A REVELATION PRINCIPLE FOR DOMINANT STRATEGY IMPLEMENTATION

by

Jesse A. Schwartz and Quan Wen

Working Paper No. 08-W19

October 2008

DEPARTMENT OF ECONOMICS
VANDERBILT UNIVERSITY
NASHVILLE, TN 37235

www.vanderbilt.edu/econ

A Revelation Principle for Dominant Strategy Implementation*

Jesse A. Schwartz[†]
Kennesaw State University

Quan Wen[‡]
Vanderbilt University

October 2008

Abstract

We introduce a *perfect price discriminating* (PPD) mechanism for allocation problems with private information. A PPD mechanism treats a seller, for example, as a perfect price discriminating monopolist who faces a price schedule that does not depend on her report. In any PPD mechanism, every player has a dominant strategy to truthfully report her private information. We establish a revelation principle for dominant strategy implementation: any outcome that can be dominant strategy implemented can also be dominant strategy implemented using a PPD mechanism. We apply this principle to derive the optimal, budget-balanced, dominant strategy mechanisms for public good provision and bilateral bargaining.

Keywords: Dominant strategy implementation, Vickrey-Clarke-Groves mechanisms, public good provision, bilateral bargaining

JEL Classification Numbers: C72 (Noncooperative Games), C78 (Bargaining Theory), D44 (Auctions), D82 (Asymmetric and Private Information), H41 (Public Goods)

*We would like to thank Brett Katzman and Herve Moulin for their comments and suggestions.

[†]Department of Economics, Finance, and Quantitative Analysis, Kennesaw State University, 1000 Chastain Road, Box 0403, Kennesaw, GA 30144, U.S.A. Email: jschwar7@kennesaw.edu

[‡]Department of Economics, Vanderbilt University, VU Station B #351819, 2301 Vanderbilt Place, Nashville, TN 37235-1819, U.S.A. Email: quan.wen@vanderbilt.edu

1 Introduction

In allocation problems with incomplete information, economists search for mechanisms that implement a desirable outcome in some equilibrium, such as Bayesian-Nash and dominant strategy equilibrium. Although dominant strategy equilibrium is more restrictive than Bayesian-Nash, it has many desirable properties. For example, dominant strategies are not sensitive to the belief of any player about the other players' information or strategies.[1] However, the best outcome under Bayesian-Nash implementation may not be implementable using dominant strategies. Take the well known example of a public good: the outcome that is both budget-balanced and efficient can be Bayesian-Nash implemented by the expected externality mechanism, but cannot be dominant strategy implemented by any mechanism.[2] If the performance shortfall is not too big, the mechanism designer may decide in favor of a dominant strategy mechanism. In this paper, we provide a revelation principle that characterizes all dominant strategy mechanisms for allocation problems, thus facilitating the search for optimal dominant strategy mechanisms.

We introduce *perfect price discriminating* (PPD) mechanisms and show that they play a defining role in dominant strategy implementation. In a PPD mechanism, each player faces a price schedule that is exogenous to her report. To a buyer, this price schedule gives the prices she would have to pay for each unit, such as \$10 for the first unit, \$15 for the second unit, etc. To a seller, this price schedule gives the prices that she would be paid for each unit. A player cannot change the prices that she would pay (or be paid) for any unit. Although a player's price schedule does not depend on her own report, it generally depends on the other players' reports. A player first reports her type to the mechanism designer, who then determines the quantity that maximizes the player's payoff, given the price schedule and the player's reported type. Thus, the PPD mechanism treats a seller as a perfect price

[1] For other advantages of dominant strategy implementation, see Mookherjee and Reichelstein (1992) and the references therein.

[2] See, for example, Fudenberg and Tirole (1992, pp 271-274) for details.

discriminating monopolist and a buyer as a perfect price discriminating monopsonist. The well known Vickrey auction is one example of a PPD mechanism. Many other mechanisms in the literature can also be formulated as PPD mechanisms, as we demonstrate in the conclusion.

As in the Vickrey auction, every player has a dominant strategy to report her type truthfully in any PPD mechanism (Proposition 4). Given the exogenous nature of the price schedule that a player faces, misreporting her type can only mess up the optimization problem to maximize this player's payoff. We establish a revelation principle for dominant strategies (Proposition 5): every outcome that can be dominant strategy implemented can also be dominant strategy implemented with a PPD mechanism. This result identifies the fundamental cause for dominant strategies that is universal to all allocation problems. We refer to this result as a revelation principle because of its similar flavor to the original Revelation Principle: every outcome that can be Bayesian-Nash implemented can also be Bayesian-Nash implemented with a direct mechanism.[3] With the original Revelation Principle, a mechanism designer may restrict attention to direct mechanisms when searching for Bayesian-Nash implementation. With our revelation principle, a mechanism designer may restrict attention to PPD mechanisms when searching for dominant strategy implementation.

Building PPD mechanisms involves constructing the price schedule for each player. Often, specific goals like budget-balancedness or specific features of the environment considerably restrict the price schedules. In other words, the special format of PPD mechanisms in itself greatly simplifies the search for dominant strategy mechanisms, and then particulars of the problem may further simplify the search. To show how to put our revelation principle to work, we build PPD mechanisms for public good provision and bilateral bargaining problems with budget-balancedness in mind. The nature of these problems impose severe quantity restrictions. In the public good problem, one player obtains the public good if and only if

[3] Myerson (2008) gives a historical perspective on mechanism design and the role of the Revelation Principle.

every player also obtains the public good. In the bargaining problem, the buyer receives the good if and only if the seller relinquishes the good. We exploit these restrictions in sections 4 and 5 to find the budget-balanced dominant strategy mechanisms that maximize the players' payoffs.

