OPTIMAL PRICING UNDER STOCHASTIC INFLATION: STATE-DEPENDENT (S,S) POLICIES

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Working Paper No. 01-W27

November 2001

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I. Introduction

Price stickiness at the firm level and its implications for aggregate behavior have been analyzed extensively. For example, several macroeconomic studies have examined the relationship between money growth, inflation, and output by assuming an economy of firms following (s,S) pricing policies. Caplin and Spulber find monetary neutrality in a simple (s,S) economy (1987), while Caplin and Leahy find that money can affect output or prices, depending on recent history (1991, 1997).

These macro findings rely on the optimality at the firm level of the (s,S) policy, in which the firm allows its relative price[1] to drift below the optimal price, to s, and then adjusts it above the optimal price, to S. This type of policy has indeed been found optimal for a firm facing a constant, positive rate of inflation and a fixed cost of adjusting price, as shown by Sheshinski and Weiss (1977, hereafter SW77).

The same authors have shown the (s,S) policy optimal when inflation is stochastic in the sense that the economy alternates between some positive rate of inflation and zero inflation according to two stochastic duration times (1983, hereafter SW83). However, the possibility of an error was acknowledged in this paper (1993). The purpose of this note is to correct this error and to characterize the true solution. We find that the (s,S) policy cannot be optimal under these circumstances of stochastic inflation. Rather, there will be *two different* (s,S) bands, corresponding to the two different inflation regimes, with the zero-inflation band contained wholly within the positive-inflation band. We then present a numerical example, showing that a higher variance of aggregate price changes increases price dispersion in states of high inflation and decreases it in states of low inflation.

II. The Model

Following SW83 we will assume that there are two states of the world. In state 0, the aggregate price level remains unchanged. In state 1, the aggregate price level is increasing at constant rate g. The world alternates between the two states stochastically. The length of any sojourn in state 1 (state 0) is distributed exponentially with parameter $\lambda_1>0$ ($\lambda_0>0$).

$\Pi(\cdot)$ is the firm's profit per unit of time as a function of the firm's log relative price. $\Pi(\cdot)$ is assumed to be continuously differentiable, strictly concave, and uniquely maximized at p^*.

The firm discounts profits at rate $r>0$. It is assumed to observe the aggregate price level and rate of change instantaneously. (Thus it can condition its policy on the

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[1] Its relative price can be written as p_i/p, where p is some aggregate price index.

aggregate inflation rate.) To change prices, the firm must pay a fixed cost of B>0. The firm's problem is then to pick optimal adjusting times and adjustment amounts for its price so as to maximize expected discounted profits net of adjustment costs from 0 to ∞, given its initial relative price.

SW83 incorrectly posits that a single (s,S) band will describe the optimal policy. In proving this, the option of changing price in state 0 is not taken into account. It turns out, however, that there will always exist a range of log relative prices from which it will be optimal to adjust in state 0 but not in state 1.

To see why this is so, it is easiest to look at the extreme case in which λ_1 and λ_0 approach zero and g is very high. Thus the chances of leaving each state are infinitesimal, so the firm's policy should look very much like that of a firm facing a constant rate of inflation forever (g or 0 in states 1 or 0, respectively.) Given the results of SW77 that a higher rate of inflation is associated with a higher S and lower s, it seems obvious that there will be different bands for the two states, with the state 0 band being contained in the state 1 band.

This intuition carries over for any values for λ_0, λ_1, and g. This can be proved[2] by showing that it is always possible to find two new critical points within (s,S), say (s*,S*), such that on (s,s*) it is a strictly dominant strategy to change price to S* while still in state 0 rather than to leave price unchanged until state 1 obtains. Thus we find that a single (s,S) band cannot be optimal for a firm facing stochastic inflation of this type.

III. THE SOLUTION

[2] Proof available on request.

The firm's choice is the amount to adjust its price (including none) at all dates and histories, states, and relative prices. Clearly the date and history will not affect the decision given the state and relative price, since the returns are not time-dependent and expectations over switching time to the next state are independent of history, due to the exponential distribution's constant hazard rate. The only information relevant to the firm's decision is its own relative price and the inflationary state of the world (0 or 1.)

It is also clear that the firm cannot adjust its price at every price and state of the world, since there exists a fixed cost of adjusting price. Thus the firm's problem can be stated as the choice of an inaction region for each state and prices to which to adjust when in each state's action region.

Let $V_i(\cdot) \equiv$ the value of a firm in state i (i=0,1) that follows the optimal pricing policy, as a function of current log relative price. At this point we will assume a bit more structure on the solution to the firm's problem. Specifically, we will assume that the value functions of both states solving the more generally posed sequence problem satisfy the following:

<u>A1</u> There exist numbers, $s_i<S_i<N_i$, i=0,1, such that $V_i(\cdot)$ is strictly increasing on (s_i,S_i), $V_i(\cdot)$ is strictly decreasing on (S_i,N_i), and $V_i(\cdot)$ is continuous and differentiable on (s_i,N_i), for i=0,1, where (s_i,N_i) are the of the inaction regions for states i=0,1.

