THE COARSE BAUM-CONNES CONJECTURE AND CONTROLLED OPERATOR K-THEORY

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Chapter 1

Introduction

An elliptic operator D on a compact manifold is a Fredholm operator, in the sense that it is invertible modulo compact operators. The Fredholm index, defined by $\mathrm{ind}D = \ker D - \mathrm{coker}D$, is a K-theory element of the algebra of all compact operators, which is a homotopy invariant and an obstruction of the invertibility of D. Atiyah-Singer index theorem computes the Fredholm index. The index often relate to the geometry and topology of the manifold. For example, the index of the Dirac operator on a spin manifold is an obstruction of the existence of positive scalar curvatures.

Elliptic operators on noncompact manifolds, however, are no longer Fredholm in the classical sense. A Dirac type operator on a complete Riemannian manifold is invertible modulo the Roe algebra (which only depends on the large-scale structure of the manifold). Hence the index lives in the K-theory of the Roe algebra. In particular, for a compact manifold, the Roe algebra is the algebra of compact operators and the index is the classical Fredholm index. This generalized index allows the Atiyah-Singer index theorem and its application to be extended to noncompact manifolds.

The coarse Baum-Connes conjecture is an algorithm to compute the index of elliptic operators on noncompact manifolds. The coarse Novikov conjecture is an algorithm of determining non-vanishing of the index. These conjectures have applications in topology and geometry, in particular to the Novikov conjecture on homotopy invariance of higher signatures, and the Gromov-Lawson conjecture on existence of positive scalar curvatures. The coarse Baum-Connes conjecture has been proved for a large class of spaces, including spaces with finite asymptotic dimensions [Y98], and more general, spaces which admit a uniform embedding into Hilbert space [Y00].

The technique used in [Y98] is a "controlled" version of Mayer-Vietoris argument. In algebraic topology, the Mayer-Vietoris sequence is an tool to compute (co)homology groups. We decompose a space into two subspaces, for which the (co)homology groups are easier to compute. The Mayer-Vietoris sequence relates the (co)homology groups of the whole space with the (co)homology groups of these subspaces and their intersection. The K-theory of Roe algebra is a large-scale "generalized" homology theory for metric spaces. It is hoped that a similar Mayer-Vietoris sequence will enable us to compute it. The difficulty is that a K-theory element for Roe algebra does not necessarily have finite propagation. But we do need finite propagations for our Mayer-Vietoris argument. As a tradeoff for controlling propagation, we have to approximate K-theory elements by quasi-projections and quasi-unitaries, and to develop results parallel to classical operator K-theory in terms of quasi-projections and quasi-unitaries, especially to establish a Mayer-Vietoris sequence. Thanks to the finite asymptotic dimension condition, we only need to decompose the space a finite number of times (which only depends on the asymptotic dimension).

In [GTY2], E. Guentner, R. Tessera and G. Yu introduced a notion of large-scale invariants, finite decomposition complexity by name, which is a generalization of the concept of finite asymptotic dimension. Roughly speaking, a metric space has finite decomposition complexity when there is an algorithm to decompose a space into nice pieces in an asymptotic way. Guentner, Tessera and Yu proved the stable Borel conjecture for every closed aspherical manifold whose fundamental group has finite decomposition complexity by controlled Mayer-Vietoris sequences in algebraic K-theory and L-theory, and suggest that coarse Baum-Connes conjecture can be proved for spaces with finite decomposition complexity by a similar Mayer-Vietoris sequence in controlled operator K-theory. In this paper, we give a detailed proof.

In [Yu10], G. Yu suggested a way to use controlled K-theory to study the elements in the image of the Baum-Connes map. Roughly speaking, for a finitely generated torsion-free group, an element is in the image of the Baum-Connes map if and only if it is equivalent to a quasi-projection(unitary) such that each of its entries is a linear combination of elements in the generating set. In this paper, we explore this method and give an interesting application.

In Chapter 2, we start with some basic techniques for computing K-theory, and use it to study the K-theory of Roe algebra. In Chapter 3, we present the formulation of coarse Baum-Connes conjecture and its applications to geometry and topology. In Chapter 4, we study the localization algebra introduced in [Y97], whose K-theory provides an alternative model for the K-homology of metric spaces. In Chapter 5, we present a detailed discussion of controlled operator K-theory and the proof of of coarse Baum-Connes conjecture for spaces with finite asymptotic dimension given in [Y98]. In Chapter 6, we prove the coarse Baum-Connes conjecture for spaces with finite decomposition complexity . In the end, we study the Baum-Connes conjecture and give a characterization of elements in the Baum-Connes map.

Chapter 2

K-theory for Roe Algebras

In this chapter, we start with two fundamental techniques for calculating K-theory group, namely the Mayer-Vietoris sequence and the Eilenberg swindle. We proceed with the study of K-theory of Roe algebras, and establish a coarse Mayer-Vietoris sequence to compute it.

Section 2.1 Mayer-Vietoris Sequences

Theorem 2.1. *If J_0 and J_1 are ideals in C^*-algebra A, with $J_0 + J_1 = A$, then there is a six-term exact sequence*

$$\begin{tikzcd}
K_1(J_0 \cap J_1) \arrow[r] & K_1(J_0) \oplus K_1(J_1) \arrow[r] & K_1(A) \arrow[d] \\
K_0(A) \arrow[u] & K_0(J_0) \oplus K_0(J_1) \arrow[l] & K_0(J_0 \cap J_1) \arrow[l]
\end{tikzcd}$$

Proof. In the following commutative diagram, we have two short exact sequences, where the vertical maps are inclusions, and the third one is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & J_1 \cap J_2 \arrow[r, "i_2"] \arrow[d, "i_1"'] & J_2 \arrow[r] \arrow[d, "j_2"'] & J_2/(J_1 \cap J_2) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & J_1 \arrow[r, "j_2"] & A \arrow[r] & A/J_1 \arrow[r] & 0
\end{tikzcd}$$

So we have the following commutative diagram in K-theory, where $p \in \{0, 1\} \cong \mathbb{Z}/2\mathbb{Z}$, the two horizontal six-term sequences are exact

$$\begin{tikzcd}
{} \arrow[r] & K_p(J_1 \cap J_2) \arrow[r, "i_{2*}"] \arrow[d, "i_{1*}"'] & K_p(J_2) \arrow[r] \arrow[d, "j_{2*}"'] & K_p(J_2/(J_1 \cap J_2)) \arrow[r] \arrow[d, "\cong"'] & K_{p-1}(J_1 \cap J_2) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & K_p(J_1) \arrow[r, "j_{2*}"] & K_p(A) \arrow[r] & K_p(A/J_1) \arrow[r] & K_{p-1}(J_1) \arrow[r] & {}
\end{tikzcd}$$

We define the $\partial_p : K_p(A) \to K_{p-1}(J_1 \cap J_2)$ by the composition of maps

$$K_p(A) \to K_p(A/J_1) \cong K_p(J_2/(J_1 \cap J_2)) \to K_{p-1}(J_1 \cap J_2)$$

The Mayer-Vietoris sequence follows easily from diagram chasing. $\square$

Lemma 2.2. *Let I and J be ideals in a C^*-algebra A. Then*

(1) *$I + J$ is closed;*

(2) *$IJ = I \cap J$.*

Proof. (1) Since $(I+J)/J \cong I/(I\cap J)$, the latter is closed.

(2) By functional calculus, every positive element in $I\cap J$ is a product of two elements in $I\cap J$. $\square$

Section 2.2 Inner Automorphisms

Let A be a unital C^*-algebra, and $u \in A$ be a unitary, then $\mathrm{Ad}_u(a) = uau^*$ defines an automorphism of A, and it is immediate from definition that this inner automorphism acts trivially on $K_0(A)$. In this section we will prove various generalization of this statement. We allow u not in A, but in a C^*-algebra containing A. In the examples of great interest to us, we take the C^*-algebra B to be the multiplier algebra of A.

Definition 2.3. *Assume that A sits as a C^*-subalgebra of $B(H)$ with nondegenerate action. An element $x \in B(H)$ is called a multiplier for A if $xA \subset A$ and $Ax \subset A$, the set of all these is a C^*-algebra called the multiplier algebra of A.*

Lemma 2.4. *Suppose that A is any C^*-algebra and that u is a unitary in the multiplier algebra $M(A)$ of A. Then Ad_u induces the identity on $K_p(A)$ for all p.*

Proof. We form a C^*-algebra $D(A) = \{m_1 \oplus m_2 \in M(A)\oplus M(A) : m_1 - m_2 \in A\}$. In the following a split short exact sequence

$$0 \longrightarrow A \xrightarrow{i} D(A) \underset{s}{\overset{q}{\rightleftarrows}} M(A) \longrightarrow 0$$

i and s are given by $i : a \to a \oplus 0$, $s : m \to m \oplus m$ and $q : m_1 \oplus m_2 \to m_2$. So we have short exact sequences in K-theory

$$0 \longrightarrow K_p(A) \xrightarrow{i_*} K_p(D(A)) \longrightarrow K_p(M(A)) \longrightarrow 0\ .$$

Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{Ad}_u"'] & D \arrow[d, "\mathrm{Ad}_w"] \\ A \arrow[r] & D \end{tikzcd}$$

where $w = u \oplus u \in D$ is a unitary. Since horizontal maps induces injections on K_p, and since Ad_w induces the identity on $K_p(D)$, we see that Ad_v induces identities on $K_p(A)$. $\square$

Let v be an isometry in the multiplier algebra $M(A)$ of A, then $\mathrm{Ad}_v(a) = vav^*$ defines an endomorphism of A. In fact, it induces the identity map on K-theory. We will prove a more general result.

Proposition 2.5. *Let $\varphi : A \to B$ be a homomorphism of C^*-algebra and let w be a partial isometry in the multiplier algebra $M(B)$ of B, such that*

$$\varphi(a)w^*w = \varphi(a) \tag{2.1}$$

for all $a \in A$. Then $(\mathrm{Ad}_w \circ \varphi)(a) = w\varphi(a)w^*$ is a $*$-homomorphism from A to B. Passing to the induced map on K-theory we have that

$$(\mathrm{Ad}_w \circ \varphi)_* = \varphi_* : K_p(A) \to K_p(B).$$

Proof. Let $j : B \to M_2(B)$ be the left-top corner inclusion. It induces identity maps on K-theory. In fact, $M_n(M_2(B)) \cong M_{2n}(B)$, each K-theory element of $M_2(B)$ can be viewed as a K-theory element of B, this map is the two-sided inverse of j_*.

Let $u = \left(\begin{smallmatrix} w & 1-ww^* \\ 1-w^*w & w^* \end{smallmatrix}\right)$. Notice u is a unitary in $M_2(M(B))$. By lemma 2.4, it induces identity map on K-theory. By given condition $\varphi(a)w^*w = \varphi(a)$ for all $a \in A$, it is easy to check that $(j \circ \mathrm{Ad}_w \circ \varphi)(a) = \mathrm{Ad}_u \circ j \circ \varphi$, i.e.

$$\begin{tikzcd}
\varphi(a) \arrow[r, "j"] \arrow[d, "\mathrm{Ad}_w"'] & \left(\begin{smallmatrix} w\varphi(a)w^* & 0 \\ 0 & 0 \end{smallmatrix}\right) \arrow[d, "\mathrm{Ad}_u"] \\
w\varphi(a)w^* \arrow[r, "j"] & \left(\begin{smallmatrix} w\varphi(a)w^* & 0 \\ 0 & 0 \end{smallmatrix}\right)
\end{tikzcd}$$

So $j_* \circ (\mathrm{Ad}_w \circ \varphi)_* = (j \circ \mathrm{Ad}_w \circ \varphi)_* = (\mathrm{Ad}_u \circ j \circ \varphi)_* = \mathrm{Ad}_{u*} \circ j_* \circ \varphi_* = \mathrm{id} \circ j_* \circ \varphi_* = j_* \circ \varphi_*$. Since j_* is an isomorphism, we conclude that $(\mathrm{Ad}_w \circ \varphi)_* = \varphi_*$. ☐

Corollary 2.6. *If v is an isometry in the multiplier algebra $M(A)$ of A then the endomorphism $\mathrm{Ad}_v(a) = vav^*$ induces identity maps on K-theory.*

Lemma 2.7. *Suppose that B is a unital C^*-algebra, and that A is a C^*-subalgebra of B. If $p \in A$ is a projection, $v \in B$, $vp, vpv^* \in A$, and $pv^*v = p$, then vpv^* is also a projection and $[vpv^*] = [p] \in K_0(A)$.*

Proof. Consider the following continuous path of map $V_t : A \to A \oplus A$, $t \in [0, 1]$

$$V_t = \begin{pmatrix} \cos\frac{\pi}{2}t & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{2}t & \sin\frac{\pi}{2}t \\ -\sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} v\cos^2\frac{\pi}{2}t + I\sin^2\frac{\pi}{2}t \\ (v - I)\cos\frac{\pi}{2}t\sin\frac{\pi}{2}t \end{pmatrix},$$

where I is the unit in B.

Since $vp, vpv^* \in A$ and $pv^*v = p$, we have that $\mathrm{Ad}_{V_t}(p)$ is a projection in A for all $t \in [0, 1]$. As

$$\mathrm{Ad}_{V_0}(p) = \begin{pmatrix} vpv^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{Ad}_{V_1}(p) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix},$$

we see that $\left(\begin{smallmatrix} vpv^* & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$ represent the same class in $K_0(M_2(A))$. It is thus clear that $[vpv^*] = [p] \in K_0(A)$ since the left-top corner inclusion induces isomorphism $K_0(A) \to K_0(M_n(A))$. ☐

Example 2.8. *If H is an infinite-dimensional Hilbert space then $K_p(B(H)) = 0$ for all p.*

Proof. Let $H' = H \oplus H \oplus H \cdots$ be the direct sum of infinitely many copies of H. Let $V_1 : H \to H'$ be the isometry $v \to (v, 0, 0, \ldots)$. By Corollary 2.6, $\alpha_1 = \mathrm{Ad}_{V_1}$ induces an isomorphism on K-theory, and Ad_{V_2} induces the two-side inverse of α_{1*} for every isometry $V_2 : H' \to H$.

Let α_2 be the homomorphism $B(H) \to B(H')$ given by $T \to 0 \oplus T \oplus T \oplus \cdots$.

Let V_3 be the isometry $H' \to H'$ given by $(v_1, v_2, v_3, \ldots) \to (0, v_1, v_2, v_3, \ldots)$.

Clearly, $\alpha_1 + \alpha_2$ is also a C^*-homomorphism, and $\alpha_2 = \mathrm{Ad}_{V_3} \circ (\alpha_1 + \alpha_2)$. By Corollary 2.6, we have that $\alpha_{1*} + \alpha_{2*} = \alpha_{2*}$; hence $\alpha_{1*} = 0$. But α_{1*} is an isomorphism, so $K_p(B(H)) = 0$. $\square$

This type of argument, which comes down to deduce 1=0 from $1 + \infty = \infty$ is called Eilenberg swindle. It will be used a number of times later.

Sometimes, it would be more convenient to represent K-theory elements by unitaries by identifying $K_p = K_1(S^{p-1}A)$. Notice that Ad_v need not to be unital; by definition, the induced K-theory map is defined to be the induced unitalized map

$$\mathrm{Ad}_{v*} = (\mathrm{Ad}_v^+)_* : K_1((S^{p-1}A)^+) \to K_1((S^{p-1}A)^+),$$

The action of Ad_v^+ on $u = u' + \lambda I$ is defined by

$$\mathrm{Ad}_v^+(u) = vu'v^* + \lambda I = vuv^* + \lambda(1 - vv^*),$$

where I is the adjoint unit, $\lambda \in \mathbb{C}$, $u' \in S^{p-1}A$.

The counterpart of Equation 2.1 in Proposition 2.5 would be

$$\varphi(u)(1 - w^*w) = 1 - w^*w, \tag{2.2}$$

and the counterpart of Lemma 2.7 is the following one.

Lemma 2.9. *Suppose that B is a C^*-algebra with unit I, and that A is a C^*-subalgebra of B, $I \notin A$. If*

(1) $u = u' + I \in A^+$, $u' \in A$;

(2) $v \in B$, $vu', u'v^*, vu'v^* \in A$, *and* $u'v^*v = u'$;

(3) $||1 - u^*u|| < \min\{1, 1/||v||^2\}$;

then $vu'v^* + I$ is invertible in A^+, and $[vu'v^* + I] = [u] \in K_1(A)$.

Proof. Take V_t as Lemma 2.7. Since vu', $u'v^*$, $vu'v^* \in A$, $u'v^*v = u'$. We see that $\mathrm{Ad}_{V_t}^+(u)$ is a continuous path in A. Since

$$||1 - \mathrm{Ad}_{V_t}^+(u)^*\mathrm{Ad}_{V_t}^+(u)|| = ||\mathrm{Ad}_{V_t}^+(1 - u^*u)|| = ||\mathrm{Ad}_{V_t}(1 - u^*u)|| \le \delta \max\{||v||, 1\}^2 < 1,$$

$\mathrm{Ad}_{V_t}^+(u)^*\mathrm{Ad}_{V_t}^+(u)$ is invertible; hence $\mathrm{Ad}_{V_t}^+(u)$ is invertible. Thus $\mathrm{Ad}_{V_0}^+(u)$ is a continuous path of invertibles in $M_2(A)$ connecting $\mathrm{Ad}_{V_0}^+(u) = \left(\begin{smallmatrix} vu'v^*+1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathrm{Ad}_{V_1}^+(u) = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, $[vu'v^* + 1] = [u]$ in $K_1(A)$. $\square$

From now on, when we talk about the adjoint action on projections, we mean Ad_v; when we talk about the adjoint action on unitaries, we mean the unitalized action Ad_v^+.

Section 2.3 Roe Algebras

In this section we will construct a C^*-algebra, Roe algebra by name, which reflects the large scale property of a metric space.

Definition 2.10. *We say a metric space X is proper if every closed ball in X is closed.*

It follows immediately from the definition that a subset of a proper metric space is compact if and only if it is closed and bounded. For every $r > 0$, X can be covered by a countable collection of subsets whose diameters are smaller than ε.

Definition 2.11. *A Borel map f from a proper metric space X to another metric space Y is called coarse if*

(1) *for any $s > 0$, there exists $r > 0$ such that for any $x_1, x_2 \in X$ and $d_X(x_1, x_2) < s$, we have $d_Y(f(x_1), f(x_2)) < r$;*

(2) *(Properness) for any $R > 0$, there exists $S > 0$ such that for any $x_1, x_2 \in X$ and $d_Y(f(x_1), f(x_2)) < R$, we have$d_X(x_1, x_2) < S$.*

Definition 2.12. *Let X be a metric space and let S be any set. Two maps $\varphi_1, \varphi_2 : S \to X$ are close if*

$$\sup_{s \in S} d(\varphi_1(s), \varphi_2(s)) < \infty.$$

Definition 2.13. *Let X, Y be proper metric spaces. X, Y are called coarsely equivalent if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is close to id_Y and $g \circ f$ is said to be close to id_X. f, g are called coarse equivalence.*

Definition 2.14. *Let X be a proper metric space. A separable Hilbert space H_X is called an X-module if there is given a representation of the C^*-algebra $C_0(X)$ on $B(H_X)$.*

It follows from the Spectral Theorem that the given representation of the continuous functions $C_0(X)$ can be canonically extended to a representation of the bounded Borel functions.

Definition 2.15. *An X-module H_X said to be nondegenerate (respectively, ample or very ample), if the representation $C_0(X) \to B(H_X)$ is nondegenerate (respectively, ample or very ample).*

Lemma 2.16. *Let H_X be an nondegenerate (respectively, ample or very ample) X-module. Let $Z \subset X$, and the interior point of Z is dense in Z. Denote H_Z be the range of the projection operator corresponding to the characteristic function of Z under Borel functional calculus. The natural representation of $C_0(Z)$ on H_Z is nondegenerate (respectively, ample or very ample).*

Proof. (i) It is easily seen that $\overline{C_0(Z)H_Z} \subset \overline{C_0(Z)}^W H_Z \subset \chi_Z H_Z = H_Z$, where the second closure is taken with respect to the weak operator topology. So H_Z is nondegenerate.

(ii) If $f \in C_0(Z)$, f acts as a compact operator on H_Z. Let z be an interior point of z, take open sets U, V of X such that $z \in \overline{U} \subset V \subset Z$. Take a continuous function g satisfies $g_{|\overline{U}} = 1$, $g_{|X \setminus V} = 0$. Then $gf \in C_0(X)$, gf acts as a compact operator on H_X. So $gf = 0$, hence $f(z) = 0$. So $f = 0$ at any interior point of Z. Therefore $f = 0$ on Z. ∎

Definition 2.17. *Let $v \in H_X$. The support of v is the complement, in X, of the union of all open subsets $U \subset X$ such that $fv = 0$ for all $f \in C_0(U)$.*

Definition 2.18. *Let $T : H_X \to H_Y$ be a bounded operator. the support of T is the complement, in $Y \times X$, of the union of all open sets $U \times V \subset Y \times X$ such that $fTg = 0$, for all $f \in C_0(U)$ and $g \in C_0(V)$.*

Definition 2.19. *For subsets $A \subset Y \times X$ and $B \subset X$, $C \subset Z \times Y$, denote $A \circ B$ the subset*

$$\{y \in Y : \exists, x \in X \text{ such that } (y, x) \in A \text{ and } x \in B\}.$$

denote $C \circ A$ the subset

$$\{(z, x) \in Z \times X : \exists, y \in Y \text{ such that } (z, y) \in C \text{ and } (y, x) \in A\}.$$

To compute support, we have the following useful lemmas.

Lemma 2.20. *For a bounded operator $T : H_X \to H_Y$, we have*

$$\mathrm{Support}(Tv) \subset \mathrm{Support}(T) \circ \mathrm{Support}(v)$$

for every compactly supported $v \in H$. Moreover, $\mathrm{Support}(T)$ *is the smallest closed subset of $Y \times X$ that has this property.*

Proof. Suppose $y \notin \mathrm{T} \circ \mathrm{Support}(v)$, we have $\{x : (y, x) \in \mathrm{support}(T)\} \cap \mathrm{support}(v) = \varnothing$. Take a bounded open set U containing $\mathrm{Support}(v)$ and $\overline{U} \cap \{x : (y, x) \in \mathrm{Support}(T)\} = \varnothing$. Take $g \in C_0(U)$ such that $g_{|\mathrm{Support}(v)} = 1$. So $(1 - g) \in C_0(X \setminus \mathrm{Support}(v))$, hence $(1 - g)v = 0$.

For any $x \in \overline{U}$, $(y, x) \notin \mathrm{Support}(T)$. So there exists open sets W_y, V_x, $W_y \times V_x \subset Y \times X$ such that $C_0(W_y)TC_0(V_x) = 0$. By compactness of $\overline{U}$, we can find open set $W \subset Y$, such that $y \in W$, $C_0(W)TC_0(V) = 0$. So for all $f \in C_0(W)$, $fTg = 0$. Hence

$$fTv = (fTg)v + fT((1 - g)v) = 0.$$

So $w \notin \mathrm{Support}(Tv)$.

For the second part, if $(y, x) \in \mathrm{Support}(T)$, i.e., for every n, there exists $g_n \subset C_0\left(B(y, \frac{1}{n})\right)$, $f_n \in C_0\left(B(x, \frac{1}{n})\right)$, such that $g_n T f_n \neq 0$. So there exists $v_n \in H_Y$, $u_n \in H_X$,

$$< \overline{g_n} v_n, T f_n u_n >=< v_n, g_n T f_n u_n >\neq 0.$$

Hence $\mathrm{Support}(g_n v_n) \cap \mathrm{Support}(T f_n u_n) \neq \varnothing$. Take $y_n \in \mathrm{Support}(f_n v) \cap \mathrm{Support}(T f_n u_n)$.

Let A be a closed subset of $Y \times X$ satisfying $\mathrm{Support}(Tv) \subset \mathrm{Support}(T) \circ \mathrm{Support}(v)$ for every compactly supported $v \in H$. Since $\mathrm{Support}(f_n u_n) \subset \overline{B(x, \frac{1}{n})}$, So

$$y_n \in \mathrm{Support}(T f_n u_n) \subset A \circ \mathrm{Support}(f_n u_n).$$

Hence there exists $x_n \in \text{Support}(f_n u_n)$, $(y_n, x_n) \in A$. Since

$$y_n \in \text{Support}(\overline{g_n} v_n) \subset \overline{B(y, \tfrac{1}{n})}, \quad x_n \in \text{Support}(f_n u_n) \subset \overline{B(x, \tfrac{1}{n})},$$

we have $(y_n, x_n) \to (y, x)$. By the closedness of A, we have $(y, x) \in A$. $\square$

Definition 2.21. *We shall say a bounded operator $T : H_X \to H_Y$ is properly supported. If the projection map from* $\text{Support}(T)$ *to X and Y are proper maps.*

Lemma 2.22. *If $T : H_X \to H_Y$ is properly supported, then for any compact supported $v \in H_X$, Tv is compactly supported in Y.*

If $S : H_Y \to H_X$ is another properly supported operator, then

$$\text{Support}(ST) \subset \text{Support}(S) \circ \text{Support}(T).$$

Proof. By Lemma 2.20, for any compactly supported $v \in H_X$,

$$\text{Support}(Tv) \subset \text{Support}(T) \circ \text{Support}(v).$$

So $\text{Support}(Tv) \subset \pi_Y(\pi_X^{-1}(\text{Support}(v)))$ is bounded.

Again by Lemma 2.20, we have

$$\text{Support}(STv) \subset \text{Support}(S) \circ \text{Support}(Tv) \subset \text{Support}(S) \circ \text{Support}(T) \circ \text{Support}(v).$$

To complete the proof, we only need to check $\text{Support}(S) \circ \text{Support}(T)$ is closed. In fact, if $\{(z_n, x_n)\} \subset \text{Support}(S) \circ \text{Support}(T)$, $(z_n, x_n) \to (z, x)$, there exists $\{y_n\} \subset Y$, such that $(z_n, y_n) \in \text{Support}(S)$, $(y_n, x_n) \in \text{Support}(T)$. Since $x_n \to x$, so $\{x_n\}$ is bounded. By properness of T, $\{y_n\}$ is also bounded, hence has convergent subsequence, denote its limit as y. So $(z, y) \in \text{Support}(S)$, $(y, x) \in \text{Support}(T)$. Hence $(z, x) \in \text{Support}(S) \circ \text{Support}(T)$. $\square$

Lemma 2.23. *If T is properly supported and S is locally compact , then (assuming the compositions make sense) the operators ST and TS are locally compact.*

Proof. We will show ST is locally compact, and the proof for TS is similar.

Let $T : H_X \to H_Y$ is properly supported, $S : H_Y \to H_Z$ is is locally compact. For any $f \in C_c(X)$. fST is compact since fS is compact. Since T is properly supported, by Lemma 2.22,

$$\text{Support}(Tf) \subset \text{Support}(T) \circ \text{Support}(f) \subset \text{Support}(T) \circ \text{supp}(f).$$

By properness of T, the set $\pi_X^{-1}(\text{Support}(f)) \supset \text{Support}(Tf)$ is bounded. So $Y_0 = \{y : \exists\, x \text{ such that } (y, x) \in \text{Support}(Tf)\}$ is bounded hence has compact support. Take $g \in C_c(Y)$ such that $g = 1$ on Y_0. We have $Tf = gTf$. So

$$STf = S(Tf) = S(gTf) = (Sg)Tf$$

is compact. $\square$

Definition 2.24. *The propagation of an bounded operator $T : H_X \to H_X$ is*

$$\text{Propagation}(T) = \sup\{d(x_1, x_2) : (x_1, x_2) \in \text{Support}(T)\}$$

Clearly, every finite propagation is properly supported. It follows from Lemma 2.22 and Lemma 2.23 that the set of locally compact finite propagation operators on H_X is a $*$-subalgebra of $B(H_X)$.

Definition 2.25. *For an X-module H, we define $C^*(X, H_X)$ to be the C^*-algebra obtained as the closure in $B(H)$ of locally compact finite propagation operator.*

Lemma 2.26. *If T is bounded operator on H_X with finite propagation, then T is a multiplier of $C^*(X, H_X)$.*

Proof. It follows immediate from Lemma 2.22 and Lemma 2.23. □

Definition 2.27. *Let $q : X \to Y$ be a coarse map. Let H_X and H_Y be non-degenerate X and Y-module. A bounded operator $V : H_X \to H_Y$ covers q if the maps π_Y and $q \circ \pi_X$, from the* Support$(V) \subset Y \times X$ *to Y, are close.*

Clearly, every such operator is properly supported.

To prove the existence of covering isometry we need a lemma of partition of space.

Lemma 2.28. *Let Y be a proper metric space. For every $\varepsilon > 0$, Y can be written as the disjoint union of countable collection of Borel subsets each having non-empty interior with diameter no more than ε.*

Proof. Since Y is proper, we can take countable open cover $\{U_n\}$ with diam$U_n \le \varepsilon$. Let $V_n = U_n \setminus (U_1 \cup \cdots \cup U_{n-1})$. Take all the n_i such that V_{n_i} has nonempty interior. Let $W_i = \overline{V_{n_i}} \setminus (\overline{V_{n_1}} \cup \cdots \cup \overline{V_{n_{i-1}}})$ We will show $\{W_i\}$ would be the desired decomposition.

(1) diam$W_i \le \varepsilon$.

Since $W_i \subset \overline{V_{n_i}} \subset \overline{U_{n_i}}$. So diam$W_i \le$ diam$\overline{U_{n_i}} \le \varepsilon$.

(2) W_i has nonempty interior.

Since $\{V_n\}$ are disjoint, V_{n_i} has nonempty interior, so interior$(V_{n_i}) \cap \overline{V_m} = \varnothing$. Hence interior$(V_{n_i}) \subset V_{n_i} \setminus (\overline{V_{n_1}} \cup \cdots \cup \overline{V_{n_{i-1}}}) = W_i$.

