APPLICATIONS OF MODULAR FORMS IN GEOMETRY AND INTERPOLATION PROBLEMS

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CHAPTER I

Introduction

The sphere packing problem asks for the densest packing of spheres into Euclidean space; i.e, what fraction of $\mathbb{R}^n$ can be covered by non-overlapping congruent balls. To begin, equip $\mathbb{R}^n$ with the standard norm $\|\|$ and Lebesgue measure Vol(). Then for $x \in \mathbb{R}^n, r > 0$ we denote $B_n(x, r)$ as the ball in $\mathbb{R}^n$ with center x and radius r. Let $X \subset \mathbb{R}^n$ be a discrete set of points such that $\|x - y\| \geq 2r$ for any distinct $x, y \in X$. Then the union

$$\mathcal{P} = \bigcup_{x \in X} B_n(x, r)$$

is a *sphere packing*. The *finite density* of a packing $\mathcal{P}$ is defined as

$$\Delta_{\mathcal{P}}(R) = \frac{\mathrm{Vol}(\mathcal{P} \cap B_n(0, R))}{\mathrm{Vol}(B_n(0, R))},$$

for $R > 0$. We define the *density* of a packing $\mathcal{P}$ as

$$\Delta\mathcal{P} = \limsup_{R \to \infty} \Delta_{\mathcal{P}}(R).$$

Then the n-dimensional sphere packing constant is the supremum over all possible packing densities

$$\Delta_n = \sup_{\mathcal{P} \subset \mathbb{R}^n} \Delta_{\mathcal{P}}.$$

All of these definitions and ideas are discussed in greater detail in Section II.2. The goal of the sphere packing problem is therefore to determine the value of Δ_n for different values of n. In several dimensions it is either known or conjectured that the optimal packing in $\mathbb{R}^n$ is a lattice, where we reall the definitions

Definition I.0.0.1. *A* ***lattice*** *is a set of the form* $\Lambda = T(\mathbb{Z}^n)$*, where* $T : \mathbb{R}^n \to \mathbb{R}^n$ *is an invertible linear transformation. The* ***dual of a lattice*** $\Lambda = T(\mathbb{Z}^n)$ *is the set* Λ^* *so that* $\Lambda^* = (T^T)^{-1}(\mathbb{Z}^n)$.

For example, if we consider the one dimensional case, we get $\Delta_1 = 1$ for $\Lambda = \mathbb{Z}$. If we consider the same problem in $\mathbb{R}^2$, the hexagonal or A_2 lattice is optimal here. This result was known for lattice packings by Lagrange as early as 1773. In 1940 Fejes Töth proved that this was optimal amongst all packings, concluding that $\Delta_2 = \frac{\pi}{\sqrt{12}}$, see [16]. Raising the dimension once more and considering $\mathbb{R}^3$, we end up with a string of results:

- In 1611, Kepler conjectured that the face centered cubic (FCC) is optimal amongst all configurations, known as *Kepler's Conjecture*, see [17].
- In 1831, Gauss proved that FCC and the hexagonal close packing (HCP) are optimal amongst lattice packings, see [16].
- In 1998, Hales announced a proof of Kepler's conjecture however it was not until 2014 that his computer assisted proof was rigorously checked giving $\Delta_3 = \frac{\pi}{\sqrt{18}}$, see [18].

Hales proof was a computer-assisted proof requiring more than one hundred thousand cases to be checked. We also note that the lack of similarity between the proofs in the two and three dimensional cases suggests that most dimensions will require their own ad hoc methods.

In 2003 Cohn and Elkies used linear programming methods and Fourier analysis (again discussed in much greater detail in Section II.2) to attack the problem.

Definition I.0.0.2. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is admissible if there exists a constant $\delta > 0$ such that $|f(x)|$ and $|\hat{f}(x)|$ are bounded above by a constant times $(1 + |x|)^{-n-\delta}$.*

Proposition I.0.0.3. *(Cohn Elkies) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an admissible function, not identically zero and satisfies the following conditions:*

- $f(0) = \hat{f}(0) > 0$
- $\hat{f}(y) \geq 0$ *for all* $y \in \mathbb{R}^n$
- $f(x) \leq 0$ *for* $|x| \geq r$

Then the sphere packing density in $\mathbb{R}^n$ is at most $\mathrm{Vol}(B_n(0, \frac{r}{2}))$.

A proof of this is given in Section II.2. As discussed in Section II.2, the proof of this proposition implies in order to show that a lattice Λ packing is optimal using the proposition, we would additionally require f to vanish on Λ and $\hat{f}$ to vanish on its dual Λ^*. For example, in the case $n = 1$, Cohn and Elkies used the function

$$f(x) = (1 - |x|)\chi_{[-1,1]}(x)$$

with Fourier transform given by

$$\hat{f}(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$$

to give another proof that $\mathbb{Z}$ is the optimal packing in $\mathbb{R}$.

The Paley-Weiner theorem however implies that it is generally difficult to control the zeros of a function and its Fourier transform simultaneously. It would therefore make sense to consider eigenfunctions of the Fourier transform. By first observing that all functions could be assumed radial, Cohn and Elkies considered the functions

$$f_+ = f + \hat{f} \tag{I.1}$$

$$f_- = f - \hat{f}. \tag{I.2}$$

This reduced the problem to finding $+1$ and -1 eigenfunctions of the Fourier transform satisfying the above extremal properties. Using lattices either known or conjectured to be optimal, Cohn and Elkies did numerical experiments using linear combinations of the functions $f(x) = e^{-\pi|x|^2}p(2\pi|x|^2)$, for appropriately chosen Laguerre polynomials p. These functions were chosen because they form a basis for the radial eigenfunctions of the Fourier transform with eigenvalues $+1$ or -1. Their estimates gave convincing evidence of 'magic functions' in the cases of $n = 2, 8, 24$ that show optimality of the A_2, E_8, and Leech lattices respectively. The authors would conjecture that these functions existed and were unique.

It was not until 2016 that Viazovska would find such a magical function for the case $n = 8$ in [2]. Motivated by the work of Cohn and Elkies, in particular equations (I.1) and (I.2), she would consider functions of the form

$$V(x) = \sin\left(\frac{\pi\|x\|^2}{2}\right)^2 \int_0^{i\infty} \psi(z) e^{i\pi\|x\|^2 z}\, dz, \tag{I.3}$$

for $x \in \mathbb{R}^8$ and appropriately defined functions $\psi(z)$. She would find that a necessary condition of V being an eigenfunction of the Fourier transform would imply that ψ would satisfy functional equations involving the transformations $z \to z + 1$ and $z \to -\frac{1}{z}$. This was suggestive of the involvement of modular forms. The function she produced would demonstrate that $\Delta_8 = \frac{\pi^4}{384}$. These ideas were subsequently applied to the case $n = 24$ in [3] by Cohn, Kumar, Miller, Radchenko, and Viazovska to give that $\Delta_{24} = \frac{\pi^{12}}{12!}$. In terms of (I.3) these solutions were given by

$$\psi_+ = \frac{(E_2E_4 - E_6)^2}{\Delta}$$
$$\psi_- = \frac{5\theta_{01}^{12}\theta_{10}^8 + 5\theta_{01}^{16}\theta_{10}^4 + 2\theta_{01}^{20}}{\Delta}$$

for the 8 dimensional case (the + and − cases in the subscripts refer to the designations from f_+ and f_-). In the 24 dimensional case the solutions were

$$\begin{aligned}\psi_+ &= \frac{25E_4^4 - 49E_6^2E_4 + 48E_6E_4^2E_2 + 25E_6^2E_2^2 - 49E_4^3E_2^2}{\Delta^2} \\ \psi_- &= \frac{7\theta_{01}^{20}\theta_{10}^8 + 7\theta_{01}^{24}\theta_{10}^4 + 2\theta_{01}^{28}}{\Delta^2}.\end{aligned}$$

The notation of both solutions is discussed in Section II.3.

From here, the idea of taking the Laplace transform of modular forms was applied to other extremal problems including those in energy optimization, interpolation, and harmonic analysis. In [8], Radchenko and Viazovska considered functions of the form

$$A(x) = \int_{-1}^{1} g(z)e^{i\pi x^2 z}dz,$$

for again an appropriately defined modular form g and real values of x. These functions were used as an interpolation basis for Schwartz functions on the real line, giving the authors the following theorem.

Theorem I.0.0.4. *There exists a collection of even Schwartz functions $a_n : \mathbb{R} \to \mathbb{R}$ with the property that for any even Schwartz function $f : \mathbb{R} \to \mathbb{R}$ and any $x \in \mathbb{R}$ we have*

$$f(x) = \sum_{n=0}^{\infty} a_n(x)f(\sqrt{n}) + \sum_{n=0}^{\infty} \hat{a}_n(x)\hat{f}(\sqrt{n}),$$

where the right-hand side converges absolutely.

Similarly, for odd functions they showed

Theorem I.0.0.5. *There exists a collection of odd Schwartz functions $d_n^+ : \mathbb{R} \to \mathbb{R}$ and $d_n^- : \mathbb{R} \to \mathbb{R}$ with the property that for any odd Schwartz function $f : \mathbb{R} \to \mathbb{R}$ and any $x \in \mathbb{R}$ we have*

$$f(x) = d_0^+(x)\frac{f'(0) + i\hat{f}'(0)}{2} + \sum_{n=1}^{\infty} c_n \frac{f(\sqrt{n})}{\sqrt{n}} - \sum_{n=1}^{\infty} \hat{c}_n \frac{\hat{f}(\sqrt{n})}{\sqrt{n}},$$

where $c_n = \frac{d_n^+(x)+d_n^-(x)}{2}$.

Since $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$, we get a general interpolation formula for real Schwartz functions. A generalization of this interpolation scheme was then applied in [14] by Cohn, Kumar, Miller, Radchenko, and Viazovska to show universal optimality (see [19] for appropriate definitions) of the E_8 and Leech Lattices in $\mathbb{R}^8$ and $\mathbb{R}^{24}$ respectively.

Finally, in [15], Cohn and Gonçalves constructed a function of the type Viazovska initially considered to demonstrate a minimizing function in the sense of an uncertainty principle they considered. More precisely, for $f:\mathbb{R}^d\to\mathbb{R}$ we say it is *eventually nonnegative* (respectively, *eventually nonpositive*) if $f(x)\geq 0$ ($f(x)\leq 0$) for all sufficiently large $|x|$. Define

$$r(f)=\inf\{R\geq 0: f(x) \text{ has the same sign for } |x|\geq R\},$$

let $\mathcal{A}_+(d)$ denote the set of functions $f:\mathbb{R}^d\to\mathbb{R}$ such that

- $f\in L^1(\mathbb{R}^d)$, $\hat{f}\in L^1(\mathbb{R}^d)$, and $\hat{f}$ is real valued
- f is eventually nonnegative while $\hat{f}(0)\leq 0$
- $\hat{f}$ is eventually nonnegative while $f(0)\leq 0$,

and let

$$\mathrm{A}_+(d)=\inf_{f\in\mathcal{A}_+(d)/\{0\}}\sqrt{r(f)r(\hat{f})}.$$

It is shown $\mathrm{A}_+(12)=\sqrt{2}$ by constructing a function f that is a $+1$ eigenfunction of the Fourier transform such that $r(f)=\sqrt{2}$.

I.1 Statement of Results

This thesis will discuss functions and transforms of the type discussed in [2] and [8]. We give a general framework for constructing $+1$ and -1 eigenfunctions of the Fourier transforms of the form (I.3) in $\mathbb{R}^d$ for d divisible by 4. Specifically, we show the following propositions and theorems

Proposition I.1.0.1. *Suppose $\psi\in L^1_{loc}(i\mathbb{R})$ is such that for some $C>0$ and constants $a_k,b_k\in\mathbb{C}$, $k=0,1,...,n$,*

$$\psi(z)=\sum_{k=0}^{n}a_ke^{-2\pi ikz}-iz\sum_{k=0}b_ke^{-2\pi ikz}+\mathcal{O}(e^{iCz})\ \text{as}\ z\to i\infty. \tag{I.4}$$

For $\mathrm{Re}(s)>2n$, let

$$W(s)=-i\int_0^{i\infty}\psi(z)e^{-2\pi isz}dz. \tag{I.5}$$

Then

$$W(s)=\sum_{k=0}^{n}\left(\frac{a_k}{\pi(s-2k)}+\frac{b_k}{\pi^2(s-2k)^2}\right)-i\int_0^{i\infty}\left(\psi(z)-\left(\sum_{k=0}^{n}a_ke^{-2\pi ikz}+z\sum_{k=0}^{n}b_ke^{-2\pi ikz}\right)\right)e^{\pi isz}dz \tag{I.6}$$

gives an analytic continuation of W to the half-plane $Re(s)>-\frac{C}{\pi}$.

Proposition I.1.0.2. *Let $\psi : \mathbb{H} \to \mathbb{C}$ be holomorphic on $\mathbb{H}$ and bounded on the angular region $R_{\alpha,\epsilon} := \{re^{it} : 0 < r < \epsilon,\, \alpha < t < \pi - \alpha\}$ for some $\epsilon > 0$ and some $0 < \alpha < \frac{\pi}{4}$. Further suppose the restriction of ψ to $i\mathbb{R}_+$ and W are as in Proposition I.1.0.1 and for $\mathrm{Re}(s) > -\frac{C}{\pi}$ let $U(s)$ be defined by*

$$U(s) = -4\sin\left(\frac{\pi}{2}s\right)^2 W(s). \tag{I.7}$$

Then $U(s)$ is holomorphic for $\mathrm{Re}(s) > -\frac{C}{\pi}$ and

$$\begin{aligned} iU(s) &= \int_{-1}^{i} \psi(Tz)e^{i\pi sz}\,dz + \int_{1}^{i} \psi(T^{-1}z)e^{i\pi sz}\,dz \\ &- 2\int_{0}^{i} \psi(z)e^{i\pi sz}\,dz + \int_{i}^{i\infty} \left(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)\right)e^{i\pi sz}\,dz, \end{aligned} \tag{I.8}$$

where the integrals are along straight line segments joining the endpoints.

Proposition I.1.0.3. *Let ψ be as in Proposition I.1.0.1 , U as in Proposition I.1.0.2 and let $F : \mathbb{R}^d \to \mathbb{C}$ be defined by*

$$F(\mathbf{x}) := U(\|\mathbf{x}\|^2), \qquad (\mathbf{x} \in \mathbb{R}^d). \tag{I.9}$$

If, in addition, ψ satisfies

$$\psi(z) = \mathcal{O}(e^{iCSz}) \quad \text{as } z \to 0 \quad \text{non-tangentially in } \mathbb{H}, \tag{I.10}$$

then F is a Schwartz function and can be written in the form

$$\begin{aligned} F(\mathbf{x}) = -i\Bigg[&\int_{-1}^{i} \psi(Tz)e^{i\pi\|\mathbf{x}\|^2 z}\,dz + \int_{1}^{i} \psi(T^{-1}z)e^{i\pi\|\mathbf{x}\|^2 z}\,dz \\ &-2\int_{0}^{i} \psi(z)e^{i\pi\|\mathbf{x}\|^2 z}\,dz + \int_{i}^{i\infty}\left(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)\right)e^{i\pi\|\mathbf{x}\|^2 z}\,dz\Bigg]. \end{aligned} \tag{I.11}$$

Consequently, the Fourier transform of F is given by

$$\begin{aligned} \hat{F}(\mathbf{t}) &= -i(-1)^{d/4}\Bigg[\int_{-1}^{i} \psi(T^{-1}Sz)e^{i\pi\|\mathbf{t}\|^2 z} z^{d/2-2}\,dz \\ &+ 2\int_{i}^{i\infty} \psi(Sz)e^{i\pi\|\mathbf{t}\|^2 z} z^{d/2-2}\,dz + \int_{1}^{i} \psi(TSz)e^{i\pi\|\mathbf{t}\|^2 z} z^{d/2-2}\,dz \\ &- \int_{0}^{i} (\psi(T^{-1}Sz) - 2\psi(Sz) + \psi(TSz))e^{i\pi\|\mathbf{t}\|^2 z} z^{d/2-2}\,dz\Bigg]. \end{aligned} \tag{I.12}$$

Proposition I.1.0.4. *Let ψ be as in Proposition I.1.0.2 , F as in Proposition I.1.0.3 and $\varepsilon \in$*

$\{-1, 1\}$. *Then* $\hat{F} = \varepsilon(-1)^{\frac{d}{4}} F$ *if and only if*

$$z^{\frac{d}{2}-2}\psi(T^{-1}Sz) = \varepsilon\psi(Tz) \tag{I.13}$$

$$2z^{\frac{d}{2}-2}\psi(Sz) = \varepsilon\left(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)\right), \tag{I.14}$$

for all $z \in \mathbb{H}$.

Proposition I.1.0.5. *Let* ψ *be as in Proposition I.1.0.1. Then the corresponding function* F *given by* (I.9) *is an eigenfunction for the Fourier transform with eigenvalue* $(-1)^{\frac{d}{4}}$, *if and only if* $z^{\frac{d}{2}-2}\psi(Sz)$ *is a quasi-modular form of weight* $4 - \frac{d}{2}$ *and depth* 2. *More precisely, there are weakly holomorphic modular forms* ψ_1, ψ_2, ψ_3 *of respective weights* $4 - \frac{d}{2}$, $2 - \frac{d}{2}$, *and* $-\frac{d}{2}$ *such that*

$$z^{\frac{d}{2}-2}\psi(Sz) = \psi_1(z) - 2E_2(z)\psi_2(z) + E_2(z)^2\psi_3(z). \tag{I.15}$$

This gives

$$\begin{aligned}\psi(z) = {} & z^2\left(\psi_1(z) - 2E_2(z)\psi_2(z) + E_2(z)^2\psi_3(z)\right) \\ & + z\frac{12i}{\pi}\left(\psi_2(z) - E_2(z)\psi_3(z)\right) - \frac{36}{\pi^2}\psi_3(z).\end{aligned} \tag{I.16}$$

Furthermore, ψ_1, ψ_2, *and* ψ_3 *have to satisfy*

$$\psi_1(z) - 2E_2(z)\psi_2(z) + E_2(z)^2\psi_3(z) = \mathcal{O}(e^{2\pi i z}) \tag{I.17}$$

for $z \to i\infty$ *in order to fulfill* (I.4) *and* (I.10).

Proposition I.1.0.6. *Let* ψ *be as in Proposition I.1.0.2. Then the corresponding function* F *given by* (III.7) *is an eigenfunction of the Fourier transform with eigenvalue* $(-1)^{\frac{d}{4}+1}$ *if and only if there exists a weakly holomorphic modular form* f *of weight* $2 - \frac{d}{2}$ *for* Γ *and* ω *a weakly holomorphic modular form of weight* $2 - \frac{d}{2}$ *for* $\Gamma(2)$ *such that*

$$\psi(z) = f(z) \cdot \mathcal{L}(z) + \omega(z), \tag{I.18}$$

$$\omega(z) = z^{\frac{d}{2}-2}\omega(Sz) + \omega(Tz), \tag{I.19}$$

where $\mathcal{L}$ *is defined in* (II.10).

Theorem I.1.0.7. *In dimensions* d *divisible by* 4, *there exists an integer* n_+ *and a radial Schwartz* *function $F_+ : \mathbb{R}^d \to \mathbb{R}$ such that*

$$F_+(\mathbf{x}) = (-1)^{\frac{d}{4}} \widehat{F}_+(\mathbf{x}) \quad \textit{for all } \mathbf{x} \in \mathbb{R}^d$$
$$F_+(\sqrt{2n_+}) = 0 \quad \textit{and } F'_+(\sqrt{2n_+}) \neq 0$$
$$F_+(\sqrt{2m}) = F'_+(\sqrt{2m}) = 0 \quad \textit{for } m > n_+, \quad m \in \mathbb{N}$$

and

Theorem I.1.0.8. *In dimensions, d divisible by 4, there exists an integer n_- and a radial Schwartz function $F_- : \mathbb{R}^d \to \mathbb{R}$ such that*

$$F_-(\mathbf{x}) = (-1)^{\frac{d+1}{4}} \widehat{F}_-(\mathbf{x}) \quad \textit{for all } \mathbf{x} \in \mathbb{R}^d$$
$$F_-(\sqrt{2n_-}) = 0 \quad \textit{and } F'_-(\sqrt{2n_-}) \neq 0$$
$$F_-(\sqrt{2m}) = F'_-(\sqrt{2m}) = 0 \quad \textit{for } m > n_-, \quad m \in \mathbb{N}.$$

We also explore their utilities within the sphere packing problem and the uncertainty principle discussed above. We also generalize the result given in [8] by showing an extension to $\mathbb{R}^2$ and $\mathbb{R}^3$

Theorem I.1.0.9. *Let $d \in \{2, 3\}$, there exists a collection of radial Schwartz functions $a_n : \mathbb{R}^d \to \mathbb{R}$ with the property that for any radial Schwartz function $f : \mathbb{R}^d \to \mathbb{R}$ and any $x \in \mathbb{R}^d$ we have*

$$f(x) = \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n=0}^{\infty} \hat{a_n}(x) \hat{f}(\sqrt{n}),$$

where the right-hand side converges absolutely.

Throughout we would like to emphasize the natural connection between the transforms, modular forms, and the underlying structures that they are meant to study. Moreover, where appropriate we give discussion about the functions and the intuition behind them. This thesis will be organized as follows. In Chapter II we give an overview of lattices, Fourier transforms, modular forms, and Riemann surfaces. Chapter III gives a discussion of functions of the form

$$V(x) = \sin\left(\frac{\pi\|x\|^2}{2}\right)^2 \int_0^{i\infty} \psi(z) e^{i\pi\|x\|^2 z}\, dz.$$

This includes conditions for analytic continuity, being Schwartz class, and being eigenfunctions of the Fourier transform. These will become functional equations for ψ and asymptotic conditions as $z \to 0$ and $z \to i\infty$. In Sections III.2.1 and III.3.1 we discuss the cases of being a +1 and

-1 eigenfunction of the Fourier transform respectively. For the former the solution will be weakly holomorphic quasi-modular forms of weight $4 - \frac{d}{2}$ and depth 2. For the latter the solutions will be weakly holomorphic modular forms for $\Gamma(2)$ of weight $2 - \frac{d}{2}$. In both cases we discuss applications to the sphere packing problems. In Chapter IV we discuss a generalization of the interpolation formula given in [8] for real Schwartz functions to radial Schwartz functions in $\mathbb{R}^2$ and $\mathbb{R}^3$. In Chapter V we give tables of the polynomials discussed in Chapter III.

CHAPTER II

Preliminary Materials

This section will be used to cover a broad spectrum of material that will assumed to be known throughout the thesis. Where appropriate we will include references to this section throughout the work.

II.1 Lattices

We equip $\mathbb{R}^n$ with its usual inner product, i.e, if $x, y \in \mathbb{R}^n, \langle x, y\rangle = y^T x$. A lattice is a set of the form $\Lambda = T(\mathbb{Z}^n)$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation. Equivilantly, we can write $\Lambda = \{\sum_{k=1}^n c_k v_k : c_k \in \mathbb{Z}\}$ and where the v_k's are the columns of T. We define a set Ω to be a *fundamental parallelepiped* for $\mathbb{R}^n/\Lambda$ if the collection of sets $\{\Omega + v : v \in \Lambda\}$ consists of pairwise disjoint sets whose union is $\mathbb{R}^n$. The canonical choice of fundamental parallelepiped will be the set $\Omega_\Lambda = \{\sum_{k=1}^n c_k v_k : c_k \in [0, 1)\}$. Its dual lattice Λ^* is defined to be the set of $v \in \mathbb{R}^n$ such that $\langle v, w\rangle \in \mathbb{Z}$ for all $w \in \Lambda$. This is equivalent to $\Lambda^* = (T^T)^{-1}(\mathbb{Z}^n)$ just by using the definition of the given inner product. With this in mind we define some standard terminology.

