)$ for all $u_1, u_2 \in U$ is by (G6). Now, most of the properties of an RC graph trivially hold for Γ', except for those related to the order on vertices. In particular, (G1)–(G6) are by construction.

Let us prove that, thus modified, the order indeed remains partial order. Through the argument, by $\trianglelefteq$ we denote the original order on Γ, and by $\trianglelefteq'$ its modification on Γ'; similarly, C' is the set of white vertices of Γ', and $S' = S$ is the set of its black vertices. Trivially, $\trianglelefteq'$ is reflexive, and $x \trianglelefteq v$ holds for no $x \in S'$ and $v \in C'$. We thus need to check transitivity and antisymmetry. Before that, let us make two straightforward observations. First, U is *convex*, that is, $u_1 \trianglelefteq v \trianglelefteq u_2$, for $u_1, u_2 \in U$, implies $v \in U$, in particular, v is white in Γ. Second, the order is not modified on black vertices, that is, for $x_1, x_2 \in S$, $x_1 \trianglelefteq' x_2$ if and only if $x_1 \trianglelefteq x_2$.

For transitivity, let us take $x, y, z \in \Gamma'$ such that $x \trianglelefteq' y$ and $y \trianglelefteq' z$, and prove that $x \trianglelefteq' z$; we can easily assume that x, y, and z are pairwise distinct, as otherwise the statement is trivial. First, suppose $u' = x$, which, in particular, means that $y,z \in C \sqcup S - U$. Then there is $u_1 \in U$ such that $u_1 \trianglelefteq y$. Suppose $y \not\trianglelefteq z$, which means that there is $u_2 \in U$ such that $y \trianglelefteq u_2 \trianglelefteq z$. But then, by convexity of U, $u_1 \trianglelefteq y \trianglelefteq u_2$ implies $y \in U$, which cannot happen. Then $y \trianglelefteq z$, but then, by transitivity of $\trianglelefteq$, $u_1 \trianglelefteq z$, and hence $x = u' \trianglelefteq z$. The case $u' = z$ is handled similarly. And, if $y = u'$, then $x,z \in C \sqcup S - U$, and there are $u_1, u_2 \in U$ such that $x \trianglelefteq u_1$ and $u_2 \trianglelefteq z$. But then $x \trianglelefteq' z$.

So we are left with the case $x,y,z \in C \sqcup S - U$. If $x \trianglelefteq y \trianglelefteq z$ then $x \trianglelefteq z$ and, consecutively, $x \trianglelefteq' z$. So either $x \not\trianglelefteq y$ or $y \not\trianglelefteq z$. Note that if it is both, then an argument similar to the one above shows $y \in U$. So let us assume $x \not\trianglelefteq y$ and $y \trianglelefteq z$, the other case is similar. Then there are $u_1, u_2 \in U$ such that $x \trianglelefteq u_1$ and $u_2 \trianglelefteq y$. But, from the latter, $u_2 \trianglelefteq z$, and hence $x \trianglelefteq' z$, finishing the proof of transitivity.

For antisymmetry, assume that for some $x \neq y$ it holds $x \trianglelefteq' y$ and $y \trianglelefteq' x$. If $x = u'$, this implies $u_1 \trianglelefteq y$ and $y \trianglelefteq u_2$, for $u_1, u_2 \in U$, which again implies $y \in U$, a contradiction; $y = u'$ case is similar. Then $x, y \in C \sqcup S - U$. If $x \not\trianglelefteq y$ and $y \not\trianglelefteq x$, then, similarly, $x, y \in U$. So suppose $x \trianglelefteq y$ and $y \not\trianglelefteq x$, that is, there are $u_1, u_2 \in U$ such that $y \trianglelefteq u_1$ and $u_2 \trianglelefteq x$. But then $x \trianglelefteq u_1$ and hence $x \in U$, a contradiction. This finishes the proof of antisymmetry.

So, $\trianglelefteq'$ is indeed a partial order, which proves a part of (G7) for Γ'. The condition $u \trianglelefteq' \eta_a(u)$ for all $u \in C'$ and $a \in \mathscr{I}$ holds trivially. To finish the proof of (G7), we need to show that $v \trianglelefteq' w$ implies $\eta_a(v) \trianglelefteq' \eta_a(w)$. The proof is only nontrivial whenever $v \not\trianglelefteq w$. If $v = u'$, then there is $u \in U$ such that $u \trianglelefteq w$. But then $\eta_a(v) = \eta_a(u) \trianglelefteq \eta_a(w)$; the case $u' = w$ is similar. Now, let us assume that there are $u_1, u_2 \in U$ such that $v \trianglelefteq u_1$ and $u_2 \trianglelefteq w$. Then $\eta_a(v) \trianglelefteq \eta_a(u_1) = \eta_a(u_2) \trianglelefteq \eta_a(w)$, and so again $\eta_a(v) \trianglelefteq \eta_a(w)$.

The proof of (G8) is similar. The closure structure remains unmodified, and so is the order on black vertices and their types. So, (G9)-(G11) remain intact. Finally, checking (E1)–(E3) is straightforward. □

Lemma 12 (No $a \to B$)**.** *Let Γ be an RC graph over $\mathscr{I}$. Then there is an RC graph Γ^*, which differs from Γ only by the sets of capital letters of its white types, such that for any white vertex u in Γ^*, its type $T = \tau(u)$ satisfies the following property*

(T) If $a \to B \in T$ then $A \in T$.

Note. We assume that the type and the extended type of a white vertex share the set of capital letters, so if it is modified for one, it is automatically modified for another.

Proof. Let Γ be an RC graph, and Γ^* its copy with the sets of capital letters of the white vertices changed as follows. For u in C, the set of capital letters $L^*(u)$ of u in Γ^* is defined as

$$L^*(u) = \{a \in \mathscr{I} \mid \text{ there is } b \in I \text{ s.t. } a \to B \in \tau(u)\},$$

in particular, $K^*(u) \supseteq K(u)$.

Now, we need to check that Γ^* is indeed an RC graph, which, trivially, amounts to checking (E2) and (E3), and that the white types of Γ^* do satisfy (T).

The white types of Γ^* *satisfy (T).* Indeed, suppose $a \to B \in \tau^*(u)$, where $\tau^*(u)$ is the type of u in Γ^*. Then $a \to b \in \tau(u)$ and there is $c \in I$ such that $b \to C \in \tau(u)$, leading to $a \to C \in \tau(u)$ and $A \in \tau^*(u)$.

Γ^* *satisfies (E2).* Let u be a white vertex and $a, b \in \mathscr{I}$ are such that $A \in \tau^*(u)$. Then there is $c \in \mathscr{I}$ such that $a \to C \in \tau(u)$ and hence, by (E2) for Γ, there is a b-b-path from u to $u^+ \trianglerighteq u$ such that $C \to b \in \tau(u^+)$. From the latter, $a \to C \to b \in \tau(u^+)$, and hence $A \to C \to b \in \tau^*(u^+)$.

Γ^* *satisfies (E3).* Let a white vertex u be such that $b \to A \in \tau^*(u)$, $x = \eta_b(u)$, and $x^+ \trianglerighteq x$. Then, for some $c \in I$, it holds $b \to a \to C \in \tau(u)$, and, by (E3) for Γ, there is $u^+ \trianglerighteq u$ such that $x^+ = \eta_b(u^+)$ and $C \in \tau(u^+)$. As $u^+ \trianglerighteq u$, $b \to a \to C \in \tau(u^+)$, and hence $b \to A \in \tau^*(u^+)$. □

Corollary 3. *If every connected* RC_k*-graph such that*

(N) no two distinct white vertices have the same neighborhood, and

(T) all white types satisfy (T),

is SSP*, then every* RC_k*-graph is* SSP.

Proof. This is just a consequtive applications of Lemma 10, Lemma 11, and Lemma 12. The key thing to observe is that if SSP fails in Γ, then it fails in one of its connected components Γ_1. Then, if it fails in Γ_1, then it fails in Γ_2, obtained from Γ_1 be consequtively collapsing its white vertices with identical neighborhoods using Lemma 11, thus imposing (N). Finally, an application of Lemma 12 to Γ_2 results in Γ_3, which is connected and satisfies (N) and (T), and which is SSP if and only if Γ_2 is. □

In view of Corollary 3, in the rest of the dissertation we will always assume that all RC graphs we consider are connected, and satisfy (N) and (T). Let us note that Γ from Example 1 is an example of such RC graph.

3.3 Relaxed RC graphs and morphisms

We will now introduce several intermediary definitions that will serve as a framework for proving SSP for RC_3-graphs. Namely, we define a *relaxed RC graph* (RRC graph for short) F over $\mathscr{I}$ as follows:

(G1) F is a bipartite graph with partitions C and S of white and black vertices;

(G2) All edges are unoriented and colored with letters from $\mathscr{I}$;

(G3*) Every black vertex has an extended black type and the corresponding black type associated with it. Every white vertex u has an associated set of capital letters $L(u) \subseteq \mathscr{I}$;

(G4*) For every white vertex u and $a \in \mathscr{I}$, there is *at most* one a-edge, adjacent to u. We denote by $\eta_a(u)$ a unique black vertex connected to u by an a-edge, and write $\eta_a(u) = \mathrm{NA}$ whenever there is no a-edge from u;

(G5) The color of an edge adjacent to a black vertex x is contained in $\tau(x)$.

As it can be seen, RRC graphs are, according to their naming, relaxed RC graphs, without closure types for white vertices, ordering, closure structure, extension conditions, and potentially lacking some edges. In particular, any RC graph is an RRC graph. For RRC graphs G and H, we define a *weak homomorphism* $\varphi\colon G \to H$ as a mapping of vertices of G such that $\varphi\colon C_G \to C_H$, $\varphi\colon S_G \to S_H$, such that:

- for any $x \in S_G$ it holds $\mathscr{T}(x) \leq \mathscr{T}(\varphi(x))$;
- for any $u \in C_G$ it holds $L(u) \subseteq L(\varphi(x))$, and $\varphi \circ \eta_a(u) = \eta_a \circ \varphi(u)$ whenever $\eta_a(u) \neq \mathrm{NA}$, for any $a \in \mathscr{I}$.

As the name suggests, we will also have homomorphisms of RC graphs that will be more important to us, and thus will be elaborated upon more thoroughly.

We define a *black set* S over the index set $\mathscr{I}$ as a set, such that every *black vertex* $x \in S$ has an extended black type $\mathscr{T}(x)$, and the corresponding black type $\tau(x)$, associated to it. Clearly, for an RRC graph F, S_F with the corresponding types is a black set, which we call the black set of F. We denote by $\underline{S}$ the set of tuples $\underline{S} = \{(x,a) \mid x \in S, a \in \tau(x)\}$. Also, for $a \in \mathscr{I}$, we define $S_a = \{x \in S \mid a \in \tau(x)\}$ and $\underline{S_a} = \{(x,a) \in \underline{S}\}$; with an abuse of notation we often identify S_a and $\underline{S_a}$ with an obvious bijection. Let us note, however, that the sets $\{S_a \mid a \in \mathscr{I}\}$ are not disjoint, while $\{\underline{S_a} \mid a \in \mathscr{I}\}$ are, moreover $\underline{S} = \bigsqcup\{\underline{S_a} \mid a \in \mathscr{I}\}$.

We define a *homomorphism* $\theta\colon G \to H$ between RRC graphs G and H sharing the same black set as a weak homomorphism between G and H, which is the identity on the black vertices. Obviously, θ then can be considered a mapping $\theta\colon C_G \to C_H$ such that for any $u \in C_G$ it holds $L(u) \subseteq L(\varphi(x))$, and $\varphi \circ \eta_a(u) = \eta_a \circ \varphi(u)$ whenever $\eta_a(u) \neq \mathrm{NA}$, for any $a \in \mathscr{I}$.

Similarly, we define subgraphs and congruences, relating RRC graphs sharing the same black set. Namely, for RRC graphs G and H such that $S_G = S_H = S$ (as black sets), we say that G is a *subgraph* of H if $C_G \subseteq C_H$ and for every $u \in C_G$ it holds $K_G(u) \subseteq K_H(u)$ and $\eta_a^G(u) = \eta_a^H(u)$ whenever $\eta_a^G(u) \neq \mathrm{NA}$, for any $a \in \mathscr{I}$. In other words, G is a subgraph of H if the identity map from C_G to C_H is a homomorphism. Finally, an equivalence relation Θ on C_G is a *congruence* if for any equivalence class $[U]$ of Θ and any $u, v \in [U]$, it holds $\eta_a(u) = \mathrm{NA}$, or $\eta_a(v) = \mathrm{NA}$, or $\eta_a(u) = \eta_a(v)$, for any $a \in \mathscr{I}$. Congruences are naturally ordered by inclusion (as relations), that is, for congruences Ω and Θ, $\Omega \leq \Theta$ if $u\Omega v$ implies $u\Theta v$. However, some properties of the congruences of RRC graphs contrast with "normal" congruences of algebras. First, if Θ is a congruence and $\Omega \leq \Theta$ is an equivalence relation, then Ω is a congruence. Second, while intersection serves as a meet, in general, there is no join and the maximal element in the poset of the congruences of RRC graphs.

Now, let us outline several standard properties of these objects. The proof is trivial and is thus omitted.

Proposition 2 (First isomorphism theorem)**.** *Let F and G be RRC graphs sharing a black set S. Then*

- *If $\theta\colon F \to G$ is a homomorphism of F to G, then the equivalence relation* $\ker\theta$ *on C_F, defined by $u \ker\theta\, v$ if and only if $\theta(u) = \theta(v)$, is a congruence of F, called a* kernel *of θ;*
- *If Ω is a congruence of F, then the* quotient of F by Ω*, denoted F/Ω, is an RRC graph with the same black set as F, whose set of white vertices is the set of equivalence classes of Ω and, for $[U] \in \Omega$ and $a \in \mathscr{I}$, $\eta_a([U]) = \eta_a(u)$ whenever there is $u \in U$ such that $\eta_a(u) \neq NA$ and $\eta_a([U]) = NA$ otherwise, and $L([U]) = \bigcup\{L(u) \mid u \in U\}$;*
- *If $\theta\colon F \to G$ and $\Omega \leq \ker\theta$, then $\theta/\Omega\colon F/\Omega \to G$ is a homomorphism, defined as $\theta/\Omega([U]) = \theta(u)$, for any $u \in [U]$;*
- *If $\theta\colon F \to G$, then $\theta/\ker\theta$ is an* embedding*, that is, a one-to-one homomorphism of $F/\ker\theta$ into G.*

We define the *size* of Θ, denoted $\|\Theta\|$, as the number of its equivalence classes.

Proposition 3. *Let F, G and H be RRC graphs such that F is a subgraph of G, and let $\theta\colon G \to H$ be a homomorphism of G to H. Let $\Theta = \ker\theta$ and let $\Theta_F = \Theta \cap C_F \times C_F$. Then Θ_F is a congruence of F and $\|\Theta_F\| \leq |C_H|$.*

In the next section, we will outline an approach toward proving $\mathrm{RC}_3 \Rightarrow \mathrm{SSP}$ by constructing, based on an RC graph Γ, a certain RRC graph F, then arguing that there is a homomorphism of F to Γ, whose kernel is thus a congruence of F. It is then sufficient to prove that, for any congruence Θ of F, $\|\Theta\| \geq |S|$. As congruences of RRC graphs will be of particular importance to us, let us elaborate on them a little. In particular, we are going to be interested in the minimal size of a congruence of F, which we denote by $\|F\|$.

Let us recall that we defined a neighborhood as a function from $\mathscr{I}$ to S. We are now going to extend this definition a little: We say that η is a *neighborhood* over a black set S if η is a function from $\mathscr{I}$ to $S \sqcup \mathrm{NA}$. Then η is a neighborhood in the old sense if $\eta(a) \neq \mathrm{NA}$, for all $a \in \mathscr{I}$, in which case we will say that η is *exact*. Trivially, for a white vertex u of an RRC graph G, η_u is a neighborhood, and we say that u is exact if and only if η_u is. For neighborhoods η and ζ, we say that $\eta \leq \zeta$ if $\eta(a) = \mathrm{NA}$ or $\eta(a) = \zeta(a)$, for all $a \in \mathscr{I}$. Thus, the set of neighborhoods over S is a poset with exact neighborhoods as its maximal elements.

Notice that, for a white vertex u of an RRC graph F, the neighborhood η_u encodes almost all information about u, except for its set of capital letters $L(u)$. Moreover, the definition of a congruence of F does not use

$L(u)$ and relies solely on the neighborhoods of its white vertices. As a matter of fact, we can say more, and, for this end, let us develop some further terminology. We start by defining $\max(F) \subseteq 2^{\mathscr{I} \to S \sqcup \mathrm{NA}}$ to be an antichain of maximal neighborhoods of white elements of F.

A natural way of defining a partial order on antichains over a poset P is either by refinement or by dual refinement; here we are interested in the former. Namely, for antichains Q and R over a poset P, we say that Q *refines* R, denoted $Q \ll R$, if and only if for any $q \in Q$ there is $r \in R$ such that $q \leq_P r$. Equivalently, $Q \ll R$ if and only if $(Q] \subseteq (R]$, where $(Q] = \{p \in P \mid p \leq q$ for some $q \in Q\}$ is called a *downward closure* of Q. It is easy to show that refinement is a partial order on the set of antichains over P.

The following easy lemma formalizes the fact that $\|F\|$ essentially depends only on $\max(F)$.

Lemma 13. *Let F and G be RRC graphs over the same black set S. Then $\|G\| \leq \|F\|$ whenever* $\max(G) \ll \max(F)$*, and $\|G\| = \|F\|$ whenever* $\max(G) = \max(F)$*. In particular, let G be obtained from F by one of the following:*

1. *Removing from C_F a nonempty set $U \subseteq C_F$ for which there is a constraint η such that $\eta_u = \eta$ for all $u \in U$, and then adding a white vertex v such that $\eta_v = \eta$; $L(v)$ can be chosen arbitrarily, but we will assume that $L(v) = \bigcup\{L(u) \mid u \in U\}$. In this case we say that U is* contracted *into u;*

2. *Removing from C_F a set $U \subseteq C_F$ such that there is a set $V \subseteq C_F$, disjoint from U, such that for any $u \in U$ there is $v \in V$ such that $\eta_u \leq \eta_v$. In this case we say that U is* eaten *by V;*

3. *Adding an a-edge between $u \in C_F$ and $x \in S_F$ whenever there is $v \in C_F$ such that $\eta_v(a) = x$ and $\eta_u \leq \eta_v$.*

Then $\|F\| = \|G\|$. Also, if G is a subgraph of F then $\|G\| \leq \|F\|$.

CHAPTER 4

RC$_3$ graphs

4.1 Free RRC graphs

To keep our and the reader's sanity, starting from this section we will assume that we deal with graphs of dimension 3, that is, $|\mathscr{I}| = 3$, unless specified otherwise. Some constructions can be generalized for arbitrary dimension in a straightforward manner, but doing so would require an unnecessary overcomplication. Moreover, we will assume that $\mathscr{I} = \{a, b, c\}$. This leads to some confusion, as a, b, and c are the names we use for variables over $\mathscr{I}$. In practice, however, these situations are perfectly distinguishable from context.

Proposition 4. *For an RC graph Γ over $\mathscr{I}$, any pairwise distinct $a, b, c \in \mathscr{I}$, and any $x \in S_\Gamma$ such that $a \in \tau(x)$, there is a weak homomorphism $\varphi = \varphi_{x,a,bc}$ of the RRC graph $F_{a,bc}$ in Figure 4.1 to Γ, such that $\varphi(\underline{a}) = x$.*

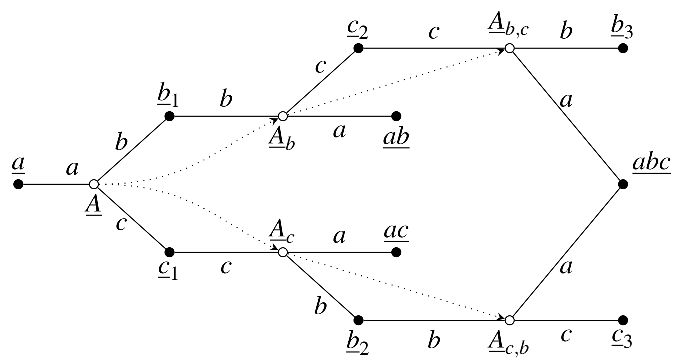


Figure 4.1: RRC graph $F_{a,bc}$.

Moreover, let us define a partial order on $F_{a,bc}$ as follows. The dotted lines indicate the basis of this order restricted to white vertices. Its basis when restricted to black vertices is $\underline{b}_1 \trianglelefteq \underline{b}_2, \underline{b}_3$, $\underline{c}_1 \trianglelefteq \underline{c}_2, \underline{c}_3$, $\underline{a}, \underline{b}_1 \trianglelefteq \underline{ab}$, $\underline{a}, \underline{c}_1 \trianglelefteq \underline{ac}$, and $z \trianglelefteq \underline{abc}$, for every black vertex z of $F_{a,bc}$. And, similarly to Example 1, the order between the white and the black vertices is given by $u \trianglelefteq z$, for $u \in C$ and $z \in S$, whenever there are $v \in C$ and $a \in \mathscr{I}$ such that $u \trianglelefteq v$ and $\eta_a(v) \trianglelefteq z$. Then φ can be chosen to respect this order, that is, $y \trianglelefteq z$ implies $\varphi(y) \trianglelefteq \varphi(z)$, for all black or white vertices y and z of $F_{a,bc}$.

The names of all black vertices in the picture are exact, that is, $\tau(\underline{a}) = a$, $\tau(\underline{b}_1) = \tau(\underline{b}_2) = \tau(\underline{b}_3) = b$, $\tau(\underline{c}_1) = \tau(\underline{c}_2) = \tau(\underline{c}_3) = c$, $\tau(\underline{ab}) = ab$, $\tau(\underline{ac}) = ac$, and $\tau(\underline{abc}) = abc$. The extended type of a black vertex is a minimal extended type compliant with its type, that is, $\mathscr{T}(\underline{a}) = \{I \subseteq \mathscr{I} \mid a \in I\}$, and similarly for all black *vertices. Finally, $L(u) = A$ for every white vertex u.*

Note. As φ can be chosen to respect the order, we will always assume that it does so.

Proof. The proof is a straightforward application of (E1) and (E2) properties of Γ: The image $\varphi(\underline{A})$ of $\underline{A}$ is obtained from x by (E1), in an arbitrary way. Then the images of $\underline{A}_b$ and $\underline{A}_c$ are obtained from $\varphi(\underline{A})$ by (E2), and, similarly, the image of $\underline{A}_{b,c}$ is obtained from $\varphi(\underline{A}_b)$, and of $\underline{A}_{c,b}$ from $\varphi(\underline{A}_c)$, again by (E2). □

Note that for $I \subseteq \mathscr{I}$, if there is an I-path between the black vertices of $F_{u,a,bc}$, then there is an I-path between their images in Γ. Proposition 1 then implies several things.

Corollary 4. *In the setup of Proposition 4*

- $c_b(x) = c_a(\varphi(\underline{b}_1)) = \varphi(\underline{ab})$, $c_c(x) = c_a(\varphi(\underline{c}_1)) = \varphi(\underline{ac})$;
- $c_{bc}(x) = c_c(\varphi(\underline{ab})) = c_b(\varphi(\underline{ac})) = \varphi(\underline{abc})$; *the latter can be trivially extended to* $c_{ac}(\varphi(\underline{b}_1)) = \varphi(\underline{abc})$, *and similarly for the remianing black vertices;*
- $c_b(\varphi(\underline{c}_i)) = c_c(\varphi(\underline{b}_j))$, *for all* $i, j = 1,2,3$;
- *Combined with* $\underline{b}_1 \trianglelefteq \underline{b}_2, \underline{b}_3$, *the latter trivially implies that if* $c \in \tau(\varphi(\underline{b}_1))$, *then* $\varphi(\underline{b}_1) = \varphi(\underline{b}_2) = \varphi(\underline{b}_3)$, *and similarly for* $b \in \tau(\varphi(\underline{c}_1))$.

Proof. This is a direct consequence of Proposition 1, that is, of the fact that $c_I(x)$ is a unique black vertex y in the same $(I \cup \tau(x))$-component, for which $I \cup \tau(x) \subseteq \tau(y)$. □

It can be noted that $\underline{b}_1 \trianglelefteq \underline{b}_2, \underline{b}_3$ and $\underline{c}_1 \trianglelefteq \underline{c}_2, \underline{c}_3$ are induced by the rule: $\eta_a(u) \trianglelefteq \eta_a(v)$ whenever $u \trianglelefteq v$; And $\underline{a}, \underline{b}_1 \trianglelefteq \underline{ab}$, $\underline{a}, \underline{c}_1 \trianglelefteq \underline{ac}$, and $z \trianglelefteq \underline{abc}$ comply with $c_b(\varphi(\underline{a})) = c_a(\varphi(\underline{b}_1)) = \varphi(\underline{ab})$ and similar properties, relating φ to the closure structure of Γ. Additionally, let us notice that $F_{a,bc}$ is parametrized with a, and $bc = \{b, c\}$, that is, $F_{a,bc}$ is the same as $F_{a,cb}$. Moreover, as $bc = \mathscr{I} - a$, $F_{a,bc}$ only depends on a, and $\varphi_{x,a,bc}$ only on a and x, and from now on we will call them F_a and $\varphi_{x,a}$ respectively.

As was noted in the beginning of the section, Proposition 4, although correct for any $\|\mathscr{I}\|$, is tailored for $|\mathscr{I}| = 3$ case: it is effectively inapplicable for $|\mathscr{I}| \leq 2$, and for $|\mathscr{I}| \geq 4$ a more general statement can be made. In fact, for, say, $|\mathscr{I}| = 4$, it is easy to guess what the graph $F_{a,bcd}$ and the corresponding properties of $\varphi_{x,a,bcd} \colon F_{a,bcd} \to \Gamma$ should be. Let us separately note that Proposition 4 is stated for "any pairwise distinct $a, b, c \in \mathscr{I}$". This setup will be typical and, effectively, means that, as variables, a, b, and c are some permutation of the letters $a, b, c \in \mathscr{I}$. Then Proposition 4 can be interpreted as stated literally for the letters a, b, and c, and then true up to any permutation of these letters.

The plan is now as follows. Having fixed the black set S of an RC graph Γ, we are going to construct an RRC graph, which "freely" admits Proposition 4 for all $x \in S$ and $a \in \tau(x)$, or, in other words, for all $(x,a) \in \underline{S}$. Here by freely we mean that the images of all white vertices of F_a are different for all applications of Proposition 4. Let us note that this construction is contingent upon the images of black vertices, which we will need to fix. Some of them are fixed already by the closure structure of Γ, namely, we know that for $\varphi = \varphi_{x,a}$ it holds $\varphi(\underline{a}) = x$, $\varphi(\underline{ab}) = c_b(x)$, $\varphi(\underline{ac}) = c_c(x)$, and $\varphi(\underline{abc}) = c_{bc}(x)$. And, to handle the images of $\underline{b}_i$ and $\underline{c}_i$, for $i = 1,2,3$, we are going to introduce an arrow structure.

Formally, an *arrow structure* $\mathscr{A}$ over a black set S is $\mathscr{A} = \mathscr{A}_1 \sqcup \mathscr{A}_2 \sqcup \mathscr{A}_3$, where $\mathscr{A}_1$ is a set of tuples of the form (x,a,b,y), for $x,y \in S$, $a \in \tau(x)$, $b \in \mathscr{I} - a$, and $b \in \tau(y)$; $\mathscr{A}_2$ is a set of tuples (x,a,b,c,y), for $x,y \in S$, $a \in \tau(x)$, $b \in \mathscr{I} - a$, $c \in \mathscr{I} - ab$, and $c \in \tau(y)$, and $\mathscr{A}_3$ is a set of tuples (x,a,b,c,b,y), for $x,y \in S$, $a \in \tau(x)$, $b \in \mathscr{I} - a$, $c \in \mathscr{I} - ab$, and $b \in \tau(y)$. A tuple (x,a,b,y) is called an *a-b-arrow from x to y*, denoted $x \xrightarrow{a-b} y$; similarly, (x,a,b,c,y) is called an *a-b-c*-arrow from x to y, denoted $x \xrightarrow{a-b-c} y$, and (x,a,b,c,b,y) an *a-b-c-b*-arrow from x to y, denoted $x \xrightarrow{a-b-c-b} y$. Moreover, for any $x \in S$, and all pairwise distinct $a,b,c \in \mathscr{I}$ such that $a \in \tau(x)$, there is exactly one *a-b*, exactly one *a-b-c*, and exactly one *a-b-c-b*-arrow from x.

Arrows from $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$ are called *degree* 1, *degree* 2, and *degree* 3 respectively. If the arrow structure in question is clear, we will write $x \xrightarrow{a-b} y$ instead of $x \xrightarrow{a-b} y \in \mathscr{A}$, and similarly for degree 2 and 3 arrows. Also, $y \xleftarrow{a-b} x$ means $x \xrightarrow{a-b} y$, and similarly for the other arrows; note that the letters above the arrow are not swapped. Note that $x \xrightarrow{a-b} y$ automatically implies $a \neq b$, and similarly $x \xrightarrow{a-b-c} y$ or $x \xrightarrow{a-b-c-b} y$ implies a, b, and c are pairwise distinct. As $x \xrightarrow{a-b} y$ implies $a \in \tau(x)$, $b \in \tau(y)$, this arrow can be considered a tuple $((x,a),(y,b))$ of elements of $\underline{S}$, which will be denoted by $(x,a) \to (y,b)$; there is no similar notation for degree 2 and 3 arrows. We define a *structure* over a black set S as a tuple $(\mathscr{C},\mathscr{A})$ of a closure structure $\mathscr{C}$ and an arrow structure $\mathscr{A}$ over S.

Lemma 14 (Structure of a graph)**.** *For an RC graph Γ, let us fix $\Phi = \{\varphi_{x,a} \mid (x,a) \in \underline{S}\}$, where $\varphi_{x,a}$ is some homomorphism from Proposition 4. Let us define an arrow structure $\mathscr{A} = \mathscr{A}(\Phi)$ on S as:*

- $x \xrightarrow{a-b} \varphi_{x,a}(\underline{b}_1)$;
- $x \xrightarrow{a-c} \varphi_{x,a}(\underline{c}_1)$;
- $x \xrightarrow{a-b-c} \varphi_{x,a}(\underline{c}_2)$;
- $x \xrightarrow{a-c-b} \varphi_{x,a}(\underline{b}_2)$;
- $x \xrightarrow{a-b-c-b} \varphi_{x,a}(\underline{b}_3)$;
- $x \xrightarrow{a-c-b-c} \varphi_{x,a}(\underline{c}_3)$.

Then $\mathscr{A}$ is an arrow structure, moreover

(AC1) *If $x \xrightarrow{a-b} y$, or there is z such that $x \xleftarrow{c-a} z \xrightarrow{c-b} y$, then $c_b(x) = c_a(y)$. In particular, if in this case $a \in \tau(y)$, then $c_b(x) = y$ and $\tau(x) \subseteq \tau(y)$, and, similarly, if $b \in \tau(x)$, then $x = c_a(y)$ and $\tau(y) \subseteq \tau(x)$;*

Note. *The first case, $x \xrightarrow{a-b} y$, implies $a \neq b$. The second case, $z \xrightarrow{c-a} x$ and $z \xrightarrow{c-b} y$, implies that $a \neq c$ and $b \neq c$. Although in the latter case $a = b$ is not forbidden, it implies that $x = y$ and, by (C2), that $c_a(x) = c_b(x) = x$. So the second case becomes trivial unless a, b, and c are pairwise distinct.*

(AC2) *If $x \xrightarrow{a-c} y$ and $b \in \tau(y)$, then $x \xrightarrow{a-b-c} y$.*

Note. While the closure system is uniquely defined by Γ, there is a certain level of arbitrariness in the choice of $\mathscr{A}$, which, in particular, we will utilize when enforcing additional properties on arrow structures in Lemma 16.

Proof. $\mathscr{A}$ is an arrow strucuture by construction, and (AC1) and (AC2) are easy reformulation of some properties from Corollary 4. Indeed, if $x \xrightarrow{a-b} y$, then $c_b(x) = c_a \circ \varphi_{x,a}(\underline{b}_1) = c_a(y)$. And $z \xrightarrow{c-a} x$ and $z \xrightarrow{c-b} y$ imply $c_b(x) = c_b \circ \varphi_{z,c}(\underline{a}_1) = c_a \circ \varphi_{z,c}(\underline{b}_1) = c_a(y)$.

Note that for the second case of (AC1) we use the mapping $\varphi_{z,c} \colon F_{c,ab} \to \Gamma$, where $F_{c,ab}$ is obtained from $F_{a,bc}$ by a permutation $[a \to c, b \to a, c \to b]$ (or, alternatively, $[a \to c, b \to b, c \to a]$), which converts $\underline{b}_1$ into $\underline{a}_1$, and $\underline{c}_1$ into $\underline{b}_1$, and transforms the mapping φ correspondingly.

