STRATEGIC BASINS OF ATTRACTION, THE PATH DOMINANCE CORE, AND NETWORK FORMATION GAMES

by

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Abstract

Given the preferences of players and the rules governing network formation, what networks are likely to emerge and persist? And how do individuals and coalitions evaluate possible consequences of their actions in forming networks? To address these questions we introduce a model of network formation whose primitives consist of a feasible set of networks, player preferences, the rules of network formation, and a dominance relation on feasible networks. The rules of network formation may range from noncooperative, where players may only act unilaterally, to cooperative, where coalitions of players may act in concert. The dominance relation over feasible networks incorporates not only player preferences and the rules of network formation but also assumptions concerning the degree of farsightedness of players. A specification of the primitives induces an abstract game consisting of (i) a feasible set of networks, and (ii) a path dominance relation defined on the feasible set of networks. Using this induced game we characterize sets of network outcomes that are likely to emerge and persist. Finally, we apply our approach and results to characterization of equilibrium of well known models and their rules of network formation, such as those of Jackson and Wolinsky (1996) and Jackson and van den Nouweland (2005).

KEYWORDS: basins of attraction, network formation games, stable sets, path dominance core, Nash networks

JEL Classifications: A14, C71, C72

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1 Introduction

1.1 *Overview of the questions, the model and the main results*

In many economic and social situations the totality of interactions between individuals and coalitions can be modeled as a network. We address the following question: given preferences of individuals and rules governing network formation, what networks are likely to emerge and persist? To address this question we introduce a model of network formation whose primitives consist of a feasible set of networks, player preferences, the rules of network formation, and a dominance relation. The rules of network formation may range from noncooperative, where players may only act unilaterally, to cooperative, where coalitions consisting of multiple players may act in concert. The dominance relation may be either direct or indirect. Under direct dominance players are concerned with immediate consequences of their network formation strategies whereas under indirect dominance players are farsighted and consider the eventual consequences of their strategies. As we will discuss, our framework can accommodate a wide variety of social and economic situations.

A specification of the primitives induces an abstract game consisting of (i) a feasible set of networks and (ii) a path dominance relation defined on the feasible set of networks. Under the path dominance relation, a network G path dominates another network G' if there is a finite sequence of networks, beginning with G and ending with G' where each network along the sequence dominates its predecessor.[1] Using this induced abstract game as our basic analytic tool we demonstrate that for any set of primitives the following results hold:

1. The feasible set of networks contains a unique, finite, disjoint collection of nonempty subsets each constituting a *strategic basin of attraction.* Given preferences and the rules of governing network formation, these basins of attraction are the absorbing sets of the process of network formation modeled via the game.

2. A stable set with respect to path dominance consists of one network from each basin of attraction.

3. The path dominance core, defined as a set of networks having the property that no network in the set is path dominated by any other feasible network, consists of one network from each basin of attraction containing a *single* network. Note that the path dominance core is contained in each stable set and is nonempty

[1] Stated formally, given feasible set of networks $\mathbb{G}$ and (a direct or indirect) dominance relation $>$, network $G' \in \mathbb{G}$ (weakly) path dominates network $G \in \mathbb{G}$, written $G' \geq_p G$, if $G' = G$ *or* if there exists a *finite* sequence of networks $\{G_k\}_{k=0}^{n}$ in $\mathbb{G}$ with $G = G_0$ and $G' = G_n$ such that for $k = 1, 2, \ldots, n$

$$G_k > G_{k-1}.$$

The path dominance relation $\geq_p$ induced by the dominance relation $>$ is sometimes referred to as the transitive closure of $>$.

if and only if there is a basin of attraction containing a single network.[2] As a corollary, we conclude that any network contained in the path dominance core is Pareto efficient. Thus, by considering the network formation game with respect to path dominance– and thus, by considering the long run - we identify networks that are *both* stable and Pareto-efficient with respect to the original dominance relation.

4. From the above results it follows that if the dominance relation is transitive, then the path dominance core is nonempty.

We also demonstrate specializations of our model to treat hedonic games and club formation games and we discuss how our results apply to these examples.

There are interesting connections between our notions of stability (basins of attraction, path dominance stable sets, and path dominance core) and some of the basic notions of stability and equilibrium found in the existing literature - such as, strong stability (Jackson and van den Nouweland 2005), pairwise stability (Jackson and Wolinsky 1996), consistency (Chwe 1994), and Nash equilibrium. We show that in general (for all primitives) the path dominance core is contained in the set of strongly stable networks. We conclude from our general results therefore that, for all primitives, the existence of at least one basin of attraction containing a single network is sufficient for the existence of a strongly stable network. We also demonstrate that, depending on how we specialize the primitives of the model, the path dominance core is equal to the set of strongly stable networks, the set of pairwise stable networks, or the set of Nash networks.

Of particular interest are the connections between the rules of network formation, the dominance relation inducing path dominance, and stability.[3] We provide a unified and systematic analysis of these connections. For example, we show that:

(a) If path dominance is induced by a direct dominance relation (as opposed to an indirect dominance relation as in Chwe 1994, for example), then the path dominance core is equal to the set of strongly stable networks.

(b) If, in addition, the rules of network formation are the Jackson-Wolinsky rules, then the path dominance core is equal to the set of pairwise stable networks.[4]

(c) If path dominance is induced by a direct dominance relation and if the rules of network formation only allow network changes brought about by individuals, then the path dominance core is equal to the set of Nash networks.

[2] Put differently, the path dominance core is empty if and only if *all* basins of attraction contain multiple networks.

[3] Although she treats a more specialized model, the questions addressed in Demange (2004) are related.

[4] Under the Jackson-Wolinsky rules arc addition is bilateral (i.e., the two players that would be involved in the arc must agree to adding the arc), arc subtraction is unilateral (i.e., at least one player involved in the arc must agree to subtract or delete the arc), and network changes take place one arc at a time (i.e., in any one play of the game, only one arc can be added or subtracted). See section 3.2.1 for a formal definition.

We then conclude from (3) above, the existence of at least one basin of attraction containing a single network is, depending on how we specialize primitives, both necessary and sufficient for either (i) the existence of a strongly stable network, or (ii) a pairwise stable network, or (iii) a Nash network.[5]

When path dominance is induced by an indirect dominance relation as in Chwe (1994), then we show that for all primitives - and in particular for all rules of network formation - each strategic basin of attraction has a nonempty intersection with the largest consistent set of networks (i.e., the Chwe set of networks, see Chwe 1994).[6] This fact, together with (2) above, implies that there always exists a path dominance stable set contained in the largest consistent set. Thus, the path dominance core is contained in the largest consistent set. In light of our results on the path dominance core and stability (both strong and pairwise), we conclude that if path dominance is induced by an indirect dominance relation, then any network contained in the path dominance core is not only consistent but also strongly stable, as well as pairwise stable.[7]

We remark that solution concepts defined using abstract dominance relations have a distinguished history in the literature of game theory. First, consider the von-Neuman-Morgenstern stable set. The vN-M stable set is defined with respect to an abstract dominance relation on a set of outcomes and consists of those outcomes that are externally and internally stable with respect to the given dominance relation. Similarly, Gilles (1959) defines the core based on a given abstract dominance relation. These solution concepts, with a few exceptions, have typically been applied to models of economies or cooperative games where the notion of dominance is based on what a coalition can achieve using only the resources owned by its members (cf., Aumann 1964) or a given set of utility vectors for each possible coalition (cf., Scarf 1967). Particularly notable exceptions are Schwartz (1974), Panzer, Kalai and Schmeidler (1976), Kalai and Schmeidler (1977) and Shenoy (1980). Their motivations are in part similar to ours in that they take as given a set of possible choices of a society and a dominance relation and, based on these, describe a set of possible or likely social outcomes called, by Kalai and Schmeidler, the admissible set. While their examples treat direct dominance, their general results have wider applications. We return to a discussion of the admissible set in our concluding section.

1.2 *A further discussion of the model*

In addition to introducing abstract games of network formation, our modeling approach contributes to the literature by extending the class of primitives used in the

[5] For Jackson-Wolinsky linking networks, Calvo-Armengol and Ilkilic (2004) provide necessary and sufficient conditions on the network link marginal payoffs such that the set of pairwise stable, pairwise Nash, and proper equilibrium networks coincide.

[6] Consistency with respect to indirect dominance and the notion of a largest consistent set were introduced by Chwe (1994) in an abstract game setting. We provide a detailed discussion of Chwe's notion in Section 5.3.

[7] Other papers on indirect dominance and consistency in games include Xue (1998), Diamantoudi and Xue (2003), and Mauleon and Vannetelbosch (2003).

analysis of network formation in three respects. These extensions, listed below, significantly broaden the set of potential applications.
1. *Directed Networks with heterogenous arcs and multiple uses of the same arc*: First, we focus on directed networks rather than on linking networks[8] and distinguish between nodes and decision making players (i.e., the set of players and the set of nodes are not necessarily the same).[9] Connections are represented by arcs and each arc possesses an orientation or direction: arc a connecting nodes i and i' must either go from node i to node i' or must go from node i' to node i.[10] For example, an individual may have a links on his web page to the web pages of all Nobel Laureates in economics but it may be that no Nobel Laureate has a link to that individual's web page. Connections between nodes (i.e., arcs), besides having an orientation, are allowed to be heterogeneous. To illustrate, if the nodes in a given network represent players, an arc a going from player i to player i' might represent a particular type and intensity of interaction (identified by the arc label a) initiated by player i towards player i'. Player i might direct great affection toward player i' as represented by arc a, but player i' may direct only lukewarm affection toward player i as represented by arc a'. Also, under our extended definition nodes are allowed to be connected by multiple, distinct arcs. Thus, we allow nodes to interact in multiple, distinct ways. For example, nodes i and i' might be connected by arcs a and a', with arc a running from node i to i' and arc a' running in the opposite direction (i.e., from node i' to node i).[11] If node i represents a seller and node i' a buyer, then arc a might represent a contract offer by the seller to the buyer, while arc a' might represent a counter offer or the acceptance or rejection of the contract offer. Finally, loops are allowed and arcs are allowed to be used multiple times in a given network.[12] For example, arc a might be used to connect nodes i and i' as well as nodes i' and i''. Thus, under our definition nodes i and i' as well as nodes i' and i'' are allowed to engage in the same type of interaction as represented by arc a.

Allowing each type of arc to be used multiple times makes it possible to distinguish coalitions by the type of interaction taking place between coalition members and to give a network representation of such coalitions. For example, if the nodes in a given network represent players, an 'a-coalition' could consist of all players i having an a-connection with at least one other player i'. Such an a-coalition would then have a network representation as the directed subnetwork consisting of pairs of nodes, i and i', connected by an a arc.

Until now, most of the economic literature on networks has focused on linking networks (see Jackson 2005 for an excellent survey). In an undirected (or linking)

[8] In particular, we focus on the notion of directed networks introduced in Page, Wooders, and Kamat (2005).

[9] Our example of club formation demonstrates a situation where the nodes are not necessarily decision-making players. In particular, some nodes are club locations.

[10] We denote arc a going from node i to node i' via the ordered pair $(a, (i, i'))$, where (i, i') is also an ordered pair. Alternatively, if arc a goes from node i' to node i, we write $(a, (i', i))$.

[11] Under our extended definition, arc a' might also run in the same direction as arc a. However, our definition does not allow arc a to go from node i to node i' multiple times.

[12] A loop is an arc going *from* a given node *to* that same node. For example, given arc a and node i, the ordered pair $(a, (i, i))$ is a loop.

network, an arc (or link) is identified with a nonempty subset of nodes consisting of exactly two distinct nodes, for example, $\{i, i'\}$, $i \neq i'$. Thus, in an undirected network, a link has no orientation and simply indicates a connection between two players. Moreover, links are typically not distinguished by type (or by label) – that is, links are homogeneous. By allowing arcs to possess direction and be used multiple times and by allowing loops and nodes to be connected by multiple arcs, our definition makes possible the application of networks to a rich set of economic environments. For example, a job opportunity market model may embody the features introduced above; individuals may have different relationships with their superiors in an organization and other individuals both within and outside of the organization. This may well affect social interactions and job opportunities.

