TOPICS IN THE PHENOMENOLOGY OF LASING AXION DISTRIBUTIONS

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Thanks to my parents, my adviser, my committee.

TABLE OF CONTENTS

LIST OF FIGURES

Plan of Discussion

In chapter I, we start with the vacuum structure in classical electrodynamics, which is the simplest theory describing one of the fundalmental fields in the universe. The contraction of the field $F^{\mu\nu}F_{\mu\nu}$ is related to the energy of the field, and the inequality $(F_{\mu\nu} - *F_{\mu\nu})^2 \geq 0$ gives a lower bound of the field energy which saturates the vacuum. As expected, this vacuum is shown to be $F_{\mu\nu} = 0$ and there is no ambiguity over what this vacuum represents (the spacetime is void of the electromagnetic field). Then looking into classical SU(2) Yang-Mills field and following similar procedures, we find that the vacuum is not uniquely determined. Instead, there are different types of vacuum categorized by a topological index q. This topological vacuum structure also appears in QCD theory since SU(2) symmetry can be embeded in SU(3). The necessary redefinition of the true vacuum or ground state gives the QCD lagrangian a new term which is a pseudoscalar and brings CP violation to QCD. However, no experimental data suggests that strong interaction violates CP. This is known as the strong CP problem, and it may jeopardize QCD as the correct theory of the strong interaction. The CP violation can be resolved if we introduce a new particle called the axion.

We discuss briefly the properties of the axion including the decay constant and the mass according to effective field theory. This analysis shows that axions are self-attracted besides gravitational effect. The bosonic nature of axions imply that a Bose-Einstein condensate of axions can result in axion clusters. The second source of axion production could be from cosmic strings. A third possibility of large scale axion production is from superradiance of primordial black holes, although there are limitations on some parameters of the black hole.

The axion also couples to the photon, thus the decay process $a \to \gamma + \gamma$ can take place which provides the basic mechanism of lasing. We then derive the kinematic relation between the momentum of the axion p^μ and the momentum of the photon k^μ. The evolution of the lasing process in a previous model was built upon the Boltzmann equation, which shows that spontaneous and stimulated decay of the axions, and back annihilation of the photons contribute to the change of the occupation number of the photons. However, the model assumed that the axion cluster is spherical symmetric in both momentum and coordinate space. A portion of our work is to investigate the lasing process when this assumption is not made.

In chapter II, we assume the axion cluster is not spherical symmetric in momentum space but it still possesses spherical symmetry in coordinate space. We expand the occupation numbers of both axions and photons in series of spherical harmonics and plug these expansions into the evolution equation. The conservation of energy and momentum allow us to integrate and simplify the equation. Since both sides of the equation are written in terms of components of spherical harmonics which are orthonormal, we arrive at equations that connect the components of occupation number of photons to the components of occupation number of axions.

As with the expansion of the occupation numbers, we can also expand the number density of the axions and the photons in series of spherical harmonics. Direct integration over the momentum space leads us to relations that tie the components of occupation number to the components of number density, for axions and photons respectively. Substitution of these relations into the equations we obtained previously gives us equations that connect the components of number density of photons to the components of number density of axions. These equations are applicable to many specific momentum distribution of axion cluster.

The first example discussed has a spherical symmetric momentum distribution and is exactly the same as the previous axion lasing model. This example should reduce to the equations given by the previous model if our calculation is not erroneous. And that is the case here which validates at least to some degree our approach of generalizing the previous model. Our second example is for axion momenta preferentially parallel to the polar axis. The photons produced from this type of distribution of axion momentum have a similar momentum distribution. Our last example is for axions with the direction of momentum parallel to the equatorial plane. Similar momentum distribution of the photons to that of axions occurs.

Chapter III is an analogy of the work done in chapter II but in coordinate space. In chapter III, we assume the axion cluster is not spherical symmetric in coordinate space, but it still possesses spherical symmetry in momentum space. We allow a nonspherically symmetric angular distribution X to modify the spherical axion spatial distribution, with the aim of finding the angular distribution Y of photons resulting from decays of axions. This aim fails to be achieved which tells us that there is no simple way to find a closed form for either X or Y. Therefore, it suggests again the series expansion approach.

Similar to what was presented in chapter II, we expand the occupation numbers and number densities of axions and photons in spherical harmonics. Then we substitute the series expansion of occupation numbers in the evolution equation and integrate the equation which is restricted by the conservation of energy and momentum. We obtain equations that connect the components of number density of photons to the components of number density of axions and show examples of application of these equations.

The first example is an axion distribution with spherical symmetry. It is expected that this example would reduce to the previous axion lasing model and this is indeed the case here. The second example shows that the distribution of the photons has the same component as that of axions which tend to concentrate more along the polar axis. The third example is also superradiance motivated and we deduce that the components of distribution of the photons are the same as those of axions.

In chapter IV, we first discuss the evolution equation of the decay process $a \to \gamma + \gamma$ in Minkowski spacetime to gain some insights on how to write the equation in curved spacetime. The equation contains integrals over several Lorentz measures which need to be expressed in covariant measure in curved spacetime. In addition, we consider the difference between coordinate time and the local time at which the decay takes place. The kinematics of the decay process is also generalized from Minkowski spacetime to curved spacetime.

After writing the evolution equation in static spacetime, we repeat the same procedures to integrate and simplify the evolution equations but with more complexity. The final results are two sets of differential equations that determine the temporal development of number densities of uniformly distributed and distorted photons and axions. We see a rapid peak in the number density of the uniformly distributed photons, we also see a rapid peak followed by a rapid trough in the number density of the distorted photons. The behavior of the uniformly distributed photons is what we expected, as it is similar to that in the previous lasing model. The unexpected behavior of the distorted photons could be a result of the gravitational tidal effect.
Chapter IV is an adaptation of the previous lasing model to static spacetime.

In chapter V, we review the effective field theory of self-interaction of the axion and propose a prescription that gives a classical potential which may describe the attraction between two axions by reverse quantization. Then we derive a continuous potential at each location if we have a continuous distribution of axions based on the classical potential between two individual axions that we proposed. A comparison between this self-attraction potential and gravitational potential is carried out. We estimate the effective range of the self-interaction potential from this comparison. Moreover, this comparison also suggests that axion stars may have inner structure made of axitons.

Using this potential, we find that the mass and radius of the axiton determine whether the axiton is relativistic or not. Even for a relativistic axiton, not all the axions are relativistic, as there are nonrelativistic regions in the axiton. The axions on the surface of an axiton could be nonrelativistic. Previous lasing model suggested that only nonrelativistic axion cluster can lase. If nonrelativistic axions do exist on the surface of an axiton, then even a relativistic axiton can lase. Finally, we show examples of lasing from axitons with parameters which were proven to be stable in recent literature. Chapter V is an application of the previous lasing model.

CHAPTER I

The strong CP problem and axion

This chapter introduces the vacuum structure of gauge boson fields with different underlying gauge symmetry groups.

I.1 Vacuum in classical electrodynamics

The consensus is that there is little structure in the classical vacuum when the underlying gauge symmetry group is only U(1). The photon field is taken here as the subject of discussion. A classical source-free photon field $F^{\mu\nu}$ satisfies the inhomogeneous Maxwell's equation $\partial_\mu F^{\mu\nu} = 0$, where $F_{\mu\nu}$ is related to 4-vector field A_μ through $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The components of $F^{\mu\nu}$ and $F_{\mu\nu}$ are

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} , \quad F_{\mu\nu} = \eta_{\mu\alpha} F^{\alpha\beta} \eta_{\beta\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} .$$

The Lorentz invariant quantity $F_{\mu\nu}F^{\mu\nu}$ can be explicitly written in terms of the 3-vector fields $\vec{E}$ and $\vec{B}$ by

$$F^{\mu\nu}F_{\mu\nu} = 2(B^2 - \frac{E^2}{c^2}) . \tag{I.1}$$

It is worth noting that this is a property of $F^{\mu\nu}F_{\mu\nu}$ in Minkowski spacetime. In 4-dimensional Euclidean spacetime, $F^{\mu\nu}F_{\mu\nu}$ actually measures the energy density $\mathscr{E}$ of the electromagnetic field. This is easy to verify as follows. In 4-dimensional Euclidean spacetime, there is no difference between $F^{\mu\nu}$ and $F_{\mu\nu}$,

$$F^{\mu\nu} = \delta_{\mu\alpha} F^{\alpha\beta} \delta_{\beta\nu} = F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} ,$$

so the minus sign in (I.1) becomes positive,

$$F^{\mu\nu}F_{\mu\nu} = 2(B^2 + \frac{E^2}{c^2}) = 2\mu_0(\frac{B^2}{\mu_0} + \varepsilon_0 E^2) = 4\mu_0 \mathscr{E} .$$

This relation may not hold in general, but in 4-dimensional Euclidean spacetime it connects energy density of the field and a quantity which is obtained from the field tensor by a simple contraction.

The components of the Hodge dual $*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and $*F_{\mu\nu} = \eta_{\mu\alpha} * F^{\alpha\beta}\eta_{\beta\nu}$ in Minkowski spacetime are

$$*F^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}, *F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix},$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol. Note that $*F_{\mu\nu} \neq \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. The contractions between these tensors are

$$*F^{\mu\nu} * F_{\mu\nu} = 2(\frac{E^2}{c^2} - B^2) = -F^{\mu\nu}F_{\mu\nu} \qquad *F^{\mu\nu}F_{\mu\nu} = -\frac{4}{c}\vec{E}\cdot\vec{B} = F^{\mu\nu} * F_{\mu\nu}$$

Consider the square of the difference between $F_{\mu\nu}$ and $*F_{\mu\nu}$,

$$(F_{\mu\nu} - *F_{\mu\nu})^2 = (F^{\mu\nu} - *F^{\mu\nu})(F_{\mu\nu} - *F_{\mu\nu}) = \frac{8}{c}\vec{E}\cdot\vec{B}\ .$$

The sign of the value of $\vec{E}\cdot\vec{B}$ is not determined in general because of the metric signature of Minkowski spacetime. In fact it is 0 for electromagnetic wave. In 4-dimensional Euclidean spacetime,

$$*F^{\mu\nu} = \delta_{\mu\alpha} * F^{\alpha\beta}\delta_{\beta\nu} = *F_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix},$$

and

$$*F^{\mu\nu} * F_{\mu\nu} = 2(\frac{E^2}{c^2} + B^2) = F^{\mu\nu}F_{\mu\nu}\ , \qquad *F^{\mu\nu}F_{\mu\nu} = F^{\mu\nu} * F_{\mu\nu} = 0\ ,$$

therefore the inequality holds regardless of the contexts,

$$(F_{\mu\nu} - *F_{\mu\nu})^2 = (F^{\mu\nu} - *F^{\mu\nu})(F_{\mu\nu} - *F_{\mu\nu}) \geq 0\ .$$

This leads to an inequality giving the lower bound of the energy density of the field

$$\mathscr{E} = \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \geq \frac{1}{4\mu_0} * F^{\mu\nu} F_{\mu\nu} \ .$$

An integration over the entire space shows the lower bound of the energy of the field

$$E = \int d^4x \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \geq \left| \int d^4x \frac{1}{4\mu_0} * F^{\mu\nu} F_{\mu\nu} \right| = |q| \ . \tag{I.2}$$

In 4-dimensional Euclidean spacetime, we ought to integrate the energy density $\mathscr{E}$ over 4 coordinates since they are all spatial coordinates and need to be treated the same. If we stayed in Minkowski spacetime, this integration results in an action instead of energy. The absolute value is allowed to be taken on the RHS (right hand side) of the inequality because similar relations can be derived from $(F_{\mu\nu} + *F_{\mu\nu})^2$.

The electromagnetic field in 4-dimensional Euclidean spacetime has a lowest possible energy, and that energy is $|q|$, where q is given by

$$q = \int d^4x \frac{1}{4\mu_0} * F^{\mu\nu} F_{\mu\nu} \ .$$

But $*F^{\mu\nu} F_{\mu\nu} = 0$ in 4-dimensional Euclidean spacetime so $q = 0$. This is expected as it merely says that the lowest possible energy of electromagnetic wave, or the energy of the vacuum state of photon, is zero.

The self-dual and anti-self-dual fields $*F^{\mu\nu} = \pm F^{\mu\nu}$ can saturate the inequality (I.2). This means that self-dual and anti-self-dual fields are vacua in electrodynamics. However, these fields imply that $\vec{E} = c\vec{B}$ so $\vec{E}$ and $\vec{B}$ are parallel vector fields. These fields do not generally satisfy wave equations $\partial_\mu F^{\mu\nu} = 0$, $\partial_\mu * F^{\mu\nu} = 0$ with the exception that $E = 0$ and $B = 0$. The vacuum in electrodynamics obtained from self-dual and anti-self-dual fields is the vacuum we expected.

Expressing the energy density of the field as simple expressions such as a single contraction between two tensors is easier in Euclidean spacetime. Had we insisted on retaining Minkowski spacetime, we would write the energy density as T^{00} , the time-time component of the energy stress tensor of the field. Another advantage of Euclidean spacetime is the positivity of the square of tensors which is absent in Minkowski spacetime. Corresponding analysis can be performed on the classical vacuum when the underlying gauge symmetry group is SU(2).

I.2 Classical SU(2) vacuum

In 1975, Belavin, Polyakov, Schwartz, and Tyupkin [1] found the solutions of the source-free classical field equation of SU(2) Yang-Mills theory in 4 dimensional Euclidean spacetime,

$$D_\mu F^{\mu\nu} = 0 \; , \; D_\mu = \partial_\mu + \text{gauge terms} \; . \tag{I.3}$$

where D_μ is the gauge covariant derivative. The Yang-Mills field tensor $F_{\mu\nu}$ has an extra commutator term when expressed in terms of the field vector A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \; , \; A_\mu = A_\mu^j \frac{\sigma_j}{2} \; ,$$

σ_j's are Pauli matrices. The inequality

$$\text{Tr} \; (F_{\mu\nu} - *F_{\mu\nu})^2 \geq 0 \; ,$$

gives the lower bound of the energy E of the field,

$$E = \frac{1}{2} \int d^4x \; \text{Tr} \; F^{\mu\nu} F_{\mu\nu} \geq \frac{1}{2} \left| \int d^4x \; \text{Tr} \; *F^{\mu\nu} F_{\mu\nu} \right| = 8\pi^2 |q| \; , \tag{I.4}$$

where

$$q = \frac{1}{16\pi^2} \int d^4x \; \text{Tr} \; *F^{\mu\nu} F_{\mu\nu} = \frac{1}{16\pi^2} \int d^4x * F_j^{\mu\nu} F_{\mu\nu}^j \; . \quad F_{\mu\nu} = F_{\mu\nu}^j \frac{\sigma_j}{2} \; ,$$

q defines the Pontryagin class of the phase space of the Yang-Mills field, and it is both a topological and a physical attribute of the field. This did not happen in classical electrodynamics where there is no ambiguity about q which is 0. Here in Yang-Mills field theory, different solutions of equation (I.3) have different energies E. The vacuum has the lowest energy so takes the equality of (I.4). Two simple but not trivial fields which satisfy both equation (I.3) and the equality of (I.4) are self-dual and anti-self-dual fields,

$$F_{\mu\nu} = \pm * F_{\mu\nu} \; .$$

These are called BPST [1] instanton (+) and anti-instanton (-), which give Pontryagin class $q = 1$ and $q = -1$ respectively. A trivial solution of equation (I.3) and which also gives the equality of (I.4) is $A_\mu = 0$ which gives Pontryagin class $q = 0$. Note that $A_\mu = 0$ is a special case of $F_{\mu\nu} = \pm * F_{\mu\nu}$ since it satisfies both.

Now at least three different fields, $F_{\mu\nu} = \pm * F_{\mu\nu}$ and $A_\mu = 0$, solve equation (I.3) and give the equality of (I.4). They all describe the vacuum SU(2) Yang-Mills field. However, they are not just some different "representations" of the same vacuum. Many reviews on this topic are available [2–4]. Although these fields all solve the classical field equation, any one of them can not be continuously deformed into another by gauge transformation. As a result, the classical vacuum of Yang-Mills theory becomes vacua and discrete, categorized by the Pontryagin class q.

The BPST (anti)instanton was solved under Euclidean spacetime and SU(2) symmetry, but it is still useful for explaining the QCD vacuum structure. Solutions from Euclidean spacetime can be adapted to the Minkowski case through Wick rotation, $dt \to idt$. SU(3) gauge theory allows BPST (anti)instanton because there is an embedding symmetry of SU(2) into SU(3) .

I.3 The QCD θ vacuum

The vacua discussed above are solutions of classical field equation. Correspondingly, there are different vacuum states characterized by the Pontryagin class q after quantization.

$$... \, |-1\rangle \, , \; |0\rangle \, , \; |1\rangle \, , \, ...$$

Unlike the classical vacuua, these quantum vacuum states can change to one another by gauge transformations G_i's.

$$G_m \, |n\rangle = |n+m\rangle \; .$$

There is no contradiction here because this is merely a trade-off. While classical vacuum solutions belong to different q can not be deformed into each other, in the quantum picture the property of lack of deformability among vacuua is absorbed into the gauge transformations themselves. For example,

$$G_0 \, |1\rangle = |1\rangle \; , \, G_1 \, |0\rangle = |1\rangle \; ,$$

where G_0 and G_1 correspond to $q = 0$ and $q = 1$, respectively, and they can not deform into each other.

The true vacuum should be a linear combination [5,6] of all the vacuum states $|i\rangle$,

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta} |n\rangle \ ,$$

$|\theta\rangle$ is constructed in this way so that it's an eigenstate of all gauge transformations G_i's,

$$G_0 |\theta\rangle = |\theta\rangle \ , \ G_1 |\theta\rangle = e^{-i\theta} |\theta\rangle \ , ..., \ G_n |\theta\rangle = e^{-in\theta} |\theta\rangle \ .$$

$|\theta\rangle$ is changed by at most a phase factor through gauge transformations and becomes the true vacuum state.

Callan et al. [5] showed that the transition amplitude between $|m\rangle$ and $|n\rangle$ is

$$\langle n| e^{-Ht} |m\rangle = \int [dA_\mu]_{n-m} \exp\Big(- \int dx^4 \mathscr{L}^{\mathrm{E}} \Big) \ ,$$

where $[\ ]_{n-m}$ denotes functional integral of gauge field of Pontryagin class $q = n - m$. This integral still assumes Euclidean spacetime so there is no i in the argument of the exponent, which renders an oscillation in Minkowski spacetime into a convergent decay in Euclidean spacetime. Transition amplitude between $|\theta'\rangle$ and $|\theta\rangle$ can be obtained similarly

$$\begin{aligned}\langle \theta' | e^{-Ht} |\theta\rangle =& \delta(\theta' - \theta) \int [dA_\mu] \exp\Big(- iq\theta - \int dx^4 \mathscr{L}^{\mathrm{E}} \Big) \\ =& \delta(\theta' - \theta) \int [dA_\mu] \exp\Big[- \int dx^4 (\mathscr{L}^{\mathrm{E}} + \mathscr{L}^{\mathrm{E}}_\theta) \Big] \ ,\end{aligned}$$

where

$$\mathscr{L}^{\mathrm{E}}_\theta = i\theta \frac{1}{32\pi^2} * F_a^{\mu\nu} F^a_{\mu\nu} \ .$$

For QCD in Minkowski spacetime, the gluon field Lagrangian, which resembles that of photon in QED is

$$\mathscr{L}_1 = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} \ .$$

Due to the vacuum topological structure, the interaction term $\mathscr{L}_\theta$ (obtained by performing Wick rotation on $\int \mathscr{L}^{\mathrm{E}}_\theta d^4x$) needs to be added to the Lagrangian $\mathscr{L}_1$ to form a new Lagrangian $\mathscr{L}_2$ for QCD,

$$\mathscr{L}_2 = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \frac{\theta}{32\pi^2} * F_a^{\mu\nu} F^a_{\mu\nu} \ .$$

$\mathscr{L}_\theta$ is an effective interaction due to vacuum structure and it should be a component of the QCD lagrangian. The reason why $\mathscr{L}_\theta$ needs to be added to the QCD lagrangian can be described as follows. The first term is the contribution to QCD from gluon coupling, however gluons are not the only players in QCD. There is also instanton coupling which presents without inquiring whether there are gluons or not.

If there weren't vacuum structure, the mechanics of gluon and QCD seem to work without referring to the added term $*F_a^{\mu\nu}F_{\mu\nu}^a$. This new term does not indicate that the properties of gluons need to be studied from scratch or that the entire QCD formalism needs to be rewritten. Adding new terms to the Lagrangian often brings new physics into being but not always. In classical mechanics, the equations of motion won't change if a total time derivative is added to the Lagrangian,

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}f \ .$$

The analogy here is that $*F_a^{\mu\nu}F_{\mu\nu}^a$ has no effects on perturbation theory [26] since it can be written as a total derivative

$$*F_a^{\mu\nu}F_{\mu\nu}^a = \partial_\rho K^\rho \ ,$$

of the Chern-Simons current K^μ. If one does a variation on the action, the contribution from this term would be at the boundary and can be set to 0.

$*F_a^{\mu\nu}F_{\mu\nu}^a$ is the pseudoscalar that brings CP violation to QCD. This is the QCD vacuum contribution to the strong CP problem. Another mechanism due to quark mass matrix from flavordynamics also contributes to the strong CP problem. The sum of the two parts forms an effective vacuum angle $\bar{\theta} = \theta_{\text{QCD}} + \theta_{\text{QFD}}$ that characterizes the strong CP problem.

In 1977, Peccei and Quinn [7] interpreted $\bar{\theta}$ as a dynamical field and the corresponding particle is called the axion. Under this new interpretation, it is proven that the potential of the whole interaction takes a minimum at $\bar{\theta} = 0$. This confirms $\bar{\theta} = 0$ as the true vacuum and solves the strong CP problem. In exchange, the proposed particle axion needs to be discovered.

I.4 Axion cluster from BEC

The axion is a candidate of dark matter. Chiral effective field theory [15] gives a relation for the mass of the axion m_a,

$$m_a f_a = [75.5 \text{ MeV}]^2 \ ,$$

where f_a(the axion decay constant) is the radius of the circle that minimize the axion potential (this is similar to that of a Mexican hat potential). Astrophysical and cosmological constraints provide a lower and a upper bound on the value of f_a : 3×10^9 eV $\lesssim f_a \lesssim 10^{12}$ eV. This gives the favored range of values for the mass of axion 10^{-4} eV $< m_a < 10^{-4}$ eV, but there is also hadronic axion [16] of mass on the order of a few eV. Below the energy scale of QCD confinement, the effective axion Lagrangian is written as,

$$\mathscr{L} = \frac{1}{2}\partial_\mu \psi \partial^\mu \psi - V(\psi) \ ,$$

where $V(\psi)$ is the axion self interaction potential. The self interaction can be expanded in a Taylor series of the axion field ψ [18],

$$V(\psi) = \frac{1}{2} m_a^2 \psi^2 + (m_a f_a)^2 \sum_{n=2}^{\infty} \frac{\lambda_{2n}}{(2n)!} (\frac{\psi}{f_a})^{2n} \ ,$$

where the values of dimensionaless coupling constants λ_{2n} are on the order 1. Higher order terms in this potential are very weak due to the large value of f_a, therefore we may ignore the self interaction all together or retain just the ψ^4 term depending on the specific situation. As identical bosons, axions can form a Bose-Einstein condensate (BEC) [17], and thus can have very high occupation number. A gravitational bound BEC is called an axion star, and a BEC formed by self attraction is called an axiton.

I.5 Axion production from superradiance

Besides BECs, axions can also be produced by superradiance of primordial black holes. When one investigates any object under some black hole background and still utilizes the metric tensor associated with just the black hole, then the object must be of low energy scale so the gravitational field produced by the object serves only as a small perturbation to the blackground. Among all the perturbations, it has been shown that nonrotating black holes are stable to scalar, electromagnetic, and gravitational perturbations [8–10]. For rotating black holes, massive perturbation has been recognized that it can leads to instability of the black hole [11]. Massive fields have bound states due to the gravitation pull from the black hole, and at the same time the amplitude can keep growing exponentially. This is due to a Penrose type process and it fills a hydrogen-like orbit with quantum numbers $(n,l,m) = (2,1,1)$ [12].

The Klein-Gordan equation $(\nabla^{\mu}\nabla_{\mu} - \frac{m_0 G}{\hbar c})\psi = 0$ in Kerr spacetime has the following types of solution, $\psi = R(r)S(\theta)e^{im\phi}e^{-i\omega t}$. If the following conditions hold,

$$\mathrm{Re}(\omega) < \frac{am}{2Mr_+} \ , \ \mu M \ll 1 \ , \ m \geq 0 \ ,$$

where a, M and r_+ are the black hole angular momentum per unit mass, mass and event horizon respectively, then there are bound states with $\mathrm{Im}(\omega) < 0$. The fastest growing mode $l = 1, m = 1$ has an e-folding time [13] of

$$\tau = 10^{-22}\mathrm{s}(\frac{\mu}{\mu_{\pi}})(\mu M)^{-8}(\frac{M}{a})$$

where $\mu_{\pi} = Gm_{\pi}$ is the standard gravitational parameter of the pion. This could be within the range of typical evaporation time of black holes $\tau_{\mathrm{evap}} = 10^{17}\mathrm{s}(\frac{M}{2\times 10^{15}\ \mathrm{g}})^3$. There are models with axions having mass from 10^{-5} eV to 10^{-3} eV. The corresponding mass limit of black holes for superradiance to occur is higher than that (2×10^{15} g) for pions. The possible scenario is low mass axion field around primordial black holes.