And, for (AC2), $b \in \tau(y) = \tau(\varphi_{x,a}(\underline{c}_1))$ implies $y = \varphi_{x,a}(\underline{c}_1) = \varphi_{x,a}(\underline{c}_2)$, and hence $x \xrightarrow{a-b-c} y$. □

We will only deal with structures provided by Lemma 14 and thus we will consider (AC1) and (AC2) to be a part of a definition of a structure. Now, for a given structure $(\mathscr{C}, \mathscr{A})$, we are going to construct the *free graph $F_{\mathscr{C},\mathscr{A}}$ of $(\mathscr{C}, \mathscr{A})$*. The construction is as follows:

(F1) $F_{\mathscr{C},\mathscr{A}}$ is an RRC graph with black set $S = S_{\mathscr{C}} = S_{\mathscr{A}}$. Moreover, $C_F = C^1 \sqcup C^2 \sqcup C^3$, where white vertices from C^1, C^2, and C^3 are called *stage-1*, *stage-2*, and *stage-3 vertices* respectively;

(F2) Stage-1 vertices are parametrized by all pairs $(\underline{a}, a) \in \underline{S}$, and denoted $u_{s1}(\underline{a}, a)$. Stage-2 vertices are parametrized by all tuples $(\underline{a}, a) \in \underline{S}$, and $b \in \mathscr{I} - a$, and denoted $u_{s2}(\underline{a}, a, b)$. Finally, stage-3 vertices are parametrized by all tuples $(\underline{a}, a) \in \underline{S}$, $b \in \mathscr{I} - a$, and $c \in \mathscr{I} - ab$, and denoted $u_{s3}(\underline{a}, a, b, c)$. Additionally, $L(u_{s1}(\underline{a}, a)) = L(u_{s2}(\underline{a}, a, b)) = L(u_{s3}(\underline{a}, a, b, c)) = a$;

(F3.1) For $u = u_{s1}(\underline{a}, a)$, the a-edge of u goes to $\underline{a}$, and, for $b \in \mathscr{I} - a$, the b-edge of u goes to $\underline{b}$ such that $\underline{a} \xrightarrow{a-b} \underline{b}$;

(F3.2) For $u = u_{s2}(\underline{a}, a, b)$, the b-edge of u goes to $\underline{b}$ such that $\underline{a} \xrightarrow{a-b} \underline{b}$, that is, to $\underline{b} = \eta_b \circ u_{s1}(\underline{a}, a)$, the a-edge of u goes to $c_b(\underline{a})$, and for $c \in \mathscr{I} - ab$, the c-edge of u goes to $\underline{c}$ such that $\underline{a} \xrightarrow{a-b-c} \underline{c}$. We call a and b-edges of u *ordinary* and its c-edge *special*;

(F3.3) For $u = u_{s3}(\underline{a}, a, b, c)$, the c-edge of u goes to $\underline{c}$ such that $\underline{a} \xrightarrow{a-b-c} \underline{c}$, that is, to $\underline{c} = \eta_c \circ u_{s2}(\underline{a}, a, b)$, the a-edge of u goes to $c_{bc}(\underline{a}) = \underline{abc}$, and the b-edge of u goes to $\underline{b}$ such that $\underline{a} \xrightarrow{a-b-c-b} \underline{b}$. Similarly, a and c-edges of u are called *ordinary* and its b-edge *special*.

Further on, every time we use the notation $u_{s1}(x,a)$, $u_{s2}(x,a,b)$, or $u_{s3}(x,a,b,c)$ we will automaticaly assume that $(x,a) \in \underline{S}$ and a, b, and c are pairwise distinct. For $u = u_{s2}(\underline{a}, a, b)$, by definition, $\eta_b(u) = \underline{b}$ and $\eta_a(u) = \underline{ab}$, for $\underline{ab} = c_a(\underline{b}) = c_b(\underline{a})$ and $\underline{b}$ such that $\underline{a} \xrightarrow{a-b} \underline{b}$. We thus say that u is a stage-2 vertex *from* $(\underline{b}, b)$ *to* $(\underline{ab}, a)$. Similarly, for $u = u_{s3}(\underline{a}, a, b, c)$ we say that u is *from* $(\underline{c}, c)$ *to* $(\underline{abc}, a)$, where $\underline{c}$ is such that $\underline{a} \xrightarrow{a-b-c} \underline{c}$. This notation emphasizes the main role of the ordinary edges of stage-2 and stage-3 vertices.

By construction, $\mathscr{C}$ and $\mathscr{A}$ capture the images of the black vertices of F_a (and of F_b and F_c, obtained from F_a by permutations of letters of $\mathscr{I}$) by the homomorphisms from Φ, fixed in Lemma 14. Also, the white vertices u_{s1}, u_{s2}, and u_{s3} of $F_{\mathscr{C},\mathscr{A}}$ correspond to the white vertices of the corresponding RRC graphs. Thus, the following lemma is nonsurprising.

Lemma 15. *Let Γ be an RC graph. Let $(\mathscr{C}, \mathscr{A})$ be the structure of Γ, where $\mathscr{A} = \mathscr{A}(\Phi)$ is constructed as in Lemma 14. Let $F = F_{\mathscr{C},\mathscr{A}}$ be the free graph of $(\mathscr{C}, \mathscr{A})$. Then there is a homomorphism $\theta \colon F \to \Gamma$.*

Note. We do not claim that θ completely restores Γ, neither in the sense of correctly restoring the capital letters of white types, nor in restoring all white vertices (that is, θ is not necessarily onto).

Proof. The statement is obvious by putting, for $(x,a) \in \underline{S}$ and $\varphi = \varphi_{x,a} \in \Phi$, $\theta(u_{s1}(x,a)) = \varphi(\underline{A})$, $\theta(u_{s2}(x,a,b)) = \varphi(\underline{A}_b)$, and $\theta(u_{s3}(x,a,b,c)) = \varphi(\underline{A}_{b,c})$; respectively, by permuting b and c, $\theta(u_{s2}(x,a,c)) = \varphi(\underline{A}_c)$, and $\theta(u_{s3}(x,a,c,b)) = \varphi(\underline{A}_{c,b})$. □

4.2 Example of a structure of an RRC$_3$ graph with trivial closures

In this and the next sections we will consider a subcase of $|\mathscr{I}| = 3$ case when an RC graph Γ has trivial closures. Recall that, assuming the connectedness of Γ, it means that all black vertices of Γ have single-element types, except for its unique top whose type is $\mathscr{I}$. As $\mathscr{I} = abc$, we denote the top by $\underline{abc}$.

In principle, we do not need to consider this subcase separately, as it falls under the general $|\mathscr{I}| = 3$ case, and the proof of RC$_3$ $\Rightarrow$ SSP does not rely on it. However, it is very illustrative. If Γ has trivial closures, the proof that it satisfies SSP becomes much easier, emphasizing the main line of reasoning without overburdening the reader with technical details. Moreover, we can then use a simplified version of arrow structures and free graphs, and thus we will be able to give illustrations for these constructions; in the general situation, the free graph becomes too bulky to even draw it. Because of the optional nature of these two sections, we will allow ourselves some level of sketchiness.

So, for the purpose of this section, we only need the *simplified arrow structure* $\mathscr{A}$, which only contains arrows of degree 1, constructed as in Lemma 14. The (AC1) condition becomes trivial, as $x \xrightarrow{a-b} y$ in case of trivial closures implies $c_b(x) = c_a(y) = \underline{abc}$. Also, (AC2) cannot be formulated, as we do not have degree-2 arrows; however, we will use a version of a free graph that will contain some proxy of (AC2) condition. Now, the *simplified free graph* $F_{\mathscr{C},\mathscr{A}}$ is defined similarly to the free graph, but without stage-3 vertices. Additionally, the definition of stage-2 vertices is relaxed as follows

(F3.2*) For $u = u_{s2}(x,a,b)$, the b-edge of u goes to y such that $x \xrightarrow{a-b} y$, that is, to $y = \eta_b \circ u_{s1}(x,a)$, and the a-edge of u goes to $c_b(x) = \underline{abc}$. We call a and b-edges of u *ordinary*. Moreover, if for $c \in \mathscr{I} - ab$ it holds $x \xrightarrow{a-c} \underline{abc}$, then there is c-edge from u to $\underline{abc}$, and this edge is called *special*; otherwise, there is no c edge from u.

The construction of special edges is a relaxed version of the omitted (AC2) condition; it is illustrated in Figure 4.2 below. Let us note that the simplified free graph $F = F_{\mathscr{C},\mathscr{A}}$ is a subgraph of the free graph, and so Lemma 15 holds for it, that is, there is a homomorphism $\theta\colon F \to \Gamma$.

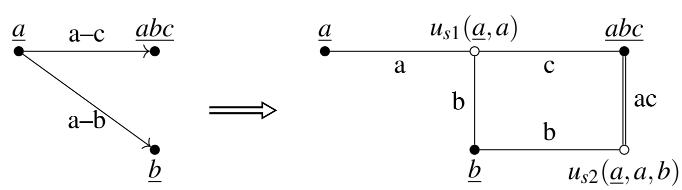


Figure 4.2: Special edges of stage-2 vertices of a simplified free graph.

Now, we claim that one of the possible simplified arrow structures for the RC graph Γ in Figure 3.1 (which has trivial closures), provided by Lemma 14, is as in Figure 4.3 below. Recall that the construction of $\mathscr{A}$ is contingent upon the choice of the mappings Φ, and, as far as we only need arrows of degree one, we only care about the choice of $\varphi_{x,a}(\underline{A})$, for $(x,a) \in \underline{S}$; the fact that it is a morphism implies that $\varphi_{x,a}(\underline{b}_1) = \eta_b \circ \varphi_{x,a}(\underline{A})$, and $\varphi_{x,a}(\underline{c}_1) = \eta_c \circ \varphi_{x,a}(\underline{A})$, and this is enough to restore the arrows of degree one. Now, $\varphi_{x,a}(\underline{A})$ should be an a-neighbor of x whose type contains A. For all $x \neq \underline{abc}$ this choice is unique, for example, $\varphi_{\underline{b},b}(\underline{B}) = \underline{A \to Bc}$; recall that the domain of $\varphi_{\underline{b},b}$ is F_b, obtained from F_a by a permutation $[a \to b, b \to c, c \to a]$, and thus $\underline{A}$ in F_a becomes $\underline{B}$ in F_b. Putting $\varphi_{\underline{abc},a}(\underline{A}) = \underline{A \to Bc}$, $\varphi_{\underline{abc},b}(\underline{B}) = \underline{[BC] \to a}$, and $\varphi_{\underline{abc},c}(\underline{C}) = \underline{C \to ab}$ gives precisely the arrows from Figure 4.3.

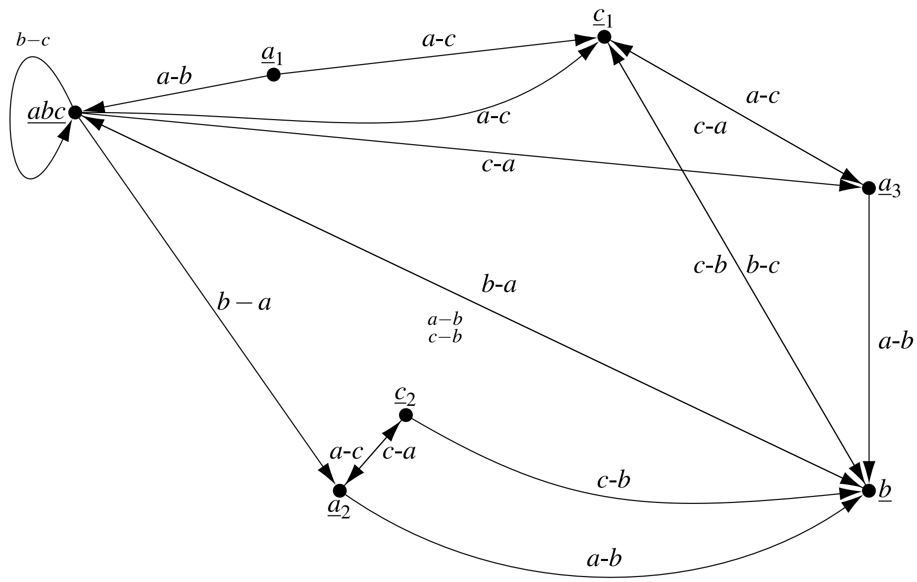


Figure 4.3: Simplified arrow structure $\mathscr{A}$ for the RC graph Γ in Figure 3.1.

The free graph $F = F_{\mathscr{C},\mathscr{A}}$ is shown in Figure 4.4 below. Stage-1 vertices are drawn in white, ordinary stage-2 vertices in yellow, and special stage-2 in red.

Finally, in Figure 4.5, we show the image of F into Γ under the morphism θ from Lemma 15, that is, the RRC graph F/Θ for $\Theta = \ker\theta$.

4.3 SSP holds for RC_3 graphs with trivial closures

Theorem 3. *Let Γ be an* RC_3*-graph with trivial closures. Then Γ is* SSP.

Proof. Let Γ be an RC_3-graph with trivial closures. Recall that we assume that Γ is connected, its top is denoted by $\underline{abc}$, and the white types of Γ satisfy (T). Let $\mathscr{A} = \mathscr{A}_\Gamma$ be any arrow structure on S. We are going to prove that for the simplified free graph $F = F_{\mathscr{C},\mathscr{A}}$, $\|F\| \geq |S|$, that is, for any congruence Θ of F it holds $\|\Theta\| \geq |S|$. In the process, we will modify F in the ways covered by Lemma 13; we will keep calling the modified graph F and use the fact that $\|F\|$ is not changed by the modification.

We define a *triangle* as three vertices $\underline{a} \in S - \underline{abc}$, for $a \in \mathscr{I}$, such that $\underline{a} \xrightarrow{a-b} \underline{b}$, for all $a, b \in \mathscr{I}$, $a \neq b$. Notice that we have explicitly excluded the top vertex, so in this case $\tau(\underline{a}) = a$, in particular, the vertices of a triangle are pairwise distinct.

Similarly, for $a, b \in \mathscr{I}$, $a \neq b$, we define an *ab-pyramid* as three vertices $\underline{a}, \underline{b} \in S - \underline{abc}$, and $\underline{c} \in S$, where c

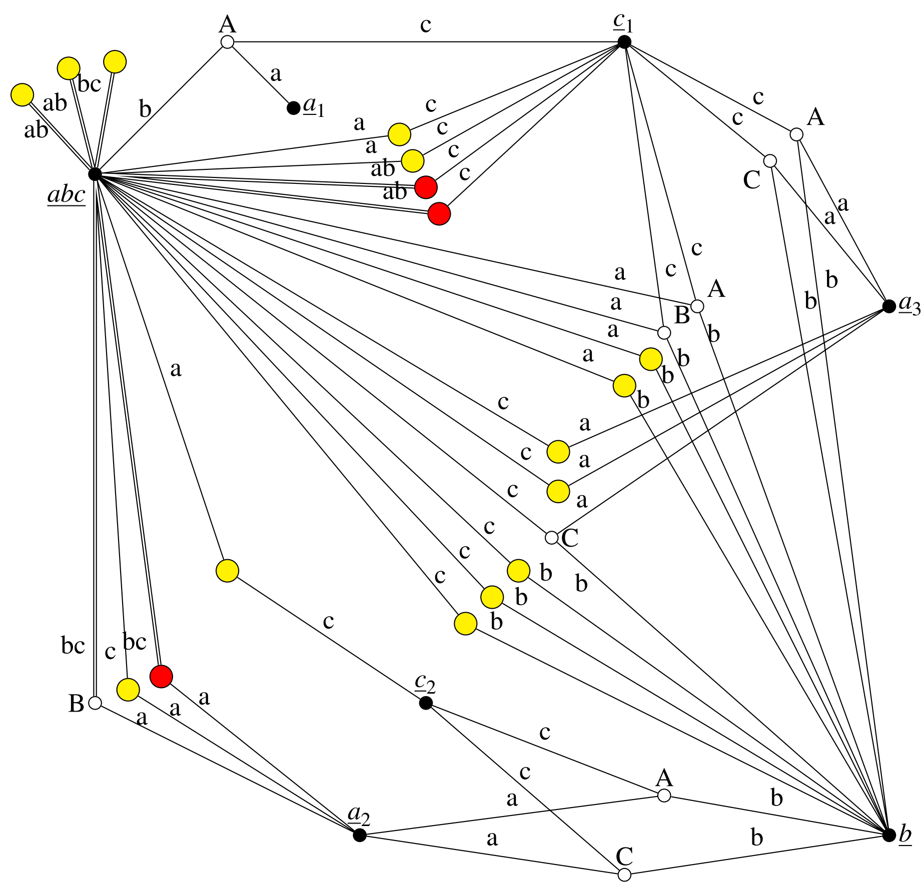


Figure 4.4: Simplified free graph $F = F_{\mathscr{C},\mathscr{A}}$ of the simplified arrow structure $\mathscr{A}$ in Figure 4.3.

is a remaining third letter from $\mathscr{I} - ab$, such that $\underline{a} \xrightarrow{a-b} \underline{b}$, $\underline{b} \xrightarrow{b-a} \underline{a}$, $\underline{a} \xrightarrow{a-c} \underline{c}$, and $\underline{b} \xrightarrow{b-c} \underline{c}$. Additionally, we require that $\underline{a}$, $\underline{b}$, and $\underline{c}$ do not form a triangle. We call the set $\{\underline{a}, \underline{b}\}$ the *base*, and $\underline{c}$ the *tip* of the pyramid; notice that the tip of a pyramid can be $\underline{abc}$. We separately note that we consider a pair a and b to be non-ordered, that is, an ab-pyramid is the same as ba-pyramid. An easy but helpful observation is that all triangles and pyramid bases are disjoint from each other.

Let t be the number of triangles, p the number of pyramids, and s the number of vertices in $S - \underline{abc}$ not in triangles or pyramid bases, which we call *singletons*. Then the total number of vertices in S is $3t + 2p + s + 1$.

For the purpose of this proof, we will exclude stage-1 vertices of $\underline{abc}$ from the set of stage-1 vertices, and will call them stage-1t. Now, let us contract (as per Lemma 13) stage-1 vertices of every triangle and every pyramid base; we still call these new vertices stage-1 and index them by the corresponding triangles and pyramids. Each singleton already has just one stage-1 vertex, which will now be indexed by this singleton. Let us denote the set of all stage-1 vertices after contractions ST1. Then $|\mathrm{ST1}| = r + p + s$, moreover, all

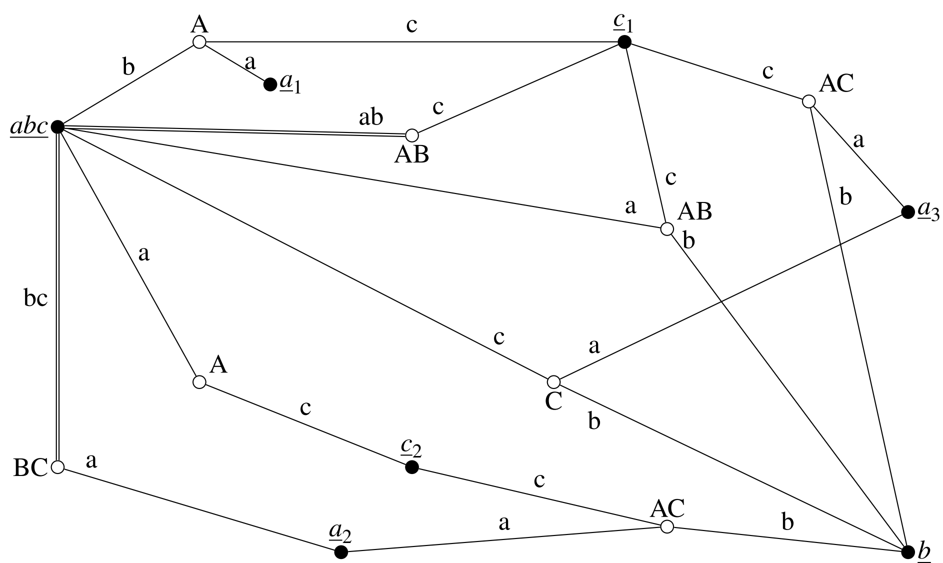


Figure 4.5: RRC graph F/Θ.

vertices in ST1 are exact and cannot be contracted with each other. That is, their neighborhoods form an antichain in the poset of neighborhoods over S.

We say that an arrow $\underline{a} \xrightarrow{a-b} \underline{b}$ is *principal* if it is an arrow inside a triangle or the base of an *ab*-pyramid. Note that $\underline{a} \xrightarrow{a-b} \underline{b}$ is principal if and only if $\underline{a}, \underline{b} \neq \underline{abc}$, $\underline{a} \xrightarrow{a-b} \underline{b}$, $\underline{b} \xrightarrow{b-a} \underline{a}$, and $\underline{a} \xrightarrow{a-c} \underline{c}$, $\underline{b} \xrightarrow{b-c} \underline{c}$ for some $\underline{c} \in S$. We say that a stage-2 vertex u is *principal* if it is generated by a principal arrow, that is, if $u = u_{s2}(\underline{a}, a, b)$ for a principal $\underline{a} \xrightarrow{a-b} \underline{b}$. Notice that stage-2 vertices generated by arrows going to the tip of a pyramid are not considered principal. We call the set of all principal stage-2 vertices by PST2; thus, $|\mathrm{PST2}| = 6t + 2p$.

Recall that, as per Lemma 13, for wite vertices u and v, u is eaten by v if and only if $\eta_u \leq \eta_v$. Let us now consider how a principal stage-2 vertex $u = u_{s2}(\underline{a}, a, b)$, generated by a principal arrow $\underline{a} \xrightarrow{a-b} \underline{b}$, might be eaten by a stage-1 vertex v. As $\underline{a} \xrightarrow{a-b} \underline{b}$ is principal, $\underline{a}, \underline{b} \neq \underline{abc}$; in particular, $\tau(\underline{a}) = a$ and $\tau(\underline{b}) = b$. The list below gives an exaustive classification of the situations when v can eat u. It is based on how v is obtained, that is, in it we just formally enumerate the cases when $\eta_a(v) = \eta_a(u) = \underline{abc}$ and $\eta_b(v) = \eta_b(u) = \underline{b}$.

(D1) If $\underline{a} \xrightarrow{a-b} \underline{b} \xrightarrow{b-a} \underline{abc}$ - u can be eaten by a stage-1 vertex $u_{s1}(\underline{b}, b)$;

(D2) If $\underline{a} \xrightarrow{a-b} \underline{b} \xleftarrow{a-b} \underline{abc}$ - u can be eaten by a stage-1t vertex $u_{s1}(\underline{abc}, a)$;

(D3.1) If $\underline{a} \xrightarrow{a-b} \underline{b} \xleftarrow{c-b} \underline{c} \xrightarrow{c-a} \underline{abc}$ - u can be eaten by a stage-1 vertex $u_{s1}(\underline{c}, c)$, for some $\underline{c} \in S - \underline{abc}$;

(D3.2) If $\underline{a} \xrightarrow{a-b} \underline{b} \xleftarrow{c-b} \underline{abc} \xrightarrow{c-a} \underline{abc}$ - u can be eaten by a stage-1t vertex $u_{s1}(\underline{abc}, c)$. Here $\underline{abc} \xrightarrow{c-a} \underline{abc}$ means that c-a arrow from $\underline{abc}$ is a loop.

As $\underline{a} \xrightarrow{a-b} \underline{b}$ is principal, (D1) cannot happen. In cases (D2) and (D3.2) u is eaten by a stage-1t vertex, which we do not consider stage-1, so these two are disregarded as well.

Thus, the only possible case is (D3.1). Notice that in this case $\underline{c}$ is a singleton: Indeed, as $\underline{c} \xrightarrow{c-a} \underline{abc}$ and $\underline{c} \xrightarrow{c-b} \underline{b}$, the only case how $\underline{c}$ can not be a singleton is that if $\underline{c}$ and $\underline{b}$ form a base of a pyramid. But then $\underline{b} \xrightarrow{b-a} \underline{abc}$. But that is impossible, as $\underline{a} \xrightarrow{a-b} \underline{b}$ is principal, and hence $\underline{b} \xrightarrow{b-a} \underline{a}$. Additionally, let us notice that in this case a stage-2 vertex $w = u_{s2}(\underline{c}, c, b)$ is special, and its constraint η_w is defined as $\eta_w(a) = \eta_w(c) = \underline{abc}$, and $\eta_w(b) = \underline{b}$; in particular, $\eta_u \leq \eta_w$. Notice that the last inequality holds even if u is itself special: In this case u has a special c-edge going to $\underline{abc}$, and hence it would simply mean that $\eta_u = \eta_v$. Then, by Lemma 13, we can add a c-edge between u and $\underline{abc}$, that is, put $\eta_u(c) = \underline{abc}$. But, after this modification, $\eta_c(u) = \underline{abc} \neq \eta_c(v) = \underline{c}$, and hence v no longer eats u. This construction is illustrated in Figure 4.6 below; The dashed arrows indicate the arrows from $\mathscr{A}$ and white types are tentatively taken from the corresponding images in Γ.

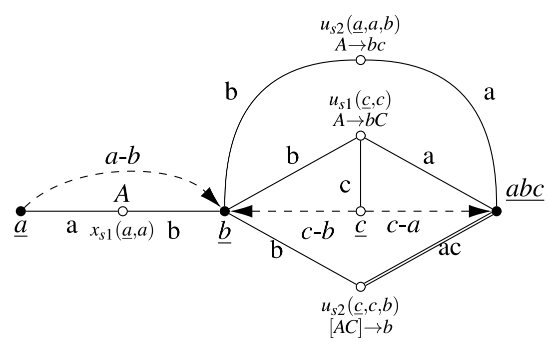


Figure 4.6: (D3.1) argument.

The maneuvring in (D3.1) case is somewhat tricky, so let us glance over it once more: It can happen that u is eaten by v this way. However, then this situation implies the existence of w that is not principal itself, but which, by Lemma 13, enables us to add a c-edge to principal u without increasing $\|F\|$, such that after it u is no longer eaten by (D3.1). As all other cases are already eliminated, we conclude that in F, after a possible modification, no PST2 vertex is eaten by ST1 vertex.

Let us call an equivalence class of an arbitrary congruence Θ of F, containing vertices of ST1, an ST1-equivalence class, and similarly for PST2-equivalence classes. Because no vertex in PST2 is eaten by ST1, and because all vertices of ST1 are exact, the sets of ST1 and PST2-equivalence classes are disjoint. Also, no two ST1-vertices can be in one equivalence class, so there are exactly $t+p+s$ ST1-equivalence classes. The next observation is crucial in addressing PST2-vertices. As long as we are only sketching the approach, the proof is omitted.

A PST2-equivalence class cannot contain more than two vertices from PST2.

Thus, there are at least $(6t+2p)/2 = 3r+p$ PST2-equivalence classes, and, consecutively, there are at least $(t+p+s)+(3t+p) = 4t+2p+s$ equivalence classes which are either ST1 or PST2. Now, if Γ is not SSP, we have

$$4t+2p+s \leq \|\Theta^*\| \leq \|\Theta\| \leq |C_\Gamma| < |S| = 3t+2p+s+1$$

where F^* is a subgraph of F containing only ST1 and PST2 vertices, and Θ^* is the restriction of Θ on F^*. Although the above inequality admits the possibility that $|C|$ is strictly smaller than $|S|$ (by at most one), it can only happen under very specific conditions. Immediately, we can infer

- $\mathscr{A}$ has no triangles;
- All PST2-equivalence classes contain exactly two PST2-vertices;
- All vertices not from ST1 or PST2 are either eaten by stage-1 vertices, or can be put into PST2-equivalence classes. This includes non-principal stage-2 vertices of non-top vertices and stage-1t vertices. We can assume that stage-2 vertices of $\underline{abc}$ are eaten by stage-1t vertices.

After that, the proof can be finished by several easy claims.

All ST1 and PST2 equivalence classes have at most one edge to $\underline{abc}$. Let θ be an ST1 or PST2 class with a and b-edges to $\underline{abc}$. If θ is PS2, then it is easy to see that the two PS2-vertices u_1 and u_2 in θ should be from some $\underline{c}$, for $\underline{c} \in S - \underline{abc}$, that is, $u_1 = u_{s2}(\underline{a},a,c)$ and $u_2 = u_{s2}(\underline{b},b,c)$, for principal arrows $\underline{a} \xrightarrow{a-c} \underline{c}$ and $\underline{b} \xrightarrow{b-c} \underline{c}$. But then $\underline{a}$, $\underline{b}$, and $\underline{c}$ form a triangle, which cannot happen.

So let θ be S1. Then a unique S1-vertex u in θ has form $u = u_{s1}(\underline{c},c)$ such that $u \xrightarrow{c-a} \underline{abc}$ and $u \xrightarrow{c-b} \underline{abc}$. But then $v = u_{s2}(\underline{c},c,a)$ is special, and, by construction, $\eta_v(a) = \eta_v(b) = \eta_v(c) = \underline{abc}$. But then v cannot be in any $S1$ or $PS2$ equivalence class, a contradiction.

No arrow in $\mathscr{A}$ goes to $\underline{abc}$ (this includes arrows from $\underline{abc}$). In particular, no S1-class has an edge to $\underline{abc}$. Indeed, if $\underline{a} \xrightarrow{a-b} \underline{abc}$, then both a and b-edges of $u = u_{s2}(\underline{a},a,b)$ go to $\underline{abc}$, and, consequently, u cannot be in any S1 or PS2 class, a contradiction.

There are no pyramids. As a consequence, there are no PS2-equivalence classes. Suppose $\underline{a},\underline{b} \in S - \underline{abc}$ form a base of an ab-pyramid with a tip $\underline{c}$. By the previous claim, $\underline{c} \neq \underline{abc}$. Then, for $u = u_{s2}(\underline{a},a,c)$, $\eta_a(u) = \underline{abc}$ and $\eta_c(u) = \underline{c}$. As no S1-class has an edge to $\underline{abc}$, u should be in a $PS2$-equivalence class, whose unique edge to $\underline{abc}$ is an a-edge. Then $\underline{c}$ should be in the base of an ac-pyramid. By a similar argument for $u_{s2}(\underline{b},b,c)$, $\underline{c}$ is in the base of a bc-pyramid. But then $\underline{c}$ is in a triangle, a contradiction.

Finally, as there are no PST2-classes, and as no edge of an ST1-class has an edge to $\underline{abc}$, there is no equivalence class of Θ that can contain $u_{s1}(\underline{abc},a)$, a contradiction. $\square$

4.4 Additional properties of RRC graph structures

This and the next two sections are dedicated to proving $\mathrm{RC}_3 \Rightarrow \mathrm{SSP}$ in full generality. Here, in particular, we elaborate on some strengthening of Proposition 4, and, consecutively, of Lemma 14. The statements are tailored for the main proof and thus might look strange. We will briefly discuss the parallel between the proof of the general case with Theorem 3 at the beginning of the next section.

We say that, in the setup of Proposition 4, $\varphi = \varphi_{x,a}$ is *maximal* if

- $\varphi(\underline{A})$ is chosen to be an a-neighbor of x with A in its type, which is maximal with respect to both the order of Γ, and to the preorder on white vertices according to the order on their closure types. That is, there is no a-neighbor u of x such that $A \in \tau(u)$ and either $\varphi(\underline{A}) \lhd u$ or $K(\varphi(\underline{A})) < K(u)$;

We note that $\varphi(\underline{A}) \lhd u$ implies $K(\varphi(\underline{A})) \leq K(u)$, so such maximal vertex can always be chosen. Indeed, let U be the finite nonempty set of a-neighbors of x with A in their types. Let K be any maximal $K(u)$ for $u \in U$, and let $U_K = \{u \in U \mid K(U) = K\}$. Finally, let us pick $v \in U_K$ that is maximal with respect to $\trianglelefteq$, and put $\varphi(\underline{A}) = v$. Now, let us pick an arbitrary $w \in U$. By construction, $K(v) \not< K(w)$. Suppose now that $v \trianglelefteq w$. But this implies $K = K(v) \leq K(w)$, and, as K is chosen to be maximal, $K = K(w)$, and so $w \in U_K$. But, as v is maximal in U_K with respect to $\trianglelefteq$, $v = w$.

- Having $\varphi(\underline{A})$, and, consecutively, $\varphi(\underline{b}_1)$ and $\varphi(\underline{c}_1)$, fixed, $\varphi(\underline{A}_b)$ is chosen as a maximal, in the same sense as before, b-neighbor of $\underline{b}_1$ such that $\varphi(\underline{A}) \trianglelefteq \varphi(\underline{A}_b)$ and $A \to b \in \tau(\varphi(\underline{A}_b))$, and similarly for the choice of $\varphi(\underline{A}_c)$;
- Having $\varphi(\underline{A}_b)$ fixed, $\varphi(\underline{A}_{b,c})$ is chosen as a maximal c-neighbor of $\underline{c}_1$ such that $\varphi(\underline{A}_b) \leq \varphi(\underline{A}_{b,c})$ and $A \to bc \in \tau(\varphi(\underline{A}_{b,c}))$, and similarly for the choice of $\varphi(\underline{A}_{c,b})$.

Similarly, $\varphi = \varphi_{x,a}$ is *type-maximal* if

- $\varphi(\underline{A})$ is chosen to be an a-neighbor of x with A in its type, which is maximal with respect to the arrows of white type. That is, there is no a-neighbor u of x such that $A \in \tau(u)$ and $A(\varphi(\underline{A})) \subsetneq A(u)$,

and similarly for the choices of $\varphi(\underline{A}_b)$ and $\varphi(\underline{A}_{b,c})$.

Let us note that φ is maximal implies φ is type-maximal, but not the other way round. For the most part of the proof, we will use type-maximality, however, in the second part of the proof of the main theorem in Section 4.6, we will have to use both maximality itself and the difference between maximality and type-maximality. We also note that, even with the requirement of maximality, the choice of $\varphi(\underline{A})$ is, in general, not unique.

Lemma 16. *If, in Lemma 14, in the construction of $\mathscr{A} = \mathscr{A}(\Phi)$ every $\varphi \in \Phi$ is type-maximal, then the following conditions hold. The conditions are described diagrammatically, and the names are not assumed to be exact. For the sake of clarity, the first condition, (AC3), is also described verbally.*

(AC3) If

$$\begin{tikzcd}
\underline{a} \arrow[rr, shift left, "a-b"] \arrow[dr, "a-c"'] & & \underline{ab} \arrow[ll, shift left, "b-a"] \arrow[dl, "b-c"] \\
& \underline{c} &
\end{tikzcd}$$

then $\underline{a} = \underline{ab}$;

That is, if there are vertices $\underline{a}, \underline{ab}$, and $\underline{c} \in S$ such that $a \in \tau(\underline{a})$, $a, b \in \tau(\underline{ab})$, $c \in \tau(\underline{c})$, $\underline{a} \xrightarrow{a-b} \underline{ab}$, $\underline{ab} \xrightarrow{b-a} \underline{a}$, $\underline{a} \xrightarrow{a-c} \underline{c}$, and $\underline{ab} \xrightarrow{b-c} \underline{c}$, then $\underline{a} = \underline{ab}$.

(AC4) If

$$\begin{tikzcd}
\underline{a} \arrow[r, "a-b"] \arrow[rr, bend left, "a-b-c"] & \underline{b} & \underline{c} \arrow[l, "c-b"'] \arrow[r, "c-a"] & \underline{abc}
\end{tikzcd}$$

then $b, c \in \tau(\underline{c})$;

(AC5) If

$$\begin{tikzcd}
\underline{a} \arrow[r, "a-b-c"] \arrow[rr, bend left, "a-b-c-b"] & \underline{c} & \underline{b} \arrow[l, "b-c"'] \arrow[r, "b-a"] & \underline{abc}
\end{tikzcd}$$

then $b, c \in \tau(\underline{b})$.

