FREE ENTROPY THEORY AND RIGID VON NEUMANN ALGEBRAS

By

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I dedicate this dissertation to the memory of a giant who walked among humans, Sundaram Balachander (1927-1990), who has opened my senses to unseen heights of artistic appreciation.

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TABLE OF CONTENTS

CHAPTER 1

Introduction

The content of this thesis is a combination of two papers written by the author with Ben Hayes and David Jekel:

1. Vanishing first cohomology and strong 1-boundedness for von Neumann algebras (B. Hayes, D. Jekel and S. Kunnawalkam Elayavalli). Preprint. October 2021.

2. Property (T) and strong 1-boundedness for von Neumann algebras (with B. Hayes, D. Jekel and S. Kunnawalkam Elayavalli). Preprint. July 2021.

1.0.1 Main results

A tracial von Neumann algebra is a pair (M,τ) of a finite von Neumann algebra and a faithful normal tracial state. Tracial von Neumann algebras have long been seen as a non-commutative analog of a probability spaces. Inspired by Bolzmann's formulation of entropy, Voiculescu [Voi94, Voi96] introduced the microstates free entropy and microstates free entropy dimension $\delta_0(x)$ associated to a tuple $x=(x_1,\cdots,x_k)$ of self-adjoint elements in a tracial von Neumann algebra, which measure the quantity of matricial approximations that x has. Microstates free entropy dimension was used by Voiculescu [Voi96] and Ge [Ge98] respectively to show that (interpolated) free group factors have no Cartan subalgebras and are prime, and since then, free entropy techniques have had many other applications to the structural properties of interpolated free group factors.

However, one limitation of microstates free entropy dimension is that $\delta_0(x)$ is not known to be an invariant of the tracial von Neumann algebra generated by x. Jung [Jun07b] offered a remedy by defining strong 1-boundedness of a finite tuple x in a tracial von Neumann algebra (M,τ). Strong 1-boundedness of x implies that is x has free entropy dimension at most 1, but unlike the case with free entropy dimension, we know that if two tuples x and y generate the same von Neumann algebra, then strong 1-boundedness of x is equivalent to strong 1-boundedness of y [Jun07b, Theorem 3.2], hence strong 1-boundedness is an invariant of a finitely generated tracial von Neumann algebra. Implicit in Jung's work, and explicitly given by the first named author in [Hay18], is the notion of the 1-bounded entropy $h(x)$ of a tuple x; strong 1-boundedness of x is equivalent to $h(x)<\infty$. Moreover, $h(x)$ only depends on the tracial von Neumann algebra generated by x, and thus $h(M)$ is well-defined for a finitely generated tracial von Neumann algebra (M,τ) [Hay18, Theorem A.9] (these definitions and results also extend to infinite tuples, see [Hay18, Definition A.2]).

The 1-bounded entropy behaves nicely under many natural operations with von Neumann algebras, which leads to free entropy proofs of various indecomposability results for free product von Neumann algebras (and other similar algebras), see Section 2.0.3. For example, h has the following subadditivity property: If P and Q are von Neumann subalgebras of (M,τ) and $P\cap Q$ is diffuse, then $h(P\vee Q)\leq h(P)+h(Q)$. Hence, a non-strongly 1-bounded von Neumann algebra M can never be generated by two strongly 1-bounded subalgebras P and Q that intersect diffusely [Jun07b, Corollary 4.2] [Hay18, Lemma A.12]. Hence, each time we find a new class of von Neumann algebras that is strongly 1-bounded or not strongly 1-bounded, we expand the applications of free entropy dimension theory. For instance, strongly 1-bounded von Neumann algebras include those that are amenable, have property Gamma, have a Cartan subalgebra, or even have a quasi-regular hyperfinite subalgebra. The main known examples of non-strongly-1-bounded von Neumann algebras are free products (or free with amalgamation over a totally atomic subalgebra) of Connes-embeddable tracial von Neumann algebras, including in particular the interpolated free group factors, which are a very important but not well understood family of von Neumann algebras. Because of the permanence properties of 1-bounded entropy given in Section 2.0.3, 1-bounded entropy can be used to prove structural results about free products, some of which are still not accessible by other methods; see for instance [Jun07b, Corollary 4.2], [Hay18, Theorem 1.3, Corollary 1.7, Theorem 1.8, Corollary 1.10].

While being strongly 1-bounded guarantees having microstates free entropy dimension at most 1 with respect to every generating set (we say that M has microstate free entropy dimension at most 1 if this holds), there is a significant difference between these two concepts. Having microstates free entropy dimension at most 1 is not known to satisfy the many permanence properties (closure under joins with diffuse intersection, normalizers, quasi-normalizers etc) that being strongly 1-bounded enjoys; see Section 2.0.3. For example, it is unknown if $L(\mathbb{F}_2)$ can be generated by two algebras with microstates free entropy dimension at most 1 with diffuse intersection (and similarly with other indecomposability results for free group factors give in [Jun07b, Hay18]). Thus being strongly 1-bounded is a serious improvement over having microstates free entropy dimension at most 1.

This thesis undertakes a comprehensive study of the relationship between strong 1-boundedness and Property (T). Property (T) is a strong rigidity property introduced first in the group context by Kazhdan [Kaž67]. It has numerous applications to geometric and measured group theory [Mar79, Gab10, Fur99a, Pop86], ergodic theory [CW80, Sch80, Sch81, GW97, Pop07, Pop06b, Pop06c, Fur99b, Ioa11a], probability [LS99, Gab05], and the existence of expander graphs [Mar73].

See [BdlHV08] for an extensive background on Property (T) groups. A generalization to von Neumann algebras was first defined by Connes [Con82, Con80] and further developed by Connes-Jones [CJ85], Popa [Pop06a], Popa-Peterson [PP05], and Peterson [Pet09a]. It has been used to great effect in the theory of von Neumann algebras, especially in Popa's deformation/rigidity theory, and is a crucial component of various general results about the structure of II_1-factors [Pop86, Oza04b, Pop06a] many of which parallel or extend the results obtained from free entropy dimension theory; see 1.0.2 for further discussion.

Several previous authors gave estimates on microstates free entropy dimension for Property (T) groups and von Neumann algebras. First, Voiculescu [Voi99] showed that the standard generators of $SL_3(\mathbb{Z})$ have free entropy dimension 1, by using the sequential commutation property. Ge and Shen in [GS02] generalized this result to all generators of $SL_3(\mathbb{Z})$, thus establishing that free entropy dimension is a von Neumann algebra invariant in this case. The next breakthrough by Jung and Shlyakhtenko [JS07] showed that a finitely generated Property (T) von Neumann algebra has free entropy dimension at most 1. Jung [Juna], and Shlyakhtenko [Shl21] showed that finitely presented sofic groups with vanishing first ℓ^2 Betti number are strongly 1-bounded, which includes finitely presented sofic Property (T) groups by the Delorme-Guichardet theorem [Del77, Gui72] (see also [BdlHV08, Section 2.12]). We make an initial observation in this paper that in fact, strong 1-boundedness of all sofic Property (T) groups follows by combining Shlyakhtenko's argument with Shalom's theorem that every Property (T) group is the quotient of a finitely presented one [Sha00]; see §5.0.3 for details.

Our main result is a complete generalization of all the above mentioned previous results. We show strong 1-boundedness assuming only that (M,τ) has a finite Kazhdan tuple, that is, some $x=(x_1,\dots,x_d)\in M^d$ such that for some constant $\gamma>0$, for every Hilbert M-M-bimodule H, we have

$$\|\xi - P_{\mathrm{central}}\xi\| \leq \frac{1}{\gamma}\left(\sum_{j=1}^{d}\|x_j\xi-\xi x_j\|_2^2\right)^{1/2},$$

where $P_{\mathrm{central}}: H\to H$ denotes the projection onto the subspace of central vectors.

Theorem 1.0.1. If (M,τ) is a tracial von Neumann algebra that admits a Kazhdan tuple $x\in M_{\mathrm{sa}}^d$, then M is strongly 1-bounded. In particular, this holds if $M=W^*(\pi(G))$ where G is a Property (T) group and $\pi\colon G\to\mathscr{U}(\mathscr{H})$ is a unitary representation given by a character on G,[1] or if M is a Property (T) tracial von Neumann algebra with finite-dimensional center. This also holds if $M=W^*(\pi(G))$

[1]A character on G is a conjugation-invariant positive-definite function $\chi: G\to\mathbb{C}$. The unitary representations of G corresponding to characters are exactly those which generate a finite von Neumann algebra.

where G is a Property (T) group, and $\pi\colon G \to \mathscr{U}(\mathscr{H})$ is a projective representation [2], provided that $W^*(\pi(G))$ is a finite von Neumann algebra.

As mentioned earlier in the introduction, this result expands the applications of free entropy in the structure theory of finite von Neumann algebras. From the subadditivity property of h, we see the following: A non-strongly 1-bounded von Neumann algebra M can never be generated by a Property (T) subfactor P and another strongly 1-bounded von Neumann subalgebra Q (e.g. amenable, Property Gamma, non-prime, having a diffuse hyperfinite quasi regular subalgebra) such that $P \cap Q$ is diffuse. We remark that in the special case when M is a non trivial free product it is possible to obtain some of these indecomposability results by combining existing techniques in Popa's deformation-rigidity theory. However, the advantage of the 1-bounded entropy approach is in the uniformity of being able to handle all the various cases at once.

It is natural to ask whether Theorem 1.0.1 extends to arbitrary tracial von Neumann algebras with Property (T). We show that this statement is equivalent to all II_1-factors with Property (T) having nonpositive 1-bounded entropy.

Proposition 1.0.2. The following are equivalent:

(i) Every tracial von Neumann algebra with Property (T) is strongly 1-bounded.

(ii) Every tracial von Neumann algebra M with Property (T) satisfies $h(M) \leq 0$.

(iii) Every II_1-factor M with Property (T) satisfies $h(M) \leq 0$.

Moreover, if M is a finite von Neumann algebra with Property (T), then there exists some faithful normal tracial state τ on M such that (M,τ) is strongly 1-bounded.

This problem of whether every II_1 factor with Property (T) satisfies $h(M) \leq 0$ does not seem to accessible by current techniques since we do not even know whether there exists a tracial von Neumann algebras with $0 < h(M) < \infty$. It is thus likely that Theorem 1.0.1 is the optimal result, see Sections 1.0.2, 3.0.3 for a more detailed discussion.

The proof of Proposition 1.0.2 is based on the behavior of 1-bounded entropy under direct sums and matrix amplifications/compressions. In particular, it uses the following formula analogous to Schreier's formula for the rank of subgroups of free groups.

Proposition 1.0.3. Suppose that M is a II_1-factor and τ is its canonical trace. For $t \in (0,\infty)$, let M^t be the amplification of M by t. Then

$$h(M^t) = \frac{1}{t^2} h(M).$$

[2] This means $\pi(g)\pi(h) = c(g,h)\pi(gh)$ for all $g,h \in G$ and some $c\colon G \times G \to S^1$.

This formula should be compared with the compression formula for free group factors due to Voiculescu [Voi90, Theorem 3.3], Dykema [Dyk94], and Radulescu [Răd94], the Connes-Shlyakhtenko formula for L^2-Betti numbers [CS05, Theorem 2.4], the similar results for L^2-Betti numbers and cost of equivalence relations due to Gaboriau [Gab00, Proposition II.1.6], [Gab02, Théorème 5.3], and Jung's results on the behavior of free entropy dimension under compression [Junb]. Moreover, it has the following consequence.

Corollary 1.0.4. Let M be a II_1 factor and let $\mathscr{F}(M) = \{t > 0 : M^t \cong M\}$ be its fundamental group. If $0 < h(M) < \infty$, then $\mathscr{F}(M) = \{1\}$. Hence, if M is a II_1 factor with Property (T), then $h(M) \leq 0$ or $\mathscr{F}(M) = \{1\}$ (possibly both).

Computation of the fundamental group is an important and challenging problem in the theory of II_1 factors. Connes' result [Con80] that a II_1 factor with Property (T) has countable fundamental group is considered the first milestone in the study of rigidity of II_1 factors. Popa achieved two major breakthroughs in this area; he gave the first example of a II_1-factor with trivial fundamental group [Pop06a] and then showed in [Pop06b] that every countable subgroup of the positive real numbers can be realized as the fundamental group of some II_1 factor. Popa conjectured that every II_1 factor with property (T) has trivial fundamental group. The first examples of II_1 factors with Property (T) and trivial fundamental group were obtained only recently in [CDHK] using small cancellation techniques from geometric group theory. As a consequence of Theorem 1.0.1 and Corollary 1.0.4, if there exists a II_1 factor M with Property (T) and $h(M) > 0$, then M must also satisfy $\mathscr{F}(M) = 1$ and hence would be another positive example of Popa's conjecture. But if there does not exist a Property (T) factor M with $h(M) > 0$, then all tracial von Neumann algebras with Property (T) are strongly 1-bounded by Proposition 1.0.2.

Our proof of Theorem 1.0.1 works directly with covering numbers of Voiculescu's matricial microstate space. It draws upon the dimension-reduction technique of Jung and Shlyakhtenko [JS07] as well as the iterative technique that Jung used in his study of groups with vanishing first ℓ^2-Betti number [Juna]. Because the strong 1-boundedness results for Property (T) groups and for groups with vanishing first ℓ^2-Betti number have overlapping ideas and applications, we take this opportunity to present an alternative proof of Shlyakhtenko's result [Shl21] that finitely presented groups with vanishing first ℓ^2-Betti number are strongly 1-bounded. Shlyakhtenko's result generalized Jung's earlier work [Juna] but with a somewhat different proof using non-microstates free entropy rather than microstates free entropy. We give a purely microstates proof that streamlines Jung's original ideas and clarifies the essential ingredients and limitations of this approach; see §1.0.3 for further discussion. The statement of the theorem is as follows; here $L(G)$ denotes the von Neumann algebra generated by the left regular representation of a discrete group G.

Theorem 1.0.5 ([Juna], [Shl21]). If G is a sofic finitely presented group with vanishing first ℓ^2-Betti number, then $L(G)$ is strongly 1-bounded.

1.0.2 Discussion of Theorem 1.0.1

Part of our motivation for Theorem 1.0.1 was to expand on the connections between free entropy dimension results and Popa's deformation/rigidity theory, in which Property (T) plays a central role [Pop06a, Pop07, Pop06b, Pop06c, Ioa11b, Ioa11a]. There has been a dynamic interchange of ideas and results between deformation/rigidity theory and free probability. For example, Voiculescu's theorem on absence of Cartan subalgebras for free group factors [Voi96], and Ge's theorem on primeness of free group factors [Ge98] are paralleled by Ozawa's solidity theorem [Oza04a] and Ozawa and Popa's results [OP10a, OP10b] on strong solidity of free group factors, as well as uniqueness of Cartan subalgebras for the group measure space construction of profinite actions of free groups (see [PV14a, PV14b] for the optimal results in this direction). Additionally, Jung's work on strongly 1-bounded algebras [Jun07b] motivates Peterson's result on primeness for von Neumann algebras of groups with positive first ℓ^2-Betti number [Pet09b]. On the other hand, the numerous works in deformation/rigidity theory on free group factors or related algebras, particularly about the structure of normalizers of subalgebras, such as in [OP10a, HS11, Pet09b] provided inspiration for the malnormality results present in [Hay18]. While free probability cannot currently be used to deduce uniqueness of Cartan results, or prove theorems about general hyperbolic groups or groups not Connes-embeddable, some malnormality results such as those in [Hay18] cannot as of yet be shown using deformation/rigidity theory.

One can draw a more precise parallel between the role of Property (T) in deformation/rigidity theory and the role of amenability in free entropy dimension theory as follows. Property (T) von Neumann algebras are a canonical class to work with in deformation/rigidity because they are characterized by rigidity–a von Neumann algebra has Property (T) if and only if it is rigid with respect to every deformation inside a larger algebra [Pop06a, Proposition 4.1]. Meanwhile, amenable tracial von Neumann algebras are a canonical class to work with in free entropy dimension theory because by [Con76, Jun07a] a separable Connes-embeddable tracial von Neumann algebra is amenable if and only if all embeddings into an ultraproduct of matrix algebras are unitarily conjugate (essentially all microstates are approximately unitarily conjugate), which implies that amenable algebras have 1-bounded entropy zero. Hence, whereas Property (T) algebras are those which are automatically rigid, amenable algebras are those which are automatically trivial in terms of matricial microstates. In this regard, compare the roles of Property (T) in [dSHHS, Theorem 1.5] and amenability in [Hay18, Corollary 1.6].

However, there is also a natural connection between Property (T) and matricial microstates. While amenability implies that the microstate spaces are trivial up to approximate unitary conjugation, Property (T) implies that they are discrete up to approximate unitary conjugation and removal of a small corner, as Jung and Shlyakhtenko realized in [JS07]. To make a direct connection between Property (T) and microstate spaces, one turns a sequence of microstates into an ultraproduct embedding and uses Property (T) to show that two embeddings that are close on a generating set have large corners that are unitarily conjugate (see Lemma 3.0.1); the argument is typical example of Popa's intertwining by bimodules technique, an important tool in deformation/rigidity theory. However, applying this estimate naively only results in bounding the free entropy dimension by 1, and so the problem of strong 1-boundedness for general Property (T) factors and group von Neumann algebras posed in [JS07, Remark 2.4] remained open. Our proof uses the Kazhdan tuple to get more explicit control over the η-covering numbers of microstate spaces in terms of ε-covering numbers for $\eta \leq \varepsilon$, and then rather than immediately taking $\eta \to 0$, we iteratively estimate the covering numbers for smaller and smaller ε as in [Juna].

Another possible route to deduce strong 1-boundedness from Property (T) would be to go through the arguments with ℓ^2-Betti numbers as in [Shl21]; we discuss this approach and its limitations in the next section. We also remark that Theorem 1.0.1 implies Jung-Shlyakhtenko's result that any finitely generated Property (T) algebra has microstates free entropy dimension at most 1 (see Section 3.0.4).

Another natural question is whether Theorem 1.0.1 generalizes to all Property (T) von Neumann algebras, rather than only those with finite-dimensional center. Of course, if a von Neumann algebra has diffuse center, then its 1-bounded entropy is automatically less than or equal to zero. Thus, the remaining case is a Property (T) von Neumann algebra that is a countable direct sum of factors. By [Pop06a, Proposition 4.7], a direct sum of tracial von Neumann algebras has Property (T) if and only if each direct summand has Property (T). We lower-bound the 1-bounded entropy in terms of the 1-bounded entropy of the direct summands. Using this, we can prove Proposition 1.0.2. Hence, strong 1-boundedness of general Property (T) von Neumann algebras is as difficult as showing 1-bounded entropy less than or equal to zero for Property (T) factors, which does not seem accessible by current techniques (if it is even true).

1.0.3 Discussion of Theorem 1.0.5

Jung [Juna] and Shlyakhtenko [Shl21] proved Theorem 1.0.5 using the following more free probabilistic result.

Theorem 1.0.6. Let (M,τ) be a tracial W*-algebra generated by some $x \in M_{\mathrm{sa}}^d$. Suppose $\|x\|_\infty < R$. Let $m \in \mathbb{N}\cup\{\infty\}$. Let $f(t_1,\dots,t_d) \in \mathbb{C}\langle t_1,\dots,t_d\rangle^{\oplus m}$ be a tuple of non-commutative polynomials such that $f(x) = 0$. Let

$$D_f(x) = \begin{pmatrix} x_1\otimes 1 - 1\otimes x_1 & \dots & x_d \otimes 1 - 1\otimes x_d \\ \partial_{x_1} f(x) & \dots & \partial_{x_d} f(x) \end{pmatrix} \in M_{m,d}(M\otimes M^{\mathrm{op}}),$$

and let $\mu_{|D_f|}$ be the spectral measure of $|D_f| = (D_f^* D_f)^{1/2}$ with respect to $\tau\otimes\tau$. If

$$\int_0^\infty |\log t|\, d\mu_{|D_f|}(t) < \infty, \tag{1.0.1}$$

then M is strongly 1-bounded.

One deduces Theorem 1.0.5 from Theorem 1.0.6 through the well-known relationship between group cocycles and derivations on the group algebra. One then parameterizes the derivations in terms of their action on a self-adjoint generating set, hence obtains a bijection between derivations and vectors z in the kernel of $\partial f(x)$. Looking at cocycles that are orthogonal to the inner cocycles results in the additional condition of $\sum_{j=1}^d [x_j, z_j] = 0$, or that z is in the kernel of the commutator operator in the first row of the matrix $D_f(x)$. Hence, the first ℓ^2-Betti number is the Murray-von Neumann dimension of the kernel of D_f.

The condition (1.0.1) is needed for the microstates argument to go through. This hypothesis is nontrivial to check in the group case, and this is where one uses the assumption of soficity. The bound (1.0.1) expresses positivity of a certain Fuglede-Kadison determinant, which is known for sofic groups [ES05]. We remark that Shlyakhtenko's results about vanishing L^2-Betti numbers have been generalized to $*$-algebras that are not group algebras [BV18], but this still requires some way of controlling the Fuglede-Kadison determinant. Thus, while Theorem 1.0.5 can be used to show strong 1-boundedness for sofic Property (T) groups, there is little hope of adapting this method to general Property (T) von Neumann algebras (or even general Property (T) groups) without some analog of soficity.

