

DEHN FUNCTIONS OF METABELIAN GROUPS

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# Chapter I

## Introduction

In this thesis, we focus on topics related to the Dehn function of a finitely presented metabelian group and relative Dehn function of a finitely generated metabelian groups. We establish a commutative algebra approach to estimate upper bounds for the Dehn function of a given finitely presented metabelian group. This approach yields much wilder results than estimating the upper bound. It first gives a uniform upper bound for Dehn functions of all finitely presented metabelian groups. Secondly, we give a similar result for Dehn functions relative to the variety of metabelian groups. For a finitely presented metabelian group, we also analyze contributions of different parts, including the metabelian part and the commutative algebra part, to its Dehn function. Finally, we use this technique to compute and estimate the Dehn functions as well as the relative Dehn functions for various examples.

### I.1 Notation and Conventions

We denote the set of rational integers by  $\mathbb{Z}$  and the set of real numbers by  $\mathbb{R}$ .  $\mathbb{N}$  indicates the set of natural numbers, where our convention is that  $0 \notin \mathbb{N}$ . In addition, we let  $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ .

If  $G$  is a group, we will denote by  $G' = [G, G]$  the derived (commutator) subgroup, by  $G_{ab} \cong G/G'$  the abelianization. For elements  $x, y \in G, n \in \mathbb{N}$ , our convention is  $x^{ny} = y^{-1}x^n y$ ,  $[x, y] = x^{-1}y^{-1}xy$ . We use double bracket  $\langle\langle \cdot \rangle\rangle_G$  to denote the normal closure of a set in the group  $G$ . Sometimes we omit the subscript when there is no misunderstanding in the context. For a set  $\mathcal{X}$ , we denote the free group generated by the set  $\mathcal{X}$  as  $F(\mathcal{X})$ . We also use  $F(\mathcal{X})$  to represent the set of reduced words in alphabet  $\mathcal{X} \cup \mathcal{X}^{-1}$ .

In addition, for a group  $G$  and a commutative ring  $K$  with  $1 \neq 0$ , we let  $KG$  be the group ring of  $G$  over  $K$ . An element  $\lambda \in KG$  is usually denoted as  $\lambda = \sum_{g \in G} \alpha_g g$ ,  $\alpha_g \in K$  where all but finitely many  $\alpha_g$ 's are 0. We also regard  $\lambda$  as a function  $\lambda : G \rightarrow K$  with finite support, where  $\lambda(g) = \alpha_g$ .

We say a group  $G$  that is an *extension* of a group  $A$  by a group  $T$  if  $A$  is a normal subgroup of  $G$  and  $T \cong G/A$ . Let  $\pi : G \rightarrow G/A$  be the natural homomorphism. We say an extension  $A$  by  $T$  *splits* if there exists a homomorphism  $s : T \rightarrow G$  such that  $\pi \circ s = id_T$ . In this case,  $G$  is a *semidirect product* of  $A$  and  $T$ , denoted by  $A \rtimes T$ . One special case of the semidirect product we consider in this thesis is the (regular) wreath product. For groups  $A$  and  $T$ , let  $B = \bigoplus_{t \in T} A^t$  be the direct sum of copies of  $A$  indexed by elements in  $T$ . Then *the wreath product*  $A \wr T$  is defined to be the semidirect product  $B \rtimes T$  where  $T$  acts on  $B$  by  $t \circ (a_\omega) = (a_{t^{-1}\omega})$ . The subgroup  $B$  is called the base group of the wreath product.

If  $A$  is abelian, then  $A$  can be considered a module over  $\mathbb{Z}T$ , and the action of  $T$  on  $A$  is given by conjugation. In this case, we also say that  $G$  is an extension of a  $T$ -module  $A$  by  $T$ . Throughout the thesis, we will use the following



notation for actions of  $\mathbb{Z}T$  on  $A$ . Let  $\lambda = \sum_{t \in T} \alpha_t t \in \mathbb{Z}T$ . Then for  $a \in A$ , we define

$$a^\lambda := \prod_{t \in T} a^{\alpha_t t}.$$

Since  $\lambda$  has finite support, the product is finite.

## I.2 The Word Problem

Let  $X$  be a set. A *word* over  $X$  is a sequence of elements of  $X$ . A *group word* over  $X$  is a sequence of elements of  $X$  and their inverses, i.e. symbols  $x^{-1}, x \in X$ . The length of a word  $w$  is denoted by  $|w|$ . We say a group word over  $X$  is *reduced* if it does not contain any subwords of the form  $xx^{-1}$  and  $x^{-1}x$  for any  $x \in X$ . The set of reduced group words over  $X$  is denoted by  $F(X)$ . It is equipped with the binary operation: the product  $uv$  of two reduced group words  $u, v$  is the result of reducing the concatenation of  $u$  and  $v$ . With this operation,  $F(X)$  is turned into a group which is called the *free group* over  $X$ . The identity element of  $F(X)$  is the empty word denoted by  $\emptyset$ . For every group  $G$ , any map  $X \rightarrow G$  extends uniquely to a homomorphism  $F(X) \rightarrow G$ : any word is mapped to the product of the images of its letter. In particular, every group  $G$  generated by at most  $|X|$  elements is a homomorphic image of  $F(X)$ . Let  $\varphi$  be one of these surjective homomorphisms. The kernel of  $\varphi$  is a normal subgroup of  $F(X)$ . Let  $R$  be a subset of  $F(X)$  such that its normal closure coincides with  $\ker \varphi$ . Then we say group  $G$  has a *presentation*  $\langle X \mid R \rangle$ , where  $X$  is called the *generating set* and  $R$  is called the *relation set* while  $r \in R$  is called a *relation* (or a *relator*).

The *word problem* in  $G$  (related to  $\varphi$  or  $\langle X \mid R \rangle$ ) is the following

**Problem I.2.1** (The word problem). Given a reduced group word  $w$  over  $X$ , decide if  $\varphi(w) = 1$  in  $G$  or not.

If the generating set is finite, the group is called *finitely generated*. In addition, if the relation set is also finite, the group is called *finitely presented*. It turns out that for finitely generated groups the solvability of the word problem does not depend on the choice of  $\varphi$  or even the choice of the finite set  $X$  [24]. Throughout this thesis, we only concern finitely generated and finitely presented groups. Therefore, in what follows, we will discuss the solvability of the word problem for a group without noting an explicit finite generating set.

## I.3 The Dehn function

Given a finitely presented group  $G$  equipped with a finite presentation  $\mathcal{P} = \langle X \mid R \rangle$ , there exists a epimorphism  $\varphi : F(X) \rightarrow G$  such that  $\ker \varphi = \langle\langle R \rangle\rangle$ . A reduced group word  $w$  over  $X$  represents the identity in  $G$  if and only if there exists  $k \in \mathbb{N}, f_1, f_2, \dots, f_k \in F(X), r_1, r_2, \dots, r_k \in R$  and  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{\pm 1\}$  such that

$$w =_{F(X)} \prod_{i=1}^k f_i^{-1} r_i^{\varepsilon_i} f_i.$$

If  $w = 1$  in  $G$ , the area of  $w$  with respect to  $\mathcal{P}$ , denoted by  $\text{Area}_{\mathcal{P}}(w)$ , is the minimal value  $k$  so that such representation of  $w$  exists. The Dehn function  $\delta_{\mathcal{P}} : \mathbb{N} \rightarrow \mathbb{N}$  with respect to  $\mathcal{P}$  is defined to be  $\delta_{\mathcal{P}}(n) = \sup\{\text{Area}_{\mathcal{P}}(w) \mid |w| \leq n\}$ .

Dehn functions are defined up to an asymptotic equivalence  $\approx$  taken on functions  $\mathbb{N} \rightarrow \mathbb{N}$  by  $f \approx g$  if and only if  $f \preceq g$  and  $g \preceq f$  where  $f \preceq g$  if and only if there exists  $C > 0$  such that  $f(n) \leq Cg(Cn) + Cn + C$  for all  $n \in \mathbb{N}$ . One can verify that  $\approx$  is an equivalence relation. This relation preserves the asymptotic nature of a function. For example, it distinguishes polynomials of different degrees and likewise polynomials and the exponential function. It also distinguishes functions like  $n^p$  and  $n^p \log n$  for  $p > 1$ . On the other hand, it identifies all polynomials of the same degree, and likewise all exponential functions, i.e.,  $a^n \approx b^n$  for  $a, b > 1$ .

Despite the dependence of Dehn function on finite presentations of a group, all Dehn functions of the same finitely presented group are equivalent under  $\approx$  [17], i.e., given a finitely presented group  $G$  with finite presentations  $\mathcal{P}$  and  $\mathcal{P}'$ , one can show that  $\delta_{\mathcal{P}} \approx \delta_{\mathcal{P}'}$ . Thus, we define the *Dehn function* of a finitely presented group  $G$ ,  $\delta_G$ , as the Dehn function of any of its finite presentation.

The Dehn function was first introduced by computer scientists Madlener and Otto to describe the complexity of the word problem of a group [24], also by Gromov as a geometric invariant of finitely presented groups [17] (see also Gersten [16] where the name ‘‘Dehn function’’ was introduced). There have been a lot of significant results about Dehn functions in the past 30 years, revealing the relationship between this geometric invariant and algebraic properties of the group. Some of the numerous examples are:

- (a) A finitely presented group has a decidable word problem if and only if its Dehn function is bounded above by a recursive function [24].
- (b) A finite generated group is hyperbolic if and only if it has sub-quadratic Dehn function [17], [30].
- (c) If  $G$  is the fundamental group of a compact Riemannian manifold  $M$ , then  $\delta_G$  is equivalent to the smallest isoperimetric function of the universal cover  $\tilde{M}$  [10].

#### I.4 Metabelian Groups

This thesis studies the Dehn function of a particular class of groups, finitely presented metabelian groups. A group  $G$  is *metabelian* if its derived subgroup  $G' = [G, G]$  is abelian. Metabelian groups are defined by the identity  $[[x, y], [z, w]] = 1$ , which implies that all commutators commutes in a metabelian group. It also follows that the class of metabelian groups form a variety. Recall that a *variety*, in the sense of B.H. Neumann [28], is a class of groups closed under subgroups, epimorphic images, and unrestricted direct products. The fact that metabelian groups form a variety allows us to define the relative Dehn function for a finitely generated metabelian group, which we will discuss in detail in Chapter VI.

Now, let us give some interesting examples of metabelian groups that we will deal with throughout this thesis.

### 1. Metabelian Baumslag-Solitar Group.

The metabelian Baumslag-Solitar group  $BS(1, n)$  has a presentation

$$BS(1, n) = \langle a, t \mid a^t = a^n \rangle,$$

where  $n$  is a positive integer. The derived subgroup  $BS(1, n)'$  is isomorphic to the additive group of  $n$ -adic rational numbers, which is not finite generated but abelian. Therefore  $BS(1, n)$  is metabelian. We can write metabelian Baumslag-Solitar groups as a semidirect product:  $BS(1, n) \cong \mathbb{Z}[1/n] \rtimes \mathbb{Z}$ , where  $\mathbb{Z}$  acts on  $\mathbb{Z}[1/n]$  by multiplication by  $n$ .

### 2. Lamplighter Group

The name of this group comes from viewing the group as acting on a doubly infinite sequence of street lamps  $\dots, l_{-2}, l_{-1}, l_1, l_2, \dots$  each of which may be on and off, and a lamplighter standing at some lamp  $l_k$ . View in wreath product we have the lamplighter group  $L_2 = \mathbb{Z}_2 \wr \mathbb{Z}$ . Recall the wreath product  $A \wr T$  is defined to be the semidirect product of  $\bigoplus_{t \in T} A^t$  by  $T$  with the conjugation action.

One standard presentation for  $L_2$  is

$$\langle a, t \mid a^2 = 1, [a, a^t] = 1, n \in \mathbb{N} \rangle.$$

We are also able to extend the idea of lamplighter group to integers greater than 2. For an integer  $m \geq 2$ , we can have “lamps” with  $m$  different status, for examples,  $m - 1$  different colors and “off”. In a similar fashion, we define general lamplighter group  $L_m$  to be  $\mathbb{Z}_m \wr \mathbb{Z}$ .

The lamplighter groups are examples of metabelian groups that are finitely generated but not finitely presented.

### 3. Baumslag's Group $\Gamma$

The Baumslag' Group  $\Gamma$  is presented by

$$\langle a, s, t \mid [a, a^t] = 1, [s, t] = 1, a^s = aa^t \rangle.$$

This is the first example of a finitely presented group with an abelian normal subgroup of infinite rank, has derived subgroup  $[\Gamma, \Gamma] = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$ . The subgroup  $\langle a, t \rangle$  of  $\Gamma$  is

$$\mathbb{Z} \wr \mathbb{Z} = \left( \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} \right) \rtimes \mathbb{Z} = \langle a, t \mid [a, a^k], k \in \mathbb{Z} \rangle.$$

Introducing the relation  $a^m = 1$  in  $\Gamma$  we have a family of metabelian groups  $\Gamma_m = \langle \Gamma \mid a^m \rangle$  [21]. It's not hard to see that  $[\Gamma, \Gamma] = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_m$  where  $\mathbb{Z}_m$  is the cyclic group of order  $m$ . Thus  $\Gamma_m$  contains a copy of  $\mathbb{Z}_m \wr \mathbb{Z}$ , the lamplighter group

$L_m$ .

## I.5 Main Results

The solvability of the word problem for finitely generated metabelian group is known for a long time [22]. As we will discuss in Section III.1, a finitely generated metabelian group is residually finite. It immediately follows that the word problem is decidable for metabelian groups [24]. One consequence is that the Dehn function of a finitely presented metabelian group is recursive. In Section III.2, we will list a few examples of finitely presented metabelian groups and their Dehn functions. One spoiler for now is that all of them are asymptotically bounded above by the exponential function. It is an intriguing question if there is a uniform upper bound for Dehn functions of finitely presented groups. We answer this question by proving the following theorem.

**Theorem A.** *Let  $G$  be a finitely presented metabelian group. Let  $k$  be the minimal torsion-free rank of an abelian group  $T$  such that there exists an abelian normal subgroup  $A$  in  $G$  satisfying  $G/A \cong T$ .*

*Then the Dehn function of  $G$  is asymptotically bounded above by*

(1) *(Theorem II.4.2)  $n^2$  if  $k = 0$ ;*

(2) *(Theorem VI.6.5)  $2^n$  if  $k = 1$ ;*

(3) *(Theorem V.1.1)  $2^{n^{2k}}$  if  $k > 1$ .*

It follows that a function of the form  $2^{h(n)}$  where  $h(n)$  is any superpolynomial function is a uniform upper bound for Dehn functions of finitely presented metabelian groups.

In general, not all finitely generated metabelian groups are finitely presented. In fact, most of them are not. But finitely generated metabelian groups satisfy the maximal condition for normal subgroups [19], if we restrict everything in the variety of metabelian groups, a finitely generated metabelian group is always relatively finitely presentable. In Section VI.2, we will extend the notion of presentation and the Dehn function to the variety of metabelian groups. Thus for a finitely generated metabelian group  $G$ , we instead study the relative Dehn function (defined in Section VI.2), denoted by  $\tilde{\delta}_G(n)$ , which exists for all finitely generated metabelian groups. Applying the same technique, we first observe that

**Theorem B** (Theorem VI.3.1). *Let  $G$  be a finitely presented metabelian group. Then*

$$\tilde{\delta}_G(n) \asymp \delta_G(n) \asymp \max\{\tilde{\delta}_G^3(n^3), 2^n\}.$$

It follows that

**Theorem C** (Theorem VI.3.6). *Let  $G$  be a finitely generated metabelian group. Let  $k$  be the minimal torsion-free rank of an abelian group  $T$  such that there exists an abelian normal subgroup  $A$  in  $G$  satisfying  $G/A \cong T$ .*

*Then the relative Dehn function of  $G$  is asymptotically bounded above by*

- (1) (Theorem II.4.2)  $n^2$  if  $k = 0$ ;
- (2) (Theorem VI.6.5) a polynomial if  $k = 1$ ;
- (3) (Theorem VI.3.6)  $2^{n^{2k}}$  if  $k > 0$ .

There is no uniform polynomial bound when  $k = 1$ . We will later show that, when  $k = 1$ , for every natural number  $l$  there exists a finitely generated metabelian group  $H_l$  such that its relative Dehn function is greater than  $n^l$ .

The general method we establish in this paper provides a way to estimate the relative Dehn function of metabelian groups. The following are some of the results.

**Theorem D.** (1) (Proposition VI.4.4) *The metabelianized Baumslag-Solitar group  $\tilde{BS}(n, m) = \langle a, t \mid (a^n)^t = a^m \rangle_{\mathcal{S}_2}$  has at most cubic relative Dehn function when  $n \neq m$  and has at most quartic relative Dehn function when  $n = m$ .*

(2) (Corollary VI.4.5) *The metabelianized Baumslag-Solitar group  $\tilde{BS}(n, m) = \langle a, t \mid (a^n)^t = a^m \rangle_{\mathcal{S}_2}$ ,  $m > 2$ ,  $m = n + 1$  has at most quadratic relative Dehn function.*

(3) (Proposition VI.4.6) *The lamplighter groups  $L_m$  have at most cubic relative Dehn function for every  $m$ .*

(4) (Proposition VI.5.1) *The lamplighter group  $L_2$  has linear relative Dehn function.*

(5) (Corollary VI.6.3) *For each  $l \in \mathbb{N}$  there exists a finitely generated metabelian group such that its relative Dehn function asymptotically is greater or equal to  $n^l$ .*

The last result follows from the following connection between subgroup distortion functions (defined in Section VI.6) and relative Dehn functions.

Let  $A$  and  $T$  be free abelian groups with bases  $\{a_1, a_2, \dots, a_m\}$  and  $\{t_1, t_2, \dots, t_k\}$  respectively. Consider the wreath product  $W := A \wr T$ . Let  $B := \langle\langle A \rangle\rangle$  be the base group, which is a  $T$ -module. For a finite subset  $\mathcal{X} = \{f_1, f_2, \dots, f_l\}$  of  $B$ , let  $H$  be the subgroup of  $W$  generated by  $\mathcal{X} \cup \{t_1, t_2, \dots, t_k\}$  and  $G$  be the group  $W / \langle\langle \mathcal{X} \rangle\rangle$ .

**Theorem E** (Theorem VI.6.1). *Let  $W, H, G$  be groups defined as above, then*

$$\Delta_H^W(n) \preceq \tilde{\delta}_G^k(n) + n^k, \tilde{\delta}_G^k(n) \preceq \max\{n^3, (\Delta_H^W(n^2))^3\}.$$

*In particular, if  $k = 1$ , then we have*

$$\Delta_H^W(n) \preceq \tilde{\delta}_G(n).$$

If we only consider the case  $k = 1$ , we have the following result.

**Theorem F** (Theorem VI.6.5). *Let  $G$  be a finitely generated metabelian group such that the minimal torsion-free rank of an abelian group  $T$  such that there exists an abelian normal subgroup  $A$  in  $G$  satisfying  $G/A \cong T$  is one. Then the relative Dehn function of  $G$  is polynomially bounded. If in addition  $G$  is finitely presented, the Dehn function of  $G$  is asymptotically bounded above by the exponential function.*

In the last section, we generalize one result in [21], and show the following.

**Theorem G** (Theorem VII.1.4). *Every wreath product of a free abelian group of finite rank with a finitely generated abelian group can be embedded into a metabelian group with exponential Dehn function. In particular, any free metabelian group of finite rank is a subgroup of a metabelian group with exponential Dehn function.*

The proof of Theorem VII.1.4 can be found in Section VII.2.

## Chapter II

### Dehn Function

#### II.1 Properties of the Dehn Function

Let  $f$  be a map from a metric space  $(X, d_X)$  to a metric space  $(Y, d_Y)$ . Then  $f$  is called a *quasi-isometry* from  $(X, d_X)$  to  $(Y, d_Y)$  if there exist constants  $A \geq 1, B \geq 0$ , and  $C \geq 0$  such that the following two conditions hold:

(1) For all pair of points  $x, y \in X$ , we have that

$$\forall x, y \in X : \frac{1}{A}d_X(f(x), f(y)) - B \leq d_Y(x, y) \leq Ad_X(x, y) + B.$$

(2) For every point  $y$  in  $Y$  is within distance  $C$  to an image point, i.e.,

$$\forall y \in Y, \exists x \in X : d_Y(y, f(x)) \leq C.$$

Two metric spaces  $(X, d_X), (Y, d_Y)$  are called *quasi-isometric* if there exists a quasi-isometry between them. It can be shown that being quasi-isometric is an equivalence relation. Two groups are called quasi-isometric if the Cayley graphs of them with respect to the word length are quasi-isometric. Quasi-isometric invariants are one major interest in geometric group theory, following the work by Gromov.

Let  $G$  be a finitely presented group in the category of all groups, namely  $G = \langle X \mid R \rangle$  where  $|X|, |R| < \infty$ .

**Theorem II.1.1** (Gromov [17]). *Let  $G$  be a finitely presented group,  $G = \langle X \rangle$ . Let  $H$  be a finitely generated group with generating set  $Y$ ,  $|X|, |Y| < \infty$ . If  $G, H$  are quasi-isometric, then  $H$  is finitely presented and  $\delta_{G, X} \approx \delta_{H, Y}$ .*

Since Cayley graphs of the same group over different generating sets are quasi-isometric, we have that the Dehn function to finite presentations of a finitely presented group is independent from the choice of the finite presentation.

**Corollary II.1.2.** *Let  $\mathcal{P}, \mathcal{P}'$  be two finite presentations of  $G$ , then*

$$\delta_{\mathcal{P}}(n) \approx \delta_{\mathcal{P}'}(n).$$

Another useful consequence of Theorem II.1.1 is that the Dehn function of a finitely presented group is preserved by taking a finite index subgroup.

**Corollary II.1.3.** *Let  $G$  be a finitely presented group and  $H \leq G$  such that  $[G : H] < \infty$ , then  $H$  is finitely presented and quasi-isometric to  $G$  hence  $\delta_H \approx \delta_G$ .*

One technique we apply a lot throughout the thesis is passing the problem to a finite index subgroup with a nicer structure, which simplifies the computation a lot.

## II.2 Van Kampen Diagram

One way to visualize the area of a given word is to consider what is called a *van Kampen diagram*. Let  $G = \langle X \mid R \rangle$  be a finitely presented group and  $w$  be a reduced word which is equal to 1. Then by the previous discussion,  $w$  has a decomposition as following:

$$w =_{F(X)} \prod_{i=1}^k r_i^{f_i} \text{ where } r_i \in R \cup R^{-1}, f_i \in F(X). \quad (\text{II.1})$$

For every decomposition (II.1), we can draw a diagram which consists of a bouquet of “lollipops”. Each “lollipop” corresponds to a factor  $r_i^{f_i}$ , the stem of which is a path labeled by  $f_i$  and the candy of which is a cycle path labeled by  $r_i$ . Going counterclockwise around the “lollipop” starting and ending at the tip of the stem, we read  $f_i^{-1} r_i f_i$ . Thus the boundary of the bouquet of “lollipops” is labeled by the word which is the right-hand side of (II.1).

Note that we obtained  $w$  from the right hand side of (II.1) by cancelling all consecutive pairs of  $xx^{-1}$  or  $x^{-1}x, x \in X$  on the boundary and removing subgraphs whose boundaries labelled by  $xx^{-1}$  or  $x^{-1}x, x \in X$  (which is a “dipole” or a sphere). In the diagram, the corresponding process is identifying two consecutive edges with the same label but different orientation on the boundary. After finitely many such reductions, we will obtain a diagram whose boundary is labeled by  $w$ .

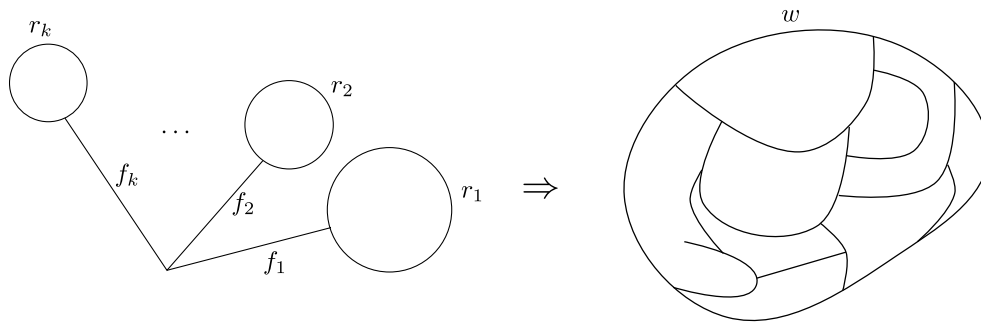


Figure II.1: a bouquet of “Lollipops” and its corresponding van Kampen Diagram

The resulting diagram is called the Van-Kampen diagram of  $w$ . The edges are labeled by elements in  $X$  and cells are (i.e. the closure of a bounded connected components of the plane minus the graph) labeled by words from  $R \cup R^{-1}$ .

For example, in group  $\langle a, b \mid [a, b] = 1 \rangle$ , the Van-Kampen diagram of  $[a^2, b] = [a, b]^a [a, b]$  looks like this.



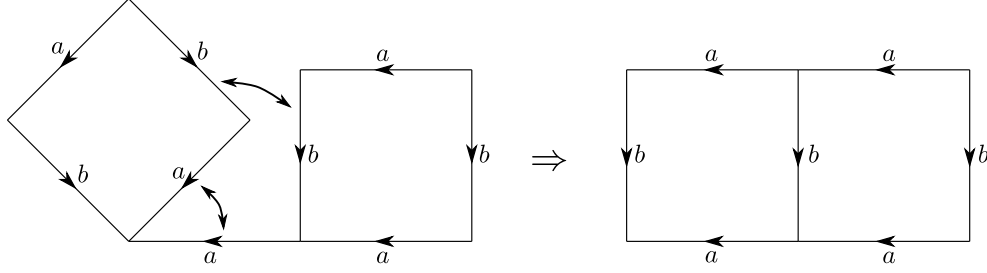


Figure II.2: the Van-Kampen diagram of  $[a^2, b] = [a, b]^a [a, b]$

The following is called the van Kampen Lemma.

**Lemma II.2.1** ([32]). *If a reduced group word  $w$  over the alphabet  $X$  is equal to 1 in  $G = \langle X \mid R \rangle$ , then there exists a van Kampen diagram over the presentation of  $G$  with boundary label  $w$ .*

*Conversely, let  $\Delta$  be a van Kampen diagram over  $G = \langle X \mid R \rangle$  where  $X = X^{-1}$  and  $R$  is closed under cyclic shifts and inverses. Let  $w$  be the boundary of  $\Delta$ . Then  $w$  is equal in the free group  $F(X)$  to a word of the form  $u_1 r_1 u_2 r_2 \dots u_k r_k u_{k+1}$  where*

- (1) *each  $r_i$  is from  $R$ ;*
- (2)  *$u_1 u_2 \dots u_{k+1} = 1$  in  $F(X)$ ;*
- (3)  *$\sum_{i=1}^{m+1} |u_i| \leq 4e$  where  $e$  is the number of edges of  $\Delta$ .*

*In particular,  $w$  is equal to 1 in  $G$ .*

We say a Van Kampen diagram is *minimal* if it has the minimal number of cells over all such diagrams of the same word. For a word  $w =_G 1$ , the area of  $w$  is the same as the number of cells of a minimal van Kampen diagram.

Other applications of the van Kampen diagram can be found in many books. For example, in the book [29], one can found the study of using van Kampen diagrams to construct groups with extreme properties such as infinite bounded torsion group, Tarski monsters, etc.

### II.3 Estimate the Upper Bound

One commonly used method to estimate the upper bound of the Dehn function of a finitely presented group is counting how many relators are cost when converting a word to the identity. To be precise, let  $G$  be a finitely presented group with a finite presentation  $\langle X \mid R \rangle$ . Let  $w_1, w_2$  be words that are equal in group  $G$ . Then the *cost* of converting  $w_1$  to  $w_2$  ( $w_2$  to  $w_1$ ) in  $G$  is defined to be the area of  $w_2^{-1} w_1$  (resp.  $w_1^{-1} w_2$ ) in  $G$ . If  $w_2$  happens to be the identity, then the cost of converting  $w_1$  to  $w_2$  coincides with the area of  $w_1$ . By the definition of the area, it is not hard to see that if  $w_1 =_G w_2 =_G w_3$  and the cost of converting  $w_1$  to  $w_2$ ,  $w_2$  to  $w_3$  is  $N_1$  and  $N_2$  respectively then the cost of converting  $w_1$  to  $w_3$  is at most  $N_1 + N_2$ .

Let us give an example. Suppose  $G = \langle a, b \mid [a, b] = 1 \rangle$ , which is a free abelian group of rank 2. Consider words  $w_1 = abab$  and  $w_2 = a^2b^2$ . We can get  $w_2$  from  $w_1$  by commuting the letter  $b$  and  $a$  in the middle. To operate this commutation, we insert the relator  $[a, b]$  into  $w_1$  as follows:

$$w_1 = abab = abab[b, a]^b = aabb = w_2.$$

The third equality above holds in the free group generated by  $a, b$ . It implies  $w_2^{-1}w_1 = [a, b]^b$ . Thus the cost of converting  $w_1$  to  $w_2$  is 1.

In general, let  $G = \langle X \mid R \rangle$  be a finitely presented group. Suppose we have a word  $w_1$  which has the form of  $uq_1v$  and there exists  $r \in R^{\pm 1}$  such that  $q_2 = q_1r$ . Then we can apply the relator  $r$  to  $w_1$  by multiply  $r^v$  on the left to  $w_1$ , that is,  $w_2 = w_1r^v = uq_2v$ . Under a sequence of such operations, we can convert a word to another word that is equal to the original word in  $G$  and by counting the number of relators we use, we can estimate the upper bound of the cost of this conversion. Therefore, given a word  $w$  which represents the identity in  $G$ , we can estimate the upper bound of its area by estimating the cost to convert it to the identity.

To demonstrate the idea, let us compute the upper bound of the Dehn function of a finitely generated abelian group. Firstly, we note that this problem can be reduced to the case of free abelian group of finite rank, where since every infinite finitely generated abelian group has a free abelian subgroup of finite index. Let  $T$  be a free abelian group of rank  $k$ . We choose our preferred presentation of  $T$ ,  $\langle t_1, t_2, \dots, t_k \mid [t_i, t_j] = 1, 1 \leq i < j \leq k \rangle$ . Under this presentation, to convert a subword of the form  $t_it_j$  to  $t_jt_i$ , we need to apply the relator  $[t_i, t_j] = 1$  if  $i < j$  or the inverse of  $[t_j, t_i]$  if  $i > j$ . Now given a reduced word  $w$  such that  $w =_T 1$  and  $|w| = 1$ , the sum of exponents of  $t_i$  is zero for all  $i$ . Let  $n_i$  be the number of the occurrence of  $t_i^{\pm 1}$  in  $w$ . It is not hard to see that  $\sum_{i=1}^k n_i = n$ . Thus we can convert  $w$  to 1 in the following way: first we gather all  $t_1$  in  $w$  to the left by commuting everything on the left side of each  $t_1^{\pm 1}$ , operating this process from the left most  $t_1^{\pm 1}$  to the right most  $t_1^{\pm 1}$ ; Then all  $t_1^{\pm 1}$  will be canceled on the left end; We repeat this process for  $t_2, t_3, \dots, t_k$  and we will end up with the identity. The cost of moving one  $t_i^{\pm 1}$  to the left most is at most  $n$ , and hence the cost of gathering all  $t_i^{\pm 1}$  to the left is at most  $n_i \cdot n$ . Therefore the total cost is at most  $n \cdot \sum_{i=1}^k n_i = n^2$ . The Dehn function of a finitely generated abelian group is at most quadratic.

