Directedness, Duality, and Parity Conditions for Embedded Graphs

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction: Representations of Cellularly Embedded Graphs

1.1 Introduction

In this dissertation we explore the relationships between various properties of cellularly embedded graphs and their related graphs. Specifically we relate parity conditions in a gem representation of a cellularly embedded graph with properties of the embedding itself, the graph embeddings generated by applying duality and Petrie duality, and the medial graph. These properties include orientability, bipartiteness, the existence of an Eulerian circuit, the existence of certain bidirections of the medial graph, and directability. We also examine how partial duality, a generalization of the concept of duality, interacts with these properties and explore the uniqueness of partial duality up to a set of desired conditions.

In Chapter 1 we discuss different representations of cellularly embedded graphs. The main way that cellularly embedded graphs are represented in this dissertation is by graph encoded maps or gems. We will show many instances where using gems, especially when interacting with partial duality and partial Petrie duality, will be more convenient than using a typical representation of a cellularly embedded graph as a ribbon graph or embedding scheme. In Chapter 2 we introduce the operations of duality and its generalization, partial duality. The operation of duality for embedded graphs is performed by placing a vertex in each face and putting a new edge across each original edge. An example is given below in Figure 1.1. The idea of partial duality was first introduced by Chmutov in 2009 using ribbon graphs and has found numerous applications in the years since. We show an equivalent way to think of partial duality using gems and discuss the basic properties of partial duality.

In Chapter 3 we will extend a result of Deng and Jin and begin to show the relationship between structural and topological properties of a cellularly embedded graph and orientations of its medial

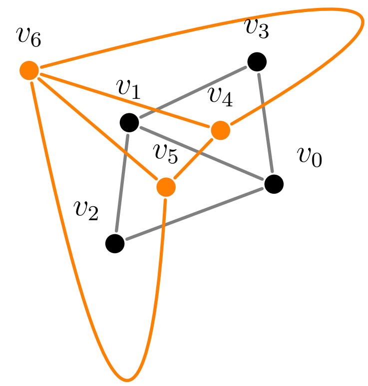


Figure 1.1: A graph G with $V(G) = \{v_0, v_1, v_2, v_3\}$ and its dual G^* with $V(G^*) = \{v_4, v_5, v_6\}$

graph. We also show how partial duality and partial Petrie duality affect certain orientations, also called directions, of the medial graph. In Chapter 4 (bi)directions of the medial graph are related to parity conditions of closed walks in gems. Furthermore, seven distinct parity conditions, which can be identified with points in the Fano plane, are each shown to correspond to embedding properties, the bipartiteness of related graphs, and bidirections of the medial graph. Each line in the Fano plane gives us a theorem relating the three properties corresponding to points in the line. We also characterize which abstract graphs have an embedding satisfying each of the seven properties. One of the seven properties gives a characterization of when an undirected embedded graph can be given an orientation in such a way that the resulting embedding of a directed graph is a directed embedding, which is shown in Chapter 5.

A directed embedding of a digraph is an embedding where the face boundary walks are directed walks. Generally, in Chapter 5 we seek to answer whether or not an embedding of an undirected graph or mixed graph can be given a direction (orientation) in such a way that the resulting directed graph is a directed embedding and if this directed embedding can have certain desired properties such as orientability or the presence of a set of prespecified faces. We give a characterization of when a pair $(G, \mathcal{W})$, where G is a mixed graph and $\mathcal{W}$ is a collection of closed directed walks, can be extended to a directed embedding of an orientation D of G where the set of existing arcs (original directed edges) of G is unchanged and $\mathcal{W}$ is a subcollection of the facial walks in the embedding. We also characterize when such an embedding can be chosen to be orientable.

Chapters 6 and 7 further explore the operations of duality and partial duality. In Chapter 6 we give a definition of duality in the setting of a generalization of cellular embeddings on surfaces to pseudocellular embeddings on pseudosurfaces and discuss the possibility of extending partial duality to this context. In Chapter 7 we explore various sets of properties in the pursuit of a result on the uniqueness of the operation of partial duality up to a set of desired properties. We discuss what makes Chmutov's definition a "natural" one and give examples of other operations that fit some of the desired properties of partial duality but seem to be more contrived. Finally, Chapter 8 gives avenues for future research.

Chapters 1, 2, and 3 mostly present background information and known results taken under a new lens. Chapters 4, 6, and 7 are joint work with Mark Ellingham. Chapter 5 is joint work with Mark Ellingham and Joanna Ellis-Monaghan.

1.2 Cellularly Embedded Graphs

An *embedded graph* $G = (V(G), E(G)) \subseteq \Sigma$ is a graph G drawn on a surface Σ such that the edges intersect only at their endpoints. By surface we mean a compact 2-manifold. Therefore all connected surfaces in this paper are S_h, a sphere with h handles added, or N_k, a sphere with k crosscaps added. The connected components of $\Sigma \setminus G$ are called the *regions* or *faces* of the embedded graph. Each face has a *boundary* consisting of edges in the embedded graph that form one or more closed walks called *facial walks*. An embedded graph G is called a *cellularly embedded graph* if each of these regions is homeomorphic to an open disc; in that case each face has only one facial walk.

In this paper we consider abstract graphs and embedded graphs up to isomorphism. An *isomorphism* from an abstract graph G to another abstract graph H is a bijection f that maps $V(G)$ to $V(H)$ and $E(G)$ to $E(H)$ such that each edge of G with endpoints u and v is mapped to an edge with endpoints $f(u)$ and $f(v)$. An *isomorphism of embedded graphs* is an abstract graph isomorphism that maps facial walks to facial walks up to reversal and/or cyclic shifts. For cellularly embedded graphs this is the same as the characterization that $G \subseteq \Sigma$ is equivalent to $G' \subseteq \Sigma'$ if there is a homeomorphism from Σ to Σ' that sends G to G'.

Now we give three standard ways to represent cellularly embedded graphs: ribbon graphs, graph encoded maps, and embedding schemes.

A *ribbon graph* $G = (V(G), E(G))$ is a (possibly nonorientable) surface with boundary represented as the union of two sets of closed discs: a set V(G) of *vertices* and a set E(G) of *edges* such that

(a) the vertices and edges intersect in disjoint line segments,

(b) each such line segment lies on the boundary of exactly one vertex and one edge, and

(c) every edge contains exactly two such line segments.

These definitions for ribbon graphs and cellularly embedded graphs define equivalent objects. If G is a cellularly embedded graph then by taking small neighborhoods around the vertices and edges of G we arrive at the corresponding ribbon graph. As long as we take small enough neighborhoods we can see that we satisfy the three criteria for vertex and edge discs. To move from a ribbon graph back to a cellularly embedded graph we can attach a disc to each boundary component of the ribbon graph; these will correspond to the faces in the embedded graph. Then we can shrink the vertex discs to points and the edge discs to line segments to arrive at a cellularly embedded graph. Thus we can think of ribbon graphs and cellularly embedded graphs in one-to-one correspondence and we consider two ribbon graphs to be equivalent if their corresponding cellularly embedded graphs are equivalent.

We now move our attention to another way to represent cellularly embedded graphs. The concept of a gem that we use here (although not the name) is due to Robertson [33] and was further developed by Lins and others. We will use a definition based on Lins [26].

An *edge-coloring* of a graph G is a labeling $\gamma_G : E(G) \to S$. If $|S| = k$ the edge-coloring is called a *k-edge-coloring*. The labels in S are called colors and the edges of a color $c_i \in S$ form a *color class* of edges, ${\gamma_G}^{-1}(c_i)$. A k-edge coloring is *proper* if adjacent edges, i.e., edges adjacent to a common vertex, have different labels; that is, if each color class is a matching.

A *graph-encoded map* or *gem* J is a finite cubic graph with a given proper 3-coloring of its edges. We will call these three colors c_v, c_f, and c_a; these are constants, not variables. As a convention, in figures we will consistently use the real colors red, blue, and yellow for c_v, c_f, and c_a, respectively. Further, we require that the components of the subgraph generated by the edges colored c_v and c_f be polygons with four edges each. We will call these polygons e-squares. We think of these three colors as vertex, facial, and auxiliary colors which will allow us to assign to our gem a unique, up to isomorphism, cellularly embedded graph. Note that this definition cannot cope with embeddings that have isolated vertices. So we also allow a component of a gem to be a "free loop" with no vertices or edges, but which we consider to be colored c_a. This free loop will represent an isolated vertex in the corresponding cellularly embedded graph.

A 2-edge-colored polygon or *bigon* in a gem J is a polygon in J whose edges are colored alternately with two of the three colors. The bigons have three types:

(a) *e-squares*, whose edges are labeled with c_v and c_f.

(b) *v-gons*, whose edges are labeled with c_v and c_a.

(c) *f-gons*, whose edges are labeled with c_f and c_a.

These three sets of bigons will correspond naturally to the edges, vertices, and faces, respectively of our cellularly embedded graph that is represented by J. A free loop is considered to be both a v-gon and an f-gon.

This definition will allow us to easily perform the operation of partial duality. First we will see how these gems correspond to ribbon graphs and therefore to cellularly embedded graphs. To move from a gem J to the corresponding ribbon graph G, we glue discs to the bigons that correspond to vertices and edges. This can be seen in the following three step process.

1. First glue a disc to each v-gon in J (so that the v-gon is identified with the boundary of the disc). These discs are the set $V(G)$ from the definition of a ribbon graph.
2. Next glue a disc to each e-square in J. These discs are the set $E(G)$ from the definition of a ribbon graph.

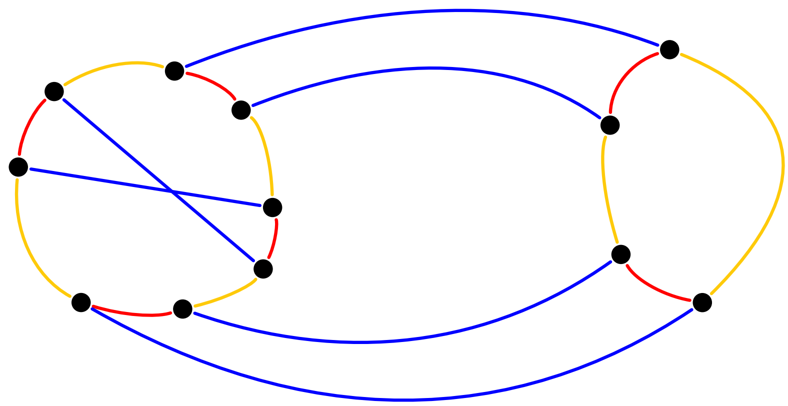


Figure 1.2: A gem J representing a cellularly embedded graph G with 2 vertices, 3 edges, and 1 face.

3. Finally remove all vertices and labeled edges from the gem J so that we are left with only two sets of discs $V(G)$ and $E(G)$ such that
   (a) the vertices and edges intersect in disjoint line segments,
   (b) each such line segment lies on the boundary of exactly one vertex and one edge, and
   (c) every edge contains exactly two such line segments.

This set of vertices and edges makes up a ribbon graph $G = (V(G), E(G))$ which has a corresponding cellularly embedded graph. Notice that the disjoint line segments where these two sets of discs intersect are precisely the edges labeled c_v in the gem J. At this stage we have shown how to move from a gem to a ribbon graph. To go from a ribbon graph to a gem we can essentially reverse the steps above while being careful where to place our labeled edges along the way. Given a ribbon graph $G = (V(G), E(G))$, we proceed as follows.

1. Label with c_v the disjoint line segments where the vertices and edges intersect and place a vertex at each end of every line segment.
2. Then label with c_f the two portions of the boundary of each disc in $E(G)$ that do not intersect with any disc in $V(G)$.
3. Then label with c_a the portions of the boundary of each disc in $V(G)$ that do not intersect with any disc in $E(G)$.

4. Now remove interiors of the discs $V(G)$ and $E(G)$ so that all that remains is a set of labeled edges which meet at vertices.

This gives a gem J. Since gems are in one-to-one correspondence with ribbon graphs they are also in one-to-one correspondence with cellularly embedded graphs. The cellularly embedded graph G associated with a gem J is the cellularly embedded graph derived from the ribbon graph corresponding with G through the process above. We will move freely between gems, ribbon graphs, and cellularly embedded graphs.

Next we give a third way to represent cellularly embedded graphs.

Given an embedded graph G, at each vertex $v \in V(G)$ the *transition graph* $T(G,v)$ is constructed as follows:

1. Create a vertex for each half-edge incident with v.
2. Join two (possibly equal) vertices by an edge for each time their corresponding half-edges occur in succession along a face in the embedding of G.

Thus at each vertex in an embedded graph G the transition graph $T(G,v)$ is isomorphic to a cycle of length $d(v)$. The *set of transition graphs of* **G**, $T(G)$, is the set $T(G) = \{T(G,v) : v \in V(G)\}$.

Next we will show that these transition graphs can be used as another method of describing an embedding. In the above definitions we described the process of how to obtain transition graphs from a given embedding. However we can also characterize cellularly embedded graphs using transition graphs, edge twists, and local orientations.

As defined by Mohar and Thomassen [32], an *embedding scheme* S is a graph G together with a clockwise cyclic orientation of the half-edges at each vertex (sometimes referred to as the *rotation* at that vertex) and a function assigning to each edge a *signature* (either $+1$ or -1).

An embedding scheme can be thought of as choosing a transition graph at each vertex, then choosing which direction along the cycle to define as clockwise, and then choosing whether each edge in G has a twist (signature). Determining an embedding scheme has the following steps:

1. Begin with an abstract graph G.
2. At each vertex $v \in V(G)$ construct a graph $T(G,v)$ by placing a vertex at each half-edge

incident with v and assign edges to $T(G, v)$ so that $T(G, v)$ is isomorphic to a cycle of length $d(v)$.

3. For each vertex $v \in V(G)$ assign a clockwise direction for the cycle $T(G, v)$.
4. For each edge $e \in E(G)$ assign a signature (untwisted $+1$ or twisted -1).

So an embedding scheme is a triple consisting of a set of transition graphs T, a consistent direction $\lambda(v)$ for each transition graph $T(G, v)$, and a signature function $\varepsilon(e)$. The first two of these determine a *rotation system*. We sometimes consider a rotation system to determine an embedding by itself, by taking all edge signatures to be $+1$.

In a cycle C of length l with cyclic vertex ordering $v_1v_2...v_l$ we can *split* a vertex v_i in C into two vertices u_1 and u_2 connected by an edge to create a new cycle C' with length $l+1$ and vertex ordering $v_1v_2...v_{i-1}u_1u_2v_{i+1}...v_l$. We will use this to translate from an embedding scheme to a gem (and thus to a cellularly embedded graph).

Let S be a embedding scheme for a graph G. We will construct a unique graph encoded map J from S. For each transition graph $T(G, v)$ in T, color the edges in $T(G, v)$ with c_a. Then split each vertex in $T(G, v)$ corresponding to a half-edge in G (half of an edge e, say) and color the new edge with c_v and associate it with the same edge e in G. Each transition graph then has been colored and split in such a way that it is a v-gon associated with the vertex v in G. These v-gons inherit the clockwise direction from S.

Then, for every edge $e \in E(G)$, we complete the two edges, e_1 and e_2, associated with e and colored c_v, to an e-square using the following procedure. If the signature associated with e is -1, then attach the vertices on the clockwise ends of e_1 and e_2 with an edge colored c_f and attach the vertices on the counterclockwise ends of e_1 and e_2 with an edge colored c_f. If the signature associated with e is $+1$, then attach the vertex on the clockwise end of e_1 with an edge colored c_f to the vertex on the counterclockwise end of e_2 and attach an edge colored c_f with endpoints being the two other ends of e_1 and e_2. This creates a unique gem associated with S.

Here are the steps to go from a gem to a (nonunique) corresponding embedding scheme. Given a gem J, select a local clockwise direction for each v-gon. If the edges colored c_f on an e-square

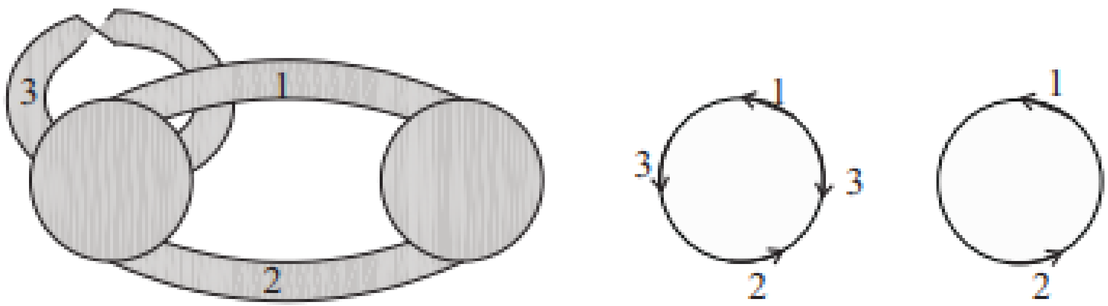


Figure 1.3: A ribbon graph G and its arrow-marked ribbon graph $G \ominus E(G)$ (Deng and Jin [11]).

associated with e connect the two clockwise ends of the edges colored c_v in e to each other (and similarly the two counterclockwise ends), then assign the signature -1 to e. Otherwise assign the signature $+1$. Then contract the edges colored c_v in J and replace them with a single edge e, but keep the direction on each v-gon as a local clockwise direction for the vertex. This gives a transition graph with direction function and signature function.

Throughout the majority of this paper we will use the gem representation of a cellularly embedded graph, though we sometimes make reference to ribbon graphs and embedding schemes.

We now give another way to represent ribbon graphs which will be used for defining partial duality.

An *arrow-marked ribbon graph* consists of a ribbon graph equipped with a collection of labeled arrows called *marking arrows* on the boundaries of its vertices. These marking arrows allow us to delete edges from the ribbon graph while maintaining the information about the position of each edge. The marking arrows are such that no marking arrow meets an edge of the ribbon graph, and there are exactly two marking arrows with each label. These two marking arrows mark the position of a missing edge and correspond to the two line segments in part (c) of the definition of a ribbon graph. Figure 1.3 shows a ribbon graph and a corresponding arrow-marked ribbon graph.

As noted, arrow-marked ribbon graphs will be useful for retaining information about an edge's placement upon deletion of that edge. Here is how that process works. Let G be a ribbon graph and $A \subseteq E(G)$. Then let $G \ominus A$ denote the arrow-marked ribbon graph obtained as follows: For each edge $e \in A$,

(a) arbitrarily orient the boundary of e,

(b) place an arrow on each of the two line segments where e meets a vertex of G that follows the orientation of the boundary of e,

(c) label these two arrows with e, then

(d) delete the edge e.

We denote by $G \oplus A$ the natural inverse of this operation where in this case A is a set of pairs of labels (pairs of marking arrows). In this process, for each label $e \in A$:

(a) take a disc and orient its boundary arbitrarily, then

(b) add this disc to the ribbon graph by identifying two non-intersecting arcs on the boundary of the disc with the two marking arrows colored e.

These discs now form edges in a new ribbon graph. From any arrow-marked ribbon graph we can repeat this process until all arrow markings are gone. Thus, we say that two arrow-marked ribbon graphs are equivalent if the ribbon graphs they describe are equivalent. We will commonly not distinguish between sets of labels and sets of edges since they represent the same objects. Recall that in a ribbon graph the disjoint line segments where the two sets of discs $V(G)$ and $E(G)$ intersect are precisely the edges labeled c_v in the gem J. Therefore, in the definition of arrow-marked ribbon graphs we see that the edges labeled c_v in the e-squares associated with an edge set A correspond directly to marking arrows in $G \ominus A$. This is important to keep in mind as we define partial duality for gems in order to see that this operation is the same as the one we defined for ribbon graphs.

1.3 Orientability

A *local orientation* at a point in a surface is an assignment of a clockwise direction do an open neighborhood homeomorphic to a disc around that point. A *curve* in a surface is a continuous image of the unit interval $[0, 1]$. A *consistent global orientation* of a surface is a local orientation at every point in the surface such that any two local orientations agree when translated along a curve.

A surface is called *orientable* if it can be given a consistent global orientation. A *local orientation* of a cellularly embedded graph is an assignment of a local orientation to disjoint neighborhoods homeomorphic to a disc around each vertex.

If the surface Σ on which a cellularly embedded graph G is embedded is orientable then we say that G is *orientable*. An equivalent definition of an orientable embedding is that G is orientable if its embedding has a *consistent global orientation*, meaning that local orientations assigned to each vertex agree when translated along an edge. Since all faces in cellular embeddings are homeomorphic to discs, a consistent global orientation of an embedded graph gives a consistent global orientation of the surface in which it is embedded. A third characterization of orientable cellularly embedded graphs is that the facial walks can be assigned directions in such a way that every edge is used once in each direction when traveling along a face. These directions can then be regarded as giving a global clockwise orientation. A cellularly embedded graph with an embedding scheme where every edge is assigned signature $+1$ is orientable. Also, an embedding where every cycle is two sided (every cycle has an even number of edges with signature -1) is orientable.

Orientability is a fundamental property of embeddings. The next theorem gives a characterization of orientable embeddings in terms of gems. This will help us observe how the property of orientability behaves under taking partial duals in a subsequent corollary.

Theorem 1.3.1 (Lins [26])**.** *Let J be a gem with a corresponding cellularly embedded graph $G \subseteq \Sigma$. Then the surface Σ, and hence the embedded graph G, is orientable if and only if J is bipartite.*

Proof. Let J be the gem corresponding to a cellularly embedded graph $G \subseteq \Sigma$. Assume that Σ is orientable. Since the embedding of G in Σ is orientable, the embedding of the gem J in the same surface Σ is orientable. Thus, there is a consistent orientation of J so that we may refer to the clockwise cyclic sequence of edges incident to a vertex in a meaningful way.

Now, under this embedding of J in Σ together with its orientation, consider the following bipartition of the vertices in J: Let X be the set of vertices whose clockwise cyclic sequence of colors is c_a, c_f, c_v. Let Y be the set of vertices whose clockwise cyclic sequence of colors is c_a,

c_v, c_f. First we must check that this is indeed a well-defined partition of $V(J)$. Indeed, by the orientability property described above, we can see that a each vertex has a unique clockwise cyclic sequence of edge labels corresponding to the three colors since J is a properly 3-edge colored cubic graph. It is also clear that each vertex is in exactly one set X or Y. So the bipartition is well defined. Now we check that each edge has an endpoint in X and an endpoint in Y. Let $e \in E(J)$. Because all bigons in J are facial walks, in order for the orientation to be consistent we must be able to orient the bigons in J so that if we traverse an oriented bigon then at each vertex we travel along the next edge in the clockwise cyclic sequence. Since all the bigons consist of two colors this means that the cyclic sequence is reversed at each vertex when traveling around a bigon. Now it becomes clear that for any edge $e \in E(J)$ one endpoint has clockwise cyclic orientation c_a, c_f, c_v and the other has c_a, c_v, c_f. Therefore we have shown that J is bipartite.

Now conversely assume that J is bipartite. Let X and Y be the two sets of the bipartition. Assign an orientation, B, of the edges of J such that all edges go from a tail in X to a head in Y. So every vertex in X is a source and every vertex in Y is a sink. Now we will show that we can orient the facial walks of J in such a way that each edge appears once in the direction of B and once in the direction opposite to B. For the v-gons we travel along the edges colored c_v in the same direction as the orientation B and along the edges colored c_a in the opposite direction to the orientation B. Then, in the f-gons we travel along the edges colored c_a in the direction of the orientation B and along the c_f edges in the opposite direction to B. Finally, we traverse the e-squares by going along the edges colored c_f in the direction of B and along the edges colored c_v in the opposite direction to B. So we traversed each face of J in such a way that each edge appears once in the direction of B and once in the direction opposite to B. Thus the embedding of J in Σ is orientable. $\square$

Chapter 2

Dualities

2.1 Duality

Given a cellularly embedded graph $G \subseteq \Sigma$, the *dual*, G^*, is formed by placing one vertex in each face of G and embedding an edge of G^* between two vertices of G^* for each edge that the corresponding faces of G share. Notice that there is a natural immersion of $G \cup G^*$ in Σ where each edge of G intersects with precisely one edge of G^* at exactly one point. We will call this the *standard immersion* of $G \cup G^*$. This immersion nicely portrays the natural bijection between $E(G)$ and $E(G^*)$ from the definition of duality. We will use e^* to denote the edge of G^* that corresponds to and intersects the edge e of G.

Recall that if we wish to change a ribbon graph G to a cellularly embedded graph we attach discs to the boundary components. These discs naturally correspond to the faces in the cellularly embedded graph. Thus if we fill in the boundary components with a set of discs, $V(G^*)$, we arrive at a surface without boundary. If we then remove the set of discs $V(G)$ we arrive at the dual $G^* = (V(G^*), E(G))$. This is how we define duality for ribbon graphs. We see clearly here the bijection between the edge sets of G and G^* because they are exactly the same.

The dual of a cellularly embedded graph G can be formed using gems. First we move from G to its associated gem J. Then the dual J^* is formed by swapping the colors c_v and c_f. Observe that this interchanges the v-gons and f-gons in the gem which, in the process described above, will cause vertices and faces to swap in the corresponding cellularly embedded graph. The dual G^* is the cellularly embedded graph associated with the gem J^*. We see that this is equivalent to the definition given for ribbon graphs. Indeed, the faces and vertices of G that correspond to the f-gons and v-gons in J are swapped by this process. The edges in the dual graph are between any two vertices in G^* that correspond to faces in G that share a common edge. This can be seen clearly by viewing the e-squares as displaying which vertices and faces are adjacent in both the graph and its dual, where these two get swapped by taking the dual.

The *Petrie dual* of a cellularly embedded graph G is obtained by giving every edge in G a twist (changing the signature of every edge in an embedding scheme representation) and is denoted by $G^{\times}$. For any edge set $A \subseteq E(G)$ the *partial Petrie dual with respect to* A is obtained from G by changing the signature of all edges in A and is denoted $G \times A$. We will show how Petrie duality and partial Petrie duality can be expressed using an extension of gems in Chapter 4.

2.2 Partial Duality

We begin with the definition for partial duality given by Chmutov [10]. In his 2009 paper, he introduced the concept of partial duality in order to generalize the dual of embedded graphs and explore properties of the Bollobás-Riordan polynomial of embedded graphs with signs attached to the edges. Though partial duality was introduced relatively recently, it has proven to be a fundamental operation on embedded graphs with applications extending well beyond Chmutov's original motivation. Partial duals of embedded graphs can have vastly different topological and graph-theoretical properties from the original embedded graph. Since we will primarily concern ourselves with characterizing embedded graphs based on the properties of the partial duality operation and are not so much concerned with graph polynomials, it will be useful for us to use the definition of Moffatt [24] rather than that of Chmutov. This definition is essentially the same as the Chmutov definition but does not use signs on edges of the graph. Both of the definitions use ribbon graphs to define partial duality.

We will now define partial duality using the definition from Moffatt [28] . Let G be a ribbon graph and $A \subseteq E(G)$. Then the partial dual G^A of G is formed by:

(a) "hiding" the edges that are not in A by replacing them with marking arrows using $G \ominus A^c$,

(b) forming the dual $(G \ominus A^c)^*$, then

(c) putting back in the edges that are not in A.

This gives us $G^A = (G \ominus A^c)^* \oplus A^c$.

Let G be a ribbon graph and $A \subseteq E(G)$. Then the *partial dual* of G with respect to A, denoted by G^A, is given by

$$G^A := (G \ominus A^c)^* \oplus A^c.$$

It becomes clear now that we are effectively taking the dual of the ribbon graph with respect to only a subset of the edges. The marking arrows are used to remove then replace edges that are not in A so that they are unaffected in some sense by taking the partial dual with respect to A.

We can also define partial duality using gems. The partial dual of a cellularly embedded graph G with respect to a subset of edges A with corresponding gem J and corresponding set of e-squares A is formed by swapping the colors c_v and c_f on the bigons corresponding to A in the gem J. This gives us the gem J^A that corresponds to the cellularly embedded graph G^A, the partial dual of G with respect to A. This appears to be a simpler way to define partial duality when compared to the definition using arrow-marked ribbon graphs, if it is indeed the same operation as in the definition using ribbon graphs. We now show that the two operations are equivalent.

Proposition 2.2.1. *Let G be a cellularly embedded graph with corresponding ribbon graph and graph encoded map. Then the partial dual of G with respect to an edge set A using the ribbon graph formulation is equivalent as a cellularly embedded graph to the partial dual of G with respect to the same edge set A using the gem formulation.*

Proof. Recall the steps to transition from a ribbon graph $G = (V(G), E(G))$ to a corresponding gem J as described above. Let us consider what happens to the gem as we take the partial dual of the ribbon graph $G = (V(G), E(G))$ with respect to a set of edges $A \subseteq E(G)$. Under the definition of partial duality for ribbon graphs, the underlying graph formed by the boundary of the discs in steps (1), (2), and (3) is unchanged. Some edges are deleted and then replaced in the same position and other edges change label when we dualize, but the underlying graph of the gem does not change. So all that remains to be seen is that the edges that change label in the gem are precisely the swaps on the e-squares for edges in A. Notice that none of the edges labeled c_a

change because boundary segments of discs in $V(G)$ that do not intersect any disc in $E(G)$ are unaffected by partial duality on the ribbon graph G. The labels on the edges corresponding to e-squares in A^c are unchanged because these discs are removed and then put back with the exact same portions intersecting vertices in the ribbon graph. The labels on the edges corresponding to e-squares in A swap when we dualize $G \ominus A^c$. Indeed, for each edge in A the portions of the edge that intersect vertices swap with the portions of the edge that do not intersect any vertex. In the gem, this corresponds to swapping the labels c_v and c_f on all of the e-squares in A and making no other changes. This is precisely the definition of partial duality using the gem formulation. □

So we have shown that the definition of partial duality using gems is equivalent to the operation defined using ribbon graphs. This definition of partial duality proves to be valuable in providing alternate methods for proving properties of the partial duals of cellularly embedded graphs. This is the main reason why we will routinely choose to use the gem representation in this paper. From the definition of partial duality using gems we obtain an immediate Corollary of Theorem 1.3.1.

Corollary 2.2.2. *G^A is orientable if and only if G is orientable.*

Partial duality preserves orientability since taking the partial dual does not affect whether or not the underlying graph of the corresponding gem is bipartite. It is also worth noting that J being bipartite is equivalent to all closed walks in J being of even length. As it turns out, there are other interesting consequences of characteristics of closed walks in J. The parity of the number of edges of each color in closed walks in a gem tells us characteristics of the corresponding cellularly embedded graph. We will discuss results of this type in Chapter 4.

2.2.1 Properties of Partial Duality

Proposition 2.2.3. *Properties of Partial Duality*

The following are properties of the operation of partial duality defined above. Here we let G be a cellularly embedded graph with $A, B \subseteq E(G)$ and $e \in E(G)$.

(a) $G^{E(G)} = G^*$.

(b) $G^{\emptyset} = G$.

(c) $(G^A)^A = G$.

(d) $(G^A)^B = (G^B)^A$.

(e) $(G^A)^B = G^{A \Delta B}$ *where* $A \Delta B = (A \cup B) \setminus (A \cap B)$ *is the symmetric difference of* A *and* B.

Proof. (a) This is clear from the way we defined partial duality. Taking the partial dual with respect to all the edges is the same as taking the dual since in both cases we swap the colors c_v and c_f on all e-squares in the gem and then $G^{E(G)} = G^*$ is the corresponding cellularly embedded graph.

(b) This follows directly from the definition since swapping colors on an empty set of e-squares leaves the gem and hence the corresponding cellularly embedded graph unchanged.

(e) Swapping colors on the e-squares for A and then swapping colors on the e-squares for B is equivalent to swapping the colors on the e-squares for $A \Delta B$. Therefore $(G^A)^B = G^{A \Delta B}$.

(c), (d) These follow directly from (e). □

Let us examine how taking the partial dual of a cellularly embedded graph with respect to a single edge impacts the graph. Taking the partial dual with respect to a single edge can be categorized by whether that edge is (a) a non-loop edge (a *link*), (b) an orientable (untwisted) loop, or (c) a nonorientable (twisted) loop. These three situations are shown in Figure 2.1. If we dualize a link, the two ends of the link become a single vertex and the link becomes an untwisted loop; if we dualize an untwisted loop the vertex to which the loop is incident splits into two vertices and the loop becomes a link; and if we dualize a twisted loop it remains a twisted loop although the vertex to which it is incident has a new cyclic order of edges.

Notice that we can define edge deletion for ribbon graphs in the natural way. Deletion of edges in a ribbon graph results in another ribbon graph which also describes a cellularly embedded graph. This will be a useful operation since in general deletion of edges from a cellularly embedded graph may result in a non-cellularly embedded graph. So, we use the following definition.