(Note we are not imposing differentiability at the boundary points of the inaction region.) In short, our assumption posits an (s,S) band for each state, possibly different. How reasonable is this assumption? Differentiability seems very likely to exist, at least everywhere but the indifference points, since the profit function is twice differentiable.

Continuity must be true given that the profit function is continuous and thus value matching must hold at the borders of inaction and action. And, given the strict concavity of the profit function, we should expect the value functions to be monotonic about a unique maximum and the inaction regions to be convex.

Note that the value of a firm must be the same everywhere in its action region, since the firm can adjust to any relative price for the same cost (B) and will thus choose the optimal one (S_i). This gives the following equation for the value of the firm in the action region:

$$V_i(x) = V_i(S_i) - B, \text{ for } x \notin (s_i, N_i),\ i=1,2 \qquad (1)$$

In the inaction region, the value of the firm must satisfy the following continuous time Bellman equations:

$$rV_i(x) = \Pi(x) + (1/dt)E[dV_i],\ i=1,2$$

Following Dixit & Pindyck (94) we can express the expected changes in value functions in this way:

$$E[dV_0] = \lambda_0 dt[V_1(x)-V_0(x)] + (1-\lambda_0 dt)\cdot 0 = \lambda_0 dt[V_1(x)-V_0(x)]$$

and

$$E[dV_1] = \lambda_1 dt[V_0(x)-V_1(x)] + (1-\lambda_1 dt)(-gdtV_1'(x))$$

$$= \lambda_1 dt[V_0(x)-V_1(x)] - gdtV_1'(x) + \lambda_1 g(dt)^2 V_1'(x)$$

Dropping the $(dt)^2$ term from $E[dV_1]$, we get the following value functions for the inaction regions of state 0 and state 1, respectively:

$$V_0(x) = \frac{\Pi(x) + \lambda_0 V_1(x)}{r + \lambda_0} \tag{2}$$

$$V_1(x) = \frac{\Pi(x) + \lambda_1 V_0(x) - gV_1'(x)}{r + \lambda_1} \tag{3}$$

Let us define $R_i \equiv r+\lambda_i$, i=0,1. The value function for the inaction region of state 1 forms a differential equation in $V_1(\cdot)$, which has the following solution:

$$V_1(x) = e^{\frac{-R_1 x}{g}} \{e^{\frac{R_1 s_1}{g}} V_1(s_1) + \int_{s_1}^{x} e^{\frac{R_1 z}{g}} [\frac{\Pi(z) + \lambda_1 V_0(z)}{g}] dz\} \tag{4}$$

We now have expressions for the value of the firm in each state in action and inaction regions. The following key facts form the basis for our analysis of the solution.

1) In the inaction region, the value functions satisfy (2) and (3), derived from the continuous time Bellman equations.

2) In the action region, the value functions satisfy (1), since the price is switched to the optimal value. In particular, the following value matching conditions hold:

$$V_i(s_i) = V_i(S_i) - B = V_i(N_i), \ i=0,1 \tag{5}$$

3) In both regimes, the value-maximizing price must be at the peak of the value function, so:

$$V_i'(S_i) = 0, \ i=0,1 \tag{6}$$

4) Smooth pasting holds at s_1; ie $V_1'(s_1) = 0$. (This is proved in lemma 1 below.)

A. Relative location of the parameters

Given these restrictions imposed by optimality, value matching, and smooth pasting, we are able to establish almost completely the relative locations of the optimally-chosen values (s_i,S_i,N_i), i=0,1. The one fact that depends on parameter values is whether S_1 is greater or less than N_0. However, if we make the following assumption, we guarantee that $S_1<N_0$:

<u>A2</u> $$B \geq \Pi(p^*)(1/\lambda_1 + 1/R_0)$$

We are now ready to state the main result:

<u>Theorem 1</u> Under the assumption A1, we have that $s_1<s_0<p^*<S_0<S_1,N_0<N_1$. Thus the inaction region of state 0 is fully contained in the inaction region of state 1. Also, we have that $\Pi(S_1)>\Pi(s_1)>\Pi(N_1)$; $\Pi(S_0)>\Pi(s_0)>\Pi(N_0)$; and $\Pi(S_0)>\Pi(S_1)$, $\Pi(s_0)>\Pi(s_1)$, and $\Pi(N_0)>\Pi(N_1)$. Under the additional assumption A2, $S_1<N_0$.

Proof. By lemmas 1-7 below.

<u>Lemma 1</u> $V_1'(s_1) = 0 = V_1'(S_1) = V_0'(S_0)$. Thus $V_1(S_1) = [\Pi(S_1)-\lambda_1 V_0(S_1)]/R_1$, and $V_1(s_1) = [\Pi(s_1)-\lambda_1 V_0(s_1)]/R_1$. S_1 (S_0) is the log relative price the firm will set whenever it changes price in state 1 (0).

Proof. That $V_1'(S_1) = V_0'(S_0) = 0$ follows directly from A1. The expression for $V_1(S_1)$ then follows from equation 3. Given A1, since S_i is the price in the inaction region that maximizes the value function in state i, firms will change price to S_i whenever they act in state i, i=1,2.