2. *The rules of network formation*: We explicitly model the rules of network formation and thus provide a systematic treatment of the relationship between rules and stability. The rules of network formation specify which players must be involved in adding, subtracting, or replacing an arc as well as how many and what types of arcs can be added, subtracted, or replaced in any one play of the game.

In much of the literature, it is assumed (sometimes implicitly) that network formation is governed by the Jackson-Wolinsky rules.[13] Other rules are possible. For example, the addition of an arc might require that a simple majority of the players agree to the addition while the deletion of an arc might require that a two-thirds majority agree to the deletion. Under our approach, such rules are allowed. We achieve this flexibility by representing the rules of network formation via a collection of coalitional effectiveness relations, $\{\to_S\}_S$, defined on the feasible set of networks. Given feasible networks G and G', if the relation $G \to_S G'$ holds, the players in coalition S can change network G to network G'. In constructing our abstract game of network formation, we will equip the feasible set of networks with a dominance relation which incorporates - or represents - *both* the preferences of individuals and coalitions *and* the rules of network formation as represented via the coalitional effectiveness relations $\{\to_S\}_S$. Thus, the stability results we obtain using the path dominance relation will reflect both preferences and rules.

3. *The Dominance Relation Defined on Feasible Networks*: We allow the path dominance relation on networks to be based on either direct dominance or indirect dominance (direct and indirect dominance are formally defined in section 3). All of our main results hold for both path dominance based on direct dominance and path dominance based on indirect dominance.

[13] Jackson-van den Nouweland (2005) focus on linking networks and assume that link addition is bilateral while link subtraction is unilateral. But in their model, network changes are not required to take place one link at a time - multiple link changes can take place in any one play of the game. We shall refer to these rules as the Jackson-van den Nouweland rules. Calvo-Armengol and Ilkilic (2004) also focus on linking networks under bilateral-unilateral rules and allow multiple link changes.

1.3 *Examples: club formation and hedonic games*

To demonstrate the flexibility of our approach as well as illustrate our notions of stability (i.e., basins of attraction, path dominance stable sets, and the path dominance core), we consider two examples.

1.3.1 *Club formation*

Our first example relates to a number of contributions in the literature, for example, Konishi, Le Breton and Weber (1998); we note other related literature in the presentation of the example. Our current formulation, taken from Page and Wooders (2005), models club structures as bipartite networks and formulates the problem of club formation as a game of network formation. In Page and Wooders (2005) we considered only indirect dominance; here we compare and contrast the results for path dominance defined with respect to direct and indirect dominance. For brevity, we will call these core concepts the direct dominance core and the indirect dominance core.

The set of stable outcomes with respect to direct dominance is not in general the same as the set of stable outcomes with indirect dominance. For the case where the total player set can be partitioned into clubs of optimal size and there are sufficiently many club locations, the indirect and direct dominance cores are equivalent and nonempty. In addition, if there are 'too few' club locations[14], so that the average number of members of a club must be larger than the optimal club size, then, while the indirect dominance core is empty, the direct dominance core is nonempty and induces a partition of players into a clubs that are all as close as possible to the same size. Moreover, the set of networks in the direct dominance core coincide with the set of strongly stable networks and the direct dominance core induces the same set of partitions as the Nash club equilibrium introduced in Arnold and Wooders (2005). This illustrates that (not surprisingly) the notion of dominance used affects the size of the path dominance core and, the greater the extent of indirect dominance allowed, the smaller the path dominance core.

If the rules of network formation allow only one player to move at a time, then the direct dominance core coincides with the set of Nash networks (which coincides with the set of strongly stable networks). This is true because only one player can move at a time and must be made better off by changing the status quo. Some further characterizations are discussed in the example.

1.3.2 *Hedonic games*

Our framework encompasses hedonic games – games where players' preferences are defined over the set of coalitions in which they are members. This is illustrated by an interesting example proposed by Salvador Barbera and Michael Maschler in private correspondence. The example also illustrates how, though indirect dominance,

[14] For example, if there are two clubs and seven players, one club would be of size three and another of size four.

outcomes in a game might move from one hedonic core point to another. From our prior results, this demonstrates that, even though the hedonic core is nonempty, the path dominance core with respect to indirect dominance is empty. In Page and Wooders (2006) we investigate relationships between cores of cooperative games (hedonic and in characteristic function form) and path dominance cores, but do not examine the question further in this paper. We remark that a much more complete investigation of indirect dominance for hedonic games appears in Diamantoudi and Xue (2003).

2 Directed Networks

2.1 The Definition

Let N be a finite set of nodes, with typical element denoted by i, and let A be a finite set of arcs, with typical element denoted by a. Arcs represent potential connections between nodes, and depending on the application, nodes can represent economic players or economic objects such as markets or firms. The following definition is from Page, Wooders, and Kamat (2001).

Definition 1 *(Directed Networks)*
Given node set N and arc set A, a directed network, G, is a nonempty subset of $A \times (N \times N)$. The collection of all directed networks is denoted by $P(A \times (N \times N))$.

A directed network $G \in P(A \times (N \times N))$ specifies how the nodes in N are connected via the arcs in A. Note that in a directed network order matters. In particular, if $(a, (i, i')) \in G$, this means that arc a *goes from* node i *to* node i'. Also, note that loops are allowed - that is, we allow an arc to go from a given node back to that given node. For example, in a network model of journal citations loops could represent self-cites.[15] Finally, an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, under our definition an arc a is not allowed to go from a node i to a node i' multiple times.

The following notation is useful in describing changes in networks and the properties of networks. Given directed network $G \in P(A \times (N \times N))$, let $G \cup (a, (i, i'))$ denote the network obtained by adding arc a from node i to node i' to network G, and let $G \backslash (a, (i, i'))$ denote the network obtained by subtracting (or deleting) arc a from node i to node i' from network G. Also, let

$$\left.\begin{array}{c} G(a) := \left\{ (i, i') \in N \times N : (a, (i, i')) \in G \right\}, \\ \text{and} \\ G(i) := \left\{ a \in A : (a, (i, i')) \in G \text{ or } (a, (i', i)) \in G \right\}. \end{array}\right\} \tag{1}$$

Thus, $G(a)$ is the *set of node pairs* connected by arc a in network G, and $G(i)$ is the *set of arcs* going from node i or coming to node i in network G.

[15] This example was suggested by a participant at the Coalition Theory Network meeting held in January 2006. Other examples could be developed. For example, in a network model of information sharing, the fact that each player knows his own information would be represented by a loop.

Note that if for some arc $a \in A$, $G(a)$ is empty, then arc a is not used in network G. Moreover, if for some node $i \in N$, $G(i)$ is empty then node i is not used in network G, and node i is said to be isolated relative to network G.

Suppose that the node set N is given by $N = \{i_1, i_2, \ldots, i_5\}$, while the arc set A is given by $A = \{a_1, a_2, \ldots, a_5, a_6, a_7\}$. Consider network G in Figure 1.

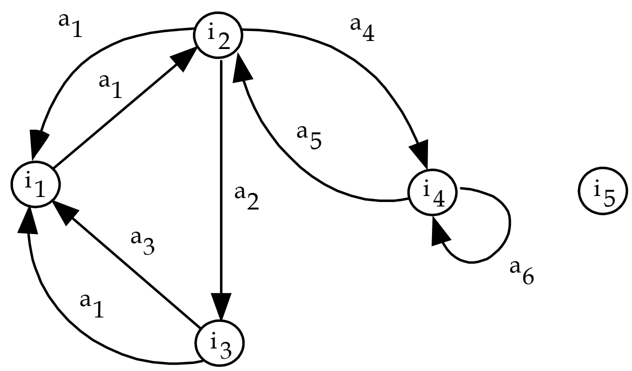

Figure 1: Network G

Note that in network G nodes i_1 and i_2 are connected by two a_1 arcs running in opposite directions and that nodes i_1 and i_3 are connected by two arcs, a_1 and a_3, running in the same directions from node i_3 to node i_1. Thus, $G(a_1) = \{(i_1, i_2), (i_2, i_1), (i_3, i_1)\}$ and $G(a_3) = \{(i_3, i_1)\}$. Observe that $(a_6, (i_4, i_4)) \in G$ is a loop. Thus, $G(a_6) = \{(i_4, i_4)\}$. Also, observe that arc a_7 is not used in network G. Thus, $G(a_7) = \emptyset$.[16] Finally, observe that $G(i_4) = \{j_4, j_5, j_6\}$, while $G(i_5) = \emptyset$. Thus, node i_5 is *isolated* relative to G.[17]

2.2 Linking Networks and Directed Graphs

Our notion of a directed network can be formally related to the notion of a linking network. As before, let N denote a finite set of nodes. A linking network, say g, consists of a finite collection of subsets of the form $\{i, i'\}$, $i \neq i'$. Thus, $\{i, i'\} \in g$ means that nodes i and i' are linked in network g. For example, g might be given by $g = \{\{i, i'\},\{i', i''\}\}$ for i, i', and i'' in N. Note that all connections or links are the same (i.e., connection types are homogeneous), direction does not matter, and loops are ruled out. Letting g^N denote the collection of all subsets of N of size 2, the collection of all linking networks given N is given by $P(g^N)$ where, recall, $P(g^N)$

[16] The fact that arc a_7 is not used in network G can also be denoted by writing

$$a_7 \notin proj_A G,$$

where $proj_A G$ denotes the projection onto A of the subset

$$G \subseteq A \times (N \times N)$$

representing the network.

[17] If the loop $(a_7, (i_5, i_5))$ were part of network G in Figure 1, then node i_5 would no longer be considered isolated under our definition. Moreover, we would have $G(i_5) = \{a_7\}$.

denotes the collection of all nonempty subsets of g^N (e.g., see the definition in Jackson and Wolinsky 1996).

A directed graph, say E, consists of a finite collection of ordered pairs $(i, i') \in N \times N$. For example, E might be given by $E = \{(i, i'), (i', i')\}$ for (i, i') and (i', i') in $N \times N$. Stated more compactly, a directed graph E is simply a subset of $N \times N$. Thus, in any directed graph connection types are again homogeneous but direction does matter and loops are allowed.

Under our definition, a directed network G is a subset of $A \times (N \times N)$, where as before A is a finite set of arcs. Thus, in a directed network, say $G \in P(A \times (N \times N))$, connection types are allowed to be heterogeneous (distinguished by arc labels), direction matters, and loops are allowed.

3 Preferences, Rules, and Dominance Relations

3.1 Preferences

Let D denote the set of players (or economic decision making units) with typical element denoted by d, and let $P(D)$ denote the collection of all coalitions (i.e., nonempty subsets of D) with typical element denoted by S. Note that, the set of players D and the set of nodes N are not necessarily the same set.

Given a feasible set of directed networks $\mathbb{G} \subseteq P(A \times (N \times N))$, we shall assume that each player's preferences over networks in $\mathbb{G}$ are specified via an *irreflexive* binary relation $\succ_d$. Thus, player $d \in D$ prefers network $G' \in \mathbb{G}$ to network $G \in \mathbb{G}$ if $G' \succ_d G$ and for all networks $G \in \mathbb{G}$, $G \not\succ_d G$ (irreflexivity). Coalition $S' \in P(D)$ *prefers network* G' *to network* G, written $G' \succ_S G$, if $G' \succ_d G$ for all players $d \in S'$.

In many applications, an player's preferences are specified via a real-valued network payoff function, $v_d(\cdot)$. For each player $d \in D$ and each directed network $G \in \mathbb{G}$, $v_d(G)$ is the payoff to player d in network G. Player d then prefers network G' to network G if $v_d(G') > v_d(G)$. Moreover, coalition $S' \in P(D)$ prefers network G' to network G if $v_d(G') > v_d(G)$ for all $d \in S'$. Note that the payoff $v_d(G)$ to player d depends on the entire network. Thus, the player may be affected by directed links between other players even when he himself has no direct or indirect connection with those players. Intuitively, 'widespread' network externalities are allowed.

