GENERALIZED GINI INDICES OF EQUALITY OF OPPORTUNITY

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Abstract. This article considers the ranking of profiles of opportunity sets on the basis of their equality. A version of the Pigou-Dalton transfer principle that is appropriate for the measurement of opportunity inequality is introduced and used to axiomatize the class of generalized Gini equality of opportunity orderings. A characterization of the class of generalized Gini social preference orderings for opportunity profiles is also provided.

Keywords and Phrases: Equity, equality of opportunity, Gini, inequality measurement, opportunity sets.

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1. Introduction

The desirability of reducing the disparities in the distribution of opportunities plays a prominent role in the platforms of many political parties and is a widely-accepted principle of distributive justice. Opportunities take many forms. For example, an individual's opportunities are enhanced with greater wealth, greater access to education, and the removal of barriers based on race, gender, or religion on the choice of one's career. The analysis of opportunity inequality is complicated by the fact that an individual's opportunities are described by a set rather than by a scalar, as is the case with income or wealth inequality. As a consequence, it was only with the seminal work of Kranich [13] that the question of how to rank different distributions of opportunities in terms of the inequality they exhibit was first addressed. There is now an extensive literature concerned with the measurement of equality of opportunity. See, for example, Arlegi and Nieto [1], Bossert, Fleurbaey, and Van de gaer [6], Herrero [10], Herrero, Iturbe-Ormaetxe, and Nieto [11], Kranich [14, 15], Ok [17], Ok and Kranich [18], and Savaglio and Vannucci [22]. Surveys of this literature may be found in Barberà, Bossert, and Pattanaik [3] and Peragine [20].

Kranich's approach to the measurement of opportunity inequality is axiomatic. He has proposed a number of properties that an equality of opportunity relation on the possible distributions (profiles) of finite opportunity sets should satisfy when the opportunities are nonrival.[1] He has shown that a subset of his axioms imply that the only relevant feature of a profile of opportunity sets is the cardinality of each individual's opportunities. For the general case in which the number of individuals is arbitrary (but at least two), Kranich has provided two axiomatizations of the class of equality of opportunity orderings that use a Kranich index of equality of opportunity to rank profiles of opportunity sets. With a Kranich index of equality of opportunity, the cardinalities of the individual opportunity sets are first rank ordered from smallest to largest and then these numbers are summed using fixed weights. The sum of these weights is zero, with positive (resp. negative) weights used for opportunity sets that have less (resp. more) than the median number of opportunities.[2]

[1]The restriction that individual opportunity sets are finite is relaxed in Kranich [14].

[2]For the case of two individuals, Kranich has also characterized the cardinality difference rule in which profiles are ranked according to the absolute value of the difference in the size of the individual opportunity sets.

Kranich's class of indices is closely related to the class of generalized Gini inequality indices introduced in Weymark [25] to measure income inequality. A generalized Gini index of equality of opportunity has the same functional form as a Kranich index, but the restriction on the signs of the weights is replaced by the requirement that the weights are nondecreasing.

The distributional sensitivity of Kranich's indices is an implication of axioms that pertain to the comparison of profiles in which the opportunity sets can be ranked in terms of set inclusion. Kranich's Progressivity (resp. Regressivity) axiom requires an expansion of an individual's opportunity set to be equality-enhancing (resp. equality-reducing) if this individual has less (resp. more) than the median number of opportunities and the expansion preserves the rank ordering of opportunity sets in terms of set inclusion.

When measuring income inequality, distributional sensitivity is achieved by the adoption of the Pigou-Dalton transfer principle, which regards a transfer of income from a richer to a poorer individual that does not reverse their ranks as being equality-enhancing. Ok and Kranich [18] have formulated a version of this principle for opportunity inequality.[3] Their principle says that if an opportunity profile is subjected to a transform that is both rank-preserving and equalizing, then the change in the distribution is equality-enhancing. In order to identify when a transformation of opportunities is rank-preserving, Ok and Kranich employ an auxiliary ordering of the individual opportunity sets to determine, for each profile, a complete ranking of the individuals according to the relative desirability of their opportunity sets. This auxiliary ordering is an extension of the partial order defined by set inclusion. Ok and Kranich [18] regard a transformation of opportunities as equalizing if it either involves a transfer of opportunities from a more to a less privileged individual (according to the auxiliary ranking of individual opportunity sets) or if it increases the number of common opportunities, subject to the constraints that the transform does not introduce new opportunities and that it does not reverse the rank order of the affected individuals' opportunity sets. The rationale for this definition of an equalizing transform when opportunities are nonrival is nicely summarized by Ok [17, p. 309]: "In general, there appears [sic] to be two distinct criteria of *equality* in comparing opportunity set vectors; one is the 'gap between the richer and poorer agents,' and the other is based on the 'similarity' of the individual opportunity sets."

[3] They only consider the case of two individuals. See Ok [17] for further discussion of this principle.

Ok [17] has argued that any transfer principle for opportunity inequality that is based on an auxiliary ranking of opportunity sets and that takes account of both ways in which opportunity sets can be equalized must rank individual opportunity sets on the basis of their cardinalities.[4] This rather nihilistic conclusion depends on Ok's definition of what constitutes an equalizing transform (which is somewhat more demanding than the one used by Ok and Kranich [18]) and on his assumption that the ranking of individual opportunity sets that is used to determine who is more privileged is complete.

In this article, I formulate a version of the Pigou-Dalton transfer principle for profiles of opportunity sets that is more limited in scope that the one proposed by Ok and Kranich. As the quotation from Ok [17] makes clear, Ok and Kranich's formulation of the Pigou-Dalton transfer principle applies to both of the ways described above that opportunity profiles can be made more equal. Equalizing opportunity sets by making them more similar is only possible because opportunities, unlike incomes, are nonrival. My version of the Pigou-Dalton transfer principle for opportunity sets only applies when there is an actual transfer of an opportunity between individuals. Thus, it is only concerned with "gap-reducing" transfers, as in the original Pigou-Dalton transfer principle for income distributions. If two individuals each have opportunities that the other doesn't, it is not possible to determine who is the more privileged individual unless one has some external standard for ranking opportunity sets, such as that provided by the cardinality ranking. Without such a basis for comparison, it is nevertheless possible to say that one individual has better opportunities than another if the former's opportunity set strictly contains the latter's. My Pigou-Dalton transfer principle only applies when individual opportunity sets are nested in this way prior to any transfer taking place. Thus, as is the case with Kranich's [13] Progressivity and Regressivity axioms, my principle does not rely on any judgements concerning the relative desirability of opportunity sets beyond that provided by set inclusion.[5]

I use my transfer principle to help provide two axiomatizations of the class of generalized Gini equality of opportunity orderings. My character-

[4]This informal statement of Ok's result is made more precise in Theorems 5.1 and 5.4 in Ok [17]. As in Ok and Kranich [18], Ok [17] only considers two-individual societies.

[5]Savaglio and Vannucci [22] have independently proposed defining transfer principles for opportunity inequality in terms of an incomplete ranking of opportunity sets. In addition to using set inclusion to compare opportunity sets, they declare all opportunity sets that fall below a minimum opportunity threshold to be indifferent to each other.

ization theorems employ all of the axioms used by Kranich except for his Progressivity and Regressivity axioms, and my proofs make extensive use of his results.

The equality exhibited by a distribution of opportunity sets is just one component of an overall evaluation of the profile. To provide a complete evaluation, one also needs to take account of the extent of the opportunities that are available. A ranking of the profiles of opportunity sets that takes both the extent and distribution of opportunities into consideration is called a social preference ordering.[6] I also provide an axiomatization of the class of generalized Gini social preferences for distributions of opportunity sets. My version of the Pigou-Dalton transfer principle for profiles of opportunity sets is used in this characterization theorem.

The rest of this article is organized as follows. In Section 2, I present the relevant background material on the measurement of income inequality. In Section 3, I introduce the Kranich and generalized Gini equality of opportunity orderings. Kranich's characterizations of the class of Kranich equality of opportunity orderings are reviewed in Section 4. My version of the Pigou-Dalton transfer principle for profiles of opportunity sets is described and discussed in Section 5. Two characterizations of the class of generalized Gini equality of opportunity orderings are presented in Section 6. In Section 7, I axiomatize the class of generalized Gini social preference orderings for opportunity profiles. Some concluding remarks appear in Section 8.