(3) $\{W_i\}$ covers Y.

We only need to show if V_k has empty interior, then $V_k \subset \bigcup_{n_i < k} \overline{V_{n_i}}$.

If $k = 1$, $V_1 = U_1$ is either empty or has nonempty interior, the claim is true.

Suppose the claim is true for all $k < l$. So

$$\bigcup_{k \le l-1} U_k = \bigcup_{k \le l-1} V_k \subset \bigcup_{n_i \le l-1} \overline{V_{n_i}}.$$

If V_l has empty interior, for every $y \in V_l$, there exists a sequence

$$\{y_n\} \subset \bigcup_{k \le l-1} U_k \subset \bigcup_{n_i \le l-1} \overline{V_{n_i}}$$

such that $\lim_{n\to\infty} y_n = y$. Hence there exists a subsequence $\{y_{n_j}\} \subset \overline{V_{n_i}}$ for some $n_i \leq l-1$. So $\lim_{j\to\infty} y_{n_j} \in \overline{V_{n_i}}$. The claim is true for $k = l$. $\square$

Lemma 2.29. *Let $f : X \to Y$ be a coarse map, H_X and H_Y be respectively ample X and Y-modules. For some $C > 0$, there exists an isometry V_f from H_X to H_Y such that*

$$\operatorname{support}(V_f) \subset \{(y, x) \in Y \times X : d(y, f(x)) \leq C\}$$

If we further assume f is uniformly continuous, then for any $\varepsilon > 0$, there exists an isometry V_f from H_X to H_Y such that

$$\operatorname{support}(V_f) \subset \{(y, x) \in Y \times X, d(y, f(x)) \leq \varepsilon\}$$

Proof. By lemma, we can find a disjoint Borel partition Y_n of Y, such that $\operatorname{diam}(Y_n) < \varepsilon/3$. Take an isometry $V_n : \chi_{f^{-1}(Y_n)}H_X \to \chi_{Y_n}H_Y$ for each n. The sum $\sum V_n\chi_{f^{-1}(Y_n)}$ converges in strong operator topology to an isometry $H_X \to H_Y$.

If $(y, x) \in \operatorname{Propagation}(V_n)$, then $x \in \overline{f^{-1}(Y_n)}$, $y \in \overline{Y_n}$. So there exists $x' \in f^{-1}(Y_n)$, $d(x, x') < 1$, $d(y, f(x')) < \operatorname{diam}(Y_n) + \varepsilon/3 = 2\varepsilon/3$. Since f is coarse, so there exists $C > 0$, such that $d(f(x_1), f(x_2)) < C - 2\varepsilon/3$, whenever $d(x_1, x_2) < 1$, so

$$d(y, f(x)) \leq d(y, f(x')) + d(f(x), f(x')) \leq \frac{2\varepsilon}{3} + C - \frac{2\varepsilon}{3}.$$

If we further assume f is uniformly continuous, we can find $\delta > 0$ such that $d(f(x_1), f(x_2)) < \varepsilon/3$ whenever $d(x_1, x_2) < \delta$. Now we can pick $x' \in f^{-1}(Y_n)$, $d(x, x') < \delta$. So

$$d(y, f(x)) \leq d(y, f(x')) + d(f(x), f(x')) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\square$

Lemma 2.30. *If an isometry $V : H_X \to H_Y$ covers a coarse map $q : X \to Y$, then Ad_V induces a homomorphism from $C^*(X, H_X)$ into $C^*(Y, H_Y)$.*

Proof. Let T be a locally compact, finite propagation operator on H_X. We will show VTV^* is also locally compact and has finite propagation. Since V and V^* are properly supported, by Lemma 2.23, VTV^* is locally compact.

If $(y_2, y_1) \in \operatorname{Propagation}(VTV)^*$, by Lemma 2.22, there exists x_2, x_1, such that $(y_2, x_2) \in \operatorname{Support}(V)$, $(x_2, x_1) \in \operatorname{Support}(T)$, $(x_1, y_1) \in \operatorname{Support}(V^*)$. So $d(y_2, f(x_2)) < C_1$, $d(x_2, x_1) < C_2$, $d(f(x_1), y_1) < C_1$ for some constant independent of x_1, x_2, y_1, y_2. Since f is coarse, there exists some C_3 such that $d(f(x), f(x')) < C_3$ whenever $d(x, x') < C_2$. Hence

$$d(y_2, y_1) \leq d(y_2, f(x_2)) + d(f(x_2), f(x_1)) + d(f(x_1), y_1) \leq C_1 + C_3 + C_1$$

So $\operatorname{Propagation}(VTV^*) < 2C_1 + C_3$. $\square$

Lemma 2.31. *Two isometries V_1 and V_2, both covering q, induce the same map on K-theory: $(\mathrm{Ad}_{V_1})_* = (\mathrm{Ad}_{V_2})_* : K_p(C^*(X, H_X)) \to K_p(C^*(Y, H_Y))$.*

Proof. Similar to the proof of Lemma 2.30, we can show $V_2V_1^*$ has finite propagation. Hence, by Lemma 2.26, $V_2V_1^*$ is a multiplier of $C^*(Y, H_Y)$. Since $\mathrm{Ad}_{V_1}(T)(V_2V_1^*)^*(V_2V_1^*) = \mathrm{Ad}_{V_1}(T)$ for all $T \in C^*(H, H_X)$, by Proposition 2.5, we conclude that

$$(\mathrm{Ad}_{V_2V_1^*} \circ \mathrm{Ad}_{V_1})_* = (\mathrm{Ad}_{V_1})_*$$

Hence $(\mathrm{Ad}_{V_2})_* = (\mathrm{Ad}_{V_1})_*$. $\square$

Corollary 2.32. *If an isometry $V : H_X \to H_Y$ covers a coarse equivalence $f : X \to Y$, then $(\mathrm{Ad}_V)_* : K_p(C^*(X, H_X)) \to K_p(C^*(Y, H_Y))$ is an isomorphism.*

Proof. Let $g : Y \to X$ satisfying that gf and fg are closed to id_X and id_Y respectively. Let W be an isometry that covers g. Then WV covers gf and hence id_X. Since id_{H_X} also covers id_X. We see that

$$\mathrm{id} = (\mathrm{Ad}_{\mathrm{id}_{H_X}})_* = (\mathrm{Ad}_{WV})_* = (\mathrm{Ad}_W)_* \circ (\mathrm{Ad}_V)_*$$

Similarly, we have that $(\mathrm{Ad}_W)_*$ is a right inverse of $(\mathrm{Ad}_V)_*$. Hence $(\mathrm{Ad}_V)_*$ is an isomorphism. $\square$

So the K-theory of $C^*(X, H_X)$ does not depend on the choice of nondegenerate X-module.

Definition 2.33. *If $q : X \to Y$ is a coarse map then we define*

$$q_* : K_p(C^*(X)) \to K_p(C^*(Y))$$

to be the map $(\mathrm{Ad}_{V_q})_*$, where $V_q : H_X \to H_Y$ is any isometry that covers q.

We can summarize what we have discussed so far in this section as the following proposition.

Proposition 2.34. *The correspondence $q \to q_*$ is a covariant functor from the category of proper metric spaces and coarse maps to the category of abelian groups and homomorphisms.*

Section 2.4 Mayer-Vietoris Sequence for K-theory of Roe algebras

In this section, we will formulate a Mayer-Vietoris sequence to compute $K_p(C^*(X))$ for certain metric spaces, including the Euclidean space $\mathbb{R}^n$.

Example 2.35. *Let $\mathbb{R}^+ = [0, +\infty)$ equip with the Euclidean metric. For all p we have*

$$K_p(C^*(\mathbb{R}^+)) = 0,$$

Proof. Let $C_0(\mathbb{R}^+)$ be represented on $H = L^2(\mathbb{R}^+)$ by multiplication operators. Clearly the representation is ample. Let H' be the direct sum of infinitely many copies of H with corresponding representations. Let V be the isometry $H \to H'$ given by $v \to (v, 0, 0, \ldots)$ which covers the identity map on $\mathbb{R}^+$. So the top corner inclusion

$$\alpha_1 = \mathrm{Ad}_V : T \to \mathrm{Ad}_V(T) = T \oplus 0 \oplus 0 \oplus \cdots$$

induces an isomorphism $\alpha_{1*} : K_p(C^*(\mathbb{R}^+, H)) \to K_p(C^*(\mathbb{R}^+, H'))$.

Let $\alpha_2 : C^*(\mathbb{R}^+, H) \to C^*(\mathbb{R}^+, H')$ given by

$$\alpha_2 : T \to 0 \oplus \mathrm{Ad}_U(T) \oplus \mathrm{Ad}_U^2(T) \oplus \cdots ,$$

where U is an isometry $H \to H$ given by

$$f(t) = \begin{cases} f(t-1) & \text{if } t \geq 1 \\ 0 & \text{if } 0 \leq t < 1. \end{cases}$$

We will show $\alpha_2(T) \in C^*(X, H_X)$.

Let $T \in C^*(\mathbb{R}^+, H)$ is locally compact with finite propagation. Notice Ad_U just translates the support of T, so does nothing to the propagation. Since the propagation of the direct sum of operators is just the supremum of each summand, hence $\alpha_2(T)$ has finite propagation.

We next consider the locally compactness. For any $f \in C_c(\mathbb{R}^+)$, suppose $\mathrm{supp} f \subset [0, N]$, then $f\mathrm{Ad}_U^m(T) = 0$ whenever $m > N$. So only finitely many summands of $\alpha_2(T) = 0 \oplus f\mathrm{Ad}_U(T) \oplus f\mathrm{Ad}_U^2(T) \oplus \cdots$ are nonzero.

Clearly U covers identity map on $\mathbb{R}^+$, so is U^n. Thus, by Lemma 2.30, $\mathrm{Ad}_U^n(T) = \mathrm{Ad}_{U^n}(T)$ is locally compact. So each summand in the above sum is compact. Hence the sum is compact.

Let W be the isometry $H' \to H'$ given by $(v_1, v_2, v_3, \ldots) \to (0, v_1, v_2, v_3, \ldots)$. W covers identity map on $\mathbb{R}^+$. Since $U^\infty = U \oplus U \oplus U \oplus \cdots : H' \to H'$ also covers identity map on $\mathbb{R}^+$, so is $U^\infty W$. Hence $\mathrm{Ad}_{U^\infty W}$ induces identity map on $K_p(C^*(\mathbb{R}^+, H'))$.

Since $\alpha_2 = \mathrm{Ad}_{U^\infty W}(\alpha_1 + \alpha_2)$. So $\alpha_{2*} = \alpha_{1*} + \alpha_{2*}$. Hence $\alpha_{1*} = 0$. Since we have shown that α_{1*} is an isomorphism, so $K_p(C^*(\mathbb{R}^+, H)) = 0$. □

By an elaborated argument, we can prove the following

Proposition 2.36. *Let Y be a proper metric space and let $X = \mathbb{R}^+ \times Y$ equipped with the product metric*

$$d((x_1, y_1), (x_2, y_2))^2 = |x_1 - x_2|^2 + d(y_1, y_2)^2$$

Then $K_p(C^*(X)) = 0$ for all p.

Proof. We will take $H = L^2(\mathbb{R}^+) \otimes H_Y$, $H' = H \oplus H \oplus H \oplus \cdots$. We would view element in H be square integrable function with valued in H, i.e.,

$$\int_0^\infty < f(t), f(t) >_{H_Y} \mathrm{d}t < \infty$$

The argument for $\mathbb{R}^+$ still works. $\square$

Let X be a proper metric space and $Y \subset X$ a closed subspace, then Y is also a proper metric space. For each $c \in \mathbb{R}^+$, let Y_c denote the closure of $\{x \in X : d(x, Y) < c\}$. We note that the inclusion map $Y \subset Y_c$ is a coarse equivalence, and that Y_c is the closure of its interior.

Definition 2.37. *A subset $S \subset X \times X$ is near Y if it is contained in $Y_n \times Y_n$ for some $n \in \mathbb{N}$. A finite propagation operator is near Y if its support is near Y.*

The set of operators near Y form an ideal of in the algebra of all finite propagation operators, and similarly the set of locally compact operators near Y form an ideal in the algebra of all locally compact finite propagation operators.

Definition 2.38. *Let Y be a closed subset of a proper metric space X. The ideal $C^*(Y; X)$ of $C^*(X)$ is by definition the norm closure of the set of all locally compact finite propagation operators near Y.*

Proposition 2.39. *There is an isomorphism*

$$K_p(C^*(Y; Z)) \cong K_p(C^*(Y))$$

between the K-theory of the ideal $C^*(Y; Z)$ and the K-theory of the C^*-algebra associated to Y as a coarse space in its own right.

Proof. Since $Y_n \subset X$, Y_n is the closure of its interior. So by lemma 2.16, H_{Y_n} is ample. $C^*(Y_n, H_{Y_n})$ can be viewed as the C^*-subalgebra of $C^*(X, H_X)$. We get an increasing sequence C^*-algebras

$$C^*(Y_1, H_{Y_1}) \hookrightarrow C^*(Y_2, H_{Y_n}) \hookrightarrow \cdots$$

whose union is dense in $C^*(Y; X)$. Since K-theory preserve direct limit, we have

$$\varinjlim_n K_p(C^*(Y_n, H_{Y_n})) \cong C^*(Y; X).$$

Notice the inclusion map $i_n : Y_n \to Y_{n+1}$ is a coarse equivalence between Y_n and Y_{n+1}. The inclusion maps $V_n : H_{Y_n} \subset H_{Y_{n+1}}$ is an isometry covering i_n. So by Corollary 2.32, $i_{n*} : K_p(C^*(Y_n, H_{Y_n})) \xrightarrow{\cong} K_p(C^*(Y_{n+1}, H_{Y_{n+1}}))$ is an isomorphism. Hence

$$K_p(C^*(Y_1, H_{Y_1})) \cong \varinjlim K_p(C^*(Y_n, H_{Y_n})) \cong K_p(C^*(Y; X)).$$

Since Y_1 and Y are coarse equivalent, by Corollary 2.32,

$$K_p(C^*(Y)) \cong K_p(C^*(Y_1, H_{Y_1})).$$

Hence

$$K_p(C^*(Y;Z)) \cong \varinjlim_n K_p(C^*(Y_n, H_{Y_n})) \cong K_p(Y_1, H_{Y_1}) \cong K_p(C^*(Y)).$$

□

Lemma 2.40. *Suppose now that X is a proper metric space which is written as a union $X = Y \cup Z$ of two closed subspaces. Then $C^*(Y;X) + C^*(Z;X) = C^*(X)$.*

Proof. Let T be a locally compact finite propagation operator on H_X. Let P be the projection operator corresponding to the characteristic function of Y. Then $T = PT + (I-P)T$, $PT \in C^*(Y;X)$, $(I-P)T \in C^*(Z;X)$. So $C^*(Y;X) + C^*(Z;X)$ is dense in $C^*(X)$. By Lemma 2.2, we get the desired result. □

It is clear that $C^*(Y \cap Z; X) \subset C^*(Y;X) \cap C^*(Z;X)$, but equality does not hold in general. It does hold, however, in several important cases.

Definition 2.41. *We say the decomposition $X = Y \cup Z$ is coarsely excisive, if for every m, there exists some n such that $Y_m \cap Z_m \subset (Y \cap Z)_n$.*

Lemma 2.42. *If the decomposition $X = Y \cup Z$ is coarsely excisive, then $C^*(Y \cap Z; X) = C^*(Y;X) \cap C^*(Z;X)$.*

Proof. By Lemma 2.2, we only need to prove if T and S are locally compact, finite propagation operators supported on $Y_m \times Y_m$ and $Z_n \times Z_n$ respectively, then $TS \in C^*(Y \cap Z; X)$. Since the decomposition is coarsely excisive, we can take k such that $Y_m \cap Z_n \subset (Y \cap Z)_k$. Then

$$\mathrm{Support}(TS) \subset (Y \cap Z)_{k+l} \times (Y \cap Z)_{k+l},$$

where $l = \max\{\mathrm{Propagation}(T), \mathrm{Propagation}(S)\}$.

□

Theorem 2.43. *Given a coarsely excisive decomposition $X = Y \cup Z$, we have the following Mayer-Vietoris Sequence.*

$$\begin{array}{ccccc} K_1(C^*(Y\cap Z)) & \longrightarrow & K_1(C^*(Y)) \oplus K_1(C^*(Z)) & \longrightarrow & K_1(C^*(X)) \\ \uparrow & & & & \downarrow \\ K_0(C^*(X)) & \longleftarrow & K_0(C^*(Y)) \oplus K_0(C^*(Z)) & \longleftarrow & K_0(C^*(Y\cap Z)) \end{array}$$

Proof. Notice $C^*(Y \cap Z; X)$, $C^*(Y;X)$, $C^*(Z;X)$ are ideals of $C^*(X)$. Since the decomposition is coarsely excisive, by Lemma 2.40 and Lemma 2.42, we have that

$$C^*(Y;X) + C^*(Z;X) = C^*(X), \quad C^*(Y;X) \cap C^*(Z;X) = C^*(Y \cap Z; X).$$

By Theorem 2.1, we have exact sequence

$$\begin{tikzcd}
K_1(C^*(Y \cap Z; X)) \arrow[r] & K_1(C^*(Y; X)) \oplus K_1(C^*(Z; X)) \arrow[r] & K_1(C^*(X)) \arrow[d] \\
K_0(C^*(X)) \arrow[u] & K_0(C^*(Y; X)) \oplus K_0(C^*(Z; X)) \arrow[l] & K_0(C^*(Y \cap Z; X)) \arrow[l]
\end{tikzcd}$$

By lemma 2.39, we get the desired result. ☐

Example 2.44. *Let $\mathbb{R}^n$ equipped with Euclidean metric, we have that*

$$K_p(C^*(\mathbb{R}^n)) = \begin{cases} \mathbb{Z} & \text{if } p \equiv n \pmod 2, \\ 0 & \text{if } p \equiv n+1 \pmod 2. \end{cases}$$

Proof. We will prove by induction. When $n = 0$, $C^*(\text{pt}) = K(H)$. The claim is true.

Suppose the claim is true for $n = k$. For the case $n = k+1$, $X = \mathbb{R}^{k+1}$, $Y = \mathbb{R}^+ \times \mathbb{R}^k$, $Z = \mathbb{R}^- \times \mathbb{R}^k$. $X = Y \cup Z$ is a coarse excisive decomposition. So we have six-term exact sequence in proposition 2.43

By Proposition 2.36, we have that $K_p(Y) = 0$, $K_p(Z) = 0$. Thus $K_p(C^*(\mathbb{R}^{k+1})) = K_{p+1}(C^*(\mathbb{R}^k))$. The claim is also true for $n = k + 1$. ☐

Chapter 3

The Coarse Baum-Connes Conjecture

In this chapter, we will review the Kasparov's K-homology [K75], Paschke Duality [P], and formulate the coarse Baum-Connes conjecture.

Section 3.1 K-homology

We use $T \sim T'$ to denote that two operators T and T' are equal up modulo compact operators.

Definition 3.1. *An ungraded Fredholm module over a separable C^*-algebra A is given by the following data:*

(1) *a separable space H,*

(2) *a representation $\rho : A \to B(H)$, and*

(3) *an operator F on H such that for all $a \in A$,*

$$(F^2 - 1)\rho(a) \sim 0, \quad (F - F^*)\rho(a) \sim 0, \quad F\rho(a) \sim \rho(a)F.$$

Definition 3.2. *An graded Fredholm module over a separable C^*-algebra A is given by the following data:*

(1) *a Hilbert space H with a direct sum decomposition $H = H^+ \oplus H^-$,*

(2) *for each $a \in A$, the operator $\rho(a)$ is even. Thus $\rho(a) = \rho^+(a) \oplus \rho^-(a)$, where $\rho^\pm$ are representations of A on $H^\pm$, and*

(3) *the operator F is odd. That is F has the form $\begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}$, where U is an operator from H^+ to H^- and V is an operator from $H^- \to H^+$.*

Definition 3.3. *Let (ρ, H, F) be a Fredholm module and let $U : H' \to H$ be a unitary isomorphism (preserving the grading, if there is one). Then $(U^*\rho U, H', U^*FU)$ is also a Fredholm module, and we say that it is unitary equivalent to (ρ, H, F).*

Definition 3.4. *Suppose that (ρ, H, F_t) is a family of Fredholm modules parameterized by $t \in [0, 1]$, in which the representation and the Hilbert space remain constant but the operator F_t varies with t. If the function $t \to F_t$ is norm continuous, then we say that the family defines an operator homotopy between the Fredholm modules (ρ, H, F_0) and (ρ, H, F_1), and that these two Fredholm modules are operator homotopic.*

Definition 3.5. *The Kasparov K-homology $K^0(A)$ (respectively $K^1(A)$) is the abelian group with one generator $[x]$ for each unitary equivalence class of graded (ungraded) Fredholm modules over A with the relations:*

(1) *if* x_0 *and* x_1 *are operator homotopic p-multigraded Fredholm modules then* $[x_0] = [x_1]$ *in* $K^p(A)$, *and*

(2) *if* x_0 *and* x_1 *are any two p-multigraded Fredholm modules then* $[x_0 \oplus x_1] = [x_0] + [x_1]$ *in* $K^p(A)$.

If A is the commutative C^*-algebra $C_0(X)$, we $K^p(C_0(X))$ may be written as $K_p(X)$.

Definition 3.6. *A Fredholm module* (ρ, H, F) *is degenerate if* $\rho(a)F = \rho(a)F^*$, $\rho(a)F^2 = \rho(a)$, *and* $[F, \rho(a)] = 0$ *for all* $a \in A$.

Lemma 3.7. *The class in* $K^p(A)$ *defined by a degenerate Fredholm module is zero.*

Proof. Let $x = (\rho, H, F)$ be a degenerate Fredholm module. We form a new Fredholm module $x' = (\rho', H', F')$, where H' is direct sum of infinitely many copies of H, similar ρ' and F' are infinite direct sum of copies of ρ and F respectively. Clearly $x \oplus x'$ is unitarily equivalent to x', so we have $[x] + [x'] = [x']$ in K-homology. Hence $[x] = 0$. ☐

For a Hilbert space H, let H^{op} denote H with the opposite grading (if it has one). Notice that the identity map $I : H \to H^{\mathrm{op}}$ then becomes an odd unitary isomorphism. If $T \in B(H)$, we shall use the notation T^{op} for the same operator considered as an element of $B(H^{\mathrm{op}})$.

Lemma 3.8. *The additive inverse in* $K^p(A)$ *of K-homology class defined by a Fredholm module* (ρ, H, F) *is the class defined by* $(\rho^{\mathrm{op}}, H^{\mathrm{op}}, -F^{\mathrm{op}})$.

Proof. We will show the direct sum the two Fredholm modules is homotopic to a degenerate. In fact, $(\rho \oplus \rho^{\mathrm{op}}, H \oplus H^{\mathrm{op}}, F_t)$

$$F_t = \begin{pmatrix} \cos(\frac{\pi}{2}t)F & \sin(\frac{\pi}{2}t)I \\ \sin(\frac{\pi}{2}t)I & -\cos(\frac{\pi}{2}t)F^{\mathrm{op}} \end{pmatrix}$$

is the homotopy of Fredholm modules connect $F \oplus F^{\mathrm{op}}$ to the degenerate $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. ☐

Corollary 3.9. *Every element of* $K^p(A)$ *can be represented by a single Fredholm module.*

In Kasparov's definition, the separable Hilbert space H and representation $\rho : A \to H$ for a Fredholm module can be arbitrary. However, it is possible to realize the whole of K-homology by Fredholm modules on a fixed Hilbert space with a fix representation of A.

Definition 3.10. *We say a representation* $A \to B(H)$ *is ample if it is nondegenerate and no nonzero element of* A *acts on* H *as a compact operator.*

An ample representation essentially absorbs any nondegenerate representation by the following theorem.

Theorem 3.11 (Voiculescu)**.** *If* ρ *and* ρ' *are nondegenerate representation of a separable unital* C^*-*algebra on separable Hilbert spaces* H, H'. *Suppose that* ρ *is ample, then there is a unitary* $U : H \to H' \oplus H$ *such that* $U\rho(a)U^* - \rho'(a) \oplus \rho(a)$ *is compact for all* $a \in A$.

In particular, we are interested in a special kind of ample representation, which will come in handy later on.

Definition 3.12. *We say a representation $\rho : A \to B(H)$ is very ample if it is the direct sum of (countably) infinitely many copies of some fixed representation.*

Very ample representations have better properties. For example, if ρ is an very ample representation, then ρ is unitarily equivalent to the direct sum of two copies or (countably) infinitely many copies of ρ.

Now let A be a C^*-algebra perhaps non-unital and fix once for all a representation $\rho_A : A \to B(H_A)$ which is the restriction to A of a very ample representation of its unitalization $\tilde{A}$. We shall call ρ_A the universal representation of A. We shall also need to consider the graded representation $\rho_A \oplus \rho_A$ of A on $H_A \oplus H_A$. From now on, when talking about a Fredholm module (ρ, H, F) over the universal representation $\rho : A \to H$, we may simply use the operator (F) denote the Fredholm module.

Lemma 3.13. *Every K-homology class can be defined by a Fredholm module over the universal representation of A.*

Proof. Let $[x] \in K^0(A)$, by corollary 3.9, $[x]$ can be represented by some Fredholm module $[(\rho, H, F)]$. Let (ρ_A, H_A, F_A) be a degenerate Fredholm module over the universal representation, for example, $F_A = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Consider the direct sum $(\rho_A \oplus \rho_A, H \oplus H_A, F \oplus F_A)$. By lemma 3.7, this sum also represent $[x]$. But according to Voiculescu's theorem, the representation $\rho \oplus \rho_A$ is essentially unitarily equivalent to ρ_A, say by a unitary $U : H_A \to H \oplus H_A$. Thus the module $(U^* \rho_A U, H_A, U^*(F \oplus F_A)U)$ also represent x. Denote $\rho'_A = U^* \rho_A U$, $F' = U^*(F \oplus F_A)U$. To complete the proof, we only need to show $[(\rho'_A, H_A, F')] = [(\rho_A, H_A, F')]$.

Consider the direct sum $(\rho_A, H_A, F') \oplus (\rho_A'^{\mathrm{op}}, H_A^{\mathrm{op}}, F'^{\mathrm{op}})$. It is homotopic to $(\rho_A \oplus \rho_A'^{\mathrm{op}}, \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right))$ by

$$F_t = \begin{pmatrix} \cos(\frac{\pi}{2}t)F' & \sin(\frac{\pi}{2}t)I \\ \sin(\frac{\pi}{2}t)I & -\cos(\frac{\pi}{2}t)F'^{\mathrm{op}} \end{pmatrix}$$

independent to the choice of F'. If we replace F' by a degenerate $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) : H_A \to H_A$. So we get a degenerate sum. Hence $[(\rho_A, H_A, F') \oplus (\rho_A'^{\mathrm{op}}, H_A^{\mathrm{op}}, F'^{\mathrm{op}})] = 0$. By lemma 3.8, we have $[(\rho_A, H_A, F')] = [(\rho'_A, H_A, F')]$. ∎

Definition 3.14. *Let $\rho : A \to B(H)$ be the universal representation for A. Denote*

$$\Psi_0(A) = \{T \in B(H) : aT - Ta \in K(H), \quad \forall\, a \in A\}$$

$$\Psi_{-1}(A) = \{T \in B(H) : aT,\, Ta \in B(H) \quad \forall\, a \in A\}$$

Every $[x] \in K^0(A)$, by Lemma 3.13, can be represented as $\left[\left(\begin{smallmatrix} 0 & V \\ U & 0 \end{smallmatrix}\right)\right]$. Clearly $U \in \Psi_0(A)$ is invertible modulo $\Psi_{-1}(A)$, and hence defines an K-theory element in $K_1(\Psi_0(A)/\Psi_{-1}(A))$.

Every $[x] \in K^1(A)$, by Lemma 3.13, can be represented as $[(T)]$, Clearly $\frac{T+I}{2} \in \Psi_0(A)$ is a projection modulo $\Psi_{-1}(A)$. and hence defines an element in $K_0(\Psi_0(A)/\Psi_{-1}(A))$.

Theorem 3.15 (Paschke Duality)**.** *The maps defines above are well-defined isomorphism*

$$K^0(A) \to K_1(\Psi_0(A)/\Psi_{-1}(A)) \qquad K^1(A) \to K_0(\Psi_0(A)/\Psi_{-1}(A))$$

Section 3.2 Coarse Baum-Connes Conjecture

We shall define an index map from $K_i(X)$ to $K_i(C^*(X))$. Recall that both groups are defined on the same fixed separable Hilbert space with the same very ample representation.

Recall that every properly supported pseudo-differential operator can be perturbed by a properly supported smoothing operator so as to have support confined to a strip near the diagonal $X \times X$. Similarly, we have the following lemma.

Lemma 3.16. $D^*(X)/C^*(X) \cong \Psi_0(X)/\Psi_{-1}(X)$

Definition 3.17. *We define the assembly map μ to be composition of the following maps*

$$K_p(X) \xrightarrow{\cong} K_{p+1}(\Psi_0(X)/\Psi_1(X)) \xrightarrow{\cong} K_{p+1}(D^*(X)/C^*(X)) \xrightarrow{\partial} K_p(C^*(M))$$

where ∂ is the K-theory boundary map.

We cannot expect the assembly map to be always isomorphism, since the right hand side depends on the large scale property of X, while the left hand side depends on the topological property. Thus it is natural to restrict our attention to spaces have no "local topology".

Definition 3.18. *We say a complete Riemannian manifold M is uniformly contractible, if for every $r > 0$, there exists $R > 0$, such that $B(x, r)$ is contractible in $B(x, R)$ for every $x \in M$.*

It follows from the definition that $\pi_n(M)$ is trivial for all $n \geq 1$. Since every complete Riemannian manifold has homotopy type of CW-complex, by Whitehead theorem, M is contractible.