Proof. (AC3). Suppose not, and let $u = \varphi_{\underline{a},a}(\underline{A})$ and $v = \varphi_{\underline{ab},b}(\underline{B})$. Then $\eta_u = \eta_v$ and hence, by (N), $u = v$. In particular, $B \to A \in \tau(u)$. Then, by (E2), there is a b-b-path from u to u^+ such that $A \to b \in \tau(u^+)$ and $u^+ \trianglerighteq u$. In particular, $A(u^+) \supseteq A(u)$, so $[Ab] \in \tau(u^+)$ and, by (E4), $\eta_a(u^+) = \eta_b(u^+) = \underline{ab}$; note also that from the fact that $\tau(u^+)$ satisfies (T) it follows that $B \in \tau(u^+)$, that is, $[AB] \in \tau(u^+)$.

Now, $v = u$, $A(u) \subseteq A(u^+)$, $B \in \tau(u^+)$, and $\eta_b(u^+) = \underline{ab}$ imply, by the type-maximality of $\varphi_{\underline{ab},b}$, that $u^+ = v$. But then $\underline{a} = \eta_a(v) = \eta_a(u^+) = \underline{ab}$. This argument is illustrated in Figure 4.7 below, the dashed arrow indicate the change of choice of $\varphi_{\underline{ab},b}(\underline{B})$ due to type-maximality.

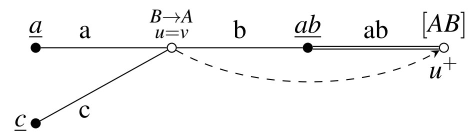


Figure 4.7: (AC3) argument.

(AC4). The proof follows along the same lines, so we adopt a proof by picture approach in Figure 4.8 below. Here $v = \varphi_{\underline{a},a}(\underline{A}_b)$, $u = \varphi_{\underline{c},c}(\underline{C})$; same as before, $u = v$. We also use the fact that $c_b(\underline{a}) = \underline{abc}$, and hence $\eta_a(v) = \underline{abc}$, which follows from the fact that $\underline{a} \xrightarrow{a-b} \underline{b}$ implies $c_b(\underline{a}) = c_a(\underline{b})$, and $\underline{b} \xleftarrow{c-b} \underline{c} \xrightarrow{c-a} \underline{abc}$ implies $c_a(\underline{b}) = c_b(\underline{abc}) = \underline{abc}$, both by (AC1). Let us separately note that $\varphi_{\underline{a},a}$ should respect the order on F_a, so, in order to infer that $v = u^+$, we also have to ensure that u^+ is an eligible candidate, that is, that $u^+ \trianglerighteq \varphi(\underline{A})$. But this is true because $u^+ \trianglerighteq u = v = \varphi(\underline{A}_b) \trianglerighteq \varphi(\underline{A})$.

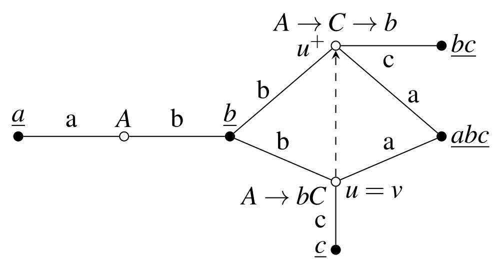


Figure 4.8: (AC4) argument.

(AC5). The proof is similar as for (AC4); still, we illustrate it in Figure 4.9. Here $v = \varphi_{\underline{a},a}(\underline{A}_{b,c})$, $u = \varphi_{\underline{b},b}(\underline{B})$, and again $u = v$; $\underline{b}'$ is just $\varphi_{\underline{a},a}(\underline{b}_1)$

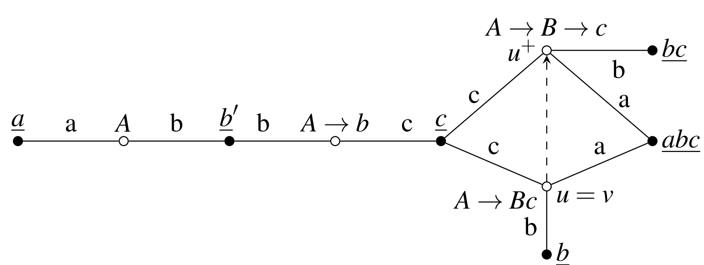


Figure 4.9: (AC5) argument.

□

As all $\varphi \in \Phi$ from which $\mathscr{A} = \mathscr{A}(\Phi)$ is constructed in Lemma 14 can, as argued, be chosen to be type-maximal, we will assume that $\mathscr{A}$ satisfies (AC3)–(AC5) on top of (AC1) and (AC2).

4.5 $\mathrm{RC}_3 \Rightarrow$SSP, part 1

Now we are going to prove the $\mathrm{RC}_3 \Rightarrow$ SSP case in general, and, as an attentive reader could have noticed, the proof is split into (two) parts, with this section covering the first one. The proof of the first part goes along the same lines as Theorem 3: The argument is concentrated on an arbitrary congruence $\|\Theta\|$ of a free graph and ends roughly with an analog (much more complicated) of $4t+2p+s \leq 3t+2p+s+1$ inequality, which proves that $\|\Theta\|$ can be smaller than $|S|$ by at most one. In Theorem 3 that was almost it, the structural properties imposed by such a tight bound were enough to finish the proof in several steps, staying inside the same setup.

The first part of the proof of a general case will follow the same pattern, which will end up with a similar inequality, proving that $\|\Theta\| \geq |S|-1$. However, it will not be enough, and for a good reason. We will give a "counterexample", that is, a connected graph Γ that satisfies all the conditions of an RC graph, except for (E3). As it can be noticed, the construction of the free graph and a congruence over it does not rely on this last condition, and thus this construction alone is not enough to finish the proof. We will thus need to go back to Γ, carry over some consequences of it being almost SSP, and use (E3) in it. This second part of the proof will be done in the next section.

Throughout the proof, for the sake of self-sufficiency, we will not be emphasizing the parallel with the trivial closures case, so let us note a few things at the beginning. Triangles and pyramids will still be of primary importance for us, however, they are now defined on $\underline{S}$ rather than on S: In trivial closures case we were able to ignore this difference by identifying all vertices in $S-\underline{abc}$ with their counterparts in $\underline{S}$. With this in mind, stage-1 vertices are dealt with in essentially the same way as before.

Similarly, we will define principal stage-2 vertices. Here, however, we will have to make some changes, in particular, the number of principal stage-2 vertices produced by triangles will, in general, drop to four. Additionally, we will need to address the principal stage-2 vertices eaten by stage-1 ones. We will be able to compensate them using stage-3 vertices and the extra properties of arrow structures, enforced by Lemma 16.

Theorem 4. *Let Γ be an* RC_3*-graph. Then Γ is* SSP.

Proof (part 1). Let Γ be an RC_3-graph, and suppose Γ is not SSP, that is, $|C|<|S|$. As argued, we can assume that Γ is connected and satisfies (T) and (N). Let Φ be chosen as in Lemma 16, that is, such that every $\varphi \in \Phi$ is type-maximal, and let $\mathscr{A}=\mathscr{A}(\Phi)$, which implies that $\mathscr{A}$ satisfies (AC1)–(AC5). Let $F=F_{\mathscr{C},\mathscr{A}}$ be the free graph, $\theta=\theta(\Phi)$ be the morphism of F into Γ, and Θ be the kernel of θ.

We will write $\underline{S}-\underline{abc}$ to denote $\underline{S}-\{(\underline{abc},a),(\underline{abc},b),(\underline{abc},c)\}$. We define a *triangle* as three elements $(\underline{a},a) \in \underline{S}-\underline{abc}$, for $a \in \mathscr{I}=abc$, called the *corners of the triangle*, such that $\underline{a} \xrightarrow{a-b} \underline{b}$, for all $a,b \in abc$, $a \neq b$. While it can be the case that $\underline{a}=\underline{b}$ for $a \neq b$, when considered as elements of $\underline{S}$, the tuples $(\underline{a},a)$ for

$a \in abc$ are pairwise distinct.

Similarly, for $a, b \in abc$, $a \neq b$, we define an *ab-pyramid* as three elements $(\underline{a}, a), (\underline{b}, b) \in \underline{S} - \underline{abc}$, and $(\underline{c}, c) \in \underline{S}$, where c is a remaining third letter from $abc - ab$, such that $\underline{a} \xrightarrow{a-b} \underline{b}$, $\underline{b} \xrightarrow{b-a} \underline{a}$, $\underline{a} \xrightarrow{a-c} \underline{c}$, and $\underline{b} \xrightarrow{b-c} \underline{c}$. Additionally, we require that $(\underline{a}, a)$, $(\underline{b}, b)$, and $(\underline{c}, c)$ do not form a triangle. We call the set $\{(\underline{a}, a), (\underline{b}, b)\}$ a *base* of the pyramid, and $(\underline{c}, c)$ its *tip*. All triangles and pyramid bases are obviously disjoint from each other. All elements in $\underline{S} - \underline{abc}$ not in a triangle or the base of a pyramid are called *singletons*.

We now exclude stage-1 vertices produced by $(\underline{abc}, a)$, for $a \in \mathscr{I}$, from the set of stage-1 elements, and call them stage-1t. Let us note that (N) implies that stage-1 vertices of every triangle and every pyramid base are mapped to the same white vertex of Γ. That is, for, say, a triangle $(\underline{a}, a)$, $(\underline{b}, b)$ and $(\underline{c}, c)$, it holds $\theta \circ u_{s1}(\underline{a}, a) = \theta \circ u_{s1}(\underline{b}, b) = \theta \circ u_{s1}(\underline{c}, c)$, and similarly for the pyramids. Now, let us contract these vertices, that is, form a congruence Ω of F whose nontrivial classes combine the stage-1 vertices of triangles and pyramids, and factor F, θ, and Θ by Ω. We will still call the corresponding objects F, θ, and Θ; additionally, we will sometimes refer to F before contraction, which we will now denote by F^*. We still use $u_{s1}(\underline{a}, a)$ notation for the stage-1 vertices, but now, if $(\underline{a}, a)$ and $(\underline{b}, b)$ are corners of a triangle or a pyramid base, then $u_{s1}(\underline{a}, a) = u_{s1}(\underline{b}, b)$. Note that, after factoring by Ω, all stage-1 vertices are exact and cannot be contracted with each other, that is, their neighborhoods form an antichain.

Let us now further classify triangles and pyramids. The classification is symbolically depicted in Figure 4.10 and Figure 4.11 and formally defined below. In this classification, we assume a, b, and c to be pairwise distinct letters from $\mathscr{I}$; recall that, alternatively, we can consider them literally the letters of $\mathscr{I}$ and then treat each case as defined up to a permutation of these letters.

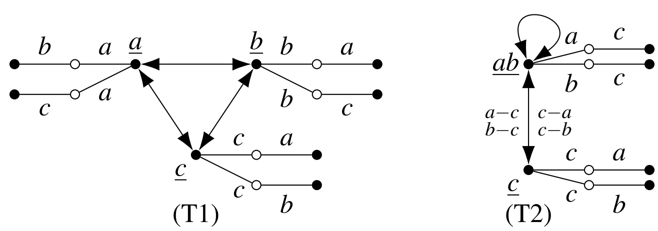


Figure 4.10: Classification of triangles. The labels on the arrows are omitted whenever they are unambiguous. Additionally, the picture shows principal stage-2 vertices produced by the given triangle.

A triangle can be one of the two (mutually exclusive) kinds:

(T1) Formed by elements $(\underline{a}, a)$, $(\underline{b}, b)$, and $(\underline{c}, c)$, with $\underline{a}$, $\underline{b}$, and $\underline{c}$ pairwise distinct, and $\tau(\underline{a}) = a$, $\tau(\underline{b}) = b$, and $\tau(\underline{c}) = c$;

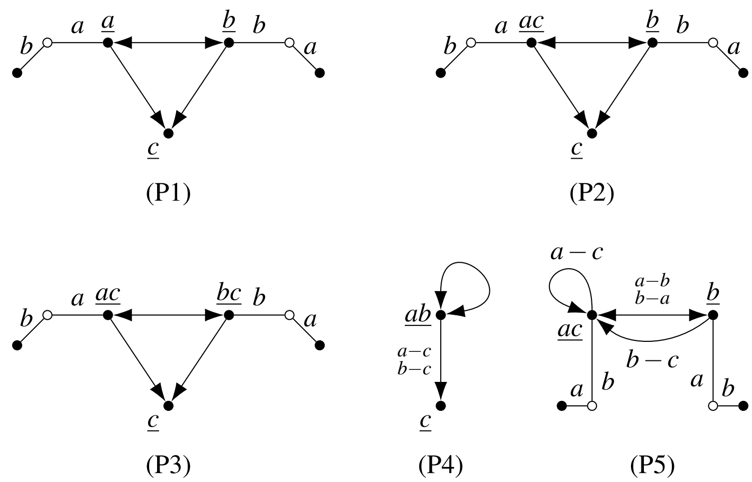


Figure 4.11: Classification of *ab*-pyramids. The notation is similar to Figure 4.10.

(T2) Formed by elements $(\underline{ab}, a)$, $(\underline{ab}, b)$, $(\underline{c}, c)$, *with* $\underline{ab} \neq \underline{c}$, $\tau(\underline{ab}) = ab$, *and* $\tau(\underline{c}) = c$.

An ab-pyramid can be one of the five (mutually exclusive) kinds:

(P1) Formed by elements $(\underline{a}, a)$, $(\underline{b}, b)$, *and* $(\underline{c}, c)$, *with* $\underline{a}$, $\underline{b}$, *and* $\underline{c}$ *pairwise distinct, and* $\tau(\underline{a}) = a$, $\tau(\underline{b}) = b$, *and* $c \in \tau(\underline{c})$*;*

(P2) Formed by elements $(\underline{ac}, a)$, $(\underline{b}, b)$, *and* $(\underline{c}, c)$, *with* $\underline{ac}$, $\underline{b}$, *and* $\underline{c}$ *pairwise distinct, and* $\tau(\underline{ac}) = ac$, $\tau(\underline{b}) = b$, *and* $c \in \tau(\underline{c})$*;*

(P3) Formed by elements $(\underline{ac}, a)$, $(\underline{bc}, b)$, *and* $(\underline{c}, c)$, *with* $\underline{ac}$, $\underline{bc}$, *and* $\underline{c}$ *pairwise distinct, and* $\tau(\underline{ac}) = ac$, $\tau(\underline{bc}) = bc$, *and* $c \in \tau(\underline{c})$*;*

(P4) Formed by elements $(\underline{ab}, a)$, $(\underline{ab}, b)$, *and* $(\underline{c}, c)$, *with* $\underline{ab} \neq \underline{c}$, $\tau(\underline{ab}) = ab$, *and* $c \in \tau(\underline{c})$*;*

(P5) Formed by elements $(\underline{ac}, a)$, $(\underline{b}, b)$, *and* $(\underline{ac}, c)$, *with* $\underline{ac} \neq \underline{b}$, $\tau(\underline{ac}) = ac$, *and* $\tau(\underline{b}) = b$.

Proof. The cases (T1) and (T2) are obviously disjoint, so we only need to show that they are also exhaustive, that is, that every triangle falls in either one or another category. So suppose the elements $(\underline{a}, a)$, $(\underline{b}, b)$, and $(\underline{c}, c)$ form a triangle. If $\underline{a}$, $\underline{b}$, and $\underline{c}$ are pairwise distinct, then, by (AC3), $\tau(\underline{a}) = a$, $\tau(\underline{b}) = b$, and $\tau(\underline{c}) = c$. Indeed, $a \in \tau(\underline{a})$ by definition, and if, say, $b \in \tau(\underline{a})$, then (AC3) implies $\underline{a} = \underline{b}$. Thus, this case falls under (T1). Note that it cannot happen that $\underline{a} = \underline{b} = \underline{c}$, as then $\tau(\underline{a}) = abc$, and hence $\underline{a} = \underline{abc}$. So we are left with the possibility that only two of $\underline{a}$, $\underline{b}$, and $\underline{c}$ coincide and, without losing generality, let us assume that $\underline{a} = \underline{b}$, and we then call this vertex $\underline{ab}$. Then $a, b \in \tau(\underline{ab})$, and, as $\underline{ab} \neq \underline{abc}$, $\tau(\underline{ab}) = ab$. But then (AC3) implies that $\tau(\underline{c}) = c$, and hence this is (T2) case.

The proof of the classification of the pyramids is along the same lines but is more meticulous. The cases (P1)–(P5) are obviously disjoint, with (P1)–(P3) corresponding to the case when all vertices are distinct, and among themselves are distinguished by the types of the vertices from the base; (P4) is the case when the vertices from the base coincide, and (P5) is when one of the vertices in the base coincides with the tip.

Now, to prove the exhaustiveness, suppose that $(\underline{a},a)$, $(\underline{b},b)$, and $(\underline{c},c)$ form an ab-pyramid. If $\underline{a}$, $\underline{b}$, and $\underline{c}$ are pairwise distinct, then, by (AC3), $b \notin \tau(\underline{a})$ and $a \notin \tau(\underline{b})$. The remaining possibilities for the types of $\underline{a}$ and $\underline{b}$ are: $\tau(\underline{a}) = a$ and $\tau(\underline{b}) = b$; $\tau(\underline{a}) = ac$ and $\tau(\underline{b}) = b$, or $\tau(\underline{a}) = a$ and $\tau(\underline{b}) = bc$; and, finally, $\tau(\underline{a}) = ac$ and $\tau(\underline{b}) = bc$, precisely correspond to the cases (P1), (P2), (P2), and (P3) respectively. Note that $\tau(\underline{a}) = a$ and $\tau(\underline{b}) = bc$ becomes precisely (P2) with $[a \to b, b \to a]$ permutation. Now, let $\underline{a} = \underline{b}$. As before, we call it $\underline{ab}$, and observe that $\underline{ab} \neq \underline{c}$ and $\tau(\underline{ab}) = ab$. This precisely corresponds to (P4).

Finally, let $\underline{c}$ be equal to either $\underline{a}$ or $\underline{b}$; without losing generality, we assume $\underline{a} = \underline{c}$, call this vertex $\underline{ac}$, and observe that $\underline{ac} \neq \underline{b}$, $\tau(\underline{ac}) = ac$, and, by (AC3), $a \notin \tau(\underline{b})$. Notice that if $\tau(\underline{b}) = b$ then this is precisely (P5), and so we only need to show that $\tau(\underline{b}) \neq bc$. But, if $\tau(\underline{b}) = bc$, then $\underline{b} \xrightarrow{b-c} \underline{ac}$ implies, by (AC1), that $\underline{b} = c_c(\underline{b}) = c_b(\underline{ac}) = \underline{abc}$, which cannot happen by definition. This ends the proof of the classification. $\square$

Additionally, we classify all singletons as either (S1), of the form $(\underline{a},a)$ with $\tau(\underline{a}) = a$, or as (S2), of the form $(\underline{ab},a)$ with $\tau(\underline{ab}) = ab$. We call this TPS-classification.

We say that an arrow $\underline{a} \xrightarrow{a-b} \underline{b}$ is *principal* if $\underline{a} \neq \underline{b}$, $\underline{b} \xrightarrow{b-a} \underline{a}$, and there is $\underline{c} \in S$ such that $\underline{a} \xrightarrow{a-c} \underline{c}$ and $\underline{b} \xrightarrow{b-c} \underline{c}$; in other words, if it is an arrow between distinct vertices inside a triangle or the base of an ab-pyramid. Note that $\underline{a} \xrightarrow{a-b} \underline{b}$ is principal if and only if $\underline{b} \xrightarrow{b-a} \underline{a}$ is. Also, from the TPS-classification, it follows that $a \notin \tau(b)$ and $b \notin \tau(a)$, in particular, $\underline{a}, \underline{b} \neq \underline{abc}$. We say that a stage-2 vertex u is *principal* if it is generated by a principal arrow, that is, if $u = u_{s2}(\underline{a},a,b)$ for a principal $\underline{a} \xrightarrow{a-b} \underline{b}$. However, we will outline a particular subcase of principal stage-2 vertices. Namely, we call a principal stage-2 vertex $u = u_{s2}(\underline{a},a,b)$ *quasi-principal*, or QP, if

(QP) $\tau(\underline{b}) = b$, $(\underline{b},b)$ is a corner of a triangle, and $\tau(c_a(\underline{b})) = ab$ and $c_c(\underline{b}) = \underline{abc}$.

The TPS-classification naturally carries over to the classification of principal stage-2 vertices. They are already drawn in Figure 4.10 for triangles, and in Figure 4.11 for pyramids; the singletons do not produce any principal vertices. However, we will now elaborate on it to add information about QP and non-QP vertices.

Note that there can be two principal stage-2 vertices from a given $(\underline{b},b)$, but this can only happen if $(\underline{b},b)$ is a corner of a triangle, moreover, $(\underline{b},b)$ should either be one of the corners of a T1-triangle, or the $(\underline{c},c)$ corner of a T2-triangle. We also note that (QP) is applicable only when the two principal stage-2 vertices from $(\underline{b},b)$ has their ordinary non-b edges going to black vertices x and y with $|\tau(x)| = 2$ and $y = \underline{abc}$; in this case, x is QP and y is non-QP. For example, if for the $(\underline{b},b)$ corner in a T1-triangle it holds $c_a(\underline{b}) = \underline{abc}$ and

$\tau(c_c(\underline{b})) = bc$, then $u_{s2}(\underline{c}, c, b)$ is QP and $u_{s2}(\underline{a}, a, b)$ is non-QP. However, if $\tau(c_a(\underline{b})) = ab$ and $\tau(c_c(\underline{b})) = bc$, then both $u_{s2}(\underline{a}, a, b)$ and $u_{s2}(\underline{c}, c, b)$ are non-QP. Let us note that we still consider a QP vertex to be principal. This decision is rather arbitrary: for the first part of the proof we will be mostly interested in the number of non-QP vertices, but for the second part we will need to address QP vertices separately, so we keep track of both of them anyway. We will write simply QP vertex instead of principal stage-2 QP vertex, and similarly for non-QP.

Naively, it looks like with (QP) in place, it might happen that all corners of the triangles, whose types has size one, will produce just one non-QP vertex and, consequently, that both T1 and T2-triangles can produce just three of them. However, we will now show that a T1-triangle produces at least four, and T2-triangle precisely four non-QP vertices. In particular, it means that (QP) is effectively inapplicable to (T2). This, together with the upcoming formal statement and the argument, is illustrated in Figure 4.12 below.

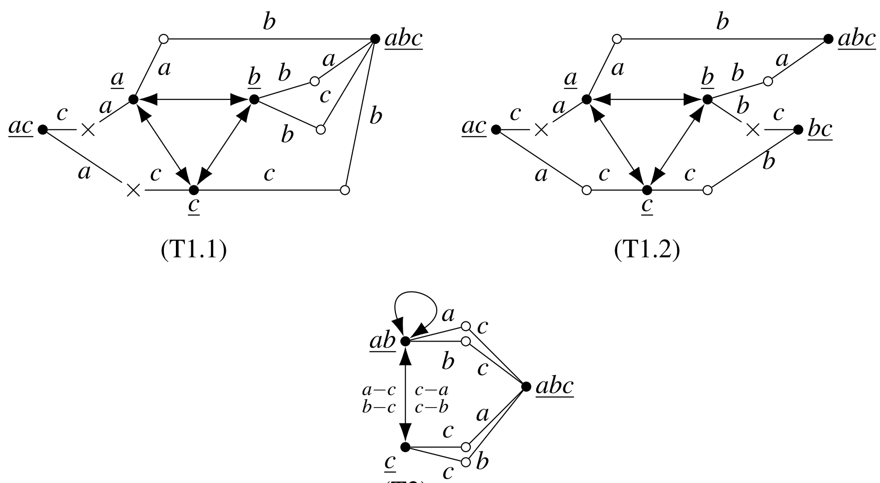


Figure 4.12: Principal stage-2 vertices of triangles. In (T1), crosses show the QP vertices.

Let $(\underline{a}, a)$, $(\underline{b}, b)$, and $(\underline{c}, c)$ form a T1-triangle. Then $c_b(\underline{a}) = c_a(\underline{b})$. In particular, the triangle produces at least four non-QP vertices, and the only two cases when exactly four non-QP vertices are produced is when, for some permutation of a, b, and c, either (T1.1) $c_b(\underline{a}) = c_a(\underline{b}) = c_c(\underline{b}) = c_b(\underline{c}) = \underline{abc}$, and $c_c(\underline{a}) = c_a(\underline{c}) = \underline{ac}$, or (T1.2) $c_b(\underline{a}) = c_a(\underline{b}) = \underline{abc}$, $c_c(\underline{a}) = c_a(\underline{c}) = \underline{ac}$, and $c_c(\underline{b}) = c_b(\underline{c}) = \underline{bc}$, where $\underline{ac}$ and $\underline{bc}$ are some vertices such that $\tau(\underline{ac}) = ac$ and $\tau(\underline{bc}) = bc$.

Proof. The fact that $c_b(\underline{a}) = c_a(\underline{b})$ is by (AC1). Now, suppose all three corners of a T1-triangle produce just one non-QP vertex. Without losing generality, let $c_b(\underline{a}) = \underline{abc}$ and $c_c(\underline{a}) = \underline{ac}$, for some $\underline{ac}$ such that $\tau(\underline{ac}) = ac$. Then $c_a(\underline{c}) = \underline{ac}$. But then, because $\underline{c}$ should also produce just one non-QP vertex, $c_b(\underline{c}) = \underline{abc}$.

But then $c_a(\underline{b}) = c_b(\underline{a}) = \underline{abc}$ and $c_c(\underline{b}) = c_b(\underline{c}) = \underline{abc}$, and hence both stage-2 vertices from the corner $\underline{b}$ are non-QP. Note that this also corresponds to (T1.1) subcase.

Now, if we assume that the triangle produces not three, but four non-QP vertices, we can, as before, assume that $c_b(\underline{a}) = \underline{abc}$ and $c_c(\underline{a}) = \underline{ac}$, and so $c_a(\underline{b}) = \underline{abc}$ and $c_a(\underline{c}) = \underline{ac}$. If $c_b(\underline{c}) = \underline{abc}$, then (T1.1) case follows, so let $c_b(\underline{c}) = \underline{bc}$, for some $\underline{bc}$ such that $\tau(\underline{bc}) = bc$. But then $c_c(\underline{b}) = \underline{bc}$. Notice that $u_{s2}(\underline{c}, c, b)$ is QP, but both stage-2 vertices from $(\underline{c}, c)$ corner, that is, $u_{s2}(\underline{a}, a, c)$ and $u_{s2}(\underline{b}, b, c)$, are non-QP. Thus, the triangle produces four non-QP vertices and corresponds to the case (T1.2). $\square$

Let $(\underline{ab}, a)$, $(\underline{ab}, b)$, and $(\underline{c}, c)$ form a T2-triangle. Then $c_c(\underline{ab}) = c_a(\underline{c}) = c_b(\underline{c}) = \underline{abc}$. In particular, both stage-2 vertices from the $(\underline{c}, c)$ corner are non-QP. Moreover, $u_{s2}(\underline{c}, c, a)$ and $u_{s2}(\underline{c}, c, b)$ are non-QP by definition, and so the trianle produces four non-QP vertices in total. The fact that $c_c(\underline{ab}) = \underline{abc}$ is trivial, and $c_a(\underline{c}) = c_b(\underline{c}) = \underline{abc}$ follows from it by (AC1). Then both stage-2 vertices from the $(\underline{c}, c)$ corner are non-QP by definition.

Recall that, by the construction of F, all stage-2 vertices are exact. However, we will mostly be interested in their non-exact ordinary parts. Formally, for a stage-2 vertex $u = u_{s2}(\underline{a}, a, b)$, we define the ordinary part η_u^* of u as a constraint such that $\eta_u^*(b) = \eta_u(b) = \underline{b}$, for $\underline{b}$ such that $\underline{a} \xrightarrow{a-b} \underline{b}$, $\eta_u^*(a) = \eta_u(a) = c_b(\underline{a})$, and $\eta_u^*(c) = \text{NA}$ for $c \in abc - ab$. Trivially, $\eta_u^* \leq \eta_u$. The following statement is obvious by construction.

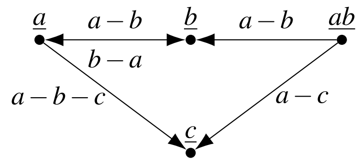
The ordinary parts of the principal stage-2 vertices are incomparable, that is, they form an antichain. Let us note that this, in general, cannot be said about the principal stage-2 vertices themselves, or about the ordinary parts of non-principal stage-2 vertices.

We say that a principal stage-2 vertex $u = u_{s2}(\underline{a}, a, b)$ is *eaten* by a stage-1 vertex $v = u_{s1}(\underline{d}, d)$, for some $d \in \mathscr{I}$, if $\eta_u = \eta_v$; here, as usual, we assume $c \in \mathscr{I} - ab$, and use d to emphasize the fact that, in general, d can be any of a, or b, or c. Note that, by (N), this implies $\theta(u) = \theta(v)$, that is, u and v are in the same class of Θ. Additionally, as, after factoring by Ω, the neighborhoods of all stage-1 vertices are incomparable, u can be eaten by at most one stage-1 vertex. Thus, if, additionally, u is eaten by $w = u_{s1}(\underline{e}, e)$, then, although $(\underline{d}, d) \neq (\underline{e}, e)$, it holds $v = u_{s1}(\underline{d}, d) = u_{s1}(\underline{e}, e) = w$. We also note that it only happens if $(\underline{d}, d)$ and $(\underline{e}, e)$ are either corners of a triangle or the base of a pyramid.

We now will classify the cases in which a principal stage-2 vertex can be eaten by a stage-1 vertex.

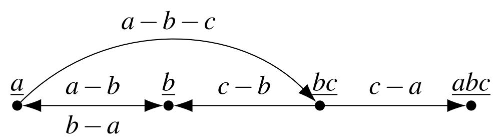
Let a, b, and c be pairwise distinct letters of $\mathscr{I}$, $\underline{a} \xrightarrow{a-b} \underline{b}$ a principal arrow, and $u = u_{s2}(\underline{a}, a, b)$ the corresponding principal stage-2 vertex from $(\underline{b}, b)$. Then, unless one of the following conditions hold, u is not eaten by any stage-1 vertex.

(K1) For $\underline{ab} = c_a(\underline{b}) = c_b(\underline{a})$, $\tau(\underline{ab}) = ab$, and a unique $\underline{c}$ such that $\underline{a} \xrightarrow{a-b-c} \underline{c}$, it holds



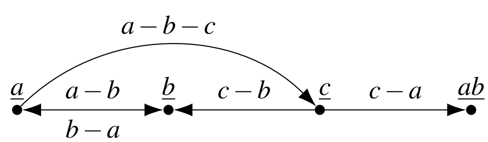
Here u is eaten by $u_{s1}(\underline{ab}, a)$. This case implies $\tau(\underline{a}) = a$ and $\tau(\underline{b}) = b$, in particular, $\underline{a}$, $\underline{b}$, and $\underline{ab}$ are pairwise distinct;

(K2) *For $\underline{abc} = c_a(\underline{b}) = c_b(\underline{a})$ and $\underline{bc} = c_c(\underline{b})$ such that $\tau(\underline{bc}) = bc$, it holds*



Here u is eaten by $u_{s1}(\underline{bc}, c)$. This case implies $\tau(\underline{a}) = a$ or ac, $\tau(\underline{b}) = b$ or bc, and $(\underline{bc}, c)$ is a singleton; the vertices $\underline{a}$, $\underline{b}$, $\underline{bc}$, and $\underline{abc}$ are pairwise distinct, except for, possibly, $\underline{b} = \underline{bc}$, which happens if and only if $\tau(\underline{b}) = bc$;

(K3) *For $\underline{ab} = c_a(\underline{b}) = c_b(\underline{a})$, $\tau(\underline{ab}) = ab$, and a unique $\underline{c}$ such that $\underline{a} \xrightarrow{a-b-c} \underline{c}$, it holds*

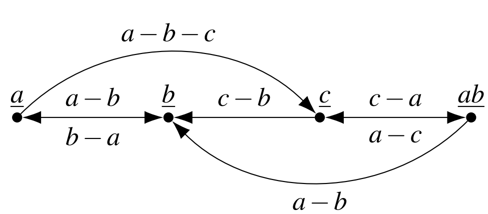


Here u is eaten by $u_{s1}(\underline{c}, c)$. This case implies $\tau(\underline{a}) = a$, $\tau(\underline{b}) = b$, and $\tau(\underline{c}) = c$ or bc; in particular, $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{ab}$ are pairwise distinct.

Additionally, the cases (K1)–(K3) are not self-compatible*, that is, in F^*, if u is eaten by v and by w, both by, say, (K1), then $v = w$, and similarly for (K2) and (K3). In F, that is, after contraction by Ω, it means that if u is eaten by (K1) by $v = u_{s1}(\underline{d}, d)$ and by $w = u_{s1}(\underline{e}, e)$, then $(\underline{d}, d) = (\underline{e}, e)$.*

Also, (K1) and (K3) are not compatible with (K2), that is, in F^*, u cannot be eaten by v by (K1) and by w by (K2), and similarly for (K3) and (K2). The cases (K1) and (K3) are compatible and this situation is described below

(K1+3) *For $\underline{ab} = c_a(\underline{b}) = c_b(\underline{a})$, $\tau(\underline{ab}) = ab$, and a unique $\underline{c}$ such that $\underline{a} \xrightarrow{a-b-c} \underline{c}$, it holds*



Here u is eaten by $u_{s1}(\underline{ab},a) = u_{s1}(\underline{c},c)$. This case implies $\tau(\underline{a}) = a$, $\tau(\underline{b}) = b$, and $\tau(\underline{c}) = c$ or bc, in particular, $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{ab}$ are pairwise distinct. Also, in this case, $(\underline{ab},a)$ and $(\underline{c},c)$ are the corners of either a P2-pyramid (in case $\tau(\underline{c}) = c$) or a P3-pyramid (in case $\tau(\underline{c}) = bc$).

Finally, if it is (K1) or (K3), but not (K1+3), then the elements $(\underline{ab},a)$ in the first case, and $(\underline{c},c)$ in the second, are singletons.

Let us note that $\underline{a} \xrightarrow{a-b} \underline{b}$ being principal also implies $\underline{a} \xrightarrow{a-c} \underline{c}'$ and $\underline{b} \xrightarrow{b-c} \underline{c}'$ arrows for some $\underline{c}' \in S$, which are not shown in the pictures.

