Fundamental Groups of Certain von Neumann Algebras

By

Krishnendu Khan

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Approved:

Jesse D. Peterson, Ph.D.

Alexander Yu. Ol'shanskiy, Ph.D.

Denis V. Osin, Ph.D.

Dietmar H. Bisch, Ph.D.

Robert J. Scherrer, Ph.D.

To my family,

for neverending support

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Chapter 1

Introduction

Motivated by their continuous dimension theory, Murray and von Neumann introduced the notion of t-by-t matrix over a II_1 factor $\mathscr{M}$, for any positive real number $t > 0$, [MvN43]. This is a II_1 factor denoted by $\mathscr{M}^t$ and called the t-amplification of $\mathscr{M}$. When $t \leq 1$ this the isomorphism class of $p\mathscr{M}p$ for a projection $p \in \mathscr{M}$ of trace $\tau(p) = t$ and when $1 < t$ it its the isomorphism class of $p(M_n(\mathbb{C}) \otimes \mathscr{M})p$ for an integer n with $t/n \leq 1$ and a projection $p \in M_n(\mathbb{C}) \otimes \mathscr{M}$ of trace $(Tr_n \otimes \tau)(p) = t/n$. One can see that up to isomorphism the $\mathscr{M}^t$ does not depend on n or p but only on the value of t.

The fundamental group, $\mathscr{F}(\mathscr{M})$, of a II_1 factor $\mathscr{M}$ is the set of all $t > 0$ such that $\mathscr{M}^t \cong \mathscr{M}$. Since for any $s, t > 0$ we have $(\mathscr{M}^s)^t \cong \mathscr{M}^{st}$ then one can see $\mathscr{F}(\mathscr{M})$ forms a subgroup of $\mathbb{R}_+$. As the fundamental group is an isomorphism invariant of the factor, its study is of central importance to the theory of von Neumann algebras. In [MvN43] Murray and von Neumann were able to show that the fundamental group of the hyperfinite II_1 factor $\mathscr{R}$ satisfies $\mathscr{F}(\mathscr{R}) = \mathbb{R}_+$. This also implies that $\mathscr{F}(\mathscr{M}) = \mathbb{R}_+$ for all McDuff factors $\mathscr{M}$. However, besides this case no other calculations were available for an extended period of time and Murray-von Neumann's original question whether $\mathscr{F}(\mathscr{M})$ could be different from $\mathbb{R}_+$ for some factor $\mathscr{M}$ remained wide open (see [MvN43, page 742] and the discussions in [Po20]).

A breakthrough in this direction emerged from Connes' discovery in [Co80] that the fundamental group of a group factor $\mathscr{F}(\mathscr{L}(G))$ reflects rigidity aspects of the underlying group G, being countable whenever G has property (T) of Kazdhan [Kaz67]. This finding also motivated him to formulate his famous Rigidity Conjecture in [Co82] along with other problems on computing symmetries of property (T) factors—that were highlighted in subsequent articles by other prolific mathematicians [Co94, Problem 2, page 551], [Jo00, Problems 8-9] and [Po13, page 9]. Further explorations of Connes' idea in [Po86, GN87, GG88, Po95] unveiled new examples of separable factors $\mathscr{M}$ with countable $\mathscr{F}(\mathscr{M})$, including examples for which $\mathscr{F}(\mathscr{M})$ contains prescribed countable sets. However despite these advances concrete calculations of fundamental groups remained elusive for more than two decades.

The situation changed radically with the emergence of Popa's deformation/rigidity theory in early 2000. Through this novel theory we have witnessed an unprecedented progress towards complete calculations of fundamental groups. The first successes in this direction were achieved by Popa and include a series of striking results: examples of factors with trivial fundamental group [Po01] which answers a long-standing open problem of Kadison [K67] (see [Ge03, Problem 3]); examples of factors that have *any* prescribed countable subgroup of $\mathbb{R}_+$ as a fundamental group [Po03]. An array of other powerful results on computations of fundamental groups were obtained subsequently [IPP05, PV06, Io06, Va07, PV08, Ho09, IPV10, BV12]. Remarkably, in [PV08] it was shown that many uncountable proper subgroups of $\mathbb{R}_+$ can be realized as fundamental groups of separable II_1 factors.

However, despite these impressive achievements, significantly less is known about the fundamental groups of property (T) factors as the prior results do not apply to these factors. In fact there is no explicit calculation of the fundamental group of any property (T) factor available in the current literature. In section 5 we make progress on this problem by providing examples of property (T) icc groups G whose factors $\mathscr{L}(G)$ have trivial fundamental group. In particular the result advances [Co94, Problem 2, page 551] and provides the first group examples satisfying the last conjecture on page 9 in Popa's list of open problems [Po13].

The first class of groups studied in 5 Γ arise as a minor variant of a construction introduced by Valette in [Va04]. We briefly describe this construction for readers' convenience. Denote by $\mathbb{H}$ the division algebra of quaternions and by $\mathbb{H}_{\mathbb{Z}}$ its lattice of integer points. Let $n \geq 2$. Recall that $\Lambda_n = \mathrm{Sp}(n,1)_{\mathbb{Z}}$ is a lattice in the rank one connected simple real Lie group $\mathrm{Sp}(n,1)$ by [BHC61]. Observe that $\mathrm{Sp}(n,1)$ acts linearly on $\mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)}$ in such a way that Λ_n preserves $(\mathbb{H}_{\mathbb{Z}})^{n+1} \cong \mathbb{Z}^{4(n+1)}$. For every $n \geq 2$, set $\Gamma_n = \mathbb{Z}^{4(n+1)} \rtimes \Lambda_n$. Throughout this document, we denote the class of these groups by $\mathcal{V}$. For $\Gamma_n \in \mathcal{V}$, we denote by $\mathscr{M}_n = \mathscr{L}(\Gamma_n)$. In combination with Gaboriau's ℓ^2-Betti numbers invariants [Ga02] and Popa–Vaes's Cartan rigidity results [PV12] we obtain that the countable family of type II_1 group factors $(\mathscr{L}(\Gamma_n))_{n\geq 2}$ with property (T), with trivial fundamental group, that possess a unique Cartan subalgebra up to unitary conjugacy, and that are pairwise stably non-isomorphic. We also show that products of finitely groups in class $\mathcal{V}$ give rise to property (T) type II_1 factors with trivial fundamental group. Our proof relies on the same strategy developed in the seminal papers by Popa and Vaes [PV11, PV12] to show that $\mathscr{F}(L^\infty(X) \rtimes \mathbb{F}_n) = 1$.

Theorem 1.0.1. *[CDHK20, Theorem A] For every $n \geq 2$, let $\Gamma_n = \mathbb{Z}^{4(n+1)} \rtimes \Lambda_n \in \mathcal{V}$, and $\mathscr{M}_n = \mathscr{L}(\Gamma_n)$. The following properties hold true.*

(i) *For every $n \geq 2$, $\mathscr{M}_n$ has trivial fundamental group.*

(ii) *The type II_1 factors $(\mathscr{M}_n)_{n\geq 2}$ are pairwise stably non-isomorphic.*

iii) *Assume that $\Gamma_{n_i} \in \mathcal{V}$ and $\Gamma = \Gamma_{n_1} \times \dots \times \Gamma_{n_k}$, where $n_i \geq 2$ for all i. Then the fundamental group satisfies $\mathscr{F}(\mathscr{L}(\Gamma)) = \{1\}$.*

The second class of groups studied in 5 G were first introduced in [CDK19] and rely on a Rips construction in geometric group theory developed by Belegradek and Osin in [BO06]. For convenience we briefly recall this construction. Using results from [Os06], it was shown in [BO06] that for every finitely generated group Q one can find a property (T) group N such that Q embeds as a finite index subgroup of $\mathrm{Out}(N)$. This gives rise to an action $\sigma : Q \to \mathrm{Aut}(N)$ such that the corresponding semidirect product group $N \rtimes_\sigma Q$ is hyperbolic relative to $\{Q\}$. When Q is torsion free one can pick N to be torsion free as well and hence both N and $N \rtimes_\sigma Q$ are icc. Moreover, when Q has property (T) then $N \rtimes_\sigma Q$ has property (T). Throughout this article this semidirect product $N \rtimes_\sigma Q$ will be called the Belegradek-Osin Rips construction and denoted by $Rip(Q)$. Our examples arise as fiber products of these Rips constructions. Specifically, consider any two groups $N_1 \rtimes_{\sigma_1} Q, N_2 \rtimes_{\sigma_2} Q \in Rip(Q)$ and form the canonical fiber product $G = (N_1 \times N_2) \rtimes_\sigma Q$ where $\sigma = (\sigma_1, \sigma_2)$ is the diagonal action. Notice that G has property (T) and the class of all these groups will be denoted by $\mathcal{S}(Q)$.

Developing a new technological interplay between methods in geometric group theory and Popa's deformation/rigidity theory which continues prior investigations in [CDK19] we show that the factors associated with groups in class $\mathcal{S}(Q)$ have trivial fundamental group. Specifically, using various technological outgrowths of prior methods [Po03, Oz03, IPP05, Io06, IPV10, Io11, PV12, CIK13, KV15, CD19, CDK19] we are able to show the following more general statement:

Theorem 1.0.2. *[CDHK20, Theorem B] Assume that Q_1, Q_2, P_1, P_2 are icc, torsion free, residually finite, hyperbolic property (T) groups. Let $Q = Q_1 \times Q_2$ and $P = P_1 \times P_2$ and consider any groups $(N_1 \times N_2) \rtimes Q \in \mathcal{S}(Q)$ and $(M_1 \times M_2) \rtimes P \in \mathcal{S}(P)$. Let $p \in \mathcal{P}(\mathscr{L}(M_1 \times M_2) \rtimes P)$ be a projection and let $\Theta : \mathscr{L}((N_1 \times N_2) \rtimes Q) \to p\mathscr{L}((M_1 \times M_2) \rtimes P)p$ be a $*$-isomorphism.*

Then $p = 1$ and one can find a $*$-isomorphism, $\Theta_i : \mathscr{L}(N_i) \to \mathscr{L}(M_i)$, a group isomorphism $\delta : Q \to P$, a multiplicative character $\eta : Q \to \mathbb{T}$, and a unitary $u \in \mathcal{U}(\mathscr{L}((M_1 \times M_2) \rtimes P))$ such that for all $\gamma \in Q$, $x_i \in \mathscr{L}(N_i)$ we have that

$$\Theta((x_1 \otimes x_2)u_\gamma) = \eta(\gamma)u(\Theta_1(x_1) \otimes \Theta_2(x_2)v_{\delta(\gamma)})u^*.$$

In particular, if we denote by $G = (N_1 \times N_2) \rtimes Q$ then the fundamental group satisfies $\mathscr{F}(\mathscr{L}(G)) = \{1\}$.

Concrete examples of countable families of pairwise non-isomorphic property (T) II_1 factors emerged from the prior fundamental works of Cowling-Hageerup [CH89] and Ozawa-Popa [OP03]. Additional examples were obtained more recently, [CDK19]. Since $\mathscr{F}(\mathscr{M})$ is countable whenever $\mathscr{M}$ is a property (T) factor [Co80, CJ85], it also follows there exist continuum many pairwise mutually non-isomorphic property (T) factors. But, however, no explicit constructions of such families exist in the literature till date. The main Theorems A and B canonically provides such examples.

Corollary 1.0.3. *[CDHK20, Corollary C] For any $G = N \rtimes Q \in \mathcal{S}(Q)$ or $G = G_1 \times ... \times G_n$ with $G_i \in \mathcal{V}$, the set of all amplifications $\{\mathscr{L}(G)^t : t \in (0,\infty)\}$ consists of pairwise non-isomorphic II_1 factors with property (T).*

While Connes' rigidity conjecture motivated to great effect a significant portion of the main developments in Popa's deformation/rigidity theory [Po03, Po04, Io11, IPV10], no example of a property (T) W^*-superrigid group is known till date. The first hard evidence towards Connes' conjecture was found by Cowling and Haagerup in [CH89], where it was shown that uniform lattices in $Sp(n,1)$ give rise to non-isomorphic factors for different values of n. Later on Ozawa and Popa were able to show in [OP03] that for any collection $\{G_n\}_n$ of hyperbolic property (T) groups (e.g. uniform lattices in $Sp(n,1)$) the group algebras $\{\mathscr{L}(\times_{i=1}^n G_i)\}_n$ are pairwise non-isomorphic. However, little is known beyond these two classes of examples. Moreover, the current literature offers an extremely limited account on what algebraic features that occur in a property (T) group are completely recognizable at the von Neumann algebraic level. For instance, besides the preservation of the Cowling-Haagerup constant [CH89], the amenability of normalizers of infinite amenable subgroups in hyperbolic property (T) groups from [OP07, Theorem 1], and the direct product rigidity for hyperbolic property (T) groups from [CdSS15, Theorem A], [CU18, Theorem A] very little is known. Therefore in order to successfully construct property (T) W^*-superrigid groups via a strategy similar to the ones used in [IPV10, BV12, B13, CI17] we believe it is imperative to identify new algebraic features of property (T) groups that survive the passage to the von Neumann algebraic regime. Any success in this direction will potentially hint to what group theoretic methods to pursue in order to address Connes' conjecture.

Motivated the aforementioned problem, in [CDK19] we showed that the semidirect product feature of groups in class $\mathscr{S}(Q)$ described above is completely reconstructible at the von Neumann algebra level. More precisely, we showed the following:

Theorem 1.0.4 (Theorem 4.0.1)**.** *Let $Q = Q_1 \times Q_2$, where Q_i are icc, biexact, weakly amenable, property (T), torsion free, residually finite groups. For $i = 1,2$ let $N_i \rtimes_{\sigma_i} Q \in Rips_T(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ the semidirect product associated with the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \curvearrowright N_1 \times N_2$. Denote by $\mathscr{M} = \mathscr{L}(\Gamma)$ be the corresponding II_1 factor. Assume that Λ is any arbitrary group and $\Theta : \mathscr{L}(\Gamma) \to \mathscr{L}(\Lambda)$ is any $*$-isomorphism. Then there exist groups action by automorphisms $H \curvearrowright^{\tau_i} K_i$ such that $\Lambda = (K_1 \times K_2) \rtimes_\tau H$ where $\tau = \tau_1 \times \tau_2 : H \curvearrowright K_1 \times K_2$*

is the diagonal action. Moreover one can find a multiplicative character $\eta : Q \to \mathbb{T}$, a group isomorphism $\delta : Q \to H$ and unitary $w \in \mathscr{L}(\Lambda)$ and $*$-isomorphisms $\Theta_i : \mathscr{L}(N_i) \to \mathscr{L}(K_i)$ such that for all $x_i \in L(N_i)$ and $g \in Q$ we have

$$\Theta((x_1 \otimes x_2)u_g) = \eta(g)w((\Theta_1(x_1) \otimes \Theta(x_2))v_{\delta(g)})w^*. \tag{1.1}$$

Here $\{u_g : g \in Q\}$ and $\{v_h, : h \in H\}$ are the canonical unitaries implementing the actions of $Q \curvearrowright \mathscr{L}(N_1)\bar{\otimes}\mathscr{L}(N_2)$ and $H \curvearrowright \mathscr{L}(K_1)\bar{\otimes}\mathscr{L}(K_2)$, respectively.

In fact, the above theorem hinted at lurking rigidity properties of group factors arising from groups in class $\mathscr{S}(Q)$. This was one of the main motivations for studying symmetry groups of these factors in [CDHK20]. We detail the proof of Theorem 1.0.4 in section 4.

The contents of this thesis are based on the joint works [CDK19] and [CDHK20].

Chapter 2

Preliminaries

In this chapter, we introduce and collect facts about several topics in von Neumann algebras which will be used throughout this thesis.

2.1 von Neumann Algebras

2.1.1 Definitions

In this section we discuss the notion of von Neumann algebra. One can think of von Neumann algebra as "non commutative analogue of measure space" as the commutative von Neumann algebra corresponds to measure spaces.

Definition 2.1.1. Let $\mathscr{H}$ be a Hilbert space. The locally convex topology on $\mathscr{B}(\mathscr{H})$, defined by semi norms $\{T \to |\langle T(\xi), \eta\rangle|\}_{\xi,\eta\in\mathscr{H}}$ is called **weak operator topology** (WOT). Equivalently, the weak operator topology has the basis of open sets $U_{T,A,B,\varepsilon}$ indexed by $T \in \mathscr{B}(\mathscr{H})$, finite subsets $A, B \subset \mathscr{H}$ and $\varepsilon > 0$

$$U_{T,A,B,\varepsilon} := \bigcap_{\eta\in A,\zeta\in B} \{S \in \mathscr{B}(\mathscr{H}) : \ |\langle S(\eta), \zeta\rangle - \langle T(\eta), \zeta\rangle| < \varepsilon\}.$$

The **strong operator topology** (SOT) on $\mathscr{B}(\mathscr{H})$ is the locally convex topology defined by the family of semi norms $\{T \to \|T(\xi)\|\}_{\xi\in\mathscr{H}}$. Equivalently, the strong operator topology has the basis of open sets $U_{T,A,\varepsilon}$ indexed by $T \in \mathscr{B}(\mathscr{H})$, finite subset $A \subset \mathscr{H}$ and $\varepsilon > 0$

$$U_{T,A,\varepsilon} := \bigcap_{\eta\in A} \{S \in \mathscr{B}(\mathscr{H}) : \ \|S(\eta) - T(\eta)\| < \varepsilon\}.$$

In particular one can think of the WOT and SOT as follows: If we have a net $(T_i)_i$ in $\mathscr{B}(\mathscr{H})$, then $T_i \to T$ in WOT if and only if for all $\eta, \zeta \in \mathscr{H}$

$$\langle T_i(\eta), \zeta \rangle \to \langle T(\eta), \zeta \rangle.$$

Similarly, $T_i \to T$ in SOT if and only if for every $\eta \in \mathscr{H}$,

$$\|T_i(\eta) - T(\eta)\| \to 0.$$

Remark 2.1.2. Since an operator $T \in \mathscr{B}(\mathscr{H})$ is normal if and only if $\|T(\xi)\| = \|T^*(\xi)\|$ for all $\xi \in \mathscr{H}$, the adjoint operator is SOT-continuous on the set of normal operators.

Lemma 2.1.3. *Let $\phi : \mathscr{B}(\mathscr{H}) \to \mathbb{C}$ be a linear functional. Then the following are equivalent;*

1. *There exists $\xi_1, \xi_2, \cdots \xi_n, \eta_1, \eta_2, \cdots, \eta_n \in \mathscr{H}$ such that $\phi(T) = \sum_{i=1}^{n} \langle T(\xi_i), \eta_i \rangle$, for all $T \in \mathscr{B}(\mathscr{H})$.*

2. *ϕ is WOT-continuous.*

3. *ϕ is SOT-continuous.*

Proof. The implication 1. ⇒ 2. and 2.⇒ 3. follows from the definition. We need to show 3. ⇒1. Suppose ϕ is SOT-continuous. The inverse image of open ball in $\mathbb{C}$ is open in SOT and hence by considering the semi norms which defines the topology we have that there exists a constant $K > 0$ and $\xi_1, \xi_2, \cdots \xi_n \in \mathscr{H}$ such that,

$$|\phi(T)|^2 \leq K \sum_{i=1}^{n} \|T(\xi_i)\|^2.$$

If we consider $\{\oplus_{i=1}^{n} T(\xi_i) : \; T \in \mathscr{B}(\mathscr{H})\} \subset \mathscr{H}^{\oplus n}$, and let $\mathscr{H}_0$ be its closure, we have that,

$$\oplus_{i=1}^{n} T(\xi_i) \to \phi(T)$$

extends to a well defined, continuous linear functional on $\mathscr{H}_0$, and hence by Riesz representation theorem, there exists $\eta_1, \cdots, \eta_n \in \mathscr{H}$ such that

$$\phi(T) = \sum_{i=1}^{n} \langle T(\xi_i), \eta_i \rangle$$

for all $T \in \mathscr{B}(\mathscr{H})$. □

Corollary 2.1.4. *Let $\mathscr{H}$ be a Hilbert space and $K \subset \mathscr{H}$ be a convex subset then $\overline{K}^{SOT} = \overline{K}^{WOT}$*[1].

Definition 2.1.5. A **von Neumann algebra** (over a Hilbert space $\mathscr{H}$) is a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ which contains 1 and is closed in the weak operator topology.

Remark 2.1.6. Since subalgebras are convex, it follows from corollary 2.1.4 that von Neumann algebras are also closed under strong operator topology.

Following lemma is a corollary of Lemma 2.1.4

Lemma 2.1.7. *Let $A \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Then $(A)_1$ is compact in the WOT.*

If $B \subset \mathscr{B}(\mathscr{H})$, the **commutant** of B (denoted by B') is defined as follows,

$$B' := \{T \in \mathscr{B}(\mathscr{H}) : \ TS = ST \ for \ all \ S \in B\} \tag{2.1}$$

We also use the notation $B'' = ((B)')'$ for the **double commutant** of B.

Theorem 2.1.8. *Let $A \subset \mathscr{B}(\mathscr{H})$ be a self-adjoint set. Then A' is a von Neumann algebra.*

Proof. We can see that A' is a self adjoint algebra containing 1. All we need to check is that A' is closed under WOT. Let $(x_i)_i$ be a net in A' such that $x_i \to x \in \mathscr{B}(\mathscr{H})$. Then for any $a \in A$ and $\xi, \eta \in \mathscr{H}$, we have

$$\langle [x,a]\xi, \eta \rangle = \langle xa\xi, \eta \rangle - \langle ax\xi, \eta \rangle$$

$$= \langle ax\xi, \eta \rangle - \langle x\xi, a^*\eta \rangle$$

[1] SOT closure and WOT closure respectively

$$= \lim_{i\to\infty} \langle x_i a\xi, \eta\rangle - \langle x_i\xi, a^*\eta\rangle = \lim_{i\to\infty} \langle [x_i, a]\xi, \eta\rangle = 0.$$

□

Corollary 2.1.9. *A self adjoint maximal abelian subalgebra $A \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra.*

Proof. Since A is maximal abelian, we have $A = A'$. □

Lemma 2.1.10. *Suppose $A \subset \mathscr{B}(\mathscr{H})$ is a self adjoint algebra containing 1. Then for all $\xi \in \mathscr{H}$, and $x \in A''$ there exists x_i such that $\lim_{i\to\infty} \|(x - x_i)\xi\| = 0$.*

Proof. Consider the closed subspace $\mathscr{K} := \overline{A\xi}$, and denote by p the projection onto this subspace. Since for all $a \in A$ we have $a\mathscr{K} \subset \mathscr{K}$, it follows that $ap = pap$. But since A is a self adjoint it then also follows that for all $a \in A$ we have $pa - (a^*p)^* = (pa^*p)^* = pap = ap$, and hence $p \in A'$.

We therefore have $xp = xp^2 = pxp$ and hence $x\mathscr{K} \subset \mathscr{K}$. Since $1 \in A$ it follows that $\xi \in A$ and hence also $x\xi \in \mathscr{K}$. □

Theorem 2.1.11 (von Neumann's double commutant theorem)**.** *Suppose $A \in \mathscr{B}(\mathscr{H})$ be a self adjoint algebra containing 1. Then $A'' = \overline{A}^{WOT}$.*

Proof. BY Theorem we have A'' is closed in the weak operator topology, and we have $A \subset A''$. We need to show that A'' is dense subset of A in the WOT.

Let $\xi_1, \cdots, \xi_n \in \mathscr{H}$ and $x \in A''$ and consider the subalgebra $\bar{A}$ of $\mathscr{B}(\mathscr{H}^n) \equiv \mathbb{M}_n(\mathscr{B}(\mathscr{H}))$ consisting of matrices with diagonal coefficients contained in A. Then the diagonal matrix whose diagonal entries are x is in $\bar{A}''$. Hence by Lemma 2.1.10 there exists a net (x_i) in A such that $\lim_{i\to\infty} \|(x - x_i)\xi_j\| = 0$ for all $1 \leq j \leq n$. This shows that $A'' \subset A$ is dense in SOT. □

Corollary 2.1.12. *Let $A \subset \mathscr{B}(\mathscr{H})$ be a self adjoint algebra. Then A is a von Neumann algebra if and only if $A = A''$.*

2.1.2 Examples

2.1.2.1 Abelian von Naumann algebra

In this subsection we will be looking at examples of von Neumann algebras. First we would like to take a look at the abelian case.

Proposition 2.1.13. *Let (X,μ) be a σ-finite measure space. Viewing $L^\infty(X,\mu) \subset \mathscr{B}(L^2(X,\mu))$ as multiplication operators. Then $L^\infty(X,\mu)' = L^\infty(X,\mu)$, i.e. $L^\infty(X,\mu)$ is a von Neumann algebra.*

Proof. Let us assume that μ is a probability measure. Then we get $L^\infty(X,\mu) \subset L^\infty(X,\mu)'$. To show the other inclusion, suppose $T \in L^\infty(X,\mu)'$. Define $f = T(1)$. We claim that $f \in L^\infty(X,\mu)$ and $\|f\|_\infty \leq \|T\|$. We have,

$$\|gf\|_2 = \|gT(1)\|_2 = \|T(g)\| \leq \|g\|_2 \|T\|$$

for all $g \in L^\infty(X,\mu)$. Suppose that , $\varepsilon > 0$ and $\mu\{x \in X : |f(x)| \geq \|T\| + \varepsilon\} > 0$. Let $f = \alpha|f|$ where α is a measurable function and $|\alpha| = 1$ a.e. Set $g = \bar{\alpha}\chi_{\{x \in X : |f(x)| \geq \|T\| + \varepsilon\}}$. Then,

$$(\|T\| + \varepsilon)\mu\{x \in X : |f(x)| \geq \|T\| + \varepsilon\}^{\frac{1}{2}} \leq \|fg\|_2 \leq \|g\|_2\|T\| = \mu\{x \in X : |f(x)| \geq \|T\| + \varepsilon\}\|g\|_2.$$

Which is a contradiction, so we have $\|f\|_\infty \leq \|T\|$.

Now let μ be a σ-finite measure. Then one can fine $\phi \in L^1(X,\mu)$ such that , $0 < \phi(x) < \infty$ for almost every x, and

$$\int \phi(x) d\mu(x) = 1.$$

Set $\nu = \phi d\mu$. Define

$$U : L^2(X,\mu) \to L^2(X,\nu), \;\; V : L^2(X,\nu) \to L^2(X,\mu)$$

as,

$$U(f) = f\phi^{-\frac{1}{2}} \text{ and } V(f) = f\phi^{\frac{1}{2}}$$

U, V are isometries and inverse to each other. Hence U is unitary. For $f \in L^\infty(X,\mu) = L^\infty(X,\nu), \xi \in L^2(X,\mu)$ we get ,

$$U(f\xi) = fU(\xi)$$

As, ν is a probability measure, by previous case we have $L^\infty(X,\nu)' = L^\infty(X,\nu)$. Pulling this identity back vis U we get,

$$L^\infty(X,\mu)' = L^\infty(X,\mu)$$

□

This is indeed the case. We would like to mention the following theorem,

Theorem 2.1.14. *Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ be a separable abelian von Neumann algebra, then there exists a separable compact Hausdorff space K with a Radon probability measure μ on K such that $\mathscr{A}$ and $L^\infty(K,\mu)$ are $*$-isomorphic.*

2.1.2.2 Tracial von Neumann algebra

Definition 2.1.15. A tracial von Neumann algebra is a pair (M,τ) where τ is a linear functional on M satisfying following,

1. $\tau(1) = 1$.
2. $\tau(x^*x) \geq 0$ and $\tau(x^*x) = 0$ if and only if $x = 0$.
3. $\tau(xy) = \tau(yx)$.
4. τ is normal, i.e. $\tau|_{\{x\in M \mid \|x\| \leq 1\}}$ is WOT continuous.

Given a tracial von Neumann algebra, we define the following inner product on M :

$$\langle x, y\rangle = \tau(y^*x)$$

for $x, y \in M$. Denote $L^2(M,\tau) = \overline{M}^{\langle\cdot,\cdot\rangle}$ be the Hilbert space completion of M with respect to this inner product. Define a $*$-representation $\lambda : M \to L^2(M,\tau)$ and a $*$-anti-representation $\rho : M \to$

$L^2(M,\tau)$ by,

$$\lambda(x)y = xy,\ \rho(x)y = yx \ for\ x,y \in M$$

Now we verify that ρ and λ are bounded function ($L^2 - L^2$ bounded). For $y \in M$ we have,

$$(xy)^*(xy) = y^*x^*xy \leq \|x\|^2 y^*y$$

$$\|xy\|_2 \leq \|x\| \|y\|_2$$

Hence,

$$\|yx\|_2^2 = \tau(x^*y^*yx) = \tau(yxx^*y^*)\ (traciality)$$

$$\leq \|x\|^2\tau(yy^*) = \|x\|^2\tau(y^*y) = \|x\|^2\|y\|_2^2$$

Which shows that λ and ρ are bounded.

Define the operator $J : L^2(M,\tau) \to L^2(M,\tau)$ densely by

$$J(x) := x^*\ for\ x \in M$$

By traciality we see that $\|J(x)\|_2 = \|x\|_2$ for all $x \in M$, hence J extends to a conjugate linear isometry to $L^2(M,\tau)$.