Conjecture 3.19 (Coarse Baum-Connes Conjecture)**.** *The assembly map for uniformly contractible manifold is an isomorphism.*

We can formulate the coarse Baum-Connes conjecture for more general spaces, but we need a process to "kill" all the "local topology".

Definition 3.20. *Let X be a locally finite discrete metric space. For each $d > 0$, we define the Rips complex $P_d(X)$ to be the simplicial polyhedron endowed with simplicial metric whose set of vertices equals to X and where a finite subset $\{x_0, \ldots, x_n\}$ spans an n-simplex in $P_d(X)$ if and only if $d(x_i, x_j) \leq d$ for all $0 \leq i, j \leq n$.*

The Rips complex $P_d(X)$ encodes the process of "killing the local topology on scale d", by squeezing everything of diameter less than d, into a single simplex. As d grows to infinity, all local topology are "smoothed out".

Recall that the simplicial metric on a simplicial complex is the unique path length metric that restricts to the standard Euclidean metric on each simplex. In some references, the spherical metric has been used. Using the simplicial metric is convenient for computation. If a simplicial complex if finite dimensional, then the simplicial metric and the spherical metric are coarse equivalent. Since we will restrict our attention to the spaces with bounded geometry, $P_d(\Gamma)$ is always finite-dimensional. The difference between the simplicial and the spherical metric is not important.

Clearly, a Rips complex $P_d(\Gamma)$ is a locally compact complete metric space. By Hopf-Rinow Theorem, $P_d(\Gamma)$ is proper and geodesic complete.

Definition 3.21. *We define the coarse K-homology for a locally discrete metric space to be*

$$KX_*(X) := \varinjlim_d K_*(P_d(X)).$$

In general, for a locally compact metric space X, we choose a locally finite net Γ, and define

$$KX_*(X) := KX_*(\Gamma),$$

where a c-net Γ for X is a locally finite discrete subspace that $d(x,\Gamma) \leq c$ for some $c > 0$ and all $x \in X$ and that $d(x,y) \geq c$ for all $x, y \in \Gamma$.

It is easy to verify that the coarse K-homology does not depend on the choice of the net. For a proper metric space X, by Zorn lemma, we can always find a net for X.

Suppose that Γ be an r-dense net in X, and that $d \geq r$. We choose a partition unity $\{\varphi_\gamma\}$ subordinate to the locally finite open cover $\{B(\gamma, d)\}_\gamma \in \Gamma$, and define $\varphi : X \to P_d(X)$ by

$$\varphi : x \to \sum_\gamma \varphi_\gamma(x)\gamma.$$

Since for each x there are only finitely many γ such that $\varphi_\gamma(x) \neq 0$, it follows that $c(x)$, in barycentric coordinates, is a point of $P_d(X)$. Passing to the inductive limit, we get a map

$$c : K_*(X) \to KX_*(X)$$

which does not depends on the choice of net and the partition of unity.

Remark 3.22. *If X be a uniformly contractible manifold with bounded geometry, then*

$$c : K_*(X) \xrightarrow{\cong} KX_*(X)$$

is an isomorphism.

Recall that a metric space has bounded geometry, if we can choose a net Γ, such that for every r, $\#B(\gamma, r) < N(r)$ for some N_r for all $\gamma \in \Gamma$. The bounded geometry condition is important here. Dranishnikov, Ferry and Weinberger have constructed an example of uniformly contractible space X for which c is not an isomorphism [DFW].

It is clear that if X has bounded geometry, $P_d(\Gamma)$ is finite-dimensional more each d. However, as d increases, the dimension of $P_d(\Gamma)$ will keep increasing, and become more complicated. For practical purposes, we sometimes need to define coarse K-homology in a more flexible way, by anti-Čech sequence. We will use it in the proof for the coarse Baum-Connes conjecture in the finite asymptotic dimension case.

Definition 3.23. *Let $\mathcal{U}$ be a locally finite and uniformly bounded cover for X. We define the nerve space $N_{\mathcal{U}}$ associated to $\mathcal{U}$ to be the simplicial complex endowed with the spherical metric whose set of vertices equals $\mathcal{U}$ and where a finite subset $\{U_0, \cdots, U_n\} \subset \mathcal{U}$ spans an n-simplex in $N_{\mathcal{U}}$ if and only if $\bigcap_{i=0}^n U_i \neq \varnothing$.*

Definition 3.24. *An anti-Čech sequence for a metric space X is a sequence $\{\mathcal{U}_i\}$ of open covers of X with the property that* $\text{Lebesgue}(\mathcal{U}_i)$ *goes to infinity, and* $\text{Diam}(\mathcal{U}_i) \leq \text{Lebesgue}(\mathcal{U}_{i+1})$.

We can define a simplicial map $f_i : N_{\mathcal{U}_i} \to N_{\mathcal{U}_{i+1}}$ such that $U \subset V$ whenever $f(U) = V$. We remark that

$$\varinjlim_j K_*(N_{\mathcal{U}_j})$$

does not depend on the choice of simplicial map above, and provides another model for $KX_*(X)$.

Remark 3.25. *Let X be a uniformly discrete space, we have that*

$$\varinjlim_j K_*(N_{\mathcal{U}_j}) \cong \varinjlim_d P_d(X)$$

If fact, let $x_0, \ldots, x_n \in X$ and let $U_i = B_r(x_i)$. If $d(x_i, x_j) < r$ then $U_0, \ldots, U_n$ have non-empty intersection. Thus for any $d < r$ we have a map

$$P_d(X) \to N_{\mathcal{U}_r}.$$

Conversely if $U_0, \ldots, U_n$ have non-empty intersection, then $d(x_i, x_j) < 2r$, so for $d \geq 2r$ we have a map

$$N_{\mathcal{U}_r} \to P_d(X).$$

Hence

$$\varinjlim_d K_*(P_d(X)) \cong \varinjlim_r K_*(N_{\mathcal{U}_r}).$$

To compare the metric d_X on X and $d_{N_{\mathcal{U}}}$ on $N_{\mathcal{U}}$, we have the following easy lemma.

Lemma 3.26. *Suppose $\mathcal{U}$ is a uniformly bounded cover of X and that the diameter of $\mathcal{U}$ is bounded by D. If $U_1, U_2 \in \mathcal{U}$ then there is a universal constant depending only on C such that*

$$d_X(U_1, U_2) \leq CDd_{N_{\mathcal{U}}}(U_1, U_2).$$

Proof. If a path of length l lying in the 1-skeleton of $N_{\mathcal{U}}$ connects U_1, U_2 then

$$l \leq 2Dd_{N_{\mathcal{U}}}(U_1, U_2).$$

A path of length l contains in the n-skeleton of $N_{\mathcal{U}}$, then we can replace γ by a path γ' that contains in the $n-1$ skeleton of $N_{\mathcal{U}}$, whose length is no more than $C\,l$, where C is a universal constant depends only on n. The result follows by induction. □

For each r, we have an assembly map $\mu : K_*(N_{\mathcal{U}_r}) \to K_*(C^*(N_{\mathcal{U}_r}))$, passing to the direct limit we get a map

$$\varinjlim_r K_*(N_{\mathcal{U}_r}) \to \varinjlim_r K_*(C^*(N_{\mathcal{U}_r}))$$

For a quasi-geodesic space, we can show that $N_{\mathcal{U}_r}$ is coarse equivalent to X for r large enough. Hence the right hand side can be identify with $K_*(C^*(X))$.

Recall that a metric space is called quasi-geodesic at scale $\bar{d}$, if $\exists \lambda \geq 0$, for every $x, y \in X$, there exists $x = x_0, x_1, \cdots, x_n = y$ such that $d(x_{i-1}, x_i) \leq \bar{d}$ and $\sum d(x_{i-1}, x_i) \leq \lambda d(x, y)$.

In general, we cannot expect $N_{\mathcal{U}_r}$ is coarse equivalent to X for some r. But we still have the following result.

Lemma 3.27. *Let X be a proper metric space. We have*

$$\varinjlim_{r} K_*(C^*(N_{\mathcal{U}_r})) \to K_*(C^*(X))$$

is an isomorphism. [W, Theorem 2.17]

Now we are ready to state the general coarse Baum-Connes conjecture.

Conjecture 3.28 (Coarse Baum-Connes Conjecture)**.** *If X is a proper metric space with bounded geometry the coarse assembly map*

$$\mu : KX_*(X) \to K_*(C^*(X)).$$

is an isomorphism

The injectivity of the conjecture is not true if we drop the bounded geometry condition [Y98]. However, there is also an counterexample for surjectivity (with coefficients) with bounded geometry condition [HLS].

Aside as a guideline to study the index of elliptic operator on noncompact manifold, the coarse Baum-Connes conjecture has many interesting applications in topology and geometry.

Theorem 3.29 (Descent Principle)**.** *Let Γ be a countable group whose classifying space $B\Gamma$ has the homotopy type of a CW-complex, then the coarse Baum-Connes conjecture for Γ as a metric space with a proper length metric implies the strong Novikov conjecture for Γ.*

Conjecture 3.30 (Gromov-Lawson)**.** *Uniformly contractible complete Riemannian manifold can not have uniformly positive scalar curvature.*

Theorem 3.31. *Coarse Baum-Connes conjecture implies Gromov-Lawson conjecture.*

Proof. Let $[D]$ be the K-homology class of the Dirac operator on M, we can show that $[D] \neq 0 \in K_*(M)$. If the coarse Baum-Connes conjecture holds, then

$$\mathrm{index}([D]) \neq 0 \in K_*(C^*(M)).$$

But if M has uniformly positive scalar curvature $k(p)$, then by Lichnerowicz formula, ([LM] Page 160)

$$D^2 = \nabla^*\nabla + \frac{1}{4}k$$

must be invertible. Hence $\mathrm{index}([D]) = 0$. □

Chapter 4

Localization Algebras

In this chapter we shall introduce the localization algebra and formulate a similar Mayer-Vietoris sequence to compute its K-theory. We shall define a local index map from K-homology to the K-theory of localization algebra, and prove that it is an isomorphism. Hence the K-theory of localization algebra provides another model for K-homology. The coarse Baum-Connes assembly map becomes K-theory homomorphism induced by an evaluation map, which is much easier to study.

Section 4.1 Localization Algebras

Definition 4.1. *Let X be a proper metric space. H_X be a nondegenerate X-module. The localization algebra $C_L^*(X, H_X)$ is defined to be the C^*-algebra generated by all bounded and uniformly continuous functions f from $[0, \infty)$ to $C^*(X, H_X)$ such that*

$$\text{Propagation}(f(t)) \to 0 \textit{ as } t \to \infty.$$

Definition 4.2. *A map g from a proper metric space X to another proper metric space Y is called Lipschitz if*

(1) *g is a coarse map;*

(2) *there exists $C > 0$ such that $d(f(x), f(y)) \leq Cd(x, y)$.*

Definition 4.3. *Suppose that $g : X \to Y$ is a Lipschtiz map. A uniformly continuous family of isometries $t \to V(t)$: $H_X \to H_Y$, $t \in [0, \infty)$ is said to cover g if there exists $c_t > 0$, $\lim\limits_{t\to\infty} c_t = 0$, such that $d(g(x), y) \leq c_t$.*

Lemma 4.4. *Let f be a bounded and uniformly continuous function from $[0, \infty)$ to $B(H_X)$, if there exist $c_t > 0$, $\lim\limits_{t\to\infty} c_t = 0$ such that* Propagation$(f(t)) < c_t$*, then f is a multiplier of $C_L^*(X, H_X)$.*

Proof. The proof is exactly similar to Lemma 2.26. ☐

Lemma 4.5. *If H_Y is very ample, then*

(1) *any Lipschitz map $g : X \to Y$ admits a covering family of isometries V_t.*

(2) *Conjugation by $\{V(t)\}$ gives $*$-homomorphisms*

$$\text{Ad}(V(t)) : C_L^*(X, H_X) \to C_L^*(Y, H_Y).$$

(3) *The corresponding maps on K-theory*

$$K_p(C_L^*(X, H_X)) \to K_p(C_L^*(Y, H_Y))$$

are independent of the choice of covering isometry, and depend functorially on g.

Proof. (a) Let f be a Lipschtiz map. Let $\{\varepsilon_k\}_k$ be a sequence of positive number such that $\varepsilon_k \to 0$ as $k \to \infty$. by Lemma 2.29, for each k, there exists an isometry V_k from H_X to H_Y such that

$$\mathrm{Support}(V_k) \subset \{(y,x) \in Y \times X : d(y, f(x)) \leq \varepsilon_k\}.$$

The following family of isometries $V_f(t)$, $t \in [0,\infty)$ from H_X to $H_Y \oplus H_Y \cong H_Y$ covers g.

$$V_f(t) = \begin{pmatrix} \cos\frac{\pi}{2}t & \sin\frac{\pi}{2}t \\ -\sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix} \begin{pmatrix} V_k & \\ & V_{k+1} \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{2}t & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} V_k\cos^2\frac{\pi}{2}t + V_{k+1}\sin^2\frac{\pi}{2}t \\ (V_{k+1}-V_k)\sin\frac{\pi}{2}t\cos\frac{\pi}{2}t \end{pmatrix}$$

where $t \in [k, k+1]$.

(b) The proof is similar to Lemma 2.30.

(c) The proof is similar to Lemma 2.31. ☐

Similar to Roe algebra, every family of isometries covers identity map on X induces an isomorphism of K-theory for localization algebras. The K-theory for the Localization algebra does not depend on the choice of very ample X-module (see Corollary 2.32). Let us take the universal X-module as the one forming K-homology group $K^p(X)$ and Roe algebra $C^*(X)$ to construct the localization algebra and denote it by $C_L^*(X)$.

Since for a very ample module H_X, we have that $H_X \cong H_X \oplus H_X \oplus \cdots \oplus H_X$ and $C_L^*(X, H_X) \cong M_n(C_L^*(X, H_X))$. Hence any element in $K_1(C_L^*(X))$ can be represented by a unitary in $(C_L^*(X))^+$.

Section 4.2 Homotopy Invariance

In this section, we will introduce a notion of coarse homotopy, namely strong Lipschitz homotopy, and prove the K-theory of localization algebras is invariant under this notion of homotopy.

Definition 4.6 (Yu97)**.** *Let X and Y be two proper metric spaces; let f and g be two Lipschitz maps from X to Y. A continuous homotopy $F(t,x)$ ($t \in [0,1]$) between f and g is said to be strongly Lipschitz if*

(1) $F(t,x)$ is a coarse map from X to Y for each t;

(2) $d(F(t,x), F(t,y) \leq Cd(x,y)$ for all $x, y \in X$ and $t \in [0,1]$, where C is a constant (called Lipschitz constant of F);

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(F(t_1,x), F(t_2,x)) < \varepsilon$ for all $x \in X$ if $|t_1 - t_2| < \delta$;

(4) $F(0,x) = f(x)$, $F(1,x) = g(x)$ for all $x \in X$.

Definition 4.7. *X is said to be strongly Lipschtiz homotopy equivalent to Y if there exists Lipschitz maps $f : X \to Y$ and $g : Y \to X$ such that gf and fg are strongly Lipschitz equivalent to id_X and id_Y.*

Theorem 4.8. *If X is strongly Lipschitz homotopy equivalent to Y, then $K_p(C_L^*(X)) \cong K_p(C_L^*(Y))$.*

Proof. Let F be the strong Lipschtiz homotopy between gf and id_X such that $F(x,0) = (gf)(x)$, $F(x,1) = x$. We need to show $(\mathrm{Ad}_{V_{gf}})_* = \mathrm{id}_*$ at the K-theory level, where id is the identity homomorphism from $C_L^*(X)$ to $C_L^*(X)$.

There exist a sequence of nonnegative numbers $\{t_{i,j}\}_{i,j=0}^{\infty}$ and a sequence of decreasing positive number $\{\varepsilon_i\}_{i=0}^{\infty}$ such that

(1) $t_{0,j} = 0$, $t_{i+1,j} \geq t_{i,j}$, $\lim_{i\to\infty} \varepsilon_i = 0$.

(2) For each j, there exists N_j such that $t_{i,j} = 1$ for all $i \geq N_j$.

(3) $d(F(x,t_{i,j}), F(x,t_{i+1,j})) \leq \varepsilon_j$ and $d(F(x,t_{i,j}), F(x,t_{i,j+1})) \leq \varepsilon_j$ for all $x \in X$.

For example, we can take

$$t_{i,j} = \begin{cases} \frac{i}{j+1} & i < j+1 \\ 1 & i \geq j+1 \end{cases}$$

Let $V_{i,j}$ be an isometry from H_X to $H_X \oplus H_X$ such that

$$\mathrm{Support}(V_{i,j}) \subset \{(x_2, x_1) \in X \times X : d(x_2, F(x_1, t_{i,j})) \leq \varepsilon_j\}$$

and $V_{i,j} = I$ if $F(x, t_{i,j}) = x$ for all $x \in X$.

Define a family of isometry from $H_X \to H_X \oplus H_X$ by

$$V_i(t) = \begin{pmatrix} V_{i,j}\cos^2 \frac{\pi}{2}t + V_{i,j+1}\sin^2 \frac{\pi}{2}t \\ (V_{i,j+1} - V_{i,j})\sin \frac{\pi}{2}t \cos \frac{\pi}{2}t \end{pmatrix} \quad \text{if } \ t \in [j, j+1]$$

where H_X is the universal H_X-module. We define elements

$$\begin{aligned} a &= \oplus_{i\geq 0}\mathrm{Ad}^+(V_i)(u)(u^{-1} \oplus I) \\ b &= \oplus_{i\geq 0}\mathrm{Ad}^+(V_{i+1})(u)(u^{-1} \oplus I) \\ c &= \oplus_{k>1}\mathrm{Ad}^+(V_k)(u)(u^{-1} \oplus I) \end{aligned}$$

where $u \in C_L^*(X, H_X)$.

We want to verify a, b, c are elements in $C_L^*(X, (H_X \oplus H_X)^\infty)^+$.

For a fixed t, we have determined j above. Once j is determined, we know that $V_{i,j} = I$ whenever $i > N_j$. Thus, in our infinite direct sum defining a, b, and c, the term for $i > N_j$ are simply given by I and hence has zero propagation. It follows that $a(t)$, $b(t)$ and $c(t)$ are in $(C^*(X, H_X \oplus H_X))^+$ for each $t \in [0, \infty)$.

From the definition of $V_i(t)$, we see that for every $\varepsilon > 0$, there exists $\delta > 0$, such that for all i,

$$||V_i(t) - V_i(t')|| < \varepsilon, \quad \text{whenever } \ |t - t'| < \delta.$$

So a, b, c are uniformly continuous.

Thus we only need to show if Propagation$(u(t)) \to 0$ as $t \to \infty$, so are a, b and c.

Fix an $\varepsilon > 0$. Let j be such that $\varepsilon_j < \frac{\varepsilon}{6}$, and pick T_0 be such that ,

$$\text{Propogation}(u(t)) < \frac{\varepsilon}{2C} \quad \text{for all } \ t > T_0.$$

Let $T_1 = \max\{j, T_0\}$. Then Propagation$(f(t_0)) \le \varepsilon$ for all $t_0 > T$. In fact, to estimate Propogation$(f(t_0))$, $t_0 \in [j, j+1]$, we only need to estimate the propagation of $V_{i,j}u(t_0)V_{i,j}^*$, $V_{i,j}u(t_0)V_{i,j+1}^*$, $V_{i,j+1}u(t_0)V_{i,j}^*$, $V_{i,j+1}u(t_0)V_{i,j+1}^*$.

Take $V_{i,j+1}u(t_0)V_{i,j}^*$ as an example. If $(x_4, x_1) \in \text{Support}(V_{i,j+1}u(t_0)V_{i,j}^*)$, then there exists x_3, x_2 such that $(x_4, x_3) \in \text{Support}(V_{i,j+1})$, $(x_3, x_2) \in \text{Support}(u(t_0))$ and $(x_2, x_1) \in \text{Support}V_{i,j}^*$. Hence $d(x_4, F(x_3, t_{i,j+1})) \le \varepsilon_j$, $d(x_3, x_2) \le \frac{\varepsilon}{6C}$ and $d(x_1, F(x_2, t_{i,j})) \le \varepsilon_j$. Since

$$\begin{aligned} d(F(x_3, t_{i,j+1}), F(x_2, t_{i,j})) &\le d(F(x_3, t_{i,j+1}), F(x_3, t_{i,j})) + d(F(x_3, t_{i,j}), F(x_2, t_{i,j})) \\ &\le \varepsilon_j + Cd(x_3, x_2) \le \varepsilon_j + \frac{\varepsilon}{2}, \end{aligned}$$

we have that

$$d(x_4, x_3) \le d(x_4, F(x_3, t_{i,j+1})) + d(F(x_3, t_{i,j+1}), F(x_2, t_{i,j})) + d(F(x_2, t_{i,j}), x_1) \le \varepsilon_j + \varepsilon_j + \frac{\varepsilon}{2} + \varepsilon_j \le \varepsilon.$$

Define

$$h(s) = \bigoplus_{i=1}^{\infty} \mathrm{Ad}^+ \begin{pmatrix} V_i \cos^2 \frac{\pi}{2}s + V_{i+1} \sin^2 \frac{\pi}{2}s \\ (V_{i+1} - V_i) \cos \frac{\pi}{2}s \sin \frac{\pi}{2}s \end{pmatrix} (u)(u^{-1} \oplus I \oplus I \oplus I)$$

By a similar estimation, we can show $h(x) \in C_L^*(X, (H_X \oplus H_X \oplus H_X \oplus H_X)^\infty)$. This time we need to estimate Support$(V_{k,l}u(t)V_{k',l'})^*$, where k, $k' \in \{i, i+1\}$, l, $l' \in \{j, j+1\}$ and $t \in [j, j+1]$.

If we identify $C_L^*(X, (H_X \oplus H_X \oplus H_X \oplus H_X)^\infty)$ and $M_2(C_L^*(X, (H_X \oplus H_X)^\infty))$, we have

$$h(0) = a \oplus I, \quad h(1) = b \oplus I$$

where I is the identity element in $C_L^*(X, (H_X \oplus H_X)^\infty)$. Hence a and b represent the same class in $K_1(C_L^*(X, (H_X \oplus H_X)^\infty))$.

Since $c = \mathrm{Ad}_W^+(b)$, where $W : (H_X \oplus H_X)^\infty \to (H_X \oplus H_X)^\infty$ given by right translation

$$W : ((v_1, v_1'), (v_2, v_2'), \ldots) \to ((0,0), (v_1, v_1'), \ldots).$$

It is clear that W covers the identity map on X, and hence that b and c represent the same class in $K_1(C_L^*(X, (H_X \oplus H_X)^\infty))$. Thus $ac^{-1} = \mathrm{Ad}_{V_0}(u)(u^{-1} \oplus I) \oplus I \oplus I \oplus \cdots$ represent $[0]$ in $K_1(C_L^*(X, (H_X \oplus H_X)^\infty))$. Since the top-left corner inclusion is given by the adjoint of an isometry family, mapping $H_X \oplus H_X$ into the first coordinate in $(H_X \oplus H_X)^\infty$, which covers covers identity map on X. So it induces isomorphism $K_1(C_L^*(X, H_X \oplus H_X)) \to K_1(C_L^*(X, (H_X \oplus H_X)^\infty))$. Therefore, $\mathrm{Ad}_{V_0}(u)$ and $u \oplus I$ represent the same class in $K_1(C_L^*(X, H_X \oplus H_X))$. Since V_0 covers gf, so $(\mathrm{Ad}V_{gf})_* = \mathrm{id}_*$. $\square$

Section 4.3 Mayer-Vietoris Sequence for K-theory of Localization Algebras

Similar to the Roe algebra, we can also develop a Mayer-Vietoris sequence for K-theory of localization algebras.

Definition 4.9. *Let Y be a closed subspace of a proper metric space X. We say $f \in C_L^*(X)$ or $(C_L^*(X))^+$is near Y if there exists $c_t > 0$ satisfying $\lim_{t\to\infty} c_t = 0$ and*

$$\mathrm{Support}(f(t)) \subset Y_{c_t} \times Y_{c_t}, \qquad \mathrm{Propagation}(f(t)) < c_t.$$

We say $f \in C([0,1]^n) \otimes C_L^*(X)$ or $(C([0,1]^n) \otimes C_L^*(X))^+$ is near Y if there exists $c_t > 0$ satisfying $\lim_{t\to\infty} c_t = 0$ and

$$\mathrm{Support}(f(s,t)) \subset Y_{c_t} \times Y_{c_t}, \qquad \mathrm{Propagation}(f(s,t)) < c_t$$

for all $s \in [0,1]^n$.

Denote by $C_L^*(Y;X)$ the closed subalgebra of $C_L^*(X)$ generated by all elements of $C_L^*(X)$ near Y.

Recall that if the interior of Y is dense in Y then Y represents very amply on the range of the projection corresponding to the characteristic function of Y under the Borel functional calculus (Lemma 2.16). Thus $C_L^*(Y)$ can be viewed as a C^*-subalgebra of $C_L^*(Y;X)$.

Lemma 4.10. *Let Y be a closed subspace of X such that*

(1) *the interior of Y is dense in Y;*

(2) *there exists $r > 0$ such that Y_r is strongly Lipschitz homotopy to Y via the inclusion $Y \hookrightarrow X$. Then the inclusion induces isomorphism $K_i(C_L^*(Y;X))$.*

Proof. (i) Surjectivity for K_1.

Every element in $K_1(C_L^*(Y;X))$ can be represented by a unitary $u = u' + I \in C_L^*(Y,X)^+$, where $u' \in C_L^*(Y,X)$. We can take $a = a' + I \in C_L^*(Y;X)$ such that a' is near Y and $||u - a|| \le \frac{1}{3}$. Thus

$$\begin{aligned}||1 - a^*a|| &= ||1 - (u^* - (u^* - a^*))(u - (u - a))|| \\ &\le ||u^*|| \cdot ||u - a|| + ||u^* - a^*|| \cdot ||u|| + ||(u^* - a^*)(u - a)|| < \frac{1}{3} + \frac{1}{3} + \frac{1}{9} < 1.\end{aligned}$$

Hence a^*a is invertible and so is a.

We note that u and a represent the same class in $K_1(C_L^*(Y;X))$. In fact, the linear homotopy, $su + (1-s)a$, $s \in [0,1]$ gives a path of invertibles, since

$$||1 - (s + (1-s)u^*a)|| = (1-s)||1 - u^*a|| = (1-s)||u^*(u-a)|| < \frac{1}{3} \le 1.$$

Since Y is a closed subspace of a proper metric space X. So for every $x \in X$, there exists some point in Y such that $d(x, g(x)) = d(x, Y)$. So $g : x \to g(x)$ defines a map $X \to X$ with range Y.

Choose a sequence $\{\varepsilon_i\}$ satisfying $\varepsilon_i > 0$ and $\lim_{i\to\infty} \varepsilon_i = 0$ in such a way that $\mathrm{Support}(a'(t)) \subset Y_{\varepsilon_i} \times Y_{\varepsilon_i}$ and $\mathrm{Propagation}(a'(t)) \le \varepsilon_i$ whenever $t \ge i - 1$. Notice that $d(g(x_1), g(x_2)) \le 2\varepsilon_i$ for any $x_1, x_2 \in Y_{\varepsilon_i}$. By a similar argument for Lemma 2.29, we can define an isometry $V_i : H_{Y_{\varepsilon_i}} \to H_Y$ such that

$$\mathrm{Support}(V_i) \subset \{(y,x) \in Y \times X : d(y, g(x)) \le 3\varepsilon_i\}.$$

V_i can also be viewed as a partial isometry $H_X \to H_X$. Let

$$V(t) = \begin{pmatrix} V_i \cos^2 \frac{\pi}{2}t + V_{i+1} \sin^2 \frac{\pi}{2}t \\ (V_{i+1} - V_i) \cos \frac{\pi}{2}t \sin \frac{\pi}{2}t \end{pmatrix}$$

We can easily check that Va', $a'V^*$, $Va'V^* \in C_L^*(Y;X)$, $a'V^*V = a'$, and $||V|| \leq 1$. By lemma, $[Va'V^* + I] = [a] = [u]$ in $K_1(C_L^*(Y;X))$. But $Va'V^* \in C_L^*(Y, H_Y)$. So i_* is onto.

(ii) Injectivity for K_1.

To make our notation simpler, as the proof of surjectivity, we will take every element in the unitalized algebra with scalar part I. Let $[a(t)] \in K_1(C_L^*(Y, H_Y))$ such that $i_*[a(t)] = I \in K_1(C_L^*(Y;X))$. Let $h(s,t)$ be a homotopy in $C_L^*(Y;X)^+$ such that $h(0,t) = a(t)$ and $h(1,t) = I$. Thus $h(s,t)$ is a unitary in $(C([0,1]) \otimes C_L^*(Y;X))^+$. We will to approximate it by an invertible element in $(C([0,1]) \otimes C_L^*(Y;X))^+$ near Y.