Proof. We start by noting that in (K1)–(K3) cases the corresponding stage-1 vertex does indeed eat u by construction. We thus need to prove that these cases are exaustive.

The fact that $\underline{a} \neq \underline{b}$, $b \notin \tau(\underline{a})$, and $a \notin \tau(\underline{b})$ is by the definition of a principal arrow. Let $\underline{ab} = c_b(\underline{a}) = c_a(\underline{b})$, then $\tau(\underline{ab})$ is either ab or abc, in particular, $\underline{ab} \neq \underline{a}, \underline{b}$, and let $\underline{c}$ be a unique black vertex such that $\underline{a} \xrightarrow{a-b-c} \underline{c}$. Note that, by definition, $\eta_u(a) = \underline{ab}$, $\eta_u(b) = \underline{b}$, and $\eta_u(c) = \underline{c}$, so, if u is eaten by a stage-1 vertex v, then $\eta_v(b) = \underline{b}$, $\eta_v(a) = \underline{ab}$ and $\eta_v(c) = \underline{c}$.

This trivially implies that v can be either i) $u_{s1}(\underline{b},b)$, or ii) $u_{s1}(\underline{ab},a)$, or iii) $u_{s1}(\underline{c},c)$. The case i) is impossible, as $\underline{a} \xrightarrow{a-b} \underline{b}$ is principal, hence $\underline{b} \xrightarrow{b-a} \underline{a}$, and so if $w = u_{s1}(\underline{b},b)$ then $\eta_w(a) = \underline{a} \neq \underline{ab}$. The case ii) falls under (K1), the only thing left to check is that $\tau(\underline{ab}) = ab$, but the only other option is $\tau(\underline{ab}) = abc$, and hence $\underline{ab} = \underline{abc}$. But it is impossible as we excluded the vertices $u_{s1}(\underline{abc},a)$ from the set of stage-1 vertices; so, $\tau(\underline{ab}) = ab$. Also, ii) implies $\tau(\underline{a}) = a$, as otherwise, that is, if $\tau(\underline{a}) = ac$, it then holds $\tau(\underline{ab}) = \tau(c_b(\underline{a})) = abc$, and hence $\underline{ab} = \underline{abc}$; similarly, $\tau(\underline{b}) = b$.

Now, let us consider the case iii), that is, $v = u_{s1}(\underline{c},c)$. This trivially implies $\underline{b} \xleftarrow{c-b} \underline{c} \xrightarrow{c-a} \underline{ab}$. Let us note that $a \notin \tau(\underline{c})$, for otherwise, by (AC1), $\underline{c} = c_a(\underline{c}) = c_c(\underline{ab}) = \underline{abc}$. But then $\underline{c}$ produces no stage-1 vertex. Thus, $\tau(\underline{c})$ is either c or bc, in particular, $\underline{c} \neq \underline{a}, \underline{ab}$.

Let us now split the case iii) according to whether $\underline{ab} = \underline{abc}$ or $\tau(\underline{ab}) = ab$, which correspond to (K2) and (K3). The fact that in the first subcase $\tau(\underline{c}) = bc$ follows from (AC4), and hence we call this vertex $\underline{bc}$. The fact that $\underline{bc} = c_c(\underline{b})$ follows from (AC1), along with the fact that $c \in \tau(\underline{b})$ implies $\underline{b} = \underline{bc}$. Finally, the second subcase, $\tau(\underline{ab}) = ab$, is precisely (K3). It implies $\tau(\underline{a}) = a$ and $\tau(\underline{b}) = b$, for otherwise, say, if $\tau(\underline{a}) = ac$, then $\underline{abc} = \tau_b(\underline{a}) = \underline{ab}$.

The fact that (K1)–(K3) are not self-compatible is trivial. Indeed, by definition of (K1), if $u = u_{s2}(\underline{a},a,b)$ is eaten by (K1), then it is eaten by (K1) by $u_{s1}(\underline{ab},a)$, where $\underline{ab} = c_b(\underline{a})$. So, if u is eaten by (K1) by $v = u_{s1}(\underline{d},d)$ and $w = u_{s1}(\underline{e},e)$, then $(\underline{d},d) = (\underline{e},e) = (\underline{ab},a)$. Self-incompatibility of (K2) and (K3) is similarly trivial. Additionally, (K1) and (K3) are not compatible with (K2) because in both of the former cases, $c_a(\underline{b}) = c_b(\underline{a}) = \underline{ab}$, with $\tau(\underline{ab}) = ab$, an in the latter $c_a(\underline{b}) = c_b(\underline{a}) = \underline{abc}$.

The (K1+3) case is just a combination of (K1) and (K3). The fact that the corresponding element in (K2), or in (K1) or (K3) but not (K1+3), is a singleton, follows from the incompatibility result. For example, in (K2), if $(\underline{bc}, c)$ is not a singleton, then $v = u_{s1}(\underline{bc}, c) = w = u_{s1}(\underline{d}, d)$ for some $(\underline{d}, d) \neq (\underline{bc}, c)$. But then, as $\eta_w = \eta_v = \eta_u$, w eats u in F^*. But u is eaten by (K2) by v, and (K2) is not compatible with (K1) or (K3) and not self-compatible. Hence, $v = w$ in F^*, and so $(\underline{bc}, c) = (\underline{d}, d)$, a contradiction.

Finally, in (K1+3), as $u_{s1}(\underline{ab}, a) = u_{s1}(\underline{c}, c)$, $(\underline{ab}, a)$ and $(\underline{c}, c)$ are corners of either a pyramid, or a triangle. But incompatibility implies that there cannot be other $(\underline{d}, d)$ such that $u_{s1}(\underline{d}, d) = u_{s1}(\underline{ab}, a) = u_{s1}(\underline{c}, c)$, and hence it cannot be a triangle. The fact that, depending on the type of $\underline{c}$, this is either a P2 or a P3-pyramid is from the clasification of pyramids. □

In the proof above, we, perhaps, overly elaborated upon incompatibility. Most of the corresponding statements are indeed self-evident. We did it because we want to be extra careful about which vertices eat which, and these cases will become more complicated as the proof progresses.

Now, in (K3) case, whenever $(\underline{c}, c)$ is a singleton, that is, whenever it is not (K1+3), we call the vertex $u = u_{s3}(\underline{a}, a, b, c)$ a *principal stage-3 vertex*. Recall that $\eta_c(u) = \eta_c(u_{s2}(\underline{a}, a, b)) = \underline{c}$, $\eta_a(u) = \underline{abc}$, and $\eta_b(u) = \underline{b}_1$, where $\underline{b}_1$ is a unique vertex such that $\underline{a} \xrightarrow{a-b-c-b} \underline{b}_1$. Figure 4.13 below illustrates how this situation might look in Γ, with $u_{s2}(\underline{a}, a, b)$ and $u_{s1}(\underline{c}, c)$ contracted. The b-edge of u is not shown.

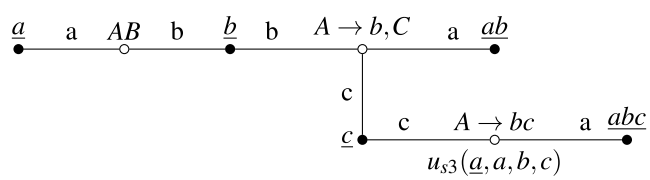


Figure 4.13: Principal stage-3 vertex.

Let us elaborate on the terminology we are going to use. We will say principal vertex to denote a principal stage-2 or principal stage-3 vertex, QP vertex to denote a principal stage-2 QP vertex, and non-QP to denote a principal stage-2 non-QP or principal stage-3 vertex.

The ordinary parts of the principal stage-3 vertices are incomparable between themselves and with the ordinary parts of the principal stage-2 vertices. That is, the ordinary parts of principal vertices form an antichain.

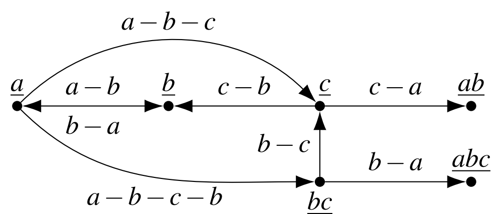
Proof. The fact that the ordinary parts of principal stage-3 vertices are incomparable with the ordinary parts of principal stage-2 vertices follows from the fact that if there is a principal stage-3 vertex from $(\underline{c}, c)$, then $(\underline{c}, c)$ is a singleton, and thus does not have principal stage-2 vertices from it.

To prove that the ordinary parts of stage-3 vertices are incomparable among themselves, let u be such vertex. Without losing generality, u is from $(\underline{c},c)$ to $(\underline{abc},a)$, and let b be a unique letter from $abc-ac$. But then, by the setup of (K3), $u = u_{s3}(\underline{a},a,b,c)$, where $\underline{a}$ is a unique vertex in $S-\underline{abc}$ such that $\underline{c} \xrightarrow{c-b} \underline{b} \xrightarrow{b-a} \underline{a}$, for some unique $\underline{b} \in S-\underline{abc}$. $\square$

Finally, let us address how a principal stage-3 vertex can be eaten by a stage-1 vertex.

Let, in the setup of (K3) with $(\underline{c},c)$ being a singleton, $u = u_{s3}(\underline{a},a,b,c)$ be the corresponding principal stage-3 vertex from $(\underline{c},c)$. Then, unless the following condition holds, u is not eaten by any stage-1 vertex.

(K4) For $\underline{bc} = c_b(\underline{c})$ it holds



Here u is eaten by $u_{s1}(\underline{bc},b)$. This case implies $\tau(\underline{a}) = a$, $\tau(\underline{b}) = b$, $\tau(\underline{ab}) = ab$, $\tau(\underline{bc}) = bc$, $\tau(\underline{c}) = c$ or bc, and $(\underline{bc},b)$ and $(\underline{c},c)$ are singletons. In particular, (K4) is not self-compatible.

Proof. As before, if u is eaten by a stage-1 vertex v, then v is either i) $u_{s1}(\underline{c},c)$, or ii) $u_{s1}(\underline{abc},a)$, or iii) $u_{s1}(\underline{b}_1,b)$ for a unique $\underline{b}_1$ such that $\underline{a} \xrightarrow{a-b-c-b} \underline{b}_1$. Recall that $\eta_u(a) = \underline{abc}$, $\eta_u(b) = \underline{b}_1$, and $\eta_u(c) = \underline{c}$. The case i) is impossible, as $\eta_a(u_{s1}(\underline{c},c)) = \underline{ab} \neq \underline{abc} = \eta_u(a)$. The case ii) is impossible because $\underline{abc}$ produces no stage-1 vertex. In case iii), $\underline{c} \xleftarrow{b-c} \underline{b}_1 \xrightarrow{b-a} \underline{abc}$, and, by (AC5), $b,c \in \tau(\underline{b}_1)$. Also, $a \notin \tau(b_1)$, as then $\underline{b}_1 = \underline{abc}$, which does not produce stage-1 vertex. So, $\tau(\underline{b}_1) = bc$, and hence we call it $\underline{bc}$. Then $\underline{c} \xleftarrow{b-c} \underline{bc}$ implies, by (AC1), $\underline{bc} = c_b(\underline{c})$.

The fact that (K4) is not self-compatible is trivial, and that $(\underline{bc},b)$ is a singleton is its direct consequence. Also, $\tau(\underline{a}) = a$, $\tau(\underline{b}) = b$, $\tau(\underline{ab}) = ab$, $(\underline{c},c)$ is a singleton, and $\tau(\underline{c}) = c$ or bc are by the setup of (K3). $\square$

Note. We do not elaborate upon the compatibility of (K4) with (K1)–(K3) because statement of this sort is meaningless: By saying that (K1) is not compatible with (K2) we mean that any stage-2 principal vertex u cannot be eaten in F^* by stage-1 vertices v, by (K1), and w, by (K2). Similarly, (K4) is not self–compatible means that any stage-3 principal vertex u cannot be eaten by stage-1 vertices v and w, distinct in F^*, by (K4). But, because (K4) and (K1) are applicable do different vertices, saying that they are or are not compatible does not make sense.

Note. As a matter of fact, it can be shown that (K4) also implies $\tau(\underline{c}) = c$, but it involves another maximality argument in the spirit of Lemma 16, so we will not prove it.

The cases (K1)–(K4) address how a principal vertex is eaten by a stage-1 vertex. Now we need to flip this construction and classify the cases when a stage-1 vertex eats one (or several) principal vertices. As a preliminary step, let us state the following.

For $(\underline{c},c)\in\underline{S}$, there cannot be a principal stage-2 vertex together with a principal stage-3 vertex from $(\underline{c},c)$; similarly, there cannot be more than one principal stage-3 vertex from $(\underline{c},c)$.

Proof. If not, then there is at least one principal stage-3 vertex from $(\underline{c},c)$, that is, $\underline{c}$ is $\underline{c}$ from (K3) case and $(\underline{c},c)$ is a singleton. Let us call $\underline{a}$, $\underline{b}$, and $\underline{ab}$ from (K3) $\underline{a}_1$, $\underline{b}_1$, and $\underline{ab}_1$ respectively. Then

$$\begin{tikzcd} \underline{a}_1 \arrow[r, shift left, "a-b"] \arrow[rr, bend left, "a-b-c"] & \underline{b}_1 \arrow[l, shift left, "b-a"] & \underline{c} \arrow[l, "c-b"'] \arrow[r, "c-a"] & \underline{ab}_1 \end{tikzcd},$$

where $\tau(\underline{b}_1)=b$, and the principal stage-3 vertex from $(\underline{c},c)$ is $u_1=u_{s3}(\underline{a}_1,a,b,c)$. As $(\underline{c},c)$ is a singleton, there is no principal stage-2 vertex from it. Recall that $\eta^*_{u_1}$ is the ordinary part of u_1 and $\eta^*_{u_1}(c)=\underline{c}$, $\eta^*_{u_1}(a)=\underline{abc}$, and $\eta^*_{u_1}(b)=\mathrm{NA}$. Suppose now that there is some other principal stage-3 vertex u_2 from $(\underline{c},c)$. As the ordinary parts of principal stage-3 vertices are incomparable, it follows that $\eta^*_{u_2}(c)=\underline{c}$, $\eta^*_{u_2}(b)=\underline{abc}$, $\eta^*_{u_2}(a)=\mathrm{NA}$, and $u_2=x_{u3}(\underline{b}_2,b,a,c)$ for some $\underline{b}_2$, $\underline{a}_2$, and $\underline{ab}_2$ such that

$$\begin{tikzcd} \underline{b}_2 \arrow[r, shift left, "b-a"] \arrow[rr, bend left, "b-a-c"] & \underline{a}_2 \arrow[l, shift left, "a-b"] & \underline{c} \arrow[l, "c-a"'] \arrow[r, "c-b"] & \underline{ab}_2 \end{tikzcd},$$

and $\tau(\underline{ab}_2)=ab$. But then $\underline{b}_1=\underline{ab}_2$ and hence $\tau(\underline{b}_1)=ab$, a contradiction. □

Let us make a remark that (K1)–(K4) effectively describes a relation between principal vertices that are eaten and stage-1 vertices of F^* that eat them. Here principal stage-2 vertices are parametrized with principal arrows and principal stage-3 vertices by (K3)-cases in which $(\underline{c},c)$ is a singleton. The natural way to parametrize stage-1 vertices of F^* is by the elements of $\underline{S}-\underline{abc}$, that is, $(\underline{a},a)$ naturally corresponds to $u_{s1}(\underline{a},a)$. We find it instructive, even if a little excessive, to redraw the cases (K1)–(K4) with renamed letters, concentrating on stage-1 vertex that eats the corresponding principal vertex. This is shown in Figure 4.14, the cases are formally renamed to (L1)–(L4). We say that $(\underline{a},a)\in\underline{S}-\underline{abc}$ is *dominating* if $u_{s1}(\underline{a},a)$ eats at least one principal vertex, which is equivalent to saying that it falls under one (or more) of (L1)–(L4) cases. In picture, the dominating element in each case is emphasized and the mapping in square brackets indicates the renaming of letters used to get this case from (K1)–(K4).

The cases (L1)–(L4) are not self-compatible*, that is, for $(\underline{ab},a)\in\underline{S}-\underline{abc}$ and $u=u_{s1}(\underline{ab},a)$, u cannot eat by (L1) more than one principal vertex, and similarly for (L2)–(L4).*

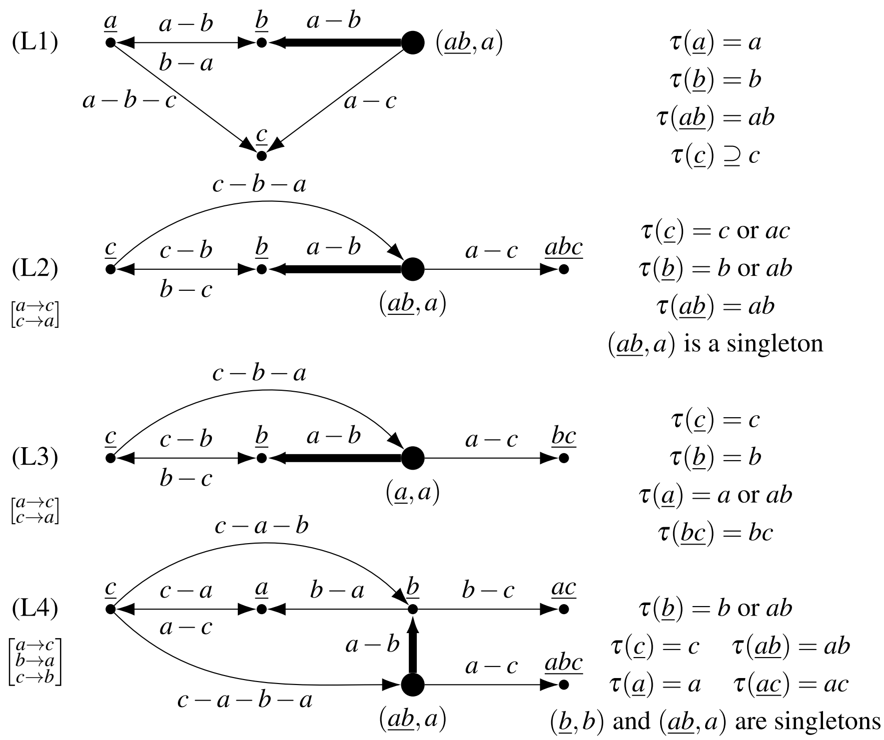


Figure 4.14: Dominating elements.

Let us note that we no longer need to specify whether it is in F or in F^*: Factoring by Ω only contracts stage-1 vertices, so a given element eats precisely the same principal vertices both in F and F^*.

Proof. This is obvious from (L1)–(L4) cases description. For example, assuming the setup of (L1), let $u = u_{s1}(\underline{ab},a)$ eat some $v = u_{s2}(\underline{d},d,e)$ by (L1), potentially after some permutation of letters. Notice that u eats v by (L1) if and only if v is eaten by u by (K1), which implies that, in F^*, $u = u_{s1}(\underline{de},d)$, with $\tau(\underline{de}) = de$. This implies $d = a$ and $e = b$, so we rename $\underline{d}$ to $\underline{a}$. Moreover, $\underline{a}$ then is a unique vertex such that $\underline{ab} \xrightarrow{a-b} \underline{b} \xrightarrow{b-a} \underline{a}$, and so v is uniquely restored from u.

The remaining cases are similar. The only potentially problematic case is (L3), as, at a glance, it looks like it can be compatible with itself after $[b \to c, c \to b]$ permutation. However, this can be ruled out by examining the types of the vertices involved in (L3) and noting that the a-b-arrow from $(\underline{a},a)$ goes to a vertex $\underline{b}$ with a type of size one (that is, b), and the a-c-arrow to a vertex with a type of size two. □

It is, however, easy to see that some of the (L1)–(L4) cases can happen simultaneously. In fact, the following is obvious from the diagrams in Figure 4.14.

Out of (L1)–(L4) cases, (L1) and (L2) are compatible, that is, an element $(\underline{a},a)$ can be dominating by both (L1) and (L2). Also, (L1) and (L3) are compatible, and all other cases are incompatible with each other.

Moreover, if u is dominating by both (L1) and (L2), then the same variables a, b, and c are used in both cases, and similarly for (L1) and (L3).

Proof. (L3) is incompatible with (L2) and (L4) because in the latter cases $(\underline{ab},a)$ has an arow from it to $\underline{abc}$, and $(\underline{a},a)$ in (L3) does not. And (L4) is incompatible with (L1) and (L2) because, in (L4), one arrow from $(\underline{ab},a)$ goes to $\underline{abc}$, and another to a singleton, which is $(\underline{b},b)$. This is obviously not the case in (L1) and (L2) cases.

The fact that (L1) is compatible with (L2) only if the same letters are used in both cases is because the values of a, b, and c are uniquely restored from the element $(\underline{ab},a)$ such that $\tau(\underline{ab}) = ab$. Same is for (L1) and (L3). $\square$

Just as we did with (K1+3), let us now draw the diagrams of (L1+2) and (L1+3) cases. Figure 4.15 is self-explanatory. The types of vertices, shown in picture, are mostly inferred by the corresponding subcases. Additionally, in (L1+2), $\tau(\underline{c}) = c$ is from the classification of triangles. Notice also that $\underline{c}$ from (L1) becomes $\underline{abc}$ in (L1+2) and $\underline{bc}$ in (L1+3).

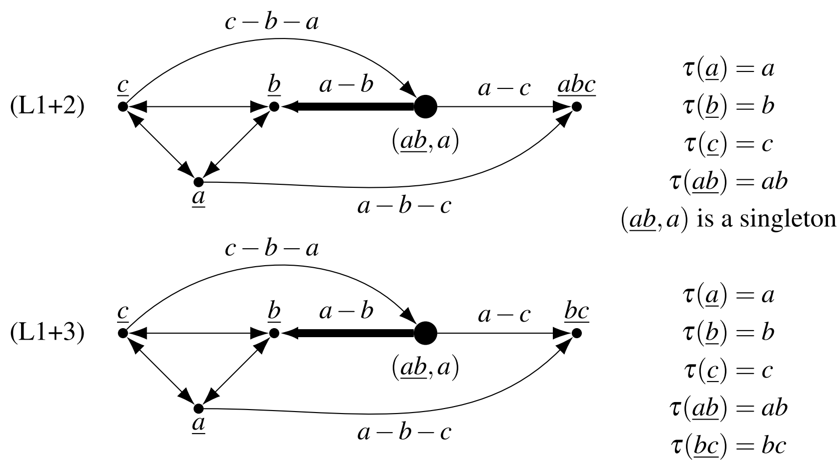


Figure 4.15: Two scenarios for an element to be dominating by two cases: by (L1) and (L2), and by (L1) and (L3).

The following classification of the elements $(\underline{a},a) \in \underline{S} - \underline{abc}$ is effectively a combination of TPS, (K1)–(K4), and (L1)–(L4) classifications.

If $(\underline{a},a) \in \underline{S} - \underline{abc}$ is dominating, then one of the following mutually exclusive conditions hold:

(M1) *$\tau(\underline{a}) = a$, $(\underline{a},a)$ is dominating only by (L3), and $(\underline{a},a)$ is an S1-singleton. Then $u_{s1}(\underline{a},a)$ eats one principal stage-2 vertex and no principal stage-3 vertices, and there is a unique principal stage-3 vertex from $(\underline{a},a)$;*

(M2) $\tau(\underline{a}) = a$, $(\underline{a}, a)$ *is dominating only by (L3), and* $(\underline{a}, a)$ *is a* $(\underline{b}, b)$ *corner of a P2-pyramid after* $[a \to c, b \to a, c \to b]$ *permutation. Then* $u_{s1}(\underline{a}, a)$ *eats one principal stage-2 vertex and no principal stage-3 vertices;*

In the remaining cases, $\tau(\underline{a}) = ab$*, so we call it* $\underline{ab}$*.*

(M3) $\tau(\underline{ab}) = ab$, $(\underline{ab}, a)$ *is dominating by (L1) and (L3), and* $(\underline{ab}, a)$ *is an S2-singleton. Then* $u_{s1}(\underline{ab}, a)$ *eats two principal stage-2 vertices and no principal stage-3 vertices, and there is a unique principal stage-3 vertex from* $(\underline{ab}, a)$*;*

(M4) $\tau(\underline{ab}) = ab$, $(\underline{ab}, a)$ *is dominating by (L1) and (L3), and* $(\underline{ab}, a)$ *is a corner of a P3-pyramid. Then* $u_{s1}(\underline{ab}, a)$ *eats two principal stage-2 vertices and no principal stage-3 vertices;*

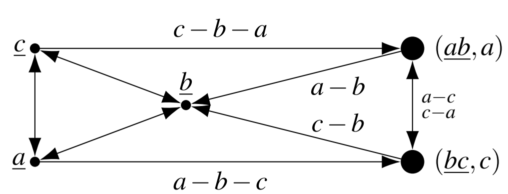
(M5) $\tau(\underline{ab}) = ab$, $(\underline{ab}, a)$ *is dominating by (L1) and (L2), and* $(\underline{ab}, a)$ *is an S2-singleton. Then* $u_{s1}(\underline{ab}, a)$ *eats two principal stage-2 vertices and no principal stage-3 vertices;*

(M6) $\tau(\underline{ab}) = ab$ *and* $(\underline{ab}, a)$ *is dominating only by one of (L1)–(L4). Moreover,* $(\underline{ab}, a)$ *is either an S2-singleton, or the* $(\underline{ac}, a)$ *corner of a P2-pyramid, or one of the corners of a P3-pyramid. Then* $u_{s1}(\underline{ab}, a)$ *eats one principal stage-2 or stage-3 vertex.*

Additionally, in (M3), (M4), and (M5) cases, both principal stage-2 vertices eaten by $u_{s1}(\underline{ab}, a)$ *are from a corner of a T1-triangle, and, in (M5), one of them is QP.*

Prior to the proof, which is just a tally of the classifications we already have, let us give an example of (M4) case that we find illustrative.

Example 2. Figure 4.16 illustrates (M4) case, that is, a situation when $(\underline{ab}, a)$ is dominating by (L1) and (L3) and is a corner of a P2-pyramid.



$$\tau(\underline{a}) = a$$
$$\tau(\underline{b}) = b$$
$$\tau(\underline{c}) = c$$
$$\tau(\underline{ab}) = ab$$
$$\tau(\underline{bc}) = bc$$

Figure 4.16: (M4) case.

Recall that $u_{s1}(\underline{ab}, a) = u_{s1}(\underline{bc}, c)$ in F implies that the second corner of the pyramid eats the same stage-2 vertices. Here, the stage-2 vertices being eaten are $v = u_{s2}(\underline{a}, a, b)$ and $w = u_{s2}(\underline{c}, c, b)$; as noted, both of them are from the same corner of a T1-triangle, which is $(\underline{b}, b)$. Note, however, that while $(\underline{ab}, a)$ eats v by (L1) and w by (L3), for $(\underline{bc}, c)$ it is the other way round: it eats v by (L3) and w by (L1).

Now, notice that (M3) and (M4) effectively say that if $(\underline{ab}, a)$ is dominating by (L1) and (L3), and it is not an S2-singleton, then it is a corner of a P3-pyramid. Why is that? Naively, simply by observing that $\tau(\underline{ab}) = ab$, we can suggest several options: Apart from being an S2-singleton, after a corresponding permutation, $(\underline{ab}, a)$ can become $(\underline{ab}, a)$ from (T2), $(\underline{ac}, a)$ from (P2), one of the corners, that is, $(\underline{ac}, a)$ or $(\underline{bc}, b)$, of (P3), $(\underline{ab}, a)$ or $(\underline{ab}, b)$ from (P4), or $(\underline{ac}, a)$ from (P5).

First of all, we can rule out (T2) case by observing that if some principal vertex u is eaten by a corner of a triangle, then it is eaten by all three corners of this triangle. But from (K1)–(K4) classification it follows that u can be eaten by at most two distinct stage-1 vertices in F^*. Note that, by this argument, a corner of a triangle cannot be dominating.

Now, if $(\underline{ab}, a)$ is in the base of a pyramid, then the second corner of this pyramid, which we denote by $(\underline{de}, d)$, eats the same vertices as $(\underline{ab}, a)$. Thus, $(\underline{de}, d)$ is dominating by either (L1+2) or (L1+3). In both cases, $|\tau(\underline{de})| = 2$, in particular, this cannot be a P2 or P5-pyramid. Finally, in $(L1+3)$ case, both arrows from $(\underline{ab}, a)$ go to a vertex different from $\underline{ab}$, that is, $\underline{de} \neq \underline{ab}$, so a P4-pyramid is also ruled out. Thus, $(\underline{ab}, a)$ is indeed a corner of a P3-pyramid.

Proof (of (M1)–(M6) classification). The cases (M1)–(M6) are clearly disjoint and cover all possibilities of (L1)–(L4) classification, with (M1) and (M2) corresponding to $\tau(\underline{a}) = a$, (M3) and (M4) corresponding to (L1+3), (M5) to (L1+2), and (M6) covering all remaining options. The (M4) case, particularly that $(\underline{ab}, b)$ is a corner of a P3-pyramid, has already been explained in Example 2. Now, we only need to address the TPS-part in (M2) and in (M6) cases.

First, suppose $\tau(\underline{a}) = a$ and $(\underline{a}, a)$ is dominating. Then, clearly, $(\underline{a}, a)$ is dominating only by (L3). If $(\underline{a}, a)$ is a singleton, then it is S1-singleton, and hence this situation falls under (M1). Now, suppose $(\underline{a}, a)$ is not an S1-singleton. We then need to prove that $(\underline{a}, a)$ is the $(\underline{b}, b)$ corner of a P2-pyramid, that is, rule out the possibilities of it being a corner of a P1-pyramid, or the $(\underline{b}, b)$-corner of a P5-pyramid. Assuming that $(\underline{a}, a)$ is a corner of a P1-pyramid, by examining the types, we infer that $(\underline{b}, b)$ from the setup of (L3) is another corner of this pyramid, and $(\underline{bc}, c)$ is its tip. But then $\underline{b} \xrightarrow{b-c} \underline{bc}$, and hence $\underline{bc} = \underline{c}$, a contradiction. And P5-pyramid case can be ruled out by observing that both degree-1 arrows from the $(\underline{b}, b)$-corner go to the same vertex $\underline{ac}$ with a type of size two, which is not the case with $(\underline{a}, a)$ in (L3). This proves that $(\underline{a}, a)$ is indeed the $(\underline{b}, b)$ corner of a P2-pyramid, and thus this case falls under (M2).

In the remaining cases, $\tau(\underline{a}) = ab$, and so we call it $\underline{ab}$. Now, suppose $(\underline{ab}, a)$ is dominating only by one of (L1)–(L4). To prove that we then fall under (M6), we need to rule out the possibility for $(\underline{ab}, a)$ to be either $(\underline{ab}, a)$ in (P4), or $(\underline{ac}, a)$ in (P5) after $[b \to c, c \to b]$ permutation; note that, by the argument in Example 2, $(\underline{ab}, a)$ cannot be a corner of a triangle, which rules out (T2). However, both (P4) and (P5) cases can be ruled out by observing that in (L1) and (L3) cases both arrows from $(\underline{ab}, a)$ go to vertices distinct from $\underline{ab}$, and in (L2) and (L4) $(\underline{ab}, a)$ is a singleton.

Suppose now that $(\underline{ab}, a)$ is dominating by (L1+3). To prove that it splits into (M3) and (M4) cases, we need to show that $(\underline{ab}, a)$ is then either an S2-singleton, or a corner of a P3-pyramid. Notice that the argument for (M6) already rules out triangles and P4 and P5-pyramids, so we only need to show that $(\underline{ab}, a)$ is not the $(\underline{ac}, a)$ corner of a P2-pyramid. But if that is the case, then, as argued, the $(\underline{b}, b)$ corner of this pyramid also eats two principal vertices, that is, falls under either (L1+2) or (L1+3), which is impossible.

Finally, we need to show that if $(\underline{ab}, a)$ is dominating by (L1+2), then it is an S2-singleton. By the same argument as for (M3) and (M4), we can rule out all other cases except for (P3). But if it is (P3), then one of the arrows from $(\underline{ab}, a)$ should go to the other corner of the pyramit, whose type has size two. But, in (L1+2), both arrows from $(\underline{ab}, a)$ go to vertices with types of size one and three, a contradiction.

The tally of how many principal elements is eaten in each case is by the definition of (L1)–(L4). The same is true about the fact that in (M1) and (M3) the corresponding cases produce a principal stage-3 vertex. Let us note that there can be a principal stage-3 vertex produced in (M6), whenever it is (L3) and $(\underline{ab}, a)$ is a singleton. However, we choose to ignore this.

In (M3), (M4), and (M5), both eaten principal stage-2 vertices are from a corner of a T1-triangle by the description of (L1+2) and (L1+3). Similarly, in (M5), from the diagram for (L1+2), $c_b(\underline{a}) = c_a(\underline{b}) = \underline{ab}$, and $c_b(\underline{c}) = c_c(\underline{b}) = \underline{abc}$, and hence $u_{s2}(\underline{a}, a, b)$ is QP. □

Let us call an equivalence class of Θ containing vertices of ST1, that is, stage-1 vertices after contraction, an ST1 equivalence class. And let us call an equivalence class containing at least one principal vertex not eaten by a stage-1 vertex, that is, not falling under one of (K1)–(K4) cases, a PST2 equivalence class. Obviously, the sets of ST1 and PST2 classes are disjoint. Now, let us prove the following

A PST2-equivalence class cannot contain more than three principal vertices. Moreover, if it contains three, then at least one of them is QP, and one of the remaining two is non-QP from the same vertex as the QP one.

Note that, effectively, this states that if we choose to count only non-QP vertices, then a PST2-class cannot contain more than two of them.

Proof. The fact that an equivalence class $[U]$ cannot contain more than three principal vertices is obvious from the fact that their ordinary parts are incomparable. Now, suppose $[U]$ contains exactly three of them, and call them u, v, and w. Recall that the ordinary parts η_u^*, η_v^*, and η_w^* are incomparable and, if $u = u_{s2}(\underline{a}, a, b)$, then $\eta_u^*(b) = \underline{b}$, for $\underline{b}$ such that $\underline{a} \xrightarrow{a-b} \underline{b}$, and $\eta_u^*(a) = \underline{ab}$, for $\underline{ab} = c_a(\underline{b}) = c_b(\underline{a})$. Also, by the definition of a principal arrow, $\underline{b} \neq \underline{ab}$, which, in particular, implies $\underline{b} \neq c_I(\underline{ab})$, for any $I \subseteq \mathscr{I}$. To sum up, η_u^* is defined on two out of three letters, the values on those letters are different, and one of them is a closure of another. The same holds if $u = u_{s3}(\underline{a}, a, b, c)$.