Our proof of Theorem 1.0.6 is longer than Shlyakhtenko's argument but it is more self-contained. Indeed, Shlyakhtenko's argument used the external fact that $\chi \le \chi^*$ and the result about strong 1-

boundedness and non-amenability sets from [Hay18, Proposition A.16]. In this paper, we generalize and streamline Jung's strategy from [Juna], which uses iteration to bound covering numbers for smaller and smaller ε with errors controlled by the integral (1.0.1). Much of the technical challenge in Jung's work had to do with converting between covering numbers with respect to different non-commutative L^p-norms on the von Neumann algebra (and in fact L^p quasinorms for $p \in (0,1)$). Our argument works mostly with L^2 norms but requires conversion between L^1 and L^2 norms at one point, and this is main time we use a significant external ingredient, Szarek's estimates for the covering numbers of Grassmannians [Sza98], which are also used for the proof of the Property (T) case. Another notable feature of the proof is the way in which the condition $\sum_j [x_j, z_j] = 0$ (which corresponded in cohomology to looking at cocycles orthogonal to inner cocycles) arises naturally in the microstate setting by considering the elements in a unitary orbit closest to a given point x.

We also remark that polynomials in Theorem 1.0.6 can be replaced more generally by power series, and even non-commutative trace C^2 functions in the sense of [JLS21]; see Remark 4.0.10.

CHAPTER 2

Preliminaries

2.0.1 Tracial von Neumann algebras and non-commutative laws

A tracial von Neumann algebra is a pair (M,τ) where M is a von Neumann algebra and $\tau\colon M\to\mathbb{C}$ is a faithful, normal, tracial state. We will primarily be interested in cases where M is diffuse, i.e. it has no nonzero minimal projections. An interesting class of tracial von Neumann algebras are the group von Neumann algebras. Given a discrete group G the left regular representation $\lambda\colon G\to\mathscr{U}(\ell^2(G))$ is given by

$$(\lambda(g)\xi)(h)=\xi(g^{-1}h) \text{ for } \xi\in\ell^2(G), g,h\in G.$$

The group von Neumann algebra $L(G)$ is defined to be

$$\overline{\operatorname{span}\{\lambda(g):g\in G\}}^{SOT}.$$

The group von Neumann algebra can be turned into a tracial von Neumann algebra by defining $\tau\colon L(G)\to\mathbb{C}$ by $\tau(x)=\langle x(\delta_1),\delta_1\rangle$. We may also view $\mathbb{M}_n(\mathbb{C})$ as a tracial von Neumann algebra with the tracial state tr_n given by

$$\mathrm{tr}_n(A)=\frac{1}{n}\sum_{i=1}^{n}A_{ii}.$$

The group von Neumann algebra is diffuse if and only if G is infinite. For a von Neumann algebra M, we use M_{sa} for the self-adjoint elements of M, and $\mathscr{U}(M)$ for the unitary elements of M.

Since tracial von Neumann algebras (M,τ) with M abelian correspond precisely to probability spaces, we may think of tracial von Neumann algebras as (a special case of) noncommutative probability spaces. Following this intuition, given a tracial von Neumann algebra (M,τ) and $1\leq p<\infty$, we define the $\|\cdot\|_p$ on M by

$$\|x\|_p=\tau(|x|^p)^{1/p}, \text{ where } |x|=(x^*x)^{1/2}.$$

It can be shown [Dix53] that this is indeed a norm on M. We use the notation $\|x\|_\infty$ for the operator norm. Moreover, the definition of the norms can be extended to tuples by

$$\|(x_1,\dots,x_d)\|_p = \begin{cases} \left(\sum_{j=1}^d \tau(|x_j|^p)\right)^{1/p}, & p \in [1,\infty), \\ \max_{j=1,\dots,d} \|x_j\|, & p = \infty. \end{cases}$$

If (M,τ) is viewed as a non-commutative probability space, then its elements maybe viewed as non-commutative random variables. In fact, a d-tuple $x=(x_1,\dots,x_d) \in M_{\text{sa}}^d$ is the non-commutative analog of an $\mathbb{R}^d$-valued random variable. Although one cannot define the law (or probability distribution) of x as a classical measure, we may define its non-commutative law as a certain linear function on a non-commutative polynomial algebra, just like the probability distribution d-tuple X of bounded classical random variables defines a map $\mathbb{C}[t_1,\dots,t_d] \to \mathbb{C}$ sending p to $\mathbb{E}[p(X)]$

For $d \in \mathbb{N}$, we let $\mathbb{C}\langle t_1,\cdots,t_d\rangle$ be the algebra of noncommutative polynomials in d formal variables t_1, $\dots$, t_d, i.e. the free $\mathbb{C}$-algebra with d-generators. We give $\mathbb{C}\langle t_1,\cdots,t_d\rangle$ the unique $*$-algebra structure which makes the t_j self-adjoint. By universality, if A is any $*$-algebra, and $x=(x_1,\cdots,x_d) \in A^d$ is a self-adjoint tuple, then there is a unique $*$-homomorphism $\mathbb{C}\langle t_1,\cdots,t_d\rangle \to A$ which sends t_j to x_j. For $p \in \mathbb{C}\langle t_1,\cdots,t_d\rangle$ we use $p(x)$ for the image of p under this $*$-homomorphism. Given a tracial von Neumann algebra (M,τ) and $x \in M_{\text{sa}}^d$, we define the law of x, denoted ℓ_x, to be the linear functional $\ell_x \colon \mathbb{C}\langle t_1,\cdots,t_d\rangle \to \mathbb{C}$ given by

$$\ell_x(f) = \tau(f(x)).$$

The non-commutative laws can be characterized as follows.

Proposition 2.0.1 (See [AGZ09, Proposition 5.2.14]). Let $\ell : \mathbb{C}\langle t_1,\dots,t_d\rangle \to \mathbb{C}$ and let $R>0$. The following are equivalent:

(i) There exists a tracial von Neumann algebra (M,τ) and $x \in M_{\text{sa}}^d$ such that $\ell = \ell_x$ and $\|x\|_\infty \le R$.

(ii) ℓ satisfies the following conditions:

- $\ell(1)=1$,
- $\ell(f^*f) \ge 0$ for $f \in \mathbb{C}\langle t_1,\dots,t_d\rangle$,
- $\ell(fg)=\ell(gf)$ for $f,g \in \mathbb{C}\langle t_1,\dots,t_d\rangle$,
- $|\ell(t_{i_1}\dots t_{i_k})| \le R^k$ for all $k \in \mathbb{N}$ and i_1, $\dots$, $i_k \in \{1,\dots,d\}$.

Remark 2.0.2. In fact, the proof of (ii) $\implies$ (i) gives an explicit description of the von Neumann algebra through the Gelfand-Naimark-Segal construction (see [AGZ09, Proposition 5.2.14]). Let $H = L^2(\ell)$ be separation-completion of $\mathbb{C}\langle t_1,\dots,t_d\rangle$ with respect to the semi-inner product $\langle f,g\rangle_\ell = \ell(f^*g)$. One can show that multiplication by t_j gives a well-defined, bounded, self-adjoint operator on $\mathscr{H}$. We take M to be the von Neumann algebra generated by $x_1, \dots, x_d$, and let τ be the state corresponding to the vector 1 in $\mathscr{H}$. In fact, we will denote M by $\mathrm{W}^*(\ell)$ and we denote by π_ℓ the unital $*$-homomorphism $\mathbb{C}\langle t_1,\dots,t_d\rangle$ sending t_j to x_j.

Let $\Sigma_{d,R}$ be the set of all linear maps $\mathbb{C}\langle t_1,\dots,t_d\rangle \to \mathbb{C}$ satisfying the equivalent conditions of Proposition 2.0.1. We equip $\Sigma_{d,R}$ with the weak* topology, that is, the topology of pointwise convergence on $\mathbb{C}\langle t_1,\dots,t_d\rangle$. It is easy to see that $\Sigma_{d,R}$ is compact and metrizable using Proposition 2.0.1 (ii).

2.0.2 Microstate spaces and 1-bounded entropy

Let (M,τ) be a diffuse tracial von Neumann algebra, and $x \in M_{\mathrm{sa}}^d$ for some $d \in \mathbb{N}$ with $W^*(x) = M$. Suppose that $\|x\|_\infty \le R$. Following [Voi94], for each open set $\mathscr{O}$ of $\Sigma_{d,R}$ and $N \in \mathbb{N}$, we define

$$\Gamma_R^{(n)}(\mathscr{O}) = \{X \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d : \ell_X \in \mathscr{O}\}.$$

When $\mathscr{O}$ is a neighborhood of ℓ_x, we call $\Gamma_R^{(n)}(\mathscr{O})$ a microstate space for x.

Given $d,n \in \mathbb{N}$, $p \in [1,\infty]$, $\varepsilon > 0$ and $\Omega, \Xi \subseteq \mathbb{M}_n(\mathbb{C})^d$ then Ξ is said to $(\varepsilon, \|\cdot\|_p)$-cover Ω if for every $A \in \Omega$, there is a $B \in \Xi$ with $\|A-B\|_p < \varepsilon$. We define the covering number of $\Omega \subseteq \mathbb{M}_n(\mathbb{C})^d$, denote $K_\varepsilon(\Omega, \|\cdot\|_p)$, to be the minimal cardinality of a set that $(\varepsilon, \|\cdot\|_p)$-covers Ω. We will use covering numbers for different values of p in Sections 3.0.1,4.0.1.

While these covering numbers are natural for many purposes, for unitarily invariant subsets of matrices, it is natural to take the orbital numbers modulo unitary conjugation. Given $n \in \mathbb{N}$, $\varepsilon > 0$ and $\Omega, \Xi \subseteq \mathbb{M}_n(\mathbb{C})^d$ we say that Ξ orbitally $(\varepsilon, \|\cdot\|_2)$-covers Ω if for every $A \in \Omega$, there is a $B \in \Xi$ and an $n \times n$ unitary matrix V so that

$$\|A - VBV^*\|_2 < \varepsilon.$$

We define the orbital covering number $K_\varepsilon^{\mathrm{orb}}(\Omega, \|\cdot\|_2)$ as the minimal cardinality of a set of Ω_0 that orbitally $(\varepsilon, \|\cdot\|_2)$-covers Ω. Since we will usually be concerned with $\|\cdot\|_2$-norms we will frequently drop $\|\cdot\|_2$ from the notation and use $K_\varepsilon^{\mathrm{orb}}(\Omega)$ instead of $K_\varepsilon^{\mathrm{orb}}(\Omega, \|\cdot\|_2)$. Let $R \in [0,\infty)$ be such that $\|x\|_\infty < R$.

The 1-bounded entropy is defined in terms of the exponential growth rate of covering numbers of $\Gamma_R^{(n)}(\mathscr{O})$ (up to unitary conjugation) as $n \to \infty$ for neighborhoods $\mathscr{O}$ of ℓ_x. For a weak*-neighborhood $\mathscr{O}$ Of ℓ_x, we define

$$h_{R,\varepsilon}(\mathscr{O}) := \limsup_{n\to\infty} \frac{1}{n^2} \log K_\varepsilon^{\mathrm{orb}}(\Gamma_R^{(n)}(\mathscr{O})),$$

$$h_{R,\varepsilon}(x) := \inf_{\mathscr{O}\ni \ell_x} h_{R,\varepsilon}(\mathscr{O}),$$

where the infimum is over all weak*-neighborhoods $\mathscr{O}$ of ℓ_x. We then define

$$h_R(x) := \sup_{\varepsilon>0} h_{R,\varepsilon}(x).$$

In [Hay18], it is shown that $h_R(x)$ is independent of R provided that $\|x\|_\infty \leq R$, and hence we may unambiguously denote it by $h(x)$. In fact, $h(x)$ only depends on the von Neumann algebra generated by x. Hence, for every finitely generated tracial von Neumann algebra M, we may define $h(M)$ as $h(x)$ for some generating tuple x. If M is not a factor, then the 1-bounded entropy depends upon the trace we choose on M. We will use $h(M,\tau)$ for a tracial von Neumann algebra (M,τ) if we wish to emphasize the dependence of the 1-bounded entropy on τ. However, we will typically suppress this from the notation and just use $h(M)$ unless there is a possibility of confusion. The definition of h can be extended even to the case where M is not finitely generated; see [Hay18]. However, our results only require a direct appeal to the definition in the finitely generated setting. The importance of 1-bounded entropy to the study of strong 1-boundedness is encapsulated by the following result.

Theorem 2.0.3 (See [Hay18, Proposition A.16]). A tracial von Neumann algebra M is strongly 1-bounded in the sense of Jung [Jun07b] if and only if $h(M) < \infty$.

Because of this result, we will not use Jung's original definition of strongly 1-bounded [Jun07b] and will instead show that algebras are strongly 1-bounded by showing that they have finite 1-bounded entropy.

2.0.3 Properties and applications of 1-bounded entropy

Since we will use general properties of 1-bounded entropy frequently, we list some of the main ones here. To state these we also need the notion of the 1-bounded entropy of N in the presence of M, denoted $h(N:M)$, defined for an inclusion $N \leq M$ of tracial von Neumann algebras. The 1-bounded entropy in the presence is defined by modifying the definition of 1-bounded entropy above to only measure the size of the space of microstates for N which have an extension to microstates for M. We will not need the precise definition, so we refer the reader to [Hay18, Definition A.2] for the definition. We also use the microstates free entropy dimension $\delta_0(x)$ due to Voiculescu [Voi96, Definition 6.1] for a tuple $x \in M_{\mathrm{sa}}^k$ in a tracial von Neumann algebra (M,τ). Here and throughout the paper, $\mathscr{R}$ refers to the hyperfinite II_1-factor. For an inclusion $N \leq M$, we let $\mathscr{N}_M^{wq}(N)$ be the set of $u \in \mathscr{U}(M)$ so that $uNu^* \cap N$ is diffuse. We assume all inclusions list below are trace preserving. We now list some general properties the 1-bounded entropy here.

1. $h(M) = h(M:M)$ for every diffuse tracial von Neumann algebra (M,τ),

2. $h(N:M) \geq 0$ if $N \leq M$ if diffuse and M embeds into an ultrapower of $\mathscr{R}$, and $h(N:M) = -\infty$ if M does not embed into an ultrapower of $\mathscr{R}$. (Exercise from the definitions).

3. $h(N_1:M_1) \leq h(N_2:M_2)$ if $N_1 \leq N_2 \leq M_2 \leq M_1$, if N_1 is diffuse. (Exercise from the definitions).

4. $h(N:M) \leq 0$ if $N \leq M$ and N is diffuse and hyperfinite. (Exercise from the definitions).

5. $h(M) = \infty$ if $M = \mathrm{W}^*(x_1,\cdots,x_n)$ where $x_j \in M_{sa}$ for all $1 \leq j \leq n$ and $\delta_0(x_1,\cdots,x_n) > 1$. For example, this applies if $M = L(\mathbb{F}_n)$, for $n > 1$. (This follows from Theorem 2.0.2 and [Jun07b, Corollary 3.5]).)

6. $h(N_1 \vee N_2 : M) \leq h(N_1:M) + h(N_2:M)$ if $N_1, N_2 \leq M$ and $N_1 \cap N_2$ is diffuse. (See [Hay18, Lemma A.12].)

7. Suppose that $(N_\alpha)_\alpha$ is an increasing chain of diffuse von Neumann subalgebras of a von Neumann algebra M. Then
$$h\left(\bigvee_\alpha N_\alpha : M\right) = \sup_\alpha h(N_\alpha : M).$$
(See [Hay18, Lemma A.10].)

8. $h(N:M) = h(N:M^\omega)$ if $N \leq M$ is diffuse, and ω is a free ultrafilter on an infinite set. (See [Hay18, Proposition 4.5].)

9. $h(\mathrm{W}^*(\mathscr{N}_M^{wq}(N)):M) = h(N:M)$ if $N \leq M$ is diffuse. Here $\mathscr{N}_M^{wq}(N) = \{u \in \mathscr{U}(M) : uNu^* = N \text{ is diffuse}\}$. (see [Hay18, Theorem 3.8 and Proposition 3.2].)

10. Let I be a countable set, and $M = \bigoplus_{i\in I} M_i$ with M_i diffuse for all i. Suppose that τ is a faithful trace on M, and that λ_i is the trace of the identity on M_i. Endow M_i with the trace $\tau_i = \frac{\tau|_{M_i}}{\lambda_i}$. Then
$$h(M,\tau) \leq \sum_i \lambda_i^2 h(M_i,\tau_i).$$

(See [Hay18, Proposition A.13].)

The above axioms imply that all the von Neumann algebras in the following list have nonpositive 1-bounded entropy:

- hyperfinite algebras,
- factors with Property Gamma,
- non-prime von Neumann algebras,
- algebras with diffuse center,
- algebras with diffuse, regular hyperfinite subalgebras (e.g. if the algebra has a Cartan subalgebra),

For proofs, see [HJNS21, Section 1.2]. The class of algebras with nonpositive 1-bounded entropy is also closed under direct sums (by item 10). Additionally, by item 9 if $N \leq M$ is regular (i.e. $W^*(\mathscr{N}_M(N)) = M$), and $h(N : M) \leq 0$, then $h(M) \leq 0$. This applies if $h(N) \leq 0$. In particular, if N is hyperfinite, or has Gamma, or is not prime, or has diffuse center, and if N is regular in M, then $h(M) \leq 0$.

2.0.4 Property (T)

Property (T) for groups, due to Kazhdan [Kaž67], is defined as follows. If $\pi : G \to \mathscr{U}(\mathscr{H})$ is a unitary representation of the group G on a Hilbert space, then a vector $\xi \in \mathscr{H}$ is said to be invariant if $\pi(g)\xi = \xi$ for all $g \in G$. We say that the group G has Property (T) if there exists a finite $F \subseteq G$ and $\delta > 0$ such that for every unitary representation π on $\mathscr{H}$, if there exists a nonzero $\xi \in \mathscr{H}$ such that $\|\pi(g)\xi - \xi\| < \delta$ for all $g \in F$, then $\mathscr{H}$ contains a nonzero invariant vector.

It turns out that if G has Property (T), then there exists a finite set $F \subseteq G$ and some $\gamma > 0$, such that for every unitary representation π of G on a Hilbert space $\mathscr{H}$, we have

$$\|\xi - P_{\text{invariant}}\xi\| \leq \frac{1}{\gamma} \sum_{g \in F} \|\pi(g)\xi - \xi\|,$$

where $P_{\text{invariant}} : H \to H$ is the projection onto the subspace of invariant vectors. Such a set F is called a Kazhdan set and γ is called the Kazhdan constant. For a proof, see [BdlHV08, Definition 1.1.3, Proposition 1.1.9]. It is sometimes notationally convenient to list the elements of F as a tuple $(g_1, \ldots, g_{|F|})$, which we call a Kazhdan tuple.

The von Neumann algebraic analogue of Property (T), introduced by Connes and Jones [CJ85] for II_1 factors and Popa [Pop06a] for general tracial von Neumann algebras, uses Hilbert bimodules over a tracial von Neumann algebra M in place of group representations.

Definition 2.0.4.

(i) If M is a von Neumann algebra, then a Hilbert M-M-bimodule is a Hilbert space equipped with normal left and right actions of M.

(ii) A vector ξ in a Hilbert M-M bimodule $\mathscr{H}$ is central if $x\xi = \xi x$ for all $x \in M$.

(iii) If (M,τ) is a tracial von Neumann algebra, then a vector ξ in an M-M-bimodule M is bitracial if $\langle \xi, x\xi \rangle = \tau(x) = \langle \xi, \xi x \rangle$ for all $x \in M$.

There is analogy between central vectors in a Hilbert M-M-bimodule and invariant vectors in a group representation motivated by the way in which representations of a discrete group G naturally give rise to bimodules over the group von Neumann algebra $L(G)$. Let λ and ρ denote the left and right regular representations of G on $\ell^2(G)$. If π is a representation of G on $\mathscr{H}$, then $\pi \otimes \lambda$ is a left representation on $H \otimes \ell^2(G)$ and $1 \otimes \rho$ is a right representation on $H \otimes \ell^2(G)$. These representations extend to normal left/right actions of $L(G)$, making $H \otimes \ell^2(G)$ into a Hilbert $L(G)$-$L(G)$-bimodule. If ξ is an invariant unit vector in $\mathscr{H}$, then $\xi \otimes \delta_e$ is a central and bitracial vector in $H \otimes \ell^2(G)$. Hence, Property (T) for tracial von Neumann algebras is defined as follows.

Definition 2.0.5 ([CJ85, Pop06a]). A tracial von Neumann algebra (M,τ) has Property (T) if for every $\varepsilon > 0$, there is a finite $F \subseteq M$ and $\delta > 0$, such that for every bitracial vector ξ in a Hilbert M-M-bimodule $\mathscr{H}$, if $\sum_{x\in F} \|x\xi - \xi x\| < \delta$, then there exists a central $\eta \in \mathscr{H}$ with $\|\xi - \eta\| < \varepsilon$.

It turns out that a group G has Property (T) if and only if $L(G)$ has Property (T) [CJ85, Theorem 2]. One can also formulate a von Neumann algebraic version of Kazhdan tuples as follows.

Definition 2.0.6. Let (M,τ) be a tracial von Neumann algebra. We say that $x = (x_1,\dots,x_d) \in M_{\mathrm{sa}}^d$ is a (self-adjoint) Kazhdan tuple if there exists $\gamma > 0$ such that for every Hilbert M-M-bimodule $\mathscr{H}$ and $\xi \in \mathscr{H}$, we have

$$\|\xi - P_{\mathrm{central}}\xi\| \leq \frac{1}{\gamma}\|x\xi - \xi x\|_{H^{\oplus d}},$$

where $x\xi = (x_1\xi,\dots,x_d\xi)$ and $\xi x = (\xi x_1,\dots,\xi x_d)$, and where P_{central} is the projection of H onto the subspace of central vectors. In this case, we call γ the Kazhdan constant associated to the Kazhdan tuple x.

Unlike the group case, every tracial von Neumann algebra with Property (T) need not admit a Kazhdan tuple (the finite set F in the definition of Property (T) may depend a priori on ε). However, it is known for several natural classes of examples.

Lemma 2.0.7.

(i) If M is a Property (T) von Neumann algebra with finite-dimensional center, then M has a Kazhdan tuple.