2. Generalized Gini inequality indices

In this section, I review some of the main features of the normative approach to inequality measurement. I also introduce the generalized Gini indices of relative and absolute inequality. Further discussion of the material in this section may be found in Blackorby, Bossert, and Donaldson [4], Bossert [5], Donaldson and Weymark [7], and Weymark [25].

For a vector $\mathbf{y} \in \mathbb{R}^n$, $y_{[i]}$ is the ith smallest component of $\mathbf{y}$, with ties broken arbitrarily, and $\mathbf{y}_{\uparrow} = (y_{[1]}, \ldots, y_{[n]})$ is a permutation of $\mathbf{y}$ in which the components of $\mathbf{y}$ have been rank ordered from smallest to largest.[7] The

[6] The social evaluation of opportunity profiles could form part of a more comprehensive evaluation of the social good. For example, the social good may also depend on individual welfare. See Peragine [20] for a discussion of this issue.

[7] $\mathbb{R}$ is the real line, $\mathbb{R}^n$ is the Euclidean n-space, and $\mathbb{R}^n_+$ is the nonnegative orthant in

mean of $\mathbf{y}$ is $\mu(\mathbf{y})$. $\mathbf{0}_n$ is the n-vector of zeros and $\mathbf{1}_n$ is the n-vector of ones.

The set of individuals is $N = \{1, \ldots, n\}$, where $n \geq 2$. An income distribution is a vector $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{D} = \mathbb{R}^n_+ \setminus \{\mathbf{0}_n\}$. Alternative income distributions are evaluated using a *social evaluation function* $W\colon \mathbb{D} \to \mathbb{R}$. It is assumed that W is continuous, increasing along the ray of equality (i.e., $W(a\mathbf{1}_n) > W(b\mathbf{1}_n)$ for all $a > b > 0$), and that each level surface of W intersects the ray of equality. The function $\xi\colon \mathbb{D} \to \mathbb{R}$ is defined implicitly by

$$W(\xi(\mathbf{y})\mathbf{1}_n) = W(\mathbf{y}), \quad \forall \mathbf{y} \in \mathbb{D}. \tag{1}$$

$\xi(\mathbf{y})$ is the *equally-distributed equivalent income* for the distribution $\mathbf{y}$. The function ξ is an increasing monotone transform of W and can therefore be used as a social evaluation function.

W is also assumed to be symmetric (i.e., invariant to permutations) and to satisfy the Pigou-Dalton transfer principle. The *Pigou-Dalton transfer principle* requires that if $\mathbf{y}^1$ is obtained from $\mathbf{y}^2$ by transferring some income from a richer to a poorer person without reversing the rank order of their incomes, holding all other incomes fixed, then $W(\mathbf{y}^1) \geq W(\mathbf{y}^2)$. Requiring W to be both symmetric and to satisfy the Pigou-Dalton transfer principle is equivalent to requiring W to be S-concave.[8]

An *inequality index* is an S-convex function $I\colon \mathbb{D} \to \mathbb{R}$. I is a *relative* index of inequality if it only depends on income shares or, equivalently, if it is homogeneous of degree zero (i.e., $I(\mathbf{y}) = I(\lambda\mathbf{y})$ for all $\mathbf{y} \in \mathbb{D}$ and all $\lambda > 0$). I is an *absolute* index of inequality if I is invariant to a common increase or decrease in each person's income (i.e., $I(\mathbf{y}) = I(\mathbf{y} + \lambda\mathbf{1}_n)$ for all all $\mathbf{y} \in \mathbb{D}$ and all $\lambda \in \mathbb{R}$ for which $\mathbf{y} + \lambda\mathbf{1}_n \in \mathbb{D}$).

In the normative approach to inequality measurement, inequality indices are constructed using an equally-distributed equivalent income function. The *Atkinson-Kolm-Sen inequality index* corresponding to ξ is the function $I^{\text{AKS}}_\xi\colon \mathbb{D} \to \mathbb{R}$ defined by

$$I^{\text{AKS}}_\xi(\mathbf{y}) = 1 - \frac{\xi(\mathbf{y})}{\mu(\mathbf{y})}, \quad \forall \mathbf{y} \in \mathbb{D}. \tag{2}$$

$\mathbb{R}^n$.

[8]A real-valued function f on $\mathbb{D}$ is *S-concave* (resp. *S-convex*) if $f(\mathbf{y}Q) \geq f(\mathbf{y})$ (resp. $f(\mathbf{y}Q) \leq f(\mathbf{y})$) for all $y \in \mathbb{D}$ and all $n \times n$ bistochastic matrices Q. A nonnegative matrix is *bistochastic* if all of its row and column sums are equal to 1.

This way of constructing an inequality index was independently proposed by Atkinson [2] and Kolm [12], and was popularized by Sen [23]. I_ξ^{AKS} is a relative index if and only if ξ is homogeneous of degree one (i.e., $\lambda\xi(\mathbf{y}) = \xi(\lambda\mathbf{y})$ for all $\mathbf{y} \in \mathbb{D}$ and all $\lambda > 0$).

The *Kolm inequality index* (see Kolm [12]) corresponding to ξ is the function $I_\xi^{\mathrm{K}} \colon \mathbb{D} \to \mathbb{R}$ defined by

$$I_\xi^{\mathrm{K}}(\mathbf{y}) = \mu(\mathbf{y}) - \xi(\mathbf{y}), \quad \forall \mathbf{y} \in \mathbb{D}. \tag{3}$$

I_ξ^{K} is an absolute index if and only if ξ is unit-translatable (i.e., $\xi(\mathbf{y}) + \lambda = \xi(\mathbf{y} + \lambda\mathbf{1}_n)$ for all $\mathbf{y} \in \mathbb{D}$ and all $\lambda \in \mathbb{R}$ for which $\mathbf{y} + \lambda\mathbf{1}_n \in \mathbb{D}$).

An equally-distributed equivalent income function $\xi \colon \mathbb{D} \to \mathbb{R}$ is a *generalized Gini* social evaluation function (see Weymark [25]) if there exists a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ with $a_1 \geq \cdots \geq a_n$ and $\sum_{i=1}^n a_i = 1$ such that

$$\xi(\mathbf{y}) = \sum_{i=1}^n a_i y_{[i]}, \quad \forall \mathbf{y} \in \mathbb{D}. \tag{4}$$

In (4), the equally-distributed equivalent income is computed by taking a weighted average of the incomes once they have been first ranked in nondecreasing order (illfare-ranked). The restriction that the weights a_i are nonincreasing ensures that ξ satisfies the Pigou-Dalton transfer principle. The corresponding generalized Gini indices of relative and absolute inequality are obtained from (4) using (2) and (3), respectively. Note that the Kolm absolute inequality index corresponding to (4) can be written as

$$I_\xi^{\mathrm{K}}(\mathbf{y}) = \sum_{i=1}^n \left(\frac{1}{n} - a_i\right) y_{[i]}, \quad \forall \mathbf{y} \in \mathbb{D}. \tag{5}$$

The coefficients in (5) are nondecreasing and sum to zero. Consequently, a function $E \colon \mathbb{D} \to \mathbb{R}$ is a generalized Gini index of absolute *equality* if there exists a vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ with $b_1 \geq \cdots \geq b_n$ and $\sum_{i=1}^n b_i = 0$ such that

$$E(\mathbf{y}) = \sum_{i=1}^n b_i y_{[i]}, \quad \forall \mathbf{y} \in \mathbb{D}. \tag{6}$$

Generalized Gini social evaluation functions are sometimes written in the form

$$\xi(\mathbf{y}) = \left(\frac{1}{\sum_{i=1}^n c_i}\right) \sum_{i=1}^n c_i y_{[i]}, \quad \forall \mathbf{y} \in \mathbb{D}, \tag{7}$$

where $c_1/\sum_{i=1}^n c_i \geq \cdots \geq c_n/\sum_{i=1}^n c_i$ and $\sum_{i=1}^n c_i \neq 0$. Given $\delta \geq 1$, by setting $c_i = (n+1-i)^\delta - (n-i)^\delta$ for all $i \in N$ in (7), ξ is a member of the class of *single-parameter Ginis* (S-Ginis) introduced by Donaldson and Weymark [7]. The Gini social evaluation function is obtained by setting $\delta = 2$.