As η_u^*, η_v^*, and η_w^* are incomparable, it then follows that, for pairwise distinct letters a, b, c, $\eta_u^*(a) = \underline{a}$, $\eta_u^*(b) = \underline{b}$, $\eta_v^*(a) = \underline{a}$, $\eta_v^*(c) = \underline{c}$, $\eta_w^*(b) = \underline{b}$, and $\eta_w^*(c) = \underline{c}$, for pairwise distinct $\underline{a}$, $\underline{b}$, and $\underline{c} \in S$. It is easy to see that the relation *there is* $I \subseteq \mathscr{I}$ *such that* $c_I(x) = y$ is a partial order on S, which we denote by $\lessdot$. As any two vertices out of $\underline{a}$, $\underline{b}$, $\underline{c}$ are comparable by this order, it follows that they are linearly ordered with respect to it and, without losing generality, let $\underline{a} \lessdot \underline{b} \lessdot \underline{c}$. Now, the fact that $\underline{a}$, $\underline{b}$, and $\underline{c}$ are pairwise distinct, together with the properties of a closure structure, implies $\tau(\underline{a}) = a$, $\tau(\underline{b}) = ab$, and $\tau(\underline{c}) = abc$. We thus call these vertices $\underline{a}$, $\underline{ab}$, and $\underline{abc}$.

Then u and v are principal vertices from $(\underline{a}, a)$. But then, because there cannot be a stage-2 and stage-3, or two stage-3 vertices from one element of $\underline{S}$, both of them are principal stage-2, and hence $(\underline{a}, a)$ is a corner of a triangle. Moreover, the ordinary edges of u provide an a-b-path between $\underline{a}$ and $\underline{ab}$, and of v between $\underline{a}$ and $\underline{abc}$. So, $c_b(\underline{a}) = \underline{ab}$, and $c_c(\underline{a}) = \underline{abc}$. But then u is QP and v is not. $\square$

Now we will count the number of vertices in S and a lower bound on $\|\Theta\|$. The latter so far will be based exclusively on the number of ST1 equivalence classes and on the lower bound on the number of PST2 equivalence classes. Moreover, for the latter we will not count QP vertices. Recall that $\underline{S} - \underline{abc}$ is divided into disjoint triangles, bases of pyramids, and singletons, which are further classified into T1 and T2-triangles, P1–P5-pyramids, and S1 and S2-singletons. Let $t1$ be the number of T1-triangles, and $t2$, $p1, \ldots, p5$, $s1$, and $s2$ are defined by analogy. To obtain $|S|$ we will now assign the weight 1 to all elements $(\underline{a}, a)$, for $\tau(\underline{a}) = a$, and $1/2$ to all elements $(\underline{ab}, a)$, for $\tau(\underline{ab}) = ab$. Summing up all those weights over $\underline{S} - \underline{abc}$ would then give us $|S - \underline{abc}|$, and we get $|S|$ by adding 1 to it. And, to compute this weighted sum, we utilize the TPS-classification. For example, the base of a P5-pyramid contains the elements $(\underline{ac}, a)$ and $(\underline{b}, b)$, and so it contributes $3/2$ to the sum. For the remaining cases, all coefficients are computed in a straightforward way, yielding

$$|S| = 3t1 + 2t2 + 2p1 + \frac{3}{2}p2 + p3 + p4 + \frac{3}{2}p5 + s1 + \frac{1}{2}s2 + 1.$$

Also, as argued, the number of stage-1 vertices, which we denote by st1, is precisely the total number of triangles, pyramids, and singletons. Thus

$$st1 = t1 + t2 + p1 + p2 + p3 + p4 + p5 + s1 + s2.$$

Let us also compute the total number of non-QP vertices pst2. Again, it uses the TPS-classification and also the fact that T1-trianles produce at least 4, and T2-triangles precisely 4 non-QP vertices. Then

$$pst2 \geq 4t1 + 4t2 + 2p1 + 2p2 + 2p3 + 2p5.$$

To count the number of principal stage-3 vertices and the number of non-QP vertices eaten by stage-1 vertices, we need to use (M1)–(M6) classification. For this we say that $s1 = s1.m1 + s1.m0$, where $s1.m1$ is the number of S1-singletons falling under (M1) and $s1.m0$ is the number of non-dominating S1-singletons. Similarly, $s2 = s2.m3 + s2.m5 + s2.m6 + s2.m0$. For a P2-pyramid, let us note that its $(\underline{b}, b)$ corner falls under (M2) if and only if $(\underline{ac}, a)$ falls under (M6), and otherwise both corners are non-dominating; thus $p2 = p2.m2m6 + p2.m0$. Similarly, for a P3-pyramid, either both of its corners fall under (M4), or (M6), or both are non-dominating, and hence $p3 = p3.m4m4 + p3.m6m6 + p3.m0$. All the remaining objects from TPS-classification do not fall under (M1)–(M6) classification.

Then the number $pst3$ of principal stage-3 vertices is estimated from below as

$$pst3 \geq s1.m1 + s2.m3,$$

and the number $epst23$ of non-QP vertices eaten by stage-1 vertices is at most

$$epst23 \leq s1.m1 + 2s2.m3 + s2.m5 + s2.m6 + p2.m2m6 + 2p3.m4m4 + p3.m6m6.$$

Notice that non-QP assumption is used to put $s2.m5$ with a coefficient 1: although (M5) element eats two principal vertices, only one of them is non-QP. Also note that although each corner of a P2 or P3-pyramid can eat a vertex, both corners eat the same vertex, which explains why $p2.m2m6$ and $p3.m6m6$ are taken with coefficient, 1 and $p3.m4m4$ with coefficient 2. Finally, although (M1)–(M6) cases give an exact number of eaten principal vertices, some of them, on top of those outlined in (M5), might be QP. So the upper bound for $epst23$ is not necessarily an equality.

Now, observe that, as argued, the number of PST2 equivalence classes is at least $\frac{1}{2}(pst2 + pst3 - epst23)$. So, $|S| > \|\Theta\| \geq st1 + \frac{1}{2}(pst2 + pst3 - epst23)$, which is better rewritten as $|S| - 1 \geq st1 + \frac{1}{2}(pst2 + pst3 - epst23)$, yields

$$
\begin{aligned}
3t1{+}2t2+2p1+\frac{3}{2}p2+p3+p4+\frac{3}{2}p5+s1+\frac{1}{2}s2 \\
\geq & t1+t2+p1+p2+p3+p4+p5+s1+s2 \\
& +\frac{1}{2}\Big(4t1+4t2+2p1+2p2+2p3+2p5+s1.m1+s2.m3 \\
& \qquad -s1.m1-2s2.m3-s2.m5-s2.m6 \\
& \qquad -p2.m2m6-2p3.m4m4-p3.m6m6\Big).
\end{aligned}
$$

With some arithmetics, we get

$$
\begin{aligned}
0 \geq & t2+\frac{1}{2}(p2-p2.m2m6)+\left(p3-p3.m4m4-\frac{1}{2}p3.m6m6\right) \\
& +\frac{1}{2}p5+\frac{1}{2}(s2-s2.m3-s2.m5-s2.m6) \\
= & t2+\frac{1}{2}p2.m0+p3.m0+\frac{1}{2}p3.m6m6+\frac{1}{2}p5+\frac{1}{2}s2.m0.
\end{aligned}
$$

As we can see, even if the right-hand side is zero and the inequality holds, it implies that $\|\Theta\|$ is at least $|S|-1$. However, proving $\|\Theta\| \geq |S|$ will require some extra effort. But, as before, by the fact that the inequality is at best tight, we get

- There are no T2-triangles and P5-pyramids;
- Any P2-pyramid eats a principal stage-2 vertex and any P3-pyramid eats two principal stage-2 vertices;
- All S2-singletons fall under one of (M3), (M5), or (M6) cases.

Additionally, several inequalities that we used should become equalities, in particular

- Any T1-triangle is either (T1.1) or (T1.2), that is, it produces exactly four non-QP vertices;
- Any PST2 equivalence class contains exactly two non-QP vertices;
- Any QP vertex is either eaten by an (M5) dominating element, or is a third principal vertex in a PST-2 class;
- All (not necessarily principal) stage-2, stage-3, or stage-1t vertices are eaten either by stage-1 vertices or by PST2 classes.

This finishes the first part of the proof. $\square$

4.6 $\mathrm{RC}_3 \Rightarrow$SSP, part 2

Prior to proceeding with the second part of the proof, let us take a look at the following example

Example 3. The graph Γ in Figure 4.17 is "almost" an RC graph, that is, it satisfies all the conditions of RC graph, except for (E3). It can be noticed that, in terms of TPS-classification, each petal of Γ contains a T1.1-triangle and a vertex like $\underline{ab}$, which constitutes two S2-singletons: $(\underline{ab}, a)$ and $(\underline{ab}, b)$. It can be further noticed that those singletons correspond to s2.m5 case.

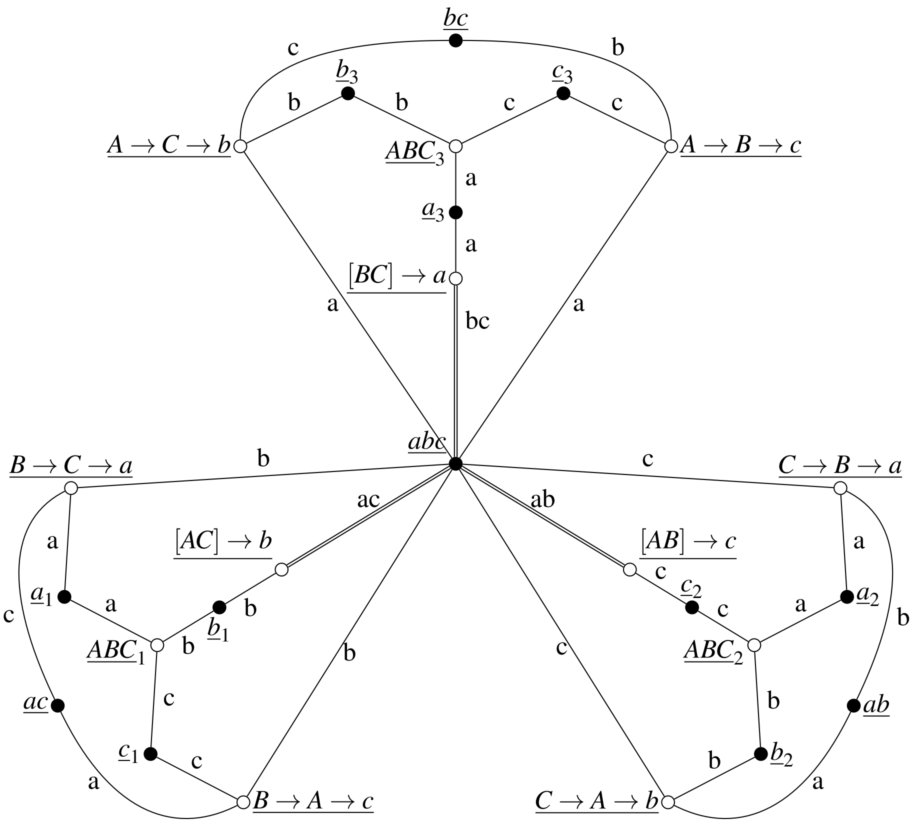


Figure 4.17: Non-SSP graph Γ, satisfying all conditions of an RC graph, except for (E3).

As usual in examples, all names are assumed to be exact. The order (or its basis) is not shown, but again, it can be easily guessed, as there are only so many candidates for witnessing (E2). Namely, $\underline{ABC}_1 \trianglelefteq \underline{B \to A \to c} \trianglelefteq \underline{[AB] \to c}$ and $\underline{ABC}_1 \trianglelefteq \underline{B \to C \to a} \trianglelefteq \underline{[BC] \to a}$, and similarly for the remaining white vertices. This gives a basis for the order on the white vertices and the order on the black vertices is inferred from this. In particular, $\underline{a}_1, \underline{a}_2 \trianglelefteq \underline{a}_3$, $\underline{b}_2, \underline{b}_3 \trianglelefteq \underline{b}_1$, and $\underline{c}_1, \underline{c}_3 \trianglelefteq \underline{c}_2$. Now we can see the problem: The closure type of $\underline{ABC}_1$ is $\{\emptyset, a, b, c, ac, abc\}$. As $\underline{a}_3 \trianglerighteq \underline{a}_1$, by (E3), there should be an a-neighbor u of $\underline{a}_3$ such that $A \in \tau(u)$ and $u \trianglerighteq \underline{ABC}_1$, in particular, $K(u) \trianglerighteq K(\underline{ABC}_1)$. As u is an a-neighbor of $\underline{a}_3$, by (G6), $K(u)$ contains I if and only if $K(\underline{a}_3)$ contains it, for any $I \subseteq \mathscr{I}$ such that $a \in I$. Then, as $K(\underline{a}_3) = \{a, abc\}$, $ab, ac \notin K(u)$. This, together with $K(u) \trianglerighteq K(\underline{ABC}_1)$, implies $K(u) \geq \{\emptyset, a, b, c, abc\}$. In particular, $u \neq \underline{ABC}_3$ and, as there are no other candidates for u, (E3) is not satisfied.

As we were saying in the foreword for Section 4.5, the construction of free graphs and congruences over them does not use (E3), and, consequently, this counterexample cannot be dealt with by examining $\|\Theta\|$ alone without going back to Γ.

Proof (of Theorem 4, part 2). So, $\|\Theta\| = |S| - 1$. Note, however, that as Γ is assumed to be non-SSP, it holds $\|\Theta\| \leq |C| < |S|$, and hence $\|\Theta\| = |C|$, that is, θ is onto. That is, all white vertices of Γ are completely restored from the congruence. In particular, each white vertex u in Γ is obtained from either ST1 or PST2 class, and we will call them ST1 and PST2 vertices respectively; we call the preimage of u under θ a *class of u*.

In Γ, there is no white vertex u such that all edges of u go to $\underline{abc}$, that is, $\eta_a(u) = \eta_b(u) = \eta_c(u) = \underline{abc}$. Indeed, whether u is ST1 or PST2 vertex, by construction, at least one of its neighbors is different from $\underline{abc}$.

There is no black vertex $\underline{a} \in S - \underline{abc}$ such that $\underline{a} \xrightarrow{a-b} \underline{abc}$ and $\underline{a} \xrightarrow{a-c} \underline{abc}$. In particular, there is no white ST1 vertex with two edges to $\underline{abc}$. Suppose not, and let $u = \theta \circ u_{s1}(\underline{a}, a)$. Then $A \in \tau(u)$ and $\eta_b(u) = \eta_c(u) = \underline{abc}$. But then $[bc] \to A \in \tau(u)$, and, applying (E2) to u, we will get a vertex $v \trianglerighteq u$ such that $[ABC] \in \tau(v)$. But then all edges of v go to abc, a contradiction.

This implies that if a white vertex u has two edges to $\underline{abc}$, then u is PST2. Moreover, the two principal stage-2 vertices u_1 and u_2 in the class of u go from the same $(\underline{b}, b)$, which can only happen if $(\underline{b}, b)$ is a $(\underline{b}, b)$-corner of a T1.1-triangle; recall that we proved that there are no T2-triangles. Then $u_1 = u_{s2}(\underline{a}, a, b)$ and $u_2 = u_{s2}(\underline{c}, c, b)$, where $(\underline{a}, a)$ and $(\underline{c}, c)$ are the other two corners of this triangle. Note that it is exactly what happens in Example 3.

So far, throughout the proof, we assumed Φ, and, consecutively, $\mathscr{A}$, θ, and Θ, to be fixed. We also assumed that each $\varphi \in \Phi$ is type-maximal. Now, we are going to suppose that, on top of it, each $\varphi \in \Phi$ is maximal. As maximality implies type-maximality, everything we have proven so far holds.

Assuming each $\varphi \in \Phi$ is maximal, there are no T1.1-triangles.

Throughout the proof, we will keep the notation similar to Example 3, so that the latter can be used as an illustration.

Proof. Suppose not, and let $(\underline{a}_1, a)$, $(\underline{b}_1, b)$, and $(\underline{c}_1, c)$ be the corners of such triangle. Moreover, without losing generality, we assume that this triangle is exactly as in Figure 4.12, that is, $c_b(\underline{a}_1) = c_a(\underline{b}_1) = c_c(\underline{b}_1) = c_b(\underline{c}_1) = \underline{abc}$, and $c_c(\underline{a}_1) = c_a(\underline{c}_1) = \underline{ac}$, with $\tau(\underline{ac}) = ac$.

Let us now consider a QP vertex $v_1 = u_{s2}(\underline{a}_1, a, c)$. As noted, it is either eaten by an (M5) dominating element, or is a third vertex in a PST2 equivalence class. In both situations, it is in the same class as the second principal stage-2 vertex from $(\underline{c}_1, c)$, that is, as $v_2 = (\underline{b}_1, b, c)$. Let $\underline{ABC}_1 = \theta \circ u_{s2}(\underline{a}_1, a) = \theta \circ u_{s2}(\underline{b}_1, b) = \theta \circ u_{s2}(\underline{c}_1, c)$ and $\underline{B \to A \to c} = \theta(v_1) = \theta(v_2)$. Then, by construction, $B \to A \to c \in \tau(\underline{B \to A \to c})$; similarly, $ABC \in \tau(\underline{ABC}_1)$, which motivates the naming of these vertices. Notice also that, from closures that hold in this T1.1-triangle, $K(\underline{ABC}_1) = \{\emptyset, a, b, c, ac, abc\}$.

From the fact that all homomorphisms from Φ respect the order of F_a from Proposition 4, it follows that $\underline{ABC}_1 \trianglelefteq \underline{B \to A \to c}$. Also, by (E2), there is a vertex $\underline{[AB] \to c} \trianglerighteq \underline{B \to A \to c}$. Then $\underline{c}_2 = \eta_c(\underline{[AB] \to c}) \trianglerighteq \underline{c}_1 = \eta_c(\underline{B \to A \to c})$. But, as both a and b-edges of $\underline{[AB] \to c}$ go to $\underline{abc}$, by what was argued earlier, $\underline{[AB] \to c}$ is PST2, moreover, in its class there are two principal stage-2 vertices from $(\underline{c}_2, c)$, where $(\underline{c}_2, c)$ is a $(\underline{b}, b)$ corner of a T1.1-triangle after either $[b \to c, c \to b]$ or $[a \to b, b \to c, c \to a]$ permutation. Let us call the remaining two corners of this triangle $\underline{a}_2$ and $\underline{b}_2$, and let $\underline{ABC}_2 = \theta \circ u_{s2}(\underline{a}_2, a) = \theta \circ u_{s2}(\underline{b}_2, b) = \theta \circ u_{s2}(\underline{c}_2, c)$. In particular, this means that $\varphi_{\underline{c}_2, c}(\underline{C}) = \underline{ABC}_2$, where $\underline{C}$ is the vertex from F_c that corresponds to $\underline{A}$ in F_a. Then $K(\underline{ABC}_2) = \{\emptyset, a, b, c, ab, abc\}$, in particular, $ab \in K(\underline{a}_2), K(\underline{b}_2)$ and, as $K(\underline{ABC}_2) \not\geq K(\underline{ABC}_1)$, $\underline{ABC}_2 \not\trianglerighteq \underline{ABC}_1$.

Now, let us use (E3) in Γ. Namely, as $\eta_c(\underline{ABC}_1) = \underline{c}_1 \trianglelefteq \underline{c}_2$, there is a c-neighbor $\underline{C}_2$ of $\underline{c}_2$ such that $\underline{C}_2 \trianglerighteq \underline{ABC}_1$. Then $K(\underline{C}_2) \geq K(\underline{ABC}_1) = \{\emptyset, a, b, c, ac, abc\}$, which, by definition of a closure type, means $K(\underline{C}_2) \subseteq \{\emptyset, a, b, c, ac, abc\}$. Also, as $c_a(\underline{c}_2) = \underline{abc}$, $ac \notin K(\underline{c}_2)$ and, consequently, $ac \notin K(\underline{C}_2)$. So $K(\underline{C}_2) \geq \{\emptyset, a, b, c, abc\} > K(\underline{ABC}_2)$.

Recall, however, that $\varphi_{\underline{c}, c}$ is chosen to be maximal, where the maximality is also by closure types. But this contradicts to the fact that $\underline{C}_2$ is a c-neighbor of $\underline{c}_2$ whose closure type is strictly greater than the one of $\underline{ABC}_2 = \varphi_{\underline{c}, c}(\underline{C})$. $\square$

As an immediate corollary, we have: *In Γ, there is no white vertex u such that at least two edges of u go to $\underline{abc}$.*

Note that the last statement is about Γ, not Φ: All that we need for it is to have some maximal Φ, which we can do. But, as long as in Γ there are no white vertices with two edges to $\underline{abc}$, we can prove the absence of T1.1-triangles without the requirement of maximality of Φ; we still assume that Φ is type-maximal.

There are no T1.1-triangles. In the proof of this statement with the assumption of maximality, before using this assumption, we showed the existence of a white vertex $\underline{[AB] \to c}$ in Γ that had two edges to $\underline{abc}$. As there is no such vertex in Γ, and as that part only used type-maximality of Φ, the proof is concluded.

The remaining part of the proof is more or less tedious but straightforward case elimination.

There is no white vertex u in Γ such that $a \to B \in \tau(u)$; note that, as Γ satisfies (T), $a \to B \in \tau(u)$ is equivalent to $A \to B \in \tau(u)$. This implies that there are no arrows from $\underline{a}$ to x such that $a \in \tau(x)$, that is,

$\underline{a} \xrightarrow{a-b} x$, $\underline{a} \xrightarrow{a-b-c} x$, *or* $\underline{a} \xrightarrow{a-b-c-b} x$ *imply* $a \notin \tau(x)$. *In particular, there are no arrows to* $\underline{abc}$. If not, then (E2) implies there is $v \trianglerighteq u$ such that $[AB] \in \tau(v)$. Applying (E2) once again, we get a white vertex $w \trianglerighteq v$ such that $[AB] \to c \in \tau(w)$. But then both a and b-edges of w go to $\underline{abc}$. Regarding the statement about the arrows, for x such that $a \in \tau(x)$, $\underline{a} \xrightarrow{a-b} x$ implies that $b \to A \in \tau(\theta \circ u_{s1}(\underline{a}, a))$, $\underline{a} \xrightarrow{a-b-c} x$ that $c \to A \to b \in \tau(\theta \circ u_{s2}(\underline{a}, a, b))$, and $\underline{a} \xrightarrow{a-b-c-b} x$ that $[AB] \to c \in \tau(\theta \circ u_{s3}(\underline{a}, a, b, c))$.

There is no $(\underline{ab}, a) \in \underline{S} - \underline{abc}$*, which is dominating by (L2) or by (L4). In particular, (M5) does not happen.* This is by observing that both (L2) and (L4) cases involve an arrow to $\underline{abc}$.

There are no P4-pyramids. Because a P4-pyramid involves a $\underline{ab} \xrightarrow{a-b} \underline{ab}$ arrow.

There are no PST2 classes with three principal vertices. Consecutively, there are no triangles, no QP vertices, and no two principal vertices from the same element of $\underline{S} - \underline{abc}$. *Moreover, there are no (L1+3), (M3), or (M4) cases, and no P3-pyramids.*

Proof. Recall that otherwise such class contains two stage-2 vertices from the same corner of a triangle, such that one of them is QP and the other is not. Without losing generality, suppose those vertices are $u_1 = u_{s2}(\underline{a}, a, b)$ and $u_2 = u_{s2}(\underline{c}, c, b)$, where u_1 is from $(\underline{b}, b)$ to $(\underline{ab}, a)$, for $\tau(\underline{ab}) = ab$, and u_2 is from $(\underline{b}, b)$ to $(\underline{abc}, c)$. But then $C \to A \to b \in \tau(u)$, for $u = \theta(u_1) = \theta(u_2)$.

Recall that, according to our tally in the end of the first part of the proof, QP vertices could either be eaten by (M5), or be added as an extra third principal vertex to PST2 classes. As both possibilities are now ruled out, there are no such vertices, which means there are no triangles. As in (L1+3) case the two eaten principal stage-2 vertices have to come from a corner of a triangle, it cannot happen, and so there are no (M3) and (M4) cases. This implies there are no P3-pyramids, as each of them should eat at least two vertices, which now cannot happen, as both (L1+2) and (L1+3) cases have been ruled out. $\square$

As the proof is getting meticulous, let us update the tally from the end of the first part of the proof, to take into account things we proved so far. So, in Γ such that $|C_\Gamma| = |S_\Gamma| - 1$, and any maximal Φ over it:

- In Γ, a white vertex can have at most one edge to $\underline{abc}$;
- In Γ, there are no white vertices such that $a \to B \in \tau(u)$;
- In $\mathscr{A}$, there are no arrows from $\underline{a}$ to x such that $a \in \tau(x)$. In particular, there are no arrows to $\underline{abc}$;
- In TPS-classification, only S1 and S2-singletons, and P1 and P2-pyramids are possible;
- In a P1 or P2-pyramid, the type of its tip $\underline{c}$ is c. This is because $\underline{c}$ has both $\underline{a} \xrightarrow{a-c} \underline{c}$ and $\underline{b} \xrightarrow{b-c} \underline{c}$ incoming arrows;
- Any P2-pyramid eats a principal stage-2 vertex;

- Any S2-singleton eats a principal vertex either by (L1) or by (L3);
- All principal vertices are non-QP;
- Out of (L1)–(L4) cases, only (L1) and (L3) are possible. Moreover, both (L1+2) and (L1+3) are impossible, so a dominating element eats precisely one principal vertex;
- Out of (M1)–(M6) cases, only (M1), (M2), and (M6) are possible;
- Any PST2 equivalence class contains exactly two principal vertices;
- All, not necessarily principal, stage-2, stage-3, or stage-1t vertices are eaten either by stage-1 vertices or by PST2 classes.

Let $u \in C_\Gamma$ be a PST2 vertex, that is, u contains two principal non-eaten vertices v_1 and v_2 in its preimage under θ. Then, up to a permutation of a, b, and c, v_1 is from $(\underline{b}, b)$ to $(\underline{abc}, a)$ and v_2 is from $(\underline{c}, c)$ to $(\underline{abc}, a)$, where $\tau(\underline{b}) = b$ or bc and $\tau(\underline{c}) = c$ or bc; in particular, $\tau(\underline{b}) = \tau(\underline{c}) = bc$ if and only if $\underline{b} = \underline{c}$. Moreover, $A \to bc \in \tau(u)$, and $B \notin \tau(u)$ and $C \notin \tau(u)$.

Proof. Recall that if v is a principal vertex, then there are $a, b \in \mathscr{I}$ such that a and b-edges of v are ordinary, and, for $\underline{b} = \eta_b(v)$, $\underline{ab} = \eta_a(v) = c_a(\underline{b})$, $\underline{ab} \neq \underline{b}$,v is from $(\underline{b}, b)$ to $(\underline{ab}, a)$, and $A \in \tau(v)$.

Now, suppose v_1 is from $(\underline{b}, b)$ to $(\underline{ab}, a)$, and v_2 if from $(\underline{e}, e)$ to $(\underline{de}, d)$, with $a \neq b$ and $d \neq e$. The notation is assumed to be as above, in particular, for v_2: $\underline{e} = \eta_e(v_2)$, $\underline{de} = \eta_d(v_2) = c_d(\underline{e})$, $\underline{de} \neq \underline{e}$, and $D \in \tau(v_2)$. As the ordinary parts of v_1 and v_2 are incomparable, it implies that one of d, e is equal to one of a, b, and the other to a unique $c \in \mathscr{I} - ab$.

Suppose first that $e = b$ and, consecutively, $d = c$. Then $\underline{b} = \underline{e}$, and hence v_1 and v_2 are principal vertices from the same $(\underline{b}, b)$, which cannot happen as we have no triangles. So let now $e = a$, consecutively, $d = c$. Then $\underline{ab} = \eta_a(v_1) = \eta_a(u) = \eta_e(v_2) = \underline{e}$ and, consequtively, $\underline{de} = c_d(\underline{e}) = c_c(\underline{ab}) = \underline{abc}$. Moreover, $A \in \tau(v_1)$, $C \in \tau(v_2)$, and $\eta_c(u) = \eta_c(v_2) = \underline{abc}$ imply $C \to A \in \tau(u)$, which cannot happen. The case $d = b$ and $e = c$ is similar.

Finally, we are left with $a = d$ and $e = c$; we thus denote $\underline{e}$ by $\underline{c}$. It also implies $\underline{ab} = \underline{de}$. But then, as $\underline{ab} = \underline{de} = c_a(\underline{c})$, $c \in \tau(\underline{ab})$. So $\underline{ab} = \underline{abc}$. Also, $c_a(\underline{b}) = \underline{abc}$ implies $a \notin \tau(\underline{b})$, and, similarly, $a \notin \tau(\underline{c})$. Additionally, $\tau(\underline{b}) = \tau(\underline{c}) = bc$ implies $\underline{b} = \underline{c}$ as there is a bc-path in Γ between them. Finally, $A \in \tau(v_1)$ implies $A \to bc \in \tau(u)$, and $B \notin \tau(u)$ because otherwise $A \to B \in \tau(u)$, and, similarly, $C \notin \tau(u)$. □

There cannot be $u \in C_\Gamma$ such that $\tau(u) = AC \to b$.

Proof. Suppose not, and let $\underline{ab} = \eta_a(u)$, $\underline{bc} = \eta_c(u)$, and $\underline{b} = \eta_b(u)$. Note that $\tau(u) = AC \to b$ implies $\tau(\underline{ab}) = ab$, $\tau(\underline{bc}) = bc$, and $\tau(\underline{b}) = b$. We claim that in Γ there is no a-neighbor v of $\underline{ab}$ such that $A \in \tau(v)$ and $A(v) \supsetneq A(u)$. Indeed, if $A(v) \supsetneq A(u)$ then either $c \to a \in \tau(v)$ or $a \to c \in \tau(v)$. The first one is impossible, as then $C \to A \in \tau(v)$, and the second one because it would imply $c \in \tau(\eta_a(v)) = \tau(\underline{ab})$.

Hence, u is a type-maximal a-neighbor of $\underline{ab}$ such that $A \in \tau(u)$. Similarly, u is a type-maximal c-neighbor of $\underline{bc}$ such that $C \in \tau(u)$. Let us then pick a type-maximal Φ' such that $\varphi_{\underline{ab},a}(\underline{A}) = \varphi_{\underline{bc},c}(\underline{C}) = u$, but otherwise arbitrary. Then, in $\mathscr{A}' = \mathscr{A}(\Phi')$, it holds $\underline{ab} \xrightarrow{a-c} \underline{bc}$, $\underline{bc} \xrightarrow{a-c} \underline{ab}$, $\underline{ab} \xrightarrow{a-b} \underline{b}$, and $\underline{bc} \xrightarrow{c-b} \underline{b}$. That is, $(\underline{ab}, a)$ and $(\underline{bc}, c)$ form the base of a P3-pyramid, a contradiction. $\square$

We note that in this proof we utilize the fact that Φ' only needs to be type-maximal. In principle, this is the only part where we need specifically type-maximality, and not maximality. As a corollary, we have

In (L3), $\tau(\underline{a}) = a$. In particular, every S2-singleton eats a principal vertex by (L1), in every base $(\underline{ac}, c)$ and $(\underline{b}, b)$ of a P2-pyramid, $(\underline{ac}, c)$ eats a principal vertex by (L1) and $(\underline{b}, b)$ by (L3), and every stage-3 principal vertex is from an S1-singleton. Suppose, in (L3), $\tau(\underline{a}) = ab$, and hence we call it $\underline{ab}$. Then the fact that $u = u_{s1}(\underline{ab}, a)$ eats $v = u_{s2}(\underline{c}, c, b)$ implies that, for $w = \theta(u) = \theta(v)$, $AC \to b \in \tau(w)$. This implies that either $\tau(w) = AC \to b$, or $\tau(w) = AC \to B$; however, both are impossible, a contradiction.

Let $(\underline{b}, b)$ be either a corner of a P1-pyramid, or the $(\underline{b}, b)$ corner of a P2-pyramid, or the $(\underline{ac}, a)$ corner of a P2-pyramid after $[a \to b, b \to a]$ permutation. Let $(\underline{c}, c)$ be the tip of this pyramid and let $\underline{bc} = c_c(\underline{b}) = c_b(\underline{c})$. Then there is a principal vertex from $(\underline{c}, c)$ to $(\underline{bc}, b)$.

This argument is illustrated in Figure 4.18 below.

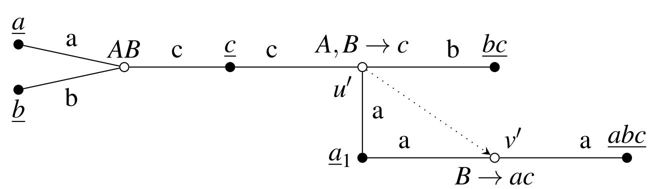


Figure 4.18: There is a principal vertex from the tip of a pyramid argument.

Proof. Although the case when $(\underline{b}, b)$ is the $(\underline{ac}, a)$ corner of a P2-pyramid after $[a \to b, b \to a]$ permutation follows by the same argument, let us address it separately, as it can be confusing. So, in this case, the base of the pyramid is $(\underline{a}, a)$ and $(\underline{bc}, b)$, and $\underline{b} = \underline{bc}$. Note that this implies $\underline{bc} = c_c(\underline{bc}) = c_b(\underline{c})$, and so this $\underline{bc}$ is the same $\underline{bc}$ from the statement of the proposition. Then $(\underline{bc}, b)$ eats a principal vertex by (L1) and, because $\underline{bc} \xrightarrow{b-c} \underline{c}$, this vertex can only be from $(\underline{c}, c)$ to $(\underline{bc}, b)$, as required. In the remaining part of the proof we thus can consider $\tau(\underline{b}) = b$, in particular, $\underline{b} \neq \underline{bc}$; however, it remains true even without this assumption.