Theorem 2.1.16. *Let (M,τ) be a tracial von Neumann algebra. Let λ,ρ be the representation and anti-representation defined above. Then,*

1. $\lambda(M)' = \rho(M)$

2. $\rho(M)' = \lambda(M)$

Proof. 1. We have $\lambda(x)\rho(y)a = \rho(y)\lambda(x)a$ for all $x,y,a \in M$. From density it follows that $\rho(M) \subset \lambda(M)'$.

For the other inclusion let $T \in \lambda(M)'$. To show thet $T \in \rho(M)$ it suffices to show that T

commutes with all $S \in \rho(M)'$ (by double commutant theorem). Set $\xi = T(1)$, then for all $x \in M$,

$$T(x) = T(\lambda(x)1) = \lambda(x)T(1) = \lambda(x)\xi.$$

$$\langle T^*(1), x\rangle = \langle 1, T(x)\rangle = \langle 1, \lambda(x)\xi\rangle = \langle x^*, \xi\rangle = \langle J\xi, x\rangle$$

Hence $T(x) = \lambda(x)\xi$ and $T^*(x) = \lambda(x)J\xi$. Similarly we can get η such that $S(x) = \rho(x)\eta$ and $S^*(x) = \rho(x)J\eta$ for any $x \in M$.

For any $x, y \in M$ we have

$$\langle TS(x), y\rangle = \langle S(x), T^*(y)\rangle = \langle \rho(x)\eta, \lambda(y)J\xi\rangle = \langle \lambda(y^*)\rho(x)\eta, J\xi\rangle = \langle \lambda(y^*)\eta, \rho(x^*)J\xi\rangle$$

as $\rho(M) \subset \lambda(M)'$. We get,

$$\langle TS(x), y\rangle = \langle \lambda(x)\xi, \rho(y)J\eta\rangle = \langle T(x), S^*(y)\rangle = \langle ST(x), y\rangle$$

This completes the proof for other inclusion. □

2.1.2.3 Group von Neumann algebra

Let Γ be a countable discrete group. Define left regular and right (anti) regular representation as,

$$\lambda : \Gamma \to \mathscr{U}(\ell^2\Gamma)$$
$$\rho : \Gamma \to \mathscr{U}(\ell^2\Gamma)$$

by

$$(\lambda(g)f)h := f(g^{-1}h) \tag{2.2}$$

$$(\rho(g)f)h := f(hg^{-1}) \tag{2.3}$$

Let $\mathscr{L}\Gamma$ be the von Neumann algebra generated by $\lambda(\Gamma)$ and $\mathscr{R}\Gamma$ be the von Neumann algebra generated by $\rho(\Gamma)$. Define,

$$\tau : \mathscr{L}\Gamma \to \mathbb{C} \tag{2.4}$$

by $\tau(x) := \langle x\delta_g, \delta_g \rangle$.

Theorem 2.1.17. *The pair $(\mathscr{L}\Gamma, \tau)$ is a tracial von Neumann algebra with $\mathscr{L}\Gamma' = \mathscr{R}\Gamma$. Let ,*

$$\mathscr{L} := \{\xi \in \ell^2\Gamma :\ \xi * f \in \ell^2\Gamma \text{ for all } f \in \ell^2\Gamma\} \tag{2.5}$$

$$\mathscr{R} := \{\xi \in \ell^2\Gamma :\ f * \xi \in \ell^2\Gamma \text{ for all } f \in \ell^2\Gamma\} \tag{2.6}$$

for $\xi \in \mathscr{L}, \eta \in \mathscr{R}$ and $f \in \ell^2\Gamma$ define,

$$\lambda(\xi)f := \xi * f \text{ and } \rho(\eta)f := f * \eta \tag{2.7}$$

Then $\lambda(\xi)$ and $\rho(\eta)$ are bounded. Furthermore,

$$\mathscr{L}\Gamma = \{\lambda(\xi) : \xi \in \mathscr{L}\} \text{ and } \mathscr{R}\Gamma = \{\rho(\xi) : \xi \in \mathscr{R}\} \tag{2.8}$$

and the map

$$\mathscr{L} \to \mathscr{L}\Gamma$$

$$\xi \to \lambda(\xi)$$

is a bijection with $\|\xi\|_2 = \|\lambda(\xi)\|_2$.

Proof. It is clear from the definition that $\tau \in \mathscr{L}\Gamma^*$, is WOT continuous and $\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g))$ for all $g, h \in \Gamma$. Let $x, y \in \mathscr{L}\Gamma$, then there exists nets $(x_i), (y_j)$ in $span\{\lambda(g) : g \in \Gamma\}$

such that $x_i \to x, y_j \to y$ in WOT. From WOT continuity of τ, we get

$$\tau(xy) = \lim_{WOT} \tau(x_i y_j) = \lim_{WOT} \tau(y_j x_i) = \tau(yx)$$

From the definition we have $\tau(x^*x) = \|x\delta_e\|_2^2$, i.e. $\tau(x^*x) \geq 0$ for all $x \in \mathscr{L}\Gamma$. Suppose $\tau(x^*x) = 0$ for some $x \in \mathscr{L}\Gamma$. One can see from the definition and WOT that $\rho(\Gamma) \subset \mathscr{L}(\Gamma)'$. Hence ,

$$\|s\delta_g\|_2 = \|x\rho(g)\delta_e\|_2 = \|\rho(g)x\delta_e\|_2 = 0$$

as $\|x\delta_e\|_2 = 0$ by assumption. Since $Span\{\delta_g : g \in \Gamma\}$ is dense in $\ell^2\Gamma$, we get $x = 0$. Hence τ is faithful. Since,

$$\tau(\lambda(h)^{-1}\lambda(g)) = \langle \delta_g, \delta_h \rangle,$$

We get a $\mathscr{L}\Gamma$-equivariant unitary $U : L^2(\mathscr{L}\Gamma, \tau) \to \ell^2\Gamma$ by,

$$U(\lambda(g)) := \delta_g \ \forall \ g \in \Gamma \tag{2.9}$$

After identifying $L^2(\mathscr{L}\Gamma, \tau)$ with $\ell^2\Gamma$ via U we get $\rho(\lambda(g)) = \rho(g)$. Thus, $\mathscr{L}\Gamma' = \mathscr{R}\Gamma$ by Theorem 2.1.16. The fact that $\lambda(\xi), \rho(\eta)$ are bounded for $\xi \in \mathscr{L}$ and for $\eta \in \mathscr{R}$ follows from closed graph theorem. Let $x \in \mathscr{L}\Gamma$, set $\xi = x(\delta_e)$,

$$x(\delta_g) = x(\rho(g)\delta_e) = \rho(g)x(\delta_e) = \xi * \delta_g.$$

Hence $x(f) = \xi * f$ for all $f \in C_c(\Gamma)$. Let $f \in \ell^2\Gamma$, choose $f_n \in C_c(\Gamma)$ with, $\|f_n - f\|_2 \to 0$. By Fatou's lemma,

$$\|\xi * f\|_2 \leq \liminf_{n\to\infty} \|\xi * f_n\|_2 \leq \liminf_{n\to\infty} \|x\|\|f_n\|_2 = \|x\|\|f\|_2$$

Thus for $\xi \in \mathscr{L}$, $\lambda(\xi)(f) = \lim_{n\to\infty} \lambda(\xi) f_n = \lim_{n\to\infty} x(f_n) = x(f)$. SO $x = \lambda(\xi)$ and $\|\xi\|_2^2 =$

$\|x\delta_e\|_2^2 = \|x\|_2^2$.

It remains to show that $\lambda(\xi) \in \mathscr{L}\Gamma$ for all $\xi \in \mathscr{L}$. By double commutant theorem it is enough to show that $\lambda(\xi)$ commutes with $\mathscr{R}\Gamma$. Since $\mathscr{R}\Gamma$ is generated by $\rho(\Gamma)$, it is enough to show that $\lambda(\xi)$ commutes with $\rho(g)$ for all $g \in \Gamma$, which is clear from the fact that $\lambda(\xi)$ is left convolution by ξ and $\rho(g)$ is right convolution by δ_g.

□

Let Γ be a countable discrete group, the the von Neumann algebra $\mathscr{L}\Gamma$ defined above is called the group von Neumann algebra associated with the group Γ.

A von Neumann algebra $\mathscr{M}$ is called a factor if it has trivial center i.e, $\mathscr{M} \cap \mathscr{M}' = \mathbb{C}$.

Proposition 2.1.18. *Let G be a countable discrete group. $\mathscr{L}G$ is a factor if and only if G is i.c.c.*[2]

Proof. Suppose $g \in G \setminus \{e\}$, such that $\mathscr{C}_g := \{h^{-1}gh | h \in G\}$ is finite. Let $x = \sum_{k \in \mathscr{C}_g} u_k$. Then $x \notin \mathbb{C}$ and ${u_g}^* x u_g = x$ for all $g \in G$. Hence $x \in \mathscr{L}G \cap \mathscr{L}G' = \mathbb{C}$. Hence LG is not a factor.
Conversely, let G is i.c.c. Let $x = \sum_{g \in G} \alpha_g u_g \in \mathscr{Z}(\mathscr{L}G) - \mathbb{C}$. Then for all $h \in G$ we have, $\sum_{g \in G} \alpha_g u_g = x = {u_h}^* x u_h = \sum_{g \in G} \alpha_g u_{h^{-1}gh} = \sum_{g \in G} \alpha_{hgh^{-1}} u_g$. Thus the Fourier coefficient for x is constant in conjugacy class. Hence $\alpha_g = 0$ for all $g \in G \setminus \{e\}$. □

2.1.3 Notations and Terminology

Throughout this document all von Neumann algebras are denoted by calligraphic letters e.g. $\mathscr{A}$, $\mathscr{B}$, $\mathscr{M}$, $\mathscr{N}$, etc. Given a von Neumann algebra $\mathscr{M}$ we will denote by $\mathcal{U}(\mathscr{M})$ its unitary group, by $\mathcal{P}(\mathscr{M})$ the set of all its nonzero projections and by $(\mathscr{M})_1$ its unit ball. Given a unital inclusion $\mathscr{N} \subseteq \mathscr{M}$ of von Neumann algebras we denote by $\mathscr{N}' \cap \mathscr{M} = \{x \in \mathscr{M} : [x, \mathscr{N}] = 0\}$. We also denote by $\mathcal{N}_{\mathscr{M}}(\mathscr{N}) = \{u \in \mathcal{U}(\mathscr{M}) : u\mathscr{N}u^* = \mathscr{N}\}$ the normalizing group. We also denote the quasinormalizer of $\mathscr{N}$ in $\mathscr{M}$ by $\mathcal{QN}_{\mathscr{M}}(\mathscr{N})$. Recall that $\mathcal{QN}_{\mathscr{M}}(\mathscr{N})$ is the set of all $x \in \mathscr{M}$ for which there exist $x_1, x_2, ..., x_n \in \mathscr{M}$ such that $\mathscr{N}x \subseteq \sum_i x_i \mathscr{N}$ and $x\mathscr{N} \subseteq \sum_i \mathscr{N} x_i$ (see [Po99, Definition 4.8]).

[2] the conjugacy class $\mathscr{C}_g := \{h^{-1}gh | h \in G\}$ is infinite for all $g \in G - \{e\}$

All von Neumann algebras $\mathscr{M}$ considered in this document will be tracial, i.e. endowed with a unital, faithful, normal linear functional $\tau : M \to \mathbb{C}$ satisfying $\tau(xy) = \tau(yx)$ for all $x, y \in \mathscr{M}$. This induces a norm on $\mathscr{M}$ by the formula $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in \mathscr{M}$. The $\|\cdot\|_2$-completion of $\mathscr{M}$ will be denoted by $L^2(\mathscr{M})$. For any von Neumann subalgebra $\mathscr{N} \subseteq \mathscr{M}$ we denote by $E_{\mathscr{N}} : \mathscr{M} \to \mathscr{N}$ the τ-preserving condition expectation onto $\mathscr{N}$. We denote the orthogonal projection from $L^2(\mathscr{M}) \to L^2(\mathscr{N})$ by $e_{\mathscr{N}}$. The Jones' basic construction [Jo83, Section 3] for $\mathscr{N} \subseteq \mathscr{M}$ will be denoted by $\langle \mathscr{M}, e_{\mathscr{N}} \rangle$.

For any group G we denote by $(u_g)_{g\in G} \subset \mathcal{U}(\ell^2 G)$ its left regular representation, i.e. $u_g(\delta_h) = \delta_{gh}$ where $\delta_h : G \to \mathbb{C}$ is the Dirac function at $\{h\}$. The weak operatorial closure of the linear span of $\{u_g : g \in G\}$ in $\mathcal{B}(\ell^2 G)$ is called the group von Neumann algebra and will be denoted by $\mathscr{L}(G)$; this is a II_1 factor precisely when G has infinite non-trivial conjugacy classes (icc). If $\mathscr{M}$ is a tracial von Neumann algebra and $G \curvearrowright^{\sigma} \mathscr{M}$ is a trace preserving action we denote by $\mathscr{M} \rtimes_{\sigma} G$ the corresponding cross product von Neumann algebra [MvN37]. For any subset $K \subseteq G$ we denote by $P_{\mathscr{M}K}$ the orthogonal projection from the Hilbert space $L^2(\mathscr{M} \rtimes G)$ onto the closed linear span of $\{xu_g \,|\, x \in \mathscr{M}, g \in K\}$. When $\mathscr{M}$ is trivial we will denote this simply by P_K.

All groups considered in this article are countable and will be denoted by capital letters A, B, G, H, Q, N ,M, etc. Given groups Q, N and an action $Q \curvearrowright^{\sigma} N$ by automorphisms we denote by $N \rtimes_{\sigma} Q$ the corresponding semidirect product group. For any $n \in N$ we denote by $\mathrm{Stab}_Q(n) = \{g \in Q : \sigma_g(n) = n\}$. Given a group inclusion $H \leqslant G$ sometimes we consider the centralizer $C_G(H)$ and the virtual centralizer $vC_G(H) = \{g \in G : |g^H| < \infty\}$. We also denote by $\langle\langle H \rangle\rangle$ the normal closure of H in G.

2.1.4 Popa's Intertwining Techniques

Over more than fifteen years ago, Popa introduced in [Po03, Theorem 2.1 and Corollary 2.3] a powerful analytic criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras. Now this is known in the literature as *Popa's intertwining-by-bimodules technique* and has played a key role in the classification of von Neumann algebras program via

Popa's deformation/rigidity theory.

Theorem 2.1.19. *[Po03] Let $(\mathscr{M},\tau)$ be a separable tracial von Neumann algebra and let $\mathscr{P},\mathscr{Q}\subseteq\mathscr{M}$ be (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:*

1. *There exist $p\in\mathcal{P}(\mathscr{P}), q\in\mathcal{P}(\mathscr{Q})$, a $*$-homomorphism $\theta: p\mathscr{P}p\to q\mathscr{Q}q$ and a partial isometry $0\neq v\in q\mathscr{M}p$ such that $\theta(x)v=vx$, for all $x\in p\mathscr{P}p$.*

2. *For any group $\mathscr{G}\subset\mathcal{U}(\mathscr{P})$ such that $\mathscr{G}''=\mathscr{P}$ there is no sequence $(u_n)_n\subset\mathscr{G}$ satisfying $\|E_{\mathscr{Q}}(xu_ny)\|_2\to 0$, for all $x,y\in\mathscr{M}$.*

3. *There exist finitely many $x_i,y_i\in\mathscr{M}$ and $C>0$ such that $\sum_i\|E_{\mathscr{Q}}(x_iuy_i)\|_2^2\geq C$ for all $u\in\mathscr{U}(\mathscr{P})$.*

If one of the three equivalent conditions from Theorem 2.1.19 holds then we say that *a corner of $\mathscr{P}$ embeds into $\mathscr{Q}$ inside $\mathscr{M}$*, and write $\mathscr{P}\prec_{\mathscr{M}}\mathscr{Q}$. If we moreover have that $\mathscr{P}p'\prec_{\mathscr{M}}\mathscr{Q}$, for any projection $0\neq p'\in\mathscr{P}'\cap 1_{\mathscr{P}}\mathscr{M}1_{\mathscr{P}}$ (equivalently, for any projection $0\neq p'\in\mathcal{Z}(\mathscr{P}'\cap 1_{\mathscr{P}}\mathscr{M}1_P)$), then we write $\mathscr{P}\prec^s_{\mathscr{M}}\mathscr{Q}$. We refer the readers to the survey papers [Po07, Va10b, Io18] for recent progress in von Neumann algebras using deformation/rigidity theory.

We also recall the notion of relative amenability introduced by N. Ozawa and S. Popa. Let $(\mathscr{M},\tau)$ be a tracial von Neumann algebra. Let $p\in\mathscr{M}$ be a projection, and let $\mathscr{P}\subseteq p\mathscr{M}p$, and $\mathscr{Q}\subseteq\mathscr{M}$ be von Neumann subalgebras. Following [OP07, Definition 2.2], we say that $\mathscr{P}$ is amenable relative to $\mathscr{Q}$ inside $\mathscr{M}$, if there exists a positive linear functional $\phi: p\langle\mathscr{M},e_{\mathscr{Q}}\rangle p\to\mathbb{C}$ such that $\phi|_{p\mathscr{M}p}=\tau$ and $\phi(xT)=\phi(Tx)$ for all $T\in\mathscr{Q}$ and all $x\in\mathscr{P}$. If $\mathscr{P}$ is amenable relative to $\mathscr{Q}$ inside $\mathscr{M}$, we write $\mathscr{P}\lessdot_{\mathscr{M}}\mathscr{Q}$.

For further use we record the following result which controls the intertwiners in algebras arsing form malnormal subgroups. Its proof is essentially contained in [Po03, Theorem 3.1] so it will be left to the reader.

Lemma 2.1.20 (Popa [Po03])**.** *Assume that $H\leqslant G$ be an almost malnormal subgroup and let $G\curvearrowright\mathscr{N}$ be a trace preserving action on a finite von Neumann algebra $\mathscr{N}$. Let $\mathscr{P}\subseteq\mathscr{N}\rtimes H$ be* *a von Neumann algebra such that $\mathscr{P} \nprec_{\mathscr{N}\rtimes H} N$. Then for every elements $x, x_1, x_2, \ldots, x_l \in \mathscr{N} \rtimes G$ satisfying $\mathscr{P}x \subseteq \sum_{i=1}^{l} x_i \mathscr{P}$ we must have that $x \in \mathscr{N} \rtimes H$.*

The following result is a mild generalization of [BV12, Lemma 2.3]. For reader's convenience we include all the details in our proof.

Theorem 2.1.21. *Let G be a group together $H \lhd G$ with a normal subgroup and assume that $G \curvearrowright (\mathscr{N}, \tau)$ is a trace preserving action on a von Neumann algebra $(\mathscr{N}, \tau)$. Consider $\mathscr{N} \rtimes G = \mathscr{M}$ the corresponding crossed product von Neumann algebra, assume that $\mathscr{A} \subset \mathscr{M}$ (possibly non-unital) and $\mathscr{G} \subseteq \mathscr{N}_{1_{\mathscr{A}}\mathscr{M}1_{\mathscr{A}}}(\mathscr{A})$ a group of unitaries such that $\mathscr{A}, \mathscr{G}'' \prec^{s}_{\mathscr{M}} \mathscr{N} \rtimes H$. Then $(\mathscr{A}\mathscr{G})'' \prec^{s}_{\mathscr{M}} \mathscr{N} \rtimes H$.*

Proof. Let $G_H \subset G$ be a section for G/H. Also denote by $\mathscr{P} = \mathscr{N} \rtimes H$. Since $\mathscr{A}, \mathscr{G}'' \prec^{s}_{\mathscr{M}} \mathscr{P}$, then by [Va10a, Lemma 2.5], for all $\varepsilon_1, \varepsilon_2 > 0$ there exist $K_{\varepsilon_1}, L_{\varepsilon_2} \subset G_H$ such that for all $a \in (\mathscr{A})_1$ and $b \in (\mathscr{G}'')_1$ we have 1) $\|P_{\mathscr{P}K_{\varepsilon_1}}(a) - a\|_2 \leq \varepsilon_1$ and 2) $\|P_{\mathscr{P}L_{\varepsilon_2}}(b) - b\|_2 \leq \varepsilon_2$. Here for every $S \subset G_H$, the map $P_{\mathscr{P}S} : L^2(\mathscr{M}) \to \overline{\text{span}}^{\|\cdot\|_2}\{\mathscr{P}u_g : g \in S\}$ is the orthogonal projection. Also notice that, for all $x \in \mathscr{M}$, $P_{\mathscr{P}S}(x) = \sum\limits_{s\in S} E_{\mathscr{P}}(xu_{s^{-1}})u_s$. In particular, for all $x \in \mathscr{M}$ we have,

$$\|P_{\mathscr{P}S}(x)\|_\infty \leq |S|\|x\|_\infty \ \text{ and } \ \|P_{\mathscr{P}S}(x)\|_2 \leq \|x\|_2. \tag{2.10}$$

Now for all $a \in (\mathscr{A})_1, b \in (\mathscr{G}'')_1$ we have

$$\begin{aligned}
\|ab - P_{\mathscr{P}K_{\varepsilon_1}}(a)P_{\mathscr{P}L_{\varepsilon_2}}(b)\|_2 &\leq \|ab - P_{\mathscr{P}K_{\varepsilon_1}}(a)b\|_2 + \|P_{\mathscr{P}K_{\varepsilon_1}}(a)b - P_{\mathscr{P}K_{\varepsilon_1}}(a)P_{\mathscr{P}L_{\varepsilon_2}}(b)\|_2 \\
&\leq \|a - P_{\mathscr{P}K_{\varepsilon_1}}(a)\|_2\|b\|_\infty + \|P_{\mathscr{P}K_{\varepsilon_1}}\|_\infty\|b - P_{\mathscr{P}L_{\varepsilon_2}}(b)\|_2 \qquad (2.11) \\
&\leq \|a - P_{\mathscr{P}K_{\varepsilon_1}}(a)\|_2 + |K_{\varepsilon_1}|\|b - P_{\mathscr{P}L_{\varepsilon_2}}(b)\|_2 \\
&\leq \varepsilon_1 + |K_{\varepsilon_1}|\varepsilon_2. \qquad (2.12)
\end{aligned}$$

So letting $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \frac{\varepsilon}{|K_{\varepsilon_1}|}$ we get that there exists $K_\varepsilon, L_\varepsilon$ finite subsets of the section G/H

such that

$$\|ab - P_{\mathscr{P}K_\varepsilon}(a)P_{\mathscr{P}L_\varepsilon}(b)\| \leq 2\varepsilon. \tag{2.13}$$

Since $H \lhd G$, then there exist a finite set $F_\varepsilon \subseteq G_H$ such that $|F_\varepsilon| \leq |K_\varepsilon||L_\varepsilon|$ and $P_{\mathscr{P}F_\varepsilon}(P_{\mathscr{P}K_\varepsilon}(a)P_{\mathscr{P}L_\varepsilon}(b)) = P_{\mathscr{P}K_\varepsilon}(a)P_{\mathscr{P}L_\varepsilon}(b)$ for all $a \in \mathcal{U}(\mathscr{A}), b \in (\mathscr{G}'')_1$. Using this fact together with (2.13) we get that $\|P_{\mathscr{P}F_\varepsilon}(ab) - P_{\mathscr{P}K_\varepsilon}(a)P_{\mathscr{P}L_\varepsilon}(b)\| \leq 2\varepsilon$ and combining with (2.13) again we get that

$$\|ab - P_{\mathscr{P}F_\varepsilon}(ab)\| \leq 2\varepsilon. \tag{2.14}$$

for all $a \in \mathcal{U}(\mathscr{A}), b \in (\mathscr{G}'')_1$. Since $(\mathcal{U}(\mathscr{A})\mathscr{G})" = (\mathscr{A}\mathscr{G})"$, this already shows that $(\mathscr{A}\mathscr{G})" \prec \mathscr{P}$. Next we argue that we actually have $(\mathscr{A}\mathscr{G})" \prec^s \mathscr{P}$. To see this fix $p \in (\mathscr{A}\mathscr{G})' \cap 1_{\mathscr{A}\vee\mathscr{G}''}\mathscr{M}1_{\mathscr{A}\vee\mathscr{G}''}$. Then there exists a finite set $G_\varepsilon \subseteq G_H$ such that

$$\|p - P_{\mathscr{P}G_\varepsilon}(p)\| \leq \frac{\varepsilon}{|K_\varepsilon||L_{\varepsilon|}}. \tag{2.15}$$

Combining (2.15) and (2.14) we get that

$$\begin{aligned}
\|abp - P_{\mathscr{P}F_\varepsilon}(ab)P_{\mathscr{P}G_\varepsilon}(p)\| &\leq \|abp - P_{\mathscr{P}F_\varepsilon}(ab)p\| + \|P_{\mathscr{P}F_\varepsilon}(ab)p - P_{\mathscr{P}F_\varepsilon}(ab)P_{\mathscr{P}G_\varepsilon}(p)\| \\
&\leq \|ab - P_{\mathscr{P}F_\varepsilon}(ab)\|_2\|p\|_\infty + \|P_{\mathscr{P}F_\varepsilon}(ab)\|_\infty\|p - P_{\mathscr{P}G_\varepsilon}(p)\|_2 \\
&\leq 4\varepsilon + |F_\varepsilon| \cdot \frac{\varepsilon}{|K_\varepsilon||L_\varepsilon|} < 5\varepsilon. \qquad (2.16)
\end{aligned}$$

Again there exists a finite set $T_\varepsilon \subset G$ such that $P_{\mathscr{P}T_\varepsilon}(P_{\mathscr{P}F_\varepsilon}(ab)P_{\mathscr{P}G_\varepsilon}(p)) = P_{\mathscr{P}F_\varepsilon}(ab)P_{\mathscr{P}G_\varepsilon}(p)$ and $|T_\varepsilon| \leq |F_\varepsilon||G_\varepsilon|$. Using this and (2.16) we get that $\|abp - P_{\mathscr{P}T_\varepsilon}(abp)\| < 10\varepsilon$ for all $a \in \mathcal{U}(\mathscr{A}), b \in \mathscr{G}$. This shows that $(\mathscr{A}\mathscr{G})" \prec^s_{\mathscr{M}} \mathscr{P}$, as desired. □

We end this section by highlighting a straightforward corollary of Theorem 2.1.21 that we will be very useful in the sequel.

Corollary 2.1.22. *Let $H \lhd G$ be a normal subgroup of G and $G \curvearrowright (\mathscr{N}, \tau)$ be a trace preserving* *action on a tracial von Neumann algebra $(\mathscr{N},\tau)$. Let $\mathscr{M}=\mathscr{N}\rtimes G$. Assume that $\mathscr{A}$, $\mathscr{B}\subseteq\mathscr{M}$ are commuting $*$-subalgebras such that $\mathscr{A}\prec^s_{\mathscr{M}}\mathscr{N}\rtimes H$ and $\mathscr{B}\prec^s_{\mathscr{M}}\mathscr{N}\rtimes H$. Then $\mathscr{A}\vee\mathscr{B}\prec^s_{\mathscr{M}}\mathscr{N}\rtimes H$.*

Proof. Follows from Theorem 2.1.21 by letting $\mathscr{G}=\mathcal{U}(\mathscr{B})$. □

We continue with the following intertwining result for group algebras which is a generalization of some previous results obtained under normality assumptions [DHI16]. For reader's convenience we also include a brief proof.

Lemma 2.1.23. *Assume that $H_1,H_2\leqslant G$ are groups, let $G\curvearrowright\mathscr{N}$ be a trace preserving action on a tracial von Neumann algebra $\mathscr{N}$ and denote by $\mathscr{M}=\mathscr{N}\rtimes G$ the corresponding crossed product. Also assume that $\mathscr{A}\prec^s\mathscr{N}\rtimes H_1$ is a von Neumann algebra such that $\mathscr{A}\prec_{\mathscr{M}}\mathscr{N}\rtimes H_2$. Then one can find $h\in G$ such that $\mathscr{A}\prec_{\mathscr{M}}\mathscr{N}\rtimes(H_1\cap hH_2h^{-1})$.*

Proof. Since $\mathscr{A}\prec^s\mathscr{N}\rtimes H_1$ then by [Va10a, Lemma 2.6] for ever $\varepsilon>0$ there exist a finite subset $S\subset G$ such that $\|P_{SH_1S}(x)-x\|_2\leqslant\varepsilon$ for all $(x\in\mathscr{A})_1$. Here for every $K\subseteq G$ we denote by P_K the orthogonal projection from $L^2(\mathscr{M})$ onto the closure of the linear span of $\mathscr{N}u_g$ with $g\in K$. Also since $\mathscr{A}\prec_{\mathscr{M}}\mathscr{N}\rtimes H_2$ then by Popa's intertwining techniques there exist a scalar $0<\delta<1$ and a finite subset $T\subset G$ so that $\|P_{TH_2T}(x)\|_2\geqslant\delta$, for all $x\in(\mathscr{A})_1$. Thus, using this in combination with the previous inequality, for every $x\in\mathcal{U}(\mathscr{A})$ and every $\varepsilon>0$, there are finite subsets $S,T\subset G$ so that $\|P_{TH_2T}\circ P_{SH_1S}(x)\|_2\geqslant\delta-\varepsilon$. Since there exist finite subsets $R,U\subset G$ such that $TH_2T\cap SH_1S\subseteq U(\cup_{r\in R}H_2\cap rH_1r^{-1}))U$ we further get that $\|P_{U(\cup_{r\in R}H_2\cap rH_1r^{-1}))U}(x)\|_2\geqslant\delta-\varepsilon$. Then choosing $\varepsilon>0$ sufficiently small and using Popa's intertwining techniques together with a diagonalization argument (see proof of [IPP05, Theorem 4.3]) one can find $r\in R$ so that $\mathscr{A}\prec\mathscr{N}\rtimes(H_2\cap rH_1r^{-1})$, as desired. □

In the sequel we need the following three intertwining lemmas, which establish that under certain conditions, intertwining in a larger algebra implies that the intertwining happens in a "smaller subalgebra".