## II.4 Estimate the Lower Bound

The lower bound of the Dehn function is usually the harder one to estimate, compared with the upper bound. There are various techniques: the distortion function in HNN extensions [1], the abelianized or centralized Dehn functions [5], and, the technique we will use in this thesis, the van Kampen diagram. The general idea is we found a sequence of words, the lower bound for areas of which can be estimated by investigating their corresponding minimal van Kampen diagrams, that induces a lower bound for the Dehn function of the group. In the following, we will demonstrate this

technique to compute the lower bound for the Dehn functions of finitely generated abelian groups with torsion-free rank greater or equal to 2.

Recall that a subgroup  $H$  of  $G$  is a *retract* if there exists an endomorphism  $\varphi : G \rightarrow G$  such that  $\varphi$  is the identity map on  $H$ , i.e.,  $\varphi(h) = h, \forall h \in H$ . We have the following lemma.

**Lemma II.4.1.** *Let  $G$  be a finitely presented group and  $H$  be a retract of  $G$ . Then  $H$  is finitely presented and  $\delta_H(n) \preceq \delta_G(n)$ .*

*Proof.* Let  $\mathcal{P} = \langle X \mid R \rangle$  be a finite presentation of  $G$  and  $m = \max\{|r|, r \in R\}$ . Let  $\varphi$  be the retraction of  $G$  onto  $H$ . And let  $Y = \{y = \varphi(x) \mid x \in X\}$  and  $S = \{s \mid s \text{ is a word in } Y \cup Y^{-1}, s =_H 1, |s| \leq m\}$ . We claim that  $\mathcal{Q} := \langle Y \mid S \rangle$  is a finite presentation for  $H$ . It is not hard to see that  $Y$  generates  $H$ . For  $w =_H 1$ ,  $w$  can be represented as  $\prod_{i=1}^k r_i^{h_i}, r_i \in R \cup R^{-1}$  in alphabet  $X \cup X^{-1}$ . Then  $w = \varphi(w) = \prod_{i=1}^k \varphi(r_i)^{\varphi(h_i)}$ . Since  $|\varphi(r_i)| \leq |r_i| \leq m$ ,  $\varphi(r_i)$  lies in  $S$ . Therefore  $\mathcal{Q}$  is a finite presentation of  $H$ . Followed by the same argument, we have that if  $w = 1$ ,  $\text{Area}_{\mathcal{Q}}(w) \leq \text{Area}_{\mathcal{P}}(w)$ . It follows immediately that  $\delta_H(n) \preceq \delta_G(n)$ .  $\square$

Now let  $G$  be a finitely generated abelian group with a torsion-free rank greater or equal to 2. Note that we can pass the problem of finding the lower bound of  $\delta_G$  to the case  $G = \mathbb{Z}^2$ , since  $\mathbb{Z}^2$  is always a retract of  $G$ . Suppose  $G = \langle a, b \mid [a, b] \rangle$ . The only cell in a van Kampen diagram over the presentation of  $G$  looks like the following.

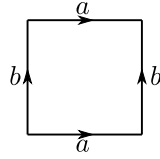


Figure II.3: the  $[a, b]$  cell

Cells form two different types of band,  $a$ -band and  $b$ -band. It is not hard to check that two bands of the same type do not intersect each other and a band does not intersect itself. Comparing the orientation of edges, an  $a$ -band cannot intersect a  $b$ -band twice, vice versa.

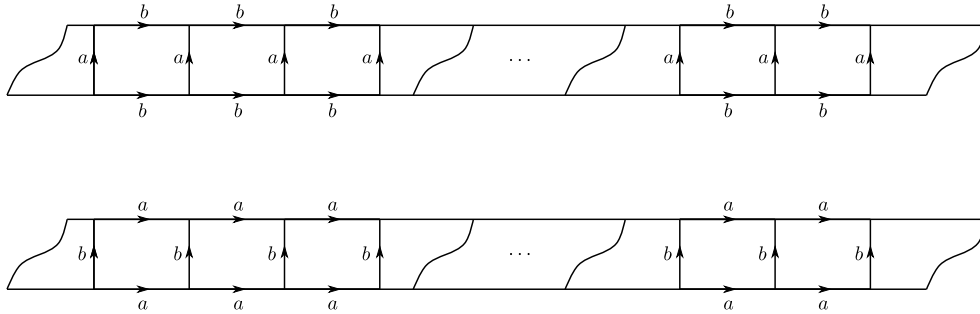


Figure II.4:  $a$ -band and  $b$ -band

We consider the minimal van Kampen diagram  $\Delta_n$  of  $w_n = [a^n, b^n]$ . An  $a$ -band starts at the boundary  $\partial\Delta_n$  will end at  $\partial\Delta_n$ . Therefore there are at least  $n$   $a$ -bands in  $\Delta_n$ . Same argument holds for  $b$ -bands. Thus in  $\Delta_n$ , we have at least  $n$   $a$ -bands and  $n$   $b$ -bands. It follows that there are at least  $n^2$  cells in  $\Delta_n$  since any pair of an  $a$ -band and a  $b$ -band intersects once. We have  $\text{Area}(w_n) \geq n^2$  and  $|w_n| = 4n$ . Hence  $\delta_G(n)$  is at least quadratic by definition.

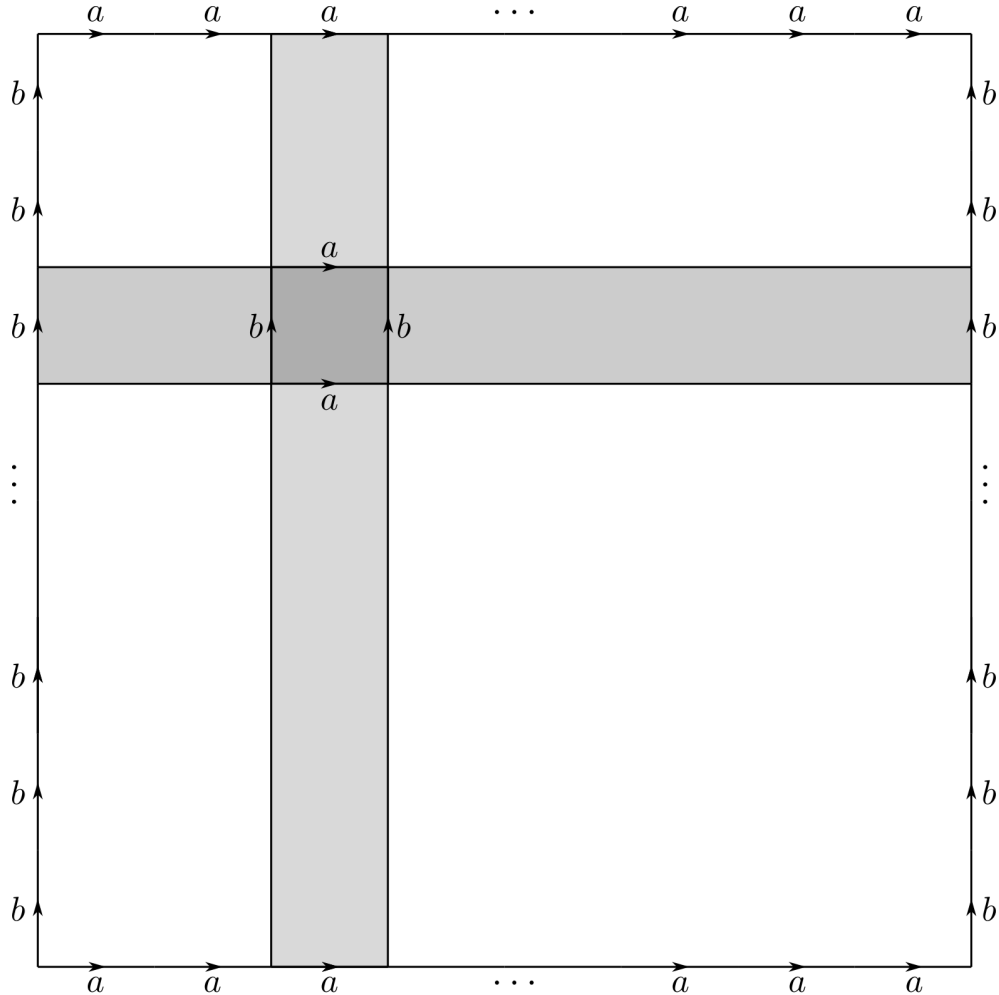


Figure II.5: The van Kampen diagram of  $w_n$

Combining the result from Section II.3, we have that finitely generated abelian groups with torsion-free rank greater or equal to 2 have quadratic Dehn function. In summary, we prove the following:

**Theorem II.4.2.** *The Dehn function of a finitely generated abelian group  $G$  is quadratic if the torsion-free rank of  $G$  is greater than one and is linear otherwise.*

This theorem is well-known and in this thesis we will use it a lot. The result will appear in the proof of Lemma V.3.1, Lemma V.4.1, and Proposition V.1.2. Moreover, this theorem directly implies the part (1) of both Theorem A and Theorem C, the first and easiest piece of our main theorem. The technique, using the van Kampen diagram to estimate the

lower bound for the Dehn function, will appear in Section VII.2.

## Chapter III

### Metabelian Groups

#### III.1 Properties of Finitely Generated Metabelian Groups

Throughout this thesis, we only consider finitely generated metabelian groups since we mainly focus on the Dehn function and the relative Dehn function. Let  $G$  be a finitely generated metabelian group. It can be written as an extension of two abelian groups  $A$  by  $T$ , by definition. It induces a short exact sequence as follows,

$$1 \longrightarrow A \longrightarrow G \longrightarrow T \longrightarrow 1 .$$

$T$  is finitely generated since  $G$  is finitely generated. It follows from a theorem by Philip Hall [19] that  $A$  is a finitely generated  $T$ -module where  $T$  acts on  $A$  by conjugation. This module structure enables us to use the commutative algebra approach to study finitely generated metabelian groups. We will discuss this idea in detail in Section V.4, where we solve the word problem for  $G$  by solving the word problem for the  $T$ -module  $A$ .

To our purpose, let us list some useful properties for finitely generated metabelian groups (most of them can be found in [22]).

- (1) Finitely generated metabelian groups are residually finite, i.e., the normal subgroups of finite index have trivial intersection.
- (2) Finitely generated metabelian groups satisfy the maximal condition for normal subgroups, i.e., any properly ascending chain of normal subgroups is finite [19]. It implies that any normal subgroup of a finitely generated metabelian group is a normal closure of a finite set.
- (3) Finitely metabelian groups are linear groups over finite products of fields (See in [31], [23], [36]), which implies that they have solvable word problems.
- (4) It follows from (2) that finitely generated metabelian groups are finitely presentable in the variety of metabelian groups.
- (5) Bieri-Strebel introduced a geometric invariant which distinguishes the finitely presented metabelian groups from the others [6]. We will sketch the proof for one direction of their main theorem in Section III.3.
- (6) A finitely generated metabelian group can be embedded into a finitely presented metabelian group [4].

### III.2 Examples of Metabelian Groups and their Dehn Functions

Before we consider finitely generated metabelian groups and their relative Dehn functions, we should first consider finitely presented ones and their Dehn functions. It is a good starting point because: (1) we know how finitely presented metabelian groups look like due to the theorem by Bieri and Strebel [6]; (2) we have more tools to study the Dehn function since the relative Dehn function highly depend on the variety; (3) The technique we established for the Dehn function can be carried to the relative case easily.

The followings are some known examples of Dehn functions of metabelian groups.

- (1) The first class of examples is the class of metabelian Baumslag-Solitar groups  $BS(1, n), n \geq 2$ , which has the presentation

$$BS(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle,$$

for any  $n \geq 2$ . It is well-known that metabelian Baumslag-Solitar groups have exponential Dehn function up to equivalence. The upper bound can be estimated by computing the cost of converting a word to the identity. The lower bound can be shown using the van Kampen diagram. The proof can be found in many places, for example, [18], [1].

- (2) For Baumslag's groups  $\Gamma$  and  $\Gamma_m$  we introduced in Section I.4, M. Kassabov and T. R. Riley [21] showed that  $\Gamma$  has an exponential Dehn function while the Dehn function of  $\Gamma_m$  is at most  $n^4$ . In particular, Y. de Cornulier and R. Tessera [13] showed that  $\Gamma_p$  has a quadratic Dehn function when  $p$  is a prime number.
- (3) The third example consists of groups that are a semidirect product of a finitely generated free abelian group and cyclic group, namely,  $\mathbb{Z}^n \rtimes \mathbb{Z}$ . Bridson and Gersten have shown that the Dehn function of such groups are either polynomial or exponential depending on the action of  $\mathbb{Z}$  on  $\mathbb{Z}^n$  [9].
- (4) Lattices in  $\mathbb{R}^n \rtimes_{\alpha} \mathbb{R}^{n-1}, n \geq 3$ , have quadratic Dehn function [17], where  $\alpha : \mathbb{R}^{n-1} \rightarrow GL(n, \mathbb{R})$  is an injective homomorphism whose image consists of all diagonal matrices with diagonal entries  $(e^{t_1}, e^{t_2}, \dots, e^{t_n})$  verifying  $t_1 + t_2 + \dots + t_n = 0$ . Drutu extends the result for the case that  $a_1 t_1 + a_2 t_2 + \dots + a_n t_n = 0$  for any fixed vector  $(a_1, a_2, \dots, a_n)$  with at least three nonzero components [14].
- (5) Let

$$G = \langle a, b, t \mid [a, b] = 1, a^t = ab, b^t = ab^2 \rangle.$$

$G$  is metabelian and polycyclic and it is also the fundamental group of a closed, orientable fibred 3-manifold. It has been shown that  $G$  has exponential Dehn function [5]. The lower bound can be proved using the structure of the second homology of  $G$ .

Note that all known examples of the Dehn functions of finitely presented metabelian groups are bounded above (up to equivalence) by the exponential function. A more general theorem from Cornuier and Tessera [11, Theorem A] implies that the Dehn function of a polycyclic group is bounded above by the exponential function. This theorem also works for metabelian Baumslag-Solitar groups, which are not polycyclic, but will fail to work for the Baumslag's group  $\Gamma$ . Those facts raise the main question we concern in this thesis, that is,

**Question III.2.1.** Is the Dehn function of a finitely presented metabelian group is bounded above (up to equivalence) by the exponential function?

We will discuss this question in Section VIII.1.

Another remark is that there exists a metabelian group with polynomially bounded but not exact polynomial Dehn function [37]. It is not known that whether there exists a finitely presented metabelian group with an intermediate Dehn function, i.e., the function is both superpolynomial and subexponential.

### III.3 Finitely Presented Metabelian Groups

In this section, we will sketch the proof of Theorem 3.1 in [6], which is a part of their main theorem [6, Theorem 5.1] characterizing all finitely presented metabelian groups. It provides us a workable presentation for an arbitrary finitely presented metabelian group. This theorem from Bieri and Strebel essentially says that a finitely generated metabelian group is finitely presentable if and only if all the metabelian relations (things like  $[[x, y], [z, w]] = 1$ ) can be generated by a finitely many relators. The following geometric lemma unveils the mechanics behind their main theorem.

Let  $\mathbb{R}^n$  be the Euclidean vector space with the usual inner product  $\langle \cdot, \cdot \rangle$ . We denote the norm induced by this inner product by  $\|x\| = \sqrt{\langle x, x \rangle}$ . If  $r > 0$ , then  $B_r$  denotes the open ball of radius  $r$ , i.e  $B_r = \{x \in \mathbb{R}^n \mid \|x\| < r\}$ .

We consider a finite collection  $\mathcal{F}$  of finite subsets  $L \subset \mathbb{R}^n$ . Say that an element  $x \in \mathbb{R}^n$  can be taken from  $B_r$  if either  $x \in B_r$  or if there exists  $L \in \mathcal{F}$  such that

$$x + L = \{x + y \mid y \in L\} \subset B_r.$$

**Lemma III.3.1** (Bieri, Strebel [6, Lemma 1.1]). *Assume that for every  $0 \neq x \in \mathbb{R}^n$ , there is  $L \in \mathcal{F}$  such that  $\langle x, y \rangle > 0$  for all  $y \in L$ . Then there exists a radius  $r_0 \in \mathbb{R}^+$  and a function  $\varepsilon : (r_0, \infty) \rightarrow \mathbb{R}^+$  with the property that for  $r > r_0$  each element of  $B_{r+\varepsilon(r)}$  can be taken from  $B_r$  by  $\mathcal{F}$ .*

*Proof.* Let  $S^{n-1} \subset \mathbb{R}^n$  be the unit sphere and consider the function  $f : S^{n-1} \rightarrow \mathbb{R}$  given by

$$f(u) = \max_L \min_y \{\langle u, y \rangle \mid y \in L \in \mathcal{F}\}, \text{ for } u \in S^{n-1}.$$

The function  $f$  is continuous. By the assumption on  $\mathcal{F}$ , we have  $f(u) > 0$  for all  $u \in S^{n-1}$ . Since  $S^{n-1}$  is compact,



the followings are well-defined:

$$C = \inf\{f(u) \mid u \in S^{n-1}\} > 0, D = \max_L \min_y \{\|y\| \mid y \in L \in \mathcal{F}\} > 0.$$

Note that  $L \subset \overline{B}_D$  for every  $L \in \mathcal{F}$ . We claim that Lemma III.3.1 holds with the following explicit choice of  $\rho_0$  and  $\varepsilon$ :

$$r_0 = \frac{D^2}{2C}, \varepsilon(r) = C - \frac{D^2}{2r}.$$

Note that  $\varepsilon(r)$  is positive and strictly increasing when  $r > r_0$ .

Let  $x \in \mathbb{R}^n$  be an element with  $\|x\| \geq r_0$ . By the definition of  $C$  there is  $L = L_x \in \mathcal{F}$  such that

$$\min_y \left\{ \left\langle \frac{-x}{\|x\|}, y \right\rangle \mid y \in L_x \right\} \geq C,$$

or equivalently

$$\max_y \left\{ \left\langle \frac{x}{\|x\|}, y \right\rangle \mid y \in L_x \right\} \geq C.$$

Therefore we have, for all  $y \in L_x$ ,

$$\|x+y\|^2 = \|x\|^2 + 2 \left\langle \frac{x}{\|x\|}, y \right\rangle \|x\| + \|y\|^2 \leq \|x\|^2 - 2C\|x\| + D^2 \leq \|x\|^2.$$

Moreover,

$$\|x+y\| - \|x\| = \frac{\|x+y\|^2 - \|x\|^2}{\|x+y\| + \|x\|} \leq \frac{-2C\|x\| + D^2}{2\|x\|} = -\varepsilon(\|x\|).$$

We can choose  $r > r_0$  such that  $r < \|x\| < r + \varepsilon(r)$  we get

$$\|x+y\| < (r + \varepsilon(r)) - \varepsilon(\|x\|) \leq r, \text{ for all } y \in L_x.$$

Hence  $x + L_x \subset B_r$ . The lemma is proved. □

Let  $T$  be a finitely generated abelian group, written multiplicatively. A (real) *character* of  $T$  is a homomorphism  $\chi : T \rightarrow \mathbb{R}$  of  $T$  into the additive group of the field of real numbers  $\mathbb{R}$ . Let  $\text{tor} T$  be the torsion subgroup of  $T$ . Then  $T/\text{tor} T \cong \mathbb{Z}^k \subset \mathbb{R}^k$  where  $k$  is the rank of  $T$ . We fix a homomorphism  $\theta : T \rightarrow \mathbb{R}^k$ . For every character  $\chi : T \rightarrow \mathbb{R}$ , there is a unique  $\mathbb{R}$ -linear map  $\tilde{\chi} : \mathbb{R}^k \rightarrow \mathbb{R}$  such that  $\chi = \tilde{\chi} \circ \theta$ . And by the Riesz representation theorem, there is a unique element  $x_\chi \in \mathbb{R}^k$  such that  $\tilde{\chi}(y) = \langle x_\chi, y \rangle, \forall y \in \mathbb{R}^k$ , whence  $\chi(t) = \langle x_\chi, \theta(t) \rangle$  [6]. Therefore each character  $\chi$  corresponds a vector  $x_\chi$  in  $\mathbb{R}^k$ . Conversely, given a vector  $x$  in  $\mathbb{R}^k$ , we can define a corresponding character by  $\chi(t) = \langle x, \theta(t) \rangle$ . This will be a useful realization for characters on  $T$ .

Every character  $\chi : T \rightarrow \mathbb{R}$  can be extended to a “character” of the group ring  $\chi : \mathbb{Z}T \rightarrow \mathbb{R} \cup \{+\infty\}$  by putting  $\chi(0) = +\infty$  and

$$\chi(\lambda) = \min\{\chi(t) \mid t \in \text{supp}(\lambda)\}, \text{ where } 0 \neq \lambda \in \mathbb{Z}T.$$

One can check that  $\chi(\lambda\mu) \geq \chi(\lambda) + \chi(\mu)$  for all  $\lambda, \mu \in \mathbb{Z}T$ . Moreover, if  $T$  is free abelian, the group ring  $\mathbb{Z}T$  has no zero divisor. It follows that  $\chi(\lambda\mu) = \chi(\lambda) + \chi(\mu)$  in this case [6].

For every  $T$ -module  $A$ , the *centralizer*  $C(A)$  of  $A$  is defined to be

$$C(A) = \{\lambda \in \mathbb{Z}T \mid \lambda \cdot a = a, \forall a \in A\}.$$

If  $A$  is a left (right)  $T$ -module then we write  $A^*$  for the right (resp. left)  $T$ -module with  $T$ -action given by  $at = t^{-1}a$  (resp.  $ta = at^{-1}$ ).

We say a  $T$ -module  $A$  is *tame* if  $A$  is finitely generated as a  $T$ -module and there is a finite subset  $\Lambda \subset C(A) \cup C(A^*)$  such that for every non-trivial character  $\chi : \mathbb{Z}T \rightarrow \mathbb{R}$  there is  $\lambda \in \Lambda$  with  $\chi(\lambda) > 0$ . Comparing this definition to Lemma III.3.1, the tameness condition gives a finite collection  $\mathcal{F}$  satisfying the hypothesis of Lemma III.3.1. Bieri and Strebel then prove that

**Theorem III.3.2** (Bieri, Strebel [6, Theorem 5.1]). *Let  $G$  be a finitely generated group and let  $A \triangleleft G$  be a normal subgroup such that both  $A$  and  $T = G/A$  are abelian. Then  $G$  is finitely presented if and only if  $A$  is tame as a  $T$ -module.*

For our purpose, let us sketch the proof of the “if” part of this theorem, more precisely

**Theorem III.3.3** (Bieri, Strebel [6, Theorem 3.1]). *If  $T$  is a finitely generated abelian group and  $A$  is a tame  $T$ -module, then every extension of  $A$  by  $T$  is finitely presented.*

To prove Theorem III.3.3, we have to introduce some preliminary concepts in order to provide a reasonable sketch. We first define ordered and semi-ordered words. Let  $F$  be the free group freely generated by  $\mathcal{F} = \{t_1, \dots, t_k\}$ . Let  $\bar{F} \subset F$  denote the subset of all *ordered* words of  $F$ , i.e.

$$\bar{F} = \{t_1^{m_1} t_2^{m_2} \dots t_k^{m_k} \mid m_1, \dots, m_k \in \mathbb{Z}\}.$$

If  $w \in F$ , we write  $\bar{w}$  as the unique word from  $\bar{F}$  representing  $w$  modulo the derived subgroup  $F'$ . In addition, a word  $w \in F$  is said to be *semi-ordered* if it is of the form

$$w = t_{\sigma(1)}^{m_1} t_{\sigma(2)}^{m_2} \dots t_{\sigma(k)}^{m_k}$$

where  $\sigma$  is a permutation of the symbols  $\{1, \dots, k\}$ .

Let  $\theta : F \rightarrow \mathbb{R}^k$  be the homomorphism given by

$$\theta(t_i) = (\delta_{i1}, \dots, \delta_{ik}),$$

for  $1 \leq i \leq k$ . For every  $w \in F$  define the *trace*  $\text{Tr}w \subset \mathbb{R}^n$  as follows: if

$$w = s_1 s_2 \dots s_m, \text{ where } s_j \in \mathcal{T} \cup \mathcal{T}^{-1},$$

is freely reduced, then

$$\text{Tr}(w) = \{\theta(s_1 \dots s_j) \mid j = 0, 1, \dots, m\},$$

The trace is the literal trace of the path of  $\theta(w)$  in  $\mathbb{R}^n$ . Next, we define a sequence of auxiliary groups. Let  $\mathcal{A}$  be a finite set and choose an assignment picking an element  $a_{ij} \in \mathcal{A}$  for every pair of integers  $(i, j)$  with  $1 \leq i < j \leq k$ . For every  $r \in \mathbb{R}^+ \cup \{+\infty\}$ , let  $H_r$  be the group generated by the set  $\mathcal{A} \cup \mathcal{T}$  with the following defining relations.

$$[t_i, t_j] = a_{ij}, \text{ for } 1 \leq i < j \leq k, \tag{III.1}$$

$$[a, b^u] = 1, \text{ for } a, b \in \mathcal{A}, u \in \bar{F} \text{ with } \|\theta(u)\| < r. \tag{III.2}$$

We have some useful properties for the group  $H_r$ .

**Proposition III.3.4.** If  $r \in \mathbb{R}^+$ , then

(a) (Bieri, Strebel [6, Lemma 3.2])  $a^{\bar{w}} = a^w$  for every  $a \in \mathcal{A}$  and every  $w \in F$  with  $\text{Tr}(w) \subset B_r$ .

(b) (Bieri, Strebel [6, Lemma 3.4a]) For  $u, v \in F$  such that

$$\text{Tr}(u) \subset B_r, \text{Tr}(v) \subset B_r, \|\theta(uv)\| < r.$$

Then  $[a, b^{uv}]$  and  $[a, b^{\bar{uv}}]$  are conjugate in  $H_r$  for every  $a, b \in \mathcal{A}$ .

(c) (Bieri, Strebel [6, Lemma 3.4b]) Assume  $r > 2k$ . Let  $u, v$  be semi-order words in  $F$  such that

$$\|\theta(u)\| \leq \frac{r}{2k}, \|\theta(v)\| \leq r + \frac{1}{2k}, \|\theta(uv)\| < r$$

are satisfied. Then  $[a, b^{uv}]$  and  $[a, b^{\bar{uv}}]$  are conjugate in  $H_r$  for every  $a, b \in \mathcal{A}$ .

For  $r = \infty$ ,  $H_\infty$  is metabelian and it is an extension of a free abelian group by another free abelian group. In fact,

by the definition of  $H_\infty$  and Proposition III.3.4 (a), we have  $[a^u, b^v] = 1$  for any  $u, v \in F, a, b \in \mathcal{A}$  and  $a^u$  is of infinite order. Then  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  is free abelian of infinite rank with basis  $\{a^u \mid a \in \mathcal{A}, u \in \bar{F}\}$  (See in Section V.2). It is also worth noting that  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  is a free module with basis  $\mathcal{A}$  over the group ring of  $H_\infty / \langle\langle \mathcal{A} \rangle\rangle$ , which is generated by the image of  $\mathcal{T}$  and is abelian since we includes all commutators  $[t_i, t_j]$  in  $\mathcal{A}$ . Each  $t_i$  is of infinite order. It follows that  $H_\infty / \langle\langle \mathcal{A} \rangle\rangle$  is also free abelian. But let us emphasize this:  $H_\infty$  is infinitely related, which can be shown by computing its second homology.

Now back to the proof of Theorem III.3.3. We first claim that the problem can be reduced to the case when  $T$  is a free abelian group. Let  $\pi : G \rightarrow T$  be the epimorphism and  $T_1 \leq T$  be a complement of the torsion subgroup of  $T$ . Then  $G_1 = \pi^{-1}(T_1)$  has finite index in  $G$  and  $G_1$  is an extension of an abelian group by a finitely generated free abelian group.  $G$  is finitely presented if and only if  $G_1$  is finitely presented. Moreover, if  $A$  is a tame  $T$ -module, then  $A$  is also a tame  $T_1$ -module [6, Proposition 2.5]. Therefore, the statement of Theorem III.3.3 is true for  $G$  if and only if it is true for  $G_1$ .

Now we assume that  $T$  is a free abelian group of rank  $k$ ,  $A$  is a tame  $T$ -module, and  $G$  is an extension of  $A$  by  $T$ . Denote  $\pi : G \rightarrow T$  to be the epimorphism such that  $A \cong \ker \pi$ .

Let  $\mathcal{T} = \{t_1, \dots, t_k\}$  be a subset of  $G$  such that  $\{\pi(t_1), \dots, \pi(t_k)\}$  forms a basis of  $T$  and  $\mathcal{A}$  be a finite subset of  $A$  containing all commutators  $a_{ij} = [t_i, t_j]$  for  $1 \leq i < j \leq k$  and generating  $A$  as a  $T$ -module. We write  $\hat{w} \in T$  for the image of  $w \in F$  under  $\pi$ .

Since  $A$  is a tame  $T$ -module. Then there is a finite subset  $\Lambda \subset C(A) \cup C(A^*)$  with the property that for every character  $\chi : T \rightarrow \mathbb{R}$ , there exists  $\lambda \in \Lambda$  such that  $\chi(\lambda) > 0$ . Recall that  $F := F(\mathcal{T})$  and  $\bar{F}$  is the set of ordered words of  $F$ . For every  $r \in (0, +\infty]$ , we define the group  $G_r$  to be given by generators  $\mathcal{A} \cup \mathcal{T}$  and defining relations

$$[t_i, t_j] = a_{ij}, \quad \text{for } 1 \leq i < j \leq k, \quad (\text{III.3})$$

$$[a, b^u] = 1, \quad \text{for } a, b \in \mathcal{A}, u \in \bar{F} \text{ with } \|\theta(u)\| < r, \quad (\text{III.4})$$

$$\prod_{u \in \bar{F}} (a^{\lambda(\hat{u})})^u = a, \quad \text{for } a \in \mathcal{A}, \lambda \in \Lambda \cap C(A), \quad (\text{III.5})$$

$$\prod_{u \in \bar{F}} (a^{\lambda(\hat{u})})^{u^{-1}} = a, \quad \text{for } a \in \mathcal{A}, \lambda \in \Lambda \cap C(A^*). \quad (\text{III.6})$$

In relations (III.5) and (III.6), we regard  $\lambda$  as a finite supported function from  $T$  to  $\mathbb{Z}$ . Hence  $\lambda(\hat{u})$  is just the value of  $\lambda$  at  $\hat{u}$ .

$G_r$  is finitely presented if  $r \neq \infty$ . If  $r = \infty$ , although the current presentation for  $G_\infty$  is not finite, it is metabelian once we realize  $G_\infty$  is a factor group of  $H_\infty$ . For each  $\lambda \in \Lambda$ ,  $\theta(\text{supp}(\lambda))$  is a finite subset of  $\mathbb{R}^k$ , denoted by  $L_\lambda$ . Let  $\mathcal{F} = \{L_\lambda \mid \lambda \in \Lambda\}$ . As previous discussion, there is a one-to-one correspondence between each character  $\chi : T \rightarrow \mathbb{R}$  and a linear functional  $\langle v_\chi, \cdot \rangle$ . Therefore if  $A$  is tame,  $\mathcal{F}$  is a collection of finite sets which satisfies assumptions of

Lemma III.3.1.