We now show the results for s_1. The idea is to partially differentiate the value function we have for state 1 with respect to the parameter s_1. If the derivative is nonzero, we can change the choice of s_1 and increase the value for some x. Let us define new functions v_i, i=0,1, where $v_i(x;s_{1p},S_{1p},N_{1p},s_{0p},S_{0p},N_{0p})$ is the value to the firm of being at log relative price x in state i and following the double (s,S) policy defined by parameters $(s_{1p},S_{1p},N_{1p},s_{0p},S_{0p},N_{0p})$. These parameters need not be chosen optimally; however we do impose equations 1, 2, and 3 (giving the value in the action and inaction regions). Thus, $V_i(x) = v_i(x;s_1,S_1,N_1,s_0,S_0,N_0)$. Below we differentiate $v_1(\cdot)$ with respect to the parameter s_{1p}, using equation (4), and set the derivative to zero. The derivatives are evaluated at the optimal parameter values.

$$\frac{\partial v_1(x;s_1,S_1,N_1,s_0,S_0,N_0)}{\partial s_{1p}} = 0 = e^{\frac{-R_1 x}{g}}\{\frac{R_1}{g}e^{\frac{R_1 s_1}{g}}V_1(s_1) - e^{\frac{R_1 s_1}{g}}[\frac{\Pi(s_1)+\lambda_1 V_0(s_1)}{g}]\} +$$

$$e^{\frac{-R_1 x}{g}}\{e^{\frac{R_1 s_1}{g}}\frac{\partial v_1(s_1;s_1,S_1,N_1,s_0,S_0,N_0)}{\partial s_{1p}} + \int_s^x \frac{e^{\frac{R_1 z}{g}}}{g}\lambda_1 \frac{\partial v_0(z;s_1,S_1,N_1,s_0,S_0,N_0)}{\partial s_{1p}}dz\}$$

If the second line of this equation is zero, then the first line gives us that and $V_1(s_1) = [\Pi(s_1)-\lambda_1 V_0(s_1)]/R_1$; and then from this and equation (3), $V_1'(s_1) = 0$. Is the second line zero? Yes, if $\partial v_1(s_1;s_1,S_1,N_1,s_0,S_0,N_0)/\partial s_{1p}$ is zero and if $\partial v_0(z; s_1,S_1,N_1,s_0,S_0,N_0)/\partial s_{1p}$ is zero on (s_1,x). The former must hold at the optimum, since otherwise we would change our choice of s_1. For the latter, note that from (2), $\partial v_0(z)/\partial s_{1p} = (\lambda_0/R_0)\partial v_1(z)/\partial s_{1p}$ on (s_0,N_0) and $\partial v_0(z)/\partial s_{1p} = (\lambda_0/R_0)\partial v_1(S_0)/\partial s_{1p}$ elsewhere. Now at the optimum, $\partial v_1(z)/\partial s_{1p}$ must be zero everywhere, since otherwise we would change our choice of s_1 at some values z, which contradicts the optimality. Thus $\partial v_0(z)/\partial s_{1p}$ is zero on (s_1,x), and we have the result. ■

<u>Lemma 2</u> $s_1<s_0<p^*$.

Proof. For $s_1<s_0$, we first show that s_1 is in the interior of the action region of state 0 if $S_0 \neq S_1$. It is sufficient to show that the gain from switching price in state 0 from s_1 to S_0, is strictly positive, since only in the action region is the gain positive and only in the interior is it strictly positive, by A1. At s_1, the payoff for keeping the same price is:

$$[\Pi(s_1)+\lambda_0(V_1(s_1))]/R_0 = [\Pi(s_1)+\lambda_0(V_1(S_1)-B)]/R_0$$

and the payoff for switching to S_1 (a payoff which is dominated by a switch to S_0, since S_0 is the optimal price for state 0) is:

$$[\Pi(S_1)+\lambda_0 V_1(S_1)]/R_0 - B$$

The gain can be calculated as $\Pi(S_1)-\Pi(s_1)-rB$. Now using the expressions for $V_1(s_1)$ and $V_1(S_1)$ from lemma 1, and the fact that $V_1(S_1)-V_1(s_1)=B$, we get that

$$\Pi(S_1)-\Pi(s_1)-rB = \lambda_1[V_0(s_1)-V_0(S_1)+B] \qquad (7)$$

Thus the gain from changing price to S_1 is nonnegative, since $V_0(x)-V_0(y)\leq B$, for all x,y. Since S_0 is the unique maximizer of $V_0(\cdot)$, $S_0 \neq S_1$ implies that $V_0(S_0)>V_0(S_1)$, so that the gain from changing price to S_0 is strictly positive. So s_1 is in the interior of the action region of state 0 if $S_0 \neq S_1$.