I.6 Kinematics of decay

Below the energy scale of QCD confinement, the effective lagrangian of axion-photon coupling is

$$\mathscr{L}_{\mathrm{em}} = \frac{c_{\gamma}\alpha}{8\pi f_a}\psi * F^{\mu\nu}F_{\mu\nu} \ ,$$

where c_{γ} is a model dependent [19] coefficient and its magnitude is on the order of 1. α is the fine structure constant. The decay rate of the process $a \to \gamma + \gamma$ is

$$\Gamma_a = \frac{c_{\gamma}^2\alpha^2 m_a^3}{256\pi^3 f_a^2} \ .$$

In the simplest KSVZ model [20] with $m_a = 10^{-4\pm 1}$ eV, the decay rate is about $\Gamma_a = 8\times 10^{-60\pm 5}$ eV which results in very long axion lifetime ($3\times 10^{36\mp 5}$ yr) [18]. In the unit system where $c = \hbar = 1$, suppose the four-momenta of an axion and the two photons produced by its decay $a \to \gamma\gamma$ are $p^{\mu} = (p^0, \vec{p})$, $k^{\mu} = (k, \vec{k})$, $k_1^{\mu} = (k_1, \vec{k}_1)$, respectively. Then conservations of momentum and energy require $p^0 = k + k_1$, $\vec{p} = \vec{k} + \vec{k}_1$. At the same time, normalization of four-momentum gives $p^0 = \sqrt{p^2 + m_a^2}$. It can be shown that the cosine of the angle between $\vec{p}$ and $\vec{k}$ is

$$\frac{\vec{p}\cdot\vec{k}}{pk} = \frac{2p^0 k - m_a^2}{2k\sqrt{(p^0)^2 - m_a^2}} ,$$

otherwise the decay can not happen. This means that the energy of the axion p^0 and the momentum of one of the photons k completely determine the angle between $\vec{p}$ and $\vec{k}$. Because this is the cosine of some angle, it restricts the value of k,

$$k_{\min} = \frac{p^0 - \sqrt{(p^0)^2 - m_a^2}}{2} < k < \frac{p^0 + \sqrt{(p^0)^2 - m_a^2}}{2} = k_{\max} . \tag{I.5}$$

I.7 Evolution equation

Details of the setup of the evolution equation of axion cluster can be found in [14]. The cluster is assumed to contain axions and photons only. No other matter is present in the cluster. For either species of particles, axion or photon, the occupation number $f(\vec{p},\vec{r},t)$ in phase space is defined as the number of particles per unit momentum volume and per unit spatial volume at time t. Number density $n(\vec{r},t)$ is an integral of occupation number over momentum space,

$$n(\vec{r},t) = \int f(\vec{p},\vec{r},t)\frac{d^3p}{(2\pi)^3} .$$

The total number of particles of any species is then

$$N(t) = \int n(\vec{r},t) d^3r .$$

Since axions are spin 0, photons are spin 1, and in the rest frame of the axion $\vec{k}_1 = -\vec{k}$, the two photons from the decay of the common axion should have the same helicity λ. The evolution equation of the decay process is

$$2k\frac{df_\lambda(\vec{k})}{dt} = \frac{4m_a\Gamma_a}{\pi}\int \frac{d^3k_1}{2k_1^0}\frac{d^3p}{2p^0}\delta^4(p-k-k_1)\times\{f_a(\vec{p})[1+f_\lambda(\vec{k})+f_\lambda(\vec{k}_1)] - f_\lambda(\vec{k})f_\lambda(\vec{k}_1)\} , \tag{I.6}$$

where $f_\lambda(\vec{k})$ is the occupation numbers of the photon of helicity λ and Γ_a is the decay rate of the axion. The terms inside the curly braces describes spontaneous decay, stimulated decay from photon of momentum $\vec{k}$, stimulated decay from photon of momentum $\vec{k}_1$ and photon back reaction, respectively.

An important assumption is made in [14]: the momentum dependences of occupation numbers are spherical symmetric. Neither axion nor photon occupation number $f(\vec{v})$ depends on the direction of the axion velocity $\vec{v}$, or in other words, $f(\vec{v}) = f(v)$. It is under this assumption that all the following calculations developed. This significantly simplifies equation (I.6),

$$2k\frac{df_\lambda(k)}{dt} = \frac{4m_a\Gamma_a}{\pi}\int \frac{d^3k_1}{2k_1^0}\frac{d^3p}{2p^0}\delta^4(p-k-k_1)\times\{f_a(p)[1+f_\lambda(k)+f_\lambda(k_1)]-f_\lambda(k)f_\lambda(k_1)\}\ .$$

Now one of the integrations can be carried out,

$$\frac{df_\lambda(k)}{dt} = \frac{m_a\Gamma_a}{k^2}\int_{\frac{m_a^2}{4k}} dk_1\times\{f_a(k+k_1)[1+f_\lambda(k)+f_\lambda(k_1)]-f_\lambda(k)f_\lambda(k_1)\}\ . \tag{I.7}$$

The evolution equations of this case(spherical symmetric momentum dependence) is solved in [14]. Here is the summary of the results. The axion number density decreases exponentially after the lasing process is initiated. The photon number density increases exponentially in the beginning and decreases later when the axion number density decrease. In the following chapters, we will investigate the lasing effect of axion clusters which may have arbitrary momentum and coordinate dependences, respectively.

CHAPTER II

Photon directional profile from axion clouds with nonspherical momentum distributions

This chapter presents a derivation of relation between the components of photon number density and those of axions with arbitrary momentum distributions. See [21] for all the details and discussions regarding this topic.

II.1 Evolution equations for individual components of photon occupation number

In the most general case, the occupation number of particle depends on the time t, three spatial coordinates $r,\ \theta,\ \phi$(spherical coordinate system), and three momentum components $p_r,\ p_\theta,\ p_\phi$. Results in [14] were based on the condition where $f_a(t,\vec{r},\vec{p}) = a(t)\Theta(p_{max}-p)\Theta(R-r)$. Here we weaken this condition as $f_a(t,\vec{r},\vec{p}) = a(t,\vec{p})\Theta(R-r)$. We neglect the trivial position dependence(Θ function) for the moment and write $f_a(\vec{p})$ as the square of a square integrable function formed in a complex spherical harmonics expansion,

$$\begin{aligned}
f_a(\vec{p}) &= [\sum_{l'm'} a_{l'}^{m'}(p,t)Y_{l'}^{m'}(\Omega_p)]^*[\sum_{lm} a_l^m(p,t)Y_l^m(\Omega_p)] \\
&= \sum_{l'lm'm} a_{l'}^{m'*}a_l^m(-1)^{m'}Y_{l'}^{-m'}(\Omega_p)Y_l^m(\Omega_p) = \sum_{l'lm'm}(-1)^{m'}a_{l'}^{-m'*}a_l^m Y_{l'}^{m'}(\Omega_p)Y_l^m(\Omega_p) \\
&= \sum_{l'lm'm}\sum_L(-1)^{m'}a_{l'}^{-m'*}a_l^m\sqrt{\frac{(2l'+1)(2l+1)}{4\pi(2L+1)}}\,C(l',l,L|0,0,0)\,C(l',l,L|m',m,m'+m)Y_L^{m'+m}(\Omega_p)\ ,
\end{aligned}$$

where the Y_l^m's are complex spherical harmonics, and the $C(l,l',l''|m,m'm'')$'s are Clebsch-Gorden coefficients. This is overkill but it shows that one can write the occupation number as a series expansion of spherical harmonic functions, even if one only knows the amplitude of the occupation number.

By regrouping and renaming coefficients, $f_a(\vec{p})$ can be written directly as a real spherical harmonic expansion,

$$f_a(\vec{p}) = \sum_{lm} a_{lm}(p,t)Y_{lm}(\Omega_p)\ ,$$

where the Y_{lm}'s are real spherical harmonics. Similarly, photon occupation numbers can also be written as

$$f_\lambda(\vec{k}) = \sum_{lm} b_{lm}(k,t)Y_{lm}(\Omega_k)\ ,\ f_\lambda(\vec{k}_1) = \sum_{lm} b_{lm}(k_1,t)Y_{lm}(\Omega_{k_1})\ .$$

The following calculations do not put any restrictions on the coefficients a_{lm}'s or b_{lm}'s, neither do the calculations require these coefficients be positive or negative. But in the real physical world, only positive occupation numbers are allowed. In short, these coefficients can be of any value, but we should scrutinize the resulting occupation numbers so that they are positive and have real world meaning. For example, occupation number $f_a(\vec{p}) = a_{10}(p,t)Y_{10}(\Omega_p)$ can not describe any real scenario since it is negative in half of phase space, but occupation number $f_a(\vec{p}) = a_{00}(p,t)Y_{00}(\Omega_p) + a_{10}(p,t)Y_{10}(\Omega_p)$ may be allowed since it can be positive everywhere by adjusting the coefficients a_{00} and a_{10}. Nevertheless the following calculations are applicable to both of these occupation numbers, whether they have real physical meaning or not.

Equation (I.6) describes the evolution process of spontaneous and stimulated decay of axions to photons, and the back reaction of photons to axions. We write down the differential d^3k_1 and the δ-function explicitly and (I.6) becomes

$$2k\frac{df_\lambda(\vec{k})}{dt} = \frac{m_a\Gamma_a}{\pi}\int \frac{(k_1)^2 dk_1 \sin\theta_{k_1} d\theta_{k_1} d\phi_{k_1}}{k_1^0}\frac{d^3p}{p^0}\frac{1}{(k_1)^2\sin\theta_{k_1}}\delta(p^0-k^0-k_1^0)\delta(|\vec{p}-\vec{k}|-k_1) \times \delta(\theta_{\vec{p}-\vec{k}}-\theta_{\vec{k}_1})\delta(\phi_{\vec{p}-\vec{k}}-\phi_{\vec{k}_1}) \times \{f_a(\vec{p})[1+f_\lambda(\vec{k})+f_\lambda(\vec{k}_1)] - f_\lambda(\vec{k})f_\lambda(\vec{k}_1)\}\ .$$

Cancel the common factors in the numerator and denominator and substitute the series expansions of $f_\lambda(\vec{k}_1)$, we find

$$2k\frac{df_\lambda(\vec{k})}{dt} = \frac{m_a\Gamma_a}{\pi}\int \frac{dk_1}{k_1^0}\frac{d^3p}{p^0}\delta(p^0-k^0-k_1^0)\delta(|\vec{p}-\vec{k}|-k_1)\delta(\theta_{\vec{p}-\vec{k}}-\theta_{\vec{k}_1})\delta(\phi_{\vec{p}-\vec{k}}-\phi_{\vec{k}_1}) \times \{f_a(\vec{p})[1+f_\lambda(\vec{k})+\sum_{lm} b_{lm}(k_1,t)Y_{lm}(\Omega_{k_1})] - f_\lambda(\vec{k})\sum_{lm} b_{lm}(k_1,t)Y_{lm}(\Omega_{k_1})\}d\theta_{k_1}d\phi_{k_1}\ .$$

The integration over θ_{k_1} and ϕ_{k_1} can be done most efficiently by changing from Ω_{k_1} to $\Omega_{\vec{p}-\vec{k}}$. Upon use of the fact that for photons $k_1^0 = k_1$ and $k^0 = k$ along with the identity $\int \delta(x-y)\delta(x-z)dx = \delta(y-z)$, we find

$$2k\frac{df_\lambda(\vec{k})}{dt} = \frac{m_a\Gamma_a}{\pi}\int \frac{d^3p}{p^0(p^0-k)}\delta[|\vec{p}-\vec{k}|-(p^0-k)] \times \{f_a(\vec{p})[1+f_\lambda(\vec{k})+\sum_{lm} b_{lm}(p^0-k,t)Y_{lm}(\Omega_{\vec{p}-\vec{k}})] - f_\lambda(\vec{k})\sum_{lm} b_{lm}(p^0-k,t)Y_{lm}(\Omega_{\vec{p}-\vec{k}})\}\ . \quad \text{(II.1)}$$

Align the z axis in $\vec{p}$ momentum space with the $\vec{k}$, then there is a differential relation from the law of cosine,

$$|\vec{p}-\vec{k}|^2 = |\vec{p}|^2 + |\vec{k}|^2 - 2\cos\theta_p|\vec{p}||\vec{k}| \quad \Rightarrow \quad |\vec{p}-\vec{k}|d|\vec{p}-\vec{k}| = |\vec{p}||\vec{k}|d(-\cos\theta_p)\ .$$

A similar differential relation arise from normalization of 4-momentum,

$$(p^0)^2 = p^2 + m_a^2 \quad \Rightarrow \quad p^0 dp^0 = p dp \ .$$

These relations convert dp and $d(-\cos\theta_p)$ in the infinitesimal momentum volume d^3p to dp^0 and $d|\vec{p}-\vec{k}|$, respectively,

$$d^3p = p^2 dp d\Omega_p = |\vec{p}|p^0 dp^0 d(-\cos\theta_p)d\phi_p = |\vec{p}|p^0 dp^0 \frac{|\vec{p}-\vec{k}|d|\vec{p}-\vec{k}|}{|\vec{p}||\vec{k}|}d\phi_p = p^0 dp^0 \frac{|\vec{p}-\vec{k}|d|\vec{p}-\vec{k}|}{|\vec{k}|}d\phi_p \ ,$$

Substitution of this relation into (II.1) gives

$$\begin{aligned} 2k\frac{df_\lambda(\vec{k})}{dt} =& \frac{m_a\Gamma_a}{\pi}\int \frac{p^0 dp^0}{p^0(p^0-k)}\frac{|\vec{p}-\vec{k}|d|\vec{p}-\vec{k}|}{|\vec{k}|}d\phi_p \ \delta[|\vec{p}-\vec{k}|-(p^0-k)] \\ & \times\{f_a(\vec{p})[1+f_\lambda(\vec{k})+\sum_{lm} b_{lm}(p^0-k,t)Y_{lm}(\Omega_{\vec{p}-\vec{k}})] - f_\lambda(\vec{k})\sum_{lm} b_{lm}(p^0-k,t)Y_{lm}(\Omega_{\vec{p}-\vec{k}})\} \ . \end{aligned}$$

The detailed calculation of the integration over ϕ_p and $|\vec{p}-\vec{k}|$ can be found in [21] and the results is

$$\begin{aligned} 2k\frac{df_\lambda(\vec{k})}{dt} =& 2m_a\Gamma_a\int \frac{dp^0}{k} \ \{f_\lambda(\vec{k})[\sum_l a_{l0}(p,t)Y_{l0}(\theta_{p0}) - \sum_l b_{l0}(p^0-k,t)Y_{l0}(\theta_{p1})] \\ & + \sum_l a_{l0}(p,t)Y_{l0}(\theta_{p0}) + \sum_{l'lm} a_{l'm}(p,t)b_{lm}(p^0-k,t)Y_{l'}^m(\theta_{p0},0)Y_l^m(\theta_{p1},0)\} \ , \end{aligned}$$

where θ_{p0} and θ_{p1} are the angles such that

$$\cos\theta_{p0} = \frac{2p^0 k - m_a^2}{2k\sqrt{(p^0)^2 - m_a^2}} \ , \ \cos\theta_{p1} = 1 - \frac{m_a^2}{2k(p^0-k)} \ .$$

Since $p_0 = k + k_1$, we are allowed to switch from integration over p_0 to k_1 (there was an integration over k_1 but we changed it to integration over k_1^0). Writing $f_\lambda(\vec{k})$ in components, we obtained the evolution equations for individual components of the occupation number,

$$\begin{aligned} \frac{db_{lm}(k,t)}{dt} =& \frac{m_a\Gamma_a}{k^2}\int_{\frac{m_a^2}{4k}} dk_1\{\delta_{l0}\delta_{m0}2\sqrt{\pi}[\sum_{l'} a_{l'0}(p,t)Y_{l'0}(\theta_{p0}) \\ & + \sum_{l'l''m'} a_{l'm'}(p,t)b_{l''m'}(k_1,t)Y_{l'}^{m'}(\theta_{p0},0)Y_{l''}^{m'}(\theta_{p1},0)] \\ & + b_{lm}(k,t)[\sum_{l'} a_{l'0}(p,t)Y_{l'0}(\theta_{p0}) - \sum_{l'} b_{l'0}(k_1,t)Y_{l'0}(\theta_{p1})]\} \ . \end{aligned} \tag{II.2}$$

II.2 Evolution equations for individual components of photon number density

The exact form of photon and axion occupation numbers are

$$f_a(\vec{p}) = \sum_{lm} a_{lm}(p,t) Y_{lm}(\Omega_p)\Theta(R-r)\ ,\ f_\lambda(\vec{k}) = \sum_{lm} b_{lm}(k,t) Y_{lm}(\Omega_k)\Theta(R-r)\ .$$

The normal way of going from occupation number to number density is by integrating $\int f p^2 dp d\Omega_p$. This gives number density but loses the information on the direction of momentum conveyed by Ω_p. Therefore, we define the photon and axion number density to be

$$n_a(t,\Omega_p,\vec{r}) = \sum_{lm} n^a_{lm}(t)\,\Theta(R-r) Y_{lm}(\Omega_p) = \int f_a(\vec{p}) p^2 dp\ ,$$
$$n_\lambda(t,\vec{r},\Omega_k) = \sum_{lm} n^\lambda_{lm}(t)\,\Theta(R-r) Y_{lm}(\Omega_k) = \int f_\lambda(\vec{k}) k^2 dk\ .$$

To focus on the analysis of the dependence on the momentum direction, we simplify the dependence on the momentum magnitude by setting

$$a_{lm}(p,t) = a_{lm}(t)\Theta(p_{\max} - p)\ ,\ b_{lm}(k,t) = b_{lm}(t)\Theta(k_+ - k)\Theta(k - k_-)\ .$$

We also assume that photons are equally likely to have helicity $\lambda = 1$ and $\lambda = -1$; thus the total photon number density component is twice that from each individual helicity state,

$$n^+_{lm}(t) = n^-_{lm}(t)\ ,\ n^\gamma_{lm}(t) = 2 n^\lambda_{lm}(t)\ .$$

In the axion cluster, the maximum momentum is $p_{\max} = m_a\beta$, where the exact value of β is based on the dynamical condition we impose on the axion cluster. For instance, in a simple model where we ignore the pressure and consider only Newtonian self gravity of axions only, then $\beta = \sqrt{\frac{2GM}{R}}$ which is the escape velocity of the axion [14]. There are always an approach to finding the maximum momentum of the axion in any model and this maximum momentum is denoted as $m_a\beta$. k_- and k_+ are the minimum and maximum momentum of the photon, or $k_{\min}$ and $k_{\max}$ when $\sqrt{(p^0)^2 - m_a^2} = m_a\beta$ in (I.5). The integral over p provides relations between components of occupation number and number density,

$$a_{lm}(t) = \frac{24\pi^3 n^a_{lm}(t)}{m_a^3\beta^3}\ ,\ b_{lm}(t) = \frac{32\pi^3 n^\lambda_{lm}(t)}{m_a^3\beta}\ . \tag{II.3}$$

Integrating equation (II.2) and substituting relations (II.3), we have equations for individual components of photon number density, regular axion number density, and sterile axion number density,

$$\begin{aligned}
\frac{dn^{\gamma}_{lm}(t)}{dt} =& \frac{2\Gamma_a}{m_a^2\beta^2}\{\delta_{l0}\delta_{m0}\sqrt{\pi}[\sum_{l'}\frac{6}{\beta}K^0_{l'}n^a_{l'0}(t)+\frac{96\pi^3}{m_a^3\beta^2}\sum_{l'l''m'}n^a_{l'm'}(t)n^{\gamma}_{l''m'}(t)K^{01}_{l'l''m'}] \quad \text{(II.4)}\\
&+\frac{16\pi^3 n^{\gamma}_{lm}(t)}{m_a^3}\times[\frac{3}{\beta^2}\sum_{l'}n^a_{l'0}(t)K^0_{l'}-2\sum_{l'}n^{\gamma}_{l'0}(t)B^1_{l'}]\}-\frac{3c}{2R}n^{\gamma}_{lm}(t)\ ,\\
\frac{dn^{a}_{lm}(t)}{dt} =& -\frac{\Gamma_a}{m_a^2\beta^2}\{\delta_{l0}\delta_{m0}\sqrt{\pi}[\sum_{l'}\frac{6}{\beta}K^0_{l'}n^a_{l'0}(t)+\frac{96\pi^3}{m_a^3\beta^2}\sum_{l'l''m'}n^a_{l'm'}(t)n^{\gamma}_{l''m'}(t)K^{01}_{l'l''m'}] \quad \text{(II.5)}\\
&+\frac{16\pi^3 n^{\gamma}_{lm}(t)}{m_a^3}\times[\frac{3}{\beta^2}\sum_{l'}n^a_{l'0}(t)K^0_{l'}-2\sum_{l'}n^{\gamma}_{l'0}(t)N^1_{l'}]\}\ ,\\
\frac{dn^{as}_{lm}(t)}{dt} =& \frac{\Gamma_a}{m_a^2\beta^2}\frac{16\pi^3 n^{\gamma}_{lm}(t)}{m_a^3}\times 2\sum_{l'}n^{\gamma}_{l'0}(t)S^1_{l'}\ . \quad \text{(II.6)}
\end{aligned}$$

We have absorbed many integrals into the newly defined constant coefficients.

$$K^0_{l'}=\sqrt{\frac{2l'+1}{4\pi}}\int dk\int_{\frac{m_a^2}{4k}}^{m_a\gamma-k}dk_1P^0_{l'}[\frac{2(k_1+k)k-m_a^2}{2k\sqrt{(k_1+k)^2-m_a^2}}].$$

$K^0_{l'}$'s are constant coefficients describing spontaneous and half of stimulated decay.

$$\begin{aligned}
K^{01}_{l'l''m'} =& \int dk\int_{\frac{m_a^2}{4k}}^{m_a\gamma-k}dk_1Y^{m'}_{l'}(\theta_{p0},0)Y^{m'}_{l''}(\theta_{p1},0)\\
=&\sqrt{\frac{(2l'+1)(l'-m')!}{4\pi(l'+m')!}}\sqrt{\frac{(2l''+1)(l''-m')!}{4\pi(l''+m')!}}\int dk\int_{\frac{m_a^2}{4k}}^{m_a\gamma-k}dk_1P^{m'}_{l'}[\frac{2(k_1+k)k-m_a^2}{2k\sqrt{(k_1+k)^2-m_a^2}}]\,P^{m'}_{l''}[1-\frac{m_a^2}{2kk_1}].
\end{aligned}$$

$K^{01}_{l'l''m'}$'s account for the other half of stimulated decay.

$$B^1_{l'}=\int dk\int_{\frac{m_a^2}{4k}}^{k_+}dk_1Y_{l'0}(\theta_{p1})=\sqrt{\frac{2l'+1}{4\pi}}\int dk\int_{\frac{m_a^2}{4k}}^{k_+}dk_1P^0_{l'}[1-\frac{m_a^2}{2kk_1}]\ .$$

The $B^1_{l'}$ are constant coefficients describing back reaction of photons, which when necessary, can be split into $N^1_{l'}$ and $S^1_{l'}$ the part of back reactions that produce normal axions and sterile axions respectively, and hence $B^1_{l'}=N^1_{l'}+S^1_{l'}$,

$$\begin{aligned}
N^1_{l'} =& \int dk\int_{\frac{m_a^2}{4k}}^{m_a\gamma-k}dk_1Y_{l'0}(\theta_{p1})=\sqrt{\frac{2l'+1}{4\pi}}\int dk\int_{\frac{m_a^2}{4k}}^{m_a\gamma-k}dk_1P^0_{l'}[1-\frac{m_a^2}{2kk_1}]\ ,\\
S^1_{l'} =& \int dk\int_{m_a\gamma-k}^{k_+}dk_1Y_{l'0}(\theta_{p1})=\sqrt{\frac{2l'+1}{4\pi}}\int dk\int_{m_a\gamma-k}^{k_+}dk_1P^0_{l'}[1-\frac{m_a^2}{2kk_1}]\ ,
\end{aligned}$$

where P_l^m's are associated legendre polynomials.

Here we should address the production of sterile axions. In our model, we focus on axions that have momentum below $p_{\max} = m_a\beta$. Axions of Higher momentum magnitude would not be included in the region $\Theta(p_{\max} - p)$ from the mathematical point of view, and would not be bounded in the region $\Theta(R-r)$ from the physical perspective (gravity alone can not restrain axions with velocity higher than the escape velocity). Thus these axions are referred to as sterile axions. An axion of momentum $m_a\beta$ decays into two photons, if one photon has momentum k_+, then the other photon has momentum k_-. If two photons both have momentum k_+ and go through $\gamma+\gamma \to a$, then the newly created axion would have momentum larger than $m_a\beta$. This is the reason why there are sterile axions.

II.3 Examples

Equations (II.4) and (II.5) are evolution equations of individual components of the number density. They can be applied to many specific cases.