There exists an $\delta > 0$ such that

$$||h(s',t) - h(s'',t)|| < \frac{1}{2}, \quad \text{whenever } |s' - s''| < \delta.$$

Take $N > \frac{1}{\delta}$ and $s_i = \frac{i}{N}$, where $i = 1, \ldots, N-1$. We can take $g_{s_i}(t) \in (C_L^*(Y;Z))^+$ for each $i = 1, \ldots, N-1$, in such a way that $||g_{s_i}(t) - h(s_i,t)|| < \frac{1}{2}$, Support$(g_{s_i}(t)) \subset Y_{c_{i,t}} \times Y_{c_{i,t}}$ and Propagation$(g_{s_i}(t))$ for some $c_{i,t} > 0$ satisfying $\lim\limits_{t\to\infty} c_{i,t} \to 0$ and for all $t \in [0, \infty)$. Define

$$h'(s,t) = \frac{s - s_{i-1}}{s_i - s_{i-1}} g_{s_i}(t) + \frac{s_i - s}{s_i - s_{i-1}} g_{s_{i-1}}(t) \quad \text{if } s \in [s_{i-1}, s_i]$$

We compute that

$$\begin{aligned} &||h'(s,t) - h(s_i,t)|| \\ \leq &\frac{s - s_i}{s_i - s_{i-1}} ||g_{s_i}(t) - h(s_i,t)|| + \frac{s_i - s}{s_i - s_{i-1}} (||g_{s_{i-1}}(t) - h(s_{i-1},t)|| + ||h(s_{i-1},t) - h(s_i,t)||) < 1 \end{aligned}$$

Thus $h'(s,t)$ is an invertible in $(C([0,1]) \otimes C_L^*(Y;Z))^+$. Let $c_t = \max_i\{c_{i,t}\}$; then c_t satisfies that $\lim\limits_{t\to\infty} c_t = 0$, Support$(h'(s,t)) \subset Y_{c_t} \times Y_{c_t}$ and Propagation$(h'(s,t)) \leq c_t$ for all $s \in [0,1]$. By uniform continuity of $a(t)$, we know that $a(t + sT_0)$, where $s \in [0,1]$, is norm continuous in s for any $T_0 > 0$. Hence $a(t)$ is equivalent to $a(t + T_0) \in K_1(C_L^*(Y;X))$. We can take T_0 large enough, such that

$$\text{Support}(h'(s, t + T_0)) \subset Y_r \times Y_r \quad \text{for all } s \in [0,1].$$

Hence $[a(t)] = [a(t + T_0)] = 0$ in $K_1(C_L^*(Y_r, H_{Y_r}))$.

We want to remark that the approximation argument also works for $(C([0,1]^n) \otimes C_L^*(Y;X))^+$. The goal is to approximate a unitary in $(C([0,1])^n \otimes C_L^*(Y;X))^+$ by an invertible element in $(C([0,1]^n) \otimes C_L^*(X))^+$ near Y. We divide $[0,1]^n$ evenly into N^n small cubes and approximate the values at vertices of small cubes by invertibles in $C_L^*(X)^+$ near Y, and extend linearly to get an invertible element in $(C([0,1]^n) \otimes C_L^*(X))^+$ near Y.

To prove the isomorphism for K_0, we will identify $K_0(A)$ by $K_1(C_0((0,1)) \otimes A)$. In the proof for surjectivity (respectively injectivity), we will deal with unitaries in $(C([0,1]) \otimes C_L^*(Y;X))^+$ (respectively, $(C([0,1]^2) \otimes C_L^*(Y;X))^+$) instead. The same "approximation" and "homotopy" arguments apply. $\square$

Definition 4.11. *Let X be a proper metric space and Y and Z be closed subspaces with $X = Y \cup Z$. Then the decomposition $X = Y \cup Z$ is said to be strongly excisive if for any $c_t > 0$ with $\lim_{t\to\infty} c_t = 0$, there exist $d_t > 0$ with $\lim_{t\to\infty} d_t = 0$ such that*

$$Y_{c_t} \cap Z_{c_t} \subset (Y \cap Z)_{d_t}$$

for all $t \in [0, \infty)$.

Lemma 4.12. *Let $X = Y \cup Z$ be a decomposition of X. Then*

$$C_L^*(Y;X) + C_L^*(Z;X) = C_L^*(X).$$

If $X = Y \cup Z$ is strongly excisive then

$$C_L^*(Y;X) \cap C_L^*(Z;X) = C_L^*(Y \cap Z;X).$$

and we have the Mayer-Vietoris sequence

$$\begin{tikzcd}
K_1(C_L^*(Y \cap Z)) \arrow[r] & K_1(C_L^*(Y)) \oplus K_1(C_L^*(Z)) \arrow[r] & K_1(C_L^*(X)) \arrow[d] \\
K_0(C_L^*(X)) \arrow[u] & K_0(C_L^*(Y)) \oplus K_0(C_L^*(Z)) \arrow[l] & K_0(C_L^*(Y \cap Z)) \arrow[l]
\end{tikzcd}$$

Proof. The proof is exactly similar to Lemma 2.40, Lemma 2.42, and Theorem 2.43. $\square$

Section 4.4 Local Index Map

We can also define a local index map from the K-homology group $K_i(X)$ to the K-theory group $K_i(C_L^*(X))$.

For each positive integer n, let $\{U_{n,i}\}$ be a locally finite open cover for X such that $\operatorname{diam}(U_{n,i})_i < 1/n$ for all i. Let $\{\varphi_{n,i}\}_i$ be a continuous partition of unity subordinate to $\{U_{n,i}\}_i$. Let (H_X, F) be a cycle for $K_0(X)$ such that H_X is an ample X-module. Define a family of operators $F(t)$ $(t \in [0, \infty)$ acting on H_X by

$$F(t) = \sum_i ((1-(t-n))\varphi_{n,i}^{1/2} F \varphi_{n,i}^{1/2} + (t-n)\varphi_{n+1,i}^{1/2} F \varphi_{n+1,i}^{1/2}),$$

for all $t \in [n, n+1]$, where the infinite sum converges in the strong operator topology. Notice that Propagation$(F(t))$ is a multiplier of $C_L^*(X)$ and $F(t)$ is a unitary modulo $C_L^*(X)$. Hence $F(t)$ gives rise to an element $[F(t)]$ in $K_0(C_L^*(X))$.

We define the local index of the cycle (H_X, F) to be $[F(t)]$. Similarly we can define the local index map from $K_1(X)$ to $K_1(C_L^*(X))$.

Theorem 4.13. *[Yu97] Let X be a simplicial complex endowed with the spherical metric. If X is finite-dimensional, then the local index map from $K_i(X)$ to $K_i(C_L^*(X))$ is an isomorphism.*

The above result has been verified for all proper metric spaces by an Eilenberg swindle argument in the work of Y. Qiao and J. Roe [QR].

Definition 4.14 (Yu97)**.** *Let X be a proper metric space. $C_{L,0}^*(X)$ is defined to be the C^*-algebra generated by all bounded and uniformly norm continuous functions f from $[0,\infty)$ to $C^*(X)$ such that* $\mathrm{Propagation}(f(t)) \to 0$ *as* $t \to \infty$, $f(0) = 0$.

Remark 4.15. *We have the following short exact sequence for any proper metric space X,*

$$0 \to C_{L,0}^*(X) \to C_L^*(X) \to C^*(X) \to 0.$$

By Theorem 4.13, we identify $K_*(X)$ with $K_*(C_L^*(X))$; thus, to prove the coarse Baum-Connes conjecture, we only need to show that

$$\varinjlim_d K_*(C_{L,0}^*(P_d(X)) = 0.$$

Chapter 5

Controlled Obstructions

The concept of controlled obstructions $QP_{\delta,r,s,k}$, $QU_{\delta,r,s,k}$ was introduced in my advisor's work on coarse Baum-Connes Conjecture for spaces with finite asymptotic dimensions [Y98]. Given a representative of $C^*_{L,0}(X)$, we can not guarantee that it has finite propagation. But we do need finite propagation to allow cutting and pasting technique to work. Given a K-theory element for $C^*_{L,0}(X)$, we will approximate it by a quasi-projection or a quasi-unitary with finite propagation. Apply functional calculus, we can easily get back the original K-theory element. In this chapter, we will study these controlled obstructions and give some more details for the results in [Y98].

Section 5.1 Controlled Projections $QP_{\delta,r,s,k}(X)$

Definition 5.1. *Let A be a C^*-algebra and δ be a positive number, an element p in A is called a δ-quasi-projection if*

$$p^* = p, \quad \text{and} \quad ||p^2 - p|| < \delta.$$

Let X be a proper metric space. Let $C^*_{L,0}(X)^+$ be the C^*-algebra obtained from $C^*_{L,0}(X)$ by adjoining an identity. Let $\delta > 0$, $r > 0$, $s > 0$, k and n be positive integers.

Definition 5.2. *We denote $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ to be the set of all continuous maps from $[0,1]^k$ to $C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ such that:*

(1) *$f(t)$ is a δ-quasi-projection for all $[0,1]^k$;*

(2) *propagation $(f(t)) \leq r$ for all $t \in [0,1]^k$;*

(3) *f is piecewise smooth in t_i and $\left\|\frac{\partial f}{\partial t_i}(t)\right\| \leq s$ for all $t \in [0,1]^k$;*

(4) *$||f(t) - p_m|| < \delta$ for all $t \in \mathrm{bd}([0,1]^k)$;*

(5) *$\pi(f(t)) = p_m$, where π is the canonical homomorphism from $C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ to $M_n(\mathbb{C})$.*

We remark that $t = (t_1, \ldots, t_n)$ can be viewed as suspension parameters. Instead of requiring $f(t) = p_m$ on the boundary of $[0,1]^n$, we allow a more flexible boundary condition, which will add some convenience to consider suspension map in section 3. By Lemma 5.6 and 5.7, we can normalize the boundary condition if we need.

Definition 5.3. *We denote $QP_{\delta,r,s,k}(X)$ to be the direct limit of $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ under the embedding: $p \to p \oplus 0$.*

Definition 5.4. *Let p and q be two elements in $QP_{\delta,r,s,k}(X)$. Now p is said to be (δ, r, s)-homotopic to q if there exists a piecewise smooth homotopy $a(t')$ ($t' \in [0,1]$) in $QP_{\delta,r,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ for some n such that* (1) *$a(0) = p$ and $a(1) = q$*
(2) *$||a'(t')|| \leq s$.*

Lemma 5.5. *Let $0 < \delta < 1$ and $p(t), q(t) \in QP_{\delta,r,s,k}(X)$ satisfying $||p(t) - q(t)|| < \delta$, then the linear homotopy $a(t') = t'p + (1 - t')q$ is a $(2\delta, r, s)$-homotopy between f and g.*

Proof. Clearly $a(t')$ is self-adjoint. We calculate that

$$\begin{aligned}||a(t')^2 - a(t')|| &= ||(t'^2 - t')(p - q) + t'(p^2 - p) + (1 - t')(q^2 - q)|| \\ &\leq (t'^2 - t')\delta^2 + t'\delta + (1 - t')\delta \leq \frac{1}{4}\delta^2 + \delta \leq 2\delta.\end{aligned}$$

$$||\frac{\partial a}{\partial t'}|| = ||p - q|| < \delta < 1, \quad ||\frac{\partial a}{\partial t}|| \leq t'||\frac{\partial p}{\partial t}|| + (1 - t')||\frac{\partial q}{\partial t}|| < s.$$

□

Lemma 5.6. *Let $0 < \delta < 10^{-k}$. Any $p \in QP_{\delta,r,s,k}(X)$ is $(10^{k+1}\delta, r, 2^k s)$-homotopic to some quasi-projection q satisfying $q(t) = \pi(q(t)) = p_m$ for some m and all $t \in \mathrm{bd}([0, 1]^k)$.*

Proof. For the case $k = 1$. Let $\varepsilon = \min\{\frac{\delta}{s}, \frac{1}{2}\}$. Let

$$p_1(t) = \begin{cases} \frac{\varepsilon - t}{\varepsilon}p_m + \frac{t}{\varepsilon}p(\varepsilon) & t \in [0, \varepsilon] \\ p(t) & t \in [\varepsilon, 1 - \varepsilon] \\ \frac{t-(1-\varepsilon)}{\varepsilon}p_m + \frac{1-t}{\varepsilon}p(1 - \varepsilon) & t \in [1 - \varepsilon, 1]. \end{cases}$$

Clearly the propagation of $p_1(t)$ is bounded by r. Since the speed of $p(t)$ is bounded by s, we have that

$$||p(t_1) - p(t_2)|| \leq \delta, \forall \qquad |t_1 - t_2| < \frac{\delta}{s}.$$

Hence

$$||p(t) - p_m|| \leq ||p(t) - p(0)|| + ||p(0) - p_m|| \leq 2\delta, \quad \forall\, t \in [0, \varepsilon]$$

and

$$||p_1(t) - p(t)|| \leq \frac{\varepsilon - t}{\varepsilon}||p_m - p(t)|| + \frac{t}{\varepsilon}||p(\varepsilon) - p(t)|| \leq \frac{\varepsilon - t}{\varepsilon} \cdot 2\delta + \frac{t}{\varepsilon} \cdot \delta \leq 2\delta.$$

Hence

$$\begin{aligned}||p_1(t)^2 - p_1(t)|| =& ||(p_1(t) - p(t))(p_1(t) - p(t) + 2p(t) - 1) + (p(t^2) - p(t))|| \\ \leq& ||p_1(t) - p(t)||(||p_1(t) - p(t)|| + 2||p(t)|| + 1 + ||p(t)^2 - p(t)||) \\ \leq& 2\delta(2\delta + 2\sqrt{1 + \delta} + 1) < 10\delta \qquad \forall\, t \in [0, \varepsilon].\end{aligned}$$

For the speed of $p_1(t)$, we have that

$$\left\|\frac{\partial}{\partial t}p_1(t)\right\| = \left\|\frac{1}{\varepsilon}(p_m - p(\varepsilon))\right\| \leq \frac{2\delta}{\varepsilon} \leq 2s \quad \forall\, t \in [0, \varepsilon].$$

We can do similar estimation for $t \in [1 - \varepsilon, 1]$. We have that $p_1(t)$ is a $(10\delta, r, 2s)$-quasi-projection.

For the case $k > 1$, repeating the above process k-times. We get a $(10^k\delta, r, 2^k s)$-quasi-projection, such that

$$||p_k(t) - p(t)|| \leq ||p_k(t) - p_{k-1}(t)|| + \cdots + ||p_1(t) - p(t)|| \leq 2 \cdot 10^{k-1}\delta + \cdots + 2\delta < 3 \cdot 10^{k-1}\delta,$$

and $\pi(p_k(t)) = p_m$ for all $t \in \mathrm{bd}[0,1]^k$. By Lemma 5.5, the linear homotopy between $p_k(t)$ and $p(t)$ is a $(10^{k+1}\delta, r, 2^k s)$-homotopy. □

Lemma 5.7. *Let $0 < \delta < 10^{-k}$. If p and q are two elements in $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ such that p is (δ, r, s)-homotopic to q, $p(t) = \pi(p(t))$ and $q(t) = \pi(q(t))$ for all $t \in \mathrm{bd}([0,1]^k)$, then there exists a $(10^{k+1}\delta, r, 2^k s)$-homotopy $a(t')$ ($t' \in [0,1]$) between p and q, such that $(a(t'))(t) = \pi((a(t'))(t))$ for all $t' \in [0,1]$ and $t \in \mathrm{bd}([0,1]^k)$.*

Proof. Let $b(t')$ be a (δ, r, s)-homotopy between p and q, since the speed of $b(t')$ is bounded by s, we have a equally spaced partition $0 = t'_0 < t'_1 < \cdots t'_{m_s} = 1$ such that

$$||b(t'_{i+1}) - b(t'_i)|| < \delta, \quad b(t') \in QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})),$$

where $m_s = [\frac{s}{\delta}] + 1$ By the proof Lemma 5.6, we can find $a(t'_i) \in QP_{5\cdot 10^k\delta,r,2^k s,k}(C^*_L(X)^+ \otimes M'_n(\mathbb{C}))$ for $i = 1, \ldots, m_s - 1$, such that $a(t'_i)(t) = \pi(a(t'_i)(t))$ for all $t \in \mathrm{bd}[0,1]^k$ and $||a(t'_i) - b(t'_i)|| < 3 \cdot 10^{k-1}\delta$. Hence

$$\begin{aligned} ||a(t'_i) - a(t'_{i+1})|| &\leq ||a(t'_i) - b(t'_i)|| + ||b(t'_{i+1}) - b(t'_{i+1})|| + ||b(t'_{i+1}) - a(t'_{i+1})|| \\ &< 3 \cdot 10^{k-1}\delta + \delta + 3 \cdot 10^{k-1}\delta < 10^k\delta. \end{aligned}$$

We define

$$a(t') = \frac{t' - t'_i}{t_{i+1} - t_i} a(t_{i+1}) + \frac{t'_{i+1} - t'}{t_{i+1} - t_i} a(t_i) \quad \text{if} \quad t' \in [t'_i, t'_{i+1}].$$

Clearly, we have that $\pi(a(t')(t)) = a(t')(t)$ for all $t' \in [0,1]$ and $t \in \mathrm{bd}[0,1]^k$. By the proof of Lemma 5.5, we know that for any $t' \in [0,1]$, $a(t')$ is a $10^{k+1}\delta$-quasi-projection with propagation no more than r.

Note that

$$\left\| \frac{\partial}{\partial t} a(t')(t) \right\| \leq 2^k s$$

and

$$\left\| \frac{\partial}{\partial t'} a(t')(t) \right\| \leq \frac{1}{t_{i+1} - t_i}(||a(t_{i+1}) - a(t_i)||) \leq \frac{\delta}{m_s} < \delta\left(\left[\frac{s}{\delta}\right] + 1\right) < s + 1.$$

We have that the speed of $a(t')$ is bounded by $2^k s$. □

By the following three lemmas, we can view $QP_{\delta,r,s,k}(X)$ as a controlled version of of $K_0(C^*_{L,0}(X) \otimes C_0((0,1)^k))$.

Lemma 5.8. *Let $0 < \delta < \frac{1}{100}$, f be a continuous function $\mathbb{R}$ satisfying $f(x) = 1$ for all $x \in [1/2, 3/2]$, and $f(x) = 0$ for all $x \in [-1/5, 1/5]$. For any $p \in QP_{\delta,r,s,k}(X)$ satisfying $\pi(p(t)) = p_m$ for some m and all $t \in \mathrm{bd}[0,1]^k$, $f(p)$ is a projection and defines an element $[f(p)]$ in $K_0((C^*_{L,0}(X) \otimes C_0((0,1)^k))^+)$.*

Proof. Let $0 < \delta < \frac{1}{100}$, $p \in QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$. Every λ in the spectrum of p is real and satisfies $|\lambda^2 - \lambda| < \delta$, hence

$$\lambda \in \left(\frac{1-\sqrt{1+4\delta}}{2}, \frac{1-\sqrt{1-4\delta}}{2}\right) \cup \left(\frac{1+\sqrt{1-4\delta}}{2}, \frac{1+\sqrt{1+4\delta}}{2}\right) \subset \left[-\frac{1}{5}, \frac{1}{5}\right] \cup \left[\frac{1}{2}, \frac{3}{2}\right].$$

So $f(p)$ is a projection in $(C^*_{L,0}(X) \otimes C_0([0,1]^k))^+ \otimes M_n(\mathbb{C})$ with $f(p)(t) = p_m$ for all $t \in \mathrm{bd}[0,1]^k$. Hence $f(p)$ can be viewed as a projection in $((C^*_{L,0}(X)^+ \otimes C_0((0,1)^k))^+ \otimes M_n(\mathbb{C}))$. $\square$

Lemma 5.9. *Let $0 < \delta < \frac{1}{100}$, p and q be element in $QP_{\delta,r,s,k}(X)$. If $a(t')$ is a (δ, r, s)-homotopy between p and q such that $\pi(a(t')(t)) = p_m$ for some m and all $t \in \mathrm{bd}[0,1]^k$, $t' \in [0,1]$, then $[f(p)] = [f(q)]$ in $K_0(C^*_{L,0}(X) \otimes C_0((0,1)^k))$.*

Proof. Given a (δ, r, s)-homotopy $a(t')$ between p and q, by the continuity of continuous functional calculus and Lemma 5.7 we have that $f(a(t'))$ is a continuous path of projections in $(C^*_{L,0}(X) \otimes C_0((0,1)^k) \otimes M_n(\mathbb{C}))^+$ connecting $f(p)$ and $f(q)$. Hence $[f(p)] = [f(q)]$ as K-theory elements. $\square$

Lemma 5.10. *For every $0 < \delta < \frac{1}{100}$, every element in $K_0(C^*_{L,0}(X) \otimes C_0((0,1)^k))$ can be represented as $[f(p_1)] - [f(p_2)]$, where $p_1, p_2 \in QP_{\delta,r,s,k}(X)$ for some $r > 0$ and $s > 0$.*

Proof. Note that every element in $K_0(C^*_{L,0}(X)) \otimes C_0((0,1)^k)$ can be represented as $[q_1] - [q_2]$, where $[q_1]$ and $[q_2]$ are projections in $(C^*_{L,0}(X) \otimes C_0((0,1)^k))^+ \otimes M_n(\mathbb{C})$ and $\pi(q_1) = \pi(q_2) = p_m$ for some m. By the approximation argument used in Lemma 4.10, we can find p_1 and p_2 in $QP_{\delta/2,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ for some $r > 0$ and $s > 0$, such that $||p_i - q_i|| < \delta/2$ for $i = 1, 2$, $\pi(p_1(t)) = \pi(p_2(t)) = p_m$ for all $t \in \mathrm{bd}[0,1]^k$. By Lemma 5.5, the linear homotopy $a_i(t')$ between p_i and q_i are (δ, r, s)-equivalences. It is clear that $\pi(a_i(t')(t)) = p_m$ for all $t \in \mathrm{bd}[0,1]^k$ and $t' \in [0,1]$. Hence by Lemma 5.9, we have $[f(p_i)] = [f(q_i)] = [q_i] \in K_0((C^*_L(X) \otimes C_0((0,1)^k))^+)$ for $i = 1, 2$. Therefore, $[q_1] - [q_2] = [f(p_1)] - [f(p_2)] \in K_0(C^*_L(X) \otimes C_0(0,1)^k)$. $\square$

Section 5.2 Controlled Unitaries $QU_{\delta,r,s,k}(X)$

In this section, we will approximate elements of $K_1(C^*_{L,0}(X))$ by quasi-unitaries. Most of the results are parallel to those in the last section.

Definition 5.11. *Let A be a C^*-algebra and δ be a positive number, an element u in A is called a δ-quasi-unitary if*

$$||u^*u - I|| < \delta, \quad \textit{and} \quad ||uu^* - I|| < \delta$$

Definition 5.12. *We denote $QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ to be the set of all continuous functions from $[0,1]^k$ to $C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ such that*

(1) *$f(t)$ is a delta-quasi-unitary for all $t \in [0,1]^k$;*

(2) *propagation$(f(t)) \leq r$ for all $t \in [0,1]^k$;*

(3) *f is piecewise smooth in t_i and $\left\|\frac{\partial f}{\partial t_i}(t)\right\| \leq s$ for all $t \in [0,1]^k$;*

(4) $||f(t) - I|| < \delta$ *for all* $t \in \mathrm{bd}([0,1]^k)$*;*

(5) $\pi(f(t)) = I$*, where* π *is the canonical homomorphism from* $C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ *to* $M_n(\mathbb{C})$

Definition 5.13. *Let* $QU_{\delta,r,s,k}(X)$ *to be the the direct limit of* $QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ *under the embedding* $u \to u \oplus I$*.*

Definition 5.14. *Let* u *and* v *be two elements in* $QU_{\delta,r,s,k}(X)$*. Now* u *is said to be* (δ, r, s)*-homotopic to* v *if there exists a piecewise smooth homotopy* $a(t')$ $(t' \in [0,1])$ *in* $QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ *such that*

(1) $a(0) = p$ *and* $a(1) = q$*;*

(2) $||a'(t')|| \leq s$*.*

Lemma 5.15. *Let* $0 < \delta < 1$ *and* u*,* $v \in QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ *satisfying* $||u - v|| < \delta$*, then the linear homotopy* $a(t') = (1-t')u + t'v$ *is a* $(2\delta, r, s)$*-homotopy between* u *and* v*.*

Proof. We calculate that

$$\begin{aligned} ||a^*(t')a(t') - I|| &= ||t'(u^*u - I) + (1-t')(v^*v - I) - (t' - t'^2)(u-v)^*(u-v)|| \\ &\leq t'\delta + (1-t')\delta + (t' - t'^2)\delta^2 \leq \delta + \frac{1}{4}\delta^2 < 2\delta. \end{aligned}$$

$$||\frac{\partial a}{\partial t'}|| = ||u - v|| < \delta < 1, \quad ||\frac{\partial a}{\partial t}|| \leq t'||\frac{\partial u}{\partial t}|| + (1-t')||\frac{\partial v}{\partial t}|| < s.$$

□

Lemma 5.16. *Let* $0 < \delta < 10^{-k}$*. Every* $u \in QU_{\delta,r,s,k}(X)$ *is* $(10^{k+1}\delta, r, 2^k s)$*-homotopic to some quasi-unitary* v *satisfying* $v(t) = \pi(v(t)) = I$ *for all* $t \in \mathrm{bd}([0,1]^k)$*.*

Proof. The proof is exactly the same as Lemma 5.6. □

Lemma 5.17. *Let* $0 < \delta < 10^{-k}$*. If* u *and* v *are two elements in* $QU_{\delta,r,s,k}(X)$ *such that* u *is* (δ, r, s)*-homotopic to* v*,* $u(t) = \pi(u(t))$ *and* $v(t) = \pi(v(t))$ *for all* $t \in \mathrm{bd}([0,1]^k)$*, then there exists a* $(10^{k+1}\delta, r, 2^k s)$*-homotopy* $a(t')$ $(t' \in [0,1])$ *between* u *and* v*, satisfying* $(a(t'))(t) = \pi((a(t'))(t))$ *for all* $t' \in [0,1]$ *and* $t \in \mathrm{bd}([0,1]^k)$*.*

Proof. The proof is exactly the same as Lemma 5.7. □

Lemma 5.18. *Every* $u \in QU_{\delta,r,s,k}(X)$ *satisfying* $u(t) = \pi(u(t)) = I$ *for all* $t \in \mathrm{bd}([0,1]^k)$ *defines an element* $[u]$ *in* $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$*.*

Proof. Since $||I - f^*f|| < \delta < 1$, so f^*f invertible. Hence f is invertible, $(f^*f)^{-1}f^* = f^*(ff^*)^{-1}$ is its inverse. So $[f]$ represent an element in $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$. □

Lemma 5.19. *Let* $0 < \delta < \frac{1}{2}$*,* u *and* v *be elements in* $QU_{\delta,r,s,k}(X)$ *satisfying* $u(t) = \pi(u(t)) = I$ *and* $v(t) = \pi(v(t)) = I$ *for all* $t \in \mathrm{bd}([0,1]^k)$*. If* u *is* (δ, r, s)*-homotopic to* v*, then* $[u] = [v]$ *in* $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$*;*

Proof. By Lemma 5.15, the elements on the linear path connecting u and v are all 2δ-unitaries, hence are all invertible. □

Lemma 5.20. *Every element in $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$ can be represented as $[u]$, where $u \in QU_{\delta,r,s,k}(X)$ for some $r > 0$ and $s > 0$ and $u(t) = \pi(u(t)) = I$ for all $t \in \mathrm{bd}([0,1]^k)$.*

Proof. Every element in $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$ can be represented as by a unitary v in $(C^*_{L,0}(X) \otimes C_0((0,1)^k) \otimes M_n(\mathbb{C}))^+$ for some n. By the approximation argument used in Lemma 4.10, we can find $u \in QU_{\delta,r,s,k}((C^*_{L,0}(X))^+ \otimes M_n(\mathbb{C}))$ for some $r > 0$ and $s > 10$ such that $||u - v|| < \delta/2$. By Lemma 5.15, u and v are (δ, r, s)-homotopic. Hence by Lemma 5.20, $[u] = [v] \in K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$. □

In K-theory, we have that $u \oplus u^*$ is homotopic to $I \oplus I$ for every unitary u. Similarly we have the following lemma.

Lemma 5.21. *Let $0 < \delta < 10^{-2}$. If $u \in QU_{\delta,r,s,k}(X)$, then there exists a $(3\delta, 2r, 8s)$-homotopy between $I \oplus I$ and $u \oplus u^*$.*

Proof. The linear homotopy between $I \oplus I$ and $uu^* \oplus I$ is a $(3\delta, 2r, 4s)$-homotopy and the rotation homotopy $(u \oplus I)R(t)(u^* \oplus I)R^*(t)$ connecting $uu^* \oplus I$ to $u \oplus u^*$ is a $(3\delta, 2r, 4s)$-homotopy. So the combination of two homotopies is a $(3\delta, 2r, 8s)$-homotopy. □

In K-theory, we know that if unitary equivalence and homotopy equivalence are stably equivalent concept for two unitaries. For the controlled obstructions, we have the following similar results.

Lemma 5.22. *Let $0 < \delta < 10^{-2}$. If p and q are elements in $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ and $u \in QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ is such that $||p - u^*qu|| < \delta$ for some small $\varepsilon > 0$, then $p \oplus 0$ is $(20\delta, 10r, 100s)$-homotopic to $q \oplus 0$, where $0 \in M_n(\mathbb{C})$.*

Proof. Let $w(t)$ denote the homotopy between $I \oplus I$ and $u \oplus u^*$. A very crude estimate yields that $w(t)(q \oplus 0)w^*(t)$ is a homotopy between $q \oplus 0$ and $uqu^* \oplus 0$, which is a $(10\delta, 10r, 100s)$-homotopy. The linear homotopy between $uqu^* \oplus 0$ and $p \oplus 0$ is a $(20\delta, 10r, 200s)$-homotopy. So the combination of two homotopies is a $(20\delta, 10r, 400s)$ -homotopy. □

Lemma 5.23. *Let $0 < \delta < 1/100$. If p and q are two (δ, r, s)-homotopic elements in $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$, then there exists $u \in QU_{\delta,C_1(\delta,s)r,C_2(s),k}(C^*_{L,0}(X)^+ \times M_n(\mathbb{C}))$ such that $||p - u^*qu|| < C_3(s)\delta$, where $C_1(\delta, s)$ depends only on δ and s, $C_2(s)$ and $C_3(s)$ depend only on s.*

Proof. The idea is that if two quasi-projections are close, then they are "quasi-unitarily equivalent". The homotopy provides us a multitude of quasi-unitaries, we will "normalized" them a little bit and take the product. The bound for the speed of homotopy helps us to control the number of quasi-unitaries in the product and it is here that we need the speed to be controlled.