3. Equality of opportunity orderings

As in Ok [17] and Ok and Kranich [18], I assume (i) that each opportunity is desirable in the sense that it enhances, or at least does not diminish, one's quality of life, (ii) that opportunities are nonrival, so that a given opportunity is potentially available to everyone simultaneously, and (iii) that opportunities are excludable, so that providing an opportunity to some individuals does not necessarily imply that everyone has this opportunity. Ok and Kranich [18] have provided a number of examples of rights, liberties, freedoms, and functionings in the sense of Sen [24] that satisfy these three properties. As they note, their assumptions rule out certain kinds of opportunities from consideration, such as ownership rights to private goods. A more comprehensive framework would be needed to consider opportunities that are rival or nonexcludable.

Formally, the universal set of *opportunities* is an infinite set L. An *opportunity set* is a nonempty, finite subset of L.[9] The set of all opportunity sets is $\mathbb{L}$. A *profile of opportunity sets* is a vector $\mathbf{O} = (O_1, \ldots, O_n)$, where $O_i \in \mathbb{L}$ for all $i \in N$. The set of all opportunity profiles is $\mathbb{L}^n$. The subdomain of $\mathbb{L}^n$ in which opportunity sets have been rank ordered in terms of set inclusion is $\bar{\mathbb{L}}^n = \{\mathbf{O} \in \mathbb{L}^n \mid O_1 \subseteq \cdots \subseteq O_n\}$.[10] The set of *egalitarian profiles* is $\mathbb{E}^n = \{\mathbf{O} \in \mathbb{L}^n \mid O_i = O_j, \forall i, j \in N\}$.

The following notational conventions are used. For $\mathbf{O}, \mathbf{O}^1, \mathbf{O}^2 \in \mathbb{L}^n$, $|\mathbf{O}| = (|O_1|, \ldots, |O_n|)$, $\mathbf{O}^1 \cup \mathbf{O}^2 = (O_1^1 \cup O_1^2, \ldots, O_n^1 \cup O_n^2)$, $\mathbf{O}^1 \cap \mathbf{O}^2 = (O_1^1 \cap O_1^2, \ldots, O_n^1 \cap O_n^2)$, $\mathbf{O}^1 \backslash \mathbf{O}^2 = (O_1^1 \backslash O_1^2, \ldots, O_n^1 \backslash O_n^2)$, and $\mathbf{O}^1 \subseteq \mathbf{O}^2$ means that $O_i^1 \subseteq O_i^2$ for all $i \in N$. For $\mathbf{O} \in \mathbb{L}^n$, $|O|_{[i]}$ is the ith component of $|\mathbf{O}|_\uparrow$. For $\mathbf{O}^1, \mathbf{O}^2 \in \bar{\mathbb{L}}^n$ and $i \in N$, $\mathbf{O}^1 \subset_i \mathbf{O}^2$ means $O_i^1 \subset O_i^2$ and $O_j^1 = O_j^2$ for all $j \neq i$. $\varnothing_n$ denotes the n-vector $(\varnothing, \ldots, \varnothing)$.

Profiles in $\mathbb{L}^n$ are ranked using the binary relation $\succeq$. In this and the

[9] Because L is infinite, there is no upper bound to the number of alternatives in an opportunity set. Some of the proofs in Kranich [13] rely on this assumption.

[10] $\subseteq$ and $\subset$ denote weak and strict set inclusion, respectively.

following three sections, $\succeq$ is an *equality of opportunity* relation and it is interpreted as meaning "at least as equal as." The asymmetric factor $\succ$ of $\succeq$ is interpreted as meaning "more equal than" and the symmetric factor $\sim$ is interpreted as meaning "exhibits the same degree of equality as."

For a vector $\mathbf{a} \in \mathbb{R}^n$ with $\sum_{i=1}^n a_i = 0$, let $E_\mathbf{a} \colon \mathbb{L}^n \to \mathbb{R}$ be the function defined by

$$E_\mathbf{a}(\mathbf{O}) = \sum_{i=1}^{n} a_i |O|_{[i]}, \quad \forall \mathbf{O} \in \mathbb{L}^n. \tag{8}$$

$E_\mathbf{a}$ is an index of equality of opportunity. The value of $E_\mathbf{a}(\mathbf{O})$ is computed by first rank ordering the cardinalities of the individual opportunity sets in $\mathbf{O}$ in nondecreasing order and then aggregating these numbers into a scalar using the linear function defined by $\mathbf{a}$. In view of (6), it is natural to call $E_\mathbf{a}$ a *generalized Gini index of equality of opportunity* if $a_1 \geq \cdots \geq a_n$. A *generalized Gini equality of opportunity ordering* is a binary relation $\succeq$ on $\mathbb{L}^n$ that can be represented by a generalized Gini index of equality of opportunity $E_\mathbf{a}$, where $E_\mathbf{a}$ represents $\succeq$ if

$$\mathbf{O}^1 \succeq \mathbf{O}^2 \leftrightarrow E_\mathbf{a}(\mathbf{O}^1) \geq E_\mathbf{a}(\mathbf{O}^2), \quad \forall \mathbf{O}^1, \mathbf{O}^2 \in \mathbb{L}^n. \tag{9}$$

$E_\mathbf{a}$ is a *Kranich index of equality of opportunity* if $a_i > 0$ for all $i < (n+1)/2$ and $a_i < 0$ for all $i > (n+1)/2$. In a Kranich index, if an opportunity set has less (resp. more) than the median number of opportunities, it is given a positive (resp. negative) weight. $\succeq$ is a *Kranich equality of opportunity ordering* if it can be represented by a Kranich index of equality of opportunity.

4. Kranich equality of opportunity orderings

Kranich [13] has provided two axiomatizations of the class of Kranich equality of opportunity orderings. His characterization theorems use the following nine properties of $\succeq$.

Ordering (ORD). $\succeq$ is a reflexive, complete, and transitive binary relation.

Anonymity (ANON). For all $\mathbf{O}^1, \mathbf{O}^2 \in \mathbb{L}^n$, if $\mathbf{O}^1$ is a permutation of $\mathbf{O}^2$, then $\mathbf{O}^1 \sim \mathbf{O}^2$.

Independence of Common Expansions (ICE). For all $\mathbf{O} \in \mathbb{L}^n$ and all $A \in \mathbb{L}$, if $A \cap O_i = \varnothing$ for all $i \in N$, then $\mathbf{O} \sim (O_1 \cup A, \ldots, O_n \cup A)$.

Assimilation (ASM). For all $\mathbf{O} \in \mathbb{L}^n$, all $x_i \in O_i$ $(i \in N)$, and all $x \in L$, if $x \notin \bigcup_{i \in N} O_i \backslash \{x_i\}$, then $((O_1 \backslash \{x_1\}) \cup \{x\}, \ldots, (O_n \backslash \{x_n\}) \cup \{x\}) \succeq \mathbf{O}$.

Independence of Rank-Preserving Expansions (IRPE). For all $\mathbf{O}^1, \mathbf{O}^2, \mathbf{O}^3 \in \bar{\mathbb{L}}^n$ for which $\mathbf{O}^1 \cap \mathbf{O}^3 = \varnothing_n$ and $\mathbf{O}^2 \cap \mathbf{O}^3 = \varnothing_n$, $\mathbf{O}^1 \succeq \mathbf{O}^2$ if and only if $\mathbf{O}^1 \cup \mathbf{O}^3 \succeq \mathbf{O}^2 \cup \mathbf{O}^3$.