Definition II.1.0.1. *Let $\Lambda = T(\mathbb{Z}^n)$ be a lattice in $\mathbb{R}^n$.*

- *Λ is* ***integral*** *if $\langle v, w\rangle$ is an integer for all $v, w \in \Lambda$ or equivalently $T^T T \in GL_n(\mathbb{Z})$.*
- *Λ is* ***even*** *if $\|v\|^2$ is an even integer for all $v \in \Lambda$.*
- *Λ is* ***unimodular*** *if $\Lambda = T(\mathbb{Z}^n)$ and $|\det T| = 1$.*
- *Λ is* ***self-dual*** *if $\Lambda = \Lambda^*$.*
- *The* ***covolume*** *$\|\Lambda\| = Vol(\mathbb{R}^n/\Lambda) = |\det T|$ is the volume of any fundamental parallelotope. It satisfies $\|\Lambda\|\|\Lambda^*\| = 1$.*
- *The* ***Gram Matrix*** *of a lattice is defined to be $T^T T$.*

Proposition II.1.0.2. *If Λ is integral and unimodular, then Λ is self dual*

Proof. Since Λ is integral it must be the case that $\Lambda \subset \Lambda^*$. So it must be the case that there exists a $W \in \mathrm{GL}_n(\mathbb{Z})$ so that $TW = T'$, where $T' = (T^T)^{-1}$. Unimodularity gives $|\det W| = 1$ and using the adjugate, we see that $W^{-1} \in GL_n(\mathbb{Z})$ too so it must be the case that $T = T'W^{-1}$. So, $\Lambda^* \subset \Lambda$, $\Lambda^* = \Lambda$ $\square$

II.1.1 Fourier Transforms and Series

Given an L^1 function $f : \mathbb{R}^n \to \mathbb{R}$, we define the *Fourier Transform* of f to be

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x,t \rangle} dx$$

Also, for each integrable, periodic f and Λ we associate a *Fourier Series*

$$\sum_{t \in \Lambda^*} c_t e^{2\pi i \langle v,t \rangle}, c_t = \frac{1}{|\Lambda|} \int_{\mathbb{R}^n/\Lambda} f(v) e^{-2\pi i \langle v,t \rangle} dv$$

Here, integrability refers to the fundamental parallelepiped and we do not imply convergence to f in any sense. This is equivalent to the standard definition of the Fourier Series on a n-dimensional torus via a change of variable.

Definition II.1.1.1. *We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is* ***admissible*** *if there is a constant $\delta > 0$ such that $|f(x)|$ and $|\hat{f}(x)|$ are bounded above by a constant times $(1 + |x|)^{-n-\delta}$.*

Remark: The Riemann-Lebesgue lemma implies that for an L^1 function f, its Fourier transform $\hat{f} \in C_0$. If we add in the additional hypothesis of f being admissible then the Fourier Inversion Formula implies that there is a function $f' \in L^1 \cap C_0$ that is equal to f almost everywhere with respect to the Lebesgue measure. With this is in mind we can without loss of generality assume f is continuous by just taking its continuous representative.

With this definition in mind we present a result that will come up throughout the course of this paper.

Theorem II.1.1.2 (Poisson Summation)**.** *Suppose f is an admissible function then for each $v \in \mathbb{R}^n$ the following holds*

$$\sum_{x \in \Lambda} f(x + v) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} e^{2\pi i \langle v,t \rangle} \hat{f}(t) \tag{II.1}$$

Proof. We first show the result when $\Lambda \equiv \mathbb{Z}^n$. In this case we first observe that given a fixed $v \in \mathbb{R}^n$ the number of lattice points, $x \in \mathbb{Z}^n$ satisfying $N < |x + v| < N + 1$ is bounded above by $c_n((N + 1)^n - (N)^n)$, where c_n is the volume of the unit n-ball, and hence $\mathrm{O}(N^{n-1})$, for sufficiently large N, since the fundamental region for this lattice has volume 1. Combining this with our assumptions about f's decay we have

$$\sum_{x \in \mathbb{Z}^n} |f(x + v)| \leq \sum_{x \in \mathbb{Z}^n} (1 + |x + v|)^{-n-\delta} \leq M \sum_{N=1}^{\infty} N^{-\delta-1}$$

where M is a positive constant and the last sum is a convergent p-series. This implies that if we put $F(v) = \sum_{x \in \mathbb{Z}^n} f(x+v)$, F is absolutely and uniformly continuous by the Weierstrass M-Test. Also, we note that F is periodic with respect to the lattice and we have the following

$$\begin{aligned}
\int_{\mathbb{R}^n/\mathbb{Z}^n} |F(v)| dv &= \int_{\mathbb{R}^n/\mathbb{Z}^n} |\sum_{x \in \mathbb{Z}^n} f(x+v)| dx \\
&\leq \int_{\mathbb{R}^n/\mathbb{Z}^n} \sum_{x \in \mathbb{Z}^n} |f(x+v)| dx \\
&= \sum_{x \in \mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n} |f(x+v)| dx \\
&= \int_{\mathbb{R}^n} |f(y)| dy < \infty
\end{aligned}$$

Here, we used a change of variables and exchanged the sum and integral by Fubini, since the sum is absolutely convergent. Then, F is integrable with respect to the fundamental parallelepiped and therefore has a well defined Fourier Series as well. We next compute the t-th Fourier coefficient of F and observe that

$$\begin{aligned}
\int_{\mathbb{R}^n/\mathbb{Z}^n} F(v) e^{-2\pi i \langle v,t \rangle} dv &= \int_{\mathbb{R}^n/\mathbb{Z}^n} \sum_{x \in \Lambda} f(x+v) e^{-2\pi i \langle v,t \rangle} dv \\
&= \sum_{x \in \mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n} f(x+v) e^{-2\pi i \langle v,t \rangle} dv \\
&= \sum_{x \in \mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n + x} f(u) e^{-2\pi i \langle u-x,t \rangle} du \\
&= \sum_{x \in \mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n + x} f(u) e^{-2\pi i \langle u,t \rangle - 2\pi i \langle x,t \rangle} du \\
&= \int_{\mathbb{R}^n} f(u) e^{-2\pi i \langle u,t \rangle} du \\
&= \hat{f}(t)
\end{aligned}$$

We justify the interchange of the integral and sum in line 2 by the absolute summability of the series. Now, we note that our first observation again with the fact that f is admissible implies the absolute convergence of the sum of the Fourier coefficients of F. Together with F's continuity this implies point wise convergence to its Fourier Series, that is to say,

$$F(v) = \sum_{x \in \mathbb{Z}^n} f(x+v) = \sum_{t \in \mathbb{Z}^n} c_t e^{2\pi i \langle v,t \rangle} = \sum_{t \in \mathbb{Z}^n} \hat{f}(t) e^{2\pi i \langle v,t \rangle}$$

as desired. In the case of the general lattice Λ we note that by definition we must have $\Lambda = W(\mathbb{Z}^n)$ for some non-singular matrix W. In this case if we define $g(x) = f(Wx)$ then $\hat{g}(x) =$

$\frac{1}{|\det(W)|}\hat{f}(W^{-T}x) = \frac{1}{|\Lambda|}\hat{f}(W^{-T}x)$ and we have the following by what we proved above

$$\begin{aligned}
\sum_{x\in\Lambda} f(x+v) &= \sum_{k\in\mathbb{Z}^n} f(x+Wk) \\
&= \sum_{k\in\mathbb{Z}^n} g(W^{-1}x+k) \\
&= \sum_{k\in\mathbb{Z}^n} \hat{g}(k)e^{2\pi ik\langle(W^{-T}k),x\rangle} \\
&= \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*} e^{2\pi i\langle v,t\rangle}\hat{f}(t),
\end{aligned}$$

as desired □

II.2 Sphere Packing

In this section we present information and definitions relevant to our understanding of the sphere packing problem. Although some of this information was presented in the introduction, we present it in greater depth and formality.

Equip $\mathbb{R}^n$ with the standard norm $\|\|$ and Lebesgue measure Vol(). Then for $x \in \mathbb{R}^n, r > 0$ we denote $B_n(x,r)$ as the ball in $\mathbb{R}^n$ with center x and radius r. Let $X \subset \mathbb{R}^n$ be a discrete set of points such that $\|x-y\| \geq 2r$ for any distinct $x, y \in X$. Then the union

$$\mathcal{P} = \bigcup_{x\in X} B_n(x,r)$$

is a *sphere packing*. If X is a lattice in $\mathbb{R}^n$ then we say that $\mathcal{P}$ is a *lattice sphere packing*. We take the *finite density* of a packing $\mathcal{P}$, given $p \in \mathbb{R}^n$, to be

$$\Delta_{\mathcal{P}}(R) = \frac{\mathrm{Vol}(\mathcal{P}\cap B_n(p,R))}{\mathrm{Vol}(B_n(p,R))}, \tag{II.2}$$

for $R > 0$. We define the *density* of a packing $\mathcal{P}$ (if the limit exists) as

$$\Delta\mathcal{P} = \lim_{R\to\infty} \Delta_{\mathcal{P}}(R).$$

It is known that when this limit exists for one p, then it exists for all $p \in \mathbb{R}^n$ and the limit is equal for all such points, [20]. If the limit exists for all p uniformly then we say $\mathcal{P}$ has *uniform density*. In this case it shown in [20] that for every compact set S that is the closure of its interior and every point p,

$$\Delta\mathcal{P} = \lim_{R\to\infty} \frac{\mathrm{Vol}\left((RS+p)\cap\mathcal{P}\right)}{\mathrm{Vol}(RS)}. \tag{II.3}$$

On the other hand, if only

$$\Delta\mathcal{P} = \limsup_{R\to\infty} \sup_{p\in\mathbb{R}^n} \frac{\mathrm{Vol}(\mathcal{P}\cap B_n(p,R))}{\mathrm{Vol}(B_n(p,R))} \tag{II.4}$$

exits, we refer to it as the *upper density.* It was shown in [20] that (II.4) always exists, the supremum of all upper densities always exists, and the supremum is achieved by a uniformly dense packing. We take the supremum over all upper densities to be Δ_n, the n-dimensional sphere packing constant.

These definitions are designed to measure the fraction of space covered by the packing by taking increasingly large subsets of n-dimensional space in the form of balls and then letting the radius become increasingly large. Equation (II.2) corresponds to our intuitive notion of packing density in the case of lattice packings and simplifies to $\frac{\mathrm{Vol}(B_n(0,\frac{r^*}{2}))}{|\Lambda|}$ (where r^* is the length of the minimal vector in Λ), that is to say, the ratio of space covered by a ball of the prescribed radius to the volume of the fundamental parallelepiped. Here, $\mathrm{Vol}(B_n(0,\frac{r^*}{2}))$ refers to the volume of the solid n-dimensional ball given by $(\frac{r^*}{2})^n \frac{\pi^{n/2}}{\Gamma(n/2+1)}$.

Not every sphere packing is a lattice packing and we can instead opt for a more general notion of packings known as *periodic packings.* In such packings, we still want the packing to be periodic under translations by Λ, however, spheres can occur anywhere in a fundamental parallelepiped of Λ not just at corners as in the case of lattice packings. Our definitions given above still carry over for such configurations. In particular for such packings we suppose that we have a collection of vectors $v_1, v_2, ..., v_N$ within the canonical fundamental domain of Λ, our packing will then be

$$\mathcal{P} = \bigcup_{v\in\Lambda} \bigcup_{i=1}^{N} B_n(v+v_i, \frac{r^*}{2}),$$

where $r^* > 0$ is the distance in the packing. Similar to the case of the lattice packing, the periodic packing's density also simplifies to an intuitive definition as well: $\frac{N\,\mathrm{Vol}(B_n(0,\frac{r^*}{2}))}{|\Lambda|}$. We also have the following lemma

Lemma II.2.0.1. *There exists a sequence of periodic packings whose densities converge to* Δ_n

Proof. Let $\mathcal{P}$ be a uniformly dense packing of density Δ_n and Ω_Λ be the fundamental parallelotope of any lattice $\Lambda \subset \mathbb{R}^n$. Then, by (II.3)

$$\Delta_n = \lim_{r \to \infty} \frac{\mathrm{Vol}(r\Omega_\Lambda \cap \mathcal{P})}{\mathrm{Vol}(r\Omega_\Lambda)}.$$

Choose $\epsilon > 0$ and observe that if we choose r sufficiently large that the volume of the spheres in $\mathcal{P}$ that lie entirely within $r\Omega_\Lambda$ is within $\epsilon \,\mathrm{Vol}(r\Omega_\Lambda)$ of $\Delta_n \,\mathrm{Vol}(r\Omega_\Lambda)$. Now define a periodic packing $\mathcal{P}'$ by taking all the spheres of $\mathcal{P}$ that lie entirely within $r\Omega_\Lambda$ and then including all translations of them by $r\Lambda$. Then this periodic packing has density at least $\Delta_n - \epsilon$. The conclusion then follows. ☐

Using Lemma II.2.0.1, it suffices to consider these cases when trying to draw general conclusions about n-dimensional packing densities.

We can further define the *center density* δ_n to be the number of sphere centers per unit volume. If we scale our packing so that unit spheres are used we have $\Delta_n = \frac{\pi^{n/2}}{(n/2)!}\delta_n$ since the unit sphere has volume $\frac{\pi^{n/2}}{(n/2)!}$, here we interpret $(n/2)! = \Gamma(n/2+1)$ for odd n. We next continue with a strong result from Cohn and Elkies from [1] that will use our new terminology and tie in some earlier ones.

Proposition II.2.0.2. *(Cohn and Elkies) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an admissible function, not identically zero and satisfies the following conditions:*

- $f(0) = \hat{f}(0) > 0$
- $\hat{f}(y) \geq 0$ *for all* $y \in \mathbb{R}^n$
- $f(x) \leq 0$ *for* $|x| \geq r$

Then the sphere packing density in $\mathbb{R}^n$ is at most $\mathrm{Vol}(B_n(0, \frac{r}{2}))$.

Proof. Lemma II.2.0.1 implies that we may without loss of generality only consider periodic packings. To this end, suppose that X is a periodic packing in $\mathbb{R}^n$ using balls of radius $\frac{r}{2}$ and lattice Λ. In this case our packing is

$$\mathcal{P}_X = \bigcup_{v \in \Lambda} \bigcup_{i=1}^{N} B_n(v + v_i, \frac{r}{2}),$$

where the vectors $v_1, v_2, ..., v_N$ are within the canonical fundamental domain of Λ. Our packing density is then

$$\delta_X = \frac{N \,\mathrm{Vol}(B_n(0, \frac{r}{2}))}{|\Lambda|}.$$

We will show that $|\Lambda| \geq N$, then, since periodic packings can be made arbitrarily close to general packings our conclusion will follow for all packings. Next, by II.1.1.2

$$\sum_{x\in\Lambda} f(x+v) = \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*} e^{-2\pi i\langle v,t\rangle}\hat{f}(t)$$

for all $v \in \mathbb{R}^n$. It follows that

$$\begin{aligned}
\sum_{1\le j,k\le N}\sum_{x\in\Lambda} f(x+v_j-v_k) =& \frac{1}{|\Lambda|}\sum_{1\le j,k\le N}\sum_{t\in\Lambda^*} e^{-2\pi i\langle v_j-v_k,t\rangle}\hat{f}(t) \\
=& \frac{1}{|\Lambda|}\sum_{1\le j,k\le N}\sum_{t\in\Lambda^*} e^{-2\pi i\langle v_j,t\rangle}e^{2\pi i\langle v_k,t\rangle}\hat{f}(t) \\
=& \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*}\hat{f}(t)\sum_{1\le j,k\le N} e^{-2\pi i\langle v_j,t\rangle}e^{2\pi i\langle v_k,t\rangle} \\
=& \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*}\hat{f}(t)(\sum_{j=1}^{N} e^{-2\pi i\langle v_j,t\rangle})\overline{(\sum_{j=1}^{N} e^{-2\pi i\langle v_j,t\rangle})} \\
=& \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*}\hat{f}(t)|\sum_{j=1}^{N} e^{-2\pi i\langle v_j,t\rangle}|^2
\end{aligned}$$

First, note that $|x+v_j-v_k| < r$ iff $x=0$ and $i=j$ since that difference represents the distance between two sphere centers in the configuration X. Therefore, the left hand side is bounded above by $Nf(0)$. On the other hand, the non-negativity of the right hand side implies that it is bounded below by $\frac{N^2}{|\Lambda|}\hat{f}(0)$. Altogether,

$$Nf(0) \ge \frac{N^2}{|\Lambda|}\hat{f}(0)$$

Rearranging by using the fact that $f(0)=\hat{f}(0)>0$ we get the desired result. $\square$

II.2.1 Remarks

We first observe that our second assumption about the Fourier transform of f in Proposition II.2.0.2 is actually too strong; in the case of a periodic configutation X we only require that $\hat{f}(y) \ge 0$ for $y \in \Lambda^*$ instead of for $y \in \mathbb{R}^n$. Moreover, we can also note that we lose no generality in assuming that f is radial because if f satisfies the desired conditions, then so does its radial part given by

$$g(x) = \int_{S^{n-1}} f(\|x\|\xi)d\omega(\xi),$$

where $d\omega(\xi)$ is the normalized Lebesgue surface measure on S^{n-1}. Noting that the Fourier Transform as well as its inverse will map radial functions to radial functions via the Hankel Transform and Bessel functions, we have no problems assuming all functions to be discussed are radial.

Next, observe that Proposition II.2.0.2 makes no mention of a lattice or periodic packing. While the theorem was used in the context of lattices in [2] and [3], the full generality of the theorem implies that we only need a function satisfying its hypotheses for some $r > 0$ to get an upper bound on the sphere packing density. On the other hand, in cases where we do have an underlying periodic configuration X, for a *tight bound* or sharp estimate, the proof of Proposition II.2.0.2 implies that f vanishes on all non-zero distances in X and $\hat{f}$ vanishes on all non-zero distances in Λ^*.

For example, in the case $n = 8$, the densest known packing of $\mathbb{R}^8$ is the E_8 lattice. To prove E_8 has the optimal packing using Proposition II.2.0.2 we would need to find a function f that satisfies the hypotheses of Proposition II.2.0.2 and such that f and $\hat{f}$ vanish on the distances in E_8 (since E_8 is self-dual). The distances in E_8 are all of the form $\sqrt{2k}$ for $k \geq 1$, explaining why the functions considered in [2] would be of the form

$$V(x) = \sin\left(\frac{\pi\|x\|^2}{2}\right)^2 \int_0^{i\infty} \psi(z) e^{i\pi\|x\|^2 z}\, dz.$$

This similarly holds for $\mathbb{R}^{24}$ where the densest packing known was given by the self-dual Leech lattice, whose distances are given by $\sqrt{2k}$ for $k \geq 2$.

To add further regularity to this problem, the authors in [1] supposed that we had radial functions f and $\hat{f}$ that satisfy Proposition II.2.0.2 and considered the functions $f_+ = f + \hat{f}$, $f_- = f - \hat{f}$. Notice that f_+ and f_- are 1 and -1 eigenfunctions of the Fourier Transform respectively, that vanish on the distances in the lattice. Uncertainty principles imply that it is difficult to simultaneously control the behavior of a function and its Fourier transform; e.g, controlling the roots of both. This implies that we can gain traction by limiting our search to radial eigenfunctions of the Fourier Transform with eigenvalues 1 or -1, that vanish on a discrete set of values. This idea will be the primary inspiration for our later work in Chapter III.

II.3 Modular Forms

We will present two approaches to understanding these functions. While equivalent, the first will emphasize their relationship with lattices, which is important because their natural connection with the things our transform seeks to deal with.

Suppose $X \subset \mathbb{R}^2$ is a lattice, then as in Section II.1 we can write $X = \{\sum_{k=1}^{2} c_k v_k : c_k \in \mathbb{Z}\}$ for some $v_1, v_2 \in \mathbb{R}^2$ or equivalently $X = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, for some $\omega_1, \omega_2 \in \mathbb{C}$. We see that we can choose representatives ω_1, ω_2 so that $\tau = \frac{\omega_1}{\omega_2} \in \mathbb{H}$. This allows us to without loss of generality write $X = \mathbb{Z}\tau + \mathbb{Z}$, for some $\tau \in \mathbb{H}$. If we let $\mathcal{L}$ denote the set of all 2-dimensional lattices, suppose we define a function

$$F : \mathcal{L} \to \mathbb{C}, \;\; F(\mathbb{Z}\tau + \mathbb{Z}) = f(\tau),$$

where for now f is just a function meant to denote the sole dependence on τ. In order to extend F to all lattices we suppose that $F(\lambda X) = \lambda^{-k} F(X)$, where $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$. This implies that in general we have

$$F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-k} F\left(\left(\mathbb{Z}\frac{\omega_1}{\omega_2}\right) + \mathbb{Z}\right) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right).$$

Furthermore, for F to be well-defined we require $F(X) = F(X')$ when X and X' are homothetic, i.e, there exists a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ so that $X' = \gamma X$. In this way, suppose that $X \cong X'$, $X = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and $X' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ for representatives $\omega_1, \omega_2, \omega_1'$, and ω_2' such that $\tau = \frac{\omega_1}{\omega_2} \in \mathbb{H}$ and $\tau' = \frac{\omega_1'}{\omega_2'} \in \mathbb{H}$. Then for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\begin{aligned}
f(\gamma\tau) = f(\tau') = f\left(\frac{\omega_1'}{\omega_2'}\right) &= F\left(\mathbb{Z}\frac{\omega_1}{\omega_2} + \mathbb{Z}\right) \\
&= F\left(\mathbb{Z}\gamma\left(\frac{\omega_1}{\omega_2}\right) + \mathbb{Z}\right) \\
&= (c\omega_1 + d\omega_2)^k F(\mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}(c\omega_1 + d\omega_2)) \\
&= (c\omega_1 + d\omega_2)^k F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \\
&= \left(c\frac{\omega_1}{\omega_2} + d\right)^k F\left(\mathbb{Z}\frac{\omega_1}{\omega_2} + \mathbb{Z}\right) \\
&= \left(c\frac{\omega_1}{\omega_2} + d\right)^k f\left(\frac{\omega_1}{\omega_2}\right) \\
&= (c\tau + d)^k f(\tau)
\end{aligned}$$

This shows that we can go back and forth between between homogeneous functions defined on lattices and functions defined on $\mathbb{H}$ that satisfy automorphic properties with respect to $\mathrm{SL}_2(\mathbb{Z})$. In a slightly different way, we can think about the group $\mathrm{SL}_2(\mathbb{Z})$ and its action on the upper half-plane $\mathbb{H}$ exclusively without ever mentioning lattices. Specifically, for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ define its action on $z \in \mathbb{H}$ to be

$$\gamma z = \frac{az + b}{cz + d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We note that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We see that the action of $\gamma, -\gamma \in \mathrm{SL}_2(\mathbb{Z})$ have the same affect on the upper half-plane so it makes sense to consider $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. Define the principal congruence subgroup of level N, $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$

$$\Gamma(N) = \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

where we consider each of the entries modulo N. We can further define a congruence subgroup, Γ, to be any subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that there exists an N such that $\Gamma(N) \subset \Gamma$. The minimal such N is defined to be the level of the subgroup Γ. Every such subgroup Γ has finite index due to the group isomorphism

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

Furthermore, we can compute explicitly that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$

where p here is prime. We also observe that given a subgroup Γ, it partitions the set $\mathbb{Q} \cup \{i\infty\}$ into equivalence classes referred to as cusps. We now consider functions on $\mathbb{H}$ with respect to such subgroups Γ but we first define some preliminaries. Given a function $f, \gamma \in \mathrm{SL}_2(\mathbb{Z}), \tau \in \mathbb{H}$ we define the factor of automorphy

$$j(\gamma, \tau) = c\tau + d, \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and the weight-k slash operator to be

$$f[\gamma]_k(\tau) = j(\gamma, \tau)^{-k} f(\gamma(\tau)),$$

for an integer k. A function on the upper-half plane will said to be weight-k invariant for Γ if $f[\gamma]_k(\tau) = f(\tau)$ for each $\tau \in \mathbb{H}$ and $\gamma \in \Gamma$. Next, observe that since Γ is a congruence subgroup there exists a minimal positive integer h so that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$. Therefore, if f is weight-k invariant for Γ and meromorphic at $i\infty$ and there exists a $c > 0$ so that f has no poles on $\{\tau \in \mathbb{H} : \mathrm{Im} > c\}$ it has a Fourier series given by

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q_h^n, \text{ if } \mathrm{Im}(c) > 0, \text{where } q_h = e^{2\pi i \tau/h}$$

We say f is meromorphic at $i\infty$ if this series truncates from the left. This definition makes sense for every cusp sense if $s \in \mathbb{Q} \cup \{i\infty\}$ then there is some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ so that $\alpha(i\infty) = s$. Moreover, $f[\alpha]_k$ is invariant under $\alpha^{-1}\Gamma\alpha$, a congruence subgroup, so by the same logic as before we get a Laurent series for $f[\alpha]_k$. We can now define a special class of functions on $\mathbb{H}$

Definition II.3.0.1. *Let Γ be congruence subgroup of $SL_2(\mathbb{Z})$ and k be an integer. A function $f : \mathbb{H} \to \mathbb{C}^*$ is an automorphic form of weight k with respect to Γ if:*

1. *f is meromorphic on $\mathbb{H}$*
2. *f is weight-k invariant under Γ*
3. *$f[\alpha]_k$ is meromorphic for all $\alpha \in SL_2(\mathbb{Z})$, i.e, f is meromorphic at the cusps*

We denote such functions by $\mathcal{A}_k(\Gamma)$ and can further refine this by considering classes of functions such that we require f to be holomorphic on $\mathbb{H}$ and at the cusps, and that f to be holomorphic on $\mathbb{H}$, holomorphic at the cusps, and that for each $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ we have $f[\alpha]_k$ vanishes at infinity. These are call called modular forms of weight-k with respect to Γ (denoted $\mathcal{M}_k(\Gamma)$) and cusps forms of weight-k with respect to Γ (denoted $\mathcal{S}_k(\Gamma)$) respectively. Finally, we can identify the action of Γ on the extended upper-half plane to get a compact Riemann surface. In our case we will will be concerned with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma = \Gamma(2)$. We will denote the surfaces as $X(1)$ and $X(2)$ respectively. A fundamental domain for Γ is given by

$$\mathcal{F}_\Gamma = \left\{ z \in \mathbb{H} : |\mathrm{Re}(z)| \leq \frac{1}{2},\ \ |z| \geq 1 \right\}. \tag{II.5}$$

We can similarly write down a fundamental domain for $\Gamma(2)$ by looking at the action the coset representatives of $\Gamma/\Gamma(2)$ have on the fundamental for Γ. The given fundamental domain for Γ is shown in Figure II.1.