Next we show that $S_0 \neq S_1$. Using the expressions for $V_1(s_1)$ and $V_1(S_1)$ from lemma 1, and equation (4), we can write:

$$e^{\frac{R_1 S_1}{g}} \frac{\Pi(S_1)+\lambda_1 V_0(S_1)}{R_1} = e^{\frac{R_1 s_1}{g}} \frac{\Pi(s_1)+\lambda_1 V_0(s_1)}{R_1} + \int_{s_1}^{S_1} e^{\frac{R_1 z}{g}} \frac{\Pi(z)+\lambda_1 V_0(z)}{g} dz$$

Integrating by parts, we obtain the following expression:

$$\int_{s_1}^{S_1} e^{\frac{R_1 z}{g}} [\Pi'(z) + \lambda_1 V_0'(z)] dz = 0 \qquad (8)$$

Now if $S_0 = S_1$, then $p^* < S_0 = S_1$, since otherwise the expression in the integral in (8) is always positive. But then

$$V_0'(S_0) = (1/R_0)[\Pi'(S_0) + \lambda_0 V_1'(S_0)] = (1/R_0)\Pi'(S_0) < 0,$$

where the first equality is from equation (2), the second from the fact that $S_0 = S_1$ and thus $V_1'(S_0) = 0$, and the third since $p^* < S_0$. $V_0'(S_0) < 0$ contradicts the optimality of S_0 for state 0. Thus $S_0 \neq S_1$.

We have that s_1 is in the interior of the action region of state 0. Thus $s_1 < s_0$ or $s_1 > N_0$. Now we will rule out the latter possibility. By equation (8), if $s_1 > N_0$, then $p^* > s_1 > N_0 (> S_0)$, since otherwise the expression in the integral will be always negative. But then

$$V_0'(S_0) = (1/R_0)[\Pi'(S_0) + \lambda_0 V_1'(S_0)] = (1/R_0)\Pi'(S_0) > 0$$

where the first equality is from equation (2), the second from the fact that S_0 is in the flat (action) region of $V_1(\cdot)$, and the third since $p^* > S_0$. $V_0'(S_0) > 0$ contradicts the optimality of S_0 for state 0. Thus $s_1 > N_0$ is impossible, and we have $s_1 < s_0$.

Finally we show $s_0 < p^*$. Using (2) to evaluate $V_0(S_0)$ and $V_0(s_0)$, and combining the two equations using the fact that $V_0(S_0) - V_0(s_0) = B$, we see that

$$\Pi(S_0) - \Pi(s_0) = rB + \lambda_1[V_0(s_0) - V_0(S_0) + B] \qquad (9)$$

The second term of the right hand side is nonnegative, since $V_0(x)-V_0(y)\leq B$, all x,y, so $\Pi(S_0)>\Pi(s_0)$. Then since $s_0<S_0$ and $\Pi(\cdot)$ is strictly concave with p* its unique maximizer, the result follows. ■

Lemma 3 $S_0<N_0<N_1$ and $\Pi(N_1)<\Pi(s_1)$.

Proof. $S_0<N_0$ by A1.

Since at both s_0 and N_0 the firm is indifferent (by value matching) between staying put and switching to price S_0, from equation (2) it must be that:

$$V_0(s_0) = (1/R_0)[\Pi(s_0)+\lambda_0 V_1(s_0)] = (1/R_0)[\Pi(N_0)+\lambda_0 V_1(N_0)] = V_0(N_0)$$

Thus

$$V_1(s_0)\geq V_1(N_0) \Leftrightarrow \Pi(s_0)\leq\Pi(N_0) \qquad (10)$$

Now assume $N_1\leq N_0$. First we will show that $N_1\leq N_0 \Rightarrow \Pi(N_1)\geq\Pi(s_1)$. Under this assumption, N_0 is at least on the border of the action region of state 1, so $V_1(N_0)=V_1(N_1)\leq V_1(s_0)$, and thus $\Pi(s_0)\leq\Pi(N_0)$ by (10). Further, we have that $\Pi(s_1)<\Pi(s_0)\leq\Pi(N_0)\leq\Pi(N_1)$, where the first inequality is because $s_1<s_0<p^*$ (lemma 2), and the last is because $p^*\leq N_1\leq N_0$. (To show that $p^*\leq N_1$, assume the opposite. Then equation (7) implies that $p^*>N_1>S_1>S_0$. But in this case, $V_0'(S_0) = (1/R_0)[\Pi'(S_0) + \lambda_0 V_1'(S_0)] > 0$, which contradicts optimality of S_0. So $p^*\leq N_1$.) Thus $\Pi(N_1)\geq\Pi(s_1)$.

Now we will show that $\Pi(N_1)\geq\Pi(s_1)$ implies a contradiction. Since s_1 and N_1 are points on the border of the action region, we know that $V_1(N_1)=V_1(s_1)$. Combining this with (5), we have:

$$V_1(s_1) = V_1(N_1) = e^{\frac{-R_1 N_1}{g}}\left[e^{\frac{R_1 s_1}{g}} V_1(s_1) + \int_{s_1}^{N_1} e^{\frac{R_1 z}{g}} \frac{\Pi(z)+\lambda_1 V_0(z)}{g} dz\right]$$

Or,

$$V_1(s_1)[1-e^{\frac{-R_1(N_1-s_1)}{g}}]=\int_{s_1}^{N_1} e^{\frac{-R_1(N_1-z)}{g}}\frac{\Pi(z)+\lambda_1 V_0(z)}{g}dz]$$