Let

$$C = \inf_{u \in S^{n-1}} \max_{\lambda \in \Lambda} \min_{y \in L_\lambda} \{\langle u, y \rangle\}, D = \max_{\lambda \in \Lambda} \min_{y \in L_\lambda} \{\|y\|\}.$$

In addition, let  $R = 2k \max\{D, D^2/2C\}$ . We have the following lemma

**Lemma III.3.5** (Bieri, Strebel [6, Lemma 3.5]). *For  $r \in [R, \infty) \cup \{\infty\}$  we have  $G_r \cong G_R$ . In particular,  $G_\infty$  is finitely presented.*

Since relations (III.3)-(III.6) hold in  $G$ , then  $G$  is a factor group of  $G_\infty$ . The epimorphism  $\varphi : G_\infty \rightarrow G$  is induced by the identity map on  $\mathcal{A} \cup \mathcal{T}$ . By the fact that the normal subgroup of a finitely generated metabelian group is normal closure of a finite set [19],  $G$  is finitely presented. Thus we finished the proof of Theorem III.3.3.

In summary, given a tame  $T$ -module  $A$ , any extension of  $A$  by  $T$  is always a factor group of  $G_\infty$ .  $G_\infty$  is finitely presented and the defining relations are given by (III.3)-(III.6) for any fixed positive real number  $r \geq R$ .

## Chapter IV

### The Membership Problem for a Submodule over a Polynomial Ring

#### IV.1 Preliminaries on Module Theory

In the chapter, we will investigate the membership problem of a submodule over a polynomial ring. Before we embark into the membership problem itself, let us first give a formal definition of a module over a ring and discuss some basic notions of modules.

Let  $R$  be a ring with 1, not necessary a polynomial ring. Let left  $R$ -module  $M$  over  $R$  consists of a abelian group  $(M, +)$  and an operation (or an action)  $\cdot : R \times M \rightarrow M$  such that for all  $r, s \in R, x, y \in M$ , we have:

$$(i) \quad r \cdot (x + y) = rx + ry;$$

$$(ii) \quad (r + s) \cdot x = r \cdot x + r \cdot y;$$

$$(iii) \quad (rs) \cdot x = r \cdot (s \cdot x);$$

$$(iv) \quad 1 \cdot x = x.$$

It generalizes the notion of a vector space over a field. The operation  $\cdot$  is called a *scalar multiplication*, and is usually written in juxtaposition, i.e., as  $rx$  instead of  $r \cdot x$ , for simplification. A right  $R$ -module is defined in a similar fashion and a bimodule is a module that is a left module and a right module such that the two multiplications are compatible. In this thesis, we only consider left  $R$ -modules. Thus we will write all the multiplication on the left. A *submodule* of an  $R$ -module  $M$  is a nonempty subset that is closed under addition and scalar multiplication.

A subset  $\{f_1, f_2, \dots, f_l\}$  of a  $R$ -module  $M$  is called a *generating set* if every  $f \in M$  is the linear combination of them, i.e, there exists  $\alpha_1, \alpha_2, \dots, \alpha_l \in R$  such that

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_l f_l.$$

A set of elements  $\{f_1, f_2, \dots, f_l\}$  of a module  $M$  is called *independent* if no nontrivial linear combination is zero, that is,

$$\text{If } \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_l f_l = 0, \text{ then } \alpha_i = 0, \text{ for } i = 1, 2, \dots, l.$$

A *basis* is an independent generating set.

One immediate example for a  $R$ -module is  $R^m$ . The addition and scalar multiplication on  $R^m$  are the following, respectively:

$$(a_1, a_2, \dots, a_m) + (b_1, b_2, \dots, b_m) = (a_1 + b_1, a_2 + b_2, \dots, a_m + b_m),$$

$$r(a_1, a_2, \dots, a_m) = (ra_1, ra_2, \dots, ra_m).$$

The module  $R^m$  is called a *free  $R$ -module of rank  $m$* . The canonical basis of  $R^m$  is  $\{e_1, e_2, \dots, e_m\}$  where  $e_i = (0, \dots, 1, \dots, 0)$  with all but the  $i$ -th entry is 0.

A submodule of the free module  $R^1$  is an ideal in the ring  $R$ .

Given a free  $R$ -module  $M$  of finite rank and a submodule  $S$  generated by a finite set  $\{f_1, f_2, \dots, f_l\}$ , the membership of a submodule  $S$  we are considering in this thesis is the following

**Problem IV.1.1.** Given an element  $f$  in  $M$ , decide whether  $f$  in  $S$ , i.e., if there exists elements  $\alpha_1, \alpha_2, \dots, \alpha_l$  such that

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_l f_l.$$

A *homomorphism*  $\varphi : M \rightarrow N$  of  $R$ -modules is a map which is compatible with the laws of composition:

$$\varphi(f + f') = \varphi(f) + \varphi(f'), \varphi(rf) = r\varphi(f)$$

for all  $f, f' \in M, r \in R$ . A bijective homomorphism is called an *isomorphism*.

Last we define the concept of quotient modules. Let  $R$  be a ring, and let  $S$  be a submodule of an  $R$ -module  $M$ . The quotient  $M/S$  is the additive group of cosets  $\bar{f} = f + S$ . And the scalar multiplication is defined by

$$r\bar{f} = \overline{rf}.$$

Thus  $M/S$  is made an  $R$ -module.

The membership problem Problem IV.1.1 can be regarded as the word problem of the quotient  $M/S$ .

## IV.2 A Well-order on a Polynomial Ring

From now on we only consider the case  $R := \mathbb{Z}[x_1, \dots, x_k]$  to be a polynomial ring over  $\mathbb{Z}$ . Given a free  $R$ -module  $M$  with basis elements  $e_1, \dots, e_m$ , a *term* in  $M$  is a product of an integer, a monomial in  $R$ , and an element from the basis. A typical term looks like  $a\mu e_i$ , where  $a \in \mathbb{Z}, \mu$  is a monomial in  $R$ . Let  $\mathcal{T}$  be the set of all terms in  $M$ . In addition, we will call  $\mu e_i$  a *module monomial* in the module  $M$ , denoted by  $\mathcal{U}$  the set of module monomials of  $M$ . The set of monomials in the polynomial ring  $R$  in the usual sense will be denoted by  $\mathcal{X}$ . For a term  $g \in \mathcal{T}$ , we denote

$C(g), M(g)$  to be the coefficient, and monomial part of  $g$  respectively. An element in  $M$  is a finite sum of terms. From now on, we only consider reduced elements in  $M$ , in the sense that no terms are sharing the same module monomial. We also denote  $\text{supp}(f)$  to be the set of module monomials with non-zero coefficients. In what follows, we will use “monomial” for both module monomials in  $\mathcal{U}$  and monomials in  $\mathcal{X}$ .

Our first goal is to put a well-order on  $\mathcal{T}$ . Recall that a *binary relation* on a set  $X$  is a subset of  $X \times X$ , that is, it is a set of ordered pair  $(x_1, x_2)$  where  $x_1, x_2 \in X$ . A binary relation  $R$  is a *partial order* if it is reflexive, antisymmetric and transitive. That is, for all  $x, y$  and  $z$  in  $X$ , it must satisfy:

- (1)  $(x, x) \in R$  (reflexivity);
- (2) if  $(x, y), (y, x) \in R$ , then  $x = y$  (antisymmetry);
- (3) if  $(x, y), (y, z) \in R$ , then  $(x, z) \in R$  (transitivity).

It is common to use the notion  $a \leq b$  instead of  $(a, b) \in R$  when dealing with partial order. A partial order  $\leq$  is a *total order* if it is connex, i.e. for all  $x, y \in X$ ,  $x \leq y$  or  $y \leq x$ . Thus any two elements in  $X$  are comparable under  $\leq$ . A *well-order* on a set  $X$  is a total order on  $X$  with the property that every non-empty subset of  $X$  has a least element in this ordering where the least element of a subset  $S$  is the element  $y \in S$  such that  $y \leq x$  for all  $x \in S$ .

To construct such an order, we have to put well-orders on  $\mathbb{Z}, \mathcal{X}$  and  $\{e_1, \dots, e_m\}$  separately. Then we will construct the lexicographical order based on all of them.

On  $\mathbb{Z}$ , we define an order  $\prec_{\mathbb{Z}}$  as following

$$0 \prec_{\mathbb{Z}} 1 \prec_{\mathbb{Z}} 2 \prec_{\mathbb{Z}} \dots \prec_{\mathbb{Z}} -1 \prec_{\mathbb{Z}} -2 \prec_{\mathbb{Z}} \dots$$

Under this order, all negative numbers are larger than any positive number. Let  $a, b \in \mathbb{Z}$  where  $a \prec_{\mathbb{Z}} b$ , then there exists unique  $q, r$  such that  $a = qb + r, 0 < r < |b|$ . Note that  $r \prec_{\mathbb{Z}} a$  whether  $a$  is positive or negative, thus we can “reduce” any number to its remainder by dividing a fix number  $b$ . One useful remark is that since the remainder is always positive, this “dividing- $b$ -reduction” can only be applied finitely many times. It is not hard to see that  $\prec_{\mathbb{Z}}$  on  $\mathbb{Z}$  is a well-order.

For monomials in  $R$ , we use the degree lexicographical order (also called shortlex or graded lexicographical order)  $\prec_R$  which is defined with respect to the convention  $x_1 \succ x_2 \succ \dots \succ x_k$ , i.e. for  $\mu_1 = x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}, \mu_2 = x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}$

$$\mu_1 \prec_R \mu_2 \text{ if } \sum_{i=1}^k |n_i| < \sum_{i=1}^k |m_i| \text{ or } \sum_{i=1}^k |n_i| = \sum_{i=1}^k |m_i|, \mu_1 \prec_{lex} \mu_2,$$



where  $\prec_{lex}$  is the usual lexicographical order which is defined in the following way

$$x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \prec_{lex} x_1^{m_1} x_2^{m_2} \dots x_k^{m_k} \text{ if } n_i < m_i \text{ for the first } i \text{ where } n_i \text{ and } m_i \text{ differ.}$$

$\prec_R$  on  $\mathcal{X}$  in fact is a well-order while  $\prec_{lex}$  might not be (See in [2]).

Finally we fix an order  $e_1 \succ e_2 \succ \dots \succ e_s$ . We now set  $\prec$  on  $T$  to be the lexicographical order based on  $\mathcal{X} \succ \{e_1, \dots, e_m\} \succ \mathbb{Z}$ . For instance,

$$7x_1^2 x_2 e_2 \prec 5x_1^3 e_1, 3x_1^3 x_2^5 e_2 \prec 3x_1^3 x_3^6 e_2, 2x_1^5 x_3^2 e_3 \prec 4x_1^5 x_3^2 e_3.$$

It is not hard to verify that  $\prec$  is a well-order on  $\mathcal{T}$ .

With the well-order  $\prec$ , we are able to compare any two terms. Consequently, for an element  $f \in M$  we can define the leading monomial  $LM(f)$  of  $f$  to be the largest monomial among  $\text{supp}(f)$ . For example,

$$LM(x_1^7 e_1 + 3x_1^3 x_2^4 e_2) = x_1^7 e_1, LM(x_2^3 e_1 + (x_2^5 x_3^2 + x_2^3 x_4^5) e_2 + x_2^5 x_3^2 e_3) = x_2^3 x_4^5 e_2.$$

Next, we define the leading coefficient of  $f$  to be the coefficient of the leading monomial, denoted by  $LC(f)$ . Then the leading term of  $f$  can be defined as

$$LT(f) := LC(f) \cdot LM(f).$$

We then extend  $\prec$  to  $M$ . For  $g, f \in M$ , we define  $g \prec f$  inductively as follows

$$g \prec f \text{ if } LT(g) \prec LT(f) \text{ or } LT(g) = LT(f), g - LT(g) \prec f - LT(f).$$

Since  $\prec$  on  $T$  is a well-order, then so is  $\prec$  on  $M$ .

Note that  $\prec$  is compatible with multiplication by elements from  $\mathcal{X}$  i.e., if  $g \prec h$ , then  $\mu g \prec \mu h$ ,  $\mu \in \mathcal{X}$ .

One remark on  $\prec$  is that it is Noetherian on  $\mathcal{U}$  as well as  $\mathcal{X}$ , the set of module monomials in  $M$ , i.e., there is no infinite descending chain of module monomials. However, the statement is not true for  $\prec$  on  $\mathcal{T}$ . Because we have an infinite descending chain for negative numbers. This issue can be avoided by what we will introduce in the next section: the polynomial reduction.

### IV.3 Gröbner Basis

Now let us define the key ingredient for the application of Gröbner bases: polynomial reduction.

For two monomials  $\mu e$  and  $\mu' e'$  from  $\mathcal{U}$ , we say  $\mu e \mid \mu' e'$  if  $\mu \mid \mu'$  and  $e = e'$ . Let  $F = \{f_1, \dots, f_l\}$  be a finite subset of  $M$  and  $S$  be the submodule generated by  $F$ . Given  $g, h \in M$ , we define the polynomial reduction  $g \rightarrow_F h$  as

follows: if there exists  $f \in F$  and a term  $g_0 \in \mathcal{T}$  of  $g$  such that  $LM(f) \mid M(g_0), LC(f) \prec C(g_0)$ , then

$$g = \frac{qM(g_0)}{LM(f)}f + h,$$

where  $C(g_0) = qLC(f) + r$ ,  $q, r$  are unique integers such that  $0 < r < |LC(f)|$ . Note that the coefficient of  $g_0$  in  $h$  is  $r$ .

For  $g \rightarrow_F h$ , read “ $g$  reduces to  $h$  modulo  $F$ ”. If there’s no such  $f$  and  $g_0$ , then we say that  $g$  is *irreducible* modulo  $F$ .

Note that we naturally have  $h \prec g$  if  $g \rightarrow_F h$ . We claim that  $\rightarrow_F$  is Noetherian, i.e., there is no infinite reduction sequence. First, note that we turn the coefficient of  $M(g_0)$  of  $h$  to a positive number after a reduction, then there are only finitely many possible reductions that can be applied to the term containing the monomial  $M(g_0)$ . Thus if we assume that  $g_0$  is the largest term that can be reduced in  $g$  modulo  $F$ , then after finitely many reductions, the monomial of the largest term that can be reduced is strictly less than the original one. Since  $\prec$  is Noetherian for monomials, we only have a reduction of finite length for any given  $g \in M$ .

Let  $\rightarrow_F^*$  be the reflexive and transitive closure of  $\rightarrow_F$ . Then for each  $g \in M$ , there exists  $h \in M$  such that  $g \rightarrow_F^* h$  and  $h$  is irreducible modulo  $F$ . We call  $h$  to be a *reduced form* of  $g$  modulo  $F$ . Unfortunately, the reduced form of an element in  $M$  may not be unique. In fact, at each step of reduction, we may have multiple choices of  $f \in F$  that can be applied to this reduction. This yields our motivation for defining *Gröbner basis*: a generating set  $F$  such that every element in  $M$  has a unique reduced form modulo  $F$ . In theoretical computer science, this property is called Church-Rosser property (See in [8]).

We denote  $g \equiv_S h$  if  $g - h \in S$ .  $\equiv_S$  defines an equivalence relation on  $M$ . We let the  $R(g)$  to be the least element in its equivalence class with respect to  $\prec$ . It is well-defined since  $\prec$  is a well-order.

**Definition IV.3.1.** Let  $M$  be a free  $R$ -module of finite rank, and  $S$  be a submodule of  $M$ . A finite generating set  $F$  of  $S$  is called a *Gröbner basis* if  $g \rightarrow_F^* R(g)$  for all  $g \in M$ .

*Remark.*  $R$  is a Noetherian ring hence  $M$  is a Noetherian module. Thus any submodule of  $M$  is finitely generated.

The Gröbner basis is a finite generating set but is not always a basis. The next theorem shows that the Gröbner basis always exists, while not all submodule has a basis.

**Theorem IV.3.2** ([34, Proposition 10.6.3]). *For any submodule of a free module of finite rank over  $R = \mathbb{Z}[x_1, \dots, x_k]$ , there exists a Gröbner basis.*

*Proof.* We consider a submodule  $S$  in the free  $R$ -module  $M$ , where  $M$  is of finite rank. Let  $S_u = \{g \in S \mid LM(g) = u\}$ ,  $u \in \mathcal{U}$  and  $L_u = \{LC(g) \mid g \in S_u\}$ . It is not hard to see that  $L_u$  is an ideal in  $\mathbb{Z}$ . Thus it is generated by the smallest element in this ideal with respect to  $\prec$ . We denote by  $h_u$  the element such that  $LC(h_u)$  generates  $L_u$  since  $\mathbb{Z}$  is a principle ideal domain. Note that the leading coefficient of  $h_u$  is always positive, since, by our definition of  $\prec$ , negative numbers are

larger than positive numbers. For our purpose, we denoted it  $c_u$  and hence  $LT(h_u) = c_u u$ . Let  $P$  be the set of all such  $h_u$  which generates  $S$  over  $\mathbb{Z}$  whenever  $h_u$  can be defined (since  $S_u$  might be empty) and  $L$  be the set of leading terms of elements of  $P$ . That is,  $L = \{c_u u \mid c_u \neq 0\}$ .

Then we claim that there is a finite subset  $F$  of  $L$  such that  $L = \mathcal{X}F := \{xf \mid x \in \mathcal{X}, f \in F\}$ . Suppose no such  $F$  exists. We choose  $c_u u$  in  $L$  with the smallest  $c_u$ . By definition,  $c_{xu} \mid c_u$  for all  $x \in \mathcal{X}$  by our choice of  $c_u u$ . Thus  $c_{xu} = c_u$  for all  $x \in \mathcal{X}$ . Let  $f_1 = c_u u, V_1 = \{xf_1 \mid x \in \mathcal{X}\}$ . It follows that  $V_1 \subset L$ . Suppose that  $f_1, f_2, \dots, f_n$  are defined and  $V_r = \{xf_i \mid x \in \mathcal{X}, i = 1, 2, \dots, r\}$ . By our assumption,  $V_n \subsetneq L$ . We choose  $c_u u$  outside  $V_n$  with the minimal coefficient and let  $f_{n+1} = c_u u$  and  $V_{n+1} = V_n \cup \{xf_{n+1} \mid x \in \mathcal{X}\}$ . By induction, we construct an infinite ascending chain of subsets of  $L$  that are closed under multiplication by an element from  $\mathcal{X}$ , i.e.,

$$V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n \subsetneq \dots,$$

where  $\mathcal{X}V_i = V_i$  for all  $i$ . Each of  $V_i$  generates a distinct submodule of the free  $R$ -module  $M$ . It is a contradiction since  $M$  is Noetherian.

Now we have a finite set  $F$  that  $\mathcal{X}F = L$ . We take a finite set  $\bar{F}$  of  $P$  such that the set of leading terms of elements in  $\bar{F}$  is  $F$ . We claim that  $\bar{F}$  is a Gröbner basis. Let  $g \in M$  and assume that  $g \xrightarrow{\bar{F}}^* h$  and  $h$  is irreducible modulo  $\bar{F}$ . If  $h \neq R(g)$ , then  $0 \neq h - R(g) \in S$ . Let  $u = LM(h - R(g))$ . We have that  $c_u \mid LC(h - NF(g))$ . It follows that there exists an element  $f \in \bar{F}$  such that  $c_u u = LT(xf)$  for some  $x \in \mathcal{X}$  and  $h$  can be reduced by  $f$ , contradicting to the irreducibility of  $h$ . Thus the theorem is proved.  $\square$

*Remark.* By our definition, the Gröbner basis is not unique since adding any element  $f \in S$  to a Gröbner basis results another Gröbner basis. Our construction in the proof of Theorem IV.3.2 has a nice property that for any  $g \in S$  there exists  $f \in \bar{F}$  such that  $g \xrightarrow{f} g'$  where  $LM(g')$  is strictly less than  $LM(g)$  with respect to  $\prec$ . In fact, every Gröbner basis satisfies this property. Suppose not, then there exists  $u \in \mathcal{U}$  such that there is no element  $f \in \bar{F}$  satisfying the following conditions:

- (1)  $LM(f) \mid u$ ,
- (2) the ideal generated by  $LC(f)$  is  $L_u$ .

Let  $g \in S_u$  such that  $LC(g) = c_u$ . Then the term  $c_u u$  in  $g$  is irreducible by any  $f \in \bar{F}$ . A contradiction, since  $R(g) = 0$ .

#### IV.4 Division Algorithm

For an element  $g \in M$ ,  $g$  can be written as a finite sum of distinct terms, i.e

$$g = c_1 u_1 + c_2 u_2 + \dots + c_d u_d,$$

where  $c_i \in \mathbb{Z}, u_i \in \mathcal{U}$  and  $u_1 \succ u_2 \succ \dots \succ u_d$ . We define the length of  $g$  to be  $|g| := \sum_{i=1}^d |c_i|$ . It is not hard to show that  $|\cdot|$  has following properties: for all  $c \in \mathbb{Z}$  and  $f, g \in M$

- (1)  $|cf| = |c||f|$ ,
- (2)  $|fg| \leq |f||g|$ ,
- (3)  $|f+g| \leq |f| + |g|$ .

And if the leading monomial of  $g$  is  $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} e_i$ , we define  $\deg(g) = \sum_{i=1}^k n_i$ . One immediate observation is that if  $g \prec h$  then  $\deg(g) \leq \deg(h)$ .

Let  $F = \{f_1, \dots, f_l\}$  be a Gröbner basis for a submodule  $S$  and  $g = c_1 u_1 + \dots + c_d u_d \in S$  such that  $\deg(g) \leq n, |g| \leq p$ . Since  $g \in S$ , then  $g \xrightarrow{*}_F 0$ . Thus there exists a finite sequence of reductions

$$g = g_0 \xrightarrow{F} g_1 \xrightarrow{F} g_2 \xrightarrow{F} g_3 \xrightarrow{F} \dots \xrightarrow{F} g_r = 0.$$

At each step, if we always choose to cancel the leading term of  $g_i$  using the polynomial reduction (this is always possible since  $g$  can be reduced to 0), we may assume that  $LM(g_0) \succ LM(g_1) \succ LM(g_2) \succ \dots \succ LM(g_r) = 0$ . Thus the number of steps of reduction is bounded by the number of monomials less or equal to  $LM(g)$ . Recall that  $m$  is the rank of the free module, then

$$r \leq |\{u \in \mathcal{U} \mid u \prec LM(g)\}| \leq mG_k(n),$$

where  $G_k(n)$  is the growth function of a free commutative monoid with a free generating set of size  $k$  (See [33, Example 3.7.1]). It is well-known that  $G_k(n)$  is a polynomial of degree  $k$ . In fact,

$$G_k(n) = \binom{n+k}{k}.$$

At the  $j$ th step of our reduction, we have

$$g_j = g_{j-1} - a_j \mu_j f_{i_j},$$

where  $a_j \in \mathbb{Z}, \mu_j \in \mathcal{X}, 1 \leq i_j \leq l$  and  $LT(g_{j-1}) = LT(a_j \mu_j f_{i_j})$ . Then  $|a_j| \leq LC(g_{j-1}) \leq |g_{j-1}|$ . Let  $C = \max\{|f_1|, |f_2|, \dots, |f_l|\}$ .

We also observe that

$$|g_j| \leq |g_{j-1}| + |a_j| |f_{i_j}| \leq |g_{j-1}| + C|a_j|. \quad (\text{IV.1})$$

Additionally, we have  $|a_1| \leq |g_0| = p$ , and

$$|a_j| \leq LC(g_{j-1}) \leq |g_{j-1}|. \quad (\text{IV.2})$$

Combine (IV.1) and (IV.2) inductively,

$$|a_j| \leq |g_{j-1}| \leq |g_{j-2}| + C|a_{j-1}| \leq |g_{j-2}|(1+C) \leq \dots \leq p(1+C)^{j-1}, j \geq 1.$$

Adding all the steps up, we have

$$g = \sum_{j=1}^r a_j \mu_j f_j = \sum_{i=1}^l \alpha_i f_i, \alpha_i \in R.$$

Note that

$$\sum_{i=1}^l |\alpha_i| = \sum_{j=1}^r |a_j| \leq p(1 + (1+C) + (1+C)^2 + \dots + (1+C)^{r-1}) \leq \frac{p((1+C)^m G_k(n) - 1)}{C}.$$

In general, one important consequence of the algorithm above is the following

**Corollary IV.4.1** (Division). *Let  $M$  be a free module over a polynomial ring  $R = \mathbb{Z}[x_1, \dots, x_k]$ . Let  $F = \{f_1, \dots, f_l\}$  be a Gröbner basis for a submodule  $S$ . Then there exists a constant  $K$  such that for every  $g \in M$ ,  $\deg(g) \leq n$ ,  $|g| \leq p$  one can write*

$$g = \sum_{i=1}^l \alpha_i f_i + r$$

with  $\alpha_i \in R, r = R(g)$  and

$$\deg(\alpha_i f_i) < \deg(g), 1 \leq i \leq l, \sum_{i=1}^l |\alpha_i| \leq pK^{n^k}.$$

*Remark.* This provides an algorithm to solve the membership problem for submodules of a finitely generated free module over polynomial rings with integral coefficients. Given  $g, f_1, \dots, f_l$ , to decide if  $g$  lies in the submodule  $S$  generated by  $f_1, \dots, f_l$  we first find a Gröbner basis for  $S$ . The algorithm which finds Gröbner bases can be found in [34]. Once we have Gröbner bases in hand, we can compute the  $R(g)$  since  $g \in S$  if and only if  $R(g) = 0$ .

Let  $T$  be the free abelian group of rank  $k$  with basis  $t_1, \dots, t_k$ . We can regard the group ring  $\mathbb{Z}T$  as a factor ring of  $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$  i.e

$$\mathbb{Z}T \cong \mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}] / \langle t_1 t_1^{-1} - 1, \dots, t_k t_k^{-1} - 1 \rangle.$$

Then a submodule generated by a finite set  $F$  over  $\mathbb{Z}T$  can be identified as a submodule generated by  $F \cup \{t_1 t_1^{-1} - 1, \dots, t_k t_k^{-1} - 1\}$  over  $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$ .

Therefore we have a similar result for group rings.

**Corollary IV.4.2.** *Let  $M$  be a free module over  $\mathbb{Z}T$  where  $T$  is the free abelian group of rank  $k$ . Let  $S$  be a submodule of  $M$ . Then there exists a finite generating set  $F = \{f_1, \dots, f_l\}$  and a constant  $K$  such that for  $g \in S$  with  $\deg(g) \leq n$ ,  $|g| \leq p$*

there exist  $\alpha_1, \dots, \alpha_l \in \mathbb{Z}T$  such that

$$g = \alpha_1 f_1 + \dots + \alpha_l f_l, \deg(\alpha_i f_i) \leq \deg(g), \sum_{i=1}^l |\alpha_i| \leq pK^{n^{2k}}.$$

*Remark.*  $\deg(g)$  and  $|g|$  for element  $g \in \mathbb{Z}T$  are inherited from  $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$ .

Corollary IV.4.2 estimates an upper bound for the complexity of the membership problem of a submodule over a group ring  $\mathbb{Z}T$ . As we will later see, it carries to an upper bound for the Dehn function of a finitely presented metabelian group.

## Chapter V

### Dehn Functions of Finitely Presented Metabelian Groups

#### V.1 Reduction Step

Given a finitely presented metabelian group  $G$  with the short exact sequence

$$1 \longrightarrow A \longrightarrow G \longrightarrow T \longrightarrow 1 .$$

such that  $A, T$  are abelian. In addition, we suppose that the torsion-free rank of  $T$  is minimized over all such short exact sequence of  $G$  where both the normal subgroup and the quotient group are abelian. The torsion-free rank of such  $T$  is denoted by  $\text{rk}(G)$ . Since  $G$  is finitely presented, in particular, it is finitely generated. Then  $T$  is a finitely generated abelian group, and  $A$  is finitely generated as a  $T$ -module (See in [19]).

The main goal of this chapter is to prove the following.

**Theorem V.1.1.** *Let  $G$  be a finitely presented metabelian group. Let  $\text{rk}(G) = k$ , i.e.,  $k$  is the minimal torsion-free rank of an abelian group  $T$  such that there exists an abelian normal subgroup  $A$  in  $G$  satisfying  $G/A \cong T$ .*

*Then the Dehn function of  $G$  is asymptotically bounded above by*

- (1)  $n^2$  if  $k = 0$ ;
- (2)  $2^{n^{2k}}$  if  $k > 0$ .

We can apply the same technique as in Section III.3 to reduce the problem to a simpler case. Denote  $\pi : G \twoheadrightarrow T$  to be the epimorphism such that  $A \cong \ker \pi$ . Let  $T_1 \leq T$  be the complement of the torsion subgroup of  $T$ .  $G_1 = \pi^{-1}(T_1)$  has finite index in  $G$  then  $G_1$  is quasi-isometric to  $G$ . It follows that  $\delta_G = \delta_{G_1}$  due to Theorem II.1.1. Therefore an upper bound of  $\delta_{G_1}$  is also an upper bound for  $\delta_G$ .

Next, we show that  $\text{rk}(G) = \text{rk}(G_1)$ . If  $G_1$  can be written as an extension of two abelian groups  $A_2$  and  $T_2$ , where the torsion-free rank of  $T_2$  is strictly less than  $k$ , consider the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{i} & G & \longrightarrow & G/G_1 \longrightarrow 1 \\ & & \downarrow \pi & & \downarrow f & & \downarrow i \\ 1 & \longrightarrow & T_2 & \xrightarrow{i} & T_2 \times G/G_1 & \longrightarrow & G/G_1 \longrightarrow 1 \end{array} ,$$

where  $f(g, h) := (\pi(g), h), g \in G_1, h \in G/G_1$ . By the snake lemma, there exists an exact sequence

$$\ker \pi = A_2 \longrightarrow \ker f \longrightarrow \ker i = 1 \longrightarrow \text{coker} \pi = 1 \longrightarrow \text{coker} f \longrightarrow \text{coker} i = 1.$$

It follows that  $\ker f$  is abelian and  $f$  is surjective. Then  $G$  can be represented as an extension of  $\ker f$  by  $T_2 \times G/G_1$  where the torsion-free rank is of  $T_2 \times G/G_1$  strictly less than  $k$ . This contradicts the minimality of  $k$ . Therefore  $k$  is preserved when passing to  $G_1$ .

Thus from now on, we shall assume that  $T$  is a free abelian group of rank  $k$  and  $G$  is an extension of a tame  $T$ -module  $A$  by the free abelian group  $T$ . Also, let us assume that  $k > 0$ , since if  $k = 0$ ,  $G$  has an finitely generated abelian subgroup of finite rank, which is not interesting to us.

Let  $\mathcal{T} = \{t_1, \dots, t_k\} \subset G$  such that  $\{\pi(t_1), \dots, \pi(t_k)\}$  forms a basis for  $T$  and  $\mathcal{A}$  be a finite subset of  $G$  such that it contains all commutators  $a_{ij} = [t_i, t_j]$  for  $1 \leq i < j \leq k$  and generates the  $T$ -module  $A$ . Then  $\mathcal{A} \cup \mathcal{T}$  is a finite generating set for the group  $G$ .

By Theorem III.3.2, since  $G$  is finitely presented,  $A$  is a tame  $T$ -module. Then there is a finite subset  $\Lambda \subset C(A) \cup C(A^*)$  such that for each character  $\chi : T \rightarrow \mathbb{R}$ , there exists  $\lambda \in \Lambda$  such that  $\chi(\lambda) > 0$ . Let  $F$  be the free group generated by  $\mathcal{T}$  and  $\bar{F}$  be the set of all ordered words in  $F$  (See Section III.3). Same as previous section, we let  $\theta : F \rightarrow \mathbb{R}^k$  be the homomorphism given by

$$\theta(t_i) = (\delta_{i1}, \dots, \delta_{ik}), 1 \leq i \leq k.$$

If  $w \in F$  we shall write  $\bar{w}$  for the unique word in  $\bar{F}$  representing  $w$  modulo  $F'$ . In addition, we denote  $\tilde{w} \in T$  for the image of  $w \in F$  under  $\pi$ .