II.3.1 Y_{00} momentum distribution

Axion momentum distribution in this example has spherical symmetry. The number density of axions is

$$n_a(t,\vec{r},\Omega_p) = \sum_{lm} n^a_{lm}(t)\,\Theta(R-r)Y_{lm}(\Omega_p) = n^a_{00}(t)\,\Theta(R-r)Y_{00}(\Omega_p)\ .$$

Then, for any $l \neq 0$, $m \neq 0$, the number density component is zero, $n^a_{lm}(t) = 0 \quad (lm \neq 00)$. This means that the evolution equations (II.5) for these components reduce to

$$\frac{dn^a_{lm}(t)}{dt} = -\frac{\Gamma_a}{m_a^2\beta^2} \times \frac{16\pi^3 n^\gamma_{lm}(t)}{m_a^3} \times [\frac{3}{\beta^2}\sum_{l'} n^a_{l'0}(t)K^0_{l'} - 2\sum_{l'} n^\gamma_{l'0}(t)N^1_{l'}] = 0 \quad (lm \neq 00)\ .$$

There are two solutions of the equation above. The first solution is $n^\gamma_{lm}(t) = \delta_{l0}\delta_{m0}n^\gamma_{00}(t)$. All the photon number density components n^γ_{lm} vanish except $lm = 00$. Thus, the photon number density becomes

$$n_\gamma(t,\vec{r},\Omega_k) = n^\gamma_{00}(t)\,\Theta(R-r)Y_{00}(\Omega_k)\ .$$

Since the photon field is fixed in the Y_{00} momentum distribution, the exact form of number density of normal axion, photon and sterile axion can be obtained by solving the evolution equations (II.4), (II.5) and (II.6) which reduce to

$$\begin{aligned}
\frac{dn^{\gamma}_{00}(t)}{dt} =& \frac{2\Gamma_a}{m_a^2\beta^2}\{\sqrt{\pi}[\frac{6}{\beta}K^0_0 n^a_{00}(t) + \frac{96\pi^3}{m_a^3\beta^2} n^a_{00}(t) n^{\gamma}_{00}(t) K^{01}_{000}] \\
&+ \frac{16\pi^3 n^{\gamma}_{00}(t)}{m_a^3} \times [\frac{3}{\beta^2} n^a_{00}(t) K^0_0 - 2n^{\gamma}_{00}(t) B^1_0]\} - \frac{3c}{2R} n^{\gamma}_{00}(t) \ . \\
\frac{dn^{a}_{00}(t)}{dt} =& -\frac{\Gamma_a}{m_a^2\beta^2}\{\sqrt{\pi}[\frac{6}{\beta}K^0_0 n^a_{00}(t) + \frac{96\pi^3}{m_a^3\beta^2} \ n^a_{00}(t) n^{\gamma}_{00}(t) K^{01}_{000}] \\
&+ \frac{16\pi^3 n^{\gamma}_{00}(t)}{m_a^3} \times [\frac{3}{\beta^2} n^a_{00}(t) K^0_0 - 2n^{\gamma}_{00}(t) N^1_0]\} \ . \\
\frac{dn^{as}_{00}(t)}{dt} =& \frac{\Gamma_a}{m_a^2\beta^2} \frac{16\pi^3 n^{\gamma}_{00}(t)}{m_a^3} \times 2n^{\gamma}_{00}(t) S^1_0 \ .
\end{aligned}$$

These equations are the same as $(34'), (37')$ and $(38')$ of [14] since

$$\begin{aligned}
&K^0_0 = \frac{m_a^2\beta^3}{6\sqrt{\pi}} = N^1_0 \ , \ K^{01}_{000} = \frac{m_a^2\beta^3}{12\pi} \ , B^1_0 = \frac{m_a^2\beta^2}{4\sqrt{\pi}}(1+\frac{2\beta}{3}) \ , \ N^1_0 = \frac{m_a^2\beta^3}{6\sqrt{\pi}} \ , \ S^1_0 = \frac{m_a^2\beta^2}{4\sqrt{\pi}} \ , \\
&n^a_{00}(t) = \frac{n_a(t)}{2\sqrt{\pi}} \ , \ n^{\gamma}_{00}(t) = \frac{n_{\gamma}(t)}{2\sqrt{\pi}} \ .
\end{aligned}$$

The second solution is

$$\frac{3}{\beta^2}\sum_{l'} n^a_{l'0}(t) K^0_{l'} - 2\sum_{l'} n^{\gamma}_{l'0}(t) N^1_{l'} = 0 \ .$$

This solution means that the effect of half of stimulated decay exactly cancels the effect of photon annihilation back into regular axions. This also expresses the $l = 0$, $m = 0$ component of axion number density as a sum of components of photon number density,

$$n^a_{00}(t) = \frac{2\beta^2}{3K^0_0}\sum_{l'} n^{\gamma}_{l'0}(t) N^1_{l'} \ .$$

Evolution equation (II.5) for $l = 0$, $m = 0$ component then becomes

$$\frac{dn^a_{00}(t)}{dt} = -\frac{6\Gamma_a\sqrt{\pi}}{m_a^2\beta^3}[K^0_0 n^a_{00}(t) + \frac{16\pi^3}{m_a^3\beta}\sum_{l''} n^a_{00}(t) n^{\gamma}_{l''0}(t) K^{01}_{0l''0}] \ .$$

Evolution equations (II.4) for any $l \neq 0$, $m \neq 0$ components of photon number density in this second case is

$$\frac{dn^{\gamma}_{lm}(t)}{dt} = -\frac{64\pi^3\Gamma_a}{m_a^5\beta^2} n^{\gamma}_{lm}(t)\sum_{l'} n^{\gamma}_{l'0}(t) S^1_{l'} - \frac{3c}{2R} n^{\gamma}_{lm}(t) \ .$$

Photon annihilation back into sterile axion and surface loss are the only mechanisms that contribute to

$n^{\gamma}_{lm}(t)(l \neq 0,\ m \neq 0)$ modes. There is no source of decay providing photons to $l \neq 0$, $m \neq 0$ number density components. Therefore $n^{\gamma}_{lm}(t)(l \neq 0,\ m \neq 0)$ modes are expected to vanish quickly. The majority of photons would be in the $n^{\gamma}_{00}(t)$ component.

In both cases, either $n^{\gamma}_{lm}(t) = 0$ or $\frac{3}{\beta^2}\sum_{l'} n^a_{l'0}(t)K^0_{l'} - 2\sum_{l'} n^{\gamma}_{l'0}(t)N^1_{l'} = 0$, if the axions are locked in a Y_{00} momentum state, then so would be the photons (or at least predominantly in the second case).

II.3.2 Y_{20} momentum distribution

This configuration indicates that the direction of momentum of axions prefers to be parallel to the polar axis instead of equatorial plane. It moderately describes the movement of axions between northern and southern hemispheres

$$n_a(t,\vec{r},\Omega_p) = \sum_{lm} n^a_{lm}(t)\,\Theta(R-r)Y_{lm}(\Omega_p) = n^a_{20}(t)\,\Theta(R-r)Y_{20}(\Omega_p)\ .$$

$n^a_{20}(t)$ is the only nonzero axion number density component, the other components are zero,

$$n^a_{lm}(t) = 0 \quad (l \neq 2 \quad m \neq 0)\ .$$

The evolution equation (II.5) for $n^a_{20}(t)$ becomes

$$\frac{dn^a_{20}(t)}{dt} = -\frac{16\pi^3\Gamma_a}{m_a^5\beta^2} n^{\gamma}_{20}(t)[\frac{3}{\beta^2} n^a_{20}(t)K^0_2 - 2\sum_{l'} n^{\gamma}_{l'0}(t)N^1_{l'}] \neq 0\ .$$

At the same time, except for $n^a_{00}(t)$ and $n^a_{20}(t)$, evolution equation (II.5) reduces to

$$\frac{dn^a_{lm}(t)}{dt} = -\frac{16\pi^3\Gamma_a}{m_a^5\beta^2} n^{\gamma}_{lm}(t)[\frac{3}{\beta^2} n^a_{20}(t)K^0_2 - 2\sum_{l'} n^{\gamma}_{l'0}(t)N^1_{l'}] = 0 \quad (lm \neq 00, 20)\ .$$

Both of these equations need to hold. Comparing the these two equations, it is easy to see that the expression inside the square bracket can not be 0.

$$n^{\gamma}_{lm}(t) = 0 \quad (lm \neq 00, 20)\ .$$

This shows that there are only two nonzero components for photon number density, $n^{\gamma}_{00}(t)$ and $n^{\gamma}_{20}(t)$.

Since the $n_{00}^{a}(t)$ component of axion number density is 0, evolution equation (II.5) reduces to

$$\begin{aligned}\frac{dn_{00}^{a}(t)}{dt} =&0=-\frac{\Gamma_a}{m_a^2\beta^2}\{\sqrt{\pi}[\frac{6}{\beta}K_2^0 n_{20}^{a}(t)+\frac{96\pi^3}{m_a^3\beta^2}\sum_{l''}n_{20}^{a}(t)n_{l''0}^{\gamma}(t)K_{2l''0}^{01}] \\ &+\frac{16\pi^3 n_{00}^{\gamma}(t)}{m_a^3}[\frac{3}{\beta^2}n_{20}^{a}(t)K_2^0-2\sum_{l'}n_{l'0}^{\gamma}(t)N_{l'}^{1}]\} \ .\end{aligned}$$

This can be substituted to simplify the photon evolution equation (II.4) for component $n_{00}^{\gamma}(t)$,

$$\begin{aligned}\frac{dn_{00}^{\gamma}(t)}{dt} =&\frac{2\Gamma_a}{m_a^2\beta^2}\{\sqrt{\pi}[\frac{6}{\beta}K_2^0 n_{20}^{a}(t)+\frac{96\pi^3}{m_a^3\beta^2}\sum_{l''}n_{20}^{a}(t)n_{l''0}^{\gamma}(t)K_{2l''0}^{01}] \\ &+\frac{16\pi^3 n_{00}^{\gamma}(t)}{m_a^3}[\frac{3}{\beta^2}n_{20}^{a}(t)K_2^0-2\sum_{l'}n_{l'0}^{\gamma}(t)B_{l'}^{1}]\}-\frac{3c}{2R}n_{00}^{\gamma}(t) \\ =&-\frac{64\pi^3\Gamma_a}{m_a^5\beta^2}n_{00}^{\gamma}(t)\sum_{l'}n_{l'0}^{\gamma}(t)S_{l'}^{1}-\frac{3c}{2R}n_{00}^{\gamma}(t) \ .\end{aligned}$$

The $n_{00}^{\gamma}(t)$ component is expected to vanish quickly since the only contributions to this mode are due to back reaction to sterile axions and surface losses. Meanwhile, the photon component $n_{20}^{\gamma}(t)$ evolves according to equation (II.4),

$$\frac{dn_{20}^{\gamma}(t)}{dt} =\frac{32\pi^3\Gamma_a}{m_a^2\beta^5}n_{20}^{\gamma}(t)[\frac{3}{\beta^2}n_{20}^{a}(t)K_2^0-2\sum_{l'}n_{l'0}^{\gamma}(t)B_{l'}^{1}]-\frac{3c}{2R}n_{20}^{\gamma}(t) \ .$$

If axions were locked in a Y_{20} momentum state, the majority of photons would be expected to be in this state, which means more photons would travel in the direction that is to some extent parallel to the polar axis. However, this is not an acceptable physical number density for particles because Y_{20} is negative in some regions, but in the appropriate combinations with Y_{00}, the total number density can be positive.

II.3.3 $Y_1^{\pm1*}Y_1^{\pm1} \sim \sin^2\theta$ momentum distribution

This configuration means that the direction of momentum of the axions prefers to be parallel to the equatorial plane instead of polar axis. It roughly describes the movement of rotating axions. The axion number density has two nonzero components, n_{00}^{a} and n_{20}^{a},

$$\begin{aligned}n_a(t,\vec{r},\Omega_p) =&\sum_{lm}n_{lm}^{a}(t)Y_{lm}(\Omega_p)\,\Theta(R-r)=n^a(t)\sin^2\theta_p\,\Theta(R-r) \\ =&n^a(t)\frac{4\sqrt{\pi}}{3}[Y_{00}(\Omega_p)-\frac{1}{\sqrt{5}}Y_{20}(\Omega_p)]\,\Theta(R-r) \ .\end{aligned}$$

This provides a relation between these two nonzero components,

$$n^a_{20}(t) = -\frac{1}{\sqrt{5}} n^a_{00}(t) \ ,$$

all other components are 0. Evolution equation (II.5) for $n^a_{20}(t)$ is

$$\frac{dn^a_{20}(t)}{dt} = -\frac{16\pi^3\Gamma_a}{m_a^5\beta^2} n^\gamma_{20}(t)[\frac{3}{\beta^2}\sum_{l'} n^a_{l'0}(t)K^0_{l'} - 2\sum_{l'} n^\gamma_{l'0}(t)N^1_{l'}] \neq 0 \ .$$

All components except for $n^a_{00}(t)$ and $n^a_{20}(t)$, evolution equation (II.5) becomes

$$\frac{dn^a_{lm}(t)}{dt} = -\frac{16\pi^3\Gamma_a}{m_a^5\beta^2} n^\gamma_{lm}(t)[\frac{3}{\beta^2}\sum_{l'} n^a_{l'0}(t)K^0_{l'} - 2\sum_{l'} n^\gamma_{l'0}(t)N^1_{l'}] = 0 \quad (lm \neq 00, 20) \ .$$

Comparing these two equations, we find that the expression inside the square bracket can not be 0, so this leads to

$$n^\gamma_{lm}(t) = 0 \quad (lm \neq 00, 20) \ ,$$

which shows that there are only two nonzero components of photon number density, $n^\gamma_{00}(t)$ and $n^\gamma_{20}(t)$. After some algebraic manipulation, we can derive a relation between the photon components $n^\gamma_{00}(t)$ and $n^\gamma_{20}(t)$,

$$\frac{d}{dt}[n^\gamma_{00}(t) + \sqrt{5} n^\gamma_{20}(t)] = -[\frac{32\pi^3\Gamma_a}{m_a^5\beta^2}\sum_{l'} n^\gamma_{l'0}(t)S^1_{l'} + \frac{3c}{2R}][n^\gamma_{00}(t) + \sqrt{5} n^\gamma_{20}(t)] \ .$$

A possible but not uniquely solution of this differential equation is $n^\gamma_{00}(t) + \sqrt{5} n^\gamma_{20}(t) = 0$, which shows that axions of $\sin^2\theta$ momentem distribution can produce (although not definitely) photons of $\sin^2\theta$ momentem distribution.

II.4 Summary and conclusions of photon directional profile from momentum distribution

Equations (II.4), (II.5), (II.6) are the main result of this chapter. We have modeled clusters of axions with spherically symmetric spacial but nonspherical momentum distributions and have studied the directional profile of photos produced in their evolution through spontaneous and stimulated axion decay via the process $a \to \gamma\gamma$. These results can be used in situations where an axion cluster is formed due to primodial density perturbations or superradiance around black holes. However, this is not a superradiance-light blast simulation. It is a decay model for scalar particles that is based on statistical counting. Three specific examples were presented, one with spherical symmetry, to make contact with previous work [14], and two others of typical but simple cases without spherical symmetry. It is straightforward to use our result to model any cluster of axions with spherically symmetric spacial but nonspherical momentum distributions.

CHAPTER III

Photon directional profile from stimulated decay of axion clouds with nonspherical axion spatial distributions

This chapter presents a derivation of relation between components of photon number density and those of axions with nonspherical spacial distributions. See [22] for a full elaboration on this topic.

III.1 Separable angular dependence

The approach is very similar to that in previous chapter. Here we allow a nonspherically symmetric spatial distribution $X(\theta,\phi)$ to modify the axion clouds model previously studied in [14], with the aim of finding the angular distribution $Y(\theta,\phi)$ of photons resulting from decays of axions, provided that there is some outside constraint (e.g., a gravitational field or self interactions) that can keep the axions in the initial spatial distribution. The assumption is that angular dependences of axion and photon occupation number are simultaneously separable from the other variables. But this is not necessarily true. For such an axion distribution, assuming it factorizes, the occupation number $f_a(p,r,\theta,t)$ and number densities $n_a(r,\theta,t)$ can be written as

$$f_a(p,r,\theta,t) = f_{ac}(t)\Theta(p_{\max}-p)\,\Theta(R-r)X(\theta) \tag{III.1}$$

and

$$n_a(r,\theta,t) = \int \frac{d^3p}{(2\pi)^3} f_a(p,r,\theta,t) = \frac{m_a^3\beta^3}{6\pi^2} f_{ac}(t)\Theta(R-r)X(\theta) = n_{ac}(t)\Theta(R-r)X(\theta)$$

where we can translate between the two with

$$f_{ac}(t) = \frac{6\pi^2}{m_a^3\beta^3} n_{ac}(t). \tag{III.2}$$

Here and elsewhere we use the short hand notation $X(\theta)$ for $X(\theta,\phi)$, likewise for Y, f and n. The photons are contained in a ball of radius R, and a momentum spherical shell of inner and outer radius $k_- = \frac{m_a\gamma}{2}(1-\beta)$ and $k_+ = \frac{m_a\gamma}{2}(1+\beta)$ respectively [14], where $\beta = v/c$. There is a similar relation between axion occupation number

$$f_\lambda(k,r,\theta,t) = f_{\lambda c}(t)\Theta(k_+-k)\Theta(k-k_-)\Theta(R-r)Y(\theta)\ , \tag{III.3}$$

and number density,

$$n_\lambda(r,\theta,t) = \int \frac{d^3k}{(2\pi)^3} f_\lambda(k,r,\theta,t) = f_{\lambda c}(t)\Theta(R-r)Y(\theta)\ \frac{V_k}{8\pi^3} = n_{\lambda c}(t)\Theta(R-r)Y(\theta)\ ,$$

with

$$f_{\lambda c}(t) = \frac{8\pi^2}{m_a^3\beta} n_{\lambda c}(t)\ . \tag{III.4}$$

where as before $\lambda = \pm 1$ represents the helicity of photon. We assume that the number density of each helicity state is the same, so the total photon number density n_γ can be written as

$$\begin{aligned} n_\gamma(r,\theta,t) &= n_{\gamma c}(t)\Theta(R-r)Y(\theta) = n_+(r,\theta,t) + n_-(r,\theta,t) = [n_{+c}(t)+n_{-c}(t)]\Theta(R-r)Y(\theta) \\ n_{+c}(t) &= n_{-c}(t) \qquad n_{\gamma c}(t) = 2n_{\lambda c}(t) \end{aligned} \tag{III.5}$$

Hence the coefficient of the total photon number density is just 2 times that of photon number density of each helicity state.

The evolution relation between axion and photon occupation numbers is (I.7)

$$\frac{df_\lambda(k)}{dt} = \frac{m_a\Gamma_a}{k^2}\int_{\frac{m_a^2}{4k}} dk_1\{f_a(k+k_1)[1+f_\lambda(k)+f_\lambda(k_1)] - f_\lambda(k)f_\lambda(k_1)\}$$

where $f_\lambda(k)$ and $f_\lambda(k_1)$ are photon occupation numbers of momentum k and k_1, respectively, and Γ_a is the spontaneous axion decay rate and $\tau_a = 1/\Gamma_a$ is the dcay time scale. Other variables in $f_\lambda(k)$ and $f_\lambda(k_1)$, i.e. r,θ,t, are the same since they share the same spacetime. $f_a(k+k_1)$ is the axion occupation number of momentum $k+k_1$. Substituting (III.1) and (III.3) into (I.7) we arrive at

$$\begin{aligned} \frac{df_\lambda(k)}{dt} =& \frac{m_a\Gamma_a}{k^2}\Bigg\{[1+f_\lambda(k)]f_{ac}\Theta(R-r)X(\theta)\int_{\frac{m_a^2}{4k}}\Theta(p_{\max}-\sqrt{(k+k_1)^2-m_a^2})dk_1 \\ &+ f_{ac}f_{\lambda c}[\Theta(R-r)]^2X(\theta)Y(\theta)\int_{\frac{m_a^2}{4k}}\Theta(p_{\max}-\sqrt{(k+k_1)^2-m_a^2})\Theta(k_+-k_1)\Theta(k_1-k_-)dk_1 \\ &- f_\lambda(k)f_{\lambda c}\Theta(R-r)Y(\theta)\int_{\frac{m_a^2}{4k}}\Theta(k_+-k_1)\Theta(k_1-k_-)dk_1\Bigg\}\ . \end{aligned}$$

The first and second integrals are the same,

$$\begin{aligned} \int_{\frac{m_a^2}{4k}}\Theta(p_{\max}-\sqrt{(k+k_1)^2-m_a^2})dk_1 =& \int_{\frac{m_a^2}{4k}}\Theta(p_{\max}-\sqrt{(k+k_1)^2-m_a^2})\Theta(k_+-k_1)\Theta(k_1-k_-)dk_1 \\ =& m_a\gamma - k - \frac{m_a^2}{4k}\ . \end{aligned}$$

The third integral is related to the back reaction of photons. It is convenient to split it into two parts

$$\int_{\frac{m_a^2}{4k}} \Theta(k_+ - k_1)\Theta(k_1 - k_-)dk_1 = \int_{\frac{m_a^2}{4k}}^{m_a\gamma - k} dk_1 + \int_{m_a\gamma-k}^{k_+} dk_1 = (m_a\gamma - k - \frac{m_a^2}{4k}) + (k - k_-) \ .$$

The first part represents back reaction resulting in axions with energy less than $m_a\gamma$ so that axions that can again participate in stimulated emission, while the second part gives the back reaction resulting in sterile axions, i.e., where the total energy of the axion $k + k_1$ is larger than $m_a\gamma$. Moving the step function $\Theta(R-r)$ in front of the curly brackets and substituting the results of the integrations, we have

$$\begin{aligned}\frac{df_\lambda(k)}{dt} = \Theta(R-r)\,\frac{m_a\Gamma_a}{k^2}\Big\{&[1+f_\lambda(k)]f_{ac}X(\theta)\,(m_a\gamma - k - \frac{m_a^2}{4k})\\ &+ f_{ac}f_{\lambda c}X(\theta)\,Y(\theta)\,(m_a\gamma - k - \frac{m_a^2}{4k}) - f_\lambda(k)f_{\lambda c}Y(\theta)\,[(m_a\gamma - k - \frac{m_a^2}{4k}) + (k - k_-)]\Big\}.\end{aligned}$$

Collecting terms $f_\lambda(k)$ can be written

$$\begin{aligned}\frac{df_\lambda(k)}{dt} =&\Theta(R-r)\Theta(k_+ - k)\Theta(k - k_-)\,\frac{m_a\Gamma_a}{k^2}\\ &\times\left\{[f_{ac}(X + 2f_{\lambda c}XY) - f_{\lambda c}^2 Y^2](m_a\gamma - k - \frac{m_a^2}{4k}) - f_{\lambda c}^2 y^2 (k - k_-)\right\} .\end{aligned}$$

The rate of change of photon number density is the integration of this equation over k space

$$\begin{aligned}\frac{dn_\lambda}{dt} =&\int \frac{df_\lambda(k)}{dt}\frac{d^3k}{(2\pi)^3}\\ =&\Theta(R-r)\frac{m_a\Gamma_a}{2\pi^2}\left\{[f_{ac}(X + 2f_{\lambda c}XY) - f_{\lambda c}^2 Y^2]\int_{k_-}^{k_+}(m_a\gamma - k - \frac{m_a^2}{4k})dk - f_{\lambda c}^2 y^2 \int_{k_-}^{k_+}(k - k_-)dk\right\} .\end{aligned}$$

Evaluating the two integrals,

$$\int_{k_-}^{k_+}(m_a\gamma - k - \frac{m_a^2}{4k})dk = \frac{m_a^2\gamma^2\beta}{2} - \frac{m_a^2}{4}\ln\left(\frac{1+\beta}{1-\beta}\right) , \qquad \int_{k_-}^{k_+}(k - k_-)dk = \frac{m_a^2\gamma^2\beta^2}{2} ,$$

gives

$$\frac{dn_\lambda}{dt} = \Theta(R-r)\frac{m_a^3\Gamma_a}{8\pi^2}\left\{[f_{ac}(X + 2f_{\lambda c}XY) - f_{\lambda c}^2 Y^2][2\gamma^2\beta - \ln\left(\frac{1+\beta}{1-\beta}\right)] - f_{\lambda c}^2 Y^2 \times (2\gamma^2\beta^2)\right\} .$$

Nonrelativistic approximations for $\beta \ll 1$

$$2\gamma^2\beta = \frac{2\beta}{1-\beta^2} \approx 2\beta(1+\beta^2) = 2\beta + 2\beta^3 , \qquad \ln\left(\frac{1+\beta}{1-\beta}\right) \approx 2\beta + \frac{2\beta^3}{3} , \qquad 2\gamma^2\beta^2 \approx 2\beta^2 ,$$

simplifies this equation to

$$\frac{dn_\lambda}{dt} = \Theta(R-r)\frac{m_a^3\Gamma_a\beta^2}{6\pi^2}\left\{f_{ac}(X+2f_{\lambda c}XY)\beta - (\beta+\frac{3}{2})f_{\lambda c}^2 Y^2\right\} .$$

Substituting the derived relations (III.2) and (III.4) into the equation above, gives

$$\frac{dn_\lambda}{dt} = \Gamma_a\Theta(R-r)\left[n_{ac}(X+\frac{16\pi^2 n_{\lambda c}}{\beta m_a^3}XY) - \frac{32\pi^2 n_{\lambda c}^2}{3m_a^3}(\beta+\frac{3}{2})Y^2\right] .$$

Taking into consideration photon surface loss [14]

$$\left(\frac{dn_\lambda}{dt}\right)_{\text{surface loss}} = -\frac{3cn_\lambda}{2R} = -\frac{3c}{2R}n_{\lambda c}\Theta(R-r)Y(\theta) ,$$

we have an equation which gives the number density for each helicity state

$$\frac{dn_\lambda}{dt} = \Theta(R-r)\times\left[\frac{n_{ac}}{\tau_a}X(\theta) + \frac{16\pi^2 n_{ac}n_{\lambda c}}{\beta m_a^3\tau_a}X(\theta)Y(\theta) - \frac{32\pi^2 n_{\lambda c}^2}{3m_a^3\tau_a}(\beta+\frac{3}{2})Y(\theta)^2 - \frac{3cn_{\lambda c}}{2R}Y(\theta)\right]$$

where we are assuming, as was shown in (III.5), that total number density of photon is twice that of the individual helicity states. Therefore the rate of change of total number density of photon is

$$\frac{dn_\gamma}{dt} = \Theta(R-r)\times\left[2\frac{n_{ac}}{\tau_a}X(\theta) + \frac{16\pi^2 n_{ac}n_{\gamma c}}{\beta m_a^3\tau_a}X(\theta)Y(\theta) - \frac{16\pi^2 n_{\gamma c}^2}{3m_a^3\tau_a}(\beta+\frac{3}{2})Y^2(\theta) - \frac{3cn_{\gamma c}}{2R}Y(\theta)\right] .$$

Since the derivative operation on n_γ is passed to its time dependence,

$$\frac{dn_\gamma}{dt} = \frac{dn_{\gamma c}}{dt}\Theta(R-r)Y(\theta) ,$$

we drop the step function $\Theta(R-r)$ and have an equation for the coefficient of total number density of photon

$$\frac{dn_{\gamma c}}{dt} = 2\frac{n_{ac}}{\tau_a}\frac{X(\theta)}{Y(\theta)} + \frac{16\pi^2 n_{ac}n_{\gamma c}}{\beta m_a^3\tau_a}X(\theta) - \frac{16\pi^2 n_{\gamma c}^2}{3m_a^3\tau_a}(\beta+\frac{3}{2})Y(\theta) - \frac{3cn_{\gamma c}}{2R} . \tag{III.6}$$

From the first to the last term on the RHS of the equation, the terms account for spontaneous decay of axions, photon stimulated decay of axions, back reaction of photons, and surface loss of photons, respectively. Following similar approach, we obtain an equation regarding the coefficient of total number density of axions

$$\frac{dn_{ac}}{dt} = -\frac{n_{ac}}{\tau_a}\frac{X(\theta)}{Y(\theta)} - \frac{8\pi^2 n_{ac}n_{\gamma c}}{\beta m_a^3\tau_a}X(\theta) + \frac{8\pi^2 n_{\gamma c}^2\beta}{3m_a^3\tau_a}Y(\theta) . \tag{III.7}$$

The third term on the RHS of (III.7) is proportional to β, while the third term on the RHS of (III.6) has a factor of $(\beta + \frac{3}{2})$. Keeping track of the two parts of axions generated from the back reacting photons, we find that the $\frac{3}{2}$ in the third term on the RHS of (III.6) represents sterile axions [14] and it should have been and was excluded in the derivation of (III.7).