For the sake of completion we present some well known modular forms for $\mathrm{SL}_2(\mathbb{Z})$. We can define

Eisenstein series E_{2k} for each even integer k.

$$E_4(\tau) = \frac{1}{2\zeta(4)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^4}$$

$$E_6(\tau) = \frac{1}{2\zeta(6)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^6}$$

We can also define a *cusp form*

$$\Delta(\tau) = \frac{E_4^3(\tau) - E_6^2(\tau)}{1728}.$$

These are forms of weight 4, 6, and 12 respectively and ζ is the Riemann zeta function. Δ has the special property that it vanishes only at $i\infty$ (and hence the cusps) and nowhere else on $\mathbb{H}$. We also define an automorphic form on $\mathcal{A}_0(\mathrm{SL}_2(\mathbb{Z}))$, the Klein j-invariant

$$j(\tau) = \frac{E_4^3(\tau)}{\Delta(\tau)}. \tag{II.6}$$

We note j is Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$, i.e, it is an isomorphism from $X(1)$ to $\mathbb{C}^*$, a proof of this will be given in Section II.4.1. Using this we can note that for any even k, $\mathcal{A}_k(\Gamma)$ is non-empty because $j'(\tau) \in \mathcal{A}_2(\Gamma)$, so, we therefore have $(j'(\tau))^{\frac{k}{2}} \in \mathcal{A}_k(\Gamma)$.

Continuing we define the following Jacobi Theta functions

$$\begin{aligned}
\theta_{01}(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 \tau} \\
\theta_{10}(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i \left(n+\frac{1}{2}\right)^2 \tau} \\
\theta_{00}(\tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}
\end{aligned}$$

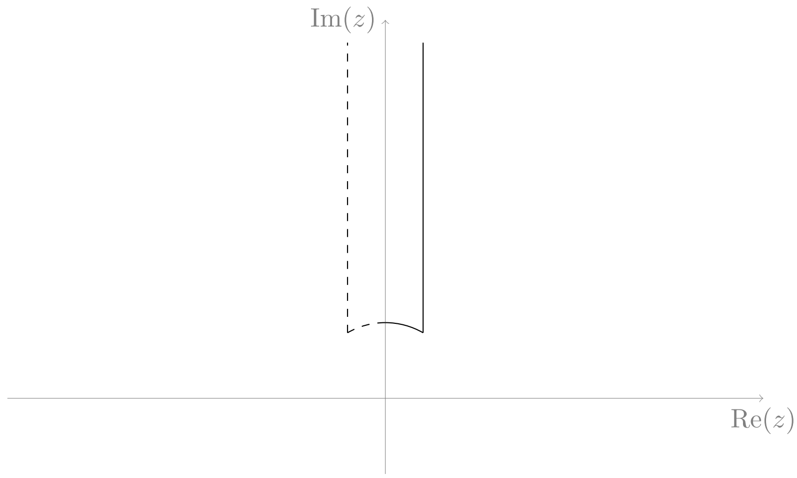

Figure II.1: A fundamental domain for Γ, $\mathcal{F}_\Gamma$

These satisfy the following identities

$$\tau^{-2}\theta_{00}^4(-\frac{1}{\tau}) = -\theta_{00}^4(\tau)$$
$$\tau^{-2}\theta_{01}^4(-\frac{1}{\tau}) = -\theta_{10}^4(\tau)$$
$$\tau^{-2}\theta_{10}^4(-\frac{1}{\tau}) = -\theta_{01}^4(\tau)$$

and

$$\theta_{00}^4(\tau+1) = \theta_{01}^4(\tau)$$
$$\theta_{01}^4(\tau+1) = \theta_{00}^4(\tau)$$
$$\theta_{10}^4(\tau+1) = -\theta_{10}^4(\tau)$$

and finally

$$\theta_{01}^4 + \theta_{10}^4 = \theta_{00}^4$$

We finally define an automorphic form for $\mathcal{A}_0(\Gamma(2))$, the modular lambda function

$$\lambda(\tau) = \frac{\theta_{10}^4(\tau)}{\theta_{00}^4(\tau)}, \tag{II.7}$$

which is Hauptmodul for $\Gamma(2)$, a proof of which is contained in Section II.4.1. It also satisfies the following identities

$$\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1} \tag{II.8}$$

$$\lambda\left(-\frac{1}{\tau}\right) = 1-\lambda(\tau). \tag{II.9}$$

It is holomorphic on $\mathbb{H}$, attains the value 1 at the origin, and has no zeros in $\mathbb{H}$ (this is all shown in Section II.4.1). Hence, we may define a holomorphic logarithm of λ, $\mathcal{L}$, by

$$\mathcal{L}(\tau) = 2\pi i \int_0^\tau \frac{\lambda'(w)}{\lambda(w)}\,dw = \pi i \int_0^\tau \theta_{01}^4(w)\,dw, \tag{II.10}$$

where the second equation follows from (II.7). We observe via direct computation with the contour integral and the properties of λ that:

$$\mathcal{L}(T^2\tau) = \mathcal{L}(\tau) + 2\pi i \tag{II.11}$$

$$2\mathcal{L}(S\tau) = \mathcal{L}(T^{-1}\tau) - 2\mathcal{L}(\tau) + \mathcal{L}(T\tau). \tag{II.12}$$

Notice that these equations imply

$$\mathcal{L}(\tau) = \mathcal{L}(T\tau) + \mathcal{L}(S\tau) + \pi i, \tag{II.13}$$

which we will need later.

Using the second equality of (II.10) we obtain the following expansion of $\mathcal{L}$ at the cusp $i\infty$:

$$\mathcal{L}(\tau) = \pi i\tau + 4\log(2) + \sum_{k=1}^{\infty}(-1)^k\frac{v_4(k)}{k}q^{\frac{k}{2}}, \tag{II.14}$$

where v_4 is given by

$$v_4(k) = |\{\mathbf{x}\in\mathbb{Z}^4 \mid \|\mathbf{x}\|^2 = k\}|.$$

Then (II.12) and (II.14) give the expansion

$$\mathcal{L}(S\tau) = -16\sum_{k=0}^{\infty}\frac{\sigma_1(2k+1)}{2k+1}q^{k+\frac{1}{2}}, \tag{II.15}$$

where σ_k is given by

$$\sigma_k(n) = \sum_{d|n} d^k.$$

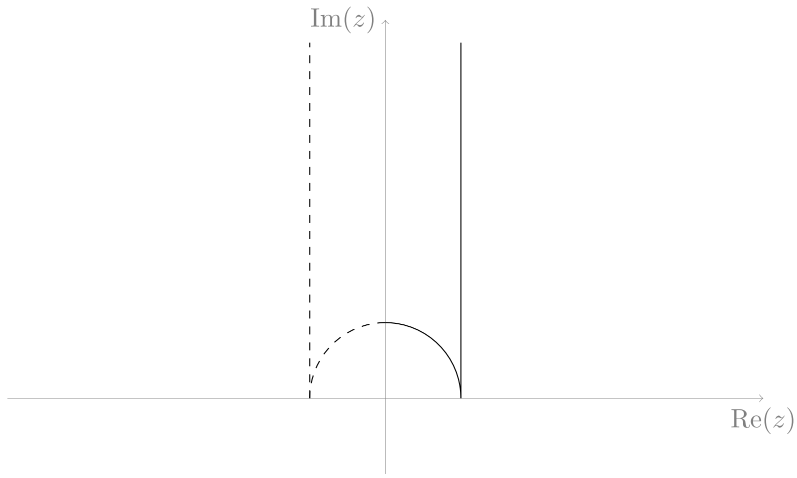

Figure II.2: A fundamental domain for Γ_θ, $\mathcal{D}$

We can further consider the Hecke subgroup Γ_θ that is generated by S and T^2 (we have $\Gamma(2) \subset \Gamma_\theta \subset \mathrm{SL}_2(\mathbb{Z})$). This is equivalent to the following matrix description

$$\Gamma_\theta = \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}.$$

This matrix description allows us to quickly compute the index of Γ_θ in $\mathrm{SL}_2(\mathbb{Z})$ to be 3 with explicit coset representatives given by $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ and two cusps given by 1 and $i\infty$. We further note that a fundamental domain for Γ_θ to be

$$\mathcal{D} = \{\tau \in \mathbb{H} : |\tau| > 1, \mathrm{Re}(\tau) \in (-1,1)\}.$$

This is shown in Figure II.2.

Finally, we define what we'll call the θ-automorphy factor on the group Γ_θ defined for $\gamma \in \Gamma_\theta$ and $z \in \mathbb{H}$

$$j_\theta(z,\gamma) = \frac{\theta(\gamma z)}{\theta(z)}.$$

The Poisson summation formula gives us $j_\theta(z,T^2) = 1$ and $j_\theta(z,S) = (-iz)^{-1/2}$, so in general we'll have that $j_\theta\left(z, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \zeta(cz+d)^{-1/2}$ for some appropriate eighth root of unity ζ, an exact formula is given [21]. Using this automorphy factor we can define the following slash operator in weight $k/2$ that acts on holomorphic functions on the upper half-plane

$$(f|_{k/2}A) = j_\theta(z,\gamma)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

for $z \in \mathbb{H}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$. Then we can more generally define a slash operator for each choice of ϵ, $|^{\epsilon}_{k/2}$ given by

$$f|^{\epsilon}_{k/2}A = \chi_\epsilon(A) f|_{k/2}A = \chi_\epsilon(A)\zeta^k (cz+d)^{-k/2} f(Az)$$

where $\chi_\epsilon : \Gamma_\theta \to \{\pm 1\}$ is homomorphism given by $\chi_\epsilon(S) = \lambda_\epsilon$ and $\chi_\epsilon(T^2) = 1$. Finally, define

$$J(z) = \frac{1}{16}\lambda(z)(1-\lambda(z)), \tag{II.16}$$

which is Hauptmodul for Γ_θ. A proof of this is contained in Section II.4.1.

Next, we define another class of functions.

Definition II.3.0.2. *A quasi-modular form of weight k and depth at most r for the group Γ to be a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$*

$$f[\gamma]_k(\tau) = \sum_{m=0}^{r} f_m(\tau)\left(\frac{c}{c\tau+d}\right)^m,$$

for some holomorphic functions $f_m(\tau)$.

For our purposes we will only be concerned with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Here the canonical example is given by

$$E_2(\tau) = \frac{1}{2\zeta(2)} \sum_{c\in\mathbb{Z}} \sum_{\substack{d\in\mathbb{Z} \\ d\neq 0}} \frac{1}{(c\tau+d)^2}$$

which satisfies

$$E_2[\gamma]_2(\tau) = E_2(\tau) + \frac{6i}{\pi}\left(\frac{c}{c\tau+d}\right),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and in particular we have

$$\tau^{-2} E_2\left(-\frac{1}{\tau}\right) = E_2(\tau) + \frac{6i}{\pi} \cdot \frac{1}{\tau}$$

The *Structure Theorem for Quasi-Modular Forms* states that a quasi-modular form f of weight $2k$ can be written as

$$f(z) = \sum_{\ell=0}^{k} E_2^{\ell}(z) f_{2k-2\ell}(z), \tag{II.17}$$

where $f_{2k-2\ell}$ is a modular form of weight $2k - 2\ell$; the term for $\ell = k - 1$ is of course trivial.

II.4 Riemann Surfaces

II.4.1 Definitions and Topology

This section will used to give brief overview of Riemann Surfaces, notably in the context of modular forms. We first give some definitions.

Definition II.4.1.1. *An **n-dimensional manifold** is a Hausdorff topological space X such that every point $a \in X$ has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$*

Definition II.4.1.2. *Let X be a two-dimensional manifold. A **complex chart** on X is a homeomorphism $\varphi : U \to V$ of an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$. Two complex charts $\varphi_i : U_i \to V_i, = 1, 2$ are said to be **holomorphically compatible** if the map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is biholomorphic.

Definition II.4.1.3. *A **complex atlas** is a system $\mathcal{U} = \{\varphi_i : U_i \to V_i, i \in I\}$ of charts that are holomorphically compatible and satisfy $\cup_{i \in I} U_i = X$*

Definition II.4.1.4. *A **complex structure** on a two-dimensional manifold X is an equivalence class of analytically equivalent atlases on X*

Definition II.4.1.5. *A **Riemann Surface** is a pair (X, Σ), where X is a connected two-dimensional manifold and Σ is a complex structure on X.*

We will almost always refer to the manifolds in question by X (or some other equivalent term when applicable) and neglect to include its complex structure. Also, for our purposes we will be concerned with Riemann Surfaces of the compact variety. Momentarily eschewing formality, we may think of such a surface as topologically a sphere with a number of punctures in it. We refer to the number of such punctures as the **genus**, given by $g \in \mathbb{N}$, for the manifold X. Now, given a nonconstant analytic map $f : X \to Y$ between compact Riemann surfaces, for each $x \in X$ we can define $e_x \in \mathbb{N}$ to be the **ramification index** or the multiplicity at which f takes 0 to 0 in local coordinates. In other words f is locally an e_x to 1 map about x. Moreover, there is a positive integer d, the **degree** of the map f, such that

$$\sum_{x \in f^{-1}(y)} e_x = d \tag{II.18}$$

holds for all $y \in Y$. Using this terminology we can state the following

Theorem II.4.1.6. *(Riemann-Hurwitz formula) Let $f : X \to Y$ be a nonconstant analytic map between compact Riemann Surfaces of degree d. Let g_X and g_Y denote the genera of X and Y respectively. Then the following formula holds*

$$2g_x - 2 = d(2g_Y - 2) + \sum_{x \in X} (e_x - 1)$$

We also show the following general lemma

Lemma II.4.1.7. *Suppose $f : X \to Y$ is a non-constant analytic map between compact Riemann surfaces, then f is surjective.*

Proof. We observe that since since f is a continuous map, the image $f(X)$ must be connected, compact, and hence closed since Y is a Hausdorff. Then, since f is an analytic map it is an open map and so $f(X)$ is open as well. Combining the fact that Y is connected we have that $f(X)$ is either the empty set or the entirety of Y. The former is impossible giving us the desired result. $\square$

As stated previously, given any subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of finite index, by considering the group action of Γ on $\mathbb{H} \cup \{\infty\} \cup \{\mathbb{Q}\}$, i.e, the extended upper half-plane and the rational numbers, we get a compact Riemann surface that we will denote by $X(\Gamma)$. Using this, if we let $f : X(\Gamma) \to X(1)$ denote the projection map (an analytical map) and d be its degree, we see that

$$d = [\mathrm{SL}_2(\mathbb{Z}) : \{\pm I\}\Gamma]$$

and moreover if we let ϵ_2, ϵ_3, and ϵ_∞ denote the number of elliptic points (points with non-trivial stabilizer group in Γ) of order 2, order 3, and the number of cusps respectively, a short computation with the Riemann-Hurwitz formula shows that

$$g = 1 + \frac{d}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{3} - \frac{\epsilon_\infty}{2}, \tag{II.19}$$

where g here is the genus of the surface of $X(\Gamma)$. Using Equation (II.19) we can compute the genus of $X(1)$ and $X(2)$ to both be 0. While it is general fact that such modular surfaces are isomorphic to $\mathbb{C}^*$, since used heavily in this, we present proofs in both cases.

Lemma II.4.1.8. *j is a biholomorphism from $X(1) \to \mathbb{C}^*$*

To show this we are required to show that j is analytic map between the surfaces, j is bijective, and that it has a holomorphic inverse. We observe that from [12], the second condition gives the third, so, it remains to prove the the first two conditions. For holomorphy, we see that the representation given Equation (II.6) gives that j is holomorphic on $X(1)$ except possibly when $\Delta = 0$. We know however that Δ vanishes only at the cusps so j is holomorphic on $\Gamma/\mathbb{H}$. We then observe that the Fourier representation

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \mathcal{O}(q^3), \tag{II.20}$$

where $q = e^{2\pi i \tau}$, shows that f is holomorphic at $i\infty$ as well. Next, we see that Equation (II.20) shows that the degree of the map j is 1 because it has a 'pole' of order 1 at $i\infty$ and so Equation (II.18) implies that j must be injective. Combining this with Lemma II.4.1.7 we get we get that j is bijective as claimed.

Lemma II.4.1.9. *λ is a biholmorphism from $X(2) \to \mathbb{C}^*$.*

Proof. We again are required to show that λ is an analytic map between the surfaces, λ is bijective, and that it has a holomorphic inverse. Again it suffices to show the first two conditions. For the first we observe that Equation (II.7) gives that λ is holomorphic on $X(2)$ except possibly if θ_{00} is 0. To this end, it's not difficult to verify that $_{00}(\tau)$ will have non-zero real and imaginary parts for any $\tau \in \Gamma(2)/\mathbb{H}$. It then just remains to verify the behavior at the cusps of $\Gamma(2)$. We note that cusps of $\Gamma(2)$ are $i\infty$, 0, and 1. For the first, we observe that the Fourier expansion of λ at $i\infty$ is given by

$$\lambda(\tau) = 16q^{\frac{1}{2}} - 128q + 704q^{\frac{3}{2}} + \mathcal{O}(q^2), \tag{II.21}$$

which shows that it vanishes at $i\infty$. On the other hand, the identities given by Equations (II.8) and (II.9) give $\lambda(0) = 1$ and $\lambda(1) = \infty$. So, it is a well defined holomorphic map as claimed. We then have Equation (II.21) shows that the degree of the map λ is 1 since it has a zero of order 1 at $i\infty$. Hence, Equation (II.18) gives that λ is injective. Combining this with Lemma II.4.1.7 we get we get that λ is bijective as claimed. □

Corollary II.4.1.10. $\mathcal{A}_0(\Gamma) = \mathbb{C}(j)$*, the set of rational functions in* j

Proof. We first note the inclusion $\mathbb{C}(j) \subset \mathcal{A}_0(\Gamma)$ by definition. On the other hand, suppose that $f \in \mathcal{A}_0(\Gamma)$ and is non-constant. Then f has a finite number of zeros and poles given by $z_1, z_2, .., z_m$ and $p_1, p_2, ..., p_n$ respectively. Such an f must only have a discrete set of each because $X(1)$ is compact so and hence any infinite sequence would lead to a limit point of zeros or poles and hence f would be constant. We moreover have $m \neq 0 \neq n$ because f must be surjective. Next, consider the function given by

$$g(\tau) = \frac{\prod_{i=1}^{m}(j(\tau) - j(z_i))}{\prod_{i=1}^{n}(j(\tau) - j(p_i))}.$$

We observe that $\frac{f}{g}$ then has no zeros or poles and by Lemma II.4.1.7 must therefore be a constant. This gives $f \in \mathbb{C}(j)$ and $\mathcal{A}_0(\Gamma) \subset \mathbb{C}(j)$. Both inclusions give the desired result. □

We can analogously show

Corollary II.4.1.11. $\mathcal{A}_0(\Gamma(2)) = \mathbb{C}(\lambda)$*, the set of rational functions in* λ*.*

Corollary II.4.1.12. $\mathcal{A}_0(\Gamma_\theta) = \mathbb{C}(J)$*, the set of rational functions in* J*.*

We now present lemmas that will be consequential to Sections III.2.1 and III.3.1.

Lemma II.4.1.13. *Suppose* f *is a weakly holomorphic modular form of weight* $2k > 0$*, that is to say,* $f \in \mathcal{A}_{2k}(\Gamma)$ *with the restriction that* f *is holomorphic on* $\Gamma/\mathbb{H}$ *but meromorphic at* $i\infty$*. Then we have that*

$$f = g \cdot P(j),$$

for some $g \in \mathcal{M}_{2k}(\Gamma)$ *and where* P *is a polynomial.*

Proof. We first note that the set $\mathcal{M}_{2k}(\Gamma)$ is non-empty because for example the $2k$-th Eisenstein series is a member. On the other hand, given that such a function g exists we have by Corollary II.4.1.10 that $\frac{f}{g} \subset \mathbb{C}(j)$. Therefore, $f = g \cdot P(j)$ for some rational function P. Next, suppose that the denominator of $P(j)$ has the form $(j-\alpha_1)(j-\alpha_2)...(j-\alpha_n)$ for some numbers $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$. On the other hand Lemma II.4.1.8 tells us that if $\alpha_i \in \mathbb{C}$ then $j - \alpha_1 = 0$ will have a unique solution in $\Gamma/\mathbb{H}$ and therefore introduce a pole somewhere other than the cusps, a contradiction. Hence, P has at most a constant denominator and is a polynomial as claimed. $\square$

We also have the analogous result for $\Gamma(2)$

Lemma II.4.1.14. *Suppose f is a weakly holomorphic modular form of weight $2k > 0$ for $\Gamma(2)$, that is to say, $f \in \mathcal{A}_{2k}(\Gamma(2))$ with the restriction that f is holomorphic on $\Gamma(2)/\mathbb{H}$ but meromorphic at the cusps. Then we have that*

$$f = g \cdot R(\lambda),$$

for some $g \in \mathcal{M}_{2k}(\Gamma)$ and where R is a rational function.

Remark: Observe that if Γ is congruence subgroup, the same method used to prove Corollary II.4.1.10 can be used to show that

$$\mathcal{A}_k(\Gamma) = f \cdot \mathbb{C}(X(\Gamma)),$$

where $f \in \mathcal{A}_k$, $\mathbb{C}(X(\Gamma))$ denotes the set of meromorphic functions on the surface $X(\Gamma)$, and we implicitly assume that $\mathcal{A}_k(\Gamma)$ is non-empty.

CHAPTER III

Laplace Transform of Modular Forms

In this section we will study functions of the form

$$U(s) = 4i \sin\left(\frac{\pi}{2}s\right)^2 \int_0^{i\infty} \psi(z) e^{i\pi s z}\, dz, \tag{III.1}$$

for real s. We analyze these functions and present an analytic continuation of these functions to a left half-plane. We then present conditions so that $U(s)$ has a well defined Fourier transform (in a classical not distributional sense), compute this transform, and then present conditions for such functions to be eigenfunctions of the Fourier transform. Throughout the course of this Chapter and the remaining ones we take the ambient dimension to be d, i.e, we are considering functions in $\mathbb{R}^d$ when we take Fourier transforms and assume that $4|d$.

III.1 Fourier Transforms and Eigenfunctions

We begin by introducing the prototypical function that we will be interested throughout this chapter. Denote the imaginary axis by $i\mathbb{R}$ and let $L^1_{\text{loc}}(i\mathbb{R})$ denote the complex valued functions that are absolutely integrable with respect to Lebesgue measure on any bounded interval $i(0, b]$.

Proposition III.1.0.1. *Suppose $\psi \in L^1_{loc}(i\mathbb{R})$ is such that for some $C > 0$ and constants $a_k, b_k \in \mathbb{C}$, $k = 0, 1, ..., n$,*

$$\psi(z) = \sum_{k=0}^{n} a_k e^{-2\pi i k z} - iz \sum_{k=0}^{n} b_k e^{-2\pi i k z} + \mathcal{O}(e^{iCz}) \;\; \text{as} \;\; z \to i\infty. \tag{III.2}$$

For $\text{Re}(s) > 2n$, *let*

$$W(s) = -i \int_0^{i\infty} \psi(z) e^{-2\pi i s z} dz. \tag{III.3}$$

Then

$$W(s) = \sum_{k=0}^{n} \left(\frac{a_k}{\pi(s-2k)} + \frac{b_k}{\pi^2(s-2k)^2} \right) - i \int_0^{i\infty} \left(\psi(z) - \left(\sum_{k=0}^{n} a_k e^{-2\pi i k z} + z \sum_{k=0}^{n} b_k e^{-2\pi i k z} \right) \right) e^{\pi i s z} dz \tag{III.4}$$

gives an analytic continuation of W to the half-plane $Re(s) > -\frac{C}{\pi}$.