Archimedean Property (ARCH). For all $\mathbf{O} \in \bar{\mathbb{L}}^n$ and for every sequence $\langle \mathbf{O}^k \rangle$ in $\bar{\mathbb{L}}^n \backslash \mathbb{E}^n$ for which $\mathbf{O}^k \cap \mathbf{O}^{k'} = \varnothing_n$ for all $k \neq k'$, there exists a $K \in \mathbb{N}$ such that $\mathbf{O} \succ (\mathbf{O}^1 \cup \cdots \cup \mathbf{O}^K)$.[11]

Archimedean Difference Property (ARCHD). For any pair of sequences $\langle \mathbf{O}^{1k} \rangle, \langle \mathbf{O}^{2k} \rangle$ in $\bar{\mathbb{L}}^n$ for which $\mathbf{O}^{1k} \cap \mathbf{O}^{1k'} = \varnothing_n$ and $\mathbf{O}^{2k} \cap \mathbf{O}^{2k'} = \varnothing_n$ for all $k \neq k'$, if $\mathbf{O}^{1k} \succ \mathbf{O}^{2k}$ for all $k \in \mathbb{N}$, then for all $\bar{\mathbf{O}}, \tilde{\mathbf{O}} \in \bar{\mathbb{L}}^n$, there exists a $K \in \mathbb{N}$ such that $(\mathbf{O}^{1k} \cup \cdots \cup \mathbf{O}^{1K} \cup \bar{\mathbf{O}}) \succeq (\mathbf{O}^{2k} \cup \cdots \cup \mathbf{O}^{2K} \cup \tilde{\mathbf{O}})$.

Progressivity (PROG). For all $\mathbf{O}^1, \mathbf{O}^2 \in \bar{\mathbb{L}}^n$ and all $i < (n+1)/2$, if $\mathbf{O}^1 \subset_i \mathbf{O}^2$, then $\mathbf{O}^2 \succ \mathbf{O}^1$.

Regressivity (REGR). For all $\mathbf{O}^1, \mathbf{O}^2 \in \bar{\mathbb{L}}^n$ and all $i > (n+1)/2$, if $\mathbf{O}^1 \subset_i \mathbf{O}^2$, then $\mathbf{O}^1 \succ \mathbf{O}^2$.

The Ordering axiom requires $\succeq$ to be a complete preorder. Anonymity is a symmetric treatment axiom that prevents $\succeq$ from paying attention to the identities of individuals. Independence of Common Expansions says that if each person's opportunity set is enlarged by the addition of a common set of new opportunities, then the new profile of opportunity sets exhibits the same degree of equality as the original profile. The Assimilation axiom says that equality of opportunity is not diminished if the profile of opportunity sets is made more similar by replacing one alternative from each opportunity set by a common opportunity.

The other axioms pertain to comparisons of profiles that are rank ordered in terms of set inclusion. In any such profile, set inclusion provides a basis for determining when one individual has more opportunities than another that is independent of how these opportunities are valued (provided that each opportunity is desirable). For a profile of opportunity sets $\mathbf{O}^1 \in \bar{\mathbb{L}}^n$, $\mathbf{O}^2$ is a *rank-preserving expansion* of $\mathbf{O}^1$ if $\mathbf{O}^2 \backslash \mathbf{O}^1 \in \bar{\mathbb{L}}^n$. In other words, if initially

[11] $\mathbb{N}$ is the set of positive integers.

individual i has every opportunity that individual j has, then i is given every new opportunity that is given to j.

Independence of Rank-Preserving Expansions requires the ranking of two rank-ordered profiles $\mathbf{O}^1$ and $\mathbf{O}^2$ to be invariant to rank-preserving expansions that use the same profile of new opportunities to enlarge both $\mathbf{O}^1$ and $\mathbf{O}^2$. A similar axiom applied to income distributions is used in Weymark's [25] axiomatization of the generalized Gini absolute inequality indices.

The first Archimedean axiom considers a sequence of rank-preserving expansions of a profile $\mathbf{O}^1 \in \bar{\mathbb{L}}^n$ in which not everybody is given exactly the same new opportunities at any stage in the sequence. The Archimedean Property requires that if $\mathbf{O}$ is also in $\bar{\mathbb{L}}^n$ and we keep expanding $\mathbf{O}^1$ in the way just described, then eventually we obtain a profile that exhibits more inequality than $\mathbf{O}$, regardless of how $\succeq$ ranks $\mathbf{O}$ and $\mathbf{O}^1$.

The second Archimedean axiom pertains to rank-preserving expansions of two opportunity profiles, $\bar{\mathbf{O}}, \tilde{\mathbf{O}} \in \bar{\mathbb{L}}^n$. The Archimedean Difference Property says that if we consider two sequences of rank-preserving expansions that enlarge these profiles by adding new profiles of opportunities to $\bar{\mathbf{O}}$ that exhibit more equality than the new profiles of opportunities that are added to $\tilde{\mathbf{O}}$, eventually the profile obtained by expanding $\bar{\mathbf{O}}$ is at least as equal as the profile obtained by expanding $\tilde{\mathbf{O}}$, regardless of how $\succeq$ ranks $\bar{\mathbf{O}}$ and $\tilde{\mathbf{O}}$.

Consider a profile $\mathbf{O} \in \bar{\mathbb{L}}^n$. The Progressivity and Regressivity axioms pertain to the expansion of a single individual's opportunity set in $\mathbf{O}$ that preserves the rank ordering of opportunity sets by set inclusion. Progressivity (resp. Regressivity) requires this expansion to be equality-enhancing (resp. equality-reducing) if this individual has less (resp. more) than the median number of opportunities in $\mathbf{O}$.

Kranich [13] has shown that ORD, ANON, ICE, and ASM jointly imply that the size of each person's opportunity set is the only information in an opportunity profile that $\succeq$ is sensitive to. In other words, if $|\mathbf{O}^1| = |\mathbf{O}^2|$, then $\mathbf{O}^1 \sim \mathbf{O}^2$. He has also shown that if $\succeq$ also satisfies PROG and REGR, then the egalitarian profiles (i.e., profiles in $\mathbb{E}^n$) maximize $\succeq$ on $\mathbb{L}^n$. Kranich's main theorems show that $\succeq$ satisfies his seven non-Archimedean axioms together with either of his Archimedean axioms if and only if $\succeq$ is a Kranich equality of opportunity ordering.

5. A generalized Pigou-Dalton transfer principle

The standard Pigou-Dalton transfer principle for income distributions requires $\mathbf{y}^2$ to be at least as equal as $\mathbf{y}^1$ if $\mathbf{y}^2$ is obtained from $\mathbf{y}^1$ by transferring some income from a richer to a poorer person without reversing the rank order of their incomes. For symmetric equality relations, this is equivalent to requiring $\mathbf{y}^2$ to be at least as equal as $\mathbf{y}^1$ if $\mathbf{y}^2$ is obtained from $\mathbf{y}^1$ by transferring some income from a richer to a poorer person in such a way that the absolute value of the difference between their incomes is diminished. If i is initially the poorer of these individuals and j the richer, this kind of transfer permits i to be the richer of the two after the transfer, but does not permit either $y_i^2 \geq y_j^1$ or $y_j^2 \leq y_i^1$. See Marshall and Olkin [16, p. 6]

In the context of income distributions, for any two individuals, it is always possible to determine which person is the richer of the two. For profiles of opportunity sets, it is natural to regard j as being more privileged than i if i's opportunity set is strictly contained in j's. When neither opportunity set is included in the other, it is not possible to determine who has better opportunities unless one has some external standard for ranking opportunity sets. However, a complete ranking of opportunity sets is not needed to formulate an analogue of the Pigou-Dalton transfer principle for opportunity sets. My version of this principle only uses the set inclusion partial ordering to determine who is more privileged.

Generalized Pigou-Dalton Transfer Principle (GPDT). For all $\mathbf{O}^1, \mathbf{O}^2 \in \mathbb{L}^n$ and all distinct $i, j \in N$, if $O_k^1 = O_k^2$ for all $k \neq i, j$, $O_i^1 \subset O_j^1$, $A \subset O_j^1 \backslash O_i^1$, $O_i^2 = O_i^1 \cup A$, and $O_j^2 = O_j^1 \backslash A$, then $\mathbf{O}^2 \succeq \mathbf{O}^1$.