Let G be a ribbon graph, $e \in E(G)$, and $v \in V(G)$. Then $G - e$ or $G \setminus e$ denotes the ribbon graph obtained from G by deleting the edge e, and $G - v$ denotes the ribbon graph obtained from G by deleting the vertex v and all of its incident edges. A ribbon graph H is a *ribbon subgraph* of

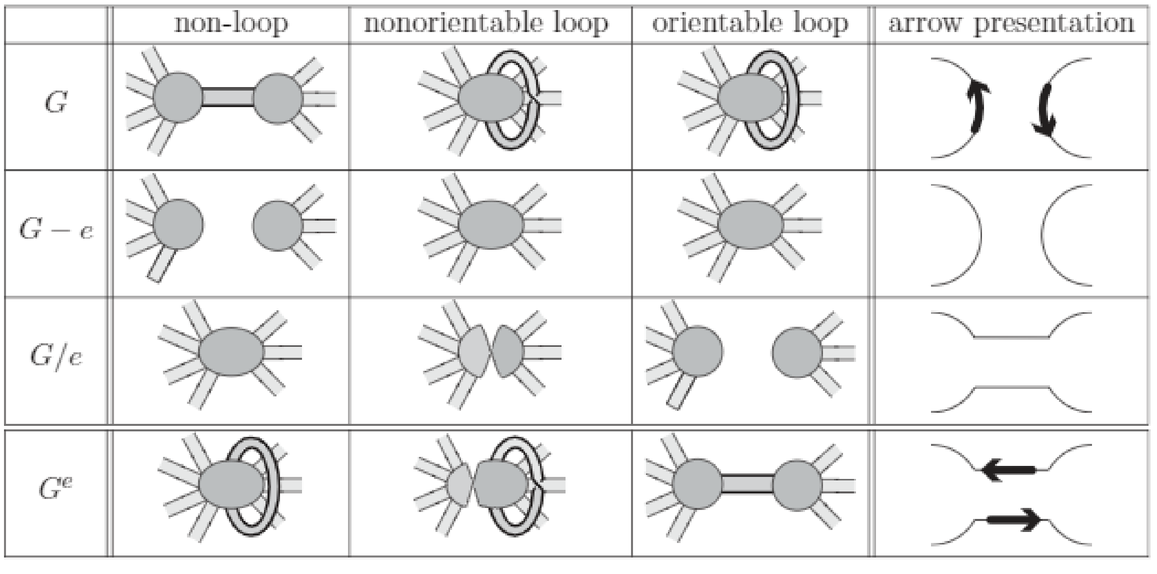


Figure 2.1: Partial Dual with a Single Edge (Moffatt [30])

G if it can be obtained from G by deleting vertices and edges. If v_1 and v_2 are the (not necessarily distinct) vertices incident to e, then G/e denotes the ribbon graph obtained as follows: consider the boundary component(s) of $e \cup v_1 \cup v_2$ as curves on G. For each resulting curve attach a disc (which will form a vertex of G/e) by identifying its boundary component with the curve. Delete e, v_1, and v_2 from the resulting complex to get the ribbon graph G/e. We say G/e is obtained from G by *contracting* e. We say that H is a *ribbon graph minor* of a ribbon graph G if H is obtained from G by a sequence of edge deletions, vertex deletions, or edge contractions. We also say that G has an *H-ribbon graph minor* if it has a ribbon graph minor equivalent to H.

The definition of ribbon graph minors can also be translated to the language of gems. To delete an edge e in a gem J we remove the edges colored c_f in the e-square for e and then replace each maximal path with internal vertices of degree 2 by a single edge of color c_a (or a cycle of degree 2 vertices by a free loop of color c_a). To contract an edge e we proceed in the same way except that we remove the edges colored c_v, not c_f, in the e-square for e.

We now have three operations on edges in cellularly embedded graphs (deletion, contraction, and partial duality) and in the following lemma we give properties of how these operations interact.

All of these properties follow directly from the definitions of the three operations.

Lemma 2.2.4. *Let G be a cellularly embedded graph and $e, f \in E(G)$ with $e \neq f$. Then*

1. $G\backslash e/f = G/f\backslash e$.
2. $G\backslash e = (G^e)/e$ *and* $G/e = (G^e)\backslash e$.
3. $(G^e)\backslash f = (G\backslash f)^e$ *and* $(G^e)/f = (G/f)^e$.

We can extend these definitions and properties to the level of sets of edges. For disjoint sets B, C we have $G\backslash B/C = G/C\backslash B$ and we can form this all at once in terms of the gem J. In this process we delete c_f edges for e-squares in B, c_v edges for e-squares in C, and then replace all maximal paths with internal vertices of degree 2 by edges colored c_a and all cycles of vertices of degree 2 by free loops colored c_a. We can also combine and extend properties (1), (2), and (3) similarly to obtain properties like $G^A\backslash B/C = (G\backslash((B\backslash A)\cup(C\cap A))/((B\cap A)\cup(C\backslash A)))^{A\backslash B\backslash C}$ for disjoint edge sets B, C and an arbitrary edge set A.

We now turn our attention to the effects of partial duality on the medial graph of a cellularly embedded graph. Let G be a cellularly embedded graph in Σ. We construct its *medial graph* $M = \mathit{Med}(G)$ by placing a vertex $v(e)$ on each edge e of G and then for each face f with boundary $v_1e_1v_2e_2...v_{d(f)}e_{d(f)}$ we draw the edges $(v(e_1), v(e_2)), (v(e_2), v(e_3)), ..., (v(e_{d(f)}), v(e_1))$ in a natural non-intersecting way inside the face f. Note that M is a 4-regular cellularly embedded graph in Σ. If we think of G as a cellularly embedded graph resulting from a gem J, we see that there is a natural bijection between the edges colored c_a in J and the edges of the medial graph. The medial graph may be obtained from the gem by contracting each e-square to a vertex.

A *checkerboard coloring* of a cellularly embedded graph is a proper 2-face coloring where we assign the color black or white to each face in such a way that adjacent faces receive different colors. A medial graph M of a cellularly embedded graph G can always be given a checkerboard coloring by assigning the color black to all faces in M that correspond to a vertex in the original graph G and coloring the remaining faces, which correspond to a face in G, white. We call this the *canonical checkerboard coloring* of M.

Although they did not explicitly state it in these terms, Corollary 3.20 in Ellis-Monaghan and

Moffatt's book [14] shows that taking the Chmutov partial dual of an edge e in a cellularly embedded graph G corresponds to flipping $v(e)$ in the checkerboard-colored, rigid-vertex, vertex-labeled medial graph M of G. This result also appears in an earlier paper from Ellis-Monaghan and Moffatt [13, Theorem 4.24]. By rigid-vertex we mean that we allow only for flips of vertices (reversals of the cyclic ordering of edges around a vertex) in the embedding of the medial graph.

Proposition 2.2.5 (Ellis-Monaghan and Moffatt [14, Corollary 3.20])**.** *Suppose G and H are embedded graphs with checkerboard-colored embedded medial graphs M and N respectively. Then $H = G^A$ for some $A \subseteq E(G)$ if and only if N is obtained from M by reversing the local clockwise ordering around each vertex of M corresponding to an edge in A, with the checkerboard coloring swapping colors around vertices corresponding to edges in A and remaining unchanged around vertices corresponding to edges in $E(G) \setminus A$.*

Proof. Swapping the colors c_v and c_f on the e-square in the gem corresponding to an edge e corresponds precisely to reversing the rotation of, and swapping the colors around, the vertex $v(e)$ in the checkerboard-colored embedded medial graph. □

2.2.2 Single-Vertex Partial Duals

We now move our attention to single-vertex embeddings, also known as bouquets. Single-vertex embeddings are an important special class of embeddings that have been studied from a number of perspectives, and have connections to other ideas such as chord diagrams. Every connected embedded graph has a partial dual that is a single-vertex embedding, namely by dualizing the edges of a spanning tree. Since every connected embedded graph has at least one single-vertex partial dual, it is natural to ask when all the partial duals are single-vertex.

Next we will give a characterization of when all partial duals of an embedded graph have only one vertex. Consider such an embedded graph G with gem J. First, recall that if an embedded graph has an untwisted loop, then taking the partial dual with respect to that loop splits the incident vertex into two vertices. So G (and all partial duals of G) contain no untwisted loops. Clearly,

G also contains no links, so all edges in G are twisted loops. Two loops e and f are called *interlaced* if they share a common vertex v and the cyclic order of edge labels around v is of the form $AeBfCeDf$ where A, B, C, and D are ordered lists of edge labels. Next, observe that if two twisted loops are interlaced, then taking the partial dual with respect to one of the two twisted loops results in the other loop becoming untwisted. So G has no interlaced twisted loops. If there are no interlaced twisted loops, then taking the partial dual with respect to a noninterlaced twisted loop keeps that edge and all other edges as noninterlaced twisted loops. So we arrive at the following characterization.

Theorem 2.2.6. *An embedded graph G with gem J is such that G^A has a single vertex for all $A \subseteq E(G)$ if and only if G has a single vertex and every edge of G is a noninterlaced twisted loop.*

A graph is *outerplanar* if it has an embedding in the plane with every vertex on the boundary of the outer face.

The *end graph* of a cellularly embedded graph G is formed by first creating a vertex for each half-edge in G. Two vertices in the end graph are adjacent if they represent half-edges of the same edge in G or if their half-edges are next to each other in cyclic order around a vertex in G. Thus the end graph is a 3-regular graph with $2E(G)$ vertices. Because the ends of G are in one-to-one correspondence with the edges colored c_v in the gem, the end graph of a cellularly embedded graph G with corresponding gem J can be formed by contracting all edges colored c_v in J, removing one of the two parallel edges colored c_f in J corresponding to an edge $e \in E(G)$, and removing the colors from the remaining edges in J.

The *colored end graph* is obtained from the gem J by the same process with the exception of the final step of removing the colors from the remaining edges in J.

Theorem 2.2.7. *All partial duals of an embedded graph G with gem J have only one vertex if and only if every edge is a twisted loop and the colored end graph can be drawn as an outerplanar graph where all outside edges are colored c_a.*

Proof. Assume that all partial duals of an embedded graph G with gem J have only one vertex.

Then G is a single vertex with no interlaced twisted loops. Then we can draw J in the plane in such a way that there is a single v-gon on the exterior of the drawing and none of the e-squares intersect with each other. In this drawing the only intersections are between two edges colored c_f on the same e-square. Contracting the edges colored c_v from J results in a cycle of edges colored c_a on the outer cycle of the drawing. Upon removal of parallel edges colored c_f we obtain the colored end graph of G. None of the edges in this drawing intersect and every vertex is on the outer cycle.

Now assume that every edge of G is a twisted loop and the colored end graph can be drawn as an outerplanar graph where all outside edges are colored c_a. Begin with such a drawing of the colored end graph of G. Replace each edge colored c_a by a path of length two where the edges are colored c_v and c_a in clockwise order around the outer cycle. Then for every path of length three with edges colored c_v, c_f, and c_v, join the two ends by an edge colored c_f to create an e-square. This is a drawing of a gem J representing G. Since all the edges are noninterlaced twisted loops, G is such that G^A has a single vertex for all $A \subseteq E(G)$. $\square$

Chapter 3

Medial Graphs and Crossing-Total Directions

The work of Huggett and Moffatt [24] was further extended by Deng and Jin [11] by using all-crossing directions of medial graphs. In this chapter we will give an overview of their main results and then point out how their results may be proved and generalized using gems. Huggett and Moffatt [24], and later Deng and Jin [11], worked to characterize partial duals of a cellularly embedded graph G with certain properties like being bipartite or Eulerian in terms of directions of the medial graph of G. Here we consider medial graph directions separately from partial duality, making it clear that they are two separate issues: what properties correspond to certain medial graph directions, and how medial graph directions are changed by taking partial duals. We will also unify this with the effects on the medial graph of taking partial Petrie duals.

3.1 Medial Graph Directions

3.1.1 Crossing-Total Directions

A *crossing-total direction* of a medial graph M of a cellularly embedded graph G is an assignment of an orientation to each edge of M in such a way that for each vertex $v(e)$ of M, the edges incident with $v(e)$ are "in, in, out, out," "in, in, in, in" or "out, out, out, out" in cyclic order with respect to $v(e)$. If M is given the canonical checkerboard coloring and a fixed crossing-total direc-

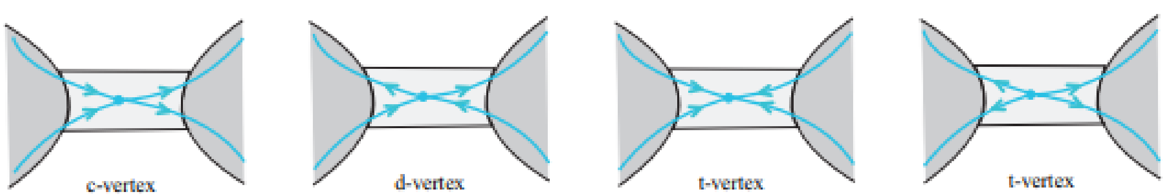


Figure 3.1: $\{c, d, t\}$-vertices (Deng and Jin [11])

tion, then we can partition vertices of M into three classes. First observe that each vertex in M is met by two black faces and two white faces in the checkerboard coloring of M, which correspond to vertices and faces respectively in G. Using this coloring, a vertex $v(e)$ in M is:

(a) a *c-vertex* if the two white faces that meet at $v(e)$ are each incident to an "in" edge and an "out" edge of $v(e)$; in this case the two black faces that meet at $v(e)$ are incident to two "in" edges or two "out" edges.

(b) a *d-vertex* if the two black faces that meet at $v(e)$ are each incident to an "in" edge and an "out" edge of $v(e)$; in this case the two white faces that meet at $v(e)$ are incident to two "in" edges or two "out" edges.

(c) a *t-vertex* if the edges incident with $v(e)$ are "in, in, in, in" or "out, out, out, out."

The corresponding edge e in G is called a *c-edge, d-edge, or t-edge* in conjunction with its vertex in the medial graph $v(e)$. Figure 3.1 depicts c-, d-, and t-vertices. A crossing-total direction with no t-vertices is called an *all-crossing direction.*

It is worth noting that $M = \mathit{Med}(G) = \mathit{Med}(G^*)$ as cellularly embedded graphs (without the checkerboard coloring) and the canonical checkerboard coloring of M considered as $\mathit{Med}(G^*)$ can be obtained from the canonical checkerboard coloring of M considered as $\mathit{Med}(G)$ by swapping the colors black and white on all faces of $M = \mathit{Med}(G)$. This allows us to directly observe the following lemma.

Lemma 3.1.1. *Let G be a cellularly embedded graph and $e \in E(G)$. Under a fixed crossing-total direction of $M = \mathit{Med}(G)$, we have:*

1. *e is a c-edge in G if and only if e is a d-edge in G^*.*
2. *e is a d-edge in G if and only if e is a c-edge in G^*.*
3. *e is a t-edge in G if and only if e is a t-edge in G^*.*

3.1.2 Graph States

Let M be a canonically checkerboard colored medial graph of G. As we noted previously, at each vertex $v(e) \in V(M)$ there are two black faces and two white faces meeting one another. We

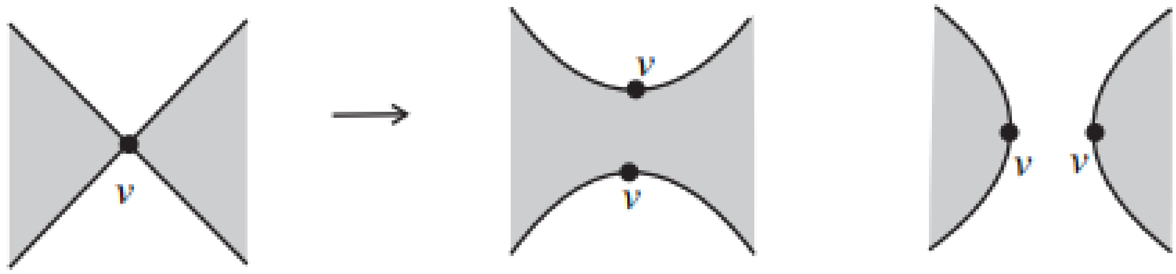


Figure 3.2: White (center) and Black (right) smoothings (Deng and Jin [11])

will consider two ways that we can split the vertex into two vertices of degree 2. We can take a *white smoothing* where we split the vertex so that the white faces remain unchanged and the black faces are combined into a single face. We can similarly take a *black smoothing* such that the two black faces remain unchanged and the two white faces become one face. These can be seen in Figure 3.2.

Let G be a cellularly embedded graph in Σ. A *state* S of G specifies a choice of either a white smoothing or a black smoothing at each vertex of M. After splitting each vertex according to the state S, we obtain a set of disjoint closed cycles in Σ. We call these disjoint closed cycles the *state circles* of S. States in the medial graph translate nicely to the language of partial duals.

Let $A \subseteq E(G)$ and $V_A = \{v(e)|e \in A\} \subseteq V(M)$. A state S_A of G is the choice of choosing white smoothing for each vertex of V_A and black smoothing for each vertex of $V(M) \setminus V_A = V_{A^c}$. This is called the state associated with A. The state associated with A has an important relationship with the partial dual of G with respect to A. This relationship can be seen in the following lemma.

Lemma 3.1.2. *Let $G \subseteq \Sigma$ be a cellularly embedded graph and $A \subseteq E(G)$. Let S_A be the graph state of G associated with A. Then there are natural bijections taking each of the following four pairs of objects to any of the other three pairs.*

(a) State circles of S_A, and their degree 2 vertices.

(b) Boundary components of $G \ominus A^c$, and their marking arrows in $(G \ominus A^c)^* \ominus A$.

(c) Vertices of G^A, and their marking arrows in $(G \ominus A^c)^* \ominus A$.

(d) v-gons in the gem of G^A, and their edges labeled c_v.

Proof. We know the bijection between (b) and (c) from the definition of partial duality for ribbon graphs, Definition 2.2. So we need only to show that there is a bijection between the pairs (c) and (d) and between the pairs (a) and (d).

For the bijection between (c) and (d) we know that v-gons in the gem of G correspond to vertices of G by the correspondence between gems and ribbon graphs. By this same correspondence edges labeled c_v correspond to marking arrows in $G \ominus E(G)$. Applying these observations with G replaced by G^A, v-gons in the gem of G^A correspond to vertices of G^A and edges labeled c_v in the gem of G^A correspond to marking arrows in $(G \ominus A^c)^* \ominus A = G^A \ominus E(G)$.

For the bijection between the pairs (a) and (d) we have already seen that the edges colored c_a in G are in one-to-one correspondence with the edges of the medial graph M. Furthermore, the vertices in M are in one-to-one correspondence with the e-squares in G. Then we can think of a black or a white smoothing at an vertex $v(e)$ in M with e-square $Q(e)$ as corresponding directly to whether or not we choose to associate the resulting split vertex with the two edges colored c_f in $Q(e)$ or the two edges colored c_v in $Q(e)$. In a white smoothing we are associating the resulting split vertex with the two edges colored c_f. In a black smoothing we are associating the resulting split vertex with the two edges colored c_v. Now, when we take the partial dual of G with respect to A we swap the colors c_v and c_f on the e-squares associated with the edges in A. So, when we take a white smoothing on each edge in A and a black smoothing on each edge in A^c to arrive at S_A, we see that we associate with each split vertex the two edges colored c_v on the e-squares in G^A. Thus we see that the edges colored c_v are in one-to-one correspondence with the vertices of degree two along the state circles of S_A. Furthermore, the state circles themselves are in one-to-one correspondence with the v-gons in G^A. □

This lemma is foundational to the following proofs, which were proved using the language of ribbon graphs in papers from Huggett, Moffatt, Deng, and Jin [11, 24]. Its first direct application is in characterizing when the partial dual of a cellularly embedded graph is Eulerian.

A graph G is *Eulerian* if it is connected and has a closed trail containing all the edges in G. The following theorem is well known.

Theorem 3.1.3. *A graph G is Eulerian if and only if it is connected and every vertex in G has even degree.*

A state is called *even* if each of its state circles is a cycle of even length.

Lemma 3.1.4. *Let G be a connected, cellularly embedded graph and $A \subseteq E(G)$. Then G^A is Eulerian if and only if S_A is even.*

Proof. We first observe that the degree of a vertex v of G^A is equal to the number of edges colored c_v in its corresponding v-gon in J^A. But, by Lemma 3.1.2 we know that the number of edges colored c_v in this v-gon is exactly equal to the number of vertices in its corresponding state circle. Thus, we see that the degree of v is even if and only if the length of its corresponding state circle is even. Therefore, G^A is Eulerian if and only if S_A is even. □

Using this lemma, Deng and Jin [11] were able to give a characterization of when the partial dual of an embedded graph is Eulerian. They made this characterization in terms of crossing-total directions of the medial graph.

Lemma 3.1.5 (Deng and Jin [11])**.** *Let G be a connected, cellularly embedded graph. Then G is Eulerian if and only if there is a crossing-total direction of its medial graph M in which all vertices are c-vertices or t-vertices.*

Proof. Assume that $G = G^\emptyset$ is Eulerian. Then by Lemma 3.1.4, we know that $S_\emptyset$ is even. Now we can give an alternating orientation to each circle of $S_\emptyset$ so that each degree two vertex either has indegree two (a sink) or outdegree two (a source). We can transfer these state circles back to G by reversing the black/white smoothing procedure done to obtain $S_\emptyset$. Let v be a vertex of M corresponding to two vertices v', v'' in $S_\emptyset$. If v' and v'' are both sources or sinks, then v is a t-vertex in M; otherwise, since all smoothings for $S_\emptyset$ were black smoothings, v is a c-vertex. So we have an orientation of M where all vertices are c- or t-vertices.

Now conversely assume that there is a crossing-total direction of M in which all vertices are c-vertices or t-vertices. Then consider the state $S_\emptyset$. We can take a black smoothing at every vertex of M and the resulting degree two vertices have indegree two or outdegree two. Thus every state circle has alternating edge orientations and therefore $S_\emptyset$ is an even state. By Lemma 3.1.4, we then know that G is Eulerian. □

In Lemma 3.1.1 we observed that c-edges and d-edges swap in the dual while t-edges remain t-edges. This lemma can be extended to the operation of partial duality as well. According to Proposition 2.2.5, if we dualize a set $A \subseteq E(G)$ and $e \in A$, then the cyclic order of the edges at $v(e)$ in M is reversed and the colors in the checkerboard coloring around $v(e)$ are swapped. From this, we see that if e was a c-edge in G it is a d-edge in G^e and if e was a d-edge in G it is a c-edge in G^e. It is clear that a t-edge e remains a t-edge in G^e because the medial graph edges at $v(e)$ remain all directed in or all directed out. This gives us the following lemma.

Lemma 3.1.6. *Let G be a cellularly embedded graph, $A \subseteq E(G)$ with $e_1 \in A$ and $e_2 \notin A$. Then under a fixed crossing-total direction of M, we have:*

1. *e_1 is a c-edge in G if and only if e_1 is a d-edge in G^A.*
2. *e_1 is a d-edge in G if and only if e_1 is a c-edge in G^A.*
3. *e_1 is a t-edge in G if and only if e_1 is a t-edge in G^A.*
4. *e_2 is a c-edge in G if and only if e_2 is a c-edge in G^A.*
5. *e_2 is a d-edge in G if and only if e_2 is a d-edge in G^A.*
6. *e_2 is a t-edge in G if and only if e_2 is a t-edge in G^A.*

The following theorem follows directly from Lemma 3.1.2 and Lemma 3.1.6.

Theorem 3.1.7 (Deng and Jin [11, Theorem 1.5])**.** *Let G be a connected, cellularly embedded graph with medial graph M and $A \subseteq E(G)$. Then G^A is Eulerian if and only if there exists a crossing-total direction of M for which A is the union of the set of all d-edges and a set of some t-edges.*

Observation 3.1.8. *Let G be a connected, cellularly embedded graph, and $A \subseteq E(G)$. Then G^A is an even-face graph if and only if G^{A^c} is Eulerian.*

The following corollary follows directly from Observation 3.1.8 and Theorem 3.1.7.

Corollary 3.1.9 (Deng and Jin [11, Corollary 3.5])**.** *Let G be a cellularly embedded graph with medial graph M and $A \subseteq E(G)$. Then G^A is an even-face graph if and only if there exists a crossing-total direction of M for which A is the union of the set of all c-edges and a set of some t-edges.*

In the same paper [11], Deng and Jin give a necessary and sufficient condition for the partial dual of a cellularly embedded graph to be bipartite. This is an extension of a result from Huggett and Moffatt for plane graphs [24]. Recall that an all-crossing direction of M is a crossing-total direction with only c-edges and d-edges.

Theorem 3.1.10 (Huggett and Moffatt [24])**.** *Let G be a plane graph with medial graph M and $A \subseteq E(G)$. Then the partial dual G^A is bipartite if and only if there exists an all-crossing direction of M for which A is the set of c-edges.*

The following theorem extends Huggett and Moffatt's result for plane graphs, Theorem 3.1.10, to all orientable cellularly embedded graphs. Deng and Jin were also able to extend this result to the case of nonorientable cellularly embedded graphs by constructing a "modified medial graph" in which they add a 2-valent vertex to each twisted edge in M. This result and proof are very similar to that of the orientable case, so for ease of notation we give the theorem with orientable embeddings and the "unmodified" medial graph. We can also extend Theorem 3.1.11 to the nonorientable case by using bidirections of the medial graph as shown in Chapter 4 (condition (010)).

Theorem 3.1.11 (Deng and Jin [11])**.** *Let G be an orientable cellularly embedded graph with medial graph M and $A \subseteq E(G)$. Then G^A is bipartite if and only if there exists an all-crossing direction of M for which A is the set of c-edges.*

We will prove Theorem 3.1.11 by proving a simpler result and combining it with Lemma 3.1.6.

Lemma 3.1.12. *Let G be an orientable cellularly embedded graph. Then G is bipartite if and only if there is an all-crossing direction of its medial graph M in which every edge is a d-edge.*

Proof. Let $G \subseteq \Sigma$ be an orientable cellularly embedded graph with corresponding gem J and medial graph M. Let B be a fixed orientation of Σ. Consider the state $S_\emptyset$ in which each vertex of the medial graph M receives a black smoothing. By Lemma 3.1.2 the state circles of $S_\emptyset$ are in one-to-one correspondence with the vertices of G that are given by the v-gons of J. The edges labeled c_a on the v-gons of J are in one-to-one correspondence with the edges of the medial graph M. If M has an all-crossing direction in which every edge is a d-edge, then the state circles of $S_\emptyset$ are directed cycles. Let X be the set of all state circles oriented clockwise relative to B and Y be the set of all state circles oriented counterclockwise relative to B. Notice that since every edge is a d-edge and black smoothings were applied to each vertex in M, each vertex in M was smoothed to a degree 2 vertex of a state circle in X and a degree 2 vertex of a state circle in Y. Then using the bijection between state circles and vertices of G we see that (X, Y) induces a bipartition of $V(G)$.

Conversely assume that G has a bipartition (X, Y). Then we can orient the v-gons representing vertices in X in the clockwise direction and the v-gons representing vertices in Y in the counterclockwise direction relative to B. The induced directions on the edges labeled c_a gives an all-crossing direction of the medial graph M. This all-crossing direction has only d-edges. □

Theorem 3.1.11 now follows from Lemma 3.1.12 and Lemma 3.1.6.

3.1.3 Balanced Directions

Let us now observe the interaction between partial Petrie duality and crossing directions of the medial graph. We first must observe that crossing-total directions of the medial graph are not fixed under Petrie duality. In addition to c-, d-, and t-edges there is a fourth type of edge involved.

A *b-vertex* in an orientation of the medial graph is a vertex $v(e)$ such that the edges around $v(e)$ alternate "in," "out," "in," "out." The edge e in G is called a *b-edge*. See Figure 3.3.

An *all-balanced direction* of the medial graph is an orientation of the medial graph such that at each vertex the number of in edges is equal to the number of out edges. An all-balanced direction of the medial graph can contain b-, c-, and d-vertices.

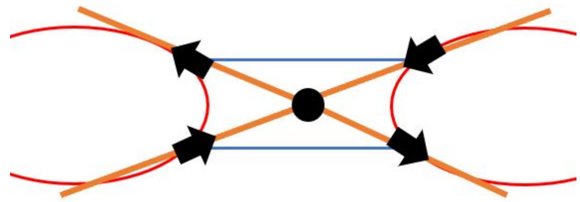


Figure 3.3: A b-vertex in the medial graph of G drawn on a b-edge in G

A *balanced-total direction* of the medial graph is an orientation of the medial graph such that at each vertex there is an even number of in edges and an even number of out edges. A balanced-total direction of the medial graph can contain b-, c-, d-, and t-vertices.

Throughout this section we will use $G * A$ to denote the partial dual of G with respect to A and $G \times A$ to denote the partial Petrie dual of G with respect to A. We can extend Lemma 3.1.6 to include b-edges.

Let us observe what happens to b-, c-, d-, and t-vertices under Petrie duality. Since each edge is being given a twist, we can think of the effect of Petrie duality on the medial graph as swapping (in cyclic order around $v(e)$) the two medial graph edges on one of the two black faces in the canonical checkerboard coloring. This is equivalent to flipping one of the sides of a vertex in the medial graph M which represents twisting an edge in the original graph G.

Now that we have this characterization of the effect of Petrie duality on the medial graph, the following lemma follows immediately.

Lemma 3.1.13. *Let G be a cellularly embedded graph, $A \subseteq E(G)$ with $e_1 \in A$ and $e_2 \notin A$. Then under a fixed balanced-total direction of M, we have:*

1. *e_1 is a b-edge in G if and only if e_1 is a d-edge in $G \times A$.*
2. *e_1 is a c-edge in G if and only if e_1 is a c-edge in $G \times A$.*
3. *e_1 is a d-edge in G if and only if e_1 is a b-edge in $G \times A$.*
4. *e_1 is a t-edge in G if and only if e_1 is a t-edge in $G \times A$.*
5. *e_2 is a b-edge in G if and only if e_1 is a b-edge in $G \times A$.*
6. *e_2 is a c-edge in G if and only if e_2 is a c-edge in $G \times A$.*

7. e_2 *is a d-edge in* G *if and only if* e_2 *is a d-edge in* $G \times A$.
8. e_2 *is a t-edge in* G *if and only if* e_2 *is a t-edge in* $G \times A$.
9. e_1 *is a b-edge in* G *if and only if* e_1 *is a b-edge in* $G * A$.
10. e_2 *is a b-edge in* G *if and only if* e_2 *is a b-edge in* $G * A$.

We can also now incorporate partial duals and partial Petrie duals together by combining Lemma 3.1.6 and Lemma 3.1.13. For example, given $e \in A \cap B$ where $A, B \subseteq E(G)$, if e is a b-edge in $G * A \times B$ then e is a c-edge in G. Note that partial duality and partial Petrie duality do not commute.

Now we can extend Theorem 3.1.7 to include both partial duality and partial Petrie duality. The following theorem follow directly from Lemma 3.1.6, Lemma 3.1.13, and Theorem 3.1.7.

Theorem 3.1.14. *Let* G *be a connected, cellularly embedded graph with corresponding medial graph* M *and* $A, B \subseteq E(G)$.

1. $G \times B * A$ *is Eulerian if and only if there exists a balanced-total direction of* M *for which* $A \setminus B$ *is the union of the set of all d-edges and a set of some t-edges, and* $A \cap B$ *is the union of the set of all b-edges and a set of some t-edges.*
2. $G * A \times B$ *is Eulerian if and only if there exists a crossing-total direction of* M *for which* A *is the union of the set of all d-edges and a set of some t-edges.*

Proof. Assume $G \times B * A$ is Eulerian. Then by Lemma 3.1.5, the medial graph $M \times B * A$ has a crossing-total direction with only c-vertices and t-vertices. Then, applying Lemma 3.1.6 by taking a partial dual with respect to A, $M \times B$ has a crossing-total direction where A is the union of the set of all d-edges and the set of some t-edges. Applying Lemma 3.1.13 by taking a partial Petrie dual with respect to B we see that M has a crossing total direction where edges in $A \cap B$ are either b-edges or t-edges. This is because partial Petrie duals swap b- and d-edges and partial duals swap c- and d-edges. All other edges remain fixed so the only edges in B that change type are those also in A. $A \setminus B$ is the union of the set of all d-edges and the set of some t-edges arising from a balanced-total direction of the medial graph M and $A \cap B$ is the union of the set of all b-edges and the set of some t-edges arising from the same balanced-total direction.

Now assume $A \setminus B$ is the union of the set of all d-edges and the set of some t-edges arising from a balanced-total direction $\mathcal{B}$ of the medial graph M and $A \cap B$ is the union of the set of all b-edges and the set of some t-edges arising from $\mathcal{B}$. Then, by Lemma 3.1.13 by taking a partial Petrie dual with respect to B, $G \times B$, $M \times B$ has a crossing-total direction of its medial graph $M \times B$ with A as the union of the set of all d-edges and the set of some t-edges. Therefore, by Theorem 3.1.7, $G \times B * A$ is Eulerian.

The proof of the result for $G * A \times B$ is comparable in nature. (Or we can just observe that $G * A \times B$ is Eulerian if and only if $G * A$ is Eulerian, and apply Theorem 3.1.7.) □

In [13], Ellis-Monaghan and Moffatt describe an action of the symmetric group S_3 on each individual edge generated by partial duality and partial Petrie duality. This action permutes the edge type among b-, c-, and d-edges, and preserves t-edges, in ways that follow from Lemmas 3.1.6 and 3.1.13. We can apply this to results like Theorem 3.1.7 to deduce many results similar to Theorem 3.1.14.

Chapter 4

Closed Walks in Gems

Next we seek to understand how characteristics of closed walks in a gem relate to properties of the underlying cellularly embedded graphs. We will fully investigate seven distinct parity conditions, which can be identified with points in the Fano plane. The parity conditions are each shown to correspond to embedding properties, the bipartiteness of related graphs, and bidirections of the medial graph. The main result of this section is that each line in the Fano plane gives us a theorem relating the three properties corresponding to points in the line. Furthermore, any three properties corresponding to points that are noncolinear imply the rest of the seven parity conditions. The first parts of this chapter are building towards these two main results, Theorems 4.2.13 and 4.2.14, and their various interpretations stemming from the characterizations of Section 4.2 which are compiled in Figure 4.2. We also give a characterization of cellularly embedded graphs that have partial duals that are all Eulerian. In this chapter we assume that all graphs are connected, but the results extend without difficulty to disconnected graphs by considering their connected components.

4.1 Notation

We will need to introduce some notation as it pertains to closed walks in a gem. Let J be a gem with a corresponding cellularly embedded graph $G \subseteq \Sigma$ and let K be a closed walk in J. We define the following functions of K:

(a) $v_J(K)$ denotes the number of edges colored c_v in K,

(b) $f_J(K)$ denotes the number of edges colored c_f in K,

(c) $a_J(K)$ denotes the number of edges colored c_a in K.

Note that we will write $v_J(K) + f_J(K) + a_J(K) = (v + f + a)_J(K)$ and make other similar notational abbreviations.

Corollary 4.1.1. *Let J be a gem with a corresponding cellularly embedded graph $G \subseteq \Sigma$. Then the surface Σ, and hence the embedded graph G, is orientable if and only if $(v + f + a)_J(K)$ is even for all closed walks K in J.*

Proof. The embedded graph G is orientable if and only if J is bipartite by Theorem 1.3.1 and J is bipartite if and only if the total length $(v + f + a)_J(K)$ is even for all closed walks K in J. □

Theorem 4.1.2. *Let J be a gem with a corresponding cellularly embedded graph $G \subseteq \Sigma$. Then G is bipartite if and only if $f_J(K)$ is even for all closed walks K in J.*

In the proof of Theorem 4.1.2 we will be converting closed walks in an embedded graph to closed walks in a related embedded graph. Here we will relate closed walks in a gem to closed walks in the corresponding cellularly embedded graphs. Elsewhere, we will use similar techniques for translating closed walks within a family of related embedded graphs.