There are several strands of literature related to our work. Green and Laffont (1977) and Holmstrom (1979) find that essentially all efficient, dominant strategy mechanisms are Vickrey-Clarke-Groves mechanisms.[4] Our Proposition 5 shows that any dominant strategy incentive compatible mechanism must be a PPD mechanism, so in a way we generalize the Vickrey-Clarke-Groves mechanisms relaxing the requirement of efficiency. Mookherjee and Reichelstein (1992) identify sufficient conditions that ensure there is no welfare loss in strengthening the equilibrium concept from Bayesian-Nash to dominant strategy. However, these sufficient conditions may fail. In Myerson and Satterthwait's (1982) bilateral bargaining, our results allow us to find the optimal dominant strategy mechanism, and for the example when the buyer's and seller's values are uniformly distributed, we quantify that the best dominant strategy implementation costs 11% of the gains available in the best Bayesian-Nash implementation. The mechanism designer would weigh this loss against the informational benefits of using dominant strategies. Our result for bilateral bargaining complements Hagerty and Rogerson (1987), who show that fixed-price mechanisms are essentially the only budget-balanced, dominant strategy mechanisms. Our revelation principle allows us immediately to show that the optimal, budget-balanced, dominant strategy mechanism is a fixed-price mechanism. Other papers have also considered sacrificing some efficiency in favor of dominant strategies, showing that losses decrease as the number of players increase. McAfee (1992) does this for double auctions, and Moulin (2007a, b) does this for single-unit and multi-unit rationing problems. Moulin (2007a, b) also contain references to several other papers in this vein.

The rest of this is organized as follows. In Section 2, we first lay out the basic problem

[4] See Milgrom (2004, pages 71-73) for a thoughtful treatment of the Green-Laffont-Holmstrom theorem.

and then introduce PPD mechanisms. In Section 3, we establish the main result: any direct mechanism is dominant strategy incentive compatible if and only if it is a PPD mechanism. Applying this principle, we solve for the optimal, budget-balanced, dominant strategy mechanisms for public good provision in Section 4 and bilateral bargaining in Section 5. In Section 6, we conclude by reformulating a number of well known dominant strategy mechanisms as PPD mechanisms.

2 Allocation Mechanisms

We consider allocation schemes among players with private values. Let $N = \{1, \ldots, n\}$ be the set of n players. Let θ_i be player i's type for $i \in N$, $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$ be the type profile of player i's opponents, and $\theta = (\theta_i, \theta_{-i})$ be the profile of player types. Assume θ is jointly distributed on $[0,1]^n$ according to distribution function $F(\cdot)$. For each $i \in N$, let $F_i(\cdot)$ denote the marginal distribution of θ_i. An outcome $(q,t) = (q_1, \ldots, q_n, t_1, \ldots, t_n) \in R^n \times R^n$ specifies each player i's quantity q_i and payment t_i. Generally, each player i's utility may depend on the entire outcome and the type profile. In this paper, we focus on the private value case where each player i has quasilinear utility that depends only on her own type θ_i and her part of the outcome (q_i, t_i):

$$u_i(q_i, t_i, \theta) = v_i(q_i, \theta_i) - t_i.$$

Assume that $v_i(0, \theta_i) = 0$ for all $\theta_i \in [0,1]$. Also assume that for all $\theta_i \in [0,1]$ and $q_i \in R$, player i's valuation function $v_i(\cdot, \theta_i) : R \to R$ satisfies the following standard conditions:

$$v_i^1(q_i, \theta_i) \equiv \partial v_i(q_i, \theta_i)/\partial q_i \geq 0, \tag{1}$$

$$v_i^1(q_i, \theta_i) \geq v_i^1(q_i, \theta_i') \quad \text{for } \theta_i > \theta_i'. \tag{2}$$

Condition (1) states that player i's valuation function is nondecreasing in quantity. Condition (2) is the single-crossing condition, which implies that player i's demand curve increases with her type $\theta_i \in [0,1]$. To apply Milgrom's (2004) envelope theorem, we also assume that

$v_i^2(q_i, \theta_i) \equiv \partial v_i(q_i, \theta_i)/\partial\theta_i$ exists almost everywhere and is bounded for each $i \in N$. As an aside, player i buys if $q_i > 0$ and sells if $q_i < 0$.

A general mechanism specifies a set of actions for each player. A player's strategy maps her type to an action. The mechanism also specifies a mapping from action profiles to outcomes. This induces a game of incomplete information. The performance (such as efficiency) of a mechanism is the compound mapping from type profiles to outcomes through the equilibrium strategy profile and the mechanism itself. By convention, a mechanism *dominant strategy* implements or *Bayesian-Nash* implements a particular performance if the equilibrium is dominant strategy or Bayesian-Nash. A direct mechanism is a special class of mechanisms where the action sets are identical to the type sets, with the interpretation that the mechanism designer asks each player to reveal her type. More specifically,

Definition 1 *In a direct mechanism, each player $i \in N$ reports her type as $\hat{\theta}_i \in [0,1]$. The mechanism maps the reported profile $\hat{\theta} \in [0,1]^n$ to an outcome $(q(\hat{\theta}), t(\hat{\theta}))$:*

$$q(\cdot) : [0,1]^n \to Q \quad \text{and} \quad t(\cdot) : [0,1]^n \to R^n,$$

where $Q \subseteq R^n$ denotes the set of permissible quantity profiles.

Given any reported type profile $\hat{\theta}$, player i will obtain quantity $q_i(\hat{\theta})$ and pay $t_i(\hat{\theta})$. Thus, any direct mechanism induces a well-defined game of incomplete information. In the rest of this paper, we simply refer to a direct mechanism by its quantity and payment rules $(q(\cdot), t(\cdot))$. In the game of incomplete information induced by a direct mechanism, a player may or may not report her type truthfully in a Bayesian-Nash equilibrium. The well known Revelation Principle states that every performance that can be Bayesian-Nash implemented can also be Bayesian-Nash implemented using a direct mechanism, where players truthfully report their types.[5] Thus, without loss of generality, we restrict our attention to direct mechanisms. We focus on direct mechanisms that are dominant strategy incentive compatible:

[5] See Milgrom (2004, pp. 39-42) for a more comprehensive description of general mechanisms, and Krishna (2002, Proposition 5.1) for a simple treatment of the Revelation Principle.

Definition 2 *A direct mechanism is dominant strategy incentive compatible (DSIC) if for all $i \in N$, and for all θ_i, $\hat{\theta}_i$ and θ_{-i},*

$$u_i\left(q_i(\theta_i, \theta_{-i}), t_i(\theta_i, \theta_{-i}), \theta_i\right) \geq u_i\left(q_i(\hat{\theta}_i, \theta_{-i}), t_i(\hat{\theta}_i, \theta_{-i}), \theta_i\right).$$

In other words, it is always optimal for a player to truthfully report her type, no matter what the other players report.