Then we are able to define a sequence of groups  $G_r$  as what we did in Section III.3, but for our purpose, we will need a larger  $R$ . Let

$$R = 2k \max\{D^2/2C, D, D^2/(4kC - 4)\},$$

where  $C, D$  are as defined before Lemma III.3.5. Since  $R > 2k \max\{D, D^2/C\}$ ,  $G_R \cong G_\infty$ , and in particular,  $G$  is a factor group of the finitely presented group  $G_\infty$ . Then we can list all defining relations of  $G_\infty$  here:

$$[t_i, t_j] = a_{ij}, \quad \text{for } 1 \leq i < j \leq k, \quad (\text{V.1})$$

$$[a, b^u] = 1, \quad \text{for } a, b \in \mathcal{A}, u \in \bar{F} \text{ with } \|\theta(u)\| < R, \quad (\text{V.2})$$

$$\prod_{u \in \bar{F}} (a^{\lambda(\hat{u})})^u = a, \quad \text{for } a \in \mathcal{A}, \lambda \in \Lambda \cap C(A), \quad (\text{V.3})$$

$$\prod_{u \in \bar{F}} (a^{\lambda(\hat{u})})^{u^{-1}} = a, \quad \text{for } a \in \mathcal{A}, \lambda \in \Lambda \cap C(A^*). \quad (\text{V.4})$$

To simplify our notation, we will write relations (V.1) and (V.2) as  $\mathcal{R}_1$  and relations (V.3) and (V.4) as  $\mathcal{R}_2$ .

Denote the epimorphism  $\varphi : G_\infty \rightarrow G$  induced by the identity map on  $\mathcal{A} \cup \mathcal{T}$ . Note that  $\varphi$  induces an isomorphism on  $G_\infty/A_\infty \cong T$ . Therefore  $\ker \varphi \leq A_\infty$  is abelian where  $A_\infty := \langle\langle \mathcal{A} \rangle\rangle_{G_\infty} \triangleleft G_\infty$ . Let  $\ker \varphi = \langle\langle \mathcal{R}_3 \rangle\rangle_{G_\infty}$ , where  $\mathcal{R}_3$  is a finite set.



Thus we obtain a finite presentation for  $G$ ,

$$G = \langle \mathcal{A} \cup \mathcal{T} \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \rangle. \quad (\text{V.5})$$

With (V.5), we have the following proposition:

**Proposition V.1.2.** If  $\text{rk}(G) > 0$ , then  $\delta_G(n) \preceq 2^{n^{2k}}$ .

Proposition V.1.2 will be proved in Section V.5.

*Proof of Theorem V.1.1.* If  $k = 0$ ,  $G$  has a finite index abelian subgroup. Therefore  $\delta_G \preceq n^2$  by Theorem II.1.1 and Theorem II.4.2.

If  $k > 0$ , the result follows directly from Proposition V.1.2 by passing the problem to a finite index subgroup.  $\square$

## V.2 The Ordered Form of Elements

For convenience, we assume that  $|\mathcal{A}| = m$  and denote  $\mathcal{A} = \{a_1, \dots, a_m\}$ .

To understand the module structure in  $G$ , we have to go back to the group  $H_\infty$  corresponding to  $G$ . As in Section III.3,  $H_\infty$  has a presentation as follows,

$$H_\infty = \langle \mathcal{A} \cup \mathcal{T} \mid [t_i, t_j] = a_{ij}, 1 \leq i < j \leq k, [a, b^u] = 1, a, b \in \mathcal{A}, u \in F(\mathcal{T}) \rangle.$$

$G$  is an epimorphic image of  $H_\infty$ , where the epimorphism is induced by the identity map on the set  $\mathcal{A} \cup \mathcal{T}$ . Let  $M$  be the free  $T$ -module with basis  $\mathcal{A}$ . We will show that  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  is isomorphic to  $M$ .

Note that we only consider words that are fully reduced in  $F(\mathcal{A} \cup \mathcal{T})$ , the free group generated by  $\mathcal{A} \cup \mathcal{T}$ . Since each group element in  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  can be also regarded as an element in the  $T$ -module  $M$ . Different words in the group might represent the same element in the module. For example,  $a_1^t a_2^t$  and  $a_2^t a_1^t$  both represent  $t_1 a_1 + t_1 a_2$  in  $M$ . We now pick the canonical element among all words represents the same element in  $M$ . This canonical element, which we will call the *ordered form* of an element in  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$ , is defined in the following way.

**Definition V.2.1.** Let  $\bar{F}$  be the set of ordered exponent words in  $F(\mathcal{T})$  (See in Section III.3) and  $\prec$  be the well-order defined in Section IV.2. Let  $f$  be an element in  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$ , then the ordered form  $\text{OF}(f)$  is of the form  $a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m}$  such that

- (1)  $\mu_i \in \mathbb{Z}T$  for  $1 \leq i \leq m$ , and each  $\mu_i$  is of the form  $\mu_i = \sum_{j=1}^{n_j} c_{ij} u_{ij}$  such that  $c_{ij} \in \mathbb{Z}, u_{ij} \in \bar{F}$  and  $u_{i1} \succ u_{i2} \succ \dots \succ u_{in_i}$ ;
- (2)  $f =_{H_\infty} \text{OF}(f)$ ,

To check the definition is well-defined we have to show that the existence and uniqueness of the ordered form.

To show the existence, let us construct the explicit algorithm that rewrites a word  $w \in \langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  to a word of the ordered form.

Let  $\pi : H_\infty \rightarrow T$  be the canonical quotient map. For  $g \in \langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$ , we have  $\pi(g) = 0$ . It follows that the sum of exponents of each  $t_i$  is 0.

Let us start with a word  $w = u_1 b_1 u_2 b_2 \dots u_s b_s u_{s+1} \in \langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  where  $u_i \in F(\mathcal{T}), b_i \in \mathcal{A}^{\pm 1}$ . Here  $u_1, u_{s+1}$  could be empty. Then

$$w = b_1^{u_1^{-1}} b_2^{(u_1 u_2)^{-1}} \dots b_s^{(u_1 u_2 \dots u_s)^{-1}} u_1 u_2 \dots u_{s+1}.$$

The equality holds in the free group generated by  $\mathcal{A} \cup \mathcal{T}$ . Note that  $u_1 \dots u_{s+1}$  is a word in  $F$ . It also has the property that the sum of exponents of each  $t_i$  is 0. We then write this word in the product of conjugates of  $\{[t_i, t_j]^{\pm 1}, i < j\}$  algorithmically in the following fashion: assume we already write  $u_1 \dots u_{s+1}$  as  $w_1 w_2$ , where  $w_1$  is a product of conjugates of  $\{[t_i, t_j]^{\pm 1}, i < j\}$  and  $w_2$  is a word in  $F$  such that the sum of exponents of each  $t_i$  is 0. Let  $t_i$  be the letter with the smallest indices among all letters in  $w_2$ . Then  $w_2$  can be written as  $w'_2 t_i^\varepsilon w''_2$ ,  $\varepsilon = \pm 1$  where  $w'_2$  does not contain any  $t_i^{\pm 1}$ . Then

$$w'_2 t_i^\varepsilon w''_2 = [t_i^\varepsilon, t_{j_1}^{\varepsilon_1}]^{(w'_2 t_{j_1}^{-\varepsilon_1})^{-1}} [t_i^\varepsilon, t_{j_2}^{\varepsilon_2}]^{(w'_2 t_{j_1}^{-\varepsilon_1} t_{j_2}^{-\varepsilon_2})^{-1}} \dots [t_i^\varepsilon, t_{j_l}^{\varepsilon_l}] t_i^\varepsilon w'_2 w''_2,$$

where  $w'_2 = t_{j_l}^{\varepsilon_l} \dots t_{j_1}^{\varepsilon_1}$ . Since the sum of exponent of  $t_i$  is 0, by repeating this process we can gather all  $t_i$  to the left and hence they will be canceled eventually. We end up with a word  $w_3 w_4$  where  $w_3$  is a product of conjugates of  $\{[t_i, t_j]^{\pm 1}, i < j\}$  and  $w_4$  is a word in  $F$  such that the sum of exponents of each  $t_i$  is 0 and of the length strictly less than  $w_2$ . Thus by repeating this algorithm, we are able to write  $g$  as a product of conjugates of  $\{[t_i, t_j]^{\pm 1}, i < j\}$  in a unique way. Now we just apply relations like  $[t_i, t_j] = a_{ij}$  to replace all the commutators by their corresponding letters in  $\mathcal{A}$ .

Since  $g$  can be written as a product of conjugates of elements in  $\mathcal{A}$ , applying commutator relations like  $[a, b^u], a, b \in \mathcal{A}, u \in F$ , we are able to commute those conjugates and hence gather all conjugates which share the same base. In addition, combining the fact  $a^u = a^{\bar{u}}$  from Proposition III.3.4 (a), we can write  $g$  in the ordered form of the following type

$$g = a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m},$$

where  $\mu_i \in \mathbb{Z}T$  and terms of  $\mu_i$  are written in the order from the high to low with respect to  $\prec$  which we define in Section IV.2. The result is a word satisfying all conditions of Definition V.2.1. Therefore the existence of the ordered form is shown.

The uniqueness of the ordered form can be justified by the fact that the set of words in the ordered form is isomorphic to the free  $T$ -module  $M$ . The isomorphism is given by the canonical map  $a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m} \mapsto \mu_1 a_1 + \mu_2 a_2 + \dots + \mu_m a_m$ .

For  $w$  in  $\langle\langle \mathcal{A} \rangle\rangle_{G_\infty}$  (or  $\langle\langle \mathcal{A} \rangle\rangle_G$ ), we define the ordered form by lifting  $w$  to  $H_\infty$ , that is, as the ordered form of  $\iota(w)$

where  $\iota : G_\infty \rightarrow H_\infty$  (resp.  $G \rightarrow H_\infty$ ) is the combinatorial map induced by identity on  $\mathcal{A} \cup \mathcal{T}$ . Note that by the way we define the ordered form, the ordered form of each word is unique. The ordered forms distinguish different elements in the  $T$ -module  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$ . In fact, two elements in  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  are equal in  $H_\infty$  if and only if they have the same ordered form. One remark is that two words which are equal in  $G$  or  $G_\infty$  may have different ordered forms, for example,  $a_1$  and  $\prod_{u \in F} (a^{\lambda(\hat{u})})^u$ ,  $\lambda \in C(A)$ .

Recall that  $G = \langle \mathcal{A} \cup \mathcal{T} \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \rangle$ . Note that both  $\mathcal{R}_2$  and  $\mathcal{R}_3$  are contained in the normal closure of  $\mathcal{A}$ . From now on we write all relators from  $\mathcal{R}_2 \cup \mathcal{R}_3$  in their ordered form.

### V.3 Main Lemmas

Before we embark on the proof of Proposition V.1.2, we shall establish some preliminary lemmas.

Now consider an arbitrary factor group  $H$  of  $G_\infty$  equipped with the presentation

$$H = \langle \mathcal{A} \cup \mathcal{T} \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R} \rangle \quad (\text{V.6})$$

where  $\mathcal{R}$  is a finite subset in  $G_\infty$ . Then  $H \cong G_\infty / \langle\langle \mathcal{R} \rangle\rangle_{G_\infty}$ . Note that if  $\mathcal{R} = \mathcal{R}_3$ ,  $H = G$ , if  $\mathcal{R} = \emptyset$ ,  $H = G_\infty$  which are two major examples we concern. We have following lemmas for  $H$ .

**Lemma V.3.1.** *Let  $H$  be a factor group of  $G_\infty$  equipped with presentation (V.6) and  $w$  be a word in  $(\mathcal{T} \cup \mathcal{T}^{-1})^*$  such that  $|w| = n$ , then*

$$w =_H \bar{w} \prod_{i=1}^p b_i^{u_i}$$

where  $p \leq n^2$ ,  $b_i \in \mathcal{A}^{\pm 1}$ ,  $u_i \in F$ ,  $\text{Tr}(\theta(u_i)) \subset B_n$ . In addition, the cost of converting LHS to RHS is bounded by  $n^2$ .

*Proof.* Since  $\bar{w} = t_1^{m_1} \dots t_k^{m_k}$  for some  $m_1, \dots, m_k \in \mathbb{Z}$  such that  $\sum_{i=1}^k |m_i| \leq n$ , to move each letter in  $w$  to the desired place, it will cost at most  $n$  commutators of the form  $[t_i, t_j]$ ,  $1 \leq i < j \leq k$ . By the discussion in Section II.3 and Section II.4, in total, we need at most  $n^2$  such commutators. That is,

$$w = \bar{w} \prod_{i=1}^p [t_{i_1}, t_{i_2}]^{\varepsilon_i u'_i}, \text{ where } u'_i \in F, p \leq n^2, 1 \leq i_1 < i_2 \leq k, \varepsilon_i \in \{\pm 1\}.$$

Moreover, since the length of  $w$  is bounded by  $n$ ,  $\text{Tr}(\theta(u'_i)) \leq n$ .

By applying relations in  $\{a_{ij} = [a_i, a_j] \mid 1 \leq i < j \leq n\}$   $p$  times we immediately have

$$w =_H \bar{w} \prod_{i=1}^p b_i^{u_i}, \text{Tr}(\theta(u_i)) \leq n.$$

The cost of relations is bounded by  $p \leq n^2$ . □

In particular, for  $w \in F$  such that  $\pi(w) = 1$ , it costs at most  $n^2$  relations in  $H$  to convert it to a product of conjugates of elements in  $\mathcal{A}$ .

**Lemma V.3.2.** *Let  $H$  be a factor group of  $G_\infty$  equipped with presentation (V.6) then there exists a constant  $K$  only depends on  $\mathcal{R}_1 \cup \mathcal{R}_2$  such that*

$$\text{Area}([a, b^u]) \leq K^n, \forall a, b \in \mathcal{A}, \|\theta(u)\| < n.$$

*Proof.* Let  $\mathcal{F} = \{\theta(\text{supp}(\lambda)) \mid \lambda \in \Lambda\}$  then  $\mathcal{F}$  is a finite collection of finite sets. By the choice of  $\Lambda$ ,  $\mathcal{F}$  satisfies the assumptions of Lemma III.3.1.

By Lemma III.3.1, each  $x \in B_{r+\varepsilon(r)}$  can be taken from  $B_r$  by  $\mathcal{F}$  for  $r > R$ . Recall that  $R$  is defined to be  $\max\{D, D^2/2C, D^2/(4kC-4)\}$  and  $\varepsilon(r) = C - D^2/2r$ , where  $C, D$  are purely determined by  $\Lambda$  hence  $\mathcal{R}_1 \cup \mathcal{R}_2$  as we stated in Lemma III.3.5.

According to our choice of  $R$ , we note that  $\varepsilon(r) \geq \varepsilon(2kD^2/(4kC-4)) = \frac{1}{2k}$  for  $r > R$ . Let  $K_1$  be the constant which is large enough such that  $f(n) \leq K_1^n$  for  $n \leq R$ , and  $K_2$  be the constant

$$K_2 := \max_{\lambda \in \Lambda} \left\{ \sum_{u \in \bar{F}} |\lambda(u)| \right\} + 2.$$

Since each  $\lambda$  has finite support,  $K_2$  is well-defined. Now let  $K := \max\{K_1, K_2^{2k}\}$ .

Suppose for  $n > R$ ,  $\text{Area}([a, b^u]) \leq K^n, \forall a, b \in \mathcal{A}, \|\theta(u)\| < n$ . We then prove our lemma by induction. Let us first consider the case  $r = n + \frac{1}{2k}$ . Fix some  $v \in \bar{F}$  satisfying  $\|\theta(v)\| < r$ . Since  $\varepsilon(n) \geq \frac{1}{2k}$ ,  $B_{n+\frac{1}{2k}}$  can be taken from  $B_n$  by  $\mathcal{F}$ . Then there is  $\lambda \in \Lambda$  with  $\theta(\text{supp}(\lambda \hat{v})) \subset B_n$  by the definition of ‘‘taken from’’.

Therefore we have two cases depending on  $\lambda \in C(A)$  or  $C(A^*)$ . Firstly assume that  $\lambda \in \Lambda \cap C(A)$ . Then by applying the commutator formula  $[x, yz] = [x, y]^{x^{-1}zx} [x, z]$ , we obtain

$$[a, b^v] =_G [a, \prod_{u \in \bar{F}} (b^{\lambda(\hat{u})})^{uv}] = \prod_{u \in \bar{F}} [a, b^{\lambda(\hat{u})uv}]^{h(u)},$$

where the  $h(u)$ 's are certain elements in  $H$  which need not concern us. Note that in the first equality above, we apply relations in (V.3) twice to replace  $b$  by  $\prod_{u \in \bar{F}} b^{\lambda(\hat{u})u}$ . Since  $\text{supp}(\lambda) \subset \bar{B}_D$  we have  $\|\theta(u)\| < D < \frac{n}{2k}$ . Additionally, we have  $\|\theta(v)\| < n + \frac{1}{2k}$  and  $\|\theta(uv)\| < n$ . It meets all assumptions of Proposition III.3.4 (c). Note that  $H$  is a factor group of  $H_n$  which we defined in Section III.3. Then  $[a, b^{\lambda(\hat{u})uv}]$  is conjugate in  $H$  to  $[a, b^{\lambda(\hat{u})\bar{u}v}]$ , the area of which is bounded by  $|\lambda(\bar{u})|K^n$ . It follows that

$$\text{Area}([a, b^v]) \leq 2 + \sum_{u \in \bar{F}} \text{Area}([a, b^{\lambda(\hat{u})uv}]) \leq 2 + \sum_{u \in \bar{F}} |\lambda(\hat{u})|K^n \leq K_2 K^n.$$

Repeating this process  $2k$  times, we obtain that

$$\text{Area}([a, b^v]) \leq K_2^{2k} K^n \leq K^{n+1}, \text{ for } v \in \bar{F}, \|\theta(v)\| < n+1.$$

If  $\lambda \in \Lambda \cap C(A^*)$ , the only different is that

$$[a, b^v] = [a^{v^{-1}}, b]^{v^{-1}} = \left[ \prod_{u \in \bar{F}} (a^{\lambda(\hat{u})})^{uv^{-1}}, b \right]^{v^{-1}}.$$

Similarly we obtain that

$$\text{Area}([a, b^v]) \leq K_2^{2k} K^n \leq K^{n+1}.$$

□

Furthermore, Lemma V.3.2 allows us to estimate the cost to commute two conjugates of elements in  $\mathcal{A}$ . Since the normal closure of  $\mathcal{A}$  in  $H$  is abelian, this lemma provides a tool to estimate the cost of converting words in  $\langle\langle \mathcal{A} \rangle\rangle_H$ , in particular, in  $G$ . Also Lemma V.3.2 reveals how much metabelianness costs in a finitely presented metabelian group. We will discuss this topic further in Section VI.1.

**Lemma V.3.3.** *Let  $H$  be a factor group of  $G_\infty$  equipped with presentation (V.6) and  $K$  be the same constant in Lemma V.3.2. Then in  $H$  we have*

$$\text{Area}(a^u a^{-\bar{u}}) \leq (2K)^n, \forall a \in \mathcal{A}, u \in F, \text{Tr}(u) \subset B_n$$

*Proof.* We prove it by an induction on  $n$ . Suppose for  $i \leq n$ , the result holds. Then for the case  $n+1$ , we write  $u = u' t_s^{\pm 1}$  then  $\text{Tr}(u) \subset B_{n+1}, \text{Tr}(u') \subset B_n$ .

$$a^u = a^{u' t_s^{\pm 1}} = (a^{\bar{u}'} t_s^{\pm 1}) v_1,$$

where  $\text{Area}(v_1) \leq (2K)^n$  by our inductive assumption. Write  $\bar{u}' = t_1^{m_1} \dots t_k^{m_k}$ , we claim that

$$\bar{u}' t_s^{\pm 1} = \bar{u} \prod_{j=1}^m c_j^{\alpha_j} \text{ where } c_j \in \{[t_s, t_l]^{\pm 1} \mid 1 \leq s < l \leq k\}, \alpha_j \in \bar{F}, m = \sum_{i=s+1}^k |m_i| \leq n.$$

We need to be really careful here. Let us first consider the case that the exponent of  $t_s$  is 1. We assume  $s < k$ , otherwise

it is trivial. Note that if  $m_k \geq 0$

$$\begin{aligned}
t_k^{m_k} t_s &= t_k^{m_k-1} t_s t_k [t_s, t_k]^{-1} = t_k^{m_k-2} t_s t_k^2 [t_s, t_k]^{-t_k} [t_s, t_k]^{-1} \\
&= t_k^{m_k-3} t_s t_k^3 [t_s, t_k]^{-t_k^2} [t_s, t_k]^{-t_k} [t_s, t_k]^{-1} \\
&\vdots \\
&= t_s t_k^{m_k} [t_s, t_k]^{-t_k^{m_k-1}} \dots [t_s, t_k]^{-t_k} [t_s, t_k]^{-1}.
\end{aligned}$$

If  $m_k < 0$ , we have

$$\begin{aligned}
t_k^{m_k} t_s &= t_k^{m_k+1} t_s t_k^{-1} [t_s, t_k]^{t_s} = t_k^{m_k+2} t_s t_k^{-2} [t_s, t_k]^{t_s t_k^{-1}} [t_s, t_k]^{t_s} \\
&= t_k^{m_k+3} t_s t_k^{-3} [t_s, t_k]^{t_s t_k^{-2}} [t_s, t_k]^{t_s t_k^{-1}} [t_s, t_k]^{t_s^{-1}} \\
&\vdots \\
&= t_s t_k^{m_k} [t_s, t_k]^{t_s t_k^{m_k+1}} \dots [t_s, t_k]^{t_s t_k^{-1}} [t_s, t_k]^{t_s}.
\end{aligned}$$

Repeating this process, we then prove the claim for the case that the exponent of  $t_s$  is 1.

On the other hand, if the exponent of  $t_s$  is  $-1$ , then similarly, consider if  $m_k \geq 0$

$$\begin{aligned}
t_k^{m_k} t_s^{-1} &= t_k^{m_k-1} t_s^{-1} t_k [t_s, t_k]^{t_k} = t_k^{m_k-2} t_s^{-1} t_k^2 [t_s, t_k]^{t_k^2} [t_s, t_k]^{t_s} \\
&= t_k^{m_k-3} t_s^{-1} t_k^3 [t_s, t_k]^{t_k^3} [t_s, t_k]^{t_k^2} [t_s, t_k]^{t_k} \\
&\vdots \\
&= t_s^{-1} t_k^{m_k} [t_s, t_k]^{t_k^{m_k-1}} \dots [t_s, t_k]^{t_k^2} [t_s, t_k]^{t_k},
\end{aligned}$$

and if  $m_k < 0$

$$\begin{aligned}
t_k^{m_k} t_s^{-1} &= t_k^{m_k+1} t_s^{-1} t_k^{-1} [t_s, t_k]^{-1} = t_k^{m_k+2} t_s^{-1} t_k^{-2} [t_s, t_k]^{-t_k^{-1}} [t_s, t_k]^{-1} \\
&= t_k^{m_k+3} t_s^{-1} t_k^{-3} [t_s, t_k]^{-t_k^{-2}} [t_s, t_k]^{-t_k^{-1}} [t_s, t_k]^{-1} \\
&\vdots \\
&= t_s^{-1} t_k^{m_k} [t_s, t_k]^{-t_k^{m_k+1}} \dots [t_s, t_k]^{-t_k^{-1}} [t_s, t_k]^{-1}.
\end{aligned}$$

Again by repeating this process, the claim is proved. Thus by induction on  $k$ , we can move  $t_s$  to the desired place.

Now we have

$$a^u = a^{\bar{u}t_s^{\pm 1}} v_1 = \left( \prod_{j=1}^m c_j^{\alpha_j} \right)^{-1} a^{\bar{u}} \left( \prod_{j=1}^m c_j^{\alpha_j} \right) v_1.$$

Apply relations from  $\{a_{ij} = [a_i, a_j] \mid 1 \leq i < j \leq n\}$   $2m$  times, we have that

$$a^u = \left( \prod_{j=1}^m d_j^{\alpha_j} \right)^{-1} a^{\bar{u}} \left( \prod_{j=1}^m d_j^{\alpha_j} \right) v_2 v_1$$

where  $d_j \in \mathcal{A}^{\pm 1}$  and  $\text{Area}(v_2) \leq 2m$  by our discussion.

Next we need to commute  $a^{\bar{u}}$  and  $d_j^{\alpha_j}$  for  $j = 1, \dots, m$  to the left and estimate the cost. Note that  $[a^{\bar{u}}, d_j^{\alpha_j}]$  is conjugate to  $[a, d_j^{\alpha_j(\bar{u})^{-1}}]$ . From the computation above,  $\alpha_j$  is either a tail of  $\bar{u}$  or a tail of  $\bar{u}$  multiplied by  $t_s^{\pm 1}$ . Therefore  $(\bar{u})^{-1}, \alpha_j, \alpha_j(\bar{u})^{-1}$  satisfy the assumption of Proposition III.3.4 (b). Thus  $[a, d_j^{\alpha_j(\bar{u})^{-1}}]$  is conjugate to  $[a, \overline{d_j^{\alpha_j u^{-1}}}]$ . Since  $\|\theta(\overline{\alpha_j u^{-1}})\| \leq n+1$ , the area of  $[a, \overline{d_j^{\alpha_j u^{-1}}}]$ , by Lemma V.3.2, is bounded by  $K^{n+1}$ .

Applying  $[a, \overline{d_j^{\alpha_j u^{-1}}}]$  to  $a^u$  and  $d_j^{\alpha_j}$  for  $j = 1, \dots, m$ , we can commute all  $d_j^{\alpha_j}$  to the left such that it cancels with  $d_j^{-\alpha_j}$ . Then we finally have

$$a^u = a^{\bar{u}} v_3 v_2 v_1,$$

where

$$\text{Area}(v_3) \leq mK^{n+1}$$

In total, the cost of converting  $a^u$  to  $a^{\bar{u}}$  is bounded by

$$\text{Area}(v_3 v_2 v_1) \leq \text{Area}(v_3) + \text{Area}(v_2) + \text{Area}(v_1) \leq (2K)^n + 2m + mK^{n+1} \leq (2K)^{n+1}.$$

Note that we use the fact that  $m \leq n$  and we can choose  $K \gg 1$ . □

Lemma V.3.3 provides a method for us to “organize” the exponent of a conjugate. In particular, combining all three lemmas introduced this section (Lemma V.3.1, Lemma V.3.2 and Lemma V.3.3), we are able to convert any word in  $\langle\langle \mathcal{A} \rangle\rangle_H$  to its ordered form. This forms the foundation of converting the word problem in group  $G$  to the membership problem of a submodule in the free  $T$ -module generated by  $\mathcal{A}$ .

#### V.4 The $T$ -module in Metabelian Groups

As we shown in Section V.2,  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  is a free  $T$ -module with the basis  $\mathcal{A}$ , where  $T$  acts on  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  by conjugation. Let  $\mathcal{A} = \{a_1, \dots, a_m\}$ . For each element  $g \in \langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$ , it can be written in its ordered form, i.e.

$$g = \prod_{i=1}^m a_1^{\lambda_1} a_2^{\lambda_2} \dots a_m^{\lambda_m} \in \mathbb{Z}T.$$

For  $\lambda_i$ , we always write its terms from high to low with respect to the order  $\prec$ . Then  $g$  can also be regarded as an element  $(\lambda_1, \dots, \lambda_m)$  in the free  $T$ -module with basis  $a_1, \dots, a_m$ . From now on, we treat an element in  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  as an element in group  $H_\infty$  as well as an element in the free  $T$ -module.

Let us first state the relation of operations between the group language and module language:

Group	Module
$(\prod_{i=1}^m a_i^{\lambda_i})(\prod_{i=1}^m a_i^{\lambda'_i}) =_{H_\infty} \prod_{i=1}^m a_i^{\lambda_i + \lambda'_i}$	$(\lambda_1, \dots, \lambda_m) + (\lambda'_1, \dots, \lambda'_m) = (\lambda_1 + \lambda'_1, \dots, \lambda_m + \lambda'_m)$
$(\prod_{i=1}^m a_i^{\lambda_i})^c =_{H_\infty} \prod_{i=1}^m a_i^{c\lambda_i}$	$c(\lambda_1, \dots, \lambda_m) = (c\lambda_1, \dots, c\lambda_m)$
$(\prod_{i=1}^m a_i^{\lambda_i})^t =_{H_\infty} \prod_{i=1}^m a_i^{t\lambda_i}$	$t(\lambda_1, \dots, \lambda_m) = (t\lambda_1, \dots, t\lambda_m)$

Table V.1: Operations in groups and modules

where  $c \in \mathbb{C}, t \in T$ .

Let  $\mathcal{X}$  be a subset of  $\langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$ . Then the normal closure of  $\mathcal{X}$  in group  $H_\infty$  coincides with the submodule generated by  $\mathcal{X}$  over  $\mathbb{Z}T$ . One direction is trivial, since by the table we have above, elements that lie in the submodule are obtained by the group operations and conjugations. Conversely, let  $g \in \langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$  then if  $h \in \mathbb{Z}T$ ,  $g^h$  can be obtained by finitely many scalar products and module operations and if  $h \in \langle\langle \mathcal{A} \rangle\rangle_{H_\infty}$ , then  $g^h = g$ . The general case is a combination of those two cases. Thus  $g^h$  must lie in the submodule generated by  $g$ . On the contrary, the subgroup generated by  $\mathcal{X}$  coincides with the submodule generated by  $\mathcal{X}$  over  $\mathbb{Z}$ .

Again we consider an arbitrary factor group  $H$  of  $G_\infty$  with the finite presentation

$$H = \langle \mathcal{A} \cup \mathcal{T} \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R} \rangle$$

where  $\mathcal{R}$  is a finite subset of  $G_\infty$ . Then  $H \cong G_\infty / \langle\langle \mathcal{R} \rangle\rangle$ . We now estimate the cost of relations in group  $H$  to make each of the module operations above. Note that notations like  $\deg(\lambda)$  and  $|\lambda|$  for element  $g \in \mathbb{Z}T$  are inherited from the polynomial ring  $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$  (See in Section IV.4).

In the following lemma,  $K$  is the same constant appeared in Lemma V.3.2, which only depends on  $\mathcal{R}_1 \cup \mathcal{R}_2$ .

**Lemma V.4.1.** *Let  $H$  be a factor group of  $G_\infty$  equipped with presentation (V.6) then we have*

(a) *Let*

$$f = \prod_{i=1}^m a_i^{\lambda_i}, g = \prod_{i=1}^m a_i^{\lambda'_i},$$

*and we denote  $P = \max\{|\lambda_i|, |\lambda'_i| \mid i = 1, \dots, m\}$ ,  $Q = \max\{\deg(\lambda_i), \deg(\lambda'_i) \mid i = 1, \dots, m\}$ . Then the cost of relations in  $H$  of converting*

$$fg =_H \prod_{i=1}^m a_i^{\lambda_i + \lambda'_i}$$

*is at most  $m^2 P^2 K^{2Q}$  where the right hand side is written in its ordered form.*



(b) Let

$$f = \prod_{i=1}^m a_i^{\lambda_i},$$

denote  $P = \max\{|\lambda_i| \mid i = 1, \dots, m\}$ ,  $Q = \max\{\deg(\lambda_i) \mid i = 1, \dots, m\}$ . For  $c \in \mathbb{Z}$  the cost of relations in  $H$  of converting  $f^c$  to  $\prod_{i=1}^m a_i^{c\lambda_i}$  is at most  $(|c| - 1)(m)^2 P^2 K^{2Q}$  where the right hand side is written in its ordered form.

(c) Let

$$f = \prod_{i=1}^m a_i^{\lambda_i},$$

denote  $P = \max\{|\lambda_i| \mid i = 1, \dots, m\}$ ,  $Q = \max\{\deg(\lambda_i) \mid i = 1, \dots, m\}$ . For  $t \in T$  the cost of relations in  $H$  of converting

$$\left(\prod_{i=1}^m a_i^{\lambda_i}\right)^t =_H \prod_{i=1}^m a_i^{t\lambda_i},$$

is bounded by  $(mP)(2K)^{k(Q+\deg t)}$ .