Remark 1 *All of our results on basins of attraction, path dominance stable sets, and the path dominance core (Theorems 1-4 below) remain valid even if coalitional preferences* $\{\succ_S\}_{S \in P(D)}$ *over networks are based on weak preference relations* $\{\succsim_d\}_{d \in D}$. *If* $G' \succsim_d G$ *then player* d *either strictly prefers* G' *to* G *(denoted* $G' \succ_d G$*) or is indifferent between* G' *and* G *(denoted* $G' \sim_d G$*). Given weak preferences* $\{\succsim_d\}_{d \in D}$, *coalition* $S' \in P(D)$ prefers *network* G' *to network* G, *written* $G' \succ_{S'} G$, *if for all players* $d \in S'$, $G' \succsim_d G$ *and if for at least one player* $d' \in S'$, $G' \succ_{d'} G$. *Note that if coalitional preferences* $\{\succ_S\}_{S \in P(D)}$ *are defined in this way (i.e., using weak preferences* $\{\succsim_d\}_{d \in D}$*), then they are irreflexive (i.e.,* $G \not\succ_S G$ *for all* $G \in \mathbb{G}$ *and* $S \in P(D)$*).*

3.2 Rules

The rules of network formation are specified via a collection of coalitional effectiveness relations $\{\rightarrow_S\}_{S\in P(D)}$ defined on the feasible set of networks $\mathbb{G}$. Each effectiveness relation $\rightarrow_S$ represents what a coalition S can do. Thus, if $G \rightarrow_S G'$ this means that under the rules of network formation coalition $S \in P(D)$ can change network $G \in \mathbb{G}$ to network $G' \in \mathbb{G}$ by adding, subtracting, or replacing arcs in G.

3.2.1 Examples of Network Formation Rules

Jackson-Wolinsky Rules: To illustrate, consider Figure 2 depicting two networks G_1 and G_2 in which the nodes represent players. Thus, $D = N = \{i_1, i_2, i_3\}$.

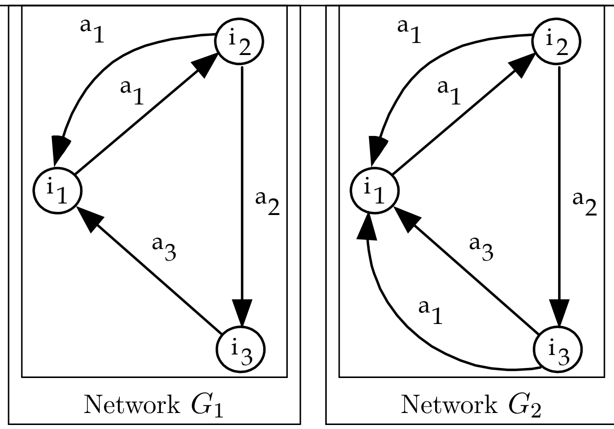

Figure 2

Observe that

$$G_2 = G_1 \cup (a_1, (i_3, i_1)) \quad \text{and} \quad G_1 = G_2 \backslash (a_1, (i_3, i_1)).$$

Assume that

(i) adding an arc a from player i to player i' requires that both players i *and* i' agree to add arc a (i.e., arc addition is bilateral);

(ii) subtracting an arc a from player i to player i' requires that player i *or* player i' agree to subtract arc a (i.e., arc subtraction is unilateral);

(iii) for any pair of networks G and G' in $\mathbb{G}$, if $G \rightarrow_S G'$, then $G \neq G'$ and

$$\text{either } G' = G \cup (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N)$$
$$\text{or}$$
$$G' = G \backslash (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N).$$

For the case $D = N$ (i.e., players = nodes), we shall refer to rules (i)-(iii) above as Jackson-Wolinsky rules. Note that rules (i) and (ii) imply that if $G \rightarrow_S G'$, then

$1 \leq |S| \leq 2$. Referring to Figure 2, the effectiveness relations over networks G and G' under Jackson-Wolinsky rules are given by

$$G_1 \underset{\{i_1,i_3\}}{\longrightarrow} G_2 \quad G_2 \underset{\{i_1,i_3\}}{\longrightarrow} G_1 \quad G_2 \underset{\{i_1\}}{\longrightarrow} G_1 \quad G_2 \underset{\{i_3\}}{\longrightarrow} G_1.$$

Under the Jackson-Wolinsky rules arc addition is bilateral, arc subtraction is unilateral, and network changes take place one arc at a time.

Jackson-van den Nouweland rules: Consider networks G_0 and G_3 depicted in Figure 3 and again suppose that nodes represent players.

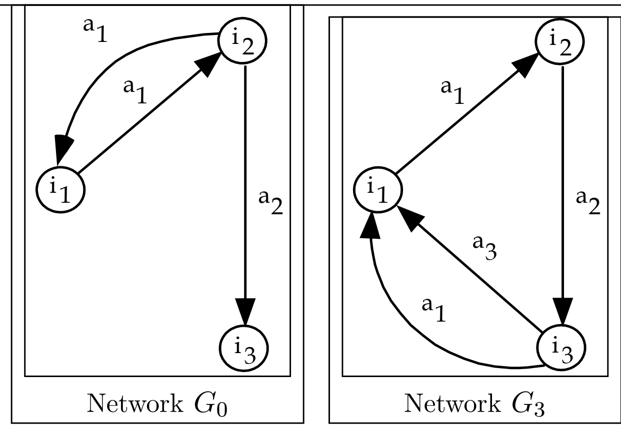

Figure 3

Observe that

$$G_3 = (G_0 \backslash (a_1, (i_2, i_1))) \cup (a_1, (i_3, i_1)) \cup (a_3, (i_3, i_1))$$
and
$$G_0 = (G_3 \backslash ((a_1, (i_3, i_1)) \cup (a_3, (i_3, i_1)))) \cup (a_1, (i_2, i_1)).$$

Assume that

(i) adding an arc a from player i to player i' requires that both players i *and* i' agree to add arc a (i.e., arc addition is bilateral);

(ii) subtracting an arc a from player i to player i' requires that player i *or* player i' agree to subtract arc a (i.e., arc subtraction is unilateral);

For the case $D = N$ (i.e., players = nodes), we shall refer to rules (i)-(ii) above as Jackson-van den Nouweland rules. Thus, the Jackson-van den Nouweland rules are the Jackson-Wolinsky rules without the one-arc-at-a-time restriction. Note that if arc addition is bilateral and arc subtraction is unilateral (i.e., if rules (i) and (ii) hold), then $G \rightarrow_S G'$ implies that G' is obtainable from G via coalition S, that is,

$$\text{(i) } (a, (i, i')) \in G' \text{ and } (a, (i, i')) \notin G$$
$$\Rightarrow \{i, i'\} \subseteq S;$$
$$\text{(ii) } (a, (i, i')) \notin G' \text{ and } (a, (i, i')) \in G$$
$$\Rightarrow \{i, i'\} \cap S \neq \emptyset.$$

Referring to Figure 3, the effectiveness relations over networks G and G' under Jackson-van den Nouweland rules are given by

$$G_0 \underset{\{i_1,i_2,i_3\}}{\longrightarrow} G_3 \quad G_0 \underset{\{i_1,i_3\}}{\longrightarrow} G_3 \quad G_3 \underset{\{i_1,i_2\}}{\longrightarrow} G_0 \quad G_3 \underset{\{i_1,i_2,i_3\}}{\longrightarrow} G_0.$$

Noncooperative Rules: Now assume that

(i) adding an arc a from player i to player i' requires only that player i agree to add the arc (i.e., arc addition is unilateral);

(ii) subtracting an arc a from node i to node i' requires only that player i agree to subtract the arc (i.e., arc subtraction is unilateral);

(iii) $G \to_S G'$ implies that $|S| = 1$ (i.e., only network changes are brought about by individual players are allowed).

We shall refer to rules (i)-(iii) as noncooperative. Note that an player i can add or subtract an arc to player i' without regard to the preferences of player i'.

Under noncooperative rules, the effectiveness relations over networks G_1 and G_2 in Figure 2 are given by

$$G_1 \underset{\{i_3\}}{\longrightarrow} G_2 \quad G_1 \underset{\{i_3\}}{\longrightarrow} G_1.$$

Note that under noncooperative rules, networks G_0 and G_3 in Figure 3 are *not* related under the effectiveness relations $\left\{\to_{\{i\}}\right\}_{i\in N}$. However, referring to the networks in Figures 2 and 3, under the noncooperative rules we have, for example, the following effectiveness relations

$$G_3 \to_{\{i_2\}} G_2 \quad G_2 \to_{\{i_3\}} G_0$$
and
$$G_0 \to_{\{i_3\}} G_2 \quad G_2 \to_{\{i_2\}} G_3.$$

$(\frac{1}{2}, \frac{2}{3})$-*Voting Rules*: All of the rules above require that arc addition and arc subtraction involve at least one player who is a party to the arc. Consider now arc addition and arc subtraction based on voting. If nodes represent players, then under certain voting rules, arcs can be imposed on players. To see this, consider the following rules for arc addition and arc subtraction.

(i) adding an arc a from player i to player i' requires a simple majority agree to add arc a;

(ii) subtracting an arc a from player i to player i' requires a $\frac{2}{3}$ majority agree to subtract arc a;

(iii) for any pair of networks G and G' in $\mathbb{G}$, if $G \to_S G'$, then $G \neq G'$ and

$$\text{either } G' = G \cup (a,(i,i')) \text{ for some } (a,(i,i')) \in A \times (N \times N)$$
or
$$G' = G \backslash (a,(i,i')) \text{ for some } (a,(i,i')) \in A \times (N \times N).$$

We shall refer to rules (i)-(iii) above as $(\frac{1}{2}, \frac{2}{3})$-voting rules. Note that rules (i) and (ii) imply that if $G \rightarrow_S G'$, then

$$G \rightarrow_S G \cup (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N)$$
$$\Leftrightarrow$$
$$G \cup (a, (i, i')) \succ_S G \text{ and } \frac{|S|}{|D|} > \frac{1}{2};$$

and

$$G \rightarrow_S G \backslash (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N)$$
$$\Leftrightarrow$$
$$G \backslash (a, (i, i')) \succ_S G \text{ and } \frac{|S|}{|D|} > \frac{2}{3}.$$

Referring to Figure 2, if players i_2 and i_3 prefer network G_2 to network G_1 while player i_1 prefers network G_1 to network G_2, then under $(\frac{1}{2}, \frac{2}{3})$-voting rules

$$G_1 \underset{\{i_2, i_3\}}{\longrightarrow} G_2,$$

where $G_2 = G_1 \cup (a_1, (i_3, i_1))$. Thus, arc a_1 from player i_3 to player i_1 is imposed on player i_1 under majority rule. Note that under $(\frac{1}{2}, \frac{2}{3})$-voting rules it is not possible to move from network G_2 back to network G_1.

Nonuniform Rules and the Network Representation of Network Formation Rules: In all of the examples above, the rules for arc addition and arc subtraction are uniform across pairs of networks. In some applications, such uniformity is not present. One very concise way to write down such nonuniform network formation rules is to use a network representation. In particular, suppose we write

$$(S, (G, G')) \text{ if and only if } G \rightarrow_S G'.$$

Thus, $(S, (G, G'))$ if and only if under the rules coalition $S \in P(D)$ can change network G to network G'. Letting the set of arcs be given by the collection of all coalitions $P(D)$ and letting the set of nodes be given by the feasible set of networks $\mathbb{G}$, the rules of network formation can be represented by a network $\mathbf{G} \subset P(D) \times (\mathbb{G} \times \mathbb{G})$. Then the set of all possible network formation rules is given by the set of all such networks.