In this paper, we establish a revelation principle for dominant strategy implementation: every performance that can be dominant strategy implemented can also be dominant strategy implemented using a perfect price discriminating mechanism. A perfect price discriminating mechanism specifies a price schedule $p_i(q_i, \hat{\theta}_{-i})$ for each player that is used in determining this player's part of the outcome. The price schedule to any player does not depend on this player's own report but may depend on the others' reports. Formally, we have

Definition 3 *A direct mechanism (q, t) is a perfect price discriminating (PPD) mechanism if for all $i \in N$ and all $\hat{\theta} \in [0,1]^n$, these exist a price schedule $p_i(\cdot, \cdot) : R_+ \times [0,1]^{n-1} \to R_+$ and a lump-sum payment $L_i(\cdot) : [0,1]^{n-1} \to R$ such that*

$$\begin{aligned} q_i(\hat{\theta}_i, \hat{\theta}_{-i}) &\in \arg\max_{q_i} \int_0^{q_i} \left[v_i^1(z, \hat{\theta}_i) - p_i(z, \hat{\theta}_{-i})\right] dz, & (3) \\ t_i(\hat{\theta}_i, \hat{\theta}_{-i}) &= \int_0^{q_i(\hat{\theta}_i, \hat{\theta}_{-i})} p_i(z, \hat{\theta}_{-i}) dz + L_i(\hat{\theta}_{-i}). & (4) \end{aligned}$$

By (4), player i pays the area under her price schedule, up to the quantity she wins, plus a lump-sum payment that is independent of her report. Given the payment rule and the reported profile $\hat{\theta}$, the PPD mechanism assumes that player i has reported truthfully and then chooses q_i to maximize her payoff, since (3) and (4) imply that

$$q_i(\hat{\theta}_i, \hat{\theta}_{-i}) \in \arg\max_{q_i} \left[v_i(q_i, \hat{\theta}_i) - t_i(\hat{\theta}_i, \hat{\theta}_{-i})\right].$$

Figure 1 illustrates player i's part of the outcome (q_i, t_i) in a PPD mechanism for the case when $L_i(\hat{\theta}_{-i}) = 0$. If we interpret $v_i^1(\cdot, \theta_i)$ as the inverse demand curve of player i of type θ_i, and if we interpret $p_i(\cdot, \hat{\theta}_{-i})$ as the inverse supply curve player i faces, then the mechanism chooses the "market-clearing" quantity. But the payment is that of a buyer who can perfectly price discriminate, paying the height of the supply curve for each unit she buys.

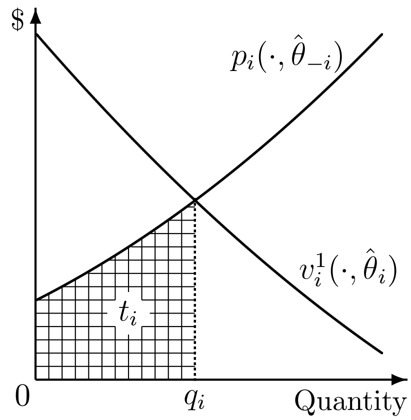

Figure 1: Player i's outcome in a PPD mechanism

3 Revelation Principle

In any PPD mechanism, each player faces a price schedule that is exogenous to her report. The player cannot change the price that she would pay for any unit. In this situation, the mechanism determines the quantity for her that maximizes her payoff, assuming she has told the truth about her type. The player could do no better if she were allowed to choose for herself how much quantity to purchase at the exogenous price schedule. In other words, the mechanism does the optimization for the player. Lying about her type can only cause the mechanism to mess up the optimization. This gives the player a dominant strategy to report her type truthfully. We next formalize this intuition with Proposition 4.

Proposition 4 *Any PPD mechanism is dominant strategy incentive compatible.*

Proof. Consider any θ_{-i} and any PPD mechanism with price schedules and lump-sum payments $\{p_i(\cdot, \cdot), L_i(\cdot)\}_{i \in N}$. Equation (4) implies that player i's report $\hat{\theta}_i$ can only affect her payment insofar as it affects the quantity player i is awarded. Given θ_{-i}, note that $q_i(\hat{\theta}_i, \theta_{-i})$ may or may not be equal to $q_i(\theta_i, \theta_{-i})$ for $\hat{\theta}_i \neq \theta_i$. Equations (3) and (4) then imply that

$$\begin{aligned} u_i(q_i(\theta_i,\theta_{-i}), t_i(\theta_i,\theta_{-i}), \theta_i) &= \max_{q_i} \int_0^{q_i} \left[v_i^1(z,\theta_i) - p_i(z,\theta_{-i})\right] dz - L_i(\theta_{-i}) \\ &\geq \int_0^{q_i(\hat{\theta}_i,\theta_{-i})} \left[v_i^1(z,\theta_i) - p_i(z,\theta_{-i})\right] dz - L_i(\theta_{-i}) \\ &= u_i(q_i(\hat{\theta}_i,\theta_{-i}), t_i(\hat{\theta}_i,\theta_{-i}), \theta_i). \end{aligned}$$

In other words, the PPD mechanism is DSIC. ■

Of course, the intuition for why a player wants to report her type truthfully is familiar from the Vickrey auction, but observe that the proof to Proposition 4 in no way depends on the allocative efficiency of the mechanism. What we show in the next in Proposition is the more astounding result that any mechanism which induces truthful revelation as a dominant strategy must be a PPD mechanism.

Proposition 5 [***Revelation Principle for Dominant Strategy Implementation***] *Any DSIC direct mechanism can be implemented with a PPD mechanism.*

Proof. Consider any DSIC direct mechanism (q, t). The idea of our proof is to construct an individual price schedule $p_i(\cdot, \cdot)$ and lump-sum payment $L_i(\cdot)$ for each player $i \in N$ such that (q, t) can be obtained using equations (3) and (4). Focus on player i. Fix an arbitrary θ_{-i} and let $q_i(\theta_i) \equiv q_i(\theta_i, \theta_{-i})$, suppressing the dependence on θ_{-i} for simplicity, and similarly throughout this proof. The incentive compatibility and the single-crossing condition (assumption 2) guarantees that $q_i(\theta_i)$ is nondecreasing (Theorem 7.2 of Fudenberg and Tirole, 1991). For all $q_i \in [q_i(0), q_i(1)]$, define

$$\overline{\theta}_i(q_i) = \begin{cases} \inf\{\theta_i \in [0,1] : q_i(\theta_i) \geq q_i\} & \text{for } q_i \leq q_i(1) \\ 1 & \text{otherwise.} \end{cases}$$