*Proof.* (a) First we consider a simpler case when  $g = a_1^{\lambda'_1}$ . Then it is essential to estimate the cost of converting LHS to RHS of

$$\left(\prod_{i=1}^m a_i^{\lambda_i}\right) a_1^{\lambda'_1} =_H (a_1^{\mu_1})(a_2^{\lambda_2} \dots a_m^{\lambda_m}), \mu_1 = \lambda_1 + \lambda'_1. \quad (\text{V.7})$$

In order to commute  $a_1^{\lambda'_1}$  with  $a_m^{\lambda_m}, \dots, a_2^{\lambda_2}$ , we apply Lemma V.3.2  $(m-1)$ -times. Each step costs at most  $PK^{2Q}$  since  $\deg(\lambda_i + \lambda'_1) \leq 2Q, |\lambda_i|, |\lambda'_1| \leq P$ . Therefore, the cost of

$$\left(\prod_{i=1}^m a_i^{\lambda_i}\right) a_1^{\lambda'_1} = (a_1^{\lambda_1} a_1^{\lambda'_1})(a_2^{\lambda_2} \dots a_m^{\lambda_m})$$

is bounded by  $(m-1)P^2 K^{2Q}$ . When it comes to the last step, i.e.,

$$a_1^{\lambda_1} a_1^{\lambda'_1} = a_1^{\mu_1},$$

the only thing we need to do is move each term  $a_1^u \in F$  to its position corresponding to  $\prec$ . We in fact sort all conjugates  $a_1^u$  in order. Note that those conjugates in  $a_1^{\lambda_1}$  and  $a_1^{\lambda'_1}$  are already in order, respectively. Thus we only need to insert each  $a_1^u$  of  $a_1^{\lambda'_1}$  into terms of  $a_1^{\lambda_1}$ . Again from Lemma V.3.2, the cost is bounded by  $P^2 K^{2Q}$ .

Therefore, the cost of (V.7) is bounded by  $mP^2 K^{2Q}$ .

In general, if  $g = \prod_{i=1}^m a_i^{\lambda'_i}$ . By repeating previous process  $m$  times, we get an upper bound  $m^2 P^2 K^{2Q}$ . We complete the proof.

(b) It follows by applying (a)  $|c| - 1$  times.

(c) Conjugating  $t$  to each term of  $a_1^{t'}$ ,  $t' \in T$ , cost zero relations. Then we basically estimate the cost of the following equation

$$a_1^{t'} = a_1^{\overline{t'}}.$$

By the result of Lemma V.3.3, since  $\text{Tr}(t't) \subset B_{(\deg t + \deg t')}$ , then the cost is bounded by  $(2K)^{(\deg t + \deg t')}$ . Notice  $\deg(t') \leq Q$ , then the total cost is at most

$$(mP)(2K)^{(\deg t + \deg t')} \leq (mP)(2K)^{(Q + \deg t)}.$$

Here we use the fact  $\text{Tr}(t) \subset B_{\deg(t)}$  since we order elements in  $\mathbb{Z}T$  degree lexicographically. □

Recall that  $G_\infty$  is a factor group of  $H_\infty$  as  $G$  is a factor group of  $G_\infty$ . Denote the epimorphism from  $H_\infty$  to  $G_\infty$  induced by identity on generating set as  $\psi$ , and then we have the following homomorphism chain:

$$H_\infty \xrightarrow{\psi} G_\infty \xrightarrow{\varphi} G.$$

Thus  $\ker \psi = \langle \langle \mathcal{R}_2 \rangle \rangle_{H_\infty}$ ,  $\ker(\varphi \circ \psi) = \langle \langle \mathcal{R}_2 \cup \mathcal{R}_3 \rangle \rangle_{H_\infty}$ . They are all normal subgroups in  $H_\infty$  as well as submodules in  $\langle \langle \mathcal{A} \rangle \rangle_{H_\infty}$ .  $H_\infty$  contains a free module structure while each of  $G$  and  $G_\infty$  contain a factor module of it. Eventually we will convert the word problem to a membership problem of a submodule in  $\langle \langle \mathcal{A} \rangle \rangle_{H_\infty}$ .

## V.5 Proof of Proposition V.1.2

Now we are ready to prove Proposition V.1.2. It is enough to show that for any given word  $w = 1$  of length  $n$ ,  $w$  can be written as a product of at most  $C^{n^{2k}}$  conjugates of relators for some constant  $C$ . Since  $G$  is a factor group of  $H_\infty$ ,  $w =_G 1$  if and only if  $w \in \ker(\varphi \circ \psi) = \langle \langle \mathcal{R}_2 \cup \mathcal{R}_3 \rangle \rangle_{H_\infty}$ . Note that  $\langle \langle \mathcal{R}_2 \cup \mathcal{R}_3 \rangle \rangle_{H_\infty} \subset \langle \langle \mathcal{A} \rangle \rangle_{H_\infty}$ . Recall that  $\langle \langle \mathcal{A} \rangle \rangle_{H_\infty}$  has a natural module structure: it is a free  $T$ -module with basis  $a_1, \dots, a_m$ . By previous discussion,  $\langle \langle \mathcal{R}_2 \cup \mathcal{R}_3 \rangle \rangle_{H_\infty}$  coincides the submodule generated by  $\mathcal{R}_2 \cup \mathcal{R}_3$  over  $\mathbb{Z}T$ . Let  $\mathcal{R}_4 = \{f_1, f_2, \dots, f_l\}$  be the Gröbner basis of the submodule generated by  $\mathcal{R}_2 \cup \mathcal{R}_3$ . We then add  $\mathcal{R}_4$  to our presentation (V.5), and in addition we assume that all relators of  $\mathcal{R}_i$ ,  $i = 2, 3, 4$  are written in their ordered form. Note that  $\mathcal{R}_2 \cup \mathcal{R}_3$  and  $\mathcal{R}_4$  generates the same submodule in  $\langle \langle \mathcal{A} \rangle \rangle_{H_\infty}$ . It implies that  $\langle \langle \mathcal{R}_2 \cup \mathcal{R}_3 \rangle \rangle_{H_\infty} = \langle \langle \mathcal{R}_4 \rangle \rangle_{H_\infty}$ . We obtained an alternating presentation of  $G$  as

$$G = \langle \mathcal{A} \cup \mathcal{T} \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \rangle. \tag{V.8}$$

Although  $\mathcal{R}_4$  is equivalent to  $\mathcal{R}_2 \cup \mathcal{R}_3$ , it is convenient to keep  $\mathcal{R}_2, \mathcal{R}_3$  in our presentation since all the estimation we have done previously are based on  $\mathcal{R}_2 \cup \mathcal{R}_3$ .

Note that  $G$  is a factor group of  $G_\infty$  and with the given presentation Lemma V.3.1, Lemma V.3.2, Lemma V.3.3 and Lemma V.4.1 all hold for  $G$ .

Since Dehn function is a quasi-isometric invariant then it enough for us to prove Proposition V.1.2 using the presentation (V.8).

*Proof of Proposition V.1.2.* We start with a word  $w \in G$  such that  $|w| = n, w =_G 1$ . WLOG we may assume

$$w = u_1 b_1 u_2 b_2 \dots u_s b_s u_{s+1}$$

where  $u_i \in F = F(\mathcal{T}), b_i \in \mathcal{A}^{\pm 1}$  and  $s + \sum_{i=1}^{s+1} |u_i| = n$ . Let  $v_i = (u_1 \dots u_i)^{-1}$  for  $i = 1, \dots, s$  and  $v = u_1 u_2 \dots u_{s+1}$ . Then we have

$$w = w_1 := b_1^{v_1} b_2^{v_2} \dots b_s^{v_s} v.$$

The equality holds in the free group generated by  $\mathcal{A} \cup \mathcal{T}$  thus the cost of relations converting  $w$  to  $w_1$  is 0. Since  $s + \sum_{i=1}^{s+1} |u_i| = n$ , in particular, we have that  $s \leq n$ . Moreover  $|v_i| = \sum_{j=1}^i |u_j| \leq n$  hence  $\text{Tr}(\theta(v_i)) \subset B_n, i = 1, 2, \dots, n$ . Next since  $w_1 =_G 1, \pi(w_1) = \pi(v_{s+1}) = 1$ . By Lemma V.3.1,

$$v = \prod_{i=s+1}^{s'} b_i^{v_i}$$

where  $s' - s \leq |v|^2 \leq n^2, b_i \in \mathcal{A}^{\pm 1}$ , and  $\text{Tr}(\theta(v_i)) \subset B_n, i = s+1, \dots, s'$ . By Lemma V.3.1, the cost of converting  $v$  to the right hand side is bounded by  $|v|^2 \leq n^2$ .

Thus we let

$$w_2 := \prod_{i=1}^{s'} b_i^{v_i}, s' \leq n^2 + n, \text{Tr}(\theta(v_i)) \subset B_n, i = 1, \dots, s'.$$

And the cost of converting  $w_2$  to  $w_1$  is bounded by  $n^2$ .

Next, note that all  $v_i$ 's are words in  $F$ . With the help of Lemma V.3.3, we are able to organize  $v_i$  to its image in  $\bar{F}$ . More precisely, we let

$$w_3 := \prod_{i=1}^{s'} b_i^{\bar{v}_i}, s' \leq n^2 + n, \|\theta(\bar{v}_i)\| \leq n.$$

Also followed by Lemma V.3.3,  $w_2 =_G w_3$ . Let us estimate the cost of converting  $w_2$  to  $w_3$ . To transform  $w_2$  to  $w_3$ , we need apply Lemma V.3.3 to each  $b_i^{v_i}$  once. Since  $\text{Area}(b_i^{v_i} b_i^{-\bar{v}_i}) \leq (2K)^n$  which provided by  $\text{Tr}(\theta(v_i)) \subset B_n$ , each transformation costs  $(2K)^n$  relations. We have in total  $s' \leq n + n^2$  many conjugates to convert therefore the cost is bounded by  $(n^2 + n)(2K)^n$ .

Now let  $w_4$  be the ordered form of  $w_3$ , which in fact is also the ordered form of  $w$ , i.e.

$$w_3 =_G w_4 := \prod_{i=1}^m a_i^{\mu_i}$$

where  $\mu_i$  are ordered under  $\prec$ . By the discussion in Section V.2, we obtain the ordered form just by rearranging all conjugates of  $\mathcal{A}^{\pm 1}$ . Note that because  $\|\theta(\bar{v}_i)\| \leq n$  for all  $i$ , it cost at most  $K^{2n}$  relations to commute any two consecutive conjugates  $b_i^{\bar{v}_i}$  and  $b_j^{\bar{v}_j}$  by Lemma V.3.2. To sort  $s'$  conjugates we need commute  $s'^2$  times. Therefore the number of relations need to commute  $w_3$  to  $w_4$  is bounded above that  $s'^2 K^{2n} \leq (n^2 + n)^2 K^{2n}$ .

The only thing remains is to compute the area of  $w_4$ . Recall that  $w_4$  can be regarded as an element in a free  $T$ -module generated by  $a_1, \dots, a_m$ .  $w_4 =_G 1$  implies that either  $w_4 =_{H_\infty} 1$  or it lies in the submodule generated by  $\mathcal{R}_4 = \{f_1, f_2, \dots, f_l\}$  which is the Gröbner basis of the submodule generated by  $\mathcal{R}_2 \cup \mathcal{R}_3$ . If  $w =_{H_\infty} 1$  then  $\mu_i = \emptyset$  for all  $i = 1, \dots, m$ . In this case  $\text{Area}(w_4) = 0$ . Thus

$$\text{Area}(w) \leq n^2 + (n^2 + n)(2K)^n + (n^2 + n)^2 K^{2n}.$$

We are done with this case.

Now let us consider the case  $w \in \langle \langle \mathcal{R}_4 \rangle \rangle_{H_\infty} \setminus \{1\}$ . Let  $K$  be a constant large enough to satisfy both Corollary IV.4.2 and Lemma V.3.2. As an element in the  $T$ -module,  $\deg w_4 \leq n$  since  $\|\theta(\bar{v}_i)\| \leq n$  for all  $i$ . Also recall that for an element  $\alpha \in \mathbb{Z}T$ ,  $|\alpha|$  is defined to be the  $l_1$ -norm of it regarded as a finite supported function from  $T$  to  $\mathbb{Z}$ . Thus  $|w_4|$  represents the number of conjugates in  $w_4$  which is  $s'$ . Then by Corollary IV.4.2 we have

$$w_4 =_{H_\infty} \prod_{i=1}^l f_i^{\alpha_i}, f_i = a_1^{\mu_{i1}} a_2^{\mu_{i2}} \dots a_m^{\mu_{im}} \in \mathcal{R}_4, \deg(f_i^{\alpha_i}) \leq n, \sum_{i=1}^l |\alpha_i| \leq s' K^{n^2} \leq (n^2 + n) K^{n^2}.$$

where  $\mu_i = \sum_{j=1}^l \alpha_j \mu_{ji}$  in  $\mathbb{Z}T$ . Note that  $f_i^{\alpha_i}$  is the product consisting of exactly  $|\alpha_i|$  many relators. In conclusion we have

$$\text{Area}\left(\prod_{i=1}^l f_i^{\alpha_i}\right) \leq \sum_{i=1}^l |\alpha_i| \leq (n^2 + n) K^{n^2}.$$

Last, let us estimate the cost of converting  $\prod_{i=1}^l f_i^{\alpha_i}$  to  $w_4$ . This process consists of two different steps: 1. converting all  $f_i^{\alpha_i}$ 's to their ordered form; 2. adding the  $l$  terms of ordered  $f_i^{\alpha_i}$ .

To transform  $f_i^{\alpha_i}$  to its ordered form, we write

$$\alpha_i = \sum_{u \in \text{supp } \alpha_i} \alpha_i(u) u.$$

Let us denote  $P = \max_{i=1}^l |f_i|, Q = \max_{i=1}^l \deg(f_i)$ . Then

$$f_i^{\alpha_i} = f_i^{\sum_{u \in \text{supp } \alpha_i} \alpha_i(u)u} = \prod_{\text{supp } \alpha_i} f_i^{\alpha_i(u)u} = \prod_{\text{supp } \alpha_i} f_{i,u} = a_1^{\mu'_1} a_2^{\mu'_2} \dots a_m^{\mu'_m} = \text{OF}(f_i), \quad (\text{V.9})$$

where  $f_{i,u}$  is the ordered form of  $f_i^{\alpha_i(u)u}$  and  $u'_{ij} = \alpha_i \mu_{ij}$  hence  $\text{OF}(f_i)$  is the ordered form of  $f_i^{\alpha_i}$ . The first two equalities above hold in the free group  $F(\mathcal{A} \cup \mathcal{T})$  thus the cost is 0. In the third equality, applying Lemma V.4.1 (b) and (c), the cost of converting  $f_i^{\alpha_i(u)u}$  to  $f_{i,u}$  is bounded by  $m|f_i|(2K)^{k(\deg f_i + \deg u)} + (|\alpha_i(u)| - 1)m^2|f_i|^2 K^{2(\deg f_i + \deg u)}$ . Here we first conjugate  $u$  to  $f_i$  then add  $|\alpha_i(u)|$  terms of  $f_i^u$ . Because  $\deg u \leq \deg \alpha_i, \sum_{u \in \text{supp } \alpha_i} |\alpha_i(u)| = |\alpha_i|, |\text{supp } \alpha_i| \leq |\alpha_i|$ . Consequently the cost of the third equality of (V.9) is bounded by

$$\begin{aligned} & \sum_{u \in \text{supp } \alpha_i} (m|f_i|(2K)^{k(\deg f_i + \deg u)} + (|\alpha_i(u)| - 1)m^2|f_i|^2 K^{2(\deg f_i + \deg u)}) \\ & \leq \sum_{u \in \text{supp } \alpha_i} (m|f_i|(2K)^{k(\deg f_i + \deg \alpha_i)} + (|\alpha_i(u)| - 1)m^2|f_i|^2 K^{2(\deg f_i + \deg \alpha_i)}) \\ & = |\text{supp } \alpha_i| m|f_i|(2K)^{k(\deg f_i + \deg \alpha_i)} + \sum_{u \in \text{supp } \alpha_i} (|\alpha_i(u)| - 1)m^2|f_i|^2 K^{2(\deg f_i + \deg \alpha_i)} \\ & = |\text{supp } \alpha_i| m|f_i|(2K)^{k(\deg f_i + \deg \alpha_i)} + (|\alpha_i| - |\text{supp } \alpha_i|)m^2|f_i|^2 K^{2(\deg f_i + \deg \alpha_i)} \\ & \leq |\alpha_i| m|f_i|(2K)^{k(\deg f_i + \deg \alpha_i)} + |\alpha_i| m^2|f_i|^2 K^{2(\deg f_i + \deg \alpha_i)} \\ & = |\alpha_i|(m|f_i|(2K)^{k(\deg f_i + \deg \alpha_i)} + m^2|f_i|^2 K^{2(\deg f_i + \deg \alpha_i)}) \\ & \leq |\alpha_i|(mP(2K)^{kn} + m^2 P^2 K^{2n}). \end{aligned}$$

The last inequality is obtained by the condition  $\deg(f_i^{\alpha_i}) \leq n$ , i.e  $\deg f_i + \deg \alpha_i \leq n$ .

The fourth equality of (V.9) is adding all  $f_{i,u}$ 's up. Since

$$\deg f_{i,u} \leq \deg f_i^{\alpha_i} \leq n, |f_{i,u}| \leq |\alpha_i(u)||f_i| \leq |\alpha_i||f_i| \leq |\alpha_i|P,$$

by Lemma V.4.1 (a), the cost of adding  $|\text{supp } \alpha_i|$  terms of  $f_{i,u}$  is bounded by  $(|\text{supp } \alpha_i| - 1)m^2(|\alpha_i|P)^2 K^{2n}$ . Here we use the fact that the size of the addition of any step is bounded by  $|f_i^{\alpha_i}|$ . Therefore the total number of relations we need to convert each  $f_i^{\alpha_i}$  to its order form  $f_i'$  is bounded by

$$\begin{aligned} & |\alpha_i|(mP(2K)^{kn} + m^2 P^2 K^{2n}) + (|\text{supp } \alpha_i| - 1)m^2(|\alpha_i|P)^2 K^{2n} \\ & \leq |\alpha_i|(mP(2K)^{kn} + (1 + |\alpha_i|^2)m^2 P^2 K^{2n}). \end{aligned}$$

In general, the cost of converting all  $f_i^{\alpha_i}$ 's to their order forms is bounded by

$$\begin{aligned}
& \sum_{i=1}^l |\alpha_i| (mP(2K)^{kn} + (1 + |\alpha_i|^2)m^2P^2K^{2n}) \\
&= (mP(2K)^{kn} + m^2P^2K^{2n}) \left( \sum_i |\alpha_i| \right) + m^2P^2K^{2n} \sum_{i=1}^l (|\alpha_i|^3) \\
&\leq (mP(2K)^{kn} + m^2P^2K^{2n})(n^2 + n)K^{n^{2k}} + m^2P^2K^{2n}(n^2 + n)^3K^{3n^{2k}}.
\end{aligned}$$

The next step, as described above, is to add all  $\text{OF}(f_i)$  up. We have that  $|\text{OF}(f_i)| \leq |\alpha_i||f_i|$  and  $\deg \text{OF}(f_i) \leq n$  for all  $i = 1, 2, \dots, l$ . Moreover, the size of any partial product  $\sum_{i=1}^{l'} \text{OF}(f_i)$ ,  $1 \leq l' \leq l$  is controlled by the following inequalities:

$$\left| \sum_{i=1}^{l'} \text{OF}(f_i) \right| \leq \sum_{i=1}^{l'} |\alpha_i||f_i| \leq P \sum_{i=1}^{l'} |\alpha_i| \leq P(n^2 + n)K^{n^{2k}}, \deg \left( \sum_{i=1}^{l'} \alpha_i f_i \right) \leq n.$$

This is similar to add  $f_{i,u}$ 's. By Lemma V.4.1 (a), the cost of the  $(|l| - 1)$  additions is bounded by

$$(l-1)m^2(P(n^2 + n)K^{n^{2k}})^2K^{2n} \leq (l-1)m^2(n^2 + n)^2P^2K^{2n^{2k}+2n}.$$

Now we need to verify the process of those steps above indeed result  $w_4$ . This is provided by the fact  $\mu_i = \sum_{j=1}^l \alpha_j \mu_{ji} = \sum_{j=1}^l \mu'_{ij}$  and eventually following Lemma V.4.1 we have

$$\prod_{i=1}^l f_i^{\alpha_i} = \prod_{i=1}^l f_i^{l'} = \prod_{i=1}^l (a_1^{\mu'_{i1}} a_2^{\mu'_{i2}} \dots a_m^{\mu'_{im}}) = \prod_{j=1}^m a_j^{\sum_{i=1}^l \mu'_{ij}} = a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m} = w_4.$$

By our estimation, the cost of the first equality is bounded by  $(mP(2K)^{kn} + m^2P^2K^{2n})(n^2 + n)K^{n^{2k}} + m^2P^2Q^{2n}(n^2 + n)^3K^{3n^{2k}}$  and the cost of the third equality is bounded by  $(l-1)m^2(n^2 + n)^2P^2K^{2n^{2k}+2n}$ . Other equalities hold in the free group hence no cost. Therefore

$$\begin{aligned}
\text{Area}(w_4) &= (n^2 + n)K^{n^{2k}} + (mP(2K)^{kn} + m^2P^2K^{2n})(n^2 + n)K^{n^{2k}} + m^2P^2K^{2n}(n^2 + n)^3K^{3n^{2k}} \\
&\quad + (l-1)m^2(n^2 + n)^2P^2K^{2n^{2k}+2n}.
\end{aligned}$$

Now we choose a constant  $C > K$  large enough such that

$$\begin{aligned}
& (n^2 + n)K^{n^{2k}} + (mP(2K)^{kn} + m^2P^2K^{2n})(n^2 + n)K^{n^{2k}} + m^2P^2K^{2n}(n^2 + n)^3K^{3n^{2k}} \\
&+ (l-1)m^2(n^2 + n)^2P^2K^{2n^{2k}+2n} \\
&\leq C^{n^{2k}}
\end{aligned}$$

It is clear that such  $C$  exists, for example we can choose  $C$  to be  $4m^2P^2QK$ . Note that  $P, Q$  only depends on  $f_1, f_2, \dots, f_l$ , hence  $\mathcal{R}_4$ , and so does  $K$ . Therefore  $C$  is independent of  $w$ .

In conclusion, we start with  $w =_G 1$  of length at most  $n$ . By converting it four times, we end up with a word  $w_4$ , of which area is bounded by  $C^{n^{2k}}$ . Thus

$$w \xrightarrow{0} w_1 \xrightarrow{\leq n^2} w_2 \xrightarrow{\leq (n+n^2)(2K)^n} w_3 \xrightarrow{\leq (n+n^2)^2 K^{2n}} w_4.$$

Summing up all the cost from  $w_1$  to  $w_4$  and with the fact  $C > K$ , we conclude that the area of  $w$  is bounded above by

$$\text{Area}(w) \leq C^{n^{2k}} + (n+n^2)^2 C^{2n} + (n+n^2)(2C)^n + n^2.$$

This completes the proof. □

## Chapter VI

### Relative Dehn Functions of Finitely Generated Metabelian Groups

#### VI.1 The Cost of Metabelianness

Metabelian groups are groups satisfying the identity  $[[x, y], [z, w]]$ . The metabelianness is provided by all relations of this form. In this chapter, we want to first estimate the area of an arbitrary metabelian relation, that is, a relation of the form  $[[x, y], [z, w]]$ , for a finitely presented metabelian group. This estimation gives us, what we call, the cost of metabelianness. Then we will “forget” the cost of metabelianness, i.e., we introduce the relative presentation in which all such relations have no cost. Also we shall note that all finitely generated metabelian group is relatively finitely presentable. Therefore we can extend Theorem V.1.1, modified for relative presentation, to all finitely generated metabelian group. Finally, we will estimate the relative Dehn function for some examples and prove the last piece of Theorem A.

First, we state an important consequence of Lemma V.3.2.

**Theorem VI.1.1.** *The metabelianness of a finitely presented metabelian group  $G$  costs at most exponentially many relations with respect to the length of the word, i.e. there exists a constant  $C$  such that*

$$\text{Area}([[x, y], [z, w]]) \leq C \cdot 2^{[[x, y], [z, w]]}, \forall x, y, z, w \in G.$$

*Proof.* Consider a finitely presented group  $G$  with a short exact sequence

$$1 \rightarrow A \hookrightarrow G \twoheadrightarrow T \rightarrow 1,$$

where  $A, T$  are abelian groups. Let  $\mathcal{A}$  be a generating set of  $A$  and  $\mathcal{T}$  be a set in  $G$  such that their image in  $T$  generates  $T$ .

Let  $\text{rk}(G) = k$ , the minimal torsion-free rank of an abelian group  $T$  such that there exists an abelian normal subgroup  $A$  in  $G$  satisfying  $G/A \cong T$ . The projection of  $G$  onto  $T$  is denoted by  $\pi : G \rightarrow T$ .

If  $k = 0$ ,  $G$  has a finitely generated abelian subgroup of finite index. Then the result follows immediately.

If  $k > 0$ , we first consider the case that  $T$  is free abelian. let  $\mathcal{T} = \{t_1, \dots, t_k\} \subset G$  such that  $\{\pi(t_1), \dots, \pi(t_k)\}$  forms a basis for  $T$  and  $\mathcal{A}$  be a finite subset of  $G$  such that it contains all commutators  $a_{ij} = [t_i, t_j]$  for  $1 \leq i < j \leq k$  and generates the  $T$ -module  $A$ . Then  $\mathcal{A} \cup \mathcal{T}$  is a finite generating set for the group  $G$ . Recall that  $G$  has a finite presentation as follows

$$G = \langle a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_k \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \rangle,$$



where

$$\begin{aligned}\mathcal{R}_1 &= \{[t_i, t_j] = a_{ij} \mid 1 \leq i < j \leq k\}; \\ \mathcal{R}_2 &= \{[a, b^u] = 1 \mid a, b \in \mathcal{A}, u \in \bar{F}, \|\theta(u)\| < R\}; \\ \mathcal{R}_3 &= \left\{ \prod_{u \in \bar{F}} (a^{\lambda(i)})^u = a \mid a \in \mathcal{A}, \lambda \in \Lambda \cap C(A) \right\} \cup \left\{ \prod_{u \in \bar{F}} (a^{\lambda(i)})^{u^{-1}} = a \mid a \in \mathcal{A}, \lambda \in \Lambda \cap C(A^*) \right\};\end{aligned}$$

and  $\mathcal{R}_4$  is the finite set generating  $\ker \varphi$ . All the notations are the same as in Section V.1.

By Lemma V.3.2, we have that there exists a constant  $C_1$  such that

$$\text{Area}[a, b^u] \leq C_1 \cdot 2^{|[a, b^u]|}, a, b \in \mathcal{A}, u \in F(\mathcal{T}).$$

Now let  $x, y, z, w$  be elements in  $G$  and  $n = |[x, y], [z, w]|$ . We use two commutator identities  $[a, bc] = [a, c][a, b]^c$  and  $[ab, c] = [a, c]^b[b, c]$  to decompose  $[x, y]$  and  $[z, w]$  into products of  $[a, b]^u$  where  $a, b \in \mathcal{A} \cup \mathcal{A}^{-1} \cup \mathcal{T} \cup \mathcal{T}^{-1}, u \in G$ . There are three cases to be considered.

1. If  $a, b \in \mathcal{A} \cup \mathcal{A}^{-1}$ ,  $[a, b]^u =_G 1$  and the cost for converting  $[a, b]^u$  to 1 is 1.
2. If  $a, b \in \mathcal{T} \cup \mathcal{T}^{-1}$ , we have two cases. If  $a, b \in \{t_i, t_i^{-1}\}$  for some  $i$ ,  $[a, b]^u =_G 1$  with no cost. If  $a \in \{t_i, t_i^{-1}\}$  and  $b \in \{t_j, t_j^{-1}\}$  where  $i \neq j$ ,  $[a, b]^u = c^{\varepsilon u'}$ , where  $c \in \mathcal{A} \cup \mathcal{A}^{-1}, \varepsilon \in \{\pm 1\}, |u'| \leq |u| + 1$ . This is due to cases like  $[t_i^{-1}, t_j] = [t_i, t_j]^{-t_i^{-1}}$ . The cost of converting  $[a, b]^u$  to  $c^{\varepsilon u'}$  is 1.
3. If  $a \in \mathcal{A} \cup \mathcal{A}^{-1}, b \in \mathcal{T} \cup \mathcal{T}^{-1}$  (or  $b \in \mathcal{A} \cup \mathcal{A}^{-1}, a \in \mathcal{T} \cup \mathcal{T}^{-1}$ ), then  $[a, b] =_G a a^b$  (resp.  $[a, b] = b^{a^{-1}} b$ ). Thus  $[a, b]^u =_G a^u a^{bu}$  (resp.  $[a, b]^u = b^{a^{-1}u} b^u$ ). The cost of converting is 0.

It follows that  $[x, y] =_G \prod_{i=1}^l b_i^{\varepsilon_i u_i}$ , where  $l \leq 2|x||y|, b_i \in \mathcal{A}, \varepsilon_i \in \{\pm 1\}, u_i \in G, |u_i| \leq |x| + |y|$ . The cost of converting  $[x, y]$  to  $\prod_{i=1}^l b_i^{\varepsilon_i u_i}$  is bounded by  $|x||y|$ .

Let  $u$  be a word in  $G$ , we claim that  $u = w_1 w_2 \prod_{i=1}^p c_i^{v_i}$  where  $w_1 \in F(\mathcal{A}), w_2 \in F(\mathcal{T}), c_i \in \mathcal{A} \cup \mathcal{A}^{-1}, v_i \in F(\mathcal{T})$ . The claim can be proved by always choosing to commute the left most pair of  $ta$  where  $t \in \mathcal{T} \cup \mathcal{T}^{-1}, a \in \mathcal{A} \cup \mathcal{A}^{-1}$ . Then for an element  $a \in \mathcal{A}$ , we have

$$a^u =_{F(\mathcal{A} \cup \mathcal{T})} a^{w_1 w_2 \prod_{i=1}^p c_i^{v_i}} =_G a^{w_2}, w_1 \in F(\mathcal{A}), w_2 \in F(\mathcal{T}).$$

Note that  $|w_2| < |u|$ . Similar to Lemma V.3.3, there exists a constant  $C_2$  such that the cost of the second equality in terms of relations is bounded  $C_2 \cdot 2^{|u|}$ .

Therefore  $[x, y] =_G \prod_{i=1}^l b_i^{\varepsilon_i u'_i}$ , where  $l \leq 2|x||y|$ ,  $b_i \in \mathcal{A}$ ,  $\varepsilon_i \in \{\pm 1\}$ ,  $u'_i \in F(\mathcal{T})$ . The cost is bounded by  $2n^2 C_2 2^n$ . Consequently,  $[[x, y], [z, w]]$  can be write as a product of at most  $8n^2$  conjugates of elements in  $\mathcal{A}$  at a cost of at most  $8n^2 C_2 2^n$ . Last, converting this product to 1 costs at most  $(8n^2)^2 C_1 2^n$  by Lemma V.3.2. The theorem is proved in this case.