Since $\Pi(\cdot)$ is strictly concave and $\Pi(N_1)\geq\Pi(s_1)$, $\Pi(x)>\Pi(s_1)$ for $x\in(s_1,N_1)$. And since s_1 is in the inaction region of state 0, $V_0(x)\geq V_0(s_1)$, for all x. So we can replace the function in the integral with constants $\Pi(s_1)+\lambda_1V_0(s_1)$ and change the equality to strict inequality:

$$V_1(s_1)[1-e^{\frac{-R_1(N_1-s_1)}{g}}]>\int_{s_1}^{N_1} e^{\frac{R_1 z}{g}}\frac{\Pi(s_1)+\lambda_1 V_0(s_1)}{g}dz]=\frac{\Pi(s_1)+\lambda_1 V_0(s_1)}{R_1}[1-e^{\frac{-R_1(N_1-s_1)}{g}}]$$

This gives that $V_1(s_1)>[\Pi(s_1)+\lambda_1V_0(s_1)]/R_1$, which contradicts lemma 1.

Since $N_1\leq N_0 \Rightarrow \Pi(N_1)\geq\Pi(s_1) \Rightarrow$ a contradiction, we have that $\Pi(N_1)<\Pi(s_1)$, and $N_1>N_0$. ■

Lemma 4 $p^*<S_0<S_1<N_1$.

Proof. $N_1>S_1$ by A1.

By equation (8), $S_1\leq S_0 \Rightarrow p^*<S_1\leq S_0$. But this contradicts optimality of S_0, since then $V_0'(S_0)<0$. So $S_1>S_0$.

Now $V_0'(S_0)=0 \Rightarrow \lambda_0V_1'(S_0) = -\Pi'(S_0)$. Since $S_0<S_1$, the left hand expression is positive, so $\Pi'(S_0)$ must be negative, that is $S_0>p^*$. ■

Lemma 5 For $x\in(s_0,N_0)$ (the common inaction region),

$$rV_1(x) = \Pi(x) - g_1V_1'(x), \text{ and} \quad (11)$$

$$rV_0(x) = \Pi(x) - g_0V_1'(x) = \Pi(x) + (g_1/R_0)\Pi'(x) - g_1V_0'(x) \quad (12)$$

where

$$g_1 \equiv g(r+\lambda_0)/(r+\lambda_0+\lambda_1), \text{ and} \quad (13)$$

$$g_0 \equiv g\lambda_0/(r+\lambda_0+\lambda_1). \quad (14)$$

Proof. By lemmas 2 and 3, inaction is strictly preferred in both states on (s_0,N_0). Thus, both equations (2) and (3) apply on this interval. Combining the two and solving for $rV_1(x)$ and $rV_0(x)$ give equations (11) and (12). ■

<u>Lemma 6</u> $\Pi(S_1)>\Pi(s_1)>\Pi(N_1)$; $\Pi(S_0)>\Pi(s_0)>\Pi(N_0)$; and $\Pi(S_0)>\Pi(S_1)$, $\Pi(s_0)>\Pi(s_1)$, and $\Pi(N_0)>\Pi(N_1)$.

Proof. $\Pi(s_0)>\Pi(s_1)$, $\Pi(N_0)>\Pi(N_1)$, and $\Pi(S_0)>\Pi(S_1)$ come from the facts that $s_1<s_0<p^*$ (lemma 2), $p^*<N_0<N_1$ (lemmas 3 and 4), and $p^*<S_0<S_1$ (lemma 4) and the fact the $\Pi(\cdot)$ is strictly concave with p^* its maximizer.

$\Pi(S_1)>\Pi(s_1)$ by equation (7), since the right hand side is nonnegative by arguments made in the proof of lemma 2. Equation (9) shows that $\Pi(S_0)>\Pi(s_0)$ by the same arguments.

$\Pi(s_1)>\Pi(N_1)$ by lemma 3. It remains to show $\Pi(s_0)>\Pi(N_0)$. Two cases will be considered. First, if $N_0<S_1$, then we have that $V_1(N_0)>V_1(s_0)$ (by A1 and lemmas 1-4). From (10), $V_1(s_0)<V_1(N_0) \Rightarrow \Pi(s_0)>\Pi(N_0)$. Thus in this case, $\Pi(s_0)>\Pi(N_0)$. Second, consider $S_1\leq N_0$. In this case, from A1 and equation (11) of lemma 6, $V_1(N_0)\geq\Pi(N_0)/r$.

Also from A1 and lemmas 2 and 4 and equation (11), $V_1(s_0)<\Pi(s_0)/r$. So if $\Pi(N_0)\geq\Pi(s_0)$, we have that $V_1(N_0) \geq \Pi(N_0)/r \geq \Pi(s_0)/r > V_1(s_0)$, that is $V_1(N_0)>V_1(s_0)$. But from (10), $V_1(N_0)>V_1(s_0) \Rightarrow \Pi(N_0)<\Pi(s_0)$. Thus we have a contradiction, and $\Pi(s_0)>\Pi(N_0)$. ■

Lemma 7 A2 implies that $S_1<N_0$.