Let $a(t')$ be a (δ, r, s)-homotopy between p and q. Since the speed of $a(t')$ is bounded by s, we have an equally spaced partition

$$0 = t'_0 < t'_1 < \cdots < t'_{m_s} = 1$$

such that
$$||a(t'_{i+1}) - a(t'_i)|| < \frac{1}{100},$$
where m_s depends only on s. Consider

$$u_i = [(2a(t'_{i+1}) - I)(2(a(t'_i) - I)) + I]/2,$$

we have that

$$\begin{aligned}
||1 - u_i|| =&||(2a(t'_{i+1}) - I)(a(t'_{i+1}) - a(t'_i)) + 2(a(t'_{i+1}) - a^2(t'_{i+1}))||\\
\leq&(2||a(t'_{i+1})|| + 1) \cdot ||a(t'_{i+1}) - a(t'_i)|| + 2||a(t'_{i+1}) - a^2(t'_{i+1})||\\
\leq&(2 \cdot 2 + 1) \cdot \frac{1}{100} + 2 \cdot \frac{1}{100} < \frac{1}{10}.
\end{aligned}$$

It follows that
$$||1 - u_i^* u_i|| = ||1 - u_i^*|| + ||u_i^*|| \cdot ||1 - u_i|| < \frac{1}{10} + 2 \cdot \frac{1}{10} = \frac{3}{10}.$$

We also have

$$\begin{aligned}
&||a(t'_{i+1})u_i - u_i a(t'_i)||\\
=&||(a(t'_{i+1})^2 a(t'_i) - 2a(t'_{i+1})a(t'_i)) + 2a(t'_{i+1})(a^2(t'_i) - a(t'_i)) - a(t'_{i+1})^2 + a(t'_{i+1}) + a(t'_i)^2 - a(t'_i)||\\
\leq&2||a(t'_{i+1})^2 - a(t'_{i+1})|| \cdot ||a(t'_i)|| + 2||a(t'_{i+1})|| \cdot ||a^2(t'_i) - a(t'_i)|| + ||a^2(t'_{i+1}) - a(t'_{i+1})|| + ||a^2(t'_i) - a(t'_i)||\\
\leq&4\delta + 4\delta + \delta + \delta = 10\delta.
\end{aligned}$$

Let $P_l(x)$ be the l-th Taylor polynomial for $\frac{1}{\sqrt{1-x}}$ at 0. Choose l_0 such that

$$\left| P_{l_0}^2(x) - \left(\frac{1}{\sqrt{1-x}}\right)^2 \right| < \frac{\delta}{4 \cdot 2^{m_s}}$$

for all $x \in [0, \frac{3}{10}]$. Let
$$w_i = u_i P_{l_0}(1 - u_i^* u_i).$$

We will show that

$$u = w_{m_{s-1}} \cdots w_0$$

is the the quasi-unitary we are looking for.

First we want to show the speed of u is at most $C_2(s)$. Since the speed of $a(t_i)$ is at most s, hence the speed of u_i is at most $10s$, so the speed of $I - u_i^* u_i$ is at most $40s$. Notice that $P_l(x)$ and $P'_l(x)$ has nonnegative coefficients, hence the sequence of $P_l(x)$ and $P'_l(x)$ is uniformly bounded by $\frac{1}{\sqrt{1-x}}$ and

$\left(\frac{1}{\sqrt{1-x}}\right)'$ respectively, and

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}P_{l_0}(I-u_i^*u_i)\right\|\\ \leq &P'_{l_0}(||1-u_i^*u_i||)\cdot\left\|\frac{\partial}{\partial t}(I-u_i^*u_i)\right\|\\ < &\sup_{x\in[0,\frac{3}{10}]}\left(\frac{1}{\sqrt{1-x}}\right)'\cdot 40s\leq 40s.\end{aligned}$$

Hence $\left\|\frac{\partial}{\partial t}w_i\right\|\leq 100s$. So

$$\left\|\frac{\partial}{\partial t}u\right\|\leq C_2 s.$$

for some $C_2(s)$ depends only on s.

Next we want to estimate the propagation of u. It is bounded by $C_1(\delta,s)r$, where C_1 depends only on l_0 and m_s, hence depends only on δ and s.

To show u is a δ-unitary, we note that

$$||1-w_i^*w_i||=||I-u_i^*P_{l_0}(I-u_i^*u_i)^2u_i||\leq\left\|u_i^*\left(\left(\frac{1}{\sqrt{1-(1-u_i^*u_i)}}\right)^2-P_l^2(I-u_i^*u_i)\right)u_i\right\|\leq\frac{\delta}{2^{m_s}}.$$

Hence

$$\begin{aligned}&||1-u^*u||\\ =&||I-w_0^*\cdots w_{m_s-1}^*w_{m_s-1}\cdots w_0||\\ \leq&||I-w_0^*\cdots w_{m_s-2}^*w_{m_s-2}\cdots w_0||+||w_0^*\cdots w_{m_s-2}^*||\cdot||1-w_{m_s-1}^*w_{m_s-1}||\cdot||w_0\cdots w_{m_s-2}||\\ \leq&||1-w_0^*\cdots w_{m_s-2}^*w_{m_s-2}\cdots w_0||+2^{m_s-1}\cdot\frac{\delta}{2^{m_s}}\leq\cdots\leq(1+\cdots+2^{m_s-1})\cdot\frac{\delta}{2^{m_s}}<\delta.\end{aligned}$$

To finish the proof, we need to check $||p-u^*qu||<C_3\delta$. Note that

$$\begin{aligned}&||u_i^*u_ia(t_i')-a(t_i')u_i^*u_i||\\ \leq&||u_i^*||\cdot||u_ia(t_i')-a(t_{i+1}')u_i||+||u_i^*a(t_{i+1}')-a(t_i')u_i^*||\cdot||u_i||\\ <&2\cdot 10\delta+10\delta\cdot 2=40\delta.\end{aligned}$$

Hence

$$\begin{aligned}&||a(t_i')(I-u_i^*u_i)^n-(I-u_i^*u_i)^na(t_i')||\\ \leq&||(a(t_i')u_i^*u_i-u_i^*u_ia(t_i'))(I-u_i^*u_i)^{n-1}||+||(I-u_i^*u_i)(a(t_i')u_i^*u_i-u_iu_i^*a(t_i'))(I-u_i^*u_i)^{n-2}||\\ &\cdots+||(I-u_i^*u_i)^{n-1}(a(t_i')u_i^*u_i-u_iu_i^*a(t_i'))||<40\delta n\left(\frac{3}{10}\right)^{n-1}.\end{aligned}$$

Therefore

$$||a(t_i')P_{l_0}(I-u_i^*u_i)-P_{l_0}(I-u_i^*u_i)a(t_i')||<40\delta P_{l_0}'\left(\frac{3}{10}\right)$$

This together with the definition of w_i, implies that

$$\begin{aligned}&||a(t_{i+1}')w_i-w_ia(t_i')||\\ \leq&||a(t_{i+1}')u_i-u_ia_i(t_i')||\cdot||P_{l_0}(I-u_i^*u_i)||+||u_i||\cdot||a(t_i')P_{l_0}(I-u_i^*u_i)-P_{l_0}(I-u_i^*u_i)a(t_i')||\\ \leq&10\delta\cdot 2+2\cdot 40\delta P_{l_0}'\left(\frac{3}{10}\right)<100\delta.\end{aligned}$$

Hence

$$\begin{aligned}||w_i^*a(t_{i+1}')w_i-a(t_i')||\leq&||w_i^*||\cdot||a(t_{i+1}')w_i-w_ia(t_i')||+||w_i^*w_i-I||\cdot||a(t_i')||\\ <&2\cdot 100\delta+\frac{\delta}{2^{m_s}}\cdot 2<300\delta.\end{aligned}$$

So

$$\begin{aligned}&||p-u^*qu||\\ =&||p-w_0^*\cdots w_{m_{s-1}}^*a(t_{m_s}')w_{m_{s-1}}\cdots w_0||\\ \leq&||p-w_0^*\cdots w_{m_{s-2}}^*a(t_{m_{s-1}}')w_{m_{s-2}}\cdots w_0||+||w_0^*\cdots w_{m_{s-2}}^*||\cdot||w_{s-1}^*a(t_{m_s}')w_{s-1}-a(t_{m_{s-1}}')||\cdot||w_{s-2}\cdots w_0||\\ \leq&||p-w_0^*\cdots w_{m_{s-2}}^*a(t_{m_{s-1}}')w_{m_{s-2}}\cdots w_0||+2^{m_{s-1}}\cdot 300\delta\\ \leq&\cdots\leq(1+\cdots+2^{m_{s-1}})300\delta<2^{m_s}\cdot 300\delta.\end{aligned}$$

We can take $C_3(s)=2^{m_s}300$. □

Remark 5.24. *Since $P_{l_0}(I-u_i^*u_i)$ is self-adjoint,*

$$\|P_{l_0}(I-u_i^*u_i)^2-I\|\leq\left\|P_{l_0}(I-u_i^*u_i)^2-\left(\frac{1}{\sqrt{I-(I-u_i^*u_i)}}\right)^2\right\|+\|I-(u_i^*u_i)^{-1}\|\leq 3\delta.$$

Hence $P_{l_0}(I-u_i^*u_i)$ is $(6\delta,2l_0r,100s)$-equivalent to I. Hence $w_i=P_{l_0}(I-u_i^*u_i)$ is $(12\delta,3l_0r,300s)$-equivalent to w_i. In general for each $\delta'>0$, we choose l_0 large enough in such a way that

$$\left|P_{l_0}(x)^2-\left(\frac{1}{\sqrt{1-x}}\right)^2\right|<\frac{\delta'}{3},$$

then u_i is $(12\delta,3l_0r,300s)$-equivalent to a δ'-quasi-unitary w_i. The trade for decreasing δ to δ' is that r increases to $3l_0r$ and s increases to $300s$, where l_0 depends only on δ'.

Remark 5.25. *As in K-theory, every homotopy of projections is implemented by a homotopy of unitaries starting from I. We can strengthen the above lemma a little bit. Let $a(t')$ be the (δ,r,s)-homotopy between p and q, then there exists a homotopy $w(t')$ in $QU_{\delta_2,C_1(\delta_3,s)r_3,C_2(s),k}(X)$ such that $w(0)=I$,*

$||p - u^*(t')a(t')u(t')|| < C_3(s)\delta_3$. *Let*

$$u_i(t') = \begin{cases} I & t' \in [0, t_i] \\ \frac{(2a(t')-I)(a(t'_i)-I)+I}{2} & t' \in [t'_i, t'_{i+1}] \\ u_i & t' \in [t'_{i+1}, 1] \end{cases}$$

We similarly have that

$$\|1 - u_i^*(t')u_i(t')\| < \frac{3}{10}, \qquad \left\|\frac{\partial}{\partial t}u\right\| \leq 2s, \qquad \left\|\frac{\partial}{\partial t'}u\right\| \leq 2s, \qquad \forall\, t' \in [0,1]$$

and that

$$\|a(t')u_i - u_i a(t'_i)\| \leq 10\delta \qquad \forall\, t \in [t'_i, t'_{i+1}]$$

We define

$$w_i(t') = u_i(t')P_{l_0}(1 - u_i^*(t')u_i(t')), \qquad u(t') = w_{m_{s-1}}(t') \cdots w_0(t').$$

We can similarly proof that $w_i(t')$ satisfies the desired property.

Section 5.3 Controlled Suspensions

In this section, we will further study how QP relates to QU. As in K-theory, K_0 and K_1 are related by the suspension map. In this section, we will demonstrate a similar result.

Definition 5.26. *For any proper metric space X, let $GQP_{\delta,r,s,k}(X)$ to be the set of formal difference $p - q$, where p, $q \in QP_{\delta,r,s,k}(X)$ for some n, and $\pi(p) = \pi(q)$.*

Definition 5.27. *Two elements $p - q$ and $p' - q'$ in $GQP_{\delta,r,s,k}(X)$ are said to be (δ, r, s)-homotopic if $p' \oplus q$ and $q' \oplus p$ are (δ, r, s)-homotopic. An element $p - q$ is said to be (δ, r, s)-homotopic to 0 if $p - q$ is (δ, r, s)-homotopic to $I \oplus 0 - I \oplus 0$ for some n that $I, 0 \in M_n(\mathbb{C})$.*

We use p_n to denote an infinite matrix with the unit in the first n places along the diagonal.

Recall that every element in $K_0(A)$ can be represented as $[x - p_n] - [p_n]$ for some n and $x \in M_{2n}(A)$.

Lemma 5.28. *Let $p - q \in GQP_{\delta,r,s,k}(X)$. Then any element $p - q \in GQP_{\delta,r,s,k}(X)$ is $(10\delta, r, 10s)$-homotopic to an element $p' - p_n$ for some nonnegative integer n and some $p' \in QP_{\delta,r,s,k}(C^*_{L,0}(X) \otimes M_{2n}(\mathbb{C}))$.*

Proof. Let p and q are in $QP_{\delta,r,s,k}(C_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$, I is the identity matrix in $M_n(\mathbb{C})$. Then

$$||(I - q)^2 - (I - q)|| = ||q^2 - q|| < \delta.$$

The homotopy $a(t) = ((I - q) \oplus 0) + R(t)(q \oplus 0)R^*(t)$ connects $I \oplus 0$ and $(I - q) \oplus q$ is a $(10\delta, r, 10s)$-homotopy. We can take a path u_t of scalar unitary matrices in $M_{2n}(\mathbb{C})$ such that $u_0 = I$, $u_1\pi(p \oplus (I - q))u_1^* = I \oplus 0$. So $u_1(p \oplus (I - q))u_1^* \oplus q$ is $(10\delta, r, 10s)$-homotopic to $(p \oplus (I \oplus 0))$. Hence $p - q$ is $(10\delta, r, 10s)$- homotopic to $u_1(p \oplus (1 - q))u_1^* - (I \oplus 0)$. ∎

For any $u \in QU_{\delta,r,s,k}(X)$, let $z_t(u)$ be homotopy connecting $I \oplus I$ to $u \oplus u^*$ demonstrated in Lemma 5.21. Let

$$e_t(u) = z_t(u)(I \oplus 0)z_t^*(u)$$

It is simply to check that $e_t(u) \in QP_{100\delta,100r,100s,k+1}(X)$ and $\pi(e_t(u)) = I \oplus 0$. So we can define map from $QU_{\delta,r,s,k}(X)$ to $GQP_{100\delta,100r,100s,k+1}(X)$ by

$$\theta(u) = e_t(u) - (I \oplus 0)$$

where t is the $(k+1)^{\text{th}}$ suspension parameter.

The following lemma shows that the suspension map is well-defined in some sense.

Lemma 5.29. *For any $0 < \delta < \frac{1}{100}$, $r > 0$, $s > 1$, there exist $0 < \delta_1 < \delta$, $0 < r_1 < r$ and $s_1 > 0$ such that if two elements u and v in $QU_{\delta_1,r_1,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ are (δ_1, r_1, s)-homotopic, then $\theta(u)$ and $\theta(v)$ are (δ, r, s_1)-homotopic, where δ_1 depends only on δ, r_1 depends only on r, and s_1 depend only on s.*

Proof. Let us, for the moment, assume that δ_1 and r_1 are small enough and have been determined, we will demonstrate a homotopy and check to see where it lives, and then pick δ_1, r_1 accordingly, s_1 is easily determined by seeing how the speed grows. We will apply this kind of argument many times in the proofs of many following facts.

Let $w(t)$ be the homotopy realizing the (δ_1, r_1, s)-homotopy between u and v. It is easy to check that the homotopy $w(t)^*u$ between u^*u and v^*u is a $(3\delta_1, 2r_1, 4s)$-homotopy. By Lemma 5.15, the linear homotopy between I and u^*u is a $(6\delta_1, 2r_1, 4s)$-homotopy. So the combination of these two homotopies is a $(6\delta_1, 2r_1, 8s)$-homotopy $b(t)$ between I and v^*u, and we denoted it by $a(t)$. We similarly define a $(6\delta_1, 2r_1, 8s)$-homotopy between I and v^*u by combining the linear homotopy between I and uu^* with the homotopy $w(t)u^*$. Define

$$x_t = z_t(v)(a(t) \oplus b(t))z_t^*(v)$$

where z_t is as in the definition of the map θ. Since by Lemma 5.21, we know that $z_t^*(v)$ is a $(3\delta_1, 2r_1, 8s)$-homotopy. It is straightforward to check that x_t is a $(21\delta_1, 6r_1, 96s)$-homotopy. We also have that $x_0 = I \oplus 0$, $||x_1 - I \oplus I|| = ||vv^*uu^* \oplus v^*vu^*u - I \oplus I|| < 3\delta$. So $x_t \in QU_{100\delta_1,100r_1,100s,k+1}(X)$. It is straightforward to check that

$$||x_t e_t(u) x_t^* - e_t(v)|| < 48\delta_1.$$

By Lemma 5.22, we can safely conclude that $e_t(u) \oplus 0_{2n}$ is $(10^5\delta_1, 10^5 r_1, 10^5 s)$-homotopic to $e_t(v) \oplus 0_{2n}$. Hence we can take $\delta_1 = \frac{\delta}{10^5}$, $r_1 = \frac{r}{10^5}$, $s_1 = 10^5 s$. ☐

The following two lemmas will show that the suspension map is injective and surjective respectively in the asymptotic sense.

Lemma 5.30. *For any $0 < \delta < \frac{1}{100}$, $r > 0$, $s > 1$, there exist $0 < \delta_2 < \delta$, $0 < r_2 < r$ and $s_2 > 0$ for which if u and v are two elements in $QU_{\delta_2,r_2,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ such that $\theta(u)$ and $\theta(v)$ are (δ_2, r_2, s)-homotopic, then u and v are (δ, r, s_2)-homotopic, where δ_2 depends only on δ and s; r_2 depends only on δ, r and s; and s_2 depends only on s.*

Proof. As we have seen many examples how to deal with quasi-projections and quasi-unitaries. We will quicken our pace a little bit. We will not determine some universal constants explicitly, but only clearly state how they depend with δ, r, s and so on.

By Lemma 5.23, there exists $x' \in QU_{\delta_2, C_1(\delta_2,s)r_2, C_2(s), k+1}(C^*_{L,0}(X)^+ \otimes M_{4n}(\mathbb{C}))$ such that

$$||x'(e_t(u) \oplus I_n \oplus 0)x'^* - (e_t(v) \oplus I_n \oplus 0)|| < C_3(s)\delta_2.$$

Since $I_n \oplus 0 = e_t(I_n)$ and that $e_t(u) \oplus e_t(I_n)$ is unitarily equivalent to $e_t(u \oplus I)$, we can find $x \in QU_{\delta_2, C_1(\delta_2,s)r, C_2(s), k+1}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ such that

$$||x(e_t(u \oplus I_n))x^* - e_t(v \oplus I_n)|| < C_3(s)\delta_2.$$

This easily implies that

$$||z_t^*(v \oplus I_n)x_t z_t(u \oplus I_n)(I_{2n} \oplus 0) - (I_{2n} \oplus 0)z_t^*(v \oplus I_n)x_t z_t(u \oplus I_n)|| < 1000C_3(s)\delta_2.$$

where x is identified with a piecewise smooth family of elements x_t in $QU_{\delta_2, C_1(\delta_2,s)r_2, C_2(s), k}(C^*_{L,0}(X)^+ \otimes M_{4n}(\mathbb{C}))$, $t \in [0,1]$.

If we write $z_t^*(v \oplus I_n)x_t z_t(u \oplus I_n) = \left(\begin{smallmatrix} c_t & g_t \\ h_t & d_t \end{smallmatrix}\right)$, we have that $||g_t|| < 1000C_3(s)\delta_2$, $||h_t|| < 1000C_3(s)\delta_2$. We can easily check that $z_t^*(v \oplus I_n)x_t z_t^*(u \oplus I_n)$ is a $100\delta_2$-quasi-unitary for every $t \in [0,1]$. So $||c_t^* c_t + h_t^* h_t - I_{2n}|| < 100\delta_2$. Hence

$$||c_t^* c_t - I_{2n}|| = ||c_t^* c_t + h_t^* h_t - I_{2n}|| + ||h_t^* h_t|| < 100\delta_2 + 10^6 C_3(s)^2\delta_2.$$

Since $||x_0 - I_{4n}|| < \delta_2$, $||x_1 - I_{4n}|| < \delta_2$. Thus

$$||z_1^*(v \oplus I_n)x_0 z_1(u \oplus I_n) - I_{4n}|| < 10\delta_2$$

$$\|z_1^*(v \oplus I_n)x_1 z_1(u \oplus I_n) - z_1^*(v \oplus I_n)z_1(u \oplus I_n)\| < 10\delta_2$$

If we write the left hand side of above inequalities in 2 by 2 matrices, and compare the left-top elements, we have that

$$||c_0 - I_{2n}|| < 10\delta_2 \quad \text{and} \quad ||c_1 - (v^* \oplus I_n)(u \oplus I_n)|| < 10\delta_2.$$

Combine the linear homotopy between I_{2n} and c_0, the homotopy c_t, the linear homotopy between c_1 and $(v^* \oplus I_n)(u \oplus I_n)$, we get a $(M_1(s)\delta_2, M_2(\delta_2, s)r_2, M_3(s))$-homotopy $a(t)$ between I_{2n} and $(v^* \oplus I_n)(u \oplus I_n)$, for some constant M_1, M_2 and M_3.

Let $b(t)$ be the linear homotopy connecting $vv^* \oplus I_n$ and I_{2n}. Then the combination of homotopies $(v \oplus I)a(t)$ and $b(t)(u \oplus I)$ is a $(\tilde{M}_1(s)\delta_2, \tilde{M}_2(\delta_2, s)r_2, \tilde{M}_3(s))$- homotopy between $(v \oplus I_n)$ and $(u \oplus I_n)$. For some constant $\tilde{M}_1$, $\tilde{M}_2$ and $\tilde{M}_3$. Now we get the desired result by picking appropriate δ_2, r_2 and s_2. ∎

Lemma 5.31. *For any $0 < \delta < \frac{1}{100}$, $r > 0$, $s > 10$, there exist $0 < \delta_3 < \delta$, $0 < r_3 < r$ and $s_3 > s$, such that for any $p - p_n \in GQP_{\delta_3, r_3, s, k}(C^*_{L,0}(X)^+ \otimes M_{2n}(\mathbb{C}))$, there exists $u \in QU_{\delta, r, s_3, k}(X)$ for which $\theta(u)$ is*

(δ, r, s_3)*-homotopic to* $p - p_m$*, where* δ_3 *depends only on* δ *and* s*,* r_3 *depends only on* δ*,* r *and* s*; and* s_3 *depends only on* s*.*

Proof. We identify p as a piecewise smooth path in $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_{2n}(\mathbb{C}))$. By Remark 5.25, there exists a homotopy $w(t)$ in $QU_{\delta_3, C_1(\delta_3,s)r_3, C_2(s),k}(C^*_{L,0}(X)^+ \otimes M_{2n}(\mathbb{C}))$ such that $w(0) = I_{2n}$, $||p(0) - w(t)^* p(t) w(t)|| < C_3(s)\delta_3$. It follows that

$$||w(1)p(0) - p(1)w(1)|| < 10C_3(s)\delta_3.$$

Since $||p(0) - I_n \oplus 0|| < \delta_3$, $||p(1) - I_n \oplus 0|| < \delta_3$, we have that

$$||w(1)(I_n \oplus 0) - (I_n \oplus 0)w(1)|| < 4\delta_3 + 10C_3(s)\delta_3.$$

So we can write $w(1) = \left(\begin{smallmatrix} u & g \\ h & v \end{smallmatrix}\right)$, where $||g|| < 4\delta_3 + 10C_3(s)\delta_3$, $||h|| < 4\delta_3 + 10C_3(s)\delta_3$. Hence u and v are $(1 + (4\delta_3 + 10C_3(s))^2)\delta_3$-quasi-unitaries. Let

$$y_t = (w(t) \oplus I_n)(I_n \oplus z_t^*(v)w^*(t))(z_t^*(u) \oplus I_n).$$

It is straightforward to check that $y_0 = I$ and there exists a universal constant $\tilde{C}_3(s)$ depend only on s such that y_t is a $\tilde{C}_3(s)$-quasi-unitary and that

$$||y_1 - I|| < \tilde{C}_3(s)\delta_3, \qquad ||y_t(z_t(u) \oplus I_n)(I_n \oplus 0)(z_t(u)^* \oplus I_n)y_t^* - (p \oplus 0)|| < \tilde{C}_3(s)\delta_3.$$

Then apply Lemma 5.22, we have that $e_t(u) \oplus 0$ and $p \oplus 0$ are homotopic, hence $e_t(u) \oplus p_n$ and $p \oplus p_n$ are homotopic. Again, by picking appropriate r_3, and s_3, we get the desired result. ☐

Section 5.4 Invariance under Strong Lipschtiz Homotopy

We have seen in Theorem 4.8 that the K-theory of localization algebra is invariant under strong Lipschitz homotopy. For the controlled obstructions, we have the following similar result.

Lemma 5.32. *Let* f *and* g *be two proper Lipschitz maps from* X *to* Y*. Assume that* f *is strongly Lipschtiz homotopic to* g*. There exists* $S_0 > 0$*,* $C_0 > 0$ *such that for any* $u \in QU_{\delta,r,s,k}(X)$*, there exists a* $(C_0\delta, C_0 r, C_0 s)$*-homotopy between* $w(0) = \mathrm{Ad}(V_f)(u) \oplus I$*, and* $w(1) = \mathrm{Ad}(V_g)(u) \oplus I$*, where* C_0 *depends only on the Lipschitz constant* C *of the strong Lipschitz homotopy between* f *and* g*.*

Proof. Let F be the strong Lipschitz homotopy between f and g such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$.

There exist a sequence of nonnegative numbers $\{t_{i,j} \mid 0 \le i \le \infty, 0 \le j < \infty\}$ and a sequence of decreasing positive number $\{\varepsilon_i\}_{i=0}^{\infty}$ such that

(1) $t_{0,j} = 0$, $t_{i+1,j} \ge t_{i,j}$, $t_{i,j+1} \le t_{ij}$.

(2) For each j, there exists N_j such that $t_{i,j} = 1$ for all $i \ge N_j$.

(3) $d(F(x, t_{i,j}), F(x, t_{i+1,j})) \le \varepsilon_j = \frac{r}{j+1}$ and $d(F(x, t_{i,j}), F(x, t_{i,j+1})) \le \varepsilon_j$ for all $x \in X$.

Let $V_{i,j}$ be an isometry from H_X to $H_X \oplus H_X$ such that

$$\text{Support}(V_{i,j}) \subset \{(x_2, x_1) \in X \times X : d(x_2, F(x_1, t_{i,j})) \leq \varepsilon_j\}$$

Define a family of isometry $H_X \to H_X \oplus H_X$ by

$$V_i(t) = \begin{pmatrix} V_{i,j} \cos^2 \frac{\pi}{2}t + V_{i,j+1} \sin^2 \frac{\pi}{2}t \\ (V_{i,j+1} - V_{i,j}) \sin \frac{\pi}{2}t \cos \frac{\pi}{2}t \end{pmatrix} \qquad \text{if} \quad t \in [j, j+1]$$

where H_X is the universal H_X-module. Consider

$$u_i(t) = V_i(t) u(t) V_i^*(t) + (I - V_i(t) V_i^*(t)),$$

where I is the identity map on $H_X \oplus H_X$. So $u = \text{Ad}^+(V_f)(u)$, $u_\infty = \text{Ad}^+(V_g)(u)$. For each i, define n_i to be the largest integer j satisfying $i \geq N_j$ if $\{j : i \geq N_j\} \neq \varnothing$, and define n_i to be 0 otherwise. We can choose $V_{i,j}$ such that $u_i(t) = u_\infty(t)$ when $t \leq n_i$ by taking $V_{i,j} = V_{\infty,j}$, whenever $t_{i,j} = 1$. Define

$$w_i(t) = \begin{cases} u_i(t)(u_\infty(t))^* & \text{if} \quad t \geq n_i \\ (n_i - t)I + (t - n_i + 1) u_i(t)(u_\infty(t))^* & \text{if} \quad n_i - 1 \leq t \leq n_i \\ I & \text{if} \quad 0 \leq t \leq n_i - 1. \end{cases}$$

Consider

$$\begin{aligned} a &= \oplus_{i=0}^\infty (w_i \oplus I) \\ b &= \oplus_{i=0}^\infty (w_{i+1} \oplus I) \\ c &= (I \oplus I) \oplus_{i=1}^\infty (w_i \oplus I) \end{aligned}$$

where I is the identity operator on $H_X \oplus H_X$. Similar to the proof of Theorem 4.8, we can verify that a, b, c are elements in $C_L^*(X, (H_X \oplus H_X \oplus H_X \oplus H_X)^\infty)^+$, and it is easily to check that a, b, c are elements in $QU_{C_1\delta, C_1 r, C_1 s, k}(Y)$ for some constant C_1 depending only on the Lipschitz constant C.