Observe that if $q_i(\cdot)$ is strictly increasing then $\overline{\theta}_i(\cdot)$ is simply the inverse of $q_i(\cdot)$, or to put it differently $\overline{\theta}_i(q_i)$ is the type of player i who would win q_i units in the mechanism. Let player i's price schedule be

$$p_i(q_i) = v_i^1(q_i, \overline{\theta}_i(q_i)). \tag{5}$$

Note that neither $\overline{\theta}_i(q_i)$ nor $p_i(q_i)$ depend on player i's reported type, although they do generally depend on θ_{-i}. Consider

$$\max_{q_i} \int_0^{q_i} \left[v_i^1(z, \theta_i) - v_i^1(z, \overline{\theta}_i(z))\right] dz.$$

For all $z < q_i(\theta_i)$, the monotonicity of $q_i(\cdot)$ guarantees that $\theta_i \geq \overline{\theta}_i(z)$. The single-crossing condition (2) then implies the integrand is nonnegative. Likewise, for all $z > q_i(\theta_i)$, the integrand is nonpositive. Therefore,

$$q_i(\theta_i) \in \arg\max_{q_i} \int_0^{q_i} \left[v_i^1(z, \theta_i) - v_i^1(z, \overline{\theta}_i(z))\right] dz,$$

thereby establishing (3) for the price schedule in (5).

Now consider the PPD mechanism (q, t^*) where

$$t_i^*(\hat{\theta}_i) = \int_0^{q_i(\hat{\theta}_i)} p_i(z) dz.$$

By Proposition 4, (q, t^*) is a DSIC direct mechanism. Applying the envelope theorem[6] to both (q, t) and (q, t^*), we have

$$v_i(q_i(\theta_i), \theta_i) - t_i(\theta_i) = v_i(q_i(0), 0) - t_i(0) + \int_0^{\theta_i} v_i^2((q_i(z), z) dz, \quad (6)$$

$$v_i(q_i(\theta_i), \theta_i) - t_i^*(\theta_i) = v_i(q_i(0), 0) - t_i^*(0) + \int_0^{\theta_i} v_i^2((q_i(z), z) dz. \quad (7)$$

From (6) and (7), we have

$$t_i(\theta_i) = t_i^*(\theta_i) - t_i^*(0) + t_i(0) = \int_0^{q_i(\theta_i)} p_i(z) dz + L_i,$$

where $L_i \equiv -t_i^*(0) + t_i(0)$ does not depend on θ_i. Thus, $t_i(\cdot)$ satisfies (4) for the price schedule in (5), so indeed, (q, t) is a PPD mechanism. ■

This proof makes familiar use of envelope theorem (Milgrom and Segal, 2002 and Milgrom 2004) once we show that the quantity rule we specify in our PPD mechanism is identical to the quantity rule in the original DSIC mechanism. The PPD mechanism we construct

[6] We make use of the envelope theorem as given in Theorem 3.1 of Milgrom (2004). See Milgrom and Segal (2002) for a more general treatment of the envelope theorem.

in the proof to Proposition 5 shares some features with the mechanism in Ausubel and Cramton (2004). However, they impose efficiency and then show that their mechanism is DSIC, whereas we show that all DSIC mechanisms must resemble the Vickrey auction.

4 A Public Good Problem

Consider a simple public good problem between two players of whether to build a public good or not. We assume this is a pure public good, so that it is not affordable to exclude anyone from using the public good. For concreteness, we may think of two island residents who consider having a bridge built. Because the amount of public good needs to be the same for every player, there are two permissible quantity profiles: $(q_1, q_2) \in Q = \{(0,0),(1,1)\}$, where $(0,0)$ means the public good will not be built and $(1,1)$ means that it will. Each player has linear utility $\theta_i q_i - t_i$. Player i's private value θ_i is continuously distributed on $[0,1]$ with distribution function $F_i(\cdot)$ and positive density $f_i(\cdot)$. The public good costs $c \in (1,2)$ so that it takes both players to pay for the public good.

A mechanism is *budget-balanced* if $t_1 + t_2 = c$ whenever $(q_1, q_2) = (1,1)$ and if $t_1 + t_2 = 0$ whenever $(q_1, q_2) = (0,0)$. A mechanism is *feasible* if $t_1 + t_2 \geq c$ whenever $(q_1, q_2) = (1,1)$ and if $t_1 + t_2 \geq 0$ whenever $(q_1, q_2) = (0,0)$. A mechanism is *individually rational* if each player's utility is nonnegative in the equilibrium for all $\theta_1 \times \theta_2 \in [0,1]^2$. An *allocatively efficient* performance requires that for all $\theta_1 \times \theta_2 \in [0,1]^2$: $(q_1, q_2) = (1,1)$ if and only if $\theta_1 + \theta_2 \geq c$. It is well known that there is no individually rational mechanism in this setting that is DSIC, allocatively efficient, and budget-balanced (or even feasible). The mechanism of Arrow (1979) and d'Aspremont and Gerard-Varet (1979a, b) forgo DSIC in order to achieve allocative efficiency and budget-balancedness.[7] Alternatively, we forgo allocative efficiency, and instead search for the DSIC mechanism that maximizes ex ante surplus among all DSIC mechanisms that are feasible. Appealing to Proposition 5, we may restrict attention to PPD

[7]See Krishna (2002) or Milgrom (2004) for more on the impossibility results for DSIC mechanisms and the possibility results for the weaker Bayes-Nash implentation.

mechanisms. Constructing PPD mechanisms requires constructing the price schedule for each player. In this public good problem, the small number of permissible quantity profiles and the feasibility constraint helps us pin down the price schedules, as we show next.