In this definition, individual j has every opportunity that i has in $\mathbf{O}^1$ and, in addition, has some opportunities that i doesn't have, so j is more privileged than i. In $\mathbf{O}^2$, i's opportunity set has been expanded by transferring some, but not all, of j's opportunities to i. Everyone else has the same opportunities in both profiles. After the transfer, the opportunity sets of i and j can't be ranked by set inclusion. Nevertheless, $O_i^2 \subset O_j^1$ and $O_i^1 \subset O_j^2$, so i is still less privileged after the transfer than j was initially and j is still more privileged after the transfer than i was initially. As a consequence, this transfer can be regarded as being an equalizing transfer. For this reason, the Generalized Pigou-Dalton Transfer Principle regards $\mathbf{O}^2$ as exhibiting at least as much equality as $\mathbf{O}^1$.

The requirement that not all of $O_j^1 \backslash O_i^1$ is transferred from j to i implicitly restricts the application of this principle to situations in which j has at least two more opportunities than i. If all of $O_j^1 \backslash O_i^1$ is transferred to i, then $\mathbf{O}^2$ is simply a permutation of $\mathbf{O}^1$. In this case, $\mathbf{O}^1$ and $\mathbf{O}^2$ exhibit the same degree of equality by the Anonymity axiom introduced in the preceding section.

As noted in the Introduction, my version of the Pigou-Dalton transfer principle for opportunity inequality differs from that of Ok [17] and Ok and Kranich [18] in a number of respects. One minor difference is that they only consider two-person societies. It is relatively straightforward to extend their principle to larger groups of individuals.

The first substantive difference is in the way that individuals with more desirable opportunity sets are identified. For each profile, I use the partial order defined by set inclusion to determine which of two individuals is more privileged. If their opportunity sets are not nested in the initial profile, then my principle does not apply to a transfer between them. In contrast, Ok and Kranich completely order the individuals in terms of the desirability of their opportunity sets. This is accomplished by using an auxiliary ordering $\mathbf{R}$ of the set of opportunity sets in $\mathbb{L}$. This ordering is assumed to be an extension of the set inclusion relation $\subseteq$. That is, for all $O^1, O^2 \in \mathbb{L}$, (i) if $O^1 \subseteq O^2$, then $O^2 \mathbf{R} O^1$ and (ii) if $O^1 \subset O^2$, then $O^2 \mathbf{P} O^1$, where $\mathbf{P}$ is the asymmetric factor of $\mathbf{R}$. Ok and Kranich's transfer principle applies to profiles in which two individuals' opportunity sets can be strictly ranked according to $\mathbf{R}$ even if the sets aren't nested.

The relative appeal of these two approaches depends on the extent to which there is consensus on how to rank individual opportunity sets. Simply knowing that all opportunities are desirable is enough to rank nested opportunity sets, which is all that is required for my transfer principle. On the other hand, if there are good reasons to single out one ordering of $\mathbb{L}$ as a desirability relation, then it seems reasonable to expand the scope of the transfer principle, as Ok and Kranich have done. For example, such an ordering may be characterized by a compelling set of axioms for a desirability relation on opportunity sets.[12] For the generalized Gini theorems presented here, it is sufficient to apply the transfer principle to nested opportunity sets.

The second substantive difference between my version of of the Pigou-Dalton transfer principle and that of Ok and Kranich is in the way in which we

[12] See Barberà, Bossert, and Pattanaik [3] for a survey of the literature on ranking opportunity sets.

define equalizing transforms. Consider a profile $\mathbf{O}^1 \in \mathbb{L}^2$ in which individual 2 has a more desirable opportunity set than individual 1 according to $\mathbf{R}$. According to Ok and Kranich [18], for $\mathbf{O}^2 \in \mathbb{L}^2$ to be an equalizing transform of $\mathbf{O}^1$, it must be the case that (i) $O_1^1 \cup O_2^1 = O_1^2 \cup O_2^2$, (ii) $O_2^2 \mathbf{R} O_1^2$, (iii) $O_1^2 \mathbf{R} O_1^1$, and (iv) $O_2^1 \mathbf{R} O_2^2$. That is, the combined set of opportunities of both individuals is the same in the two profiles, the transform does not reverse the relative desirability of the individual opportunity sets, and the first person's opportunities are no worse and the second person's opportunities are no better in $\mathbf{O}^2$ than in $\mathbf{O}^1$.

As discussed in the Introduction, Ok and Kranich's concept of an equalizing transform incorporates two distinct ways in which profiles can be made more equal. One is through a gap-reducing transfer of opportunities from a more- to a less-privileged individual. This is the kind of equalizing transform that my transfer principle applies to. The second kind of equalizing transform increases the number of alternatives that the two affected individuals have in common, thereby making their opportunity sets more similar. With this kind of equalizing transform, no actual transfer need take place. For example, if the less-privileged individual's opportunity set is expanded by simply adding some of the more-privileged individual's opportunities with no change in the latter's opportunity set, which is feasible given the nonrival nature of the opportunities, Ok and Kranich's requirements for an equalizing transform are satisfied.

When there are more than two individuals, increasing the similarity of two people's opportunity sets need not be equality-enhancing. For example, suppose that there are three people, the profile $\mathbf{O}^1$ is nested with $O_1^1 \subset O_2^1 \subset O_3^1$, and $\mathbf{O}^2 = (O_1^1, O_3^1, O_3^1)$. While the opportunity sets of individuals' 2 and 3 are more similar in $\mathbf{O}^2$ than in $\mathbf{O}^1$ (indeed, they are now equal), the opportunity sets of individuals' 1 and 2 are more dissimilar. Which effect dominates is not obvious.[13] While there is considerable appeal to the idea that making *all* opportunity sets more similar is equality-enhancing, this example suggests that it is difficult to incorporate this desiderata in a transfer principle when there are more than two people. It therefore seems better to

[13]Note that if a transfer of opportunities takes place between individuals i and j in a nested profile, then for any third individual k, if the gap between, say, i and k is increased, then the even larger initial gap between j and k is reduced, where by the 'gap' I simply mean the number of alternatives that are in one but not the other opportunity set. Thus, the preceding criticism does not apply to the kinds of transfers of opportunities considered in my version of the transfer principle.

deal with this issue in a separate axiom. Here, the equality-enhancing role of making opportunity sets more similar is captured by Kranich's Assimilation axiom.

There are a number of other contexts in which inequality is not unidimensional, including variations of the problem considered here. Not surprisingly, versions of the Pigou-Dalton principle have been developed for these problems too.

Kranich [15] has considered a model in which an individual has monetary wealth in addition to an opportunity set. For each profile, Kranich constructs an individual's shadow value of wealth by taking the difference between his or her actual monetary holdings and the money this person would have in an egalitarian benchmark in which money is distributed to offset the differences in the individuals' opportunities. These shadow values are used to provide the ranking of individuals needed to apply his transfer principle.

Gravel, Laslier, and Trannoy [9] have considered a binary relation on profiles of opportunity sets that is meant to measure the relative social freedom of the profiles. They use a transfer principle that says that social freedom is not increased if an opportunity is transferred from one individual to another. In their principle, they do not require the opportunity sets to be nested.

The problem of finding an appropriate way to rank individuals so as to apply a transfer principle also arises in the measurement of inequality for cooperative games. Einy and Peleg [8] restrict the application of their transfer principle to additive games because for such games the ranking of individuals in terms of the contributions they make to the value of a coalition is independent of the coalition considered. They use their transfer principle to help axiomatize a class of generalized Gini measures of equality for cooperative games. These indices are computed by taking a weighted sum of a rank-ordered permutation of the Shapley value.