Lemma 2.1.24. *Let $\mathscr{A},\mathscr{B}\subseteq\mathscr{N}\subseteq\mathscr{M}$ be von Neumann algebras and assume that there is a group $G\subseteq\mathscr{N}_{\mathscr{M}}(\mathscr{A})$ such that $G''=\mathscr{M}$. If $\mathscr{B}\prec_{\mathscr{M}}\mathscr{A}$ then $\mathscr{B}\prec_{\mathscr{N}}\mathscr{A}$.*

Proof. Since $\mathscr{B} \prec_{\mathscr{M}} \mathscr{A}$ then by Theorem 2.1.19 one can find $x_1, x_2 ... x_n, y_1, y_2, ..., y_n \in \mathscr{M}$ and $c > 0$ such that $\sum_{i=1}^{n} \|E_{\mathscr{A}}(x_i b y_i)\|_2^2 \geq c$, for all $b \in \mathscr{U}(\mathscr{B})$. Since $G'' = \mathscr{M}$ then using basic $\|\cdot\|_2$-approximation for x_i and y_i and shrinking $c > 0$ if necessary one can find $g_1, g_2, ..., g_l, h_1, h_2, ..., h_l \in G$ and $c' > 0$ such that for all $b \in \mathscr{U}(\mathscr{B})$ we have

$$\sum_{i=1}^{n} \|E_{\mathscr{A}}(g_i b h_i)\|_2^2 \geq c' > 0. \tag{2.17}$$

Also since G normalizes $\mathscr{A}$ we see that $E_{\mathscr{A}}(g_i b h_i) = E_{g_i \mathscr{A} g_i^*}(g_i b h_i) = g_i E_{\mathscr{A}}(b h_i g_i) g_i^*$. This combined with (2.17) and $\mathscr{A} \subseteq \mathscr{N}$ give $0 < C' \leq \sum_{i=1}^{l} \|E_{\mathscr{A}}(b h_i g_i)\|_2^2 = \sum_{i=1}^{l} \|E_{\mathscr{A}} \circ E_{\mathscr{N}}(b h_i g_i)\|_2^2 = \sum_{i=1}^{l} \|E_{\mathscr{A}}(b E_{\mathscr{N}}(h_i g_i))\|_2^2$ for all $b \in \mathscr{U}(\mathscr{B})$. Since $E_{\mathscr{N}}(h_i g_i) \in \mathscr{N}$ then using Theorem 2.1.19 this clearly shows that $\mathscr{B} \prec_{\mathscr{N}} \mathscr{A}$. □

Lemma 2.1.25. *Let Q be a group and denote by* $\operatorname{diag}(Q) = \{(q,q) \,|\, q \in Q\}$ *the diagonal subgroup of $Q \times Q$. Let $\mathscr{A}$ be a tracial von Neumann algebra and assume that $(Q \times Q) \curvearrowright^{\sigma} \mathscr{A}$ is a trace preserving action. Let $\mathscr{B} \subseteq \mathscr{A}$ be a regular von Neumann subalgebra which is invariant under the action σ. Let $\mathscr{D} \subseteq \mathscr{A} \rtimes_{\sigma} \operatorname{diag}(Q)$ be a subalgebra such that $\mathscr{D} \prec_{\mathscr{A} \rtimes_{\sigma} (Q \times Q)} \mathscr{B} \rtimes_{\sigma} \operatorname{diag}(Q)$. Then $\mathscr{D} \prec_{\mathscr{A} \rtimes_{\sigma} \operatorname{diag}(Q)} \mathscr{B} \rtimes_{\sigma} \operatorname{diag}(Q)$.*

Proof. In this proof, we let $\tilde{\mathscr{M}} = \mathscr{A} \rtimes_{\sigma} (Q \times Q)$, and $\mathscr{M} = \mathscr{A} \rtimes_{\sigma} \operatorname{diag}(Q)$. By Theorem 2.1.19, there exist $x_i, y_i \in \tilde{\mathscr{M}}$, $i = 1, ..., n$ and $c > 0$ such that:

$$\sum_{i=1}^{n} \|E_{\mathscr{B} \rtimes \operatorname{diag}(Q)}(x_i d y_i)\|^2 \geq c \text{ for all } d \in \mathscr{U}(\mathscr{D}). \tag{2.18}$$

Since $(Q \times Q) = (Q \times 1) \rtimes_{\rho} \operatorname{diag}(Q)$, where ρ is the action of $\operatorname{diag}(Q)$ on $(Q \times 1)$ by conjugation, we can $\|\cdot\|_2$-approximate x_i's (resp y_i's) inside the equation 2.18 by finite linear combinations of the form $\sum_{k=1}^{m} u_{g_k} z_k$, (resp $\sum_{k=1}^{m} z_k u_{g_k}$) where $g_k \in (Q \times 1)$ and $z_k \in \mathscr{M}$. Thus, shrinking $c > 0$ in 2.18, if necessary, we can assume that there exists finitely many $g_i, h_i \in (Q \times 1)$, $z_i, t_i \in \mathscr{M}$ and

$c>0$ such that:

$$\sum_{i=1}^{n} \|E_{\mathscr{B}\rtimes \mathrm{diag}(Q)}(u_{g_i} z_i d t_i u_{h_i})\|^2 \geq c > 0 \text{ for all } d \in \mathscr{U}(\mathscr{D}). \tag{2.19}$$

The previous equation further implies that for all $d \in \mathscr{U}(\mathscr{D})$ we have

$$0 < c \leq \sum_I \|E_{\mathscr{B}\rtimes \mathrm{diag}(Q)}(u_{g_i} z_i d t_i u_{h_i})\|_2^2 + \sum_J \|E_{\mathscr{B}\rtimes \mathrm{diag}(Q)}(u_{g_i} z_i d t_i u_{h_i})\|_2^2 \tag{2.20}$$

where $I := \{i \in \overline{1,n} \,|\, g_i \neq 1, h_i \neq 1\}$, and $J := \{i \in \overline{1,n} \,|\, g_i = 1 \text{ or } h_i = 1\}$. Let $P_{g_i \mathscr{M} h_i}$ be the orthogonal projection onto the closed subspace $\overline{span}^{\|\cdot\|_2}\{u_{g_i}\mathscr{M} u_{h_i}\}$. Note that $u_{g_i} z_i d t_i u_{h_i} \in span\{u_{g_i}\mathscr{M} u_{h_i}\}$ and therefore

$$\sum_I \|E_{\mathscr{B}\rtimes \mathrm{diag}(Q)}(u_{g_i} z_i d t_i u_{h_i})\|_2^2 = \sum_I \|E_{\mathscr{B}\rtimes \mathrm{diag}(Q)} \circ P_{g_i \mathscr{M} h_i}(u_{g_i} z_i d t_i u_{h_i})\|_2^2. \tag{2.21}$$

A direct calculation shows that $E_{\mathscr{B}\rtimes \mathrm{diag} diag(Q)} \circ P_{g_i \mathscr{M} h_i}(z) = P_{\mathscr{B}(\mathrm{diag}(Q)) \cap g_i \mathrm{diag}(Q) h_i)}(z)$, where $P_{\mathscr{B}(\mathrm{diag}(Q)) \cap g_i \mathrm{diag}(Q) h_i)}$ is the orthogonal projection onto the closed subspace $\overline{span}\{\mathscr{B} g \,|\, g \in g_i \mathrm{diag}(Q) h_i \cap \mathrm{diag}(Q)\}$. Now if $g_i \mathrm{diag}(Q) h_i \cap \mathrm{diag}(Q) \neq \emptyset$ one can find $r_i, s_i \in \mathrm{diag} Q)$ such that $h_i = r_i g_i^{-1} s_i$. Thus $g_i \mathrm{diag}(Q) h_i \cap \mathrm{diag}(Q) = (g_i \mathrm{diag}(Q) g_i^{-1} \cap \mathrm{diag}(Q)) s_i$. We now claim that $g_i \mathrm{diag}(Q) g_i^{-1} \cap \mathrm{diag}(Q) = \mathrm{diag}(\mathscr{C}_Q(g_i))$, where $\mathscr{C}_Q(g_i)$ is the centralizer of g_i in Q. To see this, let $(k,k) \in g_i \mathrm{diag}(Q) g_i^{-1} \cap \mathrm{diag}(Q)$. Then there exists $(v,v) \in \mathrm{diag}(Q)$ such that $(g_i,1)(v,v)(g_i^{-1},1) = (k,k)$. This clearly implies that $v = k$ and $g_i k g_i^{-1} = k$, thereby establishing the claim.

Therefore we have that $g_i \mathrm{diag}(Q) h_i \cap \mathrm{diag}(Q) = \mathrm{diag}(\mathscr{C}_Q(g_i)) s_i$ and hence $P_{B(\mathrm{diag}(Q)) \cap g_i \mathrm{diag}(Q) h_i)}(z) =$

$P_{\mathscr{B}\rtimes\mathrm{diag}(\mathscr{C}_Q(g_i))s_i}(z) = E_{B\rtimes(\mathrm{diag}(\mathscr{C}_Q(g_i))}(zu_{s_i^{-1}})u_{s_i}$. Using equation 2.21 we get

$$\begin{aligned}\sum_{i\in I}\|E_{\mathscr{B}\rtimes\mathrm{diag}(Q)}u_{g_i}z_idt_iu_{h_i})\|_2^2 &= \sum_i \|P_{\mathscr{B}(g_i\mathrm{diag}(Q)h_i\cap\mathrm{diag}Q))}(u_{g_i}z_idt_iu_{h_i})\|_2^2\\ &= \sum_I \|E_{\mathscr{B}\rtimes\mathrm{diag}(\mathscr{C}_Q(g_i))}(u_{g_i}z_idt_i(u_{r_ig_i^{-1}s_i})u_{s_i^{-1}})u_{s_i}\|_2^2\\ &= \sum_I \|E_{\mathscr{B}\rtimes\mathrm{diag}(\mathscr{C}_Q(g_i))}(u_{g_i}z_idt_ir_iu_{g_i^{-1}})\|_2^2 = \sum_i \|u_{g_i}E_{\mathscr{B}\rtimes\mathrm{diag}(\mathscr{C}_Q(g_i))}(z_idt_ir_i)u_{g_i^{-1}}\|_2^2\\ &= \sum_I \|E_{\mathscr{B}\rtimes\mathrm{diag}(\mathscr{C}_Q(g_i))}(z_idt_ir_i)\|_2^2.\end{aligned}$$

Combining this with (2.20) and using $\mathscr{B}\subseteq\mathscr{A}$ we see that for all $d\in\mathscr{U}(\mathscr{D})$ we have

$$\begin{aligned}0<c&\leq \sum_I \|E_{\mathscr{B}\rtimes\mathrm{diag}(Q)}(u_{g_i}z_idt_iu_{h_i})\|_2^2 + \sum_J \|E_{\mathscr{B}\rtimes\mathrm{diag}(Q)}(u_{g_i}z_idt_iu_{h_i})\|_2^2\\ &\leq \sum_{i\in I}\|E_{\mathscr{B}\rtimes\mathrm{diag}(\mathscr{C}_Q(g_i)))}(z_idt_ir_i)\|_2^2 + \sum_{i\in J, g_i=1}\|E_{\mathscr{B}\rtimes\mathrm{diag}(Q)}(z_idt_iu_{h_i})\|_2^2 + \sum_{i\in J,h_i=1}\|E_{\mathscr{B}\rtimes\mathrm{diag}(Q)}(u_{g_i}z_idt_i)\|_2^2\\ &= \sum_{i\in I}\|E_{\mathscr{B}\rtimes\mathrm{diag}(\mathscr{C}_Q(g_i))}(z_idt_ir_i)\|_2^2 + \sum_{i\in J,g_i=1}\|E_{\mathscr{B}\rtimes\mathrm{diag}(Q)}(z_idt_iE_{\mathscr{A}\rtimes\mathrm{diag}(Q)}(u_{h_i}))\|_2^2\\ &\quad + \sum_{i\in J,h_i=1}\|E_{\mathscr{B}\rtimes diag(Q)}(E_{\mathscr{A}\rtimes\mathrm{diag}(Q)}(u_{g_i})z_idt_i)\|_2^2 = \sum_{i\in I}\|E_{\mathscr{B}\rtimes\mathrm{diag}(\mathscr{C}_Q(g_i))}(z_idt_ir_i)\|_2^2\\ &+ \sum_{i\in J,g_i=1,h_i=1}\|E_{\mathscr{B}\rtimes\mathrm{diag}(Q)}(z_idt_i)\|_2^2.\end{aligned}$$

Using Theorem 2.1.19 again then above inequality establishes that $\mathscr{D}\prec_{\mathscr{A}\rtimes\mathrm{diag}(Q)}\mathscr{B}\rtimes\mathrm{diag}(Q)$, as desired. □

Lemma 2.1.26. *Let $\mathscr{C}\subseteq\mathscr{B}$ and $\mathscr{N}\subseteq\mathscr{M}$ be inclusions of von Neumann algebras. If $\mathscr{A}\subseteq\mathscr{N}\bar{\otimes}\mathscr{B}$ is a von Neumann subalgebra such that $\mathscr{A}\prec_{\mathscr{M}\bar{\otimes}\mathscr{B}}\mathscr{M}\bar{\otimes}\mathscr{C}$ then $\mathscr{A}\prec_{\mathscr{N}\bar{\otimes}\mathscr{B}}\mathscr{N}\bar{\otimes}\mathscr{C}$.*

Proof. By Theorem 2.1.19 one can find $x_i, y_i\in\mathscr{M}\bar{\otimes}\mathscr{B}$, $i=\overline{1,k}$ and a scalar $c>0$ such that

$$\sum_{i=1}^{n}\|E_{\mathscr{M}\bar{\otimes}\mathscr{C}}(x_iay_i)\|^2\geq c \text{ for all } d\in\mathscr{U}(\mathscr{A}). \tag{2.22}$$

Using $\|\cdot\|_2$-approximations of x_i and y_i by finite linear combinations of elements in $\mathscr{M}\bar{\otimes}_{\mathrm{alg}}\mathscr{B}$

together with the $\mathscr{M}\otimes 1$-bimodularity of $E_{\mathscr{M}\bar{\otimes}\mathscr{C}}$, after increasing k and shrinking $c>0$ if necessary, in (2.22) we can assume wlog that $x_i, y_i \in 1\otimes\mathscr{B}$. However, since $\mathscr{A}\subseteq\mathscr{N}\bar{\otimes}\mathscr{B}$ then in this situation we have $E_{\mathscr{M}\bar{\otimes}\mathscr{C}}(x_i a y_i) = E_{\mathscr{M}\bar{\otimes}\mathscr{C}}\circ E_{\mathscr{N}\bar{\otimes}\mathscr{B}}(x_i a y_i) = E_{\mathscr{N}\bar{\otimes}\mathscr{C}}(x_i a y_i)$. Thus (2.22) combined with Theorem 2.1.19 give $\mathscr{A}\prec_{\mathscr{N}\bar{\otimes}\mathscr{B}}\mathscr{N}\bar{\otimes}\mathscr{C}$, as desired. □

In the sequel we need the following (minimal) technical variation of [CI17, Lemma 2.6]. The proof is essentially the same with the one presented in [CI17] and we leave the details to the reader.

Lemma 2.1.27 (Lemma 2.6 in [CI17])**.** *Let $\mathscr{P},\mathscr{Q}\subseteq\mathscr{M}$ be inclusions of tracial von Neumann algebras. Assume that $\mathcal{QN}^{(1)}_{\mathscr{M}}(\mathscr{P}) = P$ and $\mathscr{Q}$ is a II_1 factor. Suppose there is a projection $z\in\mathcal{Z}(\mathscr{P})$ such that $\mathscr{P}z\prec^s\mathscr{Q}$ and a projection $p\in\mathscr{P}z$ such that $p\mathscr{P}p = p\mathscr{Q}p$. Then one can find a unitary $u\in\mathscr{M}$ such that $u\mathscr{P}zu^* = r\mathscr{Q}r$ where $r = uzu^*\in\mathcal{P}(\mathscr{Q})$.*

The next lemma is a mild generalization of [IPV10, Proposition 7.1], using the same techniques (see also the proof of [KV15, Lemma 2.3]).

Lemma 2.1.28. *Let Λ be an icc group, and let $\mathscr{M}=\mathscr{L}(\Lambda)$. Consider the comultiplication map $\Delta:\mathscr{M}\to\mathscr{M}\bar{\otimes}\mathscr{M}$ given by $\Delta(v_\lambda) = v_\lambda\otimes v_\lambda$ for all $\lambda\in\Lambda$. Let $\mathscr{A},\mathscr{B}\subseteq\mathscr{M}$ be a (unital) $*$-subalgebras such that $\Delta(\mathscr{A})\subseteq\mathscr{M}\bar{\otimes}\mathscr{B}$. Then there exists a subgroup $\Sigma<\Lambda$ such that $\mathscr{A}\subseteq\mathscr{L}(\Sigma)\subseteq\mathscr{B}$. In particular, if $\mathscr{A}=\mathscr{B}$, then $\mathscr{A}=\mathscr{L}(\Sigma)$.*

Proof. Let $\Sigma=\{s\in\Lambda : v_s\in\mathscr{B}\}$. Since $\mathscr{B}$ is a unital $*$-subalgebra, Σ is a subgroup, and clearly $\mathscr{L}(\Sigma)\subseteq\mathscr{B}$. We argue that $\mathscr{A}\subseteq\mathscr{L}(\Sigma)$.

Fix $a\in\mathscr{A}$, and let $a=\sum_\lambda a_\lambda v_\lambda$ be its Fourier decomposition. Let $I=\{s\in\Lambda : a_s\neq 0\}$. Fix $s\in I$, and consider the normal linear functional ω on $\mathscr{M}$ given by $\omega(x) = \bar{a}_s\tau(xv_s^*)$. Note that $(\omega\otimes 1)(a) = |a_s|^2\otimes v_s$ Since $\Delta(\mathscr{A})\subseteq\mathscr{M}\bar{\otimes}\mathscr{B}$, we have that $(\omega\otimes 1)\Delta(\mathscr{A})\subseteq\mathbb{C}\bar{\otimes}\mathscr{B}$. Thus, $v_s\in\mathscr{B}\Rightarrow s\in\Sigma$. Since this holds for all $s\in I$, we get that $a\in\mathscr{L}(\Sigma)$, and hence we are done. □

Finally we end this section with the following elementary result.

Lemma 2.1.29. *Let $\mathscr{M}$ be a finite von Neumann algebra and let $\mathscr{N}$ be a type II_1 factor, with $\mathscr{N}\subseteq\mathscr{M}$ a unital inclusion. If there is $p\in\mathcal{P}(\mathscr{N})$ so that $p\mathscr{N}p = p\mathscr{M}p$ then $\mathscr{N}=\mathscr{M}$.*

Proof. Without loss of generality, assume $\tau(p) = \frac{1}{n}$ for some $n \in \mathbb{N}$. Since $\mathscr{N}$ is a type II_1 factor, we can find orthogonal projections $p_i \in \mathcal{P}(N)$, and unitaries $u_i \in \mathcal{U}(N)$ (for $i = 2, \cdots, n$) such that $\sum_i p_i = 1 - p$ and $u_i p_1 u_i^* = p_i$. Then clearly we get that $p_i N p_i = p_i M p_i$ for $i = 2, \cdots, n$ as well. Let $p_1 = p$, and note that $(\mathscr{N}' \cap \mathscr{M}) p_i = \mathbb{C} p_i$ for all i. Let $z \in \mathscr{N}' \cap \mathscr{M}$. Then $z = \sum_i z p_i = \sum_i c_i p_i$ where $c_i = \frac{\tau(zp_i)}{\tau(p_i)}$. Now, $\tau(zp_i) = \tau(z u_i p_1 u_i^*) = \tau(u_i^* z u_i p_1) = \tau(zp_1)$, as $z \in \mathscr{N}' \cap \mathscr{M}$, for $i = 2, \cdots, n$. Thus, $c_i = c_1$ for all i, as $\tau(p_i) = \tau(p_1)$. So, $z \in \mathbb{C}$, as $\sum_i p_i = 1$. Hence $\mathscr{M}$ is a type II_1 factor with $\mathscr{N}$ an irreducible subfactor. As $p\mathscr{N}p = p\mathscr{M}p$, we have that $\mathscr{M} \prec_{\mathscr{M}} \mathscr{N}$. Hence by [CD18, Proposition 2.3], we get that $[\mathscr{M} : \mathscr{N}] < \infty$. In this case, $1 = [p\mathscr{M}p : p\mathscr{N}p] = [\mathscr{M} : \mathscr{N}]$, which implies that $\mathscr{M} = \mathscr{N}$. □

2.1.5 Height of elements in group von Neumann algebras

The notion of height of elements in crossed products and group von Neumann algebras was introduced and developed in [Io11] and [IPV10] and was highly instrumental in many of the recent classification results in von Neumann algebras [Io11, IPV10, KV15, CU18]. Following [IPV10, Section 3] for every $x \in \mathscr{L}(G)$ we denote by $h_G(x)$ the largest Fourier coefficient of x, i.e., $h_G(x) = \max_{g \in G} |\tau(x u_g^*)|$. Moreover, for every subset $\mathscr{G} \subseteq \mathscr{L}(\Gamma)$, we denote by $h_G(\mathscr{G}) = \inf_{x \in \mathscr{G}} h_G(x)$, the height of $\mathscr{G}$ with respect to G. Using the notion of height Ioana, Popa and Vaes proved in their seminal work, [IPV10, Theorem 3.1] that whenever G, H are icc groups such that $\mathscr{L}(G) = \mathscr{L}(H)$ and $h_G(H) > 0$, then G and H are isomorphic. The following generalization of this result to embeddings was obtained by Krogager and Vaes [KV15] and will be used in an essential way to derive our main Theorem 5.1.6 in the last section.

Theorem 2.1.30 (Theorem 4.1, [KV15])**.** *Let G be a countable group and denote by $\mathscr{M} = \mathscr{L}(G)$. Let $p \in \mathcal{P}(\mathscr{M})$ be a projection and assume that $\mathscr{G} \subseteq \mathcal{U}(p\mathscr{M}p)$ is a subgroup satisfying following properties:*

1. *The unitary representation $\{Ad\ v\}_{v \in \mathscr{G}}$ on $L^2(p(\mathscr{M}p \ominus \mathbb{C}p)$ is weakly mixing;*

2. *For any $e \neq g \in \mathscr{L}(G)$ we have $\mathscr{G}$" $\nprec_{\mathscr{M}} \mathscr{L}(C_G(g))$;*

3. *We have* $h_G(\mathscr{G}) > 0$.

Then $p = 1$ *and there exists a unitary* $u \in \mathscr{L}(G)$ *such that* $u\mathscr{G}u^* \subseteq \mathbb{T}G$.

Next we highlight a new situation when it's possible to control lower bound for height of unitary elements in the context of crossed product von Neumann algebras arising from group actions by automorphisms with no non-trivial stabilizers. Our result and its proof is reminiscent of the prior powerful techniques for Bernoulli actions introduced in [IPV10, Theorem 5.1] (see also [Io11, Theorem 6.1]) and their recent counterparts for the Rips constructions [CDK19, Theorem 5.1]. The precise statement is the following

Theorem 2.1.31. *Let* G *and* H *be countable groups and let* $\sigma : G \to Aut(H)$ *be an action by automorphisms for which there exists a scalar* $c > 0$ *satisfying* $|\mathrm{Stab}_G(h)| < c$ *for all* $h \in H \setminus \{e\}$. *Consider* $\mathscr{M} = \mathscr{L}(H \rtimes_\sigma G)$ *and let* $\mathscr{A} \subseteq \mathscr{M}$ *be a diffuse von Neumann subalgebra such that* $\mathscr{A} \prec^s_{\mathscr{M}} \mathscr{L}(H)$. *For any group of unitaries* $\mathscr{G} \subseteq \mathscr{L}(G)$ *satisfying* $\mathscr{G} \subseteq \mathscr{N}_{\mathscr{M}}(\mathscr{A})$ *we have that* $h_G(\mathscr{G}) > 0$.

Proof. For ease of exposition denote by $\mathscr{N} = \mathscr{L}(H)$. Next we prove the following property

Claim 2.1.32. *For every* $x, y \in \mathscr{L}(G)$, *every finite subsets* $K, S \subset G$, *every* $a \in \mathrm{span}\mathscr{N}K$ *with* $E_{\mathscr{L}(G)}(a) = 0$ *and every* $\varepsilon > 0$ *there exists a scalar* $\kappa_{\varepsilon,K,S,a} > 0$ *such that*

$$\|P_{\mathscr{N}S}(xay)\|_2^2 \leq \kappa_{\varepsilon,K,S,a}\|y\|_2^2\|a\|_2^2 h_G^2(x) + \varepsilon\|x\|_\infty\|y\|_\infty, \tag{2.23}$$

where $P_{\mathscr{N}S}$ *denotes the orthogonal projection from* $L^2(\mathscr{M})$ *onto* $\overline{\mathrm{span}}^{\|\cdot\|_2}(\mathscr{N}S)$.

Proof of Claim 2.1.32. First fix a finite set $L \subseteq H \setminus \{e\}$ and let $b \in \mathrm{span}(LK)$. Observe that using the Fourier decomposition of $x = \sum_g x_g u_g$ and $y = \sum y_g u_g$, where $x_g = \tau(xu_{g^{-1}})$ and $y_g = \tau(yu_{g^{-1}})$, basic calculations show that

$$\begin{aligned}\|E_{\mathcal{N}}(xby)\|_2^2 &= \|\sum_{g\in G,k\in K} x_g y_{k^{-1}g^{-1}}\sigma_g(E_{\mathcal{N}}(bu_{k^{-1}}))\|_2^2\\ &= \sum_{g_1,g_2\in G,k_1,k_2\in K} x_{g_1}y_{k_1^{-1}g_1^{-1}}\overline{x_{g_2}y_{k_2^{-1}g_2^{-1}}}\langle\sigma_{g_1}(E_{\mathcal{N}}(bu_{k_1^{-1}})),\sigma_{g_2}(E_{\mathcal{N}}(bu_{k_2^{-1}}))\rangle. \qquad (2.24)\end{aligned}$$

Furthermore, using the Fourier decomposition $b=\sum_h b_h u_h$ where $b_h=\tau(bu_{h^{-1}})$ we also see that

$$\langle\sigma_{g_1}(E_{\mathcal{N}}(bu_{k_1^{-1}})),\sigma_{g_2}(E_{\mathcal{N}}(bu_{k_2^{-1}}))\rangle = \sum_{l_1,l_2\in L} b_{k_1l_1}\overline{b_{k_2l_2}}\delta_{\sigma_{g_1}(l_1),\sigma_{g_2}(l_2)} = \sum_{l_1,l_2\in L,g_2^{-1}g_1\in S_{l_1,l_2}} b_{k_1l_1}\overline{b_{k_2l_2}}, \qquad (2.25)$$

where for every $l_1,l_2\in L$ we have denoted by $S_{l_1,l_2}=\{g\in G : \sigma_g(l_1)=l_2\}$.