If  $T$  is not free abelian, we suppose that  $\mathcal{T} = \{t_1, t_2, \dots, t_k, t_{k+1}, t_{k+2}, \dots, t_s\}$  such that the set  $\mathcal{T}_0 := \{\pi(t_{k+1}), \pi(t_{k+2}), \dots, \pi(t_s)\}$  generates a finite abelian group and  $\{\pi(t_1), \pi(t_2), \dots, \pi(t_k)\}$  generates  $\mathbb{Z}^k$ . Then we can write down a presentation of  $G$  as following:

$$G = \langle a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_s \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \rangle,$$

where

$$\mathcal{R}_1 = \{[t_i, t_j] = a_{ij}, t_l^{n_l} = a_l \mid 1 \leq i < j \leq s, k+1 \leq l \leq s\};$$

$$\mathcal{R}_2 = \{[a, b^u] = 1 \mid a, b \in \mathcal{A}, u \in \bar{F}, \|\theta(u)\| < R\};$$

$$\mathcal{R}_3 = \left\{ \prod_{u \in \bar{F}} (a^{\lambda(\hat{i})})^u = a \mid a \in \mathcal{A}, \lambda \in \Lambda \cap C(A) \right\} \cup \left\{ \prod_{u \in \bar{F}} (a^{\lambda(\hat{i})})^{u^{-1}} = a \mid a \in \mathcal{A}, \lambda \in \Lambda \cap C(A^*) \right\};$$

and  $\mathcal{R}_4$  is the finite set generating  $\ker \varphi$ . Note that  $\theta : \bar{F} \rightarrow \mathbb{R}^k$  kills all  $t_l, l > k$ . The rest of the proof is the same as the case when  $T$  is free abelian.  $\square$

## VI.2 The Relative Dehn Functions of Metabelian Groups

Recall that a set of groups form a *variety* if it is closed under subgroups, epimorphic images, and unrestricted direct products. The set of metabelian groups naturally form a variety, denoted by  $\mathcal{S}_2$ , since metabelian groups satisfy the identity  $[[x, y], [z, w]] = 1$ . Inside a variety, we can talk about relative free groups and relative presentations. Firstly, a metabelian group  $M_k$  is *free of rank k* if it satisfies the following universal property: every metabelian group generated by  $k$  elements is an epimorphic image of  $M_k$ . It is not hard to show that  $M_k \cong F(k)/F(k)''$ , where  $F(k)$  is a free group of rank  $k$  (in the variety of all groups).

Next, we shall discuss the relative presentations. Recall that the usual presentation of  $G$  consists of a free group  $F$  and a normal subgroup  $N$  such that  $G \cong F/N$ . For relative presentations, we shall replace the free group with the relative free group. Now let  $G$  be a metabelian group generated by  $k$  elements, then there exists a epimorphism  $\varphi : M_k \rightarrow G$ , where  $M_k$  is generated by  $X = \{x_1, x_2, \dots, x_k\}$ . We immediately have that  $G \cong M_k / \ker \varphi$ . Note that  $\ker \varphi$  is a normal subgroup of  $M_k$ , then it is a normal closure of a finite set. We let  $R = \{r_1, r_2, \dots, r_m\}$  to be the finite set whose normal closure is the kernel of  $\varphi$ . Therefore we obtain a *relative presentation* of  $G$  is the variety of metabelian

groups

$$G = \langle x_1, x_2, \dots, x_k \mid r_1, r_2, \dots, r_m \rangle_{\mathcal{S}_2}.$$

The notation  $\langle \cdot \rangle_{\mathcal{S}_2}$  is used to indicate that the presentation is relative to the variety of metabelian groups  $\mathcal{S}_2$ . Here, the subscript two stands for the derived length two. We denote by  $\mathcal{P}$  the relative presentation  $\langle X \mid R \rangle_{\mathcal{S}_2}$ . Note that if  $G$  is finitely presented, then the finite presentation in the usual sense is also a relative presentation, with some possible redundant relations.

Let us give an example of relation presentation of a metabelian group which is not finitely presented in the variety of all groups.  $H_\infty$ , the group we introduce in Section III.3, is a free metabelian group of rank  $k$ . It has two different relative presentations depending on how many generators we choose.

$$H_\infty = \langle t_1, t_2, \dots, t_k \rangle_{\mathcal{S}_2} = \langle a_{ij}, t_1, t_2, \dots, t_k \mid a_{ij} = [t_i, t_j], 1 \leq i < j \leq k \rangle_{\mathcal{S}_2}.$$

Back to a finitely generated metabelian group  $G$  with the finite relative presentation  $\langle X \mid R \rangle_{\mathcal{S}_2}$ . Let  $w$  be a word in  $G$  such that  $w =_G 1$ . Then  $w$  lies in the normal closure of  $R$ . Thus  $w$  can be written as

$$w =_{M_k} \prod_{i=1}^l r_i^{f_i} \text{ where } r_i \in R \cup R^{-1}, f_i \in M_k.$$

The smallest possible  $l$  is called the relative area of  $w$ , denoted by  $\widetilde{\text{Area}}_{\mathcal{P}}(w)$ . The difference between the area and the relative area is that we take the equality in different ambient groups, one in free groups and the other in free metabelian groups. Consequently, the Dehn function relative to the variety of metabelian groups with respect to the presentation  $\mathcal{P}$  is defined as

$$\tilde{\delta}_{\mathcal{P}}(n) = \sup\{\widetilde{\text{Area}}_{\mathcal{P}}(w) \mid |w|_X \leq n\}.$$

Here  $|\cdot|_X$  is the word length in alphabet  $X$ . Similar to usual Dehn functions, the relative Dehn functions are also independent of the choice of finite presentations up to equivalence, i.e.

**Proposition VI.2.1** ([15]). Let  $\mathcal{P}$  and  $\mathcal{Q}$  be finite relative presentations of the finitely generated metabelian group  $G$ . Then

$$\tilde{\delta}_{\mathcal{P}} \approx \tilde{\delta}_{\mathcal{Q}}.$$

Therefore it is valid to denote the relative Dehn function of a finitely generated metabelian group  $G$  by  $\tilde{\delta}_G$ . One remark is that every finitely generated metabelian group is finite presentable relative to the variety of metabelian groups. Thus the relative Dehn function can be defined for all finitely generated metabelian groups. Another remark is, unlike Dehn functions, it does not make sense to talk about if the relative Dehn function is a quasi-isometric

invariant. Since groups that quasi-isometric to a finitely generated metabelian group may not even be metabelian. For example,  $A_5 \wr \mathbb{Z}$  and  $\mathbb{Z}_{60} \wr \mathbb{Z}$  are quasi-isometric while the former is not metabelian.

But the relative Dehn function still inherit some nice properties from the Dehn function: one of which is that it is preserved under taking a finite index subgroup.

**Proposition VI.2.2.** Let  $G$  be a finitely generated metabelian group and  $H$  is a finite index subgroup of  $G$ . Then  $H$  is finitely generated and metabelian and the relative Dehn function of  $G$  and  $H$  are equal up to equivalence.

*Proof.* Since  $H$  is a finite index subgroup of a finitely generated metabelian group, it is finitely generated and metabelian.

Let  $\langle X \mid R \rangle_{\mathcal{S}_2}$  be a finite relative presentation of  $H$ , where  $X = \{x_1, x_2, \dots, x_n\}$ . Then we have a finite relative presentation of  $G$  as following:

$$G = \langle X \cup Y \mid R_0 \cup R_1 \cup R_2 \rangle_{\mathcal{S}_2},$$

where

$$Y = \{y_1, y_2, \dots, y_m\};$$

$$R_0 = R;$$

$$R_1 = \{y_i y_j = y_{f(i,j)} w_{i,j} y_i^{-1} = y_{g(i)} u_i\}, w_{i,j}, u_i \in (X \cup X^{-1})^*$$

$$f : \{1, 2, \dots, m\} \times \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, m\}, g : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, m\};$$

$$R_2 = \{x_l y_i = y_j v_{l,i}\}, v_{l,i} \in (X \cup X^{-1})^*.$$

We claim that there exists a constant  $L$  such that for every word  $w =_G 1$ , there exists a word  $w'$  such that  $w' = w, w' \in (X \cup X^{-1})^*$  and  $|w'| \leq L|w|$ . Moreover, it costs at most  $|w|$  relations from  $R_1 \cup R_2$  to convert  $w$  to  $w'$ .

If the claim is true, then we have that

$$\widetilde{\text{Area}}_G(w) \leq \widetilde{\text{Area}}_H(w') + |w|.$$

It immediately implies that

$$\tilde{\delta}_G(n) \leq \tilde{\delta}_H(Ln) + n.$$

Thus

$$\tilde{\delta}_G(n) \preccurlyeq \tilde{\delta}_H(n).$$

And the other direction  $\tilde{\delta}_H(n) \preccurlyeq \tilde{\delta}_G(n)$  is obvious since  $w_H = 1$  implies  $w_G = 1$ .

To prove the claim, we let  $L = \max\{|w_{i,j}|, |u_i|, |v_{l,i}| \mid 1 \leq i, j \leq m, 1 \leq l \leq n\}$ . Let  $w$  be a word such that  $w =_G 1$ .

WLOG, we assume that  $w$  has the following form:

$$w =_{F(X \cup Y)} a_1 b_1 a_2 b_2 \dots a_k b_k a_{k+1}, a_i \in (X \cup X^{-1})^*, b_i \in (Y \cup Y^{-1})^*,$$

where only  $a_1, a_{k+1}$  might be empty word. For  $b_k$ , using relations in  $R_1$  we have that

$$b_k = y_{h(k)} b'_k, h(k) \in \{1, 2, \dots, m\}, b_k \in (X \cup X^{-1})^*,$$

and  $|b'_k| \leq L|b_k|$ . Thus,

$$w = a_1 b_1 a_2 b_2 \dots a_k y_{h(k)} b'_k a_{k+1},$$

while the cost of converting is bounded by  $|b_k|$  and all relations are from  $R_1$ .

Next we commute  $y_{h(k)}$  with  $a_k$  using relations from  $R_2$ .

$$a_k y_{h(k)} = y_{h(k)} a'_k, a'_k \in (X \cup X^{-1})^*,$$

and  $|a'_k| \leq L|a_k|$ . Substituting it in, we get

$$w = a_1 b_1 a_2 b_2 \dots y_{h(k)} a'_k b'_k a_{k+1},$$

while the cost of converting is bounded by  $|a_k|$  and all relations are from  $R_2$ .

Therefore, repeating the above process, we eventually have

$$w = y_{h(1)} a'_1 b'_1 a'_2 b'_2 \dots a'_k b'_k a_{k+1}.$$

Since  $w = 1$ , thus  $y_{h(1)}$  is actually an empty word. Consequently, we have

$$w = a'_1 b'_1 a'_2 b'_2 \dots a'_k b'_k a_{k+1} \in (X \cup X^{-1})^*,$$

and the length of the left-hand side is controlled by

$$|a'_1 b'_1 a'_2 b'_2 \dots a'_k b'_k a_{k+1}| \leq \sum_{i=1}^k L(|a_i| + |b_i|) + |a_{k+1}| \leq L|w|.$$

The cost of relations is bounded by  $\sum_{i=1}^k (|a_i| + |b_i|) \leq |w|$  while all relations are from  $R_1 \cup R_2$ . The claim is proved.  $\square$

Let us consider one classic example: the Baumslag-Solitar group  $BS(1, 2)$ . The relative presentation is the same as

the usual presentation  $BS(1,2) = \langle a, t \mid a^t = a^2 \rangle_{\mathcal{S}_2}$ . But one can prove that the relative Dehn function of  $BS(1,2)$  is  $n$  instead of the usual Dehn function  $2^n$  [15]. In general, it is difficult to compute the relative Dehn function of a finitely generated metabelian group. We will list some known examples in Section VI.4.

So what is the connection between the relative Dehn function and Dehn function? On the surface, in the relative presentation, we make all metabelian relations cost 0, which should result in a significant reduce in the cost in Lemmas like Lemma V.3.2, Lemma V.4.1. In the next section, we will estimate how much cost we reduce by introducing the relative presentation.

### VI.3 Connections Between Dehn Functions and Relative Dehn Functions

The goal of the section is to prove the following theorem:

**Theorem VI.3.1.** *Let  $G$  be a finitely presented metabelian group. Then*

$$\tilde{\delta}_G(n) \preceq \delta_G(n) \preceq \max\{\tilde{\delta}_G^3(n^3), 2^n\}.$$

Before we prove the theorem, we have to introduce the third "Dehn function" in this thesis: the Dehn function of a finitely generated module (another definition can be found in [15]). Let  $R$  be the group ring  $\mathbb{Z}T$  where  $T$  is a free abelian group with basis  $\{t_1, t_2, \dots, t_k\}$  and  $A$  is a finitely generated  $R$ -module generated by  $m$  elements. Let  $M$  be a free  $R$ -module of rank  $m$ . Suppose a basis of  $M$  is  $\{a_1, a_2, \dots, a_m\}$ . Then there exists a submodule  $S$  of  $M$ , generated by  $\{f_1, f_2, \dots, f_l\}$ , such that  $A \cong M/S$ .

For an element  $f = \mu_1 a_1 + \mu_2 a_2 + \dots + \mu_m a_m$  in  $M$ , we define its length, denoted by  $\|f\|$ , to be the following:

$$\|f\| = \sum_{i=1}^l |\mu_i| + \text{reach}(f),$$

where  $\text{reach}(f)$  is the minimal length over the lengths of close loops that starts at 1 and passes through all points in  $\cup_{i=1}^l \text{supp } \mu_i$  in the Cayley graph of  $T$ . Another way to think of this length  $\|\cdot\|$  is that it is the minimal length of a group word among words that are rearranges of all conjugates of elements in  $\mathcal{A}$  in  $a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m}$ . For example, suppose  $m = k = 1$ , we have

$$\|(t_1^n + t_1^{n-1} + \dots + t_1 + 1)a_1\| = (n+1) + 2n = 3n+1,$$

because the minimal length of a loop passing  $\{1, t, t^2, \dots, t^n\}$  is  $2n$ . Note that  $a_1^{t_1^n + t_1^{n-1} + \dots + t_1 + 1} = t_1^{-n} a_1 t_1 a_1 \dots a_1 t_1 a_1$  is a group word of length  $3n+1$  in the alphabet  $\{a_1\} \cup \{t_1\}$ .

Then for every element  $f$  in  $S$ , there exists  $\alpha_1, \alpha_2, \dots, \alpha_l \in R$  such that

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_l f_l.$$

We denote by  $\widehat{\text{Area}}_A(f)$  the minimal possible  $\sum_{i=1}^l |\alpha_i|$ . Then *the Dehn function* of the  $R$ -module  $A$  is defined to be

$$\hat{\delta}_A(n) = \max\{\widehat{\text{Area}}_A(f) \mid \|f\| \leq n\}.$$

As expected, the Dehn function of a module is also independent from the choice of the finite presentation [15].

*Remark.* Now we have three different types of Dehn functions in this thesis: the Dehn function, the relative Dehn function and the Dehn function of a module. They are similar and we distinguish them by the notation. We denote by  $\delta_G(n), \text{Area}(w)$  the Dehn function of  $G$  and the area of a word  $w$ ;  $\tilde{\delta}_G(n), \tilde{\text{Area}}(w)$  the relative Dehn function and the relative area of a word  $w$ ,  $\hat{\delta}_A(n), \widehat{\text{Area}}(f)$  the Dehn function of the module  $A$  and the area of a module element  $f$ .

Let  $k = \text{rk}(G)$  be the minimal torsion-free rank of an abelian group  $T$  such that there exists an abelian normal subgroup  $A$  in  $G$  satisfying  $G/A \cong T$ .

First, if  $k > 0$  we notice that the problem can be reduced in the same way as Proposition V.1.2 does. Because for a finitely presented metabelian group  $G$  there exists a subgroup  $G_0$  of finite index such that  $G_0$  is an extension of an abelian group by a free abelian group of rank  $k$ . Most importantly, by Corollary II.1.3 and Proposition VI.2.2, their Dehn functions are equivalent as well as their relative Dehn functions. Therefore from now on, we assume that  $G$  is an extension of an abelian group  $A$  by a free abelian group  $T$ . The projection of  $G$  onto  $T$  is denoted by  $\pi : G \rightarrow T$ .

Let  $\mathcal{T} = \{t_1, \dots, t_k\} \subset G$  such that  $\{\pi(t_1), \dots, \pi(t_k)\}$  forms a basis for  $T$  and  $\mathcal{A}$  be a finite subset of  $G$  such that it contains all commutators  $a_{ij} = [t_i, t_j]$  for  $1 \leq i < j \leq k$  and generates the  $T$ -module  $A$ . Then  $\mathcal{A} \cup \mathcal{T}$  is a finite generating set for the group  $G$ .

Recall that  $G$  has a finite presentation as follows

$$G = \langle a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_k \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \rangle,$$

where

$$\mathcal{R}_1 = \{[t_i, t_j] = a_{ij} \mid 1 \leq i < j \leq k\};$$

$$\mathcal{R}_2 = \{[a, b^u] = 1 \mid a, b \in \mathcal{A}, u \in \bar{F}, \|\theta(u)\| < R\};$$

$$\mathcal{R}_3 = \left\{ \prod_{u \in \bar{F}} (a^{\lambda(i)})^u = a \mid a \in \mathcal{A}, \lambda \in \Lambda \cap C(A) \right\} \cup \left\{ \prod_{u \in \bar{F}} (a^{\lambda(i)})^{u^{-1}} = a \mid a \in \mathcal{A}, \lambda \in \Lambda \cap C(A^*) \right\};$$

and  $\mathcal{R}_4$  is the finite set that generates  $\ker \phi$  as a normal subgroup. All the notations are the same as in Section V.5.

Since we are dealing with relative Dehn function, we can reduce amount of redundant relations in  $\mathcal{R}_2$ . We set  $\mathcal{R}'_2 = \{[a, b] = 1, [a, b'] = 1 \mid a, b \in \mathcal{A}, t \in \mathcal{T}\}$ . Then we have

**Lemma VI.3.2.**  $\mathcal{R}'_2$  generates all commutative relations  $[a, b^u] = 1, a, b \in \mathcal{A}, u \in F(\mathcal{T})$  in the presentation relative to the variety of metabelian groups. Moreover, the relative area of  $[a, b^u]$  is bounded by  $4|u| - 3$ .

*Proof.* Suppose the result is proved for  $|u| \leq n$ , i.e.,  $[a, b^u] = 1$  can be written as a product of conjugates of words in  $\mathcal{R}'_2$  and metabelian relations. For metabelian relations, we mean those relations make commutators commute to each other. Note that the relative area of any metabelian relations is 0.

Now for that case  $|u| = n + 1$ , let  $u = vt, |v| = n, t \in \{t_1, t_2, \dots, t_k\}$ . By metabelian relations, we have that

$$1 = [a^{-1}a^t, b^{-t}b^u].$$

Since  $a^{-1}a^t = [a, t]$  and  $b^{-t}b^u = [b, v]^t$ . Then by inductive assumption, we are able to use relations like  $[a, b^w] = 1$  when  $a, b \in \mathcal{A}, |w| \leq n$ . In particular,  $[a, b^v] = 1$ .

And notice that

$$\begin{aligned} 1 = [a^{-1}a^t, b^{-t}b^u] &= \underbrace{a^{-t}a}_{\text{commute}} b^{-u}b^t \underbrace{a^{-1}a^t}_{\text{commute}} b^{-t}b^u \\ &= a \underbrace{a^{-t}b^{-u}}_{\text{commute}} \underbrace{b^t a^t}_{\text{commute}} a^{-1}b^{-t}b^u \\ &= ab^{-u}a^{-t}a^t \underbrace{b^t a^{-1}}_{\text{commute}} b^{-t}b^u \\ &= ab^{-u}a^{-1}b^u. \end{aligned}$$

This shows that  $[a, b^u]$  can be generated by  $\mathcal{R}'_2$  and metabelian relations. Let us count the cost. In the computation



above we use  $[a, b^v] = 1$  once (notice that  $[a^t, b^u] = [a, b^v]^t$ ),  $[a, a^t] = 1$  twice,  $[a, b] = 1$  once, and  $[a, b^t]$  once. Therefore,

$$\widetilde{\text{Area}}([a, b^u]) \leq \widetilde{\text{Area}}([a, b^v]) + 4 \leq 4(|v| + 1) - 3 = 4(n + 1) - 3.$$

This completes the proof.  $\square$

The lemma allows us to replace  $\mathcal{R}_2$  by  $\mathcal{R}_2^l$  in the relative presentation. And we immediately get the relative version of Lemma V.3.2.

**Lemma VI.3.3.** *Let  $u$  be a reduced word in  $F(\mathcal{T})$  and  $\bar{u}$  be the unique word in  $T$  representing  $u$  in the form of  $t_1^{m_1} t_2^{m_2} \dots t_k^{m_k}$ . Then we have*

$$\widetilde{\text{Area}}(a^{-u} a^{\bar{u}}) \leq 4|u|^2 + 2|u|.$$

*Proof.* The only difference of this proof to the proof of Lemma V.3.2 is that now it only costs  $4|u| - 3$  to commute conjugates of elements in  $\mathcal{A}$  every time.  $\square$

Thus in the relative sense, we save a lot of cost due to the fact we assume metabelianness is free of charge.

Our solution for the membership problem of a submodule has a better control for the degree of  $\alpha_i f_i$  than for  $\sum_{i=1}^l |\alpha_i|$ , resulting enormous upper bound for the area. When we consider  $\alpha_1, \alpha_2, \dots, \alpha_l$  that minimizes  $\sum_{i=1}^l |\alpha_i|$ , one trade-off is that we lose control of the degree of  $\alpha_i$ , sort of. The following lemma shows that in this case even though the degree of  $\alpha_i$  cannot be linearly controlled just by  $\deg f$  but can still be linearly controlled by both  $\deg f$  and  $\sum_{i=1}^l |\alpha_i|$ .

**Lemma VI.3.4.** *There exists a constant  $C$  such that for every  $f \in S$  where  $S$  is a  $T$ -submodule generated by  $\{f_1, f_2, \dots, f_l\}$ , assume that there exists  $\alpha_1, \alpha_2, \dots, \alpha_l \in \mathbb{Z}T$  such that*

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_l f_l,$$

*and  $\sum_{i=1}^l |\alpha_i|$  is minimized, then  $\deg \alpha_i f_i \leq \deg f + C \sum_{i=1}^l |\alpha_i|$  for all  $i$ .*

*Proof.* We denote by  $\Delta(f)$  the difference in the degree of the highest term and lowest term. Let

$$C := \max\{\Delta(f_1), \Delta(f_2), \dots, \Delta(f_l)\}.$$

In addition we let  $s = \sum_{i=1}^l |\alpha_i|$ .

We rewrite the sum  $\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_l f_l$  to the form

$$f = u_1 f_{i_1} + u_2 f_{i_2} + \dots + u_s f_{i_s},$$

where  $i_j \in \{1, 2, \dots, l\}$  and  $u_j$  is an element in  $\cup_{i=1}^l \text{supp } \alpha_i$ , such that  $\deg(u_j f_{i_j}) \geq \deg(u_{j+1} f_{i_{j+1}})$  for  $j = 1, 2, \dots, s-1$ .

Let  $g_n$  be the partial sum  $g_n = \sum_{j=1}^n u_j f_{i_j}$ . Assume that  $\deg(u_1 f_{i_1}) > \deg f$ , otherwise there is nothing to prove. Every term of degree greater than  $\deg f$  will be cancelled. Since we assume that  $\deg(u_j f_{i_j}) \geq \deg(u_{j+1} f_{i_{j+1}})$ , then  $\deg g_n \geq \deg g_{n+1}$  when  $\deg g_n > \deg f$ . We claim that  $\deg g_{n+1} \geq \deg g_n - Cn$  whenever  $\deg g_n > \deg f$ . If the claim is not true, let  $n_0$  be least number such that

$$\deg\left(\sum_{j=1}^{n_0} u_j f_{i_j}\right) \geq \deg(u_1 f_{i_1}) - C(n_0 - 1), \deg\left(\sum_{j=1}^{n_0+1} u_j f_{i_j}\right) \leq \deg(u_1 f_{i_1}) - Cn_0.$$

Note that  $\Delta(u_j f_{i_j}) \leq C$  for all  $j = 1, 2, \dots, s$ . The least degree among terms in  $g_{n_0}$  is greater than  $\deg(u_{n_0} f_{i_{n_0}}) - C$ . Since  $\deg(g_{n_0}) > \deg(g_{n_0+1})$ , then  $\deg(u_{n_0+1} f_{i_{n_0+1}}) = \deg(g_{n_0}) \geq \deg(u_1 f_{i_1}) - C(n_0 - 1)$ . Moreover, because  $\deg(u_{n_0} f_{i_{n_0}}) \geq \deg(u_{n_0+1} f_{i_{n_0+1}})$ , the least degree among terms in  $g_{n_0}$  is greater than  $\deg(u_1 f_{i_1}) - Cn_0$ . Therefore if  $g_{n_0+1} \neq 0$ , the least degree among terms in  $g_{n_0+1}$  is also greater than  $\deg(u_1 f_{i_1}) - Cn_0$ . It follows that  $g_{n_0+1} = 0$ . We have

$$f = \sum_{j=n_0+2}^s u_j f_{i_j}.$$

It is a contradiction to the minimality of  $s = \sum_{i=1}^l |\alpha_i|$ .

Let  $n_1$  be the largest number that  $\deg g_{n_1} > \deg f$ . By the claim, we have that  $\deg g_{n_1+1} \geq \deg(g_1) - Cn_1$ . Straight from the definition of  $n_1$ ,  $\deg g_{n_1+1} \leq \deg f$ . Combining those two inequalities, we finally have

$$\deg f \geq \deg(g_1) - Cn_1.$$

Since  $n_1 < s = \sum_{i=1}^l |\alpha_i|$  and  $\deg g_1 = \max_i \{\deg \alpha_i f_i\}$ , the lemma is proved.  $\square$

Next, we focus on the  $T$ -module  $A$ . It is not hard to see that  $A$  is the quotient of the free  $T$ -module generated by  $\mathcal{A}$  by the submodule generated by  $\mathcal{R}_3 \cup \mathcal{R}_4$ . We then replace  $\mathcal{R}_3 \cup \mathcal{R}_4$  by the Gröbner basis  $\mathcal{R}'_3 = \{f_1, f_2, \dots, f_l\}$  for the same submodule. Therefore we finally have the relative presentation of  $G$  we want:

$$G = \langle a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_k \mid \mathcal{R}_1 \cup \mathcal{R}'_2 \cup \mathcal{R}'_3 \rangle_{\mathcal{S}_2}.$$

We let  $M$  be the free  $T$ -module generated by  $\mathcal{A}$  and  $S$  be the submodule generated by  $\mathcal{R}'_3$  over  $T$ . So that  $A \cong M/S$ .

Then we have a connection between the relative Dehn function of  $G$  and the Dehn function of the submodule  $S$ .

**Lemma VI.3.5.** *Let  $G$  be a finitely generated metabelian group and  $A$  is defined as above, then*

$$\hat{\delta}_A(n) \asymp \tilde{\delta}_G(n) \asymp \max\{\hat{\delta}_A^3(n^3), n^6\}.$$

*Proof.* Now let  $w$  be a word of length  $n$  such that  $w =_G 1$ . We then estimate the cost of converting it to the ordered form. The process is exactly the same as in the proof of Proposition V.1.2. We replace the cost by the cost in relative presentation by Lemma VI.3.2 and Lemma VI.3.3. It is not hard to compute that it costs at most  $n^2 + (4n-3)(n^2+n) + (4n-3)^2(n^2+n)^2$  to convert  $w$  to its ordered form  $w' := \prod_{i=1}^m a_i^{\mu_i}$  where  $\sum_{i=1}^n |\mu_i| \leq n^2$ ,  $\deg \mu_i \leq n$  and  $|w'| \leq 2n^3$ . Since  $w'$  lies in the normal subgroup generated by  $\mathcal{R}'_3$ , then there exists  $\alpha_1, \alpha_2, \dots, \alpha_l$  such that

$$w' = \prod_{i=1}^l f_i^{\alpha_i}, \sum_{i=1}^l |\alpha_i| \leq \hat{\delta}_A(2n^3).$$

The relative area of the left hand side is less than  $\sum_{i=1}^l |\alpha_i|$ . Then we just repeat the same process of the proof of Proposition V.1.2, and compute the cost of adding  $f_i^{\alpha_i}$  up to  $w'$ . By Lemma VI.3.4,  $\deg(\alpha_i f_i) \leq n + C\hat{\delta}_A(2n^3)$  for every  $i$  and some constant  $C$ . It follows that, by Lemma VI.3.3, conjugating  $\alpha_i$  to  $f_i$  costs at most  $|\alpha_i||f_i|(4\deg^2(\alpha_i f_i) + 2\deg(\alpha_i f_i))$ . Last, we rearrange  $\sum_{i=1}^l |\alpha_i||f_i|$  terms whose degree are at most  $n + C\hat{\delta}_A(2n^3)$ , which costs at most  $\max\{\hat{\delta}_A^3(n^3), n^2\hat{\delta}_A(n^3)\}$  up to equivalence. Thus the relative area of  $w'$  is asymptotically bounded by  $\max\{\hat{\delta}_A^3(n^3), n^2\hat{\delta}_A(n^3)\}$  up to equivalence. And hence the relative area of  $w$  is bounded by  $\max\{\delta_A^3(n^3), n^6\}$ . Thus the right inequality is proved.

For the left inequality in the statement, let  $\prod_{i=1}^m a_i^{\mu_i}$  be a word of ordered form such that it realizes  $\delta_A(n)$ . The length of the word, by definition, is bounded by  $n$ . We claim that the relative area of  $\prod_{i=1}^m a_i^{\mu_i}$  is greater than  $\delta_A(n)$ . If not, by the definition of the relative area, we have that

$$\prod_{i=1}^m a_i^{\mu_i} = \prod_{j=1}^s r_j^{h_j}, r_j \in \mathcal{R}'_1 \cup \mathcal{R}'_2 \cup \mathcal{R}'_3, h_j \in M_{m+k},$$

where  $s = \text{Area}(\prod_{i=1}^m a_i^{\mu_i}) < \delta_S(n)$ . If we only keep all relations from  $\mathcal{R}'_3$  and combine the same relations together, we will get  $\prod_{i=1}^l f_i^{\alpha_i}$  and  $\sum_{i=1}^l |\alpha_i| \leq s < \delta_A(n)$ . Since canceling relations like  $[t_i, t_j] = a_{ij}$ ,  $[a, b'] = 1$  and commuting  $f_i^{h_j}$ 's do not change the value of left hand side as an element in free  $T$ -module generated by basis  $\{a_1, a_2, \dots, a_m\}$ . Therefore we eventually get

$$\sum_{i=1}^m \mu_i a_i = \sum_{j=1}^l \alpha_j f_j, \sum_{i=1}^l |\alpha_i| < \delta_A(n).$$

It leads to a contradiction. □

We have

**Theorem VI.3.6.** *Let  $G$  be a finitely generated metabelian group. Let  $k = rk(G)$ , the minimal torsion-free rank of an abelian group  $T$  such that there exists an abelian normal subgroup  $A$  in  $G$  satisfying  $G/A \cong T$ .*

*Then the relative Dehn function of  $G$  is asymptotically bounded above by*

(1)  $n^2$  if  $k = 0$ ;

(2)  $2^{n^{2k}}$  if  $k > 0$ .

*Proof.* Let  $G$  be a finitely generated metabelian group. If  $k = 0$ ,  $G$  has a finitely generated abelian subgroup of finite index. Then the relative Dehn function is asymptotically bounded by  $n^2$  by Theorem II.4.2.