Suppose that player 2 reports θ_2 and player 1 reports $\hat{\theta}_1$, and denote player 1's price schedule by $p_1(q_1, \theta_2)$. In any PPD mechanism for this public good problem, the quantity for player 1 is chosen to solve the following problem:

$$\max_{q_1 \in \{0,1\}} \int_0^{q_1} \left[\hat{\theta}_1 - p_1(z, \theta_2)\right] dz.$$

Maximization then entails comparing 0 (when $q_1 = 0$) to $\hat{\theta}_1 - \int_0^1 p_1(z, \theta_2)dz$ (when $q_1 = 1$). Without loss of generality, we may restrict to price schedules that are constants with respect to quantities by letting:

$$p_1^*(\theta_2) \equiv \int_0^1 p_1(z, \theta_2)dz \quad \text{and} \quad p_2^*(\theta_1) \equiv \int_0^1 p_2(z, \theta_1)dz.$$

Thus, in the PPD mechanism $q_1 = 1$ whenever $\theta_1 > p_1^*(\theta_2)$ and $q_1 = 0$ whenever $\theta_1 < p_1^*(\theta_2)$, and similarly for player 2. When $\theta_1 = p_1^*(\theta_2)$, the PPD mechanism may specify either $q_1 = 0$ or $q_1 = 1$. Because the only permissible outcomes are $(q_1, q_2) \in \{(0,0), (1,1)\}$, we obtain the following restrictions for the price schedules:

$$\theta_i > p_i^*(\theta_j) \Rightarrow \theta_j \geq p_j^*(\theta_i) \qquad \text{for } i \neq j \text{ and all } (\theta_i, \theta_j) \in [0,1]^2 \tag{8}$$

$$\theta_i < p_i^*(\theta_j) \Rightarrow \theta_j \leq p_j^*(\theta_i) \qquad \text{for } i \neq j \text{ and all } (\theta_i, \theta_j) \in [0,1]^2. \tag{9}$$

Lemma 6 *All price schedules $p_1^*(\theta_2)$ and $p_2^*(\theta_1)$ that satisfy restrictions (8) and (9) are nonincreasing.*

Proof. We will show that $p_1^*(\theta_2)$ is nonincreasing. To get a contradiction, suppose it is not. Then for some $\theta_2 < \theta_2'$, we have $p_1^*(\theta_2) < p_1^*(\theta_2')$. Pick any θ_1 such that $p_1^*(\theta_2) < \theta_1 < p_1^*(\theta_2')$. Restriction (9) requires that $\theta_2' \leq p_2^*(\theta_1)$.Thus, $\theta_2 < \theta_2' \leq p_2^*(\theta_1)$, which along with $\theta_1 > p_1^*(\theta_2)$ violates restriction (8) for (θ_1, θ_2). ■

We may safely restrict attention to price schedules for $i = 1, 2$ such that

$$p_1^*(1) < 1 \qquad \text{for } i = 1, 2. \tag{10}$$

Otherwise, because the price schedules are nonincreasing, for almost all realizations of (θ_1, θ_2) we would have $(q_1, q_2) = (0, 0)$ and the public good would not be built. Thus, when $(\theta_1, \theta_2) = (1, 1)$ the project is built, and feasibility requires that:

$$p_1^*(1) + p_2^*(1) \geq c \tag{11}$$

Consider the especially simple *fixed-price* mechanisms defined below.

Definition 7 *A fixed-price mechanism is a PPD mechanism such that for some $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = c$, the price schedules are*

$$\hat{p}_1(\theta_2) = \begin{cases} 1 & \textit{for } \theta_2 < \lambda_2 \\ \lambda_1 & \textit{for } \theta_2 \geq \lambda_2 \end{cases} \quad \hat{p}_2(\theta_1) = \begin{cases} 1 & \textit{for } \theta_1 < \lambda_1 \\ \lambda_2 & \textit{for } \theta_1 \geq \lambda_1. \end{cases} \tag{12}$$

The following proposition shows that we may further restrict attention to fixed-price mechanisms.

Proposition 8 *Consider any feasible PPD mechanism with $p_1^*(\cdot)$ and $p_2^*(\cdot)$ satisfying restrictions (8)-(11). Then there exists a feasible fixed-price mechanism that yields each players at least as much ex ante expected utility.*

Proof. Inequality (11) implies that either $p_1^*(1) \geq c/2$ or $p_2^*(1) \geq c/2$ or both. Without loss of generality, suppose that $p_1^*(1) \geq c/2$. Now consider the fixed-priced mechanism with the price schedules in (12) using $\lambda_1 = c - p_2^*(1)$ and $\lambda_2 = p_2^*(1)$, and with the same lump-sum payments as in the original PPD mechanism. It is straightforward to show that $\lambda_i \in [0, 1)$ for $i = 1$ and 2.

We now show that $\hat{p}_1(\theta_2) \leq p_1^*(\theta_2)$ for all $\theta_2 \in [0, 1]$. For $\theta_2 \geq \lambda_2$ we have

$$\hat{p}_1(\theta_2) = \lambda_1 = c - p_2^*(1) \leq p_1^*(1) \leq p_1^*(\theta_2),$$

where the first inequality results from restriction (11) and the second inequality results from Lemma 6. For $\theta_2 < \lambda_2$ we have $\hat{p}_1(\theta_2) = 1$. By restriction (9), $\theta_2 < \lambda_2 = p_2^*(1)$ implies that $1 \leq p_1^*(\theta_2)$, so again $\hat{p}_1(\theta_2) \leq p_1^*(\theta_2)$. Similarly, it can be shown that $\hat{p}_2(\theta_1) \leq p_2^*(\theta_1)$ for all $\theta_1 \in [0,1]$. Since the price schedules in this fixed-price mechanism never exceed the price schedules in the original PPD mechanism, the fixed-price mechanism yields at least as much expected utility for each player. The feasibility of the original mechanism and the construction of the fixed-price mechanism guarantees that the fixed-price mechanism is also feasible. ■

By Proposition 8, we can narrow our search for an optimal DSIC mechanism (maximizing the players' ex ante surplus) by searching within the class of fixed-price mechanisms, thus giving the following proposition.

Proposition 9 *Among all feasible DSIC mechanisms, the fixed-price mechanism with*

$$(\lambda_1, \lambda_2) \in \arg\max_{\lambda_1, \lambda_2} \int_{\lambda_1}^{1} \int_{\lambda_2}^{1} [\theta_1 + \theta_2 - c]\, f_2(\theta_2) f_1(\theta_1) d\theta_2 d\theta_1 \qquad \text{such that } \lambda_1 + \lambda_2 = c \quad (13)$$

and no lump sum payments is optimal.