Proof. Let $\widetilde{W}(s)$ be given by the right-hand side of (III.4). Then the local integrability of ψ and the condition (III.2) imply that $\widetilde{W}(s)$ is a well-defined meromorphic function on the half plane $\text{Re}(s) > -\frac{C}{\pi}$ with (at most) double poles at $s = 2k$, $k = 0, \ldots, n$. For an integer k and $\text{Re}(s) > 2k$, elementary computations show

$$-i\int_0^{i\infty} e^{-2\pi ikz}e^{i\pi sz}\,dz = \frac{1}{\pi(s-2k)},$$

and

$$-i\int_0^{i\infty} ze^{-2\pi ikz}e^{i\pi sz}\,dz = \frac{1}{\pi^2(s-2k)^2},$$

and hence that $\widetilde{W}(s)$ agrees with $W(s)$ for $\mathrm{Re}(s) > 2n$. $\square$

We next assume that ψ is holomorphic on the upper half-plane.

Proposition III.1.0.2. *Let $\psi : \mathbb{H} \to \mathbb{C}$ be holomorphic on $\mathbb{H}$ and bounded on the angular region $R_{\alpha,\epsilon} := \{re^{it} : 0 < r < \epsilon,\, \alpha < t < \pi - \alpha\}$ for some $\epsilon > 0$ and some $0 < \alpha < \frac{\pi}{4}$. Further suppose the restriction of ψ to $i\mathbb{R}_+$ and W are as in Proposition III.1.0.1 and for $\mathrm{Re}(s) > -\frac{C}{\pi}$ let $U(s)$ be defined by*

$$U(s) = -4\sin\left(\frac{\pi}{2}s\right)^2 W(s). \tag{III.5}$$

Then $U(s)$ is holomorphic for $\mathrm{Re}(s) > -\frac{C}{\pi}$ and

$$\begin{aligned} iU(s) = &\int_{-1}^{i} \psi(Tz)e^{i\pi sz}\,dz + \int_1^i \psi(T^{-1}z)e^{i\pi sz}\,dz \\ &- 2\int_0^i \psi(z)e^{i\pi sz}\,dz + \int_i^{i\infty} \left(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)\right)e^{i\pi sz}\,dz, \end{aligned} \tag{III.6}$$

where the integrals are along straight line segments joining the endpoints.

Proof. Starting from (III.3) we derive a second form of the analytic continuation of $-4\sin(\frac{\pi}{2}s)^2W(s)$, which is more suitable for the proof and will also be used later. We write

$$\begin{aligned} iU(s) &= \int_0^{i\infty} \psi(z)\left(e^{i\pi s(z-1)} - 2e^{i\pi sz} + e^{i\pi s(z+1)}\right)dz \\ &= \int_{-1}^{-1+i\infty} \psi(Tz)e^{i\pi sz}\,dz - 2\int_0^{i\infty} \psi(z)e^{i\pi sz}\,dz \\ &+ \int_1^{1+i\infty} \psi(T^{-1}z)e^{i\pi sz}\,dz, \end{aligned}$$

which follows by expressing the sine in terms of the exponential, expanding the square and substituting in the integral. This expression is valid for $\mathrm{Re}(s) > 2n$. Since ψ is holomorphic on $\mathbb{H}$ and bounded on $R_{\alpha,\epsilon}$, we may deform the contours of integration as follows: the path from -1 to $-1 + i\infty$ is deformed into a straight line from -1 to i and then along the imaginary axis from i to

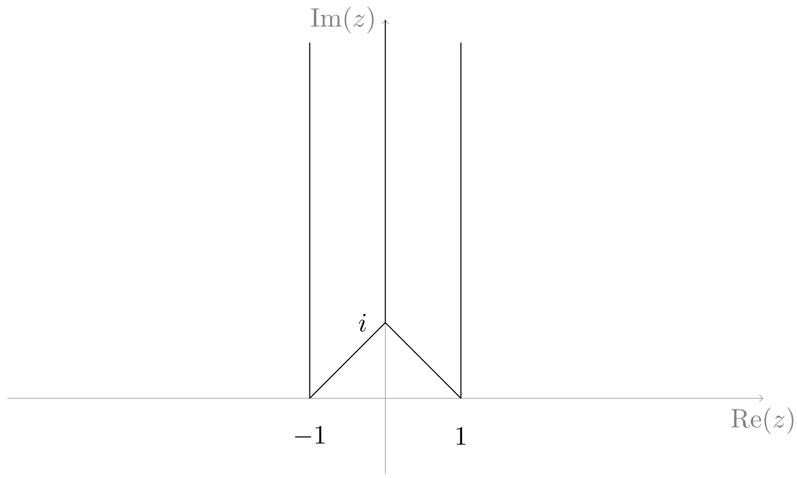

Figure III.1: Deforming the path of Integration

$i\infty$; similarly, the contour from 1 to $1 + i\infty$ is deformed into a straight line from 1 to i and then again along the imaginary axis. This path is show in Figure III.1.

Collecting terms with matching paths of integration gives (III.6) valid for $\mathrm{Re}(s) > 2n$. Since the exponential terms in the asymptotic expansion (III.2) for $z \to i\infty$ cancel in the last integral, the new expression is also valid for $\mathrm{Re}(s) > -\frac{C}{\pi}$ providing an alternative form for expressing the analytic continuation of $U(s)$. The integrals are all absolutely and uniformly convergent for $\mathrm{Re}(s) \geq 0$. $\square$

Proposition III.1.0.3. *Let ψ be as in Proposition III.1.0.1, U as in Proposition III.1.0.2 and let $F : \mathbb{R}^d \to \mathbb{C}$ be defined by*

$$F(\mathbf{x}) := U(\|\mathbf{x}\|^2), \qquad (\mathbf{x} \in \mathbb{R}^d). \tag{III.7}$$

If, in addition, ψ satisfies

$$\psi(z) = \mathcal{O}(e^{iCSz}) \quad \text{as } z \to 0 \quad \text{non-tangentially in } H, \tag{III.8}$$

then F is a Schwartz function and can be written in the form

$$\begin{aligned} F(\mathbf{x}) = -i \Bigg[&\int_{-1}^{i} \psi(Tz) e^{i\pi\|\mathbf{x}\|^2 z}\, dz + \int_{1}^{i} \psi(T^{-1}z) e^{i\pi\|\mathbf{x}\|^2 z}\, dz \\ &-2\int_{0}^{i} \psi(z) e^{i\pi\|\mathbf{x}\|^2 z}\, dz + \int_{i}^{i\infty} \big(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)\big)\, e^{i\pi\|\mathbf{x}\|^2 z}\, dz \Bigg]. \end{aligned} \tag{III.9}$$

Consequently, the Fourier transform of F is given by

$$\begin{aligned}\hat{F}(\mathbf{t}) = -i(-1)^{d/4}\Bigg[&\int_{-1}^{i} \psi(T^{-1}Sz)e^{i\pi\|\mathbf{t}\|^2 z} z^{d/2-2}\,dz \\ &+ 2\int_{i}^{i\infty} \psi(Sz)e^{i\pi\|\mathbf{t}\|^2 z} z^{d/2-2}\,dz + \int_{1}^{i} \psi(TSz)e^{i\pi\|\mathbf{t}\|^2 z} z^{d/2-2}\,dz \\ &- \int_{0}^{i} (\psi(T^{-1}Sz) - 2\psi(Sz) + \psi(TSz))e^{i\pi\|\mathbf{t}\|^2 z} z^{d/2-2}\,dz\Bigg].\end{aligned} \tag{III.10}$$

Proof. The representation (III.9) follows immediately from the definition (III.7) and the relation (III.6) of Proposition III.1.0.2. The condition (III.8) implies that ψ vanishes to arbitrary order at $z = 0$. Hence, using (III.3) it follows using well known properties of the Laplace transform that F and its derivatives all decay faster than any negative power of $\|\mathbf{x}\|$. Since U is analytic, it follows that F is a Schwartz function. Thus we can compute the Fourier transform of F by Fubini's theorem

$$\begin{aligned}\widehat{F}(\mathbf{t}) = -\,i\Bigg[&\int_{-1}^{i} \psi(Tz)e^{i\pi\|\mathbf{t}\|^2 Sz}(-iz)^{-\frac{d}{2}}\,dz \\ &+ \int_{1}^{i}\psi(T^{-1}z)e^{i\pi\|\mathbf{t}\|^2 Sz}(-iz)^{-\frac{d}{2}}\,dz - 2\int_{0}^{i}\psi(z)e^{i\pi\|\mathbf{t}\|^2 Sz}(-iz)^{-\frac{d}{2}}\,dz \\ &+ \int_{i}^{i\infty} \big(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)\big)\, e^{i\pi\|\mathbf{t}\|^2 Sz}(-iz)^{-\frac{d}{2}}\,dz\Bigg].\end{aligned}$$

Substituting Sz in this expression and collecting signs gives (III.10). $\square$

Proposition III.1.0.4. *Let ψ be as in Proposition III.1.0.1, F as in Proposition III.1.0.3 and $\varepsilon \in \{-1, 1\}$. Then $\hat{F} = \varepsilon(-1)^{\frac{d}{4}}F$ if and only if*

$$z^{\frac{d}{2}-2}\psi(T^{-1}Sz) = \varepsilon\psi(Tz) \tag{III.11}$$

$$2z^{\frac{d}{2}-2}\psi(Sz) = \varepsilon\left(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)\right), \tag{III.12}$$

for all $z \in \mathbb{H}$.

Proof. The function F is an eigenfunction of the Fourier transform for the eigenvalue $\varepsilon(-1)^{\frac{d}{4}}$, if and only if the expressions (III.10) (with $\mathbf{t}$ replaced by $\mathbf{x}$) and (III.10) are equal up to a factor of ε. By the uniqueness property of the Laplace transform this is equivalent to the fact that the integrands agree on corresponding segments of integration. This yields the equations

$$z^{\frac{d}{2}-2}\psi(T^{-1}Sz) = \varepsilon\psi(Tz) \tag{III.13}$$

$$z^{\frac{d}{2}-2}\psi(TSz) = \varepsilon\psi(T^{-1}z) \tag{III.14}$$

$$2\psi(z) = \varepsilon z^{\frac{d}{2}-2}\left(\psi(T^{-1}Sz) - 2\psi(Sz) + \psi(TSz)\right) \tag{III.15}$$

$$2z^{\frac{d}{2}-2}\psi(Sz) = \varepsilon\left(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)\right), \tag{III.16}$$

which have to hold for all $z \in \mathbb{H}$ by the holomorphy of ψ. It is immediate that (III.13) and (III.14), and (III.15) and (III.16) are equivalent by substituting $z \mapsto Sz$. □

III.2 'Positive' Eigenfunction

By 'positive' here we refer to the case $\epsilon = 1$ of Proposition III.1.0.4. We show that in this case ψ has to be a quasi-modular form and discuss what these solutions must look like.

Proposition III.2.0.1. *Let ψ be as in Proposition III.1.0.1. Then the corresponding function F given by* (III.7) *is an eigenfunction for the Fourier transform with eigenvalue $(-1)^{\frac{d}{4}}$, if and only if $z^{\frac{d}{2}-2}\psi(Sz)$ is a quasi-modular form of weight $4-\frac{d}{2}$ and depth 2. More precisely, there are weakly holomorphic modular forms ψ_1, ψ_2, ψ_3 of respective weights $4-\frac{d}{2}$, $2-\frac{d}{2}$, and $-\frac{d}{2}$ such that*

$$z^{\frac{d}{2}-2}\psi(Sz) = \psi_1(z) - 2E_2(z)\psi_2(z) + E_2(z)^2\psi_3(z). \tag{III.17}$$

This gives

$$\begin{aligned}\psi(z) = {} & z^2\left(\psi_1(z) - 2E_2(z)\psi_2(z) + E_2(z)^2\psi_3(z)\right) \\ & + z\frac{12i}{\pi}\left(\psi_2(z) - E_2(z)\psi_3(z)\right) - \frac{36}{\pi^2}\psi_3(z).\end{aligned} \tag{III.18}$$

Furthermore, ψ_1, ψ_2, and ψ_3 have to satisfy

$$\psi_1(z) - 2E_2(z)\psi_2(z) + E_2(z)^2\psi_3(z) = \mathcal{O}(e^{2\pi iz}) \tag{III.19}$$

for $z \to i\infty$ in order to fulfill (III.2) *and* (III.8).

Proof. By Proposition III.1.0.4 a function F given in the form (III.5) is an eigenfunction of the Fourier transform for the eigenvalue $(-1)^{\frac{d}{4}}$ (this is $\varepsilon = 1$) if and only if (III.11) and (III.12) hold. From (III.11) we obtain

$$\psi(z) = (z+1)^{\frac{d}{2}-2}\psi(TSTz)$$

and then

$$(z+1)^{\frac{d}{2}-2}\psi(STz) = z^{\frac{d}{2}-2}\psi(TSTSTz) = z^{\frac{d}{2}-2}\psi(Sz),$$

where we have used that $(TS)^3 = \mathrm{Id}$. Thus the function

$$\phi(z) = z^{\frac{d}{2}-2}\psi(Sz)$$

is periodic with period 1.

Now we write (III.12) as

$$\psi(Tz) - 2\psi(z) + \psi(T^{-1}z) = 2\phi(z) \tag{III.20}$$

and set

$$f(z) = \psi(Tz) - \psi(z) - (2z+1)\phi(z). \tag{III.21}$$

Then we have

$$f(z) - f(T^{-1}z) = \psi(Tz) - 2\psi(z) + \psi(T^{-1}z) - (2z+1)\phi(z) + (2z-1)\phi(T^{-1}z).$$

Using the periodicity of ϕ and (III.20) gives the periodicity of f. Now we set

$$g(z) = \psi(z) - z^2\phi(z) - zf(z). \tag{III.22}$$

We compute

$$g(Tz) - g(z) = \psi(Tz) - \psi(z) - ((z+1)^2 - z^2)\phi(z) - ((z+1) - z)f(z) = 0,$$

where we have used the periodicity of ϕ and f as well as the definition of f. This shows that also g is periodic.

Thus ψ satisfies the relation

$$\psi(z) = z^{\frac{d}{2}}\psi(Sz) + zf(z) + g(z) \tag{III.23}$$

for two (yet unknown) periodic functions f and g. We now use the definitions (III.21) and (III.22) to express g in terms of ψ

$$g(z) = (z+1)\psi(z) - z\psi(Tz) + z(z+1)z^{\frac{d}{2}-2}\psi(Sz). \tag{III.24}$$

Substituting STz and multiplying through the denominator yields

$$\begin{aligned}(z+1)^{\frac{d}{2}}g(STz) = z(z+1)(z+1)^{\frac{d}{2}-2}\psi(STz)& \\ + (z+1)(z+1)^{\frac{d}{2}-2}\psi(ST^{-1}Sz) - z\psi(Tz),&\end{aligned} \tag{III.25}$$

where we have used $TST = ST^{-1}S$. We have already established the periodicity of $\phi(z) = z^{\frac{d}{2}-2}\psi(Sz)$. This allows to replace the first and the second term to yield

$$g(STz) = (z+1)z^{\frac{d}{2}-1}\psi(Sz) + (z+1)\psi(z) - z\psi(Tz) = g(z).$$

Using the already established periodicity of g this gives

$$z^{\frac{d}{2}}g(Sz) = g(z); \tag{III.26}$$

g is a modular form of weight $-\frac{d}{2}$.

Applying $z \mapsto Sz$ to (III.23) and adding this to (III.23) (divided by $z^{\frac{d}{2}}$) yields

$$z^{\frac{d}{2}-2}f(Sz) = f(z) + \frac{2}{z}g(z); \tag{III.27}$$

f is quasi-modular of weight $2 - \frac{d}{2}$ and depth 1.

We set

$$h(z) = f(z) - \frac{\pi i}{3}E_2(z)g(z)$$

and use $z^{-2}E_2(z) = E_2(z) - \frac{6i}{\pi z}$ to obtain

$$z^{\frac{d}{2}-2}h(Sz) = h(z),$$

which together with the obvious periodicity yields that h is a modular form of weight $2-\frac{d}{2}$. Inserting this into (III.23) gives the quasi-modularity of $z^{\frac{d}{2}-2}\psi(Sz)$ with weight $4 - \frac{d}{2}$ and depth 2. By the *Structure Theorem of Quasi-Modular Forms*, Equation (II.17), $z^{\frac{d}{2}-2}\psi(Sz)$ can then be written as

(III.17), where we have set

$$\begin{aligned}
\psi_1(z) &= z^{\frac{d}{2}-2}\psi(Sz) - E_2(z)h(z) - E_2(z)^2 g(z)\\
\psi_2(z) &= -\frac{\pi i}{12}h(z)\\
\psi_3(z) &= -\frac{\pi^2}{36}g(z).
\end{aligned}$$

In order to satisfy condition (III.2), the term multiplied by z^2 in (III.18) has to tend to 0 for $z \to i\infty$, which gives (III.19). By (III.17) this implies that (III.2) and (III.8) are satisfied for any $0 < C < 2\pi$. □

III.2.1 Explicit Representations for the Sphere Packing Problem

In a next step we want to determine ψ (or equivalently ψ_1, ψ_2, ψ_3) to satisfy (III.19). Since ψ_1, ψ_2, and ψ_3 are weakly holomorphic modular forms of respective weights $-4 - \frac{d}{2}$, $2 - \frac{d}{2}$, and $-\frac{d}{2}$, we use Lemma II.4.1.13 to express these forms as

$$\begin{aligned}
\psi_1 &= \frac{1}{\Delta^\ell}\omega_{k+2}P_n^{(k)}(j)\\
\psi_2 &= \frac{1}{\Delta^\ell}\omega_{k+1}Q_n^{(k)}(j)\\
\psi_3 &= \frac{1}{\Delta^\ell}\omega_{k}R_n^{(k)}(j),
\end{aligned} \tag{III.28}$$

for $\ell \in \mathbb{N}$ chosen so that $\psi_m\Delta^\ell$ $(m = 1, 2, 3)$ are weakly holomorphic modular forms of positive weight; $P_n^{(k)}$, $Q_n^{(k)}$, and $R_n^{(k)}$ are polynomials, which have to be determined. The minimal possible choice of ℓ is then

$$\ell = \left\lceil \frac{d}{24} \right\rceil.$$

Furthermore, we set

$$k = 6\ell - \frac{d}{4},$$

which gives $0 \le k \le 5$. The forms ω_m in (III.28) are modular forms of weight $2m$ $(m = 0, \ldots, 7)$, which are given in Table III.1; these forms are uniquely determined by the requirement to be holomorphic, or to have a pole of minimal order at $i\infty$. The parameter n refers to the order of the pole of $\omega_{k+2}P_n^{(k)}(j)$, $\omega_{k+1}Q_n^{(k)}(j)$, or $\omega_k R_n^{(k)}(j)$. Notice that for $m = 1$ the form ω_m has a simple pole at $i\infty$, whereas for $m = 6, 7$ it has a simple zero there. This affects the possible degrees of the polynomials $P_n^{(k)}$, $Q_n^{(k)}$, and $R_n^{(k)}$, see Table III.2. This table also gives the dimension of the space $\mathcal{Q}_n^{(2k+2)}$ of weakly holomorphic quasi-modular forms of weight $2k + 2$ and depth 2, which have a pole of order at most n at $i\infty$. The table also gives the definition of the quantity $a(k)$, which will be needed in the sequel.

m	ω_m
0	1
1	$-j' = \frac{E_4^2 E_6}{\Delta} = \frac{E_6}{E_4} j$
2	E_4
3	E_6
4	E_4^2
5	$E_4 E_6$
6	$\Delta = \Delta\omega_0$
7	$E_4^2 E_6 = \Delta\omega_1$

Table III.1: The choices of ω_m

k	$\deg P_n^{(k)}$	$\deg Q_n^{(k)}$	$\deg R_n^{(k)}$	$\dim \mathcal{Q}_n^{(2k+2)}$	$a(k)$
0	n	$n-1$	n	$3n+2$	1
1	n	n	$n-1$	$3n+2$	1
2	n	n	n	$3n+3$	2
3	n	n	n	$3n+3$	2
4	$n+1$	n	n	$3n+4$	3
5	n	$n+1$	n	$3n+4$	3

Table III.2: Degrees of the polynomials $P_n^{(k)}$, $Q_n^{(k)}$, and $R_n^{(k)}$

In light of (III.18) and the asymptotic behaviour of ψ (III.2) used in Proposition III.1.0.1 we require that the polar order of $\psi_2(z) - E_2(z)\psi_3(z)$ (the term multiplied by z in (III.18)) is 1 less than the polar order of $\psi_3(z)$. This ensures by (III.4) that the largest real second order pole of $W(s)$ is 2 less than the largest real first order pole. Notice that condition (III.19) ensures that $W(s)$ has no third order poles in the right half plane. Together this ensures that the polar order of ψ at $i\infty$ corresponds to the desired sign change of the function F given by (III.7).

In order to achieve the behaviour of ψ described in the last paragraph, we use the degrees of freedom given by $\dim \mathcal{Q}_n^{(2k+2)}$ to first ensure that

$$\omega_{k+1} Q_n^{(k)}(j) - E_2 \omega_k R_n^{(k)}(j) = \mathcal{O}(q^{-n+1}) \tag{III.29}$$

and second to eliminate as many Laurent series coefficients of

$$\omega_{k+2} P_n^{(k)}(j) - 2E_2 \omega_{k+1} Q_n^{(k)}(j) + E_2^2 \omega_k R_n^{(k)}(j)$$

as possible. By solving the according linear equations we can achieve

$$\omega_{k+2}P_n^{(k)}(j) - 2E_2\omega_{k+1}Q_n^{(k)}(j) + E_2^2\omega_k R_n^{(k)}(j) = \mathcal{O}(q^{2n+a(k)-1}). \tag{III.30}$$

Examples of such polynomials are given in Chapter V. In order for ψ to satisfy (III.19) we have to choose n so that

$$2n + a(k) - 1 > \ell;$$

the minimal possible choice for n is then

$$n = \left\lceil \frac{\ell - a(k) + 2}{2} \right\rceil. \tag{III.31}$$

The condition (III.29) ensures that there is a sign change of $F(\mathbf{x})$ at $\|\mathbf{x}\|^2 = 2n+2\ell$ and $F(\mathbf{x}) \neq 0$ for $\|\mathbf{x}\|^2 = 2n + 2\ell - 2$. Expressing ℓ, k, and n in terms of d yields $2n + 2\ell = 2\lfloor \frac{d+4}{16} \rfloor + 2$.

Summing up, we have proved the following theorem. For the sake of simplicity, we abuse notation by writing $f(\mathbf{x}) = f(\|\mathbf{x}\|)$, whenever f is a radial function and the context is clear.

Theorem III.2.1.1. *For $d \equiv 0 \pmod 4$ set $n_+ = \lfloor \frac{d+4}{16} \rfloor + 1$. Then there exists a radial Schwartz function $F_+ : \mathbb{R}^d \to \mathbb{R}$ satisfying*

$$\begin{aligned} F_+(\mathbf{x}) &= (-1)^{\frac{d}{4}} \widehat{F}_+(\mathbf{x}) \quad \textit{for all } \mathbf{x} \in \mathbb{R}^d \\ F_+(\sqrt{2n_+}) &= 0 \quad \textit{and } F_+'(\sqrt{2n_+}) \neq 0 \\ F_+(\sqrt{2m}) &= F_+'(\sqrt{2m}) = 0 \quad \textit{for } m > n_+, \quad m \in \mathbb{N}. \end{aligned} \tag{III.32}$$

III.2.2 Examples

- $d = 8$

  In this case we have $\ell = 1, k = 4$, and $n = 0$. This gives us

  $$\psi = \frac{z^2((j-1728)\Delta - 2E_2E_4E_6 + E_2^2E_4) + z\frac{12i}{\pi}(E_4E_6 - E_2E_4) - \frac{36}{\pi^2}E_4}{\Delta},$$

  which is the same as the $+1$ eigenfunction given in [2] up to scaling.

- $d = 24$

In this case we have $\ell = 1, k = 0$, and $n = 2$. This gives us

$$\psi = \frac{z^2(E_4(175j^2 - 1840683j - 475793136) - 2E_2\frac{E_{14}}{\Delta^2}(175j + 497922) + E_2^2(175j^2 + 2534082j + 111078000))}{\Delta}$$
$$+ \frac{z\frac{12i}{\pi}(\frac{E_{14}}{\Delta^2}(175j + 497922) - E_2(175j^2 + 2534082j + 111078000))}{\Delta}$$
$$- \frac{\frac{36}{\pi^2}(175j^2 + 2534082j + 111078000)}{\Delta},$$

which is the same as the +1 Eigenfunction given [3] up to scaling.

III.3 'Negative' Eigenfunction

By 'negative' here we refer to the case $\epsilon = -1$ of Proposition III.1.0.4. We show that in this case ψ to be a weakly holomorphic modular form for $\Gamma(2)$ and discuss what these solutions must look like.