If  $k > 0$ , similarly, we can reduce the case to that  $G$  is an extension of a module  $A$  by a free abelian group  $T$  such that the torsion-free rank of  $T$  is  $k$ . Then a word  $w =_G 1$  with  $|w| \leq n$  can be converted to its ordered form  $w' := \prod_{i=1}^m a_i^{\mu_i}$  where  $|w| \leq 2n^3$ ,  $\deg(w) \leq n$ ,  $\sum_{i=1}^m |\mu_i| \leq n^2$ . Then by Corollary IV.4.2, there exists a word  $w''$  such that  $w' =_G w''$ ,  $\text{Area}(w'') \leq 2^{n^{2k}}$ . The theorem follows immediately.  $\square$

Finally, we have all the ingredients to prove Theorem VI.3.1.

*Proof of Theorem VI.3.1.* The left inequality is obvious since the finite presentation of  $G$  is also the relative finite presentation of  $G$ .

If  $k = 0$ ,  $G$  has a finitely generated abelian subgroup of finite index. The result follows immediately.

If  $k > 0$ , let  $w$  be a word of length  $n$  and  $w =_G 1$ . Then there exists  $\alpha_1, \alpha_2, \dots, \alpha_l$

$$w = \prod_{i=1}^l f_i^{\alpha_i}, \sum_{i=1}^l |\alpha_i| \leq \hat{\delta}_A(2n^3), \deg \alpha_i \leq n + C\hat{\delta}_A(2n^3). \quad (\text{VI.1})$$

According to the proof of Proposition V.1.2, adding the left hand side of (VI.1) costs at most  $\max\{\hat{\delta}_A^3(2n^3), 2^n\}$  up to equivalence. All other steps of converting cost at most exponential with respect to  $n$ . Then by the left inequality in Lemma VI.3.5,  $\text{Area}(w) \leq \max\{\tilde{\delta}_G^3(n^3), 2^n\}$ . Therefore the theorem is proved.  $\square$

#### VI.4 Estimate the Relative Dehn Function

Computing the relative Dehn function is harder than computing the Dehn function. Many techniques no longer useful for the relative case. For the variety of metabelian groups, fortunately, the structure of groups in it is not complicated. The key is to understand the natural module structure of a finitely generated metabelian group.

First, let us list some known results for relative Dehn functions, they are computed by Fuh in her thesis. Note that most of them only give the upper bound of the relative Dehn function.

**Theorem VI.4.1** (Fuh [15]). (1) *The relative Dehn function of a wreath product of two finitely generated abelian groups is polynomially bounded.*

(2) *The Baumslag-Solitar group  $BS(1,2)$  has linear Dehn function.*

(3) *Let  $G = \tilde{BS}(n,m) = \langle a, t \mid (a^n)^t = a^m \rangle_{\mathcal{F}_2}$  where  $m > 2, m = n + 1$ . Then  $\tilde{\delta}_G(n) \preccurlyeq n^3$ .*

With the technique we developed, we can improve [15, Theorem E] to the following.

**Proposition VI.4.2.** Let  $T$  be a finitely generated abelian group and let  $A$  be a finitely generated  $T$ -module. Form the semidirect product

$$G = A \rtimes T.$$

Then  $\delta_G(n) \leq \max\{n^3, \hat{\delta}_A^3(n^2)\}$ .

*Proof.* It is not hard to reduce the problem to the case when  $T$  is free abelian. Thus we just assume that  $T$  is a finitely generated free abelian group. Suppose  $\mathcal{T} = \{t_1, t_2, \dots, t_k\}$  is a basis of  $T$  and  $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$  generates the module  $A$  over  $\mathbb{Z}T$ . Let  $M$  be the free  $T$ -module generated by  $\mathcal{A}$  and  $S$  be a submodule of  $M$  generated by  $f_1, f_2, \dots, f_l$ , where  $f_i = \sum_{j=1}^m \alpha_{i,j} a_j$  for  $1 \leq i \leq l$ ,  $\alpha_{i,j} \in \mathbb{Z}T$ . Then we can write down a presentation of  $G$  as follows

$$G = \langle a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_m \mid [t_i, t_j] = 1 (1 \leq i < j \leq k), \\ [a_i, a_j^w] = 1 (1 \leq i < j \leq m, w \in \mathbb{Z}T), \prod_{j=1}^m a_j^{\alpha_{i,j}} = 1 (1 \leq i \leq l) \rangle.$$

Then, by the same discussion as in Section VI.3, we have a finite relative presentation of  $G$ :

$$G = \langle a_1, a_2, \dots, a_m, t_1, t_2, \dots, t_m \mid [t_i, t_j] = 1 (1 \leq i < j \leq k), \\ [a_i, a_j] = 1, [a_i, a_j^{t_s}] = 1 (1 \leq i < j \leq m, 1 \leq s \leq k), \prod_{j=1}^m a_j^{\alpha_{i,j}} = 1 (1 \leq i \leq l) \rangle_{\mathcal{S}_2}.$$

Now let  $w =_G 1$  and  $|w| \leq n$ . Since in this case all  $t_i, t_j$  commutes, it is much easier than the general case. Following the same process as in the proof of Proposition V.1.2,  $w$  can be convert to its ordered form  $a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m}$ , where  $\deg(\mu_i) < n, \sum_{i=1}^m |\mu_i| \leq n$ . The cost is bounded by  $n^3$ . Notice that the length of  $a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m}$  is bounded by  $n^2$ . Then there exists  $\alpha_1, \alpha_2, \dots, \alpha_l \in \mathbb{Z}T$  such that

$$a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m} = \prod_{i=1}^l f_i^{\alpha_i}, \sum_{i=1}^l |\alpha_i| \leq \hat{\delta}_A(n^2).$$

The rest of the proof is the same as the proof of Lemma VI.3.5, since it is just a special case of Lemma VI.3.5.  $\square$

Now let us estimate the relative Dehn function from above for some concrete examples.

To begin with, we consider the metabelianized Baumslag-Solitar group

$$\tilde{BS}(n, m) = \langle a, t \mid (a^n)^t = a^m \rangle_{\mathcal{S}_2}.$$

The normal subgroup generated by  $a$  is a  $\mathbb{Z}\langle t \rangle$ -module. In this case, i.e., when the module is over the Laurent polynomial of one variable and is generated by one variable, the Dehn function of the module is well-studied. The following

theorem from Davis and Olshanskiy [12] shows that the Dehn function of a  $\langle t \rangle$ -module is a polynomial.

**Theorem VI.4.3** (Davis, Olshanskiy [12, Theorem 8.6]). *Let  $M = \langle a \rangle$  is the free module of rank one over the group ring  $\mathbb{Z}\langle t \rangle$ . Let  $f = h(t)a$  where  $h(x)$  is a polynomial of the form  $d_n x^n + d_{n-1} x^{n-1} + \dots + d_0$ . Then the Dehn function of the  $\langle t \rangle$ -module  $M/\langle f \rangle$  is a polynomial. Furthermore, the degree of this polynomial is exactly one plus the maximal multiplicity of a (complex) root of  $h(x)$  having modulus one.*

Thus we have

**Proposition VI.4.4.** The metabelianized Baumslag-Solitar group  $\tilde{BS}(n, m) = \langle a, t \mid (a^n)^t = a^m \rangle_{\mathcal{S}_2}$  has at most cubic relative Dehn function when  $n \neq m$  and has at most quartic relative Dehn function when  $n = m$ .

*Proof.* We follow the same process as in Lemma VI.3.5. Note that in this case we have  $|\mathcal{A}| = |\mathcal{T}| = 1$ , which simplifies the process a lot. Give a word  $w =_G 1$  of length  $l$ . It is not hard to check that converting  $w$  to  $OF(w)$  costs at most  $(4l - 3)l^2$ . Suppose  $OF(w) = a^\mu$ , where  $|\mu| \leq l, \deg \mu \leq l$ . We can conjugate  $w$  by  $t^l$  such that  $\mu$  only have positive powers of  $t$ . Thus we assume that  $|\mu| \leq l, \deg \mu \leq 2l$ . Further, the length of  $\mu$  is bounded by  $l$  by definition.

In this case, the module  $A$  is isomorphic to  $M/S$  where  $M$  is a free  $T$ -module with basis  $a$  and  $S$  is its submodule generated by  $\{(nt - m)a\}$ . Consider the polynomial ring  $R = \mathbb{Z}[t, t^{-1}]$  and its ideal  $I = \langle nt - m, tt^{-1} - 1 \rangle$ . We have that  $A \cong R/I$ . The Gröbner basis of  $I$  is  $\{tt^{-1} - 1, nt - m, mt^{-1} - n\}$ . If we regard  $\mu$  as an element in  $I$ , it can only be reduced by  $nt - m$  since it only has positive power of  $t$ . It follows that there exists a polynomial  $v$ , which only consists of the power of  $t$ , such that

$$\mu = (nt - m)v.$$

This equality also holds in the polynomial ring  $\mathbb{Z}[t]$ . When  $n \neq m$ , the Dehn function of  $\langle t \rangle$ -module  $\mathbb{Z}T/\langle nt - m \rangle$  is linear, by Theorem VI.4.3. Thus there exists  $C$  such that  $|v| \leq C\|\mu\| + C$ . We have that

$$a^\mu =_G (a^{nt-n})^v.$$

The area of the right hand side is at most  $Cl + C$ . Converting the right hand side to its ordered form costs at most  $(4l - 3)((m + n)(Cl + C))^2$  since the degree is less than  $l$  and we have  $(m + n)(Cl + C)$  many conjugates to rearrange. Thus the upper bound of  $\widetilde{\text{Area}}(w)$  is at most  $l^3$  up to equivalence when  $n \neq m$ .

When  $n = m$ , the Dehn function of  $\langle t \rangle$ -module  $\mathbb{Z}T/\langle nt - m \rangle$  is quadratic. Following the same process, we have that the upper bound of  $\widetilde{\text{Area}}(w)$  is at most  $l^4$  up to equivalence when  $n \neq m$ . This finishes the proof.  $\square$

For the case  $n = 1$ , the group  $\tilde{BS}(1, n) \cong BS(1, n)$  is finitely presented. Following from Theorem VI.3.1, the Dehn function of  $BS(1, n)$  is at most exponential.

One special case Fuh [15, Theorem 6.1] concerned is when  $m > 2, m = n + 1$ . In this case, we have that  $a = [a^n, t]$ . Since  $a$  itself is a commutator, it follows that the relative area of words like  $[a^k, a]$  is at most 4 instead of linearly depending on  $k$ . Therefore we can improve the result in [15, Theorem 6.1] by the following corollary of Proposition VI.4.4.

**Corollary VI.4.5.** *The metabelianized Baumslag-Solitar group  $\tilde{BS}(n, m) = \langle a, t \mid (a^n)^t = a^m \rangle_{\mathcal{S}_2}, m > 2, m = n + 1$  has at most quadratic relative Dehn function.*

The lamplighter groups are another interesting class of infinite presented metabelian groups with a simple module structure. We have

**Proposition VI.4.6.** The lamplighter groups have at most cubic relative Dehn function.

*Proof.* Consider the lamplighter group  $L_m$  with the standard presentation.

$$L_m = \langle a, t \mid a^m = 1, [a, a^t] = 1, n \in \mathbb{N} \rangle.$$

By Lemma VI.3.3, we have a finite relative presentation as the following

$$L_m = \langle a, t \mid a^m = 1, [a, a^t] = 1 \rangle_{\mathcal{S}_2}.$$

The rest of the proof is the same as the proof of Proposition VI.4.4. The only difference is that the submodule is generated by  $\{m\}$ . □

This slightly improves the estimation in [15, Theorem B2].

## VI.5 Relative Dehn Function of the Lamplighter Group $L_2$

In Section VI.3 and Section VI.4, we estimate the relative Dehn function for various groups using the Dehn function of the canonical modules in those groups. Though this method works for any finitely generated group, the upper bound and lower bound given by this technique are usually different. Thus to compute the precise relative Dehn function (up to equivalence) of a finitely generated metabelian group, we need a better method to estimate, which usually means that we lose some universality. In this section, the technique we show, even though is very interesting, only works for the lamplighter group  $L_2$ .

The goal of this section is to prove the following proposition.

**Proposition VI.5.1.** The lamplighter groups  $L_2$  has linear relative Dehn function.

*Proof.* The linear lower bound is given by Theorem VI.4.3.

We choose the following relative presentation of  $L_2$ :

$$L_2 = \langle a, t \mid a^2 = 1, [a, a^t] = 1 \rangle_{\mathcal{S}_2}.$$

For the upper bound, consider a word  $w \in L_2$  that represents the identity. Thus  $w$  has the form

$$w = t^{n_1} a t^{n_2} a^{n_3} \dots t^{n_{2k}} a t^{n_{2k+1}}, \text{ where } n_2, n_3, \dots, n_{2k} \neq 0.$$

Suppose the length of  $w$  is  $n$ , combining the fact that  $w = 1$ , we have

$$2k + \sum_{i=1}^{2k+1} |n_i| = n, \sum_{i=1}^{2k+1} n_i = 0.$$

Inserting  $tt^{-1}$  or  $t^{-1}t$ , we can rewrite  $w$  as the following form:

$$w = a^{t^{-n_1}} a^{t^{-(n_1+n_2)}} \dots a^{t^{-(n_1+n_2+\dots+n_{2k})}}.$$

Thus  $w$  represents an element in  $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2$ , where the  $a^{t^i}$  is the generator of the  $i$ -th copy of  $\mathbb{Z}_2$ . Since  $w = 1$ , then every element in the set  $\{-n_1, -(n_1+n_2), \dots, -(n_1+n_2+\dots+n_{2k})\}$  occurs even many times in the sequence  $-n_1, -(n_1+n_2), \dots, -(n_1+n_2+\dots+n_{2k})$ . Our goal is to gather the conjugates of  $a$  of the same exponents together at a linear cost with respect to  $n$ .

Since  $a^{-1} = a$ , we notice that

$$a^s a^l = (a a^{t^{l-s}})^{t^s} = [a, t^{l-s}]^{t^s}, l, s \in \mathbb{Z}.$$

Thus any consecutive pair of two conjugates of  $a$  is a commutator. It follows that any such pair commutes with any other pair of this form without any cost inside the variety of metabelian groups.

For convenience, let  $m_i = \sum_{j=1}^{2k} -n_j$ . We now turn the problem of estimating the relative area of  $w$  to a problem of cancelling numbers in a sequence and estimate the cost. Consider a sequence of number

$$m_1, m_2, \dots, m_{2k}.$$

The goal is to cancel all the pairs of the same value. We have three operations allowed:

- (i) Cancel two consecutive numbers without any cost.
- (ii) Commute a pair of consecutive numbers with another pair of consecutive numbers next to it without any cost.
- (iii) Commute two consecutive numbers  $c, d$  with a cost of  $|c - d|$ .



Applying all three operations to the original sequence many times, the result might seem chaotic. To analyze the process, for a sequence of numbers, we define the  $\iota(m_i)$  to be the position of  $m_i$  in the sequence. At the beginning,  $\iota(m_i) = i$ . Then we define  $\sigma(m_i, m_j) = |\iota(m_i) - \iota(m_j)| \pmod{2}$ . So  $\sigma(m_i, m_j) = 0$  if  $m_i$  and  $m_j$  are even positions apart and  $\sigma(m_i, m_j) = 1$  if  $m_i$  and  $m_j$  are odd positions apart. We notice that

- (a) operations from (i) and (ii) do not change  $\sigma(m_i, m_j)$ ;
- (b) if  $m_j$  is next to  $m_i$ , applying the operation (iii) to commute  $m_i$  and  $m_j$  will change all values of  $\sigma(m_i, m_l), \sigma(m_j, m_l)$  for  $l \neq i, j$ .

From the above observation, we have that

- (1) if  $\sigma(m_i, m_j) = 0$  and  $i < j$ ,  $m_j$  can be moved to the position next to  $m_i$  just using operations from (ii).
- (2) if  $\sigma(m_i, m_j) = 0$ ,  $i < j$  and  $m_i = m_j$ , then  $m_i$  and  $m_j$  can be cancelled using just operations from (i) and (ii).
- (3) for  $m_i, m_j, m_l$  such that  $m_i = m_j$ ,  $\sigma(m_i, m_j) = 1$ ,  $\sigma(m_i, m_l) = 0$ , we can cancel  $m_i, m_j$  with the cost of  $|m_i - m_l|$ .

(1) can be achieved by commuting two consecutive pairs of numbers. (2) is a direct consequence of (1). Let us show how to achieve (3). By (1), we can move  $m_l$  next to  $m_i$ . Then by using operation (ii), the pair  $m_i m_l$  (or  $m_l m_i$ ) can be moved to the position next to  $m_j$ , resulting the form of  $m_i m_l m_j$  or  $m_j m_l m_i$ . Finally, we commute  $m_i$  and  $m_j$  using operation (iii) at a cost of  $|m_i - m_j|$  and cancel  $m_i m_j$ .

Now we are ready to estimate the cost to cancel the sequence  $m_1, m_2, \dots, m_{2k}$  to the empty sequence. By (2), we can assume that we have already cancelled all the pairs  $m_i, m_j$  where  $\sigma(m_i, m_j) = 0$  using operations (i) and (ii). This step costs nothing and does not change any  $\sigma(m_i, m_j)$  for  $m_i, m_j$  remaining in the resulting sequence. Let the remaining elements after cancellations be  $m_{i(1)}, m_{i(2)}, \dots, m_{i(4s)}$  for some  $2s \leq k$  and  $i(1) < i(2) < \dots < i(4s)$ . The remaining sequence satisfies the following properties:

- (a)  $\sigma(m_{i(s)}, m_{i(l)}) = i(s) - i(l) \pmod{2}$ ,
- (b) if  $m_{i(s)} = m_{i(l)}$  then  $\sigma(m_{i(s)}, m_{i(l)}) = 1$ ,
- (c)  $\sigma(m_{i(s)}, m_{i(l)}) = \sigma(m_{i(s')}, m_{i(l')})$  for  $m_{i(s)} = m_{i(s')}, m_{i(l)} = m_{i(l')}$ .

Here the property (a) is true because in the original sequence  $\iota(m_{i(s)}) = i(s)$  and we only use operation (i) and (ii) which do not change  $\sigma(m_{i(s)}, m_{i(l)})$ . (b) and (c) follow from the definition of  $\sigma$  and the remaining sequence.

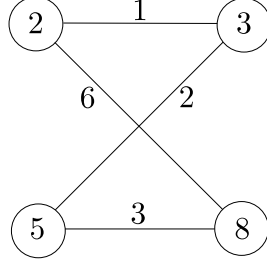


Figure VI.1: the corresponding graph of the sequence 2,3,5,3,5,8,2,8

We define the weighted graph  $\Gamma_0$  associated with  $m_{i(1)}, m_{i(2)}, \dots, m_{i(2s)}$  where the vertex set is  $\{m_{i(1)}, m_{i(2)}, \dots, m_{i(4s)}\}$  and there is an edge with weight  $|m_{i(s)} - m_{i(l)}|$  connects  $m_{i(s)}, m_{i(l)}$  if  $\sigma(m_{i(s)}, m_{i(l)}) = 0$ . Note that this graph is invariant under operations (ii) and may have multi-edge.

By (3), we are allowed to cancel  $m_{i(s)}, m_{i(l)}$  at a cost of  $|m_{i(s)} - m_{i(j)}|$  for some  $m_{i(j)}$  that  $\sigma(m_{i(s)}, m_{i(j)}) = 0$ . After the cancellation, since we use operation (iii) once,  $\sigma(m_{i(j)}, m_{i(j')})$  change to 0 for some  $m_{i(k')}$  that  $m_{i(j)} = m_{i(j')}$ . Therefore we can then cancel  $m_{i(j)}, m_{i(j')}$  without any cost. In summary, we have

(4) for  $m_{i(s)} \neq m_{i(j)}$  that  $\sigma(m_{i(s)}, m_{i(j)}) = 0$ , we can cancel a pair of number  $m_{i(s)}$  and a pair of number  $m_{i(j)}$  at a cost of  $|m_{i(s)} - m_{i(j)}|$  where  $\sigma(m_i, m_j)$  remains the same for numbers that have not been cancelled.

(4) will delete an edge of  $(m_{i(s)}, m_{i(j)})$  in the graph. If no edges connecting  $m_{i(s)}$  and  $m_{i(j)}$ , we delete the two vertices  $m_{i(s)}, m_{i(j)}$ . The cost is the weight of that edge. Let  $\mathcal{C}$  be a cancellation of  $\Gamma_0$  where  $\mathcal{C}$  consists of an ordered sequence of edges in  $\Gamma_0$ , where we cancel the edge by the order of the sequence. Thus the total cost of a cancellation  $\mathcal{C}$  to the empty graph is just a sum of the weight of edges in  $\mathcal{C}$ . Every cancellation can be associated with a path  $p_{\mathcal{C}}$  where the path passes through the sequence of edges in  $\mathcal{C}$  in the same order.

Now we delete edges in the following way. We first delete one edge  $(m_{i(1)}, m_{i(2)})$  since  $\sigma(m_{i(1)}, m_{i(2)}) = 0$ . We let the resulted graph to be  $\Gamma_1$ .

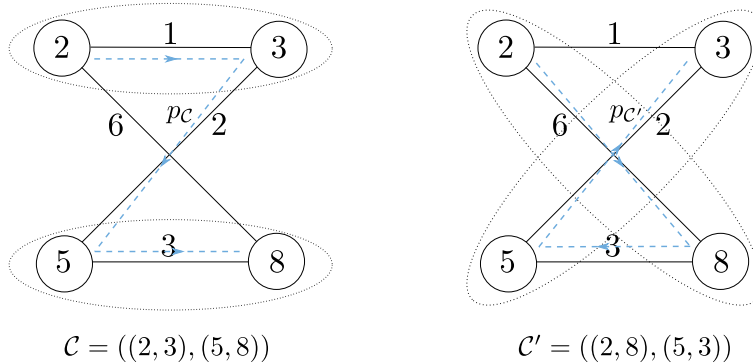


Figure VI.2: two different cancellations  $\mathcal{C}, \mathcal{C}'$  and their corresponding  $p_{\mathcal{C}}, p_{\mathcal{C}'}$ . The total cost of  $\mathcal{C}$  is 3 and the total cost of  $\mathcal{C}'$  is 8.

Inductively,  $\Gamma_{i+1}$  is obtained by deleting an edge  $(m_{i(s)}, m_{i(j)})$  where  $i, j$  are the smallest numbers remained in  $\Gamma_i$ .  $\Gamma_s$  will be an empty graph since every time we delete four numbers in the sequence.

Let us estimate the cost from  $\Gamma_0$  to  $\Gamma_s$ . Since every time we cancel pairs of numbers based on the order of the original sequence  $m_1, m_2, \dots, m_{2k}$  (always cancel the first two numbers remained). The cost is bounded by

$$\sum_{i=1}^{2k-1} |m_{i+1} - m_i| = \sum_{i=2}^{2k} |n_i| < n.$$

Let inequality can also be realized by the following interpretation: the sequence  $m_{i(1)}, m_{i(2)}, \dots, m_{i(4s)}$  defines a path  $p$  in  $\Gamma_0$  (since  $\sigma(m_{i(i)}, m_{i(j+1)}) = 0$ ) that  $p(j) = m_{i(j)}$ , the weight of the path  $p$  is bounded by  $n$  by the definition of  $m_i$ .  $p$  happens to be the path associated with this cancellation. It follows that the cost of the cancellation is bounded by the total weight of  $p$ . Thus the total cost is bounded by  $n$ .

By Lemma VI.3.2, the total cost of converting

$$a^{m_1} a^{m_2} \dots a^{m_{2k}}$$

to 0 is bounded by  $4n - 3$ . We finish the proof. □

## VI.6 Relative Dehn Functions and Subgroup Distortions

So far for all the examples considered in [15] and Section VI.4, only the upper bounds of their relative Dehn functions are estimated. Similar to the case of the Dehn function, it is genuinely much harder to estimate the lower bound. In this section, we will connect the relative Dehn function of a finitely generated metabelian group to the subgroup distortions in a wreath product of two free abelian groups. This connection provides a new method to estimate the lower bound for the relative Dehn function and yields a sequence of examples of finitely generated metabelian groups with relative Dehn function larger than  $n^k$  for arbitrary  $k \in \mathbb{N}$ . And finally, we will finish the proof of the last piece of Theorem A and Theorem C.

Let  $G$  be a finitely generated group with a finite generating set  $X$  and  $H$  be a subgroup of  $G$  with finite generating set  $Y$ . The *distortion function* of  $H$  in  $G$  is

$$\Delta_H^G(n) = \sup\{|w|_Y \mid w \in H, |w|_X \leq n\}.$$

We consider a slightly different equivalence relation for distortion functions. For non-decreasing functions  $f$  and  $g$  on  $\mathbb{N}$ , we say that  $f \preceq g$  if there exists a constant  $C$  such that  $f(n) \leq Cg(Cn)$ . Hence we say that two functions  $f$  and  $g$  are equivalent, written  $f \asymp g$ , if  $f \preceq g$  and  $g \preceq f$ . As expected, the distortion function is independent of the choice of the

finite generating set under this equivalence relation. The reason we consider  $\asymp$  rather than  $\approx$  is that if the subgroup is infinite then the distortion function is at least linear. We say a subgroup is *undistorted* if the distortion function is equivalent to the linear function.

For example, the subgroup  $\langle a \rangle$  in the Baumslag-Solitar group  $\langle a, t \mid a^t = a \rangle$  is exponentially distorted since  $a^n = a^{2^n}$ . And it not hard to check that infinite subgroups of a finitely generated abelian group are undistorted.

Let  $A$  and  $T$  be free abelian groups with bases  $\{a_1, a_2, \dots, a_m\}$  and  $\{t_1, t_2, \dots, t_k\}$  respectively. Consider the wreath product  $W := A \wr T$ . The base group  $B := \langle\langle A \rangle\rangle$  is a  $T$ -module. For a finite subset  $\mathcal{X} = \{f_1, f_2, \dots, f_l\}$  of  $B$ , let  $H$  be the subgroup of  $W$  generated by  $\mathcal{X} \cup \{t_1, t_2, \dots, t_k\}$  and  $G$  be the group  $W / \langle\langle \mathcal{X} \rangle\rangle$ . We denote by  $\pi : W \rightarrow T$  the canonical quotient map.

**Theorem VI.6.1.** *Let  $W, H, G$  be groups defined as above, then*

$$\Delta_H^W(n) \asymp \tilde{\delta}_G^k(n) + n^k, \tilde{\delta}_G(n) \asymp \max\{n^3, (\Delta_H^W(n^2))^3\}.$$

*In particular, if  $k = 1$ ,*

$$\Delta_H^W(n) \asymp \tilde{\delta}_G(n).$$

*Proof.* First we show the following lemma.

**Lemma VI.6.2.** *Let  $M$  be the  $T$ -module  $B / \langle\langle \mathcal{X} \rangle\rangle$ . Then  $\hat{\delta}_M(n) \asymp \Delta_H^W(n) \asymp \hat{\delta}_M^k(n) + n^k$ .*

*Proof.* Let  $g \in H$ . Note that  $g$  can be written as  $g_0 t$ , by adding  $t := \pi(g)$  to the end, where  $g_0 \in B, t \in T$ . Since  $|\pi(t)|_T \leq |g|_W \leq n$ ,  $|g_0|_W \leq 2|g|_W$ . Thus, we have

$$|g|_H = |g_0 t|_H \leq |g_0|_H + |t|_H \leq |g|_W + |g_0|_H.$$

Assume that the ordered form of  $OF(g_0)$  is  $a_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m}$ , let us estimate  $|g_0|_H$ . First note that  $\deg \mu_i \leq |g|_W$  for all  $i$ . Let  $\alpha_1, \alpha_2, \dots, \alpha_l$  be elements in  $\mathbb{Z}T$  such that  $g_0 = f_1^{\alpha_1} f_2^{\alpha_2} \dots f_l^{\alpha_l}$  and  $\sum_{i=1}^l |\alpha_i|$  is minimized. By Theorem 3.4 in [12],

$$|g_0|_H = \sum_{i=1}^l |\alpha_i| + \text{reach}(g_0),$$

where  $\text{reach}(g_0)$  is the length of the shortest loop starting at 0 in the Cayley graph of  $T$  that passing through all points in the set  $\cup_{i=1}^l \text{supp } \alpha_i$ . By Lemma VI.3.4, for all  $i$ ,  $\deg(\alpha_i) \leq |g|_W + C \sum_{i=1}^l |\alpha_i|$  for some constant  $C$ . It follows that  $\cup_{i=1}^l \text{supp } \alpha_i$  lies in Ball  $B_0(|g|_W + C \sum_{i=1}^l |\alpha_i|)$  of radius  $|g|_W + C \sum_{i=1}^l |\alpha_i|$  centered at 0 in the Cayley graph of  $T$ .

Since there exists a path of length  $(2(|g|_W + C \sum_{i=1}^l |\alpha_i|) + 1)^k$  passing through all the points in  $B_0(|g|_W + C \sum_{i=1}^l |\alpha_i|)$ ,

$$\text{reach}(g_0) \leq (2(|g|_W + C \sum_{i=1}^l |\alpha_i|) + 1)^k.$$

Therefore, we have

$$\sum_{i=1}^l |\alpha_i| \leq |g|_H \leq |g|_W + \sum_{i=1}^l |\alpha_i| + 2^k (|g|_W + C \sum_{i=1}^l |\alpha_i|)^k.$$

Since  $\sum_{i=1}^l |\alpha_i| = \widehat{\text{Area}}_M(g_0)$  by definition and  $\|g_0\| \leq 2|g|_W$ , we have the following estimation:

$$\widehat{\text{Area}}(2|g|_W) \leq |g|_H \leq |g|_W + \widehat{\text{Area}}(2|g|_W) + 2^k (|g|_W + C \widehat{\text{Area}}(2|g|_W))^k.$$

□

By Lemma VI.3.5, we have

$$\Delta_H^W(n) \asymp \tilde{\delta}_G^k(n) + n^k.$$

Last, by Proposition VI.4.2,

$$\tilde{\delta}_G(n) \asymp \max\{n^3, (\Delta_H^W(n^2))^3\}.$$

□

Theorem VI.6.1 connects the subgroup distortion function and the relative Dehn function, as it provides a way to estimate the relative Dehn function from the bottom. One special case is that both  $A$  and  $T$  are free abelian group of rank 1. Davis and Olshanskiy [12] show that subgroups in  $W = \langle a \rangle \wr \langle t \rangle$  have polynomial distortion functions and moreover for each  $l \in \mathbb{Z}$ , a subgroup of the form  $H_l := \langle [\dots, [a, t], t], \dots, t, t \rangle$ , where the commutator is  $(l-1)$ -fold, is isomorphic to  $\mathbb{Z} \wr \mathbb{Z}$  with  $n^l$  distortion. It follows immediately that

**Corollary VI.6.3.** *Let  $W = \langle a \rangle \wr \langle t \rangle$  be the wreath product of two infinite cyclic group. For each  $l \in \mathbb{N}$  let  $w_l = [\dots, [a, t], t], \dots, t$  be the  $(l-1)$ -fold commutator. Finally let  $H_l = W / \langle \langle w_l \rangle \rangle$ . Then we have*

$$\tilde{\delta}_{H_l} \asymp n^l.$$

Let us consider the case when the rank of  $T$  is 1, that is, when  $k = 1$ . The distortion functions in this case have been study extensively.

**Theorem VI.6.4** (Davis, Olshanskiy, [12, Theorem 1.2]). *Let  $A$  be a finitely generated abelian group.*

(1) For any finitely generated infinite subgroup  $H \leq A \wr \mathbb{Z}$  there exists  $l \in \mathbb{N}$  such that the distortion of  $H$  in  $A \wr \mathbb{Z}$  is

$$\delta_H^{A \wr \mathbb{Z}}(n) \asymp n^l.$$

(2) If  $A$  is finite, then  $l = 1$ , that is, all subgroups are undistorted.