Suppose not, that is, there is no principal vertex from $(\underline{c}, c)$ to $(\underline{bc}, b)$. Let $u = u_{s2}(\underline{b}, b, c)$, note that u is non-principal and is from $(\underline{c}, c)$ to $(\underline{bc}, b)$. Let $u' = \theta(u)$, note that $B \in \tau(u)$ and, consequently, $B \to c \in \tau(u')$, from which $C \notin \tau(u')$. Suppose u' is PST2 vertex. Then, as argued, $\underline{bc} = \underline{abc}$, $B \to ac \in \tau(u')$, and the two principal vertices v_1 and v_2 are from $(\underline{a}_1, a)$ to $(\underline{abc}, b)$ and from $(\underline{c}_1, c)$ to $(\underline{abc}, b)$ respectively. Here $\underline{a}_1$ is some black vertex such that $\eta_a(u') = \underline{a}_1$, but, as $\eta_c(u') = \underline{c}$, $\underline{c}_1 = \underline{c}$. But then v_2 is from $(\underline{c}, c)$ to $(\underline{abc}, b) = (\underline{bc}, b)$, which is impossible by assumption.

So, u' is ST1, and let v be the corresponding stage-1 vertex from the class of u. Then either $v = u_{s1}(\underline{c}, c)$, or $u_{s1}(\underline{bc}, b)$, or $u_{s1}(\underline{a}_1, a)$, for some $(\underline{a}_1, a) \in \underline{S} - \underline{abc}$. The first option is impossible, as $C \notin \tau(u')$. Let us further note that both remaining options imply $\underline{bc} \neq \underline{abc}$. Indeed, if $\underline{bc} = \underline{abc}$, then $u_{s1}(\underline{abc}, b)$ is stage-1t, not stage-1, and if $v = u_{s1}(\underline{a}_1, a)$, then $\underline{a}_1 \xrightarrow{a-b} \underline{abc}$, which is impossible. So $\tau(\underline{bc}) = bc$.

Suppose now that $v = u_{s1}(\underline{bc}, b)$. But $(\underline{bc}, b)$ is either an S2-singleton, or the $(\underline{ac}, a)$ corner of a P2-pyramid after $[a \to b, b \to a]$ permutation; in both cases, it eats a principal vertex by (L1). But then this principal vertex should be from $(\underline{c}, c)$ to $(\underline{bc}, b)$, which is impossible by assumption. So $v = u_{s1}(\underline{a}_1, a)$. In particular, $A \in \tau(u')$. Thus, $A, B \to c \in \tau(u')$. Note that this implies $a \to b \notin \tau(u')$, $b \to a \notin \tau(u')$, and $a \to c \notin \tau(u')$; the latter is because otherwise otherwise $\tau(u') = AB \to c$. This implies that $\tau(u') = A, B \to c$ and, consequtively, $\tau(\underline{a}_1) = a$.

Let us note that then $(\underline{a}_1, a)$ is an S1-singleton. Indeed, $\underline{a}_1 \xrightarrow{a-c} \underline{c}$ and $\underline{a}_1 \xrightarrow{a-b} \underline{bc}$ so, if $(\underline{a}_1, a)$ is a corner of a pyramid, then the second corner is either $(\underline{c}, c)$ or $(\underline{bc}, b)$. But the first case implies $u' = \theta(u_{s1}(\underline{c}, c))$ and the second that $u' = \theta(u_{s1}(\underline{bc}, b))$, as argued, both are impossible. Suppose now that $(\underline{a}_1, a)$ eats a principal vertex. Then it should be by (L3) after $[b \to c, c \to b]$ permutation, and the principal vertex being eaten is from $(\underline{c}, c)$ to $(\underline{bc}, b)$, again contradicting the initial assumption. Thus, $(\underline{a}_1, a)$ does not eat any principal vertex, and, consequently, there is no principal vertex from $(\underline{a}_1, a)$: There is no principal stage-2 vertex from it because $(\underline{a}_1, a)$ is a singleton, and no stage-3 principal vertex because it does not fall under (L3) case.

Let $v' \in C_\Gamma$ be obtained by (E2) applied to u'. Then $B \to ac \in \tau(v')$ and $\eta_a(v') = \underline{a}_1$. In particular, as $\tau(u') \neq \tau(v')$, $u' \neq v'$. As there is no principal vertex from $(\underline{a}_1, a)$, v' is ST1. But $A \notin \tau(v')$ and $C \notin \tau(v')$, so the stage-1 vertex in the class of v' should be $u_{s1}(\underline{abc}, b)$, which is impossible, as the latter is stage-1t. $\square$

There are no pyramids. As a consequence, there are no principal vertices, and hence no S2-singletons and no PST2 vertices in Γ. This is an easy corollary from the previous statement: Indeed, it implies that there should be two principal vertices from the tip of a pyramid, but we have already proven that it cannot happen.

Finally, let $u' \in \Gamma$ be obtained by applying (E1) to $\underline{abc}$ and a. That is, $\eta_a(u') = \underline{abc}$ and $A \to bc \in \tau(u')$. In particular, this implies $B \notin \tau(u')$ and $C \notin \tau(u')$. Then u' cannot be an ST1 vertex, because the only stage-1 vertex in its class can be $u_{s1}(\underline{abc}, a)$, but the latter is not stage-1 but stage-1t. But u' also cannot be a PST2 vertex, because we do not have them.

This contradiction concludes the proof of Theorem 4. $\square$

CHAPTER 5

Results in the direction of disproving RC⇒SSP

5.1 For RC graphs, RC_5 does not imply SSP

Now we are going to give an example of an RC_5-graph Γ_5 that is not SSP, moreover, Γ_5 has trivial closures. Note that this might point in two opposite directions: It might be the case that a graph counterexample to SSP = RC might indeed be lifted to a counterexample for the main conjecture. It also can be that our RC graphs fail to capture some essential properties of RC lattices, and thus we need to add extra conditions, or perhaps ditch the graph constructions whatsoever.

Although we might just go ahead and define Γ_5 explicitly, we will take a small detour and explain how free graphs and arrow structures can be used to come up with such counterexamples. So, let us fix $\mathscr{I} = \mathbf{k}$, where $\mathbf{k} = \{1,\dots,k\}$, let us put the base set over $\mathscr{I}$ to be $S = \mathscr{I} \sqcup \{\mathscr{I}\}$, $|S| = |\mathscr{I}|+1$, that is, for $\mathscr{I} = \mathbf{3}$, $S = \{\underline{1},\underline{2},\underline{3},\underline{123}\}$. For $u \in S$ we put $\tau(u) = u$. Notice that S has trivial closures. Now, let $\mathscr{A}$ be a simplified arrow structure over S defined as $\mathscr{A} = \{i \xrightarrow{i-j} j \mid i \neq j \in \mathscr{I}\}$; recall that a simplified arrow structure only contains arrows of degree 1, and we will also ignore the arrows from the top vertex. Still, we can construct a simplified free graph for $\mathscr{A}$ and straight away collapse all stage-1 vertices into one. After this, we need to combine stage-2 vertices into equivalence classes of Θ, trying to minimize Θ. The resulting RRC graph F/Θ will have $1+p$ white vertices, where p is the number of equivalence classes of stage-2 vertices. Our goal is then to minimize p, or at least to make it less than k.

The free graph F has $k*(k-1)$ principal stage-2 vertices $x_{s2}(\underline{a},a,b)$, for $a \neq b \in \mathscr{I}$, which we will denote by (a,b), all those stage-2 vertices are ordinary. Additionally, $\eta_{(a,b)}(a) = t$, $\eta_{(a,b)}(b) = \underline{b}$, and $\eta_{(a,b)}(c) = \mathrm{NA}$ otherwise, here t denotes the top vertex of S. Thus, elements (a_1,b_1) and (a_2,b_2) are *compatible*, that is, they can be in one equivalence class, if and only if $b_1 \neq a_2$ and $b_2 \neq a_1$. Let now T be a graph of the compatibility relation, with $V_T = \{(a,b) \mid a \neq b \in \mathscr{I}\}$ and $E_T = \{(a_1,b_1),(a_2,b_2) \mid b_1 \neq a_2, b_2 \neq a_1\}$. Then p is a clique covering number of T, that is the minimal number of cliques sufficient to cover all vertices of T. We thus are interested in the following question

Question 1. *For $k \geq 2$ let T_k be a graph with $V_T = \{(a,b) \mid a \neq b \in \mathbf{k}\}$ and $E_T = \{(a_1,b_1),(a_2,b_2) \mid b_1 \neq a_2, b_2 \neq a_1\}$. What is the clique covering number of T_k? In particular, can it be less than k?*

While the clique covering number can be hard to compute in general, it can be checked that the second part of Question 1 can be answered positively, starting with $k = 5$. For the sake of compliance with the previous naming convention of the letters of $\mathscr{I}$, let us rename 1–5 to *a*–*e*. In this case, there is a following clique covering of size 4:

- (c,a), (c,b), (d,a), (d,b), (e,a), and (e,b);
- (b,a), (b,c), (b,d), (e,c), and (e,d);
- (a,b), (a,c), (a,e), (d,c), and (d,e);
- (a,d), (b,e), (c,d), and (c,e).

This covering can be transformed, in a straightforward way, to the congruence Θ. Of course, we have no guarantees that F/Θ can be made into an RC graph. However, given a fixed choice of the clique covering, it can be checked manually. In particular, Figure 5.1 below shows an RC graph corresponding to the covering described above; the names of the vertices are exact.

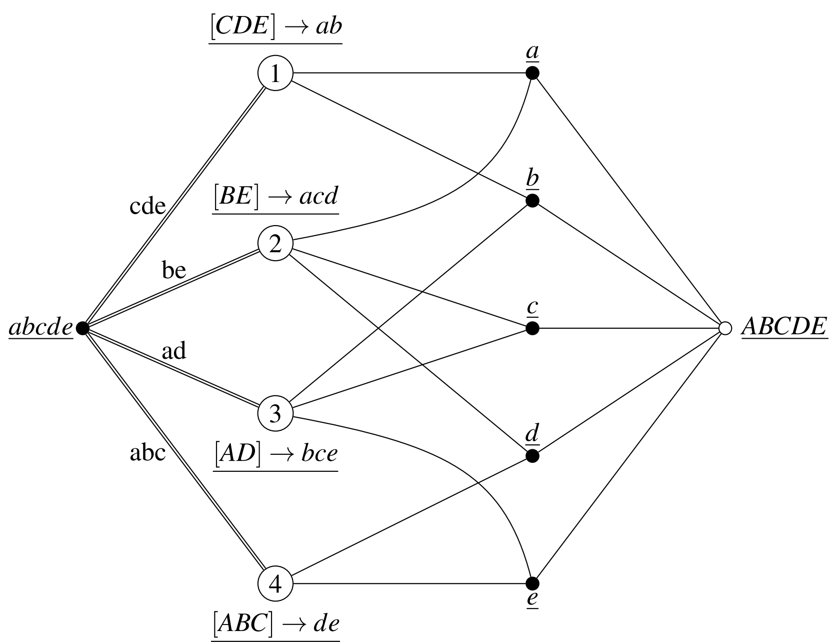


Figure 5.1: Non-SSP RC_5-graph.

The order on the vertices of Γ_5 can be easily guessed, with $\underline{abcde}$ being the top and $\underline{ABCDE}$ the bottom, and the order on the remaining vertices is induced by the edges of Γ_5, that is, $\underline{[CDE] \to ab} \leq \underline{a}, \underline{b}$, and so on. Then it is easy to check that Γ_5 is indeed an RC graph; on to of it, it even satisfies the unproven (E2*) condition.

5.2 Lattice pumping

Having constructed a counterexample to the graph version of RC⇒SSP, the next logical step would be to utilize it to try and construct a counterexample to the conjecture in general, that is, for RC lattices. However, passing from systems over lattices to RC graphs is a one-way street and, for a given RC graph, there typically are many easy reasons for why it cannot precisely correspond to a system over an RC lattice. Most typically, the graph would lack good counterparts to special elements, such as the top or the bottom of the lattices, x_i's or y_i's of the system over it, or some simple polynomials over those elements. At the same time, as we will see, those graphs can be instructive in describing how *generic* elements of such lattice should behave.

In this section we will thus make a step to the side and, without aiming at SSP=RC conjecture specifically, will design a way of constructing RC lattices that will enable us to define generic elements.

For a prime $N \geq 2$ and an integer $D \geq 0$, let $V = V(N,D)$ be a D-dimensional vector space over $\mathscr{F}_N$, which we call *ambient vector space*. We write $K \leq V$ to indicate that K is a subspace of V; the set of all linear subspaces of V is denoted by $\mathsf{Lin}\,V$. Subspaces of V are partially ordered by containment and, with this partial order, $\mathsf{Lin}\,V$ is a lattice, where $U \wedge W = U \cap W$ and $U \vee W$ is the linear span of U and W, $U \vee W = U + W = \{u+w \mid u \in U, w \in W\}$. For $K \leq V$, two cosets α and $\beta \in V/K$ are equal if they are equal as sets. For $v \in V$, we use the usual notation $v+K$ for a coset $\{v+k \mid k \in K\} \in V/K$. Thus, $v_1+K = v_2+K$ if and only if $v_1 - v_2 \in K$. Also, for $\alpha \in V/K$ and $v \in V$, $v+K = \alpha$ if and only if $v \in \alpha$, and $\alpha = v+\alpha$ if and only if $v \in K$.

Definition 1. Let L be a finite lattice and $V = V(N,D)$. Let K be a function from L to $\mathsf{Lin}\,V$. Then $L^K = L^K(L,V,K) = L^K(L,N,D,K)$ is a poset defined as

- Elements of L^K are pairs (s,σ), where $s \in L$ and $\sigma \in V/K(s)$;
- The covering relation $\prec_{L^K}$ is defined by: $(p,\pi) \prec_{L^K} (q,\theta)$ if and only if $p \prec_L q$ and $\pi \cap \theta \neq \emptyset$;
- The partial order $\leq_{L^K}$ is a reflexive transitive closure of $\prec_{L^K}$.

We call L^K a *pumping* of L. With an abuse of notation, the function K is also called *pumping*.

Note that, with thus defined $\leq_{L^K}$ and $\prec_{L^K}$, the latter is indeed the covering relation in L^K. If no ambiguity arises, we will omit the underscripts in order and covering relations. For $q \in L$, we always write K_q instead of $K(q)$, and, when no confusion arises, we denote the latter by Q, for example, $(q, v+Q) \in L^K$. Notice that L^K is equipped with a natural order-homomorphism $\kappa\colon L^K \to L$, defined by $\kappa(q,\theta) = q$. For $q \in L$, we call the set $\kappa^{-1}(q)$ a *q-bunch*, which we often denote as $L^K[q]$. The following lemma gives an explicit description of $\leq_{L^K}$.

Lemma 17. *Let L^K be as in Definition 1. Then*

1. *$(p,\pi) \leq (r,\rho)$ in L^K if and only if $p \leq r$ in L and there is a chain $p = q_1 \leq \cdots \leq q_k = r \in L$ and $\theta_i \in V/K_{q_i}$, for $i = 1,\ldots,k$, such that $\pi = \theta_1$, $\rho = \theta_k$ and $\theta_i \cap \theta_{i+1} \neq \emptyset$, for $i = 1,\ldots,k-1$;*

2. *$(p, v+P) \leq (r, w+R)$ in L^K if and only if $p \leq r$ in L and there is a chain $p = q_1 \leq \cdots \leq q_k = r \in L$ such that $v - w \in Q_1 \vee \cdots \vee Q_k$.*

Proof. (1, $\Rightarrow$). Obvious from the definition of $\leq_{L^K}$, which is a reflexive transitive closure of $\prec_{L^K}$.

(1, $\Leftarrow$). If $q_i = q_{i+1}$ for some $i \in 1,\ldots,k-1$, then, as different cosets of V/Q_i are nonintersecting, it follows that $\theta_i = \theta_{i+1}$. Hence, without loss of generality, we can assume that $q_i < q_{i+1}$, for all $i = 1,\ldots,k-1$. Now, for any $q_i < q_{i+1}$, let $q_i = q_i^1 \prec \cdots \prec q_i^l = q_{i+1}$ be any maximal chain between q_i and q_{i+1}. Take $v \in \theta_i \cap \theta_{i+1}$, then $\theta_i = v + Q_i$, $\theta_{i+1} = v + Q_{i+1}$, and $(q_i,\theta_i) = (q_i^1, v+Q_i^1) \prec \cdots \prec (q_i^l, v+Q_i^l) = (q_{i+1},\theta_{i+1})$. Then $(q_i,\theta_i) \leq (q_{i+1},\theta_{i+1})$ for all $i = 1,\ldots,k-1$, and hence $(p,\pi) \leq (r,\rho)$.

(2, $\Rightarrow$). By (1), there is $p = q_1 \leq \cdots \leq q_k = r$ and $\theta_i = u_i + Q_i$, $i = 1,\ldots,k$, such that $\theta_i \cap \theta_{i+1} \neq \emptyset$ for $i = 1,\ldots,k-1$; we can assume that $u_1 = v$ and $u_k = w$. Thus, for $i = 1,\ldots,k-1$, there are $k_i^1 \in Q_i$ and $k_{i+1}^2 \in Q_{i+1}$ such that $u_i + k_i^1 = u_{i+1} + k_{i+1}^2$; alternatively $u_i - u_{i+1} = k_{i+1}^2 - k_i^1$. By summing up from 1 to $k-1$, we get, $v - w = u_1 - u_k = -k_1^1 + (k_2^2 - k_2^1) + \cdots + (k_{k-1}^2 - k_{k-1}^1) + k_k^2$. The latter is in $Q_1 \vee \cdots \vee Q_k$.

(2, $\Leftarrow$). Let $v - w = k_1 + \cdots + k_k$, for $k_i \in Q_i$. Let $u_i = k_1 + \cdots + k_{i-1}$ and let $\theta_i = v - u_i + Q_i$, for $i = 1,\ldots,k$. Then $\theta_1 = v + P$ and $\theta_k = v - u_k + R = v - k_1 - \cdots - k_{k-1} + R = \big[\text{as } k_k \in R = Q_k\big] = v - k_1 - \cdots - k_k + R = w + R$. Also $v - u_i - k_i \in v - u_i + Q_i = \theta_i$, and $v - u_i - k_i = v - u_{i+1} \in v - u_{i+1} + Q_{i+1} = \theta_{i+1}$, that is $\theta_i \cap \theta_{i+1} \neq \emptyset$ for $i = 1,\ldots,k-1$. Hence, by (1), $(p, v+P) \leq (r, w+R)$. □

There are several relaxations of Lemma 17 that are useful for applications.

Corollary 5. *If $p \leq r \in L$, and $\pi \in V/P$ and $\rho \in V/R$ are such that $\pi \cap \rho \neq \emptyset$, then $(p,\pi) \leq (r,\rho)$.*

Corollary 6. *For $p \leq r \in L$ and $v, w \in V$, $(p,v) \leq (r,w)$ if and only if $v - w \in C_{p,r}$, where $C_{p,r}$ is defined as*

$$
\begin{aligned}
C_{p,r} &= \bigcup_{H \text{ is a maximal chain in } [p,r]} \; \bigvee_{h \in H} K_h \\
&= (P \vee R) + \left(\bigcup_{H \text{ is a maximal chain in } (p,r)} \; \bigvee_{h \in H} K_h \right) \\
&= \bigcup_{a \in A} C_{p,a} + C_{a,q}, \textit{ for any maximal antichain } A \textit{ in } [p,q].
\end{aligned}
$$

Note that $C_{p,r}$ is closed under multiplication by an element of $\mathscr{F}_N$, but not under addition, that is, in general, $C_{p,r} \notin \mathsf{Lin}\,V$.

Corollary 7. *If $(p,\pi) \le (r,\rho)$ and $v \in R$, then $(p, v+\pi) \le (r,\rho)$.*

Corollary 8. *If $(p,\pi) \le (r,\rho)$ and $p \not\preceq r$, then there is q, $p < q < r \in L$, and $\theta \in V/Q$ such that $(p,\pi) < (q,\theta) < (r,\rho)$.*

Corollary 9. *For any $p \le r \in L$ and any $\pi \in V/P$ and $\rho \in V/R$, there is some $\pi' \in V/P$ and $\rho' \in V/R$ such that $(p,\pi') \le (r,\rho)$ and $(p,\pi) \le (r,\rho')$.*

Let us now give a version of L^K construction, which can be considered a more graphic subcase of the general case. Let us fix a finite lattice L, a finite set U, which we call *a set of coordinates*, a function $\Sigma\colon L \to 2^U$, and an integer $N \ge 2$. For $s \in L$, we call the elements of $\Sigma(s)$ *expanded coordinates of s*, denoted $E_s \subseteq U$, and of $U - \Sigma(s)$ *contracted coordinates of s*, denoted C_s. For $C \subseteq U$, we denote by $\mathscr{I}[C,U,N]$ a set of functions ι from U to $\{*,1,2,\dots,N\}$, such that $\iota(v) = *$, for $v \in C$, and $\iota(v) \in \{1,\dots,N\}$, for $v \in U - C$; we denote it simply $\mathscr{I}[C]$ if U and N are clear from the context. We define a poset L^Σ as follows:

Definition 2. For a finite lattice L, finite set U, $\Sigma\colon L \to 2^U$, and $N \ge 2$, $L^\Sigma = L^\Sigma(L,U,\Sigma,N)$ is a poset defined as

- Elements of L^Σ are pairs (s,σ), where $s \in L$ and $\sigma \in \mathscr{I}[C_s]$.
- The covering relation $\preceq_{L^\Sigma}$ is defined by $(p,\pi) \preceq_{L^\Sigma} (q,\theta)$ if and only if either $(p,\pi) = (q,\theta)$, or $p \preceq_L q$ and for every $v \in V$, either $\pi(v) = *$, or $\theta(v) = *$, or $\pi(v) = \theta(v)$.
- The partial order $\le_{L^\Sigma}$ is a transitive closure of $\preceq_{L^\Sigma}$.

Note. It is easy to notice that L^Σ can be considered a subcase of L^K construction, given N is prime. Indeed, if we put V to be $|U|$-dimensional vector space over $\mathscr{F}_N$ with a basis $\{\overline{u} \mid u \in U\}$, and define $K^\Sigma\colon L \to \mathsf{Lin}\,V$ by putting $K_x = \langle \overline{u} \mid u \in C_x \rangle$, for $x \in L$, then $L^{K^\Sigma} \cong L^\Sigma$. While in general case $K[L] \subseteq \mathsf{Lin}\,V$, it can be noticed that $K^\Sigma[L] \subseteq D \subseteq \mathsf{Lin}\,V$, where D is a distributive sublattice of $\mathsf{Lin}\,V$ generated by $\{\overline{u} \mid u \in U\}$.

We use the following notation for elements of L^Σ: Let $U = \{u_1,\dots,u_k\}$, then an element (s,σ) is denoted $s_{\sigma(u_1)\dots\sigma(u_k)}$. For example, if $U = \{u_1,u_2\}$ and, for $p,q \in L$, $\pi(u_1) = *$, $\pi(u_2) = 1$, $\theta(u_1) = 2$, $\theta(u_2) = 3$, then elements (p,π) and (q,θ) are denoted p_{*1} and q_{23}.

Figure 5.2 below shows examples of this construction for $N = 2$. In the picture, the part to the left of an arrow depicts L, U, and Σ, where white and black dots next to an element s of L indicate the elements of E_s and C_s respectively. The part to the right is the corresponding poset L^Σ, where the elements are named as described above.

Just as with Lemma 17, we have the following explicit characterization of $\le_{L^\Sigma}$:

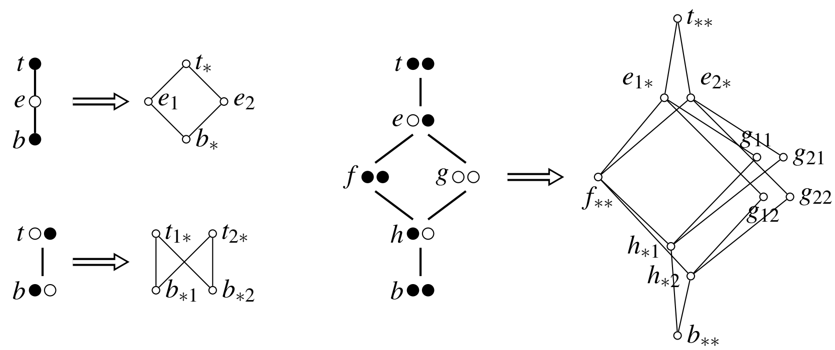


Figure 5.2: L^Σ posets.

Lemma 18. *Let L^Σ be as in Definition 2. Then $(p,\pi) \le (r,\rho) \in L^\Sigma$ if and only if $p \le r \in L$ and there is a chain $p \le q_1 \le \cdots \le q_k \le r \in L$ such that for all $v \in E_p \cap E_r$, either $\pi(v) = \rho(v)$, or there is $i \in 1,\dots,k$, such that $v \in C_{q_i}$.*

Obviously, thus constructed, L^K is not necessarily a lattice, and when it is, it is not necessarily RC. On the other hand, L^K can be an RC lattice even if L is not. The following lemma gives a necessary and sufficient condition for that.

Theorem 5. *For L, $V = V(N,D)$, and $K\colon L \to \mathsf{Lin}\,V$ as in Definition 1, $L^K = L^K(L,V,K)$ is a relatively complemented lattice if and only if the following conditions are satisfied:*

(V0) *For top and bottom elements $e,z \in L$, $K_e = K_z = V$.*

(V1) *For all $p,r \in L$, $P \wedge R \le K_{p\wedge r}, K_{p\vee r}$.*

Equivalently, for any $v \in V$, the set $B_v = \{s \in L \mid v \in S\}$ is a sublattice of L.

(V2) *For all $p \le q \le r \in L$, $Q \le P \vee R$.*

Equivalently, for any $W \in \mathsf{Lin}\,V$, the set $W_W = \{s \in L \mid S \le W\}$ is convex *in L, that is, for any $p \le q \le r \in L$, $q \in W_W$ whenever $p,r \in W_W$.*

(V3) *For all $p \le r \in L$, there is q such that $p \le q \le r$ and $Q = P \vee R$.*

(V4) *For any three-element interval $p \prec q \prec r$ of L, $Q \nleq P \wedge R$.*

(V5) *For all $q,r \le s,p \in L$, it holds*

$$(S \vee R) \wedge (P \vee Q) \le \big[(S \vee T) \wedge (Q \vee T)\big] \vee \big[(P \vee T) \wedge (R \vee T)\big],$$

for $T = (S \vee Q) \wedge (S \vee R) \wedge (P \vee Q) \wedge (P \vee R)$.

Proof. $(\Rightarrow)$. Suppose L^K is an RC lattice. For a top element $e \in L$, (e, ε) is maximal in L^K, for any $\varepsilon \in V/K_e$. As there is a unique maximum in L^K, $K_e = V$, and similarly $K_z = V$, proving (V0).

Suppose (V1) is not satisfied for some $v \in V$, and let $p, r \in L$ be such that $v \in P, R$, but $v \notin K_{p \wedge r}$, in particular, $v \neq 0$. As $v \notin K_{p \wedge r}$, $\iota_1 = 0 + K_{p \wedge r} \neq \iota_2 = v + K_{p \wedge r}$. Also, as $v \in P, R$, by Lemma 17, $(p \wedge r, \iota_1), (p \wedge r, \iota_2) \leq (p, 0+P), (r, 0+R)$. Thus, there is (s, σ) such that $(p \wedge r, \iota_1), (p \wedge r, \iota_2) \leq (s, \sigma) \leq (p, 0+P), (r, 0+R)$, but then $p \wedge r \lneq s \leq p, r$, a contradiction. Joins are handled similarly.

Let us prove a weakened version of (V3), namely:

$$\text{For all } p \leq r \in L, \text{ there is } q,\ p \leq q \leq r, \text{ such that } Q \geq P \vee R. \tag{V3*}$$

Suppose (V3*) does not hold for some $p \leq r$, that is, there is no q, such that $p \leq q \leq r$ and $Q \geq P \vee R$; in particular, P and R are incomparable, as otherwise $q = p$ witnesses (V3*). It trivially implies that $p < q$ and, additionally, we can pick p and r such that (V3*) holds for every proper subinterval of $[p, r]$. Let $v \in P - R$. Let $\rho_1 = 0 + R \neq \rho_2 = v + R$, and let $(q, \theta) = (r, \rho_1) \wedge (r, \rho_2)$. Then $q \lneq r$. Also, as $(p, 0+P) \leq (r, \rho_1), (r, \rho_2)$, it follows that $(p, 0+P) \leq (q, \theta)$, and hence $p \leq q$.

We claim that $Q \geq R$. Suppose not, that is, there is $w \in R - Q$, and let $\theta' = w + \theta \neq \theta$. Then, by Corollary 7, $(q, \theta') \leq (r, \rho_1), (r, \rho_2)$, and hence $(q, \theta') \leq (q, \theta) = (r, \rho_1) \wedge (r, \rho_2)$, which is impossible.

As $p \leq q \lneq r$, (V3*) holds for $p \leq q$ and, consequently, there is q' such that $p \leq q' \leq q \lneq r$ and $Q' \geq P \vee Q$. But $Q \geq R$ and hence $Q' \geq P \vee R$, a contradiction.

We have the following consequence of (V3*):

$$\text{If } p \prec r \in L, \text{ then } P \text{ and } R \text{ are comparable, that is, either } P \leq R \text{ or } R \leq P. \tag{$*$}$$

To prove (V4), let us take a fixed three-element interval $p \prec q \prec r$ in L, then $(p, 0+P) \prec (q, 0+Q) \prec (r, 0+R)$. As L^K is an RC lattice, it has no three-element intervals, and hence there is $v \notin Q$ such that $(p, 0+P) \prec (q, v+Q) \prec (r, 0+R)$. Then $v \in P \vee Q - Q$ and, by $(*)$, $v \in P \geq Q$. Similarly, $v \in R \geq Q$. Consequently, $Q \leq P \wedge R$ and, as $v \in P \wedge R - Q$, this inequality is strict.

Before proving (V2), we are going to prove several intermediary statements.

For all $p \leq r \in L$, $C_{p,r} \in \mathsf{Lin}\, V$. This contrasts with Corollary 6 that says that $C_{p,r}$ is not, in general, a linear subspace. Let $p \leq r$ be such that $C_{p,r} \notin \mathsf{Lin}\, V$, moreover, without loss of generality, $C_{p',r'} \in \mathsf{Lin}\, V$ for every proper subinterval $[p', r']$ of $[p, r]$. Then there are $v, w \in V$ such that $v, w \in C_{p,r}$, but $v + w \notin C_{p,r}$. Then $(p, 0+P) \leq (r, v+R), (r, w+R)$. Subtracting $v + w$ and multiplying by $-1 \in \mathscr{F}_N$, we get $(p, v+w+P) \leq$

$(r,v+R),(r,w+R)$.

Then there are $q\in L$ and $u\in V$ such that $p\leq q\leq r$ and $(p,0+P),(p,v+w+P)\leq(q,u+Q)\leq(r,v+R),(r,w+R)$; thus, $u,u-v-w\in C_{p,q}$ and $v-u,w-u\in C_{q,r}$. As $v+w\notin C_{p,r}$, it follows that $v+w\notin P$. Hence, $(p,0+P)\neq(p,v+w+P)$, and so $p<q$. Then $[q,r]$ is a proper subinterval of $[p,r]$, which means that $C_{q,r}$ is a linear subspace of V, and so $(v-u)+(w-u)=v+w-2u\in C_{q,r}$. Also, as $u\in C_{p,q}$ and $C_{p,q}$ is closed under multiplication by constants, $2u\in C_{p,q}$. However, by Corollary 6, $C_{p,r}\supseteq C_{p,q}+C_{q,r}$, and so $2u+(v+w-2u)=v+w\in C_{p,r}$, a contradiction.

Recall that, by Corollary 6,

$$C_{p,r}=\bigcup_{H-\text{a maximal chain in }[p,r]}\ \bigvee_{h\in H}K_h.$$

For a fixed $p\leq r$ in L, let us say that a linear subspace $K\subseteq C_{p,r}$ is *realizable* if there is a maximal chain H in $[p,r]$ such that $K=\bigvee_{h\in H}K_h$, and let us call K *extremal* if it is realizable and is not properly smaller than any other realizable subspace. Note that $C_{p,r}$ is a union of all extremal subspaces, however, it might be not realizable itself. Our next claim is the following:

If $K\leq\operatorname{Lin}V$ is extremal for $p\leq r\in L$, then there is $q\in L$ such that $p\leq q\leq r$ and $K=Q$. We are going to show that if there is any chain H in $[p,r]$ such that $K=\bigvee_{h\in H}K_h$, then the conclusion holds, that is, there is q, $p\leq q\leq r$ such that $K=K_q$. As K is extremal by assumption, there is a maximal chain H^* from p to r such that $K=\bigvee_{h^*\in H^*}K_{h^*}$, so the claim would follow. The proof is by induction on the length m of H. If $m=1$, then H is a one-element set and there is nothing to prove. Now, suppose it holds for $m-1\geq 1$, and let $H=h_1\leq\cdots\leq h_m$. By (V3*), there is $h'_1,\ldots,h'_{m-1}$ such that $h_1\leq h'_1\leq h_2\leq h'_2\leq\cdots\leq h_{m-1}\leq h'_{m-1}\leq h_m$ and $K_{h'_i}\geq K_{h_i}\vee K_{h_{i+1}}$, for $i=1,\ldots,m-1$. Let $H'=h'_1,\ldots,h'_{m-1}$, thus, H' is a chain in $[p,r]$ of length $m-1$. Also, $\bigvee_{h'\in H'}K_{h'}\geq\bigvee_{h\in H}K_h=K$. On the other hand, as K is extremal, it holds that $\bigvee_{h'\in H'}K_{h'}\not>K$, hence $\bigvee_{h'\in H'}K_{h'}=K$, concluding the induction and the proof.