Proposition III.3.0.1. *Let ψ be as in Proposition III.1.0.1. Then the corresponding function F given by* (III.7) *is an eigenfunction of the Fourier transform with eigenvalue $(-1)^{\frac{d}{4}+1}$ if and only if there exists a weakly holomorphic modular form f of weight $2 - \frac{d}{2}$ for Γ and ω a weakly holomorphic modular form of weight $2 - \frac{d}{2}$ for $\Gamma(2)$ such that*

$$\psi(z) = f(z) \cdot \mathcal{L}(z) + \omega(z), \tag{III.33}$$

$$\omega(z) = z^{\frac{d}{2}-2}\omega(Sz) + \omega(Tz), \tag{III.34}$$

where $\mathcal{L}$ is defined in (II.10).

Proof. By Proposition III.1.0.4 with $\epsilon = -1$, F is an eigenfunction of the Fourier transform with eigenvalue $(-1)^{\frac{d}{4}+1}$ iff ψ satisfies the two equations:

$$z^{\frac{d}{2}-2}\psi(TSz) = -\psi(T^{-1}z), \tag{III.35}$$

$$2z^{\frac{d}{2}-2}\psi(Sz) = -(\psi(Tz) - 2\psi(z) + \psi(T^{-1}z)). \tag{III.36}$$

To solve these we first consider $H(z) = z^{\frac{d}{2}-2}\psi(Sz)$ which by (III.35) gives

$$\begin{aligned} H(Tz) &= (Tz)^{\frac{d}{2}-2}\psi(STz) \\ &= (Tz)^{\frac{d}{2}-2}\psi(T^{-1}TSTz) \\ &= -(Tz)^{\frac{d}{2}-2}(TSTz)^{\frac{d}{2}-2}\psi(TSTSTz) \\ &= -z^{\frac{d}{2}-2}\psi(Sz) \\ &= -H(z), \end{aligned} \tag{III.37}$$

Where we used the property $(TS)^3 = \mathrm{Id}$ in the second to last line. Iterating this property once gives that $H(z+2) = H(z)$ and unraveling this statement in terms of ψ gives

$$(2z-1)^{\frac{d}{2}-2}\psi(ST^2Sz) = \psi(z). \tag{III.38}$$

Substituting $z \to STz$ in (III.36) and applying (III.35) repeatedly to get

$$\begin{aligned} 2\psi(Tz) &= -(Tz)^{\frac{d}{2}-2}\left(\psi(T^{-1}STz) - 2\psi(STz) + \psi(TSTz)\right) \\ &= \psi(T^2z) + 2(Tz)^{\frac{d}{2}-2}\psi(STz) + \psi(z) \\ &= \psi(T^2z) - 2z^{\frac{d}{2}-2}\psi(Sz) + \psi(z) \\ &= \psi(T^2z) + \psi(Tz) - \psi(z) + \psi(T^{-1}z). \end{aligned} \tag{III.39}$$

So, altogether we have that $\psi(T^2z) - \psi(Tz) - \psi(z) + \psi(T^{-1}z) = 0$. Defining $G(z) = \psi(Tz) - \psi(T^{-1}z)$ implies $G(z+1) = G(z)$. Furthermore using (III.35) we obtain

$$\begin{aligned} z^{\frac{d}{2}-2}G(Sz) &= z^{\frac{d}{2}-2}(\psi(TSz) - \psi(T^{-1}Sz)) \\ &= -\psi(T^{-1}z) + \psi(Tz) \\ &= G(z). \end{aligned} \tag{III.40}$$

Therefore, G is modular of weight $2 - \frac{d}{2}$ for the full modular group. Using this we define

$$\omega(z) = \psi(z) - \frac{1}{\pi i}G(z) \cdot \mathcal{L}(z) \tag{III.41}$$

and from (II.13) given we see that ω is a modular form of weight $2 - \frac{d}{2}$ for $\Gamma(2)$. Moreover, plugging this relationship into (III.35) gives

$$\omega(z) = z^{\frac{d}{2}-2}\omega(Sz) + \omega(Tz). \tag{III.42}$$

Finally, setting $f := \frac{1}{\pi i} \cdot G$ we get the desired conclusions. $\square$

III.3.1 Explicit Representations for the Sphere Packing Problem

In this step our goal will be determining ψ given its representation in terms of f and ω. We use the fact that $\mathbb{C}(\lambda)$ is a field extension of $\mathbb{C}(j)$ to characterize the solutions of (III.42). Then using linear algebra, we ensure that conditions (III.2) and (III.8) hold. We will show that due to (III.42), achieving the former condition will give the latter.

To begin, we recall f and ω are weakly holomorphic modular forms of weight $2 - \frac{d}{2}$ for the groups Γ and $\Gamma(2)$ respectively. This is because there are no modular forms of negative weight because such forms must have poles on either $\mathbb{H}$ or at the cusps. The contour integration arguments from Proposition III.1.0.2 rule out the former and so f and ω must and can only have poles at the cusps. To continue, define

$$\begin{aligned} \ell &= \left\lceil \frac{d-4}{24} \right\rceil \\ k &= 6\ell - \frac{d-4}{4}, \end{aligned}$$

which gives $0 \leq k \leq 5$. From this we set

$$f = \frac{\omega_k}{\Delta^\ell} P^{(k)}(j) \tag{III.43}$$

$$\omega = \frac{\omega_k}{\Delta^\ell} R^{(k)}(\lambda), \tag{III.44}$$

where we have that ω_k is a weakly holomorphic modular form for the full modular group of weight $2k$, $P^{(k)}$ is a polynomial associated with each k, and $R^{(k)}$ is a rational function depending on our choice of k. The values of the ω_k are detailed in Table III.1. These each follow from Lemmas II.4.1.13 and II.4.1.14. Although Lemma II.4.1.14 only gives that R is a rational function, from our contour integration argument in Proposition III.1.0.2 we see that we cannot have a pole at the origin (in fact (III.8) implies we must have a zero here), we can (in fact must) have a pole at $i\infty$, and we may have unprescribed behavior at ± 1. This implies that the most we can conclude is that the denominator of such a rational function, say $R(x)$, can only have factors of the form $x^a(1-x)^b$ because $\lambda(0) = 1$,

$\lambda(1) = \infty$, and $\lambda(i\infty) = 0$.

To continue, we will use (III.42) to analyze the possible choices for $R^{(k)}$. Combining (III.42) and (III.44) yields

$$R^{(k)}(\lambda(z)) = R^{(k)}(\lambda(Sz)) + R^{(k)}(\lambda(Tz)) \tag{III.45}$$

We note that the field of meromorphic functions $\mathbb{C}(\lambda)$ is a degree 6 field extension over the field of meromorphic functions $\mathbb{C}(j)$ with the minimal polynomial of λ over $\mathbb{C}(j)$ given by:

$$\lambda^6 - 3\lambda^5 + (6-j)\lambda^4 - (7-2j)\lambda^3 + (6-j)\lambda^2 - 3\lambda + 1 = 0 \tag{III.46}$$

Therefore, $R^{(k)}$ can be expressed in a unique way as

$$R^{(k)}(\lambda) = \sum_{m=0}^{5} R_m^{(k)}(j)\lambda^m \tag{III.47}$$

for rational functions $R_m^{(k)}$. Inserting this into (III.45) we get

$$\sum_{m=0}^{5} ((1-\lambda)^5\lambda^m - (1-\lambda)^{5+m} + (-1)^m\lambda^m(1-\lambda)^{5-m})R_m^{(k)}(j) = 0 \tag{III.48}$$

We can use the minimal polynomial (III.46) to write all powers of λ larger than 5 by linear combinations of $\{1, \lambda, \ldots, \lambda^5\}$. This gives a linear system of 6 equations for the 6 unknown functions $R_m^{(k)}$, $k = 0, \ldots, 5$. It can be checked directly that this system has rank 4 and hence has a 2 dimensional kernel. This supports an ansatz of the form

$$\omega_k R^{(k)}(\lambda) = \chi_1^{(k)} Y^{(k)}(j) + \chi_2^{(k)} Z^{(k)}(j) \tag{III.49}$$

where the $Y^{(k)}$ and $Z^{(k)}$ are polynomials and $\chi_1^{(k)}$ and $\chi_2^{(k)}$ are two linearly independent solutions of

$$\chi(z) = z^{-2k}\chi(Sz) + \chi(Tz). \tag{III.50}$$

Table III.3 gives solutions of minimal orders at $z = 0$ and $z = i\infty$.

Putting all this information together we get that ψ has the form

$$\psi = \frac{1}{\Delta^\ell}\left(X^{(k)}(j)\omega_k\mathcal{L} + \chi_1^{(k)} Y^{(k)}(j) + \chi_2^{(k)} Z^{(k)}(j)\right) \tag{III.51}$$

for polynomials $X^{(k)}, Y^{(k)}, Z^{(k)}$ that depend on the value of k. Our next step will be to choose

k	$\chi_1^{(k)}$	$\chi_2^{(k)}$
0	$\frac{(1+\lambda)(1-\lambda)(1-\lambda+\lambda^2)}{\lambda^2}$	$\frac{(1+\lambda)(1-\lambda+\lambda^2)}{\lambda(1-\lambda)}$
1	$\theta_{00}^4(1-\lambda)$	$\theta_{00}^4 \frac{(1-\lambda)^3(2+3\lambda+2\lambda^2)}{\lambda^2}$
2	$\theta_{00}^8(1-\lambda^2)$	$\theta_{00}^8 \frac{(1+\lambda)(1+3\lambda-7\lambda^2+3\lambda^3+\lambda^4)}{\lambda(1-\lambda)}$
3	$\theta_{00}^{12}(1-\lambda)(1-\lambda+\lambda^2)$	$\theta_{00}^{12} \frac{(1-\lambda+\lambda^2)(1+3\lambda-10\lambda^2+3\lambda^3+\lambda^4)}{\lambda(1-\lambda)}$
4	$\theta_{00}^{16}\lambda(1+\lambda)(1-\lambda)$	$\theta_{00}^{16} \frac{(1+\lambda)(1-\lambda+\lambda^2-\lambda^3+\lambda^4-\lambda^5+\lambda^6)}{\lambda(1-\lambda)}$
5	$\theta_{00}^{20}\lambda(1-\lambda)(1-4\lambda+\lambda^2)$	$\theta_{00}^{20} \frac{1-32\lambda^3+60\lambda^4-32\lambda^5+\lambda^8}{\lambda(1-\lambda)}$

Table III.3: The choices for the forms $\chi_1^{(k)}$ and $\chi_2^{(k)}$

the degrees of $X^{(k)}, Y^{(k)}$, and $Z^{(k)}$ and use the degrees of freedom given by the coefficients so that (III.51) satisfies (III.8). In particular this implies that we need to choose their degrees so that ψ vanishes to sufficiently large order. In particular, we want

$$\begin{aligned}\varphi(z) &:= z^{-2k}(X^{(k)}(j)\omega_k(Sz)\mathcal{L}(Sz) \\ &\quad + \chi_1^{(k)}(Sz)Y^{(k)}(j(z)) + \chi_2^{(k)}(Sz)Z^{(k)}(j(z))) = \mathcal{O}(q^{\ell+\frac{1}{2}}).\end{aligned} \tag{III.52}$$

Before continuing in this direction however, we show two short lemmas.

Lemma III.3.1.1. *Suppose $\varphi(z)$ is as in* (III.52)*. Then it has only half integer exponents in its Fourier expansion.*

Proof. Let

$$\chi(z) = \chi_1^{(k)}(z)Y^{(k)}(j(z)) + \chi_2^{(k)}(z)Z^{(k)}(j(z))),$$

denote sum of the last two terms on the right side of (III.52). Then χ satisfies (III.50) and so

$$z^{-2k}\chi(Sz) = \chi(Tz) - \chi(z),$$

which implies that all terms in the Fourier expansion of $z^{-2k}\chi(Sz)$ with integer exponents vanish. Moreover, we see from (II.15) that the expression $z^{-2k}X^{(k)}(j)\omega_k(Sz)\mathcal{L}(Sz)$ has only half integer exponents in its Fourier expansion, giving the claim. ☐

Lemma III.3.1.2. *Let ψ be given by* (III.51) *with polynomials X, Y, Z such that* (III.8) *holds. Then the principal part of ψ at $i\infty$ has only integer exponents of q.*

Proof. By our assumption $z^{\frac{d}{2}-2}\psi(Sz) = \mathcal{O}(q^{\frac{1}{2}})$. Since ψ can be written as

$$\psi(z) = \sum_{k=-m}^{\infty} a_k q^{\frac{k}{2}} - iz \sum_{k=-n}^{\infty} b_k q^k = \psi_1 + z\psi_2,$$

(III.36) implies that ψ_1 satisfies

$$\begin{aligned}\psi(Tz) - 2\psi(z) + \psi(T^{-1}z) &= \psi_1(Tz) - 2\psi_1(z) + \psi_1(T^{-1}z)\\ &= 2\psi_1(Tz) - 2\psi_1(z)\\ &= \mathcal{O}(q^{\frac{1}{2}}),\end{aligned}$$

which gives the assertion of the lemma. □

In light of Lemmas III.3.1.1 and III.3.1.2, we first assume that (III.8) holds and define the subscript n for the polynomial $X_n^{(k)}$ so that the following polar order is achieved.

$$X_n^{(k)}(j)\omega_k = \mathcal{O}(q^{-n}), \tag{III.53}$$

We note that this implies that for each $k \neq 1$ the degree of the polynomial $X_n^{(k)}$ is at most n and for $k = 1$ that it has degree at most $n - 1$. We similarly adopt the notations $Y_n^{(k)}$ and $Z_n^{(k)}$ to refer to the polynomials that give us:

$$\chi_1^{(k)}(z)Y_n^{(k)}(j(z)) + \chi_2^{(k)}(z)Z_n^{(k)}(j(z)) = \mathcal{O}(q^{-n-1}) \tag{III.54}$$

$$z^{-2k}(\chi_1^{(k)}(Sz)Y_n^{(k)}(j(z))) = \mathcal{O}(q^{-n+\frac{1}{2}}) \tag{III.55}$$

$$z^{-2k}(\chi_2^{(k)}(Sz)Z_n^{(k)}(j(z))) = \mathcal{O}(q^{-n+\frac{1}{2}}). \tag{III.56}$$

We observe that (III.54), (III.55), and (III.56) are sufficient to put upper bounds on the degrees of polynomials $Y_n^{(k)}$ and $Z_n^{(k)}$. With $b(k)$ as in Table III.4 we can use the degrees of freedom gained from the coefficients of $X_n^{(k)}$, $Y_n^{(k)}$, and $Z_n^{(k)}$ so that

$$\begin{aligned}&z^{-2k}(X_n^{(k)}(j)\omega_k(Sz)\mathcal{L}(Sz) + \chi_1^{(k)}(Sz)Y_n^{(k)}(j(z))\\ &+ \chi_2^{(k)}(Sz)Z_n^{(k)}(j(z))) = \mathcal{O}(q^{2n+\frac{b(k)}{2}}),\end{aligned} \tag{III.57}$$

which is a strengthening of our hypothesis that (III.8) is satisfied. We then observe that (III.53) and (III.54) ensure by (III.4) that the largest real second order pole of $W(s)$ is 2 less than the largest real first order pole. Altogether, this will give us the desired sign change of the function F given by (III.7). The degrees of these polynomials are also detailed in Table III.4. Lists of these polynomials in each case are given in Chapter V.

k	$\deg X_n^{(k)}$	$\deg Y_n^{(k)}$	$\deg Z_n^{(k)}$	$b(k)$
0	n	n	$n-1$	3
1	$n-1$	n	n	3
2	n	$n+1$	n	5
3	n	$n+1$	n	5
4	n	$n+2$	$n+1$	7
5	n	$n+2$	$n+1$	7

Table III.4: Degrees of the polynomials $X_n^{(k)}$, $Y_n^{(k)}$, and $Z_n^{(k)}$

We now need to choose n so that

$$2n + \frac{b(k)}{2} > \ell \tag{III.58}$$

so that (III.8) is satisfied. This then gives that the minimal choice of n is then

$$n = \left\lceil \frac{2\ell - b(k)}{4} \right\rceil . \tag{III.59}$$

Then conditions (III.53) and (III.54) ensure that there is a sign change of $F(\mathbf{x})$ at $\|\mathbf{x}\|^2 = 2n+2\ell+2$ and $F(\mathbf{x}) \neq 0$ for $\|\mathbf{x}\|^2 = 2n+2\ell$. Expressing ℓ, k, and n in terms of d yields $2n+2\ell = 2\lfloor \frac{d}{16} \rfloor$

Summing up, we have proved the following theorem. The theorem is formulated with some abuse of notation, which is justified by the fact that it discusses radial functions: we write $F_-(\mathbf{x}) = F_-(\|\mathbf{x}\|)$ and consider F_- as multivariate and univariate function as appropriate.

Theorem III.3.1.3. *For $d \equiv 0 \pmod 4$ set $n_- = \lfloor \frac{d}{16} \rfloor + 1$. Then there exists a radial Schwartz function $F_- : \mathbb{R}^d \to \mathbb{R}$ satisfying*

$$\begin{aligned} F_-(\mathbf{x}) &= (-1)^{\frac{d}{4}+1} \widehat{F}_-(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^d \\ F_-(\sqrt{2n_-}) &= 0 \quad \text{and } F'_-(\sqrt{2n_-}) \neq 0 \\ F_-(\sqrt{2m}) &= F'_-(\sqrt{2m}) = 0 \quad \text{for } m > n_-, \quad m \in \mathbb{N}. \end{aligned} \tag{III.60}$$

III.3.2 Examples

- $d = 8$

In this case we have $\ell = 1, k = 5$, and $n = -1$ giving

$$\psi = \frac{(j+1408)\chi_1^{(5)} - 256\chi_2^{(5)}}{\Delta}.$$

This is the -1 eigenfunction used in [2].

- $d = 24$

  In this case $\ell = 24, k = 1$, and $n = 0$ giving

$$\psi = \frac{\chi_2^{(1)}}{\Delta}$$

  This is the -1 eigenfunction used in [3].

III.3.3 Remarks

Here we would like to make some observations about Equation (III.51) in particular dimensions d. In particular, while the constructions we made were done specifically with the sphere packing problem in mind, the functions are interesting on their own and can also be used to study other extremal problems. To begin, we observe that the the term $\mathcal{L}$ is missing in exactly 3 cases, when $d = 8, 12, 24$, which follows from Equation (III.59). Other phenomenon includes

- $d = 4$

  Here, $\ell = 0, k = 0$, and $n = 0$. This implies that such solutions are given by a two-dimensional subspace of the form

$$\psi = C_1\mathcal{L} + C_2\chi_1^{(0)},$$

  for complex constants C_1 and C_2. In particular, if we choose $C_1 = 0$ and $C_2 = 1$, $\chi_1^{(0)}$ is a radial Fourier eigenfunction in dimension 4 with its last sign change at distance $\sqrt{2}$. In the language of [15], this implies that $A_+(4) \leq \sqrt{2}$. This follows from the following facts

  Proposition III.3.3.1. *λ is real valued on the positive imaginary axis and satisfies $0 < \lambda(iy) < 1$ for $y > 0$.*

  Proof. From Equation (II.7) we have

$$\lambda(iy) = \frac{\theta_{10}^4(iy)}{\theta_{00}^4(iy)} = \frac{\sum_{n\in\mathbb{Z}}(-1)^n e^{-\pi\left(n+\frac{1}{2}\right)^2 y}}{\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}},$$

  which of course converges to a real value for $y > 0$. On the other hand we have

$$1 - \lambda = 1 - \frac{\theta_{10}^4}{\theta_{00}^4} = \frac{\theta_{01}^4}{\theta_{00}^4} > 0,$$

  by Equation (II.7). The conclusion then follows. □

Proposition III.3.3.2. $\chi_1^{(0)}$ *is positive on the positive imaginary axis*

Proof. We have

$$\chi_1^{(0)} = \frac{(1+\lambda)(1-\lambda)(1-\lambda+\lambda^2)}{\lambda^2} = \frac{(1-\lambda^3)(1-\lambda)}{\lambda^2} > 0,$$

by the above.

$d = 12$

In this case $\ell = 1, k = 4$, and $n = -1$, giving

$$\psi = \frac{(j+768)\chi_1^{(4)} - 256\chi_2^{(4)}}{\Delta}.$$

This was the function studied in [15].

CHAPTER IV

Interpolation Theorems

IV.1 General Hypothesis

We first consider a collection of holomorphic functions, $\{g_n^+(z)\}_{n\geq 0}$, $\{g_n^-(z)\}_{n\geq 1}$ on the upper half-plane, $\mathbb{H}$, that satisfy the following conditions:

1. $g_n^\epsilon(z+2) = g_n^\epsilon(z)$

2. $g_n^\epsilon(-\frac{1}{z}) = \lambda_\epsilon(-iz)^k g_n^\epsilon(z)$

3. $g_n^+(z) = q^{-\frac{n}{2}} + O(q^{\frac{1}{2}}), z \to i\infty$

4. $g_n^-(z) = q^{-\frac{n}{2}} + O(1), z \to i\infty$

5. $g_n^\epsilon(1+\frac{i}{t}) \to 0, t \to \infty$

Here $q = e^{2\pi i z}$, k is a weight to be determined later, $\epsilon \in \{+,-\}$ is a formal symbol, and take

$$\lambda_\epsilon = \begin{cases} 1 & \epsilon = + \\ -1 & \epsilon = - \end{cases}$$

Our weight here will be determined via the Fourier transform, more specifically we write

$$b_m^\epsilon(x) = \frac{1}{2}\int_{-1}^{1} g_m^\epsilon(z) e^{i\pi\|x\|^2 z} dz,$$

for $x \in \mathbb{R}^d$. Here our contour is the circular arc from -1 to 1 in the upper half plane. More generally, we can consider a path from -1 to 1 in the upper half plane that is orthogonal to the real line at the two end points. Taking Fourier transforms (here just as a formal computation without regards to convergence) yields:

$$\widehat{b_m^\epsilon}(\xi) = \frac{1}{2}\int_{-1}^{1} g_m^\epsilon(z) e^{-\frac{i\pi\|\xi\|^2}{z}} (-iz)^{\frac{d}{2}} dz$$

Making the change of variables $u = -\frac{1}{z}$ gives us

$$\begin{aligned}
\widehat{b_m^\epsilon}(\xi) &= -\frac{1}{2}\int_{-1}^{1} g_m^\epsilon(-\frac{1}{z}) z^{-2} e^{i\pi\|\xi\|^2 z}(-iz)^{-\frac{d}{2}} dz \\
&= -\frac{1}{2}\int_{-1}^{1} \lambda_\epsilon g_m^\epsilon(z) z^{-2} e^{i\pi\|\xi\|^2 z}(-iz)^{k+\frac{d}{2}} dz
\end{aligned}$$

We seek an eigenfunction with eigenvalue λ_ϵ so this implies precisely that $k + \frac{d}{2} = 2$, which in the $d = 1$ case gives the value $k = \frac{3}{2}$ as used in [8].

IV.2 Initial General Constructions

We make general claims about functions of the form

$$g_n^+(z) = \theta^{4-d}(z) P_n^+(J^{-1}(z))$$

$$g_n^-(z) = \theta^{4-d}(z)(1 - 2\lambda(z))(P_n^-(J^{-1}(z)))$$

where here $\theta(z)$ is the Theta function of the integer lattice (i.e, θ_{00}) and the P_n^ϵ are a sequence of polynomials suitably chosen to meet the demands of the third and fourth constraints (we additionally require that $P_n^-(0) = 0$ for all n as an additional regularity). Defined in this way we have that the first two constraints will follow from the properties of the J and λ functions however the last desired constraint is in general not true for all values of d, we will study this shortly. We also show that such polynomials exist and are well defined in all such cases. We'll first formulate a general framework for such a problem and then as a corollary conclude that our situation follows

Lemma IV.2.0.1. *Given a function $Q(z)$ such that it possesses a Fourier series with no negative powers and leading constant term 1, then there always exists a polynomial P_n of degree n such that $Q(z)P_n(J^{-1}(z)) = q^{-n/2} + O(q^{1/2})$*

Proof. We first observe that if such a polynomial is to exist it has to be monic because $J^{-1}(z) = q^{-1/2} + O(1)$. Next, we suppose that

$$P_n(z) = z^n + A_{n-1}z^{n-1} + ... + A_0$$

from which it's clear that our constraints amount to solving an $(n+1)\times(n+1)$ linear system. It would therefore suffice to show that the determinant obtained from such a system is invertible.