Thus, combining (2.24) and (2.25) and using basic inequalities together with $|S_{l_1,l_2}|\leq c$ we get that

$$\begin{aligned}\|E_{\mathcal{N}}(xby)\|_2^2 &\leq \sum_{k_1,k_2\in K;l_1,l_2\in L,g_1,g_2\in G,g_2^{-1}g_1\in S_{l_1,l_2}} \left|x_{g_1}y_{k_1^{-1}g_1^{-1}}b_{k_1l_1}\overline{x_{g_2}y_{k_2^{-1}g_2^{-1}}b_{k_2l_2}}\right|\\ &\leq \sum_{k_1,k_2\in K,l_1,l_2\in L,s\in S_{l_1,l_2},g\in G,} \left|x_{gs}y_{k_1^{-1}s^{-1}g^{-1}}b_{k_1l_1}\overline{x_g y_{k_2^{-1}g}b_{k_2l_2}}\right|\\ &\leq (\max_{l_1,l_2\in L}|S_{l_1,l_2}|)|K|^2|L|^2h_G^2(x)\|y\|_2^2\|b\|_2^2 \leq c|K|^2|L|^2h_G^2(x)\|y\|_2^2\|b\|_2^2. \qquad (2.26)\end{aligned}$$

Using these estimates we are now ready to derive the proof of (2.23). To this end fix $\varepsilon>0$. Using basic approximations and $\|E_{\mathscr{L}(G)}(a)\|=0$ one can find a finite set $L\subset H\setminus\{e\}$ and $b\in \text{span}(LK)$ such that

$$\|a-b\|_2\leq \min\{\frac{\varepsilon}{2},\|a\|_2\} \text{ and } \|b\|_\infty\leq 2\|a\|_\infty. \qquad (2.27)$$

Notice that for all $z\in\mathscr{M}$ we have $P_{\mathcal{N}S}(z)=\sum E_{\mathcal{N}}(zu_{s^{-1}})u_s$ and using this formula together with estimate (2.27) and Cauchy-Schwarz inequality we get

$$\|P_{\mathcal{N}S}(xay)\|_2^2 \leq 2|S|\left(\sum_{s\in S}\|E_{\mathcal{N}}(xbyu_{s^{-1}}\|_2^2\right) + \varepsilon|x\|_\infty\|y\|_\infty.$$

Using (2.26) followed by (2.27) we further have that the last inequality above is smaller than

$$\begin{aligned} &\leq 2c|S||K|^2|L|^2(\sum_{s\in S}h_G^2(x)\|yu_{s^{-1}}\|_2^2\|b\|_2^2) + \varepsilon|x\|_\infty\|y\|_\infty \\ &\leq 4c|S|^2|K|^2|L|^2h_G^2(x)\|a\|_2^2\|y\|_2^2 + \varepsilon|x\|_\infty\|y\|_\infty. \qquad (2.28) \end{aligned}$$

Combining this with (2.27) proves the claim where $\kappa_{\varepsilon,K,S,a} = 4c|S|^2|K|^2|L|^2$. ■

In the remaining part we complete the proof of the statement. Towards this first notice that, since $\mathscr{A} \prec^s_{\mathscr{M}} \mathscr{N}$ then by [Va10a, Lemma 2.5] for every ε there exists a finite set $S \subseteq K$ such that for all $c \in \mathcal{U}(\mathscr{A})$ we have

$$\|c - P_{\mathcal{N}S}(c)\|_2 \leq \varepsilon. \qquad (2.29)$$

Next we also claim that for every finite set $S \subset G$ and every $\varepsilon > 0$ there exists $b \in \mathcal{U}(\mathscr{A})$ such that

$$\|E_{\mathscr{L}(G)} \circ P_{\mathcal{N}S}(b)\|_2 < \varepsilon. \qquad (2.30)$$

Indeed, to see this first notice that $\|E_{\mathscr{L}(G)} \circ P_{\mathcal{N}S}(b)\|_2^2 = \sum_{s\in S}|\tau(bu_{s^{-1}})|^2$. As $\mathscr{A}$ is diffuse and S is finite there exists $b \in \mathcal{U}(\mathscr{A})$ such that $\sum_{s\in S}|\tau(bu_{s^{-1}})|^2 < \varepsilon$ and the claim follows.

Now pick $b \in \mathcal{U}(\mathscr{A})$ satisfying (2.30). Since $a, gsg^{-1} \in \mathcal{U}(\mathscr{A})$ then using (2.29) two times and

(2.30) we see that

$$
\begin{aligned}
1-\varepsilon = \|gag^{-1}\|_2-\varepsilon &\leq \|P_{\mathcal{N}S}(gag^{-1}\|_2 \leq \|P_{\mathcal{N}S}(g(P_{\mathcal{N}S}(b))g^{-1})\|_2+\varepsilon \\
&\leq \|P_{\mathcal{N}S}(g(P_{\mathcal{N}S}(b)-E_{\mathcal{L}(G)}(P_{\mathcal{N}S}(b)))g^{-1})\|_2+\|E_{\mathcal{L}(G)}(P_{\mathcal{N}S}(b))\|_2+\varepsilon \\
&\leq \|P_{\mathcal{N}S}(g(P_{\mathcal{N}S}(b)-E_{\mathcal{L}(G)}(P_{\mathcal{N}S}(b)))g^{-1})\|_2+2\varepsilon. \qquad (2.31)
\end{aligned}
$$

Now, taking $a=P_{\mathcal{N}S}(b)-E_{\mathcal{L}(G)}(P_{\mathcal{N}S}(b))$ and using (2.23) we get that the last inequality above is smaller than

$$\leq \kappa_{\varepsilon,S,S,b}h_G(g)\|P_{\mathcal{N}S}(b)-E_{\mathcal{L}(G)}(P_{\mathcal{N}S}(b))\|_2+\varepsilon^{1/2}+2\varepsilon. \qquad (2.32)$$

Thus (2.31) and (2.32) further imply that $h_G(g)\geq \kappa^{-1}_{\varepsilon,S,S,b}(1-3\varepsilon-\varepsilon^{1/2})$. Since this holds for all $g\in\mathcal{G}$, letting $\varepsilon>0$ be sufficiently small we get the desired conclusion. $\square$

Chapter 3

Two Classes of Groups

3.1 A class of groups based on Belegradek-Osin Rips construction

Using the powerful technology from [Os06], Belegradek and Osin showed in [BO06, Theorem 1.1] that for every finitely generated group Q one can find a property (T) group N such that Q embeds into $\text{Out}(N)$ as a finite index subgroup. This canonically gives rise to an action $Q \curvearrowright^{\rho} N$ by automorphisms such that the corresponding semidirect product group $N \rtimes_{\rho} Q$ is hyperbolic relative to $\{Q\}$. Throughout this document the semidirect products $N \rtimes_{\rho} Q$ will be termed Belegradek-Osin's Rips construction groups. When Q is torsion free then one can pick N to be torsion free as well and hence both N and $N \rtimes_{\rho} Q$ are icc groups. Notice that the finite conjugacy radical $FC(N)$ of N is invariant under the action of Q and hence $FC(N)$ is an amenable normal subgroup G. Since G is relative hyperbolic it follows that $FC(N)$ is finite and hence it is trivial as G is torsion free; in particular N is an icc group. Also when Q has property (T) then $N \rtimes_{\rho} Q$ has property (T). Under all these assumptions we will denote by $\mathscr{R}ips_{\mathscr{T}}(Q)$ the class of these Rips construction groups $N \rtimes_{\rho} Q$.

In [CDK19, Sections 3,5] we introduced a class of property (T) groups based on the Belegradek-Osin Rips construction groups and we have proved several rigidity results for the corresponding von Neumann algebras, [CDK19, Theorem A]. Next we briefly recall this construction also highlighting its main algebraic properties that are relevant in the proofs of our main results in the next section.

Class $\mathcal{S}$. Consider any product group $Q = Q_1 \times Q_2$, where Q_i are any nontrivial, bi-exact, weakly amenable, property (T), residually finite, torsion free, icc groups. Then for every $i = 1,2$ consider a Rips construction $G_i = N_i \rtimes_{\rho_i} Q \in \mathscr{R}ips_{\mathscr{T}}(Q)$, let $N = N_1 \times N_2$ and denote by $G = N \rtimes_{\sigma} Q$ the canonical semidirect product which arises from the diagonal action $\sigma = \rho_1 \times \rho_2 : Q \to \text{Aut}(N)$, i.e. $\sigma_g(n_1,n_2) = ((\rho_1)_g(n_1),(\rho_2)_g(n_2))$ for all $(n_1,n_2) \in N$. Throughout this article the category of all these semidirect products G will denoted by **Class** $\mathcal{S}$.

Concrete examples of semidirect product groups in class $\mathcal{S}$ can be obtained if the initial groups Q_i are any uniform lattices in $Sp(n,1)$ when $n \geq 2$. Indeed one can see that required conditions on Q_i's follow from [Oz03, CH89].

For further reference we record some algebraic properties of groups in class $\mathcal{S}$. For their proofs the reader may consult [CDK19, Sections 3,4,5] and the references within.

Theorem 3.1.1. *For any $G = N \rtimes_\sigma Q \in \mathcal{S}$ the following hold*

a) G is an icc, torsion free, property (T) group;

b) Q is malnormal subgroup of G, i.e. $gQg^{-1} \cap Q = \{e\}$ for every $g \in G \setminus Q$;

c) The stabilizer $\mathrm{Stab}_Q(n) = \{e\}$ for every $n \in N \setminus \{e\}$;

d) The virtual centralizer satisfies $vC_G(N) = 1$;

e) G is the fiber product $G = G_1 \times_Q G_2$; thus embeds into $G_1 \times G_2$ where Q embeds diagonally into $Q \times Q$.

Finally we conclude this section with a folklore lemma related to the calculation of centralizers of elements in products of hyperbolic groups. We include some details for readers' convenience.

Lemma 3.1.2. *Let $Q = Q_1 \times Q_2$, where Q_is are non-elementary torsion free, hyperbolic groups. For any $e \neq g \in Q$ the centralizer $C_Q(g)$ is of one of the following forms: A, $A \times Q_2$ or $Q_1 \times A$, where A is an amenable group.*

Proof. Let $g = (g_1, g_2) \in Q$ where $g_i \in Q_i$ and notice that $C_Q(g) = C_{Q_1}(Q_1) \times C_{Q_2}(g_2)$. Therefore to get our conclusion it suffices to show that for every $g_i \in Q_i$ either $C_{Q_i}(g_i) = Q_i$ or $C_{Q_i}(g_i)$ is an elementary group. However this is immediate once we note that for every $g_i \neq e$ the centralizer satisfies $C_{Q_i}(g_i) \leqslant E_{Q_i}(g_i)$, where $E_{Q_i}(g_i)$ is maximal elementary subgroup containing g_i of the torsion free icc hyperbolic group Q_i, see for example [Ol91]. □

3.2 A class of groups $\mathcal{V}$ arising from Valette's examples

We describe a construction of group pairs with property (T) due to Valette [Va04]. Denote by $\mathbb{H}$ the division algebra of quaternions and by $\mathbb{H}_{\mathbb{Z}}$ its lattice of integer points. Let $n \geq 2$. Recall that $\mathrm{Sp}(n,1)$ is the rank one connected simple real Lie group defined by

$$\mathrm{Sp}(n,1) = \{A \in \mathrm{GL}_{n+1}(\mathbb{H}) \ | A^*JA = J\}$$

where $J = \mathrm{Diag}(1,\dots,1,-1)$. Since the subgroup $\mathrm{Sp}(n,1)$ is the set of real points of an algebraic $\mathbb{Q}$-group, the group of integer points $\Lambda_n = \mathrm{Sp}(n,1)_{\mathbb{Z}}$ is a lattice in $\mathrm{Sp}(n,1)$ by Borel–Harish-Chandra's result [BHC61]. Observe that $\mathrm{Sp}(n,1)$ acts linearly on $\mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)}$ in such a way that Λ_n preserves $(\mathbb{H}_{\mathbb{Z}})^{n+1} \cong \mathbb{Z}^{4(n+1)}$. For every $n \geq 2$, set $\Gamma_n = \mathbb{Z}^{4(n+1)} \rtimes \Lambda_n$. Throughout this document, we denote the class of these groups by $\mathcal{V}$. For $\Gamma_n \in \mathcal{V}$, we denote by $\mathscr{M}_n = \mathscr{L}(\Gamma_n)$, and by $\mathscr{A}_n = \mathscr{L}(\mathbb{Z}^{4(n+1)})$. Note that $\mathscr{M}_n = \mathscr{A}_n \rtimes \Lambda_n$.

For further use we record some algebraic properties of groups in $\mathcal{V}$, and the von Neumann algebras $\mathscr{M}_n$.

Theorem 3.2.1. *Let $\Gamma_n \in \mathcal{V}$. Then the following hold true:*

(i) *For every $n \geq 2$, Γ_n is an infinite icc countable discrete group with property $\mathbb{T}$ so that $\mathscr{M}_n$ is a type II_1 factor with property $\mathbb{T}$.*

(ii) *For every $n \geq 2$, $\mathscr{A}_n \subseteq \mathscr{M}_n$ is the unique Cartan subalgebra, up to unitary conjugacy.*

Proof. (i) We use the notation $g = (\gamma, a) \in \mathbb{Z}^{4(n+1)} \rtimes \Lambda_n = \Gamma_n$. Since the lattice $\Lambda_n/\{\pm\mathrm{id}\}$ in the adjoint Lie group $\mathrm{Sp}(n,1)/\{\pm\mathrm{id}\}$ is icc, the exact same proof as [Va04, Theorem 4, Step 3] shows that the conjugacy class of any element of the form $g = (\gamma, a)$ in Γ_n with $\gamma \notin \{\pm\mathrm{id}\}$ is infinite. Since the $\mathbb{Z}^{4(n+1)}$-conjugacy class of any element of the form $g = (-\mathrm{id}, a)$ in Γ_n is also clearly infinite, it follows that Γ_n is an infinite icc countable discrete group. By [Va04, Proposition 1], the group pair $(\mathbb{R}^{4(n+1)} \rtimes \mathrm{Sp}(n,1), \mathbb{R}^{4(n+1)})$ has relative property (T). Since both $\mathbb{Z}^{4(n+1)} \rtimes \Lambda_n < \mathbb{R}^{4(n+1)} \rtimes \mathrm{Sp}(n,1)$ and $\mathbb{Z}^{4(n+1)} < \mathbb{R}^{4(n+1)}$ are lattices, the group pair $(\mathbb{Z}^{4(n+1)} \rtimes \Lambda_n, \mathbb{Z}^{4(n+1)})$ also has property (T). Since $\mathrm{Sp}(n,1)$ has property (T) by Kostant's result, so does its lattice $\Lambda_n < \mathrm{Sp}(n,1)$. Altogether, this implies that Γ_n has property (T). Hence $\mathscr{M}_n = \mathscr{L}(\Gamma_n)$ has property (T) by [CJ85].

(ii) We first show that $\mathscr{A}_n \subseteq \mathscr{M}_n$ is a Cartan subalgebra. Note that it suffices to show that $\mathscr{A}_n \subseteq \mathscr{M}_n$ is maximal abelian. To this end, it's enough to show that the $\mathbb{Z}^{4(n+1)}$-conjugacy class in Γ_n of any element of the form $g = (\gamma, 0)$ with $\gamma \neq \mathrm{id}$ is infinite. Indeed, if $\gamma \in \Lambda_n$ is such that the $\mathbb{Z}^{4(n+1)}$-conjugacy class of $g = (\gamma, 0)$ in Γ_n is finite, since $\mathbb{Z}^{4(n+1)}$ is torsion-free, this forces γ to act trivially on $\mathbb{Z}^{4(n+1)}$ and so necessarily $\gamma = \mathrm{id}$.

Since $L^\infty(\mathbb{T}^{4(n+1)}) = \mathscr{A}_n \subset \mathscr{M}_n = L^\infty(\mathbb{T}^{4(n+1)}) \rtimes \Lambda_n$ is a Cartan subalgebra and since $\mathscr{M}_n$ is a type II_1 factor, the probability measure-preserving action $\Lambda_n \curvearrowright \mathbb{T}^{4(n+1)}$ is essentially free and ergodic. Then [PV12, Theorem 1.1] shows that $\mathscr{A}_n \subset \mathscr{M}_n$ is the unique Cartan subalgebra, up to unitary conjugacy. □

Chapter 4

von Neumann Algebraic Rigidity of Semidirect Product

An impressive milestone in the classification of von Neumann algebras was the emergence over the past decade of the first examples of groups G that can be completely reconstructed from their von Neumann algebras $\mathscr{L}(G)$, i.e. W^**-superrigid groups* [IPV10, BV12, CI17]. The strategies used in establishing these result share a common key ingredient, namely, the ability to first reconstruct from $\mathscr{L}(G)$ various algebraic feature of G such as its (generalized) wreath product decomposition in [IPV10, BV12], and respectively, its amalgam splitting in [CI17, Theorem A]. This naturally leads to a broad and independent study, specifically identifying canonical group algebraic features of a group that pass to its von Neumann algebra. While several works have emerged recently in this direction [CdSS15, CI17, CU18] the surface has been only scratched and still a great deal of work remains to be done.

A difficult conjecture of Connes predicts that all icc property (T) groups are W^*-superrigid. Unfortunately, not a single example of such group is known at this time. Moreover, in the current literature there is an almost complete lack of examples of algebraic features occurring in a property (T) group that are recognizable at the von Neumann algebraic level. For instance, besides the preservance of the Cowling-Haagerup constant [CH89], the amenablity of normalizers of infinite amenable subgroups in hyperbolic property (T) groups from [Oz03, Theorem 1] and the product rigidity for hyperbolic property (T) groups from [CdSS15, Theorem A] very little is known. Therefore in order to successfully construct property (T) W^*-superrigid groups via a strategy similar to [IPV10, CI17] we believe it is imperative to first identify a comprehensive list of algebraic features of property (T) groups that survive the von Neumann algebraic structure. Any success in this direction will potentially hint to what group theoretic methods to pursue in order to address Connes' conjecture.

In this section we investigate a new class of property (T) groups that appears as natural fiber products of Belegradek-Osin Rips type constructions. Specifically, consider any two groups $N_1 \rtimes Q, N_2 \rtimes Q \in \mathscr{R}ips_T(Q)$ and form the canonical fiber product $G = (N_1 \times N_2) \rtimes Q$. Notice that since property (T) is closed under extensions [BdlHV00, Section 1.7] it follows that G has property (T). Then for a fairly large family of groups Q we show that the semidirect product feature of G is an algebraic property completely recoverable from the von Neumann algebraic regime. In addition, we also have a complete reconstruction of the acting group Q. The precise statement is the following

Theorem 4.0.1 (Theorem 1.0.4)**.** *Let $Q = Q_1 \times Q_2$, where Q_i are icc, torsion free, biexact, property (T), weakly amenable, residually finite groups. For $i = 1,2$ let $N_i \rtimes_{\sigma_i} Q \in \mathscr{R}ip_T(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ the semidirect product associated with the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \curvearrowright N_1 \times N_2$. Denote by $\mathscr{M} = \mathscr{L}(\Gamma)$ be the corresponding II_1 factor. Assume that Λ is any arbitrary group and $\Theta : \mathscr{L}(\Gamma) \to \mathscr{L}(\Lambda)$ is any $*$-isomorphism. Then there exist groups action by automorphisms $H \curvearrowright^{\tau_i} K_i$ such that $\Lambda = (K_1 \times K_2) \rtimes_\tau H$ where $\tau = \tau_1 \times \tau_2 : H \curvearrowright K_1 \times K_2$ is the diagonal action. Moreover one can find a multiplicative character $\eta : Q \to \mathbb{T}$, a group isomorphism $\delta : Q \to H$, a unitary $w \in \mathscr{L}(\Lambda)$, and $*$-isomorphisms $\Theta_i : \mathscr{L}(N_i) \to \mathscr{L}(K_i)$ such that for all $x_i \in L(N_i)$ and $g \in Q$ we have*

$$\Theta((x_1 \otimes x_2)u_g) = \eta(g)w((\Theta_1(x_1) \otimes \Theta(x_2))v_{\delta(g)})w^*. \tag{4.1}$$

Here $\{u_g \,|\, g \in Q\}$ and $\{v_h \,|\, h \in H\}$ are the canonical unitaries implementing the actions of $Q \curvearrowright \mathscr{L}(N_1)\bar{\otimes}\mathscr{L}(N_2)$ and $H \curvearrowright \mathscr{L}(K_1)\bar{\otimes}\mathscr{L}(K_2)$, respectively.

From a different perspective our theorem can be also seen as a von Neumann algebraic superrigidity result regarding conjugacy of actions on noncommutative von Neumann algebras. Notice that very little is known in this direction as well, as most of the known superrigidity results concern algebras arising from actions of groups on probability spaces.

We continue with a series of preliminary results that are essential to derive the proof of Theorem 4.0.1 at the end of the section. First we present a location result for commuting diffuse property

(T) subalgebras inside a von Neumann algebra arising from products of relative hyperbolic groups.

Theorem 4.0.2. *For $i = 1,...,n$ let $H_i < G_i$ be an inclusion of infinite groups such that H_i is residually finite and G_i is hyperbolic relative to H_i. Denote by $H = H_1 \times ... \times H_n < G_1 \times ... \times G_n = G$ the corresponding direct product inclusion. Let $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{L}(G)$ be two commuting von Neumann subalgebras with property (T). Then for every $k \in \overline{1,n}$ there exists $i \in \overline{1,2}$ such that $\mathcal{N}_i \prec \mathcal{L}(\hat{G}_k \times H_k)$, where $\hat{G}_k := \times_{j \neq k} G_j$.*

Proof. Our proof relies heavily on the use of two powerful results in geometric group theory due to Osin [Os06] and Dahmani-Guirardel-Osin [DGO11] regrading Dehn filling constructions for relatively hyperbolic groups. Specifically, since H_i is residually finite then using Theorem 2.27 there is a short exact sequence

$$1 \to ker(\pi_i) \hookrightarrow G_i \xrightarrow{\pi_i} F_i \to 1,$$

where F_i is a non-elementary hyperbolic group and $\ker(\pi_i) = \langle\langle H_i^0 \rangle\rangle = *_{t \in T_i} (H_i^0)^t$, for some subset $T \subset G_i$ and a finite index normal subgroup $H_i^0 \lhd H_i$.

Following [CIK13, Notation 3.3] we now consider the von Neumann algebraic embedding corresponding to π_i, i.e. $\Pi_i : \mathcal{L}(G_i) \to \mathcal{L}(G_i) \bar{\otimes} \mathcal{L}(F_i)$ given by $\Pi_i(u_g) = u_g \otimes v_{\pi_i(g)}$ for all $g \in G$; here u_g's are the canonical unitaries of $\mathcal{L}(G)$ and v_h are the canonical unitaries of $\mathcal{L}(F_i)$. Notice we canonically have the following embedding $\Pi = \bar{\otimes}_{i=1}^n \Pi_i : \mathcal{L}(G) \to \mathcal{L}(G) \bar{\otimes} \mathcal{L}(\times_{i=1}^n F_i) = \tilde{\mathcal{M}}$. From the hypothesis we have that $\Pi(\mathcal{N}_1), \Pi(\mathcal{N}_2) \subset \mathcal{L}(G) \otimes \mathcal{L}(\times F_i)$ are commuting property (T) subalgebras. Fix $\mathcal{A} \subset \Pi(\mathcal{N}_1)$ any diffuse amenable von Neumann subalgebra. Also fix $k \in \overline{1,n}$ and notice that $\tilde{\mathcal{M}} = \mathcal{L}(G) \otimes \mathcal{L}(\times_{j \neq k} F_j) \bar{\otimes} \mathcal{L}(F_k)$. Using [PV12, Theorem 1.4] we have either

a) $\mathcal{A} \prec_{\tilde{\mathcal{M}}} \mathcal{L}(G) \bar{\otimes} \mathcal{L}(\times_{j \neq k} F_j)$, or

b) $\Pi(\mathcal{N}_2)$ is amenable relative to $\mathcal{L}(G) \bar{\otimes} \mathcal{L}(\times_{j \neq k} F_j)$ inside $\tilde{\mathcal{M}}$.

Since the $\mathcal{N}_i$'s have property (T) then so do the $\Pi(\mathcal{N}_i)$'s. Thus part b) above implies that $\Pi(\mathcal{N}_2) \prec_{\tilde{\mathcal{M}}} \mathcal{L}(G) \bar{\otimes} \mathcal{L}(\times_{j \neq k} F_j)$. On the other hand, if case a) above were to hold for all $\mathcal{A}$'s then by [BO08, Corollary F.14] we would get that $\Pi(\mathscr{N}_1) \prec_{\tilde{\mathscr{M}}} \mathscr{L}(G)\bar{\otimes}\mathscr{L}(\times_{j\neq k}F_j)$. Therefore we can always assume that $\Pi(\mathscr{N}_i) \prec \mathscr{L}(G)\bar{\otimes}\mathscr{L}(\times_{j\neq k}F_j)$ for $i=1$ or 2.

Due to symmetry we only treat $i=1$. Using [CK15, Proposition 8.8] we get that $\mathscr{N}_1 \prec \mathscr{L}(\Pi^{-1}(\times_{j\neq k}F_j)) = \mathscr{L}(\hat{G}_k \times ker(\pi_k))$. Thus there exist nonzero projections $p \in \mathscr{N}_1$, $q \in \mathscr{L}(\hat{G}_k \times ker(\pi_k))$, nonzero partial isometry $v \in \mathscr{M}$ and a $*$-isomorphism $\phi : p\mathscr{N}_1 p \to \mathscr{B} := \phi(p\mathscr{N}_1 p) \subset q\mathscr{L}(\hat{G}_k \times \ker(\pi_k))q$ on the image such that

$$\phi(x)v = vx \text{ for all } x \in p\mathscr{N}_1 p. \tag{4.2}$$

Also notice that since $\mathscr{N}_1$ has property (T) then so does $p\mathscr{N}_1 p$ and therefore $\mathscr{B} \subseteq q\mathscr{L}(\hat{G}_k \times \ker(\pi_k))q$ is a property (T) subalgebra. Since $ker(\pi_k) = *_{t\in T}(H_k^0)^t$ then by further conjugating q in the factor $\mathscr{L}(\hat{G}_k \times \ker(\pi_k))$ we can assume that there exists a unitary $u \in \mathscr{L}(\hat{G}_k \times \ker(\pi_k))$ and a projection $q_0 \in \mathscr{L}(\hat{G}_k)$ such that $\mathscr{B} \subseteq u(q_0\mathscr{L}(\hat{G}_k)q_0)\bar{\otimes}\mathscr{L}(\ker(\pi_k))u^*$. Using property (T) of $\mathscr{B}$ and [IPP05, Theorem] we further conclude that there is $t_0 \in T$ such that $\mathscr{B} \prec_{u(q_0\mathscr{L}(\hat{G}_k)q_0\bar{\otimes}\mathscr{L}(\ker(\pi_k)))u^*} u(q_0\mathscr{L}(\hat{G}_k)q_0 \otimes \mathscr{L}((H_k^0)^{t_0}))u^*$. Composing this intertwining with ϕ we finally conclude that $\mathscr{N}_1 \prec_{\mathscr{M}} \mathscr{L}(\hat{G}_k \times H_k^0)$, as desired. $\square$

Theorem 4.0.3. *Under the same assumptions as in Theorem 4.0.2 for every $k \in \overline{1,n}$ one of the following must hold*

1) there exists $i \in 1,2$ such that $\mathscr{N}_i \prec_{\mathscr{M}} \mathscr{L}(\hat{G}_k)$;

2) $\mathscr{N}_1 \vee \mathscr{N}_2 \prec_{\mathscr{M}} \mathscr{L}(\hat{G}_k \times H_k)$.

Proof. From Theorem 4.0.2 there exists $i \in \overline{1,2}$ such that $\mathscr{N}_i \prec \mathscr{L}(\hat{G}_k \times H_k)$. For convenience assume that $i=1$. Thus there exist nonzero projections $p \in \mathscr{N}_1$, $q \in \mathscr{L}(\hat{G}_k \times H_k)$, nonzero partial isometry $v \in \mathscr{M}$ and a $*$-isomorphism $\phi : p\mathscr{N}_1 p \to \mathscr{B} := \phi(p\mathscr{N}_1 p) \subset q\mathscr{L}(\hat{G}_k \times H_k)q$ on the image such that

$$\phi(x)v = vx \text{ for all } x \in p\mathscr{N}_1 p. \tag{4.3}$$

Notice that $q \geq vv^* \in \mathscr{B}' \cap q\mathscr{M}q$ and $p \geq v^*v \in p\mathscr{N}_i p' \cap p\mathscr{M}p$. Also we can pick v such that

$s(E_{\mathscr{L}(\hat{G}_k\times H_k)}(vv^*)) = q$. Next we assume that $\mathscr{B} \prec_{L(\hat{G}_k\times H_k)} L(\hat{G}_k)$. Thus there exist nonzero projections $p' \in \mathscr{B}$, $q' \in \mathscr{L}(\hat{G}_k)$, nonzero partial isometry $w \in q'\mathscr{L}(\hat{G}_k \times H_k)p'$ and a $*$-isomorphism $\psi : p'\mathscr{B}p' \to\subset q'\mathscr{L}(\hat{G}_k)q'$ on the image such that

$$\psi(x)w = wx \text{ for all } x \in p'\mathscr{B}p'. \tag{4.4}$$

Notice that $q \geq p' \geq ww^* \in (p'\mathscr{B}p')' \cap p'\mathscr{M}p'$ and $q' \geq w^*w \in \psi(p'\mathscr{B}p')' \cap q'\mathscr{M}q'$. Using (4.3) and (4.4) we see that

$$\psi(\phi(x))wv = w\phi(x)v = wvx \text{ for all } x \in p_0\mathscr{N}_ip_0, \tag{4.5}$$

where $p_0 \in \mathscr{N}_i$ is a projection picked so that $\phi(p_0) = p'$. Also we note that if $0 = wv$ then $0 = wvv^*$ and hence $0 = E_{\mathscr{L}\hat{G}_k\times H_k}(wvv^*) = wE_{\mathscr{L}(\hat{G}_k\times H_k)}(vv^*)$. This further implies that $0 = ws(E_{\mathscr{L}(\hat{G}_k\times H_k)}(vv^*)) = wq = w$ which is a contradiction. Thus $wv \neq 0$ and taking the polar decomposition of wv we see that (4.5) gives 1).