The left hand sides (LHS) of (III.6) and (III.7) have no θ dependence, but the RHS does. $X(\theta) = Y(\theta)$ will not make (III.6) and (III.7) valid simultaneously. So even if there is some outside constraint which can keep the axions in the $X(\theta)$ distribution fixed, the photons cannot have the same distribution, i.e., $Y(\theta) \neq X(\theta)$.

There is no simple way to find a closed form for $Y(\theta)$ because the LHS of the equations (III.6) and (III.7) have no θ dependence, while the θ dependences on the RHS of these equations are different. This suggests the possibility that $Y(\theta)$ may be found as a series expansion in $X(\theta)$. As a first test of this idea we replaced the general form $X(\theta)$ with $\sin\theta$ to study the distribution with more axions accumulated near the equatorial plane with few near the polar area, aiming at matching orders of $\sin\theta$ on each side of equations. But this fails as it turns out that $\sin^n\theta$ $(n \in \mathbb{Z})$ is not an orthogonal set of functions and thus the calculation leads to contradictions. Therefore, we must expand the occupation numbers and number density in terms of a full set of orthogonal functions. We do this in the next section where we choose the set to be the real spherical harmonics.

III.2 Real spherical harmonics expansion

The set-up here is similar to the previous discussion except that the axion and photon occupation numbers and number densities have coefficients labeled by order index l and m. Occupation number is treated as homogeneous in the radial direction. This formalism does not fully incorporate the physical mechanism of superradiance or gravity. It is more a decay model for scalar particles that is built on statistics than a superradiance-blast simulation model. Axion occupation number and number density are

$$f_a(p,r,\Omega,t) = \sum_{lm} f_{alm}(t) Y_{lm}(\Omega) \Theta(p_{\max} - p)\, \Theta(R-r)\ , \qquad n_a(r,\Omega,t) = \sum_{lm} n_{alm}(t) Y_{lm}(\Omega)\, \Theta(R-r)\ ,$$

respectively. The relation between components of axion occupation number and number density is

$$f_{alm}(t) = \frac{6\pi^2}{m_a^3 \beta^3} n_{alm}(t)\ .$$

Photon occupation number and number density are

$$f_\lambda(k,r,\Omega,t)=\sum_{lm} f_{\lambda lm}(t)Y_{lm}(\Omega)\Theta(R-r)\Theta(k_+-k)\Theta(k-k_-)\ ,\quad n_\lambda(r,\Omega,t)=\sum_{lm} n_{\lambda lm}(t)Y_{lm}(\Omega)\Theta(R-r)\ ,$$

respectively. The relation between components of photon occupation number and number density is

$$f_{\lambda lm}(t)=\frac{8\pi^2}{m_a^3\beta}n_{\lambda lm}(t)\ .$$

The number density of each helicity state is assumed to be the same, so the total photon number density n_γ can be written as

$$\begin{aligned} n_\gamma(r,\Omega,t)=&\sum_{lm}[n_{+lm}(t)+n_{-lm}(t)]Y_{lm}(\Omega)\Theta(R-r)=\sum_{lm} n_{\gamma lm}(t)Y_{lm}(\Omega)\Theta(R-r)\\ n_{+lm}(t)=&n_{-lm}(t)\ ,\qquad n_{\gamma lm}(t)=2n_{\lambda lm}(t)\end{aligned}$$

Hence the component of the total photon number density is just 2 times that of component of photon number density of each helicity state. Following the procedures from the previous section, we have an equation similar to (III.6) for each choice of lm

$$\frac{dn_{\gamma lm}(t)}{dt}=2\frac{n_{alm}}{\tau_a}+\frac{16\pi^2}{\beta m_a^3\tau_a}E_{lm}-\frac{16\pi^2}{3m_a^3\tau_a}(\beta+\frac{3}{2})F_{lm}-\frac{3c}{2R}n_{\gamma lm}(t)\ , \tag{III.8}$$

where E_{lm} and F_{lm} are defined through

$$n_a(\Omega,t)n_\gamma(\Omega,t)=\sum_{l'm'l''m''} n_{al'm'}n_{\gamma l''m''}Y_{l'm'}Y_{l''m''}=\sum_{lm} E_{lm}Y_{lm} \tag{III.9}$$

and

$$[n_\gamma(\Omega,t)]^2=\sum_{l'm'l''m''} n_{\gamma l'm'}n_{\gamma l''m''}Y_{l'm'}Y_{l''m''}=\sum_{lm} F_{lm}Y_{lm}\ , \tag{III.10}$$

respectively. We also have equations similar to equation (III.7) for each choice of lm with regard to the changing number density of axions. The equation includes components representing spontaneous decay, stimulated decay and back reaction with sterile axions excluded

$$\frac{dn_{alm}(t)}{dt}=-\frac{n_{alm}}{\tau_a}-\frac{8\pi^2}{\beta m_a^3\tau_a}E_{lm}+\frac{8\pi^2\beta}{3m_a^3\tau_a}F_{lm}\ . \tag{III.11}$$

The sterile axions evolve according to

$$\frac{dn_{slm}(t)}{dt} = \frac{4\pi^2}{m_a^3 \tau_a} F_{lm} \ . \tag{III.12}$$

The rate of change of photon number density component can be expressed in terms of the changing components of normal axion and sterile axion, and the components of surface loss

$$\frac{dn_{\gamma lm}(t)}{dt} = -2[\frac{dn_{alm}(t)}{dt} + \frac{dn_{slm}(t)}{dt}] - \frac{3c}{2R} n_{\gamma lm}(t) \ . \tag{III.13}$$

III.3 Examples

Equations (III.8) and (III.11) are the main results based on which we explore some example choices of initial axion distributions.

III.3.1 Y_{00} distribution

Consider the spherical symmetric axion distribution where the only nonzero component of axion number density is n_{a00},

$$n_a = \Theta(R-r) n_{a00} Y_{00}(\Omega) \ ,$$

then all components other than n_{a00} are zero,

$$n_{alm} = 0 \quad (lm \neq 00) \ .$$

This simplifies equation (III.9) to

$$n_a(\Omega,t) n_\gamma(\Omega,t) = \sum_{lm} n_{a00} Y_{00} n_{\gamma lm} Y_{lm} \ ,$$

which tells a relationship between E_{lm} and $n_{\gamma lm}$,

$$E_{lm} = n_{a00} Y_{00} n_{\gamma lm} \ . \tag{III.14}$$

Equation (III.11) is also simplified for $lm \neq 00$ to

$$0 = 0 - \frac{8\pi^2}{\beta m_a^3 \tau_a} E_{lm} + \frac{8\pi^2 \beta}{3 m_a^3 \tau_a} F_{lm} \ ,$$

which reduces to

$$F_{lm} = \frac{3}{\beta^2} E_{lm} \quad (lm \neq 00) \ . \tag{III.15}$$

Substitute (III.14) and (III.15) into equation (III.10) gives, upon splitting of 00 pieces, the two forms of (III.10)

$$[n_\gamma(\Omega,t)]^2 = F_{00}Y_{00} + \frac{3}{\beta^2} n_{a00} Y_{00} \sum_{lm \neq 00} n_{\gamma lm} Y_{lm} \ , \tag{III.16}$$

and

$$[n_\gamma(\Omega,t)]^2 = \frac{n_{\gamma 00} n_{\gamma 00}}{2\sqrt{\pi}} Y_{00} + 2 n_{\gamma 00} Y_{00} \sum_{lm \neq 00} n_{\gamma lm} Y_{lm} + \sum_{l'm'l''m'' \neq 0000} n_{\gamma l'm'} n_{\gamma l''m''} Y_{l'm'} Y_{l''m''} \ . \tag{III.17}$$

The most conspicuous solution to the equation is

$$F_{00} = \frac{n_{\gamma 00} n_{\gamma 00}}{2\sqrt{\pi}} \ , \qquad n_{\gamma lm} = 0 \quad (lm \neq 00) \ .$$

where the only nonzero component of photon number density is $n_{\gamma 00}$. So if there is spherical symmetry in the axion distribution, then spherical symmetry also exists in the photon distribution. These yield all the coupling coefficients E_{lm} and F_{lm},

$$E_{lm} = \frac{n_{a00} n_{\gamma 00}}{2\sqrt{\pi}} \delta_{l0} \delta_{m0} \ , \qquad F_{lm} = \frac{n_{\gamma 00} n_{\gamma 00}}{2\sqrt{\pi}} \delta_{l0} \delta_{m0} \ .$$

Equations (III.8), (III.11) and (III.12) reduce to the equations (34'), (37'), (38') in [14] given that

$$n_{\gamma 00} = 2\sqrt{\pi} n_\gamma \ , \qquad n_{a00} = 2\sqrt{\pi} n_a \ ,$$

Hence we have arrived at the spherically symmetric model results given in [14] .

III.3.2 Y_{20} distribution

For a Y_{20} axion distribution the only nonzero component of the axion number density is n_{a20},

$$n_a = \Theta(R - r) n_{a20} Y_{20}(\Omega) \ ,$$

so that

$$n_{alm} = 0 \quad (lm \neq 20)\ .$$

Equation (III.11) is simplified to

$$0 = 0 - \frac{8\pi^2}{\beta m_a^3 \tau_a} E_{lm} + \frac{8\pi^2 \beta}{3 m_a^3 \tau_a} F_{lm}\ ,$$

which reduces to

$$F_{lm} = \frac{3}{\beta^2} E_{lm} \quad (lm \neq 20)\ . \tag{III.18}$$

The nonzero component n_{a20} of axion number density evolves via

$$\frac{dn_{a20}(t)}{dt} = -\frac{n_{a20}}{\tau_a} - \frac{8\pi^2}{\beta m_a^3 \tau_a} E_{20} + \frac{8\pi^2 \beta}{3 m_a^3 \tau_a} F_{20}\ .$$

The photon number density component $n_{\gamma 20}$ growth rate is

$$\frac{dn_{\gamma 20}(t)}{dt} = 2\frac{n_{a20}}{\tau_a} + \frac{16\pi^2}{\beta m_a^3 \tau_a} E_{20} - \frac{16\pi^2}{3 m_a^3 \tau_a} (\beta + \frac{3}{2}) F_{20} - \frac{3c}{2R} n_{\gamma 20}(t)\ ,$$

while the other photon number density component $n_{\gamma lm}(lm \neq 20)$ evolve as

$$\frac{dn_{\gamma lm}(t)}{dt} = -\frac{8\pi^2}{m_a^3 \tau_a} F_{lm} - \frac{3c}{2R} n_{\gamma lm}(t)\ .$$

Since no spontaneous decay from axion feeds into these components, they are negligible. This example is not physical because a number density of the form Y_{20} becomes negative in some regions. It is included here for demonstration purposes. Even though this example is not physical, it is needed because it is a component of the next example which is physical and motivated by superradiance.

III.3.3 $Y_1^{\pm 1 *} Y_1^{\pm 1} \sim \sin^2\theta$ distribution

A $sin^2\theta$ distribution is positive definite everywhere, and hence can represent a physical distribution of particles. The only nonzero components of the axion number density are n_{a00} and n_{a20}, so we can write $n_a(r,\theta,t)$ in several useful forms

$$n_a = \Theta(R-r) n_a(t) \sin^2\theta = \Theta(R-r) n_a(t) \frac{4\sqrt{\pi}}{3} (Y_{00} - \frac{1}{\sqrt{5}} Y_{20}) = \Theta(R-r)[n_{a00}(t) Y_{00} + n_{a20}(t) Y_{20}]\ .$$

The relation between n_{a00} and n_{a20} is

$$n_{a20}(t) = -\frac{n_{a00}(t)}{\sqrt{5}} \ . \tag{III.19}$$

Similar to previous examples, we find that for components other than 00 and 20

$$F_{lm} = \frac{3}{\beta^2} E_{lm} \quad (lm \neq 00, 20) \ ,$$

so that the components of photon number density evolve as

$$\frac{dn_{\gamma lm}(t)}{dt} = -\frac{8\pi^2}{m_a^3 \tau_a} F_{lm} - \frac{3c}{2R} n_{\gamma lm}(t) \quad (lm \neq 00, 20) \ .$$

Since no spontaneous decay from axions feed into these components, they are negligible, as in the previous example.

The nonzero axion number density components evolve according to

$$\frac{dn_{a00}(t)}{dt} + \frac{n_{a00}}{\tau_a} = -\frac{8\pi^2}{\beta m_a^3 \tau_a} E_{00} + \frac{8\pi^2 \beta}{3 m_a^3 \tau_a} F_{00} \ , \qquad \frac{dn_{a20}(t)}{dt} + \frac{n_{a20}}{\tau_a} = -\frac{8\pi^2}{\beta m_a^3 \tau_a} E_{20} + \frac{8\pi^2 \beta}{3 m_a^3 \tau_a} F_{20} \ .$$

Because of (III.19), we can derive a relation from these two equations,

$$-\frac{8\pi^2}{\beta m_a^3 \tau_a} E_{20} + \frac{8\pi^2 \beta}{3 m_a^3 \tau_a} F_{20} = \frac{1}{\sqrt{5}} \left(\frac{8\pi^2}{\beta m_a^3 \tau_a} E_{00} - \frac{8\pi^2 \beta}{3 m_a^3 \tau_a} F_{00} \right) \ . \tag{III.20}$$

The photon number density components $n_{\gamma 00}$ and $n_{\gamma 20}$ grow as

$$\frac{dn_{\gamma 00}(t)}{dt} = 2\frac{n_{a00}}{\tau_a} + \frac{16\pi^2}{\beta m_a^3 \tau_a} E_{00} - \frac{16\pi^2}{3 m_a^3 \tau_a} (\beta + \frac{3}{2}) F_{00} - \frac{3c}{2R} n_{\gamma 00}(t)$$

and

$$\frac{dn_{\gamma 20}(t)}{dt} = 2\frac{n_{a20}}{\tau_a} + \frac{16\pi^2}{\beta m_a^3 \tau_a} E_{20} - \frac{16\pi^2}{3 m_a^3 \tau_a} (\beta + \frac{3}{2}) F_{20} - \frac{3c}{2R} n_{\gamma 20}(t) \ .$$

Because of (III.19) and (III.20), we can combine the previous two equations and write

$$\frac{dn_{\gamma 00}(t)}{dt} + \frac{3c}{2R} n_{\gamma 00}(t) = 2\frac{n_{a00}}{\tau_a} + \frac{16\pi^2}{\beta m_a^3 \tau_a} E_{00} - \frac{16\pi^2}{3 m_a^3 \tau_a} (\beta + \frac{3}{2}) F_{00} \ ,$$

and

$$
\begin{aligned}
\frac{dn_{\gamma 20}(t)}{dt} + \frac{3c}{2R} n_{\gamma 20}(t) =& 2\frac{n_{a00}}{\tau_a}(\frac{-1}{\sqrt{5}}) + (\frac{16\pi^2}{\beta m_a^3 \tau_a} E_{00} - \frac{16\pi^2 \beta}{3m_a^3 \tau_a} F_{00})(\frac{-1}{\sqrt{5}}) - \frac{8\pi^2}{m_a^3 \tau_a} F_{20} \\
=& \frac{-1}{\sqrt{5}}[\frac{dn_{\gamma 00}(t)}{dt} + \frac{3c}{2R} n_{\gamma 00}(t)] - \frac{8\pi^2}{m_a^3 \tau_a} F_{20} \ .
\end{aligned}
$$

We observe that if the part of back reaction that results in sterile axions is neglected, then

$$
n_{\gamma 20}(t) = -\frac{n_{\gamma 00}(t)}{\sqrt{5}} \ ,
$$

so the photons would remain in a $\sin^2 \theta$ distribution.

In the previous section, we found that $X \neq Y$, i.e. the angular dependences of axion and photon are not equal. $X = Y$ would have contradicted the results from the previous section but that is not what this example indicates. This example shows that the combination of 00 and 20 components of axion and photon have the same angular dependence. This is not the same as $X = Y$. There are photon components other than 00 and 20. These components are neglected under circumstances that there is no source feeding these components but they are not equal to 0. In fact, the presence of these negligible components verifies that the angular dependences can not be written in a closed-form such as $X(\theta)$ or $Y(\theta)$. X is approximately Y but not exactly.

Here we have a similar result as that in [21], which is not a coincidence. We are trying to solve the same type of problems, i.e., finding photon distribution from corresponding axion distribution. If the axion distribution is given, then we always have equations for axion components that are absent in axion distribution. Equations (III.11) still applies, even if the specific axion component is 0. The solution of these equations with absent axion components is that the back scattering of photons somewhat cancels the effect of stimulated decay of axion, which prompt simplified equations for the corresponding photon components. There seems to be no source feeding these photon components and they would become absent also.

III.3.4 General distribution

Suppose that we have an axion number density

$$
n_a = \Theta(R - r) \sum n_{alm} Y_{lm}(\Omega) \ .
$$

For any $n_{alm} = 0$, then according to (III.11) this leads to

$$F_{lm} = \frac{3}{\beta^2} E_{lm} \quad (n_{alm} = 0) \ .$$

Substituting this condition into equation (III.8), we have

$$\frac{dn_{\gamma lm}(t)}{dt} = -\frac{16\pi^2}{3m_a^3\tau_a}(\frac{3}{2})F_{lm} - \frac{3c}{2R} n_{\gamma lm}(t) \quad (n_{alm} = 0) \ .$$

Hence there is no source feeding those photon components. The parts of the back reaction that results in sterile axions and surface loss are the only terms that contribute to these components. It is expected that these components die out quickly and thus have no effect on lasing. So

$$n_\gamma = \Theta(R-r) \sum n_{\gamma lm} Y_{lm}(\Omega) \ , \quad \text{where } n_{\gamma lm} \approx 0 \ \text{ when } n_{alm} = 0 \ ,$$

i.e., the photon field has the same spherical harmonic components as the axion field, as other components die out quickly due to lack of sources. Neither spontaneous decay nor stimulated decay contributes to the harmonic components of photons that are not present in the axions.

Suppose we are given a distribution,

$$n_{alm} = \alpha_{lm} n_{al_0m_0},$$

where all of these axion components are nonzero, and α_{lm} are numbers and $n_{al_0m_0}$ is the fiducial component to which all other components are proportional. Then

$$\frac{8\pi^2\beta}{3m_a^3\tau_a} F_{lm} - \frac{8\pi^2}{\beta m_a^3\tau_a} E_{lm} = \frac{dn_{alm}(t)}{dt} + \frac{n_{alm}}{\tau_a}$$

and

$$\frac{dn_{\gamma lm}(t)}{dt} + \frac{3c}{2R} n_{\gamma lm}(t) = 2\frac{n_{alm}}{\tau_a} + \frac{16\pi^2}{\beta m_a^3\tau_a} E_{lm} - \frac{16\pi^2}{3m_a^3\tau_a}(\beta + \frac{3}{2})F_{lm} = -2\frac{dn_{alm}(t)}{dt} - \frac{8\pi^2}{m_a^3\tau_a} F_{lm}$$

If the part of the back reaction that results in sterile axions is neglected, then

$$\frac{dn_{\gamma lm}(t)}{dt} + \frac{3c}{2R} n_{\gamma lm}(t) = -2\frac{dn_{alm}(t)}{dt} = -2\alpha_{lm}\frac{dn_{al_0m_0}(t)}{dt} = \alpha_{lm}[\frac{dn_{\gamma l_0m_0}(t)}{dt} + \frac{3c}{2R} n_{\gamma l_0m_0}(t)] \ ,$$

where the last step utilized (III.12) and (III.13). This means that nonzero components of photon number density are also proportional to each other,

$$n_{\gamma lm} = \alpha_{lm} n_{\gamma l_0 m_0} \ .$$

Hence the distribution of photons would keep the same shape as that of the axions if sterile axions were neglected.

III.4 Discussion of Photon directional profile from spatial distribution

In the example of a $\sin^2\theta$ distribution, the 00 and 20 components of the given axion number density are “filled”(not 0) but all the other components are “empty”(0). Our results show that the “filled’ photon number density components are also 00 and 20, and the other components are almost “empty”(under the circumstance discussed above). This relation is not only pertinent to this $\sin^2\theta$ example. In the general distribution, if axions number density is only “filled” for certain component $l_0 m_0$ with the other components “empty”, then the photon number density also has these almost “empty” components, with only $l_0 m_0$ components “filled”. This correlation between components of photon and axions also occurs in [21] where the momentum dependences of occupation numbers were considered.

CHAPTER IV

Stimulated Radiation from Axion Cluster Evolution in Static Spacetimes

This chapter presents a generalization of the axion decay model of the previous chapters to curved spacetime, specifically static spacetime. [23] presents a complete analysis of this subject.