(ii) If G is a Property (T) group, and $\pi\colon G \to \mathscr{U}(\mathscr{H})$ is a projective unitary representation such that $W^*(\pi(G))$ is finite, then $W^*(\pi(G))$ has a finite Kazhdan tuple.

Proof. (i): Cutting M down by the minimal central projections in M, we see that M is a direct sum of II_1-factors. Each such factor has Property (T) by [Pop06a, Proposition 4.7.2], and thus admits a Kazhdan tuple by [CJ85, Proposition 1]. Combining these Kazhdan tuples gives a Kazhdan tuple for M.

(ii): Choose a Kazhdan set $\{g_1, \cdots, g_k\} \subseteq G$ for G. Let

$$x = (\mathrm{Re}(\pi(g_1)), \mathrm{Re}(\pi(g_2)), \ldots, \mathrm{Re}(\pi(g_k)), \mathrm{Im}(\pi(g_1)), \ldots, \mathrm{Im}(\pi(g_k))) \in M_{\mathrm{sa}}^{2k}.$$

Even though π may not be an honest representation, note that if $\mathscr{H}$ is an M-M bimodule, then we do have a representation of G on $\mathscr{H}$ given by conjugating by $\pi(G)$. Applying [BdlHV08, Proposition 1.1.9] to this representation, we see that x is a Kazhdan tuple for $W^*(\pi(G))$. $\square$

It is a standard exercise to show that a Kazhdan tuple for a group G must generate the group. A well-known result of Popa [Pop86, Theorem 4.4.1] shows that the same is true for Kazhdan tuples associated to tracial von Neumann algebras. For the reader's convenience, we recall the proof here.

Lemma 2.0.8. If (M, τ) is a tracial von Neumann algebra and $x = (x_1, \ldots, x_d)$ is a Kazhdan tuple, then x generates M as a von Neumann algebra.

Proof. Consider the standard (GNS) representation of M on $L^2(M, \tau)$. Let $N = \mathrm{W}^*(x) \subseteq M$. Let $e \in B(L^2(M, \tau))$ be the orthogonal projection onto $L^2(N, \tau|_N) \subseteq L^2(M, \tau)$. The von Neumann algebra $M_1 = \langle M, e \rangle \subseteq B(L^2(M, \tau))$ is called the basic construction for $N \subseteq M$. It is well known that $M \cap \{e\}' = N$ and that M has a semi-finite trace Tr such that $\mathrm{Tr}(aeb) = \tau(E_N(a)E_N(b))$ for $a, b \in M$ by [Jon83]. Since M embeds into M_1, we may regard $L^2(M_1, \mathrm{Tr})$ as an M-M bimodule. The element $e \in L^2(M_1, \mathrm{Tr})$ satisfies $ae = ea$ for $a \in N$ and in particular this holds for $a = x_1, \ldots, x_d$. Since X is a Kazhdan tuple for M, we have $ae = ea$ for all $a \in M$, which implies that $M = N$. $\square$

2.0.5 Ultraproducts of matrix algebras

At several points, we use ultraproducts of matrix algebras, so we recall the relevant background as well as the connection between ultraproducts and microstate spaces.

Definition 2.0.9. Let J be an infinite set. A free ultrafilter is a unital homomorphism

$$\omega\colon \ell^{\infty}(J)/c_0(J) \to \mathbb{C}.$$

For every $(a_n)_{n\in J} \in \ell^{\infty}(J)$ we will use $\lim_{n\to\omega} a_n$ for $\omega((a_n)_{n\in J} + c_0(J))$.

It follows from [Con90, Proposition VIII.1.12] that $\lim_{n\to\omega}$ is a $*$-homomorphism. It thus preserves inequalities, and commutes with complex conjugation, by definition $\lim_{n\to\omega}$ preserves sums and products. By Gelfand-Naimark duality [Con90, Theorem VIII.2.1], free ultrafilters exist in abundance (in fact, there are enough to separate points in $\ell^{\infty}(J)/c_0(J)$). We use $\beta J\setminus J$ for the space of free ultrafilters on J.

Recall that if ω is a free ultrafilter on $\mathbb{N}$, then the tracial ultraproduct of $\mathbb{M}_n(\mathbb{C})$ with respect to ω is given by

$$\prod_{n\to\omega}\mathbb{M}_n(\mathbb{C}) = \frac{\{(x_k)_n \in \prod_n \mathbb{M}_n(\mathbb{C}) : \sup_n \|x_n\|_{\infty} < \infty\}}{\{(x_n)_n \in \prod_n \mathbb{M}_n(\mathbb{C}) : \sup_n \|x_n\|_{\infty} < \infty, \lim_{n\to\omega}\|x_n\|_2 = 0\}}.$$

If $(x_n)_n \in \prod_n \mathbb{M}_n(\mathbb{C})$ and $\sup_n \|x_n\|_{\infty} < \infty$, we let $[x_n]_n$ be the image of $(x_n)_n$ under the natural quotient map

$$\{(x_n)_n \in \prod_n \mathbb{M}_n(\mathbb{C}) : \sup_n \|x_n\|_{\infty} < \infty\} \to \prod_{n\to\omega}\mathbb{M}_n(\mathbb{C}).$$

It can be shown (see [BO08, Lemma A.9]) that this is a tracial von Neumann algebra, with trace given by

$$\tau_{\omega}((x_n)_{n\to\omega}) = \lim_{k\to\omega}\operatorname{tr}_n(x_n).$$

We say that a tracial von Neumann algebra (M,τ) is Connes-embeddable if every von Neumann subalgebra with separable predual admits a trace-preserving embedding into a tracial ultraproduct of matrices.

Suppose that $d\in\mathbb{N}$, and $R>0$. If $(\ell_n)_n$ is a sequence in $\Sigma_{d,R}$, and $\ell\in\Sigma_{d,R}$ we say that $\lim_{n\to\omega}\ell_n = \ell$ if $\lim_{n\to\omega}\ell_n(P) = \ell(P)$ for all $P\in\mathbb{C}\langle t_1,\cdots,t_d\rangle$. Suppose (M,τ) is a tracial von Neumann algebra and $x\in M_{\mathrm{sa}}^d$. If $X^{(n)}\in\mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d$ satisfies $\lim_{n\to\omega}\ell_{X^{(n)}} = \ell_x$, then there is a unique trace-preserving embedding

$\pi\colon M \to \prod_{n\to\omega} \mathbb{M}_n(\mathbb{C})$ which satisfies

$$\pi(x_j) = [X_j^{(n)}]_n \text{ for } j = 1, \cdots, d;$$

(see for instance [GJNS21, Lemma 5.10] for further detail). Because of this, we may heuristically think of the microstate spaces $\Gamma_R^{(n)}(\mathscr{O})$ as $\mathscr{O} \to \ell_x, n \to \infty$ as parameterizing the space of embeddings of M into an ultraproduct of matrices.

See [Jun07a] and [AKE21, Sections 1.2, 1.3] for further discussion of the connections between ultraproducts and microstate spaces. These concepts also relate to random matrix theory; see for instance [Voi94, Voi98, DJS05, BDJ08, HJNS21].

CHAPTER 3

Strong 1-boundedness and Property (T)

3.0.1 Proof of Theorem 1.0.1

As in Theorem 1.0.1, we consider a tracial von Neumann algebra (M,τ) together with a Kazhdan tuple $x=(x_1,\dots,x_d)$. The following lemma is a refinement of [JS07, Theorem 1.1]. It shows roughly that microstates for our Kazhdan tuple are unitarily conjugate except for a piece that is under a projection of small trace. The core idea for the proof of this lemma is the corresponding statement about embeddings into ultraproducts of matrices: if two embeddings of a Property (T) algebra are close on x, then they are unitarily conjugate except on a small corner of M. Further, the size of that corner can be controlled in terms of how close they are on x.

Lemma 3.0.1. Let (M,τ) be a tracial von Neumann algebra and let $x=(x_1,\dots,x_d)\in M_{\mathrm{sa}}^d$ be a Kazhdan tuple with Kazhdan constant γ and suppose $\|x\|_\infty<R$. Then for every $\varepsilon>0$ and $\delta>0$, there exists a neighborhood $\mathscr{O}$ of ℓ_x and $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, if $Y,Z\in\Gamma_R^{(n)}(\mathscr{O})$ with $\|Y-Z\|_2\leq\varepsilon$, then there exists a unitary U in $\mathbb{M}_n(\mathbb{C})$ and a projection P such that

$$\|(U^*YU-Z)(1-P)\|_2<\delta, \qquad \operatorname{tr}_N(P)<\frac{\varepsilon}{\gamma}\left(2+\frac{\varepsilon}{\gamma}\right)+\delta.$$

Proof. Write

$$\Delta(Y,Z,U,P)=\max\left(\|(U^*YU-Z)(1-P)\|_2,\operatorname{tr}_N(P)-(\varepsilon/\gamma)(2+\varepsilon/\gamma)\right).$$

Fix $\varepsilon>0$ and $\delta>0$. Suppose for contradiction that the claim fails. Fix a sequence of neighborhoods $\mathscr{O}_k$ shrinking to μ. By our assumption, for each k there must exist arbitrarily large $N\in\mathbb{N}$ such that the claim about Y, $Z\in\Gamma_R^{(n)}(\mathscr{O}_k)$ fails. Hence, we can choose $n_k\in\mathbb{N}$ with $n_{k+1}>n_k$ such that there exist $Y^{(n_k)}$, $Z^{(n_k)}\in\Gamma_R^{(n_k)}(\mathscr{O}_k)$ with $\|Y^{(n_k)}-Z^{(n_k)}\|_2\leq\varepsilon$ such that

$$\liminf_{k\to\infty}\inf\{\Delta(Y^{(n_k)},Z^{(n_k)},U,P): U \text{ unitary}, P \text{ projection in } \mathbb{M}_n(\mathbb{C})\}\geq\delta.$$

Now choose a free ultrafilter $\omega\in\beta\mathbb{N}\setminus\mathbb{N}$ so that $\lim_{n\to\omega}1_{\{n_k:k\in\mathbb{N}\}}(n)=1$. For notational convenience, extend $Y^{(n_k)}$ and $Z^{(n_k)}$ to sequences $Y^{(n)}$ and $Z^{(n)}$ defined for all $n\in\mathbb{N}$ (however, the values for

$n \notin \{n_k : k \in \mathbb{N}\}$ will not matter). Then

$$\lim_{n\to\omega} \inf\{\Delta(Y^{(n)}, Z^{(n)}, U, P) : U \text{ unitary, } P \text{ projection in } \mathbb{M}_n(\mathbb{C})\} \geq \delta.$$

and

$$\lim_{n\to\omega} \lambda_{Y^{(n)}} = \lim_{n\to\omega} \lambda_{Z^{(n)}} = \ell_x.$$

Let

$$(\mathscr{M}, \mathrm{tr}) = \prod_{n\to\omega} (\mathbb{M}_n(\mathbb{C}), \mathrm{tr}_n),$$

that is, the tracial W* ultraproduct of matrix algebras with respect the ultrafilter ω. Let $y = [Y^{(n)}]_{n\in\mathbb{N}}$ and $z = [Z^{(n)}]_{n\in\mathbb{N}}$ in $\mathscr{M}_{\mathrm{sa}}^d$. Then y and z have the same law as x and therefore, there are trace-preserving embeddings $\pi_1, \pi_2 : (M, \tau) \to (\mathscr{M}, \mathrm{tr})$ such that $\pi_1(x) = y$ and $\pi_2(x) = z$. We can make $L^2(\mathscr{M}, \mathrm{tr})$ into an M-M bimodule by letting M act on the left by π_1 and on the right by π_2. Let $T : L^2(\mathscr{M}, \mathrm{tr}) \to L^2(\mathscr{M}, \mathrm{tr})^d$ be the operator $T(a) = ya - az$. By definition of the Kazhdan constant γ, we have

$$\|1 - P_{\ker(T)}(1)\|_2 \leq \frac{1}{\gamma}\|T(1)\|_2 = \frac{1}{\gamma}\|y - z\|_2 \leq \frac{\varepsilon}{\gamma}.$$

Let $a = P_{\ker(T)}(1) \in L^2(\mathscr{M}, \mathrm{tr})$, so that $ya = az$ by definition of T. Here we consider a as an affiliated operator to $\mathscr{M}$; (see [BO08, Appendix F]) and as such it has a polar decomposition $a = v|a|$ where $|a| = (a^*a)^{1/2} \in L^2(\mathscr{M}, \mathrm{tr})$ and v is a partial isometry with range equal to the closure of the range of a (see [RS80, Theorem VII.32]). It is not hard to show that $yv = vz$ (see [Pop04, Lemma 5]), and it follows that v^*v commutes with z. Furthermore,

$$\begin{aligned}
1 - \mathrm{tr}(v^*v) &= \mathrm{tr}(P_{\ker(a)}) \\
&\leq \|a^*a - 1\|_1 \\
&\leq \|(a^* - 1)a\|_1 + \|a - 1\|_1 \\
&\leq \|a^* - 1\|_2(1 + \|a - 1\|_2) + \|a - 1\|_2 \\
&\leq \frac{\varepsilon}{\gamma}\left(2 + \frac{\varepsilon}{\gamma}\right).
\end{aligned}$$

Let $p = 1 - v^*v$. Since $\mathscr{M}$ is a finite von Neumann algebra, there exists a partial isometry sending $1 - v^*v$ to $1 - vv^*$, and therefore, there is a unitary u such that $v = u(1 - p)$. Since $1 - p = v^*v$

commutes with z, we have

$$(u^*yu - z)(1-p) = u^*yu(1-p) - u^*u(1-p)z = u^*(yv - vz) = 0.$$

It is a standard exercise to show that there exists a sequence of projections $(P^{(n)})_{n\in\mathbb{N}}$ and a sequence of unitaries $(U^{(n)})_{n\in\mathbb{N}}$ such that $p = [P^{(n)}]_{n\in\mathbb{N}}$ and $u = [U^{(n)}]_{n\in\mathbb{N}}$. We also have

$$\lim_{n\to\omega} \|((U^{(n)})^* Y^{(n)} U^{(n)} - Z^{(n)})(1 - P^{(n)}\|_2 = \|(u^*yu - z)(1-p)\|_2 = 0$$

and

$$\lim_{n\to\omega} \operatorname{tr}_n(P^{(n)}) = \operatorname{tr}(p) \le \frac{\varepsilon}{\gamma}\left(2 + \frac{\varepsilon}{\gamma}\right).$$

Therefore,

$$\lim_{n\to\omega} \Delta(Y^{(n)}, Z^{(n)}, U^{(n)}, P^{(n)}) = 0,$$

which contradicts our choice of $Y^{(n)}$ and $Z^{(n)}$, and thus the argument is complete. □

Now that we know that microstates are conjugate up to a small projection, in order to control the covering number of the microstate space, we need to estimate the covering numbers of the space of these projections. We rely on an estimate of Szarek [Sza98] on the covering numbers of Grassmannians, which we state in the following form.

Lemma 3.0.2. There exists a universal constant C such that for $t \ge 0$,

$$\begin{aligned} K_\varepsilon(\{P \in \mathbb{M}_n(\mathbb{C}) \text{ projection, } \operatorname{tr}_n(P) \le t\}, \|\cdot\|_\infty) &= K_\varepsilon(\{P \in \mathbb{M}_n(\mathbb{C}) \text{ projection, } \operatorname{tr}_n(P) \ge 1-t\}, \|\cdot\|_\infty) \\ &\le (1+nt)\left(\frac{C}{\varepsilon}\right)^{2n^2 t}. \end{aligned}$$

Proof. First, the two covering numbers are equal because we can make the substitution $P \mapsto 1 - P$. Thus, it suffices to estimate the first one. For $\ell \in \mathbb{N}\cup\{0\}$, let $G(\ell, n-\ell)$ be the subset of $\mathbb{M}_n(\mathbb{C})$ consisting of rank ℓ orthogonal projections. By [Sza98, Theorem 8 and Remark (ii) below it], there is a uniform $C > 0$ so that

$$K_\varepsilon(G(\ell, n-\ell), \|\cdot\|_\infty) \le \left(\frac{C}{\varepsilon}\right)^{2\ell(n-\ell)}.$$

Note that

$$\{P \in \mathbb{M}_n(\mathbb{C}) \text{ projection } \operatorname{tr}_n(P) \le t\} = \bigcup_{\substack{\ell\in[n] \\ \ell \le nt}} G(\ell, n-\ell).$$

The number of terms in the union is the ceiling of $n\min(1,t)$ which is bounded by $1+nt$. The covering number of each of the individual sets can be bounded by $(C/\varepsilon)^{2\ell(n-\ell)} \leq (C/\varepsilon)^{2n^2t}$ since $\ell \leq nt$ and $n-\ell \leq n$. □

Next, we combine Lemma 3.0.1 and Lemma 3.0.2 to obtain the following estimate, which bounds the η-covering numbers for microstate spaces in terms of the ε-covering numbers for $\eta \leq \varepsilon$. We will then conclude the proof of the theorem by iterating this estimate.

Lemma 3.0.3. Let (M,τ) be a tracial von Neumann algebra and suppose $x \in M_{\mathrm{sa}}^d$ is a Kazhdan tuple with $\|x\|_\infty < R$ and with Kazhdan constant γ. Let $0 < \eta \leq \varepsilon < \gamma/2$. Then

$$h_{R,\eta}(x) \leq h_{R,\varepsilon}(x) + \frac{12(d+1)\varepsilon}{\gamma} \log \frac{CRd^{1/2}}{\eta}$$

where C is a positive constant.

Proof. Fix $\delta > 0$ with $\delta < \eta/3$ and $(2\varepsilon/\gamma)(2+2\varepsilon/\gamma)+\delta < 6\varepsilon/\gamma$ (the latter being possible because $\varepsilon < \gamma$). Let $\mathscr{O}$ and n_0 be as in Lemma 3.0.1 for the constants δ and 2ε (rather than δ and ε), and assume that $n \geq n_0$. For ease of notation, let $t = 6\varepsilon/\gamma$.

By Lemma 3.0.2, choose a set Ξ of projections of rank less than or equal to nt such that every projection P with $\mathrm{tr}_n(P) \leq t$ satisfies $\|P-Q\|_\infty < \eta/(6Rd^{1/2})$, such that $|\Xi| \leq (1+nt)(6C_1Rd^{1/2}/\eta)^{2n^2t}$. For each $Q \in \Xi$ with rank ℓ, the space $\mathbb{M}_n(\mathbb{C})^dQ$ has real dimension $2n\ell d \leq 2n^2td$ and therefore there exists $E_Q \subseteq \mathbb{M}_n(\mathbb{C})^dQ$ that $(\eta/3, \|\cdot\|_2)$-covers $\overline{B}_{\mathbb{M}_n(\mathbb{C})^dQ,\|\cdot\|_2}(0,2Rd^{1/2})$ and satisfies

$$|E_Q| \leq \left(\frac{6C_2Rd^{1/2}}{\eta}\right)^{2n^2td}.$$

Finally, fix $\Omega \subseteq \Gamma_R^{(n)}(\mathscr{O})$ that orbitally $(2\varepsilon, \|\cdot\|_2)$-covers $\Gamma_R^{(n)}(\mathscr{O})$ and satisfies $|\Omega| \leq K_\varepsilon(\Gamma_R^{(n)}(\mathscr{O}), \|\cdot\|_2)$.

We claim that $\Omega' = \bigcup_{Q\in\Xi}(\Omega + E_Q)$ is an $(\eta, \|\cdot\|_2)$-covering of $\Gamma_R^{(n)}(\mathscr{O})$. To see this, suppose $Y \in \Gamma_R^{(n)}(\mathscr{O})$. Then there exists $Z \in \Omega$ and a unitary V such that $\|Y - VZV^*\|_2 < 2\varepsilon$, hence $\|V^*YV - Z\|_2 < 2\varepsilon$. By our choice of n_0 and $\mathscr{O}$, there exists a projection P and a unitary U such that

$$\|(U^*V^*YVU - Z)(1-P)\|_2 < \delta < \frac{\eta}{3}, \qquad \mathrm{tr}_n(P) < (2\varepsilon/\gamma)(2+2\varepsilon/\gamma)+\delta \leq t.$$

To simplify notation, let us rename VU to U, so that $\|(U^*YU - Z)(1-P)\|_2 < \eta/3$. Fix a projection

$Q \in \Xi$ such that $\|P-Q\|_\infty < \eta/(6Rd^{1/2})$. Note that

$$\|(U^*YU-Z)(P-Q)\|_2 \le \|U^*YU-Z\|_2\|P-Q\|_\infty \le 2Rd^{1/2}\|P-Q\|_\infty < \frac{\eta}{3}.$$

Moreover,

$$\|(U^*YU-Z)Q\|_2 \le d^{1/2}\|U^*YU-Z\|_\infty\|Q\|_\infty \le 2Rd^{1/2},$$

hence there exists $W \in E_Q$ such that

$$\|(U^*YU-Z)Q-W\|_2 < \frac{\eta}{3}$$

It follows that

$$\begin{aligned}\|U^*YU-(Z+W)\|_2 &\le \|(U^*YU-Z)(1-P)\|_2 + \|(U^*YU-Z)(P-Q)\|_2 + \|(U^*YU-Z)Q-W\|_2 \\ &< \frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3} = \eta.\end{aligned}$$

This shows that Ω' orbitally $(\eta, \|\cdot\|_2)$-covers $\Gamma_R^{(n)}(\mathscr{O})$ as desired.