3.3 Dominance Relations

We will consider two types of dominance relations on the feasible set of networks $\mathbb{G} \subseteq P(A \times (N \times N))$, direct and indirect dominance.

3.3.1 Direct Dominance

Network $G' \in \mathbb{G}$ *directly dominates* network $G \in \mathbb{G}$, sometimes written $G' \rhd G$, if for some coalition $S \in P(D)$,

$$G \prec_S G'$$
$$\text{and}$$
$$G \underset{S}{\longrightarrow} G'.$$

Thus, network G' directly dominates network G if some coalition S prefers G' to G *and* if under the rules of network formation coalition S has the power to change G to G'.

3.3.2 Indirect Dominance

Network $G' \in \mathbb{G}$ *indirectly dominates* network $G \in \mathbb{G}$, written $G' \rhd\rhd G$, if there is a *finite* sequence of networks,

$$G_0, G_1, \ldots, G_h,$$

with $G = G_0$, $G' = G_h$, and $G_k \in \mathbb{G}$ for $k = 0, 1, \ldots, h$, and a corresponding sequence of coalitions,

$$S_1, S_2, \ldots, S_h,$$

such that for $k = 1, 2, \ldots, h$

$$G_{k-1} \underset{S_k}{\longrightarrow} G_k,$$
$$\text{and}$$
$$G_{k-1} \prec_{S_k} G_h.$$

Note that if network G' indirectly dominates network G (i.e., if $G' \rhd\rhd G$), then what matters to the initially deviating coalition S_1, as well as all the coalitions along the way, is that the ultimate network outcome $G' = G_h$ be preferred. Thus, for example, the initially deviating coalition S_1 will not be deterred from changing network G_0 to network G_1 even if network G_1 is not preferred to network $G = G_0$, as long as the ultimate network outcome $G' = G_h$ is preferred to G_0, that is, as long as $G_0 \prec_{S_1} G_h$.[18]

3.3.3 Path Dominance

Each dominance relation $>$, whether it be direct or indirect (i.e., whether $> = \rhd$ or $> = \rhd\rhd$), induces a path dominance relation on the set of networks. In particular, corresponding to dominance relation $>$ on networks $\mathbb{G}$ there is a corresponding path dominance relation $\geq_p$ on $\mathbb{G}$ specified as follows: network $G' \in \mathbb{G}$ (weakly) path dominates network $G \in \mathbb{G}$ with respect to $>$ (i.e., with respect to the underlying dominance relation $>$), written $G' \geq_p G$, if $G' = G$ or if there exists a *finite* sequence of networks $\{G_k\}_{k=0}^{h}$ in $\mathbb{G}$ with $G_h = G'$ and $G_0 = G$ such that for $k = 1, 2, \ldots, h$

$$G_k > G_{k-1}.$$

We refer to such a finite sequence of networks as a *finite domination path* and we say network G' is $>$*-reachable* from network G if there exists a finite domination path from G to G'. Thus,

$$G' \geq_p G \text{ if and only if } \begin{cases} G' \text{ is } > \text{-reachable from } G, \text{ or} \\ G' = G. \end{cases} \tag{2}$$

[18] In order to capture the idea of farsightedness in strategic behavior, Chwe (1994) analyzes abstract games equipped with indirect dominance relations in great detail, introducing the equilibrium notions of consistency and largest consistent set. The basic idea of indirect dominance goes back to the work of Guilbaud (1949) and Harsanyi (1974).

If network G is reachable from network G, that is, if there is a finite domination path from G back to G then we call this path a *circuit*. Finally, if network G is *not* reachable from any network in $\mathbb{G}$ and if no network in $\mathbb{G}$ is reachable from G, then network G is *isolated* (i.e., network $G \in \mathbb{G}$ is isolated if there does not exist a network $G' \in \mathbb{G}$ with $G' \geq_p G$ or $G \geq_p G'$).

3.3.4 The Directed Graph of a Dominance Relation

It is often useful to represent the dominance relation over networks using a directed graph. For example, Figure 3 depicts the graph of a direct dominance relation $>$ on the feasible set of networks $\mathbb{G} = \{G_0, G_1, \ldots, G_7\}$.

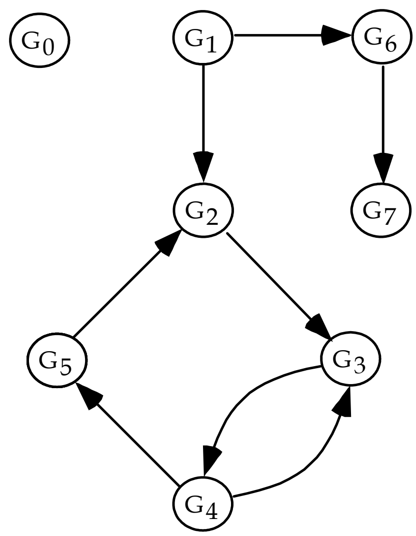

Figure 3: Directed Graph of Dominance Relation $>$

The arrow (or $>$-arc) from network G_3 to network G_4 in Figure 3 indicates that G_4 dominates G_3. Given primitives $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$ and given that $>$ is a *direct* dominance relation, the $>$-arc from network G_3 to network G_4 means that for some coalition S, G_4 is preferred to G_3 and more importantly, that coalition S has the power to change network G_3 to network G_4. Thus, $G_3 \prec_S G_4$ and $G_3 \rightarrow_S G_4$. But notice also that there is a $>$-arc in the opposite direction, from network G_4 to network G_3. Thus, G_3 also dominates G_4, and thus for some other coalition S' distinct from coalition S, that is, some coalition S' with $S' \cap S = \emptyset$, $G_4 \prec_{S'} G_3$ and $G_4 \rightarrow_{S'} G_3$.

Note that network G_3 is $>$-reachable from network G_3 via two paths. Thus, the graph of dominance relation $>$ depicted in Figure 3 contains two circuits. Defining the *length* of a domination path to be the number of $>$-arcs in the path, these two circuits are of length 4 and length 2.

Because networks G_2 and G_5 in Figure 3 are on the same circuit, G_5 is $>$-reachable from G_2 *and* G_2 is $>$-reachable from G_5. Thus, G_5 path dominates G_2 (i.e., $G_5 \geq_p G_2$) *and* G_2 path dominates G_5 (i.e., $G_2 \geq_p G_5$). The same cannot be said of networks

G_1 and G_5 in Figure 3. In particular, while $G_5 \geq_p G_1$, it is not true that $G_1 \geq_p G_5$ because G_1 is not $>$-reachable from G_5. Finally, note that network G_0 is isolated.

4 Network Formation Games and Stability

We can now present our main results. Using the abstract network formation game with respect to path dominance given by the pair

$$(\mathbb{G}, \geq_p) \tag{3}$$

and induced by primitives

$$(\mathbb{G}, \{\succ_S\}, \{\to_S\}, >)_{S \in P(D)}, \tag{4}$$

we introduce and characterize the notions of (i) strategic basins of attraction, (ii) path dominance stable sets, and (iii) the path dominance core. All of the results presented in this section hold for path dominance relations induced by direct dominance relations or by indirect dominance relations.

4.1 Networks Without Descendants

If $G_1 \geq_p G_0$ and $G_0 \geq_p G_1$, networks G_1 and G_0 are *equivalent,* written $G_1 \equiv_p G_0$. If networks G_1 and G_0 are equivalent then either networks G_1 and G_0 coincide or G_1 and G_0 are on the same circuit (see Figure 3 above for a picture of a circuit). If $G_1 \geq_p G_0$ but G_1 and G_0 are not equivalent (i.e., not $G_1 \equiv_p G_0$), then network G_1 is a *descendant* of network G_0 and we write

$$G_1 >_p G_0. \tag{5}$$

Referring to Figure 3, observe that network G_5 is a descendant of network G_1, that is, $G_5 >_p G_1$.

Network $G' \in \mathbb{G}$ *has no descendants in* $\mathbb{G}$ if for any network $G \in \mathbb{G}$

$$G \geq_p G' \text{ implies that } G \equiv_p G'.$$

Thus, if G' has no descendants then $G \geq_p G'$ implies that G and G' coincide or lie on the same circuit.[19]

In attempting to identify those networks which are likely to emerge and persist, networks *without descendants* are of particular interest. Here is our main result concerning networks without descendants.

Theorem 1 *(All path dominance network formation games have networks without descendants)*

Let $(\mathbb{G}, \geq_p)$ *be a network formation game. For every network* $G \in \mathbb{G}$ *there exists a network* $G' \in \mathbb{G}$ *such that* $G' \geq_p G$ *and* G' *has no descendants.*

[19] Note that any isolated network is by definition a network without descendants (e.g., network G_0 in Figure 3).

Proof. Let G_0 be any network in $\mathbb{G}$. If G_0 has no descendants then we are done. If not choose G_1 such that $G_1 >_p G_0$. If G_0 has no descendants then we are done. If not, continue by choosing $G_2 >_p G_1$. Proceeding iteratively, we can generate a sequence, $G_0, G_1, G_2, \ldots$. Now observe that in a finite number of iterations we must come to a network $G_{k'}$ without descendants. Otherwise, we could generate an infinite sequence, $\{G_k\}_k$ such that for all k,

$$G_k >_p G_{k-1}.$$

However, because $\mathbb{G}$ is finite this sequence would contain at least one network, say $G_{k'}$, which is repeated an infinite number of times. Thus, all the networks in the sequence lying between any two consecutive repetitions of $G_{k'}$ would be on the same circuit, contradicting the fact that for all k, G_k is a descendant of G_{k-1} (i.e., $G_k >_p G_{k-1}$). ■

By Theorem 1, in any network formation game $(\mathbb{G}, \geq_p)$, corresponding to any network $G \in \mathbb{G}$ there is a network $G' \in \mathbb{G}$ without descendants which is $>$-reachable from G. Thus, in any network formation game the set of networks without descendants is nonempty. Referring to Figure 3, the set of networks without descendants is given by

$$\{G_0, G_2, G_3, G_4, G_5, G_7\}.$$

We shall denote by $\mathbb{Z}$ the set of networks without descendants.

4.2 Basins of Attraction

Stated loosely, a basin of attraction is a set of *equivalent* networks to which the strategic network formation process represented by the game might tend and from which there is no escape. Formally, we have the following definition.

Definition 2 *(Basin of Attraction)*

Let $(\mathbb{G}, \geq_p)$ be a network formation game. A set of networks $\mathbb{A} \subseteq \mathbb{G}$ is said to be a basin of attraction for $(\mathbb{G}, \geq_p)$ if

1. *the networks contained in $\mathbb{A}$ are equivalent (i.e., for all G' and G in $\mathbb{A}$, $G' \equiv_p G$), and*
2. *no network in $\mathbb{A}$ has descendants (i.e., there does not exist a network $G' \in \mathbb{G}$ such that $G' >_p G$ for some $G \in \mathbb{A}$).*

As the following characterization result shows, there is a very close connection between networks without descendants and basins of attraction.