We will construct a homotopy between a and b by constructing a homotopy $w_{i,i+1}(t')$ between $w_i(t) \oplus I$ and $w_{i+1}(t) \oplus I$ for each i. The idea is simple enough, we will replace $V_i(t)$ in $w_i(t) \oplus I$ by $V_{i+1}(t)$ through a "rotation" homotopy, then joining with $w_{i+1}(t)$ linearly. To be more precise we define that

$$u_{i,i+1}(t') = \begin{pmatrix} V_i \cos^2 \pi t' + V_{i+1} \sin^2 \pi t' \\ (V_i - V_{i+1}) \cos \pi t' \sin \pi t' \end{pmatrix} u \begin{pmatrix} V_i \cos^2 \pi t' + V_{i+1} \sin^2 \pi t' \\ (V_i - V_{i+1}) \cos \pi t' \sin \pi t' \end{pmatrix}^*$$

$$w_{i,i+1}(t') = \begin{cases} I \oplus I & \text{if} \quad 0 \leq t \leq n_i - 1 \\ (n_i - t)(I \oplus I) - (t - n_i + 1) u_{i,i+1}(t)(t')(u_\infty(t) \oplus I)^* & \text{if} \quad n_{i-1} \leq t \leq n_i \\ u_{i,i+1}(t)(t')(u_\infty^*(t) \oplus I) & \text{if} \quad t \geq n_i \end{cases}$$

$$w_{i,i+1}(t') = (2t' - 1)(w_{i+1} \oplus I) + (2 - 2t') w_{i,i+1}\left(\frac{1}{2}\right) \qquad \text{if} \quad 0 \leq t' \leq \frac{1}{2}$$

$$h_1(t') = \oplus_{i=0}^{\infty} w_{i,i+1}(t') \qquad \text{if} \quad 0 \le t' \le 1.$$

Similar to the proof of Theorem 4.8, we can check that for each $t' \in [0, \frac{1}{2}]$, $h_1(t') \in C_{L,0}(X, (H_X \oplus H_X \oplus H_X \oplus H_X)^{\infty})$. It is easily to check that $h_1(t)$ is a $(C_2\delta, C_2 r, C_2 s)$-homotopy for some $C_2 > C_1$, where C_2 depends only on C.

Next we will construct a homotopy between b and c, where I is the identity map on $H_X \oplus H_X$. Let V be the isometry on $(H_X \oplus H_X)^{\infty}$ by right translation, then $VbV^* = c$, consider the homotopy

$$h_2(t') = R(t') \begin{pmatrix} I & \\ & V \end{pmatrix} R(t')^* \begin{pmatrix} b & \\ & I \end{pmatrix} R(t') \begin{pmatrix} I & \\ & V^* \end{pmatrix} R(t')^*$$

where I is the identity operator on $(H_X \oplus H_X)^{\infty}$. It is easily to check that $h_2(t')$ is a $(C_3\delta, C_3 r, C_3 s)$-homotopy for some constant C depends only on C.

Finally, we define $w(t')$ to be the homotopy obtained by combining the following homotopies:

(1) The linear homotopy between

$$(u_0 \oplus I) \oplus_{i=1}^{\infty} (I \oplus I) \quad \text{and} \quad c^* a((u_\infty \oplus I) \oplus_{i=1}^{\infty} (I \oplus I)).$$

(2) $h_2^*(1-t')a((u_\infty \oplus I) \oplus_{i=1}^{\infty} (I \oplus I))$.

(3) $h_1^*(1-t')a((u_\infty \oplus I) \oplus_{i=1}^{\infty} (I \oplus I))$.

(4) The linear homotopy between

$$a^* a((u_\infty \oplus I) \oplus_{i=1}^{\infty} (I \oplus I)) \quad \text{and} \quad (u_\infty \oplus I) \oplus_{i=1}^{\infty} (I \oplus I).$$

□

Section 5.5 Controlled Cutting and Pasting

In this section, we will study the cutting and pasting techniques for controlled obstructions.

Definition 5.33. *Let X be a proper metric space and let X_i for $i = 1, 2$ be metric subspaces. The triple $(X; X_1, X_2)$ is said to satisfy excision condition if*

1. *$X = X_1 \cup X_2$ where X_i is closed subset of X with interior of X_i dense in X_i.*
2. *For any $r > 0$, $\mathrm{bd}_r(X_1) \cap \mathrm{bd}_r(X_2) = \mathrm{bd}_r(X_1 \cap X_2)$.*

We remark that if X is geodesic complete proper metric space. In particular, X is a locally compact simplicial polyhedron, we have that for any decomposition $X = X_1 \cup X_2$, where X_i is closed, the condition 2 in the above definition always holds.

Given $(X; X_1, X_2)$ satisfying the excision condition, we might want to construct a boundary map $\partial : QU_{\delta,r,s,k}(X) \to GQP_{\delta',r',s',k}(X_1 \cap X_2)$. However, in order to account for the propagation, we would "fatten" $X_1 \cap X_2$ a little bit. To be more precise, for $u \in QU_{\delta,r,s,k}(X)$ define $u_{X_1} = \chi_{X_1} u \chi_{X_1} + \chi_{X - X_1}$. Then let

$$w = \begin{pmatrix} I & u_{X_1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -u_{X_1}^* & I \end{pmatrix} \begin{pmatrix} I & u_{X_1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and again note that

$$w = \begin{pmatrix} u_{X_1} & 0 \\ 0 & u^*_{X_1} \end{pmatrix} + \begin{pmatrix} u_{X_1}(I - u^*_{X_1}u_{X_1}) & u_{X_1}u^*_{X_1} - I \\ I - u^*_{X_1}u_{X_1} & 0 \end{pmatrix}$$

A very rough estimate yields $||w|| \leq 10$. Hence $||w^*w|| \leq 100$. So $w_1^*w_1 \leq 100$. Note that w is invertible, hence w^*w is positive and invertible. For

$$w^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & -u_{X_1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ u^*_{X_1} & I \end{pmatrix} \begin{pmatrix} I & -u_{X_1} \\ 0 & I \end{pmatrix}$$

We similarly have $||w^{-1}|| \leq 10$, $||w^{*-1}w^{-1}|| \leq 100$, $w^{*-1}w^{-1} \leq 100$. So $w^*w \geq \frac{1}{100}$. Hence

$$0 \leq I - \frac{w^*w}{100} \leq 1 - \frac{1}{100^2}.$$

This observation allow us to convert w into a δ-quasi-unitary. In fact, let $P_l(x)$ be the l-th Taylor polynomial for $\frac{1}{10\sqrt{1-x}}$. Choose $l(\delta)$ to be the smallest integer such that

$$\left| P_l(x) - \frac{1}{10\sqrt{1-x}} \right| < \frac{\delta}{10^4}, \qquad \forall\, x \in \left[0, 1 - \frac{1}{100^2}\right],$$

which easily implies that

$$\left| P_l(x)^2 - \left(\frac{1}{10\sqrt{1-x}}\right)^2 \right| < \frac{\delta}{100}, \qquad \forall\, x \in \left[0, 1 - \frac{1}{100^2}\right].$$

Let $w_u = wP_l(I - \frac{w^*w}{100})$, then

$$\begin{aligned} \|I - w_u^*w_u\| &= \left\| I - w^*P_l\left(I - \frac{w^*w}{100}\right)^2 w \right\| \\ &= \left\| w^* \left(\left(\frac{1}{10\sqrt{I - (I - \frac{w^*w}{100})}} \right)^2 - P_l^2\left(I - \frac{w^*w}{100}\right) \right) w \right\| \leq \delta \end{aligned}$$

Hence w_u is a δ-quasi-unitary. To estimate the propagation of w_u, we notice that the largest power of x in $P_l(x)$ is l, so the propagation of $P_l(I - \frac{w^*w}{100})$ is at most $2lr$, and hence that the propagation of w_u is at most $3lr$. Thus $w_u(I \oplus 0)w_u^*$ will have propagation no larger than $10lr$. We will enlarge the region $X_1 \cap X_2$ for the propagation of $w_u(I \oplus 0)w_u^*$. For a closed subset A of X, such that the interior of A is dense in A and $\mathrm{bd}_{10lr}(X_1 \cap X_2) \subset A$. We define $\partial_0(u) = \chi_A w_u(I \oplus 0)w_u^*\chi_A$.

Lemma 5.34. *$\partial_0(u) \in QP_{N_0\delta, N(\delta)r, N_0 s, k}(A)$ for some universal constat $N_0 \geq 1$.*

Proof. We first verify that $\partial_0(u)$ is quasi-projection on H_A. In fact,

$$\begin{aligned}
&||\partial_0(u)^2 - \partial_0(u)|| \\
=&||\chi_A((w_u(I\oplus 0)w_u^*)^2 - w_u(I\oplus 0)w_u^*)\chi_A - \chi_A w_u(I\oplus 0)w_u^*\chi_{X-A}w_u(I\oplus 0)w_u^*\chi_A|| \\
\leq&||(w_u(I\oplus 0)w_u^*)^2 - w_u(I\oplus 0)w_u^*|| + ||\chi_A w_u(I\oplus 0)w_u^*\chi_{X-A}w_u(I\oplus 0)w_u^*\chi_A|| \\
\leq&||w_u(I\oplus 0)(w_u^*w_u - I)(I\oplus 0)w_u^*|| + 2||\chi_A w_u(I\oplus 0)w_u^*\chi_{X-A}|| \\
<&2\delta + 2||\chi_A w_u(I\oplus 0)w_u^*\chi_{X_1-A}|| + 2||\chi_A w_u(I\oplus 0)w_u^*\chi_{X_2-A}||.
\end{aligned}$$

To estimate $||\chi_A w_u(I\oplus 0)w_u^*\chi_{X_2-A}||$, we note that $(X_2 - A)\cap X_1 = \varnothing$. Thus χ_{X_2-A} commute with u_{X_1} hence u and w_u. Therefore

$$||\chi_A w_u(I\oplus 0)w_u^*\chi_{X_2-A}|| = 0$$

To estimate $||\chi_A w_u(I\oplus 0)w_u^*\chi_{X_1-A}||$. The key observation is that by the excision condition, the distance between $X_1 - A$ and X_2 is at least $10lr$, recall that the propagation of $w_u(I\oplus 0)w_u^*$ is less than $10lr$, so $X_1 - A$ is far away enough from X_2, hence $w_u(I\oplus 0)w_u^*$ cannot move it out of X_1. So we can replace u_{X_1} by u in the construction of w_u if we just consider the restriction on $X_1 - A$. In this case, we can calculate that

$$w(I\oplus 0)w^* = \begin{pmatrix} uu^* + 2u(I-u^*u)u^* + u(I-u^*u)^2u^* & u(I-u^*u) + u(I-u^*u)^2 \\ (I-u^*u)u^* + (I-u^*u)^2u^* & (1-u^*u)^2 \end{pmatrix}$$

and thus that $||w(I\oplus 0)w^* - (I\oplus 0)|| < 10\delta$. Notice that $P_l'(x)$ has nonnegative coefficient, hence $\{P_l'(x)\}_l$ is uniformly bounded, we can apply a similar estimation used in the proof of Lemma 5.23, to conclude that there exists a universal constant N' such that

$$||(w_u(I\oplus 0)w_u^* - (I\oplus 0))\chi_{X_1-A}|| < N'\delta$$

Therefore, $\partial_0(u)$ is a $N_1\delta$-quasi-projection, where N_1 is a universal constant.

We have already seen that the propagation of $\partial_0(u)$ is at most $10lr$. Since l depends on δ, so we can take $N(\delta)$ as $10l$.

To estimate the speed of $\partial(u)$, we can apply a similar argument used in the proof of Lemma 5.23, to conclude that it is bounded by $N_2 s$ for some universal constant N_2. Now we may take $N_0 = \max\{N_1, N_2\}$. $\square$

Remark 5.35. *Similar to the estimation of $||\chi_A w_u(I\oplus 0)w_u^*\chi_{X_1-A}||$, we can show that*

$$||(w_u - w)\chi_{X_1-A}|| < N\delta,$$

$$||\chi_{X_1-A}(w_u - w)|| < N\delta$$

for some universal constant N. In fact,

$$\left\|\left(w - \begin{pmatrix} u & \\ & u^* \end{pmatrix}\right)\chi_{X_1-A}\right\| = \left\|\begin{pmatrix} u(1-u^*u) & uu^*-1 \\ 1-u^*u & 0 \end{pmatrix}\right\| < 4\delta.$$

Hence

$$||(I - w^*w)\chi_{X_1-A}|| < 20\delta.$$

Thus

$$\left\|\left(I - \frac{w^*w}{100}\right)\chi_{X_1-A}\right\| = \left\|\left(\frac{99}{100}I + \left(\frac{I - w^*w}{100}\right)\right)\chi_{X_1-A}\right\| < \frac{992}{1000}$$

Since $P_l(x)$ has nonnegative coefficients, hence uniformly bounded. Applying a similar estimation used in the proof of Lemma 5.23, we can show that

$$\left\|\left[P_l\left(I - \frac{w^*w}{100}\right) - P_l\left(I - \frac{I}{100}\right)\right]\chi_{X_1-A}\right\| < N'\delta$$

for some universal constant. Hence

$$\begin{aligned}
&\left\|\left(P_l\left(I - \frac{w^*w}{100}\right) - I\right)\chi_{X_1-A}\right\| \\
=&\left\|\left[P_l\left(I - \frac{w^*w}{100}\right) - P_l\left(I - \frac{I}{100}\right)\right]\chi_{X_1-A}\right\| + \left\|\left(P_l\left(I - \frac{I}{100}\right) - I\right)\chi_{X_1-A}\right\| \\
<&N'\delta + \frac{1}{10^4}\delta
\end{aligned}$$

So

$$||(w_u - w)\chi_{X_1-A}|| = \left\|\left(wP_l\left(I - \frac{w^*w}{100}\right) - w\right)\chi_{X_1-A}\right\| < 2\left(N'\delta + \frac{1}{10^4}\right)\delta.$$

Hence

$$||(w_u - (u \oplus u^*))\chi_{X_1-A}|| \le ||(w_u - w)\chi_{X_1-A}|| + ||(w - (u \oplus u^*))\chi_{X_1-A}|| < 2\left(N' + \frac{1}{10^4}\right)\delta + 4\delta = N\delta,$$

where $N = 2(N' + \frac{1}{10^4}) + 4$ is a universal constant.

Similarly, we have that

$$||(w_u^* - (u^* \oplus u))\chi_{X_1-A}|| < N\delta.$$

Hence

$$||\chi_{X_1-A}(w_u - (u \oplus u^*))|| < N\delta.$$

Definition 5.36. *We define the boundary of u by $\partial(u) = \partial_0(u) - (I \oplus 0)$.*

Definition 5.37. *We define $j : QU_{\delta,r,s,k}(X_1) \oplus QU_{\delta,r,s,k}(X_2) \to QU_{\delta,r,s,k}(X)$ by*

$$j(u_1 \oplus u_2) = (u_1 + \chi_{X_2-A}) \oplus (u_2 + \chi_{X_1-A}).$$

Lemma 5.38. *Let $(X; X_1, X_2)$ be as in Definition 5.33. For any $0 < \delta < \frac{1}{100}$, $r > 0$, $s > 10$, there exists $0 < \delta_1 < \delta$, $0 < r_1 < r$, $s_1 > s$ such that $\partial j(u_1 \oplus u_2)$ is (δ, r, s_1)-homotopic to 0 for any $u_i \in QU_{\delta_1,r_1,s,k}(X_i)$ $(i = 1, 2)$ where δ_1 depends only on δ, r_1 depends only on r and δ, s_1 depends only on s.*

Proof. We will consider each part of the direct sum separately. For $u_1 \in QU_{\delta_1,r_1,s,k}(X_1)$, $(j_1(u_1))_{X_1} = j_1(u_1)$ which is a δ_1-quasi-unitary. By the same estimate as for $||\chi_A w_u(I \oplus 0)w_u^*\chi_{X_2-A}||$ in Lemma 5.34, we have that $||\partial_0 j_1(u_1) - (I \oplus 0)|| < N'\delta_1$ for some universal constant N'. Hence if we take $\delta_1 \leq \frac{\delta}{2N_1}$, $r_1 \leq \frac{r}{N(\delta_1)}$, $s_1 \geq N_1 s$, where N_1 as Lemma 5.34 Then by Lemma 5.5, the linear homotopy between $\partial_0(j_1(u))$ and $I \oplus 0$ is a (δ, r, s_1)-homotopy.

In the definition of $\partial_0(u)$, we first "chop" u by χ_{X_1} to get $u_{X_1} = \chi_{X_1} u \chi_{X_1} + \chi_{X-X_1}$. Now we will chop u by χ_{X-X_2}, and define $u_{X-X_2} = \chi_{X-X_2} u \chi_{X-X_2} + \chi_{X_2}$. Using u_{X-X_2}, we do the same construction, we will get w' and w'_u. Let $a(t)$ be the linear homotopy connecting w_u and w'_u. Applying the same argument in Lemma 5.34, we have that for any $t \in [0,1]$, $\chi_A a(t) \oplus (I \oplus 0) a^*(t) \chi_A \in QP_{2N_0\delta_1, N(\delta)r_1, N_0 s, k}(A)$. In fact, we only need to note that $\chi_{X_2 - A}$ commute with w'_u and $a(t)(I \oplus 0)a^*(t)$ can not move $X_1 - A$ out of $X - X_2$. Hence $\chi_A a(t)(I \oplus 0) a^*(t) \chi_A$ is a $(2N_0\delta_1, N(\delta)r_1, N_0 s)$-homotopy. Since $(j_2(u))_{X-X_2} = I$, hence $w' = I$, $w'_u = P_l(\frac{99}{100})$ which is $(\delta_1, 0, 1)$-homotopic to $(I \oplus 0)$. Hence if we choose $\delta_1 \leq \frac{\delta}{2N_0}$, $r_1 \leq \frac{r}{N(\delta)}$, $s_1 \geq 2N_0 s$, we have that $\partial_0(j_2(u))$ is (δ, r, s_1)-homotopic to $(I \oplus 0)$. ☐

Lemma 5.39. *Let $(X; X_1, X_2)$ be as Definition 5.33. For any $0 < \delta < \frac{1}{100}$, $r > 0$, $s > 10$, there exists $0 < \delta_2 < \delta$, $0 < r_2 < r$, $s_2 > s$ such that if u is an element in $QU_{\delta_2, r_2, s, k}(X)$ for which $\partial(u)$ is (δ_2, r_2, s)-homotopic to 0 in $GQP_{\delta_2, r_2, s, k}(X_1 \cap X_2)$ then there exists $u_i \in QU_{\delta, r, s_2, k}(X_i)$ $(i = 1, 2)$ such that $j(u_1 \oplus u_2)$ is (δ, r, s_2)-homotopic to u. Where δ_2 depends only on δ and s; r_2 depends only on δ, r, s; and s_2 depends only on s.*

Proof. Since $\partial(u)$ is (δ_2, r_2, s)-homotopic to 0, and hence by definition, we have that $\chi_A w_{u\oplus I}(I \oplus 0) w^*_{u\oplus I} \chi_A$ is (δ_2, r_2, s)-homotopic to $I \oplus 0$. By Proposition 5.22, we have that there exists a $y \in QU_{\delta_2, C_1(\delta_2, s), C_2(s), k}(A)$ such that

$$||y \chi_A w_{u\oplus I}(I \oplus 0) w^*_{u\oplus I} \chi_A y^* - (\chi_A \oplus 0)|| < C_3(s)\delta_2 \tag{5.1}$$

Let $x = y + \chi_{X_1 - A}$. Then x is a δ_2-quasi-unitary on $X_1 \cup A$. Since $\chi_{X_1 \cup A}$ commute with w, hence $w_{u\oplus I}$, we compute that

$$\begin{aligned}
&||x w_{u\oplus I}(\chi_{X_1\cup A} \oplus 0) w^*_{u\oplus I} x^* - (\chi_{X_1 \cup A} \oplus 0)|| \\
=&||x w_{u\oplus I}(I \oplus 0) w^*_{u\oplus I} x^* - (\chi_{X_1 \cup A} \oplus 0)|| \\
=&||y \chi_A w_{u\oplus I}(I \oplus 0) w^*_{u\oplus I} \chi_A y^* - (\chi_A \oplus 0)|| + ||\chi_{X_1 - A} w_{u\oplus I}(I \oplus 0) w^*_{u\oplus I} \chi_A y^*|| \\
&+ ||y \chi_A w_{u\oplus I}(I \oplus 0) w^*_{u\oplus I} \chi_{X_1 - A}|| + ||\chi_{X_1 - A} w_{u\oplus I}(I \oplus 0) w^*_{u\oplus I} \chi_{X_1 - A} - (\chi_{X_1 - A} \oplus 0)||.
\end{aligned}$$

Since in the proof Lemma 5.34, we have already demonstrated that

$$||(w_u(I \oplus 0) w^*_u - I \oplus 0)\chi_{X_1 - A}|| < N'\delta_2.$$

Hence the sum of last three terms is no more than $10N'\delta_2$. This together with (5.1) implies that

$$||x w_{u\oplus I}(\chi_{X_1\cup A} \oplus 0) w^*_{u\oplus I} x^* - (\chi_{X_1\cup A} \oplus 0)|| < (10N' + C_3(s))\delta_2.$$

This easily implies that

$$||x w_{u\oplus I}(\chi_{X_1\cup A} \oplus 0) - (\chi_{X_1\cup A} \oplus 0) x w_{u\oplus I} \chi_{X_1\cup A}|| < 10(10N' + C_3(s))\delta_2$$

which in turn implies that $xw_{u\oplus I}\chi_{X_1\cup A}$ is a matrix of the form $\left(\begin{smallmatrix} v_1 & b \\ c & d \end{smallmatrix}\right)$ with

$$||b|| \leq 10(10N' + C_3(s_2))\delta_2, \quad ||c|| \leq 10(10N' + C_3(s_2))\delta_2.$$

This allows us to conclude that v_1 is a $(M_1(s)\delta_2, M_2(\delta_2, s)r_2, M_3 s)$-quasi-unitary on $H_{X_1\cup A}$. By Remark 5.35, we have that

$$\|\chi_{X_1-A}(w_{u\oplus I} - (u\oplus I)\oplus(u^*\oplus I))\| < N\delta_2$$

for some universal constant N. Since $\chi_{X_1-A}xw_{u\oplus I}\chi_{X_1\cup A} = \chi_{X_1-A}w_{u\oplus I}\chi_{X_1\cup A} = \chi_{X_1-A}w_{u\oplus I}$, and since v_1 is the left-top element in $xw_{u\oplus I}$, we have that

$$\|\chi_{X_1-A}(v_1 - u\oplus I)\| < N\delta_2$$

Hence

$$\begin{aligned} ||\chi_{X_1-A}((u\oplus I)v_1^* - I)|| &\leq ||\chi_{X_1-A}((u\oplus I) - v_1)v_1^*|| + ||\chi_{X_1-A}v_1v_1^* - \chi_{X_1-A}|| \\ &< (2N + M_1(s))\delta_2 \end{aligned}$$

and

$$\begin{aligned} ||((u\oplus I)v_1^* - I)\chi_{X_1-A}|| &\leq ||(u\oplus I)(v_1^* - (u\oplus I)^*)\chi_{X_1-A}|| + ||((u\oplus I)(u\oplus I)^* - I)\chi_{X_1-A}|| \\ &\leq (2N + M_1(s))\delta_2. \end{aligned}$$

We now need a quasi-unitary that lives on $X_2\cup A$. The basic idea is that we want to "divide out" what's left of $u\oplus I$, as we already know that v_1 is the "quasi"-adjoint of $u\oplus I$ on $X_1 - A$.

We now define v_2 as follow

$$v_2 = \chi_{A\cup X_2}(u\oplus I)\bar{v}_1^*\chi_{A\cup X_2}$$

where $\bar{v}_1 = v_1 + \chi_{X_2-A}$. We calculate that

$$\begin{aligned} &||v_2v_2^* - \chi_{A\cup X_2}|| \\ =&||\chi_{A\cup X_2}(u\oplus I)\bar{v}_1^*\bar{v}_1(u\oplus I)^*\chi_{A\cup X_2} - \chi_{A\cup X_2}|| + ||\chi_{A\cup X_2}(u\oplus I)\bar{v}_1^*\chi_{X_1-A}\bar{v}_1(u\oplus I)^*\chi_{A\cup X_2}|| \\ <&3M_1(s)\delta_2 + 4(2N + M_1(s))\delta_2 \end{aligned}$$

Similarly, we can show that $||v_2^*v_2 - \chi_{A\cup X_2}||$ has the same bound. Hence v_2 is a $(M_1'(s)\delta_2, M_2'(\delta_2, s)r_2, M_3(s))$-quasi-unitary for some universal constant $M_1'(s)$, $M_2'(\delta_2, s)$ and $M_3'(s)$.

To finish the proof, we only need to check that $(v_1 + \chi_{X_2-A})\oplus(\chi_{X_1-A}\oplus v_2)$ is homotopic to 0. Since

$$\begin{aligned} &\|(\chi_{X_1-A} + v_2)(v_1 + \chi_{X_2} - A)\oplus I - (u\oplus I)\| \\ =&||\chi_{X_1-A}v_1 + \chi_{A\cup X_2}(u\oplus I)\bar{v}_1^*\bar{v}_1 - \chi_{A\cup X_2}(u\oplus I)\bar{v}_1^*\chi_{X_1-A}\bar{v}_1 - (u\oplus I)|| \\ \leq&||\chi_{X_1-A}(v_1 - u\oplus I)|| + ||\chi_{A\cup X_2}(u\oplus I)(\bar{v}_1^*\bar{v}_1 - 1)|| + ||\chi_{A\cup X_2}(u\oplus I)\bar{v}_1^*\chi_{X_1-A}v_1^*|| \\ \leq&N\delta_2 + 2M_1(s_1)\delta_2 + (2N + M_1(s))\delta_2. \end{aligned}$$

Hence by combining the rotation homotopy $R(t)(I\oplus(v_2+\chi_{X_1-A}))R(t)^*((v_1+\chi_{X_2-A})\oplus I)$ and the linear homotopy between $(\chi_{X_1-A}+v_2)(v_1+\chi_{X_2-A})\oplus I$ and $(u\oplus I)\oplus I$, and picking appropriate δ_2, r_2, s_2, we get the desired result. $\square$

Remark 5.40. *If we further require that* $\exists\, R_0>0$, $C_0>0$ *for each* $X'=X_1, X_2, X_1\cap X_2$, *and any* $r<r_0$, $\mathrm{bd}(X')$ *is strongly Lipschitz homotopy equivalent to* X' *with* c_0 *as Lipschitz constant. Assume that* $r<r_0/(1=N(\delta))$, *let* f *be the proper strong Lipschitz map from* $\mathrm{bd}_{N(\delta)r}10\cap\mathrm{bd}_{N(\delta)r}/10(X_2)$ *to* $X_1\cap X_2$ *realizing the strong Lipschitz homotopy equivalence. Let* $V_f(t)$ *be the family of isometries as the proof of Lemma 4.5, where* $\{\varepsilon_k\}$ *is chosen in such a way that* $\sup_k\varepsilon_k<\frac{r}{10}$. *We may redefine the boundary map* u *by*

$$\partial(u)=\mathrm{Ad}(V_f)(\partial_0(u))-(I\oplus 0).$$

By Lemma 5.32, we have that the following sequence

$$QU_{\delta,r,s,k}(X_1)\oplus QU_{\delta,r,s,k}(X_2)\xrightarrow{j}QU_{\delta,r,s,k}(X)\xrightarrow{\partial}GQP_{N_0\delta,N(\delta)r,N_0s,k}(X_1\cap X_2)$$

is asymptotic in the sense of Lemma 5.38 and 5.39. In this case, δ_2, r_2 *and* s_2 *in Lemma 5.39 also depends on* r_0 *and* c_0.

Together with controlled suspension, we have the following asymptotic exact sequence

$$QU_{\delta,r,s,k}(X_1)\oplus QU_{\delta,r,s,k}(X_2)\to QU_{\delta,r,s,k}(X)\to QU_{\delta,r,s,k-1}(X_1\cap X_2).$$

Proposition 5.41. *Let* X *be a simplicial complex with finite dimension* m. *For any* $k>m+1$, $0<\delta<\frac{1}{100}$, $r>0$, $s\geq 0$, *there exist* $0<\delta_1\leq\delta$, $0<r_1\leq r$, $s_1\geq s$ *such that every element* u *in* $QU_{\delta_1,r_1,s,k}(X)$ *is* (δ,r,s_1)*-equivalent to* I, *where* δ_1 *depends only on* δ,s,k *and* m*;* r_1 *depends only on* δ,r,s,k *and* m*; and* s_1 *depends only on* s,k *and* m.

Section 5.6 Finite Asymptotic Dimension

In this section, we will prove the coarse Baum-Connes conjecture for spaces with finite asymptotic dimensions. Asymptotic dimension is a large-scale analogy to the Lebesgue covering dimension. We will see that, with the asymptotic dimension condition, we can choose an anti-Čech sequence with nice properties.

Definition 5.42. *The asymptotic dimension of a metric space the smallest integer* n *such that for every* $r>0$ *there exists a uniformly bounded open cover* $\mathcal{U}$, *such that the the* r*-multiplicity of* $\mathcal{U}$ *does not exceed* $n+1$, *i.e., every ball of diameter* r *intersects at most* $n+1$ *members of* $\mathcal{U}$.

Let X be a proper metric space with finite asymptotic dimension m. We construct an anti-Čech sequence inductively. We start from some positive number R_0. Suppose now we have chosen R_k. We can choose a uniformly bounded cover $\mathcal{V}_{k+1}$ with R_k-multiplicity at most $m+1$. Denote the bound of the diameters of members in $\mathcal{V}_k$ to be C_{k+1}, we choose R_{k+1} such that $R_{k+1}>4R_k$, and $R_{k+1}>4C_k$. Denote $\mathcal{U}_k=\{B(V,R_k)\mid V\in\mathcal{V}_k\}$, where $B(V,R_k)=\{x\in X\mid d(x,\mathcal{V})<R_k\}$.

Lemma 5.43. $\{U_k\}$ *is an anti-Čech sequence, and the dimension of* $N_{\mathcal{U}_k}$ *is no more than* m *for every* k.

Proof. It is easy to check that any x belongs to at most $m+1$ members of $\mathcal{U}_k$ for each k. The Lebesgue number of $\mathcal{U}_{k+1}$ is at least R_{k+1} which is greater than the bound C_k+2R_k for diameters of members in $\mathcal{U}_k$, and $R_k \to \infty$. $\square$

Next we will define a sequence of metric shrinking maps, which allows us to change an operator with finite propagation into an operator with arbitrarily small propagation.