Proof. Notice that in closed walks in gems, traveling on an edge colored c_a or an edge colored c_v corresponds to traveling around a vertex in G. Traveling on an edge colored c_f corresponds to traveling along an edge in G. For this reason we can associate the pair of edges colored c_f in an e-square in J with the corresponding edge in G. This gives us a full description of how a closed walk in a gem can be mapped to a walk in the associated cellularly embedded graph. We will formalize this relationship with the following function. Given a walk K in J, define $\pi(K)$ to be the corresponding walk in G, where for each edge colored c_a or c_v we do not move in $\pi(K)$, but for each edge colored c_f we follow the corresponding edge of G. So the length of $\pi(K)$ is equal to $f_J(K)$. Further we see that if K is closed then so is $\pi(K)$.

Conversely, for each walk K' in G there is some (nonunique) walk K in J with $\pi(K) = K'$, and furthermore K can be chosen to be closed if K' is closed. In other words π is a surjective map from closed walks in J to closed walks in G. Given a closed walk $K' = v_0e_0v_1e_1...v_ke_kv_0$ in G we construct a closed walk K in J as follows. Begin with the initial edge e_0 in K'. Lift e_0 to an edge in J by selecting either of the two edges colored c_f in J corresponding to e_0. Then travel along edges colored c_v and c_a around the v-gon corresponding to v_1 until the e-square corresponding to

e_1 is reached. Then travel along an edge colored c_f corresponding to e_1. Repeat this procedure from v_1 and e_1 for each vertex and edge pair $v_i e_i$ until v_0 is ultimately reached. Then travel along edges colored c_v and c_a around the v-gon corresponding to v_0 until the initial edge selected in the lift of e_0 is reached, thus closing the walk. This closed walk K, constructed as described, satisfies $\pi(K) = K'$ and it is clear that the length of K' is equal to $f_J(K)$. We also observe that K is not uniquely chosen.

So G is bipartite if and only if all closed walks K' in G are of even length; and all closed walks K' in G are of even length if and only if $f_J(K)$ is even for all closed walks K in J. Therefore, G is bipartite if and only if $f_J(K)$ is even for all closed walks K in J. $\square$

4.1.1 Jewels and Petrie Duality

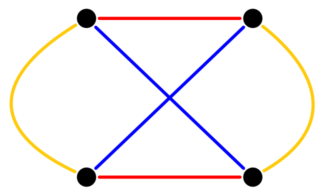
In order to continue to extrapolate information from the closed walks in gems, it will be convenient to extend gems to jewels.

Let J be a gem. The *jewel* L is obtained from J by adding two edges across the diagonals of every e-square in J and coloring these edges with a new color denoted c_z. In depictions of jewels we will use the real color green for the color c_z as a convention. This e-square, together with the two adjacent edges colored c_z, is called an *e-simplex* in L. So L is a 4-regular, properly 4-edge-colored graph associated with J. The edge labeling function is denoted $\gamma_L : E(L) \to \{c_v, c_f, c_a, c_z\}$. The gem J is $L \setminus {\gamma_L}^{-1}(c_z)$. We call the bigons in L whose edges are alternately colored c_z and c_a the *z-gons* of L. We call this color c_z because the z-gons correspond to what are commonly known as zig-zag walks (or Petrie walks) in the underlying cellularly embedded graph.

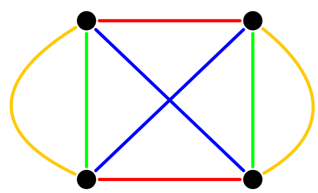
Let J be a gem with associated jewel L and with associated cellularly embedded graph G. The *Petrie dual* of L, denoted $L^\times$, is the jewel obtained from L by swapping the colors c_f and c_z. The *Petrie dual* of the gem J is obtained by removing the edges colored c_f and then drawing new edges labeled c_f which join opposite corners of the original e-squares in J, resulting in a new gem which we denote $J^\times$. This is equivalent to twisting (changing the signature of) all of the edges in G in the embedding scheme representation of the embedded graph G. We write $G^\times$ to denote the cellularly embedded graph corresponding to $J^\times$ and $G^\times$ is called the *Petrie dual* of G.

Remark 4.1. *We note that $(J^\times)^\times = J$ so the operation of Petrie duality is an involution, as is the operation of duality. We also observe that, in general, these two involutions do not commute as operations on the gem J, as seen in the following example.*

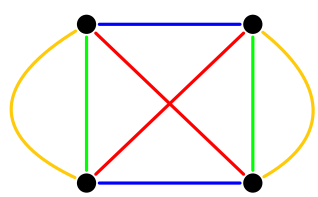
Consider an embedding of a single vertex with a twisted loop. The corresponding gem J is drawn below. Recall, we are using yellow as the color for c_a, red for c_v, and blue for c_f.



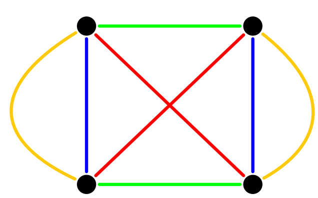
We can extend J to a jewel L by adding two edges across the e-square to create an e-simplex. We are using green for the color c_z.



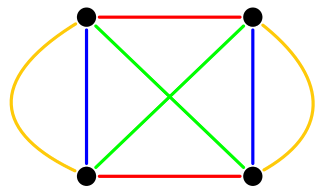
Taking the dual of this jewel swaps red and blue resulting in the following jewel L^*. The corresponding gem J^* (obtained by removing the green edges) still represents a single vertex with a twisted loop.



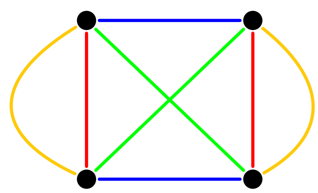
Then taking the Petrie dual of L^* swaps green and blue resulting in the following jewel $(L^*)^\times$. The corresponding gem represents a single vertex with an untwisted loop.



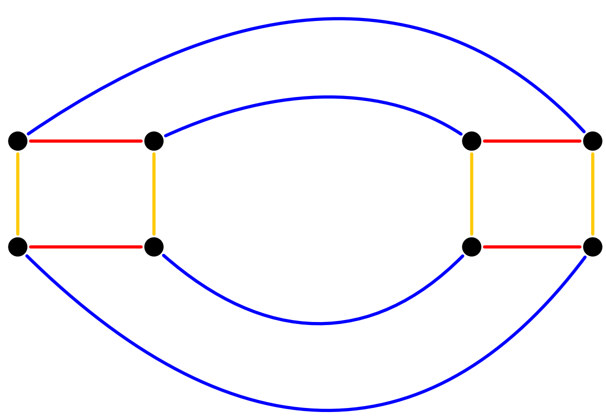
If instead we first apply the Petrie dual to L by swapping the colors blue and green we obtain the jewel $L^\times$ whose corresponding embedded graph is a vertex with an untwisted loop.



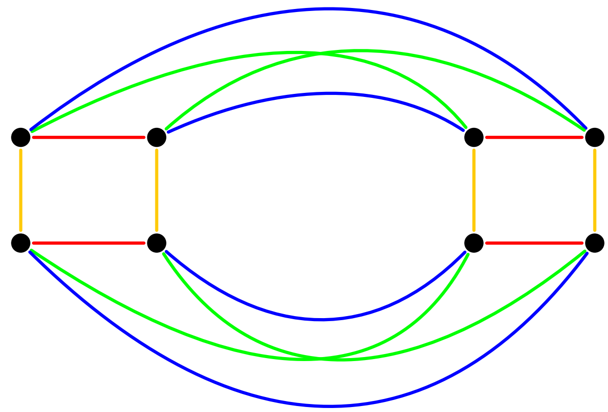
Taking the dual of $L^\times$ by swapping red and blue gives the jewel $(L^\times)^*$ whose corresponding embedded graph is two vertices connected by an edge. So we see that $(G^\times)^*$ and $(G^*)^\times$ are not the same and so duality and Petrie duality do not commute.



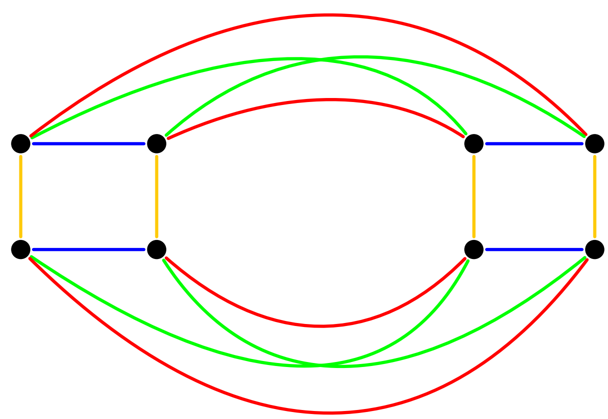
There are, however, situations where the involutions do commute, as seen in the following example. Begin with a gem J. Again, we are using yellow as the color for c_a, red for c_v, and blue for c_f. So the gem below represents two vertices with two parallel edges cellularly embedded on a sphere.



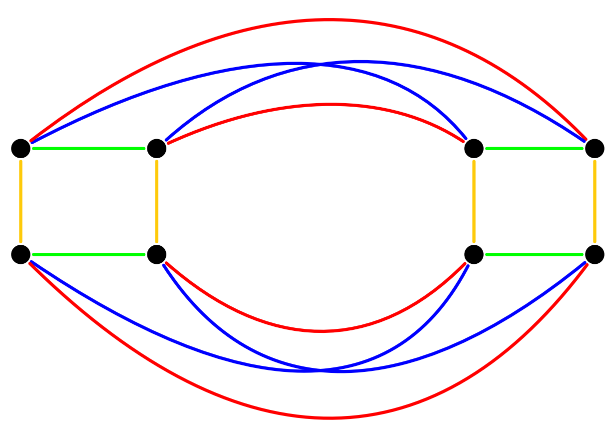
We can extend J to a jewel L by adding two edges in each e-square to form an e-simplex. Here we are using green for the color c_z.



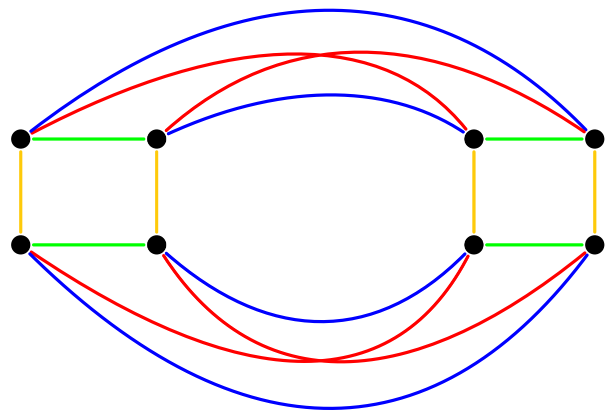
The jewel L^* is obtained by swapping the colors c_v and c_f (red and blue).



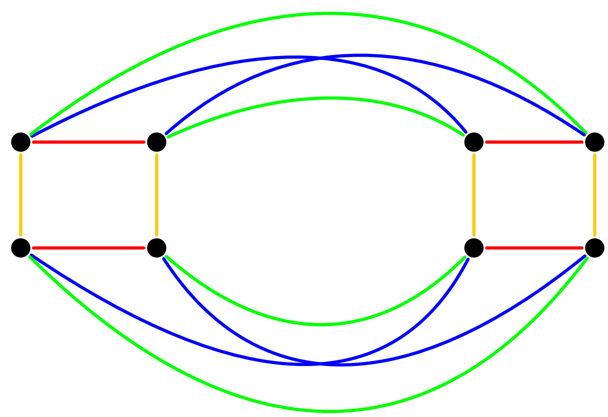
The jewel $(L^*)^\times$ is obtained from L^* by swapping c_f and c_z (blue and green).



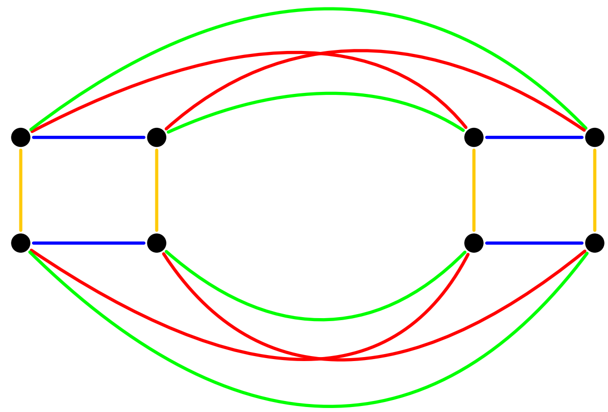
Swapping c_v and c_f (red and blue) again gives us $((L^*)^\times)^* = ((L^\times)^*)^\times$.



The jewel $L^\times$ is obtained from L by swapping c_f and c_z (blue and green).



The jewel $(L^\times)^*$ is formed from $L^\times$ by swapping c_v and c_f (red and blue).



Removing the c_z (green) edges gives us the gems J, J^*, $J^\times$, $(J^*)^\times$, $(J^\times)^*$, and $((J^\times)^*)^\times = ((J^*)^\times)^*$. In this example all six of these gems represent an orientable embedding of two vertices with two parallel edges on a sphere. However, as we noted, in general these two operations do not commute.

If we begin with a jewel L, with gem J and cellularly embedded graph G, and allow ourselves only to apply the operations of taking the Petrie dual and taking the dual, there are six possible jewels (with corresponding gems and cellularly embedded graphs) that arise. Let L' be one of these jewels with corresponding gem J' and cellularly embedded graph G'.

The jewel L' can be fully described as a permutation of the colors c_v, c_f, and c_z applied to L. Since the group S_3 has six elements, there are six possible jewels.

(a) L

(b) $L^* = (c_v c_f)(L)$

(c) $L^\times = (c_f c_z)(L)$

(d) $(L^*)^\times = (c_f c_z)(c_v c_f)(L) = (c_v c_z c_f)(L)$

(e) $(L^\times)^* = (c_v c_f)(c_f c_z)(L) = (c_v c_f c_z)(L)$

(f) $((L^*)^\times)^* = (c_v c_f)(c_f c_z)(c_v c_f)(L) = (c_v c_z)(L) = (c_f c_z)(c_v c_f)(c_f c_z)(L) = ((L^\times)^*)^\times$

The gem J' can be described fully by:

(a) edges of which color, c_v, c_f, or c_z, has its edges removed from L,

(b) edges of which color, c_v, c_f, or c_z, becomes c_v in J' and corresponds, together with c_a, to the vertices in G,

(c) edges of which color, c_v, c_f, or c_z, becomes c_f in J' and corresponds, together with c_a, to the faces in G.

Notice that the operations of duality and Petrie duality are defined by swapping the colors (b) and (c) and swapping (a) and (c) respectively. Since there are three colors, c_v, c_f, and c_z, to choose from for these three categories there are six total possible gems. For instance we could describe J as $J = (L \setminus c_z, c_v, c_f)$ to denote that J is obtained by removing edges with the color c_z, attributing c_v to vertices, and attributing c_f to faces. Using this notation, here are the six gems:

(a) $J = (L \setminus c_z, c_v, c_f)$ which we call the gem, reference gem, or original gem.

(b) $J^* = (L \setminus c_z, c_f, c_v)$ which we call the dual gem.

(c) $J^\times = (L \setminus c_f, c_v, c_z)$ which we call the *skew* gem or *Petrie* gem.

(d) $(J^*)^\times = (L \setminus c_v, c_f, c_z)$ the *dual-skew* gem.

(e) $(J^\times)^* = (L \setminus c_f, c_z, c_v)$ the *skew-dual* gem.

(f) $((J^\times)^*)^\times = ((J^*)^\times)^* = (L \setminus c_v, c_z, c_f)$ which we call the *phial* gem.

Notice that we do not mention the color c_a since its role is the same in all six gems. The six gems arise from permutations of the three colors c_v, c_f, and c_z. So, there is an action of the symmetric group S_3 on the six gems, generated by duality and Petrie duality.

4.2 Results

4.2.1 Petrie Orientability, Bipartiteness, and Applications to Partial Duals

Using the above notation, similar to that used in [26] with jewels, we are prepared to further understand the consequences of the characteristics of closed walks in a gem on the properties of the underlying cellularly embedded graphs. We have shown that G is orientable if and only if $(v + a + f)_J(K)$ is even for all closed walks K in J. We also know that G is bipartite if and only if $f_J(K)$ is even for all closed walks K in J.

We now explore what happens if $(v + a)_J(K)$ is even for all closed walks K in J. We will relate this and other walk parity parameters to embedding properties, give a proof of a result that is already known (Theorem 4.2.3) using parities of walks, and generalize a result of Deng and Jin.

We will begin by examining a particular theorem from Ellis-Monaghan and Moffatt [13, Proposition 4.30] (also [14, Proposition 3.27]). Here we state the result as written in [13, Proposition 4.30], show that part of this result implies our Theorem 4.2.3, and point out some flaws in their result.

(Not Entirely Correct) Theorem 4.2.1 (Ellis-Monaghan and Moffatt [13, Proposition 4.30])**.** *Let G be an embedded graph. Then*

1. *$|Orb_{(\tau)}(G)|$ is bounded above by the number of cycles in G.*
2. *If G is bipartite, then $G^{\tau(E(G))} = G$. Furthermore, (Wilson [42]) if G is an orientable regular embedded graph, then $G^{\tau(E(G))} = G$ if and only if G is bipartite.*

Theorem 4.2.3 follows from this particular proposition of Ellis-Monaghan and Moffatt [13, Proposition 4.30(2)]. They cite a paper of Wilson in which a special case of one part of the theorem for regular maps is stated without proof [42]. Their proof of our Theorem 4.2.3 is correct, but there are errors in the other parts of their Proposition 4.30.

First, in Proposition 4.30(1), they state that the number of distinct edge-labeled cellularly embedded graphs under the action of partial Petrie duality ($|Orb_{(\tau)}(G)|$ in their notation) is bounded above by the number of cycles of G. This is not true. For example, let G the embedding of a single twisted loop. This embedding has one face and one cycle. The Petrie dual, $G^{\times}$, is a single vertex with an untwisted loop with two faces. Thus there are two distinct cellularly embedded graphs under the action of partial Petrie duality, but there is only one cycle in G. There are many other counterexamples.

The statement of [13, Proposition 4.30(1)] was modified in [14, Proposition 3.27(1)] to say that $|Orb_{(\tau)}(G)|$ is bounded above by 2 to the power of the number of cycles in G. This is still not correct. Consider a tree T consisting of five edges $e_1 = uv, e_2 = uw, e_3 = ux, e_4 = xy, e_5 = xz$ embedded in the plane so that the single clockwise facial walk is $ue_1ve_1ue_2we_2ue_3xe_4ye_4xe_5ze_5xe_3u$ (see Figure 4.1). Then the partial Petrie dual $T \times e_3$ has a facial walk $ue_1ve_1ue_2we_2ue_3xe_5ze_5x$

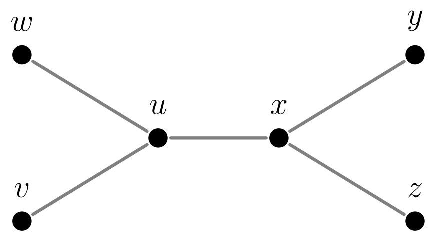
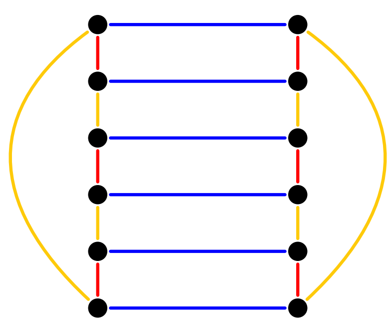


Figure 4.1: A tree with 2 distinct partial Petrie duals.

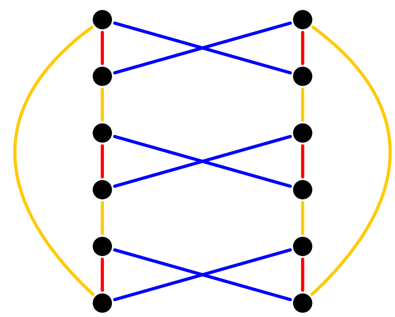
$e_4ye_4xe_3u$, which uses the edges in a different order (even up to reversal). These are therefore different as edge-labeled embeddings so there are 2 distinct Petrie duals, even though the number of cycles is 0.

Second, in Proposition 4.30(2), they also state that if G is bipartite then $G^{\tau(E(G))} = G$, or in our notation $G^{\times} = G$. This is also not true. Take for example a planar embedding G of two vertices joined by three parallel edges. This embedding has three faces. The Petrie dual, $G^{\times}$ has only one face. So G is bipartite but $G^{\times} \neq G$. This error was continued in [14, Proposition 3.27(2)] but is mentioned in the online errata for [14].

Below is a drawing of the gem J for G with c_v as red, c_f as blue, and c_a as yellow. There are three f-gons in J that alternate between blue and yellow edges so there are three faces in G.



Then the Petrie dual of G, $G^{\times}$, represented by the gem $J^{\times}$ is given below. In $J^{\times}$ there is a single f-gon so $G^{\times}$ has only one face.



We now explore what happens if $(v+a)_J(K)$ is even for all closed walks K in J.

Theorem 4.2.2. *Given a gem J with a corresponding cellularly embedded graph G, $(v+a)_J(K)$ is even for all closed walks K in J if and only if $G^\times$ is orientable.*

Proof. By Corollary 4.1.1, G is orientable if and only if $(v+f+a)_J(K)$ is even for all closed walks K in J. Thus the statement of Theorem 4.2.2 is equivalent to: given a gem J, (1) $(v+a)_J(K)$ is even for all closed walks K in J if and only if (2) $(v+a+f)_{J^\times}(K')$ is even for all closed walks K' in $J^\times$. From the discussion above, closed walks in J and $J^\times$ are in one-to-one correspondence with closed walks in L restricted to the colors $\{c_v, c_f, c_a\}$ and $\{c_v, c_z, c_a\}$, respectively. Since J is a color preserving subgraph of L, (1) is equivalent to saying that $(v+a)_L(K)$ is even for all closed walks K in L that only use the colors c_v, c_a, and c_f. Similarly, since $v_{J^\times} = v_L$, $a_{J^\times} = a_L$, and $f_{J^\times} = z_L$, (2) is equivalent to saying that $(v+a+z)_L(K')$ is even for all closed walks K' in L that only use the colors c_v, c_a, and c_z.

We will show that for each closed walk K in L restricted to $\{c_v, c_f, c_a\}$ there is a closed walk K' in L restricted to $\{c_v, c_z, c_a\}$ where $(v+a)_L(K)$ and $(v+a+z)_L(K')$ have the same parity, and we will show that for each closed walk K' in L restricted to $\{c_v, c_z, c_a\}$ there is a closed walk K in L restricted to $\{c_v, c_f, c_a\}$ where $(v+a)_L(K)$ and $(v+a+z)_L(K')$ have the same parity. Let K be a closed walk in L on $\{c_v, c_f, c_a\}$. Now for each edge colored c_f in K consider the e-simplex it is in. On this e-simplex we arbitrarily take a pair of edges labeled c_v and c_z that connect the endpoints of the edge colored c_f. After repeating this process for all edges labeled c_f in K we arrive at a closed walk K' in L restricted to $\{c_v, c_z, c_a\}$. By this process we see that $v_L(K') = (v+f)_L(K)$, $z_L(K') = f_L(K)$, and $a_L(K') = a_L(K)$ so that $(v+a+z)_L(K') = (v+a+2f)_L(K)$ which clearly has the same parity as $(v+a)_L(K)$. So for each closed walk K in L restricted to $\{c_v, c_f, c_a\}$ there is a closed walk K' in L restricted to $\{c_v, c_z, c_a\}$ where $(v+a)_L(K)$ and $(v+a+z)_L(K')$ have the same parity.

Now conversely let K' be a closed walk in L restricted to $\{c_v, c_z, c_a\}$. Then, for each edge that is colored c_z in K' consider the e-simplex associated with that edge. On this e-simplex we arbitrarily take a pair of edges labeled c_v and c_f that connect the endpoints of the edge colored c_z. We repeat this process for all edges colored c_z in K'. After completing the process we arrive at a closed walk K in L which only uses the colors c_a, c_f, and c_v, and thus corresponds to a closed walk in J. By this interchange of edges from K' to K we see that $v_L(K) = (v+z)_L(K')$ and $a_L(K) = a_L(K')$ so that $(v+a+z)_L(K') = (v+a)_L(K)$.

We have shown that for each closed walk K in L restricted to $\{c_v, c_f, c_a\}$ there is a closed walk K' in L restricted to $\{c_v, c_z, c_a\}$ where $(v+a)_L(K)$ and $(v+a+z)_L(K')$ have the same parity and vice versa. So, there is a closed walk K in L restricted to $\{c_v, c_f, c_a\}$ where $(v+a)_L(K)$ is odd if and only if there is a closed walk K' in L restricted to $\{c_v, c_z, c_a\}$ where $(v+a+z)_L(K')$ is odd. Therefore, $(v+a)_J(K)$ is even for all closed walks K in J if and only if $(v+a+f)_{J^\times}(K')$ is even for all closed walks K' in $J^\times$. □

Now we have three characterizations of cellularly embedded graphs based on parity conditions for closed walks in gems. As usual let J be a gem with a corresponding cellularly embedded graph G. Then, as a summary of Corollary 4.1.1, Theorem 4.1.2, and Theorem 4.2.2:

(a) G is orientable if and only if $(v+a+f)_J(K)$ is even for all closed walks K in J.

(b) G is bipartite if and only if $f_J(K)$ is even for all closed walks K in J.

(c) $G^\times$ is orientable if and only if $(v+a)_J(K)$ is even for all closed walks K in J.

The following theorem (Theorem 4.2.3) follows from a proposition of Ellis-Monaghan and Moffatt [13, Proposition 4.30(2)], as mentioned earlier. The technique of translating embedded graph properties into properties of closed walks in gems allows us to use walk parities to give a different proof of Theorem 4.2.3.

Theorem 4.2.3 (Ellis-Monaghan and Moffatt, Wilson [13, 42])**.** *Any two of the following properties imply the third:*

1. *G is orientable.*
2. *G is bipartite.*
3. *$G^{\times}$ is orientable.*

Proof. It is clear that any two of the following properties imply the third:

1. $(v + a + f)_J(K)$ is even for all closed walks K in J.
2. $f_J(K)$ is even for all closed walks K in J.
3. $(v + a)_J(K)$ is even for all closed walks K in J.

Therefore by our summary of Corollary 4.1.1, Theorem 4.1.2, and Theorem 4.2.2, the three equivalent conditions in the statement of the theorem are also such that any two imply the third. □

We say that G is *Petrie orientable* if $G^{\times}$ is orientable. So we can restate the three properties as follows:

1. G is orientable.
2. G is bipartite.
3. G is Petrie orientable.

We now show how Theorem 4.2.3 can be used to extend the results of Deng and Jin [11]. In Lemma 3.1.12 and Theorem 3.1.11 they gave characterizations of bipartite duals and partial duals using all-crossing directions of the medial graph M. Using Theorem 4.2.2 these can be extended to characterize when G and G^A are Petrie orientable. In the orientable case Theorem 4.2.3 shows that this reduces to G and G^A being bipartite.

Theorem 4.2.4. *Let G be a cellularly embedded graph. Then G is Petrie orientable if and only if there is a direction of its medial graph M in which every edge of G is a d-edge.*

Proof. Any direction of M gives an orientation of the edges of J colored c_a in a natural way. This gives a partition of the vertices of the gem J into two sets, the set X of vertices whose incident c_a

edge goes in, and the set Y of vertices whose incident c_a edge goes out. Suppose the direction of M is a direction in which all edges of G are d-edges.

Each edge $e \in E(G)$ corresponds to an e-square $Q(e)$ in J and a vertex $v(e)$ in M. Since all edges are d-edges, in the checkerboard coloring of M corresponding to G, what we see as we go around $v(e)$ is an incoming arc, a white region, an incoming arc, a black region, an outgoing arc, a white region, an outgoing arc, and a black region. Then as we go around $Q(e)$ in J we will see a vertex in X, an edge colored c_f, a vertex in X, an edge colored c_v, a vertex in Y, an edge colored c_f, a vertex in Y, and an edge colored c_v. Thus, edges colored c_a and c_v always join vertices in different sets X and Y while edges colored c_f join vertices in the same set. Hence $(v+a)_J(K)$ is even for every closed walk K in J since closed walks begin and end with the same vertex. So by Theorem 4.2.2, G is Petrie orientable.

Conversely, assume that G is Petrie orientable. Then by Theorem 4.2.2 $(v+a)_J(K)$ is even for every closed walk K in J. Then starting by assigning an arbitrary vertex of J to X, we can partition the vertices of J into two sets X and Y so that edges colored c_v and c_a join a vertex in X and a vertex in Y and edges colored c_f either join two vertices in X or join two vertices in Y. Since $(v+a)_J(K)$ is always even this can be done consistently. Assign a direction to all edges colored c_a from X to Y (which we can translate to a direction of the medial graph M). Then, for every edge $e \in E(G)$, as we go around $Q(e)$ in J we will see a vertex in X, an edge colored c_f, a vertex in X, an edge colored c_v, a vertex in Y, an edge colored c_f, a vertex in Y, and an edge colored c_v. So in the checkerboard coloring of M corresponding to G, what we see as we go around $v(e)$ is an incoming arc, a white region, an incoming arc, a black region, an outgoing arc, a white region, an outgoing arc, and a black region. This is exactly the characterization of a d-edge. So we have a direction of M in which every edge of G is a d-edge. $\square$

Recall that c-edges become d-edges when (partially) dualized and vice versa. In an all-crossing direction of the medial graph of a cellularly embedded graph, every edge is either a c-edge or a d-edge. Theorem 4.2.4 therefore implies the following.

Theorem 4.2.5. *Let G be a cellularly embedded graph with medial graph M and $A \subseteq E(G)$. Then G^A is Petrie orientable if and only if A is the set of c-edges arising from an all-crossing direction of M.*

By applying Theorem 4.2.3 we get the result from Deng and Jin.

Corollary 4.2.6 (Deng and Jin [11])**.** *Let G be an orientable cellularly embedded graph with medial graph M and $A \subseteq E(G)$. Then G^A is bipartite if and only if A is the set of c-edges arising from an all-crossing direction of M.*

By taking duals in Theorem 4.2.5 we obtain the following corollary.

Corollary 4.2.7. *Let G be a cellularly embedded graph with medial graph M. Then G^* is Petrie orientable if and only if there is a direction of M in which every edge of G is a c-edge.*

4.2.2 Parity Conditions for Closed Walks in Gems and Jewels

We have shown parity conditions of colors of closed walks in gems to be representative of important structural and topological graph theoretical properties of the underlying embedded graph. We will now systematically investigate the interpretation of parity conditions for colors of closed walks in gems and jewels.

We note that all of these parity conditions for closed walks in gems are equivalent to the parity conditions for cycles in gems. We will show that if the number of edges of a certain color(s) S in a closed walk is odd, then there is a cycle where the number of edges of color S is odd. We do this by proving a more general result.

A walk is *trivial* if it has no edges and *nontrivial* otherwise.

Lemma 4.2.8. *In a graph G (with loops and multiple edges allowed), a nontrivial closed walk $W = v_0e_1v_1e_2v_2...e_k(v_k = v_0)$ with no repeated vertex ($v_i \neq v_j$ for $0 \leq i < j < k$) is either a cycle of has the form $v_0ev_1ev_0$.*

Theorem 4.2.9. *Let G be a graph with $S \subseteq E(G)$. Let $s(W)$ denote the number of edges of a closed walk W (counted with multiplicity) that belong to S. If $s(W)$ is odd then there is a cycle C that is a (not necessarily consecutive) subsequence of W for which $s(C)$ is odd.*

Proof. We will induct on the length of W. First assume that W is trivial. Then $s(W) = 0$ and the result holds. Now assume W is nontrivial and $s(W)$ is odd. If W has no repeated vertex then W cannot have the form $v_0ev_1ev_0$ because then $s(W)$ would be 0 or 2, so by Lemma 4.2.8 W is a cycle and the result holds. If W has a repeated vertex, then we can split W at the repeated vertex into two shorter nontrivial closed walks W_1 and W_2 with $s(W) = s(W_1) + s(W_2)$. Since $s(W)$ is odd one of $s(W_1)$ or $s(W_2)$ must be odd and by induction the result follows. □

2-color walks

A *k-color* walk in an edge-colored graph is a walk that travels along edges of at most k colors.

We begin with 2-color closed walks in a gem. By Theorem 4.2.9 we may restrict our attention to 2-color cycles. These are the v-gons, f-gons, and e-squares of a gem J. All e-squares K have $a_J(K) = 0$ and $v_J(K) = f_J(K) = 2$, which are all even. So we consider only the v-gons and f-gons. On each v-gon the number of edges colored c_a is equal to the number of edges colored c_v. Similarly, on each f-gon, the number of edges colored c_a is equal to the number of edges colored c_f. We also know that v-gons in the gem J of G are f-gons in the gem J^* of G^*. The following three theorems are immediately clear because the degree of a vertex in a cellularly embedded graph G is equal to the number of edges colored c_v on the corresponding v-gon.