Theorem 2 *(A characterization of basins of attraction)*

Let $(\mathbb{G}, \geq_p)$ be a network formation game and let $\mathbb{A}$ be a subset of networks in $\mathbb{G}$. The following statements are equivalent:

1. *$\mathbb{A}$ is a basin of attraction for $(\mathbb{G}, \geq_p)$.*

2. There exists a network without descendants, $G \in \mathbb{Z}$, such that

$$\mathbb{A} = \left\{G' \in \mathbb{Z} : G' \equiv_p G\right\}.$$

Proof. (1) implies (2): Because the sets $\mathbb{A}$ and $\{G' \in \mathbb{Z} : G' \equiv_p G\}$, $G \in \mathbb{Z}$, are equivalence classes, $\mathbb{A} \neq \{G' \in \mathbb{Z} : G' \equiv_p G\}$ implies that

$$\mathbb{A} \cap \left\{G' \in \mathbb{Z} : G' \equiv_p G\right\} = \emptyset \text{ for all } G \in \mathbb{Z}.$$

Thus, if (2) fails, this implies that $\mathbb{A}$ contains a network with descendants. Thus, $\mathbb{A}$ cannot be a basin of attraction for $(\mathbb{G}, \geq_p)$, and thus, (1) implies (2).[20]

(2) implies (1): Suppose now that

$$\mathbb{A} = \left\{G' \in \mathbb{Z} : G' \equiv_p G\right\}$$

for some network $G \in \mathbb{Z}$. If $\mathbb{A}$ is not a basin of attraction, then for some network $G'' \in \mathbb{G}$, $G'' >_p G'$ for some $G' \in \mathbb{A}$. But now $G'' >_p G'$ and $G' \equiv_p G$ imply that $G'' >_p G$, contradicting the fact that $G \in \mathbb{Z}$. Thus, (2) implies (1). ■

In light of Theorem 2, we conclude that in any network formation game $(\mathbb{G}, \geq_p)$, $\mathbb{G}$ contains a *unique*, finite, disjoint collection of basins of attraction, say $\{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_m\}$, where for each $k = 1, 2, \ldots, m$ $(m \geq 1)$

$$\mathbb{A}_k = \mathbb{A}_G := \left\{G' \in \mathbb{Z} : G' \equiv_p G\right\}$$

for some network $G \in \mathbb{Z}$. Note that for networks G' and G in $\mathbb{Z}$ such that $G' \equiv_p G$, $\mathbb{A}_{G'} = \mathbb{A}_G$ (i.e. the basins of attraction $\mathbb{A}_{G'}$ and $\mathbb{A}_G$ coincide). Also, note that if network $G \in \mathbb{G}$ is isolated, then $G \in \mathbb{Z}$ and

$$\mathbb{A}_G := \left\{G' \in \mathbb{Z} : G' \equiv_p G\right\} = \{G\}$$

is, by definition, a basin of attraction - but a very uninteresting one.

Example 1 *(Basins of attraction)*

In Figure 3 above the set of networks without descendants is given by

$$\mathbb{Z} = \{G_0, G_2, G_3, G_4, G_5, G_8\}.$$

Even though there are six networks without descendants, because networks G_2, G_3, G_4, and G_5 are equivalent, there are only three basins of attraction:

$$\mathbb{A}_1 = \{G_0\}, \ \mathbb{A}_2 = \{G_2, G_3, G_4, G_5\}, \text{ and } \mathbb{A}_3 = \{G_7\}.$$

Moreover, because G_2, G_3, G_4, and G_5 are equivalent,

$$\mathbb{A}_{G_2} = \mathbb{A}_{G_3} = \mathbb{A}_{G_4} = \mathbb{A}_{G_5} = \{G_2, G_3, G_4, G_5\}.$$

[20] Note that if $G \in \mathbb{Z}$ and $G' \equiv_p G$, then $G' \in \mathbb{Z}$.

4.3 Stable Sets with Respect to Path Dominance

The formal definition of a $\geq_p$-stable set is as follows.[21]

Definition 3 *(Stable Sets with Respect to Path Dominance)*
Let $(\mathbb{G}, \geq_p)$ be a network formation game. A subset $\mathbb{V}$ of networks in $\mathbb{G}$ is said to be a stable set for $(\mathbb{G}, \geq_p)$ if

(a) (internal $\geq_p$ -stability) whenever G_0 and G_1 are in $\mathbb{V}$, with $G_0 \neq G_1$, then neither $G_1 \;\geq_p G_0$ nor $G_0 \;\geq_p G_1$ hold, and

(b) (external $\geq_p$ -stability) for any $G_0 \notin \mathbb{V}$ there exists $G_1 \in \mathbb{V}$ such that $G_1 \;\geq_p G_0$.

In other words, a nonempty subset of networks $\mathbb{V}$ is a stable set for $(\mathbb{G}, \geq_p)$ if G_0 and G_1 are in $\mathbb{V}$, with $G_0 \neq G_1$, then G_1 is not reachable from G_0, nor is G_0 reachable from G_1, and if $G_0 \notin \mathbb{V}$, then there exists $G_1 \in \mathbb{V}$ reachable from G_0.

We now have our main results on the existence, construction, and cardinality of stable sets.[22]

Theorem 3 *(Stable sets: existence, construction, and cardinality)*
Let $(\mathbb{G}, \geq_p)$ be a network formation game, and without loss of generality assume that $(\mathbb{G}, \geq_p)$ has basins of attraction given by

$$\{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_m\},$$

where basin of attraction $\mathbb{A}_k$ contains $|\mathbb{A}_k|$ many networks (i.e., $|\mathbb{A}_k|$ is the cardinality of $\mathbb{A}_k$). Then the following statements are true:

1. *$\mathbb{V} \subseteq \mathbb{G}$ is a stable set for $(\mathbb{G}, \geq_p)$ if and only if $\mathbb{V}$ is constructed by choosing one network from each basin of attraction, that is, if and only if $\mathbb{V}$ is of the form*

$$\mathbb{V} = \{G_1, G_2, \ldots, G_m\},$$

where $G_k \in \mathbb{A}_k$ for $k = 1, 2, \ldots, m$.

2. *$(\mathbb{G}, \geq_p)$ possesses*

$$|\mathbb{A}_1| \cdot |\mathbb{A}_2| \cdot \cdots \cdot |\mathbb{A}_m| := M$$

many stable seats and each stable set, $\mathbb{V}_q$, $q = 1, 2, \ldots, M$, has cardinality

$$|\mathbb{V}_q| = |\{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_m\}| = m.$$

[21] By equipping the abstract network formation game with the path dominance relation rather than the original dominance relation, we entirely avoid the famous Lucas (1968) example of a game with no stable set.

[22] These results can be viewed as applications of some classical results from graph theory to the theory of network formation games (e.g., see Berge 2001, Chapter 2).

Proof. It suffices to prove (1). Given (1), the proof of (2) is straightforward. To begin, let

$$\mathbb{V} = \{G_1, G_2, \ldots, G_m\},$$

where $G_k \in \mathbb{A}_k$ for $k = 1, 2, \ldots, m$, and suppose that for G_k and $G_{k'}$ in $\mathbb{V}$, $G_{k'} \geq_p G_k$. Since $G_k \in \mathbb{A}_k$ has no descendants, this would imply that $G_{k'} \equiv_p G_k$. But this is a contradiction because $G_k \in \mathbb{A}_k$ and $G_{k'} \in \mathbb{A}_{k'}$ and the basins of attraction $\mathbb{A}_k$ and $\mathbb{A}_{k'}$ are disjoint. Thus, $\mathbb{V}$ is internally $\geq_p$-stable. Now suppose that network G is not contained in $\mathbb{V}$. By Theorem 1, there exists a network $G' \in \mathbb{G}$ such that $G' \geq_p G$, and $\Gamma_{>_p}(G') = \emptyset$ (i.e., G' is a network without descendants). By Theorem 2, G' is contained in some basin of attraction $\mathbb{A}_k$ and therefore $G' \equiv_p G_k$ where G_k is the k^{th} component of $\{G_1, G_2, \ldots, G_m\}$. Thus, we have $G_k \geq_p G' \geq_p G$ implying that $G_k \geq_p G$, and thus $\mathbb{V}$ is externally $\geq_p$-stable.

Suppose now that $\mathbb{V} \subseteq \mathbb{G}$ is a stable set for $(\mathbb{G}, \geq_p)$. First note that each network G in $\mathbb{V}$ is a network without descendants. Otherwise there exists $G' \in \mathbb{G} \backslash \mathbb{V}$ such that $G' >_p G$. But then because $\mathbb{V}$ is externally $\geq_p$-stable, there exists $G'' \in \mathbb{V}$, $G'' \neq G$, such that $G'' \geq_p G'$ implying that $G'' \geq_p G$ and contradicting the internal $\geq_p$-stability of $\mathbb{V}$. Because each $G \in \mathbb{V}$ is without descendants, it follows from Theorem 2 that each $G \in \mathbb{V}$ is contained in some basin of attraction $\mathbb{A}_k$. Moreover, because $\mathbb{V}$ is internally $\geq_p$-stable and because all networks contained in any one basin of attraction are equivalent, no two distinct networks contained in $\mathbb{V}$ can be contained in the same basin of attraction. It only remains to show that for each basin of attraction, $\mathbb{A}_k$, $k = 1, 2, \ldots, m$,

$$\mathbb{V} \cap \mathbb{A}_k \neq \emptyset.$$

Suppose not. Then for some k', $\mathbb{V} \cap \mathbb{A}_{k'} = \emptyset$. Because all networks in $\mathbb{A}_{k'}$ are without descendants, for no network $G \in \mathbb{A}_{k'}$ is it true that there exists a network $G' \in \mathbb{V}$ such that $G' \geq_p G$. Thus, we have a contradiction of the external $\geq_p$-stability of $\mathbb{V}$. ■

Example 2 *(Basins of attraction and stable sets)*

Referring to Figure 3, it follows from Theorem 3 that because

$$|\mathbb{A}_1| \cdot |\mathbb{A}_2| \cdot |\mathbb{A}_3| = 1 \cdot 4 \cdot 1 = 4,$$

the network formation game $(\mathbb{G}, \geq_p)$ has 4 stable sets, each with cardinality 3. By examining Figure 3 in light of Theorem 3, we see that the stable sets for $(\mathbb{G}, \geq_p)$ are given by

$$\begin{aligned} \mathbb{V}_1 &= \{G_0, G_2, G_7\}, \\ \mathbb{V}_2 &= \{G_0, G_3, G_7\}, \\ \mathbb{V}_3 &= \{G_0, G_4, G_7\}, \\ \mathbb{V}_4 &= \{G_0, G_5, G_7\}. \end{aligned}$$

4.4 The Path Dominance Core

Definition 4 *(The Path Dominance Core)*

Let $(\mathbb{G}, \geq_p)$ be a network formation game. A subset $\mathbb{C}$ of networks in $\mathbb{G}$ is said to be the path dominance core of $(\mathbb{G}, \geq_p)$ if for each network $G \in \mathbb{C}$ there does not exist a network $G' \in \mathbb{G}$, $G' \neq G$, such that $G' \geq_p G$.

Our next results give necessary and sufficient conditions for the path dominance core of a network formation game to be nonempty, as well as a recipe for constructing the path dominance core.

Theorem 4 *(Path dominance core: nonemptiness and construction)*

Let $(\mathbb{G}, \geq_p)$ be a network formation game, and without loss of generality assume that $(\mathbb{G}, \geq_p)$ has basins of attraction given by

$$\{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_m\},$$

where basin of attraction $\mathbb{A}_k$ contains $|\mathbb{A}_k|$ many networks. Then the following statements are true:

1. *$(\mathbb{G}, \geq_p)$ has a nonempty path dominance core if and only if there exists a basin of attraction containing a single network, that is, if and only if for some basin of attraction $\mathbb{A}_k$, $|\mathbb{A}_k| = 1$.*

2. *Let*
$$\{\mathbb{A}_{k_1}, \mathbb{A}_{k_2}, \ldots, \mathbb{A}_{k_n}\} \subseteq \{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_m\},$$
be the subset of basins of attraction containing all basins having cardinality 1. Then the path dominance core $\mathbb{C}$ of $(\mathbb{G}, \geq_p)$ is given by
$$\mathbb{C} = \{G_{k_1}, G_{k_2}, \ldots, G_{k_n}\},$$
where $G_{k_i} \in \mathbb{A}_{k_i}$, for $i = 1, 2, \ldots, n$.