In the literature on multidimensional inequality, an individual's economic status is described by a vector in $\mathbb{R}^m$, where m is the number of dimensions in which inequality is measured.[14] In the univariate case, the requirement that the absolute value of the difference between two individuals' incomes not be increased by a transfer is equivalent to requiring that (i) their post-transfer incomes lie in the convex hull of their pretransfer incomes and (ii) the sums of the pre- and post-transfer incomes are equal. Equivalently, the post-transfer income distribution is obtained by multiplying the initial income distribu-

[14] For a recent survey of this literature, see Savaglio [21].

tion by a T-transform.[15] This same construct can be applied component by component to define a multivariate version of a Pigou-Dalton transfer. More precisely, a (possibly different) T-transform is applied to each component of income for which a transfer takes place between the two individuals and an identity transform is applied to the other components of income.[16]

An opportunity set can be described by an infinite-dimensional vector whose kth component is 1 if the kth opportunity is in the set and whose kth component is 0 otherwise. In this notation, nesting of opportunity sets corresponds to vector dominance. My definition of a Pigou-Dalton transfer from individual i to individual j involves changing some of the 1 entries to 0s in i's opportunity vector and changing the corresponding 0 entries to 1s in j's opportunity vector. In other words, some, but not all, of the components of their opportunity vectors are permuted. In this discrete framework, the only T-transforms that are feasible are the identity matrix and the matrices that permute two individuals' incomes. Thus, my definition of a Pigou-Dalton transfer for opportunity sets is equivalent to the multivariate Pigou-Dalton transfer defined above if a common T-transform is applied to components of the initial opportunity vector corresponding to the opportunities that are transferred and an identity transform is applied to the other components of this vector.[17]

6. Generalized Gini equality of opportunity orderings

In this section, I provide two axiomatizations of the class of generalized Gini equality of opportunity orderings. My characterizations of this class of orderings are obtained by replacing Kranich's Progressivity and Regressivity axioms with the Generalized Pigou-Dalton Transfer Principle and, for the axiomatization in which ARCH is used, a new axiom that helps orient the direction in which equality of opportunity is increased.

[15] A T-transform is either an identity matrix or a square bistochastic matrix with exactly two nonzero off-diagonal elements.

[16] There is not a unique way of generalizing the univariate definition of a Pigou-Dalton transfer to the multivariate case. The use of component-specific T-transforms is a special case of the construction Marshall and Olkin [16, p. 432] use to define rowwise majorization. They have also considered an alternative multivariate transfer principle in which a common T-transform is applied to all components of income. See Marshall and Olkin [16, p. 430].

[17] For further dicussion of T-transforms in a discrete setting, see Marshall and Olkin [16, pp. 134–137].

This new axiom is Weak Egalitarian Dominance. Weak Egalitarian Dominance requires any egalitarian profile of opportunity sets to be at least as equal as any profile that is obtained from it by providing a common set of new opportunities to some, but not all, individuals.

Weak Egalitarian Dominance (WED). For all $\mathbf{O}^1 \in \mathbb{E}^n$ and all $\mathbf{O}^2 \in \mathbb{L}^n \backslash \mathbb{E}^n$, if there exists an $A \in \mathbb{L}$ such that for all $i \in N$, either $O_i^1 = O_i^2$ or $O_i^2 \backslash O_i^1 = A$, then $\mathbf{O}^1 \succeq \mathbf{O}^2$.

Lemma 1 shows that ORD and ICE imply that all egalitarian profiles exhibit the same degree of equality.

Lemma 1. *If $\succeq$ satisfies ORD and ICE, then for all $\mathbf{O}^1, \mathbf{O}^2 \in \mathbb{E}^n$, $\mathbf{O}^1 \sim \mathbf{O}^2$.*

Proof. Because $\succeq$ is reflexive, we only need to consider the case in which $\mathbf{O}^1$ and $\mathbf{O}^2$ are distinct profiles in $\mathbb{E}^n$. If $\mathbf{O}^1 \cap \mathbf{O}^2 = \varnothing_n$, then ICE implies that $\mathbf{O}^1 \sim \mathbf{O}^1 \cup \mathbf{O}^2$ and $\mathbf{O}^2 \sim \mathbf{O}^1 \cup \mathbf{O}^2$. Transitivity then implies that $\mathbf{O}^1 \sim \mathbf{O}^2$. If $\mathbf{O}^1 \subseteq \mathbf{O}^2$, the conclusion that $\mathbf{O}^1 \sim \mathbf{O}^2$ follows from ICE by noting that $\mathbf{O}^2 = \mathbf{O}^1 \cup (\mathbf{O}^2 \backslash \mathbf{O}^1)$. The case in which $\mathbf{O}^2 \subseteq \mathbf{O}^1$ can be dealt with using a symmetric argument. If $\mathbf{O}^1 \cap \mathbf{O}^2 \neq \varnothing_n$, but neither $\mathbf{O}^1 \subseteq \mathbf{O}^2$ nor $\mathbf{O}^2 \subseteq \mathbf{O}^1$, then ICE implies that $\mathbf{O}^1 \cap \mathbf{O}^2 \sim \mathbf{O}^1$ and $\mathbf{O}^1 \cap \mathbf{O}^2 \sim \mathbf{O}^2$ because $\mathbf{O}^1 \cap \mathbf{O}^2 \subseteq \mathbf{O}^1$ and $\mathbf{O}^1 \cap \mathbf{O}^2 \subseteq \mathbf{O}^2$. Transitivity then implies that $\mathbf{O}^1 \sim \mathbf{O}^2$. □

As noted above, Kranich [13] (see his Lemma 4) has shown that ORD, ANON, ICE, ASM, PROG, and REGR jointly imply that any egalitarian profile exhibits at least as much equality as any other profile. Lemma 2 shows that the same conclusion follows if WED and GPDT are substituted for PROG and REGR.

Lemma 2. *If $\succeq$ satisfies ORD, ANON, ICE, ASM, WED, and GPDT, then for all $\mathbf{O}^1 \in \mathbb{E}^n$ and all $\mathbf{O}^2 \in \mathbb{L}^n$, $\mathbf{O}^1 \succeq \mathbf{O}^2$.*

Proof. By Lemma 1, we only need to consider the case in which $\mathbf{O}^2 \in \mathbb{L}^n \backslash \mathbb{E}^n$. Let $\mathbf{O}^3$ be a permutation of $\mathbf{O}^2$ in which $|O_1^3| \leq \cdots \leq |O_n^3|$. By ANON, $\mathbf{O}^2 \sim \mathbf{O}^3$. Kranich [13] (see his Lemma 2) has shown that ORD, ANON, ICE, and ASM imply that for all $\mathbf{O}, \mathbf{O}' \in \mathbb{L}^n$, $\mathbf{O} \sim \mathbf{O}'$ if $|\mathbf{O}| = |\mathbf{O}'|$. Hence, there exists a $\mathbf{O}^4 \in \bar{\mathbb{L}}^n \backslash \mathbb{E}^n$ with $|\mathbf{O}^3| = |\mathbf{O}^4|$ such that $\mathbf{O}^3 \sim \mathbf{O}^4$. There are two cases to consider.

Case 1. Suppose that there exists a $k \in \{1, \dots, n-1\}$ such that $|O_1^4| = \cdots = |O_k^4|$ and $|O_{k+1}^4| = \cdots = |O_n^4| = |O_1^4| + 1$. Let $\bar{\mathbf{O}} = (O_1^4, \dots, O_1^4)$. WED implies that $\bar{\mathbf{O}} \succeq \mathbf{O}^4$. Lemma 1 and transitivity then imply that $\mathbf{O}^1 \succeq \mathbf{O}^2$.

Case 2. If Case 1 does not hold, there exist $k, k' \in N$ with $k < k'$ such that $|O_k^4| + 1 < |O_{k'}^4|$. Consider any $x \in O_{k'}^4 \backslash O_k^4$. Define $\mathbf{O}'$ by setting $O_i' = O_i^4$ for all $i \neq k, k'$, $O_k' = O_k^4 \cup \{x\}$, and $O_{k'}' = O_{k'}^4 \backslash \{x\}$. By GPDT, $\mathbf{O}' \succeq \mathbf{O}^4$. Because $O_k' \nsubseteq O_{k'}'$, $\mathbf{O}' \notin \bar{\mathbb{L}}^n$. However, $|O_k'| \leq |O_{k'}'|$. Therefore, if we define $\mathbf{O}^5$ by setting $O_i^5 = O_i'$ for all $i \neq k$ and $O_k^5 = (O_k' \backslash \{x\}) \cup \{y\}$ for some $y \in O_{k'}' \backslash O_k'$, $\mathbf{O}^5 \in \bar{\mathbb{L}}^n \backslash \mathbb{E}^n$ and, by Kranich's Lemma 2, $\mathbf{O}^5 \sim \mathbf{O}'$.