Theorem 4.2.10. *Let G be a connected, cellularly embedded graph with corresponding gem J. Then G is Eulerian if and only if $v_J(K)$ is even for all 2-color closed walks K in J.*

Theorem 4.2.11. *Let G be a connected, cellularly embedded graph with corresponding gem J. Then G^* is Eulerian, i.e., G is an even-faced embedding, if and only if $f_J(K)$ is even for all 2-color closed walks K in J.*

Theorem 4.2.12. *Let G be a connected, cellularly embedded graph with corresponding gem J. Then G and G^* are both Eulerian if and only if $a_J(K)$ is even for all 2-color closed walks K in J.*

3-color walks

Now we move on to three color, i.e. arbitrary, closed walks in the gem. There are seven possible nontrivial parity conditions. Let $(\alpha_v \alpha_f \alpha_a)$, where each $\alpha_x \in \{0, 1\}$, represent the condition that $(\alpha_v v + \alpha_f f + \alpha_a a)_J(K)$ is even for all closed walks K in a gem J corresponding to a cellularly embedded graph G. The condition (000) corresponds to the trivially true statement that $(0v+0f+0a)_J(K) = 0$ is even for all K, so we focus on the other seven triples. We have already related the properties (111), (010), and (101) to properties of the embedded graph G in Subsection 4.2.1. Here we will examine the seven properties in detail.

First we observe that the seven conditions are in a natural one-to-one correspondence with the nonzero elements of $\mathbb{Z}_2^3$. It will be helpful to interpret the seven parity conditions as points in the projective plane over $\mathbb{Z}_2$, the *Fano plane*. The points in the Fano plane are nonzero elements of $\mathbb{Z}_2^3$ and three points $\{u.v, w\}$ are on a line in this projective plane if $u+v+w = 0$. This interpretation will be helpful because, as in the proof of Theorem 4.2.3, any two of the seven parity conditions imply the third parity condition that is on the same line in the Fano plane.

Theorem 4.2.13. *Let $(\alpha_v \alpha_f \alpha_a)$, where each $\alpha_x \in \{0, 1\}$, represent the condition that $(\alpha_v v + \alpha_f f + \alpha_a a)_J(K)$ is even for all closed walks K in a gem J corresponding to a cellularly embedded graph G. Then any two distinct conditions $(\alpha_v \alpha_f \alpha_a)$ and $(\beta_v \beta_f \beta_a)$, neither equal to (000), imply the third condition $(\gamma_v \gamma_f \gamma_a)$ such that $(\alpha_v \alpha_f \alpha_a) + (\beta_v \beta_f \beta_a) = (\gamma_v \gamma_f \gamma_a)$ in $\mathbb{Z}_2^3$, or equivalently $(\alpha_v \alpha_f \alpha_a) + (\beta_v \beta_f \beta_a) + (\gamma_v \gamma_f \gamma_a) = 0$ in $\mathbb{Z}_2^3$.*

Proof. Suppose $(\alpha_v \alpha_f \alpha_a) + (\beta_v \beta_f \beta_a) = (\delta_v \delta_f \delta_a)$ in $\mathbb{Z}^3$. Assume $(\alpha_v v + \alpha_f f + \alpha_a a)_J(K)$ is even for all closed walks K in a gem J and $(\beta_v v + \beta_f f + \beta_a a)_J(K)$ is even for all closed walks K in J. Then $(\delta_v v + \delta_f f + \delta_a a)_J(K)$ is even for all closed walks K in J. Let $\gamma_x = \delta_x \text{mod } 2$ for $x \in \{v, f, a\}$ so $(\gamma_v \gamma_f \gamma_a) \in \mathbb{Z}_2^3$. Then $(\gamma_v v + \gamma_f f + \gamma_a a)_J(K)$ has the same parity as $(\delta_v v + \delta_f f + \delta_a a)_J(K)$ for all closed walks K in J. Therefore, $(\gamma_v v + \gamma_f f + \gamma_a a)_J(K)$ is even for all closed walks K in

J. Since $(\gamma_v\gamma_f\gamma_a)$ is the unique element of $\mathbb{Z}_2^3$ where $(\alpha_v\alpha_f\alpha_a) + (\beta_v\beta_f\beta_a) = (\gamma_v\gamma_f\gamma_a)$, the result follows. $\square$

So we have a theorem corresponding to any three colinear points in the Fano plane. We also have a theorem for any three noncolinear points in the Fano plane, i.e., three conditions, $(\alpha_v\alpha_f\alpha_a)$, $(\beta_v\beta_f\beta_a)$, $(\gamma_v\gamma_f\gamma_a)$ where $(\alpha_v\alpha_f\alpha_a) + (\beta_v\beta_f\beta_a) + (\gamma_v\gamma_f\gamma_a) \neq 0$ in $\mathbb{Z}_2^3$. In this case $\{(\alpha_v\alpha_f\alpha_a), (\beta_v\beta_f\beta_a), (\gamma_v\gamma_f\gamma_a)\}$ is a basis of $\mathbb{Z}_2^3$ and so any three such properties imply the other four.

Theorem 4.2.14. *Let* $(\alpha_v\alpha_f\alpha_a)$*, where each* $\alpha_x \in \{0,1\}$*, represent the condition that* $(\alpha_v v + \alpha_f f + \alpha_a a)_J(K)$ *is even for all closed walks* K *in a gem* J *corresponding to a cellularly embedded graph* G*. Then every three distinct conditions* $(\alpha_v\alpha_f\alpha_a)$*,* $(\beta_v\beta_f\beta_a)$*, and* $(\gamma_v\gamma_f\gamma_a)$ *none of which is* (000) *and such that* $(\alpha_v\alpha_f\alpha_a) + (\beta_v\beta_f\beta_a) + (\gamma_v\gamma_f\gamma_a) \neq 0$ *in* $\mathbb{Z}_2^3$ *imply every condition* $(\delta_v\delta_f\delta_a)$ *in* $\mathbb{Z}_2^3\backslash\{(000)\}$.

The seven cases of color combinations being even are explored in the following theorems. Three of the conditions correspond to a pair of the six embedded graphs generated by duality and Petrie duality being orientable. Another three correspond to a pair of the six embedded graphs generated by duality and Petrie duality being bipartite. The final parity condition corresponds to bipartiteness of the medial graph. We will show that all seven of these parity conditions can be interpreted as some related graph (or sometimes several related graphs) being bipartite. All seven of these conditions can also be interpreted as the existence of a specific type of direction or bidirection of the medial graph. We also characterize which classes of abstract graphs have an embedding with each of the seven properties.

We will go through the conditions by starting with the ones that have already shown a characterization in Subsection 4.2.1. Whenever two conditions can be thought of as duals of one another we characterize them consecutively. So the order we will use is (111), (101), (011), (010), (100), (110), and finally (001).

Condition (111)

First we will examine the condition (111), namely $(v+f+a)_J(K)$ is even for all closed walks K in J. We showed in Corollary 4.1.1 that (111) holds if and only if G is orientable. Since G is orientable if and only if G^* is orientable, (111) holds if and only if G and G^* are orientable. We also know by Theorem 1.3.1 that (111) holds if and only if J is bipartite. Next we will characterize (111) by a direction of its medial graph.

Theorem 4.2.15. *Let G be a cellularly embedded graph with corresponding gem J and medial graph M. Then $(v + f + a)_J(K)$ is even for all closed walks K in J if and only if there is a direction of M where every edge of G is a b-edge.*

Proof. Let G be a cellularly embedded graph with corresponding gem J and medial graph M. We know that $(v + f + a)_J(K)$ is even for all closed walks K in J if and only if G is orientable. So we will show G is orientable if and only if there is a direction of M where every edge of G is a b-edge.

Assume G is orientable. Then there is a consistent global orientation of the v-gons in J such that each edge has signature $+1$. Direct the edges in each v-gon in accordance with the clockwise direction of this global orientation. The direction of the v-gons admits a natural direction of M (using the corresponding direction of the edges colored c_a in J). Then as we travel around a vertex $v(e)$ in the cannonical 2-face-coloring of M we see an incoming arc, a white region, an outgoing arc, a black region, an incoming arc, a white region, an outgoing arc, and a black region. This is exactly the characterization of $v(e)$ as a b-vertex in M which means e is a b-edge in G.

Conversely, assume there is a direction of M where every edge of G is a b-edge. Then this admits a direction of the edges colored c_a in J. Since every edge is a b-edge, each edge colored c_f in J has an incoming edge colored c_a and an outgoing edge colored c_a. So we can extend this direction to a direction S of the f-gons in J. On an e-square in J, the two edges colored c_f are directed in opposite directions. Thus S gives an orientation of the faces in G such that every edge is used once in each direction; therefore, G is orientable. $\square$

As a final note for condition (111), since every abstract graph has an orientable embedding, every abstract graph has an embedding satisfying (111).

Condition (101)

Next we will give equivalent characterizations of the condition (101), namely $(v+a)_J(K)$ is even for all closed walks K in J. We have already shown in Theorem 4.2.2 that (101) holds if and only if $G^{\times}$ is orientable. Since $G^{\times}$ is orientable if and only if $(G^{\times})^*$ is orientable, condition (101) is equivalent to $G^{\times}$ and $(G^{\times})^*$ being orientable.

Next we wish to show that condition (101) is equivalent to a graph related to G being bipartite. At this point we will define three graphs related to the cellularly embedded graph G. We will then show that these each of these graphs being bipartite is equivalent to one of the conditions (110), (101), and (011).

Every edge in an embedded graph has 2 *sides* (right and left) as we travel with a particular direction and a particular local orientation of the surface. The *face pullback graph* or *side graph* of a cellularly embedded graph G is formed by creating a vertex for each side of an edge in G. Then two vertices in the side graph are adjacent if they represent a side of the same edge or if they are next to each other in cyclic order around a face in G. The side graph is 3-regular with $2E(G)$ vertices. Because the sides of G are in one-to-one correspondence with the edges colored c_f in the gem, the side graph of a cellularly embedded graph G with corresponding gem J can be formed by contracting all edges colored c_f in J, removing one of the two parallel edges colored c_v in J corresponding to an edge $e \in E(G)$, and removing the colors from the remaining edges in J.

A *corner* in an embedded graph is a place where a face and a vertex come together. Every vertex of degree d has d corners. The *corner graph* of a cellularly embedded graph G is formed by creating a vertex for each corner in G then joining two vertices if their corresponding corners appear consecutively around a vertex or face in G. Because corners are in one-to-one correspondence with edges colored c_a in the gem, notice that the corner graph of a cellularly embedded graph G with corresponding gem J can be formed by contracting all edges colored c_a in J and removing the colors from the remaining edges in J.

Recall that the end graph of a cellularly embedded graph G is formed by first creating a vertex for each half-edge in G. Two vertices in the end graph are adjacent if they represent half-edges of the same edge in G or if their half-edges are next to each other in cyclic order around a vertex in G. Thus the end graph is a 3-regular graph with $2E(G)$ vertices. Because the ends of G are in one-to-one correspondence with the edges colored c_v in the gem, the end graph of a cellularly embedded graph G with corresponding gem J can be formed by contracting all edges colored c_v in J, removing one of the two parallel edges colored c_f in J corresponding to an edge $e \in E(G)$, and removing the colors from the remaining edges in J.

The *colored side graph*, *colored corner graph*, and *colored end graph* are obtained from the gem J by the same process with the exception of the final step of removing the colors from the remaining edges in J.

Note that in the above definitions we are considering these graphs as embedded graphs in the same surface as the original cellularly embedded graph G. As we observed, the side, corner, and end graphs of a cellularly embedded graph G with gem J can be obtained by contracting the edges colored c_f, c_a, and c_v, respectively, possibly removing some parallel edges, and then removing the other colors from the remaining edges in J. So, for example, a closed walk K in the gem J can be projected onto a closed walk K' in the corner graph by contracting all the edges colored c_a in the walk K and removing all edge colors. This walk has length $(v+f)_J(K)$ in the corner graph. A walk K' in the corner graph can be lifted to a closed walk K in the gem by lifting each edge and attaching the appropriate edges colored c_a. Similar relationships exist for closed walks in the end and side graphs. These relationships give us the following theorems.

Theorem 4.2.16. *Let G be a cellularly embedded graph with corresponding gem J. Then the side graph of G is bipartite if and only if $(v+a)_J(K)$ is even for all closed walks K in J.*

Theorem 4.2.17. *Let G be a cellularly embedded graph with corresponding gem J. Then the corner graph of G is bipartite if and only if $(v+f)_J(K)$ is even for all closed walks K in J.*

Theorem 4.2.18. *Let G be a cellularly embedded graph with corresponding gem J. Then the end graph of G is bipartite if and only if $(f + a)_J(K)$ is even for all closed walks K in J.*

So we see that (101) holds if and only if the side graph of G is bipartite.

We have shown in Theorem 4.2.4 that G is Petrie orientable if and only if there is a direction of the medial graph in which every edge of G is a d-edge. Then we obtain the following characterization of (101) in terms of a direction of the medial graph.

Corollary 4.2.19. *Let G be a cellularly embedded graph with corresponding gem J and medial graph M. Then $(v + a)_J(K)$ is even for all closed walks K in J if and only if there is a direction of M in which every edge of G is a d-edge.*

Finally, we observe that since all abstract graphs have an orientable embedding, all abstract graphs also have a Petrie orientable embedding obtained by taking the Petrie dual of an orientable embedding (using the fact that $(G^\times)^\times = G$). So every abstract graph has an embedding where (101) holds.

Condition (011)

To begin characterizing the condition (011) observe that this condition is in some sense dual to condition (101) since the colors c_v and c_f are swapped when taking the dual. So we see that (011) holds in G if and only if (101) holds in G^*. Therefore condition (011) is equivalent to $(G^*)^\times$ and $((G^*)^\times)^*$ being orientable.

Theorem 4.2.20. *Let G be a cellularly embedded graph with corresponding gem J and medial graph M. Then $(f + a)_J(K)$ is even for all closed walks K in J if and only if $(G^*)^\times$ is orientable.*

We also showed in Theorem 4.2.18 that (011) is equivalent to the bipartiteness of the end graph. By the fact that (011) holds in G if and only if (101) holds in G^* and using the dual relationship of c-edges and d-edges we obtain the following theorem.

Corollary 4.2.21. *Let G be a cellularly embedded graph with corresponding gem J and medial graph M. Then $(f+a)_J(K)$ is even for all closed walks K in J if and only if there is a direction of M in which every edge of G is a c-edge.*

In Chapter 5, specifically Theorem 5.2.9, we will show the equivalence of (011) and the existence of an assignment of a direction on the edges such that the embedding is a directed embedding (all facial walks are directed walks). By Corollary 5.2.6, such an embedding only exists if a graph is Eulerian so the only abstract graphs with embeddings such that (011) holds are Eulerian graphs. Furthermore, all Eulerian graphs have such an embedding.

Condition (010)

We have already observed in Theorem 4.1.2 that (010) is equivalent to G being bipartite. Since $G^{\times}$ is bipartite if and only if G is bipartite, we see that (010) is equivalent to the bipartiteness of a pair (G and $G^{\times}$) of the six graphs generated by duality and Petrie duality. Since G is bipartite if and only if G^* is 2-face-colorable, we also see that (010) is equivalent to G^* and $(G^{\times})^*$ being 2-face colorable.

In order to give a description relating (010) to the medial graph we need to extend the notion of directions to bidirections.

A *bidirection of an edge* e in a graph G is an assignment of a direction to each half-edge of e that either points toward or away from the middle of e. A *bidirection of a graph* G is a bidirection of every edge of G. A *bidirected graph* is a graph together with such a bidirection. There are three types of bidirected edges:

1. those whose directions both point toward the middle which we call *introverted*,
2. those whose directions both point toward the ends which we call *extraverted*,
3. and those in which one direction points toward the middle and the other points toward the end which we call *directed*.

Notice that we can consider the bidirected edges that we are calling "directed" here to just be directed edges in the normal sense. So in this way a directed graph is a bidirected graph with all directed edges. We call a bidirection of an edge that is not directed an *antidirection*. An *antidirection of a graph* G is an antidirection of every edge of G. An *antidirected graph* is a graph together with such an antidirection. An antidirected graph can be regarded as a signed graph where extraverted edges get the sign $+$ and introverted edges get the sign $-$. We next consider special cases of antidirections of the medial graph of a cellularly embedded graph G. These antidirections are analogous to different types of directions of the medial graph.

If M is given the canonical checkerboard coloring and a fixed antidirection such that at each vertex the number of introverted edges is equal to the number of extraverted edges, then we can partition vertices of M into three classes. First observe that each vertex in M is met by two black faces and two white faces in the checkerboard coloring of M, which correspond to vertices and faces respectively in G. Using this coloring, a vertex $v(e)$ in M is:

(a) a *b′-vertex* is a vertex where the edges in order around the vertex alternate introverted, extraverted, introverted, extraverted.

(b) a *c′-vertex* is a vertex where the edges and faces in order around the vertex are introverted, black, introverted, white, extraverted, black, extraverted, white.

(c) a *d′-vertex* is a vertex where the edges and faces in order around the vertex are introverted, black, extraverted, white, extraverted, black, introverted, white.

The corresponding edge e in G is called a *b′-edge*, *c′-edge*, or *d′-edge* in conjunction with its vertex in the medial graph $v(e)$. An *all-balanced antidirection* of the medial graph is an antidirection of the medial graph such that at each vertex the number of introverted edges is equal to the number of extraverted edges. An all-balanced antidirection of the medial graph can contain b′-, c′-, and d′-vertices. An *all-crossing antidirection* is an all-balanced antidirection where all vertices are c′- or d′-vertices.

Theorem 4.2.22. *Let G be a cellularly embedded graph with corresponding gem J and medial graph M. Then $f_J(K)$ is even for all closed walks K in J if and only if there is an antidirection of M in which every edge of G is a c′-edge.*

Proof. Let G be a cellularly embedded graph with corresponding gem J and medial graph M.

Assume that $f_J(K)$ is even for all closed walks K in J. Then G is bipartite with partite sets X and Y. Assign an antidirection to the edges in the medial graph such that an edge is introverted if the corresponding edge colored c_a in J is along a v-gon corresponding to a vertex in X and an edge is extraverted if the corresponding edge colored c_a in J is along a v-gon corresponding to a vertex in Y. Then around a vertex $v(e) \in V(M)$ in the checkerboard-colored medial graph we see the two black faces that meet at $v(e)$ are incident to two introverted edges or two extraverted edges and the two white faces that meet at $v(e)$ are incident to an introverted edge and an extraverted edge. This is exactly the characterization of a c′-vertex in the medial graph. Thus every edge is a c′-edge in this antidirection.

Now assume that there exists an antidirection B of M such that every edge is a c′-edge. Apply the corresponding antidirection to the edges colored c_a in G. Then each half-edge in G corresponds to two introverted edges or two extraverted edges in B, so around a v-gon in J, edges colored c_a are either all introverted or all extraverted. Let X be the set of vertices in G whose corresponding v-gons have introverted edges colored c_a and let Y be the set of vertices in G whose corresponding v-gons have extraverted edges colored c_a. Clearly all edges in G have an end in X and an end in Y so G is bipartite. Therefore, by Theorem 4.1.2, $f_J(K)$ is even for all closed walks K in J. □

The class of abstract graphs that have an embedding satisfying condition (010) is the set of all bipartite graphs; any embedding satisfies the condition.

Condition (100)

Similarly to the relationship between condition (011) and condition (101), we see that (100) holds in G if and only if (010) holds in G^*. Therefore, G^* and $(G^*)^\times$ are bipartite if and only if condition (100) holds. Equivalently we see that (100) holds if and only if G and $((G^*)^\times)^*$ are 2-face-colorable.

The dual relationship between c′-edges and d′-edges is analogous to the relationship between c-edges and d-edges shown in Lemma 3.1.1. And b′-edges are unchanged under duality, in a way analogous to b-edges in part 9 of Lemma 3.1.13.

Lemma 4.2.23. *Let G be a cellularly embedded graph with corresponding medial graph M and let $e \in E(G)$. Under a fixed antidirection of M, we have:*

1. *e is a b'-edge in G if and only if e is a b'-edge in G^*.*
2. *e is a c'-edge in G if and only if e is a d'-edge in G^*.*
3. *e is a d'-edge in G if and only if e is a c'-edge in G^*.*

Therefore, by the duality of the properties (100) and (010) we obtain the following result.

Theorem 4.2.24. *Let G be a cellularly embedded graph with corresponding gem J and medial graph M. Then $v_J(K)$ is even for all closed walks K in J if and only if there is an antidirection of M in which every edge of G is a d'-edge.*

Finally, we show that an abstract graph has an embedding G with corresponding gem J such that $v_J(K)$ is even for all closed walks K in J if and only if the abstract graph is Eulerian. First, it is clear that if $v_J(K)$ is even for all closed walks K in J, then all vertices in G have even degree and so G is Eulerian. Now, we will show in Chapter 5, specifically Corollary 5.2.6, that every Eulerian graph can be given a directed embedding. Furthermore, we will show that this embedding can be chosen to be orientable. This means that every Eulerian graph has an embedding satisfying properties (011) and (111), respectively. By our interpretation of the parity conditions as points on the Fano plane where on any line two of the conditions imply the third, we see that there is an embedding of every Eulerian graph that satisfies (100).

Condition (110)

Next we characterize the condition (110). We will show that this condition is equivalent to a pair of the six cellularly embedded graphs generated by duality and Petrie duality being bipartite.

Theorem 4.2.25. *Let G be a cellularly embedded graph with corresponding gem J. Then $(v+f)_J(K)$ is even for all closed walks K in J if and only if $(G^\times)^*$ is bipartite.*

Proof. By Theorem 4.2.17 we know that condition (110) holds if and only if the colored corner graph of G is bipartite with partite sets X and Y. Then around each 4-cycle in the corner graph corresponding to an e-square in J, the vertices are in the sets X, Y, X, and Y in cyclic order. Lift this partition of the vertices into J so that edges colored c_a join two vertices in the same set while edges colored c_v and c_f join two vertices in opposite sets. Then, taking the Petrie dual of G (with the same vertex partition), we see that edges colored c_a and c_f in $J^\times$ join vertices in the same set while edges colored c_v join vertices in opposite sets. Then $G^\times$ is 2-face-colorable by our characterization of condition (100). Therefore, $(G^\times)^*$ is bipartite.

Now assume that $(G^\times)^*$ is bipartite. Then we can partition the vertices in $(J^\times)^*$ into two sets X and Y so that edges colored c_f join vertices in different sets while edges colored c_v and c_a join vertices in the same set. Then in $J^\times$ edges colored c_v join vertices in different sets while edges colored c_f and c_a join vertices in the same set. Taking the Petrie dual, in J we see that edges colored c_v and c_f join vertices in different sets while edges colored c_a join vertices in the same set. Therefore, for all closed walks K in J, $(v+f)_J(K)$ is even. $\square$

Since G is bipartite if and only if $G^\times$ is bipartite we see that condition (110) holds if and only if $(G^\times)^*$ and $((G^\times)^*)^\times$ are bipartite. Equivalently condition (110) holds if and only if $G^\times$ and $(G^*)^\times$ are 2-face-colorable.

We have already shown that condition (110) is equivalent to the corner graph of G being bipartite, and there is a fourth graph related to G whose bipartiteness is equivalent with condition (110). Let G be a cellularly embedded graph with gem J and medial graph M. The medial graph *of* M (*Med*(M)) is equivalent as an embedded graph to the corner graph of G. Indeed, vertices in the medial graph of M (edges in M) are in one-to-one correspondence with edges colored c_a in J and two edges in M occur in succession along a facial walk in M if and only if their corresponding edges colored c_a are joined by an edge colored c_v or c_f. So the condition (110) holds if and only if the medial graph of the medial graph of G is bipartite.

Next we move on to a characterization of (110) based on an antidirection of the medial graph. The Petrie dual relationship between b′-edges and d′-edges is analogous to the relationship between b-edges and d-edges shown in Lemma 3.1.13.

Lemma 4.2.26. *Let G be a cellularly embedded graph with corresponding medial graph M and let $e \in E(G)$. Under a fixed antidirection of M, we have:*

1. *e is a b′-edge in G if and only if e is a d′-edge in $G^{\times}$.*
2. *e is a c′-edge in G if and only if e is a c′-edge in $G^{\times}$.*
3. *e is a d′-edge in G if and only if e is a b′-edge in $G^{\times}$.*

Therefore, applying Lemma 4.2.26 and Theorem 4.2.24 we obtain the following result.

Theorem 4.2.27. *Let G be a cellularly embedded graph with corresponding gem J and medial graph M. Then $(v + f)_J(K)$ is even for all closed walks K in J if and only if there is an antidirection of M in which every edge of G is a b′-edge.*

Proof. We know that condition (110) is equivalent to $(G^{\times})^*$ being bipartite. This in turn is equivalent to the existence of an antidirection of the medial graph for $(G^{\times})^*$ in which every edge of $(G^{\times})^*$ is a c′-edge. Then using Lemma 4.2.23, condition (110) is equivalent to the existence of an antidirection of the medial graph for $G^{\times}$ in which every edge of $G^{\times}$ is a d′-edge. Then by Lemma 4.2.26, condition (110) holds if and only if there is an antidirection of the medial graph for G in which every edge of G is a b′-edge. □

Since we showed that all Eulerian graphs have an embedding where (100) is satisfied (equivalently, where G^* is bipartite), by taking the Petrie dual of such an embedding we obtain an embedding where (110) is satisfied (equivalently where $(G^{\times})^*$ is bipartite).

Condition (001)

Let G be a cellularly embedded graph with corresponding gem J and medial graph M. As we have noted, edges in the medial graph are in one-to-one correspondence with edges colored c_a in J so condition (001) holds if and only if M is bipartite. Now we will show that the equivalence with the bipartiteness of M corresponds to a direction of M.

Theorem 4.2.28. *Let G be a cellularly embedded graph with corresponding gem J and medial* *graph M. Then $a_J(K)$ is even for all closed walks K in J if and only if there is a direction of M in which every edge of G is a t-edge.*

Proof. Assume that $a_J(K)$ is even for all closed walks K in J. Then M is bipartite with partite sets X and Y. We can assign a direction to every edge in M so that it becomes an arc from a vertex in X to a vertex in Y. Then under this direction of M every vertex in X has four incident half-arcs directed out and every vertex in Y has four incident half-arcs directed in. Thus every vertex is a t-vertex and corresponds to a t-edge in G.

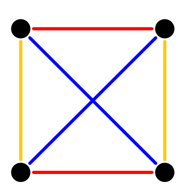
Conversely, assume that there is a direction of M in which every edge of G is a t-edge. Let X be the set of t-vertices in M with outdegree 4. Let Y be the set of t-vertices in M with indegree 4. Then every arc in M goes from a vertex in X to a vertex in Y. So M is bipartite with partite sets X and Y. Therefore, $a_J(K)$ is even for all closed walks K in J. □

Next we will give a characterization of when an abstract graph can be embedded in such a way that the medial graph is bipartite. We begin with the following observation.

Observation 4.2.29. *If a cellularly embedded graph G with gem J has a bipartite medial graph M then G and G^* are Eulerian.*

Proof. As we have noted, edges in M are in one-to-one correspondence with edges colored c_a in J. Then all v-gons and f-gons in J have an even number of edges colored c_a. So in G all vertices have even degree and all faces have even length meaning G and G^* are Eulerian. □

However, the converse of this statement is not true. For example, consider the embedded graph G with a single vertex and one twisted loop. Then G^* is also a single vertex with a twisted loop. So G and G^* are Eulerian, but the medial graph of G is a one-vertex 4-regular graph, which must consist of a vertex with two loops, which is not bipartite. Below we have drawn the gem for G with yellow representing c_a, red representing c_v, and blue representing c_f.



Lemma 4.2.30. *If a graph G is Eulerian and $|E(G)|$ is even, then G has a 2-coloring of its edges such that at each vertex there are an equal number of incident half-edges of each color.*

Proof. Since G is Eulerian, there exists an Eulerian circuit in G. Because G has an even number of edges we can color the edges alternately with white and black. Coloring the edges in this way creates an equal number of incident half-edges of each color at each vertex. □

Theorem 4.2.31. *A graph G has an embedding with a bipartite medial graph if and only if G is Eulerian and $|E(G)|$ is even. Moreover, the embedding can be chosen to be orientable.*

Proof. Assume that G has an embedding with a bipartite medial graph M. Let J be the corresponding gem for G. Since M is 4-regular, if it is bipartite then there are an equal number of vertices in each partite set. Then $|E(G)|$ is even since $|E(G)| = |V(M)|$. Since M is bipartite, v-gons in J have an even number of edges colored c_a and therefore every vertex in G has even degree. Thus G is Eulerian.

Now assume that G is Eulerian and $|E(G)|$ is even. By Lemma 4.2.30 there is a 2-coloring of the edges in G such that at each vertex there are an equal number of incident half-edges of each color. Then we can choose a rotation around each vertex so that the half-edges alternate in color. This coloring of $E(G)$ gives a bipartition of the vertices in the medial graph.

Note that the choice of edge signatures does not alter the medial graph (as an abstract graph). So we can create an orientable embedding if we so choose by choosing all edge signatures to be $+1$. □

Overview of Parity Condition Results

In the table in Figure 4.2 we give a full overview of the results in this section. Recall that any three conditions whose sum is zero in $\mathbb{Z}_2^3$ (meaning they occur on a line in the associated Fano plane) satisfy the property that any two of the three conditions imply the third condition. So each of the seven lines from the Fano plane corresponds to a theorem stating that any two of the three corresponding properties implies the third. The seven conditions either correspond to two of the six gems generated by duality and Petrie duality being bipartite, two of the six gems being orientable, or to the medial graph being bipartite. So we call these conditions the three bipartiteness conditions, the three orientability conditions, and the special condition (condition (001)). Theorems from the seven lines of the Fano plane fit into three classes:

1. There is one line with three bipartiteness conditions (the line containing $\{(100),(010),(110)\}$).
2. There are three lines with two orientability conditions and one bipartiteness condition (the lines containing $\{(111),(100),(011)\}$, $\{(111),(010),(101)\}$, and $\{(110),(101),(011)\}$). The theorems corresponding to the lines $\{(111),(100),(011)\}$ and $\{(110),(101),(011)\}$ can be obtained from the theorem corresponding to the line $\{(111),(010),(101)\}$, namely Theorem 4.2.3, by replacing G by G^* and $(G^*)^\times$, respectively.
3. There are three lines with one orientability condition, a bipartiteness condition, and the special condition (the lines containing $\{(111),(110),(001)\}$, $\{(100),(101),(001)\}$, and $\{(010),(011),(001)\}$). The theorems corresponding to the lines $\{(100),(101),(001)\}$ and $\{(010),(011),(001)\}$ can be obtained from the theorem corresponding to the line $\{(111),(100),(011)\}$ by replacing G by $G^\times$ and $(G^*)^\times$, respectively.

Recall that we have a theorem corresponding to any three colinear points in the Fano plane by Theorem 4.2.13, and we also have a theorem for any three noncolinear points in the Fano plane by Theorem 4.2.14. Using Figure 4.2 as a reference, we can interpret these types of theorems in many different ways. For example, a directed, orientable embedding of a bipartite, Eulerian digraph satisfies all seven properties. We can also apply partial duality and partial Petrie duality to the characterizations listed in Figure 4.2 using techniques similar to those used in the proof of Theorem 3.1.14. The results in Figure 4.2 can be extended to embeddings that are not connected by regarding Eulerian as meaning that every vertex has even degree (but not necessarily that the graph is connected).

Point in Fano Plane	Parity Condition: _______ is even for all closed walks K in J	Embedding Properties	Bipartite Related Graph	Bidirection of M in which every edge is a ...	Abstract graphs with an embedding with these properties
(100)	v_J	G^* bipartite, $(G^*)^\times$ bipartite, G 2-f-c, $((G^*)^\times)^*$ 2-f-c	G^*, $(G^*)^\times$	d'-edge	Eulerian
(010)	f_J	G bipartite, $G^\times$ bipartite, G^* 2-f-c, $(G^\times)^*$ 2-f-c	G, $G^\times$	c'-edge	bipartite graphs
(001)	a_J	M bipartite	M	t-edge	Eulerian with an even number of edges
(110)	$(v+f)_J$	$(G^\times)^*$ bipartite, $((G^\times)^*)^\times$ bipartite, $G^\times$ 2-f-c, $(G^*)^\times$ 2-f-c	corner, medial of M, $(G^\times)^*$, $((G^\times)^*)^\times$	b'-edge	Eulerian
(101)	$(v+a)_J$	$G^\times$ orientable, $(G^\times)^*$ orientable	side graph	d-edge	all graphs
(011)	$(f+a)_J$	$(G^*)^\times$ orientable, $((G^*)^\times)^*$ orientable	end graph	c-edge	Eulerian
(111)	$(v+f+a)_J$	G orientable, G^* orientable	J	b-edge	all graphs

Figure 4.2: Overview of Parity Condition Results

4.2.3 Eulerian Partial Duals

A *vertex-face walk* in a gem J is a closed walk that alternates between edges colored c_a and edges colored either c_v or c_f. A *vertex-face cycle* or *v/f-gon* in a gem J is a cycle that alternates between edges colored c_a and edges colored either c_v or c_f. These v/f-gons represent a vertex (and equivalently a face) in some partial dual of the cellularly embedded graph G represented by J.

Theorem 4.2.32. *Let G be a connected, cellularly embedded graph with corresponding gem J. All partial duals of G are Eulerian if and only if all v/f-gons have length 0 (mod 4).*

Proof. Let G be a connected, cellularly embedded graph with corresponding gem J. First assume that all partial duals of G are Eulerian. Then for any $A \subseteq E(G)$, the v-gons of J^A must have an even number of edges colored c_v. This means that the total length of the v-gon must be equal to 0 (mod 4). But, every v/f-gon in J represents a v-gon in J^A for some edge set $A \subseteq E(G)$. To see this, begin with a v/f-gon C in J. Note that since C is a cycle, it cannot cross an e-square in J along both an edge colored c_v and an edge colored c_f. Using both an edge colored c_v and an edge colored c_f in the same e-square causes a vertex-face walk to use all three edges incident with a vertex, thus making it not a cycle. So, for each e-square e in J, C crosses e along at most one color (it may use one or both edges of that color, but it may only use one of the two colors). Let A be the set of edges in G corresponding to e-squares in J that C crosses along edges colored c_f. Then consider G^A. In G^A, C becomes a v-gon since it is a v/f-gon with only edges colored c_a and c_v. Therefore C must have an even number of edges colored c_v since G^A is Eulerian by assumption and so C has length 0 (mod 4). Thus, all v/f-gons have length 0 (mod 4).