Proof. By Proposition 5, we can search an optimal mechanism within PPD mechanisms, and by Proposition 8, we can further restrict attention to fixed price mechanisms. Note that (13) maximizes the sum of the players' ex ante surpluses over all fixed-price mechanisms. Feasibility requires that the sum of the lump sum payments exceeds zero, and so having nonzero lump sum payments cannot increase the sum of the player surpluses. ■

As an example, when each θ_i is uniformly distributed on $[0,1]$, i.e., $f_1(\cdot) = f_2(\cdot) = 1$, the integral in (13) becomes

$$\begin{aligned} & \frac{1}{2}(1-\lambda_2)\left(1-\lambda_1^2\right) + \frac{1}{2}(1-\lambda_1)\left(1-\lambda_2^2\right) - c(1-\lambda_1)(1-\lambda_2) \\ = \; & \frac{1}{2}(1-\lambda_1)(1-\lambda_2)(2-c) = \frac{2-c}{2}(1-\lambda_1)(1-c+\lambda_1), \end{aligned}$$

which is maximized when $\lambda_1 = \lambda_2 = c/2$. This result has a simple interpretation: the public good is built if and only if each player values the good at least half of its cost, in which case, the cost is equally shared. Evaluating (13) at $\lambda_1 = \lambda_2 = c/2$ yields ex ante surplus of $\left(1 - \frac{1}{2}c\right)^3$.

In contrast, the mechanism of d'Aspremont and Gerard-Varet (1979a, b) and Arrow (1979)—commonly referred to as the expected externality mechanism—Bayesian-Nash implements the allocatively efficient outcome, but their mechanism is not DSIC.[8] The fully efficient surplus

$$\int_{c-1}^{1}\int_{c-\theta_1}^{1} [\theta_1 + \theta_2 - c]\, d\theta_2 d\theta_1 = \frac{4}{3}\left(1 - \frac{1}{2}c\right)^3.$$

Thus in this example, the optimal DSIC mechanism captures 75% of the total surplus available. In other words, 25% of the total surplus is the cost of adopting the best DSIC mechanism instead of the best Bayesian incentive compatible mechanism.

5 A Bargaining Problem

Consider Myerson and Satterthwaite's (1983) bargaining problem between a buyer (player 1) and a seller (player 2). Each player has linear utility $\theta_i q_i - t_i$. Each θ_i is continuously distributed on $[0, 1]$ with distribution function $F_i(\cdot)$ and positive density function $f_i(\cdot)$. When a trade occurs, the buyer obtains the object and the seller forfeits the object. Thus, there are only two permissible quantity profiles: $(q_1, q_2) \in Q = \{(0, 0), (1, -1)\}$, where $(0, 0)$ means no trade and $(1, -1)$ means trade.

A mechanism is *budget-balanced* if $t_1 + t_2 = 0$ in all outcomes, and *feasible* if $t_1 + t_2 \geq 0$ for all outcomes. A mechanism is *individually rational* if each player's utility is nonnegative in the equilibrium for all $\theta_1 \times \theta_2 \in [0, 1]^2$. An *allocatively efficient* performance requires that for all $\theta_1 \times \theta_2 \in [0, 1]^2$, $(q_1, q_2) = (1, -1)$ if and only if $\theta_1 \geq \theta_2$. Myerson and Satterthwaite (1983) show that there is no individually rational mechanism in this setting that is DSIC, allocatively efficient, and budget-balanced. Instead, they find the surplus-maximizing mechanism among all budget-balanced mechanisms that can be Bayesian-Nash implemented. Alternatively, we

[8] See Fudenberg and Tirole (1992), pages 273-275, for details.

forgo allocative efficiency, and instead search for the DSIC mechanism that maximizes ex ante surplus among all DSIC mechanisms that are feasible. By Proposition 5, we may restrict attention to PPD mechanisms. Constructing PPD mechanisms requires constructing the price schedules for the buyer and the seller. In this bargaining problem, the small number of permissible quantity profiles and the feasibility constraint helps us pin down the price schedules, as we shown next.

Similar to the public good problem considered in the previous section, we may safely restrict ourselves to price schedules for the buyer that are constant with respect to quantities by letting:

$$p_1^*(\theta_2) \equiv \int_0^1 p_1(z, \theta_2) dz.$$

Now consider the seller's price schedule. Suppose that the buyer reports θ_1 and the seller reports $\hat{\theta}_2$, and denote the seller's price schedule by $p_2(q_2, \theta_1)$. In any PPD mechanism for this bargaining problem, the quantity for the seller is chosen to solve the following problem:

$$\max_{q_2 \in \{0,-1\}} \int_0^{q_2} \left[\hat{\theta}_2 - p_2(z, \theta_1)\right] dz.$$

Maximization then entails comparing 0 (when $q_2 = 0$) to $\int_{-1}^0 p_2(z, \theta_1)dz - \hat{\theta}_2$ (when $q_2 = -1$). Without loss of generality, we may restrict the seller's price schedules to be constant with respect to quantity by letting:

$$p_2^*(\theta_1) \equiv \int_{-1}^0 p_2(z, \theta_1) dz$$

with the interpretation that $p_2^*(\theta_1)$ is the payment the seller will receive if there is a trade, given the buyer's reported value of θ_1. In the PPD mechanism:

$$q_1 = \begin{cases} 1 & \text{if } \theta_1 > p_1^*(\theta_2) \\ 0 & \text{if } \theta_1 < p_1^*(\theta_2) \end{cases} \quad \text{and} \quad q_2 = \begin{cases} -1 & \text{if } \theta_2 < p_2^*(\theta_1) \\ 0 & \text{if } \theta_2 > p_2^*(\theta_1). \end{cases}$$

When $\theta_1 = p_1^*(\theta_2)$, the PPD mechanism may specify either $q_1 = 0$ or $q_1 = 1$ and when $\theta_2 = p_2^*(\theta_1)$ the PPD mechanism may specify either $q_2 = 0$ or $q_2 = -1$. Because the only permissible outcomes here are $(q_1, q_2) \in \{(0,0), (1,-1)\}$, the price schedules must satisfy the following restrictions:

$$\theta_i > p_i^*(\theta_j) \Rightarrow \theta_j \leq p_j^*(\theta_i) \qquad \text{for } i \neq j \text{ and all } (\theta_i, \theta_j) \in [0,1]^2 \tag{14}$$

$$\theta_i < p_i^*(\theta_j) \Rightarrow \theta_j \geq p_j^*(\theta_i) \qquad \text{for } i \neq j \text{ and all } (\theta_i, \theta_j) \in [0,1]^2 . \tag{15}$$