Therefore,

$$\begin{aligned}K_\eta^{\mathrm{orb}}(\Gamma_R^{(n)}(\mathscr{O})) &\le |\Omega| \sum_{Q\in\Xi} |E_Q| \\ &\le K_\varepsilon^{\mathrm{orb}}(\Gamma_R^{(n)}(\mathscr{O}))(1+nt)\left(\frac{6C_1Rd^{1/2}}{\eta}\right)^{2n^2t}\left(\frac{6C_2Rd^{1/2}}{\eta}\right)^{2n^2td} \\ &\le K_\varepsilon^{\mathrm{orb}}(\Gamma_R^{(n)}(\mathscr{O}))(1+nt)\left(\frac{C_3Rd^{1/2}}{\eta}\right)^{2n^2t(d+1)}\end{aligned}$$

where $C_3 = 6\max(C_1,C_2)$. After applying $\limsup_{n\to\infty}(1/n^2)\log$ to both sides, we obtain

$$h_{R,\eta}(\mathscr{O}) \le h_{R,\varepsilon}(\mathscr{O}) + 2t(d+1)\log\frac{C_3Rd^{1/2}}{\eta}$$

Substitute back $t = 6\varepsilon/\gamma$ and letting $\mathscr{O}$ shrink to ℓ_x, we obtain the asserted result. □

Now we can finish the proof of Theorem 1.0.1 by iterating Lemma 3.0.3.

Proof of Theorem 1.0.1. Let (M,τ) be a tracial W*-algebra with a Kazhdan tuple $x=(x_1,\dots,x_d)$ with Kazhdan constant γ and $\|x\|_\infty < R$. For $\varepsilon < \min(\gamma,1)$, we can take $\eta = \varepsilon^2$ in the previous lemma and obtain

$$h_{R,\varepsilon^2}(x) \leq h_{R,\varepsilon}(x) + \frac{12(d+1)\varepsilon}{\gamma} \log \frac{CRd^{1/2}}{\varepsilon^2}.$$

Iterating this estimate, we have that for $\varepsilon < \min(1,\gamma)$,

$$h_{R,\varepsilon^{2^k}}(x) \leq h_{R,\varepsilon}(x) + \frac{12(d+1)}{\gamma} \sum_{j=0}^{k-1} \varepsilon^{2^j} \log \frac{CRd^{1/2}}{\varepsilon^{2^{j+1}}}.$$

Fix $\varepsilon \in (0, \min(1,\gamma))$. As $k \to \infty$, we have $h_{R,\varepsilon^{2^k}}(x) \to h(M)$. Therefore,

$$h(M) \leq h_{R,\varepsilon}(x) + \frac{12(d+1)}{\gamma} \sum_{j=0}^{\infty} \varepsilon^{2^j} \left(\log(CRd^{1/2}) + 2^{j+1} \log \frac{1}{\varepsilon} \right).$$

Since the sum on the right-hand side converges, we have $h(M) < \infty$, and thus M is strongly 1-bounded. □

3.0.2 Direct sums and strong 1-boundedness

In order to prove Propositions 1.0.2 and 1.0.3, we must understand the behavior of h under direct sums of tracial von Neumann algebras. Let J be a countable index set and $(\lambda_j)_{j\in J} \in (0,\infty)^J$ such that $\sum_{j\in J} \lambda_j = 1$, and for each $j \in J$, let (M_j, τ_j) be a tracial von Neumann algebra. Then

$$\bigoplus_{j\in J} \lambda_j (M_j, \tau_j)$$

is defined as the tracial von Neumann algebra (M, τ) where

$$M = \{(x_j)_{j\in J} : \sup_j \|x_j\| < \infty\}$$

and

$$\tau((x_j)_{j\in J}) = \sum_{j\in J} \lambda_j \tau_j(x_j).$$

It is shown in [Hay18, Proposition A.13(i)] that

$$h(M,\tau) \leq \sum_{j\in J} \lambda_j^2 h(M_j, \tau_j). \tag{3.0.1}$$

Our goal in this section is to give a corresponding lower bound for $h(M,\tau)$ (see Lemma 3.0.5). As with many results in free entropy theory, we run into the issue that we do not know whether using a $\liminf$ instead of $\limsup$ in the definition of h would yield the same result. Thus, we will need to use the "$\liminf$ version" of 1-bounded entropy $\underline{h}$ in our formula. It is also convenient for the proof to use the description of 1-bounded entropy in terms of relative microstates rather than unitary orbits. We therefore recall the following definitions.

Definition 3.0.4. Let (M,τ) be a tracial von Neumann algebra, $l \in \mathbb{N}$ and $y \in M_{\mathrm{sa}}^{l}$. A microstates sequence for y is a sequence $(Y^{(n)}) \in \prod_n \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^{l}$ with

$$\sup_n \|Y^{(n)}\|_\infty < \infty,$$

$$\ell_{Y^{(n)}} \to_{n\to\infty} \ell_y.$$

Suppose that $d \in \mathbb{N}$, that $x \in M_{\mathrm{sa}}^{d}$ and that $Y^{(n)}$ is a microstates sequence for y as above. Fix

$$R > \max(\|x\|_\infty, \sup_n \|Y^{(n)}\|_\infty)$$

For $n \in \mathbb{N}$, and $\mathscr{O}$ a weak*-neighborhood of $\ell_{x,y}$ we let

$$\Gamma_R^{(n)}(\mathscr{O}|A^{(n)}) = \{B \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^{r} : (B, Y^{(n)}) \in \Gamma^{(n)}(\mathscr{O})\}.$$

We let $K_\varepsilon(\Omega, \|\cdot\|_2)$ be the minimal cardinality of a subset of $\mathbb{M}_n(\mathbb{C})^d$ which $(\varepsilon, \|\cdot\|_2)$-covers Ω. Let (M,τ) be a tracial von Neumann algebra, $d,s,l \in \mathbb{N}$, and let $x \in M_{\mathrm{sa}}^{d}, y \in M_{\mathrm{sa}}^{s}, a \in M_{\mathrm{sa}}^{l}$. Suppose that $(A^{(n)})_n$ is a microstates sequence for a. Fix $R \in (0,\infty)$ with $R > \max(\|x\|_\infty, \|y\|_\infty, \sup_n \|A^{(n)}\|_\infty)$. For $n \in \mathbb{N}$, let $\pi_r \colon M_n(\mathbb{C})_{\mathrm{sa}}^{d+s} \to M_n(\mathbb{C})_{\mathrm{sa}}^{d}$ be the projection onto the first d-coordinates. For a weak*-neighborhood $\mathscr{O}$ Of $\ell_{x,y,a}$, and $n \in \mathbb{N}$, we set

$$\Gamma_R^{(n)}(x|A^{(n)} : \mathscr{O}) = \pi_r(\Gamma_R^{(N)}(\mathscr{O}|A^{(k)})).$$

We will heuristically refer to the collection of spaces $(\Gamma_R^{(n)}(x|A^{(n)} : \mathscr{O}))_{n,\mathscr{O}}$ as the microstates spaces for x relative to a in the presence of y. Roughly speaking, the usual microstates spaces correspond to embeddings of $W^*(x,a)$ into an ultraproduct of matrices. As in Section 2.0.5 relative microstates spaces in the presence correspond to embeddings of $W^*(x)$ into ultraproducts of matrices which both restrict to the embedding of $W^*(a)$ given by $(A^{(n)})_n$, and have an extension to $W^*(x,y,a)$.

We now define

$$\underline{h}_{R,\varepsilon}(x|(A^{(n)})_n : \mathscr{O}) := \liminf_{n\to\infty} \frac{1}{n^2} \log K_\varepsilon(\Gamma_R^{(n)}(x|A^{(n)}; \mathscr{O}), \|\cdot\|_2),$$
$$\underline{h}_{R,\varepsilon}(x|(A^{(n)})_n : y) := \inf_{\mathscr{O}} \underline{h}_{R,\varepsilon}(x|(A^{(n)})_n : \mathscr{O}),$$

where the infimum is over all weak*-neighborhoods $\mathscr{O}$ of $\ell_{x,y,a}$. If $W^*(a)$ is diffuse and hyperfinite, and if $W^*(a) \subseteq W^*(x)$, we then define

$$\underline{h}(x : y) := \sup_{\varepsilon>0} \underline{h}_{R,\varepsilon}(x|(A^{(n)})_n : y).$$

By standard methods, it can be shown that $\underline{h}(x:y)$ does not depend upon the choice of R, and this justifies dropping it from the notation. If $W^*(a)$ is diffuse and hyperfinite, then the above quantity is independent of a and the choice of $(A^{(n)})_n$ by the same arguments in [Hay18, Lemma A.5, Corollary A.6]. In [Hay18], this is only shown when a is a single element which generates a diffuse, abelian von Neumann algebra, but the same argument works for any tuple generating a diffuse, hyperfinite von Neumann algebra. By the same arguments as in [Hay18, Theorem A.9] one can show that if $d', s' \in \mathbb{N}$ and $x' \in M_{\mathrm{sa}}^{d'}, y \in M_{\mathrm{sa}}^{s'}$ satisfies $W^*(x') = W^*(x)$, $W^*(x,y) = W^*(x',y')$, and if $W^*(x), W^*(y)$ are diffuse, then $\underline{h}(x:y) = \underline{h}(x':y')$. Thus if $M_1 \leq M_2 \leq M$ and M_1, M_2 are finitely generated and diffuse, we can set

$$\underline{h}(M_1 : M_2) = \underline{h}(x : y),$$

where $x_j, j = 1,2$ are finite self-adjoint tuples in M with $W^*(x_1) = M_1$, $W^*(x_1,x_2)$. We then set

$$\underline{h}(M_1 : M) = \inf_{Q} \underline{h}(M_1 : Q),$$

where the infimum is over all finitely generated subalgebras Q of M which contain M_1. Finally, for $N \leq M$, we let

$$\underline{h}(N : M) = \sup_{P} \underline{h}(P : M),$$

where the supremum is over all finitely generated, diffuse subalgebras P of N. We call $\underline{h}(N:M)$ the lower 1-bounded entropy of N in the presence of M, and we set $\underline{h}(M) = \underline{h}(M:M)$. The above definition using relative microstates can intuitively be thought of describing the 1-bounded entropy as a measurement of how many embeddings N has into an ultraproduct of matrices which both restrict to a given embedding of $W^*(a)$ and have an extension to M. The definition of 1-bounded entropy using unitary conjugation orbits can be thought of as a measurement of how many embeddings M has into an ultraproduct of matrices modulo unitary conjugation. It is a fact that when $N = M$, these measurements are the same [Hay18, Lemma A.5].

We remark that the properties of h given in Section ?? are satisfied, mutatis mutandis, for $\underline{h}$ with the exception that the analogue of item 6 is that

$$\underline{h}(N_1 \vee N_2 : M) \leq \underline{h}(N_1 : M) + h(N_2 : M) \text{ if } N_1 \cap N_2 \text{ is diffuse.}$$

The lower bound for the 1-bounded entropy of direct sums is as follows.

Lemma 3.0.5. Suppose that J is a countable index set, and that $((M_j, \tau_j))_{j \in J}$ are diffuse tracial von Neumann algebras. Let $(\lambda_j)_{j \in J}$ be positive numbers with $\sum_j \lambda_j = 1$ and set $(M, \tau) = \bigoplus_j \lambda_j (M_j, \tau_j)$. Suppose that $N_j \leq M_j, j \in J$ are diffuse and set $N = \bigoplus_j N_j$. Then

$$\underline{h}(N : M) \geq \sum_j \lambda_j^2 \underline{h}(N_j : M_j),$$

$$h(N : M) \geq \lambda_i^2 h(N_i : M_i) + \sum_{j \neq i} \lambda_j^2 \underline{h}(N_j : M_j) \text{ for all } i \in J.$$

Proof. If there is a $j \in J$ so that M_j is not Connes-embeddable, then both sides are $-\infty$, and there is nothing to prove. So we will assume without further comment throughout the proof that all algebras involved are Connes-embeddable. The proofs for h and $\underline{h}$ are nearly identical, so we will only give the proof for $\underline{h}$. First, consider the case where $J = \{1, 2\}$. Fix tracial von Neumann algebras $(M_j, \tau_j), j = 1, 2$ and diffuse $N_j \leq M_j$. Without loss of generality, we may assume that N_j, M_j are finitely generated, so let $r_j, s_j \in \mathbb{N}, j = 1, 2$ and let $x_j \in (N_j)_{\mathrm{sa}}^{r_j}, y_j \in (M_j)_{\mathrm{sa}}^{s_j}$ be generating sets. Let $z_j, j = 1, 2$ be the identity of M_j, regarded as a projection in $M_1 \oplus M_2$. Let $a_j \in (N_j)_{\mathrm{sa}}, j = 1, 2$ be an element with diffuse spectrum. We will use microstates relative to $b = (a_1 \oplus 0, 0 \oplus a_1, z_1, z_2)$ to compute the entropy. So let $C^{(n)} \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^4$ be a microstates sequence for b. To ease notation set $A_j^{(n)} = C_j^{(n)}, j = 1, 2$ and $Z_j^{(n)} = C_{j+2}^{(n)}, j = 1, 2$. We may, and will, assume that

- $Z_1^{(n)}, Z_2^{(n)}$ are diagonal projections with $Z_1^{(n)} + Z_2^{(n)} = 1$ for all n,
- $n \operatorname{tr}_n(Z_1^{(n)}) = \lfloor \tau(z_1) n \rfloor$ for all n,
- $Z_j^{(n)} A_j^{(n)} Z_j^{(n)} = A_j^{(n)}$ for all n and all $j = 1, 2$,
- $A_j^{(n)}$ are diagonal for all n, and all $j = 1, 2$.

Set $b_j = (a_j, z_j), j = 1,2$, and let $B_j^{(n)} = (A_j^{(n)}, \mathrm{id}_{\ell_{n,j}})$ where we regard $A_j^{(n)}$ as an element of $M_{\ell_{n,j}}$ where $\ell_{n,j} = n\tau(Z_j^{(n)})$ Set $x = (x_1 \oplus 0, 0 \oplus x_2)$, and $y = (y_1 \oplus 0, 0 \oplus y_2)$. Finally, fix

$$R > \max(\|x\|_\infty, \|y\|_\infty, \sup_n \|B^{(n)}\|_\infty).$$

Given a neighborhood $\mathscr{O}$ of $\ell_{x,y,b}$ we may find neighborhoods $\mathscr{O}_j, j = 1,2$ of ℓ_{x_j,y_j,b_j} so that

$$\Gamma_R^{(n)}(x|B^{(n)} : \mathscr{O}) \supseteq \Gamma_R^{(\lfloor \tau(z_1)n \rfloor)}(x_1|B_1^{(\lfloor \tau(z_1)n \rfloor)}) : \mathscr{O}_1) \oplus \Gamma_R^{(\lceil \tau(z_2)n \rceil)}(x_2|B_2^{(\lceil \tau(z_2)n \rceil)}; \mathscr{O}_2) \text{ for all sufficiently large } n.$$

Given $\varepsilon > 0$, let $\Omega_{n,j} \subseteq \Gamma^{(\ell_{n,j})}(x_j|B_j^{(\ell_{n,j})}; \mathscr{O}_j)$ be ε-separated with respect to $\|\cdot\|_2$ (meaning that distinct points in $\Omega_{n,j}$ are distance at least ε from each other). Then for all sufficiently large n, we have that $\Omega_{n,1} \oplus \Omega_{n,2}$ is an ε-separated subset of $\Gamma^{(n)}(x|B^{(n)}; \mathscr{O})$ and thus has cardinality at most $K_{\varepsilon/2}(\Gamma_R^{(n)}(x|B^{(n)}; \mathscr{O}), \|\cdot\|_2)$.

Thus for all $\varepsilon > 0$, and for all sufficiently large n,

$$K_{\varepsilon/2}(\Gamma_R^{(n)}(x|B^{(n)} : \mathscr{O}), \|\cdot\|_2) \geq K_\varepsilon(\Gamma_R^{(\lfloor \tau(z_1)n \rfloor)}(x_1|B_1^{(n)} : \mathscr{O}_1), \|\cdot\|_2) K_\varepsilon(\Gamma_R^{(\lceil \tau(z_2)n \rceil)}(x_2|B_2^{(n)} : \mathscr{O}_2), \|\cdot\|_2).$$

Since $\{\lfloor \tau(z_1)n \rfloor : n \in \mathbb{N}\}$ and $\{\lceil \tau(z_2)n \rceil : n \in \mathbb{N}\}$ have finite complement in $\mathbb{N}$, and since limit infimums are super additive, the above inequality shows that for all $\varepsilon > 0$ we have

$$\underline{h}_{R,\varepsilon/2}(x|(B^{(n)})_n : \mathscr{O}) \geq \tau(z_1)^2 \underline{h}_{R,\varepsilon}(x_1|B_1^{(n)} : y_1) + \tau(z_2)^2 \underline{h}_{R,\varepsilon}(x_2|B_2^{(n)} : y_2).$$

Letting $\mathscr{O}$ decrease to $\ell_{x,y,b}$ and letting $\varepsilon \to 0$ completes the proof of the case where $J = \{1,2\}$.

The case where J is finite follows by induction. Now suppose that J is infinite and assume without loss of generality that $J = \mathbb{N}$. Let $(N_j \leq M_j)_{j \in J}$ be as in the statement of the lemma. For each $j \in J$, fix a diffuse abelian $A_j \leq N_j$. For $r \in \mathbb{N}$, let

$$N_{\leq r} = \bigoplus_{j=1}^{r} N_j \oplus \bigoplus_{j=r+1}^{\infty} A_j.$$

Since $\bigoplus_{j=r+1}^{\infty} A_j$ has 1-bounded entropy zero, we have by the case of finite J that

$$\underline{h}(N_{\leq r} : M) \geq \sum_{j=1}^{r} \lambda_j^2 \underline{h}(N_j : M_j).$$

Since $N_{\leq r}$ are an increasing sequence of subalgebras of N and $\bigvee N_{\leq r} = N$, we have that

$$\underline{h}(N:M) = \sup_r \underline{h}(N_{\leq r}:M) \geq \sum_{j=1}^{\infty} \lambda_j^2 \underline{h}(N_j:M_j). \qquad \square$$

3.0.3 Amplification and strong 1-boundedness

This section will show that $h(M^t) = t^{-2}h(M)$ for II_1 factor M (Proposition 1.0.3), and then conclude the proof of Proposition 1.0.2. To prove Proposition 1.0.3 for $t \in (0,1)$, we consider a projection $p \in M$ with trace t, and analyze the 1-bounded entropy of certain subalgebras of M related to pMp. At a key point, we use the fact that if N is a von Neumann subalgebra of M, then for $h(N:M)$ to equal $h(M)$, it is sufficient that every sequence of matricial microstates for N extends to a sequence of matricial microstates for M. We state this property more precisely as follows.

Definition 3.0.6. Let (M,τ) be a tracial von Neumann algebra, and $N \leq M$. We say the inclusion $N \leq M$ has the microstates extension property if for every free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, for every $N_0 \leq N, M_0 \leq M$ so that N_0, M_0 have separable predual and $N_0 \leq M_0$, and for every trace-preserving embedding $\Theta\colon N_0 \to \prod_{k\to\omega} \mathbb{M}_k(\mathbb{C})$ into a tracial ultraproduct of matrix algebras $\mathbb{M}_k(\mathbb{C})$, there exists a trace-preserving embedding $\widetilde{\Theta}\colon M_0 \to \prod_{k\to\omega} \mathbb{M}_k(\mathbb{C})$ with $\widetilde{\Theta}\big|_{M_0} = \Theta$.

The following lemma shows that the microstates extension property can be formulated in terms of microstates spaces. The proof is an exercise in understanding the definitions and is left to the reader.

Lemma 3.0.7. Let (M,τ) be a tracial von Neumann algebra, and let $N \leq M$. Then the inclusion $N \leq M$ has the microstates extension property if and only if for every $d, s \in \mathbb{N}$, every $x \in N_{\mathrm{sa}}^d, y \in M_{\mathrm{sa}}^s$, every $R > \max(\|x\|_\infty, \|y\|_\infty)$ and every neighborhood $\mathscr{V}$ of $\ell_{x,y}$ there is a neighborhood $\mathscr{O}$ of ℓ_x so that

$$\Gamma_R^{(n)}(\mathscr{O}) \subseteq \pi_d(\Gamma_R^{(n)}(\mathscr{V})) \text{ for all sufficiently large } n$$

where $\pi_d\colon \mathbb{M}_k(\mathbb{C})_{\mathrm{sa}}^{d+s} \to \mathbb{M}_k(\mathbb{C})_{\mathrm{sa}}^d$ is the projection onto the first d coordinates.

The following lemma gives some examples and also explain the relevance to 1-bounded entropy.

Lemma 3.0.8.

(i) Suppose that (M,τ) is a tracial von Neumann algebra, that $N \leq Q \leq M$, that N is diffuse, and that the inclusion $Q \leq M$ has the microstates extension property. Then $h(N:Q) = h(N:M)$ and $\underline{h}(N:Q) = \underline{h}(N:M)$.

(ii) If (M,τ) is Connes embeddable, and $N \leq M$ is hyperfinite, then the inclusion $N \leq M$ has the microstates extension property.

(iii) If $(M_j,\tau_j), j=1,2$ are Connes-embeddable, then $M_1 * 1 \leq M_1 * M_2$ has the microstates extension property.

We remark that (i) makes sense from our intuitive description of 1-bounded entropy. The quantity $h(N:Q)$ (resp. $h(N:M)$) is supposed to be a measurement of "how many" embeddings there are of N into an ultraproduct of matrices which have an extension to Q (resp. M). If $Q \leq M$ has the microstates extension property, then every embedding of Q extends to M and thus the quantities $h(N:Q)$,$h(N:M)$ should be the same.

Proof. (i): This is an exercise from Lemma 3.0.7.