Fix a positive integer n_0, for each $n > n_0$, let $r_n = \frac{R_n}{CR_{n_0+1}} - 2$, where C is as Lemma 3.26, depending only on m. Since $R_{k+1} > 4R_k$, there exists $n_1 > n_0$ such that $r_n > 1$ if $n > n_1$, and there exists a sequence of nonnegative smooth functions $\{\chi_n\}_{n>n_1}$ on $[0,\infty)$ for which (1) $\chi_n(t) = 1$ for all $0 \leq t \leq 1$, and $\chi_n(t) = 0$ for all $t \geq r_n$; (2) There exists a sequence of positive number $\varepsilon_n \to 0$ satisfying $|\chi_n'(t)| < \varepsilon_n \leq 1$ for all $n > n_1$. For each $V \in \mathcal{V}_n$ $(n > n_1)$, we define

$$V' = \{U \in N_{\mathcal{U}_{n_0}} \mid U \in \mathcal{U}_{n_0}, U \cap V \neq \varnothing\},$$

where $U \in N_{\mathcal{U}_{n_0}}$ is a vertex of $N_{\mathcal{U}_{n_0}}$ corresponding to $\mathcal{U}_{n_0}$. We define a map G_n from $N_{\mathcal{U}_{n_0}}$ to $N_{\mathcal{U}_n}$ by

$$G_n(x) = \sum_{V \in \mathcal{V}_n} \frac{\chi(d(x,V'))}{\sum_{W \in \mathcal{V}_n} \chi_n(d(x,W'))} B(V, R_n)$$

for all $x \in N_{\mathcal{U}_{n_0}}$. The following lemma shows that $G_n(x)$ is indeed in $N_{\mathcal{U}_n}$.

Lemma 5.44. *G_n is a proper Lipschitz map from $N_{\mathcal{U}_{n_0}}$ to $N_{\mathcal{U}_n}$ with a Lipschitz constant depending only on m.*

For any $\varepsilon > 0$, $R > 0$, there exists $K > 0$ such that $d(G_n(x), G_n(y)) < \varepsilon$ whenever $n > K$ and $d(x,y) \leq R$.

Proof. We note that if $x = \sum t_i B(V_i, R_{n_0})$ with $t_i \neq 0$ and $V_i \cap V \neq \varnothing$, then $B(V_i, R_{n_0}) \in V'$ and $d(x, B(V_i, R_{n_0})) \leq 1$. Thus $\chi_n(d(x,V')) = 1$, hence $\sum_{V \in V_n} \chi(d(x,V')) \geq 1$.

Let W be an element in $\mathcal{V}_n$ such that $\chi_n(d(x,W')) \neq 0$ for some $x \in N_{\mathcal{U}_0}$. By property (1) of χ_n, we have that $d(x,W') < r_n$. Thus $d(x,U) < r_n$ for some $U \in W'$. Let $x = \sum t_i B(V_i, R_{n_0})$, where $t_i > 0$, $\sum t_i = 1$ and $V_i \in N_{\mathcal{V}_0}$. Hence

$$d(B(V_i, R_{n_0}), U) \leq d(x,U) + d(x, B(V_i, R_{n_0})) < r_n + 1.$$

By Lemma **??**, we have that $d_X(U, B(V_i, R_{n_0})) \leq C(r_n+1)R_{n_0+1}$. Thus

$$d_X(W, V_i) \leq d_X(U, V_i) \leq C(r_n+2)R_{n_0+1} = \frac{R_n}{2}.$$

So $V_i \subset B(W, R_n)$, and hence

$$\bigcap_{W : \chi_n(d(x,W')) \neq 0} B(W, R_n) \neq \varnothing.$$

Therefore $G_n(x)$ is indeed in $N_{\mathcal{U}_n}$ for all $x \in N_{\mathcal{U}_0}$. The above observation also implies the properness of $G_{n_0 n}$.

To verify the G_n is a Lipschitz map, we only need to consider two points in the same simplex. We just calculate one coordinate,

$$\begin{aligned}
&\left|\frac{\chi_n(d(x,V'))}{\sum_{W\in\mathcal{V}_n}\chi_n(d(x,W'))}-\frac{\chi_n(d(y,V'))}{\sum_{W\in\mathcal{V}_n}\chi_n(d(y,W'))}\right| \\
\le&\left|\left(\sum_{W\in\mathcal{V}_n}\chi_n(d(y,W'))\right)\chi_n(d(x,V'))-\chi_n(d(y,V'))\left(\sum_{W\in\mathcal{V}_n}\chi_n(d(x,W'))\right)\right| \\
\le&\sum_{W\in\mathcal{V}_n}\left|\chi_n(d(y,W'))\right|\cdot\left|\chi_n(d(x,V'))-\chi_n(d(y,U'))\right|+\left|\chi_n(d(y,V'))\right|\cdot\left|\sum_{W\in\mathcal{V}_n}(\chi_n(d(x,W'))-\chi_n(d(y,W')))\right| \\
\le&(m+1)\varepsilon_n|d(x,V')-d(y,V')|+\varepsilon_n\sum_{W\in\mathcal{V}_n}\left|d(x,W')-d(y,W')\right| \\
\le&2\varepsilon_n(m+1)d(x,y)\le 2(m+1)d(x,y)
\end{aligned}$$

Take into account all coordinates, we have that $d(G_n(x),G_n(y))\le 2(m+1)^{3/2}d(x,y)$.

□

Let $n>n_1$. We can choose a simplicial map i_{n_0n} from $N_{\mathcal{U}_{n_0}}$ to $N_{\mathcal{U}_n}$ in such a way that, for each $V\in\mathcal{V}_{n_0}$,

$$i_{n_0n}(B(V,R_{n_0}))=B(W,R_n)$$

for some $W\in\mathcal{V}_n$ satisfying $W\cap V\neq\varnothing$.

If $x=\sum t_iB(V_i,R_{n_0})$ and $i_{n_0n}(x)=\sum t_iB(W_i,R_{n_0})$ with $t_i>0$, then $d(x,B(V_i,R_{n_0}))\le 1$ and $B(W_i,R_{n_0})\in W'$. So $d(x,W')\le 1$. Hence $\chi_n(d(x,W'))=1$. Thus $i_{n_0n}(x)$, and $G_n(x)$ belongs to the same simplex with vertices $\{B(W,R_n)\mid W\in\mathcal{V}_n,\chi_n(d(x,W'))\neq 0\}$.

Remark 5.45. *It is not hard to see that $G_n(x)$ and $i_{n_0n}(x)$ are strong Lipschitz equivalent. Since i_{n_0n} is a simplicial map and hence is 1-Lipschitz, the linear homotopy between $G_n(x)$ and $i_{n_0n}(x)$ is a $2(m+1)$-strong Lipschitz equivalence.*

Theorem 5.46. *The coarse Baum-Connes conjecture holds for proper metric spaces with finite asymptotic dimension.*

Proof. Let X be a proper metric space with finite asymptotic dimension m. We choose $\mathcal{U}_n$ as above, so the dimension of $N_{\mathcal{U}_n}$ is no more than m for all n. By Remark 4.15, we only need to prove that

$$\varinjlim_n K_i(C^*_{L,0}(N_{\mathcal{U}_n}))=0$$

By Lemma 5.20, every element in $K_i(C^*_{L,0}(N_{\mathcal{U}_n}))$ can be represented by some $QU_{\delta,r,s/100,k}$ for some $k>m+1$. By Remark 5.24, the K-theory element can also be represented by some $QU_{\delta_1,r,s,k}$, where δ_1 is as in Proposition 5.41.

Let $u_n=\mathrm{Ad}^+(V_{G_n})(u)$, where G_n is as in Lemma 5.44, $\mathrm{Ad}^+(V_{G_n})$ is as in Lemma 4.5, and r_1 is as in Proposition 5.41. By Lemma 5.44, there exists $K>0$ such that u_n has propagation at most r_1 for some $n>K$. Thus, by Proposition 5.41, we see that u_n is (δ,r,s_1)-equivalent to I in $QU_{\delta,r,s_1,k}(N_{\mathcal{U}_n})$ for $n>K$, where s_1 is as in Proposition 5.41. By Lemma 5.19, it follows that u_n represent 0 in

$K_i(C^*_{L,0}(N_{\mathcal{U}_n}))$. By Theorem 4.8 and Remark 5.45, we have that u_n is equivalent to $\mathrm{Ad}^+(V_{i_{n_0}n})(u)$ in $K_i(C^*_{L,0}(N_{\mathcal{U}_n}))$. Hence $[u]=0$ in $\varinjlim_n K_i(C^*_{L,0}(N_{\mathcal{U}_n}))$. $\square$

Chapter 6

Finite Decomposition Complexities and the Coarse Baum-Connes Conjecture

Inspired by the concept of finite asymptotic dimension, Guentner, Tessera and Yu introduced the concept of finite decomposition complexity [GTY2]. Roughly speaking, if a metric space has finite decomposition complexity, then we have an algorithm to decompose the space into well-separated families until uniformly bounded families, which allows us to prove isomorphism conjectures, e.g. bounded Borel conjecture, inductively by the Mayer-Vietoris argument. In this chapter, we will prove the coarse Baum-Connes conjecture for spaces with finite decomposition complexities.

Section 6.1 Finite Decomposition Complexities

Definition 6.1. *A metric family $\mathcal{X}$ is r-decomposable over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits an r-decomposition*

$$X = X_0 \cup X_1, \quad X_i = \bigsqcup_{r-disjoint} X_{ij},$$

where each $X_{ij} \in \mathcal{Y}$. We introduce the notation $\mathcal{X} \xrightarrow{r} \mathcal{Y}$ to indicate that $\mathcal{X}$ is r-decomposable over $\mathcal{Y}$.

Definition 6.2. *Let $\mathfrak{U}$ be a collection of metric families. A metric family $\mathcal{X}$ is decomposable over $\mathfrak{U}$ if, for every $r > 0$, there exists a metric family $\mathcal{Y} \in \mathfrak{U}$ and an r-decomposition of $\mathcal{X}$ over $\mathcal{Y}$. The collection $\mathfrak{U}$ is stable under decomposition if every metric family which decomposes over $\mathfrak{U}$ actually belongs to $\mathfrak{U}$.*

Definition 6.3 (GTY2)**.** *The collection $\mathfrak{D}$ of metric families with finite decomposition complexity is the minimal collection of metric families containing the bounded metric families and stable under decomposition. We abbreviate the membership in $\mathfrak{D}$ by saying that a metric family in $\mathfrak{D}$ has FDC.*

It is shown in [GTY3] that finite decomposition complexity is a coarse invariant, a metric space having finite asymptotic dimension has finite decomposition complexity, and that one having finite decomposition complexity has Property A.

The most interesting example of metric space is countable discrete groups with a proper left invariant metric. In [GTY3], a large class of groups are verified to satisfy finite decomposition complexity, which includes all countable linear groups, countable subgroups of almost connected Lie groups, elementary amenable groups and hyperbolic groups. The class of groups with finite decomposition complexity are closed under the following operations

(1) subgroups,

(2) direct products,

(3) extensions,

(4) free and amalgamated products,

(5) HNN-extensions,

(6) direct limits.

Up to now, the only group known not satisfying finite decomposition complexity is Gromov's random group.

Section 6.2 Rips Complexes for Metric Families

To work with metric families, we need some preparation of Rips complex for families.

Definition 6.4. *Let Σ be a subset of Γ. For $1 \leq a \leq b$ we define the relative Rips complex $P_{ab}(\Gamma, \Sigma)$ to be the simplicial polyhedron with vertex set Γ and in which a finite subset $\{\gamma_0, \dots, \gamma_n\}$ spans a simplex if one of the following conditions hold:*

(1) $d(\gamma_i, \gamma_j) \leq a$ for all i and j;

(2) $d(\gamma_i, \gamma_j) \leq b$ for all i, j, and $\gamma_i \in \Sigma$ for all i.

The relative Rips complex is equipped with the simplicial metric.

We can extend the definition of the relative Rips complex to families. For families $\mathcal{C} = \{C\}$ and $\mathcal{W} = \{W\}$ with each $C \subset \Gamma$ and each $W \subset \Sigma$ we define

$$P_{ab}(\mathcal{C}, \mathcal{W}) = \bigcup_{C \in \mathcal{C}} P_a(C) \cup \bigcup_{W \in \mathcal{W}} P_b(W),$$

as subspaces of $P_{ab}(\Gamma, \Sigma)$. If Σ is not explicitly specified, then Σ is understood to be the union of all W in $\mathcal{W}$. In the special case $a = b$ we have $P_{aa}(\Gamma, \Sigma) = P_a(\Gamma)$, and more generally $P_{aa}(\mathcal{C}, \mathcal{W}) = P_a(\mathcal{C} \cup \mathcal{W})$. As for the standard Rips complex, we have the elementary equalities

$$P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W}) = P_{ab}(\mathcal{C}, \mathcal{W}) \cup P_{ab}(\mathcal{D}, \mathcal{W}), \quad P_{ab}(\mathcal{C} \cap \mathcal{D}, \mathcal{W}) = P_{ab}(\mathcal{C}, \mathcal{W}) \cap P_{ab}(\mathcal{D}, \mathcal{W})$$

as subspaces of $P_{ab}(\Gamma, \Sigma)$.

Lemma 6.5 (Comparison Lemma)**.** *Let $a \geq 1$, and let $P_a(\Gamma)$ be equipped as usual with the simplicial metric. For x and $y \in \Gamma$ we have*

$$d_\Gamma(x, y) \leq a \, \alpha \, d_{P_a(\Gamma)}(x, y),$$

for some constant α depending only on the dimension of $P_a(\Gamma)$.

Lemma 6.6. *Let C be a subspace of Γ and let $\varepsilon \geq 1$ and $a \geq 1$. There exists $\beta \geq 1$ depending only on the dimension of $P_a(\Gamma)$ such that the following statements are true. Viewing $P_a(C)$ as a subspace of $P_a(\Gamma)$ we have*

$$N_\varepsilon(P_a(C)) \cap \Gamma \subset N_{a\varepsilon\beta}(C),$$

Similarly for the relative Rips complex, viewing $P_b(C)$ as a subspace of $P_{ab}(\Gamma, C)$ $(b \geq a)$ we have

$$N_\varepsilon(P_b(C)) \cap \Gamma \subset N_{a\varepsilon\beta}(C).$$

Lemma 6.7 (Neighborhood Lemma)**.** *Let $C \subset \Gamma$, $\varepsilon \geq 1$ and $a \geq 1$. Viewing $P_a(C) \subset P_a(\Gamma)$ we have*

$$N_\varepsilon(P_a(C)) \subset P_a(N_{a\varepsilon\beta(C)}),$$

for some constant β depending only on the dimension of $P_a(\Gamma)$. Similarly for the relative Rips complex, viewing $P_b(C) \subset P_{ab}(\Gamma, C)$ $(b \geq a)$ we have

$$N_\varepsilon(P_b(C)) \subset P_{ab}(N_{a\varepsilon\beta}(C), C).$$

Proof. For every $x \in N_\varepsilon(P_a(C))$, suppose x belongs to a simplex spanned by $K = \{\gamma_0, \ldots, \gamma_n\}$, then $K \subset N_{L+\varepsilon}(P_a(C)) \cap \Gamma$. L as the proof of previous lemma. Take β' as the β in the previous lemma, we have

$$K \subset N_{L+\varepsilon}(P_a(C)) \cap \Gamma \subset N_{a(L+\varepsilon)\beta'}(C) \subset N_{a\varepsilon(L+1)\beta'}(C).$$

So $x \in P_a(K) \subset P_a(N_{a\varepsilon(L+1)\beta'}(C))$. We can take $\beta = (L+1)\beta'$.

The case of relative Rips complex is exactly the same argument. □

Section 6.3 Coarse Baum-Connes Conjecture for spaces with FDC

In this section, we will prove the coarse Baum-Connes conjecture using controlled K-theory and cutting-pasting methods.

Theorem 6.8. *The coarse Baum-Connes conjecture is true for Γ for every locally compact proper metric space with bounded geometry and finite decomposition complexity.*

Let $A = C^*_{L,0}(X)$, $r > 0$

$p \in A$ is a r-quasi-projection if $p \in A$ with propagation no more than r, $p = p^*$ and $\|p^2 - p\| < \frac{1}{100}$.

$u \in A$ is a r-quasi-unitary if $u \in A$ with propagation no more than r, $\|u^*u - I\| < \frac{1}{100}$ and $\|uu^* - I\| < \frac{1}{100}$.

$P^r(A)$ is the set of r-quasi-projections of A.

$U^r(A)$ is the set of r-quasi-unitaries of A.

$P^r_\infty(A)$= $\bigcup_{n\in\mathbb{N}} P^r(M_n(A))$ for $P^r(M_n(A)) \hookrightarrow P^r(M_{n+1}(A))$; $x \to \mathrm{diag}(x, 0)$.

$U^r_\infty(A)$= $\bigcup_{n\in\mathbb{N}} U^r(M_n(A))$ for $U^r(M_n(A)) \hookrightarrow U^r(M_{n+1}(A))$; $x \to \mathrm{diag}(x, 1)$

For $A = C^*_{L,0}(X)$, $r > 0$, we define the equivalence relations of $P^r_\infty(A) \times \mathbb{N}$ and on $U^r_\infty(A)$.

$(p, l) \sim (q, l')$ if there is $k \in \mathbb{N}$ and $h \in P^r_\infty(C[0,1], A)$ such that $h(0) = \mathrm{diag}(p, I_{k+l'})$ and $h(1) = \mathrm{diag}(q, I_{k+l})$.

$u \sim v$ if there is $h \in U^{3r}_\infty(C[0,1], A)$ such that $h(0) = u$ and $h(1) = v$.

Definition 6.9. $K^{r,n}_0(A)$= $P^r(A \otimes C_0(0,1)^n)/\sim$ *and* $[p,l]_r$ *is the class of* $(p,l) \mod \sim$.

$K^{r,n}_1(A)$= $U^r(A \otimes C_0(0,1)^n)/\sim$ *and* $[u]_r$ *is the class of* $u \mod \sim$.

Proposition 6.10.

$$\lim_{r\to\infty} K^r_p(A) = K_p(A).$$

The main ingredients for the Mayer-Vietoris argument are the following asymptotic exact sequence and an asymptotic version of Bott periodicity.

Proposition 6.11. *Let X be a locally compact and finite dimensional polyhedron with the simplicial metric and $X = Y \cup Z$, where Y and Z are closed subsets of X, and the interior of X and Y are respectively dense in Y and Z, then there exists a universal constant $c \geq 1$ such that the following* *sequence is asymptotically exact:*

$$\begin{array}{ccccc} K_1^{r,n+1}(C^*_{L,0}(Y\cap Z)) & \xrightarrow{i} & K_1^{r,n+1}(C^*_{L,0}(Y))\oplus K_1^{r,n+1}(C^*_{L,0}(Z)) & \xrightarrow{j} & K_1^{r,n+1}(C^*_{L,0}(X)) \\ & & & & \downarrow \partial \\ K_1^{cr,n}(C^*_{L,0}(N_{cr}(X))) & \xleftarrow{j} & K_1^{cr,n}(C^*_{L,0}(N_{cr}(Y))\oplus K_1^{cr,n}(C^*_{L,0}(N_{cr}(Z))) & \xleftarrow{i} & K_1^{cr,n}(C^*_{L,0}(N_{cr}(Y)\cap N_{cr}(Z))) \end{array}$$

in the sense that

(1) $j\circ i=0$;

(2) *the kernel of* $j: K_1^{r,n}(C^*_{L,0}(Y))\oplus K_1^{r,n}(C^*_{L,0}(Z))\to K_1^{r,n}(C^*_{L,0}(X))$ *in* $K_1^{c^2r,n}(C^*_{L,0}(Y))\oplus K_1^{c^2r,n}(C^*_{L,0}(Z))$ *is contained in the image of* $i: K_1^{c^2r,n}(C^*_{L,0}(Y\cap Z))\to K_1^{c^2r,n}(C^*_{L,0}(Y))\oplus K_1^{c^2r,n}(C^*_{L,0}(Z))$;

(3) $\partial\circ j=0$;

(4) *the kernel of* ∂ *in* $K_1^{c^2r,n+1}(C^*_{L,0}(X))$ *is contained in the image of* $j: K_1^{c^2r,n+1}(C^*(N_{cr}(Y)))\oplus K_1^{c^2r,n+1}(C^*_{L,0}(N_{cr}(Z)))\to K_1^{c^2r,n+1}(C^*_{L,0}(X))$;

(5) $i\circ\partial=0$;

(6) *the kernel of* $i: K_1^{r,n}(C^*(Y\cap Z))\to K_1^{rn}(C^*_{L,0}(Y))\oplus K_0^{r,n}(C^*_{L,0}(Z))$ *in* $K_1^{c^2r,n}(C^*_{L,0}(N_{cr}(Y)\cap N_{cr}(Z)))$ *is contained in the image of* $\partial: K_1^{cr,n}(C^*_{L,0}(X))\to K_1^{c^2r,n}(C^*_{L,0}(N_{cr}(Y)\cap N_{cr}(Z))$.

In the finite asymptotic dimension case, we only need to decompose the space finitely many times (no more than the asymptotic dimension). Hence all the parameters are easily controlled. In the finite decomposition complexity case, the decomposition needs not to stop in finite steps; hence we need more careful work to control the parameters.

To prove Theorem 6.8 by showing $\varinjlim K_n(C^*_{L,0}(P_d(\Gamma)))=0$ for some $n\geq 0$. It suffices to show that for all $r>0$, $a>1$ there exists $b>a$ such that the map $K_1 *^{r,n}(C^*_{L,0}P_a(\Gamma))\to K_1^{r,n}(C^*_{L,0}(P_b(\Gamma)))$ is 0. We will define a collection $\mathscr{F}$ of metric subspaces of Γ to be a vanishing family if for some $N>0$ and every $n\geq N$, $r>1$,$a>1$, $t>1$ there exists $b>a$ such that for every $Z\subset N_t(X)$ the homomorphisms

$$K_1^{n,r}(C^*_{L,0}(P_a(Z)))\to K_1^{n,r}(C^*_{L,0}(P_b(Z))) \tag{6.1}$$

are zero for all $n\geq N$. We want to show that the collection of vanishing families contains bounded families and is stable under decomposition. That bounded families are vanishing families follows from the fact that if a subspace $Y\subset\Gamma$ has diameter at most $b\geq 0$ then $P_b(Y)$ is strong Lipschitz homotopy equivalent to a point. To show that a family of subspaces of Γ is a vanishing family, we will decompose it over two well-separated vanishing families, say $\mathcal{C}$ and $\mathcal{D}$. The following diagram will demonstrate our strategy. However, the diagrams should be carefully interpreted in terms of controlled operator K-theory.

$$\begin{array}{ccccc} K_1(P_a(\mathcal{C}))\oplus K_1(P_a(D)) & \longrightarrow & K_1(P_a(\mathcal{C})\cup P_a(\mathcal{D})) & \longrightarrow & K_0(P_a(\mathcal{C})\cap P_a(\mathcal{D})) \\ \downarrow & & \downarrow & & \downarrow i \\ K_1(P_b(\mathcal{C}))\oplus K_1(P_b(\mathcal{D})) & \longrightarrow & K_1(P_b(\mathcal{C})\cup P_b(\mathcal{D})) & \longrightarrow & K_0(P_b(\mathcal{C})\cap P_b(\mathcal{D})) \\ j\downarrow & & \downarrow & & \downarrow \\ K_1(P_c(\mathcal{C}))\oplus K_1(P_c(\mathcal{D})) & \longrightarrow & K_1(P_c(\mathcal{C})\cup P_c(\mathcal{D})) & \longrightarrow & K_0(P_c(\mathcal{C})\cap P_c(\mathcal{D})) \end{array}$$

The induction hypothesis applies to the first and third column. Given a we can choose b large enough such that $i = 0$; then we can choose c large enough such that $j = 0$. By a simple diagram chase, we have that the composite of two maps in the middle column is 0.

Proof of Theorem 6.8. A uniformly bounded family of subspaces of Γ is a vanishing family follows Lemma 5.32, since we notice that if a subspace $Y \subset \Gamma$ has diameter at most b for some $b \geq 0$ then $P_c(Z)$ is strong Lipschitz homotopy equivalent to a point with Lipschitz constant one for $c \geq b$.

Now let $\mathcal{F}$ be a family of subspaces of Γ and assume that $\mathcal{F}$ is decomposable over the collection of vanishing family. Precisely there exists $b = b(i, t, a, \delta, \mathcal{F})$ such that for every $X \in \mathcal{F}$ and every $Z \subset N_t(X)$ the maps (6.1) are zero.

Set $\epsilon = \epsilon(t, a, \delta, \lambda)$ sufficiently large, to be specified later. Obtain an ε-decomposition of $\mathcal{F}$ over a vanishing family $\mathcal{G} = \mathcal{G}(\varepsilon, \mathcal{F})$. Let $X \in \mathcal{F}$, we obtain a decomposition

$$X = A \cup B, \qquad A = \bigsqcup_{\varepsilon} A_i, \qquad B = \bigsqcup_{\varepsilon} B_j$$

for which all A_i and $B_j \in \mathcal{G}$. Let $Z \subset N_t(X)$ setting $C_i = Z \cap N_{t+a}(A_i)$ and $D_j = Z \cap N_{t+a}(B_j)$ we obtain an analogous decomposition

$$Z = C \cup D, \qquad C = \bigsqcup_{\varepsilon - 2(t+a)} C_i, \qquad D = \bigsqcup_{\varepsilon - 2(t+a)} D_j.$$

Denote $\mathcal{C} = \{C_i\}$ and $\mathcal{D} = \{D_j\}$. By separation hypothesis we have $\varepsilon - 2(t + a) > a$, so that $P_a(\mathcal{C}) = P_a(C)$ and $P_a(\mathcal{D}) = P_a(D)$. Further, $P_a(Z) = P_a(C) \cup P_a(D) = P_a(\mathcal{C} \cup \mathcal{D})$. We intend to compare the Mayer-Vietoris sequence for certain subspaces of appropriate relative Rips complex. We enlarge the intersection $\mathcal{C} \cap \mathcal{D} = \{C_i \cap D_j\}$ by setting

$$W = N_{a\beta\lambda r}(C) \cap N_{a\beta\lambda r}(D) \cap Z = (N_{a\beta\lambda r}(C) \cap D) \cup (C \cap N_{a\beta\lambda r}(D)) = \bigsqcup_{\varepsilon - 2(t+a\beta\lambda r)} W_{ij}.$$

where all the neighborhoods are in Γ and

$$W_{ij} = N_{a\beta\lambda} r(C_i) \cap N_{a\beta\lambda r}(D_j) \cap Z$$

and where β is the constant appearing in neighborhood Lemma. Observe that $C_i \cap D_j \subset W_{ij}$ so that denoting $\mathcal{W} = \{W_{ij}\}$ we have $\mathcal{C} \cap \mathcal{D} \subset \mathcal{W}$. Provided $a \leq b$ we have a commuting diagram

$$\begin{tikzcd}
K_1^{r,n+1}(C^*_{L,0}(P_a(\mathcal{C} \cup \mathcal{D}))) \arrow[r] \arrow[d] & K_1^{\lambda r,n}(C^*_{L,0}(N_{\lambda\varepsilon}(P_a(\mathcal{C} \cap \mathcal{D})))) \arrow[d] \\
K_1^{r,n+1}(C^*_{L,0}(P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W}))) \arrow[r] & K_1^{\lambda r,n}(C^*_{L,0}(N_{\lambda\varepsilon}(P_b(\mathcal{W}))))
\end{tikzcd}$$

The horizontal maps are boundary maps in controlled Mayer-Vietoris sequences. In the top row the neighborhood is taken in $P_a(\mathcal{C} \cup \mathcal{D})$ and all subspaces are given the subspace metric from $P_{ab}(\Gamma, \mathcal{W})$. The vertical maps are induced from the proper contraction $P_a(\Gamma) \to P_{ab}(\Gamma, \mathcal{W})$. In fact, the right hand vertical map factors as the composite

$$N_{\lambda r}(P_a(\mathcal{C}\cap\mathcal{D}))\subset P_a(\mathcal{W})\to P_b(\mathcal{W})\subset N_{\lambda r}(P_b(\mathcal{W}))$$

in which the first two spaces are subspaces of $P_a(\mathcal{C}\cup\mathcal{D})\subset P_a(\Gamma)$ and the last two are subspaces of $P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W})\subset P_{ab}(\Gamma,\mathcal{W})$. The first inclusion follows from

$$N_{\lambda r}(P_a(\mathcal{C}\cap\mathcal{D}))=\bigcup_{i,j}N_{\lambda r}(P_a(C_i\cap D_j))\subset\bigcup_{ij}P_a(N_{a\beta\lambda r}(C_i)\cap N_{a\beta\lambda r}(D_j))\subset\bigcup_{ij}P_a(W_{ij})=P_a(\mathcal{W})$$

where we have applied neighborhood lemma for the first inclusion - keep in mind that the neighborhoods on the first line are taken in $P_a(\mathcal{C}\cup\mathcal{D})$.