Next we assume that $\mathscr{B} \nprec_{\mathscr{L}(\hat{G}_k\times H_k)} \mathscr{L}(\hat{G}_k)$. Since G_k is hyperbolic relative to H_k then by Lemma 2.1.20 we have that for all $x, x_1x_2, \ldots, x_l \in M$ such that $\mathscr{B}x \subseteq \sum_{i=1}^{l} x_i\mathscr{B}$ we must have that $x \in \mathscr{L}(\hat{G}_k \times H_k)$. Hence in particular we have that $vv^* \in \mathscr{B}' \cap q\mathscr{M}q \subseteq \mathscr{L}(\hat{G}_k \times H_k)$ and thus relation (4.3) implies that $\mathscr{B}vv^* = v\mathscr{N}_iv^* \subseteq \mathscr{L}(\hat{G}_k \times H_k)$. Also for every $c \in \mathscr{N}_{i+1}$ we can see that

$$\begin{aligned} \mathscr{B}vcv^* &= \mathscr{B}vv^*vcv^* = v\mathscr{N}_iv^*vcv^* = vv^*vc\mathscr{N}_iv^* \\ &= vc\mathscr{N}_iv^* = vc\mathscr{N}_iv^*vv^* = vcv^*v\mathscr{N}_iv^* = vcv^*\mathscr{B}vv^* = vcv^*\mathscr{B}. \end{aligned} \tag{4.6}$$

Therefore by Lemma 2.1.20 again we have that $vcv^* \in \mathscr{L}(\hat{G}_k \times H_k)$ and hence $v\mathscr{N}_{i+1}v^* \subseteq \mathscr{L}(\hat{G}_k \times H_k)$. Thus $v\mathscr{N}_i\mathscr{N}_{i+1}v^* = vv^*v\mathscr{N}_i\mathscr{N}_{i+1}v^* = v\mathscr{N}_iv^*v\mathscr{N}_{i+1}v^* \subseteq \mathscr{L}(\hat{G}_k \times H_k)$, which by Popa's intertwining techniques implies that $\mathscr{N}_1 \vee \mathscr{N}_2 \prec \mathscr{L}(\hat{G}_k \times H_k)$, i.e. 2) holds. □

We now proceed towards proving the main result of this chapter. To simplify the exposition we first introduce a notation that will be used throughout the section.

Notation 4.0.4. *Denote by $Q = Q_1 \times Q_2$, where Q_i are infinite, residually finite, biexact, property (T), icc groups. Then consider $\Gamma_i = N_i \rtimes Q \in \mathscr{R}ip_T(Q)$ and consider the semidirect product $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ arising from the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \to Aut(N_1 \times N_2)$, i.e. $\sigma_g(n_1, n_2) = ((\sigma_1)_g(n_1), (\sigma_2)_g(n_2))$ for all $(n_1, n_2) \in N_1 \times N_2$. For further use we observe that Γ is the fiber product $\Gamma = \Gamma_1 \times_Q \Gamma_2$ and thus embeds into $\Gamma_1 \times \Gamma_2$ where Q embeds diagonally into $Q \times Q$. Over the next proofs when we refer to this copy we will often denote it by $diag(Q)$. Also notice that Γ is an icc group with property (T) as it arises from an extension of property (T) groups.*

Theorem 4.0.5. *Let Γ be a group as in Notation 4.0.4 and assume that Λ is a group such that $\mathscr{L}(\Gamma) = \mathscr{L}(\Lambda) = \mathscr{M}$. Let $\Delta : \mathscr{M} \to \mathscr{M} \bar{\otimes} \mathscr{M}$ be the "comultiplication along Λ" i.e. $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$. Then the following hold:*

3) for all $j \in 1,2$ there is $i \in 1,2$ such that $\Delta(\mathscr{L}(N_i)) \prec_{\mathscr{M} \bar{\otimes} \mathscr{M}} \mathscr{M} \bar{\otimes} \mathscr{L}(N_j)$, and

4) a) for all $j \in 1,2$ there is $i \in 1,2$ such that $\Delta(\mathscr{L}(Q_i)) \prec_{\mathscr{M} \bar{\otimes} \mathscr{M}} \mathscr{M} \bar{\otimes} \mathscr{L}(N_j)$ or

b) $\Delta(\mathscr{L}(Q)) \prec_{\mathscr{M} \bar{\otimes} \mathscr{M}} \mathscr{M} \bar{\otimes} \mathscr{L}(Q)$; moreover in this case for every $j \in 1,2$ there is $i \in 1,2$ such that $\Delta(\mathscr{L}(Q_j)) \prec_{\mathscr{M} \bar{\otimes} \mathscr{M}} \mathscr{M} \bar{\otimes} \mathscr{L}(Q_i)$

Proof. Let $\tilde{\mathscr{M}} = \mathscr{L}(\Gamma_1 \times \Gamma_2)$. Since $\Gamma < \Gamma_1 \times \Gamma_2$ we notice the following inclusions $\Delta(\mathscr{L}(N_1)), \Delta(\mathscr{L}(N_2)) \subset \mathscr{M} \bar{\otimes} \mathscr{M} = \mathscr{L}(\Gamma \times \Gamma) \subset \mathscr{L}(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2)$. Since Γ_i is hyperbolic relative to Q then using Theorem 4.0.3 we have either

5) there exists $i \in 1,2$ such that $\Delta(\mathscr{L}(N_i)) \prec_{\tilde{\mathscr{M}} \bar{\otimes} \tilde{\mathscr{M}}} \mathscr{M} \bar{\otimes} \mathscr{L}(\Gamma_1)$, or

6) $\Delta(\mathscr{L}(N_1 \times N_2)) \prec_{\tilde{\mathscr{M}} \bar{\otimes} \tilde{\mathscr{M}}} \mathscr{M} \bar{\otimes} \mathscr{L}(\Gamma_1 \times Q)$

Assume 5). Since $\Delta(\mathscr{L}(N_i)) \subset \mathscr{M} \bar{\otimes} \mathscr{L}(\Gamma)$ then by Lemma 2.1.23 there is a $h \in \Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2$ so that $\Delta(\mathscr{L}(N_i)) \prec_{\tilde{\mathscr{M}} \bar{\otimes} \tilde{\mathscr{M}}} \mathscr{L}(\Gamma \times \Gamma \cap h(\Gamma_1 \times \Gamma_2 \times \Gamma_1)h^{-1})) = \mathscr{L}(\Gamma \times (\Gamma \cap \Gamma_1)) = \mathscr{M} \bar{\otimes} \mathscr{L}((N_1 \times N_2) \rtimes diag(Q)) \cap (N_1 \rtimes Q \times 1)) = \mathscr{M} \bar{\otimes} \mathscr{L}(N_1)$. Note that since $\Delta(\mathscr{L}(N_i))$ is regular in $\mathscr{M} \bar{\otimes} \mathscr{M}$, using Lemma 2.1.24, we get that $\Delta(\mathscr{L}(N_i)) \prec_{\mathscr{M} \bar{\otimes} \mathscr{M}} \mathscr{M} \bar{\otimes} \mathscr{L}(\Gamma_1)$, thereby establishing 3).

Assume 6). Since $\Delta(\mathscr{L}(N_1 \times N_2)) \subset \mathscr{L}(\Gamma \times \Gamma)$ then by Lemma 2.1.23 there is $h \in \Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2$ such that $\Delta(\mathscr{L}(N_1 \times N_2)) \prec \mathscr{L}(\Gamma \times \Gamma \cap h(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times Q)h^{-1})) = \mathscr{L}(\Gamma \times (\Gamma \cap$

$(\Gamma_1 \times h_4 Q h_4^{-1}))) = \mathscr{M}\bar{\otimes}\mathscr{L}((N_1 \times N_2) \rtimes diag(Q)) \cap (N_1 \rtimes Q \times h_4 Q h_4^{-1}))$. Since $h_4 \in \Gamma_2 = N_2 \rtimes Q$ we can assume that $h_4 \in N_2$. Notice that $((N_1 \times N_2) \rtimes diag(Q)) \cap (N_1 \rtimes Q \times h_4 Q h_4^{-1}) = h_4(N_1 \times N_2) \rtimes diag(Q)) \cap (N_1 \rtimes Q \times Q) h_4^{-1} = h_4(N_1 \times 1) \rtimes diag(Q)) h_4^{-1}$ and hence $\Delta(\mathscr{L}(N_1 \times N_2)) \prec_{\tilde{\mathscr{M}}\bar{\otimes}\tilde{\mathscr{M}}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q))$. Moreover using Lemma 2.1.25 we further have that $\Delta(\mathscr{L}(N_1 \times N_2)) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q))$.

In conclusion, there exist a $*$-isomorphism on its image $\phi : p\Delta(\mathscr{L}(N_1 \times N_2))p \to \mathscr{B} := \phi(p\Delta(\mathscr{L}(N_1 \times N_2))p) \subseteq q\mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q))$ and $0 \neq v \in q\mathscr{M}\bar{\otimes}\mathscr{M}p$ such that

$$\phi(x)v = vx \text{ for all } x \in p\Delta(\mathscr{L}(N_1 \times N_2))p. \tag{4.7}$$

Next assume that 3) doesn't hold. Thus proceeding as in the first part of the proof of Theorem 4.0.3, we get

$$\mathscr{B} \nprec_{\mathscr{M}\bar{\otimes}(N_1 \rtimes diag(Q))} \mathscr{M}\bar{\otimes}\mathscr{L}(N_1) =: \mathscr{M}_1. \tag{4.8}$$

Next we observe the following inclusions

$$\begin{aligned} &\mathscr{M}_1 \rtimes_{1\otimes\sigma} diag(Q) = \mathscr{M}\bar{\otimes}\mathscr{L}(N_1) \rtimes_{1\otimes\sigma} diag(Q) = \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes_\sigma diag(Q)) \\ &\subset \mathscr{M}\bar{\otimes}\mathscr{L}((N_1 \times N_2) \rtimes_\sigma diag(Q)) = \mathscr{M}\bar{\otimes}\mathscr{L}(N_1)\bar{\otimes}\mathscr{L}(N_2) \rtimes diag(Q) = \mathscr{M}_1 \rtimes_{1\otimes\sigma} N_2 \rtimes diag(Q) \end{aligned} \tag{4.9}$$

Also since Q is malnormal in $N_2 \rtimes Q$ it follows from Lemma 2.1.20 that $vv^* \in \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q))$ and hence $\mathscr{B}vv^* \subset \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q))$. Pick $u \in \mathscr{QN}_{p(\mathscr{M}\bar{\otimes}\mathscr{M})p}(p\Delta(\mathscr{L}(N_1 \times N_2))p)$ and using (4.7) we see that there exist $n_1, n_2, ..., n_s \in p(\mathscr{M}\bar{\otimes}\mathscr{M})p$ satisfying

$$
\begin{aligned}
\mathscr{B}vuv^* &= \mathscr{B}vv^*vuv^* = vp(\Delta(\mathscr{L}(N_1\times N_2)))pv^*vnv^* = vp(\Delta(\mathscr{L}(N_1\times N_2)))pnv^*\\
&\subseteq \sum_{i=1}^{s} vn_ip(\Delta(\mathscr{L}(N_1\times N_2)))pv^* = \sum_{i=1}^{s} vn_ip(\Delta(\mathscr{L}(N_1\times N_2)))pv^*vv^* \\
&= \sum_{i=1}^{s} vn_ipv^*v(\Delta(\mathscr{L}(N_1\times N_2)))pv^* = \sum_{i=1}^{s} vn_ipv^*\mathscr{B}vv^* = \sum_{i=1}^{s} vn_ipv^*\mathscr{B}.
\end{aligned}
\tag{4.10}
$$

Then by Lemma 2.1.20 again we must have that $vuv^* \in \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q))$. Hence we have shown that

$$
v\mathscr{QN}_{p(\mathscr{M}\bar{\otimes}\mathscr{M})p}(p\Delta(\mathscr{L}(N_1\times N_2))p)v^* \subseteq \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q)). \tag{4.11}
$$

Since $v^*v \in p\Delta(\mathscr{L}(N_1\times N_2))p' \cap p\mathscr{M}\bar{\otimes}\mathscr{M}p \subset \mathscr{QN}_{p(\mathscr{M}\bar{\otimes}\mathscr{M})p}(p(\Delta(\mathscr{L}(N_1\times N_2)))p$ then (4.11) further implies that

$$
v\mathscr{QN}_{p(\mathscr{M}\bar{\otimes}\mathscr{M})p}(p(\Delta(\mathscr{L}(N_1\times N_2)))p)''v^* \subseteq \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q)). \tag{4.12}
$$

Here for every inclusion of von Neumann algebras $\mathscr{R}\subseteq\mathscr{T}$ and projection $p\in\mathscr{R}$ we used the formula $\mathscr{QN}_{p\mathscr{T}p}(p\mathscr{R}p)'' = p\mathscr{QN}_{\mathscr{T}}(\mathscr{R})''p$ [Po03, Lemma 3.5]. As $vp\Delta(\mathscr{M})pv^* \subseteq v\mathscr{QN}_{p(\mathscr{M}\bar{\otimes}\mathscr{M})p}(p(\Delta(\mathscr{L}(N_1\times N_2)))p)''v^*$ we conclude that $\Delta(\mathscr{M}) \prec \mathscr{L}(N_1 \rtimes Q)$ which contradicts the fact that N_2 is infinite. Thus 3) must always hold.

Next we derive 4). Again we notice that $\Delta(\mathscr{L}(Q_1)), \Delta(\mathscr{L}(Q_2)) \subset \Delta(\mathscr{M}) \subset \mathscr{M}\bar{\otimes}\mathscr{M} = \mathscr{L}(\Gamma\times\Gamma) \subset \mathscr{L}(\Gamma_1\times\Gamma_2\times\Gamma_1\times\Gamma_2)$. Using Theorem 4.0.3 we must have that either

7) $\Delta(\mathscr{L}(Q_i)) \prec_{\tilde{\mathscr{M}}\bar{\otimes}\tilde{\mathscr{M}}} \mathscr{M}\bar{\otimes}\mathscr{L}(\Gamma_1)$, or

8) $\Delta(\mathscr{L}(Q)) \prec_{\tilde{\mathscr{M}}\bar{\otimes}\tilde{\mathscr{M}}} \mathscr{M}\bar{\otimes}\mathscr{L}(\Gamma_1\times Q)$.

Proceeding exactly as in the previous case, and using Lemma 2.1.24, we see that 7) implies $\Delta(\mathscr{L}(Q_i)) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_1)$ which in turn gives $4a)$. Also proceeding as in the previous case, and using Lemma 2.1.25, we see that 8) implies

$$\Delta(\mathscr{L}(diag(Q)) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q)). \tag{4.13}$$

To show the part 4b) we will exploit (4.13). Notice that there exist nonzero projections $r \in \Delta(\mathscr{L}(Q)), t \in \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q))$, nonzero partial isometry $w \in r(\mathscr{M}\bar{\otimes}\mathscr{M})t$ and $*$-isomorphism onto its image $\phi : r\Delta(\mathscr{L}(Q))r \to \mathscr{C} := \phi(r\Delta(\mathscr{L}(Q))r) \subseteq t(\mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes diag(Q)))t$ such that

$$\phi(x)w = wx \text{ for all } x \in r\Delta(\mathscr{L}(Q))r. \tag{4.14}$$

Since $\mathscr{L}(Q)$ is a factor we can assume without loss of generality that $r = \Delta(r_1 \otimes r_2)$ where $r_i \in \mathscr{L}(Q_i)$. Hence $\mathscr{C} = \phi(r\Delta(\mathscr{L}(Q))r) = \phi(\Delta(r_1\mathscr{L}(Q_i)r_2))\bar{\otimes}r_2\mathscr{L}(Q_2)r_2 =: \mathscr{C}_1 \vee \mathscr{C}_2$ where we denoted by $\mathscr{C}_i = \phi(\Delta(r_i\mathscr{L}(Q_i))r_i) \subseteq t(\mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes \text{diag}(Q)))t$. Notice that $\mathscr{C}_i$'s are commuting property (T) subfactors of $\mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \rtimes \text{diag}(Q))$. Since $N_i \rtimes Q$ is hyperbolic relative to $\{Q\}$ and seeing $\mathscr{C}_1 \vee \mathscr{C}_2 \subseteq \mathscr{M}\bar{\otimes}\mathscr{L}(N_i \rtimes \text{diag}(Q)) \subset \mathscr{L}(\Gamma_1 \times \Gamma_2 \times (N_1 \rtimes \text{diag}(Q)))$ then by applying Theorem 4.0.3 we have that there exits $i \in 1,2$ such that

9) $\mathscr{C}_1 \prec_{\tilde{\mathscr{M}}\bar{\otimes}\mathscr{L}(N_1 \rtimes \text{diag}(Q))} \mathscr{L}(\Gamma_1 \times \Gamma_2)$ or

10) $\mathscr{C}_1 \vee \mathscr{C}_2 \prec_{\tilde{\mathscr{M}}\bar{\otimes}\mathscr{L}(N_1 \rtimes \text{diag}(Q))} \mathscr{L}(\Gamma_1 \times \Gamma_2 \times \text{diag}(Q))$.

Since $\mathscr{C}_1 \subset \mathscr{M}\bar{\otimes}\mathscr{M}$ then 9) and Lemma 2.1.26 imply that $\mathscr{C}_1 \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M} \otimes 1$ which by [Io11, Lemma 9.2] further implies that $\mathscr{C}_1$ is atomic, which is a contradiction. Thus we must have 10). However since $\mathscr{C}_1 \vee \mathscr{C}_2 \subset \mathscr{M}\bar{\otimes}\mathscr{M}$ then 10) and Lemma 2.1.26 give that $\mathscr{C}_1 \vee \mathscr{C}_2 \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(\text{diag}(Q))$ and composing this intertwining with ϕ (as done in the proof of the first case in Theorem 4.0.3) we get that $\Delta(\mathscr{L}(Q)) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(\text{diag}(Q))$. Now we show the moreover part. So in particular the above intertwining shows that we can assume from the beginning that $\mathscr{C} = \mathscr{C}_1 \vee \mathscr{C}_2 \subset t(\mathscr{M}\bar{\otimes}\mathscr{L}(\text{diag}(Q)))t$. Since Q_i are biexact, weakly amenable then by applying [PV12, Theorem 1.4] we must have that either $\mathscr{C}_1 \prec \mathscr{M}\bar{\otimes}\mathscr{L}(\text{diag}(Q_1))$ or $\mathscr{C}_2 \prec \mathscr{M}\bar{\otimes}\mathscr{L}(\text{diag}(Q_1))$ or $\mathscr{C}_1 \vee \mathscr{C}_2$ is amenable relative to $\mathscr{M}\bar{\otimes}\mathscr{L}(\text{diag}(Q_1))$ inside $\mathscr{M}\bar{\otimes}\mathscr{M}$. However since $\mathscr{C}_1 \vee \mathscr{C}_2$ has property (T) the last case above still entails that $\mathscr{C}_1 \vee \mathscr{C}_2 \prec \mathscr{M}\bar{\otimes}\mathscr{L}(\mathrm{diag}(Q_1))$ which completes the proof. $\square$

Theorem 4.0.6. *Let Γ be a group as in Notation 4.0.4 and assume that Λ is a group such that $\mathscr{L}(\Gamma) = \mathscr{L}(\Lambda) = \mathscr{M}$. Let $\Delta : \mathscr{M} \to \mathscr{M}\bar{\otimes}\mathscr{M}$ be the "commultiplication along Λ" i.e. $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$. Also assume for every $j \in 1,2$ there is $i \in 1,2$ such that either $\Delta(\mathscr{L}(Q_i)) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(Q_j)$ or $\Delta(\mathscr{L}(Q_i)) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_j)$. Then one can find subgroups $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ such that*

1. *Φ_1, Φ_2 are infinite, commuting, property (T), finite-by-icc groups;*

2. *$[\Phi : \Phi_1\Phi_2] < \infty$ and $\mathscr{QN}^{(1)}_{\Lambda}(\Phi) = \Phi$;*

3. *there exist $\mu \in \mathscr{U}(\mathscr{M})$, $z \in \mathscr{P}(\mathscr{Z}(\mathscr{L}(\Phi)))$, $h = \mu z \mu^* \in \mathscr{P}(\mathscr{L}(Q))$ such that*

$$\mu \mathscr{L}(\Phi) z \mu^* = h \mathscr{L}(Q) h. \tag{4.15}$$

Proof. For the proof of this result we built upon the strategy used in the proof of [CU18, Claim 5.2]. We encourage the reader to consult this result beforehand as we will focus mainly on the new aspects of the technique. By hypothesis, using [DHI16, Theorem 4.1] (see also [Io11, Theorem 3.1] and [CdSS15, Theorem 3.3]), one can find a subgroup $\Sigma < \Lambda$ with $C_\Lambda(\Sigma)$ non-amenable such that $\mathscr{L}(Q_1) \prec_{\mathscr{M}} \mathscr{L}(\Sigma)$. Also recall that $Q < \Gamma$ is malnormal and has property (T). Let $\Omega = vC_\Lambda(\Sigma)$. Let $\{\mathscr{O}_1, ..., \mathscr{O}_k, ...\}$ be a countable enumeration of the finite orbits under conjugation by Σ, and note that $\cup_k \mathscr{O}_k = \Omega$. Finally, let $\Omega_k = \langle \mathscr{O}_1, ..., \mathscr{O}_k \rangle \leqslant \Lambda$, and note that $\Omega_k \nearrow \Omega$. Then using the same argument from [CU18, Claim 5.2] one can find nonzero projections $a \in \mathscr{L}(Q_2)$, $q \in \mathscr{L}(\Omega_k)$ a nonzero partial isometry $w \in \mathscr{L}(Q)$ a subalgebra $\mathscr{D} \subseteq \eta q \mathscr{L}(\Omega_k) q z \mu^*$ and a $*$-isomorphism $\phi : a\mathscr{L}(Q_2)a \to \mathscr{D}$ such that

4) $\mathscr{D} \vee \mathscr{D}' \cap \eta q \mathscr{L}(\Omega_k) q z \eta^* \subseteq \eta q \mathscr{L}(\Omega_k) q z \eta^*$ is finite index and

5) $\phi(x)w = wx$ for all $x \in a\mathscr{L}(Q_2)a$.

Let $r=\eta qz\eta^*$, $ww^*\in\mathscr{D}'\cap r\mathscr{L}(Q)r$, $w^*w\in a\mathscr{L}(Q_2)a'\cap a\mathscr{L}(Q)a=\mathscr{L}(Q_1)\otimes\mathbb{C}a$. Thus exist $b\in\mathscr{L}(Q_1)$ projection st $w^*w=b\otimes a$. Pick $c\in U(\mathscr{L}(Q))$such that $w=c(b\otimes a)$ then (4) gives that

$$\mathscr{D}ww^*=w\mathscr{L}(Q_2)w^*=c(\mathbb{C}b\otimes aL(Q_2)a)c^*. \tag{4.16}$$

Moreover, the same argument from the proof of [CU18, Claim 5.2] shows that we can assume the following is a finite inclusion of II_1 factors

$$\mathscr{D}\subseteq\eta q\mathscr{L}(\Omega_k)qz\eta^* \tag{4.17}$$

Thus if we denote by $\Xi=\mathscr{QN}_\Lambda(\Omega_k)$ using (4.16) and (4.17) above we see that

$$c(b\otimes a)L(Q)(b\otimes a)c^*=ww^*\eta qz\mathscr{QN}_{\mathscr{L}(\Lambda)}(\mathscr{L}(\Omega_k))''qz\eta^*ww^*=ww^*\eta qzL(\Xi)qz\eta^*ww^* \tag{4.18}$$

and also

$$\begin{aligned}c(b\mathscr{L}(Q_1)b\otimes\mathbb{C}a)c^*&=(c(\mathbb{C}b\otimes a\mathscr{L}(Q_2)a)c^*)'\cap c(b\otimes a)\mathscr{L}(Q)(b\otimes a)c^*\\&=(\mathscr{D}ww^*)'\cap ww^*\eta qz\mathscr{L}(\Xi)qz\eta^*ww^*=ww^*(\mathscr{D}'\cap\eta qz\mathscr{L}(\Xi)qz\eta^*)ww^*.\end{aligned} \tag{4.19}$$

Also, using (4.17) and [Po02, Lemma 3.1] we have that

$$\mathscr{D}\vee(\eta qz\mathscr{L}(\Omega_k)zq\eta^*)'\cap\eta qz\mathscr{L}(\Xi)zq\eta^*\subseteq^f\mathscr{D}\vee\mathscr{D}'\cap\eta qz\mathscr{L}(\Xi)zq\eta^*\subseteq\eta qz\mathscr{L}(\Xi)zq\eta^*, \tag{4.20}$$

where the symbol $\subseteq^f$ above means inclusion of finite index.

Relation (4.16) also shows that

$$
\begin{aligned}
\mathscr{D}\vee(\eta qz\mathscr{L}(\Omega_k)zq\eta^*)'\cap\eta qz\mathscr{L}(\Xi)zq\eta^* &\subseteq^f \eta qz\mathscr{L}(\Omega_k)zq\eta^*\vee(\eta qz\mathscr{L}(\Omega_k)zq\eta^*)'\cap\eta qz\mathscr{L}(\Xi)zq\eta^*\\
&\subseteq \eta qz\mathscr{L}(\Omega_k vC_\Lambda(\Omega_k))zq\eta^* \quad (4.21)\\
&\subseteq \eta qz\mathscr{L}(\Xi)zq\eta^* \quad (4.22)
\end{aligned}
$$

Here $vC_\Lambda(\Omega_k)=\{\lambda\in\Lambda : |\lambda^{\Omega_k}|<\infty\}$ is the virtual centralizer of Ω_k in Λ.

If we let $\Phi=\mathscr{QN}_\Lambda^{(1)}(\Xi)$ then the same argument as in [CU18, Claim 5.2] shows that $\Xi\leqslant\Phi$ has finite index.

Combining (4.19), (4.16) (4.18) notice that

$$
ww^*(\mathscr{D}\vee\mathscr{D}'\cap\eta qz\mathscr{L}(\Xi)zq\eta^*)ww^*=ww^*\eta qz\mathscr{L}(\Xi)zq\eta^*ww^*=ww^*\eta qz\mathscr{L}(\Phi)zq\eta^*ww^* \quad (4.23)
$$

In particular (4.23) shows that $\eta qz\mathscr{L}(\Xi)zq\eta^*\prec_{\eta qz\mathscr{L}(\Xi)zq\eta^*}\mathscr{D}\vee\mathscr{D}'\cap\eta qz\mathscr{L}(\Xi)zq\eta^*$ and using the finite index condition in (4.20) we get that $\eta qz\mathscr{L}(\Xi)zq\eta^*\prec_{\eta qz\mathscr{L}(\Xi)zq\eta^*}\mathscr{D}\vee(\eta qz\mathscr{L}(\Omega_k)zq\eta^*)'\cap \eta qz\mathscr{L}(\Xi)zq\eta^*$. Thus, by (4.21) we further get that $\eta qz\mathscr{L}(\Xi)zq\eta^*\prec_{\eta qz\mathscr{L}(\Xi)zq\eta^*}\eta qz\mathscr{L}(\Omega_k vC_\Lambda(\Omega_k))zq\eta^*$ and since $\Omega_k vC_\Lambda(\Omega_k)\leqslant\Phi$ and $[\Phi:\Xi]<\infty$ then using [CI17, Lemma 2.6] we get that $[\Phi:\Omega_k vC_\Lambda(\Omega_k)]<\infty$.