IV.1 $a \leftrightarrow \gamma + \gamma$ process in Minkowski spacetime

Two photons emitted by decay of a spin zero particle have the same helicity, as required by angular momentum conservation. The change in the number density of photons of a given helicity $\lambda = \pm 1$ within the axion cluster, due to the process $a \leftrightarrow \gamma + \gamma$ in Minkowski spacetime, is

$$\frac{dn_\lambda}{dt} = \int dX^{(3)}_{\text{LIMS}} [f_a(1+f_{1\lambda})(1+f_{2\lambda}) - f_{1\lambda} f_{2\lambda}(1+f_a)] |M(a \to \gamma(\lambda)\gamma(\lambda))|^2 \,, \tag{IV.1}$$

where f_a, $f_{1\lambda}$ and $f_{2\lambda}$ are the occupation numbers of the axion and the two photons and $M = M(a \to \gamma(+)\gamma(+)) = M(a \to \gamma(-)\gamma(-))$ is the decay amplitude determined by the Abelian chiral anomaly [24,25] and is related to the spontaneous axion decay constant by

$$\tau_a^{-1} = \Gamma_a = \frac{1}{8\pi}(\frac{1}{2m_a})\frac{1}{2}\sum_{\lambda=\pm} |M(a \to \gamma(\lambda)\gamma(\lambda))|^2 \,. \tag{IV.2}$$

The three body Lorentz invariant momentum space is

$$\int dX^{(3)}_{\text{LIMS}} = \int \frac{d^3p}{(2\pi)^3 2p^0} \int \frac{d^3k_1}{(2\pi)^3 2k_1^0} \int \frac{d^3k_2}{(2\pi)^3 2k_2^0} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2) \,.$$

One obtains eq. (10) of [14] by writing the three body Lorentz invariant momentum space explicitly,

$$2k\frac{df_\lambda(\vec{k})}{dt} = \frac{4m_a\Gamma_a}{\pi} \int \frac{d^3k_1}{2k_1^0}\frac{d^3p}{2p^0} \delta^4(p-k-k_1)\{f_a(\vec{p})[1+f_\lambda(\vec{k})+f_\lambda(\vec{k}_1)] - f_\lambda(\vec{k}) f_\lambda(\vec{k}_1)\} \,. \tag{IV.3}$$

which became the starting point for studying the lasing of nonrelativistic spherically symmetric axion clouds. Here we need to discuss how the Lorentz invariant phase space originates in Minkowski space in order to generalize it to curved space. The one body Lorentz invariant momentum space measure is

$$\int dX_{\text{LIMS}} = \int \frac{d^3p}{(2\pi)^3 2p^0} \,.$$

In Minkowski spacetime, the quantity $dp^0dp^1dp^2dp^3$ which equals $\sqrt{|\eta|}dp^0dp^1dp^2dp^3$ is Lorentz invariant since $|\eta| = 1$, where η is the determinant of the Minkowski metric. For a particle in special relativity, when we count how many possible momentum states it possesses, p^0, p^1, p^2, p^3 can not just take any values. These values have to satisfy the normalization condition $p^\mu p_\mu = m^2$. The first component is just the energy of the particle, so in addition, we need to require that $p^0 > 0$. Therefore, a quantity that is both Lorentz invariant and counts the number of all possible momentum states is

$$\int_{p_0=-\infty}^{p_0=+\infty} dp^0dp^1dp^2dp^3\delta(p^\mu p_\mu - m^2)\Theta(p^0) = \frac{d^3p}{2\sqrt{\vec{p}^2+m^2}} = \frac{d^3p}{2p^0} = (2\pi)^3 dX_{\text{LIMS}} \ .$$

The quantity $\sqrt{|\eta|}dx^0dx^1dx^2dx^3 = dx^0dx^1dx^2dx^3$ is a Lorentz invariant pseudoscalar. The small change $d(\frac{\tau}{m})$ is also a Lorentz invariant, where τ and m are the proper time and the rest mass of the particle, respectively. Hence we have another Lorentz invariant quantity

$$m\frac{dx^0}{d\tau}dx^1dx^2dx^3 = p^0d^3r \ .$$

Finally, from the product of two pseudoscalars $dx^0dx^1dx^2dx^3$ and $dp^0dp^1dp^2dp^3$, which is a scalar, the Lorentz invariant phase space measure arises naturally,

$$\frac{2m}{d\tau}dx^0dx^1dx^2dx^3\int dp^0dp^1dp^2dp^3\delta(p^\mu p_\mu - m^2)\Theta(p^0) = 2p^0d^3r\frac{d^3p}{2p^0} = d^3rd^3p = dX_{\text{LIPS}} \ .$$

This is the quantity that we will generalize to curved space in the following section. In polar coordinates, the infinitesimal line element in Minkowski metric using polar coordinates reads $ds^2 = -dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$, and the particle number density is given by

$$\begin{aligned} n(\vec{r},t) &= \int f(\vec{p},\vec{r},t)\frac{d^3p}{(2\pi)^3} = \int \frac{f(\vec{p},\vec{r},t)}{(2\pi)^3}dp^xdp^ydp^z = \int \frac{f(\vec{p},\vec{r},t)}{(2\pi)^3}\frac{\partial(p^x,p^y,p^z)}{\partial(p^r,p^\theta,p^\phi)}dp^rdp^\theta dp^\phi \\ &= \int \frac{f(\vec{p},\vec{r},t)}{(2\pi)^3}r^2\sin\theta dp^rdp^\theta dp^\phi = \int \frac{f(\vec{p},\vec{r},t)}{(2\pi)^3}\sqrt{|\eta_{\text{E}}|}dp^rdp^\theta dp^\phi \ , \end{aligned}$$

where η_{E} is the determinant of the 3 dimensional Euclidean metric in spherical coordinates. The total number of the particle is

$$N(t) = \int n(\vec{r},t)r^2\sin\theta drd\theta d\phi = \int n(\vec{r},t)\sqrt{|\eta_{\text{E}}|}drd\theta d\phi \ .$$

IV.2 Covariance measures in Static spacetime

The concept of Lorentz transformation is not very useful in Schwarzschild spacetime because there is no global inertial frame. At a single event, the spacetime can be treated locally as flat and special relativity is still applicable, Lorentz transformation still provides the relationship between two nearby observers from two local inertial frames having relative velocity in that infinitesimal region. As soon as an observer, for example, moves forward along the geodesic a finite distance, the previous established Lorentz transformation loses its meaning. But even if the functionality of Lorentz transformation between inertial frames is irrecoverable in curved spacetime, we can embrace the general coordinate transformation under which general relativity demonstrates general covariance. The invariance provided by Lorentz transformation in flat spacetime can be emulated by writing quantities in a general covariant form in curved spacetime. For example, in Schwarzschild spacetime, the coordinates can be expressed as Schwarzschild coordinates, or Kruskal-Szekeres coordinates, or other equivalent transformations, but the general covariant quantities are valid without referring to the specific coordinate system.

For any species of particles, the occupation number $f(\vec{p},\vec{r},t)$ in Minkowski spacetime should change to $f(p^i,x^\alpha)$ in curved spacetime, where p^i is the three spatial component of the 4-momentum of the particle and x^α is the spacetime coordinate of the particle. The occupation number does not depend on the first component p^0 of the 4-momentum of the particle since normalization of the 4-momentum $g_{\mu\nu}p^\mu p^\nu = -m^2$ still holds in curved spacetime. A stationary particle has 4-momentum $p^\mu = (p^0,0,0,0)$, and its 4-velocity is $v^\mu = (p^0/m,0,0,0)$. A co-stationary observer at the location of the particle would measure the energy of the particle to be $m = -g_{\mu\nu}p^\mu v^\nu = -g_{00}\frac{(p^0)^2}{m}$. $p^0 = \pm\frac{m}{\sqrt{-g_{00}}}$, where we choose p^0 such that it reduces to m in the flat spacetime limit. So in addition, we may as well require that $p^0 > 0$. Moreover, $p^0 > 0$ implies that $\frac{dt}{d\tau} > 0$, or the proper time and coordinate time flow in the same direction. Similar to the Lorentz invariant momentum space dX_{LIMS}, now there is a general covariant momentum space dX_{GCMS},

$$dX_{\text{GCMS}} = \int \sqrt{|g|}dp^0dp^1dp^2dp^3\delta(g_{\mu\nu}p^\mu p^\nu + m^2)\Theta(p^0) \ .$$

The general covariant 4-momentum volume element is $\sqrt{|g|}dp^0dp^1dp^2dp^3$, regardless of the metric or the coordinates that give a specific form to the metric, where g is the determinant of the metric. This abstract form of momentum space is as far as we can go without implementing the knowledge of a specific metric. It is applicable not only to static spacetime such as Schwarzschild spacetime, but to any atationary spacetime, including the Kerr spacetime.

Let us consider static spacetime where $g_{0\mu} = g_{00}\delta_{0\mu}$. In this case we can have a static covariant momentum space dX_{SCMS} which is a 4-momentum volume measure that is compatible with all static spacetimes and their coordinates transformations which keep $g_{0\mu} = g_{00}\delta_{0\mu}$.

$$
\begin{aligned}
& dX_{\text{SCMS}} \\
=& \sqrt{|g|}\int_{p_0=0}^{+\infty} dp^0 dp^1 dp^2 dp^3 \delta[g_{00}(p^0)^2 + g_{ij}p^i p^j + m^2] \\
=& \sqrt{|g|}dp^1 dp^2 dp^3 \int_0^{+\infty} dp^0 \delta[g_{00}(p^0 - \sqrt{\frac{g_{ij}p^i p^j + m^2}{-g_{00}}})(p^0 + \sqrt{\frac{g_{ij}p^i p^j + m^2}{-g_{00}}})] \\
=& \sqrt{|g|}dp^1 dp^2 dp^3 \int_0^{+\infty} dp^0 [\frac{1}{-2g_{00}\sqrt{\frac{g_{ij}p^i p^j + m^2}{-g_{00}}}}\delta(p^0 - \sqrt{\frac{g_{ij}p^i p^j + m^2}{-g_{00}}}) \\
& + \frac{1}{-2g_{00}\sqrt{\frac{g_{ij}p^i p^j + m^2}{-g_{00}}}}\delta(p^0 + \sqrt{\frac{g_{ij}p^i p^j + m^2}{-g_{00}}})] \\
=& \frac{\sqrt{|g|}dp^1 dp^2 dp^3}{2\sqrt{-g_{00}}\sqrt{g_{ij}p^i p^j + m^2}} = \frac{\sqrt{-g_{00}||g_{ij}||}dp^1 dp^2 dp^3}{2\sqrt{-g_{00}}\sqrt{-g_{00}(p^0)^2}} = \frac{\sqrt{||g_{ij}||}dp^1 dp^2 dp^3}{2\sqrt{-g_{00}}p^0} \ ,
\end{aligned}
$$

where $|g_{ij}|$ is the determinant of the metric of 3-surface $dx^0 = 0$ in the static spacetime of $ds^2 = g_{00}(dx^0)^2 + g_{ij}dx^i dx^j$.

A general covariant 4-volume element is $\sqrt{|g|}dx^0 dx^1 dx^2 dx^3$. A small change $d\frac{\tau}{m}$, where τ is the proper time and m the rest mass of the particle is also general covariant. So the quantity

$$
\sqrt{|g|}m\frac{dx^0}{d\tau}dx^1 dx^2 dx^3 = \sqrt{|g|}p^0 dx^1 dx^2 dx^3
$$

is also general covariant. Combining this with the static covariant momentum space, we have the static covariant phase space element

$$
dX_{\text{SCPS}} = \sqrt{|g|}p^0 dx^1 dx^2 dx^3 \frac{\sqrt{||g_{ij}||}dp^1 dp^2 dp^3}{\sqrt{-g_{00}}p^0} = ||g_{ij}||dx^1 dx^2 dx^3 dp^1 dp^2 dp^3 \ .
$$

If we adopt notations

$$
d^3x = \sqrt{||g_{ij}||}dx^1 dx^2 dx^3 \qquad d^3p = \sqrt{||g_{ij}||}dp^1 dp^2 dp^3 \ ,
$$

then static covariant phase space has an identical form to Lorentz invariant phase space

$$
dX_{\text{SCPS}} = d^3x d^3p \ .
$$

In a static spacetime, the number density given by

$$n(x^\alpha) = \int f(p^i, x^\alpha) \frac{d^3p}{(2\pi)^3} = \int f(p^i, x^\alpha) \frac{\sqrt{||g_{ij}||}}{(2\pi)^3} dp^1 dp^2 dp^3 \ ,$$

and the total number of the particle is

$$N(t) = \int n(x^\alpha) d^3x = \int n(x^\alpha) \sqrt{||g_{ij}||} dx^1 dx^2 dx^3 \ ,$$

where $|g_{ij}|$ is the determinant of the 3-surface metric $ds^2 = g_{ij} dx^i dx^j$ at constant coordinate time t.

IV.3 $a \leftrightarrow \gamma + \gamma$ process in Static spacetime

It is now straightforward to generalize equation (IV.1) to find the change in the number density of photons of a given helicity $= \pm 1$ within the axion cluster, due to the process $a \leftrightarrow \gamma + \gamma$ in static spacetime, which is

$$\frac{dn_\lambda}{d\tau} = \int dX^{(3)}_{\text{SCMS}} [f_a(1 + f_{1\lambda})(1 + f_{2\lambda}) - f_{1\lambda} f_{2\lambda}(1 + f_a)] |M(a \to \gamma(\lambda)\gamma(\lambda))|^2 \ . \tag{IV.4}$$

The one body static covariant momentum space is

$$\int dX_{\text{SCMS}} = \int \frac{\sqrt{||g_{ij}||} dp^1 dp^2 dp^3}{(2\pi)^3 2\sqrt{-g_{00}} p^0} \ ,$$

and the three body static invariant phase space is

$$\int dX^{(3)}_{\text{SCMS}} = \int \frac{\sqrt{||g_{ij}||} dp^1 dp^2 dp^3}{(2\pi)^3 2\sqrt{-g_{00}} p^0} \frac{\sqrt{||g_{ij}||} dk_1^1 dk_1^2 dk_1^3}{(2\pi)^3 2\sqrt{-g_{00}} k_1^0} \frac{\sqrt{||g_{ij}||} dk_2^1 dk_2^2 dk_2^3}{(2\pi)^3 2\sqrt{-g_{00}} k_2^0} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2) \ .$$

Equation (IV.2) describes the axion decay constant τ_a in a local inertial frame that is comoving with the axion. The relation between the time in the comoving frame τ with the axion and the coordinate time t in static spacetime is

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{-g_{00}}} \sqrt{g_{ij} \frac{p^i p^j}{m^2} + 1} \ .$$

where $g_{ij} p^i p^j$ is the square of the magnitude of the 3-momentum. The decay constant τ_a in the comoving frame would change to the decay constant t_a in lab frame

$$t_a = \tau_a \frac{1}{\sqrt{-g_{00}}} \sqrt{g_{ij} \frac{p^i p^j}{m^2} + 1} \ ,$$

and the decay rate Γ_a changes

$$\Gamma_a \quad \to \quad \Gamma_a \sqrt{-g_{00}} (g_{ij} \frac{p^i p^j}{m^2} + 1)^{-1/2} .$$

Multiplying equation (IV.4) by $\frac{d\tau}{dt}$, we have an equation for the rate of change in the number density of photons measured by coordinate time t,

$$\frac{dn_\lambda}{dt} = \int dX^{(3)}_{\text{SCMS}} \{[f_a[1 + f_\lambda(k^i) + f_\lambda(k^i_1)] - f_\lambda(k^i) f_\lambda(k^i_1)\} \times 16\pi m_a \Gamma_a \frac{d\tau}{dt} .$$

Writing $\frac{dn_\lambda}{dt}$ in terms of f_a we have one static covariant momentum space measure dX_{SCMS} on the LHS of this equation that we use to cancel one of the three static covariant momentum space measures in $dX^{(3)}_{\text{SCMS}}$ on the RHS. There remains two static covariant momentum space measures, one for the axion and one for a photon, plus the δ function.

$$2\sqrt{-g_{00}} k^0 \frac{df_\lambda}{dt} = \int \frac{\sqrt{||g_{ij}||} dp^1 dp^2 dp^3}{(2\pi)^3 2\sqrt{-g_{00}} p^0} \frac{\sqrt{||g_{ij}||} dk^1_1 dk^2_1 dk^3_1}{(2\pi)^3 2\sqrt{-g_{00}} k^0_1} (2\pi)^4 \delta^4(p^\alpha - k^\alpha - k^\alpha_1) \{f_a[1 + f_\lambda(k^i) + f_\lambda(k^i_1)] - f_\lambda(k^i) f_\lambda(k^i_1)\} 16\pi m_a \Gamma_a \sqrt{-g_{00}} (g_{ij} \frac{p^i p^j}{m_a^2} + 1)^{-1/2} .$$

After simplification, we obtained the evolution equation

$$2k^0 \frac{df_\lambda}{dt} = \frac{4m_a \Gamma_a}{\pi} \int \frac{\sqrt{||g_{ij}||} dp^1 dp^2 dp^3}{2\sqrt{-g_{00}} p^0} \frac{\sqrt{||g_{ij}||} dk^1_1 dk^2_1 dk^3_1}{2\sqrt{-g_{00}} k^0_1} \{f_a[1 + f_\lambda(k^i) + f_\lambda(k^i_1)] - f_\lambda(k^i) f_\lambda(k^i_1)\} \delta^4(p^\alpha - k^\alpha - k^\alpha_1) (g_{ij} \frac{p^i p^j}{m_a^2} + 1)^{-1/2} . \tag{IV.5}$$

which resembles equation (IV.3), and reduces to it in the flat space limit.

IV.4 Integration over k_1

We write last equation as

$$2k^0 \frac{df_\lambda(k^i)}{dt} = \frac{4m_a \Gamma_a}{\pi} \int (g_{ij} \frac{p^i p^j}{m_a^2} + 1)^{-1/2} \frac{\sqrt{||g_{ij}||} dp^1 dp^2 dp^3}{2\sqrt{-g_{00}} p^0} \int \frac{\sqrt{||g_{ij}||} dk^1_1 dk^2_1 dk^3_1}{2\sqrt{-g_{00}} k^0_1} \frac{1}{\sqrt{|g|}} \delta(p^0 - k^0 - k^0_1) \times \delta(p^1 - k^1 - k^1_1) \delta(p^2 - k^2 - k^2_1) \delta(p^3 - k^3 - k^3_1) \times \{f_a(p^i)[1 + f_\lambda(k^i) + f_\lambda(k^i_1)] - f_\lambda(k^i) f_\lambda(k^i_1)\} ,$$

to emphasize the k^i_1 part of the equation. The only function that depends on k^1_1, k^2_1 and k^3_1 is $f_\lambda(k^i_1) = f_\lambda(k^1_1, k^2_1, k^3_1)$. Therefore, after the three δ function are integrated, $f_\lambda(k^i_1)$ will change to $f_\lambda(p^1 - k^1, p^2 -$

$k^2, p^3 - k^3) = f_\lambda(p^i - k^i)$. k_1^0 is an implicit function of k_1^1, k_1^2 and k_1^3 as given by the normalization of photon 4-momentum. After integration k_1^0 would change to

$$k_1^0 \quad \rightarrow \quad \sqrt{\frac{g_{ij}(p^i - k^i)(p^j - k^j)}{-g_{00}}} \ ,$$

Because $g = -g_{00}|g_{ij}|$, integration over k_1 gives us

$$\begin{aligned} 2k^0 \frac{d f_\lambda(k^i)}{dt} =& \frac{4 m_a \Gamma_a}{\pi} \int (g_{ij} \frac{p^i p^j}{m_a^2} + 1)^{-1/2} \frac{\sqrt{||g_{ij}||} dp^1 dp^2 dp^3}{2\sqrt{-g_{00}} p^0} \times \frac{1}{2\sqrt{g_{ij}(p^i - k^i)(p^j - k^j)}} \frac{1}{\sqrt{-g_{00}}} \\ & \times \delta[p^0 - k^0 - \sqrt{\frac{g_{ij}(p^i - k^i)(p^j - k^j)}{-g_{00}}}] \\ & \times \{f_a(p^i)[1 + f_\lambda(k^i) + f_\lambda(p^i - k^i)] - f_\lambda(k^i) f_\lambda(p^i - k^i)\}. \end{aligned}$$

Normalization of axion 4-momentum allows us to write

$$(g_{ij} \frac{p^i p^j}{m_a^2} + 1)^{-1/2} = \frac{m_a}{\sqrt{-g_{00}} p^0} \ .$$

We use this to simplify the equation to

$$\begin{aligned} 2k^0 \frac{d f_\lambda(k^i)}{dt} =& \frac{4 m_a \Gamma_a}{\pi(-g_{00})} \int \frac{m_a}{p^0} \frac{\sqrt{||g_{ij}||} dp^1 dp^2 dp^3}{2\sqrt{-g_{00}} p^0} \times \frac{1}{2\sqrt{g_{ij}(p^i - k^i)(p^j - k^j)}} \\ & \times \delta[p^0 - k^0 - \sqrt{\frac{g_{ij}(p^i - k^i)(p^j - k^j)}{-g_{00}}}] \\ & \times \{f_a(p^i)[1 + f_\lambda(k^i) + f_\lambda(p^i - k^i)] - f_\lambda(k^i) f_\lambda(p^i - k^i)\} \ . \end{aligned} \qquad \text{(IV.6)}$$

IV.5 Integration over p^i

We proceed to integrate if occupation numbers are assumed to be isotropic. Occupation numbers depend on the 3-momentum only through the norm,

$$f_a(p^i) = f_a(\sqrt{g_{ij} p^i p^j}) \qquad f_\lambda(k^i) = f_\lambda(\sqrt{g_{ij} k^i k^j}) \ .$$

Applying the isotropic assumption to equation (IV.6) results in

$$2k^0\frac{df_\lambda(\sqrt{g_{ij}k^ik^j})}{dt}=\frac{4m_a\Gamma_a}{\pi(-g_{00})}\int\frac{m_a}{p^0}\frac{\sqrt{||g_{ij}||}dp^1dp^2dp^3}{2\sqrt{-g_{00}}p^0}\times\frac{1}{2\sqrt{g_{ij}(p^i-k^i)(p^j-k^j)}} \tag{IV.7}$$
$$\times\,\delta[p^0-k^0-\sqrt{\frac{g_{ij}(p^i-k^i)(p^j-k^j)}{-g_{00}}}]\times\{f_a(\sqrt{g_{ij}p^ip^j})[1+f_\lambda(\sqrt{g_{ij}k^ik^j})$$
$$+f_\lambda\left(\sqrt{g_{ij}(p^i-k^i)(p^j-k^j)}\right)]-f_\lambda(\sqrt{g_{ij}k^ik^j})f_\lambda\left(\sqrt{g_{ij}(p^i-k^i)(p^j-k^j)}\right)\}\ .$$

This equation is valid at event $\mathscr{P}_0(t_{\mathscr{P}_0},x^i_{\mathscr{P}_0})$. On the 3-surface $ds^2=g_{ij}dx^idx^j$, at point $\mathscr{P}_0$ we employ Riemann normal coordiantes x^i_{R} on an infinitesimal small patch around $\mathscr{P}_0$. We write the metric on the infinitesimal 3-patch in these coordinates as

$$ds^2=\mathscr{G}_{ij}(x_{\text{R}})dx^i_{\text{R}}dx^j_{\text{R}}\ .$$

The coordinate labels have to be changed from (t,x^i) to $\mathscr{P}_0$, to remind us that we are focusing on the physics only at $\mathscr{P}_0$. Hence we will make the following replacements

$$k^0(t,x^1,x^2,x^3)\quad\Rightarrow\quad k^0\Big|_{\mathscr{P}_0}\qquad p^0(t,x^1,x^2,x^3)\quad\Rightarrow\quad p^0\Big|_{\mathscr{P}_0}\qquad g_{00}(t,x^1,x^2,x^3)\quad\Rightarrow\quad g_{00}\Big|_{\mathscr{P}_0}\ .$$

The isotropic assumption is required so that the spatial quantites in equation (IV.7) can be written in general covariant form (general covariant only on the 3-surface). General covariant quantities keep their form when switching to Riemann normal coordinates x^i_{R},

$$\sqrt{||g_{ij}||}dp^1dp^2dp^3\Rightarrow\sqrt{||\mathscr{G}_{ij}||}dp^1_{\text{R}}dp^2_{\text{R}}dp^3_{\text{R}}\Big|_{\mathscr{P}_0}\qquad\sqrt{g_{ij}(p^i-k^i)(p^j-k^j)}\Rightarrow\sqrt{\mathscr{G}_{ij}(p^i_{\text{R}}-k^i_{\text{R}})(p^j_{\text{R}}-k^j_{\text{R}})}\Big|_{\mathscr{P}_0}$$
$$\sqrt{g_{ij}p^ip^j}\Rightarrow\sqrt{\mathscr{G}_{ij}p^i_{\text{R}}p^j_{\text{R}}}\Big|_{\mathscr{P}_0}\qquad\sqrt{g_{ij}k^ik^j}\Rightarrow\sqrt{\mathscr{G}_{ij}k^i_{\text{R}}k^j_{\text{R}}}\Big|_{\mathscr{P}_0}$$

With these substitutions equation (IV.7) becomes

$$
\begin{aligned}
&2k^0\Big|_{\mathscr{P}_0} \frac{d f_\lambda(\sqrt{\mathscr{G} k^i_{\rm R} k^j_{\rm R}}\Big|_{\mathscr{P}_0}, \mathscr{P}_0)}{dt} \\
=&\frac{4m_a\Gamma_a}{\pi(-g_{00}\Big|_{\mathscr{P}_0})} \int \frac{m_a}{p^0\Big|_{\mathscr{P}_0}} \frac{\sqrt{||\mathscr{G}_{ij}||} dp^1_{\rm R} dp^2_{\rm R} dp^3_{\rm R}\Big|_{\mathscr{P}_0}}{2\sqrt{-g_{00}\Big|_{\mathscr{P}_0}} p^0\Big|_{\mathscr{P}_0}} \times \frac{1}{2\sqrt{\mathscr{G}_{ij}(p^i_{\rm R}-k^i_{\rm R})(p^j_{\rm R}-k^j_{\rm R})}\Big|_{\mathscr{P}_0}} \\
&\times \delta[p^0\Big|_{\mathscr{P}_0} - k^0\Big|_{\mathscr{P}_0} - \sqrt{\frac{\mathscr{G}_{ij}(p^i_{\rm R}-k^i_{\rm R})(p^j_{\rm R}-k^j_{\rm R})}{-g_{00}}}\Big|_{\mathscr{P}_0}] \\
&\times \{f_a(\sqrt{\mathscr{G}_{ij} p^i_{\rm R} p^j_{\rm R}}\Big|_{\mathscr{P}_0})[1+f_\lambda(\sqrt{\mathscr{G}_{ij} k^i_{\rm R} k^j_{\rm R}}\Big|_{\mathscr{P}_0}) + f_\lambda\left(\sqrt{\mathscr{G}_{ij}(p^i_{\rm R}-k^i_{\rm R})(p^j_{\rm R}-k^j_{\rm R})}\Big|_{\mathscr{P}_0}\right)] \\
&- f_\lambda(\sqrt{\mathscr{G}_{ij} k^i_{\rm R} k^j_{\rm R}}\Big|_{\mathscr{P}_0}) f_\lambda\left(\sqrt{\mathscr{G}_{ij}(p^i_{\rm R}-k^i_{\rm R})(p^j_{\rm R}-k^j_{\rm R})}\Big|_{\mathscr{P}_0}\right)\}\ .
\end{aligned}
$$

The metric $\mathscr{G}_{ij}$ at point $\mathscr{P}_0$ in Riemann normal coordinates $x^i_{\rm R}$ is $\mathscr{G}_{ij}\Big|_{\mathscr{P}_0} = (+1,+1,+1)$, which means at point $\mathscr{P}_0$ on the 3-surface, the space is Euclidean. So the momentum differential volume at point $\mathscr{P}_0$ can be expressed as

$$
\sqrt{||\mathscr{G}_{ij}||} dp^1_{\rm R} dp^2_{\rm R} dp^3_{\rm R}\Big|_{\mathscr{P}_0} = |\vec{p}_{\rm R}|^2 d|\vec{p}_{\rm R}| d(-\cos\theta_{\rm R}) d(\phi_{\rm R})\Big|_{\mathscr{P}_0}\ .
$$

where $|p_{\rm R}|$ is the magnitude of the 3-momentum, which is invariant under the coordinate change

$$
|\vec{p}_{\rm R}|^2\Big|_{\mathscr{P}_0} = \mathscr{G}_{ij} p^i_{\rm R} p^j_{\rm R}\Big|_{\mathscr{P}_0} = g_{ij} p^i p^j\Big|_{\mathscr{P}_0}\ .
$$

Rewriting other quantities using the Euclidean notation, gives

$$
\sqrt{\mathscr{G}_{ij} p^i_{\rm R} p^j_{\rm R}}\Big|_{\mathscr{P}_0} = |\vec{p}_{\rm R}|\Big|_{\mathscr{P}_0} \qquad \sqrt{\mathscr{G}_{ij} k^i_{\rm R} k^j_{\rm R}}\Big|_{\mathscr{P}_0} = |\vec{k}_{\rm R}|\Big|_{\mathscr{P}_0} \qquad \sqrt{\mathscr{G}_{ij}(p^i_{\rm R}-k^i_{\rm R})(p^j_{\rm R}-k^j_{\rm R})}\Big|_{\mathscr{P}_0} = |\vec{p}_{\rm R}-\vec{k}_{\rm R}|\Big|_{\mathscr{P}_0}\ ,
$$

The law of cosines still hold at the event $\mathscr{P}_0$,

$$
|\vec{p}_{\rm R}-\vec{k}_{\rm R}|^2\Big|_{\mathscr{P}_0} = |\vec{p}_{\rm R}|^2\Big|_{\mathscr{P}_0} + |\vec{k}_{\rm R}|^2\Big|_{\mathscr{P}_0} - 2\cos\theta_{\rm R}\Big|_{\mathscr{P}_0} |\vec{p}_{\rm R}|\Big|_{\mathscr{P}_0} |\vec{k}_{\rm R}|\Big|_{\mathscr{P}_0}\ . \tag{IV.8}
$$

The photon 3-momentum $\vec{k}_{\rm R}$ is an independent variable in the 3-Euclidean space around the event $\mathscr{P}_0$. As far as the integration process is concerned, we have the freedom to choose that in this 3-Euclidean space, the angle formed by $\vec{p}_{\rm R}$ and $\vec{k}_{\rm R}$ is $\theta_{\rm R}$, or in other words, $\vec{k}_{\rm R} = |\vec{k}_{\rm R}|\vec{e}_z$.