(ii): Without loss of generality we may, and will, assume that N,M have separable predual. Fix a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Let $\Theta\colon N \to \prod_{k\to\omega} \mathbb{M}_k(\mathbb{C})$ be a trace-preserving embedding. Since M is Connes-embeddable, there exists a trace-preserving embedding $\Psi\colon M \to \prod_{k\to\omega} \mathbb{M}_k(\mathbb{C})$. By [Con76, Jun07a] there exists a unitary $u \in \prod_{k\to\omega} \mathbb{M}_k(\mathbb{C})$ so that $\Psi\big|_N = \mathrm{ad}(u) \circ \Theta$. Set $\widetilde{\Theta} = \mathrm{ad}(u^*) \circ \Psi$. Then $\widetilde{\Theta}$ is the desired extension.

(iii): Without loss of generality we may, and will, assume that M_1,M_2 have separable predual. Our desired result is now a consequence of [Voi98, Theorem 2.4], [Pop14, Corollary 0.2]. Namely, fix a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, and let $\mathscr{M} = \prod_{k\to\omega} \mathbb{M}_k(\mathbb{C})$. Then, by assumption, there exists trace-preserving embeddings $\Theta_j\colon M_j \to \mathscr{M}, j=1,2$. By [Voi98, Theorem 2.4], [Pop14, Corollary 0.2] there exists a unitary $u \in \mathscr{M}$ which is Haar distributed and freely independent of $\Theta_1(M_1) \vee \Theta_2(M_2)$. Then $\Theta_1(M_1), u\Theta_2(M_2)u^*$ are freely independent, and this produces an extension $\widetilde{\Theta}\colon M_1 * M_2 \to \mathscr{M}$ of Θ. $\square$

Now we are ready to prove Proposition 1.0.3, and in fact we also give the analogous result for $\underline{h}$. We remark that the point $h(M^t) \leq t^{-2}h(M)$ for $t \in (0,1)$ was already shown in [Hay18, Propostion A.13(ii)].

Proposition 3.0.9. Suppose that M is a II_1-factor and τ is its canonical trace. For $t \in (0,\infty)$, let M^t be the t^{th} compression of M. Then

$$h(M^t) = \frac{1}{t^2}h(M) \quad \text{and} \quad \underline{h}(M^t) = \frac{1}{t^2}\underline{h}(M).$$

Proof. We first handle the case when $t \in (0,1)$. Fix a hyperfinite II_1-subfactor R of M. Let $p \in R$ be a projection with $\tau(p) = t$. Observe that $M^t \cong pMp$. Let $N = pMp + (1-p)R(1-p)$ Since R is a factor, we may find partial isometries $v_1, \cdots, v_n \in R$ so that $v_j^* v_j \leq p$ and $\sum_j v_j v_j^* = 1$. Thus

$$N \vee R \supseteq W^*(v_1, v_2, \cdots, v_n) \vee (pMp + (1-p)) = M.$$

So

$$N \vee R = M. \tag{3.0.2}$$

Step 1. We show that $\underline{h}(M^t) \geq \frac{1}{t^2}\underline{h}(M)$. By (3.0.2) and the fact that $N \cap R = pRp + (1-p)R(1-p)$ is diffuse, we have

$$\underline{h}(M) = \underline{h}(N \vee R) \leq \underline{h}(N) + h(R) = \underline{h}(N) \leq t^2 \underline{h}(pMp) + (1-t)^2 h((1-p)R(1-p)) = t^2 \underline{h}(pMp) = t^2 \underline{h}(M^t).$$

Here we use the analogues of [Hay18, Lemma A.12 and Proposition A.13 (i)] for $\underline{\mathrm{h}}$.

Step 2. We prove that $\underline{h}(M^t) \leq \frac{1}{t^2}\underline{h}(M)$. We start with the following claim.

Claim. $N \leq M$ has the microstates extension property. Suppose that ω is a free ultrafilter on the natural numbers, and let $\mathscr{M} = \prod_{k\to\omega} \mathbb{M}_k(\mathbb{C})$ be the tracial ultraproduct of $\mathbb{M}_k(\mathbb{C})$. Let $\Theta \colon N \to \mathscr{M}$ be a trace-preserving embedding. Since $N \cap R$ is hyperfinite, we know that $N \cap R \leq R$ has the microstates extension property by Lemma 3.0.8 (ii). So there exists a trace-preserving embedding $\Psi \colon R \to \mathscr{M}$ with $\Psi\big|_{N\cap R} = \Theta\big|_{N\cap R}$. Let $v_1, \cdots, v_n$ be as before the proof of Step 1. Define $\widetilde{\Theta} \colon M \to \mathscr{M}$ by

$$\widetilde{\Theta}(x) = \sum_{i,j} \Psi(v_i)\Theta(v_i^* x v_j)\Psi(v_j)^*.$$

Observe that $v_i^* M v_j \subseteq pMp \subseteq N$ for all i, j, so the formula above makes sense. For all $1 \leq i, j, k, l \leq n$ and all $x, y \in M$ we have

$$\begin{aligned}
\Psi(v_i)\Theta(v_i^* x v_j)\Psi(v_j)^*\Psi(v_k)\Theta(v_k^* y v_l)\Psi(v_l)^* &= \Psi(v_i)\Theta(v_i^* x v_j)\Psi(v_j^* v_k)\Theta(v_k^* y v_l)\Psi(v_l)^* \\
&= \delta_{j=k}\Psi(v_i)\Theta(v_i^* x v_j)\Theta(v_j^* v_j)\Theta(v_j^* y v_l)\Psi(v_l)^* \\
&= \delta_{j=k}\Psi(v_i)\Theta(v_i^* x v_j v_j^* v_j v_j^* y v_l)\Psi(v_l)^* \\
&= \delta_{j=k}\Psi(v_i)\Theta(v_i^* x v_j v_j^* y v_l)\Psi(v_l)^*.
\end{aligned}$$

From here, it is direct to show that $\widetilde{\Theta}(xy) = \widetilde{\Theta}(x)\widetilde{\Theta}(y)$ for all $x, y \in M$. It is also direct to show that $\widetilde{\Theta}$ preserves adjoints. Finally, for all $x \in M$:

$$\tau_\omega(\widetilde{\Theta}(x)) = \sum_{i,j} \tau_\omega(\Theta(v_i^* x v_j)\Psi(v_j^* v_i)) = \sum_i \tau_\omega(\Theta(v_i^* x v_i v_i^* v_i)) = \sum_i \tau(x v_i v_i^*) = \tau(x).$$

This proves the claim.

By (3.0.2) and the fact that $N \cap R$ is diffuse, we have that

$$\underline{h}(M) = \underline{h}(N \vee R : M) = \underline{h}(N : M) = \underline{h}(N),$$

the last step following by the claim and Lemma 3.0.8 (i). By Lemma 3.0.5,

$$\underline{h}(M) = \underline{h}(N) \geq t^2 \underline{h}(pMp).$$

This completes the proof of Step 2 and hence also the proof for $\underline{h}$ when $t \in (0,1)$. The case $t = 1$ is trivial, whereas if $t > 1$, then $\frac{1}{t} \in (0,1)$ and so

$$\underline{h}(M) = \underline{h}((M^t)^{\frac{1}{t}}) = t^2 \underline{h}(M^t),$$

For the case of h, it was already shown in [Hay18, Proposition A.13(ii)] that $h(M^t) \leq \frac{1}{t^2} h(M)$ for $t \in (0,1)$, and the proof of the opposite inequality proceeds in a similar manner to Step 2 above. □

With Lemma 3.0.5 on direct sums and Proposition 3.0.9 on amplifications in hand, we are ready to finish the proof of Proposition 1.0.2 showing that strong 1-boundedness of all tracial von Neumann algebras with Property (T) is equivalent to $h(M) \leq 0$ for all II_1 factors with Property (T).

Proof of Proposition 1.0.2. (iii) $\implies$ (ii). Assume that every II_1 Property (T) factor has nonpositive 1-bounded entropy. Let M be a tracial von Neumann algebra with Property (T), and we will show that $h(M) \leq 0$. Decomposing the center of M into diffuse and atomic parts, we see that there is a countable index set J (potentially empty) so that

$$M = M_0 \oplus \bigoplus_{j \in J} M_j,$$

where M_0 either has diffuse center or is $\{0\}$, and each M_j is a factor. Since M_0 either has diffuse center or is $\{0\}$, we know $h(M_0) \leq 0$. Since M has Property (T), each M_j has Property (T) by [Pop06a, Proposition 4.7.2]. Thus, by (3.0.1), we have $h(M) \leq 0$.

(ii) $\implies$ (i). If every tracial von Neumann algebra with Property (T) satisfies $h(M) \leq 0$, then it is strongly 1-bounded since strong 1-boundedness is equivalent to $h(M) < \infty$ by [Hay18, Proposition A.16].

(i) $\implies$ (iii). Proceeding by contraposition, assume that N is a Property (T) factor with $h(N) > 0$, and we will show that there is a tracial von Neumann algebra (M,τ) with Property (T) such that $h(M,\tau)=\infty$. Let $(N^{4^{-k}},\tau_k)$ be the compression of M, and let

$$(M,\tau)=\bigoplus_{k\in\mathbb{N}}2^{-k}(N^{4^{-k}},\tau_k).$$

By [Pop06a, Proposition 4.7.1] we know that M has Property (T). By Lemma 3.0.5 and Proposition 3.0.9, for each $j\in\mathbb{N}$, we have

$$h(M,\tau)\geq 4^{-j}h(N^{4^{-j}})+\sum_{k\neq j}4^{-k}\underline{h}(N^{4^{-j}})=4^{j}h(N)+\sum_{k\neq j}4^{k}\underline{h}(N).$$

Since $h(N)>0$, we know that N is Connes-embeddable, and thus $\underline{h}(N)\geq 0$. So

$$h(M)\geq 4^{j}h(N)\text{ for all } j\in\mathbb{N}.$$

Letting $j\to\infty$ we see that $h(M)=\infty$, i.e. M is not strongly 1-bounded.

Finally, we show that for each finite von Neumann algebra M with Property (T), there exists a faithful normal tracial state τ such that (M,τ) is strongly 1-bounded. As in (iii) $\implies$ (ii), write $M=\bigoplus_{j=0}^{\infty}M_j$ such that M_0 is zero or has diffuse center, and M_j is a factor for $j\geq 1$. Let τ_j be the unique tracial state on M_j. Since $h(M_0)\leq 0$ and since $h(M_j)<\infty$ for $j\geq 1$ by Theorem 1.0.1, we may choose nonnegative constants $(\lambda_j)_{j=0}^{\infty}$ such that $\sum_{j=0}^{\infty}\lambda_j=1$, $\sum_{j\in\mathbb{N}_0}\lambda_j^2 h(M_j)<\infty$, and $\lambda_j>0$ if and only if $M_j\neq 0$. Let τ be the faithful normal tracial state on M given by $\bigoplus_{j=0}^{\infty}\lambda_j\tau_j$. It follows by (3.0.1) that $h(M,\tau)<\infty$. □

3.0.4 Direct sums and free entropy dimension

In this section, we show that Theorem 1.0.1 implies Jung and Shlyakhtenko's result that a Property (T) tracial von Neumann algebra has microstates free entropy dimension at most 1 with respect to every finite generating tuple [JS07]. In fact, at the end of the section, we also sketch how to generalize the argument to show that $\delta_0(x)\leq 1$ for any infinite generating tuple x. First, we recall the definition of Voiculescu's microstates free entropy dimension.

Definition 3.0.10. Let (M,τ) be a tracial von Neumann algebra, $x\in M_{\mathrm{sa}}^{d}$. Fix $R>\|x\|_\infty$. For $\varepsilon>0$, and a weak*-neighborhood $\mathscr{O}$ of ℓ_x, we set

$$\delta_{R,\varepsilon}(\mathscr{O})=\limsup_{k\to\infty}\frac{\log K_\varepsilon(\Gamma_R^{(k)}(\mathscr{O}),\|\cdot\|_2)}{k^2|\log(\varepsilon)|},$$

$$\delta_{R,\varepsilon}(x) = \inf_{\mathscr{O}} \delta_\varepsilon(\mathscr{O}),$$

where the infimum is over all weak*-neighborhoods $\mathscr{O}$ of ℓ_x. We then set

$$\delta_0(x) = \limsup_{\varepsilon \to 0} \delta_{R,\varepsilon}(x).$$

We call $\delta_0(x)$ the microstates free entropy dimension of x.

By standard methods, $\delta_0(x)$ does not depend upon the choice of R and this justifies dropping it from the notation. This is not the original definition in [Voi96], however by [Jun03a, Corollary 2.4] they are the same. The following lemma, based on previous work of Jung, describes the behavior free entropy dimension under direct sums.

Lemma 3.0.11. Let (M, τ) be a tracial von Neumann algebra, $d \in \mathbb{N}$, and $x \in M_{\mathrm{sa}}^d$. Let J be a countable set and $(z_j)_{j \in J}$ be central projections in M with $\sum_{j \in J} z_j = 1$. For $j \in J$, let x_{z_j} be xz_j regarded as an element of Mz_j. Then

$$\delta_0(x) - 1 \leq \sum_{j \in J} \tau(z_j)^2 (\delta_0(x_{z_j}) - 1).$$

Proof. Fix $R > \|x\|_\infty$. We first handle the case where J is finite. By induction, to handle the case of finite J it suffices to handle the case where $J = \{1, 2\}$. In this case we use z for z_1. Let $P_n \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}$ be microstates for z such that each P_n is an orthogonal projection. By [Jun06, Lemma 3.2 and Corollary 4.3], we have

$$\delta_0(x) = \delta_0(x, z) = \delta_0(z) + \limsup_{\varepsilon \to 0} \frac{h_{R,\varepsilon}(x|(P_k))}{\log(1/\varepsilon)} = 2\tau(z)(1 - \tau(z)) + \limsup_{\varepsilon \to 0} \frac{h_{R,\varepsilon}(x|(P_k))}{\log(1/\varepsilon)},$$

where in the last step we use [Jun03b, Corollary 5.8]. It follows from the proof of [Hay18, Proposition A.13(i)] that

$$h_{R,6\varepsilon}(x|(P_k)_k) \leq \tau(z)^2 h_{R,\varepsilon}(x_z) + (1 - \tau(z))^2 h_{R,\varepsilon}(x_{1-z}), \text{ for all sufficiently small } \varepsilon.$$

Dividing by $\log(1/\varepsilon)$ and letting $\varepsilon \to 0$ we obtain that

$$\delta_0(x) \leq 2\tau(z)(1 - \tau(z)) + \tau(z)^2 \delta_0(x_z) + (1 - \tau(z))^2 \delta_0(x_{1-z}),$$

and by direct computation this is equivalent to the desired inequality.

We now handle the case of infinite J. We may, and will, assume that $J = \mathbb{N}$. For $n \in \mathbb{N}$, let

$z_{\leq n} = \sum_{j=1}^{n} z_j$. Then, by the case of finite J:

$$\delta_0(x) - 1 \leq (1 - \tau(z_{\leq n}))^2(\delta_0(x_{1-z_{\leq n}}) - 1) + \sum_{j=1}^{n} \tau(z_j)^2 \delta_0(x_{z_j}).$$

We have that $\delta_0(x_{1-z_{\leq n}}) \leq d$, and thus

$$\delta_0(x) - 1 \leq (1 - \tau(z_{\leq n}))^2(d - 1) + \sum_{j=1}^{n} \tau(z_j)^2 \delta_0(x_{z_j}).$$

The proof is thus completed by letting $n \to \infty$.

□

Since being strongly 1-bounded implies microstates free entropy dimension at most 1 with respect to any set of generators, the preceding lemma automatically implies the following.

Corollary 3.0.12. Let J be a countable set and $((M_j, \tau_j))_{j\in J}$ tracial von Neumann algebras. Suppose that $(\lambda_j)_{j\in J} \in (0,1]^J$ with $\sum_{j\in J} \lambda_j = 1$. Let $(M, \tau) = \bigoplus_j \lambda_j (M_j, \tau_j)$. Assume each (M_j, τ_j) is strongly 1-bounded. Then for any $x \in M_{\mathrm{sa}}^d$ with $W^*(x) = M$ we have $\delta_0(x) \leq 1$.

We now recover the results of Jung-Shlyakhtenko.

Corollary 3.0.13. Let (M, τ) be a Property (T) von Neumann algebra which is finitely generated. Suppose that $x \in M_{\mathrm{sa}}^d$ satisfies $W^*(x) = M$. Then $\delta_0(x) \leq 1$.

Proof. We may write $M = M_0 \oplus \bigoplus_{j\in J} M_j$ where J is a (potentially empty) countable set, each M_j is a Property (T) factor, and M_0 is either $\{0\}$ or an algebra with diffuse center. Since M_0 either has diffuse center or is $\{0\}$, we have $h(M_0) \leq 0$. If J is finite, we see that M is strongly 1-bounded by Theorem 1.0.1, and by how 1-bounded entropy behaves under direct sums. If J is infinite, then we may apply Corollary 3.0.12 to complete the proof.

□

We remark that if one works carefully with free entropy dimension in the presence in Lemma 3.0.11, then a proof of the Corollary 3.0.12 can be given for infinite tuples as well. Using this, one can show that if M is a Property (T) algebra, then $\delta_0(x) \leq 1$ for every tuple which generates M, even if x is infinite. We will not give the full proof here, but sketch the details for the interested reader.

Given a tracial von Neumann algebra (M, τ) and finite tuples $x \in M_{\mathrm{sa}}^d, y \in M_{\mathrm{sa}}^s$ with $d, s \in \mathbb{N}$, an

$R > \max(\|x\|_\infty, \|y\|_\infty)$, a weak*-neighborhood $\mathscr{O}$ of $\ell_{x,y}$, and $\varepsilon > 0$ we set

$$\delta_{R,\varepsilon}(x : \mathscr{O}) = \limsup_{n\to\infty} \frac{K_\varepsilon(\Gamma_R^{(n)}(x : \mathscr{O}), \|\cdot\|_2)}{\log(1/\varepsilon)},$$

$$\delta_{R,\varepsilon}(x : y) = \inf_{\mathscr{O}} \delta_{R,\varepsilon}(x : \mathscr{O}),$$

where the infimum is over all weak*-neighborhoods $\mathscr{O}$ of $\ell_{x,y}$. We then define the free microstates free entropy dimension of x in the presence of y by

$$\delta_0(x : y) = \limsup_{\varepsilon\to 0} \delta_{R,\varepsilon}(x : y),$$

by standard methods we can show that this is independent of R, and this justifies dropping R from the notation. One can show that if $a \in M_{\mathrm{sa}}^t$ for some $t \in \mathbb{N}$ and $W^*(x,y) = W^*(x,a)$, then $\delta_0(x : y) = \delta_0(x : a)$. So we set

$$\delta_0(x : W^*(x,y)) = \delta_0(x : y),$$

and this does not depend upon the choice of y.

Now suppose that $x = (x_j)_{j\in J} \in M_{\mathrm{sa}}^J$ for some set J. For a finite $F \subseteq J$, let $x_F \in M_{\mathrm{sa}}^F$ be given by $x_F = (x_j)_{j\in F}$. We then set

$$\delta_0(x_F : M) = \inf_Q \delta_0(x_F : Q),$$

$$\delta_0(x : M) = \sup_F \delta_0(x_F : M),$$

where the infimum is over all finitely generated $Q \leq M$ with $x_F \in Q_{\mathrm{sa}}^F$, and the supremum is over all finite subsets F of J. We set $\delta_0(x) = \delta_0(x : W^*(x))$.

To generalize Corollary 3.0.12 to infinite tuples, one first proves a modification of Lemma 3.0.11. Namely if x is a self-adjoint tuple in M and $(z_j)_{j\in J}$ are projections in $Z(M) \cap W^*(x)$, then

$$\delta_0(x : M) - 1 \leq \sum_j \tau(z_j)^2(\delta_0(x_{z_j} : Mz_j) - 1) \tag{3.0.3}$$

To prove (3.0.3) one first handles the case x is a finite tuple, and $J = \{1,2\}$. The proof of (3.0.3) in this case is a minor modification of Lemma 3.0.11. Namely, one modifies the proof of [Jun06, Lemma 3.2 and Corollary 4.3] to show that if $z \in Z(M) \cap W^*(x)$ is a central projection, then

$$\delta_0(x, z : M) = 2\tau(z)(1 - \tau(z)) + \limsup_{\varepsilon\to 0} \frac{h_{R,\varepsilon}(x : M)}{\log(1/\varepsilon)}.$$

One then modifies the proof of [Jun03b, Corollary 5.8] to say that $\delta_0(x:M) = \delta_0(x,z:M)$. After changing the results in [Jun06, Jun03b] to work for free entropy dimension in the presence, the proof of (3.0.3) in the case that x is a finite tuple and $J = \{1,2\}$ proceeds exactly as in Lemma 3.0.11. The proof of the general case of (3.0.3) from this special case also follows precisely as in Lemma 3.0.11.

The inequality (3.0.3) automatically shows that if M is a direct sum of strongly 1-bounded algebras and if x is any self-adjoint tuple in M, then $\delta_0(x) \leq 1$, and from this one deduces a version of Corollary 3.0.13 where x is any self-adjoint tuple (finite or not).

CHAPTER 4

Strong 1-boundedness and vanishing cohomology

4.0.1 Proof of Theorem 1.0.6

In order to prove strong 1-boundedness, or equivalently that $h_R(x) < \infty$, we will estimate $h_{R,\varepsilon}(x)$ iteratively for smaller and smaller values of ε in a similar manner to Jung [Jun07b]. In particular, if $\eta \leq \varepsilon$, then we want to estimate $h_{R,\eta}(x)$ in terms of $h_{R,\varepsilon}(x)$ by covering a $(\varepsilon, \|\cdot\|_2)$-ball in the microstate space by an $(\eta, \|\cdot\|_2)$-balls.