Conversely, assume that all v/f-gons have length 0 (mod 4). Let A be an arbitrarily chosen subset of $E(G)$. Let $v \in G^A$ be given. Then v is represented by a v-gon in G^A which is itself a v/f-gon in J up to possibly a different coloring on the edges colored c_v and c_f. The length of the v-gon representing v is 0 (mod 4) and thus v has even degree. Since v and A were chosen arbitrarily, we see that every vertex in a partial dual of G has even degree. Therefore, all partial duals of G are Eulerian. □

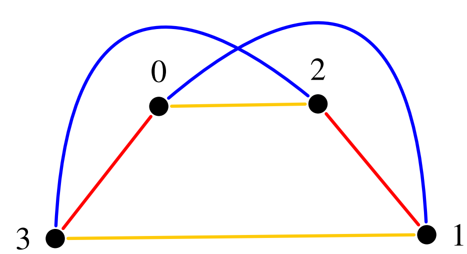
Corollary 4.2.33. *Let G be a connected, cellularly embedded graph with corresponding gem J. If for each cycle C in J either $(v+f)_J(C)$ is even or $a_J(C)$ is even, then all partial duals of G are Eulerian.*

Proof. Assume that $(v+f)_J(C)$ is even or $a_J(C)$ is even for all cycles C in J. Then for each v/f-gon C in J, since v/f-gons are themselves cycles in J, $(v+f)_J(C)$ or $a_J(C)$ is even. However, in any v/f-gon $(v+f)_J(C) = a_J(C)$. Thus the total length of C is equal to 0 (mod 4). Therefore, by Theorem 4.2.32, all partial duals of G are Eulerian. □

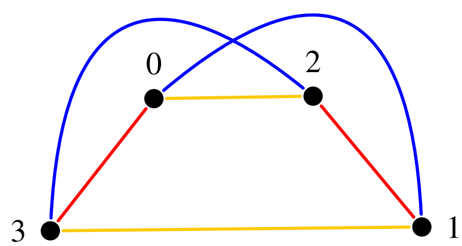
Corollary 4.2.34. *Let G be a connected, cellularly embedded graph with corresponding gem J and medial graph M. If $G^\times$ is 2-face-colorable or if M is bipartite then all partial duals of G are Eulerian.*

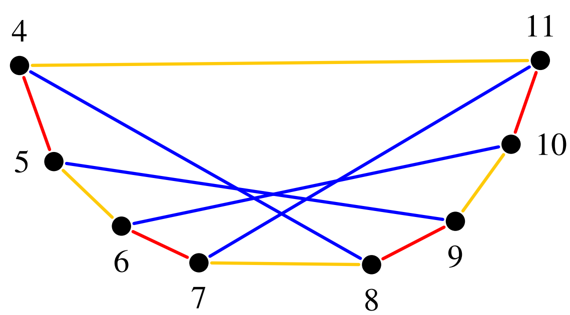
Note that the converse of Corollary 4.2.33 is not true. We will construct a counterexample by first finding an embedded graph with all partial duals Eulerian and a cycle C in its gem J such that $a_J(C)$ is odd. Then we will find an embedded graph with all partial duals Eulerian and a cycle with $(v+f)_J(C)$ odd. Combining these two will give us a counterexample for the converse of Corollary 4.2.33.

First, consider an embedded graph G with a single vertex and a single twisted edge e. The gem J representing this graph is unchanged up to color-preserving automorphism by taking the partial dual with respect to e. Thus, the graph and its only partial dual are both a single vertex of degree 2. Therefore G and all its partial duals are Eulerian. The gem J contains a cycle C with $a_J(C) = 1$. One such cycle is $(0, 1, 2)$.

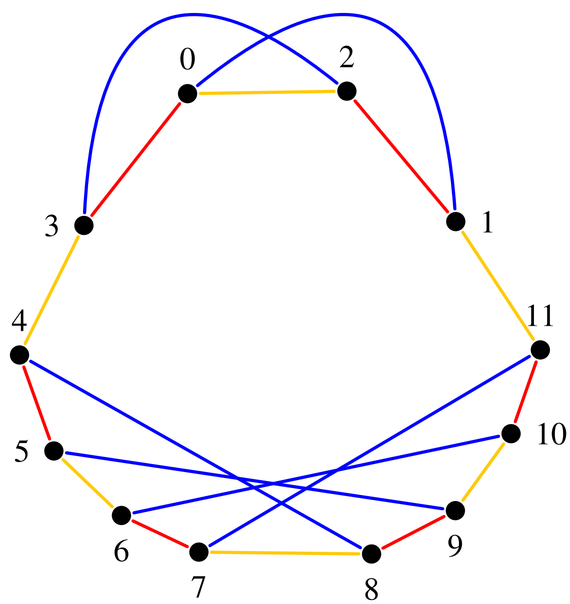


Similarly we can construct a cellularly embedded graph G_1 and corresponding gem J_1 such that all partial duals of G_1 are Eulerian but there exists a closed walk K such that $(v+f)_{J_1}(K)$ is odd. Recall that two loops e and f are called interlaced if they share a common vertex v and the cyclic order of edge labels around v is of the form $AeBfCeDf$ where A, B, C, and D are ordered lists of edge labels. Let G_1 be an embedded graph with one vertex v_1 of degree 4 with two interlaced twisted loops e_1 and e_2. Then we see that $G_1^{e_1}$ (and symmetrically $G_1^{e_2}$) are embedded graphs with a single vertex of degree 4 and two interlaced loops with one twisted and the other untwisted. G_1^* has two vertices of degree 2. Thus all partial duals of G_1 are Eulerian. Observe that G_1 contains a cycle C_1 of length five with two edges colored v_{J_1} and an edge colored f_{J_1}. One such cycle is $(4, 8, 9, 10, 11)$. Below we have drawn J and J_1.





We can combine these two graphs and consider a graph with a single vertex, one isolated twisted loop, and two interlaced twisted loops. This connected graph has partial duals that are either a single vertex of degree 6 or two vertices of degree 4 and 2 respectively. So all partial duals are Eulerian, but in this graph there exists a cycle with $a_J(C)$ odd and $(v+f)_J(C)$ odd. One such cycle is $(0, 1, 11, 10, 6, 5, 4, 3)$ with three edges colored c_a, two edges colored c_f, and three edges colored c_v.


0
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Chapter 5

Directed Embeddings

A directed embedding is a digraph embedded in a surface in such a way that all the faces of the embedding are bounded by directed walks. This is equivalent to an embedding of a digraph where half-arcs alternate between in and out around a vertex. We seek to answer whether or not an embedding of a mixed graph can be given an orientation in such a way that the resulting digraph is a directed embedding, and if this directed embedding can have certain desired properties such as orientability or the presence of a predetermined set of facial walks. First, we will give a brief history of the study of directed embeddings.

5.1 History of Directed Embeddings

While embedded graphs have a long history of study, the study of directed embeddings has begun in earnest only over the past few decades. In general the depth of research on directed graphs pales in comparison to the research on undirected graphs. The first modern book on graph theory striving to cover the theory of directed graphs in detail was published in 2000 [3].

Tutte [38] related 3-face-colorable, orientable, cellular embeddings of cubic bipartite graphs to alternating dimaps (orientable, cellular, directed embeddings). Given a 3-face-colored, orientable, cellular embedding of a bicubic graph G with partite sets X and Y,

1. Assign a direction to G so that every edge becomes an arc from X to Y.
2. Select a color, R, and contract all the faces, $f_i \in F_R$, of that color to a vertex R_i.

Call the resulting embedded digraph D. After doing so, since G was bipartite, the half-edges around R_i alternate "in" and "out." Every orientable directed embedding can be constructed in this manner. An example is given in Figure 5.1 where the partite sets are labeled with $+$ and $-$ and the 3-face-coloring has colors $\{R, Y, B\}$. In this example we chose to contract faces colored R,

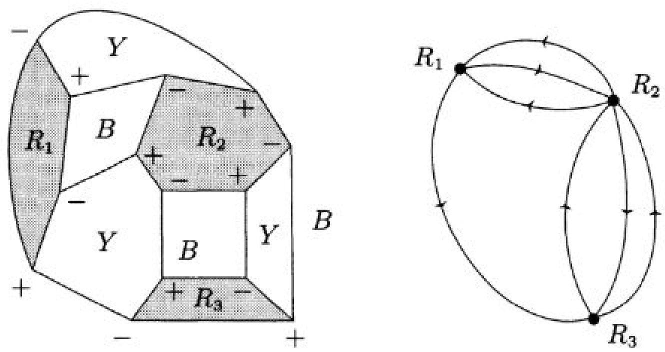


Figure 5.1: A bicubic 3-face-colored embedding and a related directed embedding (from Tutte [38]

but we can apply the same procedure for any of the three colors. Thus for each 3-face-colored, orientable, cellular embedding of a bicubic graph there are three related directed embeddings. These three digraphs are have a *triality* or *trinity* relationship. The three digraphs are called a *trial set*. An *arborescence* is a directed, rooted tree. Tutte's Tree Trinity Theorem [40] showed that for a trial set $\{G_1, G_2, G_3\}$, $T(G_1) = T(G_2) = T(G_3)$ where $T(G_i)$ is the number of spanning arborescences in G_i (rooted at any vertex; the choice of root vertex does not change the value). The Trinity Theorem was first proved by Tutte in [39] and an alternative approach was given in [40]. Brooks, Smith, Stone, and Tutte [7] gave a survey of the correspondence between 3-face-colorable, orientable, bicubic maps and alternating dimaps and gave another way of proving the Trinity Theorem.

Berman [4] and Farr [16] expanded on the work of Tutte. Berman gave a new proof of Tutte's Trinity Theorem based on a determinant formula for the number of spanning arborescences of a digraph. Farr developed minor operations for orientable directed embeddings and showed that these minor operations have a triality relation in a manner similar to the duality between deletion and contraction for cellularly embedded graphs. These results that stem from Tutte's Trinity Theorem do not seem to be related to other directed embedding results.

A more recent study of directed embeddings of Eulerian directed graphs began in 2002 with Bonnington, Conder, Morton, and McKenna [5]. They explored embeddings of directed graphs on orientable surfaces. They showed that the genus range for directed embeddings of a fixed digraph is continuous, and that the range can have cardinality 2 (with values g and $g+1$) or be arbitrarily large. They also showed that every regular tournament (Eulerian orientation of a complete graph of odd order) has a directed embedding with only two faces, the minimum number of faces, so this has the maximum possible directed genus (it is *upper-embeddable*) among all directed embeddings of that tournament. The questions that they posed in [5] inspired much of the recent work on directed embeddings.

Bonnington, Hartsfield, and Širáň [6] studied obstructions to directed embeddings of Eulerian digraphs in the plane. They proved Kuratowski-type theorems giving minimal excluded configurations for directed planar embeddings. Later Archdeacon, Bonnington, and Mohar [1] gave minimal obstructions for directed planar embeddings of $(2,2)$-regular digraphs in terms of the cycle removal operation.

Hao [20] characterized when complete tripartite graphs have minimum genus orientable embeddings that can be oriented to give a directed embedding. She also provided a sufficient condition for a complete graph to have an embeding of this type which was later extended to a characterization (necessary and sufficient condition) by Liu and Hao [27]. In a result more closely related to our work, Hao [20] stated a general result on when an orientable embedding can be oriented to give a directed embedding (Theorem 5.2.7 below).

It is often asked whether results for embeddings of undirected graphs have analogous results for directed embeddings. Archdeacon, Devos, Hannie, and Mohar [2] showed an analogous theorem to Whitney's theorem on different embeddings of the same graph being related by flips for embeddings of $(2,2)$-regular directed graphs. In [5], Bonnington et al. asked if the directed genus distribution of a directed graph is (strongly) unimodal. This is still an open conjecture, as is the analogous conjecture for undirected genus distributions.

Hales and Hartsfield [19] found the minimal orientable directed genus of the de Bruijn graph. The directed genus distributions of several classes of digraphs are known. Hao and Liu [21] found the directed genus distribution for a class of digraphs that they called antiladders. Hao, Liu, Zhang, and Xu [22] found the directed genus distributions of two classes of $(2,2)$-regular digraphs. Then

Chen, Gross, and Hu [9] extended their work by finding a formula for the directed genus distribution for (2,2)-regular outerplanar digraphs, and showing that this distribution is log-concave and hence unimodal. They also introduced the method of Eulerian splitting of a vertex in a digraph. They proved a splitting theorem for directed genus distributions that is analogous to a theorem for undirected genus distributions. Chen, Gao, and Huang [8] found explicit formulas for unlabeled directed genus distributions of directed bouquets of cycles, directed dipoles, and a class of regular tournaments.

The study of directed embeddings has found applications in mathematical biology, specifically for polypeptide nanostructures in DNA [17, 41]. The strong trace model has been used to model these polypeptide nanostructures as embedded graphs. Fijavž, Pisanski, and Rus [17] showed that a graph G admits a parallel strong trace, meaning a 1-face directed embedding, which is necessarily nonorientable, if and only if G is Eulerian.

Research has been done regarding various types of embeddings of directed graphs, not all of which are directed embeddings. For example, Sneddon and Bonnington [37] characterized clustered planar graphs in a Kuratowski-type theorem. A *clustered* embedding has all the "in" half-arcs occurring sequentially in the cyclic ordering around each vertex as opposed to alternating "in" and "out" for a directed embedding.

5.2 Extending Partial Orientations of Eulerian Graphs to Digraph Embeddings

In this section we will be working with directed embeddings, but our work will be quite different in nature from the work discussed above. In general, our work is on whether or not an embedding of an undirected graph or mixed graph can be given a direction (orientation) in such a way that the resulting directed graph is a directed embedding. First, we give a characterization of when an undirected embedded graph can be given a direction in such a way that the resulting embedding of a directed graph is a directed embedding. This characterization fits in nicely with the work that we did in Chapter 4.

Next we give a characterization of when a pair $(G, \mathcal{W})$, where G is a mixed graph and $\mathcal{W}$

is a collection of closed directed walks, can be extended to a directed embedding of a direction (orientation) D of G where the orientation on the original set of arcs (directed edges) in G is unchanged and $\mathcal{W}$ is a subcollection of the facial walks in the embedding. This characterization is done in two separate steps. First we restate a result of Ford and Fulkerson and give a proof of when an Eulerian graph G with a subset of edges S given a direction can be extended to an Eulerian digraph D. Second, we characterize when a fixed collection of walks $\mathcal{W}$ in D can be a subcollection of the facial walks in a directed embedding of D. We will show that these two steps can be done independently. Furthermore, we will determine when the directed embedding of a direction D of G (where the orientation on S is unchanged and $\mathcal{W}$ is a subcollection of the facial walks in the embedding) can be chosen to be orientable.

5.2.1 Basic Definitions

A *directed graph* or *digraph* D is a triple consisting of a vertex set $V(D)$, an arc set $A(D)$, and a function mapping each arc to an ordered pair of vertices. The first vertex of the ordered pair is the *tail* of the arc, and the second is the *head*. The arc goes from its tail to its head. The *underlying graph* of a digraph D is the graph G that is obtained from D by taking each arc associated with an ordered pair of vertices (u, v) and replacing it with an edge associated with the unordered pair $\{u, v\}$.

A digraph D can be obtained from a graph G by assigning a direction to each edge in $E(G)$ thus associating each edge with an ordered pair of vertices making the edge an arc. In this case D is a *direction* of G. Note that typically D is referred to as an 'orientation' of G, but to avoid confusion with the topological meaning of orientation we will refer to this as a direction of G.

A *directed path* is a simple digraph whose vertices can be linearly ordered so that there is an arc with tail u and head v if an only if v immediately follows u in the vertex ordering. In other words, the underlying graph is a path and the arcs are given a consistent overall direction. A *directed cycle* is a simple digraph whose vertices can be cyclically ordered so that there is an arc with tail u and head v if and only if v immediately follows u in the vertex ordering. A *directed walk* is a list $v_0, e_1, v_1, e_2, ..., e_k, v_k$ of vertices and arcs such that, for $1 \leq i \leq k$, the arc e_i has tail v_{i-1} and head v_i. A *directed trail* is a directed walk with no repeated arc. A directed trail or directed walk is called *closed* if it begins and ends at the same vertex. For closed directed walks and trails we often consider two closed directed walks (or trails) to be equivalent if one is a cyclic shift of the other. A closed trail is also called a *circuit*.

A *mixed graph* G is a quintuple consisting of a vertex set $V = V(G)$, a set of arcs $A = A(G)$, a set of edges $E = E(G)$ (with A and E disjoint), a function ϕ mapping each arc in A to an ordered pair of vertices, and a function ψ mapping each edge in E to an unordered pair of vertices. Note that a directed graph is a mixed graph with $E = \emptyset$ and an undirected graph is a mixed graph with $A = \emptyset$. We write $G = (V, A, E, \phi, \psi)$ or just $G = (V, A, E)$.

The *underlying graph* of a mixed graph G is the graph G' that is obtained from G by replacing each directed arc with an undirected edge in the natural way.

In a mixed graph G, the *degree* of a vertex $v \in V(G)$, denoted $d(v)$, is the total number of all half-edges and half-arcs incident with v. The *undirected degree* of v, denoted $d^0(v)$, is the total number of all half-edges incident with v. The *indegree* of v, denoted $d^-(v)$, is the number of arcs with head v. The *outdegree* of v, denoted $d^+(v)$, is the number of arcs with tail v. So we have $d(v) = d^0(v) + d^-(v) + d^+(v)$.

A *directed embedding* of a digraph D is an embedding such that every facial walk in the embedding is a directed walk.

A graph G is *Eulerian* if it is connected and has a closed trail containing all the edges in G. A directed graph D is *Eulerian* if it is connected and D has a directed closed trail containing all the arcs in D. We call such a closed trail an *Eulerian circuit* or a *Eulerian directed circuit*.

A mixed graph G is called *undirected-Eulerian* if the underlying undirected graph of G is Eulerian. A mixed graph G is called *Eulerian-directable* if each undirected edge can be given a direction so that the resulting directed graph is Eulerian. A mixed graph or directed graph is called *balanced* if $d^+(v) = d^-(v)$ for every vertex v.

The following two theorems are well known.

Theorem 5.2.1. *A graph G is Eulerian if and only if it is connected and every vertex in $V(G)$ has even degree.*

Theorem 5.2.2. *A directed graph D is Eulerian if and only if it is connected and balanced.*

Observation 5.2.3. *Every connected Eulerian undirected graph is Eulerian-directable.*

Proof. Let G be an Eulerian undirected graph. Then G has an Eulerian circuit W. Choose a direction for this closed walk W and direct every edge in the direction W goes through it. □

Lemma 5.2.4. *If a mixed graph G is balanced and undirected-Eulerian, then G is Eulerian-directable.*

Proof. Assume that $G = (V, A, E)$ is a balanced, undirected-Eulerian mixed graph and let U be the undirected graph with vertex set V and edge set E. Since G is balanced and undirected-Eulerian, $d^0(v)$ is even for all $v \in V$. Thus every component of U is Eulerian. By Observation 5.2.3 each component of U can be given an Eulerian direction so that U becomes a digraph D so that each component of D is Eulerian. Adding in the arcs in A to D results in a connected, balanced digraph D' which is Eulerian by Theorem 5.2.2. Then D' is a direction of G so G is Eulerian-directable. □

We also observe that if a mixed graph G is Eulerian-directable, then it is undirected-Eulerian. A mixed graph that is Eulerian-directable may or may not be balanced.

Proposition 5.2.5. *If a connected digraph D has a directed embedding then D is Eulerian. Conversely, every Eulerian digraph D can be given a directed embedding. This embedding can be chosen to be orientable.*

Proof. If D has a directed embedding then because the arcs alternate between inward and outward at every vertex, D is balanced and hence Eulerian.

Let D be an Eulerian digraph. Then D is connected and balanced by Theorem 5.2.2. Then we can choose a rotation scheme for D such that the half-arcs alternate in and out as we travel in the clockwise direction around every vertex. This rotation scheme, together with the choice of $+1$ for every edge signature, gives an orientable directed embedding for D. □

The following corollary follows directly from Proposition 5.2.5 and Observation 5.2.3.

Corollary 5.2.6. *If a connected graph G has a direction that has a directed embedding, then G is Eulerian. Conversely, there is a direction D of every Eulerian graph G such that there is an embedding of D that is a directed embedding. This embedding can be chosen to be orientable.*

5.2.2 Embeddings Extending to Directed Embeddings

In 2018, Rong-Xia Hao gave a characterization of when an orientable embedded graph can be given a direction resulting in a directed embedding, as follows.

Theorem 5.2.7 (Hao [20, Lemma 4.1])**.** *For an orientable cellularly embedded graph G, there exists a direction D of G such that D is a directed embedding if and only if G is 2-face-colorable.*

As our main result in this section, we extend this characterization to arbitrary (not necessarily orientable) embedded graphs. We will then be able to show, in Theorem 5.2.9, the relationship between this property and other properties for embedded graphs discussed in Chapters 3 and 4.

Theorem 5.2.8. *For a cellularly embedded graph G, there exists a direction D of G such that D is a directed embedding if and only if $(G^*)^\times$ is orientable.*

Proof. Assume that there exists a direction D of an embedded graph G such that D is a directed embedding. So we have an orientation for all the faces in G (following the direction of the arcs in D) which gives us a clockwise orientation for all the vertices in G^*. Call this local clockwise orientation S. In D, faces that share an arc go through the arc in the same direction since D is a directed embedding. This means that in G^*, if we travel along an edge e between two vertices x and y, and translate the local orientation at x along e, when we reach y the translated orientation is opposite to the orientation at y. If we put a twist on every edge in G^* by taking the Petrie dual $(G^*)^\times$, when we translate the orientation from x to y along e, the translated orientation always agrees with the local orientation at y. In other words, S gives a consistent global orientation of $(G^*)^\times$, so $(G^*)^\times$ is orientable.

Now assume that $(G^*)^\times$ is orientable. Let S be a consistent global orientation of $(G^*)^\times$. Then in G^*, S gives a local orientation such that when we tranlate the local orientation from a vertex x along an edge e to a vertex y, the translated orientation does not agree with the local orientation at y. This means that in G, the two faces f and g that share the edge e, corresponding to x and y in G^*, both use the edge e in the same direction. Therefore, giving this uniquely defined direction to each edge in G results in a directed embedding because the faces of G all agree with this direction. □

Combining this theorem with our previous results for closed walks in gems gives the following theorem.

Theorem 5.2.9. *Let J be a gem with corresponding cellularly embedded graph G. The following are equivalent:*

1. $(G^*)^\times$ *is orientable.*
2. *There exists an orientation D of G such that D is a directed embedding.*
3. $(f + a)_J(K)$ *is even for all closed walks K in J.*
4. *There exists an all-crossing direction of the medial graph of G such that every edge is a c-edge.*
5. *The end graph of G is bipartite.*

Proof. Theorem 5.2.8 shows the equivalence of 1 and 2. Theorem 4.2.7 shows the equivalence of 1 and 4. The equivalence of 1 and 3 is given by Theorem 4.2.20. Finally the equivalence of 3 and 5 is given by Theorem 4.2.18. □

Note that Theorem 5.2.7 is a corollary of Theorem 5.2.9 and Theorem 4.2.3 from our work on parity conditions in closed walks. It fits as one of the lines in the Fano plane mentioned in Chapter 4. We have shown that for a cellularly embedded graph G with gem J,

1. G is orientable if and only if $(v + a + f)_J(K)$ is even for all closed walks K in J.
2. G is 2-face-colorable if and only if $v_J(K)$ is even for all closed walks K in J.

3. There exists a direction D of G such that D is a directed embedding if and only if $(a+f)_J(K)$ is even for all closed walks K in J.

So Theorem 5.2.7 is an immediate consequence of Theorem 4.2.13 applied to the line $\{(100), (011), (111)\}$ in the Fano plane.

In the following theorem we give a direct proof of the equivalence of 2 and 4 of Theorem 5.2.9 to further illustrate the relationship between directed embeddings and the other properties above.

Theorem 5.2.10. *For an embedded graph G, there exists a direction D of G such that D is a directed embedding if and only if there exists an all-crossing direction of the medial graph of G such that every edge of G is a c-edge.*

Proof. Assume that there exists a direction D of an embedded graph G such that D is a directed embedding. Let J be the gem associated with G and let M be the medial graph of G. The direction on the edges of D yields a natural direction of the edges colored c_f in J. Since D is a directed embedding, all faces in D must be directed walks and so the direction of the edges also extends to all the f-gons in J and thus to the edges colored c_a. Since edges colored c_a are in one-to-one correspondence with edges in M this gives us a direction M' of M.

Now we partition the vertices in J into two types. Let X be the set of vertices in J with incident edge colored c_a directed in and let Y be the set of vertices in J with incident edge colored c_a directed out. Clearly edges colored c_a are directed from a vertex in Y to a vertex in X. Edges colored c_f are directed from a vertex in X to a vertex in Y since all facial walks are directed. Edges colored c_v either join a vertex in X to another vertex in X or a vertex in Y to another vertex in Y.

Each edge $e \in E(G)$ corresponds to an e-square $Q(e)$ in J and a vertex $v(e)$ in M. As we go around $Q(e)$ in J we will see an X vertex, an edge colored c_v, an X vertex, an edge colored c_f, a Y vertex, an edge colored c_v, a Y vertex, and an edge colored c_f. Translating this to the medial graph, what we see as we go around $v(e)$ in M, equipped with the checkerboard coloring corresponding to G, is an incoming arc, a black region, an incoming arc, a white region, an outgoing arc, a black region, an outgoing arc, and a white region. This is exactly the characterization of $v(e)$ as a c-vertex in M which means e is a c-edge in G.

Now assume that there exists an all-crossing direction of the medial graph of G with all c-edges. Translate this to the natural direction of the edges colored c_a in J. By our characterization of c-edges this means that the edges colored c_f are between vertices in X and vertices in Y. Assign a direction to each edge colored c_f so that it is an arc from a vertex in X to a vertex in Y. Then every f-gon is a directed walk in J. The characterization of c-edges also means that the edges colored c_v are between two vertices in X or between two vertices in Y. Then on an e-square $Q(e)$, both edges colored c_f correspond to the same direction of the edge e in G. Assigning this direction for each edge e of G creates an embedded digraph D. Since all f-gons are directed walks in J, all facial walks in D are directed. □

We can also characterize when an embedding of a mixed $G = (V, A, E)$ graph can be given a full direction (directing the edges in E without changing the directions on A or changing the underlying embedding scheme) in such a way that this direction D of G is a directed embedding. Observe that if G can be directed to give a directed embedding, is connected and has an arc then the direction on this arc determines the direction on all the other edges and arcs in the embedding. Indeed, the directions on half-arcs around a vertex in a directed embedding must alternate so a single arc in a directed embedding determines the directions on all edges and arcs adjacent to that edge. If G is connected, a single arc determines all the other directions. Using this, we can assign a direction to every arc or edge, checking that we do not create any contradictions to the directions of the already existing arcs, and verifying that the final direction does in fact give a directed embedding.

We can also characterize this scenario as follows. Such a direction of a mixed graph G exists if and only if G^* is Petrie orientable and along any sequence $u_1, f_1, u_2, f_2, ..., f_k, u_k$ where the facial walk for the face f_i contains the edge(s) and/or arc(s) u_i and u_{i+1}, the (tentative) directions for the edges and the directions already assigned for the arcs must alternate between left and right. If there are any discrepancies between assignments, or an edge can receive two opposing tentative assignments by following different sequences, then there is no direction D of G that is a directed embedding.

5.3 Eulerian-Directable Mixed Graphs

In the next three theorems (Theorems 5.3.1, 5.3.2, and 5.3.3) we present results of Ford and Fulkerson. Theorem 5.3.1 is well known so it is simply stated. Theorem 5.3.2 has been slightly modified to account for the possibility of parallel arcs in a network, and we present a proof that works in that situation. In the proof of Theorem 5.3.3 we fill in some details that are omitted in the original proof by Ford and Fulkerson.

A *network* is a directed graph with a non-negative *capacity* $c(a)$ on each arc a and a distinguished *supply* vertex s and *target* vertex t. A *flow* f assigns a real value $f(a)$ to each arc a. We write $f^+(v)$ for the total flow on the arcs leaving v and $f^-(v)$ for the total flow on arcs entering v. A flow is *feasible* if it satisfies the capacity constraints $0 \leq f(a) \leq c(a)$ for each arc and the conservation constraints $f^+(v) = f^-(v)$ for each vertex $v \notin \{s,t\}$. In some networks there are non-negative lower bounds $l(a)$ and in this case feasibility requires the capacity constraints $0 \leq l(a) \leq f(a) \leq c(a)$ to be satisfied for each arc in addition to the conservation constraints $f^+(v) = f^-(v)$ for each vertex $v \notin \{s,t\}$. The quantity $f^+(s) - f^-(s) = f^-(t) - f^+(t)$ is called the *value* of a feasible flow. A *circulation* is a flow in which there is not a distinguished supply or target vertex, in other words a circulation is a flow that satisfies the conservation constraint $f^+(v) = f^-(v)$ at each vertex v.

Let N be a network with $X, Y \subseteq V(N)$ and let f be a function defined on the arcs of N. We write $f(X,Y)$ to denote the sum of the values of $f(a)$ for all arcs a from X to Y, i.e., with tail in X and head in Y (X and Y are not necessarily disjoint). Similarly $f(x,Y)$ denotes the sum of the values of $f(a)$ for all arcs from the vertex x to the set Y, and $f(Y,x)$ denotes the sum of the values of $f(a)$ for all arcs from the set Y to the vertex x.

A *supply/target* cut $[S,T]$ in a network with supply s and target t consists of the arcs from a set S to a set T where S and T partition the set of vertices, with $s \in S$ and $t \in T$. The quantity $c(S,T)$ is called the *capacity of the cut* $[S,T]$.

The following is the well-known Max-Flow Min-Cut Theorem.

Theorem 5.3.1 (Ford and Fulkerson [18]). *In every network, the maximum value of a feasible flow equals the minimum capacity of a supply/target cut. Furthermore, if all capacities are integral, a feasible flow of maximum value may also be chosen to be integral.*

The following theorem has been modified slightly from how it was stated by Ford and Fulkerson so that it applies for networks that can have multiple arcs.

Theorem 5.3.2 (Ford and Fulkerson [18, Theorem 3.1, p. 51]). *Suppose N is a network with a nonnegative lower bound $l(a)$ for each arc a. Then there exists a circulation f satisfying $l(a) \leq f(a) \leq c(a)$ for all $a \in A(N)$ if and only if $c(X, \bar{X}) \geq l(\bar{X}, X)$ for all $X \subseteq V(N)$. Moreover, if c and l are integral and f exists, f may be chosen to be integral.*

Proof. Let N be a network with lower and upper bound functions defined on $A(N)$. Let $V = V(N)$. Extend N to N' by adding two vertices s and t and an arc sx from s to x and an arc xt from x to t for every $x \in V$. Remove the lower bound constraints in N' ($l'(a) = 0$ for all $a \in A(N')$). The capacity function defined on $A(N')$ is given by:

1. $c'(a) = c(a) - l(a)$ for all $a \in A(N)$,
2. $c'(sx) = l(V, x)$ for all $x \in V(N)$,
3. $c'(xt) = l(x, V)$ for all $x \in V(N)$.

Then a feasible circulation f, meaning a circulation satisfying $l(a) \leq f(a) \leq c(a)$ for all $a \in A(N)$, in N generates a flow f' from s to t in N' by the assignment:

1. $f'(a) = f(a) - l(a)$ for all $a \in A(N)$,
2. $f'(sx) = l(V, x)$ for all $x \in V(N)$,
3. $f'(xt) = l(x, V)$ for all $x \in V(N)$.

From these assignments we see that $f^+(s) - f^-(s) = f^-(t) - f^+(t) = l(V, V)$ and it can easily be verified that f' satisfies the conservation constraints for all vertices except s and t. So we see that if there is a feasible circulation in N then there is a feasible flow in N' of value $l(V, V)$.

Assume now that there is a feasible flow f' in N' of value $l(V, V)$ where the capacities for N' are as above. Define a flow f on N by

1. $f(a) = f'(a) + l(a)$.

For each $x \in V(N)$, the flow of $l(V', x)$ coming into x along sx is replaced by additional flow along the arcs $a \in (V, x)$, and the flow of $l(x, V)$ leaving x along xt is replaced by additional flow along the arcs $a \in (x, V)$, so the conservation condition still holds at x. Moreover, $l(a) \leq f(a) = f'(a) + l(a) \leq c'(a) + l(a) = c(a)$ for all $a \in A(N)$. Thus, f is a feasible circulation in N.

Therefore there is a feasible circulation in N if and only if there is a feasible flow in N' of value $l(V, V)$. By Theorem 5.3.1 there exists a feasible flow from s to t of value $l(V, V)$ if and only if all cut capacities in N' are greater than or equal to $l(V, V)$.

Let $(X', \bar{X}')$ be a supply/target cut in N'. Let $X = X' \setminus s$ and $\bar{X} = \bar{X}' \setminus t$. Then, using the fact that $l(V, V) = l(\bar{X}, \bar{X}) + l(X, V) + l(\bar{X}, X)$ for the final step, we see that:

$$\begin{aligned}
c'(X', \bar{X}') &= c'(X \cup s, \bar{X} \cup t) \\
&= c'(X, \bar{X}) + c'(s, \bar{X}) + c'(X, t) \\
&= c(X, \bar{X}) - l(X, \bar{X}) + l(V, \bar{X}) + l(X, V) \\
&= c(X, \bar{X}) + l(\bar{X}, \bar{X}) + l(X, V) \\
&= c(X, \bar{X}) + l(V, V) - l(\bar{X}, X)
\end{aligned}$$

Therefore, $c'(X', \bar{X}') \geq l(V, V)$ if and only if $c(X, \bar{X}) \geq l(\bar{X}, X)$. Thus there exists a circulation in a network N satisfying $l(a) \leq f(a) \leq c(a)$ for all $a \in A(N)$ if and only if $c(X, \bar{X}) \geq l(\bar{X}, X)$ for all $X \subseteq V(N)$.