For $p\leq r\in L$, there is q, $p\leq q\leq r$ such that $C_{p,r}=Q$. Suppose not, and let us pick $p\leq r$ such that the statement holds for all proper subintervals of $[p,r]$. Let K be an extremal subspace of $C_{p,r}$. Then, by the previous claim, there is $q\in[p,r]$ such that $K=Q\lneq C_{p,r}$. Note that, as $P\vee R\leq\bigvee_{h\in H}K_h$ for any maximal H, then $P\vee R\leq K'$ for any realizable $K'\in\operatorname{Lin}V$. Thus, $P\vee R\leq Q$, but also notice that if $Q=P\vee R$, then any realizable subspace contains Q and, as Q is extremal, it follows that Q is the only realizable, and hence the only extremal subspace. Then, as $C_{p,r}$ is a union of extremal subspaces, $C_{p,r}=Q$, which cannot happen. Hence $P\vee R\lneq Q\lneq C_{p,r}$, in particular, $q\neq p,r$. Let us take $v\in C_{p,r}-Q$ and $u\in Q-P\vee R$. Then $(p,0+P)\neq(p,u+P)$ and $(p,0+P),(p,u+P)\leq(q,0+Q),(r,v+R)$, here $(p,u+P)\leq(r,v+R)$ follows from the fact that $u,v\in C_{p,r}$, and so, as $C_{p,r}$ is a linear subspace of V, $u-v\in C_{p,r}$. Hence, there is $t\in L$ and $w\in V$ such that $p<t\leq q,r$ and $(p,0+P),(p,u+P)\leq(t,w+T)\leq(q,0+Q),(r,v+R)$. Note that, as

Q is extremal in $[p,r]$, it is also extremal in $[p,q]$ and in $[t,r]$. However, by the assumption of minimality of $[p,r]$, there is $q* \in [p,q]$ such that $C_{p,q} = Q^*$. Hence, there is a unique extremal subspace in $[p,q]$, and so this subspace is Q. Consequently, $Q = C_{p,q}$ and, similarly, $Q = C_{t,r}$. Finally, $(p, 0+P) \leq (t, w+T)$ implies $w \in C_{t,q} \leq C_{p,q} = Q$; similarly, $v - w \in C_{t,r} = Q$. Thus, $v = w + (v-w) \in Q$, a contradiction.

We illustrate this part of the proof in Figure 5.3 below. There $C_{f,h} = V = V(2,2)$, however, all extremal subspaces of $[f,h]$ are one-dimensional. Then, following the proof, there are $u, v \in V$ such that $(f, 0+F), (f, u+F) < (g, 0+G), (h, v+H)$, but there is no element between them in L^K, and so L^K is not a lattice. Recall that we are now working towards proving (V2), and, in terms of (V0)–(V5) properties, L^K satisfies all of them except for (V2) and (V3). It does, however, satisfy (V3*), and it can be noticed that (V3) is a simple corollary of (V3*) and (V2). This illustrates that here the failure of (V2) is the essential reason for L^K not being an RC lattice.

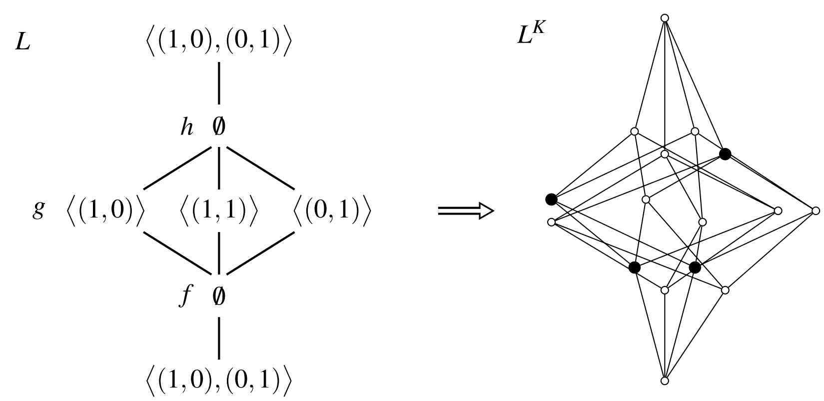


Figure 5.3: L^K poset breaking (V2) for the interval $[f,h]$.

Now, suppose (V2) does not hold for some q, $p \leq q \leq r$, that is, $Q \not\leq P \vee R$; in particular, $p < q < r$. By the previous statement, without loss of generality, we might assume that $Q = C_{p,r} \gneq P \vee R$. Thus, there is $v \in Q - P \vee R$, and let $(s, u+S)$ be a complement of $(q, 0+Q)$ in the interval $[(p, 0+P), (r, v+R)]$. Note that in this case $u \in C_{p,s} \leq C_{p,r} = Q$. We claim that s is a complement of q in $[p,r]$. Indeed, if $p < q \wedge s$, then $(q \wedge s, u + K_{q\wedge s}) \leq (q, u+Q) = (q, 0+Q)$ and $(q \wedge s, u + K_{q \wedge s}) \leq (s, u+S)$, however $(q \wedge s, u + K_{q\wedge s}) \not\leq (p, 0+P)$, a contradiction. The proof of $q \vee s = r$ is similar.

Now, $u \in C_{p,s}$, and hence, by (V1), there is $p^* \in [p,s]$ such that $u \in P^*$. Then $p = q \wedge p^*$, $u \in P^*, Q$ and hence, by (V1), $u \in K_{q \wedge p^*} = P$. Similarly, there is $r^* \in [s,r]$ such that $v - u \in R^*$, and $v - u \in K_{q \vee r^*} = R$. But then $v = u + (v-u) \in P + R = P \vee R$, a contradiction. Note that (V3) is a direct consequence of (V3*) and (V2). Also, for all $p \leq r$, $C_{p,q} = P \vee R$.

Before proving (V5), we prove the following intermediary statements, in which we do not assume that

L^K is a relatively complemented lattice, but rely on (V1), (V2) and (V3) instead.

Let $q, r \le s$, then from (V1), (V2), and (V3) it follows that

$$S \vee K_{q\vee r} = (S\vee Q)\wedge(S\vee R).$$

As $q\vee r \in [q,s]$, it follows from (V2) that $K_{q\vee r} \le S\vee Q$ and, similarly, $K_{q\vee r} \le Q\vee R$. Hence $S\vee K_{q\vee r} \le (S\vee Q)\wedge(S\vee R)$. On the other hand, by (V3), there is $q^* \in [q,s]$ and $r^* \in [r,s]$ such that $Q^* = S\vee Q$ and $R^* = S\vee R$. Then, by (V1), $K_{q^*\vee r^*} \ge Q^*\wedge R^* = (S\vee Q)\wedge(S\vee R)$. However, $q^*\vee r^* \in [q\vee r, s]$, and thus, by (V2), $S\vee K_{q\vee r} \ge K_{q^*\vee r^*} \ge (S\vee Q)\wedge(S\vee R)$, proving he claim.

Let $q, r \le s, p$, then from (V1), (V2), and (V3) it follows that

$$K_{s\wedge p}\vee K_{q\vee r} = (S\vee Q)\wedge(S\vee R)\wedge(P\vee Q)\wedge(P\vee R).$$

By the previous statement, $S\vee K_{q\vee r} = (S\vee Q)\wedge(S\vee R)$ and $P\vee K_{q\vee r} = (P\vee Q)\wedge(P\vee R)$. Hence

$$\begin{aligned}
K_{s\wedge p}\vee K_{q\vee r} &= [\text{apply previous claim to } q\vee r \le s,p]\\
&= (K_{q\vee r}\vee S)\wedge(K_{q\vee r}\vee P)\\
&= [\text{apply previous claim to } q\vee r\le s \text{ and } q\vee r \le p]\\
&= (S\vee Q)\wedge(S\vee R)\wedge(P\vee Q)\wedge(P\vee R).
\end{aligned}$$

Finally, to prove (V5), let us take $q, r \le s, p$ in L and $w \in (S\vee R)\wedge(P\vee Q)$. Then, by construction, $(q, 0+Q), (r, w+R) \le (s, 0+S), (p, w+P)$ and so there is $t \in [q\vee r, s\wedge p]$ and $u \in V$ such that $(q, 0+Q), (r, w+R) \le (t, u+K_t) \le (s, 0+S), (p, w+P)$. The latter holds if and only if $u \in Q\vee K_t$, $u \in S\vee K_t$, $w-u \in R\vee K_t$, and $w-u \in P\vee K_t$. Without losing generality, we may take t for which $K_t = K_{s\wedge p}\vee K_{q\vee r}$. By the previous claim, $K_{s\wedge p}\vee K_{q\vee r} = (S\vee Q)\wedge(S\vee R)\wedge(P\vee Q)\wedge(P\vee R) = T$, where T is as defined in the statement of the claim. Hence, $K_t = T$, in line with our notation. But notice that $u \in (Q\vee T)\wedge(S\vee T)$ and $w-u \in (R\vee T)\wedge(P\vee T)$ implies $w \in \big[(S\vee T)\wedge(Q\vee T)\big]\vee\big[(P\vee T)\wedge(R\vee T)\big]$, as needed.

($\Leftarrow$). Recall that, by (V2) and (V3), for any $p \le r$ there is q such that, $p \le q \le r$ and $C_{p,r} = P\vee R = Q$. In particular, $C_{p,r}$ is a linear subspace.

We start by showing that L^K is a lattice. Let us prove that for $q, r \le s, p$ in L and for $v_q, v_r, v_s, v_p \in V$ such that $(q, v_q+Q), (r, v_r+R) \le (s, v_s+S), (p, v_p+P)$ there is $t \in L$ and $v_t \in V$ such that $(q, v_q+Q), (r, v_r+R) \le (t, v_t+T) \le (s, v_s+S), (p, v_p+P)$, in particular, $q, r \le t \le s, p$. Together with (V0), this would prove the existence of arbitrary meets and joins in L^K. We argue that it is enough to prove it in the case when $v_s = v_q = 0$

and $v_p = v_r$. Indeed, $(q, v_q + Q), (r, v_r + R) \le (s, v_s + S), (p, v_p + P)$ if and only if $v_s - v_q \in C_{s,q} = S \vee Q$, $v_s - v_r \in C_{s,r} = S \vee R$, $v_p - v_q \in C_{p,q} = P \vee Q$, and $v_p - v_r \in C_{p,r} = P \vee R$. Then those inequalities hold if and only if they hold for arbitrary $v'_s \in v_s + S, \ldots, v'_r \in v_r + R$. And, as $v_s - v_q \in S \vee Q$, we can pick pick them so that $v'_s = v'_q$ and, similarly, $v'_p = v'_r$. Then we need to prove $(q, v'_s + Q), (r, v'_r + R) \le (t, v_t + T) \le (s, v'_s + S), (p, v'_r + P)$, which is equivalent to $(q, v'_s - v'_s + Q), (r, v'_r - v'_s + R) \le (t, v_t - v'_s + T) \le (s, v'_s - v'_s + S), (p, v'_r - v'_s + P)$, as needed.

So let us assume that $(q, 0 + Q), (r, w + R) \le (s, 0 + S), (p, w + P)$, for some $w \in V$. Then $w \in (S \vee R) \wedge (P \vee Q)$. Let $t \in [q \vee r, s \wedge p]$ be such that $T = K_{s \wedge p} \vee K_{q \vee r}$. As argued in the ($\Leftarrow$) part, from (V1), (V2), and (V3) it follows that $T = K_{s \wedge p} \vee K_{q \vee r} = (S \vee Q) \wedge (S \vee R) \wedge (P \vee Q) \wedge (P \vee R)$. Then, from (V5), $w \in \big[(S \vee T) \wedge (Q \vee T)\big] \vee \big[(P \vee T) \wedge (R \vee T)\big]$, that is, $w = u + v$ for $u \in (S \vee T) \wedge (Q \vee T)$ and $v \in (P \vee T) \wedge (R \vee T)$. But then $(q, 0 + Q), (r, w + R) \le (t, u + T) \le (s, 0 + S), (p, w + P)$, as needed.

Now we will show that L^K does not have three-element intervals, and hence is RC. Suppose not, and let $(s, v_s + S) \prec (q, v_q + Q) \prec (r, v_r + R)$ be such three-element interval. It follows from Corollary 8 that $s \prec q \prec r$ in L. If $s \prec q \prec r$ is a three-element interval in L then, by (V4), there is $w \in S \wedge P - Q$. Hence, $(s, v_s + S) \prec (q, w + v_q + Q) \prec (r, v_r + R)$ and $(q, v_q + Q) \neq (q, w + v_q + Q)$.

Now suppose the interval $[s, r]$ is not three-element in L, that is, there is p, $s < p < r$, $p \neq q$. By (V2), $v_s - v_r \in S \vee R$. So there is $v'_s \in v_s + S$ and $v'_r \in v_r + R$ such that $v'_s = v'_r$. But then $(s, v_s + S) \prec (p, v'_s + P) \prec (r, v_r + R)$. $\square$

We note that the condition (V5) is necessary, as shown in the following example:

Example 4. Let N be a big enough prime, let V be a 3-dimensional space over $\mathscr{F}_N$, and let $\bar{x}, \bar{y}$, and $\bar{z}$ be a basis of V. Let L^K and $q, r \le p, s \in L$ be as in Figure 5.4. Then it can be checked that (V0)–(V4) hold, however

$$
\begin{aligned}
LHS &= (S \vee R) \wedge (P \vee Q) = \langle \overline{x + y + z} \rangle, \\
RHS &= \big[(S \vee T) \wedge (Q \vee T)\big] \vee \big[(P \vee T) \wedge (R \vee T)\big] = \emptyset.
\end{aligned}
$$

Thus, $LHS \not\le RHS$ and L^K is not a lattice. It can be also explicitly checked that

$$
\begin{aligned}
T &= (S \vee Q) \wedge (S \vee R) \wedge (P \vee Q) \wedge (P \vee R) \\
&= \langle \bar{x}, \bar{z} \rangle \wedge \langle \bar{x}, \overline{y + z} \rangle \wedge \langle \bar{y}, \overline{x + z} \rangle \wedge \langle \bar{y}, \bar{z} \rangle \wedge = \emptyset,
\end{aligned}
$$

as it should be.

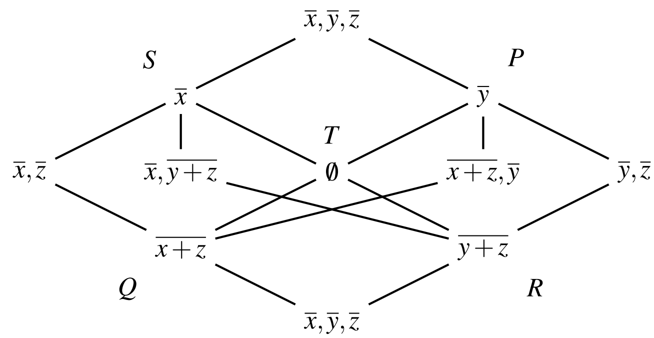


Figure 5.4: L^K from Example 4.

Condition (V5) in Theorem 5 is, in a way, cheating: using it we effectively enforce L^K to have meets and joins we need. Hence, the question:

Question 2. *Is there an alternative, "better" condition, equivalent to (V5)? For example, will this suffice:*

$$(S \vee P \vee Q) \wedge (S \vee P \vee R) \leq S \vee P \vee T, \textit{ and}$$

$$(Q \vee R \vee S) \wedge (Q \vee R \vee P) \leq Q \vee R \vee T,$$

for $T = (S \vee Q) \wedge (S \vee R) \wedge (P \vee Q) \wedge (P \vee R)$*?*

Note that, for L^K construction from Example 4, we have

$$(S \vee P \vee Q) = (S \vee P \vee R) = (Q \vee R \vee S) = (Q \vee R \vee P) = \langle \overline{x}, \overline{y}, \overline{z} \rangle, \text{ and}$$

$$S \vee P \vee T = \langle \overline{x}, \overline{y} \rangle,$$

$$Q \vee R \vee T = \langle \overline{x+z}, \overline{y+z} \rangle,$$

hence the condition from Question 2 is not satisfied.

Theorem 5 is closely mimicked, but in a simplified way, in the following criterion for L^Σ to be RC.

Theorem 6. *For L, U,* Σ*, and N as in Definition 2,* $L^\Sigma = L^\Sigma(L, U, \Sigma, N)$ *is a relatively complemented lattice if and only if the following conditions are satisfied:*

(W0) For top and bottom elements $e, z \in L$, $C_e = C_z = U$.

(W1) For any $p, r \in L$, $C_p \cap C_r \subseteq C_{p \wedge r}, C_{p \vee r}$.

Equivalently, for any $u \in U$*, the set* $B_u = \{s \in L \mid u \in C_s\}$ *is a sublattice of L.*

(W2) *For any* $p \leq q \leq r \in L$, $C_q \subseteq C_p \cup C_r$.

Equivalently, for any $u \in U$, *the set* $W_u = \{s \in L \mid u \in E_s\}$ *is convex in* L.

(W3) *For any* $p \leq r \in L$, *there is* $q \in L$ *such that* $p \leq q \leq r$ *and* $C_q = C_p \cup C_r$.

(W4) *For any three-element interval* $p \prec q \prec r$ *of* L, $C_q \subsetneq C_p \cap C_r$.

Notice that, except for (V5), conditions (W0)–(W4) perfectly correspond to (V0)–(V4) of Theorem 5.

Proof. ($\Rightarrow$). Follows directly from Theorem 5.

($\Leftarrow$). In this direction, we only need to show that (V5) automatically holds for L^{Σ^K}. As we mentioned before, $\Sigma^K[L]$ lies in a distributive sublattice of $\mathsf{Lin}\,V$, which we will now use. Note that, using the distributive identity, we get

$$LHS = (S \vee R) \wedge (P \vee Q) = (S \wedge P) \vee (S \wedge Q) \vee (P \wedge R) \vee (Q \wedge R), \text{ and}$$
$$T = (S \vee Q) \wedge (S \vee R) \wedge (P \vee Q) \wedge (P \vee R) = (S \wedge P) \vee (Q \wedge R).$$

And then

$$\begin{aligned} RHS &= \big[(S \vee T) \wedge (Q \vee T)\big] \vee \big[(P \vee T) \wedge (R \vee T)\big] \\ &= \big[(S \wedge Q) \vee T\big] \vee \big[(P \wedge R) \vee T\big] = (S \wedge Q) \vee (P \wedge R) \vee T \\ &= (S \wedge Q) \vee (P \wedge R) \vee (S \wedge P) \vee (Q \wedge R) = LHS. \end{aligned}$$

$\square$

We call a function $K\colon L \to \mathsf{Lin}\,V$ satisfying (V0)–(V5), or a function $\Sigma\colon L \to 2^U$ satisfying (W0)–(W4), an *RC-pumping*. Let us now state several corollaries from Theorem 5 for L^K construction, but they also hold for L^Σ construction.

Corollary 10. *Let* $L^K = L^K(L,V,K,N)$ *be an RC lattice. Then* $(p,\pi) \leq (r,\rho)$ *in* L^K *if and only if* $p \leq r$ *and* $\pi \cap \rho \neq \emptyset$. *In particular, if* $(p,\pi) \leq (r,\rho)$, *then for every* $q \in L$ *such that* $p \leq q \leq r$ *there is* $\theta \in V/Q$ *such that* $(p,\pi) \leq (q,\theta) \leq (r,\rho)$.

Corollary 11. *If* L *is an RC lattice, then* $L^K = L^K(L,V,K,N)$ *is an RC lattice if and only if the conditions (V0)–(V3) and (V5) from Theorem 5 are satisfied.*

Corollary 12. *Let* L^K *be an RC lattice and, for* $p \leq r$ *in* L, *let* $q \in [p,r]$ *be an element witnessing (V3) for* $[p,r]$, *that is, such that* $Q = P \vee R$. *Then* q *has a complement* s *in* $[p,r]$, *and* $S \leq P \wedge R$.

Proof. As L^K is an RC lattice, let (s, v_s) be a complement of $(q,0)$ in $[(p,0),(r,0)]$. Suppose s is not a complement of q in $[p,r]$, say, $s \vee q = r^* < r$. As $(s,v_s) \leq (r,0)$, it holds that $v_s \in S \vee R \leq P \vee R$. But then $(s,v_s),(q,0) \leq (r^*, v_s)$, which is a contradiction, as $(r,0) \not\leq (r^*, v_s)$.

Now, by (V1), $R = K_{s\vee q} \geq S \wedge Q = S$. Similarly, $P \geq S$, and hence $S \leq P \wedge R$. □

We want to use pumpings to construct not only RC lattices, but also systems over them. Hence, let us now give the following definition.

Definition 3. For a system $\mathscr{E} = \{(x_i, y_i) \mid i \in \mathscr{I}\}$ over $\mathscr{I}$ in L and a pumping $K\colon L \to \mathsf{Lin}\,V$, a pumping $\mathscr{E}^K$ of $\mathscr{E}$ is a system $\mathscr{E}^K$ over $\mathscr{I}^K$ in L^K, where

$$\mathscr{I}^K = \{(i,\pi,\theta) \mid i \in \mathscr{I}, \pi \in V/K(x_i), \theta \in V/K(y_i)$$
$$\text{such that } (x_i,\pi) \geq (y_i,\theta) \text{ in } L^K\};$$

and

$$x_{(i,\pi,\theta)} = (x_i,\pi); \qquad\qquad y_{(i,\pi,\theta)} = (y_i,\theta).$$

It is obvious that, for thus defined $\mathscr{E}^K$, it holds that $S(\mathscr{E}^K) = \kappa^{-1}\left[S(\mathscr{E})\right]$, however, in general, for $C(\mathscr{E}^K)$ it only holds that $C(\mathscr{E}^K) \supseteq \kappa^{-1}\left[C(\mathscr{E})\right]$, with Example 5 giving an improper inclusion. We are thus interested in conditions on K which assert that $C(\mathscr{E}^K) = \kappa^{-1}\left[C(\mathscr{E})\right]$. We outline these conditions in Lemma 19 below. There we will use the following notation: for a system $\mathscr{E} = (X_\mathscr{E}, Y_\mathscr{E})$ and $A \subseteq X_\mathscr{E}$, we define $C_A(\mathscr{E})$ as $C_A(\mathscr{E}) = \bigcup_{x\in A} C_x$. Thus, $C(\mathscr{E}) = C_{X_\mathscr{E}}(\mathscr{E})$.

Example 5. Let L and $\mathscr{E} = \{(x,y)\}$ be as in Figure 5.5. Let U be a one-element set and $\Sigma\colon L \to 2^U$ be an RC pumping shown in Figure 5.5, with notation for Σ as in Figure 5.2. It can be seen that $C(\mathscr{E}^\Sigma) \supsetneq \kappa^{-1}\left[C(\mathscr{E})\right]$.

Lemma 19. *Let $\mathscr{E}$ be a system over $\mathscr{I}$ in L, and let $K\colon L \to \mathsf{Lin}\,V$ be an RC pumping of L. Then, for all $(x,y) \in \mathscr{E}$, $C_{\kappa^{-1}(x)}(\mathscr{E}^K) = \kappa^{-1}\left[C_x(\mathscr{E})\right]$ if and only if $X \geq Y$.*

In particular, if, for all $(x,y) \in \mathscr{E}$, $X \geq Y$, then $C(\mathscr{E}^K) = \kappa^{-1}\left[C(\mathscr{E})\right]$.

Proof. $(\Rightarrow)$. Let us take $(x,y) \in \mathscr{E}$ and suppose that $X \not\geq Y$, that is, $X \vee Y > X$, and let $v \in X \vee Y - X$. By (V3), there is $t \in [y,x]$ such that $T = X \vee Y$; in particular, $v \in T$. Then, by (V1), we can assume that t is a unique maximal element such that $T = X \vee Y$ and, by Corollary 12, there is a complement s of t in $[y,x]$. As $X < T$, $x \neq t$, that is, $t < x$. Consequently, $y < s$. Let q be an atom in $[y,s]$, in particular, $q \not\leq t$. Note that

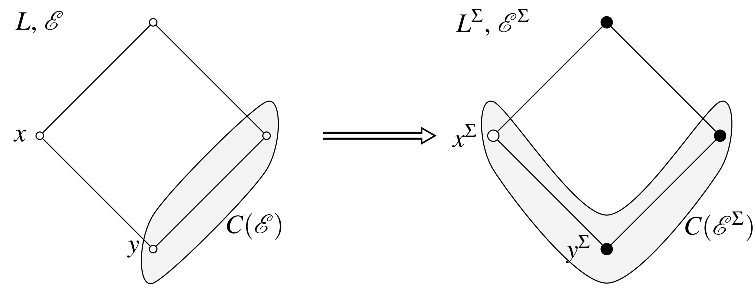


Figure 5.5: RC-pumping Σ, for which $C(\mathscr{E}^\Sigma) \supsetneq \kappa^{-1}\left[C(\mathscr{E})\right]$.

$X \vee Q < T = X \vee Y$, as otherwise there is $r \in [q,x]$ such that $R = X \vee Q = T$, but then $t < r \vee t \leq x$, while $K_{r\vee t} \geq R \wedge T = T$, contradicting the maximal choice of t.

Now, take $v \in X \vee Y - X \vee Q$. Then $(q, v+Q) \not\leq (x, 0+X)$, but $(y,v) \leq (q, v+Q), (x, 0+X)$. As $y \prec q$, this implies $(y, v+Y) = (q, v+Q) \wedge (x, 0+X)$, hence $(q, v+X) \in C_{(x,0+X)}(\mathscr{E}^K)$. Then, if $C_{\kappa^{-1}(x)}(\mathscr{E}^K) = \kappa^{-1}\left[C_x(\mathscr{E})\right]$, it follows that $q \in C_x(\mathscr{E})$, which is impossible, as $q \in [y,x]$ and $q \neq y$, and, consequently, $x \wedge q = q \neq y$.

($\Leftarrow$). Let us take $(x,y) \in \mathscr{E}$ such that $X \geq Y$ and suppose $C_{\kappa^{-1}(x)}(\mathscr{E}^K) \supsetneq \kappa^{-1}\left[C_x(\mathscr{E})\right]$, that is, there is $q \in L$ and $v_q, v_x, v_y \in V$, such that $q \wedge x \neq y$ and $(q, v_q + Q) \wedge (x, v_x + X) = (y, v_y + Y)$. Note that, in this case, $q \wedge x = y^* > y$, $v_q - v_y \in Q \vee Y$, and $v_x - v_y \in X \vee Y = X$. Then $v_q - v_y = w_q - w_y$ for some $w_q \in Q$ and $w_y \in Y$. We claim that $(y^*, v_q - w_q + Y^*) \leq (q, v_q + Q), (x, v_x + X)$. Indeed, $v_q - (v_q - w_q) = w_q \in Q$, hence $(y^*, v_q - w_q + Y^*) \leq (q, v_q + Q)$. Also, $v_x - (v_q - w_q) = v_x - (v_y - w_y) = (v_x - v_y) + w_y \in X \vee Y = X$, hence $(y^*, v_q - w_q + Y^*) \leq (x, v_x + X)$. But then $(y^*, v_q - w_q + Y^*) \leq (q, v_q + Q) \wedge (x, v_x + X)$, in particular, $(q, v_q + Q) \wedge (x, v_x + X) \neq (y, v_y + Y)$, a contradiction. $\square$

5.3 SSP-like properties that hold for Boolean, but not for RC lattices

Recall that, for an RC lattice L, the fact that L satisfies SSP_1 and SSP_2 was equivalent to showing that, for a one-dimensional system $\mathscr{E} = \{(x_a, y_a)\}$, $|S_a| \leq |C_a|$. And that, for a two-dimensional system $\mathscr{E} = \{(x_a, y_a), (x_b, y_b)\}$, $|S_a \cup S_b| \leq |C_a \cup C_b|$. Let us note that, in fact, these properties hold for arbitrary system $\mathscr{E}$ such that $(x_a, y_a), (x_b, y_b) \in \mathscr{E}$.

Sets of the form S_a and C_a and relations between them are, thus, essential objects in our approach to this problem. We then can ask if, perhaps, some other, similarly looking properties hold for such sets, first in Boolean case, and then in RC case. Indeed, even for systems of dimension two, in Boolean case we can find several examples of such properties.

Proposition 5. *For a Boolean lattice L with a system $\mathscr{E} = \{(x_a, y_a), (x_b, y_b)\}$ over it, it holds:*

1. $|S_{ab}| \geq |C_{ab}|$;
2. $|S_{a-b}| \leq |C_{a-b}|$ *and* $|S_{b-a}| \leq |C_{b-a}|$;
3. $|S_{a\triangle b}| \leq |C_{a\triangle b}|$,

where $S_{ab} = S_a \cap S_b$, $S_{a-b} = S_a - S_b$, *and* $S_{a\triangle b} = S_a \triangle S_b$ *(symmetric difference), and similarly for* C.

Proof. (1). Let $L = 2^U$, for a finite base set U, be the Boolean lattice in question. As $x_a, x_b, y_a, y_b \in L$ are subsets of U, let us denote them by X_a, X_b, Y_a, and Y_b respectively. Recall that, by the definition of a system, $Y_a \subseteq X_a$ and $Y_b \subseteq X_b$. Also,

$$S_{ab} = \{w \in L \mid w \geq x_a, x_b\} = \{W \subseteq U \mid W \supseteq X_a \cup X_b\},$$

and, similarly,

$$C_{ab} = \{W \subseteq U \mid W \cap X_a = Y_a \text{ and } W \cap X_b = Y_b\}.$$

It then can be easily noticed that $|S_{ab}| = 2^{|U|-|X_a \cup X_b|}$ and $|C_{ab}| = 2^{|U|-|X_a \cup X_b|}$ whenever $X_a \cap X_b \cap Y_a = X_a \cap X_b \cap Y_b$, and $|C_{ab}| = 0$ otherwise. In both cases, $|S_{ab}| \geq |C_{ab}|$ holds.

(2). In the same setup,

$$S_{a-b} = \{W \subseteq U \mid W \supseteq X_a \text{ and } W \not\supseteq X_b\},$$

and

$$C_{a-b} = \{W \subseteq U \mid W \cap X_a = Y_a \text{ and } W \cap X_b \neq Y_b\}.$$

Then $W \in S_{a-b}$ if and only if W contains all elements of X_a, misses at least one element of $X_b - X_a$, and is arbitrary otherwise. From this,

$$|S_{a-b}| = \left(2^{|X_b - X_a|} - 1\right) 2^{|U|-|X_a \cup X_b|}.$$

Similarly, for W to be in C_{a-b}, two conditions have to be satisfied. First, for $x \in X_a$, $x \in W$ if and only if $x \in Y_a$. This ensures that $W \cap X_a = Y_a$. Note that, if $X_a \cap X_b \cap Y_a \neq X_a \cap X_b \cap Y_b$, then the second condition, $W \cap X_b \neq Y_b$, is satisfied automatically. Otherwise, we need to require that at least one element of $X_b - X_a$ distinguishes W from Y_a. Thus

$$|S_{a-b}| = \begin{cases} \left(2^{|X_b - X_a|} - 1\right) 2^{|U|-|X_a \cup X_b|} & \text{whenever } X_a \cap X_b \cap Y_a \neq X_a \cap X_b \cap Y_b, \\ 2^{|X_b - X_a|} \cdot 2^{|U|-|X_a \cup X_b|} = 2^{|U|-|X_a|} & \text{otherwise.} \end{cases}$$

As before, we get $|S_{a-b}| \leq |C_{a-b}|$. The $|S_{a-b}| \leq |C_{a-b}|$ inequality is similar.

Finally, (3) is a straightforward consequence of (2). $\square$

These properties, however, illustrate a stark contrast between Boolean and RC case. Indeed, the following example in Figure 5.6 below shows that properties (1) and (2) of Proposition 5 do not hold for RC lattices. Here $|S_{ab}| = 1 < |C_{ab}| = 4$ and $|S_{a-b}| = 3 > |C_{a-b}| = 1$. Notice that for the system $\mathscr{E} = \{(x_a, y_a), (x_b, y_b)\}$ here it holds $y_a = y_b$.

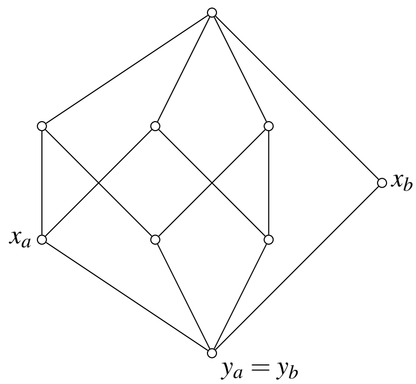


Figure 5.6: Properties (1) and (2) of Proposition 5 do not hold for RC lattices.

Property (3) of Proposition 5 for RC lattices, however, is more tricky. Let us first prove a positive result of this sort.

Lemma 20. *For an RC lattice L and a system $\mathscr{E} = \{(x_a, y_a), (x_b, y_b)\}$ over it, let $y_a = y_b = y$. Then $|S_{a\triangle b}| \leq |C_{a\triangle b}|$.*

Proof. Notice that $y_a = y_b = y$ implies, in particular, that $y \leq x_a, x_b$, and thus $y \leq x_a \wedge x_b$. Now, let us split S_{a-b} into two disjoint subsets, $S_{a-b} = X_a^b \sqcup Y_a^b$, where

$$Y_a^b = \{u \in S_{a-b} \mid u \wedge x_b > x_a \wedge x_b\};$$
$$X_a^b = \{u \in S_{a-b} \mid u \wedge x_b = x_a \wedge x_b\}.$$

Let us define a mapping $\psi_a^b \colon S_a \to [y)$ by taking $\psi_a^b(u) = w$, where

$$v = c(u \wedge x_b, (u \wedge x_b) \vee x_a, u);$$
$$z = c(y, x_a \wedge x_b, u \wedge x_b);$$
$$w = c(z, u \wedge x_b, v).$$

The construction of ψ_a^b and φ_a^b, to be defined later, is illustrated in Figure 5.7 below, dotted lines indicate complementation.

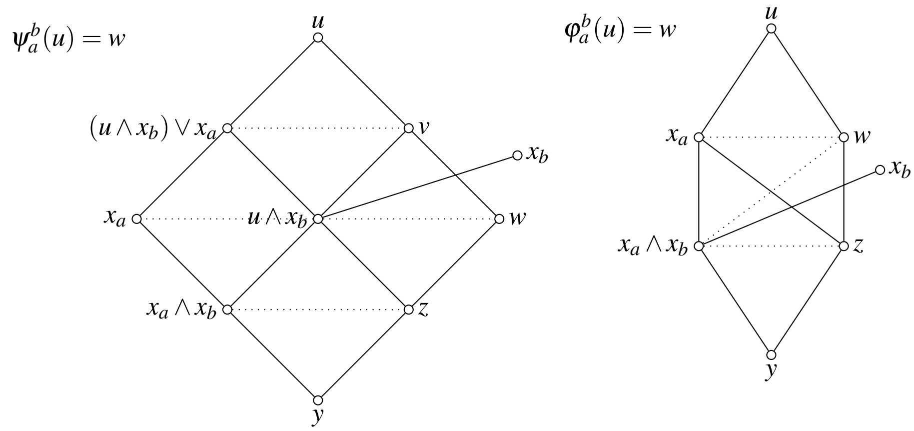


Figure 5.7: Functions ψ_a^b and φ_a^b

We prove the following simple property about gluing complementation squares in order to use it in the argument about ψ_a^b and φ_a^b. *If $e, f, g, h, i, j \in L$ are such that f and g are complements in $[e,h]$, and h and i are complements in $[g, j]$, then f and i are complements in $[e, j]$.* Indeed, $f \vee i = f \vee (g \vee i) = h \vee i = j$, and the argument for the meet is similar.