Proof. It suffices to show that a network G is contained in the path dominance core $\mathbb{C}$ if and only if $G \in \mathbb{A}_k$ for some basin of attraction $\mathbb{A}_k$, $k = 1, 2, \ldots, m$, with $|\mathbb{A}_k| = 1$. First note that if G is in the path dominance core, then G is a network without descendants. Thus, $G \in \mathbb{A}_k$ for some basin of attraction $\mathbb{A}_k$. If $|\mathbb{A}_k| > 1$, then there exists another network $G' \in \mathbb{A}_k$ such that $G' \equiv_p G$. Thus, $G' \geq_p G$ contradicting the fact that G is in the path dominance core. Conversely, if $G \in \mathbb{A}_k$ for some basin of attraction $\mathbb{A}_k$ with $|\mathbb{A}_k| = 1$, then there does not exist a network $G' \neq G$ such that $G' \geq_p G$. ■

Remark 2 *If coalitional preferences $\{\succ_S\}_{S \in P(D)}$ over networks are based on weak preference relations $\{\succsim_d\}_{d \in D}$ rather than on strong preference relations $\{\succ_d\}_{d \in D}$ (see Remark 1 above), then the corresponding path dominance core - the weak path dominance core - is contained in the path dominance core (i.e., the path dominance core based on strong preference relations).*

Example 3 *(Basins of attraction and the path dominance core)*

It follows from Theorem 4 that the path dominance core of the network formation game $(\mathbb{G}, \geq_p)$ with feasible set

$$\mathbb{G} = \{G_0, G_1, \ldots, G_7\}$$

and path dominance relation $\geq_p$ induced by the dominance relation depicted in Figure 3 is

$$\mathbb{C} = \{G_0, G_7\}.$$

Figure 4 contains the graph of a different dominance relation on $\mathbb{G} = \{G_0, G_1, \ldots, G_7\}$.

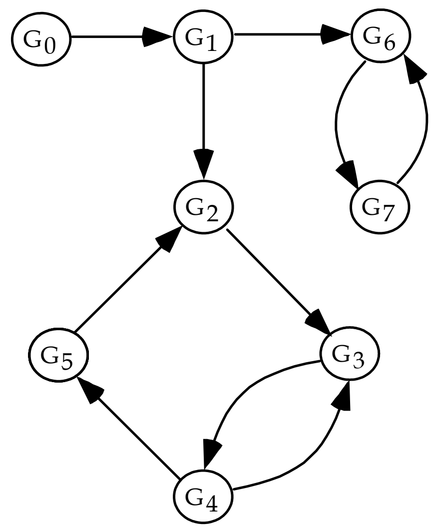

Figure 4: Graph of a different dominance relation $\gtrdot$

Denoting the new dominance relation by $\gtrdot$,the network formation game $\left(\mathbb{G}, \underline{\gtrdot}_p\right)$ with respect to the path dominance relation $\underline{\gtrdot}_p$ induced by the dominance relation $\gtrdot$ has 3 circuits and 2 basins of attraction,

$$\mathbb{A}_1 = \{G_2, G_3, G_4, G_5\} \text{ and } \mathbb{A}_2 = \{G_6, G_7\}.$$

Because $|\mathbb{A}_1| = 4$ and $|\mathbb{A}_2| = 2$, by Theorem 4 the path dominance core of $\left(\mathbb{G}, \underline{\gtrdot}_p\right)$ is empty. By Theorem 3, $\left(\mathbb{G}, \underline{\gtrdot}_p\right)$ has 8 stable sets each containing 2 networks (i.e., each with cardinality 2). These stable sets are given by

$$\begin{aligned}
\mathbb{V}_1 &= \{G_2, G_6\}, \\
\mathbb{V}_2 &= \{G_3, G_6\}, \\
\mathbb{V}_3 &= \{G_4, G_6\}, \\
\mathbb{V}_4 &= \{G_5, G_6\}, \\
\mathbb{V}_5 &= \{G_2, G_7\}, \\
\mathbb{V}_6 &= \{G_3, G_7\}, \\
\mathbb{V}_7 &= \{G_4, G_7\}, \\
\mathbb{V}_8 &= \{G_5, G_7\}.
\end{aligned}$$

4.4.1 The Path Dominance Core and Pareto Efficiency

Given primitives $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$, we say that a network $G \in \mathbb{G}$ is Pareto Efficient if there does not exist another network $G' \in \mathbb{G}$ such that (i) $G \rightarrow_S G'$ for some coalition $S \in P(D)$ and (ii) $G \prec_d G'$ for *all* players $d \in D$. Let $\mathbb{E}$ denote the set of Pareto efficient networks and let $\mathbb{C}$ denote the path dominance core of network formation game $(\mathbb{G}, \geq_p)$. It is easy to see that $\mathbb{C} \subseteq \mathbb{E}$.

5 Other Stability Notions for Network Formation Games

5.1 Strongly Stable Networks

We begin with a formal definition of strong stability for abstract network formation games.

Definition 5 *(Strong Stability)*
Given primitives $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$ *and network formation game* $(\mathbb{G}, \geq_p)$, *network* $G \in \mathbb{G}$ *is said to be strongly stable in* $(\mathbb{G}, \geq_p)$ *if for all* $G' \in \mathbb{G}$ *and* $S \in P(D)$, $G \rightarrow_S G^{'}$ *implies that* $G \not\prec_S G^{'}$.

Thus, a network is strongly stable if whenever a coalition has the power to change the network to another network, the coalition will be deterred from doing so because not *all* members of the coalition are made better off by such a change.[23] If nodes represent players and arc addition is bilateral while arc subtraction is unilateral, then our definition of strong stability is essentially that of Jackson-van den Nouweland but for directed networks rather than linking networks. Note that under our definition of strong stability a network $G \in \mathbb{G}$ that cannot be changed to another network by any coalition is strongly stable.

We now have our main result on the path dominance core and strong stability. Denote the set of strongly stable networks by $\mathbb{SS}$.

Theorem 5 *(The path dominance core and strong stability)*
Given primitives $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$ *and network formation game* $(\mathbb{G}, \geq_p)$, *the following statements are true.*

1. *If the path dominance core* $\mathbb{C}$ *of* $(\mathbb{G}, \geq_p)$ *is nonempty, then* $\mathbb{SS}$ *is nonempty and* $\mathbb{C} \subseteq \mathbb{SS}$.

2. *If the dominance relation* $>$ *underlying* $\geq_p$ *is a direct dominance relation, then* $\mathbb{C} = \mathbb{SS}$ *and* $\mathbb{SS}$ *is nonempty if and only if there exists a basin of attraction containing a single network.*

Proof. 1. Let $\mathbb{C} \subseteq \mathbb{G}$, $\mathbb{C} \neq \emptyset$, be the path dominance core of $(\mathbb{G}, \geq_p)$ and let network G be contained in $\mathbb{C}$. Then there does not exist a network $G' \in \mathbb{G}$, $G' \neq G$, such that $G' \geq_p G$. If for some coalition S and some network $G' \in \mathbb{G}$, $G \rightarrow_S G'$ *and* $G \prec_S G'$,

[23] Our definition of a strongly stable network differs slightly from the definition given in Jackson-van den Nouweland (2005). In particular, under their definition, a network is strongly stable if whenever a coalition has the power to change the network to another network, the coalition will be deterred from doing so because at least one member of the coalition is made *worse off* by the change.

then $G' \geq_p G$ trivially, a contradiction. Thus, for G contained in $\mathbb{C}$, $G \to_S G'$ for coalition S implies that $G \not\prec_S G^{'}$, and thus $G \in \mathbb{C}$ implies $G \in \mathbb{SS}$.

2. To see that $\mathbb{SS} \subseteq \mathbb{C}$ if the dominance relation $>$ underlying $\geq_p$ is a direct dominance relation, consider the following. If $G \notin \mathbb{C}$, then there exists a network $G' \neq G$ which path dominates G, that is, $G' \geq_p G$. This implies that there exists a network G'' such that $G' \geq_p G'' > G$. Because $>$ is a direct dominance relation, for some coalition S we have $G \to_S G''$ and $G \prec_S G''$. Thus, $G \notin \mathbb{SS}$. By part 1 of Theorem 4, $\mathbb{C} = \mathbb{SS}$ is nonempty if and only if there exists a basin of attraction containing a single network. ■

Note that the set of strongly stable networks is contained in the set of Pareto efficient networks. Thus, $\mathbb{C} \subseteq \mathbb{SS} \subseteq \mathbb{E}$.

5.2 Pairwise Stable Networks

The following definition is a formalization of Jackson-Wolinsky (1996) pairwise stability for abstract network formation games.

Definition 6 *(Pairwise Stability)*

Given networks $P(A\times(N \times N))$ where nodes represent players (i.e., $N = D$) and given feasible networks $\mathbb{G} \subseteq P(A\times(N\times N))$ and primitives $(\mathbb{G}, \{\succ_S\}, \{\to_S\}, >)_{S\in P(D)}$, network $G \in \mathbb{G}$ is said to be pairwise stable in network formation game $(\mathbb{G}, \geq_p)$ if for all $(a,(i,i')) \in A \times (N \times N)$,

1. *$G \to_{\{i,i'\}} G \cup (a,(i,i'))$ implies that $G \not\prec_{\{i,i'\}} G \cup (a,(i,i'))$;*
2. *(a) $G \to_{\{i\}} G\diagdown(a,(i,i'))$ implies that $G \not\prec_{\{i\}} G\diagdown(a,(i,i'))$, and*

   (b) $G \to_{\{i'\}} G\diagdown(a,(i,i'))$ implies that $G \not\prec_{\{i'\}} G\diagdown(a,(i,i'))$.

Thus, a network is pairwise stable if there is no incentive for any pair of players to add an arc to the existing network and there is no incentive for any player who is party to an arc in the existing network to dissolve or remove the arc. Note that under our definition of pairwise stability a network $G \in \mathbb{G}$ that cannot be changed to another network by any coalition, or can only be changed by coalitions of size greater than 2, is pairwise stable.

Let $\mathbb{PS}$ denote the set of pairwise stable networks. It follows from the definitions of strong stability and pairwise stability that $\mathbb{SS} \subseteq \mathbb{PS}$. Moreover, if the full set of Jackson-Wolinsky rules are in force, then $\mathbb{SS} = \mathbb{PS}$.

We now have our main result on the path dominance core and pairwise stability.

Theorem 6 *(The path dominance core and pairwise stability)*

Given primitives $(\mathbb{G}, \{\succ_S\}, \{\to_S\}, >)_{S\in P(D)}$ where nodes represent players (i.e., $N = D$) and given network formation game $(\mathbb{G}, \geq_p)$, the following statements are true.

1. *If the path dominance core $\mathbb{C}$ of $(\mathbb{G}, \geq_p)$ is nonempty, then $\mathbb{PS}$ is nonempty and $\mathbb{C} \subseteq \mathbb{PS}$.*

2. *If the dominance relation $>$ underlying $\geq_p$ is a direct dominance relation and if the Jackson-Wolinsky rules hold, then $\mathbb{C} = \mathbb{PS}$ and $\mathbb{PS}$ is nonempty if and only if there exists a basin of attraction containing a single network.*

Proof. The proof of part 1 follows from part 1 of Theorem 5 and the fact that $\mathbb{SS} \subseteq \mathbb{PS}$. For the proof of part 2, note that under the Jackson-Wolinsky rules $\mathbb{SS} = \mathbb{PS}$. Thus, we have $\mathbb{C} \subseteq \mathbb{SS} = \mathbb{PS}$. If in addition the path dominance relation is induced by a direct dominance relation, then we have $\mathbb{PS} = \mathbb{SS} \subseteq \mathbb{C}$. Thus, if the path dominance is induced by a direct dominance and if the Jackson-Wolinsky rules hold, then we have $\mathbb{C} = \mathbb{SS} = \mathbb{PS}$. By part 1 of Theorem 4, $\mathbb{C} = \mathbb{SS} = \mathbb{PS}$ is nonempty if and only if there exists a basin of attraction containing a single network. ■

Theorem 6 can be viewed as an extension of a result due Jackson and Watts (2002) on the existence of pairwise stable linking networks for network formation games induced by Jackson-Wolinsky rules. In particular, Jackson and Watts (2002) show that for this particular class of Jackson-Wolinsky network formation games, if there does not exist a closed cycle of networks, then there exists a pairwise stable network. Our notion of a strategic basin of attraction containing *multiple* networks corresponds to their notion of a closed cycle of networks. Thus, stated in our terminology, Jackson and Watts show that for this class of network formation games, if there does not exist a basin of attraction containing multiple networks, then there exists a pairwise stable network. Following our approach, if we specialize to this class of Jackson-Wolinsky network formation games, then by part 2 of Theorem 6 the existence of *at least one* strategic basin containing a single network is both necessary and sufficient for the existence of a pairwise stable network.