Proof. If $S_1\geq N_0$, then

$$V_0(S_1) = V_0(s_0) = V_0(S_0)-B = [\Pi(S_0)+\lambda_0 V_1(S_0)-R_0B]/R_0$$

Using this and lemma 1, we obtain:

$$V_1(S_1) = [\Pi(S_1)+\lambda_1 V_0(S_1)]/R_1 = [R_0\Pi(S_1)+\lambda_1\Pi(S_0)+\lambda_0\lambda_1 V_1(S_0)-\lambda_1 R_0 B]/R_0R_1, \text{ or}$$

$$R_0R_1V_1(S_1)-\lambda_0\lambda_1V_1(S_0) = R_0\Pi(S_1)+\lambda_1\Pi(S_0)-\lambda_1R_0B.$$

Since $V_1(S_1)>V_1(S_0)$ and $R_0R_1>\lambda_0\lambda_1$, the left hand side is strictly positive. Thus the right hand side must also be. This gives the following restriction on B:

$$B < \Pi(S_1)/\lambda_1 + \Pi(S_0)/R_0,$$

which is necessary for $N_0\leq S_1$. Since $\Pi(S_1)<\Pi(S_0)<\Pi(p^*)$, if we make the following assumption we are guaranteed that $S_1<N_0$:

$$B \geq \Pi(p^*)(1/R_0 + 1/\lambda_1)$$ ■

B. Finding the parameters

Using our theorem and the lemmas, we can now derive expressions for the value functions in both regions that depend only on border values and not on the value function for the other state. In particular, equations (11) and (12) of lemma 5 enable us to do so.

First we will comment on the certainty *equivalence* rate of inflation concept stressed in SW83. There the authors make the point that the value of the firm in this setup with stochastic inflation is the same as that of a firm facing a constant rate of inflation, say g_{eq}, equal to some function of g, the λ_i's, and r. In fact, the equivalent constant rate of inflation they derive for our setup is exactly g_1 (compare our equation (13) with SW83 equation 6). The asymptotic average inflation rate can be calculated as $g_{avg} = g\lambda_0/(\lambda_0+\lambda_1)$. Note that $g_1>g_{avg}$. This positive difference between the equivalent certainty rate of inflation and the average rate was interpreted in SW83 as a risk premium the firm was willing to pay in order to have a constant rate of inflation. That is, the firm was allegedly willing to accept a higher constant rate of inflation than the average rate in the stochastic case, because uncertainty is costly.

However, the mistake in this risk premium interpretation is that the asymptotic average rate ignores the initial state, whereas g_1 applies to a firm currently in state 1. Thus we would expect $g_1>g_{avg}$, since starting in state 1 increases the average inflation over any finite interval. Further, this explains the effect of the interest rate on g_1-g_{avg}. As $r\to 0$, so that there is no discounting and the initial state does not matter, then $g_1\to g_{avg}$. On the other hand, as $r\to\infty$, $g_1\to g$, since all that matters in this case of extreme discounting is the current state. Similarly, $g_0<g_{avg}$, $g_0\to g_{avg}$ as $r\to 0$, and $g_0\to 0$ as $r\to\infty$. (The risk premium would actually be negative if the firm started in state 0.) These all confirm the interpretation of the relative positions of g_1, g_{avg}, and g_0 not as risk premia, but simply as accounting for the initial state of the world.

We now derive expressions for the value function in the mutual inaction region, that is, on (s_0,N_0). Equation (11) gives:

$$V_1(x) = e^{\frac{-rx}{g_1}}\{e^{\frac{rs_0}{g_1}}V_1(s_0) + \int_{s_0}^{x} e^{\frac{rz}{g_1}} \frac{\Pi(z)}{g_1}dz\} \tag{15}$$

And (12) yields:

$$V_0(x) = e^{\frac{-rx}{g_1}}\{e^{\frac{rs_0}{g_1}}V_0(s_0) + \int_{s_0}^{x} e^{\frac{rz}{g_1}}[\frac{\Pi(z)}{g_1} + \frac{\Pi'(z)}{R_0}]dz\} \tag{16}$$

We now turn to the regions that lead to action in state 0 but inaction in state 1: $(s_1,s_0]$ and $[N_0,N_1)$. We will look only at $V_1(\cdot)$, since $V_0(\cdot)$ just equals $V_0(s_0)=V_0(S_0)-B$ everywhere in this region. Using this fact in (3), we get the following differential equation which describes $(s_1,s_0]$ and $[N_0,N_1)$:

$$R_1V_1(x) = \Pi(x) + \lambda_1V_0(s_0) - gV_1'(x) \tag{17}$$

Using lemma 1 and the fact that $V_0(s_1)=V_0(s_0)$, since $s_1<s_0$, we see that $\lambda_1V_0(s_0)=R_1V_1(s_1)-\Pi(s_1)$. Thus we can rewrite (17):

$$R_1[V_1(x)-V_1(s_1)] = \Pi(x) - \Pi(s_1) - g[V_1(x)-V_1(s_1)]' \tag{18}$$

where $[V_1(x)-V_1(s)]' \equiv \partial[V_1(x)-V_1(s)]/\partial x = V_1'(x)$. This says that the difference in value as we move away from s_1 (or N_1) evolves in the same way as the value under a constant rate of inflation g and interest rate R_1, but with instantaneous profit as the profits in excess over $\Pi(s)$.