Applying the inclusion hypothesis we claim that for sufficiently large b the right hand vertical map is zero. Indeed, the components $W_{ij}\in\mathcal{W}$ are contained in the neighbourhood $N_{t+a\beta\lambda r}(A_i)$ and also of $N_{t+a\beta\lambda r}(B_j)$ and we can apply the hypothesis with appropriate choices of the parameters $t'=t+a\beta\lambda r$, $r'=\lambda r$, $a'=a$, etc. In detail, if n large enough

$$K_1^{\lambda r,n}(C^*_{L,0}(P_a(W)))\xrightarrow{\cong}\prod K_1^{\lambda r,n}(C^*_{L,0}(P_a(W_{ij})))\xrightarrow{0}\prod K_1^{\lambda r,n}(C^*_{L,0}(P_b(W_{ij})))\longrightarrow K_{\lambda r,n-1}(C^*_{L,0}(P_b(\mathcal{W})))$$

as the spaces $P_a(W_{ij})$ and $P_a(\mathcal{W})$ are given the subspace metric from $P_a(\Gamma)$ and the individual W_{ij} are well separated, the first map is an isomorphism since various $P_a(W_{ij})$ are separated by at least λr. The spaces $P_b(W_{ij})$ are given the simplicial metric from $P_{ab}(\Gamma,\mathcal{W})$ and the last map is induced by proper contractions $P_b(W_{ij})\subset P_b(W)$ onto disjoint subspaces. Having chosen $b=b(n,r,a,t,\mathcal{G})$ we extend the diagram to incorporate the relax control map from the bottom sequence

We conclude from the above discussion and controlled Mayer-Vietoris sequence that the image of $K_1^{n,r}(P_a(\mathcal{C}\cup\mathcal{D}))$ under composition of the two vertical map is contained in the bottom of the horizontal map. It remains to apply the induction hypothesis to $\mathcal{C}$ and $\mathcal{D}$. The case of $\mathcal{D}$ being analogous. We concentrate on $\mathcal{C}$ and shall show that for sufficiently large $c\geq b$, the composite

$$P_{ab}(\mathcal{C},\mathcal{W})\cup N_{\lambda r}(P_b(\mathcal{W}))\subset P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W})\to P_b(Z)\to P_c(Z)$$

in which the arrows are induced by proper contractions $P_{ab}(\Gamma,\mathcal{W})\cup\mathcal{P}_b(\Gamma)\to P_c(\Gamma)$ is 0 on $\lambda^2 r$-controlled K-theory. We have, as subspaces of $P_{ab}(\mathcal{C}\cup D,\mathcal{W})\subset P_{ab}(\Gamma,\mathcal{W})$

$$P_{ab}(\mathcal{C},\mathcal{W})\cup N_{\lambda r}(P_b(\mathcal{W}))=\bigcup_i(P_a(C_i)\cup\bigcup_j N_{\lambda r}(P_b(W_{ij})))$$

in which the spaces comprising the union over i are well separated by Separation Lemma (which guarantees $\lambda^2 r$-separation). Further for fixed i and j we have

$$N_{\lambda r}(P_b(W_{ij}))\subset P_{ab}(N_{a\beta\lambda r}(W_{ij}),W_{ij})\to P_b(N_{a\beta\lambda r}(W_{ij}))\subset P_b(N_{2a\beta\lambda r}(C_i))$$

where we have applied neighborhood lemma for the first containment. For each fixed i, we have

$$P_a(C_i) \cup \bigcup_j P_b(N_{a\beta r\delta}(W_{ij})) \to P_b(N_{2a\beta\lambda\delta}(C_i)).$$

Now we apply our induction hyperthesis a second time with appropriate choices of the parameters $t'' = t + a2\beta\lambda r$, $r'' = \lambda^2 r$, $a'' = b$ etc, noting that $N_{2a\beta\lambda r}(C_i) \subset N_{t+2a\beta\lambda r}(A_i)$, we get $c = c(n, r, a'', r'', \mathcal{G})$ and analyze

$$\begin{aligned} K_1^{\lambda^2 r,n} C^*_{L,0}((P_{ab}(\mathcal{C}, \mathcal{W}) \cup N_{\lambda\delta}(P_b(\mathcal{W})))) &\cong \prod K_1^{\lambda^2,r}(C^*_{L,0}(P_a(C_i) \cup P_b(N_{a\beta\lambda\delta}(W_{ij})))) \\ \to \prod K_1^{\lambda^2 r}(C^*_{L,0}(P_b(N_{2a\beta\lambda r}(C_i)))) &\to \prod K_1^{\lambda^2 r}(C^*_{L,0} P_c(N_{2a\beta\lambda r}(C_i))) \\ &\to K_1^{\lambda^2 r}(C^*_{L,0}(P_c(Z))) \to K_1^{r,n}(C^*_{L,0}(P_{\lambda^2 c}(\mathcal{Z}))) \end{aligned}$$

the $\cong$ follows from the well-separatedness, the second arrow is 0, checking the dependence of contant c, we find $c = c(n, r, t, a, \mathcal{F})$. $\square$

Chapter 7

A Characterization of the Image of the Baum-Connes Map

In this chapter, we will apply the controlled K-theory to study the Baum-Connes conjecture and give a characterization of K-theory elements in the image of Baum-Connes map. In particular, we prove the coarse Baum-Connes conjecture is true for a class of spaces which have not been verified by other methods. This section is joint work with Oyono-Oyono and Yu

Section 7.1 Equivariant Controlled K-theory

Let H be a Hilbert space with a Γ-action and let φ be a $*$-homomorphism from $C_0(X)$ to $B(H)$ such that it is covariant in the sense that $\varphi(\gamma f)h = (\gamma(\varphi(f))\gamma^{-1})h$ for all $\gamma \in \Gamma$, $f \in C_0(X)$ and $h \in H$. Such a triple $(C_0(X), \Gamma, \varphi)$ is called a covariant system.

Definition 7.1. *We define the covariant system $(C_0(X), \Gamma, \varphi)$ be admissible if*

(1) the Γ-action on X is proper and cocompact;

(2) $\varphi(f)$ is noncompact for any nonzero function $f \in C_0(X)$;

(3) for each $x \in X$, the action of the stabilizer group Γ_x on H is regular in the sense that it is isomorphic to the action Γ_x on $l^2(\Gamma_x) \otimes W$ for some infinite dimensional Hilbert space W, where the Γ_x action on $l^2(\Gamma_x)$ is regular and the Γ_x action on W is trivial.

Definition 7.2. *Let $(C_0(X), \Gamma, \varphi)$ be admissible covariant system. We define $\mathbb{C}(\Gamma, X, H)$ to be the algebra of Γ-invariant locally compact operators acting on H with finite propagation. The C^*-algebra $C^*_{\mathrm{red}}(\Gamma, X, H)$ is the operator norm closure of $\mathbb{C}(\Gamma, X, H)$.*

We remark that if $(C_0(X), \Gamma, \varphi)$ be an admissible covariant system, then $C^*_{\mathrm{red}}(\Gamma, X, H)$ is $*$-isomorphic to $C^*_{\mathrm{red}}\Gamma \otimes K$, where $C^*_{\mathrm{red}}\Gamma$ is the reduced group C^*-algebra and K is the algebra of all compact operators. Let X be a locally compact and finite dimensional simplicial polyhedron. We endow X with the simplicial metric. Let $(C_0(X), \Gamma, \varphi)$ be an admissible covariant system introduced in the previous section, where φ is a $*$-homomorphism from $C_0(X)$ to $B(H)$ for some Hilbert space H.

Definition 7.3. *The algebraic localization algebra $C^*_{L,\mathrm{alg}}(\Gamma, X, H)$ is defined to the algebra of all bounded and uniformly continuous functions $f : [0, \infty) \to \mathbb{C}(\Gamma, X, H)$ such that the propagation of $f(t)$ goes to 0 as $t \to \infty$. The localization algebra $C^*_L(\Gamma, X, H)$ is the norm closure of $C^*_{L,\mathrm{alg}}(\Gamma, X, H)$ with respect to the following norm:*

$$\|f\| = \sup_{t\in[0,\infty)} \|f(t)\|.$$

Definition 7.4. *Let X_1 and X_2 be two metric spaces with proper cocompact Γ-actions. Assume that $(C_0(X_k), \Gamma, \varphi_k)$ is an admissible covariant system for each $k = 1, 2$, where φ_k is a $*$-homomorphism from $C_0(X_k)$ to $B(H_k)$ for some Hilbert space H_k. A map is called a Lipschitz map if there exits a constant $C > 0$ satisfying $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X_1$, where C is called the Lipschitz constant. A homotopy $F : X_1 \times [0, 1] \to X_2$ is called a Lipschitz homtopy if $F(\cdot, t)$ is Lipschitz with the same Lipschitz constant.*

We denote

$$P^r_\infty(C^*_{\text{red}}(\Gamma, X, H)) = \left\{p \in M_n(C^*_{\text{red}}(\Gamma, X, H)) \mid \|p^2 - p\| < \frac{1}{100}, \quad \text{Prop}(p) \le r\right\}$$

$$U^r_\infty(C^*_{\text{red}}(\Gamma, X, H)) = \left\{p \in M_n(C^*_{\text{red}}(\Gamma, X, H)) \mid \|u^2 - I\| < \frac{1}{100}, \quad \text{Prop}(u) \le r\right\}.$$

We can similar define the controlled K-theory for $C^*_{\text{red}}(\Gamma, X, H)$ [OY2].

Definition 7.5. *We define an equivalent relation on $P^r_\infty(C_{\text{red}}(\Gamma, X, H)) \times \mathbb{N}$ as following:*

$(p, l) \sim (q, l')$ if there is $k \in \mathbb{N}$ such that $h \in P^r_\infty(C[0,1], C^*_{\text{red}}(\Gamma, X, H))$ such that $h(0) = \text{diag}(p, I_{k+l'})$ and $h(1) = \text{diag}(q, I_{k+l})$.

We denote $K^r_0(C^*_{\text{red}}(\Gamma, X, H)) = P^r_\infty(C^*_{\text{red}}(\Gamma, X, H)) \times \mathbb{N}/\sim$.

Definition 7.6. *We define an equivalent relation on $U^{3r}_\infty(C^*_{\text{red}}(\Gamma, X, H))$ as following:*

$u \sim v$ if there is $h \in U^r_\infty(C[0,1], C^*_{\text{red}}(\Gamma, X, H))$ such that $h(0) = u$ and $h(1) = v$.

We denote $K^r_1(C^*_{\text{red}}(\Gamma, X, H)) = U^{3r}_\infty(C^*_{\text{red}}(\Gamma, X, H))/\sim$.

Proposition 7.7. *If F is a Γ-equivariant Lipschitz homotopy from X_1 to X_2 with Lipschitz constant C, then*

$$F(c\cdot, 0)_* = F(\cdot, 1)_* : K^r_*(C^*_{red}(\Gamma, X_1, H_1)) \to K^{10Cr}_*(C^*_{\text{red}}(\Gamma, X_2, H_2)).$$

Proof. The proof is similar to Lemma 5.32. The result also holds for localization algebras. □

For any $r > 0$, we can define a quantitative Baum-Connes map:

$$\mu_r : K^\Gamma_*(X) \to K^r_*(C^*_{\text{red}}(\Gamma, X, H)).$$

Proposition 7.8. *The local Baum-Connes map μ_L is an isomorphism from $K^\Gamma_*(X)$ to $K_*(C^*_L(\Gamma, X, H))$ if X is a finite dimensional simplicial polyhedron.*

Proof. The proof is using a standard cutting-pasting techniques, which is similar to the case without group actions. □

Proposition 7.9. *Let X be a locally compact and finite dimensional polyhedron with simplicial metric and $X = Y \cup Z$, where Y and Z are closed subsets of X. Assume that $(C_0(X), \Gamma, \varphi)$ is an admissible covariant system where φ is a $*$-homomorphism from $C_0(X) \to B(H)$ for some Hilbert space H. If Y and Z are Γ-invariant, $\text{int}(Y)$ and $\text{int}(Z)$ are respectively dense in Y and Z, then there exists a universal constant $c \ge 1$ such that the following sequence is asymptotically exact:*

$$\begin{aligned} K^r_1(C^*_{\text{red}}(\Gamma, Y \cap Z, H)) &\xrightarrow{i} K^r_1(C^*_{\text{red}}(\Gamma, Y, H) \oplus K^r_1(C^*_{\text{red}}(\Gamma, Z, H))) \xrightarrow{j} K^r_1(C^*_{\text{red}}(\Gamma, X, H)) \\ &\xrightarrow{\partial} K^{cr}_0(C^*_{\text{red}}(\Gamma, N_{cr}(Y) \cap N_{cr}(Z), H)) \\ &\xrightarrow{i} K^{cr}_0(C^*_{\text{red}}(\Gamma, N_{cr}(Y), H)) \oplus K^{cr}_0(C^*_{\text{red}}(\Gamma, N_{cr}(Z), H)) \xrightarrow{j} K^{cr}_0(C^*_{\text{red}}(\Gamma, X, H)), \end{aligned}$$

in the sense that

(1) $j \circ i = 0$;

(2) *the kernel of* $j : K_1^r(C^*_{\text{red}}(\Gamma, Y, H)) \oplus K_1^r(C^*_{\text{red}}(\Gamma, Z, H)) \to K_1^r(C^*_{\text{red}}(\Gamma, X, H))$ *is contained in the image of* $i : K_1^{c^2r}(C^*_{\text{red}}(\Gamma, Y \cap Z, H)) \to K_1^{c^2r}(C^*_{\text{red}}(\Gamma, Y, H)) \oplus K_1^{c^2r}(C^*_{\text{red}}(\Gamma, Z, H))$

(3) $\partial \circ j = 0$;

(4) *the kernel of* ∂ *in* $K_1^{c^2r}(C^*_{\text{red}}(\Gamma, X, H))$ *is contained in the image of* $j : K_1^{c^2r}(C^*_{\text{red}}(\Gamma, N_{cr}(Y), H)) \oplus K_1^{c^2r}(C^*_{\text{red}}(\Gamma, N_{cr}(Z), H)) \to K_1^{c^2r}(C^*_{\text{red}}(\Gamma, X, H))$;

(5) $i \circ \partial = 0$;

(6) *the kernel of* $i : K_0^r(C^*_{\text{red}}(\Gamma, Y \cap Z, H)) \to K_0^r(C^*_{\text{red}}(\Gamma, Y, H)) \oplus K_0^r(C^*_{\text{red}}(\Gamma, Z, H))$ *in* $K_0^{c^2r}(C^*_{\text{red}}(\Gamma, N_{cr}(Y) \cap N_{cr}(Z), H))$ *is contained in the image of* $\partial : K_1^{cr}(C^*_{\text{red}}(\Gamma, X, H)) \to K_0^{c^2r}(C^*_{\text{red}}(\Gamma, N_{cr}(Y) \cap N_{cr}(Z), H))$.

The similar result is true for the localization algebras.

The controlled Bott periodicity was introduced in [OY2].

Proposition 7.10. *We have a controlled isomorphism between* $\beta : K_*^r(A) \to K_*^{\lambda r}(S^2 A)$ *in the sense that,*

(1) *if* $x \in K_*^r(A)$*, then* $\beta(x) = 0 \in K_*^{\lambda r}(S^2A)$*, then* $x = 0 \in K_*^{\lambda r}(A)$;

(2) *if* $y \in K_*^{\lambda r}(S^2A)$*, then there exists some* $x \in K_*^{\lambda r}(A)$*, such that* $\beta(x) = y \in K_*^{\lambda^2 r}(S^2A)$*, where* λ *is a universal constant does not depends on* A.

Section 7.2 A Characterization of the Image of the Baum-Connes Map

In this section, we will give a characterization of the K-theory elements in the image of the Baum-Connes map.

Theorem 7.11. *Let* X *be a locally compact and finite polyhedron with simplicial metric and dimension* n *and let* $(C_0(X), \Gamma, \varphi)$ *be an admissible covariant system. Then there exists* $r_n > 0$ *such that the quantitative Baum-Connes map* μ_r *is an isomorphism for all positive* $r \le r_n$.

Proof. Let e be the evaluation map $C_L^*(\Gamma, X, H) \to C^*_{\text{red}}(\Gamma, X, H)$ defined by $e(f) = f(0)$, for all $f \in C_L^*(\Gamma, X, H)$. We have $\mu = e_* \circ \mu_L$. $\ker e$ consists all f with $f(0) = 0$. We denote such algebra by $C^*_{L,0}(\Gamma, X, H)$. We will show that for all $K_1^r(C^*_{L,0}(\Gamma, X, H)) = 0$, where ε depends only on the dimension of X.

We let $X^{(n)}$ to be the n-skeleton of X and prove the theorem by induction on n. In what follows, we will consider the homotopy variable to be $t' \in [0, 1]$ and we will consider $t \in [0, \infty)$ to be the localization variable. If $n = 0$, we have a discrete space. Thus the choice of r' is simple. We let $r' = \min\{r, 1\}$, then any element $u \in K_*^r(C_L^*(\Gamma, X, H))$ has propagation 0. Given $t_0 \in [0, \infty)$, define

$$u_{t_0} = \begin{cases} I & 0 \le t \le t_0 \\ u(t - t_0) & t_0 \le t \le \infty \end{cases}.$$

We consider

$$w(t') = (\oplus_{k\geq 0}(u_k \oplus I)) \cdot ((I \oplus I) \oplus_{k\geq 1} (u_{k-t'}^{-1} \oplus I)).$$

for $t' \in [0,1]$. Note that $w(t')$ acts on the standard nondegenerate X-module $\oplus_{t\geq 0}(H_X \oplus H_X)$.

Now $w(0) = (u\oplus I)\oplus_{k\geq 1}(I\oplus I)$, and $w(1) = (\oplus_{k\geq 0}(u_k\oplus I))\cdot((I\oplus I)\oplus_{t\geq 1}(u_{t-1}^{-1}\oplus I))$. We now construct a homotopy from $((I \oplus I) \oplus_{k\geq 1} (u_{k-1}^{-1} \oplus I))$ to $\oplus_{k\geq 0}(u_k^{-1} \oplus I)$. We know that $((I \oplus I) \oplus_{k\geq 1} (u_{k-1}^{-1} \oplus I))$ is isomorphic to $I \oplus_{t\geq 1} (I \oplus u_{t-1}^{-1})$ and that

$$v_1(t') = I \oplus_{k\geq 1} (R(t')(I \oplus u_{k-1}^{-1})R^*(t'))$$

will give a homotopy between $I \oplus_{k\geq 1} (I \oplus u_{k-1}^{-1})$ and $I \oplus_{k\geq 1} (u_{k-1}^{-1} \oplus I)$. Since $I \oplus_{k\geq 1} (u_{k-1}^{-1} \oplus I)$ is isomorphic to $\oplus_{k\geq 0}(I \oplus u_k^{-1})$. we see that

$$v_2(t') = \oplus_{k\geq 0}(R(t')(I \oplus u_k^{-1})R^*(t'))$$

will yield the homotopy between $((I \oplus I) \oplus_{k\geq 1} (u_{k-1}^{-1} \oplus I))$ and $\oplus_{k\geq 0}(u_k^{-1} \oplus I)$. Denote the homotopy by v.

We now define

$$F(t') = \begin{cases} w(2t') & 0 \leq t' \leq \frac{1}{2} \\ (\oplus_{k\geq 0}(u_k \oplus I))v(2t'-1) & \frac{1}{2} \leq t' \leq 1. \end{cases}$$

Then $F(0) = w(0) = (u\oplus I)\oplus_{k\geq 1}(I\oplus I)$ and $F(1) = (\oplus_{k\geq 0}(u_k\oplus I))\cdot\oplus_{t\geq 0}(u_t^{-1}\oplus I) = (I\oplus I)\oplus_{t\geq 1}(I\oplus I)$. Thus, the result holds for $n = 0$.

Assume now the theorem holds when $n = l-1$. Let $r > 0$ be small. For each simplex $\triangle$ of dimension l in X, define

$$\triangle_1 = \{x \in \triangle : d(x, c(\triangle)) \leq r\}$$

$$\triangle_2 = \{x \in \triangle : d(x, c(\triangle)) \geq r\}$$

where $c(\triangle)$ is the center of $\triangle$. Define $X = \cup\triangle_i$ $(i = 1, 2)$ where the union is taken over all simplices of dimension l in X.

We then notice that the X_i are G-subspaces as distances are preserved by the action of G. It is also clear that since r is small. X_1 is strongly Lipschitz G-homotopy equivalent to the collection of $c(\triangle)$ for all l-dimensional simplices $\triangle$ in X and hence by the Lipschitz homotopy theorem we have the result holds for X_1.

Similarly, we note that X_2 is strongly Lipschitz Γ-homotopy equivalent to $X^{(l-1)}$. By the Lipschitz homotopy theorem and induction hypothesis, we see that the result also holds for X_2. Now, it is clear that $X^{(l)} = X_1 \cup X_2$. Thus we need only look at $X_1 \cap X_2$. For, if the result holds for $X_1 \cap X_2$, then the proof is done by appealing to the controlled cutting pasting and the five lemma.

Since $X_1 \cap X_2$ is strongly Lipschitz Γ-homotopy equivalent to the disjoint union of the boundaries of all l-dimensional simplices in X, we have the desired result by the induction hypothesis. $\square$

Theorem 7.12. *An element $[p]$ in $K_0(C^*_{\text{red}}\Gamma)$ is in the image of the Baum-Connes map if and only if* *there exists an admissible covariant system $(C_0(X), \Gamma, \varphi)$ for some locally compact and finite dimensional simplicial polyhedron with the simplicial metric and dimension n such that $[p]$ is equivalent to $[q] - [p_0]$ and q is a quasi-projection in $M_{k_n}(C^*_{\text{red}}(\Gamma, X, H))$ for some natural number k_n with propagation at most r_n, where k_n depends only on n and r_n is a positive constant depending only on n, $p_0 = I \oplus 0$, and the propagation of an element in $M_k(C^*_{\text{red}}(\Gamma, X, H))$ is defined to be the maximal propagation of its entries.*

Proof. The "only if" part follows from the construction of the Baum-Connes map. The "if" part follows from Theorem 7.12. □

Corollary 7.13. *Let Γ be a finitely generated torsion-free group with a finite generating set S. Every element in the image of the Baum-Connes map in $K_0(C^*_{\text{red}}\Gamma)$ is equivalent to $[q] - [p_0]$ such that q is a quasi-projection in $M_k(C^*_{\text{red}}\Gamma)$ such that each of its entries is a linear combination of elements in $S \cup \{e\}$, where e is identity of Γ.*

Proof. We have $C^*_{\text{red}}(\Gamma, X, H) \cong C^*_{\text{red}}(\Gamma) \otimes K$. Since Γ is torsion free, small propagation in $C^*_{\text{red}}(\Gamma, X, H)$ implies propagation at most 1 in $C^*_{\text{red}}\Gamma \otimes K$ with respect to the word metric of Γ. □

Theorem 7.14. *An element $[u]$ in $K_1(C^*_{\text{red}})$ is in the image of the Baum-Connes map if and only if there exists an admissible covariant system $(C_0(X, \Gamma, \varphi))$ for some locally compact and finite dimensional simplicial polyhedron with the simplicial metric and dimension n such that u is equivalent to a quasi-unitary in $M_{k_n}(C^*_{\text{red}}(\Gamma, X, H)^+)$ for some natural number k_n with propagation at most r_n, where k_n depends only on n and r_n is a positive constant depending only on n.*

Proof. The "only if" part follows from the construction of the Baum-Connes map. The "if" part follows from Theorem 7.12. □

Corollary 7.15. *Let Γ be a finitely generated torsion free group with a finite generating set S. Every element in the image of the Baum-Connes map in $K_1(C^*_{\text{red}}\Gamma)$ is equivalent to $[v]$ such that v is a quasi-unitary in $M_k(C^*_{\text{red}}\Gamma)$ and each of its entries is a linear combination of elements in $S \cup \{e\}$, where e is the identity of Γ.*

Proof. We have $C^*_{\text{red}}(\Gamma, X, H) \cong C^*_{\text{red}}(\Gamma) \otimes K$. Since Γ is torsion free, small propagation in $C^*_{\text{red}}(\Gamma, X, H)$ implies propagation at most 1 in $C^*_{\text{red}}\Gamma \otimes K$ with respect to the word metric of Γ. □

In particular, if the classifying space for the torsion free group Γ is finite dimensional, then the matrix size k in the above corollary depends only on the dimension of the classifying space.

Section 7.3 Applications

In this section, we prove the coarse Baum-Connes conjecture for some special spaces using controlled K-theory.

Let $\varphi(n) : \mathbb{N} \to \mathbb{R}$ be an increasing function, and let X be the disjoint union of S^{2n} $(n = 1, 2, 3, \ldots)$, where S^{2n} is the sphere of dimension 2n. Endow a metric d on X such that

(1) $d_{|S^{2n}} = \varphi(n)^{-1} d_s$, where $d_{|S^{2n}}$ is the restriction of d to S^{2n}, and d_s is the standard Riemannian metric on the sphere S^{2n} with radius 1.

(2) if $n' < n$, then $d(S^{2n}, S^{2n'}) > 100n$.

Let D_n be the Dirac operator on S^{2n}. Define $D = \oplus_{n=1}^{\infty} D_n$. D gives rise to a K-homology class $[D]$ in $K_*(X)$.

If $\varphi(n) = n$, this gives a counterexample for the coarse Baum-Connes conjecture without bounded geometry condition [Yu98].

If $\varphi(n)$ grows very fast, we can verify the coarse Baum-Connes conjecture. Generalizing such phenomenon will lead to a very interesting way to work on the coarse Baum-Connes conjecture and the Baum-Connes conjecture.

Theorem 7.16. *There exists some constant $C > 0$, such that if $\varphi(n) > \exp(Cn)$ then the coarse Baum-Connes conjecture is true for X.*

Denote X_n to be the subspace $\bigoplus_{i=1}^{n} S^{2i}$ of X. For a subspace Y of X, we define $Q_m(Y)$ to be the quotient space $Y/(X_m \cap Y)$, so

$$\lim_{m\to\infty} K_*(Q_m(X)) \cong KX_*(X), \qquad \lim_{m\to\infty} K_*(C^*(Q_m(X))) \cong K_*(C^*(X)).$$

We have the following result.

Lemma 7.17. *For every $n > \max\{r, m\}$, we have*

$$K_*^r(C^*(Q_m(X))) \cong K_*^r(C^*(Q_m(X_n))) \oplus \prod_{i=n+1}^{\infty} K_*^r(C^*(Q_m(S^{2i}))).$$

Proof. Since the element in K^r are represented by elements with propagations at most r, and the constituent subspaces $Q_m(X_n), Q_m(S^{2(n+1)}), Q_m(S^{2(n+2)}), \ldots$ are at distance more than r, we can decompose a quasi-projection (quasi-unitary) in $K_*^r(Q_m(X))$ into a product of quasi-projections (quasi-unitaries) restricted on the subspaces. We only need to check the equivalence relations on both sides are equivalent. Clearly, two equivalent quasi-projections on the left hand side are also equivalent on the right hand side, since the restriction of a homotopy in the whole space gives rises to a collection of homotopies in the subspaces.

If two quasi-projections (quasi-unitaries) are equivalent on the right hand side, we want to take the product of the collection of homotopies in subspaces to a homotopy in the whole space. However, the product of a collection of continuous maps need not to be continuous. A standard trick is to make the collection of homotopies uniformly continuous by increasing the size of matrices of K-theory elements (See Proposition 1.29 [OY2]). $\square$

Proof of Theorem 7.16. Consider the following diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] \\
& K_*(Q_m(X_n)) \oplus \bigoplus_{i=n+1}^{\infty} K_*(Q_m(S^{2i})) \arrow[r] \arrow[d] & K_*(C^*(Q_m(X_n))) \oplus \bigoplus_{i=n+1}^{\infty} K_*^r(C^*(Q_{2m}(S^{2i}))) \arrow[d] \\
& K_*(Q_m(X)) \arrow[r] \arrow[d] & K_*^r(C^*(Q_m(X))) \arrow[d] \\
& \frac{\prod_{i=1}^{\infty} K_*(Q_m(S^{2i}))}{\bigoplus_{i=1}^{\infty} K_*(Q_m(S^{2i}))} \arrow[r, "\cong"] \arrow[d] & \frac{\prod_{i=1}^{\infty} K_*^r(C^*(Q_m(S^{2i})))}{\bigoplus_{i=1}^{\infty} K_*^r(C^*(Q_m(S^{2i})))} \arrow[d] \\
& 0 & 0
\end{tikzcd} \tag{7.1}$$

Note that $Q_m(S_{2i}) = S^{2i}$ if $i > m$. If $\varphi(i)$ grow very fast, then for every $r > 0$, we have

$$\varphi(i)r \leq r_{2i} \quad \text{if} \quad n \quad \text{large enough}$$

where r_{2i} as in Theorem 7.12. So for every $r > 0$, we have

$$\frac{\prod K_*(Q_m(S^{2i}))}{\bigoplus K_*(Q_m(S^{2i}))} \cong \frac{\prod K_*(S^{2i})}{\bigoplus K_*(S^{2i})} \cong \frac{\prod K_*^{\varphi(i)r}(C^*(S^{2i}, d_s))}{\bigoplus K_*^{\varphi(i)r}(C^*(S^{2i}, d_s))} \cong \frac{\prod K_*^r(C^*(S^{2i}, d))}{\bigoplus K_*^r(C^*(S^{2i}, d))} \cong \frac{\prod K_*^r(C^*(Q_m(S^{2i})))}{\bigoplus K_*^r(C^*(Q_m(S^{2i})))}$$

where the product and sum are taken from $i = 1$ to ∞. Hence we have checked the isomorphism of the lower horizontal map in 7.1.

To show the isomorphism of first horizontal map in 7.1 as $m \to \infty$. We notice that all the nonzero entries are absorbed to the first entry as m increasing. The problem reduces to the isomorphism of the controlled assembly map μ_r when the space is a singleton, which is clearly true.

Therefore, the middle horizontal map in 7.1 is also an isomorphism. Hence the coarse Baum-Connnes conjecture holds if $\varphi(n)$ grows fast enough. □

We can generalize Theorem 7.16 to simplicial complexes.

Theorem 7.18. *Let $(X_n, \varphi(n)d)$ be a sequence of simplicial complexes with $\dim X_n \leq n$ and let the metric of X_n be the standard simplicial metric multiplied by $\varphi(n)$. Let $X = \bigsqcup_{n=1}^{\infty} X_n$ be the disjoint union of $(X_n, \varphi(n)d)$, where $d(X_n, X_{n'}) \geq \varphi(n)$ whenever $n' < n$. There exists some constant $C > 0$, such that if $\varphi(n) > \exp(Cn)$ then the coarse Baum-Connes conjecture is true for X.*

Proof. The proof is similar to the Theorem 7.16. □

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