The statement about the integrality of f follows from Theorem 5.3.1. □

The next theorem is another result of Ford and Fulkerson that we have adapted to better serve our purposes in the following section. We have restated their theorem and filled in a few minor details that Ford and Fulkerson leave out. Before the theorem we introduce some notation. For a mixed graph G with $X, Y \subseteq V(G)$ we denote by $e(X, Y)$ and $a(X, Y)$ the number of edges and arcs, respectively, from X to Y.

Theorem 5.3.3 (Ford and Fulkerson [18, Theorem 7.1, p. 60])**.** *A mixed graph G is Eulerian-directable if and only if it is undirected-Eulerian and for every $X \subseteq V(G)$, $e(X, \bar{X}) + a(X.\bar{X}) \geq a(\bar{X}, X)$.*

Proof. Assume that G is Eulerian-directable. Then clearly it is undirected-Eulerian. Let $X \subseteq V(G)$. Since an Eulerian circuit must traverse every arc and travel from X to $\bar{X}$ the same number of times as it travels from $\bar{X}$ to X, the maximum number of times it could travel from X to $\bar{X}$, namely $e(X, \bar{X}) + a(X.\bar{X})$, must be greater than or equal to the minimum number of times it must travel from $\bar{X}$ to X, namely $a(\bar{X}, X)$.

Now we prove sufficiency. Assume that G is undirected-Eulerian and for every $X \subseteq V(G)$, $e(X, \bar{X}) + a(X.\bar{X}) \geq a(\bar{X}, X)$. First replace each edge of G by a pair of oppositely directed arcs, thus creating a directed graph G_1 from the mixed graph G. Define the lower bounds and capacities for arcs $a \in A(G_1)$ by

1. $c(a) = 1$ for all $a \in A(G_1)$,
2. $l(a) = 1$ for all $a \in A(G)$,
3. $l(a) = 0$ for all $a \in A(G_1) \setminus A(G)$.

Using these assignments and the fact that $e(X, \bar{X}) + a(X.\bar{X}) \geq a(\bar{X}, X)$ in G, we see that $e(X, \bar{X}) + a(X, \bar{X}) = c(X, \bar{X}) \geq l(\bar{X}, X) = a(\bar{X}, X)$ for all $X \subseteq V(G_1)$. So by Theorem 5.3.2 there exists a feasible integral circulation f in G_1 so that $f(a) = 0$ or $f(a) = 1$ for all $a \in A(G_1)$. Now we will assign a direction to some of the edges in G that we replaced by two arcs in G_1. Suppose e is an edge in G between x and y, which becomes two arcs e' from x to y and e'' from y to x. If $f(e') = f(e'')$ leave e untouched, otherwise replace e in G by $e_1 \in \{e', e''\}$ satisfying $f(e_1) = 1$. This creates a new mixed graph G_2. Because G_2 was obtained from G by applying a feasible flow in G_1, G_2 is balanced.

So G_2 is a balanced, undirected-Eulerian mixed graph and hence by Lemma 5.2.4 we know that G_2 is Eulerian-directable. Since G_2 was itself a (partial) direction of G this means that G is Eulerian-directable. □

From this result of Ford and Fulkerson we have a full characterization of when a mixed graph can be given a direction so that it is an Eulerian digraph. In what follows we give a characterization of when a fixed collection of walks $\mathcal{W}$ in an Eulerian digraph D can be a subcollection of the facial walks in a directed embedding of D. Putting these together we will be able to determine whether a pair $(G, \mathcal{W})$, where G is a mixed graph and $\mathcal{W}$ is a collection of closed directed walks in G, can be extended to a directed embedding of a direction D of G where the direction on $A(G)$ is unchanged and $\mathcal{W}$ is a subcollection of the facial walks in the embedding. We will show that these two steps can be done independently. Namely, we can first direct a mixed graph to form a directed graph then find an embedding with $\mathcal{W}$ as a subcollection of the facial walks.

5.4 Extending Partial Directed Embeddings to Directed Embeddings

In Chapter 1 we defined transition graphs for embedded graphs. We will now look at transition graphs in more generality. Note that the situation in Chapter 1 corresponds to the case where the collection of closed walks $\mathcal{W}$ is the collection of facial walks of a graph embedding.

Given a graph G and a collection (i.e., multiset) of closed walks $\mathcal{W}$ in G, at each vertex $v \in V(G)$ the *transition graph* or *partial transition graph induced by* $\mathcal{W}$, $T(G, \mathcal{W}, v)$, is constructed as follows:

1. Create a vertex for each half-edge incident with v.
2. Join two (possibly equal) vertices by an edge for each time their corresponding half-edges occur in succession along a walk in $\mathcal{W}$. The *set of transition graphs induced by* $\mathcal{W}$, $T(G, \mathcal{W})$, is the set $T(G, \mathcal{W}) = \{T(G, \mathcal{W}, v) : v \in V(G)\}$.

Note that every half-edge incident with v occurs in the transition graph $T(G, \mathcal{W}, v)$ even if this half-edge does not occur along a walk in $\mathcal{W}$; in this case it is an isolated vertex. It is also worth noting that this set of induced transition graphs may or may not be able to be extended into a set of transition graphs representing an embedding of G with $\mathcal{W}$ as a subcollection of the facial walks in the embedding.

Next we extend the idea of transition graphs to digraphs. Given an Eulerian digraph D and a collection of closed directed walks $\mathcal{W}$ in D, at each vertex $v \in V(D)$ the *transition graph* or *partial transition graph induced by* $\mathcal{W}$, $T(D, \mathcal{W}, v)$, is constructed as follows:

1. Create a vertex for each half-arc incident with v.
2. Join two (possibly equal) vertices by an edge for each time their corresponding half-arcs occur in succession along a walk in $\mathcal{W}$. The *set of transition graphs induced by* $\mathcal{W}$, $T(D, \mathcal{W})$, is the set $T(D, \mathcal{W}) = \{T(D, \mathcal{W}, v) : v \in V(G)\}$.

Again, note that every half-arc incident with v occurs in the transition graph. Also, this set of induced transition graphs may or may not be able to be extended into a set of transition graphs representing a directed embedding of D with $\mathcal{W}$ as a subcollection of the facial walks in the embedding.

An ordered pair $(G, \mathcal{W})$, where G is a graph and $\mathcal{W}$ is a collection of closed walks in G, is called *extendable* if there is an embedding of G where $\mathcal{W}$ is a subcollection of the facial walks in the embedding. An ordered pair $(D, \mathcal{W})$, where D is a digraph and $\mathcal{W}$ is a collection of closed directed walks in D, is called *extendable* if there is a directed embedding of D where $\mathcal{W}$ is a subcollection of the facial walks in the embedding.

A *linear forest* is a graph consisting of a disjoint union of paths (some of which may just be isolated vertices).

The next set of theorems is a characterization of when these pairs of graphs and walks are extendable. We first give a result of Širáň and Škoviera for extending undirected graphs. In this case the obvious necessary conditions are also sufficient.

Theorem 5.4.1 (Širáň and Škoviera [35])**.** *Given a graph G and a collection of closed walks $\mathcal{W}$ in G, there exists an embedding of G with $\mathcal{W}$ as a subcollection of the facial walks if and only if*

1. *edges in G occur no more than twice along the walks in $\mathcal{W}$, and*
2. *for each vertex $v \in V(G)$ the partial transition graph $T(G, \mathcal{W}, v)$ is a cycle of length $d(v)$ or a linear forest, i.e. $T(G, \mathcal{W}, v)$ is isomorphic to a subgraph of a cycle of length $d(v)$.*

Širáň and Škoviera also give a characterization of when an ordered pair $(G, \mathcal{W})$ is extendable and the embedding of G with $\mathcal{W}$ as a subcollection of the facial walks can be chosen to be orientable.

Theorem 5.4.2 (Širáň and Škoviera [36]). *Given a graph G and a collection of closed walks $\mathcal{W}$ in G, there exists an orientable embedding of G with $\mathcal{W}$ as a subcollection of the faces if and only if*

1. *we can assign directions to the walks in $\mathcal{W}$ in such a way that each edge of G is used at most once in each direction, and*
2. *for each vertex $v \in V(G)$ the partial transition graph $T(G, \mathcal{W}, v)$ is a cycle of length $d(v)$ or a linear forest, i.e. $T(G, \mathcal{W}, v)$ is isomorphic to a subgraph of a cycle of length $d(v)$.*

We now introduce some notation and results needed for dealing with transition graphs for digraphs. First we will work in a more general situation, but eventually X and Y will represent the sets of inward and outward half-arcs at a vertex. If a linear forest L is a subgraph of a complete bipartite graph $K_{n,n}$ with partite sets X and Y, then the components in L can be partitioned naturally into three sets. Let $\mathcal{X}$ be the set of components that begin and end with a vertex in X. Similarly let $\mathcal{Y}$ be the set of components that begin and end with a vertex in Y. Let $\mathcal{U}$ be the set of components that have one end in X and the other end in Y. Note that we consider isolated vertices in L to be paths of length 0. So an isolated vertex in X is in $\mathcal{X}$ and an isolated vertex in Y is in $\mathcal{Y}$. Each component in $\mathcal{U}$ includes an equal number of X and Y vertices, each component in $\mathcal{X}$ has one more X vertex than Y vertex, and each component in $\mathcal{Y}$ has one more Y vertex than X vertex. Since $|X| = |Y|$, it follows that $|\mathcal{X}| = |\mathcal{Y}|$, for any such linear forest L.

Lemma 5.4.3. *Any bipartite linear forest G with partite sets X and Y of equal order can be completed to a cycle that respects the bipartition.*

Proof. Let G be a bipartite linear forest with partite sets X and Y with $|X| = |Y| = n$. We will proceed by induction on the number of components in such a graph.

Suppose that the number of components of G is 1. Then G is a Hamiltonian path. In this Hamiltonian path with $2n - 1$ edges one end must be in X and the other end must be in Y. Connecting these two vertices by an edge results in a cycle respecting the bipartition.

Now assume that all graphs of this type with up to c components can be completed to a cycle respecting the bipartition. Suppose G has $c + 1$ components. Select a component with an end in

X and a different component with an end in Y. First we show that two such components exist. If there is a component in $\mathcal{X}$ then there is a component in $\mathcal{Y}$ since $|\mathcal{X}| = |\mathcal{Y}|$ so in this case we can choose a component in $\mathcal{X}$ and a component in $\mathcal{Y}$. If there is not a component in $\mathcal{X}$, then there must be two distinct components $U_1, U_2 \in \mathcal{U}$ so we can choose U_1 and U_2. Connecting these two components by an edge joining an end in X in the first component with an end in Y from the second component creates a bipartite graph G' with c components. By the inductive hypothesis, we can complete G' to a cycle respecting the bipartition. Therefore, any bipartite graph G with partite sets X and Y of equal order whose components are paths can be completed to a cycle that respects the bipartition. □

In the next theorem we give a characterization of when a pair $(D, \mathcal{W})$ is extendable. We will show that the same conditions as Theorem 5.4.1 characterize extendability in the directed case with the added assumption that the collection of closed walks contains only directed walks. During the construction of the extension, more care must be taken to ensure that the embedding is a directed embedding.

Theorem 5.4.4. *Given an Eulerian digraph D and a collection of closed directed walks $\mathcal{W}$ in D, there exists an embedding of D with $\mathcal{W}$ as a subcollection of the faces if and only if*

1. *arcs in D occur no more than twice along the walks in $\mathcal{W}$, and*
2. *for each vertex $v \in V(D)$ the partial transition graph $T(D, \mathcal{W}, v)$ is a cycle of length $d(v)$ or a linear forest, i.e. $T(D, \mathcal{W}, v)$ is isomorphic to a subgraph of a cycle of length $d(v)$.*

Notice that we can treat the first condition as a special case of the second condition. Indeed, if an arc a in D occurs more than twice along the walks in W, then the half-arcs of e become vertices in the transition graphs with degree greater than two. This means that a vertex v incident with e does not have a transition graph $T(D, \mathcal{W}, v)$ isomorphic to a subgraph of a cycle. However, it is still worth stating and considering the first condition separately because in Theorem 5.4.2 the corresponding undirected condition makes the parallels between the general and orientable cases clearer. Here the first condition makes the parallels between the undirected and directed cases clearer.

Proof. Let D be an Eulerian digraph and $\mathcal{W}$ a collection of closed walks in D. Assume that there is a directed embedding of D with $\mathcal{W}$ as a subcollection of the faces in the embedding, $\mathcal{W} \subseteq F(D)$. Let $\mathcal{W}^*$ be the collection of facial directed walks in the directed embedding. The facial walks of a directed embedding use every arc twice, and every transition graph $T(D, \mathcal{W}^*, v)$ is a cycle. Since $\mathcal{W}$ is a subcollection of $\mathcal{W}^*$, the two conditions for $\mathcal{W}$ and $T(D, \mathcal{W}, v)$ hold.

Now we show sufficiency. Assume that arcs in D occur no more than twice along the walks in $\mathcal{W}$ and that for every $v \in V(D)$, $T(D, \mathcal{W}, v)$ is isomorphic to a subgraph of a cycle of length $d(v)$. At each vertex v we can partition the vertices in $T(D, \mathcal{W}, v)$ into two sets. Let $X(v)$ be the set of vertices in $T(D, \mathcal{W}, v)$ corresponding to half-arcs entering v. Let $Y(v)$ be the set of vertices in $T(D, \mathcal{W}, v)$ corresponding to half-arcs leaving v. Since the edges of $T(D, \mathcal{W}, v)$ are determined by directed walks in $\mathcal{W}$, every edge must be between a vertex in $X(v)$ and a vertex in $Y(v)$. So every transition graph $T(D, \mathcal{W}, v)$ is bipartite with the two parts being $X(v)$ and $Y(v)$. Since either $T(D, \mathcal{W}, v)$ is a cycle or the components of $T(D, \mathcal{W}, v)$ are paths and $T(D, \mathcal{W}, v)$ is bipartite with partite sets of equal order, by Lemma 5.4.3 we can then complete $T(D, \mathcal{W}, v)$ to a cycle Z_v that respects this bipartition.

At each vertex v, assign a clockwise direction for Z_v. Use this clockwise direction along Z_v to determine the clockwise ordering of half-arcs around v in an embedding. Observe that since we do not care whether or not the embedding is orientable, the choices for local clockwise orientation are arbitrary. Since the cycle Z_v alternates between half-arcs directed in and half-arcs directed out, this embedding will be a directed embedding, regardless of what choices we make for the edge signatures.

Once the clockwise directions have been selected, we then determine whether or not each arc should have a twist. For each arc $a \in A(D)$ that belongs to a walk in $\mathcal{W}$ with incident vertices v_1 and v_2 we observe the preceding and succeeding arc(s) along walk(s) in $\mathcal{W}$. If the previous arc a_1 in a walk $W \in \mathcal{W}$ is on the clockwise side of the half-arc of a along Z_{v_1} and the next arc a_2 of W is also on the clockwise side of the other half-arc of a along Z_{v_2}, then assign the signature

-1 to a. Likewise if a_1 and a_2 are on the counterclockwise sides, assign the signature -1 to a. Assign the signature $+1$ to a if one of a_1 or a_2 is on the clockwise side and the other is on the counterclockwise side. Notice that if two walks in $\mathcal{W}$ travel through the arc a, then the two choices of signatures in this process agree.

If an arc a is not traversed by a walk in $\mathcal{W}$, then we may arbitrarily choose the assignment of $+1$ or -1 to a. Now we have a full embedding scheme that gives a corresponding embedding of D. Furthermore, it is clear from the process outlined above that in this embedding $\mathcal{W}$ is a subcollection of the facial walks of the embedding (if we trace faces in the embedding, we make the same turns as the walks in $\mathcal{W}$). So we have found an embedding of D with $\mathcal{W}$ as a subcollection of the facial walks. □

Notice that in the proof of this theorem, there was quite a bit of leeway in the construction of the embedding. Any extension of the partial transition graphs to full transition graphs could then be extended into a full embedding scheme. The edge twists (signatures) were also arbitrary choices so long as they were not already determined by the walks in $\mathcal{W}$. Much more care would need to be taken if the embedding were required to be orientable. In fact, stronger conditions must be met in order to ensure an orientable embedding. Applying the same modification to Theorem 5.4.4 as we did to Theorem 5.4.1 to obtain Theorem 5.4.2 does not yield a correct result for orientable directed embeddings. In this way extensions of directed embeddings are more complicated than they are for ordinary embeddings. We now begin with a few preliminary definitions and observations as we set out to characterize when there exists an orientable embedding of D with $\mathcal{W}$ as a subcollection of the facial walks.

If a linear forest L is given a proper 2-edge-coloring with colors a and b, then we can refine the partition of the components into $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{U}$ by splitting $\mathcal{U}$ into two parts. Observe that all components in $\mathcal{X}$ and $\mathcal{Y}$ are even length and thus alternate between edges colored a and edges colored b with an equal number of edges of each color. Components in $\mathcal{U}$ either begin and end with an edge colored a or else begin and end with an edge colored b. Let $\mathcal{A}$ be the set of components beginning and ending with an a edge. Let $\mathcal{B}$ be the set of components that begin and end with a b

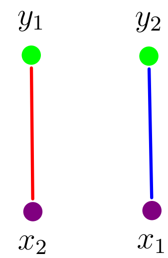


Figure 5.2: A non-extendable properly 2-edge colored linear forest.

edge.

We wish to characterize when L can be extended to a properly 2-edge-colored Hamiltonian cycle H in $K_{n,n}$. By extended we mean that the proper 2-edge-coloring of L is inherited and extended by H and the original bipartition of $K_{n,n}$ is respected by the extension.

A properly 2-edge-colored linear forest L that is a spanning subgraph of $K_{n,n}$ is called *extendable* if L can be extended to a properly 2-edge-colored Hamiltonian cycle in $K_{n.n}$.

First we note that there are situations where L cannot be extended to a properly 2-edge-colored Hamiltonian cycle H in $K_{n,n}$. Consider a graph $L \subseteq K_{2,2}$ with two disjoint edges (Figure 5.2). Let $X = \{x_1 x_2\}$ and $Y = \{y_1, y_2\}$ be the partite sets and color the edges with with two different colors a (blue) and b (red). There is no way to extend L to a properly 2-edge-colored Hamiltonian cycle in $K_{2,2}$.

Lemma 5.4.5. *A properly 2-edge-colored linear forest that is a spanning subgraph of $K_{n,n}$ can be extended to a properly 2-edge-colored Hamiltonian cycle in $K_{n,n}$ if and only if at least one of the following conditions is satisfied:*

1. *There is at least one component in $\mathcal{X}$.*
2. *There is at least one component in $\mathcal{Y}$.*
3. *There are no components in $\mathcal{A}$.*
4. *There are no components in $\mathcal{B}$.*

Corollary 5.4.6. *A properly 2-edge-colored linear forest that is a spanning subgraph of $K_{n,n}$ cannot be extended to a properly 2-edge-colored Hamiltonian cycle in $K_{n,n}$ if and only if every component belongs to $\mathcal{U} = \mathcal{A} \cup \mathcal{B}$ and there is at least one component in each of $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Let L be a properly 2-edge-colored linear forest that is a spanning subgraph of $K_{n,n}$. Let X and Y be the partite sets of size n in $K_{n,n}$. Let a and b be the colors in the 2-edge-coloring of L. Recall that $|\mathcal{X}| = |\mathcal{Y}|$.

We will think of extending L by adding new edges one at a time. In order to remain a linear forest, each edge added to L must combine two components into a new component by adding an edge between a terminal vertex in one component and a terminal vertex in the other.

Now we observe which types of components can be joined together by an edge while remaining properly 2-edge-colored and respecting the bipartition. A component in $\mathcal{A}$ can be joined to another component in $\mathcal{A}$ by connecting the terminal vertex in X from one of the two components with the terminal vertex in Y from the other component and coloring this new edge with b. This creates a single component in $\mathcal{A}$. Similarly a component in $\mathcal{B}$ can be joined to another component of $\mathcal{B}$ by connecting the terminal vertex in X from one of the two components with the terminal vertex in Y from the other component and coloring this new edge with a. This creates a single component in $\mathcal{B}$.

A component in $\mathcal{A}$ cannot be joined to a component in $\mathcal{B}$. Indeed, the terminal edges of a component in $\mathcal{A}$ are colored with a while the terminal edges of a component in $\mathcal{B}$ are colored with b. If the two components are joined by an edge, there is no way that this edge can be assigned a color while inheriting the prior proper 2-coloring.

A component in $\mathcal{A}$ can be joined to a component in $\mathcal{X}$ in a unique way. The terminal vertex in Y from the $\mathcal{A}$ component must be joined to the terminal vertex in the $\mathcal{X}$ component that is incident to the terminal edge colored a. This new edge must be colored b.

Similarly, a component in $\mathcal{A}$ can be joined to a component in $\mathcal{Y}$ in a unique way. The terminal vertex in X from the $\mathcal{A}$ component must be joined to the terminal vertex in the $\mathcal{X}$ component that is incident to the terminal edge colored a. This new edge must be colored b.

From these two unique ways to join a component in $\mathcal{A}$ to a component in $\mathcal{X}$ or $\mathcal{Y}$, we see that joining a component in $\mathcal{A}$ to a component in $\mathcal{X}$ results in a component in $\mathcal{X}$ and joining a component in $\mathcal{A}$ to a component in $\mathcal{Y}$ results in a component in $\mathcal{Y}$.

Connecting components in $\mathcal{B}$ and $\mathcal{X}$ or connecting components in $\mathcal{B}$ and $\mathcal{Y}$ works in an analogous way. Joining a component in $\mathcal{B}$ to a component in $\mathcal{X}$ results in a component in $\mathcal{X}$ and joining a component in $\mathcal{B}$ to a component in $\mathcal{Y}$ results in a component in $\mathcal{Y}$.

A component in $\mathcal{X}$ cannot be joined to another component in $\mathcal{X}$ without adding an edge between two vertices of X, violating the bipartition. Similarly, we cannot join two components in $\mathcal{Y}$.

There are two distinct ways to join a component in $\mathcal{X}$ to a component in $\mathcal{Y}$. We can join the terminal vertex in X incident to an edge colored a to the terminal vertex in Y incident to an edge colored a. The new edge will then be colored b. Likewise we can join the terminal vertex in X incident to an edge colored b to the terminal vertex in Y incident to an edge colored b. The new edge will then be colored a. The first option will result in a component in $\mathcal{B}$ while the second option will result in a component in $\mathcal{A}$.

This gives a complete picture of which types of components can be joined and what results after adding the additional edge.

Now, we begin by showing that if at least one of the conditions is satisfied, then we can extend L. Assume there is at least one component in $\mathcal{X}$. Then there is also a component in $\mathcal{Y}$. If there is more than one component in $\mathcal{X}$, and hence more than one component in $\mathcal{Y}$, select a component in $\mathcal{X}$ and a component in $\mathcal{Y}$ and join them to create a component in $\mathcal{A}$ or $\mathcal{B}$. The choice of which type of component to create may be made arbitrarily. Iterate this process until there is only one component in $\mathcal{X}$ and one component in $\mathcal{Y}$. Next, if there is a component in $\mathcal{A}$ combine it with the component in $\mathcal{X}$, resulting in a new single component in $\mathcal{X}$. Iterate until there are no components in $\mathcal{A}$. Similarly, if there is a component in $\mathcal{B}$ combine it with the component in $\mathcal{Y}$, resulting in a new single component in $\mathcal{Y}$. Iterate until there are no components in $\mathcal{B}$.

What remains is one component in $\mathcal{X}$ and one component in $\mathcal{Y}$. Connect the two ends of the component in $\mathcal{X}$ to the two ends of the component in $\mathcal{Y}$ by connecting the two terminal vertices incident to edges colored a with an edge colored b and connecting the two terminal vertices incident to edges colored b with an edge colored a. The result is a 2-edge-colored Hamiltonian cycle in

$K_{n,n}$. So we have shown that the result holds for the condition that L has at least one component in $\mathcal{X}$ (or equivalently the condition that L has at least one component in $\mathcal{Y}$).

Now suppose there are no components in $\mathcal{X} \cup \mathcal{Y}$. Assume that there are either no components in $\mathcal{A}$ or no components in $\mathcal{B}$. Without loss of generality, assume that there are no components in $\mathcal{A}$. Then L consists only of components in $\mathcal{B}$. Join pairs of components in $\mathcal{B}$ until there is only one component in $\mathcal{B}$. This component must be a Hamiltonian path in $K_{n,n}$. Connect the two terminal vertices of the component by an edge of color a to create the desired Hamiltonian cycle.

Now we show that if none of the conditions are met then there is not an extension of L to a Hamiltonian cycle. Assume that none of the conditions are met. Then L has no components in $\mathcal{X}$, no components in $\mathcal{Y}$, at least one component in $\mathcal{A}$, and at least one component in $\mathcal{B}$ (as in the statement of Corollary 5.4.6). Recall that there is no way to join a component in $\mathcal{A}$ and a component in $\mathcal{B}$. So the only choice we have for adding edges to L is by joining two components in $\mathcal{A}$ to create another component in $\mathcal{A}$ or joining two components in $\mathcal{B}$ to create another component in $\mathcal{B}$. After continuing this process we eventually have one component in $\mathcal{A}$ and one component in $\mathcal{B}$. Then there are no edges that can be added, and thus we cannot extend to a Hamiltonian cycle. $\square$

A *walk coloring* of a collection of walks or directed walks $\mathcal{W}$ is an assignment of a color to each walk $W \in \mathcal{W}$. A walk coloring is *proper* if any two walks in $\mathcal{W}$ that share an edge or arc are assigned a different color. Note, if $\mathcal{W}$ is a collection of facial walks, this definition is equivalent to the definition of (proper) face coloring.

Given an Eulerian digraph D and a collection of colored closed directed walks $\mathcal{W}$ in D, at each vertex $v \in V(D)$ the *induced partial colored transition graph* or *colored transition graph* $T(D, \mathcal{W}, v)$ is constructed as follows:

1. Place a vertex at each half-arc incident with v.
2. Join two vertices by an edge if their corresponding half-arcs occur in succession along a walk $W \in \mathcal{W}$.
3. Color this edge with the color assigned to W. This coloring of $T(D, \mathcal{W}, v)$ is the *inherited*

coloring from the walk coloring of $\mathcal{W}$. If the coloring of $\mathcal{W}$ is proper then the coloring of $T(D, \mathcal{W}, v)$ is also proper.

Theorem 5.4.7. *Given an Eulerian digraph D and a collection of closed directed walks $\mathcal{W}$ in D, there exists an orientable directed embedding of D with $\mathcal{W}$ as a subcollection of the facial walks if and only if there exists a proper 2-coloring of $\mathcal{W}$ such that the colored transition graph $T(D, \mathcal{W}, v)$ is either a cycle or an extendable linear forest subgraph of the complete bipartite graph Q_v with partite sets corresponding to incoming and outgoing half-arcs at v for every $v \in V(D)$.*

In other words we can find the orientable directed embedding if and only if $\mathcal{W}$ can be colored so that each $T(D, \mathcal{W}, v)$ is either a cycle or satisfies the conditions of Lemma 5.4.5.

Proof. Let D be an Eulerian digraph and $\mathcal{W}$ be a collection of closed walks in D. Assume that there exists a proper 2-coloring of $\mathcal{W}$ with colors a and b such that the colored transition graph $T(D, \mathcal{W}, v)$ is a cycle or an extendable linear forest subgraph of Q_v for every $v \in V(D)$. Proceed by extending each $T(D, \mathcal{W}, v)$ to a Hamiltonian cycle Z_v in Q_v. Now each cycle Z_v is a properly 2-edge-colored cycle containing $T(D, \mathcal{W}, v)$ as a subgraph, respecting the bipartition of the half-arcs at v, and extending the edge coloring of $T(D, \mathcal{W}, v)$ inherited from the coloring of $\mathcal{W}$.

As seen in the proof of Theorem 5.4.4, we can use this set of cycles Z_v to construct a embedding scheme corresponding to a directed embedding of D with $\mathcal{W}$ as a subcollection of the facial walks and with Z_v as the transition graph at each vertex v. Once we assign local clockwise directions to the vertices (or equivalently to each Z_v) this procedure automatically chooses an edge signature for each arc belonging to $\mathcal{W}$. We need to be careful in the assignment of clockwise directions and edge signatures not automatically assigned to ensure that this embedding is orientable.

Let $X(v)$ be the set of half-arcs entering v and $Y(v)$ be the set of half-arcs leaving v. Assign a direction, representing the local clockwise orientation, to Z_v in the following way. Orient the edges in Z_v colored a from the end in set $X(v)$ to the end in the set $Y(v)$. Orient the edges in Z_v colored b from the end in the set $Y(v)$ to the end in the set $X(v)$. Since Z_v is a properly 2-edge-colored cycle respecting the bipartition of the arcs at v, this orientation of Z_v is consistent and induces a local clockwise orientation of Z_v. Then if c is an arc from v_1 to v_2 used by a walk W in $\mathcal{W}$ colored

a, this means that W also uses the arc counterclockwise from c at v_1, and the arc clockwise from c at v_2, so the procedure in the proof of Theorem 5.4.4 assigns a signature of $+1$ to c. Similarly, c is assigned a signature of $+1$ if it is used by a walk W of $\mathcal{W}$ colored b.

By assigning a signature of $+1$ to the edges non used by $\mathcal{W}$, all signatures are $+1$. We therefore have the desired orientable directed embedding. □

The condition for this particular type of 2-edge-coloring of $\mathcal{W}$ given in Theorem 5.4.7 can be checked by constructing a related graph and determining whether or not it is bipartite. The issue is that the walks in $\mathcal{W}$ may not all be 'connected together,' where 'connected together' means that for every two walks $U, V \in \mathcal{W}$ there is a list $W_1W_2...W_k$ with $U = W_1$, $V = W_k$, and each $W_i \in \mathcal{W}$, such that W_i and W_{i+1} share an arc for $i = 1, ..., k-1$. If they are connected together, then there are either zero or two proper 2-colorings of $\mathcal{W}$. If there are two, they are equivalent up to swapping colors, and we can check the condition of Theorem 5.4.7 for one of those 2-colorings. However, if the walks are not connected together, then each 'walk component' may have a 2-coloring, but we need to check whether we can find 2-colorings for all the components that are mutually consistent. The next theorem gives a way to check whether we can find consistent 2-colorings in polynomial time, and thus shows that we can check whether or not the condition of Theorem 5.4.7 is satisfied in polynomial time.

Theorem 5.4.8. *Given an Eulerian digraph D and a collection of closed directed walks $\mathcal{W}$ in D, let $X(v)$ be the set of vertices corresponding to half-arcs entering v and $Y(v)$ be the set of vertices corresponding to half-arcs leaving v for all $T(D, \mathcal{W}, v)$. Let $\mathcal{X}(v)$ be the set of components of $T(D, \mathcal{W}, v)$ that begin and end with a vertex in $X(v)$. Similarly let $\mathcal{Y}(v)$ be the set of components of $T(D, \mathcal{W}, v)$ that begin and end with a vertex in $Y(v)$. Let $\mathcal{U}(v)$ be the set of components of $T(D, \mathcal{W}, v)$ that have one end in $X(v)$ and the other end in $Y(v)$. Let $V_1 = \{v \in V(D) \mid |\mathcal{X}(v)| = |\mathcal{Y}(v)| = 0\} \subseteq V(D)$. Define H to be a graph with $V(H) = \mathcal{W} \cup V_1$. Connect the vertices in H corresponding to two walks $W_1, W_2 \in \mathcal{W}$ if they share an arc. For each $v \in V_1$ and component C of $\mathcal{U}(v)$ choose one terminal edge e of C and add an edge from v to the walk $W \in \mathcal{W}$ that corresponds to e.*

Suppose that every $T(D, \mathcal{W}, v)$ is a cycle or a linear forest. Then there exists an orientable directed embedding of D with $\mathcal{W}$ as a subcollection of the facial walks if and only if H is bipartite.

Proof. We will show that H is bipartite if and only if there exists a proper 2-coloring of $\mathcal{W}$ such that the colored transition graph $T(D, \mathcal{W}, v)$ is an extendable linear forest subgraph of the complete bipartite graph Q_v with partite sets corresponding to incoming and outgoing half-arcs at v for every $v \in V(D)$.

Suppose that H is bipartite. Choose a 2-coloring of H with colors a and b. Color each walk in $\mathcal{W}$ according to its color in H. Assume that there exists a transition graph $T(D, \mathcal{W}, v)$ that is not extendable for this proper 2-coloring of $\mathcal{W}$. Then by Corollary 5.4.6, every component of $T(D, \mathcal{W}, v)$ is in $\mathcal{U}(v)$, so that $v \in V_1$, and $\mathcal{A}(v)$ and $\mathcal{B}(v)$ are nonempty, meaning there is a walk colored a and a walk colored b adjacent to v in H. Then v cannot be given a color in the bipartition of H, giving a contradiction.

Now suppose that there exists a proper 2-coloring of $\mathcal{W}$ such that $T(D, \mathcal{W}, v)$ is extendable for every vertex v. Color the vertices in H corresponding to walks in $\mathcal{W}$ according to this proper 2-coloring. By Lemma 5.4.5 every vertex $v \in V(D)$ with $|\mathcal{X}(v)| = |\mathcal{Y}(v)| = 0$ either has all components in $\mathcal{A}(v)$ or all components in $\mathcal{B}(v)$ using the inherited coloring. For $v \in V_1$ color v with a if all components of $T(D, \mathcal{W}, v)$ are in $\mathcal{B}(v)$ and color v with b if all components are in $\mathcal{A}(v)$. This gives a 2-coloring of H. □

Now we can combine the results of Theorems 5.4.4 and 5.4.7 with Theorem 5.3.3 to determine when, given a mixed graph $G = (V, A, E)$ and a collection of directed walks $\mathcal{W}$ in the mixed graph, there exists a directed embedding of a direction of the mixed graph G such that $\mathcal{W}$ is a subcollection of the facial walks.

The following results follow directly from Theorems 5.4.4 and 5.4.7 with Theorem 5.3.3, noting that determining a direction for a mixed graph and determining an embedding for the resulting digraph can be done in two independent steps. Indeed, since walks in $\mathcal{W}$ are directed, adding directions to the edges of a mixed graph does not alter the partial transition graphs $T(D, \mathcal{W}, v)$ for the resulting digraph D. So the conditions for applying Theorem 5.4.4 or Theorem 5.4.7 do not depend on the exact direction provided by Theorem 5.3.3.