5.3 Consistent Networks

We begin with a formal definition of farsighted consistency (Chwe 1994).

Definition 7 *(Consistent Sets)*

Let $(\mathbb{G}, \geq_p)$ be a network formation game where path dominance $\geq_p$ is induced by an indirect dominance relation $\triangleright\triangleright$. A subset $\mathbb{F}$ of directed networks in $\mathbb{G}$ is said to be consistent in $(\mathbb{G}, \geq_p)$ if

$$\begin{gathered} \text{for all } G_0 \in \mathbb{F}, \\ G_0 \rightarrow_{S_1} G_1 \text{ for some } G_1 \in \mathbb{G} \text{ and some coalition } S_1 \text{ implies that} \\ \text{there exists } G_2 \in \mathbb{F} \\ \text{with } G_2 = G_1 \text{ or } G_2 \triangleright\triangleright G_1 \text{ such that,} \\ G_0 \not\prec_{S_1} G_2. \end{gathered}$$

In words, a subset of directed networks $\mathbb{F}$ is said to be consistent in $(\mathbb{G}, \geq_p)$ if given any network $G_0 \in \mathbb{F}$ and any deviation to network $G_1 \in \mathbb{G}$ by coalition S_1 (via adding, subtracting, or replacing arcs in accordance with effectiveness relations $\rightarrow_S$), there exists further deviations leading to some network $G_2 \in \mathbb{F}$ where the initially deviating coalition S_1 is not better off - and possibly worse off. A network $G \in \mathbb{G}$ is said to be consistent if $G \in \mathbb{F}$ where $\mathbb{F}$ is a consistent set in $(\mathbb{G}, \geq_p)$. There can be many consistent sets in $(\mathbb{G}, \geq_p)$. We shall denote by $\mathbb{F}^*$ is largest consistent set (or simply, the *largest consistent set*). Thus, if $\mathbb{F}$ is a consistent set, then $\mathbb{F} \subseteq \mathbb{F}^*$.

We now have our main result on the relationship between basins of attraction, stable sets, the path dominance core, and the largest consistent set.

Theorem 7 *(Basins of attraction, the path dominance core, and the largest consistent set)*

Given primitives $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$ *and given network formation game* $(\mathbb{G}, \geq_p)$*, where path dominance is induced by an indirect dominance relation* $\rhd\rhd$*, assume without loss of generality that* $(\mathbb{G}, \geq_p)$ *has nonempty largest consistent set given by* $\mathbb{F}^*$ *and basins of attraction given by*

$$\{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_m\}.$$

Then the following statements are true:

1. *Each basin of attraction* $\mathbb{A}_k$*,* $k = 1, 2, \ldots, m$*, has a nonempty intersection with the largest consistent set* $\mathbb{F}^*$*, that is*

$$\mathbb{F}^* \cap \mathbb{A}_k \neq \emptyset, \text{ for } k = 1, 2, \ldots, m.$$

2. *If* $(\mathbb{G}, \geq_p)$ *has a nonempty path dominance core* $\mathbb{C}$*, then*

$$\mathbb{C} \subseteq \mathbb{F}^*.$$

Proof. In light of Theorem 4, (2) easily follows from (1). Thus, it suffices to prove (1). Suppose that for some basin of attraction $\mathbb{A}_{k'}$

$$\mathbb{F}^* \cap \mathbb{A}_{k'} = \emptyset.$$

Let G' be a network in $\mathbb{A}_{k'}$. Because $\mathbb{F}^*$ is externally stable with respect to the indirect dominance relation $\rhd\rhd$, $G' \notin \mathbb{F}^*$ implies that there exists some network $G^* \in \mathbb{F}^*$ such that $G^* \rhd\rhd G'$. Thus, $G^* \geq_p G'$. Because the networks in $\mathbb{A}_{k'}$ are without descendants, it must be true that $G' \geq_p G^*$. But this implies that $G^* \equiv_p G'$, and therefore that $G^* \in \mathbb{A}_{k'}$, a contradiction. ■

Remark 3 *Recently, Herings, Mauleon, and Vannetelbosch (2005) introduced a notion of pairwise farsighted stability. If in our model coalitional preferences* $\{\succ_S\}_{S \in P(D)}$ *over networks are based on weak preference relations* $\{\succsim_d\}_{d \in D}$ *(see Remark 1 above), if nodes represent players (i.e.,* $N = D$*), and if the dominance relation underlying the path dominance relation is indirect, then under Jackson-Wolinsky rules the corresponding weak path dominance core is contained in the set of pairwise farsightedly stable networks.*

5.4 Nash Networks

Definition 8 *(Nash Networks)*

Given primitives $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$ *and network formation game* $(\mathbb{G}, \geq_p)$, *network* $G \in \mathbb{G}$ *is said to be a Nash network in* $(\mathbb{G}, \geq_p)$ *if for all* $G' \in \mathbb{G}$ *and* $S \in P(D)$ *such that* $|S| = 1$, $G \rightarrow_S G^{'}$ *implies that* $G \not\prec_S G^{'}$.

Thus, a network is Nash if whenever an individual player has the power to change the network to another network, the player will have no incentive to do so. We shall denote by $\mathbb{NE}$ the set of Nash networks. Note that our definition of a Nash network does not require that the network formation rules, as represented via the effectiveness relations $\{\rightarrow_S\}_{S \in P(D)}$, be noncooperative (see subsection 3.2.1). Also, note that under our definition any network that cannot be changed to another network by a coalition of size 1 is a Nash network. Finally, note that the set of strongly stable networks $\mathbb{SS}$ is contained in the set of Nash networks $\mathbb{NE}$.

We now have our main result on the path dominance core and strong stability.

Theorem 8 *(The path dominance core and Nash equilibrium)*

Given primitives $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$ *and network formation game* $(\mathbb{G}, \geq_p)$, *the following statements are true.*

1. *If the path dominance core* $\mathbb{C}$ *of* $(\mathbb{G}, \geq_p)$ *is nonempty, then* $\mathbb{NE}$ *is nonempty and* $\mathbb{C} \subseteq \mathbb{NE}$.

2. *If the dominance relation* $>$ *underlying* $\geq_p$ *is a direct dominance relation and if the rules of network formation are such that* $G \rightarrow_S G^{'}$ *implies that* $|S| = 1$, *then* $\mathbb{C} = \mathbb{NE}$ *and* $\mathbb{NE}$ *is nonempty if and only if there exists a basin of attraction containing a single network.*

Proof. The proof of part 1 follows from part 1 of Theorem 5 and the fact that $\mathbb{SS} \subseteq \mathbb{NE}$. For the proof of part 2, note that if the rules of network formation are such that $G \rightarrow_S G^{'}$ implies that $|S| = 1$, then $\mathbb{SS} = \mathbb{NE}$. Thus, we have $\mathbb{C} \subseteq \mathbb{SS} = \mathbb{NE}$. If in addition the path dominance relation is induced by a direct dominance relation, then we have $\mathbb{NE} = \mathbb{SS} \subseteq \mathbb{C}$, and we conclude that $\mathbb{C} = \mathbb{SS} = \mathbb{NE}$. Thus, if the path dominance is induced by a direct dominance and if the rules are such that $G \rightarrow_S G^{'}$ implies that $|S| = 1$, then we have $\mathbb{C} = \mathbb{SS} = \mathbb{NE}$. By part 1 of Theorem 4, $\mathbb{C} = \mathbb{SS} = \mathbb{NE}$ is nonempty if and only if there exists a basin of attraction containing a single network. ■

We close this section by noting that if the dominance relation $>$ underlying $\geq_p$ is a direct dominance relation and if the rules of network formation are such that $G \rightarrow_S G^{'}$ implies that $|S| = 1$, then the set of Nash networks $\mathbb{NE}$ is contained in the set of Pareto efficient networks $\mathbb{E}$. Thus, for this case we have $\mathbb{C} = \mathbb{SS} = \mathbb{NE} \subseteq \mathbb{E}$.

6 Examples

6.1 Club Networks

The following version of a club network formation game is taken from Page and Wooders (2005) on club formation games. There are a number of models in the literature that use some of the same specification of primitives. Our network club model can be viewed as a network version of the local public good games analyzed by, among others, Konishi, Le Breton and Weber (1997a, 1998). Konishi et al. (1997a), and, for an even more general model including external effects of group formation on non–members, Hollard (2000), prove existence of Nash equilibrium when players unilaterally chose clubs. Holzman and Law–Yone (1997), Konishi et al. (1997b), and Milchtaich (1996) are concerned with the special case of congestion games, where each player's payoff is non–increasing in the number of players choosing the same strategy as himself. The latter two articles also provide conditions for the existence of strong Nash equilibria. Club games with positive externalities are analyzed in Konishi et al. (1997b).

For noncooperative network formation rules, the setup of our model is closely related to the model of Konishi, Le Breton and Weber (1997a). They define a *free mobility equilibrium* of a local public goods economy as an assignment of players to clubs (locations, or jurisdictions) that partitions the population and has the property that no individual can gain by either moving to any other existing club, or creating his own club.[24] The partition derived from the players' strategy choices is thus stable against unilateral deviations by individuals. For our network club model, one possible specification of the rules of the game is to allow only individual deviations and it is for this specification that our model most closely resembles that of Konishi, Le Breton and Weber (1997a).

For a specification of the rules of network formation that allows all subsets of decision makers to form coalitions and act in concert, our model is also closely related to models of economies with clubs or local public goods as in, for example, Conley and Wooders (2001) and papers cited therein, that study both price-taking equilibrium and the core of the cooperative game derived from their underlying economic principles. When there are 'enough' clubs, our model is closely related to the particular case, treated in Wooders (1980) and other papers, where crowding is anonymous (that is, only club size is relevant and not the characteristics of club members) but is more restrictive in that in our model, all players have identical preferences.[25]

Since we allow coalitional moves, our research is also related to Arnold and Wooders (2005), who consider a dynamic formulation of a game arising from the same sort of primitives as described below. Arnold and Wooders restrict coalitions to subsets of members of clubs, but also show the set of 'Nash club equilibrium' outcomes is equivalent to the set of strong equilibrium outcomes.

[24] Such an equilibrium is sometimes called a free entry equilibrium.

[25] A possible modification of the model is to allow multiple types of players – heterogeneous players. With such a modification, we could then examine additional questions, for example, whether equilibrium club networks generate homogenous clubs.

Overall, our club example illustrates several relationships between concepts shown in this paper and also indicates the importance of the network formation rules and the nature of the dominance relation (direct or indirect) in determining the equilibrium which emerges from the process of network formation represented by the game. In the case of 'too few' clubs, our example also highlights that our model and results cannot be presented in the framework of cooperative games or hedonic games. With too few clubs, a group of players larger than a singleton can only ensure themselves the worst possible outcome, the outcome which is least preferred. In the case of 'too few' clubs, it also highlights that our model and results cannot be presented in the framework of cooperative games or hedonic games. With too few clubs, a group of players larger than a singleton can only ensure themselves the worst possible outcome, the outcome which least preferred.