Lemma 10 *If $p_1^*(\theta_2)$ and $p_2^*(\theta_1)$ satisfy restrictions (14) and (15), then $p_1^*(\theta_2)$ and $p_2^*(\theta_1)$ are nondecreasing.*

Proof. The proof is similar to that of Lemma 6. ■

We may safely restrict attention to price schedules such that

$$p_1^*(0) < 1 \text{ and } p_2^*(1) > 0. \tag{16}$$

Otherwise, if either of these conditions did not hold, for almost all realizations of (θ_1, θ_2) we would have $(q_1, q_2) = (0, 0)$ and trade would not occur. Thus, when $(\theta_1, \theta_2) = (1, 0)$ trade occurs, and feasibility requires that:

$$p_1^*(0) \geq p_2^*(1) \tag{17}$$

Consider the PPD mechanism where, whenever trade occurs, the buyer pays a fixed-price and the seller receives a fixed-price. Specifically, consider the *fixed-price* mechanism defined below.

Definition 11 *A fixed-price mechanism is a PPD mechanism such that for some $\lambda \in [0,1]$, the price schedules are*

$$\hat{p}_1(\theta_2) = \begin{cases} \lambda & \text{for } \theta_2 \leq \lambda \\ 1 & \text{for } \theta_2 > \lambda \end{cases} \quad \hat{p}_2(\theta_1) = \begin{cases} 0 & \text{for } \theta_1 \leq \lambda \\ \lambda & \text{for } \theta_1 > \lambda. \end{cases} \tag{18}$$

The following proposition shows that we may further restrict attention to fixed-price mechanisms.

Proposition 12 *Consider any feasible PPD mechanism with $p_1^*(\cdot)$ and $p_2^*(\cdot)$ satisfying restrictions (14)-(17). There exists a feasible fixed-price mechanism that yields the players at least as much ex ante expected utility.*

Proof. Consider the fixed-price mechanism with price schedules in (18) using $\lambda = p_1^*(0)$, and with the same lump-sum payments as in the original PPD mechanism. By (16) and (17) we have $0 < \lambda < 1$.

We now show that $\hat{p}_1(\theta_2) \leq p_1^*(\theta_2)$ for all $\theta_2 \in [0,1]$. For $\theta_2 \leq \lambda$ we have $\hat{p}_1(\theta_2) = \lambda = p_1^*(0) \leq p_1^*(\theta_2)$, where the inequality results from Lemma 10. For $\theta_2 > \lambda$ we have $\hat{p}_1(\theta_2) = 1$. Using (17), we obtain $\theta_2 > \lambda = p_1^*(0) \geq p_2^*(1)$. Restriction (14) then implies that $1 \leq p_1^*(\theta_2)$, so again $\hat{p}_1(\theta_2) \leq p_1^*(\theta_2)$. Similarly, it can be shown that $\hat{p}_2(\theta_1) \geq p_2^*(\theta_1)$ for all $\theta_1 \in [0,1]$. Since the buyer's price schedule in this fixed-price mechanism never exceeds the price schedule in the original PPD mechanism, and since the seller's price schedule in this fixed-price mechanism always weakly exceeds the price schedule in the original PPD mechanism, the fixed-price mechanism yields at least as much expected utility for the players. The feasibility of the original mechanism and the construction of the fixed-price mechanism guarantees that the fixed-price mechanism is also feasible. ■

By Proposition 12, we can search for an optimal mechanism (maximizing the players ex ante surplus) by searching within the class of fixed-price mechanisms, thus giving the following proposition.

Proposition 13 *Among all feasible DSIC mechanisms, the fixed-price mechanism with*

$$\lambda \in \arg\max_{\lambda} \int_{\lambda}^{1} \int_{0}^{\lambda} [\theta_1 - \theta_2]\, f_2(\theta_2) f_1(\theta_1) d\theta_2 d\theta_1 \tag{19}$$

and no lump sum payments is optimal.

Proof. By Proposition 5, we can search for an optimal mechanism within the class of PPD mechanisms, and by Proposition 12, we can further restrict attention to fixed price mechanisms. Note that (19) maximizes the sum of the players' ex ante surpluses over all fixed-price mechanism. Feasibility requires that the sum of the lump sum payments exceeds zero, and so having nonzero lump sum payments cannot increase the sum of the player surpluses. ■

When θ_i is uniformly distributed on $[0,1]$, the integral in (19) becomes

$$\int_{\lambda}^{1}\int_{0}^{\lambda}[\theta_1-\theta_2]\,d\theta_2 d\theta_1=\frac{1}{2}\lambda(1-\lambda),$$

which is maximized when $\lambda = 1/2$. This result also has a simple interpretation: trade occurs if and only if the buyer's value excesses 1/2 and the seller's value is less than 1/2. Evaluating (19) at $\lambda = 1/2$ yields ex ante surplus of $\frac{1}{8}$.

In contrast, Myerson and Satterthwaite (1983) derive the optimal Bayesian mechanism and show that trade occurs if and only if $\theta_1 - \theta_2 \geq 1/4$, thus giving ex ante surplus

$$\int_{1/4}^{1}\int_{0}^{\theta_1-\frac{1}{4}}(\theta_1-\theta_2)\,d\theta_2 d\theta_1=\frac{9}{64}.$$

In this bargaining problem, the optimal DSIC mechanism captures 8/9 of the total surplus. In other words, about 11% of the total surplus would the "cost" of adopting any DSIC mechanism.

Our PPD characterization of DSIC mechanisms complements Hagerty and Rogerson (1987) who use a completely different approach to show that fixed-price mechanisms are the only DSIC, budget-balanced, individually rational mechanism for bilateral bargaining.