Theorem 5.4.9. *Given a mixed graph $G = (V, A, E)$ and a collection of closed directed walks $\mathcal{W}$ in G, there exists a directed embedding of a direction of G with $\mathcal{W}$ as a subcollection of the facial walks if and only if*

1. *G is underlying-Eulerian.*
2. *For every $X \subseteq V(G)$, $e(X, \bar{X}) + a(X.\bar{X}) \geq a(\bar{X}, X)$.*
3. *Arcs in D occur no more than twice along the walks in $\mathcal{W}$.*
4. *For each vertex $v \in V(D)$ the partial transition graph $T(D, \mathcal{W}, v)$ is a cycle of length $d(v)$ or a linear forest. In other words $T(D, \mathcal{W}, v)$ is isomorphic to a subgraph of a cycle of length $d(v)$.*

Theorem 5.4.10. *Given a mixed graph $G = (V, A, E)$ and a collection of closed directed walks $\mathcal{W}$ in G, there exists an orientable, directed embedding of a direction of G with $\mathcal{W}$ as a subcollection of the facial walks if and only if*

1. *G is underlying-Eulerian.*
2. *For every $X \subseteq V(G)$, $e(X, \bar{X}) + a(X.\bar{X}) \geq a(\bar{X}, X)$.*
3. *There exists a proper 2-coloring of $\mathcal{W}$ such that the colored transition graph $T(D, \mathcal{W}, v)$ is a cycle or an extendable linear forest subgraph of Q_v for every $v \in V(G)$, or in other words, such that each $(T, D, \mathcal{W})$ is a cycle or satisfies the conditions of Lemma 5.4.5.*

In the case of mixed graphs there may be some (undirected) half-edges at a given vertex v. So there are some vertices of $T(D, \mathcal{W}, v)$ that are not initially assigned to one partite set or the other. However, these unassigned vertices are all isolated vertices of $T(D, \mathcal{W}, v)$, and by condition 2 we can assign them to $X(v)$ or $Y(v)$ so as to end up with $|X(v)| = |Y(v)|$. Upon this assignment $T(D, \mathcal{W}, v)$ is a vertex-2-colored subgraph of Q_v that is well defined up to isomorphism. Then condition 3 can be checked in the same manner that we showed in Theorem 5.4.8. We can define $\mathcal{X}(v)$, $\mathcal{Y}(v)$, and $\mathcal{U}(v)$ as before.

Chapter 6

Introduction to Pseudosurface Duality

6.1 Definitions

A pseudosurface is a closed, connected topological space obtained by identifications of finite sets of points on a disjoint union of surfaces. We use the formal definition from Heidema [23]:

Let $\Sigma_1, ..., \Sigma_k$ be pairwise disjoint surfaces and let $X_1, ..., X_t$ be a collection of finite, nonempty, pairwise disjoint subsets of $\cup_{i=1}^{k} \Sigma_i$. Let $\sim$ be an equivalence relation defined by $x \sim y$ if and only if $x = y$ or $x, y \in X_j$ for some $j = 1, ..., t$. If the space $\cup_{i=1}^{k} \Sigma_i / \sim$ is a connected topological space then it is called a *pseudosurface*. Other names for these spaces are *pinched manifolds* and *pseudomanifolds*. The finite number of points where the pseudosurface fails to be a surface are in one-to-one correspondence with the sets $X_1, ..., X_t$ and are called *singular points* or *pinchpoints* (see Figure 6.1). The *multiplicity* of each of these singular points, respectively, is $|X_j|$ for $j = 1, ..., t$.

We are interested in embeddings of graphs into pseudosurfaces. The definition for embedding graphs in pseudosurfaces is the same as the definition for embedding graphs in surfaces with the added condition that each pinchpoint must be the image of a vertex in the graph.

Let G be an abstract graph and let $\Gamma(G)$ be an embedding of the graph G such that each singular point of G is the image under Γ of a vertex in G. This is a *pseudosurface embedding* of G. As is common we will abuse notation and often think of G and its image $\Gamma(G)$ as the same object. As with embeddings in surfaces the components of $\Theta - \Gamma(G)$ are called regions or faces in the

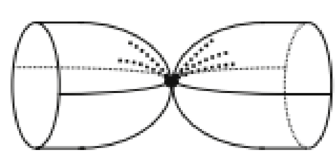


Figure 6.1: A vertex at a pinchpoint in a pseudosurface (Ellis-Monaghan and Moffatt [15]).

embedding and a cellular embedding is an embedding where each component of $\Theta - \Gamma(G)$ is homeomorphic to a disc.

We wish to generalize this concept of a cellular embedding and under this generalization define duality for graph embeddings on pseudosurfaces. A *pseudocellular embedding* is an embedding where each component of $\Theta - \Gamma(G)$ is homeomorphic to a sphere with finitely many punctures.

Note that a cellular embedding is also a pseudocellular embedding under this definition. We now wish to define duality for pseudocellular embeddings of graphs on pseudosurfaces. We will see in this definition the need to look at pseudocellular embeddings instead of cellular embeddings in pseudosurfaces. Namely, the dual of a vertex at a pinchpoint is a face that is not homeomorphic to a disc. Because of this, there are cellular embeddings in pseudosurfaces that have duals that are not cellular.

The *dual* of a pseudocellular embedding G in a pseudosurface Θ is obtained as follows. We will use a generalization of gems. A pseudocellular pseudosurface embedding can be represented by a gem with two added equivalence relations. We need an equivalence relation on the v-gons that tells us which vertices get identified together into a pinchpoint, and we need an equivalence relation on the f-gons to denote which sets of faces get deleted and replaced with a single punctured sphere. Then the operation of duality is performed in the same manner as for cellularly embedded graphs. We swap the colors c_v and c_f on all e-squares in the generalized gem and the former equivalence relation on v-gons becomes the new equivalence relation on f-gons, and vice versa. Using this gem representation it is unclear how the equivalence relations should be handled when trying to dualize with respect to a subset of the edges. This problem, among other issues, will come up as we seek to define the operation of partial duality on pseudosurfaces in the next section.

An example is given in Figures 6.2 and 6.3. Figure 6.2 shows two gems and Figure 6.3 shows the corresponding embeddings. The gem J on the left represents two loops embedded in spheres. If we give this gem an equivalence relation on the v-gons where both v-gons are equivalent and an equivalence relation on the f-gons where f-gons are only equivalent to themselves, then we obtain a pseudosurface embedding with one vertex embedded at a pinchpoint between two spherical com-

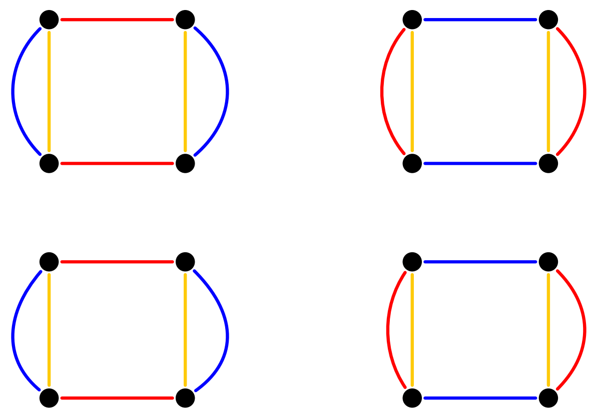


Figure 6.2: A gem J (left) and its dual gem J^* (right)

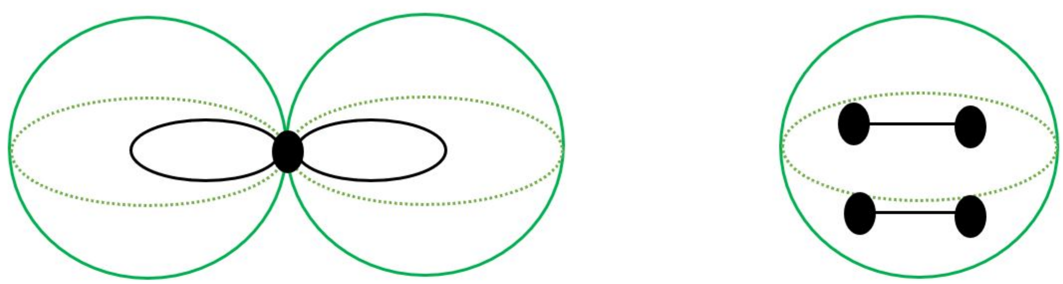


Figure 6.3: A pseudocellular, pseudosurface embedding corresponding to J and a pseudocellular embedding corresponding to J^*

ponents and a loop edge in each spherical component. In the dual, the two f-gons are equivalent and the four v-gons are only equivalent to themselves. This gives an embedding of two disjoint edges in a sphere. Notice that the gem J with its equivalence relations yields a cellular embedding while its dual gem J^* gives an embedding that is pseudocellular (but not cellular).

Remark 6.1. *Duality of Faces and Vertices*

If a vertex $v \in G$ has multiplicity m, then the corresponding face of G^* is homeomorphic to a sphere with m punctures, and vice versa.

6.2 Partial Duality on Pseudosurfaces

Since we are able to define duality for pseudocellular embeddings on pseudosurfaces, we would also like to be able to extend our definition for partial duality to these embeddings. Again we would like similar properties to hold as in our discussion of the uniqueness of the operation of partial duality (Chapter 7). We hope that there is a way where we can take the dual of a part of the graph in such a way that if we dualize several parts of the graph that together make up the "whole graph" we should arrive at the dual of the original graph. We notice that since vertices and faces are being interchanged in duality, the only consistent way to define parts of the graph is by using the edges.

However, unlike the situation for cellularly embedded graphs on surfaces there are nontrivial duals for pseudocellular embeddings on pseudosurfaces that can be formed for graphs without edges. For example, consider the pseudosurface arising from identifying two points of a sphere. Then take the pseudocellular embedding of a graph with two vertices and no edges with one vertex at the pinchpoint and one vertex somewhere else on the pseudosurface. This embedding has two vertices of multiplicities one and two respectively and one face (region), which is homeomorphic to a sphere with three punctures. Thus the dual of this embedded graph is a single vertex of multiplicity three with two adjacent faces that are homeomorphic to spheres with one and two punctures respectively. This dual embedding is on the pseudosurface obtained by beginning with two spheres and identifying two points of the first sphere with one point of the second sphere. The embedding is a single vertex at the pinchpoint. So, we see from this example that a graph with no edges can have a meaningful and distinct dual. Thus, there is no way for us to define partial duality solely with respect to an edge set.

This problem persists in any case where one of the surface components does not have an edge. Since edges are the only consistent way to divide up the graph into parts to dualize, we do not see any sensible way to define partial duality for pseudosurfaces. However, an area for further research is the search for a restricted class of pseudocellular pseudosurface embeddings where we can obtain a consistent and meaningful definition of partial duality.

Remark 6.2. *An objection may be made that the same issue arises in the definition of partial duality for cellular embeddings on surfaces. Namely we could have a component without an edge. However, the only example of a cellular embedding of a nonempty graph with no edges on a connected surface is the embedding of one vertex on the sphere. This graph is self-dual, so taking the dual of the graph is trivial.*

Chapter 7

Uniqueness of Partial Duality

Since taking partial duals seems to be a fundamental operation for cellularly embedded graphs, we would like to know if this is indeed the only way that it can be "naturally defined." By this we mean that we would like to prove that Chmutov's original definition of partial duality is the unique operation satisfying certain desired properties. Here we will elaborate on what we mean by "naturally defined" as well as describe these desired properties. To this point we have been unable to obtain a result of this type, however, we have made interesting discoveries as to why a result of this type is difficult to pin down. Here we will describe our approach and what we have found thus far.

7.1 Basic Properties

Since taking the dual of a graph requires the graph to be embedded in some topological space, it only makes sense for partial duality to also have this requirement. Furthermore, duality is an operation on cellularly embedded graphs and so we will also require partial duality to be an operation defined for cellularly embedded graphs. With this in mind we now turn our attention to outlining which properties should naturally be desired for such an operation on cellularly embedded graphs.

The concept of "partial duality" was defined out of a desire to be able to take the dual of part of the graph. In other words we wanted some set of operations (and inverses for these operations) that each dualize part of the graph. When we perform these operations on parts of the graph that together make up the entire graph we should arrive at the dual of the original graph. This is the first observation of what must be included in a natural definition for partial duality. There must be some sense in which taking several partial duals is the same as taking the dual.

Also, if the two parts of the graph that we dualize are disjoint, then the order in which we dualize them should not matter. So, at least for disjoint parts, partial duality should be a commutative operation. (It will become clear that partial duality should be a commutative operation for arbitrary parts after making the following observations about what the "parts" are and how to maintain the meaning of "duality.")

Since we want to take a dual of a part of the graph we must have a way to define which part we are talking about. We notice in the operation of duality that faces and vertices are exchanged, so in any definition of partial duality we cannot have fixed faces or vertices to refer to. However, there is a nice one-to-one correspondence between the edges in the original graph and its dual. Thus, we may regard an embedded graph G and its dual as having the same set of edges. Hence, it makes sense to assume that there is a single edge set that represents the set of edges in G and all of its partial duals. This means that all partial duals have the same edges even though the faces and vertices change. Moreover, we should define partial duality as an operation that uses parts to mean sets of edges. This is the second observation of what seems reasonable for a natural definition of partial duality.

A third observation is that partial duality should inherit the meaning of "dual." When we take the partial dual of any part of the graph twice, we should arrive at the original graph. Thus we will require that the operation of partial duality be defined by a set of involutions i.e., self-inverse operations, in order to preserve this property.

In what follows we need to be precise about what we mean by equality for edge labeled cellularly embedded graphs. To do this, we introduce the following equivalence relation for cellularly embedded graphs. We regard a graph as consisting of vertices and edges, which are atomic objects, with an incidence relation between them. For two embedded graphs $G_1 \subseteq \Sigma_1$ and $G_2 \subseteq \Sigma_2$ with the same edge set E we write $G_1 \sim G_2$ if there is a homeomorphism from the surface Σ_1 to the surface Σ_2 that maps G_1 to G_2 and moreover maps each edge e in G_1 to the same edge e in G_2. Thus, the homeomorphism must preserve edge labels but not necessarily vertex or face labels. Equivalently, if two cellularly embedded graphs G_1 and G_2 have corresponding gems J_1 and J_2, then $G_1 \sim G_2$ if $J_1 \sim J_2$, by which we mean that J_1 and J_2 are isomorphic as edge-colored, e-square labeled graphs with exact color preservation (colors cannot be nontrivially permuted). In this chapter we will consider $\sim$ to be the natural equivalence relation for cellularly embedded graphs when dealing with partial duals, and so when we write $G_1 = G_2$ (or $J_1 = J_2$) we always mean $G_1 \sim G_2$ (or $J_1 \sim J_2$).

When Chmutov defined partial duality he was working in the context of ribbon graphs. In the ribbon graph definition of duality G and G^* have exactly the same set of edges; it is not just a one-to-one correspondence. We also note that since he was working with ribbon graphs having the same edge sets, when Chmutov wrote $=$ he generally meant $\sim$. Moreover, in the gem definition of duality, the gem of G and the gem of G^* have the same e-squares (with the colors swapped). This gives further credence to the idea that it is natural to work with the same set of edges for partial duality and to consider $\sim$ as the appropriate definition of when two cellularly embedded graphs are the same.

Now we give a more precise explanation for what we mean by a partial duality operation that inherits the meaning of dual and uses edge sets for the parts. Given a fixed edge set $E = \{e_1, ..., e_m\}$ we require partial duality to be a set of m operations $\delta_{e_1}, ..., \delta_{e_m}$ that are defined on all cellularly embedded graphs with this edge set. Let C_E denote all cellularly embedded graphs with edge set labeled by E, where we consider $G_1, G_2 \in C_E$ to be identical if $G_1 \sim G_2$. In other words, C_E is a set of equivalence classes under $\sim$. The vertices and faces in elements of C_E are considered to be unlabeled, but the edges are labeled as specific elements of E. Each of these m operations $\delta_{e_1}, ..., \delta_{e_m}$ is an involution on C_E. We use the notation $\delta_{e_i}(G) = G^{e_i}$ where $G \in C_E$. We will write $G^{e_i e_j} = \delta_{e_j}(\delta_{e_i}(G))$ and, since the operations are involutions, $G^{e_i e_i} = G$.

Then from the first observation, if $E(G) = \{e_1, e_2, ..., e_m\}$ we must have that $G^{e_1 e_2 ... e_m} = G^*$. Furthermore it should be true that for any permutation σ of the set $1, 2, ..., m$ we have $G^{e_{\sigma(1)} e_{\sigma(2)} \cdots e_{\sigma(m)}} = G^*$, which leads us to our next observation.

We can show that this set of involutions must commute with one another in order to maintain a natural meaning of partial duality. Since for any permutation σ of the set $1, 2, ..., m$ we have $G^{e_{\sigma(1)} e_{\sigma(2)} \cdots e_{\sigma(m)}} = G^*$, we know that $G^{e_1 e_2 e_3 ... e_m} = G^* = G^{e_2 e_1 e_3 ... e_m}$, i.e., $(G^{e_1 e_2})^{e_3 ... e_m} =$

$(G^{e_2e_1})^{e_3\ldots e_m}$. But, for any partial dual of a cellularly embedded graph $H \subseteq C_E$, $H^{e_3\ldots e_m e_m \ldots e_3} = H$ since $H^{e_ie_i} = H$. So $G^{e_1e_2} = G^{e_2e_1}$. Since the labels were arbitrary, any two of the m operations commute.

Since the operations commute, we can define partial duality with respect to a set of edges. For $A \subseteq E(G)$ we write G^A to denote the partial dual of G with respect to A, meaning we take the partial dual of G with respect to each edge in A in any order. The commutativity properties give us that, for two edge sets $A, B \subseteq E(G)$ with $A \cap B = \emptyset$, $(G^A)^B = (G^B)^A = G^{A\cup B}$. Furthermore, using the involution property $G^{e_ie_i} = G$ and commutativity, we see that $(G^A)^A = G$.

We also want this operation of partial duality to be invariant under isomorphism. Recall that two cellularly embedded graphs, G_1 and G_2, are isomorphic as cellularly embedded graphs, written $G_1 \cong G_2$, if there is a homeomorphism between their surfaces that carries one embedding to the other; equivalently, their corresponding gems J_1 and J_2 are isomorphic as abstract graphs and this isomorphism exactly preserves edge colors. Notice that if $G_1 \sim G_2$ then $G_1 \cong G_2$ so $\sim$ is a stronger relation than $\cong$. Isomorphism invariance means that taking a partial dual should commute with renaming the edges of an embedded graph, i.e., applying an isomorphism. Specifically, taking a partial dual with respect to a a fixed edge set A and then applying an isomorphism σ should be the same as applying σ and then taking partial duality with respect to the image of A, $\sigma(A)$. Thus, for any two cellularly embedded graphs G and H with $\sigma : G \to H$ an isomorphism, for any $A \subseteq E(G)$ there exists an isomorphism $\phi : G^A \to \sigma(G)^{\sigma(A)} = H^{\sigma(A)}$ such that $\phi(e) = \sigma(e)$ for all $e \in E(G)$. So ϕ and σ agree on edges but not necessarily on vertices and faces (since, as we noted before, we cannot clearly identify vertices and faces, and we consider them to be unlabeled in graphs in C_E). We can also state this condition in terms of the gems. Given two gems J_1 and J_2 corresponding to cellularly embedded graphs G_1 and G_2, if there is an isomorphism $\sigma : J_1 \to J_2$ that preserves edge colors, then for any $A \subseteq E(G_1)$ there is an isomorphism $\phi : J_1^A \to \sigma(J_1)^{\sigma(A)}$ that preserves edge colors and such that $\phi(K) = \sigma(K)$ for each e-square K.

These properties allow us to assume a fixed edge set and to only consider cellularly embedded graphs up to isomorphism. So at this point we arrive at a preliminary set of properties that we require for any definition of partial duality for cellularly embedded graphs. For cellularly embedded graphs G and H with $A, B \subseteq E(G)$ those properties are as follows.

(a) $G^{E(G)} = G^*$.

(b) $(G^A)^B = G^{A\cup B}$ if A and B are disjoint.

(c) $(G^A)^A = G$.

(d) If $\sigma : G \to H$ is an isomorphism, for any $A \subseteq E(G)$ there exists an isomorphism $\phi : G^A \to \sigma(G)^{\sigma(A)}$ such that $\phi(e) = \sigma(e)$ for all $e \in E(G)$.

We notice that properties (b) and (c) together are equivalent to the property $(G^A)^B = G^{A\Delta B}$ for any $A, B \subseteq E(G)$, where $A\Delta B$ is the symmetric difference of the sets A and B. Indeed, assuming properties (b) and (c) if we let $C = A\backslash B$, $D = A \cap B$, and $E = B\backslash A$, then C, D, and E are disjoint, so:

$$(G^A)^B = (G^{(C\cup D)})^{(D\cup E)} = (((G^C)^D)^D)^E = (G^C)^E = G^{C\cup E} = G^{A\Delta B}$$

It is clear that $(G^A)^B = G^{A\Delta B}$ implies properties (b) and (c). This also tells us that under the three properties above, $(G^A)^B = (G^B)^A$ for any two edge sets $A, B \subseteq E(G)$. If we wish, we can restate these properties as:

1. $G^{E(G)} = G^*$.
2. $(G^A)^B = G^{A\Delta B}$.
3. If $\sigma : G \to H$ is an isomorphism, for any $A \subseteq E(G)$ there exists an isomorphism $\phi : G^A \to \sigma(G)^{\sigma(A)}$ such that $\phi(e) = \sigma(e)$ for all $e \in E(G)$.

We now show that these properties are not sufficient for a unique definition of partial duality (considered as an operation on C_E for each finite set E).

Proposition 7.1.1. *The set of properties below does not yield a unique definition for partial duality.*

1. $G^{E(G)} = G^*$.

2. $(G^A)^B = G^{A\Delta B}$ *for all* $A, B \subseteq E(G)$.

3. If $\sigma : G \to H$ *is an isomorphism, for any* $A \subseteq E(G)$ *there exists an isomorphism* $\phi : G^A \to \sigma(G)^{\sigma(A)}$ *such that* $\phi(e) = \sigma(e)$ *for all* $e \in E(G)$.

In this proof, and for the rest of the results in this chapter, we will use G^e and G^A to denote the Chmutov partial dual with respect to e and A, respectively. We will use $G *' e$ and $G *' A$ to denote an alternate partial dual with respect to e and A, respectively. Here G always denotes a cellularly embedded graph with gem J.

Proof. As we showed earlier, Chmutov's definition of partial duality satisfies all of these properties. Now we will give an alternate definition for partial duality that satisfies these properties as well and is not equivalent to the Chmutov definition. We will denote this operation by $*'$ and write $G *' e$ to mean the alternate partial dual with respect to an edge e and similarly $G *' A$ to mean the alternate partial dual with respect to the set of edges A. For each edge $e_0 \in E(G)$, e_0 represents a vertex in the medial graph M. In the medial graph the vertex $v(e_0)$ has four neighbors that correspond to four edges e_1, e_2, e_3, e_4 in G. Note that these edges e_0, e_1, e_2, e_3, and e_4 need not be distinct. Let $A_{e_0} = \{e_0\}\Delta\{e_1\}\Delta\{e_2\}\Delta\{e_3\}\Delta\{e_4\}$. Now we define a new operation $*'$ so that $G *' e_0 = G^{e_0e_1e_2e_3e_4}$, i.e., the Chmutov partial dual of G with respect to A_{e_0}. Even if the edges e_0, e_1, e_2, e_3, and e_4 are not distinct, we still are taking the Chmutov partial dual with respect to five edges. We will show that this operation satisfies the four properties above.

Since each edge e in G corresponding to a vertex $v(e)$ in M has exactly four neighbors, we see each edge occurs five times in the sets A_e for every $e \in E(G)$. Note here that we are counting with multiplicity. For example, if we are taking the alternate partial dual of a graph with a single edge e, then e occurs five times in A_e. In general, $G *' E(G) = G^{E(G)E(G)E(G)E(G)E(G)} = ((((G^*)^*)^*)^*)^* = G^*$. Thus property (a) is satisfied by the alternate partial dual.

Properties (b), (c), and (d) are inherited by alternate partial duality directly from Chmutov partial duality. Indeed we see that each alternate operation corresponding to an edge e is an involution. These involutions commute with each other since they correspond to five Chmutov involutions. Thus, we see that $G *' A *' B = G *' (A\Delta B)$. We also see that the bijection between edges in G and edges in G^* or any alternate partial dual is held intact by the operation of alternate partial duality.

It is clear that this operation is not equivalent to Chmutov partial duality. Therefore, Chmutov partial duality is not the only operation satisfying the above properties. □

Even though we have shown an alternate definition of partial duality satisfying these properties, this new definition definitely is not convincing as a natural way to define partial duality. The alternate definition limits which parts of the graph we can dualize by forcing five edges to be dualized (in the Chmutov sense) at a time. There are other ways that we could similarly define an operation that satisfies the above properties but is not equivalent to Chmutov partial duality. For example, for any integer s, let $\mathcal{W}$ be the set of all walks starting at $v(e)$ in the medial graph of length s, let $t(W)$ be the final vertex of each path $W \in \mathcal{W}$, let $A_e = \{e\}\Delta(\Delta_{W\in\mathcal{W}} t(W))$, and define $G *' e = G^{A_e}$ for each edge e. The operation defined in the proof of Proposition 7.1.1 corresponds to the case $s = 1$. This gives us infinitely many distinct operations that also satisfy these conditions.

However, each of these operations is a modification of Chmutov partial duality where the part of the graph that is altered becomes larger and the operation becomes less local. We do not wish to include these formulations as other "natural" ways to define partial duality. So in order for Chmutov's definition of partial duality to be the unique operation satisfying certain properties, there must be another condition included. We suspect that such a condition would need to restrict partial duality to being a local operation. As we saw previously, under the Chmutov partial dual operation we have some compatibility of minors and partial duals, namely $G/e = (G^e)\backslash e$. Perhaps this would be a natural condition to include for any definition of partial duality.

We also could require that all partial duals of G have the same rigid-vertex, vertex-labeled medial graph. By rigid-vertex we mean that we allow only for flips of vertices (reversals of the cyclic ordering of edges around a vertex) in the embedding of the medial graph. Recall that by Proposition 2.2.5 taking the Chmutov partial dual of an edge e in a cellularly embedded graph G corresponds to flipping $v(e)$ in the checkerboard-colored, rigid-vertex, vertex-labeled medial graph M of G. So perhaps this condition should be included in a list of properties of partial duality. The property is that for any partial dual of an edge-labeled cellularly embedded graph G with vertex-labeled medial graph M, the vertex-labeled medial graph of $G * A$ where $A \subseteq E(G)$ is equivalent to M up to a reversal in the cyclic edge ordering at the vertices in M. Equivalently, the gem of $G * A$ may be obtained from the gem of G by swapping the colors c_v and c_f on some subset of the e-squares. Notice that this property is also satisfied by the family of operations defined above in the proof of Proposition 7.1.1 and the generalization thereafter to similar operations for longer walks in the medial graph. Indeed, since these operations are defined in terms of a set of Chmutov partial duals, the vertex-labeled medial graph is fixed up to a reversal in the cyclic edge ordering at the vertices in M.

7.2 Minor Conditions

An alternative approach that may provide a locality condition is applying the compatibility of minors and partial duals. The simplest minor-related conditions say that deletion and contraction are duals. In other words, recalling that by $=$ we mean $\sim$, for every edge e we have:

1. $G * e/e = G\backslash e$.
2. $G * e\backslash e = G/e$.

In verifying whether or not two cellularly embedded graphs are equivalent under $\sim$, it will be helpful to understand the effects of ribbon graph minor operations on gems. To delete an edge e in a gem J we remove the edges colored c_f in the e-square for e and then replace each maximal path with at least one internal vertex of degree 2 by a single edge of color c_a, or a cycle of degree 2 vertices by a free loop of color c_a. To contract an edge e we proceed in the same way except that we remove the edges colored c_v, not c_f, in the e-square for e. In the process of deletion or contraction, any labels on edges colored c_a that were replaced by a new single edge or free loop of color c_a are transferred to this new c_a edge.

To add an edge e to a graph G represented by a gem J we select two edges colored c_a.

- If the two edges colored c_a are distinct, replace each by a path with colors c_a then c_v then c_a.
- If the two edges colored c_a are the same and this edge is not a free loop, replace this edge with a path of length 5 with linear colors c_a, c_v, c_a, c_v, and c_a.
- If the two edges colored c_a are the same and this edge is a free loop, replace this edge with

a cycle of length 4 with linear colors c_a, c_v, c_a, and c_v.

Then we add two edges colored c_f connecting ends of the two edges labeled c_v so as to create an e-square labeled e. Note that there are two choices for this connection and under a fixed drawing of J in the plane (with crossing edges allowed) we can describe the new edge e as twisted or untwisted. We will write $G + e$ to denote "G add e."

To uncontract an edge e in a graph G represented by a gem J we select two edges colored c_a.

- If the two edges colored c_a are distinct, replace each by a path with colors c_a then c_f then c_a.
- If the two edges colored c_a are the same and this edge is not a free loop, replace this edge with a path of length 5 with linear colors c_a, c_f, c_a, c_f, and c_a.
- If the two edges colored c_a are the same and this edge is a free loop, replace this edge with a cycle of length 4 with linear colors c_a, c_f, c_a, and c_f.

Then we add two edges colored c_v connecting ends of the two edges labeled c_f so as to create an e-square labeled e. Note that there are two choices for this connection and under a fixed drawing of J in the plane (with crossing edges allowed) we can describe the new edge e as twisted or untwisted. We write $G \times e$ to denote "G uncontract e." (We previously used $G \times e$ to denote the partial Petrie dual with respect to e, but here the notation is exclusively used for uncontraction).

During the uncontraction process, if the two edges colored c_f of the new e-square are inserted in an untwisted way on the same v-gon of the original gem, then the new e-square represents a link and the number of vertices increases by one. If they are inserted in a twisted way on the same v-gon, then the new e-square represents a twisted loop and the number of vertices is unchanged. If they are inserted on different v-gons, then the new e-square represents an untwisted loop and the number of vertices decreases by one.

Observe that unlike the operations of deletion and contraction, there is not a unique way to add or uncontract an edge labeled e. However, if $G\backslash e = G_1$ and $G/e = G_2$ then there is a way to add an edge labeled e to G_1 and uncontract an edge labeled e in G_2 so that $G = G_1 + e$ and $G = G_2 \times e$.

The statement that the two minor conditions uniquely determine the partial dual can be restated as follows. If H is a cellularly embedded graph such that (1) $G\backslash e = H/e$ and (2) $G/e = H\backslash e$ then

$H = G^e$.

Next we explore whether or not the Chmutov definition of partial duality is the unique operation for cellularly embedded graphs satisfying these minor related conditions.

Proposition 7.2.1. *The set of minor related properties (1 and 2 above) does not yield a unique definition of partial duality (considered as an operation on C_E for each finite set E).*

An *isolated loop* in a cellularly embedded graph G is a loop e incident with a vertex v_0 such that in the gem J corresponding to G the two edges colored c_v in the e-square associated with e occur in succession along the v-gon associated with v_0. An isolated loop can be either twisted or untwisted. An isolated twisted loop remains an isolated twisted loop in all Chmutov partial duals of G. In particular, for an isolated twisted loop e we see that $G^e = G$ and $G \backslash e = G/e$.

Proof of Proposition 7.2.1. We define the following operation $*'$ for cellularly embedded graphs. Again we will use G^e to denote the Chmutov partial dual with respect to e. Begin with a cellularly embedded graph G. First, for every isolated twisted loop e label the edge colored c_a in between the two edges colored c_v associated with e with the label a_e. Then arbitrarily select another distinct edge colored c_a that is neither incident with the e-square associated with e in J nor incident with an e-square in J associated with any other twisted loop in G, and label it with a_e in such a way that all edges colored c_a are labeled with at most one edge label. We call an edge labeled with a_e colored c_a *active* if it is between two edges colored c_v associated with e. We call an edge labeled with a_e colored c_a *dormant* if it is not active. So, for each twisted loop e, there is exactly one active and exactly one dormant edge colored c_a and labeled a_e.

Now we define $*'$ as follows. For any edge that is not an isolated twisted loop, $G *' e = G^e$. For any edge e that is a twisted loop, $G *' e$ is formed by (1) performing the gem operation associated with deleting e in G so that the active c_a edge labeled a_e becomes dormant and (2) adding an isolated twisted loop labeled e encompassing the previously dormant c_a edge labeled a_e so that it becomes active. By encompassing we mean that we replace the dormant c_a edge by a path of length 5 with edges colored c_a, c_v, c_a, c_v, and c_a. Then we complete the two edges colored c_v to an e-square representing a twisted loop labeled e. The middle edge labeled c_a is now the active c_a edge labeled a_e.

In this way the definition of $*'$ is simply taking Chmutov partial duals with the exception being that isolated twisted loops are allowed to "float" between their active and dormant locations. We now verify that the desired conditions are satisfied.

Since there is not an overlap between dormant and active edges, we see that for all edges $e_1, e_2 \in E(G)$, $G *' e_1 *' e_2 = G *' e_2 *' e_1$. We also observe that $G *' e_1 *' e_1 = G$. Therefore, for all $A, B \subseteq E(G)$, $G *' A *' B = G *' (A \Delta B)$.

As we previously observed, for an isolated twisted loop e, $G^e = G$. By the definition of $*'$, $G *' e *' e = G = G^e$.

For all edges that are not isolated twisted loops, the minor related conditions are inherited directly from Chmutov partial duality. For the isolated twisted loops, we previously noted that $G \backslash e = G/e$. This is because in both cases after removing the edges colored either c_f in the case of deletion or c_v in the case of contraction, there forms a single maximal path of length five with internal vertices of degree two. The endpoints of this maximal path are the same in the case of deletion and contraction, so when it is replaced by a single edge colored c_a, the resulting gem is the same.