If there exists a $k \in \{1, \dots, n-1\}$ such that $|O_1^5| = \cdots = |O_k^5|$ and $|O_{k+1}^5| = \cdots = |O_n^5| = |O_1^5| + 1$, Case 1 applies to $\mathbf{O}^5$. If not, by repeating the argument in the preceding paragraph a finite number of times, we obtain a profile for which Case 1 does apply. Transitivity and Lemma 1 then imply that $\mathbf{O}^1 \succeq \mathbf{O}^2$. □

My first axiomatization of the class of generalized Gini equality of opportunity orderings uses the Archimedean Difference Property. The only difference between my set of axioms and those used in Kranich's [13] Theorem 3 to characterize the class of Kranich equality of opportunity orderings is my use of GPDT instead of PROG and REGR.

Theorem 1. *$\succeq$ satisfies ORD, ANON, ICE, ASM, IRPE, ARCHD, and GPDT if and only if $\succeq$ can be represented by a generalized Gini index of equality of opportunity.*

Proof. It is readily checked that a generalized Gini index of equality of opportunity satisfies all of these axioms. To establish the sufficiency part of the theorem, note that the first part of the proof of Lemma 3 in Kranich [13] shows that if $\succeq$ satisfies ORD, ANON, ICE, ASM, IRPE, and ARCHD, then there exists an $\mathbf{a} \in \mathbb{R}^n$ with $\sum_{i=1}^n a_i = 0$ such that $\succeq$ can be represented by the function $E_{\mathbf{a}}$, where $E_{\mathbf{a}}$ is defined in (8). GPDT implies that $a_1 \geq \cdots \geq a_n$. □

My second axiomatization of the class of generalized Gini equality of opportunity orderings uses the Archimedean Property instead of the Archimedean Difference Property and adds WED to the list of axioms. Kranich [13] has shown that the substitution of ARCH for ARCHD can be used in his Theorem 3 to obtain a second characterization of his class of equality of opportunity orderings.

Theorem 2. *$\succeq$ satisfies ORD, ANON, ICE, ASM, IRPE, ARCH, WED, and GPDT if and only if $\succeq$ can be represented by a generalized Gini index of equality of opportunity.*

Proof. Necessity is easy to verify, so I only consider the sufficiency of the axioms. Using my Lemma 2 instead of Kranich's [13] Lemma 4, the argument in the proof of his Theorem 3′ shows that if $\succeq$ satisfies the axioms of Theorem 2, then there exists an $\mathbf{a} \in \mathbb{R}^n$ with $\sum_{i=1}^n a_i = 0$ such that $\succeq$ can be represented by the function $E_{\mathbf{a}}$. As in the proof of Theorem 1, GPDT implies that $a_1 \geq \cdots \geq a_n$. □

If the conclusion in GPDT is strengthened by requiring that $\mathbf{O}^2 \succ \mathbf{O}^1$, the weights $a_1, \ldots, a_n$ in both theorems must all be distinct. If, instead, the conclusion of WED is strengthened by requiring that $\mathbf{O}^1 \succ \mathbf{O}^2$, not all of the weights in Theorem 2 can be equal. This can be seen by considering a profile $\mathbf{O}^1 \in \mathbb{E}^n$ and an $x \notin O_n^1$. Let $\mathbf{O}^2$ by defined by setting $O_i^2 = O_i^1$ for all $i \neq n$ and $O_n^2 = O_n^1 \cup \{x\}$. By the strengthened form of WED, $\mathbf{O}^1 \succ \mathbf{O}^2$, which implies that $a_n < 0$. Similarly, adding the strengthened form of WED to the axioms in Theorem 1 restricts the weights in that theorem from being all equal.

7. Generalized Gini social preference orderings for opportunity profiles

An overall evaluation of opportunity profiles needs to take account of the extent of the opportunities available, not just their distribution. In this section, I consider social preference orderings on the set $\mathbb{L}^n$ of all profiles of opportunity sets and axiomatize the class of generalized Gini social preferences on $\mathbb{L}^n$.

A binary relation $\succeq$ on $\mathbb{L}^n$ is now interpreted as meaning "socially weakly preferred to." Similarly, $\succ$ now means "socially preferred to" and $\sim$ now means "is socially indifferent to."

For a vector $\mathbf{a} \in \mathbb{R}^n$ with $\sum_{i=1}^n a_i = 1$, let $W_{\mathbf{a}} \colon \mathbb{L}^n \to \mathbb{R}$ be the function defined by

$$W_{\mathbf{a}}(\mathbf{O}) = \sum_{i=1}^n a_i |O|_{[i]}, \quad \forall \mathbf{O} \in \mathbb{L}^n. \tag{10}$$

$W_{\mathbf{a}}$ is a social evaluation function for *profiles of opportunity sets*. $W_{\mathbf{a}}$ is a *generalized Gini social evaluation function* if $a_1 \geq \cdots \geq a_n$. Note that $W_{\mathbf{a}}$ is defined on $\mathbb{L}^n$, while the generalized Gini social evaluation function for income distributions defined in (4) has $\mathbb{D}$ as its domain. A *generalized Gini social preference ordering for opportunity profiles* is a binary relation $\succeq$ on $\mathbb{L}^n$ that can be represented by a generalized Gini social evaluation function $W_{\mathbf{a}}$.

Many of the axioms used in my characterizations of the class of generalized Gini equality of opportunity orderings are equally natural for a social preference relation on $\mathbb{L}^n$. This is the case with Ordering, Anonymity, the Generalized Pigou-Dalton Transfer Principle, Independence of Rank-Preserving Expansions, and the Archimedean Difference Property. For example, because the Generalized Pigou-Dalton Transfer Principle is only concerned with the distribution of opportunities, an increase in equality is a social improvement.

Independence of Common Expansions is not appropriate for a social preference relation because a common expansion of everyone's opportunity set would generally be regarded as being a social improvement. If we expand $\mathbf{O}^1$ by adding new opportunities in the way described in the definition of the Archimedean Principle, one would expect that if enough new opportunities are added, the resulting profile would be socially preferred to the profile $\mathbf{O}$, at least for some choices of the sequence $\langle \mathbf{O}^k \rangle$. For this reason, the Archimedean Principle is not appropriate for a social preference relation. The Weak Egalitarian Dominance axiom is not appropriate either because for some choices of $\mathbf{O}^1$ and $\mathbf{O}^2$, the expansion of opportunities in the latter profile would more than compensate for the increase in inequality.

Whether the Assimilation axiom is a desirable property of a social preference relation is less clear. It is not used in my characterization theorem.

While a social preference ordering on the set of opportunity profiles should respond positively to an increase in opportunities in some circumstances, not all expansions are social improvements. For example, suppose that $\mathbf{O}^1, \mathbf{O}^2 \in \bar{\mathbb{L}}^n$ and $\mathbf{O}^1 \subset_i \mathbf{O}^2$. If the expansion in i's opportunities substantially increases the inequality of the distribution, one might not regard $\mathbf{O}^2$ as being socially preferred to $\mathbf{O}^1$. I only require that a common expansion of everyone's opportunity set from an initial egalitarian distribution is socially improving.

Egalitarian Monotonicity (EMON). For all $\mathbf{O}^1, \mathbf{O}^2 \in \mathbb{E}^n$, if there exists an $A \in \mathbb{L}$ such that $O_i^2 = O_i^1 \cup A$ for all $i \in N$, then $\mathbf{O}^2 \succ \mathbf{O}^1$.

My final axiom is a strengthening of Independence of Rank-Preserving Expansions. It makes use of an ordering $\mathbf{R}$ (with asymmetric and symmetric factors $\mathbf{P}$ and $\mathbf{I}$, respectively) of the set of possible opportunity sets $\mathbb{L}$ that reflects their relative social desirability. This ordering is assumed to be an extension of the set inclusion relation $\subseteq$. As in Ok [17] and Ok and Kranich [18], $\mathbf{R}$ can be thought of as embodying the views of a social planner or a representative agent.