Note that (4.18) also shows that

$$
c(b\otimes a)\mathscr{L}(Q)(b\otimes a)c^*=ww^*\eta qz\mathscr{L}(\Xi)zq\eta^*ww^*=ww^*\eta qz\mathscr{L}(\Phi)zq\eta^*ww^*. \quad (4.24)
$$

As Q has property (T) then by [CI17] so is Φ (or Ξ) and hence $\Omega_k vC_\Lambda(\Omega_k)$ as well. Let $\{\mathscr{O}_n\}_n$ be an enumeration of all the orbits in Λ under conjugation by Ω_k. Denote by $\Sigma_l:=\langle\mathscr{O}_1,\dots,\mathscr{O}_l\rangle$. Clearly $\Sigma_l\leqslant\Sigma_{l+1}$ and Ω_k normalizes Σ_l for all l. Notice that $\Sigma_l\Omega_k\leqslant\Sigma_{l+1}\Omega_k$ for all l and in fact $\Sigma_l\Omega_k\nearrow\Omega_k vC_\Lambda(\Omega_k)$. Since $\Omega_k vC_\Lambda(\Omega_k)$ has property (T) there exists l_0 such that $\Sigma_{l_0}\Omega_k=\Omega_k vC_\Lambda(\Omega_k)$. In particular there exists a finite index subgroup $\Omega_k'\leqslant\Omega_k$ such that $[\Omega_k',\Sigma_{l_0}]=1$ thus

$\Omega_k', \Sigma_{l_0} \leqslant^f \Omega_k v C_\Lambda(\Omega_k) \leqslant^f \Phi$ are commuting subgroups. Moreover if $t = z(ww^*)$ is the central support of ww^* in $\eta z L(\Phi) qz\eta^*$ then by (4.24) we also have that $\mathscr{L}(Q) \supseteq \eta_0 qz\mathscr{L}(\Xi) qz\eta_0^* t$. Now since the Q_i's are biexact the same argument from [CdSS15] shows that the finite conjugacy radical of Φ is finite. Hence Φ is a finite-by-icc group and this canonically implies that $\Phi_1 := \Omega_k'$ and $\Phi_2 := \Sigma_{l_0}$ are also finite-by-icc. As Φ has property (T) then so are the Φ_i's. To this end we have shown there exist subgroups $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ satisfying the following properties:

1. Φ_1, Φ_2 are infinite, commuting, property (T), finite-by-icc groups;
2. $[\Phi : \Phi_1 \Phi_2] < \infty$ and $\mathscr{QN}_\Lambda^{(1)}(\Phi) = \Phi$;
3. there exist $\mu \in \mathscr{U}(M)$, $d \in \mathscr{P}(\mathscr{L}(\Phi))$, $h = \mu d \mu^* \in \mathscr{P}(\mathscr{L}(Q))$ such that

$$\mu d \mathscr{L}(\Phi) d \mu^* = h \mathscr{L}(Q) h. \tag{4.25}$$

In the last part of the proof we show we can actually "bump" d to its central support in $\mathscr{L}(Q)$ and all the required relations in the statement still hold. Since $\mathscr{L}(Q)$ is a factor then using (4.25) one can find $v \in \mathscr{U}(\mathscr{M})$ such that $v\mathscr{L}(\Phi) z v^* \subseteq \mathscr{L}(Q)$ where $z := z_{\mathscr{L}(Q)}(d)$ and hence $v\mathscr{L}(\Phi) z v^* \subseteq r\mathscr{L}(Q) r$, where $r = vzv^*$. Fix $e \leqslant z$ and $f \leqslant d$ projections (in the factor $\mathscr{L}(\Phi)z$) such that $\tau(f) \geqslant \tau(e)$. Notice that from (4.25) we have $\mu f \mathscr{L}(\Phi) f \mu^* = l\mathscr{L}(Q) l$ and $ve\mathscr{L}(\Phi) ev^* \subseteq r_o \mathscr{L}(Q) r_o$ where $r_o = vev^*$ and $l = \mu f \mu^*$. Let $v_o \in \mathscr{L}(Q)$ be a unitary such that $r_o \leqslant v_o l v_o^*$. Thus $ve\mathscr{L}(\Phi) ev^* \subseteq r_o \mathscr{L}(Q) r_o \subseteq v_o l \mathscr{L}(Q) l v_o^* = v_o \mu f \mathscr{L}(\Phi) f \mu^* v_o^*$ and hence

$$\mu^* v_o^* v e \mathscr{L}(\Phi) e \subseteq f \mathscr{L}(\Phi) f \mu^* v_o^* v \subset \mathscr{L}(\Phi) \mu^* v_o^* v. \tag{4.26}$$

Next let $e + p_1 + p_2 + \ldots + p_s = z$ where $p_i \in \mathscr{L}(\Phi) z$ are mutually orthogonal projection such that $e \sim_{\mathscr{L}(\Phi)z} p_i$ for all $i \in \overline{1, s-1}$ and $p_s \prec_{\mathscr{L}(\Phi)} e$. Next let u_i be unitaries in $\mathscr{L}(\Phi) z$ such that $u_i p_i u_i^* = e$ for all $i \in \overline{1, s-1}$ and $u_s p_s u_s^* = z_o' \leqslant e$. Combining this with relation (4.26) we get $\mu^* v_o^* v e \mathscr{L}(\Phi) p_i = \mu^* v_o^* v e \mathscr{L}(\Phi) u_i^* e u_i = \mu^* v_o^* v e \mathscr{L}(\Phi) e u_i \subseteq \mathscr{L}(\Phi) \mu^* v_o^* v u_i$ for all $i \in \overline{1, s-1}$. Similarly we get $\mu^* v_o^* v e \mathscr{L}(\Phi) p_s = \mu^* v_o^* v e \mathscr{L}(\Phi) u_s^* z_o' u_s = \mu^* v_o^* v e \mathscr{L}(\Phi) z_o' u_s \subseteq \mu^* v_o^* v e \mathscr{L}(\Phi) e u_s \subset$

$\mathscr{L}(\Phi)\mu^* v_o^* v u_s$. Using these relations we conclude that

$$\begin{aligned}\mu^* v_o^* v e \mathscr{L}(\Phi) = \mu^* v_o^* v e \mathscr{L}(\Phi) z = \mu^* v_o^* v e \mathscr{L}(\Phi)(e + p_1 + p_2 + \ldots + p_s) \\ \subseteq \mu^* v_o^* v e \mathscr{L}(\Phi) e + \sum_{i=1}^{s} \mu^* v_o^* v e \mathscr{L}(\Phi) p_i \\ \subseteq \mathscr{L}(\Phi)\mu^* v_o^* v + \sum_{i=1}^{s} \mathscr{L}(\Phi)\mu^* v_o^* v u_i.\end{aligned} \tag{4.27}$$

In particular this relation shows that $\mu^* v_o^* v e \in \mathscr{QN}^{(1)}_{\mathscr{L}(\Lambda)}(\mathscr{L}(\Phi))$ and since $\mathscr{QN}^{(1)}_{\mathscr{L}(\Lambda)}(\mathscr{L}(\Phi))'' = \mathscr{L}(\Phi)$ by 2. then we conclude that $\mu^* v_o^* v e \in \mathscr{L}(\Phi)$. Thus one can check that $ve\mathscr{L}(\Phi)ev^* = vev^* v_o^* \mu\mu^* v_o v e \mathscr{L}(\Phi) \ldots . = vev^* v_o^* \mu f \mathscr{L}(\Phi) f \mu^* v_o v e v^* = v z_o v^* v_o^* l \mathscr{L}(\Phi) l v_o v e v^* = r_o \mathscr{L}(Q) r_o$.

In conclusion we have proved that $v\mathscr{L}(\Phi)zv^* \subseteq r\mathscr{L}(Q)r$ and for all $e \leqslant z$ and $f \leqslant d$ projections in the factor $\mathscr{L}(\Phi)z$ such that $\tau(f) \geqslant \tau(e)$ we have $ve\mathscr{L}(\Phi)ev^* = r_o\mathscr{L}(Q)r_o$ where $r_o \leqslant r = vzv^*$. By Lemma 2.1.29 this clearly implies that $v\mathscr{L}(\Phi)zv^* = r\mathscr{L}(Q)r$ which finishes the proof. □

Lemma 4.0.7. *Let* Γ *be a group as in Notation 4.0.4 and assume that* Λ *is a group such that* $\mathscr{L}(\Gamma) = \mathscr{L}(\Lambda) = M$. *Also assume there exists a subgroup* $\Phi < \Lambda$, *a unitary* $\mu \in \mathscr{U}(\mathscr{M})$ *and projections* $z \in \mathscr{Z}(\mathscr{L}(\Phi))$, $r = \mu z \mu^* \in \mathscr{L}(Q)$ *such that*

$$\mu \mathscr{L}(\Phi) z \mu^* = r \mathscr{L}(Q) r. \tag{4.28}$$

For every $\lambda \in \Lambda \setminus \Phi$ *so that* $\left|\Phi \cap \Phi^{\lambda}\right| = \infty$ *we have* $z u_\lambda z = 0$. *In particular, there is* $\lambda_o \in \Lambda \setminus \Phi$ *so that* $\left|\Phi \cap \Phi^{\lambda_o}\right| < \infty$.

Proof. Notice that since $Q < \Gamma = (N_1 \times N_2) \rtimes Q$ is almost malnormal then we have the following property: for every sequence $\mathscr{L}(Q) \ni x_n \to 0$ weakly and every $x, y \in M$ such that $E_{\mathscr{L}(Q)}(x) = E_{\mathscr{L}(Q)}(y) = 0$ we have

$$\|E_{\mathscr{L}(Q)}(x x_k y)\|_2 \to 0, \text{ as } k \to \infty. \tag{4.29}$$

Using basic approximations and the $\mathscr{L}(Q)$-bimodularity of the expectation we see that it suffices to check (4.29) only for elements of the form $x = u_n$ and $y = u_m$ where $n, m \in (N_1 \times N_2) \setminus \{1\}$.

Consider the Fourier decomposition $x_n = \sum_{h\in Q} \tau(x_k u_{h^{-1}}) u_h$ and notice that

$$\begin{aligned} \|E_{\mathscr{L}(Q)}(xx_k y)\|_2^2 &= \|\sum_{h\in Q} \tau(x_k u_{h^{-1}})\delta_{nhm,Q} u_{nhm}\|_2^2 \\ &= \|\sum_{h\in Q} \tau(x_k u_{h^{-1}})\delta_{n\sigma_h(m)h,Q} u_{n\sigma_h(m)h}\|_2^2 = \sum_{h\in Q, \sigma_h(m)=n^{-1}} |\tau(x_k u_{h^{-1}})|^2. \end{aligned} \tag{4.30}$$

Since the action $Q \curvearrowright N_i$ has finite stabilizers one can easily see that the set $\{h \in Q : \sigma_h(m) = n^{-1}\}$ is finite and since $x_n \to 0$ weakly then $\sum_{h\in Q,\sigma_h(m)=n^{-1}} |\tau(x_k u_{h^{-1}})|^2 \to 0$ as $k \to \infty$ which concludes the proof of (4.29). Using the conditional expectation formula for compression we see that (4.29) implies that for every sequence $\mathscr{L}(Q) \ni x_n \to 0$ weakly and every $x, y \in r\mathscr{M}r$ so that $E_{r\mathscr{L}(Q)r}(x) = E_{r\mathscr{L}(Q)r}(y) = 0$ we have $\|E_{r\mathscr{L}(Q)r}(xx_k y)\|_2 \to 0$, as $k \to \infty$. Thus using the formula 4.28 we get that for all $\mu\mathscr{L}(\Phi)z\mu^* \ni x_n \to 0$ weakly and every $x, y \in \mu z\mathscr{M}z\mu^*$ so that $E_{\mu\mathscr{L}(\Phi)z\mu^*}(x) = E_{\mu\mathscr{L}(\Phi)z\mu^*}(y) = 0$ we have $\|E_{\mu\mathscr{L}(\Phi)z\mu^*}(xx_k y)\|_2 \to 0$, as $k \to \infty$. This entails that for all $\mathscr{L}(\Phi)z \ni x_n \to 0$ weakly and every $x, y \in z\mathscr{M}z$ satisfying $E_{\mathscr{L}(\Phi)z}(x) = E_{\mathscr{L}(\Phi)z}(y) = 0$ we have

$$\|E_{\mathscr{L}(\Phi)z}(xx_k y)\|_2 \to 0, \text{ as } k \to \infty. \tag{4.31}$$

Fix $\lambda \in \Lambda \setminus \Phi$ so that $|\Phi \cap \Phi^\lambda| = \infty$. Hence there are infinite sequences $\lambda_k, \omega_n \in \Lambda$ so that $\lambda \omega_k \lambda^{-1} = \lambda_k$ for all integers k. Since $\lambda \in \Lambda \setminus \Phi$ then $E_{\mathscr{L}(\Phi)}(u_\lambda z) = E_{\mathscr{L}(\Phi)z}(zu_{\lambda^{-1}}) = 0$. Also we have that $u_{\omega_k} z \to 0$ weakly as $k \to \infty$. Using these calculations we have that

$$\begin{aligned} \|E_{\mathscr{L}(\Phi)}(zu_\lambda zu_{\lambda^{-1}}z)\|_2^2 &= \|E_{\mathscr{L}(\Phi)}(u_\lambda zu_{\lambda^{-1}}z)\|_2^2 = \|u_{\lambda\omega_k\lambda^{-1}} E_{\mathscr{L}(\Phi)}(u_\lambda zu_{\lambda^{-1}}z)\|_2^2 \\ &= \|E_{\mathscr{L}(\Phi)}(u_{\lambda\omega_k} zu_{\lambda^{-1}}z)\|_2^2 = \|E_{\mathscr{L}(\Phi)z}(zu_\lambda zu_{\omega_k} zu_{\lambda^{-1}}z)\|_2^2 \to 0 \text{ as } k \to \infty. \end{aligned} \tag{4.32}$$

Also using (4.32) the last quantity above converges to 0 as $k \to \infty$ and hence $E_{\mathscr{L}(\Phi)}(zu_\lambda zu_{\lambda^{-1}}z) = 0$ which entails that $zu_\lambda z = 0$, as desired. For the remaining part notice first that since $[\Gamma : Q] = \infty$ then (4.28) implies that $[\Lambda : \Phi] = \infty$. Assume by contradiction that for all $\lambda \in \Lambda \setminus \Phi$ we have $zu_\lambda z = 0$. As $[\Lambda : \Phi] = \infty$ then for every positive integer l one can construct inductively $\lambda_i \in \Lambda \setminus \Phi$

with $i \in \overline{1,l}$ such that $\lambda_i\lambda_j^{-1} \in \Lambda \setminus \Phi$ for all $i > j$ such that $i, j \in \overline{1,l}$. But this implies that $0 = zu_{\lambda_i\lambda_j^{-1}}z = zu_{\lambda_i}u_{\lambda_j^{-1}}z$ and hence $u_{\lambda_i^{-1}}zu_{\lambda_i}$ are mutually orthogonal projections when $i = \overline{1,l}$. This is obviously false when l sufficiently large. □

Theorem 4.0.8. *Assume the same conditions as in Theorem 4.0.6. Then one can find subgroups* $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ *so that*

1. Φ_1, Φ_2 *are infinite, icc, property (T) groups so that* $\Phi = \Phi_1 \times \Phi_2$*;*

2. $\mathscr{QN}_{\Lambda}^{(1)}(\Phi) = \Phi$*;*

3. *There exists* $\mu \in \mathscr{U}(\mathscr{M})$ *such that* $\mu\mathscr{L}(\Phi)\mu^* = \mathscr{L}(Q)$.

Proof. From Theorem 4.0.6 there exist subgroups $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ such that

1. Φ_1, Φ_2 are, infinite, commuting, finite-by-icc, property (T) groups so that $[\Phi : \Phi_1\Phi_2] < \infty$;

2. $\mathscr{QN}_{\Lambda}^{(1)}(\Phi) = \Phi$;

3. There exist $\mu \in \mathscr{U}(\mathscr{M})$ and $z \in \mathscr{P}(\mathscr{Z}(\mathscr{L}(\Phi)))$ with $h = \mu z \mu^* \in \mathscr{P}(\mathscr{L}(Q))$ satisfying

$$\mu\mathscr{L}(\Phi)z\mu^* = h\mathscr{L}(Q)h. \tag{4.33}$$

Next we show that in (4.33) we can pick $z \in \mathscr{Z}(\mathscr{L}(\Phi))$ maximal with the property that for every projection $t \in \mathscr{Z}(\mathscr{L}(\Phi)z^{\perp})$ we have

$$L(\Phi_i)t \nprec_{\mathscr{M}} \mathscr{L}(Q) \text{ for } i = 1, 2. \tag{4.34}$$

To see this let $z \in \mathscr{F}$ be a maximal family of mutually orthogonal (minimal) projections $z_i \in \mathscr{Z}(\mathscr{L}(\Phi))$ such that $\mathscr{L}(\Phi)z_i \prec_{\mathscr{M}} \mathscr{L}(Q)$. Note that since Φ has finite conjugacy radical it follows that $\mathscr{F}$ is actually finite. Next let $z \leqslant \sum z_i := a \in \mathscr{Z}(\mathscr{L}(\Phi))$ and we briefly argue that $\mathscr{L}(\Phi)a \prec^s_{\mathscr{M}} \mathscr{L}(Q)$. Indeed since $\mathscr{L}(\Phi)a' \cap a\mathscr{M}a = a(\mathscr{L}(\Phi)' \cap \mathscr{M})a = \mathscr{Z}(\mathscr{L}(\Phi))a$ and the later is finite dimensional then for every $r \in \mathscr{L}(\Phi)a' \cap a\mathscr{M}a$ there is $z_i \in \mathscr{F}$ such that $rz_i = z_i \neq 0$. Since

$\mathscr{L}(\Phi)z_i \prec_{\mathscr{M}} \mathscr{L}(Q)$ and then $\mathscr{L}(\Phi)r \prec_{\mathscr{M}} \mathscr{L}(Q)$ as desired. Thus applying Lemma 2.1.27, after perturbing μ to a new unitary we get $\mu\mathscr{L}(\Phi)a\mu^* = h_o\mathscr{L}(Q)h_o$. Finally, we show (4.34). Assume by contradiction there is $t_o \in \mathscr{Z}(\mathscr{L}(\Phi)z^\perp)$ so that $\mathscr{L}(\Phi_i)t_o \prec_{\mathscr{M}} \mathscr{L}(Q)$ for some $i = 1,2$. Thus there exist projections $r \in \mathscr{L}(\Phi)t_o$, $q \in \mathscr{L}(Q)$, a partial isometry $w \in \mathscr{M}$ and a $*$-isomorphism on the image $\phi : r\mathscr{L}(\Phi)r \to \mathscr{B} := \phi(r\mathscr{L}(\Phi)r) \subseteq q\mathscr{L}(Q)q$ such that $\phi(x)w = wx$. Notice that $w^*w \in t_o(\mathscr{L}(\Phi_i)' \cap \mathscr{M})t_o$ and $ww^* \in \mathscr{B}' \cap q\mathscr{M}q$. But since $Q < \Gamma$ is malnormal it follows that $\mathscr{B}' \cap q\mathscr{M}q \subseteq q\mathscr{L}(Q)q$ and hence $ww^* \in q\mathscr{L}(Q)q$. Using this in combination with previous relations we get that $wr\mathscr{L}(\Phi_i)rw^* = \mathscr{B}ww^* \subseteq \mathscr{L}(Q)$ and extending w to a unitary u we have that $ur\mathscr{L}(\Phi_i)ru^* \subseteq \mathscr{L}(Q)$. Since $\mathscr{L}(Q)$ is a factor we can further perturb the unitary u so that $u\mathscr{L}(\Phi_i)r_ou^* \subseteq \mathscr{L}(Q)$ where $r \leqslant r_o \leqslant t_o$ is the central support of r in $\mathscr{L}(\Phi_i)t_o$. Using malnormality of Q again we further get $r_o(\mathscr{L}(\Phi_i) \vee \mathscr{L}(\Phi_i)' \cap \mathscr{M})r_ou^* \subseteq \mathscr{L}(Q)$ and perturbing u we can further assume that $(\mathscr{L}(\Phi_i) \vee \mathscr{L}(\Phi_i)' \cap \mathscr{M})s_ou^* \subseteq \mathscr{L}(Q)$ where $r_o \leqslant s_o$ is the central support or r_o in $\mathscr{L}(\Phi_i) \vee \mathscr{L}(\Phi_i)' \cap \mathscr{M})$. In particular, $u(\mathscr{L}(\Phi)s_ou^* \subseteq \mathscr{L}(Q)$ and hence $\mathscr{L}(\Phi)s_o \subseteq u^*\mathscr{L}(Q)u$. Since $r \leqslant r_o \leqslant s_o$ and $r \leqslant t_o$ the previous containment implies that there is a minimal projection $s' \in \mathscr{L}(\Phi)a^\perp$ so that $\mathscr{L}(\Phi)s' \prec \mathscr{L}(Q)$ which contradicts the maximality assumption on $\mathscr{F}$. Finally replacing z with a in our statement, etc our claim follows.

Next fix $t \in \mathscr{Z}(\mathscr{L}(\Phi)z^\perp)$. Since $\mathscr{L}(\Phi_1)t$ and $\mathscr{L}(\Phi_2)t$ are commuting property (T) von Neumann algebras then using the same arguments as in the first part of the proof of Theorem 4.0.5 there are two possibilities: either i) there exists $j \in 1,2$ such that $\mathscr{L}(\Phi_j)t \prec_{\mathscr{M}} \mathscr{L}(N_2)$ or ii) $\mathscr{L}(\Phi)t \prec_{\mathscr{M}} \mathscr{L}(N_2 \rtimes Q)$. Next we briefly argue ii) is impossible. Indeed, assuming ii), Theorem 4.0.2 for $n = 1$ would imply the existence of $j \in 1,2$ so that $\mathscr{L}(\Phi_j)t \prec_{\mathscr{M}} \mathscr{L}(Q)$ which obviously contradicts the choice of z. Thus we have i) and passing to the relative commutants intertwining we have that $\mathscr{L}(N_1) \prec \mathscr{L}(\Phi_j)t' \cap t\mathscr{M}t = t(\mathscr{L}(\Phi_j)' \cap \mathscr{M})t$. Using the relations the Φ_j's we see that $t(\mathscr{L}(\Phi_j)' \cap \mathscr{M})t \subset t\mathscr{L}(\Phi_j) \vee \mathscr{L}(\Phi_j)' \cap \mathscr{M})t \subseteq t\mathscr{L}(\Phi_j v C_\Lambda(\Phi_j))t \subseteq t\mathscr{L}(\Phi)t$. In conclusion, we have

$$\mathscr{L}(N_1) \prec_{\mathscr{M}} t\mathscr{L}(\Phi)t, \text{ for all } t \in \mathscr{Z}(\mathscr{L}(\Phi)z^\perp). \tag{4.35}$$

Let $A = \{\lambda \in \Lambda : |\Phi \cap \Phi^\lambda| < \infty\}$ and $B = \{\lambda \in \Lambda : |\Phi \cap \Phi^\lambda| = \infty\}$. Note that $A \cup B = \Lambda$ and

$A \neq \emptyset$. Since N_1 is infinite then for every $\lambda \in A$ we have that $\mathscr{L}(N_1) \nprec_{\mathscr{M}} \mathscr{L}(\Phi \cap \Phi^{\lambda})z^{\perp}$. Thus using (4.35) together with the same argument from the proof of [PV06, Theorem 6.16], working under $z^{\perp}$, we get $z^{\perp}E_{\mathscr{L}(\Phi)}(u_{\lambda}z^{\perp}xz^{\perp}) = 0$ for all $x \in \mathscr{M}$. This further implies that $z^{\perp}u_{\lambda}z^{\perp} = 0$ for all $\lambda \in A$ and hence $u_{\lambda}z^{\perp}u_{\lambda^{-1}} \leqslant z$.

On the other hand by Lemma 4.0.7 we have that for all $\lambda \in B$ we get $zu_{\lambda}z = 0$ and hence $u_{\lambda}zu_{\lambda^{-1}} \leqslant z^{\perp}$. So if $B \neq \emptyset$ we obviously have equality in the previous two relations, i.e. $u_{\lambda}zu_{\lambda^{-1}} = z^{\perp}$ for all $\lambda \in B$ and $u_{\lambda}z^{\perp}u_{\lambda^{-1}} = z$ for all $\lambda \in A$. These further imply there exist $a_o \in A$ and $b_0 \in B$ such that $A = a_0C_{\Lambda}(z^{\perp})$ and $B = b_oC_{\Lambda}(z)$; here $C_{\Lambda}(z) \leqslant \Lambda$ is the subgroup of all elements of Λ that commute with z and similarly for $C_{\Lambda}(z^{\perp})$. Thus $\Lambda = A \cup B = a_oC_{\Lambda}(z^{\perp}) \cup b_oC_{\Lambda}(z)$. Thus we can assume, without loss of generality, that $[\Lambda : C_{\Lambda}(z)] < \infty$. But since Λ is icc this implies that $z = 1$. The rest of the statement follows. $\square$

Theorem 4.0.9. *In the Theorem 4.0.5 we cannot have case 4a).*

Proof. Assume by contradiction that for all $j \in 1,2$ there is $i \in 1,2$ such that $\Delta(\mathscr{L}(Q_i)) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_j)$. Using [DHI16, Theorem 4.1] and the property (T) on N_j one can find a subgroup $\Sigma < \Lambda$ such that $\mathscr{L}(Q_i) \prec_{\mathscr{M}} \mathscr{L}(\Sigma)$ and $\mathscr{L}(N_j) \prec_{\mathscr{M}} \mathscr{L}(C_{\Lambda}(\Sigma))$. Since $\mu\mathscr{L}(\Phi)\mu^* = \mathscr{L}(Q)$ and Q_i are biexact then by the product rigidity in [CdSS15] one can assume there is a unitary $u \in \mathscr{L}(Q)$ such that $u\mathscr{L}(Q_1)u^* = \mathscr{L}(\Phi_1)^t$ and $u\mathscr{L}(Q_2)u^* = \mathscr{L}(\Phi_2)^{1/t}$. Thus we get that $\mathscr{L}(\Phi_i) \prec_{\mathscr{M}} \mathscr{L}(\Sigma)$ and hence $[\Phi_i : g\Sigma g^{-1} \cap \Phi_i] < \infty$. So working with $g\Sigma g^{-1}$ instead of Σ we can assume that $[\Phi_i : \Sigma \cap \Phi_i] < \infty$.In particular $\Sigma \cap \Phi_i$ is infinite and since Φ is almost malnormal in Λ it follows that $C_{\Lambda}(\Sigma \cap \Phi_i) < \Phi$. Thus we have that $\mathscr{L}(N_j) \prec_{\mathscr{M}} \mathscr{L}(C_{\Lambda}(\Sigma)) \subseteq \mathscr{L}(C_{\Lambda}(\Sigma \cap \Phi_i)) \subset \mathscr{L}(\Phi) = \mu^*\mathscr{L}(Q)\mu$ which is obviously a contradiction. $\square$

Theorem 4.0.10. *Let Γ be a group as in Notation 4.0.4 and assume that Λ is a group such that $\mathscr{L}(\Gamma) = \mathscr{L}(\Lambda) = \mathscr{M}$. Let $\Delta : \mathscr{M} \to \mathscr{M}\bar{\otimes}\mathscr{M}$ be the comultiplication "along Λ" i.e. $\Delta(v_{\lambda}) = v_{\lambda} \otimes v_{\lambda}$. Then the following hold:*

i) $\Delta(\mathscr{L}(N_1)), \Delta(\mathscr{L}(N_2)), \Delta(\mathscr{L}(N_1 \times N_2) \prec^{s}_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{L}(N_1 \times N_2)\bar{\otimes}\mathscr{L}(N_1 \times N_2)$, and

ii) there is a unitary $u \in \mathscr{M}\bar{\otimes}\mathscr{M}$ such that $u\Delta(\mathscr{L}(Q))u^* \subseteq \mathscr{L}(Q)\bar{\otimes}\mathscr{L}(Q)$.

Proof. First we show i). From Theorem 4.0.5 we have that for all $j \in 1,2$ there is $j_i \in 1,2$ such that $\Delta(\mathscr{L}(N_{j_i})) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_j)$. Notice that since $\mathscr{N}_{\mathscr{M}\bar{\otimes}\mathscr{M}}\Delta(\mathscr{L}(N_i))'' \supset \Delta(\mathscr{M})$ and $\Delta(\mathscr{M})' \cap \mathscr{M}\bar{\otimes}\mathscr{M} = \mathbb{C}1$ then by [DHI16, Lemma 2.4 part (3)] we actually have $\Delta(\mathscr{L}(N_{j_i})) \prec^s_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_j)$. Notice that for all $i \neq k$ we have $j_i \neq j_k$. Otherwise we would have $\Delta(\mathscr{L}(N_{j_i}) \prec^s \mathscr{M}\bar{\otimes}\mathscr{L}(N_1)$ and $\Delta(\mathscr{L}(N_{j_i}) \prec^s \mathscr{M}\bar{\otimes}\mathscr{L}(N_2)$ which by [DHI16, Lemma 2.8 (2)] would imply that $\Delta(\mathscr{L}(N_{j_i}) \prec^s \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \cap N_2) = \mathscr{M} \otimes 1$ which is a contradiction. Furthermore using the same arguments as in [Is16, Lemma 2.6] we have that $\Delta(\mathscr{L}(N_1 \times N_2) \prec^s_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}\mathscr{L}(N_1 \times N_2)$. Then working on the left side of the tensor we get that $\Delta(\mathscr{L}(N_1 \times N_2) \prec^s_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{L}(N_1 \times N_2)\bar{\otimes}\mathscr{L}(N_1 \times N_2)$.

Next we show ii). First we claim there is unitary $u \in \mathscr{M}\bar{\otimes}\mathscr{M}$ such that $u\Delta(\mathscr{L}(Q))u^* \subseteq \mathscr{M}\bar{\otimes}\mathscr{L}(Q)$. To see this notice that 4b) in Theorem 4.0.5 implies that there is $\phi : p\Delta(\mathscr{L}(Q)p \to \mathscr{C} := \phi(p\Delta(\mathscr{L}(Q))p) \subseteq q(\mathscr{M}\bar{\otimes}\mathscr{L}(Q))q$ a $*$-isomorphism so that

$$\phi(x)v = vx \text{ for all } x \in p\Delta(\mathscr{L}(Q))p. \tag{4.36}$$

We also have $vv^* \in \mathscr{C}' \cap q(\mathscr{M}\bar{\otimes}\mathscr{M})q$ and $v^*v \in p\Delta(\mathscr{L}(Q))p' \cap p\mathscr{M}\bar{\otimes}\mathscr{M}p$ and moreover we can assume that $s(E_{\mathscr{M}\bar{\otimes}\mathscr{L}(Q)}(vv^*)) = q$. If $\mathscr{C} \prec_{\mathscr{M}\bar{\otimes}\mathscr{L}(Q)} \mathscr{M} \otimes 1$ then using the same argument form the first part of the proof of Theorem 4.0.3 we would get that $\Delta(\mathscr{L}(Q)) \prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M} \otimes 1$ which contradicts [IPV10, Proposition 7.2.2]; hence $\mathscr{C} \not\prec_{\mathscr{M}\bar{\otimes}\mathscr{L}(Q)} \mathscr{M} \otimes 1$. Since Q is malnormal in Γ then by Lemma 2.1.20 we have that $vv^* \in \mathscr{C}' \cap q(\mathscr{M}\bar{\otimes}\mathscr{M})q \subseteq \mathscr{C}' \cap q(\mathscr{M}\bar{\otimes}\mathscr{L}(Q))q$ and hence relation (4.36) implies that $vp\Delta(\mathscr{L}(Q))pv^* = \mathscr{C}vv^* \subseteq \mathscr{M}\bar{\otimes}\mathscr{L}(Q)$ since $\mathscr{M}\bar{\otimes}\mathscr{L}(Q)$ is a factor there is a unitary $w \in \mathscr{M}\bar{\otimes}\mathscr{M}$ such that $w\Delta(\mathscr{L}(Q))w^* \subseteq \mathscr{M}\bar{\otimes}\mathscr{L}(Q)$, as desired.