With these replacements, the evolution equation becomes

$$
\begin{aligned}
& 2k^0\Big|_{\mathscr{P}_0} \frac{d f_\lambda(|\vec{k}_{\rm R}|\big|_{\mathscr{P}_0}, \mathscr{P}_0)}{dt} \\
=& \frac{4 m_a \Gamma_a}{\pi(-g_{00}\big|_{\mathscr{P}_0})} \int \frac{m_a}{p^0\big|_{\mathscr{P}_0}} \frac{|\vec{p}_{\rm R}|^2 d|\vec{p}_{\rm R}| d(-\cos\theta_{\rm R}) d(\phi_{\rm R})\Big|_{\mathscr{P}_0}}{2\sqrt{-g_{00}\big|_{\mathscr{P}_0}}\, p^0\big|_{\mathscr{P}_0}} \\
& \times \frac{1}{2|\vec{p}_{\rm R} - \vec{k}_{\rm R}|\big|_{\mathscr{P}_0}} \times \delta\Big(p^0\Big|_{\mathscr{P}_0} - k^0\Big|_{\mathscr{P}_0} - \frac{|\vec{p}_{\rm R} - \vec{k}_{\rm R}|}{\sqrt{-g_{00}}}\Big|_{\mathscr{P}_0}\Big) \\
& \times \{ f_a(|\vec{p}_{\rm R}|\big|_{\mathscr{P}_0})[1 + f_\lambda(|\vec{k}_{\rm R}|\big|_{\mathscr{P}_0}) + f_\lambda(|\vec{p}_{\rm R} - \vec{k}_{\rm R}|\big|_{\mathscr{P}_0})] - f_\lambda(|\vec{k}_{\rm R}|\big|_{\mathscr{P}_0}) f_\lambda(|\vec{p}_{\rm R} - \vec{k}_{\rm R}|\big|_{\mathscr{P}_0}) \} \; .
\end{aligned}
$$

$\phi_{\rm R}\Big|_{\mathscr{P}_0}$ is directly integrated to yield

$$
\begin{aligned}
& 2k^0\Big|_{\mathscr{P}_0} \frac{d f_\lambda(|\vec{k}_{\rm R}|\big|_{\mathscr{P}_0}, \mathscr{P}_0)}{dt} \\
=& \frac{4 m_a \Gamma_a}{(-g_{00}\big|_{\mathscr{P}_0})} \int \frac{m_a}{p^0\big|_{\mathscr{P}_0}} \frac{|\vec{p}_{\rm R}|^2 d|\vec{p}_{\rm R}| d(-\cos\theta_{\rm R})\Big|_{\mathscr{P}_0}}{\sqrt{-g_{00}\big|_{\mathscr{P}_0}}\, p^0\big|_{\mathscr{P}_0}} \\
& \times \frac{1}{2|\vec{p}_{\rm R} - \vec{k}_{\rm R}|\big|_{\mathscr{P}_0}} \times \delta\Big(p^0\Big|_{\mathscr{P}_0} - k^0\Big|_{\mathscr{P}_0} - \frac{|\vec{p}_{\rm R} - \vec{k}_{\rm R}|}{\sqrt{-g_{00}}}\Big|_{\mathscr{P}_0}\Big) \\
& \times \{ f_a(|\vec{p}_{\rm R}|\big|_{\mathscr{P}_0})[1 + f_\lambda(|\vec{k}_{\rm R}|\big|_{\mathscr{P}_0}) + f_\lambda(|\vec{p}_{\rm R} - \vec{k}_{\rm R}|\big|_{\mathscr{P}_0})] - f_\lambda(|\vec{k}_{\rm R}|\big|_{\mathscr{P}_0}) f_\lambda(|\vec{p}_{\rm R} - \vec{k}_{\rm R}|\big|_{\mathscr{P}_0}) \} \; .
\end{aligned}
\tag{IV.9}
$$

The condition of normalization at point $\mathscr{P}_0$ reads

$$
m^2 + \mathscr{G}_{ij} p^i_{\rm R} p^j_{\rm R}\Big|_{\mathscr{P}_0} = m^2 + |\vec{p}_{\rm R}|^2\Big|_{\mathscr{P}_0} = -g_{00}(p^0)^2\Big|_{\mathscr{P}_0} \; ,
$$

which gives a differential relation at $\mathscr{P}_0$

$$
|\vec{p}_{\rm R}| d|\vec{p}_{\rm R}|\Big|_{\mathscr{P}_0} = -g_{00} p^0 dp^0\Big|_{\mathscr{P}_0} \; .
$$

From the law of cosines (IV.8), there is another differential relation which tells us how the magnitude $|\vec{p}_{\rm R} - \vec{k}_{\rm R}|$ changes when we change the angle $\theta_{\rm R}$ formed by $\vec{p}_{\rm R}$ and $\vec{k}_{\rm R}$ at $\mathscr{P}_0$,

$$
|\vec{p}_{\rm R} - \vec{k}_{\rm R}| d|\vec{p}_{\rm R} - \vec{k}_{\rm R}|\Big|_{\mathscr{P}_0} = |\vec{p}_{\rm R}||\vec{k}_{\rm R}| d(-\cos\theta_{\rm R})\Big|_{\mathscr{P}_0} \; .
$$

Replacing $d(-\cos\theta_{\text{R}})$ with $d|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|$, $|\vec{p}_{\text{R}}|d|\vec{p}_{\text{R}}|$ with $-g_{00}p^0dp^0$, and changing the argument of the δ function, equation (IV.9) turns into

$$
\begin{aligned}
&2k^0\Big|_{\mathscr{P}_0}\frac{df_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0},\mathscr{P}_0)}{dt}\\
=&\frac{4m_a\Gamma_a}{(-g_{00}\Big|_{\mathscr{P}_0})}\int\frac{m_a}{p^0\Big|_{\mathscr{P}_0}}\frac{(-g_{00})p^0dp^0\quad|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|d|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0}}{\sqrt{-g_{00}\Big|_{\mathscr{P}_0}}p^0\Big|_{\mathscr{P}_0}|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0}}\\
&\times\frac{1}{2|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0}}\times\sqrt{-g_{00}}\Big|_{\mathscr{P}_0}\times\delta(|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0}-\sqrt{-g_{00}}(p^0-k^0)\Big|_{\mathscr{P}_0})\\
&\times\{f_a(|\vec{p}_{\text{R}}|\Big|_{\mathscr{P}_0})[1+f_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0})+f_\lambda(|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0})]-f_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0})f_\lambda(|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0})\}\ .
\end{aligned}
$$

After canceling common factors in numerators and denominators, integrating the $|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|$ part, this equation becomes

$$
\begin{aligned}
&k^0\Big|_{\mathscr{P}_0}\frac{df_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0},\mathscr{P}_0)}{dt}\\
=&m_a\Gamma_a\int\frac{m_a}{p^0\Big|_{\mathscr{P}_0}}\frac{dp^0\Big|_{\mathscr{P}_0}}{|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0}}\times\{f_a(|\vec{p}_{\text{R}}|\Big|_{\mathscr{P}_0})[1+f_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0})+f_\lambda\left(\sqrt{-g_{00}}(p^0-k^0)\Big|_{\mathscr{P}_0}\right)]\\
&-f_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0})f_\lambda\left(\sqrt{-g_{00}}(p^0-k^0)\Big|_{\mathscr{P}_0}\right)\}\ .
\end{aligned}
$$

The rate of change of photon occupation number at $\mathscr{P}_0$ is then

$$
\begin{aligned}
&\frac{df_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0},\mathscr{P}_0)}{dt}\\
=&\frac{m_a\Gamma_a}{k^0|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0}}\int\frac{m_a}{p^0\Big|_{\mathscr{P}_0}}dp^0\Big|_{\mathscr{P}_0}\times\{f_a(|\vec{p}_{\text{R}}|\Big|_{\mathscr{P}_0})[1+f_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0})+f_\lambda\left(\sqrt{-g_{00}}(p^0-k^0)\Big|_{\mathscr{P}_0}\right)]\\
&-f_\lambda(|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0})f_\lambda\left(\sqrt{-g_{00}}(p^0-k^0)\Big|_{\mathscr{P}_0}\right)\}\ .
\end{aligned}
\tag{IV.10}
$$

We converte from Riemann normal coordinates x^i_{R} back to general coordinates, via the following substitutions,

$$
\begin{aligned}
&\sqrt{g_{ij}p^ip^j}\Big|_{\mathscr{P}_0}\Leftarrow\sqrt{\mathscr{G}_{ij}p^i_{\text{R}}p^j_{\text{R}}}\Big|_{\mathscr{P}_0}=|\vec{p}_{\text{R}}|\Big|_{\mathscr{P}_0}\qquad\sqrt{g_{ij}k^ik^j}\Big|_{\mathscr{P}_0}\Leftarrow\sqrt{\mathscr{G}_{ij}k^i_{\text{R}}k^j_{\text{R}}}\Big|_{\mathscr{P}_0}=|\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0}\\
&\sqrt{g_{ij}(p^i-k^i)(p^j-k^j)}\Big|_{\mathscr{P}_0}\Leftarrow\sqrt{\mathscr{G}_{ij}(p^i_{\text{R}}-k^i_{\text{R}})(p^j_{\text{R}}-k^j_{\text{R}})}\Big|_{\mathscr{P}_0}=|\vec{p}_{\text{R}}-\vec{k}_{\text{R}}|\Big|_{\mathscr{P}_0}\ .
\end{aligned}
$$

With $k^0\Big|_{\mathscr{P}_0} = \sqrt{\frac{g_{ij}k^ik^j}{-g_{00}}}\Big|_{\mathscr{P}_0}$, equation (IV.10) changes to

$$\begin{aligned}\frac{df_\lambda(\sqrt{g_{ij}k^ik^j}\Big|_{\mathscr{P}_0},\mathscr{P}_0)}{dt} =& \frac{m_a\Gamma_a\sqrt{-g_{00}}\Big|_{\mathscr{P}_0}}{g_{ij}k^ik^j\Big|_{\mathscr{P}_0}}\int\frac{m_a}{p^0\Big|_{\mathscr{P}_0}}dp^0\Big|_{\mathscr{P}_0}\times\{f_a(\sqrt{g_{ij}p^ip^j}\Big|_{\mathscr{P}_0})[1+f_\lambda(\sqrt{g_{ij}k^ik^j}\Big|_{\mathscr{P}_0})\\ &+f_\lambda\left(\sqrt{-g_{00}}(p^0-k^0)\Big|_{\mathscr{P}_0}\right)]-f_\lambda(\sqrt{g_{ij}k^ik^j}\Big|_{\mathscr{P}_0})f_\lambda\left(\sqrt{-g_{00}}(p^0-k^0)\Big|_{\mathscr{P}_0}\right)\}\ .\end{aligned}$$

The reason we can use Riemann normal coordinate and convert back is that, the equations do not depend on the specific Riemann normal coordinates we are using, and where the event point is located. All the relevant variables can be written in covariant form. We can go through the same process at events $\mathscr{P}_1(t_{\mathscr{P}_1},x^i_{\mathscr{P}_1})$, and $\mathscr{P}_2(t_{\mathscr{P}_2},x^i_{\mathscr{P}_2})$ using Riemann normal coordinates x^i_{R1} and x^i_{R2} respectively, and then obtain equations of the same form. If the event $\mathscr{P}_0(t_{\mathscr{P}_0},x^i_{\mathscr{P}_0})$ is not at a special point in spacetime, then we should have this equation at any location x^α,

$$\begin{aligned}\frac{df_\lambda(\sqrt{g_{ij}k^ik^j})}{dt} =& \frac{m_a\Gamma_a\sqrt{-g_{00}}}{g_{ij}k^ik^j}\int\frac{m_a}{p^0}dp^0\times\{f_a(\sqrt{g_{ij}p^ip^j})[1+f_\lambda(\sqrt{g_{ij}k^ik^j})+f_\lambda\left(\sqrt{-g_{00}}(p^0-k^0)\right)]\\ &-f_\lambda(\sqrt{g_{ij}k^ik^j})f_\lambda\left(\sqrt{-g_{00}}(p^0-k^0)\right)\}\ .\end{aligned} \tag{IV.11}$$

We note the factor of $\sqrt{-g_{00}}$ is from the gravitational redshift which corrects the time difference between the clock in the lab and the clock at the location the axion. The factor $\frac{m_a}{p^0}$ is due to special relativity correction.

IV.6 Kinematics of decay in static spacetime

Utilizing the normalization of momentum, the law of cosine (IV.8) can be written as

$$\begin{aligned}(-g_{00})(p^0-k^0)^2\Big|_{\mathscr{P}_0} =& [(-g_{00})(p^0)^2-m_a^2]\Big|_{\mathscr{P}_0}+(-g_{00})(k^0)^2\Big|_{\mathscr{P}_0}\\ &-2\cos\theta_{\text{R}}\sqrt{(-g_{00})(p^0)^2-m_a^2}\sqrt{(-g_{00})(k^0)^2}\Big|_{\mathscr{P}_0}\ .\end{aligned}$$

$\cos^2\theta_{\text{R}}\Big|_{\mathscr{P}_0}\leq 1$ requires that

$$p^0\Big|_{\mathscr{P}_0}\geq k^0\Big|_{\mathscr{P}_0}+\frac{m_a^2}{4k^0(-g_{00})}\Big|_{\mathscr{P}_0}\quad\text{or}\quad k^0_{\text{min}}\Big|_{\mathscr{P}_0}\leq k^0\Big|_{\mathscr{P}_0}\leq k^0_{\text{max}}\Big|_{\mathscr{P}_0}\ ,$$

where

$$k^0_{\text{max/min}}\Big|_{\mathscr{P}_0} = \frac{1}{2}[\, p^0 \pm \sqrt{(p^0)^2 - \frac{m_a^2}{-g_{00}}}\,]\Big|_{\mathscr{P}_0} = \frac{1}{2}(p^0 \pm \sqrt{\frac{g_{ij}p^ip^j}{-g_{00}}})\Big|_{\mathscr{P}_0} .$$

This equation and equation (IV.8) provides a relationship between momentum of the axions and the photons, all located at a single point where Riemann normal coordinates were applied. But in general, neither the law of cosines nor the trigonometric relation $\cos^2\theta \le 1$ holds in curved spacetime over an extended region. We find that at event $\mathscr{P}_0$, there are upper and lower limits on the 0^{th} component of photon momentum. This is true for every event at any point in spacetime. The bounds are

$$p^0 \ge k^0 + \frac{m_a^2}{4k^0(-g_{00})} \qquad k^0_{\text{max/min}} = \frac{1}{2}[\, p^0 \pm \sqrt{(p^0)^2 - \frac{m_a^2}{-g_{00}}}\,] = \frac{1}{2}(p^0 \pm \sqrt{\frac{g_{ij}p^ip^j}{-g_{00}}})$$
$$\sqrt{g_{ij}k^ik^j}_{\text{max/min}} = \frac{1}{2}(\sqrt{-g_{00}}p^0 \pm \sqrt{g_{ij}p^ip^j}) . \qquad \text{(IV.12)}$$

IV.7 From occupation number f to number density n

Switching from variable $\sqrt{g_{ij}p^ip^j}$ to p^0, the evolution equation (IV.11) becomes

$$\frac{df_\lambda(\sqrt{g_{ij}k^ik^j})}{dt} = \frac{m_a\Gamma_a\sqrt{-g_{00}}}{g_{ij}k^ik^j}\int_{k^0-\frac{m_a^2}{4k^0g_{00}}} \frac{m_a}{p^0}dp^0\{f_a\left(\sqrt{(-g_{00})(p^0)^2 - m_a^2}\right)$$
$$\times[1 + f_\lambda(\sqrt{g_{ij}k^ik^j}) + f_\lambda\left(\sqrt{-g_{00}}(p^0 - k^0)\right)] - f_\lambda(\sqrt{g_{ij}k^ik^j})f_\lambda\left(\sqrt{-g_{00}}(p^0 - k^0)\right)\} .$$

Sincet $p^0 = k^0 + k_1^0$ and $\sqrt{-g_{00}}k_1^0 = \sqrt{g_{ij}k_1^ik_1^j}$, this equation can also be rewritten as an integral over $\sqrt{g_{ij}k_1^ik_1^j}$,

$$\frac{df_\lambda(\sqrt{g_{ij}k^ik^j})}{dt} = \frac{m_a\Gamma_a\sqrt{-g_{00}}}{g_{ij}k^ik^j}\int_{\frac{m_a^2}{4\sqrt{g_{ij}k^ik^j}}} \frac{m_a d(\sqrt{g_{ij}k_1^ik_1^j})}{\sqrt{g_{ij}k^ik^j} + \sqrt{g_{ij}k_1^ik_1^j}} \times\{f_a\left(\sqrt{(\sqrt{g_{ij}k^ik^j} + \sqrt{g_{ij}k_1^ik_1^j})^2 - m_a^2}\right)$$
$$\times[1 + f_\lambda(\sqrt{g_{ij}k^ik^j}) + f_\lambda(\sqrt{g_{ij}k_1^ik_1^j})] - f_\lambda(\sqrt{g_{ij}k^ik^j})f_\lambda(\sqrt{g_{ij}k_1^ik_1^j})\} .$$

At any event x^α, the number density is given by

$$n(x^\alpha) = \int f(p^i, x^\alpha)\frac{\sqrt{||g_{ij}||}}{(2\pi)^3}dp^1dp^2dp^3 ,$$

which also holds at event $\mathscr{P}_0$ in Riemann normal coordinates with $\mathscr{G}_{ij}\Big|_{\mathscr{P}_0} = \eta_{ij} = (+1,+1,+1)$,

$$\begin{aligned}
\frac{dn_\lambda}{dt}\Big|_{\mathscr{P}_0} =& \frac{dn_\lambda}{dt}\Big|_{x^\alpha = t_{\mathscr{P}_0}, x^i_{\mathscr{P}_0}} = \int \frac{df_\lambda(\sqrt{g_{ij}k^ik^j})}{dt} \frac{\sqrt{||g_{ij}||}}{(2\pi)^3} dp^1 dp^2 dp^3 \Big|_{\mathscr{P}_0} \\
&= \int \frac{df_\lambda(\sqrt{\mathscr{G}_{ij}k^ik^j})}{dt} \frac{\sqrt{||\mathscr{G}_{ij}||}}{(2\pi)^3} dk^1 dk^2 dk^3 \Big|_{\mathscr{P}_0} = \int \frac{df_\lambda(|\vec{k}_{\text{R}}|)}{dt} \frac{|\vec{k}_{\text{R}}|^2}{(2\pi)^3} d|\vec{k}_{\text{R}}| d(-\cos\theta_{\text{R}}) d(\phi_{\text{R}}) \Big|_{\mathscr{P}_0} \\
&= \int \frac{df_\lambda(\sqrt{g_{ij}k^ik^j})}{dt} \frac{g_{ij}k^ik^j}{(2\pi)^3} d(\sqrt{g_{ij}k^ik^j}) d(-\cos\theta_{\text{R}}) d(\phi_{\text{R}}) \Big|_{\mathscr{P}_0} \\
&= \int \frac{df_\lambda(\sqrt{g_{ij}k^ik^j})}{dt} \frac{g_{ij}k^ik^j}{2\pi^2} d(\sqrt{g_{ij}k^ik^j}) \Big|_{\mathscr{P}_0} .
\end{aligned}$$

The rate of change in photon number density is then

$$\begin{aligned}
\frac{dn_\lambda}{dt} =& \frac{m_a \Gamma_a \sqrt{-g_{00}}}{2\pi^2} \int \int_{k^0 - \frac{m_a^2}{4k^0 g_{00}}} \frac{m_a}{p^0} \Big\{ f_a \left(\sqrt{(-g_{00})(p^0)^2 - m_a^2} \right) [1 + f_\lambda(\sqrt{g_{ij}k^ik^j}) \\
&+ f_\lambda(\sqrt{-g_{00}}p^0 - \sqrt{g_{ij}k^ik^j})] - f_\lambda(\sqrt{g_{ij}k^ik^j}) f_\lambda(\sqrt{-g_{00}}p^0 - \sqrt{g_{ij}k^ik^j}) \Big\} dp^0 d(\sqrt{g_{ij}k^ik^j}) ,
\end{aligned} \tag{IV.13}$$

or alternatively

$$\begin{aligned}
\frac{dn_\lambda}{dt} =& \frac{m_a \Gamma_a \sqrt{-g_{00}}}{2\pi^2} \int \int_{\frac{m_a^2}{4\sqrt{g_{ij}k^ik^j}}} \frac{m_a d(\sqrt{g_{ij}k_1^ik_1^j}) d(\sqrt{g_{ij}k^ik^j})}{\sqrt{g_{ij}k^ik^j} + \sqrt{g_{ij}k_1^ik_1^j}} \Big\{ f_a \left(\sqrt{(\sqrt{g_{ij}k^ik^j} + \sqrt{g_{ij}k_1^ik_1^j})^2 - m_a^2} \right) \\
&\times [1 + f_\lambda(\sqrt{g_{ij}k^ik^j}) + f_\lambda(\sqrt{g_{ij}k_1^ik_1^j})] - f_\lambda(\sqrt{g_{ij}k^ik^j}) f_\lambda(\sqrt{g_{ij}k_1^ik_1^j}) \Big\} .
\end{aligned}$$

IV.8 Setup of simple cluster model

Assume that the dependences of the axion occupation number are separable, and of the form

$$f_a(\sqrt{g_{ij}p^ip^j}, r, t) = \Theta(p_{\max} - \sqrt{g_{ij}p^ip^j})[f_{ac}(t)\Theta(r_+ - r)\Theta(r - r_-) + f_{ad}(t)d(r)] . \tag{IV.14}$$