This however is clear simply by expanding $P_n(J^{-1}(z))$ and getting a system of the form

$$\begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \\ a_{(N+1)1} & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ A_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}.$$

That is to say, we are necessarily interested in inverting an upper triangle matrix with all diagonal entries being 1. It's clear that such a matrix is invertible because the determinant is easily seen to be 1, from which the conclusion follows. ☐

Corollary IV.2.0.2. *For $d \geq 0$ we have that $\theta^{4-d}(z)$ meets the criteria of $Q(z)$ in the above lemma.*

Proof. For $d \leq 4$ this is clear from the definition of $\theta(z)$. On the other hand, for $d > 4$ this follows readily from the fact that $\theta(z)$ has neither zeros in the upper half plane nor at $i\infty$ (we can see this from the representation $\theta(z) = \frac{\eta^5(z)}{\eta^2(z/2)\eta^2(2z)}$, where $\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the classical Dedekind eta function). This implies that the Fourier expansion of $\theta^{-1}(z)$ has only non-negative powers of q and moreover an explicit computation yields that the constant term of θ^{-1} is 1 so the conclusion follows. ☐

The lemma generalizes for the case of g_n^- and can be given an analogous proof. We now study the analytical properties of the Theta function for the integers

Lemma IV.2.0.3. *We have that*

$$\theta\left(1+\frac{i}{t}\right) \to 0$$

as $t \to \infty$

Proof. First, we fix $t > 0$ and note by direct substitution that

$$\theta\left(1+\frac{i}{t}\right) = \sum_{n\in\mathbb{Z}} (-1)^n e^{-\frac{\pi n^2}{t}} \tag{IV.1}$$

Observe that (IV.1) is absolutely convergent since we have the following

$$\sum_{n\in\mathbb{Z}} (-1)^n e^{-\frac{\pi n^2}{t}} \leq \sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{t}} \leq 1 + 2\sum_{n\geq 1}\frac{t}{\pi n^2}$$

where the second inequality follows from the estimate $e^{-x} \leq x^{-1}$ for $x \geq 0$ and the last sum of (IV.2) is convergent as a p-series. This implies that we can rearrange (IV.1) and consider it as a difference of two sums

$$\sum_{n\in\mathbb{Z}} e^{-\frac{4\pi n^2}{t}} - \sum_{n\in\mathbb{Z}} e^{-\frac{\pi(2n+1)^2}{t}}$$

We then have the following by the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} e^{-\frac{4\pi n^2}{t}} = \frac{\sqrt{t}}{2}\sum_{n\in\mathbb{Z}} e^{-\frac{t\pi n^2}{4}}$$

and

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\pi(2n+1)^2}{t}} = \frac{\sqrt{t}}{2}\sum_{n\in\mathbb{Z}}(-1)^n e^{-\frac{t\pi n^2}{4}}$$

This implies that

$$\sum_{n\in\mathbb{Z}} e^{-\frac{4\pi n^2}{t}} - \sum_{n\in\mathbb{Z}} e^{-\frac{\pi(2n+1)^2}{t}} = \frac{\sqrt{t}}{2}\sum_{n\in\mathbb{Z}} e^{-\frac{t\pi n^2}{4}} - \frac{\sqrt{t}}{2}\sum_{n\in\mathbb{Z}}(-1)^n e^{-\frac{t\pi n^2}{4}} = \sqrt{t}\sum_{n\in\mathbb{Z}} e^{-\frac{t\pi(2n+1)^2}{4}}$$

which further gives that (IV.1) is actually non-negative. Next, we note that Gaussian is a monotonically decreasing function on the positive real axis hence we have the following estimate

$$\sqrt{t}\sum_{n\in\mathbb{Z}} e^{-\frac{t\pi(2n+1)^2}{4}} \leq 2\sqrt{t}\int_0^\infty e^{\frac{-t\pi(2x+1)^2}{4}}\,dx \tag{IV.2}$$

A change of variables, $u = \sqrt{t}(2x+1)/2$, allows us to rewrite (IV.2) as

$$2\sqrt{t}\int_0^\infty e^{\frac{-t\pi(2x+1)^2}{4}}\,dx = 2\int_{\frac{\sqrt{t}}{2}}^\infty e^{-\pi u^2}\,du$$

Allowing $t \to \infty$, applying the fact that (IV.1) is non-negative, and using the Lebesgue dominated convergence theorem shows that

$$0 \leq \lim_{t\to\infty} \theta\left(1 + \frac{i}{t}\right) \leq 0$$

With this, our desired conclusion follows. $\square$

Corollary IV.2.0.4. *For $d < 4$ we have that g_n^ϵ satisfies our fifth hypothesis for each fixed n*

Proof. By Lemma IV.2.0.3 it suffices to show that $P_n^+(J^{-1}(1+\frac{i}{t}))$ and $(1-2\lambda(1+\frac{i}{t}))(P_n^-(J^{-1}(1+$

$\frac{i}{t}$))) to converge to a constant for each fixed n. For the former this follows from asymptotic

$$J^{-1}\left(1-\frac{1}{z}\right) = -4096q - 98304q^2 + O(q^3)$$

implying that it vanishes at 1 so so we just get the constant term of the polynomial P_n^+, as desired. For the latter case it remains to study the behavior of the lambda invariant. We make use of the identities

- $\lambda(-\frac{1}{z}) = 1 - \lambda(z)$

- $\lambda(z+1) = \frac{\lambda(z)}{\lambda(z)-1}$

- $\lambda(z) = \frac{\theta_{10}^4(\tau)}{\theta_{00}^4(\tau)}$,

These in turn give us the following

$$1 - 2\lambda\left(1-\frac{1}{z}\right) = \frac{1}{8}q^{-1/2} + \frac{5}{2}q^{1/2} + O(q^{3/2}),$$

implying that this term is $O(e^{\pi t})$. On the other hand, by our normalization assumption chosen for the polynomial P_n^- we have $P_n^-(0) = 0$ for all n. This implies that the overall term $(1-2\lambda(1+\frac{i}{t}))(P_n^-(J^{-1}(1+\frac{i}{t})))$ is $O(e^{-\pi t})$, a positive function that decreases monotonically to 0 as $t \to 0$, as needed. □

IV.3 Interpolation Baseis

Returning to what was discussed in Section IV.1 we consider the functions

$$b_m^\epsilon(x) = \frac{1}{2}\int_{-1}^{1} g_m^\epsilon(z) e^{i\pi\|x\|^2 z} dz,$$

where the contour being integrated over a semi-circle in the upper half plane connecting -1 and 1. For the rest of this Chapter we assume that $d \in \{2, 3\}$. We now show some properties of this function:

Proposition IV.3.0.1. *For $x \in \mathbb{R}^d$ we have that b_m^ϵ is a convergent real valued function.*

Proof. Since our path of integration has finite length, to show our integral is well defined it would suffice to show boundedness of the integrand on the path of integration by this follows directly from Corollary IV.2.0.2. To show our integral is real valued we observe that it suffices to show that each integral of the form

$$\varphi_n(x) = \frac{1}{2}\int_{-1}^{1} \theta^{4-d}(z)J^{-n}(z)e^{i\pi\|x\|^2 z}dz$$

is real valued for each $n \in \mathbb{N}$. Combining the well definedness on the integrand with Corollary IV.2.0.4 we see it is enough to show that

$$\int_{-1}^{1} J^{-n}(z)e^{i\pi m z}e^{i\pi\|x\|^2 z}dz$$

is real valued for each $m \in \mathbb{N}_0$. To this end we have the following computations, where we parameterize by $z = e^{it}$ for $t \in [0, \pi]$

$$\begin{aligned}\int_{-1}^{1} J^{-n}(z)e^{i\pi m z}e^{i\pi\|x\|^2 z}dz &= \int_{-1}^{i} J^{-n}(z)e^{i\pi m z}e^{i\pi\|x\|^2 z}dz + \int_{i}^{1} J^{-n}(z)e^{i\pi m z}e^{i\pi\|x\|^2 z}dz \\ &= i\int_{\pi}^{\frac{\pi}{2}} J^{-n}(e^{it})e^{i\pi m e^{it}}e^{i\pi\|x\|^2 e^{it}}e^{it}dt + i\int_{\frac{\pi}{2}}^{0} J^{-n}(e^{it})e^{i\pi m e^{it}}e^{i\pi\|x\|^2 e^{it}}e^{it}dt \\ &= i\int_{0}^{\frac{\pi}{2}} J^{-n}(e^{it})e^{i\pi m e^{-it}}e^{i\pi\|x\|^2 e^{-it}}e^{-it}dt - i\int_{0}^{\frac{\pi}{2}} J^{-n}(e^{it})e^{i\pi m e^{it}}e^{i\pi\|x\|^2 e^{it}}e^{it}dt\end{aligned}$$

Here we performed a change of variables in the first integral, $t' = \pi - t$ and used the fact that $J(-\frac{1}{z}) = J(z)$. Furthermore, using the fact that J^{-1} is a real number between 0 and 64 on the arc from -1 to 1 we observe the integrals are the conjugates of one another. Therefore their difference is purely imaginary and multiplying by i gives that our integral is precisely real valued as desired. □

We now verify assertions made about b_m^ϵ in Section IV.1

Proposition IV.3.0.2. *The function $b_m^\epsilon : \mathbb{R}^d \to \mathbb{R}$ is an even Schwartz class function that satisfies*

$$\widehat{b_m^\epsilon}(x) = \epsilon b_m^\epsilon(x) \tag{IV.3}$$

and

$$b_m^\epsilon(\|x\|) = \delta_{n,m}, n \geq 1, m \geq 0, x \in \mathbb{R}^d, \|x\| = \sqrt{n} \tag{IV.4}$$

Proof. First, it's clear that b_m^ϵ is an even function and moreover to verify (IV.3) it suffices to verify that b_m^ϵ is Schwartz class. This is because we can just exchange the integral that defines b_m^ϵ and the integral of the Fourier transform and apply our computation from Section IV.1. We'll focus on the $\epsilon = +$ case because the $\epsilon = -$ case is analogous. It's enough to show that for each $n \in \mathbb{N}$ we have that the integral given by

$$\varphi_n(x) = \frac{1}{2}\int_{-1}^{1} \theta^{4-d}(z)J^{-n}(z)e^{i\pi\|x\|^2 z}dz$$

is Schwartz class because P_n^+ is a polynomial for each n. We again use the fact that on the circle arc -1 and 1 the function $J^{-1}(z)$ takes real values between 0 and 64. Uniform convergence (due to the fat that our contour of integration is finite) implies that

$$\frac{\partial^k \varphi_n}{\partial x^\alpha}(x) = \frac{1}{2}\int_{-1}^{1} \theta^{4-d}(z)J^{-n}(z)P_\alpha(x,z)e^{i\pi\|x\|^2 z}dz,$$

where α here is a multi-index such that $|\alpha| = k$ and $P_\alpha(x,z)$ is a polynomial obtained form the differentiation of the term $e^{i\pi\|x\|^2 z}$. The triangle inequality and Arithmetic-Geometric inequality implies that there is a constant C_α such that

$$|P_\alpha(x,z)| \leq C_\alpha(1+\|x\|^2)^k(1+|z|^2)^k \tag{IV.5}$$

Parametrizing $z = e^{2\pi it}$ for $t \in (0, \frac{1}{2})$ and estimating using (IV.5) gives us

$$\begin{aligned}
\left|\frac{\partial^k \varphi_n}{\partial x^\alpha}(x)\right| &= \left|\pi C_\alpha \int_0^{1/2} \theta^{4-d}(e^{2\pi it})J^{-n}(e^{2\pi it})P_\alpha(x, e^{2\pi it})e^{i\pi\|x\|^2 e^{2\pi it}}e^{2\pi it}dt\right| \\
&\leq 2^k C_\alpha \pi(1+\|x\|^2)^k \int_0^{1/2} J^{-n}(e^{2\pi it})\left|\theta^{4-d}(e^{2\pi it})\right| e^{-\pi\|x\|^2\sin(2\pi t)}dt \\
&= 2^{k+1} C_\alpha \pi(1+\|x\|^2)^k \int_0^{1/4} J^{-n}(e^{2\pi it})\left|\theta^{4-d}(e^{2\pi it})\right| e^{-\pi\|x\|^2\sin(2\pi t)}dt \\
&< 2^{k+5-d} C_\alpha \pi(1+\|x\|^2)^k \int_0^{1/4} J^{-n}(e^{2\pi it})e^{-\pi\|x\|^2\sin(2\pi t)}dt
\end{aligned}$$

here we used the facts that J^{-1} is a real number between 0 and 64 on the circle arc, $J(-\frac{1}{z}) = J(z)$, and $\left|\theta(e^{2\pi it})\right| < 2$ for $t \in (0, \frac{1}{4})$. We now finish in an approach that is exactly the same as the one-dimensional approach. Using the facts that $4t < \sin(2\pi t)$ for $t \in (0, \frac{1}{4})$ and and that $J^{-1}(1-\frac{1}{z}) = O(e^{-2\pi t})$ as $t\to\infty$ gives

$$\begin{aligned}\int_0^{1/4} J^{-n}(e^{2\pi it})e^{-\pi\|x\|^2\sin(2\pi t)}dt &= \int_0^{\delta} J^{-n}(e^{2\pi it})e^{-\pi\|x\|^2\sin(2\pi t)}dt + \int_{\delta}^{1/4} J^{-n}(e^{2\pi it})e^{-\pi\|x\|^2\sin(2\pi t)}dt\\ &\le C\delta e^{-4/\delta} + 64^n e^{-4\pi\|x\|^2\delta}(\frac{1}{4}-\delta)\\ &\le C\delta e^{-4/\delta} + 64^n e^{-4\pi\|x\|^2\delta}\end{aligned}$$

Here $\delta\in(0,\frac{1}{4})$ was chosen to homogenize the exponentials in last line. We can take $\delta = \frac{1}{\|x\|\sqrt{\pi}}$ which is of course a valid choice for each $x\in\mathbb{R}^d$ such that $\|x\| > \frac{4}{\sqrt{\pi}}$. This gives that the last line is bounded above by $e^{-4\sqrt{\pi}\|x\|}(C/(\|x\|)+64^n)$. Altogether, we have that this implies

$$\sup_{x\in\mathbb{R}^d}\left|x^\beta\frac{\partial^k\varphi_n}{\partial x^\alpha}(x)\right| < \infty$$

for each multi-index α,β as required. With this the conclusion for (IV.3) follows. For (IV.4) we have

$$b_m^\epsilon(\|x\|) = \frac{1}{2}\int_{-1}^{1} g_m^\epsilon(z)e^{i\pi nz}dz,$$

when $\|x\| = \sqrt{n}$, is just the coefficient of $q^{-n/2}$ in the q-expansion of g_m^ϵ. This gives precisely that $b_m^\epsilon(\|x\|) = \delta_{n,m}, n\ge 1, m\ge 0, x\in\mathbb{R}^d, \|x\| = \sqrt{n}$ as claimed. □

IV.3.1 Remarks

We note by Poisson summation that for any lattice $\Lambda\subset\mathbb{R}^d$ we have

$$\sum_{v\in\Lambda} b_m^-(v) = \frac{1}{\|\Lambda\|}\sum_{v\in\Lambda^*}\widehat{b_m^-}(v) = -\frac{1}{\|\Lambda\|}\sum_{v\in\Lambda^*} b_m^-(v)$$

In the one-dimensional case this is can be used to show that

$$b_m^-(0) = \begin{cases}-2, & m\ge 1 \text{ is a square}\\ 0, & \text{otherwise}\end{cases}$$

by taking the only feasible selection of $\Lambda = \mathbb{Z}\subset\mathbb{R}$. For $\mathbb{R}^d$ we can also take the natural choice of $\Lambda = \mathbb{Z}^d$ giving us

$$b_m^-(0) = -v_d(m),$$

where $v_d(m)$ id the number of solutions to the equation $\|x\|^2 = m$ for $x \in \mathbb{Z}^d$. However, in other dimensions it is not immediately obvious what other lattices would provide useful information or side conditions about our b_m^ϵ. In particular what can be said about isodual, integral lattices in $\mathbb{R}^d$?

Proposition IV.3.1.1. *In $\mathbb{R}^2$ there is up to orthogonal trandformation only one isodual lattice with the property that the square of its distances are always integers, namely $\mathbb{Z}^2$.*

Proof. Suppose we have a lattice given by $\Lambda = A\mathbb{Z}^2$ for some non-singular 2×2 matrix A. Then its Gram matrix, G will be given by $G = A^T A$ and the Gram matrix, G^*, of the dual of Λ, Λ^*, will then be given by $G^* = G^{-1}$. Then let the quadratic forms generated by G and G^{-1} be $Q_1(x) = x^T G x$ and $Q_2(x) = x^T G^{-1} x$ respectively, where $x \in \mathbb{Z}^2$. Our hypothesis is that our lattice be isodual, hence, the quadratic forms generated by G and G^{-1} being equivalent over $\mathbb{Z}$, i.e, there exists some $\phi \in \mathrm{GL}_2(\mathbb{Z})$ so that $Q_2(x) = Q_1(\phi(x))$. This condition can then be rewritten as

$$G^{-1} = \phi^T G \phi$$

This yields the fact that we must have $G \in \mathrm{GL}_2(\mathbb{Z})$. Additionally, we get the equivalent relationship using the fact that G is symmetric

$$(\phi G)^{-1} = (G\phi)^T$$

Putting $\phi = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ and $G = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ gives us the system of equations

$$\begin{aligned} aw &= cz \\ -c(x+y) &= b(y+w) \\ -a(x+y) &= 2bz \end{aligned}$$

We can solve this system in the integers primarily using the fact that $\mathrm{Det}(\phi) = \mathrm{Det}(G) = \pm 1$. This gives 6 solutions for G

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

all of which are orthogonally equivalent to $\mathbb{Z}^2$ as claimed. $\square$

We can further ask what about $\mathbb{R}^3$?

Proposition IV.3.1.2. *In $\mathbb{R}^3$ there is up to orthogonal trandformation only one isodual lattice with the property that the square of its distances are always integers, namely $\mathbb{Z}^3$.*

Proof. Suppose our lattice Λ is isodual, then it is known from [11] that its Gram matrix must be similar via orthogonal transforamation to one of two matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -h \\ 0 & -h & \beta \end{pmatrix} \tag{IV.6}$$

where $\alpha\beta - h^2 = 1$ and $0 \leq 2h \leq \alpha \leq \beta$ or

$$\frac{1}{2-\alpha\beta} \begin{pmatrix} \frac{2\alpha}{\beta} & -\alpha\beta & -\alpha(2-\beta) \\ -\alpha\beta & \frac{2\beta}{\alpha} & \frac{2\beta(1-\alpha)}{\alpha} \\ -\alpha(2-\beta) & \frac{2\beta(1-\alpha)}{\alpha} & \frac{\alpha^2\beta+2\alpha+2\beta-4\alpha\beta}{\alpha} \end{pmatrix} \tag{IV.7}$$

for $0 < \alpha \leq \beta < 1$. In either case, since we require our lattice to integral we therefore require integral entries of each matrix. In the case of (IV.6), the inequalities provided imply that $1 \geq 3h^2$, hence $h = 0$ and $\alpha\beta = 1$. This gives either $\alpha = \beta = 1$ or $\alpha = \beta = -1$. The former represents the situation $\mathbb{Z}^3$ while the latter can be ruled out for violating the second set of inequalities for (IV.6). For (IV.7) we have that $2 - \alpha\beta$ divides $-\alpha(2-\beta)$ so that $|\alpha(2-\beta)| \geq |2-\alpha\beta|$. Using the inequalities given in this case we can rewrite that inequality as $\alpha \geq 1$, a contradiction. Hence, no integral isodual lattice exists in this case and with this the conclusion follows. $\square$

IV.4 Generating Functions

Proposition IV.4.0.1. *The generating functions for* $\{g_n^+(z)\}_{n\geq 0}$ *and* $\{g_n^-(z)\}_{n\geq 1}$ *are given by*

$$\sum_{n=0}^{\infty} g_n^+(z)e^{i\pi n\tau} = \frac{\theta^d(\tau)(1-2\lambda(\tau))\theta^{4-d}(z)J(z)}{J(z)-J(\tau)} = K_+(\tau,z)$$
$$\sum_{n=0}^{\infty} g_n^-(z)e^{i\pi n\tau} = \frac{\theta^d(\tau)J(\tau)\theta^{4-d}(z)(1-2\lambda(z))}{J(z)-J(\tau)} = K_-(\tau,z)$$

Proof. We show the proof only for K_+ because the proof for K_- is analogous. Our argument will follow along the lines of Lemma 2 in [10], where we will show that the Fourier series of both sides are the same. To begin, observe that by Cauchy's theorem we have

$$P_n^+(\zeta) = \frac{1}{2\pi i}\oint_C \frac{P_n^+(J^{-1})}{J^{-1}-\zeta}dJ^{-1},$$

where C is a sufficiently small (counterclockwise) circle around 0 in the J^{-1}-plane and J^{-1} here is implicitly a function of τ. Recalling that $g_n^+(z) = q_z^{-n/2} + \mathcal{O}(q_z^{1/2})$ and $g_n^+(z) = \theta^{4-d}(z)P_n^+(J^{-1}(z))$ we have

$$\begin{aligned} P_n^+(\zeta) &= \frac{1}{2\pi i}\oint_C \frac{P_n^+(J^{-1})}{J^{-1}-\zeta}dJ^{-1} \\ &= \frac{1}{2\pi i}\oint_C \frac{g_n^+(\tau)}{\theta^{4-d}(\tau)(J^{-1}-\zeta)}dJ^{-1} \\ &= \frac{1}{2\pi i}\oint_C \frac{q_\tau^{-n/2}}{\theta^{4-d}(\tau)(J^{-1}-\zeta)}dJ^{-1} \end{aligned}$$

Combining the identity

$$-q_\tau^{1/2}J^2\frac{dJ^{-1}}{dq_{\tau^{1/2}}} = q_\tau^{1/2}\frac{dJ}{dq_{\tau^{1/2}}} = \frac{J'(\tau)}{\pi i} = \theta^4(\tau)(1-2\lambda(\tau))J(\tau)$$

with the fact that J is Hauptmodul for Γ_θ gives a well defined change of variables. In particular

$$\begin{aligned} P_n^+(\zeta) &= \frac{1}{2\pi i}\oint_C \frac{q_\tau^{-n/2}}{\theta^{4-d}(\tau)(J^{-1}-\zeta)}dJ^{-1} \\ &= -\frac{1}{2\pi i}\oint_{\tilde{C}} \frac{q_\tau^{-n/2-1/2}\theta^d(\tau)(1-2\lambda(\tau))}{(J^{-1}-\zeta)}dq_{\tau^{1/2}} \end{aligned}$$

where $\tilde{C}$ is a clockwise circle (the orientation is reversed because $J(\tau) = \mathcal{O}(\frac{1}{q_\tau})$). We conclude by then writing

$$P_n^+(J^{-1}(z)) = -\frac{1}{2\pi i}\oint_{\tilde{C}} \frac{q_\tau^{-n/2-1/2}\theta^d(\tau)(1-2\lambda(\tau))}{(J^{-1}-J^{-1}(z))} dq_{\tau^{1/2}}$$

and rearranging gives the desired expression. $\square$

Next, we outline some identities for $K_\epsilon(x,\tau)$ that will be used in the next proposition:

$$\begin{aligned}
K_\epsilon(\tau, -\frac{1}{z}) &= \lambda_\epsilon(-iz)^{(4-d)/2}K_\epsilon(\tau, z)\\
K_\epsilon(-\frac{1}{\tau}, z) &= -\lambda_\epsilon(-i\tau)^{d/2}K_\epsilon(\tau, z)\\
\mathrm{Res}_{z=\tau}K_\epsilon(\tau, z) &= \frac{1}{\pi i}
\end{aligned}$$

While the first two are clear from the definitions of $K_\epsilon(\tau, z)$ the last follows from the identity

$$\frac{1}{\pi i} = \frac{\theta^4(\tau)(1-2\lambda(\tau))J(\tau)}{J'(\tau)}$$

This gives

$$\begin{aligned}
\mathrm{Res}\ K_+(\tau, z)\Big|_{z=\tau} &= \lim_{z\to\tau}(z-\tau)\frac{\theta^d(\tau)(1-2\lambda(\tau))(\theta^{4-d}(z))(J(z))}{J(z)-J(\tau)}\\
&= \lim_{z\to\tau}\frac{\theta^d(\tau)(1-2\lambda(\tau))(\theta^{4-d}(z))(J(z))}{J'(\tau)}\\
&= \frac{\theta^d(\tau)(1-2\lambda(\tau))(\theta^{4-d}(\tau))(J(z))}{J'(\tau)}\\
&= \frac{1}{\pi i},
\end{aligned}$$

where the second line follows from L'Hopital's rule and the proof for $K_-(\tau, z)$ is exactly the same. We next define a function $F_\epsilon(\tau, x)$ on the set

$$\mathcal{S} = \{\tau \in \mathbb{H} : \forall k \in \mathbb{Z}, |\tau - 2k| > 1\}$$

given by

$$F_\epsilon(\tau, x) = \frac{1}{2}\int_{-1}^{1} K_\epsilon(\tau, x)e^{i\pi\|x\|^2 z}dz$$

where the contour of integration here is over the arc of the circle going from -1 to 1. Observe that because $\mathcal{S}$ consists of all integer translates of $\mathcal{D}$ (where $F_\epsilon(\tau, x)$ is well defined) along the real axis, it's clear that the translation invariance of $K_\epsilon(\tau, z)$ gives the well definedness of $F_\epsilon(\tau, x)$ in $\mathcal{S}$. We further note that when $\mathrm{Im}(\tau) > 1$ we have

$$F_\epsilon(\tau, x) = \sum_{n=0}^{\infty} b_n^\epsilon(x) e^{i\pi n\tau} \tag{IV.8}$$

and moreover that this series converges absolutely. We will now show that this identity holds for all $\tau \in \mathbb{H}$.