Equation (18) yields the following expression for $V_1(\cdot)$ on $(s_1,s_0]$:

$$V_1(x) = V_1(s_1) + \int_{s_1}^{x} e^{\frac{-R_1(x-z)}{g}} \frac{\Pi(z)-\Pi(s_1)}{g}dz \tag{19}$$

and since $V_1(N_1)=V_1(s_1)$, on $[N_0,N_1)$:

$$V_1(x) = V_1(N_1) - \int_x^{N_1} e^{\frac{R_1(z-x)}{g}} \frac{\Pi(z)-\Pi(s_1)}{g} dz \qquad (20)$$

Equations (15), (16), (19), and (20) completely describe the value functions for both states in their respective inaction regions. The value in action regions is given by equation (1). Given a profit function satisfying the assumptions, we can solve for the four parameters (s_0,s_1,S_0,S_1) and boundary value $V_1(s_0)$ (from which $V_0(s_0)$ is known given equation (2)). This is done using five conditions imposed on the equations above: $V_i'(S_i)=0$, i=1,2, $V_1'(s_1)=0$, and $V_i(S_i)=V_i(s_i)+B$, i=1,2. (N_0,N_1) can then be found by imposing $V_i(N_i)=V_i(s_i)$, i=1,2 on the above equations. Thus the full solution is readily solved for.

C. Numerical Example and Comparative Statics

Using the above techniques, we now solve for and graph the value functions for both states, given a quadratic profit function and equal hazard rates. Figure 1 shows $V_i(x)$, i=0,1, and for comparison, $\Pi(x)/r$, the value of staying at relative price x forever.

Figure 2 shows what happens to (s_i,S_i,N_i), i=0,1, as we decrease the λ_i (thus prolonging expected stays in each state). As SW83 shows, decreasing the λ_i's by the same factor corresponds to a mean-preserving increase in variance. The result of this experiment is clear – the state 0 bands tighten while the state 1 bands widen. The testable implication for price dispersion is that an increase in inflation variance will increase price

FIGURE 1. Value functions in states 0 and 1, compared with value of certain profits