6 Conclusion

In this paper, we characterize all DSIC mechanisms as PPD mechanisms. Many DSIC mechanisms in the literature can be formulated immediately as PPD mechanisms. For example, consider various games among players with single-unit demand and linear utility $\theta_i q_i - t_i$. The Vickrey (1961)—i.e., second price—auction is a PPD mechanism where player i's price schedule is

$$p_i(q_i,\theta_{-i})=\max_{j\neq i}\theta_j \quad \text{for } 0\leq q_i\leq 1,$$

i.e., the highest report of player i's opponents. When each distribution $F_i(\cdot)$ of θ_i is regular, meaning that the marginal revenue $MR_i(\theta_i) = \theta_i - [1 - F_i(\theta_i)] / f_i(\theta_i)$ is increasing, Myerson's (1981) optimal auction is implemented by Bulow and Roberts' (1989) "second marginal revenue" auction, where player i's price schedule is

$$p_i(q_i, \theta_{-i}) = \max_{j \neq i} \left\{ MR_i^{-1}(0), MR_i^{-1}(MR_j(\theta_j) \right\} \quad \text{for } 0 \leq q_i \leq 1.$$

Consider McAfee's (1992) dominant strategy double auction among m buyers who each demands one unit and n sellers who each can supply one unit. Each player has linear utility with type bounded between $[0, 1]$. Players report their types, the buyers' reports are sorted such that $b_1 \geq b_2 \geq \cdots \geq b_m$, sellers' reports are sorted such that $s_1 \leq s_2 \leq \cdots \leq s_n$. To ensure the mechanism is well defined, McAfee artificially appends $b_{m+1} = 0$ and $s_{n+1} = 1$. Let k be the the number of efficient trades, i.e., the smallest k such that $b_{k+1} < s_{k+1}$. Define $p = \frac{1}{2}(b_{k+1} + s_{k+1})$. If $p \in [s_k, b_k]$ then the first k buyers purchase from the first k sellers at price p. Otherwise, the mediator buys a unit from each of the first $k - 1$ sellers at price s_k and sells a unit to each of the first the first $k - 1$ buyers at price b_k, pocketing the difference for herself. McAfee shows that this double auction is a DSIC mechanism.

Alternatively, we can reformulate McAfee's auction as a PPD mechanism with the price schedules determined as follows. Consider any arbitrary buyer and rename her buyer 0. Sort and rename the other $m - 1$ buyers such that $b_2 \geq b_3 \geq \cdots \geq b_m$. Artificially set $b_1 = 1$. Again, let k be the smallest number such that $b_{k+1} < s_{k+1}$ and define $p = \frac{1}{2}(b_{k+1} + s_{k+1})$. Then buyer 0's price schedule (defined over $q_0 \in [0, 1]$) is

$$p_0(q_0, \theta_{-0}) = \begin{cases} b_k & \text{if } p \notin [s_k, b_k] \\ p & \text{otherwise.} \end{cases}$$

A similar construction provides the seller's price schedule. Our construction is similar to that in McAfee's proof; however, our price schedule makes more explicit that the price schedule a buyer or seller faces in no way depends on her own report.

In the more general, multiple-unit Vickrey auction (see Vickrey, 1961, pp 66-71) among players with quasilinear utility functions, to find the height of buyer i's price schedule, one needs to find the aggregate demand of the other buyers and the aggregate supply of the other sellers built from their reports, and then find the equilibrium price in the market that excludes player i. The height of buyer i's price schedule at quantity q_i is the price at which a residual supply of q_i becomes available. Ausubel (2004) illustrates the price schedule that a buyer faces in a Vickrey auction, showing a figure remarkably like Figure 1 in this paper. Likewise, the inverse of the residual demand is the seller's price schedule.

In the third price auction, the highest bidder wins the object and pays the third highest bid. At first blush, one might think a third price auction is a PPD mechanism, since a player's own report does not affect the price schedule she faces (the second highest bid of her opponents). However, the third price auction is not a PPD mechanism since the player who has the second highest bid receives zero units even though her report exceeds the price schedule she faces, violating condition (3) in the definition of PPD mechanisms.[9]

References

[1] Arrow, K. (1979): "The Property Right Doctrine and Demand Revelation under Incomplete Information," in M. Boskin (ed.), *Economics and Human Welfare*, New York: Academic Press.

[2] Ausubel, L.M. (2004): "An Efficient Ascending Auction for Multiple Objects," *American Economic Review*, 94, 1452–1475.

[3] Ausubel, L.M., and P.C. Cramton (2004) "Vickrey Auction with Reserve Pricing," *Economic Theory*, 23, 493–505.

[4] Bulow, J., and J. Roberts (1989): "The Simple Economics of Optimal Auctions," *Journal of Political Economy*, 97, 1060–1090.

[5] d'Aspremont, C., and L.A. Gerard-Varet (1979a): "Incentives and Incomplete Information," *Journal of Public Economics*, 11, 25–45.

[9] We would like to thank Brett Katzman for pointing out this counter-example.

[6] d'Aspremont, C., and L. A. Gerard-Varet (1979b): "On Bayesian Incentive Compatible Mechanisms," in J.-J. Laffont (ed.), *Aggregation and Revelation of Preferences*, Amsterdam: North-Holland.

[7] Fudenberg, D., and J. Tirole (1992): Game Theory, Cambridge, MA: MIT Press.

[8] Green, J., and J.-J. Laffont (1977): "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods," *Econometrica*, 45, 427–438.

[9] Hagerty, K.M., and W.P. Rogerson (1987): "Robust Trading Mechanisms," *Journal of Economic Theory*, 42, 94–107.

[10] Holmstrom, B. (1979): "Groves' Scheme on Restricted Domains," *Econometrica*, 47, 1137–1144.

[11] Krishna, V. (2002): *Auction Theory*, New York: Academic Press.

[12] McAfee, R.P. (1992): "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56, 434–450.

[13] Milgrom, P. (2004): *Putting Auction Theory to Work*, Cambridge, UK: Cambridge University Press.

[14] Milgrom, P., and I. Segal (2002): "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70, 583–601.

[15] Mookherjee, D., and S. Reichelstein (1992): "Dominant Strategy Implementation of Bayesian Incentive Compatible Allocation Rules," *Journal of Economic Theory*, 56, 378–399.

[16] Moulin, H. (2007a): "Auctioning or Assigning an Object: Some Remarkable VCG Mechanisms," manuscript, Rice University.

[17] Moulin, H. (2007b): "Efficient, Strategy-Proof and Almost Budget-Balanced Assignment," manuscript, Rice University.

[18] Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

[19] Myerson, R. (2008): "Perspectives on Mechanism Design in Economic Theory," *American Economic Review*, 98, 586-603.

[20] Myerson, R., and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

[21] Vickrey, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.