We let $r_+ \sim r_-$ both be far beyond the event horizon of the host star or black hole, with the small distortion $d(r)$ in the region $(r_- \leq r \leq r_+)$ away from a uniform distribution. We define the maximum 3-momentum of an axion to be $p_{\max} = [\sqrt{g_{ij}(r)p^ip^j}]_{\max} = m_a\beta'$. [23] provides several ways of calculating the value of β' based on circumstances. For Schwarzschild space time,

$$\beta'_{\text{sch}} = 2 \times 10^{-3} \sqrt{\frac{MR_\odot}{M_\odot r_+}} \sqrt{\frac{r_+}{r_-} - 1} ,$$

where $M_\odot$ and $R_\odot$ are the solar mass and solar radius respectively, and M is the mass of host star or black hole. Suppose that the photon occupation number which correspondences to that of axions (IV.14), is

$$f_\lambda(\sqrt{g_{ij}k^ik^j},r,t) = [f_{\lambda c}(t)\Theta(r_+ - r)\Theta(r-r_-) + f_{\lambda d}(t)d(r)] \times \Theta(\sqrt{g_{ij}k^ik^j}_+ - \sqrt{g_{ij}k^ik^j})\Theta(\sqrt{g_{ij}k^ik^j} - \sqrt{g_{ij}k^ik^j}_-) \ , \tag{IV.15}$$

$\sqrt{g_{ij}k^ik^j}_{+/-}$ are $\sqrt{g_{ij}k^ik^j}_{\text{max/min}}$ in (IV.12) when $\sqrt{g_{ij}p^ip^j}$ takes the value $m_a\beta'$.

$$\begin{aligned}\sqrt{g_{ij}k^ik^j}_\pm &= \frac{1}{2}(\sqrt{-g_{00}}p^0 \pm \sqrt{g_{ij}p^ip^j})\Big|_{\sqrt{g_{ij}p^ip^j}=m_a\beta'} = \frac{1}{2}[\sqrt{m_a^2+(m_a\beta')^2} \pm \sqrt{(m_a\beta')^2}] \\ &= \frac{m_a}{2}(\sqrt{1+\beta'^2} \pm \beta') \ .\end{aligned}$$

We integrate equation (IV.13) over $|k_1^i|$ and $|k^i|$. The integration process is long and tedious, see [23] for detailed steps toward the results presented here.

$$\begin{aligned}\frac{dn_\lambda}{dt} =& \frac{m_a\Gamma_a}{2\pi^2}\sqrt{-g_{00}}\{\frac{m_a^2\beta'^3}{3}[f_{ac}(t)\Theta(r_+-r)\Theta(r-r_-) + f_{ad}(t)d(r) + 2f_{ac}(t)f_{\lambda c}(t)\Theta(r_+-r)\Theta(r-r_-) \\ &+ 2(f_{ac}(t)f_{\lambda d}(t) + f_{ad}(t)f_{\lambda c}(t))d(r) - f_{\lambda c}^2(t)\Theta(r_+-r)\Theta(r-r_-) - 2f_{\lambda c}(t)f_{\lambda d}(t)d(r)] \\ &- \frac{m_a^2\beta'^2}{2}[f_{\lambda c}^2(t)\Theta(r_+-r)\Theta(r-r_-) + 2f_{\lambda c}(t)f_{\lambda d}(t)d(r)]\} \ ,\end{aligned} \tag{IV.16}$$

Using Riemann normal coordinates, the axion number density at event x^α is an integration of $f_a(\sqrt{g_{ij}p^ip^j},r)$ over p^i,

$$\begin{aligned}n_a(t,x^i) =& \int_0^{m_a\beta'}\Theta(p_{\max} - \sqrt{g_{ij}p^ip^j})\frac{g_{ij}k^ik^j}{2\pi^2}d(\sqrt{g_{ij}k^ik^j})[f_{ac}(t)\Theta(r_+-r)\Theta(r-r_-) + f_{ad}(t)d(r)] \\ =& \frac{(m_a\beta')^3}{6\pi^2}[f_{ac}(t)\Theta(r_+-r)\Theta(r-r_-) + f_{ad}(t)d(r)] = [n_{ac}(t)\Theta(r_+-r)\Theta(r-r_-) + n_{ad}(t)d(r)] \ .\end{aligned}$$

The occupation numbers can be converted to number densities with

$$f_{ac}(t) = \frac{6\pi^2}{(m_a\beta')^3}n_{ac}(t) \ , \qquad f_{ad}(t) = \frac{6\pi^2}{(m_a\beta')^3}n_{ad}(t) \ .$$

Again, using Riemann normal coordinates, we calculated the volume of the shell $\Theta(\sqrt{g_{ij}k^ik^j}_+ - \sqrt{g_{ij}k^ik^j})$ $\times\Theta(\sqrt{g_{ij}k^ik^j} - \sqrt{g_{ij}k^ik^j}_-)$ to be

$$V_k \approx 4\pi[\frac{1}{2}(\sqrt{g_{ij}k^ik^j}_+ + \sqrt{g_{ij}k^ik^j}_-)]^2 \times (\sqrt{g_{ij}k^ik^j}_+ - \sqrt{g_{ij}k^ik^j}_-) = \pi m_a^3\beta'(1+\beta'^2) \ .$$

The number density of photons

$$n_\lambda(t,x^i) = n_{\lambda c}(t)\Theta(r_+ - r)\Theta(r - r_-) + n_{\lambda d}(t)d(r) \tag{IV.17}$$

is an integration of photon occupation number f_λ over k^i. The coefficients of occupation number and number density are related by

$$f_{\lambda c}(t) = \frac{(2\pi)^3 n_{\lambda c}(t)}{V_k} = \frac{8\pi^2}{m_a^3\beta'} n_{\lambda c}(t) , \qquad f_{\lambda d}(t) = \frac{(2\pi)^3 n_{\lambda d}(t)}{V_k} = \frac{8\pi^2}{m_a^3\beta'} n_{\lambda d}(t) ,$$

which we note is different from the axion relations.

IV.9 Radial distribution approximation

Because of the factor $\sqrt{-g_{00}}$ in (IV.13), neither axions nor photons can maintain a uniform radial distribution and it is this fact that causes the distortion $d(r)$ to arise in those quantities. We assume that the distortion is the displacement of $\sqrt{-g_{00}}$ from 1,

$$\sqrt{-g_{00}} = 1 + d(r) .$$

In the case of Schwarzschild spacetime, the Maclaurin series for the correction factor $\sqrt{-g_{00}}$ is

$$\sqrt{1 - \frac{2M}{r}} = 1 - \frac{1}{2}\frac{2M}{r} - \frac{1}{8}(\frac{2M}{r})^2 - \ldots$$

In the case of Reissner-Nordström spacetime it is

$$\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} = 1 - \frac{1}{2}\frac{2M}{r} + [\frac{1}{2}\frac{Q^2}{r^2} - \frac{1}{8}(\frac{2M}{r})^2] - \ldots$$

From the Maclaurin series, we can see that all the r dependent terms contribute to the unevenness of the distribution, i.e., to $d(r)$. Returning to the general form, we consider the time derivative of (IV.17). Neglecting terms which are second or higher order of $d(r)$, we have

$$\begin{aligned}\frac{dn_\lambda}{dt} =& \frac{dn_{\lambda c}(t)}{dt}\Theta(r_+ - r)\Theta(r - r_-) + \frac{dn_{\lambda d}(t)}{dt}d(r) = \frac{m_a\Gamma_a}{2\pi^2}[1+d(r)]\{\frac{m_a^2\beta'^3}{3}[f_{ac}(t)\Theta(r_+ - r)\Theta(r - r_-)\\ &+ f_{ad}(t)d(r) + 2f_{ac}(t)f_{\lambda c}(t)\Theta(r_+ - r)\Theta(r - r_-) + 2f_{ac}(t)f_{\lambda d}(t)d(r) + 2f_{ad}(t)f_{\lambda c}(t)d(r)\\ &- f_{\lambda c}^2(t)\Theta(r_+ - r)\Theta(r - r_-) - 2f_{\lambda c}(t)f_{\lambda d}(t)d(r)]\\ &- \frac{m_a^2\beta'^2}{2}[f_{\lambda c}^2(t)\Theta(r_+ - r)\Theta(r - r_-) + 2f_{\lambda c}(t)f_{\lambda d}(t)d(r)]\} .\end{aligned}$$

which defines the uniform and distorted parts of the number density, $n_{\lambda c}(t)$ and $n_{\lambda d}(t)$ respectively, as well as the uniform and distorted parts of the of the corresponding occupation number. We match terms according to whether the radial distribution is uniform $\Theta(r_+ - r)\Theta(r - r_-)$ or distorted $d(r)$. The equation for uniform part of the photon distribution is

$$\frac{dn_{\lambda c}(t)}{dt} = \frac{m_a\Gamma_a}{2\pi^2}\{\frac{m_a^2\beta'^3}{3}[f_{ac}(t) + 2f_{ac}(t)f_{\lambda c}(t) - f_{\lambda c}^2(t)] - \frac{1}{2}m_a^2\beta'^2 f_{\lambda c}(t)^2\} \ .$$

while the equation for the deformed part is

$$\begin{aligned}\frac{dn_{\lambda d}(t)}{dt} =& \frac{m_a\Gamma_a}{2\pi^2}\{\frac{m_a^2\beta'^3}{3} \times [f_{ac}(t) + f_{ad}(t) + 2f_{ac}(t)f_{\lambda c}(t) + 2f_{ac}(t)f_{\lambda d}(t) \\ &+ 2f_{ad}(t)f_{\lambda c}(t) - f_{\lambda c}^2(t) - 2f_{\lambda c}(t)f_{\lambda d}(t)] - \frac{m_a^2\beta'^2}{2}[f_{\lambda c}^2(t) + 2f_{\lambda c}(t)f_{\lambda d}(t)]\} \ .\end{aligned}$$

Finally we replace all the occupation number coefficents with number density coefficents and simplify to find

$$\frac{dn_{\lambda c}}{dt} = \Gamma_a(n_{ac} + \frac{16\pi^2}{m_a^3\beta'}n_{ac}n_{\lambda c} - \frac{32\pi^2\beta'}{3m_a^3}n_{\lambda c}^2 - \frac{16\pi^2}{m_a^3}n_{\lambda c}^2)$$

which is the same as equation (32) of [14], as expected. In addition, we have the simplified equation that accounts for radial distortion,

$$\begin{aligned}\frac{dn_{\lambda d}(t)}{dt} =& \Gamma_a(n_{ac} + n_{ad} + \frac{16\pi^2}{m_a^3\beta'}n_{ac}n_{\lambda c} + \frac{16\pi^2}{m_a^3\beta'}n_{ac}n_{\lambda d} + \frac{16\pi^2}{m_a^3\beta'}n_{\lambda c}n_{ad} \\ &- \frac{32\pi^2\beta'}{3m_a^3}n_{\lambda c}^2 - \frac{64\pi^2\beta'}{3m_a^3}n_{\lambda c}n_{\lambda d} - \frac{16\pi^2}{m_a^3}n_{\lambda c}^2 - \frac{32\pi^2}{m_a^3}n_{\lambda c}n_{\lambda d}) \ .\end{aligned}$$

IV.10 Surface loss and total photon density

The surface loss of photon at $r = r_+$ is

$$(dn_\lambda)_{r_+ \text{surface loss}} = \frac{1}{2} \times \frac{-dN_\lambda}{V} = -\frac{1}{2} \times \frac{n_\lambda dV}{\int \sqrt{|g_{ij}|}dx^i dx^2 dx^3} \ ,$$

where $\frac{1}{2}$ accounts for the probability that in the tangent space of an event at the surface, the momentum of the photon has positive radial component. In general, the surface loss rate Γ_s (at both r_+ and r_-) is proportional to the number density,

$$(\frac{dn_\lambda}{dt})_{r_+ \text{surface loss}} + (\frac{dn_\lambda}{dt})_{r_- \text{surface loss}} = -\Gamma_s n_\lambda \ .$$

For Schwarzschild spacetime,

$$(dn_\lambda)_{r_+\text{surface loss}} = -\frac{n_\lambda Sc\,(dt)}{\int\sqrt{(1-\frac{2M}{r})^{-1}r^2r^2\sin^2\theta drd\theta d\phi}}\times\frac{1}{2} = -\frac{n_\lambda Sc\,(dt)}{4\pi\int_{r_-}^{r_+}\frac{r^2}{\sqrt{1-\frac{2M}{r}}}dr}\times\frac{1}{2}$$
$$\approx -\frac{n_\lambda}{2}\frac{4\pi r_+^2}{\frac{4\pi r_+^3}{3}-\frac{4\pi r_-^3}{3}}c\,(dt)\ .$$

Here since $r \gg 2M$, (working to lowest order in $d(r)$) we used the approximation $\sqrt{1-\frac{2M}{r}}\sim 1$. Then the surface loss at r_+ is

$$(\frac{dn_\lambda}{dt})_{r_+\text{surface loss}} = -\frac{3cr_+^2 n_\lambda}{2(r_+^3-r_-^3)} = -\frac{3cr_+^2}{2(r_+^3-r_-^3)}[n_{\lambda c}(t)\Theta(R-r)+n_{\lambda d}(t)d(r)]\ .$$

The surface loss of photon at $r = r_-$ can be neglected because photons would go back to the cluster unless being captured by the black hole. $r_- \gg 2M$ limits the possibility of incidents that photon falling into black hole. Define the surface loss rate,

$$\Gamma_s = -\frac{3cr_+^2}{2(r_+^3-r_-^3)}\ .$$

With surface loss included, the photon number density rate equation becomes

$$\begin{aligned}\frac{dn_{\lambda c}}{dt} =& \Gamma_a[n_{ac}+\frac{16\pi^2}{m_a^3\beta'}n_{ac}n_{\lambda c}-\frac{32\pi^2}{3m_a^3}(\beta'+\frac{3}{2})n_{\lambda c}^2]-\Gamma_s n_{\lambda c}\ ,\\ \frac{dn_{\lambda d}}{dt} =& \Gamma_a[n_{ac}+n_{ad}+\frac{16\pi^2}{m_a^3\beta'}(n_{ac}n_{\lambda c}+n_{ac}n_{\lambda d}+n_{\lambda c}n_{ad})-\frac{32\pi^2}{3m_a^3}(\beta'+\frac{3}{2})n_{\lambda c}^2\\ &-\frac{64\pi^2\beta'}{3m_a^3}(\beta'+\frac{3}{2})n_{\lambda c}n_{\lambda d}]-\Gamma_s n_{\lambda d}\ .\end{aligned}$$

Assuming that all reactions create and/or annihilate equal number of photons from each helicity state, $n_{+c} = n_{-c}$ and $n_{+d} = n_{-d}$. Hence $n_{\lambda c} = \frac{1}{2}n_{\gamma c}$ and $n_{\lambda d} = \frac{1}{2}n_{\gamma d}$, and we find

$$\begin{aligned}\frac{dn_{\gamma c}}{dt} =& \Gamma_a[2n_{ac}+\frac{16\pi^2}{m_a^3\beta'}n_{ac}n_{\gamma c}-\frac{16\pi^2}{3m_a^3}(\beta'+\frac{3}{2})n_{\gamma c}^2]-\Gamma_s n_{\gamma c}\ ,\\ \frac{dn_{\gamma d}}{dt} =& \Gamma_a[2n_{ac}+2n_{ad}+\frac{16\pi^2}{m_a^3\beta'}(n_{ac}n_{\gamma c}+n_{ac}n_{\gamma d}+n_{\gamma c}n_{ad})-\frac{16\pi^2}{3m_a^3}(\beta'+\frac{3}{2})n_{\gamma c}^2\\ &-\frac{32\pi^2}{3m_a^3}(\beta'+\frac{3}{2})n_{\gamma c}n_{\gamma d}]-\Gamma_s n_{\gamma d}\ .\end{aligned}$$

The uniform and distorted axion number density rate equations are minus one half of the photon number density rate equation, excluding sterile axions

$$\begin{aligned}\frac{dn_{ac}}{dt} &= -\Gamma_a[+n_{ac} + \frac{8\pi^2}{m_a^3\beta'}n_{ac}n_{\gamma c} - \frac{8\pi^2}{3m_a^3}\beta' n_{\gamma c}^2] \ , \\ \frac{dn_{ad}}{dt} &= -\Gamma_a[+n_{ac} + n_{ad} + \frac{8\pi^2}{m_a^3\beta'}(n_{ac}n_{\gamma c} + n_{ac}n_{\gamma d} + n_{\gamma c}n_{ad}) - \frac{8\pi^2}{3m_a^3}\beta' n_{\gamma c}^2 - \frac{16\pi^2}{3m_a^3}\beta' n_{\gamma c}n_{\gamma d}] \ .\end{aligned}$$

Counting time in units of axion decay constant $1/\Gamma_a$, then defining a new dimensionless variable $t_a = t\Gamma_a$ and measuring volume in unit of axion Compton volume $\frac{16\pi^2}{m_a^3}$, the system of rate equations can be expressed as

$$\begin{aligned}\frac{dn_{\gamma c}^{\mathrm{C}}}{dt_a} &= [2n_{ac}^{\mathrm{C}} + \frac{1}{\beta'}n_{ac}^{\mathrm{C}}n_{\gamma c}^{\mathrm{C}} - \frac{1}{3}(\beta' + \frac{3}{2})(n_{\gamma c}^{\mathrm{C}})^2] - \frac{\Gamma_s}{\Gamma_a}n_{\gamma c}^{\mathrm{C}} \ , \\ \frac{dn_{ac}^{\mathrm{C}}}{dt_a} &= [-n_{ac}^{\mathrm{C}} - \frac{1}{2\beta'}n_{ac}^{\mathrm{C}}n_{\gamma c}^{\mathrm{C}} + \frac{\beta'}{6}(n_{\gamma c}^{\mathrm{C}})^2] \ , \\ \frac{dn_{\gamma d}^{\mathrm{C}}}{dt_a} &= 2n_{ac}^{\mathrm{C}} + 2n_{ad}^{\mathrm{C}} + \frac{1}{\beta'}(n_{ac}^{\mathrm{C}}n_{\gamma c}^{\mathrm{C}} + n_{ac}^{\mathrm{C}}n_{\gamma d}^{\mathrm{C}} + n_{\gamma c}^{\mathrm{C}}n_{ad}^{\mathrm{C}}) - \frac{1}{3}(\beta' + \frac{3}{2})(n_{\gamma c}^{\mathrm{C}})^2 - \frac{2}{3}(\beta' + \frac{3}{2})n_{\gamma c}^{\mathrm{C}}n_{\gamma d}^{\mathrm{C}} - \frac{\Gamma_s}{\Gamma_a}n_{\gamma d}^{\mathrm{C}} \ , \\ \frac{dn_{ad}^{\mathrm{C}}}{dt_a} &= -n_{ac}^{\mathrm{C}} - n_{ad}^{\mathrm{C}} - \frac{1}{2\beta'}(n_{ac}^{\mathrm{C}}n_{\gamma c}^{\mathrm{C}} + n_{ac}^{\mathrm{C}}n_{\gamma d}^{\mathrm{C}} + n_{\gamma c}^{\mathrm{C}}n_{ad}^{\mathrm{C}}) + \frac{\beta'}{6}(n_{\gamma c}^{\mathrm{C}})^2 + \frac{\beta'}{3}n_{\gamma c}^{\mathrm{C}}n_{\gamma d}^{\mathrm{C}} \ .\end{aligned}$$

These equations are our main results. They can be applied in many circumstances.

IV.11 Example and discussion: lasing axions clustered near a solar mass black hole

We have numerically solved the above system of rate equations for the case of a one solar mass, $M = M_\odot$ Schwarzschild black hole. (Although the mass of sun is below the minimum value required for a star to form a black hole, primordial black holes are allowed to have solar mass.) We assume there is a hadronic axion ($\sim$ 3 eV) cluster (Again, this may not be realistic because hadronic axions are not favored currently.) of diameter 600 m with roughly standard ice density 900 kg/m^3 which is equivalent to an initial axion number density of about 7.56×10^{18} times the unit axion Compton number density. If the cluster is placed at 40 AU (this corresponds to the radius of the Kuiper belt in solar system) from the black hole, the relativity index becomes $\beta' = 2.156 \times 10^{-10}$. The uniform photon density $n_{\gamma c}$ grows exponentially on a time scale of $10^{-28}/\Gamma_a$. Since the distortion factor $d(r) \sim -2.465 \times 10^{-10}$ is very small, the total photon and axion number density are affected very little. The detailed photon radiation outcome, such as growth time and pulse height, are highly dependent on the initial axion number density, as the surface loss would affect low density axion clusters more noticeably than high density ones.

For the numerical calculation we assume the axion cluster is approximately a cylinder rather than spherical. Actually it is a section of a cone of height 600 m, i.e., a frustum, which we get by including the

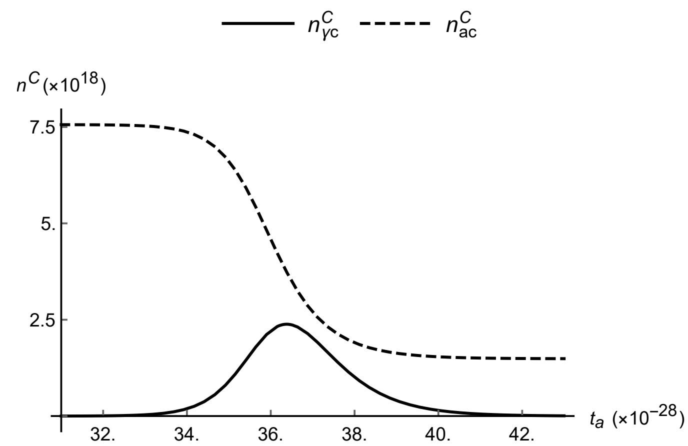

Figure IV.1: The uniform axion density n_{ac} decreases exponentially on the same temporal scale as the photos.

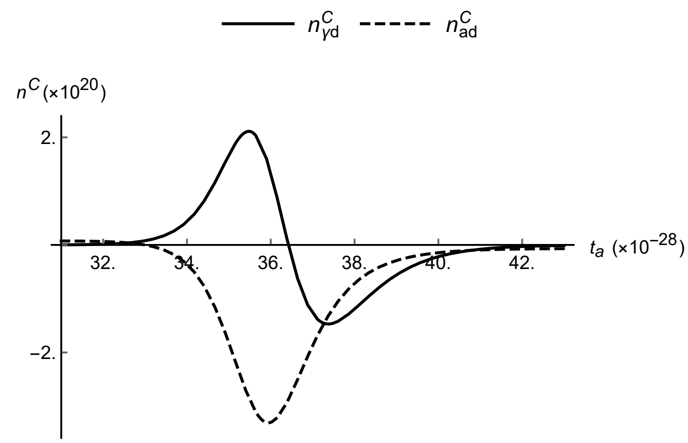

Figure IV.2: The distorted photon density formed a sharp pulse. The distorted axion density also formed a sharp pulse with the amplitude being the opposite of that of distorted photon.

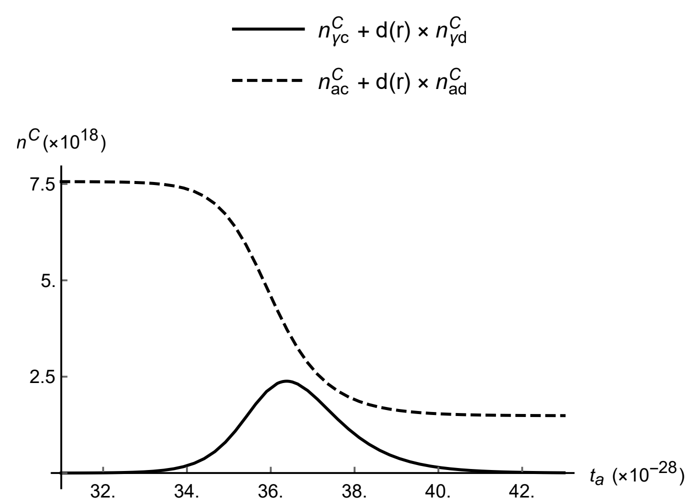

Figure IV.3: The total particle densities factors $\Theta(\theta)\Theta(10^{-8}-\theta)\Theta(\phi)\Theta(10^{-8}-\phi)$ to constrain the axion and photon occupation numbers (IV.14) and (IV.15), since g_{00} in equation (IV.16) does not depend on the θ or ϕ if the black hole is of Schwarzschild or Reissner-Nordström type. Including these factors makes our results applicable to asteroid size axion cluster. These factors will not change β'. During the lasing time scale, axions move only a few micrometer if the previous β' is the maximum velocity of the axions. This means that the axions would be confined in the region described by these Θ factors during lasing. Surface loss terms may need to be changed here, therefore it is not guaranteed that every point in the cluster would lase as the figures suggest, but some part of the cluster will. The purpose of the example presented here is only to demonstrate the applicability of the method we have developed but not to provide an example of a realistic physical system. Other examples are easily handled by this approach, for instance, ring type axion clusters are also able to be described by our results by adding similar angular factors and changing surface loss terms as far as a spherical symmetric metric is concerned.

CHAPTER V

Analysis of the possibility of lasing from axitons and axion stars

This chapter explores the some properties of axitons and axion stars based on the classical analogy with axion self interaction lagrangians and investigates the conditions of possible lasing from these objects.