Proposition IV.4.0.2. *For any ϵ and $x \in \mathbb{R}^d$, the function $F_\epsilon(\tau, x)$ admits an analytic continuation to $\mathbb{H}$. These continuations satisfy the functional equations:*

$$F_\epsilon(\tau, x) - F_\epsilon(\tau + 2, x) = 0 \tag{IV.9}$$

$$F_\epsilon(\tau, x) + \lambda_\epsilon(-i\tau)^{-d/2} F_\epsilon\left(-\frac{1}{\tau}, x\right) = e^{i\pi\tau\|x\|^2} + \lambda_\epsilon(-i\tau)^{-d/2} e^{i\pi(-1/\tau)\|x\|^2} \tag{IV.10}$$

Proof. We observe that is enough to show that there exists an analytic continuation to some open set Ω containing the boundary of $\mathcal{D}$ on which equations (IV.9) and (IV.10) hold. Then choosing Ω such that

$$\mathcal{D} \subset \Omega \subset \mathcal{D} \cup S\mathcal{D} \cup T^2\mathcal{D} \cup T^{-2}\mathcal{D},$$

from which since $\cup_{g\in\Gamma_\theta} g\Omega = \mathbb{H}$ ($\mathcal{D}$ is a fundamental domain) we can construct a continuation by repeatedly using equations (IV.9) and (IV.10). Then by the Monodromy theorem, since $\mathbb{H}$ is simply connected, this would give a well defined extension to all of $\mathbb{H}$. Next, we observe that equation (IV.9) will always be satisfied since the integrand that defines F_ϵ is a two-periodic function on $\mathcal{S}$, which contains the boundary of $\mathcal{D}$. In this case it remains only to deal with equation (IV.10). We have an analytic continuation of F_ϵ to some neighborhood of $\{z \in \mathbb{H} : |z| = 1, z \neq i\}$ by the following computations

$$\begin{aligned} 2F_\epsilon(\tau, x) &= \int_{-1}^{i} K_\epsilon(\tau, z) e^{i\pi\|x\|^2 z} dz + \int_{i}^{1} K_\epsilon(\tau, z) e^{i\pi\|x\|^2 z} dz \\ &= \int_{-1}^{i} K_\epsilon(\tau, z) e^{i\pi\|x\|^2 z} dz - \int_{-1}^{i} K_\epsilon(\tau, -\frac{1}{z}) e^{i\pi\|x\|^2(-\frac{1}{z})} z^{-2} dz \\ &= \int_{-1}^{i} K_\epsilon(\tau, z)(e^{i\pi\|x\|^2 z} + \lambda_\epsilon(-iz)^{-d/2} e^{i\pi\|x\|^2(-1/z)}) dz \end{aligned} \tag{IV.11}$$

Observe that if $\tau \in \overline{\mathcal{D}} \cup \overline{S\mathcal{D}}$ then the only poles of $K_\epsilon(\tau, z)$ inside $\overline{\mathcal{D}} \cup \overline{S\mathcal{D}}$ are $z = \tau$ and $z = -\frac{1}{\tau}$. Let γ_1 denote the circle arc from -1 to i and let γ_2 be a smooth simple path from

-1 to i that lies inside $\overline{SD}$ and strictly below γ_1. Denote by $\mathcal{F}$ the region enclosed between γ_1 and γ_2. We can build a continuation of $F_\epsilon(x,\tau)$ to $\mathcal{F}$ and show that it satisfies equation (IV.9). We take

$$\tilde{F}_\epsilon(\tau,x) = \frac{1}{2}\int_{\gamma_2} K_\epsilon(\tau,z)(e^{i\pi\|x\|^2 z} + \lambda_\epsilon(-iz)^{-d/2}e^{i\pi\|x\|^2(-1/z)})dz$$

For τ with sufficiently large imaginary part, it is clear by construction that $F_\epsilon = \tilde{F}_\epsilon$. So then for $\tau \in \mathcal{F}$ we have

$$\begin{aligned}
\tilde{F}_\epsilon(\tau,x) + \frac{\lambda_\epsilon}{(-iz)^{d/2}}F_\epsilon\left(-\frac{1}{\tau},x\right) &= \tilde{F}_\epsilon(\tau,x) - \frac{1}{2}\int_{\gamma_1} K_\epsilon(\tau,z)(e^{i\pi\|x\|^2 z} + \lambda_\epsilon(-iz)^{d/2}e^{i\pi\|x\|^2(-1/z)})dz \\
&= \frac{1}{2}\int_{\partial\mathcal{F}} K_\epsilon(\tau,z)(e^{i\pi\|x\|^2 z} + \lambda_\epsilon(-iz)^{d/2}e^{i\pi\|x\|^2(-1/z)})dz \\
&= i\pi\sum_{z\in\mathcal{F}} \mathrm{Res}_{z=\tau}K_\epsilon(\tau,z)(e^{i\pi\|x\|^2 z} + \lambda_\epsilon(-iz)^{d/2}e^{i\pi\|x\|^2(-1/z)}) \\
&= e^{i\pi\tau\|x\|^2} + \lambda_\epsilon(-i\tau)^{-d/2}e^{i\pi(-1/\tau)\|x\|^2}
\end{aligned}$$

which is exactly the equation we desired. We can do the exact same computation on the arc from i to 1 as well. We need only now check that $\tau = i$ is not a pole. For $\epsilon = +$ this follows directly from equation (IV.10) and for $\epsilon = -$ we again have that $e^{i\pi z\|x\|^2} + \lambda_\epsilon(-iz)^{-d/2}e^{i\pi(-1/z)\|x\|^2}$ and $1-2\lambda(z)$ both vanish at $z=i$ so they cancel the double pole at i coming from $J(z)-J(i)$ so equation (IV.11) converges at $\tau = i$. $\square$

This result then implies that (IV.8) converges for all $\tau \in \mathbb{H}$, as desired. For the remainder of this, we note that we will occasionally use the notation $F_\epsilon(\tau) = F_\epsilon(\tau,x)$ when it simplifies notation. The context will generally be clear.

We now present two ideas that will prove crucial to proving the polynomial growth of the b_n^ϵ:

Proposition IV.4.0.3. *Let $\lambda > 0$ and $\alpha \geq 0$. Suppose $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function such that it possesses the Fourier series given by:*

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda}$$

Further, suppose that $f(\tau) = \mathcal{O}(y^{-\alpha})$ uniformly for all $x \in \mathbb{R}$ as $y \to 0^+$, where x and y are *the real and the imaginary parts of τ respectively. Then we have that*

$$a_n = O((\frac{n\pi e}{\alpha})^{\alpha})$$

as $n \to \infty$.

Proof. We present a proof adapted from [9]. We have the following computation for any $\tau_0 \in \mathbb{H}$

$$\begin{aligned} a_n &= \frac{1}{\lambda}\int_{\tau_0}^{\tau_0+\lambda} f(\tau)e^{-2\pi i n\tau/\lambda}d\tau \\ &= \frac{1}{\lambda}\int_{\mathrm{Re}(\tau_0)}^{\mathrm{Re}(\tau_0)+\lambda} f(t+i\mathrm{Im}(\tau_0))e^{-\frac{2\pi i n}{\tau}t}e^{\frac{2\pi n}{\tau}\ \mathrm{Im}(\tau_0)}dt \end{aligned}$$

Hence,

$$\begin{aligned} |a_n| &\leq \frac{1}{\lambda}\int_{\mathrm{Re}(\tau_0)}^{\mathrm{Re}(\tau_0)+\lambda} |f(t+i\mathrm{Im}(\tau_0))|e^{\frac{2\pi n}{\tau}\ \mathrm{Im}(\tau_0)}dt \\ &\leq \frac{1}{\lambda}e^{\frac{2\pi n}{\tau}\ \mathrm{Im}(\tau_0)}\int_{\mathrm{Re}(\tau_0)}^{\mathrm{Re}(\tau_0)+\lambda} Cn^{\alpha}dt \\ &= Ce^{\frac{2\pi n}{\tau}\ \mathrm{Im}(\tau_0)}n^{\alpha} \end{aligned}$$

Letting $\mathrm{Im}(\tau_0) = \frac{\alpha}{ne\pi}$ and allowing $n \to \infty$ gives the desired result. □

We also have the following

Proposition IV.4.0.4. *For each multi-index α, β there exists an absolute constant $C_{\alpha,\beta} > 0$ such that the inequality*

$$\left|x^{\alpha}\frac{\partial F^n}{\partial x^{\beta}}(\tau, x)\right| \leq C_{\alpha,\beta}(1 + Im(\tau)^{-m-n-\frac{1}{2}}),$$

where $|\alpha| = m$ and $|\beta| = n$.

Proof. Suppose $\tau \in \mathcal{D}$ is arbitrary. From equation (IV.8) and Proposition IV.3.0.1 we have that $F_{\epsilon}(it)$ is real valued for $t > 0$ and furthermore the Schwartz reflection principle gives that

$$F_{\epsilon}(\overline{\tau}) = \overline{F_{\epsilon}(-\tau)}$$

This symmetric property enables us to only have to consider when $\tau \in \mathcal{D}_1 = \{\tau \in \mathcal{D} : \mathrm{Re}(\tau) < 0\}$. Moreover, this combined with the facts that $\mathrm{Im}(J(\tau)) < 0$ for $\tau \in \mathcal{D}_1$ for τ with sufficiently large imaginary part (this can be observed by looking at its Fourier series) and $J : \mathcal{D} \to \mathbb{C}$ is Hauptmodul we have that $\mathrm{Im}(J(\tau)) < 0$ for all $\tau \in \mathcal{D}_1$. We define

$$L = \{w \in \mathbb{C} : \mathrm{Re}(w) = J(i) = 1/64, \ \mathrm{Im}(w) > 0\}$$

and let $\mathcal{L}$ be the preimage of L under J. Then by the above $\mathcal{L}$ is a smooth path in $\mathcal{D}/\mathcal{D}_1$ from i to 1. Let $\gamma = S\mathcal{L} \cup \mathcal{L}$, a path from -1 to 1 in $\mathcal{D}$. We observe that $|z|$ and $|z|^{-1}$ are bounded on γ, this follows from the fact that γ is a smooth path that avoids the origin. As in Proposition IV.3.0.2 we let $P_\beta(x,z)$ be the polynomial obtained by differentiating the term $e^{i\imath\|x\|^2 z}$, where here β is a multi-index and $|\beta| = n$. Again, as in Proposition IV.3.0.2 we have

$$\begin{aligned} x^\alpha \frac{\partial^k F_\epsilon}{\partial x^\beta}(\tau, x) &= \frac{1}{2}\int_{-1}^{1} K_\epsilon(\tau,z) x^\alpha P_\beta(x,z) e^{i\pi\|x\|^2 z} dz \\ &= \frac{1}{2}\int_{i}^{1} K_\epsilon(\tau,z) x^\alpha \left(P_\beta(x,z)e^{i\pi\|x\|^2 z} + \lambda_\epsilon(-iz)^{-d/2} P_\beta(x, -\frac{1}{z}) e^{i\pi\|x\|^2(-\frac{1}{z})}\right) dz, \end{aligned}$$

where we used our established properties of $K_\epsilon(\tau, z)$. Next, using the fact that $|z|$ is bounded for all $z \in \gamma$ we have that $z^s x^\delta$, with $|\delta| < |\beta|$, satisfies $z^s x^\delta = O(1 + \|x\|^{2n})$. This gives us

$$\begin{aligned} \left| x^\alpha \frac{\partial^k F_\epsilon}{\partial x^\beta}(\tau, x) \right| &= \mathcal{O}\left(\int_{\mathcal{L}} |K_\epsilon(\tau,z) x^\alpha| \left| P_\beta(x,z) e^{i\pi\|x\|^2 z} + \lambda_\epsilon(-iz)^{-d/2} P_\beta(x, -\frac{1}{z}) e^{i\pi\|x\|^2(-\frac{1}{z})} \right| |dz| \right) \\ &= \mathcal{O}\left(\int_{\mathcal{L}} |K_\epsilon(\tau,z)| (1 + \|x\|^{2n+2m}) \left(e^{-\pi\|x\|^2 \mathrm{Im}(z)} + |z|^{-d/2} e^{-\pi\|x\|^2 \mathrm{Im}\left(-\frac{1}{z}\right)} \right) |dz| \right) \end{aligned}$$

Note that since the Gaussian is in the Schwartz class there is a constant $C_{n,m} > 0$ so that

$$(1 + \|x\|^{2n+2m}) e^{-\pi\|x\|^2 \mathrm{Im}(z)} \leq C_{n,m}(1 + \mathrm{Im}(z)^{-m-n})$$

Altogether, this then gives

$$\begin{aligned} \left| x^\alpha \frac{\partial^k F_\epsilon}{\partial x^\beta}(\tau, x) \right| &= \mathcal{O}\left(\int_{\mathcal{L}} |K_\epsilon(\tau,z)| (1 + \mathrm{Im}(z)^{-m-n} + |z|^{-\frac{d}{2}-2} \mathrm{Im}(-\frac{1}{z})^{-m-n}) |dz| \right) \\ &= \mathcal{O}\left(\int_{\mathcal{L}} |K_\epsilon(\tau,z)| (1 + \mathrm{Im}(z)^{-m-n} + |z|^{-\frac{d}{2}-2+2m+2n} \mathrm{Im}(z)^{-m-n}) |dz| \right) \\ &= \mathcal{O}\left(\int_{\mathcal{L}} |K_\epsilon(\tau,z)| (1 + \mathrm{Im}(z)^{-m-n}) |dz| \right) \end{aligned}$$

At this point we can finish with the same estimates as in [8]. $\square$

IV.5 Cocycle Relations

We now define a Γ_θ relation, $\{\phi_A(\tau)\}_{A\in\Gamma_\theta}$, by writing

$$\phi_{T^2}(\tau) = 0$$
$$\phi_S(\tau) = e^{i\pi\tau\|x\|^2} + \lambda_\epsilon(-i\tau)^{-d/2}e^{i\pi(-1/\tau)\|x\|^2}$$

We define $\phi_{AB} = \phi_B + \phi_A|B$, where $|$ refers to the notation $|^{-\epsilon}_{d/2}$ given in Section II.3, and observe that $\phi_S + \phi_S|S = 0$. This implies that for any $A \in \Gamma_\theta$ it uniquely has a representative ϕ_A. Moreover we have from Proposition IV.4.0.2 that

$$F_\epsilon(\tau) - (F_\epsilon|^{-\epsilon}_{d/2}A)(\tau) = \phi_A(\tau). \tag{IV.12}$$

We then have the following lemma from [8]

Lemma IV.5.0.1. *Suppose we have a Γ_θ relation $\{\phi_A\}_{A\in\Gamma_\theta}$ satisfying*

$$\phi_T(\tau) = 0$$
$$|\phi_S(\tau)| \leq |\tau|^\alpha + Im(\tau)^{-\beta}$$

for some $\alpha, \beta \geq 0$. Let $\tau' \in \mathcal{D}$, $A \in \Gamma_\theta$, $\tau = A\tau'$, and suppose $Im(\tau) \leq 1$. then we have that

$$|\phi_A(\tau')| \leq |\tau|^\alpha + Im(\tau)^{-\alpha-1} + 2Im(\tau)^{-\beta-1}$$

We now finally prove our polynomial bound for b_n^ϵ

Proposition IV.5.0.2. *We have $b_n^\epsilon(x) = O(n^{\frac{3d+5}{4}})$*

Proof. First, let $\tau \in \mathbb{H}$ be an arbitrary point satisfying $\mathrm{Im}(\tau) \leq 1$ that doesn't lie on the boundary of $\mathcal{D}$ or any of the elements of its orbit in Γ_θ. Let $\tau = \frac{a\tau'+b}{c\tau'+d}$, where $\tau \in \mathcal{D}$ and

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$. Observe that

$$\begin{aligned} |\phi_S(\tau)| &= |e^{i\pi\tau\|x\|^2} + \lambda_\epsilon(-i\tau)^{-d/2}e^{i\pi(-1/\tau)\|x\|^2}| \\ &\leq 1 + \mathrm{Im}(\tau)^{-d/2}. \end{aligned}$$

Recall that equation (IV.12) gives

$$\chi_{-\epsilon}(A)j_\theta^d(\tau', A)F_\epsilon(\tau) = F_\epsilon(\tau') - \phi_A(\tau').$$

Combining this with Lemma IV.5.0.1 ($\alpha = 0$, $\beta = \frac{d}{2}$) with Propostion IV.4.0.4, give us

$$\begin{aligned} |F_\epsilon(\tau)| &\leq \frac{\mathrm{Im}(\tau')^{d/4}}{\mathrm{Im}(\tau)^{d/4}}|F_\epsilon(\tau')| + \frac{\mathrm{Im}(\tau')^{d/4}}{\mathrm{Im}(\tau)^{d/4}}|\phi_A(\tau')| \\ &\leq C_{0,0}\frac{\mathrm{Im}(\tau')^{1/4} + \mathrm{Im}(\tau')^{(d-2)/4}}{\mathrm{Im}(\tau)^{d/4}} + \mathrm{Im}(\tau')^{d/4}(1 + \mathrm{Im}(\tau)^{(-d-4)/4} + \mathrm{Im}(\tau)^{(-2d-5)/4}), \end{aligned}$$

where $C_{0,0}$ is the constant from Proposition IV.4.0.4. We also used the facts that $j_\theta\left(z, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \zeta(cz+d)^{-1/2}$ for some appropriate eighth root of unity ζ and $|c\tau' + d|^2\mathrm{Im}(\tau) = \mathrm{Im}(\tau')$. If $c = 0$ then we have $\mathrm{Im}(\tau') = \mathrm{Im}(\tau)$ thus giving

$$|F_\epsilon(\tau)| \leq C_{0,0}(\mathrm{Im}(\tau)^{(1-d)/4} + \mathrm{Im}(\tau)^{-1/2}) + \mathrm{Im}(\tau)^{1/4} + \mathrm{Im}(\tau)^{-1} + 2\mathrm{Im}(\tau)^{(-d-5)/4}.$$

In the other case $c > 0$ then we have $\mathrm{Im}(\tau) < \mathrm{Im}(\tau')$ and therefore

$$\mathrm{Im}(\tau)\mathrm{Im}(\tau') = \frac{\mathrm{Im}(\tau')^2}{|c\tau' + d|^2} \leq 1.$$

This gives

$$|F_\epsilon(\tau)| \leq C_{0,0}(\mathrm{Im}(\tau)^{(-d-1)/4} + \mathrm{Im}(\tau)^{(-d+1)/4}) + \mathrm{Im}(\tau)^{-d/4} + \mathrm{Im}(\tau)^{(-d-2)/2} + \mathrm{Im}(\tau)^{(-3d-5)/4}.$$

So Proposition IV.4.0.3 gives

$$b_n^\epsilon(x) = O(n^{\frac{3d+5}{4}}),$$

as claimed. $\square$

IV.6 Summation Formulae

Define the following

$$a_n(x) = \frac{b_n^+(x) + b_n^-(x)}{2}$$

and by construction this gives

$$\widehat{a}_n(x) = \frac{b_n^+(x) - b_n^-(x)}{2}.$$

Polynomial growth of the $b_n^\epsilon(x)$ uniformly in n implies that

$$\sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n=0}^{\infty} \widehat{a}_n(x) \widehat{f}(\sqrt{n}) \tag{IV.13}$$

converges absolutely for all radial Schwartz class functions, here $f(\sqrt{n})$ refers to the value of $f(v)$ at any $v \in \mathbb{R}^d$ such that $\|v\|^2 = n$. We can defne the linear functional ω_x on radial Schwartz functions

$$\omega_x(f) = f(x) - \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) - \sum_{n=0}^{\infty} \widehat{a}_n(x) \widehat{f}(\sqrt{n}). \tag{IV.14}$$

Polynomial growth of the b_n^ϵ again implies that ω_x is a tempered distribution and Proposition IV.4.0.2 implies that (IV.14) is 0 when $f(x) = e^{i\pi\|x\|^2\tau}$ for any $\tau \in \mathbb{H}$ so it must be the case that ω_x vanishes on the linear span of $\left\{e^{i\pi\|x\|^2\tau}\right\}_{\tau\in\mathbb{H}}$. An approximation of the identity argument shows that the space of compactly supported radial smooth functions is dense in the radial Schwartz class functions so it therefore suffices to show that ω_x vanishes for all compactly supported, radial smooth functions.

Proposition IV.6.0.1. *We have that the space of compactly supported radial functions are in the kernel of the tempered distribution given by*

$$\omega_x(f) = f(x) - \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) - \sum_{n=0}^{\infty} \widehat{a}_n(x) \widehat{f}(\sqrt{n})$$

Proof. Suppose f is such a function, define $e_\tau(x) = e^{i\pi\|x\|^2\tau}$, and observe that we can assume that

$$f(x) = F(\|x\|^2) e_i(x)$$

for some $F \in C^\infty(\mathbb{R})$ with compact support. Observe that we have $\widehat{F}$ is a Schwartz class function so we can apply the Fourier inversion formula to give

$$f(x) = F(\|x\|^2)e_i(x) = \int_{-\infty}^{\infty} \widehat{F}(\xi)e_{i+2\xi}(x)d\xi,$$

where here we treat F as a one-dimensional function and considered its one-dimensional transform. If we define

$$p_T = \int_{-T}^{T} \widehat{F}(\xi)e_{i+2\xi}(x)d\xi$$

it's clear by Lebesgue dominated convergence that

$$\|f - p_T\|_{\alpha,\beta} \to 0$$

for all fixed multi-indices α, β defining a seminorm with respect to the Schwartz class as $T \to \infty$. Hence,

$$\omega_x(f - p_T) \to 0,$$

$T \to \infty$. Combining this with the fact that $\omega_x(p_T) = 0$ we have that $\omega_x(f) = 0$ for all compactly supported, radial smooth functions f as desired. $\square$

We now state the above as a theorem

Theorem. *Let $d \in \{2, 3\}$, there exists a collection of radial Schwartz functions $a_n : \mathbb{R}^d \to \mathbb{R}$ with the property that for any radial Schwartz function $f : \mathbb{R}^d \to \mathbb{R}$ and any $x \in \mathbb{R}^d$ we have*

$$f(x) = \sum_{n=0}^{\infty} a_n(x)f(\sqrt{n}) + \sum_{n=0}^{\infty} \hat{a_n}(x)\hat{f}(\sqrt{n}),$$

where the right-hand side converges absolutely.

Let $\mathcal{S}$ denote the vector space of all rapidly decaying sequences of real numbers; i.e, sequences $\{x_n\}_{n \geq .0}$ such that for all $k \geq 0$ we have $n^k x_n \to 0$ as $n \to \infty$ and we can let $\mathbb{S}_R$ denote the space of radial Schwartz class functions on $\mathbb{R}^d$. In the spirit of [8] we can also define the linear functional $L : \mathcal{S} \bigoplus \mathcal{S} \to \mathbb{R}$

$$L(\{x_n\}_{n\geq 0}, \{y_n\}_{n\geq 0}) = \sum_{m\geq 0} v_d(m)x_m - \sum_{m\geq 0} v_d(m)y_m$$

and the linear functional $\Psi : \mathbb{S}_R \to \mathcal{S} \bigoplus \mathcal{S}$ given by

$$\Psi(f) = \left(\left(f(\sqrt{n}) \right)_{n \geq 0} \right) \bigoplus \left(\left(\widehat{f}(\sqrt{n}) \right)_{n \geq 0} \right)$$

From the Poisson Summation formula, valid for all $f \in \mathbb{S}_R$, we have

$$\sum_{v \in \mathbb{Z}^d} f(v) = \sum_{v \in \mathbb{Z}^d} \widehat{f}(v)$$

or equivalently,

$$\sum_{m \geq 0} v_d(m) f(\sqrt{m}) = \sum_{m \geq 0} v_d(m) \widehat{f}(\sqrt{m}),$$

which implies $L \circ \Psi(f) = 0$.