Consider the $(\varepsilon, \|\cdot\|_2)$-ball centered at some microstate $X \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d$ with $\|X\|_\infty \leq R$. Let $D_f(X)$ denote the matrix of tensors as in the theorem statement with x replaced by X. If Y is a microstate in the $(\varepsilon, \|\cdot\|_2)$-ball of X, then by Taylor expansion $f(Y) - f(X)$ is approximately $D_f(X)\#(Y - X)$. By taking a high degree of approximation for our microstate space, we can make $f(Y) - f(X)$ arbitrarily small, and thus arrange that $Y - X$ is in the approximate kernel of $\partial f(X)$. Furthermore, because we are only considering microstates up to unitary orbits, we can assume without loss of generality that Y is the closest point in its unitary orbit to X, which implies that $\sum_{j=1}^m [X_j, Y_j] = 0$ (see Lemma 4.0.3). Hence, $Y - X$ is in the approximate kernel of $D_f(X)$. Because $\int |\log t| \, d\mu_{|D_f(x)|}(t) < \infty$ and $|D_f(X)|$ converges in distribution to $|D_f(x)|$, the dimension of the kernel of $D_f(X)$ vanishes in comparison to n^2, and we can use standard estimates on covering numbers of approximate kernels to get a bound on the η-covering number.

4.0.2 Background on non-commutative derivatives and Taylor expansion

First, we recall Voiculescu's free difference quotient. Consider the d-variable non-commutative polynomial algebra $\mathbb{C}\langle t_1, \dots, t_d \rangle$. Let $\partial_j : \mathbb{C}\langle t_1, \dots, t_d \rangle \to \mathbb{C}\langle t_1, \dots, t_d \rangle \otimes \mathbb{C}\langle t_1, \dots, t_d \rangle$ be the unique linear map satisfying

$$\partial_j [t_{i_1} \dots t_{i_k}] = \sum_{\alpha=1}^k \delta_{i_\alpha = j} t_{i_1} \dots t_{i_{\alpha-1}} \otimes t_{i_{\alpha+1}} \dots t_{i_k}.$$

The map ∂_j can also be characterized as the unique derivation $\mathbb{C}\langle t_1, \dots, t_d \rangle \to \mathbb{C}\langle t_1, \dots, t_d \rangle \otimes \mathbb{C}\langle t_1, \dots, t_d \rangle$ satisfying $\partial_j(t_i) = \delta_{i=j}(1 \otimes 1)$. Here, when we describe ∂_j as a "derivation," we are viewing $\mathbb{C}\langle t_1, \dots, t_d \rangle \otimes \mathbb{C}\langle t_1, \dots, t_d \rangle$ as a bimodule over $\mathbb{C}\langle t_1, \dots, t_d \rangle$ using the multiplication operations

$$p(f \otimes g) = pf \otimes g, \qquad (f \otimes g)p = f \otimes gp.$$

If $f = (f_1, \dots, f_m) \in \mathbb{C}\langle t_1, \dots, t_d \rangle^m$, then

$$\partial f \in \mathbb{M}_{m,d}(\mathbb{C}\langle t_1, \dots, t_d \rangle \otimes \mathbb{C}\langle t_1, \dots, t_d \rangle)$$

will denote the matrix whose (i, j) entry is $\partial_j f_i$. This matrix plays a similar role to the derivative of a function $\mathbb{R}^d \to \mathbb{C}^m$, in that it furnishes the first-order term in a non-commutative Taylor expansion for the evaluation of f on elements of a tracial von Neumann algebra.

Recall that if (M, τ) is a tracial von Neumann algebra and $f \in \mathbb{C}\langle t_1, \dots, t_d \rangle$ and $x = (x_1, \dots, x_d) \in M_{\text{sa}}^d$, then the evaluation of $f(x)$ is the image of f under the unique unital $*$-homomorphism $\mathbb{C}\langle t_1, \dots, t_d \rangle \to M$ given by $t_j \mapsto x_j$. The evaluation of $f = (f_1, \dots, f_m)$ on $x = (x_1, \dots, x_d)$ is defined by $(f_1(x), \dots, f_m(x))$. Moreover, $f, g \in \mathbb{C}\langle t_1, \dots, t_d \rangle$, we set

$$(f \otimes g)(x) = f(x) \otimes g(x)^{\text{op}} \in M \otimes M^{\text{op}},$$

where M^{op} denotes the opposite algebra of M[1], and $\otimes$ is the algebraic tensor product. By extending this operation linearly, we can define $F(x) \in \mathbb{M}_{m,d}(M \otimes M^{\text{op}})$ for $F \in \mathbb{C}\langle t_1, \dots, t_d \rangle \otimes \mathbb{C}\langle t_1, \dots, t_d \rangle$, and in fact for $F \in \mathbb{M}_{m,d}(\mathbb{C}\langle t_1, \dots, t_d \rangle \otimes \mathbb{C}\langle t_1, \dots, t_d \rangle)$.

For $a, b, x \in M$, we define

$$(a \otimes b^{\text{op}}) \# x = axb.$$

This extends to a bilinear map $(M \otimes M^{\text{op}}) \times M \to M$. If $A \in \mathbb{M}_{m,d}(M \otimes M^{\text{op}})$ and $x \in M_{\text{sa}}^d$, we define $A \# x \in M^m$ as the vector with entries

$$(A \# x)_i = \sum_{j=1}^{d} A_{i,j} \# x_j.$$

The first-order Taylor approximation is as follows. Note that in contrast with the classical Taylor approximation where the error estimates are typically given in the Euclidean norm or 2-norm on $\mathbb{R}^m$, we have to mix different non-commutative p-norms of $y - x$ in the estimates.

Lemma 4.0.1. Let $f \in \mathbb{C}\langle t_1, \dots, t_d \rangle^m$ and let $R > 0$. Then there exists a constant A_f, B_f, C_f depending only on f and R, such that for every tracial von Neumann algebra (M, τ) and $x, y \in M_{\text{sa}}^d$ with

[1] M^{op} is an algebra with the same addition and $*$-operation but the order of multiplication is reversed; note that M^{op} is a tracial von Neumann algebra

$\|x\|_\infty, \|y\|_\infty \leq R$, we have

$$\|f(x)\|_\infty \leq A_f \tag{4.0.1}$$

$$\|f(y) - f(x)\|_2 \leq B_f \|y - x\|_2 \tag{4.0.2}$$

$$\|f(y) - f(x) - \partial f(x)\#(y - x)\|_1 \leq C_f \|y - x\|_2^2. \tag{4.0.3}$$

Proof. The case of general m will follow from applying the $m = 1$ case componentwise. For the $m = 1$ case, to verify the claims for every non-commutative polynomial f, it suffices to check them for $f(t_1, \dots, t_d) = t_j$ and show that they are preserved under linear combinations and products.

(1) For $f(t_1, \dots, t_d) = t_j$, the claims hold with $A_f = R$, $B_f = 1$, $C_f = 0$ since $\partial_i f = \delta_{i=j}(1 \otimes 1)$.

(2) If f and g satisfy the claims and α, $\beta \in \mathbb{C}$, then $\alpha f + \beta g$ satisfies the claims with $A_{\alpha f + \beta g} = |\alpha| A_f + |\beta| A_g$ and the same for the B's and C's.

(3) Suppose $f, g \in \mathbb{C}\langle t_1, \dots, t_d \rangle$ satisfy the conclusions of the lemma. Then fg satisfies (4.0.1) with $A_{fg} = A_f A_g$. Moreover, by writing $(fg)(y) - (fg)(y) = (f(y) - f(x))g(y) + f(x)(g(y) - g(x))$ and using the L^2-L^∞-Hölder inequality, fg satisfies (4.0.2) with $B_{fg} = B_f A_g + A_f B_g$. Similarly, using algebraic manipulations and the fact that ∂_j is a derivation,

$$\begin{aligned}(fg)(y) - (fg)(x) - \partial(fg)(x)\#(y - x) = {} & [f(y) - f(x) - \partial f(x)\#(y - x)]g(x) \\ & + f(x)[g(y) - g(x) - \partial g(x)\#(y - x)] \\ & + (f(y) - f(x))(g(y) - g(x)).\end{aligned}$$

We estimate the first two terms by the L^1-L^∞ Hölder inequality and the third term by the L^2-L^2 Hölder inequality and thus obtain that fg satisfies (4.0.3) with $C_{fg} = C_f A_g + C_g A_f + B_f B_g$. $\square$

The following lemma will be needed to show that the spectral measures of certain operators on $\mathbb{M}_n(\mathbb{C})^d$ associated to matricial microstates for $x \in M_{\mathrm{sa}}^d$ converge as $n \to \infty$ to the spectral measures of corresponding operators from a tracial von Neumann algebra. In the following, for a tracial von Neumann algebra M, we denote by $M \overline{\otimes} M^{\mathrm{op}}$ the tracial von Neumann algebraic tensor product of M, equipped with the trace $\tau_M \otimes \tau_{M^{\mathrm{op}}}$. If $M \overline{\otimes} M^{\mathrm{op}}$ is represented on the Hilbert space H, then $\mathbb{M}_{m,d}(M \overline{\otimes} M^{\mathrm{op}})$ are represented as operators $H^d \to H^m$. Also, $\mathbb{M}_d(M \overline{\otimes} M^{\mathrm{op}})$ is a tracial von Neumann algebra and can be equipped with the normalized trace $\mathrm{tr}_d \otimes \tau_M \otimes \tau_{M^{\mathrm{op}}}$ where tr_d is the normalized trace on $\mathbb{M}_d(\mathbb{C})$. Moreover, $\mathscr{P}(\mathbb{R})$ denotes the space of probability measures on $\mathbb{R}$ equipped with the weak* topology as linear functionals on $C_0(\mathbb{R})$.

Lemma 4.0.2. Let $d, m \in \mathbb{N}$, $f \in \mathbb{C}\langle t_1, \ldots, t_d \rangle^m$, and $R > 0$. For $\ell \in \Sigma_{d,R}$, let $\pi_\ell \colon \mathbb{C}\langle t_1, \cdots, t_d \rangle \to W^*(\ell)$ be the GNS construction corresponding to ℓ as in Remark 2.0.2. Let

$$F \in \mathbb{M}_{m,d}(\mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle),$$

consider $F(\pi_\ell(t_1, \ldots, t_d)) \in \mathbb{M}_{m,d}(\mathrm{W}^*(\ell) \otimes \mathrm{W}^*(\ell)^{\mathrm{op}})$, and let $\mu_{|F(\pi_\ell(t_1,\ldots,t_d))|}$ be the spectral measure of $|F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))| = (F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))^* F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d)))^{1/2}$ as an element of $\mathbb{M}_d(\mathrm{W}^*(\ell) \overline{\otimes} \mathrm{W}^*(\ell)^{\mathrm{op}})$. Then the map $\Sigma_{d,R} \to \mathscr{P}(\mathbb{R}) : \ell \mapsto \mu_{|F(\pi_\ell(t_1,\ldots,t_d))|}$ is weak*-weak* continuous.

Proof. Because $F(x)$ is a linear combination of simple tensors of polynomials, there is some universal constant C depending on F and R such that $\|F(x)\|_{\mathbb{M}_{m,d}(M \overline{\otimes} M^{\mathrm{op}})} \leq K$ for every tuple of operators with $\|x\|_\infty \leq R$. In particular, the spectral measure of $|F(x)|$ is supported on $[0, K]$. Hence, it suffices to show that for every $\phi \in C([0,K])$, the map

$$\ell \mapsto (\mathrm{tr}_d \otimes \tau_{\mathrm{W}^*(\ell)} \otimes \tau^{\mathrm{op}}_{\mathrm{W}^*(\ell)})(\phi(|F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))|))$$

is continuous. By the Stone-Weierstrass theorem, it suffices to consider the case when $\phi(s) = \psi(s^2)$ where ψ is a polynomial. In this case,

$$\phi(|F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))|)) = \psi(F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))^* F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d)).$$

The right-hand side is just an element of $\mathbb{M}_d(\mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle)$ applied to the operators $\pi_\ell(t_1)$, $\ldots$, $\pi_\ell(t_d)$. Hence,

$$(\mathrm{tr}_d \otimes \tau_{\mathrm{W}^*(\ell)} \otimes \tau^{\mathrm{op}}_{\mathrm{W}^*(\ell)})(\phi(|F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))|)) = (\tau_{\mathrm{W}^*(\ell)} \otimes \tau^{\mathrm{op}}_{\mathrm{W}^*(\ell)})(G(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))),$$

where $G \in \mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle$ is $1/d$ times the sum of the diagonal entries of this matrix of tensors of polynomials. Since G is a linear combination of simple tensors, it suffices to show the continuity of the map

$$\ell \mapsto (\tau_{\mathrm{W}^*(\ell)} \otimes \tau^{\mathrm{op}}_{\mathrm{W}^*(\ell)})((f \otimes g)(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))),$$

where $f, g \in \mathbb{C}\langle t_1, \ldots, t_d \rangle$. But the right-hand side is equal to $\ell(f)\ell(g)$, and $\ell \mapsto \ell(f)\ell(g)$ is continuous by definition of the weak* topology. $\square$

4.0.3 Covering the microstate space

We now give the details of the argument sketched in §??. We begin with the orbital optimization trick. This lemma also appears in [GJNS21, Lemma 1.14], where it is related with non-commutative optimal transport theory.

Lemma 4.0.3. Let X, $Y \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d$. There exists a unitary matrix U that minimizes $\|X - UYU^*\|_2$, and any such unitary satisfies $\sum_{j=1}^d [X_j, UY_jU^*] = 0$.

Proof. A minimizer exists because the unitary group is compact and the function $U \mapsto \|X - UYU^*\|_2$ is continuous. Suppose U is a minimizer and let $A \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}$. Then

$$\begin{aligned} 0 &\le \|X - e^{itA}UYU^*e^{-itA}\|_2^2 - \|X - UYU\|_2^2 \\ &= 2\langle X, e^{itA}UYU^*e^{-itA} - UYU^*\rangle. \end{aligned}$$

Differentiating at $t = 0$, we get

$$0 = \sum_{j=1}^d \mathrm{tr}_n(X_j i[A, UY_jU^*]) = \sum_{j=1}^d \mathrm{tr}_n(i[UY_jU^*, X_j]A).$$

Because A was arbitrary, we have $\sum_{j=1}^d [UY_jU^*, X_j] = 0$. □

Next, we will give an initial form of the iterative estimate in terms of an auxiliary quantity $\Psi_{R,\eta,\delta,\varepsilon}(x, f)$ measuring the size of approximate kernels of $D_f(X)$. For a neighborhood $\mathscr{O}$ of ℓ_x in $\Sigma_{d,R}$, define

$$\Psi_{R,\eta,\delta,\varepsilon}(\mathscr{O}, f) = \limsup_{n\to\infty} \frac{1}{n^2} \sup_{X \in \Gamma_R^{(n)}(\mathscr{O})} \log K_\eta\left(\{Z : \|Z\| \le 2R,\ \|Z\|_2 < \delta,\ \|D_f(X)\#Z\|_1 < \varepsilon\},\right).$$

Note that $\Psi_{\eta,\delta,\varepsilon,R}(\mathscr{O}, f, \|\cdot\|_p, \|\cdot\|_q)$ is monotone in $\mathscr{O}$. We define

$$\Psi_{R,\eta,\delta,\varepsilon}(x, f) = \inf_{\mathscr{O}} \Psi_{R,\eta,\delta,\varepsilon}(\mathscr{O}, f)$$

At this point, the reader may be wondering why we use $\|D_f(X)\#Z\|_1 < \varepsilon$ instead of $\|D_f(X)\#Z\|_2 < \varepsilon$. The reason is that the error estimate in the non-commutative Taylor expansion requires the 1-norm rather than the 2-norm, that is, $\|f(Y) - f(X) - \partial f(X)\#(Y - X)\|_1 \le C\|Y - X\|_2^2$. Later we will work to estimate this in terms of the approximate kernel with the error measured in 2-norm.

Lemma 4.0.4. With the set up of Theorem 1.0.6, there is a constant $C > 0$ (depending only upon f and R) so that for all $\varepsilon, \eta > 0$ we have that

$$h_{R,\eta}(x) \leq h_{R,\varepsilon}(x) + \Psi_{R,\eta/2,2\varepsilon,C\varepsilon^2,R}(x,f).$$

Proof. Fix the neighborhood

$$\mathscr{U} = \left\{ \ell : \sum_{j=1}^{m} \ell(f_j^* f_j)^{1/2} < \varepsilon^2 \right\} \subseteq \Sigma_{d,R}.$$

In order to estimate $h_{R,\eta}(x, \|\cdot\|_2)$, pick a neighborhood $\mathscr{O}$ of ℓ_x, and then we will cover the microstate space $\Gamma^{(n)}(\mathscr{O} \cap \mathscr{U})$ by orbital $(\eta, \|\cdot\|_2)$-balls. Recall that if a set can be covered by a certain number of ε-balls with centers not necessarily in that set, then it can be covered with the same number of 2ε-balls with centers in the set. Hence, there exists a set $\Omega \subseteq \Gamma_R^{(n)}(\mathscr{O} \cap \mathscr{U})$ of cardinality at most $K_\varepsilon(\Gamma_R^{(n)}(\mathscr{O} \cap \mathscr{U}), \|\cdot\|_2)$ such that the $(2\varepsilon, \|\cdot\|_2)$-balls centered at X in Ω cover $\Gamma_R^{(n)}(\mathscr{O} \cap \mathscr{U})$.

We want to cover each of the orbital $(\varepsilon, \|\cdot\|_2)$-balls by orbital $(\eta, \|\cdot\|_2)$-balls. If Y is in the orbital $(2\varepsilon, \|\cdot\|_2)$-ball around X, then because we only need to cover Y up to unitary equivalence, we can assume without loss of generality that Y is the element of its orbit that is closest to X in $\|\cdot\|_2$, and thus $\sum_j [X_j, Y_j] = 0$ by Lemma 4.0.3. Recall by Lemma 4.0.1,

$$f(Y) - f(X) = \partial f(X) \# (Y - X) + \Delta_f(X,Y),$$

where the error term $\Delta_f(X,Y)$ satisfies

$$\|\Delta_f(X,Y)\|_1 \leq C_f \|Y - X\|_2^2 \leq 4C_f \varepsilon^2$$

for a constant C_f depending only on f and R. By our choice of $\mathscr{U}$, we have

$$\|f(X)\|_1 \leq \sum_{j=1}^{\infty} \mathrm{tr}_n(f_j(X)^* f_j(X))^{1/2} < \varepsilon^2,$$

and similarly $\|f(Y)\|_1 < \varepsilon^2$. It follows that

$$\|\partial f(X) \# (Y - X)\|_1 < (2 + 4C_f)\varepsilon^2.$$

Let $Z = Y - X$. Note that $\sum_{j=1}^{d} [X_j, Z_j] = \sum_{j=1}^{d} [X_j, Y_j] = 0$. Also, $\|Z\|_\infty \leq 2R$. Of course, the number of

$(\eta, \|\cdot\|_2)$-balls needed to cover the set of Z's obtained in this way is at most

$$\sup_{X \in \Gamma_R^{(n)}(\mathscr{O})} K_\eta \left(\left\{ Z : \|Z\| \le 2R,\ \|Z\|_2 < 2\varepsilon,\ \|D_f(X)\#Z\|_1 < (2+4C_f)\varepsilon^2 \right\}, \|\cdot\|_2 \right).$$

It follows that

$$K_\eta^{\mathrm{orb}}(\Gamma_R^{(n)}(\mathscr{O} \cap \mathscr{U}), \|\cdot\|_2) \le \frac{1}{N^2} K_\varepsilon^{\mathrm{orb}}(\Gamma_R^{(n)}(\mathscr{O} \cap \mathscr{U}), \|\cdot\|_2)$$
$$\sup_{X \in \Gamma_R^{(n)}(\mathscr{O})} K_{\eta/2} \left(\left\{ Z : \|Z\| \le 2R,\ \sum_{j=1}^d [Z_j, X_j] = 0,\ \|Z\|_2 < 2\varepsilon,\ \|\partial f(X)\#Z\|_1 < (2+C_f)\varepsilon^2 \right\}, \|\cdot\|_2 \right).$$

Apply $\limsup_{n\to\infty}(1/n^2)\log$ to obtain

$$h_{R,\eta}(\mathscr{O} \cap \mathscr{U}, \|\cdot\|_2) \le h_{R,\varepsilon}(\mathscr{O} \cap \mathscr{U}) + \Psi_{2R,\eta/2,\varepsilon,(2+C_f)\varepsilon^2}(\mathscr{O} \cap \mathscr{U}, f).$$

Because all the covering numbers are monotone in the "$\mathscr{O}$" variable, taking the infimum over all $\mathscr{O}$ yields the same result whether or not we intersect with $\mathscr{U}$ first. Thus, upon taking the infimum with respect to $\mathscr{O}$, we obtain the asserted result. □

4.0.4 Covering the approximate kernel

In order to convert our estimate with the $\|\cdot\|_1$-approximate kernel to an estimate with the $\|\cdot\|_2$-approximate kernel, we will estimate in Lemma 4.0.5 the $\|\cdot\|_2$-covering number of the intersection of a $\|\cdot\|_1$-ball and a $\|\cdot\|_\infty$ ball. We employ Szarek's covering estimate in a similar way to Lemma 3.0.3.