We argue that ψ_a^b on Y_a^b is one-to-one into C_{a-b}. Indeed, gluing four complementation squares in Figure 5.7, we see that $w \in C(y, x_a, u)$, hence the injectivity. Also, $w \wedge x_b = w \wedge u \wedge x_b = z$. As $x_a \wedge x_b < u \wedge x_b$ for $u \in Y_a^b$, by the definition of z we have $y < z = w \wedge x_b$.

Now, let us define $\varphi_a^b \colon X_a^b \to [y)$ as $\varphi_a^b(u) = w$, where

$$z = c(y, x_a \wedge x_b, x_a);$$

$$w = c(z, x_a, u).$$

Gluing the squares, we get, $w \in C(y, x_a \wedge x_b, u)$, proving the injectivity. Now, using the fact that $u \in X_a^b$, $w \wedge x_b = w \wedge u \wedge x_b = w \wedge x_a \wedge x_b = y$. Also $x_a \wedge x_b < x_a$ implies $y < z = w \wedge x_a$. All in all, $\varphi_a^b[X_a^b] \subseteq C_{b-a}$.

Now, let us define X_b^a, Y_b^a, ψ_b^a and φ_b^a by swapping x_a with x_b in the definitions. Thus, ψ_b^a and φ_b^a are one-to-one into C_{b-a} and C_{a-b} correspondingly. To finish the proof, we need to show that the sets $\psi_a^b[Y_a^b]$, $\varphi_a^b[X_a^b]$, $\psi_b^a[Y_b^a]$, and $\varphi_b^a[X_b^a]$ are disjoint, for which it is clearly enough to show that $\psi_a^b[Y_a^b]$ is disjoint from $\varphi_b^a[X_b^a]$, disjointness of $\psi_b^a[Y_b^a]$ and $\varphi_a^b[X_a^b]$ follows by a symmetric, that is, with a and b swapped, argument.

To do this, we note that for $u \in Y_a^b$, $\psi_a^b(u) \vee x_a = u \not\geq x_b$. Now, let us take $u' \in X_a^b$; Yes, we consider

$u' \in X_a^b$ and φ_a^b to align better with the picture, and will then symmetrize the argument to get a statement about $\varphi_b^a[X_b^a]$. Then

$$\varphi_a^b(u') \vee x_b = \varphi_a^b(u') \vee z \vee (x_b \wedge x_a) \vee x_b = \varphi_a^b(u') \vee x_a \vee x_b \geq x_a.$$

Symmetrizing, we get that for $u' \in X_b^a$, $\varphi_b^a(u') \vee x_a \geq x_b$. As $\psi_a^b(u) \vee x_a \not\geq x_b$ for $u \in Y_a^b$, it follows that $\psi_a^b[Y_a^b]$ and $\varphi_b^a[X_b^a]$ are disjoint. $\square$

Now, let us prove that property (3) of Proposition 5 does not, in general, hold for RC lattices. For this, we are going to use pumping, and so we also use it as an illustration for how pumping can be used to construct RC lattices, whose generic elements follow a template provided by an RC graph.

Lemma 21. *Properties (1)–(3) of Proposition 5 do not, in general, hold for a system $\mathscr{E} = \{(x_a, y_a), (x_b, y_b)\}$ over an RC lattice.*

Proof. Properties (1) and (2) were already shown to fail for the example in Figure 5.6. We thus need to disproof (3), that is, to construct an RC lattice L and a system $\mathscr{E} = \{(x_a, y_a), (x_b, y_b)\}$ over it, such that $|S_{a\triangle b}| > |C_{a\triangle b}|$.

Figure 5.8 below shows a lattice L together with a function $\Sigma\colon L \to 2^U$, where $|U| = 1$; the notatin is as in Figure 5.2, that is, black vertices stand for contracted, and white for expanded coordinates. We note that L is a direct product of a lattice P with a two-element interval B_1. In picture, all covers are drawn, but those corresponding to $(p,0) \prec (p,1)$, for $p \in P$, are dotted in order to not litter the picture. We claim that, for $N \geq 2$, the lattice $L^\Sigma = L^\Sigma(L,U,\Sigma,N)$ is RC.

Indeed, by Theorem 6, we need to check (W0)–(W4). Moreover, it can be noticed that P is RC, and then, as a direct product of RC lattices, so is $L = P \times B_1$. So, (W4) holds automatically. (W0) simply requires that all coordinates of the top and the bottom vertices are contracted, which is true. Finally, for (W1), we need to check that the set $B_u = \{s \in L \mid u \text{ is contracted in } s\}$ is a sublattice of L, for every $u \in U$. As there is only one element in U, we, basically, need to check that in Figure 5.8 the black elements form a sublattice. The task can be made even easier by observing that the same elements are black in both of the copies of P in L, so we need to check it only for one of them, and this is straightforward. Similarly, for (W2), we need to check that the white elements of L form a convex set, which is also obvious.

The elements x_a, x_b, y_a, and y_b are indicated in L in figure. Notice that all of them correspond to elements of L, all of whose coordinates are contracted. Every one of them thus corresponds to a unique element in its bunch L^Σ, which we will consider a corresponding element of $\mathscr{E}$ over L^Σ. We now claim that this $\mathscr{E}$ breaks (3) of Theorem 6, that is, in L^Σ, $|S_{a\triangle b}| > |C_{a\triangle b}|$. However, let us remark that we only claim this asymptotically,

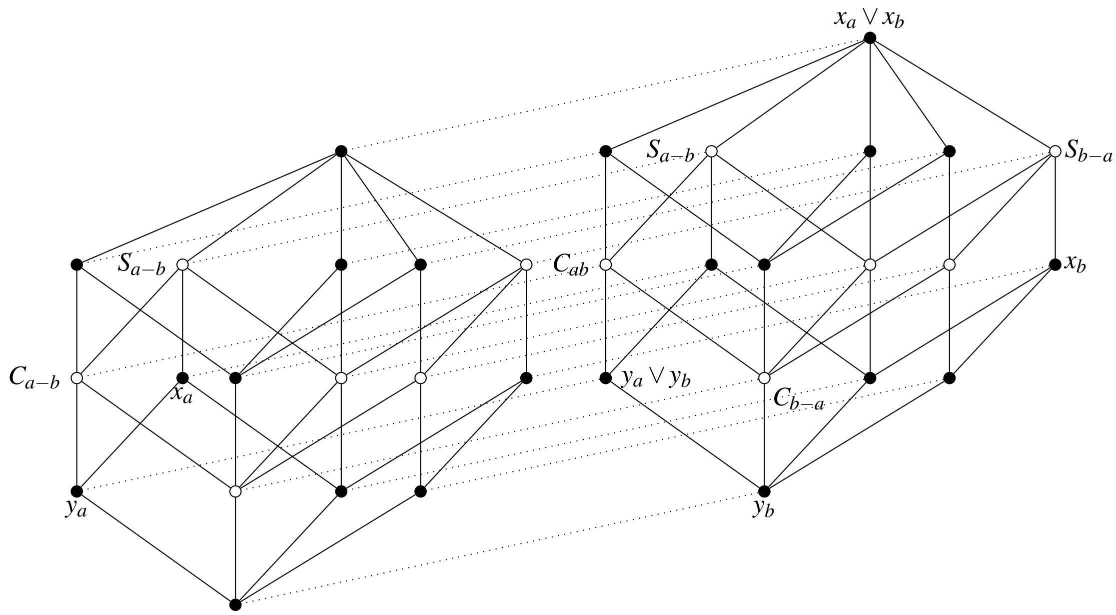


Figure 5.8: RC lattice L^{Σ} and a system over it, invalidating (3) of Proposition 5.

that is, for N arbitrarily large.

As one can notice, some elements in Figure 5.8 are indicated as C_{a-b}, S_{a-b}, etc. Moreover, all those elements are white. It can be easily seen that all elements in the bunches of the corresponding elements in L do satisfy the indicated property. That is, if u is an element indicated by C_{a-b} in L, then all N elements in u-bunch in L^{Σ} are in C_{a-b}. Moreover, those properties are exaustively indicated for all elements of L, that is, for a white element u of L, and any v in u-bunch in L^{Σ}, $v \in C_{a-b}$ if and only if u is marked as C_{a-b}. However, we chose to ignore indicating those properties for black vertices. For example, y_a is obviously in C_{a-b}, which is not shown in figure.

This omission, however, is deliberate: We claim that irregardless of how the elements in bunches of black elements are classified with respect to belonging to C_{a-b}, S_{a-b}, etc., L^{Σ} will invalidate the required property for big N. Indeed, in L^{Σ}, $|S_{a\triangle b}| = |S_{a-b}| + |S_{b-a}| = 2N + c_1 + N + c_2$, where c_1 and c_2 are some constants, independent of N. Similarly, $|C_{a\triangle b}| = |C_{a-b}| + |C_{b-a}| = N + c_3 + N + c_4$. And so, $|S_{a\triangle b}| = 3N + c_1 + c_2 > |C_{a\triangle b}| = 2N + c_3 + c_4$, as needed. $\square$

While in the proof of Lemma 21 we used pumping to construct, technically, a family of lattices, breaking a given property of interest, it might be not clear how did we come up with L and Σ that do the job. Our point here is that RC graphs might be a good tool for that: By design, they only abstract the essential properties of RC lattices, and thus are easier to deal with and, at the same time, properties related to counting vertices satisfying S_a, C_a, etc., are easily expressible with them in terms of types. Then, having constructed an RC graph with the desired properties, we want to try to construct a pumping that, in some sense, corresponds to it. Moreover, when going from graphs to lattices, we might be loose with how we define RC graphs: After all, the justification of their usefulness comes from our ability or inability to construct the corresponding pumping.

To illustrate this, in Figure 5.9 below, we show an "RC graph", which serves as a template for L and Σ in Lemma 21.

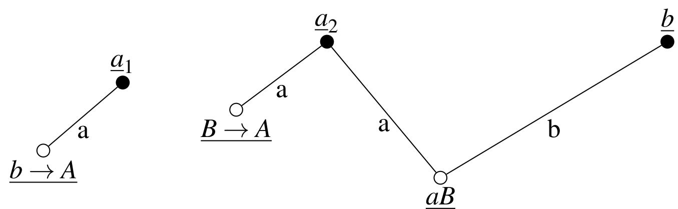


Figure 5.9: RC graph template for L in Lemma 21.

Let us note several things here. First, the graph in Figure 5.9 is not really RC: It obviously lacks some edges from white vertices, an order, and, on top of it, there is no $\underline{ab}$ element, so a closure structure cannot be defined. But, as noted, we are happy with whatever relaxations we need, as long as it helps with coming up with a right lattice to pump.

Second, the graph in Figure 5.9 corresponds to L and Σ in Figure 5.8 in the following sense: All six vertices of the graph are in one-to-one correspondence with the white elements in L which are in either $S_{\mathscr{E}}$ or in $C_{\mathscr{E}}$. Moreover, this correspondense preserves the types of the vertices, and the edges of the graph, which become joins with x_a and x_b in L^{Σ}. For example, the element $\underline{b \to A}$ corresponds to the white vertex u indicated with C_{a-b}. Here the fact that, in the type, a is capital and b is not, precisely corresponds to the fact that $u \in C_a$ and $u \notin C_b$. Moreover, it can be noticed that $u \vee x_b \geq x_a$, which corresponds to the $b \to a$ arrow in the type. Finally, $u \vee x_a = v$, where $v \in L$ is indicated with S_{a-b}, and corresponds to $\underline{a}_1$ in the graph.

Finally, all elements of L without a counterpart in the graph are either black, which means that, asymptotically, their contribution to the property of interest is negligible, or they do not lie in either $S_{\mathscr{E}}$ or $C_{\mathscr{E}}$, and thus they are unimportant for this property. We can also notice that the missing edges of the graph correspond to joins that end up in black vertices of L.

Although it is possible to give a formal definition of an RC graph template, and what does it mean for a pumping to correspond to it, we will not do it. This definition is going to be bulky and very much ad hoc, and will tell nothing more than that the elements of the template should bijectively correspond to the elements of the lattice L with maximal codimention with respect to the mapping K.

5.4 Obstructions to RC-pumping in dealing with SSP=RC conjecture

As we saw in the example in Figure 5.8, with respect to the ability of RC-pumpings to address SSP-like properties of RC lattices in general, and the SSP=RC conjecture in particular, we are interested in the (relative) sizes of q-bunches, arising under these constructions. It is easy to see that, for a given $q \in L$, this size is $N^{\operatorname{codim}(V)} = N^{\dim(V)-\dim(Q)}$ for L^K, and $N^{|E_q|}$ for L^{Σ}. This motivates the following definition.

Definition 4. For a finite lattice L, we say that a function $\phi\colon L \to \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of nonnegative rationals, is *realizable*, if there is an integer $M > 0$, a vector space V over $\mathscr{F}_N$, and a function $K\colon L \to \mathsf{Lin}\,V$ satisfying (V0)–(V5), that is, if K is an RC-pumping, such that $\phi(x) = \operatorname{codim}(K_x)/M$, for all $x \in L$.

For realizable ϕ, the corresponding K is called its *realization*.

Lemma 22. *Let L be a finite lattice, $\mathscr{E} = \{(x_i, y_i) \mid i = 1, \ldots, k\}$ a system in L, $\phi\colon L \to \mathbb{Q}^+$ a K-realizable function, and $K\colon L \to \mathsf{Lin}\,V$ its realization. Moreover, suppose K can be chosen such that $X_i \geq Y_i$, for all $(x_i, y_i) \in \mathscr{E}$, thus satisfying the conditions of Lemma 19.*

Then

$$\left|C(\mathscr{E}^K)\right| = \sum_{x \in C(E)} \overline{N}^{\phi(x)}, \text{ and}$$

$$\left|S(\mathscr{E}^K)\right| = \sum_{x \in S(E)} \overline{N}^{\phi(x)},$$

where $\overline{N} = N^M$, for N, M from Definition 4 of a K-realizable function.

To use Lemma 22 effectively we want to ensure that if $\phi\colon L \to \mathbb{Q}^+$ is realizable, then it is realizable for $\overline{N}$ "large enough". In case of Σ-realizability, it is trivial, as, by definition, Σ does not depend on the choice of N. For K-realizability, however, we need to provide an additional argument.

Lemma 23. *Let L, $\mathscr{E}$, and K be as in Lemma 22. Then, for any integer $m > 0$, $K^m = K\colon L \to V^m$ is a realization. Moreover, $\dim K^m(x) = m \cdot \dim K(x)$, and hence*

$$\left|C(\mathscr{E}^{K^m})\right| = \sum_{x \in C(E)} \left(\overline{N}^m\right)^{\phi(x)}, \text{ and}$$

$$\left|S(\mathscr{E}^{K^m})\right| = \sum_{x \in S(E)} \left(\overline{N}^m\right)^{\phi(x)}.$$

This motivates the following definition:

Definition 5. For L, $\mathscr{E}$ and ϕ as in Lemma 22, we define the polynomials

$$C_{L,\mathscr{E},\phi}(t) = \sum_{x\in C(\mathscr{E})} t^{\phi(x)}, \text{ and}$$
$$S_{L,\mathscr{E},\phi}(t) = \sum_{x\in S(\mathscr{E})} t^{\phi(x)}.$$

Corollary 13. *If there are L, $\mathscr{E}$, and ϕ as in Lemma 22, such that $C_{L,\mathscr{E},\phi}(x)$ is asymptotically smaller than $S_{L,\mathscr{E},\phi}(x)$, then SSP≠RC.*

As we mentioned in the introduction, the lattices with nonvanishing Möbius function (NMF) satisfy SSP. Now we will elaborate upon this result to show that RC-pumping of NMF lattices cannot help with proving SSP≠RC. The following lemma is from [22, Lemma 5.2]:

Lemma 24 (P.Hall)**.** *For $a,b\in L$, $\mu_L(a,b)=\sum(-1)^{l(C)}$, where the sum is over all chains C in L with minimal element a and maximal element b.*

Here $l(C)$ is a length of C and μ_L is a Möbius function of L. We can use Lemma 24 to calculate Möbius function of L^K.

Lemma 25. *For L, ϕ, and K as in Lemma 22, let $a,b\in L$ and $u,v\in V$ be such that $a\le b$ and $(a,u+A)\le(b,v+B)$. Then*

$$\mu_{L^K}\big((a,u+A),(b,v+B)\big)=\sum(-1)^{l(C)}\cdot N^{P_C}=\sum(-1)^{l(C)}\cdot\overline{N}^{P_C/M},$$

where

$$\begin{aligned}P_C &= \sum_{i=0,\dots,l}\dim K_i-\sum_{i=0,\dots,l-1}\dim K_i\wedge K_{i+1}-\dim A\vee B\\ &= \sum_{i=1,\dots,l-1}\dim K_i-\sum_{i=0,\dots,l-1}\dim K_i\wedge K_{i+1}+\dim A\wedge B\end{aligned}$$

and the sum is over all chains $C=(a=k_0,\dots,k_l=b)$ in L. In particular, $\mu_{L^K}\big((a,u+A),(b,v+B)\big)$ is a polynomial with rational powers in $\overline{N}$.

Proof. For the purpose of the proof, we denote $\alpha=(a,u+A)$ and $\beta=(b,v+B)$, $\alpha,\beta\in L^K$.

The second expression follows from the first one by noticing that $\dim K_0+\dim K_l=\dim A+\dim B=\dim A\vee B+\dim A\wedge B$. Thus, we only need to prove the first one. It will follow from Lemma 25 once we prove that $D_C=N^{P_C}$, where D_C is the number of chains of the form $\alpha=\theta_1,\dots,\theta_l=\beta$ in L^K, for $\theta_i\in\kappa^{-1}(k_i)$, that is, such that $\theta_i=(k_i,w_i+K_i)$. The proof is by induction on the length of C. For $|C|=2$, C is the chain $a=$

$k_0, k_1 = b$. Then $P_C = \dim K_0 + \dim K_1 - \dim K_0 \wedge K_1 - \dim A \vee B = \dim A + \dim B - \dim A \wedge B - \dim A \vee B = 0$. Thus, $N^{P_C} = 1 = D_C$, as there is only one such chain in L^K, namely α, β.

Now, suppose $D_C = N^{P_C}$, for all C such that $|C| = l$, that is, for C of the form $a = k_0, \ldots, k_l = b$. We can additionally assume that it holds for chains between arbitrary elements, not just between a and b. Then, if C^* is $a = k_0, \ldots, k_{l+1} = b$, we have:

$$\log_N D_{C^*} = \log_N \left(\sum_{\substack{\theta_l \in \kappa^{-1}(k_l) \\ \alpha < \theta_l < \beta}} D_{C^*[\alpha, \theta_l]} \right),$$

where $D_{C^*[\alpha,\theta_l]}$ is the number of chains from α to θ_l of the form $\alpha = \theta_1', \ldots, \theta_l' = \theta_l$, for $\theta_i' \in \kappa^{-1}(k_i)$. The induction hypothesis is applicable to $D_{C^*[\alpha,\theta_l]}$, so

$$\begin{aligned} \log_N D_{C^*} = \sum_{i=0,\ldots,l} \dim K_i - \sum_{i=0,\ldots,l-1} \dim K_i \wedge K_{i+1} - \dim A \vee K_l \\ + \log_N \Big| \{\theta_l \in \kappa^{-1}(k_l) \mid \alpha < \theta_l < \beta\} \Big|. \end{aligned}$$

Now,

$$\begin{aligned} &\log_N \Big| \{\theta_l \in \kappa^{-1}(k_l) \mid \alpha < \theta_l < \beta\} \Big| \\ &\qquad = \log_N \big| \{w \in V \mid u - w \in A \vee K_l, v - w \in B \vee K_l\} \big| - \dim K_l \\ &\qquad = \log_N \big| (u + A \vee K_l) \cap (v + B \vee K_l) \big| - \dim K_l. \end{aligned}$$

As $(a,u+A) \le (b,v+B)$, $u-v \in A \vee B$, that is, $u+u_a = v+v_b$ for $u_a \in A$ and $v_b \in B$. Then

$$\begin{aligned}
&\log_N \big|(u+A\vee K_l)\cap(v+B\vee K_l)\big| - \dim K_l \\
&\quad= \log_N \big|(u+u_a+A\vee K_l)\cap(v+v_b+B\vee K_l)\big| - \dim K_l \\
&\quad= \log_N \big|(A\vee K_l)\wedge(B\vee K_l)\big| - \dim K_l \\
&\quad= \dim(A\vee K_l)\wedge(B\vee K_l) - \dim K_l = \big[\text{as } K_l \le A\vee B \text{ by (V2)}\big] \\
&\quad= \dim A\vee K_l + \dim B\vee K_l - \dim A\vee B - \dim K_l \\
&\quad= \big[\text{as } \dim B\vee K_l - \dim K_l = \dim B - \dim B\wedge K_l\big] \\
&\quad= \dim B - \dim B\wedge K_l + \dim A\vee K_l - \dim A\vee B \\
&\quad= \dim K_{l+1} - \dim K_l\wedge K_{l+1} + \dim A\vee K_l - \dim A\vee B.
\end{aligned}$$

Combining, we get,

$$\begin{aligned}
\log_N D_{C^*} &= \sum_{i=0,\dots,l} \dim K_i - \sum_{i=0,\dots,l-1} \dim K_i\wedge K_{i+1} - \dim A\vee K_l \\
&\quad + \dim K_{l+1} - \dim K_l\wedge K_{l+1} + \dim A\vee K_l - \dim A\vee B \\
&= \sum_{i=0,\dots,l+1} \dim K_i - \sum_{i=0,\dots,l} \dim K_i\wedge K_{i+1} - \dim A\vee B,
\end{aligned}$$

as needed. □

Definition 6. For L and K as in Lemma 22, we define the *Möbius polynomial* $\mathscr{M}_{L,K}\colon \{(x,y) \mid x,y\in L, x\le y\} \to \mathsf{Poly}(t)$ as

$$\mathscr{M}_{L,K}(a,b) = \sum_C (-1)^{l(C)} \cdot t^{P_C},$$

where C and P_C are as in Lemma 25.

Corollary 14. *Let L and ϕ be as in Lemma 22. Then, if there is a realization K of ϕ such that $\mathscr{M}_{L,K}$ is nonvanishing, then $C_{L,\mathscr{E},\phi}(t)$ is not asymptotically smaller than $S_{L,\mathscr{E},\phi}(t)$, for any system $\mathscr{E}$ in L.*

In particular, if L is NMF, then $C_{L,\mathscr{E},\phi}(t)$ is not asymptotically smaller than $S_{L,\mathscr{E},\phi}(t)$.

Here $\mathscr{M}_{L,K}$ is nonvanishing means that $\mathscr{M}_{L,K}(a,b) \ne 0$, as a polynomial, for all $a \le b \in L$.

Proof. Indeed, if $C_{L,\mathscr{E},\phi}(t) \le S_{L,\mathscr{E},\phi}(t)$, for all $t_0 \ge T$, then, as NMF implies SSP, for every $t \ge T$ there is $a \le b \in L$ such that $\mathscr{M}_{L,K}(a,b)(t_0) = 0$. But then there are a,b such that $\mathscr{M}_{L,K}(a,b)(t_0)$ is zero for infinitely many values of t. As $\mathscr{M}_{L,K}(a,b)(t_0)$ is a polynomial, it implies $\mathscr{M}_{L,K}(a,b)(t_0) = 0$. In particular, by Lemma 24,

$$\mu_L(a,b) = \sum_C (-1)^{I(C)} = \sum_C (-1)^{I(C)} \cdot 1^{P_C} = \mathscr{M}_{L,K}(a,b)(1) = 0.$$

So, L is NMF. $\square$

Let us now pay a closer attention to realizability. For this purpose, we are going to introduce several relaxations of this notion. For a finite lattice L, and $\phi : L \to \mathbb{Q}^+$, we say that

- ϕ is $K^{(1)}$*-realizable*, if $\phi(x) = \operatorname{codim}(K_x)/M$, for $K : L \to \mathsf{Lin}\, V$ satisfying (V0), (V1), and (V2);
- ϕ is Σ*-realizable*, if $\phi(x) = |E_x|/M$, for $\Sigma : L \to 2^V$ satisfying (W0)–(W4);
- ϕ is $\Sigma^{(1)}$*-realizable*, if $\phi(x) = |E_x|/M$, for $\Sigma : L \to 2^V$ satisfying (W0), (W1), and (W2);
- ϕ is *any-realizable* if it is K, or $K^{(1)}$, or Σ, or $\Sigma^{(1)}$-realizable.

In line with this notation , ϕ is K*-realizable* if and only if it is realizable.

It is easy to see that Σ-realizability implies K-realizability and $\Sigma^{(1)}$-realizability, and any of these two, in turn, imply $K^{(1)}$-realizability. Thus, ϕ is any-realizable if and only if it is $K^{(1)}$-realizable; we will use the former definition when we need to stress the fact that we do not care about a specific form of realizability.

Let us now give several easy corollaries of Theorem 6 that are relevant to the discussion of realizability.

Corollary 15. *If L is a lattice containing elements $s \prec p \prec q \prec r$ such that both $s \prec p \prec q$ and $p \prec q \prec r$ are three-element intervals, then $L^K = L^K(L,V,K,N)$ is not an RC lattice for any V, K and N. For example, this holds if L has an interval, isomorphic to a four-element chain, or to a diamond lattice N_5.*

Corollary 16. *Let L^K be an RC lattice, and let $A, B \subseteq L$ be such that $a \le b$ for all $a \in A$, $b \in B$. Then*

$$K_{\bigvee A} \vee K_{\bigwedge B} = \bigwedge_{a \in A, b \in B} (K_a \vee K_b).$$

Corollary 17. *If L^K is an RC lattice then, for $t \in L$, it holds:*

$$T = \bigwedge_{p \le t \le r} (K_p \vee K_r) = \bigwedge_{\substack{j \in J, j \le t \\ m \in M, m \ge t}} (K_j \vee K_m) \tag{V6}$$

$$= \bigwedge_{j \in \mathscr{J}, m \in \mathscr{M}} (K_j \vee K_m), \textit{ for any } \mathscr{M} \in MMR(t),\ \mathscr{J} \in MJR(t).$$

where $M = M(L)$ and $J = J(L)$ are the sets of meet-irreducible and join-irreducible elements of L respectively, and $MMR(t) \subseteq 2^M$ and $MJR(t) \subseteq 2^J$ are the sets of minimal meet representations and of minimal join representations of t.

As it turns out, a necessary condition to any-realizability is *submodularity*.

Definition 7. We say that a function $\zeta : L \to Q^+$ is *submodular* if:

1. $\zeta(e) = \zeta(z) = 0$, where e and z are the top and bottom elements of L;

2. $\zeta(x) + \zeta(y) \geq \zeta(x \vee y) + \zeta(x \wedge y)$.

Note. Typically, submodular functions are defined as functions from 2^S, that is, from a Boolean lattice, to $\mathbb{R}$, satisfying (2). Here, we bend this definition to fit our needs.

Theorem 7. *For a finite lattice L, if a function $\phi : L \to \mathbb{Q}^+$ is any-realizable, then it is submodular.*

Proof. As was said before, we need to argue for ϕ being $K^{(1)}$-realizable. Let us fix M, V, K, and $\phi = \phi_{M,V,K}$ as in Definition 4, and let $x, y \in L$. By (V1), $X \wedge Y \leq K_{x\vee y}, K_{x\wedge y}$. Also, by (V2), $X, Y \leq K_{x\vee y} \vee K_{x\wedge y}$. Hence

$$\begin{aligned}
\dim(X) + \dim(Y) &= \big[\text{by modular law for subspace lattices}\big] \\
&= \dim X \vee Y + \dim X \wedge Y \\
&\leq \dim K_{x\vee y} \vee K_{x\wedge y} + \dim K_{x\vee y} \wedge K_{x\wedge y} \\
&= \dim(K_{x\vee y}) + \dim(K_{x\wedge y}).
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(x) + \phi(y) &= (\dim(V) - \dim(X) + \dim(V) - \dim(Y))/M \\
&\geq (\dim(V) - \dim(K_{x\vee y}) + \dim(V) - \dim(K_{x\wedge y}))/M \\
&= \phi(x \vee y) + \phi(x \wedge y).
\end{aligned}$$

□

The entailment between submodularity and different realizabilities is shown in Figure 5.10 below.

Proposition 6. *Submodular functions on L are closed under $\mathbb{Q}^+$-weighted sums, that is, if $\zeta_1, \ldots, \zeta_k : L \to \mathbb{Q}^+$ are submodular and $q_1, \ldots, q_k \in \mathbb{Q}^+$, then $q_1\zeta_1 + \cdots + q_k\zeta_k : L \to \mathbb{Q}^+$ is submodular. In particular, the constant zero function is submodular.*

$$
\begin{tikzcd}
 & \Sigma^{(1)} \arrow[dr] & & \\
\Sigma \arrow[ur] \arrow[dr] & & K^{(1)} \arrow[rr] & & \text{submodular} \\
 & K \arrow[ur] & &
\end{tikzcd}
$$

Figure 5.10: Entailment of realizabilities.

Proposition 7. *If $\zeta : L \to \mathbb{Q}^+$ is submodular, then its zeros form a sublattice.*

Proof. Let $x, y \in L$ be such that $\zeta(x) = \zeta(y) = 0$. Then $0 \le \zeta(x \vee y) + \zeta(x \wedge y) \le \zeta(x) + \zeta(y) = 0$. Hence, $\zeta(x \vee y) = \zeta(x \wedge y) = 0$. $\square$

Corollary 18. *Let $\zeta : L \to \mathbb{Q}^+$ be submodular. Then for every $x \in L$ there is a unique minimal $x^u \ge x$ and a unique maximal $x^l \le x$ such that $\zeta(x^u) = \zeta(x^l) = 0$.*

Corollary 19. *Let $\zeta : L \to \mathbb{Q}^+$ be submodular. If $\zeta(x) \ne 0$, then there is $m \in M(L)$, $m \ge x$, and $j \in J(L)$, $j \le x$, such that $\zeta(m) > 0$ and $\zeta(j) > 0$.*

Corollary 20. *Let $\phi : L \to \mathbb{Q}^+$ be any-realizable and let $K : L \to \mathsf{Lin}\,V$ be its realization. Then, for any x, y for which $\phi(x) + \phi(y) = \phi(x \vee y) + \phi(x \wedge y)$, it holds: $X \vee Y = K_{x \vee y} \vee K_{x \wedge y}$ and $X \wedge Y = K_{x \vee y} \wedge K_{x \wedge y}$.*

The examples below show the limitations on realizability of submodular functions, and, to an extent, justifies the introduction of L^K as opposed to an easier construction of L^Σ, as well as the study of $K^{(1)}$-realizability as opposed to K-realizability.

Example 6. Let L and ϕ be as in Figure 5.11. Then ϕ is K-realizable, in particular, it is submodular, however, as we prove in Lemma 26 below, it is not $\Sigma^{(1)}$-realizable.

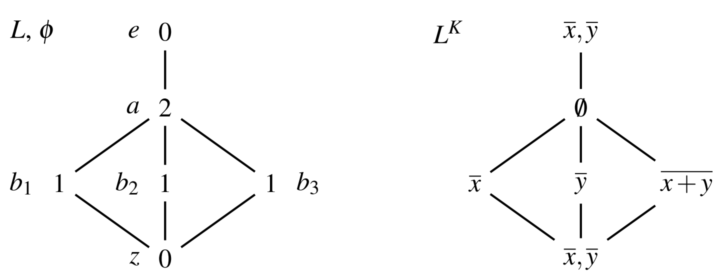


Figure 5.11: Submodular function ϕ that is K-realizable, but not $\Sigma^{(1)}$-realizable.

Lemma 26. *For L, ϕ as in Figure 5.11, ϕ is not $\Sigma^{(1)}$-realizable.*

Proof. Given Σ, we consider $K = K^{\Sigma}$ as in the note to Definition 2, and use the fact that $K[L]$ is a distributive sublattice in $\mathsf{Lin}\, V$. As $\phi(z) = 0$, $Z = V$, and, by Corollary 20, $A = A \wedge Z = B_i \wedge B_j$ and $V = Z = A \vee Z = B_i \vee B_j$, for $i \neq j$. But then

$$\begin{aligned} Z &= (B_1 \vee B_2) \wedge (B_1 \vee B_3) \wedge (B_2 \vee B_3) \\ &= (B_1 \wedge B_2) \vee (B_1 \wedge B_3) \vee (B_2 \wedge B_3) = A, \end{aligned}$$

a contradiction. □

Example 7. Let L and ϕ be as in Figure 5.12. It can be checked that ϕ is submodular, however, it is not $K^{(1)}$-realizable, as we prove in Lemma 27 below.

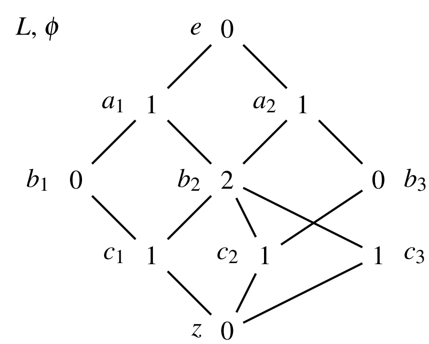


Figure 5.12: Submodular function ϕ that is not $K^{(1)}$-realizable.

Lemma 27. *For L, ϕ as in Figure 5.12, ϕ is not $K^{(1)}$-realizable.*

Proof. Note that, as $\phi(b_1) = \phi(b_3) = \phi(e) = \phi(z) = 0$, it follows that $B_1 = B_3 = E = Z = V$. Also, by Corollary 20, $C_3 = B_1 \wedge C_3 = A_1 \wedge Z = A_1$. And, similarly, $C_2 = A_1 = C_3 = A_2 = C_1$. Also, $C_1 = C_1 \wedge C_2 = B_2 \wedge Z = B_2$. But that is impossible as $\phi(b_2) \neq \phi(c_1)$. □

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