6.1.1 Basic Ingredients and Assumptions

Let D be a finite set of players with typical element denoted by i and let C be a finite set of club types - or alternatively, a set of club labels or club locations - with typical element denoted by c. Assume that the set of nodes is given by $N = D \cup C$, while the set of arcs is given by a singleton, $A = \{1\}$. The set of all club networks consists of the collection of all nonempty subsets of $A \times (D \times C)$, a collection denoted by $P(A \times (D \times C))$. Note that $P(A \times (D \times C))$ is a subset of $P(A \times (N \times N))$. For any club network $G \in P(A \times (D \times C))$

$$(1,(i,c)) \in G \text{ means that player } i \text{ is a member of club } c.$$

Given club network $G \in P(A \times (D \times C))$,

$$G(c) := \{i \in D : (1,(i,c)) \in G\}$$

(i.e., the section of G at c) is the set of members of club c in network G, while the set

$$G(i) := \{c \in C : (1,(i,c)) \in G\}$$

(i.e., the section of G at i) is the set of clubs to which player i belongs in network G.

We shall maintain the following assumptions throughout:

A-1 (single club membership) The feasible set of club networks, $\mathbb{G} \subset P(A \times (D \times C))$, is given by

$$\mathbb{G} := \{G \in P(A \times (D \times C)) : |G(i)| = 1 \text{ for all } i \in D\}.$$

Thus, in each feasible club network $G \in \mathbb{G}$ each player is a member of one and only one club. Note that under assumption [A-1] the collection $\{G(c) : c \in C\}$ forms a partition of the set of players.

A-2 (identical payoff functions depending on club size) Players have identical payoff functions, $u(\cdot)$ and payoffs are a function of club size only. In general, given any club network G, $\left|G^2(i)\right|$ denotes the total number of club members in the club to which player i belongs in network G. In particular, $G(i) = c$ denotes the single club to which player i belongs and $G(G(i)) := G^2(i)$ is the set of members of the club to which player i belongs. Thus, $\left|G^2(i)\right|$ is the total number of members in the club to which player i belongs in club network G and

$$u(\left|G^2(i)\right|) = \text{ the payoff to player } i \text{ in club network } G.$$

A-3 (single-peaked payoffs) There exists a club size s^* with $1 \leq s^* < |D|$ such that payoffs are increasing in club size up to club size s^* and decreasing thereafter.

A-4 (free mobility) Each player can move freely and unilaterally from one club to another. This means that an player can drop his membership in any given club and join any other club without bargaining with or seeking the permission of any player or group of players. Put differently, an player i can unilaterally change his 1-arc from player node i to club node c to a 1-arc from player node i to any other club node c'. Moreover, any number of players acting unilaterally and noncooperatively can change the existing or status quo club network by switching their arcs (i.e., by switching their club memberships).

6.1.2 Preferences, Rules, and Dominance

Given club networks G_0 and G_1 in $\mathbb{G}$, we say that players $i \in S$ prefer G_1 to G_0, denoted $G_0 \prec_S G_1$,

$$\text{if } u(\left|G_0^2(i)\right|) < u(\left|G_1^2(i)\right|) \text{ for players } i \in S.$$

We say that players $i \in S$ can change G_0 to G_1, denoted $G_0 \rightarrow_S G_1$, if the move from G_0 to G_1 only involves a change in club memberships by players in S, leaving unchanged the memberships of players outside group S, that is,

$$\text{if } G_0(i) = G_1(i) \text{ for all players } i \in N \backslash S \text{ (i.e, } i \text{ not contained in } S\text{)}.$$

Given these preferences and rules, equip the feasible set of club networks with the indirect dominance relation $\rhd\rhd$ (i.e., $> = \rhd\rhd$). Thus, club network $G' \in \mathbb{G}$ indirectly dominates club network $G \in \mathbb{G}$, that is $G \lhd\lhd G'$, if there exists a finite sequence of club networks, $G_0, \ldots, G_n$ in $\mathbb{G}$, with $G := G_0$ and $G' := G_n$, and a corresponding sequence of groups of players, $S_1, \ldots, S_n$, such that for $k = 1, 2, \ldots, n$,

$$G_{k-1} \rightarrow_{S_k} G_k \text{ and } G_n \succ_{S_k} G_{k-1}.$$

6.1.3 Nonemptiness of the Path Dominance Core with Indirect Dominance

Given primitives $(\mathbb{G}, \{\succ_S\}, \{\to_S\}, >)_{S \in P(D)}$ as specified above, Page and Wooders (2005) consider the network club formation game $(\mathbb{G}, \geq_p)$. Under assumptions [A-1]-[A-4], they show that if

$$|C| \geq \frac{|D|}{s^*} \text{ and } |D| = rs^* + l \text{ for nonnegative integers } r \text{ and } l, l < s^*,$$

then the path dominance core of $(\mathbb{G}, \geq_p)$ is nonempty if and only if $l = 0$ or $u(l) \geq u(s^* + 1)$. Moreover, they show that club network G^* is contained in the path dominance core if and only if G^* has r clubs of size s^* and one club of size l. Thus, if there are enough clubs (or club locations) to allow for the formation of all possible clubs of optimal size (i.e., $|C| \geq \frac{|D|}{s^*}$), then a necessary and sufficient condition for nonemptiness of the path dominance core is that no player in a club of suboptimal size has an incentive to join an already existing club of optimal size (i.e., $u(l) \geq u(s^*+1)$). In addition, if the maximum number of clubs of optimal size is r, then a club network is contained in the path dominance core if and only if it has r clubs of optimal size and one club containing the "left overs."

6.1.4 Nonemptiness of the Path Dominance Core with Direct Dominance

One can also define path dominance with respect to direct dominance. In this case, the result above continues to hold. Moreover, if there are 'too few' clubs, that is, if

$$|C| < \frac{|D|}{s^*} \text{ and } |D| = rs^* + l \text{ for positive integer } r \text{ and nonnegative integer } l, l < s^*,$$

then the path dominance core defined with respect to direct dominance is nonempty. The induced partition of players into clubs will have the property that all clubs are as close to the same size as possible. For example, if there are two clubs, $s^* = 5$, and there are 17 players, outcomes in the path dominance core induce partitions of the player set into two clubs, one with eight members and the other with nine members. In this case the set of 'core club structures' coincides with the set of 'Nash club equilibrium' clubs in Arnold and Wooders (2005) and also with the set of Nash equilibrium outcomes as defined in their model.

6.2 Hedonic games

In the abstract game ($\mathbb{G}, \geq_p$) that we have considered, the set of outcomes $\mathbb{G}$ is a set of networks. However, our main results, Theorems 1-4, hold for any finite set of outcomes. With this in mind, consider the following hedonic eight-person game where $\mathbb{G}$ consists of coalition structures (where each coalition structure is a partition of the total player set) proposed to us by Salvador Barbara and Michael Mashler (2006). A move from one coalition structure to another can be brought about by any group of players defecting from the original structure, but unlike the club example above, free entry is not assumed. The example illustrates that our framework encompasses hedonic games and that with indirect dominance the hedonic core is not necessarily equivalent to the path dominance core.

Let the players be denoted by $D = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Player preferences over coalitions is as follows:

player 1	$(1,2,3,4)$	$(1,2,3)$	$(1,2)$	(1)	$\ldots$
player 2	$(1,2,3,4)$	$(1,2,3)$	$(1,2)$	(2)	$\ldots$
player 3	$(1,2,3,4)$	$(3,4,5,6)$	$(1,2,3)$	(3)	$(3,6)$
player 4	$(3,4,5,6)$	$(1,2,3,4)$	$(4,5)$	(4)	$\ldots$
player 5	$(3,4,5,6)$	$(5,6,7,8)$	$(4,5)$	(5)	$\ldots$
player 6	$(5,6,7,8)$	$(3,4,5,6)$	$(6,7,8)$	(6)	$(3,6)$
player 7	$(5,6,7,8)$	$(6,7,8)$	$(7,8)$	(7)	$\ldots$
player 8	$(5,6,7,8)$	$(6,7,8)$	$(7,8)$	(8)	$\ldots$

Players' Preferences Over Coalitions

Consider the row for player 1 in the table above. The interpretation is that 1 prefers the coalition $(1, 2, 3, 4)$ to the coalition $(1, 2, 3)$, to the coalition $(1, 2)$, and so on. Player 1's preferences over the remaining coalitions are irrelevant to the following example so they are not specified. The same interpretation applies to the rows corresponding to other players.

A coalition structure is in the *hedonic core* if there does not exist a coalition that is preferred by all its members to their coalitions of membership in the original coalition structure. Consider the coalition structure $((1, 2, 3, 4), (5, 6, 7, 8)) \in \mathbb{G}$. This is a core point for the hedonic game because the only coalition that is preferred by players 5 and 6 is $(3, 4, 5, 6)$ but two members of this coalition, 3 and 4, do not prefer it. If players 4 and 5 are farsighted, however, and domination is indirect, 4 and 5 can decide to form a coalition $(4, 5)$ - thus bringing about the coalition structure $((1, 2, 3), (4, 5), (6, 7, 8))$. Now players $3, 4, 5$, and 6 could all benefit from forming a coalition. This brings us to the coalition structure $((1, 2), (3, 4, 5, 6), (7, 8))$ a hedonic core point in which 4 and 5 are better off than in the original hedonic core point.

But the story is not finished. Starting from $((1, 2), (3, 4, 5, 6), (7, 8))$, players 3 and 6 can separate and form their own coalition. Using an argument similar to the one above, this move by 3 and 6 can then lead back to the original coalition structure.

We see here that, even though the hedonic core is nonempty, the path dominance core, defined with respect to indirect dominance, is empty. (Note that, in contrast, if the path dominance core is defined with respect to direct dominance then the path dominance core is nonempty and is equivalent to the hedonic core). Another point illustrated is that for path dominance, it is only necessary that a coalition perceive *some* path that would lead to a preferred situation; it is not required that a coalition perceive some preferred *final* (and presumably stable) outcome. The example also suggests for those special cases of cooperative games and hedonic games that if the core (or the hedonic core) is non-empty and not a singleton, then the path dominance core with respect to indirect dominance is empty while the path dominance core with respect to direct dominance is equivalent to the core of the hedonic game. We investigate this further in Page and Wooders 2006.

7 Conclusions

From the viewpoint of the path-dominance core with direct or indirect dominance, there are a number of potential questions to be addressed. For example, what is the relationship, if any, between basins of attraction and the path dominance core and partnered (or separating) collections of coalitions, as in for example Page and Wooders (1995), Reny and Wooders (1997) or Maschler and Peleg (1967) and Maschler, Peleg and Shapley (1971)? Or what is relationship between basins of attraction and the path dominance core and the inner core, as in Qin (1993,1994)?

To conclude, we return to the prior research introducing concepts similar to the abstract game defined in this paper and the union of basins of attractions; see Schwartz (1974), Panzer, Kalai and Schmeidler (1976), Kalai and Schmeidler (1977) and Shenoy (1980).[26] For specificity, we focus on Kalai and Schmeidler (1977). These authors take as given a set of feasible alternatives, denoted by S, a dominance relation, denoted by M, and the transitive closure of M, denoted by $\widehat{M}$. Their admissible set is the set $A(S,M) := \{x \in S : y \in S \text{ and } y\widehat{M}x \text{ imply } x\widehat{M}y\}$.[27] Besides non-emptiness of the admissible set, they also shown that the admissible set is equal to the union of certain subsets – in our terminology, basins of attraction. While Kalai and Schmeidler apply their concept to cooperative games and games in normal (strategic) form, they do not consider networks, the focus of our research. Once our model of network formation is developed, then our abstract game is a particular case of the abstract game of these earlier authors. Our contribution differs in that we develop the network framework and characterize several equilibrium concepts from network theory in terms of their relationships to each other and to basins of attraction and the path dominance core. In addition, we characterize the set of von-Neumann-Morgenstern solutions and the path-dominance core (a case of the abstract core notion introduced in Gilles 1959) in terms of their relationships to basins of attraction. It may well be that the insightful examples developed by these authors will lead to new sorts of examples for networks, a question we are currently addressing. Also, Kalai and Schmeidler (1977) allow an infinite set of possibilities, which, in a network framework, introduces a host of new questions. We plan to address some of these in future research.

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