So, in this case it also does not make a difference whether the twisted loop was moved and then deleted or simply deleted. In other words $G *' e \backslash e = G \backslash e = G/e$. Similarly, $G *' e/e = G/e = G \backslash e$. So the minor related conditions are satisfied. □

As with the first example in Proposition 7.1.1, this sort of bouncing around of isolated twisted loops seems artificial in nature and so this alternate definition of a partial dual does not seem convincing for something that should fit under what would be considered "natural" partial duality.

First of all, this definition is not intrinsically isomorphism invariant if the active and dormant locations of each isolated twisted loop are decided arbitrarily. However, we can restrict the definition of $*'$ from Proposition 7.2.1 to the case when the gem of G minus all isolated twisted loops has a unique nontrivial automorphism of order 2 so that each orbit of c_a edges corresponds to at most one isolated twisted loop. We use the automorphism to swap active and dormant locations. In any other case we simply perform Chmutov partial duality. This creates an isomorphism invariant operation.

In addition we observe that under $*'$ it is not guaranteed that $G*' E(G) = G^*$. So this operation does not satisfy the basic conditions from Proposition 7.1.1. In particular this means that the minor related conditions do not imply the basic conditions. This leads us to two new questions.

7.3 Combined Conditions

In the case of isolated twisted loops, we have $G\backslash e = G/e$. The first question is therefore (1) do the minor conditions force any alternate definition of partial duality $*'$ to satisfy $G *' e = G^e$ for edges e that do not have $G\backslash e = G/e$? In particular, are isolated twisted loops the only "bad" cases. It turns out they are not.

The second question is (2) does the set of minor properties *and* basic properties yield a unique definition of partial duality? We will refer to these conditions as the *combined conditions*. We will first answer this question while still allowing for isolated twisted loops.

So, the full list of proposed conditions would be:

1. $G * E(G) = G^*$.
2. $G * A * B = G * (A \Delta B)$ for all $A, B \subseteq E(G)$.
3. If $\sigma : G \to H$ is an isomorphism, for any $A \subseteq E(G)$ there exists an isomorphism $\phi : G * A \to \sigma(G) * \sigma(A)$ such that $\phi(e) = \sigma(e)$ for all $e \in E(G)$.
4. $G * e/e = G\backslash e$ for all $e \in E(G)$.
5. $G * e\backslash e = G/e$ for all $e \in E(G)$.

Proposition 7.3.1. *The set of conditions listed above does not yield a unique definition of partial duality.*

Proof. Let T_n be the embedded graph G consisting of a cycle $v_1e_1v_2e_2...v_ne_nv_1$ of length n together with two isolated twisted loops t_1 and t_2 next to each other in the clockwise cyclic ordering of the half-edges at v_1. The cyclic order of the edges corresponding to the half-edges at v_1 is $t_1t_1t_2t_2e_1e_n$. Let $\mathcal{T}$ be the set of all cellularly embedded graphs isomorphic to T_n or one of its partial duals for some n. The embedded graphs in $\mathcal{T}$ are readily recognizable from their gems, and they all have exactly two twisted loops that occur next to each other in the rotation around some vertex.

Define $*'$ to be the operation where $G *' e$ is obtained by simply swapping the labels of the two twisted loops if $G \in \mathcal{T}$ and e is a twisted loop, and $G *' e = G^e$ otherwise. Since the labels are swapped it is clear that $T_n *' t_i \neq T_n^{t_i}$ for $n \geq 2$. The minor related properties hold by the same logic used in Proposition 7.2.1 since $U *' t_i \backslash t_i = U \backslash t_i = U/t_i$, $U *' t_i/t_i = U/t_i = U \backslash t_i$ for any partial dual U of T_n, and the minor properties for the other edges are inherited from the Chmutov definition. In this case it is clear that $G *' E(G) = G^*$. Commutativity is clear since the isolated twisted loops do not impact the operation of $*'$ on the other edges and vice versa. The definition of $*'$ did not make use of edge labels so it is isomorphism invariant. □

Now we look to address question (1). Twisted loops played an important role in the examples in Propositions 7.2.1 and 7.3.1, so it would be reasonable to look at situations where there are no twisted loops. However, every nonorientable embedding has a (Chmutov) partial dual with at least one twisted loop. Indeed, for a cellularly embedded graph G, if we take the partial dual with respect to the edges of a spanning tree $A \subseteq E(G)$, then G^A is a bouquet (a single-vertex embedding). Since Chmutov partial duals do not alter orientability, if G is nonorientable then G^A has a twisted loop. Therefore, in order to exclude twisted loops we need to consider only orientable embeddings.

Because of this, we could next explore if the minor conditions provide added locality for the operation of partial duality with the added condition of orientability. In the case of orientable graphs, progress is made as shown in Theorem 7.3.5. However, in the orientable case, the combined properties do not yield a unique definition for partial duality, answering question (2).

Observation 7.3.2. *Note that to show $G = H$ for two orientable cellularly embedded graphs we need only to show that G and H have the same cyclic edge labelings around their vertices. Indeed, since there are no twisted edges i.e., all edge signatures are $+1$, the rotation scheme (here*

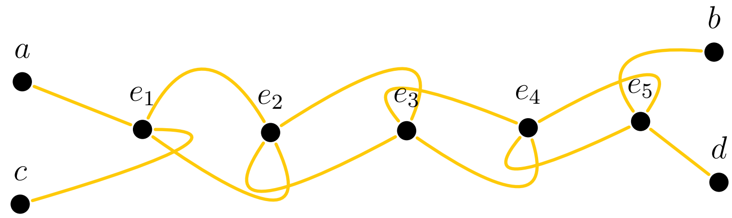


Figure 7.1: A chain of 5 vertices in the medial graph

described by the cyclic edge labels around vertices) completely determines the embedding.

Proposition 7.3.3. *The set of basic properties plus minor related properties above does not yield a unique definition of partial duality for the class of orientable cellularly embedded graphs.*

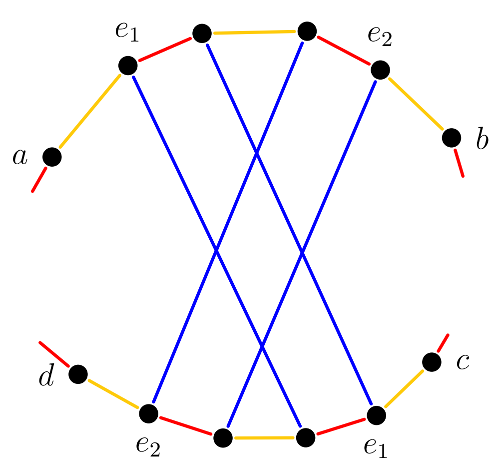
For the proof of Proposition 7.3.3 we will need to note a special class of edges in a cellularly embedded graph G whose vertices have a distinct structure in the medial graph that is invariant under Chmutov partial duality. Let G be a cellularly embedded graph with corresponding gem J. If there are two distinct lists of edge labels in the clockwise rotation around a single vertex or around two separate vertices in G, which may overlap or wrap around a vertex, of the form $a, e_1, e_2, e_3, ..., e_k, b$ and $c, e_1, e_2, e_3, ..., e_k, d$ where $e_i \neq e_j$ for $i \neq j$, then the edges $e_1, ..., e_k$ form a *chain*. Furthermore, if $a \neq c$, $b \neq d$, and $a, b, c, d \neq e_i$ for all i, then the edges $e_1, ..., e_k$ form a *strict chain*. Note that a strict chain is also a maximal chain. We call the edges $e_1, ..., e_k$ *chain edges* because of their structure in the medial graph that we will now describe.

Another way to think of chain edges is by observing the structure of the medial graph of G. In the medial graph M, the vertices $v(e_1), ..., v(e_k)$ are such that there are two parallel edges between $v(e_i)$ and $v(e_{i+1})$ for $i = 1, ..., k-1$. Furthermore, the half-edges corresponding to the two parallel edges are opposite one another in the cyclic ordering of edge labels around $v(e_i)$ and $v(e_{i+1})$ for $i = 1, ..., k-1$. We call such a structure a *chain* in M and we call the vertices $v(e_1), ..., v(e_k)$ *chain vertices*. In the medial graph, if $v(e_1)$ has three distinct neighbors $v(a)$, $v(c)$, and $v(e_2)$, and similarly $v(e_k)$ has three distinct neighbors $v(b)$, $v(d)$, and $v(e_{k-1})$ then $v(e_1), ..., v(e_k)$ forms a *strict chain* in M. The order of such a chain or strict chain is the number of chain vertices, k.

Proof of Proposition 7.3.3. Let $*'$ be an operation on orientable cellularly embedded graphs defined as follows. Let G be a cellularly embedded graph with corresponding gem J. For every edge $e \in E(G)$ define $G *' e = G^e$ unless the vertex $v(e)$ is a part of a *strict* chain of even order $2k$ in the medial graph M. In this case, $G *' e_i$ for $i = 1, ..., 2k$ is obtained by swapping the edge labels e_i and e_{i+1} if i is odd or swapping the edge labels e_i and e_{i-1} if i is even and then taking the Chmutov partial dual with respect to e_i. Since G is orientable, there is a consistent global clockwise direction so this operation is well-defined. The operation also does not depend on the choice of global clockwise direction.

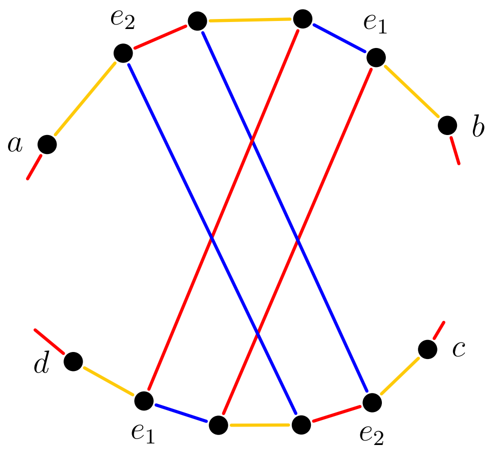
Note that under both $*'$ and Chmutov partial duality the rigid vertex medial graph is unchanged (though vertices can be flipped and can change label). So chains cannot be extended or broken under either operation. Therefore, the property of being a chain edge is invariant under $*'$ and Chmutov partial duality.

Next we observe that under this definition of $*'$ for i odd, $G *' e_i *' e_{i+1} = G *' e_{i+1} *' e_i = G^{e_i e_{i+1}} = G$ and $G *' e_i *' e_i = G$. We will give an example for the case $k = 1$ that captures the general argument. Below is a partial drawing of the gem J representing G with the edges colored c_v (red) labeled. We use blue for c_f and yellow for c_a. The three figures below are partial drawings of gems, but they give a complete picture of the changes made by $*'$ and Chmutov partial duality. The first drawing is of the gem J representing G.

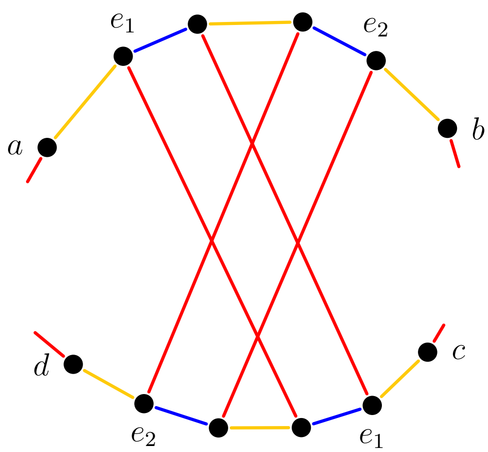


Then in the gem for $G*'e_1$ we have the following (noting that in general this is not an equivalent gem to that of G^{e_1}). Also, we see that by applying $*'e_2$ to the gem below, we get the gem for G

above, so $G *' e_1 *' e_2 = G$. A symmetric argument shows that $G *' e_2 *' e_1 = G$. Remember that '=' means equivalent under $\sim$, but actually the gems here are identical, not just related by $\sim$.



Then the partial gem for $G *' e_1 *' e_1$ is given below. This is the same partial gem as for $G^{e_1 e_2}$. Similarly, $G *' e_2 *' e_2 = G^{e_1 e_2}$. Again, the gems are identical.



We also observe that if the cyclic edge labels around the vertex or vertices incident to e_1 and e_2 in G contain ae_1e_2b and ce_1e_2d, then the cyclic edge labels around the vertex or vertices in $G *' e_1 *' e_1$ also contain ae_1e_2b and ce_1e_2d. None of the other cyclic edge labels around other vertices were changed so by Observation 7.3.2 we see that $G = G^{e_1 e_2}$, where '=' here means $\sim$. So we get $G = G *' e_1 *' e_1 = G *' e_2 *' e_2 = G *' e_1 *' e_2 = G *' e_2 *' e_1 = G^{e_1 e_2}$. In general, for arbitrary values of k and i odd, this same effect is happening locally for any pair $\{e_i, e_{i+1}\}$, so we have $G = G *' e_i *' e_i = G *' e_{i+1} *' e_{i+1} = G *' e_i *' e_{i+1} = G *' e_{i+1} *' e_i = G^{e_i e_{i+1}}$.

We now show that the operation is commutative. Recall that the property of being a chain edge is invariant under $*'$. We have shown that for two chain edges e_i and e_{i+1} that are paired up in the label swaps of $*'$, the operation $*'$ is commutative. In the following situations there is no overlap in the effects on the gem so $G *' e *' f = G *' f *' e$ holds when e and f are two chain edges that never swap labels, a chain edge and a non-chain edge, or two non-chain edges. Since the identification of strict chains does not rely on edge labels, $*'$ is isomorphism invariant. Since $G *' e_i *' e_{i+1} = G^{e_i e_{i+1}}$ for a chain edge e_i with i odd and $G *' f = G^f$ for a non-chain edge f, and using the fact that the operation is commutative, we see that $G *' E(G) = G^{E(G)} = G^*$. So the basic properties hold for $*'$.

Now we show that the minor related conditions are satisfied. We need only to check chain edges. In what follows we will only focus on vertices that (possibly) differ amongst those in G, $G *' e_i$, $G \backslash e_i$, and G/e_i and will not mention those whose clockwise edge labels remain unchanged.

Suppose e_i is a link and a chain edge that is part of two distinct edge label lists $e_1 e_2 e_3 ... e_{2k} S$ and $e_1 e_2 e_3 ... e_{2k} T$ at two distinct vertices, where S and T are ordered edge lists. Assume without loss of generality that i is odd. Then $G \backslash e_i$ has vertices cyclically labeled $e_1 ... e_{i-1} e_{i+1} ... e_{2k} S$ and $e_1 ... e_{i-1} e_{i+1} ... e_{2k} T$. In G/e_i, there is a single vertex with edge labels $e_1 ... e_{i-1} e_{i+1} ... e_{2k} T e_1 ... e_{i-1}$ $e_{i+1} ... e_{2k} S$. In $G *' e_i$ the edge label around the single vertex is $e_1 ... e_{i-1} e_{i+1} e_i e_{i+2} ... e_{2k} T e_1 ... e_{i-1}$ $e_{i+1} e_i e_{i+2} ... e_{2k} S$. From this we see that $G *' e_i \backslash e_i$ has the same edge labels around vertices as G/e_i. Similarly $G *' e_i / e_i$ has the same edge labels around vertices as $G \backslash e_i$. Since all of these graph embeddings are orientable, by Observation 7.3.2 $G *' e_i \backslash e_i = G/e_i$ and $G *' e_i / e_i = G \backslash e_i$ as desired. The proof for the case when e_i is a loop is similar.

We have shown that the operation $*'$ satisfies all of the basic properties as well as the minor related properties. It is also clear that this operation is not equivalent in general to Chmutov partial duality. Therefore Chmutov partial duality is not the unique operation satisfying the listed properties for orientable cellularly embedded graphs. □

Thus far we have shown three negative results concerning the uniqueness of partial duality. Next, we give a positive result, in the sense that there is a restriction as to how an alternate partial

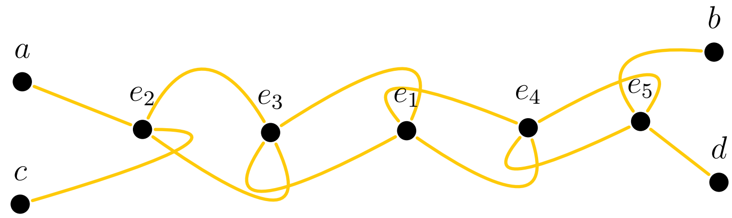


Figure 7.2: A chain shift $\sigma = (e_1e_2e_3)$ of e_1 applied to the chain in Figure 7.1

dual can affect the cellularly embedded graph when requiring the minor related conditions.

A *pendant* edge is an edge that is incident to at least one vertex of degree one. The dual of a pendant edge is an isolated untwisted loop.

Given a chain in the medial graph of an orientable cellularly embedded graph with vertices $v_1, ..., v_k$ in order along the chain and corresponding labels $e_1, ..., e_k$, a *chain shift of* e_1 is the permutation of the edge labels $\sigma = (e_1e_2...e_k)$. The *magnitude* of the chain shift is $k-1$. If $k = 1$, then the permutation and the chain shift are called *trivial*. Otherwise, the chain shift is *nontrivial*. Then σ is a permutation of edge labels in G and a permutation of e-square labels in J corresponding with this chain shift. We write $\sigma(G)$ to denote this permutation of the edge labels applied to G. Figure 7.2 shows the effects of the chain shift $\sigma = (e_1e_2e_3)$ on the chain in Figure 7.1.

For a sequence of edge labels A, we will use $|A|$ to denote the length of the sequence, and we denote an empty sequence by $\emptyset$.

Lemma 7.3.4. *Suppose A and B are cyclic edge labels around a vertex with $A, B \neq \emptyset$. If $AB = BA$ then $A = B$.*

Proof. Assume that $AB = BA$ with $A = a_1, ..., a_k$ and $B = b_1, ..., b_l$ where $k, l > 0$. Considering the first label in $AB = BA$, we see that $a_1 = b_1$ and this label occurs nowhere else in AB because no edge label occurs more than twice. Thus, if $A = a_1A'$ and $B = b_1B'$ we get $a_1A'a_1B' = AB = BA = a_1B'a_1A'$ from which $A' = B'$ and hence $A = B$. □

Theorem 7.3.5. *Suppose G is an orientable cellularly embedded edge labeled graph and $e \in E(G)$ with $G\backslash e = H/e$ and $G/e = H\backslash e$ for some cellularly embedded graph H. If e is not an* *isolated (untwisted) loop or pendant edge, then $H = \sigma(G)^e$ if e is a loop and $H = \sigma(G^e)$ if e is a link where σ is a chain shift of e. If σ is a trivial chain shift then $H = G^e$.*

Note that in the proof of Proposition 7.3.3 for every chain edge e_i, $G *' e_i = \sigma(G)^{e_i} = \sigma(G^{e_i})$ where σ is a chain shift of e_i of magnitude 1. In this case the result of a chain shift followed by a partial dual is the same as the result of a partial dual followed by a chain shift, but this is not true in general.

For sequences of edge labels A and B we will write $(A) = (B)$ to mean that A is a cyclic shift of B and $A = B$ to mean that they are exactly equal.

Proof. Let G be an orientable cellularly embedded edge labeled graph with e-square-labeled gem J. The cyclic order of the edge labels around vertices in G is equivalent to the cyclic order of the e-square labels of the edges colored c_v on the v-gons in J. Assume $G \backslash e = H/e$ and $G/e = H \backslash e$ for some edge $e \in E(G)$ that is not an isolated loop or pendant.

Let us first recall how G^e is obtained from G in terms of edge deletions, additions, contractions, and uncontractions. We assume G is orientable. In this case, $G^e = G \backslash e \times e$ where the two choices of edges labeled c_a are exactly edges labeled c_a that replaced the maximal path or cycle of vertices of degree 2 in the edge deletion process. Then the edges labeled e colored c_f are connected by edges labeled c_v to form an untwisted e-square. Similarly $G^e = G/e + e$ where the two choices of edges labeled c_a are exactly edges labeled c_a that replaced the maximal path or cycle of vertices of degree 2 in the edge contraction process. Then the edges labeled e colored c_v are connected by edges labeled c_f to form an untwisted e-square.

Consider the previously chosen edge e. Then e is either a loop that is not isolated or a link with both incident vertices having degree at least 2. Let us first assume that e is a loop with incident vertex v. Then the two occurrences of the label e on edges colored c_v in J occur on the same v-gon that represents v in J. Let $ed_1d_2 \ldots d_ked_{k+1} \ldots d_l$ be the cyclic ordering of edge labels around v. Let $A = d_1d_2 \ldots d_k$ and $B = d_{k+1} \ldots d_l$ where A and B are nonempty since e is not isolated. Then $G \backslash e$ has a vertex v_{AB} with edge labels AB by definition of $G \backslash e$. Similarly G/e has two distinct vertices v_A and v_B with edge labels A and B, respectively, by definition of G/e. Since

$G\backslash e = H/e$ and $G/e = H\backslash e$ we must be able to uncontract an edge labeled e from $G\backslash e$ and add an edge labeled e to G/e in such a way that $G\backslash e \times e = H$ and $G/e + e = H$.

Let $n = V(G)$. Then $G\backslash e$ has n vertices and G/e has $n+1$ vertices. Adding an edge to G/e does not change the number of vertices so H must have $n+1$ vertices. This means that we must add a vertex by uncontracting an edge from $G\backslash e$ to form H. Uncontracting an edge can only add a vertex if the two edges labeled c_a that we select at the beginning of the uncontraction process occur on the same v-gon. Furthermore, in this case we must uncontract the edge in the untwisted manner otherwise the number of vertices is unchanged. This means H has $n+1$ vertices and is orientable. So we would now like to show that the two edges labeled c_a we select when uncontracting an edge e from $G\backslash e$ must be from a vertex with edge labels AB.

Suppose to the contrary that the two edges colored c_a chosen are from a v-gon with different edge labels. Then the vertex v_{AB} in $G\backslash e$ (where $A, B \neq \emptyset$) survives the uncontraction process unaffected and so H has the vertex v_{AB} labeled AB. But then v_{AB} must have been present in G/e since $G/e = H\backslash e$ and so v_{AB} is unchanged by deleting e from H. Furthermore, since A and B are nonempty v_{AB} must be distinct from the vertices v_A and v_B in G/e. So v_{AB} must have been present in G since it was unchanged going from G to G/e. So G must have vertices labeled $eAeB$ and AB. This means that $G\backslash e$ has two vertices with edge labels AB, the vertex we named v_{AB} originally and another vertex we call u_{AB}. Since by our assumption the two edges colored c_a chosen in the uncontraction of e from $G\backslash e$ do not occur on a vertex with labels AB, this means H must have two vertices with edge label AB, v_{AB} and u_{AB}. Recall that G/e has two vertices v_A and v_B with edge labels A and B respectively. Since H is obtained from G/e by the addition of an edge labeled e, H also has two distinct vertices $v_{A'}$ and $v_{B'}$ with edge labels A' and B' (A and B with the possible addition of an edge(s) labeled e to one or both of these vertices). Thus H has three occurrences of the label d_i on its vertices. This is a contradiction. Therefore, the two edges in the gem of $G\backslash e$ colored c_a during the uncontraction creating H from $G\backslash e$ must be chosen from a vertex labeled AB. Note that if two vertices in $G\backslash e$ have edge labels AB then they are indistinguishable, and so we can assume this vertex is v_{AB} from above.

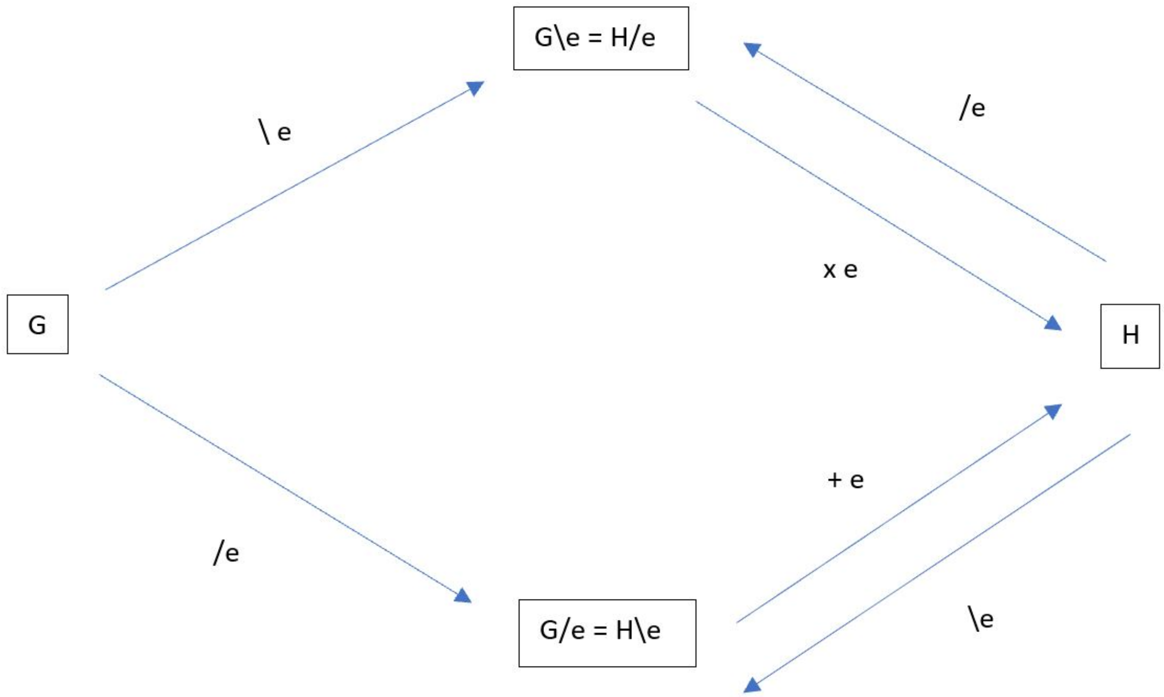


Figure 7.3: Diagram for the relationship between G, H, and the minor operations.

Next we observe possible candidates for the two gem edges colored c_a, g_1 and g_2, from the v-gon associated with the vertex v_{AB}. As noted above, H has two distinct vertices $v_{A'}$ and $v_{B'}$ with edge labels A' and B'. Since g_1 and g_2 are chosen from the same v-gon in $G\backslash e$, the two edge labels for e on edges colored c_v occur on different v-gons in $H = G\backslash e \times e$. Thus, one occurrence must be on $v_{A'}$ and one occurrence must be on $v_{B'}$, so $d(v_{A'}) = k+1$ and $d(v_{B'}) = l-k+1$. Thus g_1 and g_2 must be k edges apart in the edge label ordering around v_{AB}, meaning that there are k edge labels on edges colored c_v between the two chosen c_a edges g_1 and g_2. Without loss of generality suppose $|A| \leq |B|$. Then at most one of g_1 and g_2 can be interior to A (chosen between the edge labels d_i and d_{i+1} for $1 \leq i \leq k-1$). However, if $|B| \geq |A|+2$, both g_1 and g_2 can be interior to B (both chosen between the edge labels d_i and d_{i+1} for $k+1 \leq i \leq l-1$). This gives us two cases.

Case 1: If both g_1 and g_2 are interior to B then we can write $B = B_1B_2B_3$ where g_1 and g_2 are between B_1 and B_2 and between B_2 and B_3. Then B_1 and B_3 are nonempty, $|B_2| = k = |A|$, and

H has vertices with labels eB_2 and eB_3AB_1, where $|B_3AB_1| = |B| > k$. Then in $H\backslash e = G/e$, these two vertices correspond to v_A and v_B so $(B_2) = (A)$ and $(B_3AB_1) = (B_1B_2B_3)$. Note that all the edge labels in A occur once in A and again in B_2, so none of the edge labels in A occur in B_1 or B_3. Since $(A) = (B_2)$, we can write $A = A_1A_2$ and $B = A_2A_1$ where, without loss of generality, $A_1 \neq \emptyset$ but it is possible that A_2 is empty.

Then $(B_3AB_1) = (B_1B_2B_3)$ implies $(B_3A_1A_2B_1) = (B_1A_2A_1B_3)$. Since edges in A do not occur in B_1 or B_3 and $A_1 \neq \emptyset$, we have $A_1A_2B_1B_3 = A_1B_3B_1A_2$. Eliminating A_1 from both sides we get $A_2B_1B_3 = B_3B_1A_2$. Therefore, $A_2 = \emptyset$, so $B_1B_3 = B_3B_1$ and $A = B_2$. Since $B_1B_3 = B_3B_1$, by Lemma 7.3.4 we have $B_1 = B_3$. In G^e we have a vertex with labels $eB_1B_2B_3$ and a vertex with labels eA, which are exactly the same as the vertices labeled eB_3AB_1 and eB_2 in H since $B_1 = B_3$ and $A = B_2$. These are the only two vertices that change in the minor operations so we see that $H = G^e$.

Case 2: If g_1 and g_2 are not both interior to B, then we may assume that $A = A_1A_2$ where g_1 occurs between A_1 and A_2, and $B = B_1B_2$ where g_2 occurs between B_1 and B_2. Here A_1, A_2, B_1, or B_2 may be empty. During the uncontraction of e, we create two vertices in $H = G\backslash e \times e$ with edge labels eA_2B_1 and eB_2A_1. In G/e ($H\backslash e$) these two vertices have edge labels B_1A_2 and A_1B_2 and correspond to the pair of vertices v_A and v_B and therefore have the same cyclic edge labels (up to a cyclic shift). Without loss of generality suppose that $(B_1A_2) = (B_1B_2)$ and $(A_1B_2) = (A_1A_2)$.

Suppose first that $A_1 \neq \emptyset$. Consider the first edge label d_1 of A_1, which can occur at most once elsewhere amongst the edge labels in $A_1, A_2, B_1,$ and B_2. The edge label d_1 appears in A_2 if and only if it also occurs in B_2 since $(A_1A_2) = (A_1B_2)$. Therefore, it occurs in neither A_2 nor B_2 since d_1 already occurs in A_1 and it cannot occur three times. Thus, if d_1 occurs only once in A_1, not only do we have $(A_1A_2) = (A_1B_2)$, but also $A_1A_2 = A_1B_2$ so $A_2 = B_2 = S$, say. If d_1 occurs twice in A_1 then we can write $A_1 = d_1Cd_1D$ so $A_1A_2 = d_1Cd_1DA_2$ and $A_1B_2 = d_1Cd_1DB_2$. Since $(A_1A_2) = (A_1B_2)$ either (1) $d_1Cd_1DA_2 = d_1Cd_1DB_2$ (meaning $A_2 = B_2 = S$) or (2) $d_1Cd_1DA_2 = d_1DB_2d_1C$. From (2) we see that $C = DA_2 = DB_2$ so $A_2 = B_2 = S$. Therefore we have that the edge labels for v can be written as $eAeB = eA_1SeB_1S$ meaning that the vertices in the medial graph corresponding to the edges Se form a chain. Therefore, H can be formed from G by first performing a chain shift of e of magnitude $|S|$ and then taking the Chmutov partial dual with respect to e, as desired.

Now suppose that $A_1 = \emptyset$. If we also have $B_1 = \emptyset$ then g_1 and g_2 are the edges colored c_a formed in the deletion process of e to obtain $G\backslash e$, and so $H = G^e$. Therefore, we may suppose that $B_1 \neq \emptyset$. Then we may repeat the argument for $A_1 \neq \emptyset$, swapping A_1 with B_1 and A_2 with B_2. This does not alter the conditions $(B_1A_2) = (B_1B_2)$ and $(A_1B_2) = (A_1A_2)$, and the argument did not depend on the fact that $|B| \geq |A|$.

For the case where e is a link that is not a pendant we know that e is a nonisolated loop in G^e. If $G\backslash e = H/e$ and $G/e = H\backslash e$ then $G^e\backslash e = H^e/e$ and $G^e/e = H^e\backslash e$ so by the above argument, $H^e = \sigma(G^e)^e$. Therefore, $H = \sigma(G^e)$. □

As previously mentioned another possible condition to consider is forcing the same rigid-vertex, vertex-labeled medial graph for every partial dual of G. We note that the counterexample from Proposition 7.1.1 satisfied this property, but the counterexamples from Propositions 7.2.1 and 7.3.3 do not satisfy this property (at least in general). We will note this and other remaining questions in the next chapter.

Chapter 8

Future Directions

Many of the clear future directions for research stem from our endeavors to prove that Chmutov's original definition of partial duality is the unique operation satisfying certain desired properties. The next natural step seems to be to examine the following set of conditions. The addition of condition 6 below eliminates the possibility of chain shifts (up to equivalence under $\sim$) and so in the case of orientable embeddings, Theorem 7.3.5 gives us that $G * e = G^e$ for every edge e that is not an isolated loop or pendant. We view this as a sort of "kitchen sink" type approach to this problem in search of a positive result.

1. $G * E(G) = G^*$
2. $G * A * B = G * A\Delta B$
3. If $\sigma : G \to H$ is an isomorphism, for any $A \subseteq E(G)$ there exists an isomorphism $\phi : G * A \to \sigma(G) * \sigma(A)$ such that $\phi(e) = \sigma(e)$ for all $e \in E(G)$.
4. $G * e/e = G\backslash e$
5. $G * e\backslash e = G/e$
6. The vertex labeled medial graphs for G and $G * e$ are equivalent up to reversals in the clockwise cyclic ordering at each vertex.

Other possible conditions include more restrictive locality conditions. One idea is to have conditions involving more than one edge. For example, we may include the property that $G * e\backslash f = G\backslash f * e$ for distinct edges e and f. Another idea is to have a condition that somehow fixes vertices and faces that are not incident with the edge being partially dualized. This would most likely take the form of an adaptation of the isomorphism invariance property to incorporate labels on vertices and faces that are not expected to change. We could also consider the inclusion of a condition regarding the interaction between partial duality and partial Petrie duality, namely $G * e \times e * e \times e * e \times e = G$.

Another area for future study is a structural description of important subclasses of the classes of embedded graphs that we classified in Chapter 4. Especially for the properties that are partial dual invariant, we would like to be able to describe the bouquets (one-vertex embeddings) in these embedding classes.

In our work on pseudosurfaces, we would like to be able to find a method for applying minor operations. In addition, we would like to find a class of pseudosurface embeddings where partial duality can be naturally defined.

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