To this end we notice that the same arguments as above (in all theorems involved) while working on the left tensor one can show there is a unitary $v \in \mathscr{M}\bar{\otimes}\mathscr{M}$ such that $v\Delta(\mathscr{L}(Q))v^* \subseteq \mathscr{L}(Q)\bar{\otimes}\mathscr{M}$. Combining this with the claim above and using [DHI16, Lemma 2.8(2)] we get that $\Delta(\mathscr{L}(Q)) \prec^s_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{L}(Q)\bar{\otimes}\mathscr{L}(Q)$. As $\Delta(\mathscr{L}(Q)) \not\prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{M}\bar{\otimes}1, 1\bar{\otimes}\mathscr{M}$ then one can iterate the same argument as in the proof of the claim above to show that one can find a new unitary

$u \in \mathscr{M}\bar{\otimes}\mathscr{M}$ such that $u\Delta(\mathscr{L}(Q))u^* \subseteq \mathscr{L}(Q)\bar{\otimes}\mathscr{L}(Q)$. □

4.1 Proof of Theorem 4.0.1

Proof. We divide the proof into separate parts to improve the exposition.

4.1.1 Reconstruction of the Acting Group Q

To accomplish this we will use the notion of height for elements in group von Neumann algebras as introduced in [IPV10, Io11]). From the previous theorem recall that $u\Delta(\mathscr{L}(Q))u^* \subseteq \mathscr{L}(Q)\bar{\otimes}\mathscr{L}(Q)$. Let $\mathscr{A} = u\Delta(\mathscr{L}(N_1))u^*$. Next we claim that

$$h_{Q\times Q}(u\Delta(Q)u^*) > 0. \tag{4.37}$$

For every $x, y \in \mathscr{L}(Q)\bar{\otimes}\mathscr{L}(Q)$ and every $a \in \mathscr{A}\bar{\otimes}\mathscr{A}$ supported on a finite set $F \subset N = N_1 \times N_2$ we have that

$$\begin{aligned}
\|E_{\mathscr{A}\bar{\otimes}\mathscr{A}}(xay)\|_2^2 &= \|\sum_{q,l} \tau(xu_{q^{-1}})\tau(yu_l)E_{\mathscr{A}\bar{\otimes}\mathscr{A}}(u_q a u_{l^{-1}})\|_2^2 \\
&= \|\sum_{q,l} \tau(xu_{q^{-1}})\tau(yu_l)E_{\mathscr{A}\bar{\otimes}\mathscr{A}}(\sigma_q(a)u_{ql^{-1}})\|_2^2 \\
&= \|\sum_{q} \tau(xu_{q^{-1}})\tau(yu_l)\sigma_q(a)\|_2^2 \\
&= \|\sum_{q\in Q, n\in N^2} \tau(xu_{q^{-1}})\tau(yu_l)\tau(au_{n^{-1}})u_{\sigma_q(n)}\|_2^2 \\
&= \sum_{r\in N^2} |\sum_{\sigma_q(n)=r} \tau(xu_{q^{-1}})\tau(yu_l)\tau(au_{n^{-1}})|^2 \\
&\leqslant h^2_{Q\times Q}(x) \sum_{r\in N^2} (\sum_{q\in Q:\, \sigma_{q^{-1}}(r^{-1})\in F} |\tau(yu_l)||\tau(au_{\sigma_{q^{-1}}(r)}|)^2 \\
&\leqslant h^2_{Q\times Q}(x)\|y\|_2^2\|a\|_2^2 \max_{r\in N^2} |\{q \in Q : \sigma_{q^{-1}}(r^{-1}) \in F\}|.
\end{aligned} \tag{4.38}$$

This estimate leads to the following property: for every finite sets $K, S \subset Q$, every $a \in span\{\mathscr{A}\bar{\otimes}\mathscr{A}u_g :$

$g \in K\}$ and all $\varepsilon > 0$ there exist a scalar $C > 0$ and a finite set $F \subset N^2$ such that for all $x, y \in \mathscr{L}(Q)\bar{\otimes}\mathscr{L}(Q)$ we have

$$\|P_{\sum_{s\in S}\mathscr{A}\bar{\otimes}\mathscr{A}u_s}(xay)\|_2^2 \leqslant |K||S|C(h^2_{Q\times Q}(x)\|y\|_2^2\|a\|_2^2 \max_{r\in N^2}|\{q\in Q : \sigma_{q^{-1}}(r^{-1})\in F\}|) + \varepsilon\|x\|_\infty\|y\|_\infty \tag{4.39}$$

Note this follows directly from (4.38) after we decompose the a and the projection $P_{\sum_{s\in S}\mathscr{A}\bar{\otimes}\mathscr{A}u_s}$.

Next we use (4.39) to prove our claim. Fix $\varepsilon > 0$. Since $\Delta(\mathscr{A}) \nprec \mathscr{M}\otimes 1, \quad 1\otimes\mathscr{M}$ then by Theorem 2.1.19 one can find a finite subset $F_o \subset N^2 \setminus N\times 1 \cup 1\times N$ such that $a_{F_o} \in \mathscr{A}\bar{\otimes}\mathscr{A}$ is supported on F_o and $\|a - a_{F_o}\|_2 \leqslant \varepsilon$. Since $\Delta(\mathscr{A}) \prec^s \mathscr{A}\bar{\otimes}\mathscr{A}$ there is a finite $S \subseteq Q\times Q$ such that

$$\|P_{\sum_{s\in S}\mathscr{A}\bar{\otimes}\mathscr{A}u_s}(a) - a\|_2 \leqslant \varepsilon \text{ for all } a \in \Delta(\mathscr{A}). \tag{4.40}$$

Assume by contradiction (4.37) doesn't hold. Thus there is a sequence $g_n \in Q$ such that $h_{Q\times Q}(t_n) = h_{Q\times Q}(u\Delta(u_{g_n})u^*) \to 0$ as $n\to\infty$. As t_n normalizes $\Delta(\mathscr{A})$ then one can see that

$$\begin{aligned} 1-\varepsilon = \|t_n a t_n^*\|_2^2 - \varepsilon &\leqslant \|P_{\sum_{s\in S}\mathscr{A}\bar{\otimes}\mathscr{A}u_s}(t_n a t_n^*)\|_2^2 \\ &\leqslant \||P_{\sum_{s\in S}\mathscr{A}\bar{\otimes}\mathscr{A}u_s}(t_n a t_n^*)\|_2^2 + \varepsilon \\ &\leqslant |F_o||S|C(h^2_{Q\times Q}(t_n)\|t_n\|_2^2\|a_{F_o}\|_2^2 \max_{r\in N^2}|\{q\in Q : \sigma_{q^{-1}}(r^{-1})\in F_o\}|) + \varepsilon\|t_n\|_\infty^2 \\ &\leqslant |F_o||S|C(h^2_{Q\times Q}(t_n)\max_{r\neq 1}|Stab_Q(r)||F_o|) + 2\varepsilon. \end{aligned} \tag{4.41}$$

Since the stabilizers sizes are uniformly bounded we get a contradiction if $\varepsilon > 0$ is arbitrary small. To this end we notice that the height condition together with Theorem 4.0.8 and [CU18, Lemmas 2.4,2.5] already imply that $h_Q(\mu\Phi\mu^*) > 0$ and by [IPV10, Theorem 3.1] there is a unitary $\mu_0 \in \mathscr{M}$ such that $\mathbb{T}\mu_0\Phi\mu_0^* = \mathbb{T}Q$.

4.1.2 Reconstruction of a Core Subgroup and its Product Feature

From Theorem 4.0.10 have that $\Delta(\mathscr{L}(N_1\times N_2)) \prec^s_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1\times N_2)$. Proceeding exactly as in the proof of [CU18, Claim 4.5] we can show that $\Delta(\mathscr{A})\subseteq \mathscr{A}\bar{\otimes}\mathscr{A}$, where $\mathscr{A}=u\mathscr{L}(N_1\times N_2)u^*$. By Lemma 2.1.28, there exists a subgroup $\Sigma<\Lambda$ such that $\mathscr{A}=\mathscr{L}(\Sigma)$. The last part of the proof of [CU18, Theorem 5.2] shows that $\Lambda=\Sigma\rtimes\Phi$. In order to reconstruct the product feature of Σ, we need a couple more results.

Claim 4.1.1. *For every $i=1,2$ there exists $j=1,2$ such that*

$$\Delta(\mathscr{L}(N_j)) \prec^s \mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_i). \tag{4.42}$$

Proof of Claim. We prove this only for $i=1$ as the other case is similar. Also notice that since $\mathscr{N}_{\mathscr{M}\otimes\mathscr{M}}(\Delta(\mathscr{L}(N_j)))''\supseteq\Delta(\mathscr{M})$ and $\Delta(\mathscr{M})'\cap\mathscr{M}\bar{\otimes}\mathscr{M}=\mathbb{C}1$ then to establish (4.42) we only need to show that $\Delta(\mathscr{L}(N_j))\prec\mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_i)$. From above we have $\Delta(\mathscr{L}(N_1\times N_2)\prec_{\mathscr{M}\bar{\otimes}\mathscr{M}} \mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1\times N_2)$. Hence there exist nonzero projections $a_i\in\Delta(\mathscr{L}(N_i))$ and $b\in\mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1\times N_2)$, a partial isometry $v\in\mathscr{M}\bar{\otimes}\mathscr{M}$ and an $*$-isomorphism on the image $\Psi: a_1\otimes a_2\Delta(\mathscr{L}(N_1\times N_2))a_1\otimes a_2\to\Psi(a_1\otimes a_2\Delta(\mathscr{L}(N_1\times N_2)a_1\times a_2):=\mathscr{R}\subseteq b\mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1\times N_2)b$ such that $\Psi(x)v=vx$ for all $x\in a_1\otimes a_2\Delta(\mathscr{L}(N_1\times N_2))a_1\otimes a_2$.

Denote by $\mathscr{D}_i:=\Psi(a_i\Delta(\mathscr{L}(N_i)))a_i)\subseteq b\mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1\times N_2)b$ and notice that $\mathscr{D}_1$ and $\mathscr{D}_2$ are commuting property (T) diffuse subfactors. Since the group N_2 is $(\mathbb{F}_\infty)$-by-(non-elementary hyperbolic group) then by [CIK13, CK15] it follows that there is $j=1,2$ such that $\mathscr{D}_j\prec_{\mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1\times N_2)} \mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1\times\mathbb{F}_\infty)$. Since $\mathbb{F}_\infty$ has Haagerup's property and $\mathscr{D}_j$ has property (T) this further implies that $\mathscr{D}_j\prec_{\mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1\times N_2)} \mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1)$. Composing this intertwining with Ψ we get $\Delta(\mathscr{L}(N_j))\prec\mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1)$, as desired.

Also, we note that $j_1\neq j_2$. Otherwise we would have that $\Delta(\mathscr{L}(N_j))\prec^s\mathscr{L}(N_1\times N_2)\bar{\otimes}\mathscr{L}(N_1)\cap\mathscr{L}(N_2)=\mathscr{L}(N_1\times N_2)\bar{\otimes}1$, which obviously contradicts [IPV10, Proposition 7.2.1]. ■

Let $\mathscr{A}=u\mathscr{L}(N_1))u^*$. Thus, we get that $\Delta(\mathscr{A})\prec^s\mathscr{L}(N_1\times N_2)\otimes\mathscr{L}(N_i)$ for some $i=1,2$.

This implies that for every $\varepsilon > 0$, there exists a finite set $S \subset u^*Qu$, containing e, such that $\|d - P_{S\times S}(d)\|_2 \leq \varepsilon$ for all $d \in \Delta(\mathscr{A})$. However, $\Delta(\mathscr{A})$ is invariant under the action of u^*Qu, and hence arguing exactly as in [CU18, Claim 4.5] we get that $\Delta(\mathscr{A}) \subset (\mathscr{L}(\Sigma)\bar{\otimes}u\mathscr{L}(N_i)u^*)$. We now separate the argument into two different cases:

Case I: $i = 1$.

In this case, $\Delta(\mathscr{A}) \subseteq \mathscr{L}(\Sigma)\bar{\otimes}\mathscr{A}$. Thus by Lemma 2.1.28 we get that there exists a subgroup $\Sigma_0 < \Sigma$ with $\mathscr{A} = \mathscr{L}(\Sigma_0)$. Now, $\mathscr{A}' \cap \mathscr{L}(\Sigma) = u\mathscr{L}(N_2)u^*$. Thus, $\mathscr{L}(\Sigma_0)' \cap \mathscr{L}(\Sigma) = u\mathscr{L}(N_2)u^*$. Note that Σ and Σ_0 are both icc property (T) groups. This implies that $\mathscr{L}(\Sigma_0)' \cap \mathscr{L}(\Sigma) = \mathscr{L}(vC_\Sigma(\Sigma_0))$, where $vC_\Sigma(\Sigma_0)$ denotes the *virtual centralizer* of Σ_0 in Σ. Proceeding as in [CdSS17] we can show that $\Sigma = \Sigma_0 \times \Sigma_1$.

Case II: $i = 2$.

Let $\mathscr{B} = u\mathscr{L}(N_2)u^*$. In this case, $\Delta(\mathscr{A}) \subseteq \mathscr{L}(\Sigma)\bar{\otimes}\mathscr{B}$. However, Lemma 2.1.28 then implies that $\mathscr{A} \subseteq \mathscr{B}$, which is absurd, as $\mathscr{L}(N_1)$ and $\mathscr{L}(N_2)$ are orthogonal algebras. Hence this case is impossible and we are done. □

Remarks. 1) There are several immediate consequences of the Theorem 4.0.1. For instance one can easily see the von Neumann algebras covered by this theorem are non-isomorphic with the ones arising from any irreducible lattice in higher rank Lie group. Indeed, if Λ is any such lattice satisfying $\mathscr{L}(\Gamma) \cong \mathscr{L}(\Lambda)$, then Theorem 4.0.1, would imply that Λ must contain an infinite normal subgroup of infinite index which contradicts Margulis' normal subgroup theorem.

2) While it well known there are uncountable many non-isomorphic group II_1 factors with property (T) [Po07] little is known about producing concrete examples of such families. In fact the only currently known infinite families of pairwise non-isomorphic property (T) groups factors are $\{\mathscr{L}(G_n)\,|\,n \geq 2\}$ for G_n uniform latices in $Sp(n,1)$ [CH89] and $\{\mathscr{L}(G_1 \times G_2 \times \cdots \times G_k)\,|\,k \geq 1\}$ where G_k is any icc property (T) hyperbolic group [OP03]. Theorem 4.0.1 makes new progress in this direction by providing a new explicit infinite family of icc property (T) groups which gives rise to pairwise non-isomorphic II_1 factors. For instance, in the statement one can simply Q_i to vary in any infinite family of non-isomorphic uniform lattices in $Sp(n,1)$ for any $n \neq 2$. Unlike

the other families ours consists of factors which are not solid, do not admit tensor decompositions [CdSS17], and do not have Cartan subalgebras, [CIK13].

3) We notice that Theorem 4.0.1 still holds if instead of $\Gamma = (N_1 \times N_2) \rtimes (Q_1 \times Q_2)$ one considers any finite index subgroup of Γ of the form $\Gamma_{s,r} = (N_1 \times N_2) \rtimes (Q_1^s \times Q_2^r) \leqslant \Gamma$, where $Q_1^s \leqslant Q_1$ and $Q_2^r \leqslant Q_2$ are arbitrary finite index subgroups. One can verify these groups still enjoy all the algebraic/geometric properties used in the proof of Theorem 4.0.1 (including the fact that $N_1 \rtimes Q_1^s$ is hyperbolic relative to Q_1^s and $N_1 \rtimes Q_2^r$ is hyperbolic relative to Q_2^r) and hence all the von Neumann algebraic arguments in the proof of Theorem 4.0.1 apply verbatim. The details are left to the reader.

Chapter 5

Fundamental Groups of Property (T) type II_1 Factors

5.1 Fundamental Group of Factors Arising from Groups in Class $\mathcal{S}$

In this section we prove our main result describing isomorphisms of amplifications of property (T) group factors $\mathscr{L}(G)$ associated with groups $G \in \mathcal{S}$. These factors were first considered in [CDK19], where various rigidity properties were established. For instance, in [CDK19, Theorem A] it was shown that the semidirect product decomposition of the group $G = N \rtimes Q$ is a feature that's completely recoverable from $\mathscr{L}(G)$. In this section we continue these investigations by showing in particular that these factors also have trivial fundamental group (see Theorem 5.1.6 and Corollary 5.1.10). In order to prepare for the proof of our main theorem we first need to establish several preliminary results on classifying specific subalgebras of $\mathscr{L}(G)$. Some of the theorems will rely on results proved in [CDK19]. We recommend the reader to consult these results beforehand as we will focus mostly on the new aspects of the techniques. Throughout this section we shall use the notations introduced in Section 3.1.

Our first result classifies all diffuse, commuting property (T) subfactors inside these group factors.

Theorem 5.1.1. *Let $N \rtimes Q \in \mathcal{S}$. Also let $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{L}(N \rtimes Q) = \mathscr{M}$ be two commuting, property (T), type II_1 factors. Then for all $k \in \{1,2\}$ one of the following holds:*

1. *There exists $i \in \{1,2\}$ such that $\mathscr{A}_i \prec_{\mathscr{M}} \mathscr{L}(N_k)$;*
2. *$\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\mathscr{M}} \mathscr{L}(N_k) \rtimes Q$.*

Proof. Let $G_k = N_k \rtimes Q$ for $k \in \{1,2\}$. Notice that by part e) in Theorem 3.1.1 we have that $N \rtimes Q \leqslant G_1 \times G_2 = G$ where Q is embedded as $\mathrm{diag}(Q) \leqslant Q \times Q$. Notice that $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{L}(N) \rtimes Q \subseteq \mathscr{L}(G_1 \times G_2) =: \tilde{\mathscr{M}}$. By [CDK19, Theorem 5.3] there exists $i \in \{1,2\}$ such that

a) $\mathscr{A}_i \prec_{\tilde{\mathscr{M}}} \mathscr{L}(G_k)$, or

b) $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\tilde{\mathscr{M}}} \mathscr{L}(G_k \times Q)$.

Assume a). Since $\mathscr{A}_1 \vee \mathscr{A}_2 \subseteq \mathscr{L}(N) \rtimes Q$, by using [CDK19, Lemma 2.3] we further get that $\mathscr{A}_i \prec_{\tilde{\mathscr{M}}} \mathscr{L}(G \cap hG_k h^{-1}) = \mathscr{L}(((N_1 \times N_2) \rtimes \operatorname{diag}(Q)) \cap (N_k \rtimes Q)) = \mathscr{L}(N_k)$ and thus we have that *c)* $\mathscr{A}_i \prec_{\tilde{\mathscr{M}}} \mathscr{L}(N_k)$.

Assume b). Then $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\tilde{\mathscr{M}}} \mathscr{L}(\Gamma \cap h(\Gamma_k \times Q)h^{-1}) = \mathscr{L}(h(N_k \rtimes \operatorname{diag}(Q))h^{-1})$. This implies that *d)* $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\tilde{\mathscr{M}}} \mathscr{L}(N_k) \rtimes Q$.

Note that by using [CDK19, Lemma 2.5] case d) already implies that $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\mathscr{M}} \mathscr{L}(N_k) \rtimes Q$ which gives possibility 2. in the statement.

Next we show that c) gives 1. To accomplish this we only need to show that the intertwining actually happens in $\mathscr{M}$. By Popa's intertwining techniques c) implies there exist finitely many $x_i \in \tilde{\mathscr{M}}$, and $c > 0$ such that

$$\sum_{i=1}^{n} \|E_{\mathscr{L}(N_k)}(ax_i)\|_2^2 \geq c \text{ for all } a \in \mathscr{U}(\mathscr{A}_i). \tag{5.1}$$

Using basic approximations of x_i's and increasing $n \in \mathbb{N}$ and decreasing $c > 0$, if necessary, we can assume that $x_i = u_{g_i}$ where $g_i \in \hat{G}_k \times Q$. Now observe that $E_{\mathscr{L}(N_k)}(ax_i) = E_{\mathscr{L}(N_k)}(au_{g_i}) = E_{\mathscr{L}(N_k)}(E_{\mathscr{M}}(au_{g_i})) = E_{\mathscr{L}(N_k)}(aE_{\mathscr{M}}(u_{g_i}))$. Thus (5.1) becomes

$$\sum_{i=1}^{n} \|E_{\mathscr{L}(N_k)}(aE_{\mathscr{M}}(u_{g_i}))\|_2^2 \geq c \; for \; all \; u \in \mathscr{U}(\mathscr{A}_i)$$

and hence $\mathscr{A}_i \prec_{\mathscr{M}} \mathscr{L}(N_k)$ as desired. □

Next we show that actually the intertwining statements in the previous theorem can be made much more precise.

Theorem 5.1.2. *Let $N \rtimes Q \in \mathcal{S}$. Also let $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{L}(N \rtimes Q) = \mathscr{M}$ be two commuting, property (T), type* II_1 *factors. Then for every $k \in \{1,2\}$ one of the following holds:*

1. *There exists* $i \in \{1,2\}$ *such that* $\mathscr{A}_i \prec_{\mathscr{M}} \mathscr{L}(N_k)$;

2. $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\mathscr{M}} \mathscr{L}(Q)$.

Proof. Using Theorem 5.1.1 the statement will follow once we show that $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\mathscr{M}} \mathscr{L}(N_k) \rtimes Q$ implies $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\mathscr{M}} \mathscr{L}(Q)$, which we do next. Since $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\mathscr{M}} \mathscr{L}(N_k) \rtimes Q$, there exists

$$\psi : p(\mathscr{A}_1 \vee \mathscr{A}_2)p \to \psi(p(\mathscr{A}_1 \vee \mathscr{A}_2)p) = \mathscr{R} \subseteq q(\mathscr{L}(N_k) \rtimes Q)q \tag{5.2}$$

$*$-homomorphism, nonzero partial isometry $v \in q\mathscr{M}p$ such that

$$\psi(x)v = vx \text{ for all } x \in p(\mathscr{A}_1 \vee \mathscr{A}_2)p. \tag{5.3}$$

Notice that we can pick v such that the support projection satisfies $s(E_{\mathscr{L}(N_k \rtimes Q)}(vv^*)) = q$. Moreover, since $\mathscr{A}_i$'s are factors we can assume that $p = p_1 p_2$ for some $p_i \in \mathcal{P}(\mathscr{A}_i)$.

Next let $\mathscr{R}_i = \psi(p_i \mathscr{A}_i p_i)$. Note that $\mathscr{R}_1$, $\mathscr{R}_2$ are commuting property (T) subfactors such that $\mathscr{R}_1 \vee \mathscr{R}_2 = \mathscr{R} \subseteq q(\mathscr{L}(N_k) \rtimes Q)q$. Using the Dehn filling technology from [Os06, DGO11], we see that there exists a short exact sequence $1 \to \underset{\gamma_j}{*} Q_0^{\gamma_j} \to N_k \rtimes Q \to H \to 1$ where H is a hyperbolic, property (T) group and $Q_0 \leqslant Q$ is a finite index subgroup. Then using [PV12, CIK13] in the same way as in the proof of [CDK19, Theorem 5.2] we have either a) $\mathscr{R}_i \prec_{\mathscr{L}(N_k) \rtimes Q} \mathscr{L}(\underset{\gamma_j}{*} Q_0^{\gamma_j})$, for some i, or b) $\mathscr{R} = \mathscr{R}_1 \vee \mathscr{R}_2 \lessdot_{\mathscr{L}(N_k) \rtimes Q} \mathscr{L}(\underset{\gamma_j}{*} Q_0^{\gamma_j})$. Since $\mathscr{R}_i$'s have property (T) then by [Po01, Proposition 4.6] so does $\mathscr{R}$ and hence possibility b) entails $\mathscr{R} \prec_{\mathscr{L}(N_k) \rtimes Q} \mathscr{L}(\underset{\gamma_j}{*} Q_0^{\gamma_j})$. Summarizing, cases a)-b) imply that $\mathscr{R}_i \prec_{\mathscr{L}(N_k) \rtimes Q} \mathscr{L}(\underset{\gamma_j}{*} Q_0^{\gamma_j})$, for some i. Then using [IPP05, Theorem 4.3] this further implies $\mathscr{R} \prec_{\mathscr{L}(N_k) \rtimes Q} \mathscr{L}(Q_0^{\gamma_j})$ and hence $\mathscr{R}_i \prec_{\mathscr{L}(N_k) \rtimes Q} \mathscr{L}(Q_0) \subseteq \mathscr{L}(Q)$. As $Q \leqslant N_k \rtimes Q$ is malnormal, using the same arguments as in the proof of [CDK19, Theorem 5.3] one can show that $\mathscr{R} \prec_{\mathscr{L}(N_k) \rtimes Q} \mathscr{L}(Q)$. Indeed, let $\phi : r\mathscr{R}_i r \to \phi(r\mathscr{R}_i r) := \tilde{\mathscr{R}} \subseteq q_1 \mathscr{L}(Q) q_1$ be a unital $*$-homomorphism, and let $w \in q_1 \mathscr{L}(N_k \rtimes Q) r$ be a nonzero partial isometry such that

$$\phi(x)w = wx \text{ for all } x \in r\mathscr{R}_i r. \tag{5.4}$$

Note that $ww^* \in \mathscr{L}(Q)$ by Lemma 2.1.20 and hence $\tilde{\mathscr{R}}ww^* = w\mathscr{R}_i w^* \subseteq \mathscr{L}(Q)$. For every $u \in \mathscr{R}_{i+1}$ we have

$$\begin{aligned}\tilde{\mathscr{R}}wuw^* &= \tilde{\mathscr{R}}ww^*wuw^* = w\mathscr{R}_i w^*wuw^* = ww^*wu\mathscr{R}_i w^* = wu\mathscr{R}w^* \\ &= wu\mathscr{R}_i w^*ww^* = wuw^*w\mathscr{R}_i w^* = wuw^*\tilde{\mathscr{R}}ww^* = wuw^*\tilde{\mathscr{R}}.\end{aligned}$$

Thus Lemma 2.1.20 again implies that $wuw^* \in \mathscr{L}(Q)$. Altogether these show that $w\mathscr{R}_{i+1}w^* \subseteq \mathscr{L}(Q)$. Combining with the above we get $w\mathscr{R}w^* = w\mathscr{R}_i\mathscr{R}_{i+1}w^* = ww^*w\mathscr{R}_i\mathscr{R}_{i+1}w^* = w\mathscr{R}_i w^*w\mathscr{R}_{i+1}w^* \subseteq \mathscr{L}(Q)$. From relation (5.4) we have that $w^*w \in \mathscr{R}$. Also by (5.3) we have $\mathscr{R}v = vp(\mathscr{A}_1 \vee \mathscr{A}_2)p$ and hence $v^*\mathscr{R}v = v^*vp(\mathscr{A}_1 \vee \mathscr{A}_2)p$. Hence there exists $p_0 \in \mathcal{P}(p(\mathscr{A}_1 \vee \mathscr{A}_2)p)$ so that $v^*w^*wv = v^*vp_0$. Next we argue that $wvp_0 \neq 0$. Indeed, otherwise we would have $wv = 0$ and hence $wvv^* = 0$. As $w \in \mathscr{L}(N_k \rtimes Q)$ this would imply that $wE_{\mathscr{L}(N_k \rtimes Q)}(vv^*) = 0$ and hence $w = wq = ws(E_{\mathscr{L}(N_k \rtimes Q)}(vv^*)) = 0$, which is a contradiction. To this end, combining the previous relations we have $wvp(\mathscr{A}_1 \vee \mathscr{A}_2)pp_0 \subseteq wvp(\mathscr{A}_1 \vee \mathscr{A}_2)pv^*vp_0 = wvp(\mathscr{A}_1 \vee \mathscr{A}_2)pv^*w^*wv = w\mathscr{R}vv^*w^*wv = w\mathscr{R}w^*wv \subseteq \mathscr{L}(Q)wv$. Since the partial isometry $wv \neq 0$ the last relation clearly shows that $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\mathscr{M}} \mathscr{L}(Q)$, as desired. □

Theorem 5.1.3. *Let $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{L}(N) \rtimes Q = \mathscr{M}$ be two commuting, property (T), type* II_1 *factors such that $(\mathscr{A}_1 \vee \mathscr{A}_2)' \cap r(\mathscr{L}(N) \rtimes Q)r = \mathbb{C}r$. Then one of the following holds:*

a) $\mathscr{A}_1 \vee \mathscr{A}_2 \prec^s_{\mathscr{M}} \mathscr{L}(N)$, or

b) $\mathscr{A}_1 \vee \mathscr{A}_2 \prec^s_{\mathscr{M}} \mathscr{L}(Q)$.

Proof. Fix $k \in \{1,2\}$. By Theorem 5.1.2 we get that either

i) $i_k \in \{1,2\}$ such that $\mathscr{A}_{i_k} \prec_{\mathscr{M}} \mathscr{L}(N_k)$, or

ii) $\mathscr{A}_1 \vee \mathscr{A}_2 \prec_{\mathscr{M}} \mathscr{L}(Q)$.