IV.6.1 Remarks

We discuss the relationship between the transforms discussed in Chapter III and Chapter IV. More precisely between

$$V(x) = \sin\left(\frac{\pi \|x\|^2}{2}\right)^2 \int_0^{i\infty} \psi(z) e^{i\pi \|x\|^2 z} dz \tag{IV.15}$$

and

$$f(x) = \frac{1}{2} \int_{-1}^{1} g(z) e^{i\pi \|x\|^2 z} dz \tag{IV.16}$$

for appropriate weakly holomorphic modular forms ψ and g. As discussed in [8] we can obtain an alternate form for f with contour integration shown in Figure IV.1. We can make $T > 0$ as large as we like because g has no poles on $\mathbb{H}$. Cauchy's theorem then gives

$$\begin{aligned} f(x) &= \frac{1}{2} \int_{-1}^{-1+iT} g(z) e^{i\pi \|x\|^2 z} dz + \frac{1}{2} \int_{-1+iT}^{1+iT} g(z) e^{i\pi \|x\|^2 z} dz + \frac{1}{2} \int_{1+iT}^{1} g(z) e^{i\pi \|x\|^2 z} dz \\ &= \frac{i}{2} \int_0^T g(-1+is) e^{i\pi \|x\|^2 (-1+is)} ds + \frac{1}{2} \int_{-1}^{1} g(s+iT) e^{i\pi \|x\|^2 (s+iT)} ds + \frac{i}{2} \int_T^0 g(1+is) e^{i\pi \|x\|^2 (1+is)} ds \\ &= \sin\left(\pi \|x\|^2\right) \int_0^T g(1+is) e^{-\pi \|x\|^2 s} ds + \frac{e^{-\pi \|x\|^2 T}}{2} \int_{-1}^{1} g(s+iT) e^{i\pi s \|x\|^2} ds. \end{aligned}$$

Allowing $T \to \infty$ we see that for all $\|x\|^2$ larger than the order of the pole of g at $i\infty$ we have

$$f(x) = \sin\left(\pi \|x\|^2\right) \int_0^{\infty} g(1+is) e^{-\pi \|x\|^2 s} ds. \tag{IV.17}$$

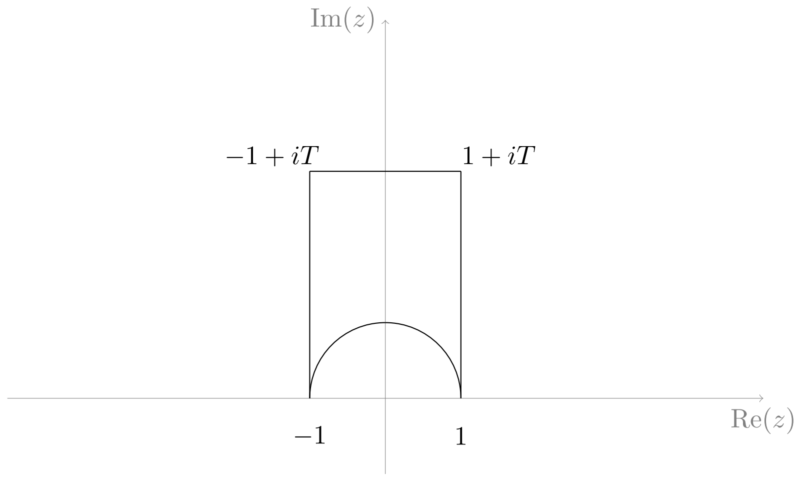

Figure IV.1: Shifting the contour for f

Using the Fourier expansion of g at $i\infty$, we can perform an argument similar to Proposition III.1.0.1 to analytically extend f to an entire function. Similar results exist for real analogues, for example the following proposition.

Proposition IV.6.1.1. *Suppose $g \in C[-1,1]$ and define*

$$f(z) = \int_{-1}^{1} g(t)e^{itz}dt.$$

We then have that

- *f is an entire function*
- *f has infinitely many zeros*

Proof. For the first statement we need to to show holomorphy at any point $z_0 \in \mathbb{C}$. Since g is continuous on the compact set $[-1,1]$, there is a number M such that $|g(x)| \le M$ for all x on

$[-1,1]$. We have

$$\begin{aligned}
\left|\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\right| &= \left|\lim_{h\to 0}\frac{\int_{-1}^{1} g(t)e^{it(z_0+h)}dt - \int_{-1}^{1} g(t)e^{itz_0}dt}{h}\right| \\
&= \left|\lim_{h\to 0}\frac{\int_{-1}^{1} g(t)e^{itz_0}(e^{ith}-1)dt}{h}\right| \\
&\le M\lim_{h\to 0}\int_{-1}^{1} e^{-t\mathrm{Im}(z_0)}\frac{|e^{ith}-1|}{h}dt \\
&= 2M\lim_{h\to 0}\int_{-1}^{1} e^{-t\mathrm{Im}(z_0)}\frac{|\sin(\frac{th}{2})|}{h}dt \\
&= M\int_{-1}^{1} e^{-t\mathrm{Im}(z_0)}|t|dt.
\end{aligned}$$

For z_0 with non-zero imaginary part the last integral is bounded above by

$$M\left(\frac{-e^{-\mathrm{Im}(z_0)}+e^{\mathrm{Im}(z_0)}}{\mathrm{Im}(z_0)}\right),$$

which is finite. When z_0 is real, the last integral is seen to be finite too. With this we conclude f is entire. For the second assertion assume by contradiction that f has finitely many zeros. We then have the following estimate for $z\in\mathbb{C}$ with sufficiently large modulus

$$\begin{aligned}
|f(z)| &\le M\int_{-1}^{1} e^{-t\mathrm{Im}(z)}dt \\
&= M\frac{e^{|\mathrm{Im}(z)|}-e^{-|\mathrm{Im}(z)|}}{|\mathrm{Im}(z)|} \\
&\le Me^{|z|}.
\end{aligned}$$

This implies that f has order at most 1. Combining our assumption that f has finitely many zeros with Hadamard's theorem implies that

$$f(z) = Cz^M e^{p(z)}\prod_{k=1}^{N}\left(1-\frac{z}{a_k}\right) = e^{p(z)}g(z), \tag{IV.18}$$

where C is a non-zero complex number, the a_k are the non-zero zeros of f listed according to multiplicity and $p(z)$ is a polynomial of degree at most 1. Observe also that the Riemann Lebesgue lemma implies that $\lim_{z\to\pm\infty} f(z)=0$. We now consider two cases

Case 1 p is a constant

In this case (IV.18) implies that $\lim_{z\to\pm\infty} f(z)\neq 0$ because $\lim_{z\to\pm\infty}|g(z)|=\infty$ for any non-zero polynomial g.

Case 2 p is linear

Suppose $p(z) = az + b$ for complex numbers a and b. In this case (IV.18) gives

$$\lim_{z\to\pm\infty} |f(z)| = Ce^{\mathrm{Re}(b)} \lim_{z\to\pm\infty} e^{\mathrm{Re}(a)z}|g(z)|. \tag{IV.19}$$

If $\mathrm{Re}(a) > 0$ then (IV.19) gives

$$\lim_{z\to\infty} |f(z)| = \infty,$$

a contradiction. On the other hand, if $\mathrm{Re}(a) < 0$ we similarly have

$$\lim_{z\to-\infty} |f(z)| = \infty,$$

a contradiction.

With both of these cases done, we conclude that f has infinitely many roots as claimed. □

CHAPTER V

Polynomials

In this chapter we present examples of the polynomials in Chapter III. We begin with those mentioned Section III.2.1 first and then those in Section III.3.1.

n	$P_n^{(0)}(w)$	$Q_n^{(0)}(w)$	$R_n^{(0)}(w)$
1	$w - 3528$	1	$w + 1800$
2	$175w^2 - 1840638w - 475793136$	$175w + 497922$	$175w^2 + 2534082w + 111078000$
3	$28028w^3 - 529158959w^2 - 743163984060w - 36431480423520$	$28028w^2 + 313867225w + 64418011860$	$28028w^3 + 1108461025w^2 + 543950742180w + 5541859144800$
4	$1524237w^4 - 42145350931w^3 - 152149668189990w^2 - 44927306881285200w - 786633729801847200$	$1524237w^3 + 39704513165w^2 + 32461802436810w + 1951924212447600$	$1524237w^4 + 118920495725w^3 + 170487912830970w^2 + 22905812156084400w + 88841543950288800$

Table V.1: Choices for $P_n^{(0)}(w), Q_n^{(0)}(w), R_n^{(0)}(w)$

n	$P_n^{(1)}(w)$	$Q_n^{(1)}(w)$	$R_n^{(1)}(w)$
1	$w - 1008$	$w - 1368$	1
2	$25w^2 - 167286w - 10456992$	$25w^2 - 18966w - 41044752$	$25w + 172554$
3	$308w^3 - 4466219w^2 - 3475841460w - 42141677760$	$308w^3 + 1438141w^2 - 3248268900w - 269661057120$	$308w^2 + 7874725w + 1924882860$
4	$401115w^4 - 9290647703w^3 - 22763759548386w^2 - 3803898729081360w - 16560710101091520$	$401115w^4 + 6483236137w^3 - 8811218303346w^2 - 6117821433048720w - 146625660982251360$	$401115w^3 + 22950246697w^2 + 21877018488510w + 1467229443673200$

Table V.2: Choices for $P_n^{(1)}(w), Q_n^{(1)}(w), R_n^{(1)}(w)$

n	$P_n^{(2)}(w)$	$Q_n^{(2)}(w)$	$R_n^{(2)}(w)$
0	1	1	1
1	$w - 5628$	$w + 420$	$w + 4740$
2	$21w^2 - 277373w - 147949620$	$21w^2 + 104155w + 2942940$	$21w^2 + 449395w + 62398380$
3	$6435w^3 - 140254351w^2 - 282318350967w - 30019840201260$	$6435w^3 + 99024689w^2 + 35786965905w + 274637022660$	$6435w^3 + 327184049w^2 + 245151611865w + 8521836402420$
4	$2032316w^4 - 62069814983w^3 - 286418906608260w^2 - 122843458826869680w - 4811816447479782000$	$2032316w^4 + 66185583145w^3 + 74093275280940w^2 + 8042313072870000w + 25260590226128400$	$2032316w^4 + 190929139225w^3 + 355109361032220w^2 + 74992208896198800w + 1031257846302829200$

Table V.3: Choices for $P_n^{(2)}(w), Q_n^{(2)}(w), R_n^{(2)}(w)$

n	$P_n^{(3)}(w)$	$Q_n^{(3)}(w)$	$R_n^{(3)}(w)$
0	1	1	1
1	$w-2548$	$w-1588$	$w+1100$
2	$7w^2-63953w-13216476$	$7w^2+3079w-26138316$	$7w^2+82207w+2838660$
3	$9009w^3-156206287w^2-190598031705w-8023599855180$	$9009w^3+70099793w^2-133001882625w-25965745982460$	$9009w^3+311973425w^2+133133324055w+1154553988500$
4	$18290844w^4-477240504257w^3-1552265260337700w^2-412158967113855600w-6424175460048418800$	$18290844w^4+393037853263w^3-422667127582740w^2-520978390810425360w-2874281910524302680$	$18290844w^4+1294922789215w^3+1681874776577340w^2+202659994747486800w+695925427610595600$

Table V.4: Choices for $P_n^{(3)}(w), Q_n^{(3)}(w), R_n^{(3)}(w)$

n	$P_n^{(4)}(w)$	$Q_n^{(4)}(w)$	$R_n^{(4)}(w)$
0	$w-1728$	1	1
1	$5w^2-39879w-3302208$	$5w+6741$	$5w+44721$
2	$539w^3-8627782w^2-7880390700w-114190352640$	$539w^2+4167770w+396226740$	$539w^2+16031930w+4608398100$
3	$364650w^4-9016810139w^3-24757015920428w^2-4655529290734140w-23107967582918400$	$364650w^3+7413443701w^2+4189697279620w+112988498908740$	$364650w^3+23213582341w^2+24687385132660w+1870654853648580$
4	$151915621w^5-5071000280643w^4-29247957518248095w^3-17244571366685860020w^2-1223043935443094430900w-27752437987409169216000$	$151915621w^4+6062124318525w^3+8919878506072545w^2+1544144937621803100w+17500353569626344300$	$151915621w^4+16932738724605w^3+39664347599006625w^2+12164471420869968300w+37555037971521990030$

Table V.5: Choices for $P_n^{(4)}(w), Q_n^{(4)}(w), R_n^{(4)}(w)$

n	$P_n^{(5)}(w)$	$Q_n^{(5)}(w)$	$R_n^{(5)}(w)$
0	1	$w-864$	1
1	$w-4473$	$w^2-1413w-453600$	$w+3375$
2	$49w^2-575942w-254965620$	$49w^3+109498w^2-330901140w-7628100480$	$49w^2+879610w+100694220$
3	$21450w^3-434056333w^2-770521453516w-71729320315380$	$21450w^4+248877347w^3-404044911676w^2-155029475117940w-1028241012839040$	$21450w^3+968876627w^2+642106262420w+19426107195660$
4	$600457w^4-17413947261w^3-72970303098615w^2-28414747354211820w-1002557443945508100$	$600457w^5+16573766499w^4-12583512934935w^3-29100262763915100w^2-3006413385227590500w-84324443009403168000$	$600457w^4+51599069955w^3+87712885387785w^2+16790085749805300w+207128509258621500$

Table V.6: Choices for $P_n^{(5)}(w), Q_n^{(5)}(w), R_n^{(5)}(w)$

n	$X_n^{(0)}(w)$	$Y_n^{(0)}(w)$	$Z_n^{(0)}(w)$
1	$120(7w+384)$	$63w+171776$	91392
2	$2520(143w^2+84480w+983040)$	$10725w^2+120772096w+23220584448$	$256(169455w+62000384)$
3	$360360(323w^3+548352w^2+82575360w+352321536)$	$1851759w^3+49401907328w^2+38926309097472w+2213781090336768$	$256(56346381w^2+75292332160w+6573528121344)$
4	$15315302185w^4+7383552w^3+3451650048w^2+210386288640w+425201762304)$	$33096195w^4+1644933485408w^3+3009329278672896w^2+738847600435789824w+18179994719864487936$	$512(820050165w^3+2384500618256w^2+838993381785600w+28645861563039744)$
5	$465585120(310155w^5+1748073600w^4+1697439744000w^3+337979113472000w^2+10382718365859840w+11556416963739648)$	$973308260925w^5+78719422877414656w^4+26226312144152808652 8w^3+154187768864184886886400w^2+17019494697241218880897024w+219611165287835282272419840$	$256(71228468185875w^4+361358305202975744w^3+291993337330975309824w^2+426812956399609986641664w+718996664772793320079360)$

Table V.7: Choices for $X_n^{(0)}(w), Y_n^{(0)}(w), Z_n^{(0)}(w)$

n	$X_n^{(1)}(w)$	$Y_n^{(1)}(w)$	$Z_n^{(1)}(w)$
0	0	0	1
1	840	$-840w + 514304$	$63w + 131584$
2	$110880(13w + 3840)$	$-32(45045w^2 - 18526200w - 5341896704)$	$32175w^2 + 371685632w + 39092682752$
3	$180180(323w^2 + 365568w + 27525120)$	$-4(14549535w^3 + 5879887104w^2 - 8326334765056w - 370066358009856)$	$617253w^3 + 18360757472w^2 + 11306427441152w + 324717024116736$
4	$232792560(115w^3 + 291456w^2 + 90832896w + 2768240640)$,	$-16(1673196525w^4 + 3009763070160w^3 - 1421847294355456w^2 - 605234559377473536w - 956513932168239513 6)$	$165480975w^4 + 9518793732992w^3 + 15132309418754048w^2 + 2766389035294261248w + 328466758401408368 64$
5	$10708457760(13485w^4 + 60802560w^3 + 44281036800w^2 + 5877897625600w + 90284507529216)$	$-32(4512611027925w^5 + 17010213344003880w^4 + 9995783626578964 48w^3 - 6562315960254172495872w^2 - 7959440423846479396864 00w - 5957334388285995388764160)$	$583984956555w^5 + 55770750508732928w^4 + 16966065686898448793 6w^3 + 8363147783762030572339 2w^2 + 6681822444597248471859200w + 403674311469670372212736 00$

Table V.8: Choices for $X_n^{(1)}(w), Y_n^{(1)}(w), Z_n^{(1)}(w)$

n	$X_n^{(2)}(w)$	$Y_n^{(2)}(w)$	$Z_n^{(2)}(w)$
0	6144	$5w + 8192$	-1280
1	$53760(3w + 512)$	$33w^2 + 202688w + 117014528$	$-256(33w + 88256)$
2	$14192640(13w^2 + 11648w + 458752)$,	$7017w^3 + 287157760w^2 + 318152900608w + 47113022996480$	$-256(17017w^2 + 177480192w + 36305240064)$
3	$46126080(2261w^3 + 4961280w^2 + 1169817600w + 17616076800)$	$5460315w^4 + 189789291328w^3 + 367295135350784w^2 + 157796929026129920w + 8319769568776028160$	$-256 \times (5460315w^3 + 132052395328w^2 + 107242597384192w + 6452654238597120)$
4	$119189790720(1035w^4 + 4209920w^3 + 2586574848w^2 + 251909898240w + 1842540969984)$	$4159088505w^5 + 251393130327552w^4 + 798055211699077120w^3 + 632914387743732137984w^2 + 120787993383067707244544w + 29097574176311405759692 80$	$-256(4159088505w^4 + 185453507000832w^3 + 344647665809293312w^2 + 87554788870491996160w + 22630209414371601285 12)$

Table V.9: Choices for $X_n^{(2)}(w), Y_n^{(2)}(w), Z_n^{(2)}(w)$

n	$X_n^{(3)}(w)$	$Y_n^{(3)}(w)$	$Z_n^{(3)}(w)$
0	1536	$5w - 9856$	640
1	$215040(3w + 128)$	$231w^2 - 26752w - 1267400704$	$128(231w + 1002752)$
2	$7096320(13w^2 + 6656w + 65536)$	$12155w^3 + 89270912w^2 - 271330017280w - 53238862708736$	$128(12155w^2 + 201748992w + 45519863808)$
3	$92252160(323w^3 + 496128w^2 + 66846720w + 251658240)$	$2028117w^4 + 43705241600w^3 - 83647820742656w^2 - 87515623734640640w - 4918832488186380288$	$128(2028117w^3 + 76877475968w^2 + 68261619335168w + 4356410519322624)$
4	$7449361920(1035w^4 + 3238400w^3 + 1392771072w^2 + 77510737920w + 141733920768)$	$319929885w^5 + 13839620317152w^4 - 10573527793258496w^3 - 5672242019047152025 6w^2 - 147254039470153337405 44w - 3571514025413465226608 64$	$128(319929885w^4 + 22082073232992w^3 + 446115730536939 52w^2 + 1196477923829638758 4w + 3228157113094027345 92)$

Table V.10: Choices for $X_n^{(3)}(w), Y_n^{(3)}(w), Z_n^{(3)}(w)$

n	$X_n^{(4)}(w)$	$Y_n^{(4)}(w)$	$Z_n^{(4)}(w)$
-1	0	$w+768$	256
0	7864320	$-7w^2-14080w-3670016$	$256(7w+8704)$
1	$660602880(11w+3840)$	$-2145w^3-18151424w^2-10765860864w-513701576704$	$256(2145w^2+16504064w+1442906112)$
2	$11808276480(17w^2+21504w+1835008)$	$-29393w^4-631750304w^3-700880797696w^2-170140507308032w-2724018046107648$	$256(29393w^3+609176480w^2+329551011840w+8310294052864)$
3	$802962800640(437w^3+1203840w^2+410910720w+13841203200)$	$-30644625w^5-1284428619904w^4-2602053588762624w^3-1289000609952825344w^2-137186753819160084480w-1003585773474781593600$	$256(30644625w^4+1260893547904w^3+1805118063624192w^2+299979977879715840w+3216411012303421440)$
4	$30512586424320(10005w^4+48222720w^3+37720883200w^2+5407665815552w+90284507529216)$	$-17696513835w^6-1244402571623168w^5-4215555713699020800w^4-3542821624051128074240w^3-904667406337659165999104w^2-48655357241931748717101056w-193311568802774178842279936$	$256(17696513835w^5+1230811648997888w^4+3420612063394922496w^3+1560023852214403465216w^2+115291143987952747544576w+640226338100622957477888)$

Table V.11: Choices for $X_n^{(4)}(w), Y_n^{(4)}(w), Z_n^{(4)}(w)$

n	$X_n^{(5)}(w)$	$Y_n^{(5)}(w)$	$Z_n^{(5)}(w)$
-1	0	$w+1408$	-256
0	55050240	$-35w^2-19456w+89587712$	$256(35w-29824)$
1	$289013760(11w+1536)$	$-429w^3-1764160w^2+1176043520w+2334566383616$	$256(429w^2+1160128w-973922304)$
2	$330631741440(17w^2+13440w+458752)$	$-323323w^4-4435796480w^3-605044539392w^2+7780900527931392w+1333999196264464384$	$256(323323w^3+3980557696w^2-2003804553216w-569750497263616)$
3	$1405184901120\times(437w^3+875520w^2+186777600w+2516582400)$	$-19501125w^5-591920574272w^4-567799302914048w^3+1087844837081743360w^2+822610678048163364864w+4353278907748938940416 0$	$256(19501125w^4+564462990272w^3+91251038625792w^2-299092243346620416w-18730093313481768960)$
4	$213588104970240(10005w^4+37889280w^3+21554790400w^2+1931309219840w+12897786789888)$	$-42977247885w^6-2350682955705856w^5-5039996005091246080w^4+3076582372212857634816w^3+804589088891094963765248w^2+18658759896145791389049815 04w+435765122514249405956934860 80$	$256(42977247885w^5+2290170990683776w^4+2908554618349289472w^3-1901567036589334331392w^2-738869329451114326654976w-188099039688179492428185 60)$

Table V.12: Choices for $X_n^{(5)}(w), Y_n^{(5)}(w), Z_n^{(5)}(w)$

BIBLIOGRAPHY

[1] H. Cohn, N. Elkies, New Upper Bounds on Sphere Packings I, Ann. of Math. (2) 157 (2003), no. 2, 689 - 714. http://annals.math.princeton.edu/wp-content/uploads/annals-v157-n2-p09.pdf.

[2] M.S. Viazovska, The Sphere Packing Problem in Dimension 8, to appear in Annals of Mathematics. https://arxiv.org/pdf/1603.04246.pdf.

[3] H. Cohn, A. Kumar, S. Miller, D. Radchenko, and M.S. Viazovska, The Sphere Packing Problem in Dimension 24. https://arxiv.org/abs/1603.06518.

[4] D. Zaiger, Elliptic Modular Forms and Their Application, In: The 1-2-3 of Modular Forms, (K. Ranestad, ed.) Norway, Springer Universitext, 2008.

[5] F. Diamond, J. Shurman, A First Course in Modular Forms, Springer New York, 2005

[6] H. Cohn, A Conceptual Breakthrough in Sphere Packing, https://arxiv.org/pdf/1611.01685.pdf

[7] D.De Laat, F. Vallentin, A Breakthrough in Sphere Packing: The Search for Magic Functions, https://arxiv.org/pdf/1607.02111.pdf

[8] D. Radchenko, M. Viazovska, Fourier Interpolation on the Real Line, https://arxiv.org/abs/1701.00265, 2017

[9] B.C. Berndt, M.I. Knopp, Hecke's Theory of Modular Forms and Dirichlet Series, World Scientific, 2008

[10] W. Duke, P. Jenkins, On the zeros and coefficients of certain weakly holomorphic modular forms, Pure Appl. Math. Q. 4 pp 1327 1340, 2008

[11] J.H. Conway N.J.A. Sloane, The Optimal Isodual Lattice Quantizer in Three Dimensions, https://arxiv.org/pdf/math/0701080.pdf, 2006

[12] W. Rudin (1974). Real and Complex Analysis. New York: McGraw-Hill.

[13] J. Hanke, Revised Notes from the 2009 Arizona Winter School – "Quadratic Forms and Automorphic Forms", https://arxiv.org/pdf/1105.5759.pdf, 2009

[14] H. Cohn, A. Kumar, S. Miller, D. Radchenko, M. VIazovska, Universal Optimality of the E_8 and Leech Lattices and Interpolation Formulas, https://arxiv.org/abs/1902.05438

[15] H. Cohn, F. Gonçalves, An Optimal Uncertainty Principle in Twelve Dimensions via Modular Forms, https://arxiv.org/abs/1712.04438

[16] H. Chang, L. Wang, A Simple Proof of Thue's Theorem on Circle Packing, https://arxiv.org/pdf/1009.4322.pdf

[17] J. Kepler, Six Cornered Snowflake, https://larouchepac.com/snowflake

[18] T. Hales, et al, A Formal Proof of the Kepler Conjecture, https://arxiv.org/abs/1501.02155

[19] H. Cohn, A. Kumar, Universally optimal distribution of points on spheres, https://arxiv.org/abs/math/0607446

[20] H. Groemer, Existenzs¨atze f¨ur Lagerungen im Euklidischen Raum, Math. Z. 81(1963), 260–278

[21] D. Mumford, Tata Lectures on Theta: Jacobian theta functions and differential equations, Progress in Mathematics, Birkhäuser, 1983