Independence of R-Equivalent Rank-Preserving Expansions ($\mathbf{R}$-IRPE). For all $\mathbf{O}^1, \mathbf{O}^2, \mathbf{O}^3, \mathbf{O}^4 \in \bar{\mathbb{L}}^n$ for which $\mathbf{O}^1 \cap \mathbf{O}^3 = \varnothing_n$, $\mathbf{O}^2 \cap \mathbf{O}^4 = \varnothing_n$, and $O_i^3 \mathbf{I} O_i^4$ for all $i \in N$, $\mathbf{O}^1 \succeq \mathbf{O}^2$ if and only if $\mathbf{O}^1 \cup \mathbf{O}^3 \succeq \mathbf{O}^2 \cup \mathbf{O}^4$.

Recall that Independence of Rank-Preserving Expansions requires the ordering of two rank-ordered profiles $\mathbf{O}^1$ and $\mathbf{O}^2$ to be invariant to rank-preserving expansions that use the same profile of new opportunities to enlarge both $\mathbf{O}^1$ and $\mathbf{O}^2$. Independence of $\mathbf{R}$-Equivalent Rank-Preserving Expansions strengthens this axiom by dropping the requirement that it is the same opportunities that are added to individual i's opportunity sets in both profiles. Instead, it requires that the new opportunities added to O_i^1 are indifferent according to $\mathbf{R}$ to the new opportunities added to O_i^2. Because the new opportunities added to O_i^1 are regarded as being equally good as the new opportunities added to O_i^2 for every individual $i \in N$ and because the rank orders of the profiles in terms of set inclusion have not been altered, $\mathbf{R}$-IRPE requires these expansions to preserve the relative social ranking of the two profiles.[18]

A generalized Gini social preference ordering for opportunity profiles only takes account of the cardinality of the opportunity sets and their distribution in ranking profiles. Thus, to axiomatize this class of orderings, the underlying ordering of the opportunity sets used in $\mathbf{R}$-IRPE needs to be the *cardinality ordering* $\mathbf{R}^\#$ defined by

$$O^1 \mathbf{R}^\# O^2 \leftrightarrow |O^1| \geq |O^2|, \quad \forall O^1, O^2 \in \mathbb{L}. \tag{11}$$

[18] The use of the ordering $\mathbf{R}$ is subject to the same reservations raised in Section 5 about the use of an auxiliary ordering in Ok and Kranich's version of the Pigou-Dalton transfer principle. I do not know of a way to axiomatize a generalized Gini social preference ordering for opportunity profiles without the use of an auxiliary ordering of individual opportunity sets.

Ok [17] and Ok and Kranich [18] have used $\mathbf{R}^{\#}$ as the ordering in their definition of an equalizing transform. $\mathbf{R}^{\#}$ has been axiomatized by Pattanaik and Xu [19] for the case in which the universal set is finite.

$\mathbf{R}^{\#}$-IRPE forces the social preference relation $\succeq$ to take account of the cardinality of opportunity sets, but only in limited circumstances. Lemma 3 shows that when this axiom is combined with ORD and ANON, the only relevant features of an opportunity profile are the cardinalities of its opportunity sets.

Lemma 3. *If $\succeq$ satisfies ORD, ANON, and $\mathbf{R}^{\#}$-IRPE, then for all $\mathbf{O}^1, \mathbf{O}^2 \in \mathbb{L}^n$, $|\mathbf{O}^1| = |\mathbf{O}^2| \rightarrow \mathbf{O}^1 \sim \mathbf{O}^2$.*

Proof. By ANON, there is no loss in generality is assuming that $\mathbf{O}^1, \mathbf{O}^2 \in \bar{\mathbb{L}}^n$. Consider a third profile $\mathbf{O}^3 \in \bar{\mathbb{L}}^n$ for which $|\mathbf{O}^3| = |\mathbf{O}^1|$, $\mathbf{O}^3 \cap \mathbf{O}^1 = \varnothing_n$, and $\mathbf{O}^3 \cap \mathbf{O}^2 = \varnothing_n$. Because $\succeq$ is reflexive, $\mathbf{O}^3 \sim \mathbf{O}^3$. Because $|\mathbf{O}^1| = |\mathbf{O}^2|$, we also have $\mathbf{O}^3 \cup \mathbf{O}^1 \sim \mathbf{O}^3 \cup \mathbf{O}^2$ by $\mathbf{R}^{\#}$-IRPE. $\mathbf{R}^{\#}$-IRPE also implies that $\mathbf{O}^1 \sim \mathbf{O}^2$ if and only if $\mathbf{O}^1 \cup \mathbf{O}^3 \sim \mathbf{O}^2 \cup \mathbf{O}^3$. Hence, $\mathbf{O}^1 \sim \mathbf{O}^2$. □

Theorem 3 is my axiomatization of the class of generalized Gini social preference orderings for opportunity profiles.

Theorem 3. *$\succeq$ satisfies ORD, ANON, $\mathbf{R}^{\#}$-IRPE, ARCHD, EMON, and GPDT if and only if $\succeq$ can be represented by a generalized Gini social evaluation function for profiles of opportunity sets.*

Proof. The necessity of the axioms is straightforward to verify, so I only establish the sufficiency part of the theorem. With the following modifications, the proof of Theorem 3 in Kranich [13] can be used to show that if $\succeq$ satisfies the axioms of Theorem 3, then there exists an $\mathbf{a} \in \mathbb{R}^n$ such that $\succeq$ can be represented by the function $W_{\mathbf{a}}$ defined in (10). First, Kranich's Lemma 3 is modified by replacing his hypotheses with ORD, ANON, and $\mathbf{R}^{\#}$-IRPE, with my Lemma 3 used instead of his Lemma 2 to show that his equation (2) holds. Second, this variant form of Kranich's Lemma 3 is used instead of his Lemma 3 in the rest of the proof. Third, IRPE is not assumed directly, but instead follows from $\mathbf{R}^{\#}$-IRPE.

EMON implies that $\sum_{i=1}^{n} a_i > 0$. Without loss of generality, this sum can be set equal to 1. GPDT implies that $a_1 \geq \cdots \geq a_n$. □

8. Concluding remarks

I have only considered comparisons of opportunity profiles for a fixed population. A possible extension of my analysis would be to compare the opportunities available to societies with different numbers of individuals. This would make it possible to rank opportunity profiles for different geographic regions, different demographic groups, and for different time periods. Assuming that the identities of individuals are not of social concern, the domain of opportunity profiles would now be $\mathbb{L}^* = \bigcup_{n\in\mathbb{N}} \mathbb{L}^n$. Social preference relations and equality of opportunity orderings would then be defined on $\mathbb{L}^*$. The axioms I have considered could be used when comparing distributions for equal-sized populations.

The class of single-series Ginis introduced by Donaldson and Weymark [7] to measure income inequality when the population size is variable is a class of generalized Gini indices that contains the class of single-parameter Ginis. Bossert [5] and Donaldson and Weymark [7] have characterized the single-series and single-parameter Ginis, respectively. Their axiomatizations make use of the *principle of population*, which requires inequality to be invariant to replications of the income distribution. It would be of interest to use the variable-population framework described above to axiomatize the analogues for profiles of opportunity sets of the single-series and single-parameter Ginis. It is not possible to simply reinterpret the results of Bossert, Donaldson, and Weymark in terms of opportunity orderings because the cardinality of an opportunity set can only take on integer values and one of the axioms Bossert uses in his axiomatization of the single-series Ginis makes essential use of the fact that incomes are divisible.[19]

As Kranich [13] and others have noted, the cardinality of an opportunity set is a rather crude way of measuring the extent of an individual's opportunities. Some of the other axiomatic analyses of opportunity inequality cited in the Introduction use alternative metrics to rank individual opportunity sets. My Generalized Pigou-Dalton Transfer Principle is compatible with the use of any metric whose underlying ordering is an extension of the set inclusion relation. Thus, some of the shortcomings of the cardinality-based approach to measuring opportunities can be alleviated by using one of these metrics without giving up the distributional sensitivity provided by the Generalized

[19] Donaldson and Weymark restricted attention to the class of single-series Ginis, so Bossert's axiom is needed for a complete characterization of the single-parameter Ginis.

Pigou-Dalton Transfer Principle.

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