Note that case ii) together with the assumption $(\mathscr{A}_1 \vee \mathscr{A}_2)' \cap r(\mathscr{L}(N) \rtimes Q)r = \mathbb{C}r$ and [DHI16, Lemma 2.4] already give $b)$. So assume that case i) holds. Hence for all $k \in \{1,2\}$, there exists $i_k \in \{1,2\}$ such that $\mathscr{A}_{i_k} \prec_{\mathscr{M}} \mathscr{L}(N_k)$. Using [DHI16, Lemma 2.4], there exists $0 \neq z \in$

$\mathscr{Z}(\mathscr{N}_{r\mathscr{M}r}(\mathscr{A}_{i_k})' \cap r\mathscr{M}r)$ such that $\mathscr{A}_{i_k} z \prec^s_{\mathscr{M}} \mathscr{L}(N_k)$. Since $\mathscr{A}_1 \vee \mathscr{A}_2 \subseteq \mathscr{N}_{r\mathscr{M}r}(\mathscr{A}_{i_k})''$, then $\mathscr{N}_{r\mathscr{M}r}(\mathscr{A}_1 i_k)' \cap r\mathscr{M}r \subseteq (\mathscr{A}_1 \vee \mathscr{A}_2)' \cap r\mathscr{M}r = \mathbb{C}r$. Thus we get that $z = r$. In particular

$$\mathscr{A}_{i_k} \prec^s_{\mathscr{M}} \mathscr{L}(N_k). \tag{5.5}$$

We now briefly argue that $k \neq l \Rightarrow i_k \neq i_l$. Assume by contradiction that $i_1 = i_2 = i$. Then (5.5) implies that $\mathscr{A}_i \prec^s_{\mathscr{M}} \mathscr{L}(N_1)$ and $\mathscr{A}_i \prec^s_{\mathscr{M}} \mathscr{L}(N_2)$. By [DHI16, Lemma 2.6], this implies that $\mathscr{A}_i \lessdot_{\mathscr{M}} \mathscr{L}(N_1)$ and $\mathscr{A}_i \lessdot_{\mathscr{M}} \mathscr{L}(N_2)$. Note that $\mathscr{L}(N_i)$ are regular in $\mathscr{M}$ and hence by [PV11, Proposition 2.7] we get that $\mathscr{A}_i \lessdot_{\mathscr{M}} \mathscr{L}(N_1) \cap \mathscr{L}(N_2) = \mathbb{C}$, which implies that $\mathscr{A}_i$ is amenable. This contradicts our assumption that $\mathscr{A}_i$ has property (T). Thus $i_k \neq i_l$ whenever $k \neq l$. Therefore we have that $\mathscr{A}_{i_1} \prec^s_{\mathscr{M}} \mathscr{L}(N_1) \subseteq \mathscr{L}(N)$ and $\mathscr{A}_{i_2} \prec^s_{\mathscr{M}} \mathscr{L}(N_2) \subseteq \mathscr{L}(N)$. Using Corollary 2.1.22 we get that $\mathscr{A}_1 \vee \mathscr{A}_2 \prec^s_{\mathscr{M}} \mathscr{L}(N)$, which completes the proof. $\square$

Our next result concerns the location of the "core" von Neumann algebra.

Theorem 5.1.4. *Let $N \rtimes Q, M \rtimes P \in \mathcal{S}$. Let $p \in \mathscr{L}(M \rtimes P)$ be a projection and assume that $\Theta : \mathscr{L}(N \rtimes Q) \to p\mathscr{L}(M \rtimes P)p$ is a $*$-isomorphism. Then there exists a unitary $v \in \mathcal{U}(p\mathscr{L}(M \rtimes P)p)$ such that $\Theta(\mathscr{L}(N)) = vp\mathscr{L}(M)pv^*$.*

Proof. From assumptions there are Q_1, Q_2, P_1, P_2 icc, torsion free, residually finite, hyperbolic property (T) groups so that $Q = Q_1 \times Q_2$ and $P = P_1 \times P_2$. We also have that $N = N_1 \times N_2$ and $M = M_1 \times M_2$ where N_i's and M_i's have property (T). Denoting by $\mathscr{M} = \mathscr{L}(M \rtimes P)$, $\mathscr{A} = \Theta(\mathscr{L}(N))$ and $\mathscr{A}_i = \Theta(\mathscr{L}(N_i))$ we see that $\mathscr{A}_1$ and $\mathscr{A}_2$ are commuting property (T) subalgebras of $p\mathscr{M}p$. Using part b) in Theorem 5.1.3 we have that $\{\mathscr{A}_1 \vee \mathscr{A}_2\}' \cap \mathscr{N} = \Theta(\mathscr{L}(N)' \cap \mathscr{L}(N \rtimes Q)) = \mathbb{C}\Theta(1) = \mathbb{C}p$. Using Theorem 5.1.3 we get either

a) $\mathscr{A} \prec^s_{\mathscr{M}} \mathscr{L}(M)$ or,

b) $\mathscr{A} \prec^s_{\mathscr{M}} \mathscr{L}(P)$.

Assume case b) above holds. Then there exists projections $r \in \mathscr{A}$, $q \in \mathscr{L}(P)$, a nonzero partial isometry $v \in q\mathscr{M}r$, and a $*$-homomorphism $\psi : r\mathscr{A}r \to \psi(r\mathscr{A}r) \subseteq q\mathscr{L}(P)q$ such that $\psi(x)v = vx$

for all $x \in r\mathscr{A}r$. Arguing exactly as in the proof of [CDK19, Theorem 5.5], we can show that $v\mathscr{QN}_{r\mathscr{M}r}(r\mathscr{A}r)''v^* \subseteq q\mathscr{L}(P)q$.

Now, $\mathscr{QN}_{r\mathscr{M}r}(r\mathscr{A}r)'' = r\mathscr{M}r$, using [Po03, Lemma 3.5]. Thus, $\mathscr{M} \prec_{\mathscr{M}} \mathscr{L}(P)$ and hence $\mathscr{L}(P)$ has finite index in $\mathscr{M}$ by [CD18, Theorem 2.3], which is a contradiction. Hence we must a), i.e. $\mathscr{A} \prec^s_{\mathscr{M}} p\mathscr{L}(M)p$.

Repeating the above argument verbatim, we get that $p\mathscr{L}(M)p \prec_{p\mathscr{M}p} \mathscr{A}$. Let $\mathscr{N} = \mathscr{L}(N \rtimes Q)$ and $\mathscr{B} = p\mathscr{L}(M)p$. Note that $\mathscr{A} \subseteq \Theta(\mathscr{N})$ and $\mathscr{B} \subseteq p\mathscr{M}p$ are amplifications of genuine crossed product inclusions. Also by part d) in Theorem 3.1.1 $\mathscr{A}$ is regular irreducible subfactor of $\Theta(\mathscr{N}) = p\mathscr{M}p$, while $\mathscr{B}$ is a quasi-regular irreducible subfactor of $p\mathscr{M}p$ (as $\mathscr{QN}_{p\mathscr{M}p}(p\mathscr{B}p)'' = p\mathscr{QN}_{\mathscr{M}}(\mathscr{L}(M))p$). Thus, we are in the setting of the first part of the proof of [IPP05, Lemma 8.4] and using the same arguments there we conclude one can find $r \in \mathcal{P}(\mathscr{A})$, a unital $*$-isomorphism $\psi : r\mathscr{A}r \to \mathscr{R} := \psi(r\mathscr{A}r) \subseteq p\mathscr{L}(M)p$, and a partial isometry $v \in p\mathscr{M}p$ satisfying $v^*v = r$, $vv^* \in R' \cap p\mathscr{M}p$ and $\psi(x)v = vx$ for all $x \in r\mathscr{A}r$. Moreover, we have that $\mathscr{R} \subseteq p\mathscr{L}(M)p$ has finite index, and $\mathscr{R}' \cap p\mathscr{L}(M)p = \mathbb{C}p$. Notice that by [Po02, Lemma 3.1], we have that $[R' \cap p\mathscr{M}p : (p\mathscr{L}(M)p)' \cap p\mathscr{M}p] \leq [p\mathscr{L}(M)p : \mathscr{R}]$. As $(p\mathscr{L}(M)p)' \cap p\mathscr{M}p = \mathbb{C}$, we conclude that $\mathscr{R}' \cap p\mathscr{M}p$ is finite dimensional.

Let $x \in \mathscr{R}' \cap p\mathscr{M}p$. Since $xr = rx$ for all $r \in \mathscr{R}$ we have that $r\sum_g x_g u_g = \sum_g x_g u_g r$, where $x = \sum_{g\in P} x_g u_g$ is the Fourier decomposition of x in $\mathscr{M} = \mathscr{L}(M) \rtimes P$. Thus $\sum_g r x_g u_g = \sum_g x_g \sigma_g(r) u_g$ and hence $r x_g = x_g \sigma_p(r)$ for all g in r . In particular this entails that

$$x_g x_g^* \in \mathscr{R}' \cap p\mathscr{L}(M)p = \mathbb{C}p \tag{5.6}$$

$$x_g u_g \in \mathscr{R}' \cap p\mathscr{M}p. \tag{5.7}$$

From (5.6) we see that x_g is a scalar multiple of a unitary in $p\mathscr{M}p$. Hence by normalization we may assume that each x_g is itself either a unitary or zero.

Let K be the set of all $g \in P$ for which there exists $x_g \in \mathcal{U}(p\mathscr{L}(M)p)$ such that $x_g u_g \in \mathcal{U}(\mathscr{R}' \cap p\mathscr{M}p)$ and notice that K is a subgroup of P. Note that $\{x_g u_g\}_{g\in K}$ is a τ-orthogonal family in

$\mathscr{R}' \cap p\mathscr{M}p$. As $\mathscr{R}' \cap p\mathscr{M}p$ is finite dimensional, we get that K is a finite subgroup of P. As P is torsion free (see part a) in Theorem 3.1.1) then $K = \{e\}$. In particular this shows that $\mathscr{R}' \cap p\mathscr{M}p = \mathscr{R}' \cap p\mathscr{L}(M)p = \mathbb{C}p$ which implies $vv^* = p$ and since $v^*v = r \leq p$ we get $r = p$ and $v \in \mathcal{U}(p\mathscr{M}p)$. Thus $\psi(x) = vxv^*$ for all $x \in r\mathscr{A}r$ and hence $\mathscr{R} = vr\mathscr{A}rv^* = v\mathscr{A}v^* \subseteq p\mathscr{L}(M)p$. Let $v = \Theta(w_0)$, where $w_0 \in \mathscr{U}(\mathscr{L}(N \rtimes Q))$. Thus, we get that $\mathscr{L}(N) \subseteq w_0^*\Theta^{-1}(p\mathscr{L}(M)p)w_0 \subseteq \mathscr{L}(N) \rtimes Q$. By, [Ch78] (see also [CD19, Corollary 3.8]), we deduce that there exists a subgroup $L \leqslant Q$ such that $w_0^*\Theta^{-1}(p\mathscr{L}(M)p)w_0 = \mathscr{L}(N) \rtimes L$. As $[w_0^*\Theta^{-1}(p\mathscr{L}(M)p)w_0 : \mathscr{L}(N)]$ is finite, we must have that L is a finite subgroup of the torsion free group Q. Thus $L = \{e\}$ which gives that $\Theta(\mathscr{L}(N)) = \mathscr{A} = v^*p\mathscr{L}(M)pv$. $\square$

Next we show that in the previous result we can also identify up to corners the algebras associated with the acting groups. The proof relies heavily on the classification of commuting property (T) subalgebras provided by 5.1.3 and the malnormality of the acting groups.

Theorem 5.1.5. *Let $N \rtimes Q, M \rtimes P \in \mathcal{S}$. Let $p \in \mathscr{L}(M \rtimes P)$ be a projection and assume that $\Theta : \mathscr{L}(N \rtimes Q) \to p\mathscr{L}(M \rtimes P)p$ is a $*$-isomorphism. Then the following hold*

1. *There exists $v \in \mathcal{U}(p\mathscr{L}(M \rtimes P)p)$ such that $\Theta(\mathscr{L}(N)) = vp\mathscr{L}(M)pv^*$, and*

2. *There exists $u \in \mathcal{U}(\mathscr{L}(M \rtimes P))$ such that $\Theta(\mathscr{L}(Q)) = pu^*\mathscr{L}(P)up$.*

Proof. As part 1. follows directly from Theorem 5.1.4 we only need to show part 2.

Recall that $Q = Q_1 \times Q_2$, $P = P_1 \times P_2$, $N = N_1 \times N_2$ and $M = M_1 \times M_2$ where Q_i, P_i, N_i and M_i are icc, property (T) groups. Denote by $\mathscr{M} = \mathscr{L}(M \rtimes P)$, $\mathscr{A} = \Theta(\mathscr{L}(N))$, $\mathscr{B} = \Theta(\mathscr{L}(Q))$ and $\mathscr{B}_i = \Theta(\mathscr{L}(Q_i))$. Then we see that $\mathscr{B}_1, \mathscr{B}_2 \subset p\mathscr{M}p$ are commuting property (T) subalgebras such that $\mathscr{B}_1 \vee \mathscr{B}_2 = \mathscr{B}$. Moreover, by part d) in Theorem 3.1.1 we have that $\{\mathscr{B}_1 \vee \mathscr{B}_2\}' \cap p\mathscr{M}p = \mathscr{B}' \cap \Theta(\mathscr{L}(N \rtimes Q)) = \mathbb{C}\theta(1) = \mathbb{C}p$. Hence by Theorem 5.1.3, we either have that a) $\mathscr{B} \prec^s_{\mathscr{M}} \mathscr{L}(M)$, or b) $\mathscr{B} \prec^s_{\mathscr{M}} \mathscr{L}(P)$. By part 1. we also know that $\mathscr{A} \prec^s_{\mathscr{M}} \mathscr{L}(M)$. Thus, if a) holds, then Theorem 2.1.21 implies that $p\mathscr{M}p = \Theta(\mathscr{L}(N \rtimes Q)) \prec_{\mathscr{M}} \mathscr{L}(M)$. In turn this implies that Q is finite, a contradiction. Hence b) must hold, i.e. $\mathscr{B} \prec^s_{\mathscr{M}} \mathscr{L}(P)$.

Thus there exist projections $q \in \mathscr{B}$, $r \in \mathscr{L}(P)$, a nonzero partial isometry $v \in \mathscr{M}$ and a $*$-homomorphism $\psi : q\mathscr{B}q \to \mathscr{R} := \psi(q\mathscr{B}q) \subseteq r\mathscr{L}(P)r$ such that $\psi(x)v = vx$ for all $x \in q\mathscr{B}q$. Note that $vv^* \in R' \cap r\mathscr{M}r$. Since $\mathscr{R} \subseteq r\mathscr{L}(P)r$ is diffuse, and $P \leqslant M \rtimes P$ is a malnormal subgroup (part c) in Theorem 3.1.1), we have that $\mathfrak{QN}_{r\mathscr{M}r}(\mathscr{R})'' \subseteq r\mathscr{L}(P)r$. Thus $vv^* \in r\mathscr{L}(P)r$ and hence $vq\mathscr{B}qv^* = \mathscr{R}vv^* \subseteq r\mathscr{L}(P)r$. Extending v to a unitary v_0 in $\mathscr{M}$ we have that $v_0q\mathscr{B}qv_0^* \subseteq \mathscr{L}(P)$. As $\mathscr{L}(P)$ and $\mathscr{B}$ are factors, after perturbing v_0 to a new unitary u, we may assume that $u\mathscr{B}u^* \subseteq \mathscr{L}(P)$. This further implies that $upu^* \in \mathscr{L}(P)$ and since $\Theta(1) = p$ we also have

$$\mathscr{B} = p\mathscr{B}p \subseteq pu^*\mathscr{L}(P)up. \tag{5.8}$$

Next we claim that

$$pu^*\mathscr{L}(P)up \prec_{\mathscr{M}} \mathscr{B} \tag{5.9}$$

To see this first notice that, since P is malnormal in $M \rtimes P$ and P is icc (see parts a) and c) in Theorem 3.1.1) then $(pu^*\mathscr{L}(P)up)' \cap \Theta(\mathscr{L}(N \rtimes Q)) = (pu^*\mathscr{L}(P)up)' \cap p\mathscr{M}p = u^*(\mathscr{L}(P)' \cap \mathscr{L}(M \rtimes Q))up = \mathbb{C}p$. Thus using Theorem 5.1.3 we have either a) $pu^*\mathscr{L}(P)up \prec^s_{p\mathscr{M}p} \mathscr{A}$ or b) $pu^*\mathscr{L}(P)up \prec^s_{p\mathscr{M}p} \mathscr{B}$. Assume a) holds. By part 1. we have $pu^*\mathscr{L}(P)up \prec^s_{p\mathscr{M}p} \mathscr{A} = vp\mathscr{L}(M)pv^*$; in particular, this implies that $\mathscr{L}(P) \prec_{\mathscr{M}} \mathscr{L}(M)$ but this contradicts the fact that $\mathscr{L}(M)$ and $\mathscr{L}(P)$ are diffuse algebras that are τ-perpendicular in $\mathscr{M}$. Thus b) holds which proves the claim.

Using (5.9) together with malnormality of $\Theta(\mathscr{L}(Q))$ inside $\Theta(\mathscr{L}(N \rtimes Q))$ and arguing exactly as in the proof of relation (5.8) we conclude that there exists $w \in \mathscr{U}(p\mathscr{M}p)$ such that

$$wpu^*\mathscr{L}(P)upw^* \subseteq \mathscr{B}. \tag{5.10}$$

Combining (5.8) and (2.19) we get that $w\mathscr{B}w^* \subseteq wpu^*\mathscr{L}(P)upw^* \subseteq \mathscr{B}$ and hence $w \in \mathfrak{QN}_{p\mathscr{M}p}(\mathscr{B})$" $=$

$\mathscr{B}$. Thus we get

$$pu^*\mathscr{L}(P)up \subseteq w^*\mathscr{B}w = \mathscr{B}. \tag{5.11}$$

Combining (5.8) and (5.11) we get the theorem. □

Finally, we are now ready to derive the main result of this paper.

Theorem 5.1.6. *[CDHK20, Theorem 4.6] Let $N \rtimes Q, M \rtimes P \in \mathcal{S}$ with $N = N_1 \times N_2$ and $M = M_1 \times M_2$. Let $p \in \mathscr{L}(M \rtimes P)$ be a projection and assume that $\Theta : \mathscr{L}(N \rtimes Q) \to p\mathscr{L}(M \rtimes P)p$ is a $*$-isomorphism. Then $p = 1$ and one can find $*$-isomorphisms, $\Theta_i : \mathscr{L}(N_i) \to \mathscr{L}(M_i)$, a group isomorphism $\delta : Q \to P$, a multiplicative character $\eta : Q \to \mathbb{T}$, and a unitary $u \in \mathcal{U}(\mathscr{L}(M \rtimes P))$ such that for all $g \in Q$, $x_i \in N_i$ we have that*

$$\Theta((x_1 \otimes x_2)u_g) = \eta(g)u(\Theta_1(x_1) \otimes \Theta_2(x_2)v_{\delta(g)})u^*.$$

Proof. Throughout this proof we will denote by $\mathscr{M} = \mathscr{L}(N \rtimes Q)$. Using Theorem 5.1.4, and replacing Θ by $\Theta \circ Ad(v)$ if necessary, we may assume that $\Theta(\mathscr{L}(N)) = p\mathscr{L}(M)p$. By Theorem 5.1.5, there exists $u \in \mathcal{U}(\mathscr{M})$ such that $\Theta(\mathscr{L}(Q)) \subseteq u^*\mathscr{L}(P)u$, where $\mathscr{M} = \mathscr{L}(M \rtimes P)$. Moreover $\Theta(1) = p$, $upu^* \in \mathscr{L}(P)$ and also $\Theta(\mathscr{L}(Q)) = pu^*\mathscr{L}(P)up$. Next we denote by $\Gamma = u^*Pu$ and by $\mathscr{G} = \{\Theta(u_g) : g \in Q\}$. Using these notations we show the following

Claim 5.1.7. $h_\Gamma(\mathscr{G}) > 0$.

Proof of Claim 5.1.7. Notice that $\mathscr{G} \subseteq \mathscr{L}(\Gamma)$ is a group of unitaries normalizing $\Theta(\mathscr{L}(N))$. Moreover, by Theorem 3.1.1 we can see that the action $\sigma : P \to \mathrm{Aut}(M)$ satisfies all the conditions in the hypothesis of Theorem 2.1.31 and thus using the conclusion of the same theorem we get the claim. ■

Claim 5.1.8. *Let $e \neq g \in \Gamma$. Then $\mathscr{G}'' \nprec \mathscr{L}(C_\Gamma(g))$.*

Proof of Claim 5.1.8. Since Γ is isomorphic to the product of two biexact groups, say $\Gamma_1\times\Gamma_2$, by Lemma 3.1.2 we get that $C_\Gamma(g)=A$, $\Gamma_1\times A$, or $A\times\Gamma_2$ for an amenable group A. If $C_\Gamma(g)=A$ then since $\mathscr{G}$ is non-amenable we clearly have $\mathscr{G}''\nprec\mathscr{L}(C_\Gamma(g))$. Next assume $C_\Gamma(g)=A\times\Gamma_2$ and assume by contradiction that $\mathscr{G}''\prec\mathscr{L}(C_\Gamma(g)$. As $Q=Q_1\times Q_2$ for Q_i property (T) icc group, then $\mathscr{G}''=\Theta(\mathscr{L}(Q_1))\bar{\otimes}\Theta(\mathscr{L}(Q_2))$ is a II_1 factor with property (T). Since $\mathscr{G}''\prec\mathscr{L}(A\times\Gamma_2)=\mathscr{L}(A)\bar{\otimes}\mathscr{L}(\Gamma)$ and $\mathscr{L}(A)$ is amenable then it follows that $\mathscr{G}''\prec\mathscr{L}(\Gamma)$. However by [Oz03, Theorem 1] this is impossible as $\mathscr{L}(\Gamma_2)$ is solid and $\mathscr{G}''$ is generated by two non-amenable commuting subfactors. The case $C_\Gamma(g)=\Gamma_1\times A$ follows similarly. ■

Claim 5.1.9. *The unitary representation $\{\text{Ad}(v)\}_{v\in\mathscr{G}}$ on $L^2(p\mathscr{L}(\Gamma)p\ominus\mathbb{C}p)$ is weakly mixing.*

Proof of Claim 5.1.9. First note we have that $\Theta(\mathscr{L}(Q))=\mathscr{G}''=p\mathscr{L}(\Gamma)p$. Also since Q is icc then using [CSU13, Proposition 3.4] the representation $\text{Ad}(Q)$ on $L^2(\mathscr{L}(Q)\ominus\mathbb{C})$ is weak mixing. Combining these two facts, we get that the representation $\mathscr{G}$ on $L^2(p\mathscr{L}(\Gamma)p\ominus\mathbb{C}p)$ is weak mixing, as desired. ■

Claims 2-4 above together with Theorem 2.1.30 show that $p=1$ and moreover there exists unitary $w\in\mathscr{L}(M\rtimes P)$, a group isomorphism $\delta:Q\to P$ and a multiplicative character $\eta:Q\to\mathbb{T}$ such that $\Theta(u_g)=\eta(g)wv_{\delta(g)}w^*$ for all $g\in Q$. Since $\Theta(\mathscr{L}(N))=\mathscr{L}(M)$ then the same argument as in proof of [CD19, Theorem 4.5] (lines 10-27 on page 25) shows that i) $w^*\mathscr{L}(M)w\subseteq\mathscr{L}(M)$. However re-writing the previous relation as $v_h=\overline{\eta(g)}w^*\Theta(u_{\delta^{-1}(h)})w$ for all $h\in P$ and applying the same argument as above for the decomposition $\mathscr{M}=\Theta(\mathscr{L}(N))\rtimes\Theta(Q)$ we get that ii) $w\Theta(\mathscr{L}(N))w^*\subseteq\Theta(\mathscr{L}(N))$. Then combining i) and ii) we get that $w^*\mathscr{L}(M)w=\mathscr{L}(M)$. Now from the above relations it follows clearly that the map $\Psi=\text{ad}(w^*)\circ\Theta:\mathscr{L}(N)\to\mathscr{L}(M)$ is a $*$-isomorphism, and $\Theta(xu_g)=\eta(g)w(\Psi(x)u_{\delta(g)})w^*$ for all $x\in\mathscr{L}(N)$. Finally, proceeding as in the proof of [CDK19, Theorem 5.1] one can further show that the isomorphism Ψ arises from a tensor of $*$-isomorphisms $\Phi_i:\mathscr{L}(N_i)\to\mathscr{L}(M_i)$. We leave these details to the reader. □

Corollary 5.1.10. *For any $G=N\rtimes Q\in\mathcal{S}$ the fundamental group of $\mathscr{L}(G)$ is trivial, i.e. $\mathscr{F}(\mathscr{L}(G))=1$.*

While it is well known that there exist many families of pairwise non-isomorphic II_1 factors with property (T), much less is known about producing concrete such examples. Our Corollary 5.1.10 shades new light in this direction.

Corollary 5.1.11. *For any $G = N \rtimes Q \in \mathcal{S}(Q)$ or $G = G_1 \times ... \times G_n$ with $G_i \in \mathcal{V}$ then the set of all amplifications $\{\mathscr{L}(G)^t : t \in (0,\infty)\}$ consists of pairwise non-isomorphic II_1 factors with property (T).*

Proof. The statement follows trivially from Corollary 5.1.10, Theorem 5.2.1 and the definition of fundamental group. □

5.2 Fundamental Group of Factors Arising from Class $\mathcal{V}$

In this section we describe another class of examples of property (T) factors with trivial fundamental group. These factors arise as group von Neumann algebras $\mathscr{L}(\Gamma_n)$, with $\Gamma_n \in \mathcal{V}$. We refer the reader to section 3.2 for elementary properties of these groups, and their von Neumann algebras. These factors are a minor variant of group factors studied in [Va04]. In combination with Gaboriau's ℓ^2-Betti numbers invariants [Ga02] and Popa–Vaes's Cartan rigidity results [PV12] we obtain a countable family of type II_1 group factors $(\mathscr{L}(\Gamma_n))_{n\geq 2}$ with property (T), with trivial fundamental group, that possess a unique Cartan subalgebra up to unitary conjugacy (see Theorem 3.2.1), and that are pairwise stably non-isomorphic. We also show that products of finitely groups in class $\mathcal{V}$ give rise to property (T) type II_1 factors with trivial fundamental group.

Theorem 5.2.1. *[CDHK20, Theorem 5.1] For every $n \geq 2$, let $\Gamma_n = \mathbb{Z}^{4(n+1)} \rtimes \Lambda_n \in \mathcal{V}$, and $\mathscr{M}_n = \mathscr{L}(\Gamma_n)$. The following properties hold true.*

(i) *For every $n \geq 2$, $\mathscr{M}_n$ has trivial fundamental group.*

(ii) *The type II_1 factors $(\mathscr{M}_n)_{n\geq 2}$ are pairwise stably non-isomorphic.*

iii) *Assume that $\Gamma_{n_i} \in \mathcal{V}$ and $\Gamma = \Gamma_{n_1} \times ... \times \Gamma_{n_k}$, where $n_i \geq 2$ for all i. Then the fundamental group satisfies $\mathscr{F}(\mathscr{L}(\Gamma)) = \{1\}$.*

Proof. (i) Denote by $\mathscr{R}_n$ the orbit equivalence relation induced by the essentially free ergodic probability measure-preserving action $\Lambda_n \curvearrowright \mathbb{T}^{4(n+1)}$. Then we have $L(\mathscr{R}_n) = \mathscr{M}_n$ and [PV12, Theorem 1.4] implies that $\mathscr{F}(\mathscr{M}_n) = \mathscr{F}(\mathscr{R}_n)$. Using Borel's result [Bo83], the n-th ℓ^2-Betti number of Λ_n is nonzero and finite. Then a combination of [Ga02, Corollaire 3.16] and [Ga02, Corollaire 5.7] implies that $\mathscr{F}(\mathscr{R}_n) = \{1\}$. This further implies that $\mathscr{F}(\mathscr{M}_n) = \{1\}$.

(ii) Let $m, n \geq 2$ and $t > 0$ so that $(\mathscr{M}_n) \cong (\mathscr{M}_m)^t$. Then [PV12, Theorem 1.4] implies that $\mathscr{R}_n \cong (\mathscr{R}_m)^t$. Then [Ga02, Corollaire 0.4] (see also [CZ88]) further implies that $m = n$.

(iii) Using the Kunneth formula for ℓ^2-Betti numbers, we see that the n-th ℓ^2-Betti number of $\Lambda_{n_1} \times \cdots \times \Lambda_{n_k}$ is nonzero and finite. Arguing exactly as in the proof of (i), we get that $\mathscr{F}(\mathscr{L}(\Gamma)) = \{1\}$. $\square$

Let us point out that we could have directly applied [Va04, Theorem 4] to the adjoint group of $Sp(n,1)$ in order to obtain examples of icc groups that satisfy the conclusion of the theorem. Instead, we adapted the explicit and simpler construction given in [Va04, Example 1, (a)] to the case of $Sp(n,1)$.

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