(3) If  $A$  is infinite, then for every  $l \in \mathbb{N}$  there is a 2-generated subnormal subgroup  $H$  of  $A \wr \mathbb{Z}$  having distortion function

$$\Delta_H^{A \wr \mathbb{Z}}(n) \asymp n^l.$$

It follows that

**Theorem VI.6.5.** *Let  $G$  be a finitely generated metabelian group such that  $\text{rk}(G) = 1$ . Then the relative Dehn function of  $G$  is polynomially bounded. If in addition  $G$  is finitely presented, the Dehn function of  $G$  is asymptotically bounded above by the exponential function.*

*Proof.* By passing to a finite index subgroup, we can assume that there exists a short exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1,$$

where  $A$  is abelian.

We denote by  $T = \langle t \rangle$  the  $\mathbb{Z}$  in the short exact sequence. Since every short exact sequence  $1 \rightarrow A \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  splits,  $G$  is isomorphic to the semidirect product  $A \rtimes T$ .

Note that  $A$  is a normal subgroup of  $G$ , then it is finitely generated as a  $T$ -module. Thus, there exists a free  $T$ -module  $M$  of rank  $m$  and a submodule  $S = \langle f_1, f_2, \dots, f_l \rangle$  such that  $A \cong M/S$ . We have that

$$G \cong (M/S) \rtimes T \cong (M \rtimes T) / \langle \langle f_1, f_2, \dots, f_l \rangle \rangle.$$

Let  $\bar{A}$  be a free abelian group of rank  $m$  and  $W := \bar{A} \wr T$  be the wreath product of  $\bar{A}$  and  $T$ . Then there is an isomorphism  $\varphi : M \rtimes T \rightarrow W$ . We have

$$G \cong W / \langle \langle \varphi(f_1), \varphi(f_2), \dots, \varphi(f_l) \rangle \rangle.$$

Let  $H$  be the subgroup in  $W$  generated by  $\{\varphi(f_1), \varphi(f_2), \dots, \varphi(f_l), t\}$ . By Theorem VI.6.1, we have that

$$\tilde{\delta}_G(n) \asymp \max\{n^3, (\Delta_H^W(n^2))^3\}.$$

By Theorem VI.6.4,  $\Delta_H^W(n)$  is a polynomial. Therefore the relative Dehn function  $\tilde{\delta}_G(n)$  of  $G$  is polynomially bounded.

We are done for the relative Dehn function case.

If  $G$  is finitely presented, by Theorem VI.3.1, if the relative Dehn function is polynomially bounded,  $\delta_G(n)$  is bounded above by the exponential function.  $\square$

This theorem gives the exponential upper bound of Dehn functions for many examples we introduced in Section III.2, including the metabelian Baumslag-Solitar groups and  $\mathbb{Z}^n \rtimes_{\phi} \mathbb{Z}$  where  $\phi \in GL(n, \mathbb{Z})$ . And it is also the final piece of Theorem A and Theorem C for the case  $\text{rk}(G) = 1$ .

## Chapter VII

### Embedding Problems

#### VII.1 Motivations

Another interesting topic for finitely generated metabelian groups is the embedding problem. It is well-known that every recursively generated group can be embedded into a finitely presented group [20]. If we restrict to a variety of group  $\mathcal{V}$ , we can ask if a recursively generated group in  $\mathcal{V}$  can be embedded into a finitely presented group (in the absolute sense) from  $\mathcal{V}$ . It turns out the only non-trivial positive result so far is when  $\mathcal{V} = \mathcal{S}_2$ , the variety of metabelian groups.

**Theorem VII.1.1** (Baumslag [4]). *A finitely generated metabelian group embeds into a finitely presented metabelian group.*

Recall that

**Theorem VII.1.2** (Birget, Olshanskii, Rips, Sapir [7]). *The word problem of a finitely generated group  $G$  is in NP if and only if  $G$  embeds into a finitely presented group with polynomial Dehn function.*

It is known that any finitely generated metabelian group can be represented by matrices over finite products of fields (See in [31], [23], [36]). It follows that the word problem for any finitely generated metabelian group is in *LSAPCE*, in particular in *NP*. Therefore a finitely generated metabelian group can be embedded into a finitely presented group with polynomial Dehn function.

Many results show that, in some cases, a finitely generated metabelian group embeds into a finitely presented metabelian group with polynomial Dehn function. For example, the metabelian Baumslag-Solitar groups embed into finitely presented metabelian groups with quadratic Dehn function [13], and embed quasi-isometrically into finitely presented metabelian groups with cubic Dehn function [1], and the lamplighter groups embed into finitely presented metabelian group with at most quartic [21].

G. N. Arzhantseva and D. Osin asked the following question [1].

**Problem VII.1.3.** *Is any finitely generated metabelian group embedded into a finitely presented metabelian group with polynomial Dehn functions?*

The Dehn functions of finitely presented metabelian groups constructed in Theorem VII.1.1 are mostly unknown. M. Kassabov and T. R. Riley computed the Dehn function of  $\Gamma = \langle a, s, t \mid [a^t, a] = 1, a^s = a^{t+1}, [s, t] = 1 \rangle$ , which is the group contains a free metabelian group of rank 1 (it is  $\mathbb{Z}$ ) in Baumslag's construction. But, for example, the Dehn function of finitely presented groups in [3] contains free metabelian groups of rank other than 1 are unknown.



We answer this question by computing the Dehn functions for a wider class of finitely presented metabelian groups Baumslag's constructed.

**Theorem VII.1.4.** *Every wreath product of a free abelian group of finite rank with a finitely generated abelian group can be embedded into a metabelian group with exponential Dehn function.*

The theorem will be proved in the next section. One immediate consequence of this theorem is the following.

**Corollary VII.1.5.** *A free metabelian group of finite rank embeds into a finitely presented metabelian group with exponential Dehn function.*

The question if such groups can be embedded into a finitely presented metabelian group with polynomial Dehn function remains, even in the case of  $\mathbb{Z} \wr \mathbb{Z}$ . We shall discuss Problem VII.1.3 in Section VIII.2.

## VII.2 Embeddings of Wreath Product of Abelian Groups

The class of examples we investigate in this section was introduced by Baumslag in 1973 [4]. Let  $A$  be a free abelian group of finite rank freely generated by  $\{a_1, a_2, \dots, a_r\}$ . Furthermore let  $T$  be a finitely generated abelian group with basis  $\{t_1, t_2, \dots, t_k, \dots, t_l\}$ , where  $t_1, \dots, t_k$  are of infinite order and  $t_{k+1}, \dots, t_l$  are respectively of finite order  $m_{k+1}, \dots, m_l$ . Finally let  $F = \{f_1, f_2, \dots, f_k\}$  be a set of element  $f_i$  from  $\mathbb{Z}T$ , where each  $f_i$  is of the form

$$f_i = 1 + c_{i,1}t_i + c_{i,2}t_i^2 + \dots + c_{i,d_i-1}t_i^{d_i-1} + t_i^{d_i}, d_i \geq 1, c_{i,j} \in \mathbb{Z}.$$

Now let us define a group  $W_F$  corresponds to  $F$ . The generating set is the following

$$X = \{a_1, a_2, \dots, a_r, t_1, t_2, \dots, t_l, u_1, \dots, u_k\},$$

where  $r, k, l$  are the same integers as above.

The defining relations of  $W_F$  are of four kinds. First we have the power relations

$$t_i^{m_i} = 1, i = k+1, \dots, l.$$

Next we have the commutativity relations

$$\begin{cases} [u_i, u_j] = 1, & 1 \leq i, j \leq k; \\ [t_i, t_j] = 1, & 1 \leq i, j \leq l; \\ [t_i, u_j] = 1, & 1 \leq i \leq l, 1 \leq j \leq k; \\ [a_i, a_j] = 1, & 1 \leq i, j \leq r. \end{cases}$$

Thirdly we have the commutativity relations for the conjugates of the generators  $a_i$ :

$$[a_i^u, a_j^w] = 1, 1 \leq i, j \leq r,$$

where  $u, w \in \{t_1^{\alpha_1} t_2^{\alpha_2} \dots t_l^{\alpha_l} \mid 0 \leq \alpha_i \leq d_i \text{ for } i = 1, \dots, k, 0 \leq \alpha_i < m_i \text{ for } i = k+1, \dots, l\}$ . Finally we have relations defining the action of  $u_j$  on  $a_i$ :

$$a_i^{u_j} = a_i^{f_j}, 1 \leq i \leq r, 1 \leq j \leq k.$$

It is not hard to show that  $W_F$  is metabelian [4]. Moreover, Baumslag showed the following:

**Proposition VII.2.1** (Baumslag [4]). Given a free abelian group  $A$  of finite rank and a finite generated abelian group  $T$ , there exists  $F$  such that  $A \wr T \hookrightarrow W_F$ .

In particular, if  $r = k = l$  and we let  $f_i = 1 + t_i$  for all  $i$ ,  $W_F$  contains a copy of the free metabelian group of rank  $r$ .

We claim that

**Proposition VII.2.2.** If  $k > 0$ ,  $W_F$  has exponential Dehn function.

Note that when  $i = j = k = 1$ ,  $f_1 = 1 + t_1$ ,  $W_F$  is the Baumslag group  $\Gamma = \langle a, s, t \mid [a, a^t] = 1, [s, t] = 1, a^s = aa^t \rangle$ .

The exponential Dehn function of this special case is proved in [21].

We need a few lemmas before we prove Proposition VII.2.2. First, let us denote the abelian groups generated by  $\{t_1, t_2, \dots, t_l\}$  and  $\{u_1, u_2, \dots, u_k\}$  by  $T$  and  $U$  respectively.

**Lemma VII.2.3.** Let  $M$  be a free  $(U \times T)$ -module with basis  $e_1, \dots, e_r$ . Let  $S$  be the submodule of  $M$  generated by  $\{(u_i - f_i)e_j \mid 1 \leq i \leq k, 1 \leq j \leq r\}$ . If  $h = h_1 e_1 + h_2 e_2 + \dots + h_r e_r \in S$  such that  $h_i \in \mathbb{Z}T$  for all  $i$ . Then  $h = 0$ .

*Proof.* If  $k = 1$ , then  $h \in S$  means there exists  $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{Z}(U \times T)$  such that

$$h = \alpha_1(u_1 - f_1)e_1 + \alpha_2(u_1 - f_1)e_2 + \dots + \alpha_r(u_1 - f_1)e_r.$$

Since  $h = h_1 e_1 + h_2 e_2 + \dots + h_r e_r$ , then  $h_i = \alpha_i(u_1 - f_1)$ . Note that  $h_i \in \mathbb{Z}T$ . It follows that  $\alpha_i(u_1 - f_1)$  does not have

any term involves  $u_1$ . Suppose  $\alpha_i \neq 0$  for some  $i$ . Because  $f_1 \in \mathbb{Z}T$ ,  $\deg_{u_1}(\alpha_i u_1) > \deg_{u_1}(\alpha_i f_1)$ . Thus  $\alpha_i(u_1 - f_1)$  has at least one term contains  $u_1$ , that leads to a contradiction.

If the statement of  $k = n$  has been proved, for  $k = n + 1$ , we have

$$h = \sum_{i=1}^r \sum_{j=1}^{n+1} \alpha_{i,j}(u_j - f_j)e_i.$$

We choose an integer  $N$  large enough such that  $u_1^N \alpha_{i,j}$  does not have any negative power of  $u_1$  for all  $i, j$ . Then

$$u_1^N h = \sum_{i=1}^r \sum_{j=1}^{n+1} f_1^N \alpha_{i,j}(u_j - f_j)e_i =: \sum_{i=1}^r \sum_{j=1}^{n+1} \beta_{i,j}(u_j - f_j)e_i,$$

where  $\beta_{i,j} = u_1^N \alpha_{i,j}$ . We regard  $\beta_{i,j}(u_1)$  as a polynomial of  $u_1$ . Replacing  $u_1$  by  $f_1$ , we have

$$f_1^N h = \sum_{i=1}^r \sum_{j=2}^{n+1} \beta_{i,j}(f_1)(u_j - f_j)e_i.$$

Note that  $f_1^N h_i \in \mathbb{Z}T$  for  $i = 1, \dots, r$ , then by the inductive assumption,  $f_1^N h_i = 0$  for all  $i$ . Since  $f_1 = 1 + c_{1,1}t_1 + c_{1,2}t_1^2 + \dots + c_{i,d_1-1}t_1^{d_1-1} + t_1^{d_1}$  and  $t_1$  has infinite order, then  $f_1$  is not a zero divisor in  $\mathbb{Z}(U \times T)$ . Thus  $h_i = 0$  for all  $i$ .

Therefore  $h = 0$ . The induction finishes the proof.  $\square$

It follows that if  $a_1^{h_1} a_2^{h_2} \dots a_r^{h_r} =_{W_F} 1$  such that  $h_i \in \mathbb{Z}T$  for all  $i$ , then  $h_i = 0$  as an element in  $\mathbb{Z}(U \times T)$  for every  $i$ . To convert it to 1, we only need those metabelian relations to commute all the conjugates of  $a_i$ 's. By Theorem VI.1.1, it will cost at most exponentially many relations with respect to the length of the word to kill the word.

Next, let  $w =_{W_F} 1$  and consider the minimal van Kampen diagram  $\Delta$  over  $W_F$ . There are two types of relations contain  $u_i$ : (1) commutative relations  $[u_i, u_j] = 1, [u_i, t_s] = 1, j \neq i, 1 \leq s \leq l$ ; (2) action relations  $a_j^{u_i} = a_j^{f_i}, 1 \leq j \leq r$ . Those cells, in the van Kampen diagram, form a  $u_i$ -band.

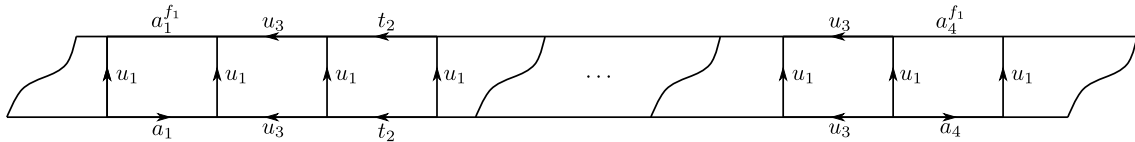


Figure VII.1: an example of a  $u_1$ -bands

We have some properties for  $u_i$ -bands in a van Kampen diagram over  $W_F$ .

**Lemma VII.2.4.** (i) *The top (or bottom) path of a  $u_i$ -band is a word  $w$  that all  $t_s, u_j$  for  $s, j \neq i$  are in the same orientation, i.e. the exponents of each letter  $t_s, u_j$ 's are either all 1 or all  $-1$ . In particular,*

$$w =_{W_F} a_1^{h_1} a_2^{h_2} \dots a_r^{h_r} t_1^{\alpha_1} \dots t_l^{\alpha_l} u_1^{\beta_1} \dots u_k^{\beta_k},$$

where  $h_i \in \mathbb{Z}(U \times T)$ ,  $\text{sgn}(\alpha_i) = \text{sgn}(\beta_j)$  for all  $i, j$ , and  $\alpha_s$  (or  $\beta_j$ ) is equal to the number of times of  $t_s$  (resp.  $u_j$ ) appears in  $w$  for  $s, j \neq i$ .

(ii)  $u_i$ -bands do not intersect each other. In particular, a  $u_i$ -band does not self-intersect.

(iii) If  $i \neq j$ , a  $u_i$ -band intersects a  $u_j$ -band at most one time.

*Proof.* (i) By the definition of a  $u_i$ , all letters  $t_s, u_j, s, j \neq i$  of the top (or bottom) path must share the same direction.

The second half of the statement can be proved basically the same way as we did for the ordered form (See ??).

(ii) Because there is no  $u_i$  on the top or the bottom path of a  $u_i$ -band, two  $u_i$ -bands cannot intersect each other.

(iii) If  $i \neq j$  and a  $u_i$ -band intersects a  $u_j$ -band. Since the van Kampen diagram is a planer graph, by comparing the orientation, it is impossible for a  $u_i$ -band to intersect a  $u_j$ -band twice (or more). □

Last, we have

**Lemma VII.2.5.** *Let  $f(t) = t^d + c_{d-1}t^{d-1} + \dots + c_1t + 1 \in \mathbb{Z}[t], c_i \in \mathbb{Z}, d > 0$ . Then there exists  $\alpha > 1$  such that  $|(f(t))^n| > \alpha^n$  for all  $n$ .*

*Proof.* We denote that  $(f(t))^n = \sum_{i=0}^{nd} c_{n,i}t^i$ .

Consider the corresponding holomorphic function  $g(z) = z^d + c_{1,d-1}z^{1,d-1} + \dots + c_{1,1}z + 1$ . If  $\exists z_0, |z_0| = 1$  such that  $|g(z_0)| > 1$ , we have

$$|g(z_0)|^n = |(g(z_0))^n| = \left| \sum_{i=0}^{nd} c_{n,i}z_0^i \right| \leq \sum_{i=0}^{nd} |c_{n,i}| = |f^n|.$$

Then we are done.

Now suppose  $|g(z)| \leq 1$  for all  $|z| = 1$ . Then by Cauchy's integral formula we have

$$1 = g(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{g(z)}{z} dz.$$

Take modulus on both sides:

$$1 = \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{g(z)}{z} dz \right| \leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} |g(e^{i\theta})| d\theta \leq 1.$$

Therefore  $|g(z)| = 1$  for  $|z| = 1$  almost everywhere. Let  $z = e^{i\theta}$ . We have

$$g(e^{i\theta}) = \left( \sum_{j=0}^d c_{1,j} \cos(j\theta) \right) + i \left( \sum_{j=0}^d c_{1,j} \sin(j\theta) \right).$$

Then

$$\left( \sum_{j=0}^d c_{1,j} \cos(j\theta) \right)^2 + \left( \sum_{j=0}^d c_{1,j} \sin(j\theta) \right)^2 = \sum_{h=0}^d c_{1,h}^2 + 2 \sum_{j < k} c_{1,j} c_{1,k} \cos((k-j)\theta) = 1,$$

holds for all  $\theta$ . But  $\cos((k-j)\theta)$  is a polynomial with respect to  $\cos \theta$ , i.e.  $\cos((k-j)\theta) = T_{k-j}(\cos \theta)$ , where  $T_m(x)$  is the  $m$ -th Chebyshev polynomial. The leading term of  $T_m(x)$  is  $2^{m-1}x^m$ . Thus

$$\sum_{h=0}^d c_{1,j}^2 + 2 \sum_{j < k} c_{1,j} c_{1,k} T_{k-j}(\cos \theta) = 1, \forall \theta.$$

Note the leading term of the left-hand side is  $2^{d-1} \cos^d \theta$ . That leads a contradiction since the equation above has at most  $d$  solutions for  $\cos \theta$ .  $\square$

*Proof of Proposition VII.2.2.* First, we show that the lower bound is exponential. Consider the word  $w = [a_1^{u_1^n}, a_1]$ .  $w$  is of length  $2n+4$  and  $w =_{W_F} 1$ . Let  $\Delta$  be a minimal Van-Kampen diagram with boundary label  $w$ . By comparing the orientation,  $u_1$ -bands starting at the top left of  $\Delta$  will end at either bottom left or top right. By Lemma VII.2.4,  $u_1$ -bands do not intersect each other, then we can suppose at least half of the  $u_1$ -bands starting at the top left end at the top right. See in Figure 4, the shaded areas are  $u_1$ -bands.

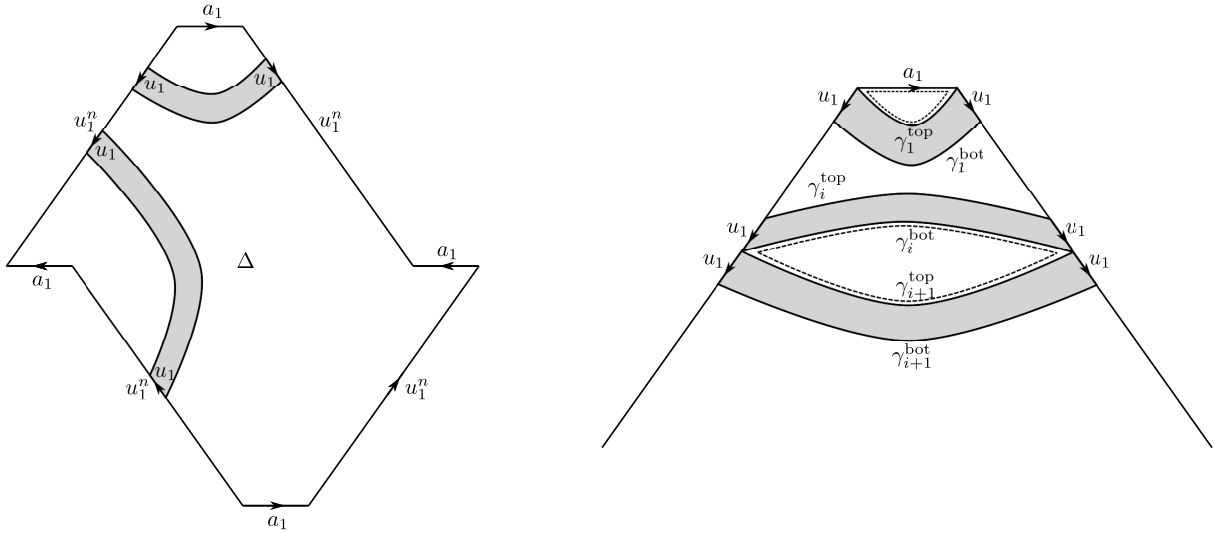


Figure VII.2:  $u_1$ -bands in  $\Delta$

We first claim that there are no cells containing  $t_s, u_j, s, j > 1$  on each  $u_1$ -band. We denote the top and bottom path of the  $i$ -th  $u_1$ -band from the top by  $\gamma_i^{\text{top}}$  and  $\gamma_i^{\text{bot}}$ , where  $i = 1, 2, \dots, m, m > \frac{n}{2}$ . Assuming a  $u_i$ -band intersect one of the  $u_1$ -band, again by Lemma VII.2.4 (ii), (iii), it can neither intersect a  $u_1$ -band twice nor intersect itself. Thus it has to end all the way to the boundary of  $\Delta$ . A contradiction.

Then if there exists a cell containing  $t_s$  for  $s > 1$  in the top most  $u_1$ -band, then by Lemma VII.2.4 (i),  $\gamma_1^{\text{top}}$  is a word  $a_1^{h_1} a_2^{h_2} \dots a_r^{h_r} t_1^{\alpha_1} \dots t_l^{\alpha_l}$ . Thus  $\gamma_1^{\text{top}}$  and  $a_1$  form a cycle  $\gamma$ . We have

$$a_1^{h_1+1} a_2^{h_2} \dots a_r^{h_r} t_1^{\alpha_1} \dots t_l^{\alpha_l} = 1, \alpha_i \neq 0.$$

It leads to a contradiction since the image of the left hand side in  $U \times T$  is not trivial. And by definition of a  $u_1$ -band, if  $\gamma_i^{\text{top}}$  does not have any  $t_s, s > 1$ , neither does  $\gamma_i^{\text{bot}}$ . Next consider two consecutive  $u_1$ -bands. If  $\gamma_i^{\text{bot}}$  does not have  $t_s$ , then by the same argument, neither does  $\gamma_{i+1}^{\text{top}}$ . Therefore the claim is true.

Denote the words of  $\gamma_i^{\text{top}}$  and  $\gamma_i^{\text{bot}}$  by  $w_i^{\text{top}}$  and  $w_i^{\text{bot}}$  respectively. Such words only consist of  $a_i$ 's and  $t_1$ . Note that  $w_i^{\text{bot}} = w_{i+1}^{\text{top}}$  for  $i = 1, \dots, m-1$ . Since  $w_1^{\text{bot}} = a_1^{-f_1}$ , by the same discussion above,  $w_i^{\text{top}} = a_1^{-f_1^{i-1}}, w_i^{\text{bot}} = a_1^{-f_1^i}$  (See in Figure 4). Next we focus on the number of  $a_1$  in each  $w_i^{\text{bot}}$ , which is at least  $|f_1^i|$ . By Lemma VII.2.5, there exists  $\alpha > 1$  such that  $|f_1^i| > \alpha^i$ . Therefore, the number of  $a_1$  in  $w_m^{\text{top}}$  is at least  $\alpha^{m-1}$ . Since  $m > \frac{n}{2}$ , the number of cells in the  $m$ -th  $u_1$ -band is at least  $\alpha^{\lfloor \frac{n}{2} \rfloor}$ . Thus the area of  $[a_1^{\frac{n}{2}}, a_1]$  is at least  $\alpha^{\lfloor \frac{n}{2} \rfloor}$ . It follows that the lower bound is exponential.

For the upper bound, as Theorem VI.1.1 suggests, all we need is to consider how to solve the membership problem of the submodule  $S$  where  $S$  is generated by  $\{(u_i - f_i)e_j \mid 1 \leq i \leq k, 1 \leq j \leq r\}$ . Suppose  $w = 1$  with  $|w| \leq n$ , the  $w$  has a ordered form as

$$w =_{W_F} a_1^{g_1} a_2^{g_2} \dots a_r^{g_r}, g_i \in \mathbb{Z}(U \times T).$$

And the cost of converting  $w$  to its ordered form is exponential with respect to  $n$  as we showed in Section V.4. Also note that  $\deg(g_i), |g_i| \leq n$  for all  $i$ . WLOG, we assume that the all exponents of  $u_i$ 's are positive. The corresponding module element of  $w$  is

$$g_1 e_1 + g_2 e_2 + \dots + g_r e_r.$$

For each term  $t_1^{\alpha_1} t_2^{\alpha_2} \dots t_l^{\alpha_l} u_1^{\beta_1} u_2^{\beta_2} \dots u_k^{\beta_k}$ ,  $\alpha_i \in \mathbb{Z}, \beta_i \geq 0$ , we replace  $u_i$  by  $u_i - f_i + f_i$ . Then we convert  $t_1^{\alpha_1} t_2^{\alpha_2} \dots t_l^{\alpha_l} u_1^{\beta_1} u_2^{\beta_2} \dots u_k^{\beta_k}$  to a form

$$\sum_{i=1}^k \eta_i (u_i - f_i) + \tau, \eta_i \in \mathbb{Z}(U \times T), \tau \in \mathbb{Z}T.$$

If  $|\alpha_1| + \dots + |\alpha_l| + |\beta_1| + \dots + |\beta_k| < n$ , then  $\deg(\eta_i), \deg(\tau) < Dn, |\eta_i|, |\tau| < D^n$ , where  $D = \max\{d_1, \dots, d_k, |f_1|, \dots, |f_k|\}$ .

Therefore, replacing  $u_i$  by  $u_i - f_i + f_i$  in every term of  $w$ , we have

$$g_1 e_1 + g_2 e_2 + \dots + g_r e_r = \sum_{i=1}^r \sum_{j=1}^k \mu_{i,j} (u_j - f_j) e_i + \rho, \mu_{i,j} \in \mathbb{Z}(U \times T), \rho \in \mathbb{Z}T.$$

Since  $w = 1$ , then  $\rho$  lies in the submodule  $S$ . By Lemma VII.2.3,  $\rho = 0$ . Also note that  $\deg(\mu_{i,j}), \deg(\rho) < Dn, |\mu_{i,j}|, |\rho| < nD^n$ . It follows from Lemma V.4.1 that all module computations in the process cost exponentially many relations with respect to  $n$ . And it also cost at exponentially many relations to convert  $\rho$  to 0. Therefore

$$w =_{W_F} \prod_{i=1}^r \prod_{j=1}^k a_i^{\mu_{i,j}(u_j - f_j)},$$

and the cost of converting is exponential with respect to  $n$ . And the area of the right hand side is bounded by  $\sum_{i,j} |\mu_{i,j}| \leq rknD^n$ . The upper bound is exponential.  $\square$

Theorem VII.1.4 follows immediately from Proposition VII.2.2.

*Proof.* Let  $A$  be a free abelian group of finite rank and  $T$  be a finitely generated abelian group. If the torsion-free rank of  $T$  is greater than 0, by Proposition VII.2.2,  $A \wr T$  embeds into  $W_F$ , which has exponential Dehn function. If the torsion-free rank is 0,  $A \wr T$  can be first embedded into  $A \wr (T \times \mathbb{Z})$ . Then the problem is reduced to the first case.  $\square$

## Chapter VIII

### Further Discussions

#### VIII.1 Tight upper bound for Dehn functions of metabelian groups

The upper bound we obtained in this thesis is slightly bigger than the exponential function for the case  $rk(G) > 1$  while all known examples are bounded by the exponential function. The answer to the following question remains unknown.

**Problem VIII.1.1.** Is the (relative) Dehn function of any finitely presented metabelian group bounded above by the exponential function?

In this thesis, we analyze the contribution of two different parts to the Dehn function of a finitely presented metabelian group: the metabelian part and the module structure part. We show that the metabelian part will never exceed the exponential cost. Furthermore, we prove in Lemma VI.3.5 and Theorem VI.3.1 that the Dehn function of the module structure gives a lower bound of the Dehn function of the group. So if there exists a finitely presented metabelian group with super-exponential Dehn function, it must contain a module structure with super-exponential Dehn function. It means that there exists a membership problem of a submodule over a group ring of the free abelian group such that its complexity (measured in the Dehn function of the factor module) is larger than exponential.

The first obstruction for us is the existence of such a membership problem. There is already a lot of study of the polynomial ideal membership problem, which is a special case for the membership problem over modules. For example, Mayr and Meyer showed that the lower space bound of a general polynomial ideal membership problem is exponential [26], though in their work the input of the problem including the generating set of the ideal, which is different from the problem we concern. Other results can be found several surveys, such as [25], [27]. But it remains unknown whether there exists an integral coefficient polynomial ideal such that the time complexity of its membership problem is harder than exponential.

The second obstruction comes from the finitely-presentedness. Recall that a finitely generated metabelian group is finitely presented if and only if the module structure is tame [6]. Thus even if we manage to find a complicated enough membership problem in some submodule, it may not give us a finitely presented metabelian group unless the module is tame.

Our estimation in Section IV.4 might not be tight at all. For example, when we compute the relative Dehn function of the metabelianized Baumslag-Solitar groups we solve the membership problem of the module structure in linear time (See in Section VI.4), while the division algorithm in Section IV.4 suggests an exponential upper bound. But it is still possible to find a more efficient algorithm to solve the membership problem of a submodule and hence we find a



better upper bound for the Dehn function of a finitely presented metabelian group.

However, there might be an alternative way to give an affirmative answer to Problem VIII.1.1. Yves Cornuier suggests that the technique in [11] can be used to show that the Dehn function of split finitely metabelian groups are either polynomially or exponential bounded. If this is true, combining results from [6] and [35], every finitely presented metabelian group has either a polynomial or exponential Dehn function. But so far, their technique cannot be applied to the Baumslag's group  $\Gamma$ , even though it has been proved that  $\Gamma$  has exponential Dehn function [21].

### **VIII.2 Embeddings of finitely generated metabelian groups**

In Chapter VII, we show that some wreath products of abelian groups embeds into finitely presented metabelian groups with exponential Dehn function. It can be asked if those groups cannot be embedded into finitely generated metabelian groups with polynomial Dehn function. For example, it is not known even for the case  $\mathbb{Z} \wr \mathbb{Z}$  while it embeds into a finitely presentation metabelian group with exponential Dehn function (the Baumslag's group  $\Gamma$ ) and embeds into a finitely presented group with quadratic Dehn function (the Thompson group  $F$ ).

If we want to give a negative answer to Problem VII.1.3, we have to prove that there exists a finitely generated metabelian group that cannot be embedded into a finitely presented metabelian group with polynomial Dehn function. One possible way to prove such a thing is that construct a finitely generated metabelian group with a module structure that has exponential time complexity for its membership problem. Thus any finitely presented metabelian group it embeds into might inherit the module structure with exponential time complexity, and consequently has at least exponential Dehn function.

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