Lemma 4.0.5. There is a universal constant C such that for $t > 0$ and $\varepsilon \le 3R$,

$$\limsup_{n\to\infty} \frac{1}{n^2} \log K_\varepsilon(B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,R) \cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,t\varepsilon), \|\cdot\|_\infty) \le 12t \log \frac{CR}{\varepsilon}.$$

Proof. By Lemma 3.0.2, there exists a set Ξ of projections of rank at least $n(1-3t)$ such that every projection P of rank at least $n(1-3t)$ satisfies $\|P-Q\|_\infty < \varepsilon/3R$ for some $Q \in \Xi$ and such that

$$|\Xi| \le (1+nt)\left(\frac{6C_1R}{\varepsilon}\right)^{6n^2t}.$$

Next, for each $Q \in \Xi$, observe that $(1-Q)\mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}$ is a Hilbert space of real dimension at most $6n^2t$, and hence for some constant C_2,

$$K_{\varepsilon/3}((1-Q)B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,R),\|\cdot\|_\infty) \leq \left(\frac{3C_2R}{\varepsilon}\right)^{6n^2t}.$$

Therefore, we may choose a set Ω_Q with $|\Omega_Q| \leq \left(\frac{3C_2R}{\varepsilon}\right)^{2n^2t}$ that $(\varepsilon/3,\|\cdot\|_\infty)$-covers $(1-Q)B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,R)$.

We claim $\Omega = \bigcup_{Q\in\Xi}\Omega_Q$ is an $(\varepsilon,\|\cdot\|_\infty)$-covering of $B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,R)\cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,\delta\varepsilon/3)$. Let $A \in B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,R)\cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,t\varepsilon)$, and let $\mu_{|A|}$ be the spectral measure of $|A|$, which is supported on $[0,R]$. Let $P = 1_{[0,\varepsilon/3)}(|A|)$. Note that

$$\mu_{|A|}([\varepsilon/3,\infty)) \leq \frac{3}{\varepsilon}\int_\varepsilon^\infty x\,d\mu_{|A|}(x) \leq \frac{3\|A\|_1}{\varepsilon} \leq 3t.$$

Therefore,

$$\operatorname{rank}(P) = n\mu_{|A|}([0,\varepsilon/3)) \geq n(1-3t).$$

Choose $Q \in \Xi$ such that $\|P-Q\|_\infty < \varepsilon/3R$. There is some $B \in \Omega_Q$ such that $\|B-QA\|_\infty < \varepsilon/3$. Observe that

$$\begin{aligned}\|A-B\| &\leq \|PA\|_\infty + \|(P-Q)A\|_\infty + \|QA-B\|_\infty \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3R}R + \frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

It follows that

$$K_\varepsilon(B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,R)\cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,t\varepsilon)),\|\cdot\|_\infty) \leq (1+3nt)\left(\frac{6C_1R}{\varepsilon}\right)^{6n^2t}\left(\frac{3C_2R}{\varepsilon}\right)^{6n^2t}.$$

Let $C = \max(6C_1, 3C_2)$. Then

$$\frac{1}{n^2}\log K_\varepsilon(B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,R)\cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,t\varepsilon)) \leq \frac{1}{n^2}\log(1+3nt) + 12t\log\frac{CR}{\varepsilon}.$$

Taking $n\to\infty$, we obtain the desired estimate. $\square$

The second ingredient for estimating $\Psi_{R,\eta,\delta,\varepsilon}(x,f)$ is the following standard estimate for covering numbers of approximate kernels of operators on a Hilbert space. Of course, we will apply this lemma to the operator $D_f(X)\#$ from the Hilbert space $\mathbb{M}_n(\mathbb{C})^d$ with the normalized Hilbert-Schmidt norm $\|\cdot\|_2$ to the Hilbert space $\mathbb{M}_n(\mathbb{C})^m$ with $\|\cdot\|_2$. We remark that $B(\mathbb{M}_n(\mathbb{C})^d)$ is isomorphic to $\mathbb{M}_d(\mathbb{M}_n(\mathbb{C})\otimes$

$\mathbb{M}_n(\mathbb{C})^{\mathrm{op}}$) acting on $\mathbb{M}_n(\mathbb{C})^d$ with the # operation, and the normalized trace on $B(\mathbb{M}_n(\mathbb{C})^d)$ corresponds to $\mathrm{tr}_d \otimes \tau_{\mathbb{M}_n(\mathbb{C})} \otimes \tau_{\mathbb{M}(\mathbb{C})^{\mathrm{op}}}$.

Lemma 4.0.6. There is a universal constant $C > 0$ with the following property. Let $\mathscr{H}, \mathscr{K}$ be (complex) Hilbert spaces with $\mathscr{H}$ finite-dimensional, and let $T \in B(\mathscr{H}, \mathscr{K})$. Fix $R > 0$. For any $\delta, \varepsilon, \eta > 0$ with $\eta < 3$ we have that,

$$K_\eta(\{\xi \in \mathscr{H} : \|\xi\| < \delta, \|T\xi\| < \varepsilon) \leq \left(\frac{C\delta}{\eta}\right)^{2\dim(\mathscr{H})\mu_{|T|}\left(\left[0,\frac{2\varepsilon}{\eta}\right]\right)}.$$

Where $\mu_{|T|}$ is the spectral measure of $|T|$ with respect to the normalized trace on $B(\mathscr{H})$.

Proof. Let $P = 1_{\left[0,\frac{2\varepsilon}{\eta}\right]}(|T|)$. Suppose $\xi \in \mathscr{H}$ and $\|T\xi\| < \varepsilon$. Then, by functional calculus,

$$\|\xi - P\xi\| = \|1_{\left(\frac{2\varepsilon}{\eta},\infty\right)}(|T|)\xi\| \leq \frac{\eta}{2\varepsilon}\|T\xi\| < \frac{\eta}{2}.$$

Thus, $\{\xi \in \mathscr{H} : \|\xi\| < \delta, \|T\xi\| < \varepsilon\}$ is contained in the $\eta/2$-neighborhood of $B_{P(\mathscr{H})}(0,\delta)$. Thus,

$$K_\eta(\{\xi \in \mathscr{H} : \|\xi\| < \delta, \|T\xi\| < \varepsilon\}) \leq K_{\eta/2}(B_{P(\mathscr{H})}(0,\delta)) \leq \left(\frac{C\delta}{\eta}\right)^{2\dim(P\mathscr{H})},$$

since the real dimension of $P\mathscr{H}$ is twice the complex dimension. Then note that

$$\dim(P\mathscr{H}) = \mathrm{tr}_{\dim(\mathscr{H})}(P)\dim(\mathscr{H}) = \mu_{|T|}([0,2\varepsilon/\eta])\dim(\mathscr{H}). \qquad \square$$

Lemma 4.0.7. Let $t \in (0,1/3]$ and suppose that $Rt \geq \varepsilon$. Then for some constants C_2 and C_3 depending on f, we have

$$\Psi_{R,\eta,\delta,\varepsilon}(x,f) \leq \mu_{|D_f(x)|}([0,2\varepsilon/t\eta])\log\frac{C_2\delta}{\eta} + 12mt\log\frac{C_3Rmt}{\varepsilon}$$

Proof. Let $\mathscr{O}$ be a neighborhood of ℓ_x and $X \in \Gamma^{(n)}(\mathscr{O})$. We want to estimate the $(\eta, \|\cdot\|_2)$ covering number of

$$B_{\mathbb{M}_n(\mathbb{C})^d_{\mathrm{sa}},\|\cdot\|_\infty}(0,R) \cap B_{\mathbb{M}_n(\mathbb{C})^d_{\mathrm{sa}},\|\cdot\|_2}(0,\delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_1}(0,\varepsilon)).$$

There exists a constant C_1 depending on f such that

$$\|D_f(X)\#Z\|_\infty \leq C_1\|Z\|_\infty,$$

and in particular, this is bounded by $2C_1R$ when $\|Z\|_\infty \leq 2R$. Hence, it suffices to estimate the

$(\eta, \|\cdot\|_2)$-covering number of

$$B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,\delta) \cap D_f(X)^{-1}\left(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_\infty}(0,2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_1}(0,\varepsilon)\right),$$

where we use $D_f(X)$ to denote the linear transformation $D_f(X)\# : \mathbb{M}_n(\mathbb{C})^d \to \mathbb{M}_n(\mathbb{C})^m$. Fix a set $\Omega \subseteq \mathbb{M}_n(\mathbb{C})^m$ that $(\frac{\varepsilon}{2t}, \|\cdot\|_2)$-covers $B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_\infty}(0,2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_1}(0,\varepsilon)$ and satisfies

$$\begin{aligned}|\Omega| &\leq K_{\varepsilon/2t}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_\infty}(0,2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_1}(0,\varepsilon), \|\cdot\|_2) \\ &\leq K_{\varepsilon/2mt}(B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,\varepsilon), \|\cdot\|_\infty)^m,\end{aligned}$$

where for the last several steps we used that $\|\cdot\|_2 \leq m\|\cdot\|_\infty$ on $\mathbb{M}_n(\mathbb{C})^m_{\mathrm{sa}}$ and that $B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_\infty}(0,2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_1}(0,\varepsilon)$ is contained in the product of m copies of $B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,\varepsilon)$. Then

$$\begin{aligned}&B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,\delta) \cap D_f(X)^{-1}\left(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_\infty}(0,C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_1}(0,\varepsilon)\right) \\ &\qquad\qquad \subseteq \bigcup_{Y \in \Omega} B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,\delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_2}(Y, \tfrac{\varepsilon}{2t})).\end{aligned}$$

For each $Y \in \Omega$, if $B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,\delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_2}(Y,\varepsilon/2t))$ is non-empty, then pick some Z_Y in this set, so that

$$\begin{aligned}&B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,\delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_2}(Y, \tfrac{\varepsilon}{2t})) \\ &\qquad \subseteq B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,\delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_2}(D_f(X)\#Z_Y, \tfrac{\varepsilon}{2t})) \\ &\qquad \subseteq Z_Y + \left(B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,2\delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_2}(0, \tfrac{\varepsilon}{2t}))\right).\end{aligned}$$

By Lemma 4.0.6,

$$K_\eta\left(B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,2\delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_2}(0, \tfrac{\varepsilon}{2t})), \|\cdot\|_2\right) \leq \left(\frac{C_2\delta}{\eta}\right)^{2dn^2\mu_{|D_f(X)|}([0,2\varepsilon/t\eta])}.$$

In particular,

$$\begin{aligned}&\frac{1}{n^2}\log K_\eta\left(B_{\mathbb{M}_n(\mathbb{C})^d_{\mathrm{sa}},\|\cdot\|_\infty}(0,R) \cap B_{\mathbb{M}_n(\mathbb{C})^d_{\mathrm{sa}},\|\cdot\|_2}(0,\delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m,\|\cdot\|_1}(0,\varepsilon)), \|\cdot\|_2\right) \\ &\leq 2d\left(\sup_{X \in \Gamma^{(n)}(\mathscr{U})} \mu_{|D_f(X)|}([0,2\varepsilon/t\eta])\right)\log\frac{C_2\delta}{\eta} + \frac{m}{n^2}\log K_{\varepsilon/2mt}(B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,\varepsilon), \|\cdot\|_\infty)\end{aligned}$$

By Lemma 4.0.5,

$$\limsup_{n\to\infty} \frac{1}{n^2} \log K_{\varepsilon/2mt}(B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_\infty}(0,2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C}),\|\cdot\|_1}(0,\varepsilon), \|\cdot\|_\infty) \leq 12t \log \frac{C_3Rmt}{\varepsilon}$$

Now observe that as $\mathscr{O}$ shrinks to $\{\ell_x\}$, the measures $\mu_{|D_f(X)|}$ for $X \in \Gamma^{(n)}(\mathscr{O})$ converge uniformly in distribution to $\mu_{|D_f(x)|}$ using Lemma 4.0.2. Thus, we have

$$\limsup_{n\to\infty} \sup_{X\in\Gamma^{(n)}(\mathscr{U})} \mu_{|D_f(X)|}([0,2\varepsilon/t\eta]) \leq \mu_{D_f(x)}([0,2\varepsilon/t\eta]).$$

Thus, when we take the $\limsup$ as $n \to \infty$, we obtain the assertion of the theorem. □

4.0.5 Iteration of the estimates

By combining Lemmas 4.0.4 and 4.0.7, we obtain the following bounds.

Corollary 4.0.8. Let $t \in (0,1/3]$ and $\eta \leq \varepsilon$ and $Rt \geq \varepsilon$. Then

$$h_{R,\eta}(x) \leq h_{R,\varepsilon}(x) + \mu_{|D_f(x)|}\left(\left[0, \frac{C_1\varepsilon^2}{t\eta}\right]\right) \log \frac{C_2\varepsilon}{\eta} + 12mt \log \frac{C_3Rmt}{\varepsilon^2}. \tag{4.0.4}$$

In particular, if ε is sufficiently small (depending on R and f), we can take $\eta = \varepsilon^{4/3}$ and $t = \varepsilon^{1/3}$ to get

$$h_{R,\varepsilon^{4/3}}(x,\|\cdot\|_2) \leq h_{R,\varepsilon}(x,\|\cdot\|_2) + \mu_{|D_f(x)|}([0,C_1\varepsilon^{1/3}])\log(C_2\varepsilon^{-1/3}) + 12m\varepsilon^{1/3}\log(C_3Rm\varepsilon^{-5/3}). \tag{4.0.5}$$

Proof of Theorem 1.0.6. Fix some ε sufficiently small that we can apply (4.0.5). By repeated application of that estimate,

$$\begin{aligned} &h_{R,\varepsilon^{4^k/3^k}}(x) \\ &\quad \leq h_{R,\varepsilon}(x) + \sum_{j=0}^{k-1} \left(\mu_{|D_f(x)|}([0,C_1\varepsilon^{4^j/3^{j+1}}])\log(C_2\varepsilon^{-4^j/3^{j+1}}) + 12m\varepsilon^{4^j/3^{j+1}}\log(C_3Rm\varepsilon^{-5\cdot4^j/3^{j+1}})\right). \end{aligned}$$

Recall that $h_{R,\eta}(x,\|\cdot\|_2)$ decreases to $h(x)$ as $\eta \to 0$. Thus,

$$h(M) = h_R(x) \leq h_{R,\varepsilon}(x) + \sum_{j=0}^{\infty} \left(\mu_{|D_f(x)|}([0,C_1\varepsilon^{4^j/3^{j+1}}])\log(C_2\varepsilon^{-4^j/3^{j+1}}) + 12m\varepsilon^{4^j/3^{j+1}}\log(C_3Rm\varepsilon^{-5\cdot4^j/3^{j+1}})\right).$$

Of course, because $\Gamma_R^{(n)}(\mathscr{O})$ is always contained in $B_{\mathbb{M}_n(\mathbb{C})^d,\|\cdot\|_2}(0,R)$, the first term $h_{R,\varepsilon}(x)$ is automatically finite. The summability in j of the term $12m\varepsilon^{4^j/3^{j+1}}\log(C_3Rm\varepsilon^{-5\cdot4^j/3^{j+1}})$ in the series is straightforward: $t\log(1/t^5)$ is bounded by a constant times $t^{3/2}$, hence we can estimate the terms by a constant times $\varepsilon^{4^j/3^j}$ which is in turn bounded by a geometric series. Thus, to complete the argument, it suffices to show the summability of the first term. We rewrite

$$\sum_{j=0}^{\infty} \mu_{|D_f(x)|}([0, C_1\varepsilon^{4^j/3^{j+1}}])\log(C_2\varepsilon^{-4^j/3^{j+1}}) = \int_0^{\infty} \phi(t)\, d\mu_{|D_f(x)|}(t),$$

where

$$\phi(t) = \sum_{j=0}^{\infty} \log(C_2\varepsilon^{-4^j/3^{j+1}})\, 1_{[0,C_1\varepsilon^{4^j/3^{j+1}}]}(t).$$

We claim that $\phi(t) \leq A + B\log(1/t)$ for some constants A and B (depending on ε and all the parameters in the theorem), and this claim is sufficient to complete the proof because $\mu_{|D_f(x)|}$ is a compactly supported probability measure and we assumed that $\int_0^{\infty} \log(1/t)\, d\mu_{|D_f(x)|}(t) < \infty$. For every $t \in [0, C_1\varepsilon)$, there exists a unique $k \in \mathbb{N}$ such that

$$C_1\varepsilon^{4^k/3^{k+1}} < t \leq C_1\varepsilon^{4^{k-1}/3^k}.$$

Then

$$\begin{aligned}
\phi(t) &= \sum_{j=0}^{k-1} \log(C_2\varepsilon^{-4^j/3^{j+1}}) \\
&= \sum_{j=0}^{k-1} \left(\log C_2 + \frac{4^j}{3^{j+1}} \log\frac{1}{\varepsilon}\right) \\
&\leq \left(\log C_2 + \frac{1}{3}\log\frac{1}{\varepsilon}\right) \sum_{j=0}^{k-1} \left(\frac{4}{3}\right)^j \\
&\leq 12\left(\log C_2 + \frac{1}{3}\log\frac{1}{\varepsilon}\right) \frac{4^{k-1}}{3^k} \\
&\leq 12\left(\log C_2 + \frac{1}{3}\log\frac{1}{\varepsilon}\right) \frac{\log(1/t) + \log C_1}{\log(1/\varepsilon)} \\
&= A + B\log\frac{1}{t},
\end{aligned}$$

for some constants A and B. □

Remark 4.0.9. Given the apparent freedom to choose parameters in (4.0.4), one might wonder whether it is possible to improve the argument to allow a weaker hypothesis on $\mu_{|D_f(x)|}$ than integrability of the logarithm. But in fact, this hypothesis is necessary for any argument based on (4.0.4) to bound $h(x)$. Indeed, suppose we choose a sequence ε_k decreasing to zero and $t_k \in (\varepsilon_k/R, 1/3)$ and suppose that

$$\sum_{k=0}^{\infty} \mu_{|D_f(x)|}\left(\left[0, \frac{C_1 \varepsilon_n^2}{t_k \varepsilon_{k+1}}\right]\right) \log \frac{C_2 \varepsilon_k}{\varepsilon_{k+1}} < \infty.$$

Since ε_k is decreasing and $t_k \leq 1/3$, we have $C_1 \varepsilon_k^2 / t_k \varepsilon_{n+1} \geq 3 C_2 \varepsilon_k$. Since $\varepsilon_k < 1$, we have $\log(C_2 \varepsilon_k / \varepsilon_{k+1}) \geq \log(C_2 / \varepsilon_{k+1})$. Hence,

$$\begin{aligned} \mu_{|D_f(x)|}\left(\left[0, \frac{C_1 \varepsilon_k^2}{t_k \varepsilon_{k+1}}\right]\right) \log \frac{C_2 \varepsilon_k}{\varepsilon_{k+1}} &\geq \mu_{|D_f(x)|}\left((3C_1 \varepsilon_{k+1}, 3C_1 \varepsilon_k]\right) \log(C_2/\varepsilon_{k+1}) \\ &\geq \int_{(3C_1 \varepsilon_{k+1}, 3C_1 \varepsilon_k]} \log(3C_1 C_2/t)\, d\mu_{|D_f(x)|}(t). \end{aligned}$$

Hence, if the sum converges, then $\int_0^1 \log(1/t)\, \mu_{|D_f(x)|}(t) < \infty$.

Remark 4.0.10. Although we have stated Theorem 1.0.6 only for polynomial f for simplicity, the same argument works for more general non-commutative functions. Indeed, it only requires that f has a Taylor expansion and error estimate as in Lemma 4.0.1 and that the spectral measure of $|\partial f|$ is the large-n limit of the spectral measures of corresponding operators on $M_n(\mathbb{C})^d$ as in Lemma 4.0.2. This holds for instance if f is given by a non-commutative power series with radius of convergence $R' > R$ as in [Voi94, Section 3.3]. More generally, it applies to the non-commutative C^2 functions of [JLS21] (as well as those of [DGS16]). Roughly speaking, [JLS21, §3.2] defines a space $C_{\mathrm{tr}}^k(\mathbb{R}^{*d})$ consisting of functions f that can be evaluated on self-adjoint d-tuples $(x_1, \ldots, x_d)$ from every tracial von Neumann algebra (M, τ), such that f is a Fréchet C^k map $M_{\mathrm{sa}}^d \to M$, and the Fréchet derivatives of order $j \leq k$, viewed as multilinear maps $(M_{\mathrm{sa}}^d)^j \to M$, satisfy

$$\|\partial^j f(x)[y_1, \ldots, y_j]\|_p \leq \mathrm{constant}(f, j, R) \|y_1\|_{p_1} \ldots \|y_j\|_{p_j}$$

whenever $1/p = 1/p_1 + \cdots + 1/p_j$ and $\|x\|_\infty \leq R$. In particular, the space is cooked up so that Taylor expansions with error estimates inspired by the non-commutative Hölder's inequality, such as Lemma 4.0.1, will hold. Furthermore, [JLS21, §4.4] describes a trace (as well as a log-determinant for invertible elements) on the algebra $C_{\mathrm{tr}}^{k-1}(\mathbb{R}^{*d}, \mathscr{M}^1)$ in which the first derivatives $\partial_j f$ of a trace C^k function f live. Extending this trace to $d \times d$ matrices over $C_{\mathrm{tr}}^{k-1}(\mathbb{R}^{*d}, \mathscr{M}^1)$ enables us to make sense of the spectral measure of $\partial f(x)^* \partial f(x)$. This also applies to the operator $D_f f(x)$ in Theorem 1.0.6 since the $t_j \otimes 1 - 1 \otimes t_j$ defines an element of $C_{\mathrm{tr}}^{k-1}(\mathbb{R}^{*d}, \mathscr{M}^1(\mathbb{R}^{*1}))$ for each j. Furthermore, thanks to the way that the trace on $C_{\mathrm{tr}}^{k-1}(\mathbb{R}^{*d}, \mathscr{M}^1)$ describes the asymptotic behavior of traces on matrices (see [JLS21, §4.5]), Lemma 4.0.2 generalizes to this setting. Hence, mutatis mutandis Theorem 1.0.6 generalizes to $f \in C_{\mathrm{tr}}^2(\mathbb{R}^{*d})^m$.

We have now completed the proof of Theorem 1.0.6. We refer the reader to §5.0.2 for a proof that Theorem 1.0.6 implies Theorem 1.0.5.