V.1 Guesstimate of interaction potential between two axions

We want to find a classical potential that can characterize the interaction between two axions similar to the way Newtonian gravity does. In [18], the axion interaction potential is expressed by

$$V(\phi) = \frac{1}{2}m_a^2\phi^2 + (m_a f_a)^2 \sum_{n=2}^{\infty} \frac{\lambda_{2n}}{(2n)!}(\frac{\phi}{f_a})^{2n} , \tag{V.1}$$

where f_a is the decay constant of axion and m_a is the mass of axion. Astrophysical and cosmological constraints put the value of f_a between 3×10^9 GeV and 10^{12} GeV. The product of m_a and f_a was given in [15] as

$$m_a f_a = [75.5 \text{ MeV}]^2 .$$

Based on this relation, the mass of axion m_a has a typical value of $10^{-4\pm1}$ eV. $V(\phi)$ is the potential density of the axion field with dimension of $[ML^{-3}] = [M^4]$. This means that the field ϕ has the same dimension as energy. The lagrangian density of axion field with lowest two orders of interaction is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_a^2\phi^2 - \frac{\lambda_4}{4!}(\frac{m_a}{f_a})^2\phi^4 - \frac{\lambda_6}{6!}(\frac{m_a^2}{f_a^4})\phi^6 .$$

The quantum field lagrangian density is obtained from the classical particle lagrangian through the canonical quantization, $p \to \partial_\mu\phi, \quad x \to \phi$. By reversing the quantization process,

$$\partial_\mu\phi \to p, \quad \phi \to x ,$$

we might guess a classical potential that roughly describes the interaction between individual axions. Under this analogy, $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_a^2\phi^2$ becomes the kinetic energy. The exact prescription of obtaining the classical potential is

$$\phi \to m_a \frac{r}{\frac{2\pi}{m_a}} = m_a r_a \text{ , then} \times (\frac{2\pi}{m_a})^3 \text{ ,} \tag{V.2}$$

where r_a is the distance r between two axions measured in unit of axion Compton wavelength $\ell_a = \frac{2\pi}{m_a}$. The field ϕ needs to have a dimension of energy and the only relevant energy scale here is the mass of axion. At the same time, the field ϕ needs to be replaced by the coordinate x but it can not take the dimension of length. Then a possible solution is to associate ϕ to distance measured in terms of Compton length of axion $\frac{2\pi}{m_a}$. At last, we have to multiply the potential density by a volume, in order to make the classical potential having the correct dimension. Under this prescription, the self interaction between axions is approximated to the lowest order by the potential energy

$$V_4(r_a) \approx \frac{|\lambda_4|\pi^3}{3}(\frac{m_a}{f_a})^2 m_a r_a^4 \text{ .} \tag{V.3}$$

This becomes an attractive potential when the absolute value of λ_4 is taken.

It was said in [18] that in the axion field potential (V.1), $\frac{m_a^2}{f_a^2}$ is a quantum loop factor and higher orders of interaction are suppressed by $\frac{m_a^2}{f_a^2}$. Under prescription (V.2), the next lowest order term in the potential density (V.1) becomes

$$V_6(r_a) \sim \frac{\lambda_6 \pi^3}{90}(\frac{m_a}{f_a})^4 m_a r_a^6 \text{ .}$$

This shows that the second lowest order potential energy V_6 is $\frac{m_a^2}{f_a^2}$ times smaller than V_4, which is consistent with the argument presented in [18]. As a side note, higher order potentials gain an amplification factor of r_a^2. As long as this factor is smaller than the suppression factor $\frac{m_a^2}{f_a^2}$, the higher order potentials such as V_6, V_8 could be ignored. That is to say,

$$r_a < \sqrt{30|\frac{\lambda_4}{\lambda_6}|}\frac{f_a}{m_a} \text{ .}$$

If we take $f_a = \sqrt{3 \times 10^9 \times 10^{12}}$ GeV, this suggests that the higher order potential can be dropped if the distance r between axions is less than 2.1×10^6 ly. In the following study, we won't encounter events of galactic size, so higher order potentials will be neglected.

The energy scale of potential (V.3) is very small at small distance due to the supression factor $\frac{m_a^2}{f_a^2}$. Two axions separated from each other with a distance of 1 μm have $V_4 \approx 6.61 \times 10^{-68}$ eV. By comparison, the same distance between two axions provides a gravitational potential of $V_{gr} \approx 1.43 \times 10^{-65}$ eV. At closer range, the attraction between axions would be even weaker, and much weaker than gravity. So potential (V.3) doesn't contradict the current consensus that the self interaction between axions is very weak.

At larger distance, if potential (V.3) is still applicable, it indicates a stronger attraction between axions via self interaction than gravity. For instance, self interaction at a distance of 1 cm has a value of $V_4 \approx 6.61 \times 10^{-52}$ eV, which is still weak but much stronger than that from gravity $V_{gr} \approx 1.43 \times 10^{-69}$ eV.

V.2 Continuous potential of an axion cluster

Potential (V.3) shows the energy between two individual axions. If we have a continuous distribution of axions, the potential energy at each location can be obtained by integration over the entire distribution. Consider a sphere of radius R with the center of the sphere sitting at the origin. Two arbitrary points inside the sphere have coordinates $\vec{r}_1$ and $\vec{r}_2$. We can arrange the coordinate system so that the z-axis align with $\vec{r}_1$ which results in

$$\vec{r}_1 = r_1\vec{e}_z \,, \qquad \vec{r}_2 = r_2 \cos\theta\,\vec{e}_z + r_2 \sin\theta \cos\varphi\,\vec{e}_x + r_2 \sin\theta \sin\varphi\,\vec{e}_y \,, \qquad |\vec{r}_1 - \vec{r}_2| = \sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2} \,.$$

The gravitational potential at $\vec{r}_1$ is given by

$$U_{gr}(r_1) = \int -\frac{G\rho r_2^2 dr_2 \sin\theta d\theta d\varphi}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}} = \begin{cases} \frac{GM}{2R^3}(r_1^2 - 3R^2)\,, & r_1 < R \\ -\frac{GM}{r_1}\,, & r_1 > R \end{cases}$$

where ρ is the constant mass density of the axion sphere of mass M. By similar calculation we can find the continuous axion self interaction potential

$$U_4(r_1) = \int n_a(\vec{r}_2) V_4(\vec{r}_1 - \vec{r}_2) r_2^2 dr_2 \sin\theta d\theta d\varphi \,,$$

where $n_a(\vec{r}_2)$ is the number density of axions at $\vec{r}_2$. For a constant mass density ρ, the number density also becomes constant $n_a(\vec{r}_2) = n_{a0} = \frac{3M}{4\pi R^3 m_a} = \frac{\rho}{m_a}$. The self interaction potential at $\vec{r}_1$ becomes

$$U_4(r_1) = \int n_{a0} \frac{|\lambda_4|\pi^3}{3} (\frac{m_a}{f_a})^2 m_a \left(\frac{|\vec{r}_1 - \vec{r}_2|}{\ell_a}\right)^4 r_2^2 dr_2 \sin\theta d\theta d\varphi = \frac{\pi^3|\lambda_4|}{3\ell_a^4}(\frac{m_a}{f_a})^2 (Mr_1^4 + 2Mr_1^2R^2 + M\frac{3R^4}{7})$$

We can write the continuous potential in terms of dimensionless length $r_a = \frac{r}{\ell_a}$,

$$U_4(r_a) = \pi^3 |\lambda_4| (\frac{m_a}{f_a})^2 M (\frac{r_a^4}{3} + \frac{2r_a^2R_a^2}{3} + \frac{R_a^4}{7}) \,. \tag{V.4}$$

The value of potential (V.4) increases as the distance from the center increases, this is similar to the gravitational potential. Unlike gravity, the zero point of U_4 is not at $r_a = \infty$, but is at $r_a = 0$. The exact potential energy at any location r_a should be $U_4(r_a) - U_4(0)$.

V.3 Comparison of self interaction and gravity for axiton and axion star

A Newtonian gravitational potential is not guaranteed to bound a particle as long as the particle has enough kinetic energy. If potential (V.3) is applicable at a long distance, then it seems that a particle can never escape the quartic barrier, regardless of its kinetic energy. This implies that if potential (V.3) is applicable at relative long distance, axions inside an axiton formed by self interaction can only be relativistic. There is a critical radius of the axion cluster beyond which the self interaction potential is larger than the gravitational potential at the surface R:

$$U_4(R) - U_4(0) > m_a[U_{gr}(\infty) - U_{gr}(R)] \ , \qquad \Rightarrow \qquad R_{\text{self-dominate}} > \sqrt[5]{\frac{\ell_a^4}{\pi^3|\lambda_4|}(\frac{f_a}{m_a})^2\frac{Gm_a}{c^4}} \approx 2.35\mu\text{m} \ .$$

Stable axion clusters of 1 cm in size can exist according to [18], which may verify that the axion self interaction has an effective range longer than 2.35 μm. Then the self interaction potential may have an applicable range from a few μm to a few cm. At the same time, we expect the potentials (V.3) and (V.4) to break down at some distance $R_{\text{b}d} > 1$ cm. The first reason is that the prescription (V.2) is not an accurate calculation, and the potentials (V.3) and (V.4) trap axions inside an infinite high potential well which is not physically achievable. According to [18], there is no stable axiton of $\gg 1$ cm in size.

For axitons, gravity can be ignored since the self interaction dominates in the range from 2.35 μm to 1 cm. Therefore, we only need to consider axion self interaction when we investigate axitons. However, the corresponding argument probably is not true for axion stars. In theory, stable axion stars of roughly 100 km in size can exist. Based on the discussion so far, inside this type of axion stars, it's possible that in the range from μm to cm, self interaction of axiton dominates, which results in many small granular axitons or pebble axitons. Thus the inner structure of axion stars may be that, on a short length scale axions form granular axitons because of the stronger self interaction; on a long length scale, as the self interaction breaks down and gravitational potential grows stronger, those granular axitons are bounded by gravity and form the axion star. If a dilute axion star of mass $1.2 \times 10^{-14} M_\odot$ and radius $2.3 \times 10^{-3} R_\odot$ is made up of granular axitons of mass $2.0 \times 10^{-19} M_\odot$, radius $1.75 \times 10^{-11} R_\odot$, then the number density of granular axiton is about 1 per 2.87×10^5 km^3. There is very long distance between granular axitons. On the other hand, if a dense axion star of mass $1.0 \times 10^{-12} M_\odot$, radius $5 \times 10^{-10} R_\odot$ is made up of the same granular axitons, then the number

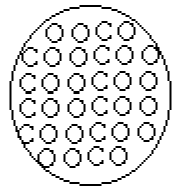

Figure V.1: axion star inner structure: granular axitons

density of granular axiton is about 28 per cm^3. There is significant overlap between the granular axitons.

V.4 Axiton mechanics

Consider an axion cluster of mass M and radius R and ignore gravity, an axion initially sitting at the surface of the cluster would have the maximum possible kinetic energy when it travels down to the center of the cluster from the surface. Let β denote the maximum possible speed of the axion when it is at the origin. Then ignoring gravity, conservation of energy gives us

$$m_a(\frac{1}{\sqrt{1-\beta^2}}-1) = U_4(R) - U_4(0) = \pi^3|\lambda_4|(\frac{m_a}{f_a})^2 M R_a^4 \,. \tag{V.5}$$

Different value combinations of the mass M and radius R of the axiton would make the resident axions either relativistic or nonrelativistic. We plotted figure V.2 that shows the conditions the mass M and radius R of the axiton need to meet in order to keep those axions nonrelativistic, semirelativistic, and relativistic, respectively. For an axiton of mass $2.0 \times 10^{-20} M_\odot$, radius $1.75 \times 10^{-11} R_\odot$, which may be a stable axiton as suggested in [18], the maximum axion speed is found to be larger than $0.999c$, if we assume the axiton has a uniform density. This does not necessarily mean that most of the axions in this axiton are highly relativistic. It only requires that an axion needs an initial speed of $> 0.999c$ to escape from the center to the surface of the axiton. So even an initial highly relativistic axion will not always be relativistic. A common scenario is that a highly relativistic axion initially at the center moves towards the outside, losing kinetic energy quickly and then becoming nonrelativistic. When it barely touches the surface of the axiton, it exhausts all the kinetic energy and moves back toward the center. The axions essentially oscillate between the surface and the center. Our calculation shows that the average number density of this axiton is a few times higher than the central number density given in [18]. This could possibly confirms that relativistic axions can exist in the center of the axiton. If we assume the mass density is same everywhere in the axiton, then a lower number density requires the axiton to be heavier.

Along the track of an axion bouncing between the center and the surface of the axiton, we can find the

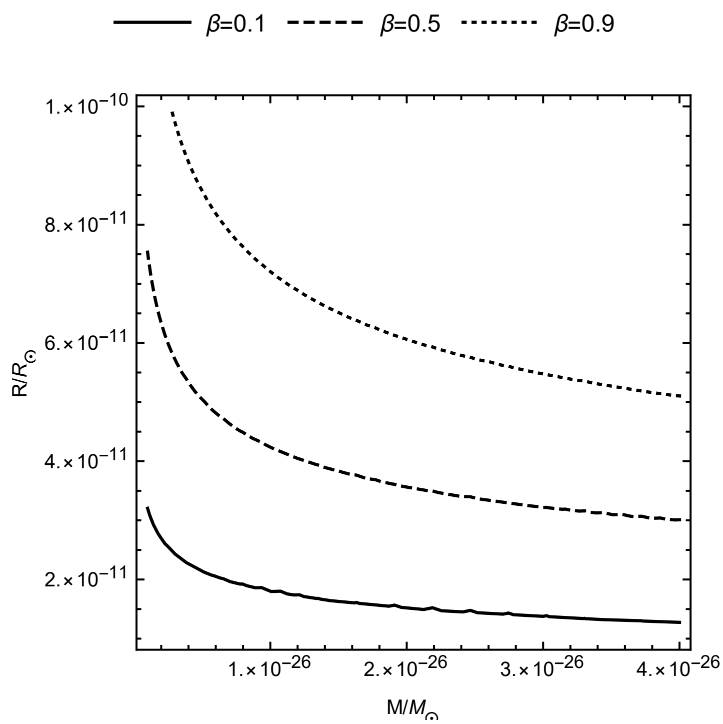

Figure V.2: Relativistic category of axitons formed of 10^{-4} eV axions

velocity β_1 of the axion when it is at r_1.

$$
\begin{aligned}
m_a(\frac{1}{\sqrt{1-\beta^2}}-1) =& U_4(R)-U_4(0)=\frac{\pi^3|\lambda_4|}{\ell_a^4}(\frac{m_a}{f_a})^2 MR^4 \\
m_a(\frac{1}{\sqrt{1-\beta_1^2}}-1) =& U_4(R)-U_4(r_1)=\frac{\pi^3|\lambda_4|}{3\ell_a^4}(\frac{m_a}{f_a})^2 M(3R^4-r_1^4-2r_1^2R^2) \\
\Rightarrow (\frac{1}{\sqrt{1-\beta_1^2}}-1) =& (\frac{1}{\sqrt{1-\beta^2}}-1)(1-\frac{r_1^4}{3R^4}-\frac{2r_1^2}{3R^2})
\end{aligned}
$$

The relationship between β_1 and r_1 for surface axions with zero angular momentum inside nonrelativistic, semirelativistic, and relativistic axiton are plotted in figure V.3. For a nonrelativistic axiton of $\beta=0.1$, the maximum velocity of an axion is 0.1c and that happens when a surface axion at $r_1=R$ falls down and arrives at the center $r_1=0$. The entire axiton is nonrelativistic as the $\beta=0.1$ curve shows. For a semirelativistic axiton of $\beta=0.5$, the maximum velocity of an axion is 0.5c and that happens when a surface axion at $r_1=R$ falls down and arrives at the center $r_1=0$. It is not that all the axions are semirelativistic. As the $\beta=0.5$ curve shows, in the shell region from $0.95R$ to R, the axions are nonrelativistic. For a relativistic axiton of

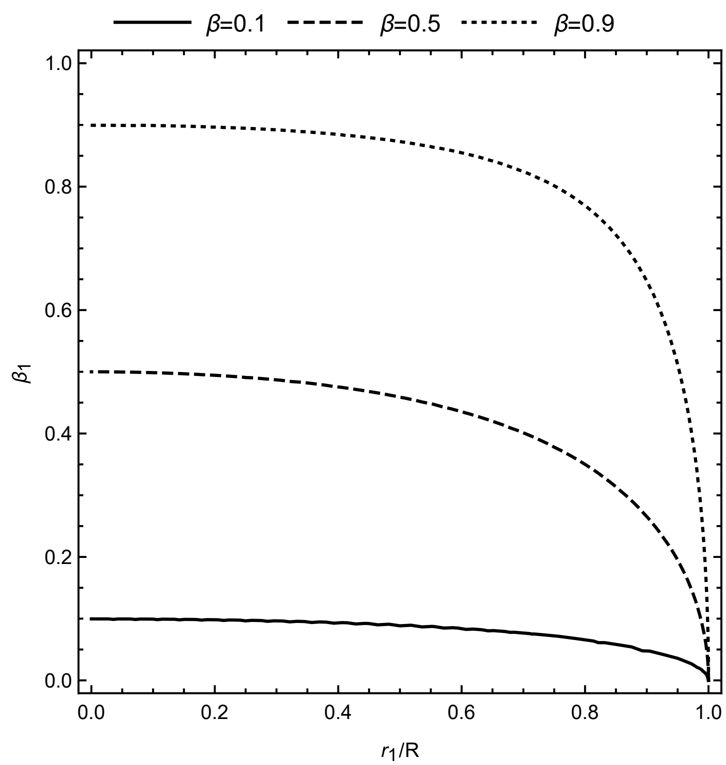

Figure V.3: Relativistic regions of axitons

$\beta = 0.9$, the maximum velocity of an axion is 0.9c and that happens when a surface axion at $r_1 = R$ falls down and arrives at the center $r_1 = 0$. Indeed, most of the axions inside this type of axiton are highly relativistic. But in the thin layer from $0.99R$ to R, these "exhausted" axions can be treated as nonrelativistic.

V.5 Relativistic axiton surface lasing model

According to [14], lasing of axion clusters requires two conditions. The first condition is that the initial axion density is high enough. The second condition is that these axions are nonrelativistic. We think this second condition would be satisfied in many cases. For a highly relativistic axiton with $\beta \sim 1$, its thin surface layer of axions consists of nonrelativistic "exhausted" axions. And if the density of those nonrelativistic axion is high enough, this surface layer of axions would lase first. As this lasing finishes, there would be a new type of axiton with less mass and shorter radius. Equation (V.5) says the maximum velocity β of the new axiton is less than the axiton before lasing. Therefore this new axiton would be less relativistic than the previous axiton, and have a thicker layer of nonrelativistic axions on its surface. This may leads to another lasing event if the density of the axions is high enough. In short, even in the case of highly relativistic axiton, there could be a cascade of lasing events that continues to reduce the mass and shrink the size of axiton, that renders the axiton less relativistic, that gradually gains thicker layer of nonrelativistic axions at its surface which could

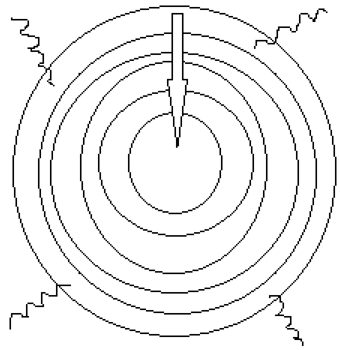

Figure V.4: Relativistic axiton lases through layers of nonrelativistic axions

produce a stronger lasing event. Relativistic axitons showly shreds its nonrelativistic layers of axions through lasing, which causes its inner semirelativistic layers of axions to become less relativistic and starts another lasing. The process may not stop until the entire axiton has self annihilated. This process could also be continuous with lasing always taking place at the surface of the axiton.

V.6 Axiton and axion star lasing

[14] proposed equations for the evolution of axion cluster lasing,

$$\frac{dn_\gamma^C}{dt_a} = 2n_a^C + \frac{1}{\beta}n_a^C n_\gamma^C - (\frac{\beta}{3}+\frac{1}{2})(n_\gamma^C)^2 - \frac{\Gamma_s}{\Gamma_a}n_\gamma^C , \qquad \frac{dn_a^C}{dt_a} = -n_a^C - \frac{1}{2\beta}n_a^C n_\gamma^C + \frac{\beta}{6}(n_\gamma^C)^2 . \tag{V.6}$$

n_a^C and n_γ^C are the dimensionless number density of axion and photon, respectively. β is the maximum velocity of the axions and t_a is the dimensionless time which measures real time t in units of axion decay time $1/\Gamma_a$. For a typical nonrelativistic axiton of mass $M = 2\times10^{-26}M_\odot$, radius $R = 1.75\times10^{-11}R_\odot$ in figure V.2, equations (V.6) shows that it may not lead to lasing. The reason is that the axion density is too low, the photons from spontaneous decays escaped the axiton before they can further stimulate other axions to decay. For a typical relativistic axiton of mass $M = 2\times10^{-26}M_\odot$, radius $R = 8\times10^{-11}R_\odot$ in the figure V.2, the axion density is even lower, and equations (V.6) shows that it would not lase either.

A stable axiton of mass $M = 2.0\times10^{-20}M_\odot$ and radius $R = 1.75\times10^{-11}R_\odot$ is relativistic, as we discussed before.See figure V.5. It will not lase if all the axions are relativistic. But there are nonrelativistic axions in outer regions of the axiton. These nonrelativistic axions have near zero velocity along the radius direction due to the self interaction potential. The lasing model proposed in [14] takes the escape velocity of the gravitational bound as the maximum velocity of the lasing axions. In the current case, although gravity is weaker than the self interaction, it does not prohibit us from using the gravitational escape velocity as the

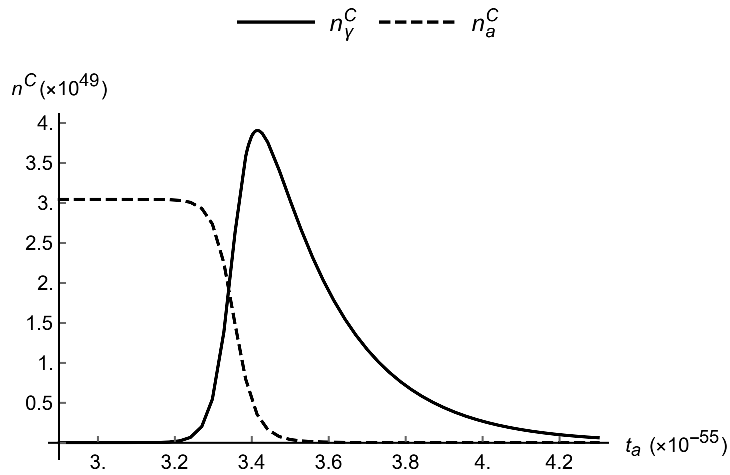

Figure V.5: Surface lasing from axiton of $M = 2.0 \times 10^{-20} M_\odot$, $R = 1.75 \times 10^{-11} R_\odot$, $m_a = 10^{-4}$ eV

maximum velocity for nonrelativistic axions on the surface of the axiton. This gravitational escape velocity is found to be $\sqrt{\frac{2GM}{R}} = 20.87$ m/s. On the thin layer near the surface of the axiton, it is dynamically allowed to have axions moving with ~ 20 m/s perpendicular to the radius direction even in the presence of the much stronger self interaction. These axions would fall down quickly and the thin layer can be constantly supplied by other incoming axions from the center. Therefore, there is a sustained density of the nonrelativistic axions close to the surface. Taking all this information as initial conditions for equations (V.6), we can plot the relation between the density of photons and axions with respect to time as follows. The nonrelativistic axions on the surface could decay in an extremely short peoriod of time. For a 10 times heavier axiton of mass $M = 2.0 \times 10^{-19} M_\odot$, radius $R = 1.75 \times 10^{-11} R_\odot$, see figure V.6, which is also stable as indicated by [18], equations (V.6) also suggests rapid surface decay of axions and photon production. The evolution of the process is even quicker than that of the lighter axiton.

For dilute axion stars with mass $M = 9.0 \times 10^{-14} M_\odot$, radius $R = 1.75 \times 10^{-4} R_\odot$ and mass $M = 1.2 \times 10^{-14} M_\odot$, radius $R = 2.3 \times 10^{-3} R_\odot$, we did not find axion decay induced lasing by solving equations (V.6). The density of axion is too low and photons escaped the cluster region before encoutering significant number of axions along the path to induce lasing. This may verify the claim that these type of axion stars are stable [18] and it means that lasing can not be prompted from axion stars formed by only gravitational binding. But as we discussed before, axion stars may not be a cluster type object but have inner structure, the granular axitons. These granular axitons can induce lasing if there are high axion occupation numbers.

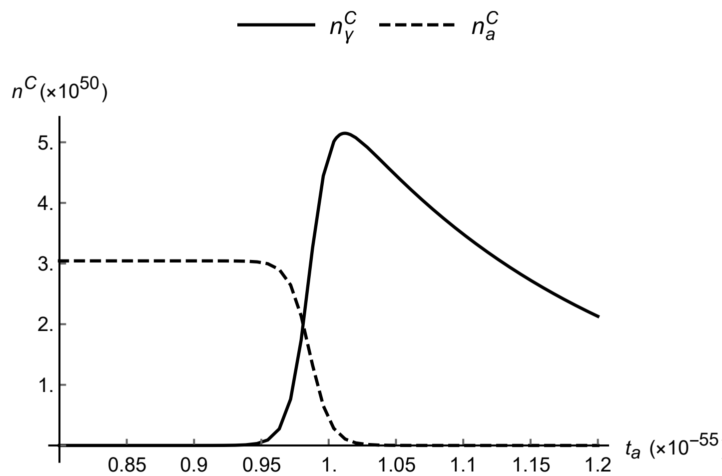

Figure V.6: Surface lasing from axiton of $M = 2.0 \times 10^{-19} M_{\odot}$, $R = 1.75 \times 10^{-11} R_{\odot}$, $m_a = 10^{-4}$ eV

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