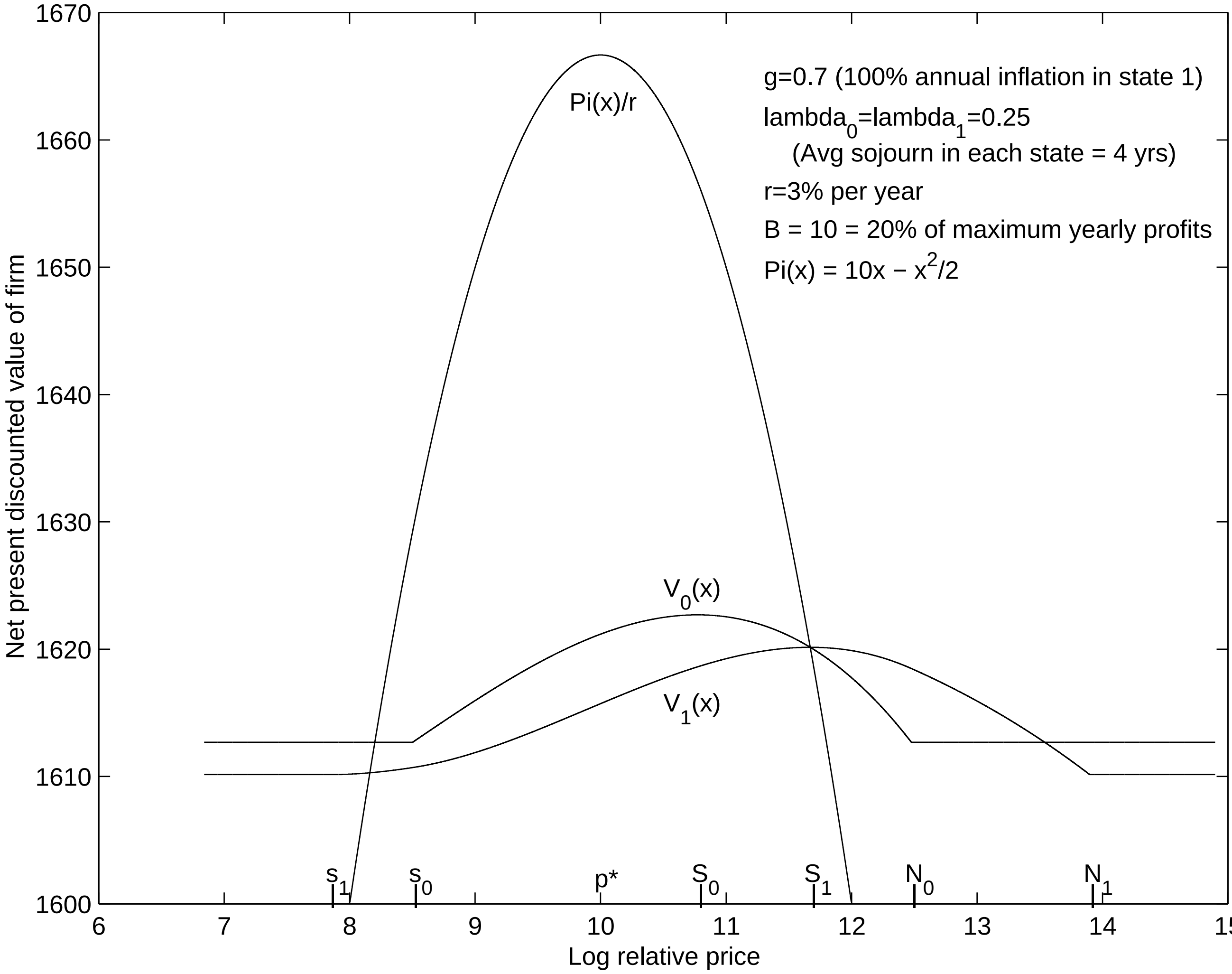
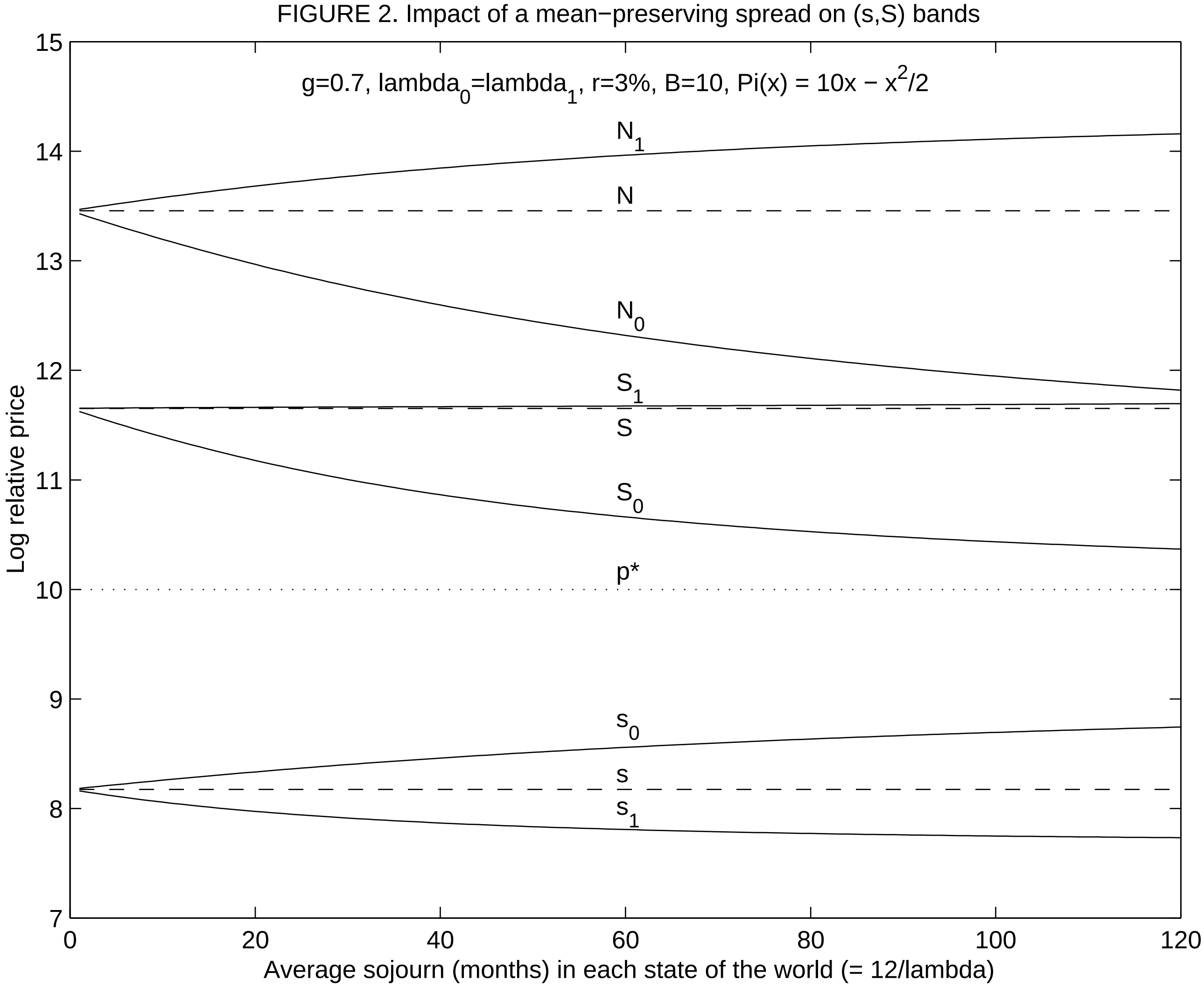


FIGURE 2. Impact of a mean−preserving spread on (s,S) bands



dispersion in highly inflationary states and decrease it in low inflationary states of the world.

IV. Conclusion

In this paper we have established that the optimal policy of the firm facing the proposed two states of inflation must be state-dependent, and we have characterized the solution that satisfies certain plausible assumptions. This policy involves a different (s,S) policy for each state of the world, with the zero-inflation band being contained within the positive inflation band. Increases in the variance of inflation lead the firm to keep its relative price within a narrower range in non-inflationary periods and to allow it to vary more widely during positive-inflation periods.

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