CHAPTER 5

Connections to L^2-invariants of sofic groups

In this section, we recall the connection between ℓ^2 cohomology and the non-commutative difference quotient (§5.0.1) exploited by Shlyakhtenko [Shl21] as well as his argument why Theorem 1.0.6 implies Theorem 1.0.5 (§5.0.2). Then we show how the argument for Theorem 1.0.5, together with Shalom's result [Sha00], furnishes an alternative proof of strong 1-boundedness for the von Neumann algebras associated to sofic Property (T) groups (§5.0.3).

5.0.1 Cocycles, derivations, and the free difference quotient

This subsection describes how to translate from group cohomology to derivations on the group algebra to the kernel of the free difference quotient ∂f for a function f associated to a group presentation, following [CS05, MS05, Tho08, Shl21].

For a $*$-algebra A and an A-A bimodule $\mathscr{H}$, let $\operatorname{Der}(A,\mathscr{H})$ denote the set of derivations $\delta\colon A\to\mathscr{H}$. If (M,τ) is a tracial von Neumann algebra, and $A\subseteq M$ is a weak*-dense $*$-subalgebra, then one bimodule of interest is $L^2(M)\otimes L^2(M)$, where A acts on the left by left multiplying by $a\otimes 1$ and on the right by right multiplying by $1\otimes a$. We have a commuting action of $M\overline{\otimes}M^{\mathrm{op}}$ on $L^2(M)\otimes L^2(M)$ where $a\otimes b^{\mathrm{op}}$ acts on $c\otimes d$ by sending it to $cb\otimes ad$. We use $\#_{\mathrm{in}}$ for this action, so

$$(a\otimes b^{\mathrm{op}})\#_{\mathrm{in}}(\xi)=(1\otimes a)\xi(b\otimes 1),$$

it is straightforward to verify that this action extends to a normal representation of $M\overline{\otimes}M^{\mathrm{op}}$ on $L^2(M)\otimes L^2(M)$. Moreover, for all $x\in M\overline{\otimes}M^{\mathrm{op}}$, all $a,b\in M$, and all $\xi\in L^2(M)\otimes L^2(M)$,

$$x\#_{\mathrm{in}}((a\otimes 1)\xi(1\otimes b))=(a\otimes 1)(x\#_{\mathrm{in}}\xi)(1\otimes b).$$

This produces an action of $M\overline{\otimes}M^{\mathrm{op}}$ on $\operatorname{Der}(A,L^2(M)\otimes L^2(M))$ by

$$(x\delta)(a)=x\#_{\mathrm{in}}(\delta(a))$$

for all $x\in M\overline{\otimes}M^{\mathrm{op}}, a\in A$. So we may regard $\operatorname{Der}(A,L^2(M)\otimes L^2(M))$ as a module over $M\overline{\otimes}M^{\mathrm{op}}$, and so it makes sense by [Lüc98] to consider

$$\dim_{M\overline{\otimes}M^{\mathrm{op}}}(\mathrm{Der}(A,L^2(M)\otimes L^2(M))).$$

We have a special class of derivatives called the inner derivations. We say that δ is inner if there is a $\xi \in L^2(M)\otimes L^2(M)$ with $\delta(a) = [a,\xi]$. We let $\mathrm{Inn}(A,L^2(M)\otimes L^2(M))$ be the inner derivations, and let

$$H^1(A,\tau) = \frac{\mathrm{Der}(A,L^2(M)\otimes L^2(M))}{\mathrm{Inn}(A,L^2(M)\otimes L^2(M)}.$$

We define the first L^2-Betti number of A by

$$\beta^1_{(2)}(A,\tau) = \dim_{M\overline{\otimes}M^{\mathrm{op}}}(H^1(A,\tau))$$

This definition is due to Connes-Shlyakhtenko [CS05].

Proposition 5.0.1. Let G be a countable, discrete group, let τ be the canonical trace, and set $M = L(G)$. Then

(i) $\beta^1_{(2)}(G) = \beta^1_{(2)}(\mathbb{C}[G],\tau)$. In particular, if G is infinite, then

$$\beta^1_{(2)}(G)+1 = \dim_{M\overline{\otimes}M^{\mathrm{op}}}(\mathrm{Der}(\mathbb{C}[G],L^2(M)\otimes L^2(M)))$$

(ii) Suppose that G is finitely generated, and suppose $g_1,\cdots,g_r$ is a finite generating set. Set

$$x = (\mathrm{Re}(g_1),\mathrm{Im}(g_1),\mathrm{Re}(g_2),\mathrm{Im}(g_2),\cdots,\mathrm{Re}(g_k),\mathrm{Im}(g_k)) \in (\mathbb{C}[G]_{\mathrm{sa}})^{2r}.$$

where $\mathrm{Re}(a) = \frac{a+a^*}{2}, \mathrm{Im}(a) = \frac{a-a^*}{2i}$ for all $a \in \mathbb{C}[G]$. Let J be the kernel of the homomorphism

$$\mathrm{ev}_x\colon \mathbb{C}\langle t_1,\cdots,t_{2r}\rangle \to \mathbb{C}[G].$$

Then G is finitely presented if and only if J is finitely generated as a two-sided ideal.

Proof. (i): This is [CS05, Proposition 2.3],[MS05, Corollary 3.6], [Tho08, Section 4].

(ii): Let $\mathbb{F}_r$ be the free group on letters $a_1,\cdots,a_r$. Consider the surjective homomorphism $q\colon \mathbb{F}_r \to$

G so that $q(a_j) = g_j$, we continue to use q to denote the linear extension $q\colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G]$. Let

$$y = (\mathrm{Re}(a_1), \mathrm{Im}(a_1), \mathrm{Re}(a_2), \mathrm{Im}(a_2), \cdots, \mathrm{Re}(a_k), \mathrm{Im}(a_k)) \in (\mathbb{C}[\mathbb{F}_r])^{2r}_{\mathrm{sa}},$$

so $\mathrm{ev}_x = q \circ \mathrm{ev}_y$. Let B be the ideal in $\mathbb{C}\langle t_1, \cdots, t_{2k}\rangle$ generated by $\{[t_{2j-1}, t_{2j}] : j = 1, \cdots, k\} \cup \{t_{2j-1}^2 + t_{2j}^2 - 1 : j = 1, \cdots, k\}$, and let

$$\pi\colon \mathbb{C}\langle t_1, \cdots, t_k\rangle \to \mathbb{C}\langle t_1, \cdots, t_k\rangle / B$$

be the quotient map. Then the kernel of ev_y contains π and so ev_y descends to a map

$$\overline{\mathrm{ev}}_y\colon \mathbb{C}\langle t_1, \cdots, t_{2k}\rangle / B \to \mathbb{C}[\mathbb{F}_r]$$

with $\mathrm{ev}_y = \overline{\mathrm{ev}}_y \circ \pi$. For every $1 \leq j \leq k$, the element $u_j = t_{2j-1} + it_{2j} + B \in \mathbb{C}\langle t_1, \cdots, t_{2k}\rangle / B$ is unitary, and so there is a unique map $\phi\colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}\langle t_1, \cdots, t_{2k}\rangle / B$ which satisfies $\phi(a_j) = u_j$. Routine calculations verify that $\phi, \overline{\mathrm{ev}}_y$ are mutual inverses to each other, and so $\phi \circ \mathrm{ev}_y = \pi$.

First suppose that G is finitely presented, and let F be a finite subset of the kernel of $q : \mathbb{F}_r \to G$ so that $\ker(q)$ is the smallest normal subgroup containing F. It is direct to verify that the kernel of $q : \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G]$ is the smallest ideal in $\mathbb{C}[\mathbb{F}_r]$ containing $\{w - 1 : w \in F\}$. For $w \in F$, let $Q_w \in \mathbb{C}\langle t_1, \cdots, t_{2k}\rangle$ be any element so that $\pi(Q_w) = \phi(w)$. We leave it as an exercise to show that J is generated as a two-sided ideal by

$$\{Q_w - 1 : w \in F\} \cup \{[t_{2j-1}, t_{2j}] : j = 1, \cdots, k\} \cup \{t_{2j-1}^2 + t_{2j}^2 - 1 : j = 1\}.$$

This shows that J is finitely generated as a two-sided ideal.

Now suppose that J is finitely generated as a two-sided ideal, say by $F_1, \cdots, F_k$. Set $N = \ker(q\colon G \to \mathbb{F}_r)$, and $Q_j = \mathrm{ev}_y(F_j)$. Then $\pi(F_1), \cdots, \pi(F_k)$ generate $\ker(q \circ \overline{\mathrm{ev}}_y)$ as a two-sided ideal. Since $\overline{\mathrm{ev}}_y$ is an isomorphism, it follows that $Q_1, \cdots, Q_k$ generate $\ker(q\colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G])$ as a two-sided ideal. Observe that $\ker(q\colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G])$ is generated as a two-sided ideal by $\{x - 1 : x \in N\}$. So for $j = 1, \cdots, k$ we can find a finite $F_j \subseteq N$ so that Q_j is in the two-sided ideal generated by $\{x - 1 : x \in F_j\}$. Let $F = \bigcup_{j=1}^k F_j$, and let I be the two-sided in $\mathbb{C}[\mathbb{F}_r]$ generated by $\{x - 1 : x \in F\}$. Then $Q_j \in I$ for all j, and so $I = \ker(q\colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G])$. If $\widetilde{N}$ is the normal subgroup of G generated by R, then I is the kernel of the natural quotient map

$$\mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[\mathbb{F}_r / \widetilde{N}].$$

But $\widetilde{N} \leq N$, and we saw above that I is the kernel of the natural quotient map

$$\mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[\mathbb{F}_r/N].$$

So $N = \widetilde{N}$, and this establishes that G is finitely presented. $\square$

The following may be argued exactly as in [Shl21, Lemma 3.1].

Proposition 5.0.2. Let (M,τ) be a tracial von Neumann algebra and let $x \in M_{\mathrm{sa}}^k$ be such that $W^*(x) = M$. Let A be the $*$-algebra generated by x, and let J be the kernel of $\mathrm{ev}_x \colon \mathbb{C}\langle t_1,\cdots,t_k\rangle \to A$. Suppose that $(F_j)_{j=1}^{\infty}$ is a sequence which generates J as a two-sided ideal in $\mathbb{C}\langle t_1,\cdots,t_k\rangle$. Then the map

$$\delta \mapsto (\delta(x_j))_{j=1}^{k}$$

is an M-M bimodular isomorphism

$$\mathrm{Der}(A, L^2(M)\otimes L^2(M)) \to \bigcap_{j=1}^{\infty} \ker((\partial F_j)(x)\#).$$

5.0.2 Strong 1-boundedness from vanishing ℓ^2-Betti numbers

We have explained how to get from ℓ^2 Betti number conditions as in Theorem 1.0.5 to conditions on ∂f for some tuple f of non-commutative polynomials as in Theorem 1.0.6. The other main ingredient needed to prove Theorem 1.0.5 is positivity of Fuglede-Kadison determinant. The following theorem of Elek and Szabo is the main way we know of to guarantee positivity of Fuglede-Kadison determinants.

Theorem 5.0.3 (Theorem 5 in [ES05]). Let G be a countable, discrete, sofic group, and $m,n \in \mathbb{N}$. Fix $A \in \mathbb{M}_{m,n}(\mathbb{Z}(G))$. Then

$$\det{}^{+}_{L(G)}(A) \geq 1.$$

Note that if G is as in the statement of the above theorem, and $A \in \mathbb{M}_{m,n}(\mathbb{Q}(G))$ for some $m,n \in \mathbb{N}$, then there is a $q \in \mathbb{N}$ so that $qA \in M_{m,n}(\mathbb{Z}(G))$. Thus

$$\det{}^{+}_{L(G)}(A) = \frac{1}{q}\det{}^{+}_{L(G)}(qA) \geq \frac{1}{q} > 0.$$

Having collected the appropriate background material on derivations and L^2-Betti numbers, we now discuss why Theorem 1.0.6 implies Theorem 1.0.5.

Proof of Theorem 1.0.5 from Theorem 1.0.6. Let $G = \langle g_1, \cdots, g_s | w_1, \cdots, w_l \rangle$ be a finite presentation of G. For $1 \leq j \leq s$, set

$$x_{2j-1} = \frac{g_j + g_j^{-1}}{2}, \;\; x_{2j} = \frac{g_j - g_j^{-1}}{2i},$$

and set $x = (x_1, x_2, \cdots, x_{2s}) \in (\mathbb{C}[G]_{\text{sa}})^{2k}$. Let $q \colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G]$ and $y \in \mathbb{C}[\mathbb{F}_r]_{\text{sa}}^{2k}$ be as in the proof of Proposition 5.0.1 (ii). For $j = 1, \cdots, l + 2s$ define $f_j \in \mathbb{C}\langle t_1, \cdots, t_{2k} \rangle$ by

$$f_j = \begin{cases} w_j(t_1 + it_2, t_3 + it_4, \cdots, t_{2s-1} + it_{2s}), & \text{if } 1 \leq j \leq l \\ t_{2j-1}t_{2j} - t_{2j}t_{2j-1}, & \text{if } l+1 \leq j \leq l+s \\ t_{2j-1}^2 + t_{2j}^2 - 1, & \text{if } l+s+1 \leq j \leq l+2s. \end{cases}$$

By the proof of Proposition 5.0.1 (ii), we see that the kernel of $\text{ev}_x \colon \mathbb{C}\langle t_1, \cdots, t_{2s} \to \mathbb{C}[G]$ is generated (as an ideal) by

$$\{f_1, f_2, \cdots, f_{l+2s}\}.$$

Set $f = (f_1, \cdots, f_{l+2s})$. Let D_f be as in the statement of Theorem 1.0.6. We leave it as an exercise to verify that $D_f \in M_{l+2s+1,2s}(\mathbb{Q}(G \times G))$. By Theorem 5.0.3, we have that $\det^+_{L(G)}(D_f) > 0$, i.e. $\int_{(0,\infty)} \log(t)\, d\mu_{|D_f|}(t) > -\infty$. All that remains is to verify that D_f is injective. Recall that the $(1, j)$ entry of D_f is $x_j \otimes 1 - 1 \otimes x_j$ and the remaining rows are given by the matrix of partial derivatives ∂f discussed in §4.0.2. Suppose that $\xi \in [L^2(M) \otimes L^2(M)]^{2s}$ and $D_f \# \xi = 0$. This implies that $(\partial f)(x) \# \xi = 0$. By Proposition 5.0.2 we see that there is a derivation $\delta \colon \mathbb{C}[G] \to L^2(M) \otimes L^2(M)$ so that $\xi_j = \delta(x_j)$ for $j = 1, \cdots, 2s$. By Proposition 5.0.1 and the fact that $\beta^1_{(2)}(G) = 0$, we find that δ is approximately inner. Thus we may choose a sequence $\zeta_n \in L^2(M) \otimes L^2(M)$ so that for all $j = 1, \cdots, 2s$

$$\xi_j = \lim_{n\to\infty} [x_j, \zeta_n] = \lim_{n\to\infty} (x_j \otimes 1 - 1 \otimes x_j^{op}) \# \zeta_n.$$

Since $D_f \# \xi = 0$, we have that $(x_j \otimes 1 - 1 \otimes x_j^{op}) \# \xi_j = 0$ for all $j = 1, \cdots, 2s$. Thus, for all $j = 1, \cdots, 2s$:

$$\|\xi_j\|_2^2 = \lim_{n\to\infty} \langle \xi_j, (x_j \otimes 1 - 1 \otimes x_j^{op}) \# \zeta_n \rangle = \lim_{n\to\infty} \langle (x_j \otimes 1 - 1 \otimes x_j^{op}) \# \xi_j, \zeta_n \rangle = 0.$$

So we have shown that $\xi = 0$. Thus D_f is injective, and this completes the proof. $\square$

More generally, the same proof shows that if (A, τ) is any tracial $*$-algebra and

- $\beta^1_{(2)}(A, \tau) = 0$,

- there exists a generating tuple $x \in A_{\mathrm{sa}}^d$ and $f \in \mathbb{C}\langle t_1, \cdots, t_d\rangle^{\oplus m}$ so that $\{f_1, \cdots, f_m\}$ generates ev_x as an ideal, and with $\det_A((\partial f)(x)) > 0$,

then $W^*(A,\tau)$ is strongly 1-bounded. This recovers the case $n = \mathrm{rank}((\partial F)(x))$ of [Shl21, Theorem 2.5].

5.0.3 Strong 1-boundedness of Property (T) sofic groups from Theorem 1.0.5

We now use results of Shalom [Sha00] and Shlyakhtenko [Shl21] to give a short proof that sofic groups with Property (T) are strongly 1-bounded. We will need to use the Delorme-Guichardet Theorem [Del77, Gui72], which is about cohomology of groups with values in a unitary representation. Let G be a countable, discrete group and $\pi\colon G \to \mathscr{U}(\mathscr{H})$ a unitary representation. A cocycle for π is a map $\beta\colon G \to \mathscr{H}$ which satisfies

$$\beta(gh) = \pi(g)\beta(h) + \beta(g) \text{ for all } g, h \in G.$$

We say that β is inner if there is a $\xi \in \mathscr{H}$ so that $\beta(g) = \pi(g)\xi - \xi$. The Delorme-Guichardet theorem says that G has (T) if and only if for every cocycle on G with values in a unitary representation is inner. See [BdlHV08, Section 2.12] for a proof.

Lemma 5.0.4. Let $\widetilde{G}, G$ be Property (T) groups and let $q\colon \widetilde{G} \to G$ be a surjective homomorphism. Let $\mathscr{H}$ be an $L(G) - L(G)$ bimodule, and view $\mathscr{H}$ as a bimodule over $\mathbb{C}[\widetilde{G}]$ via q. Then every derivation $\delta\colon \mathbb{C}[\widetilde{G}] \to \mathscr{H}$ is inner.

Proof. Suppose that $\delta\colon \mathbb{C}[\widetilde{G}] \to \mathscr{H}$ is a derivation. Define $\beta\colon \widetilde{G} \to \mathscr{H}$ by $\beta(x) = \delta(x)u_{q(x)}^{-1}$. The fact that δ is a derivation implies, by a direct calculation, that β is a cocycle for π. By the Delorme-Guichardet theorem and the fact that $\widetilde{G}$ has Property (T) we know that β is inner, i.e. there is a $\xi \in \mathscr{H}$ so that $\beta(x) = u_{q(x)}\xi u_{q(x)}^{-1} - \xi$ for all $x \in \widetilde{G}$. So for all $x \in \widetilde{G}$

$$\delta(x) = \beta(x)u_{q(x)} = u_{q(x)}\xi - \xi u_{q(x)},$$

and this verifies that δ is inner.

□

We will primarily interested in the following special case of the above lemma.

Corollary 5.0.5. Let $\widetilde{G}, G$ be infinite Property (T) groups and let $q\colon \widetilde{G} \to G$ be a surjective homomorphism. Set $M = L(G)$. Then every derivation $\delta\colon \mathbb{C}[\widetilde{G}] \to L^2(M) \otimes L^2(M)$ is inner.

We now show that Property (T) sofic groups are strongly 1-bounded. This proof is different than the one we give for Theorem 1.0.1, and we argue directly from [Shl21] using a Theorem of Shalom on the structure of Property (T) groups.

Corollary 5.0.6. Let G be an infinite Property (T) sofic group. Then $L(G)$ is strongly 1-bounded.

Proof. Since G has Property (T), it is finitely generated. By a theorem of Shalom [Sha00, Theorem 6.7], there is a finitely presented Property (T) group $\widetilde{G}$ and a surjective homomorphism $q\colon \widetilde{G} \to G$. It may be that $\widetilde{G}$ is not sofic. However, we will still be able to use soficity of G to apply Shlyakhtenko's results to our setting.

Let $\widetilde{S}$ be a finite generating set of $\widetilde{G}$ and set $S = q(\widetilde{S})$. Then there is a finite set R of words in S so that $\widetilde{G}$ has a presentation $\langle S|R\rangle$. Use S to build self-adjoint generators $x = (x_1, \cdots, x_m)$ of $\mathbb{C}[G]$ which have lifts $\widetilde{x} = (\widetilde{x}_1, \cdots, \widetilde{x}_r)$ to generators of $\widetilde{G}$. Now use the relations R to produce $F_1, \cdots, F_r \in \mathbb{Q}[i]\langle t_1, \cdots, t_m\rangle$ with the property that if J is the ideal generated by $F_1, \cdots, F_r$, then the natural map $\mathbb{C}\langle t_1, \cdots, t_r\rangle \to \mathbb{C}[\widetilde{G}]$ given by $F \mapsto F(\widetilde{x})$ has kernel J. Let $F = (F_1, \cdots, F_r)$. By the proof of Proposition 5.0.2, we have that

$$\ker((\partial F)(x)\#) \cong \mathrm{Der}(\mathbb{C}[\widetilde{G}], L^2(M) \otimes L^2(M))$$

with $M = L(G)$. By the preceding corollary, it follows that $\ker((\partial F)(x))$ corresponds under this isomorphism to the inner derivations $\mathbb{C}[\widetilde{G}] \to L^2(M) \otimes L^2(M)$, and since M is diffuse

$$\dim_{M\overline{\otimes}M^{op}}(\ker((\partial F)(x)\#)) = \dim_{M\overline{\otimes}M^{op}}(\mathrm{Inn}(\mathbb{C}[\widetilde{G}], L^2(M) \otimes L^2(M))) = 1.$$

Further, since $F_1, \cdots, F_r \in \mathbb{Q}[i]\langle t_1, \cdots, t_r\rangle$, we know from soficity of G and Theorem 5.0.3 that $\det^+_M((\partial F)(x)) > 0$. Thus a theorem of Shlyakhtenko [Shl21] implies that M is strongly 1-bounded (this also follows from our proof of Theorem 1.0.5 from Theorem 1.0.6, see the discussion at the end of the previous subsection). □

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