Skein theory of planar algebras and some applications

By

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TABLE OF CONTENTS

Chapter 1

Introduction

Modern subfactor theory was initiated by Jones in [Jon83]. In this paper, Jones proposed the definition of the index for subfactors and showed that the possible index values lie in the set

$$\{4\cos^2(\frac{\pi}{n}) : n \geq 3\} \cup [4, +\infty]$$

Furthermore, for every λ in the above set, there exists a subfactor with index λ. Subfactors can be studied using the index as a measurement of complexity. This led to the classification of subfactors with small index. Following this approach, Haagerup discovered the "exotic" Haagerup subfactor [Haa94]. The best classification has been achieved with up to index 5 and even higher index with other restrictions [JMSa][JMSb][MS12][MPPS12][IJMS12][PT12][LMP].

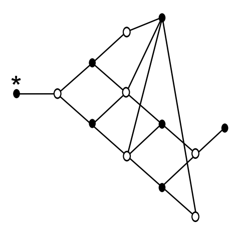
Such classification results are achieved by the analysis of the "standard invariant" of subfactors. A deep theorem of Popa states that the standard invariant completely classifies strongly amenable subfactors [Pop94]. There are three axiomatizations of the standard invariant: Ocneanu's paragroups for finite depth subfactors [Ocn88]; Popa's standard λ-lattice [Pop95]; Jones' subfactor planar algebras [Jon]. In this paper, we focus on Jones' planar algebras.

The planar algebras axiomatize the standard invariant and captures the topological properties. This topological perspective suggests us to study subfactors via their skein theory, which is analogous to the presentation theory for groups via generators and relations. While considering the skein theory as the measurement of complexity, the simplest subfactor planar algebras are Temperley-Lieb-Jones planar algebras. For every subfactor planar algebra, the Temperley-Lieb-Jones algebra (with the same index) sits as a planar subalgebra. The skein theory of Temperley-Lieb-Jones planar algebras involves no generators and no relations.

A planar algebra $\mathscr{P}_\bullet$ consists of graded $*$-algebras $(\mathscr{P}_{n,\pm})$, called the n-box spaces, and skein theory provides sufficient relations to evaluate the partition function of each closed diagram labeled by the generators. With such an evaluation algorithm, the planar algebra is uniquely determined by its skein theory. However, to construct a subfactor planar algebra from a skein theory, there are still two more main difficulties: consistency and positivity. For a diagram in the planar algebra, one might evaluate it in multiple ways. To prove the consistency of a skein theory, one needs to show the evaluations of a diagram in all different ways end up with the same result. Positivity means that each $\mathscr{P}_{n,\pm}$ is a C*-algebra.

From this perspective, one can ask for a classification of subfactor planar algebras via skein theory. Planar algebras generated by a single 1-box were completely analyzed by Jones [Jon]. The classification of subfactor planar algebras generated by a non-trivial 2-box was initiated by Bisch and Jones [BJ97][BJ03][DBL17]. The motivation for such a classification was the example of BWM algebras with the Yang-Baxter relation [BW89][Mur87]. The complete classification of planar algebras singly generated by a non-trivial 2-box with the Yang-Baxter relation was finished in [Liu15]. The Yang-Baxter relation is forced to hold by a certain dimension restriction of the 3-box space. However, when the dimension of the 3-box space reaches 15, the Yang-Baxter relation is an "extra" condition. One natural question is whether one can extend the classification results to planar algebras whose 3-box space is 15 dimensional. The answer to the question is no and we have the following counterexample.

Theorem 1.0.1. *The subgroup subfactor planar algebra $S_2 \times S_3 \subset S_5$ is generated by a single 2-box and the dimension of the 3-box space is 15 without the Yang-Baxter relation. The subfactor has the following principal graph*



The subgroup subfactor is viewed as a "classical" case as opposed to a "quantum" one since the index is an integer. However, very little is known about the planar algebra and in particular, its skein theory. The example above admits a so-called "one-way" Yang-Baxter relation and satisfies the dimension restrictions required. However, this relation is not sufficient to completely determine the whole planar algebra and thus, it is difficult to show that the planar algebra is generated by its 2-box space. We overcome the difficulty by providing another skein theory for this planar algebra.

Following the spirit of such a classification, we also consider the planar algebra generated by a single 3-box. The Jones-Wenzl subfactors from the representation category of the quantum $SU(N)$

with the HOMFLY-PT skein relations are shown to be generated by a single 3-box [Jon]. Later Thurston provided an intrinsic skein theory with respect to the 3-box generator [Thu17], called the Thurston relation. Therefore, we expect a classification of planar algebras generated by a single 3-box with Thurston relation. Consistency forces such planar algebras to be parameterized by two parameters δ and γ. To prove positivity, we embed the planar algebra into the Hecke algebra $H(r,q)$ with r,q determined by δ and γ. From the discussion of the positivity of the Hecke algebra, we deduce the description of the parameter δ and γ such that the planar algebra has positivity. Therefore, we have the following theorem.

Theorem 1.0.2. *Any singly generated Thurston-relation planar algebra is either E_6 or (the semisimple quotient of) $\mathscr{P}_\bullet^H(q,r)$. Moreover, $r = q^N$ for some $N \in \mathbb{N}$, $N \geq 3$, and $q = e^{\frac{i\pi}{N+l}}$ for some $l \in \mathbb{N}$, $l \geq 3$, or $q \geq 1$.*

Subfactors are also well known to have connections with other fields of mathematics. With the help of the planar algebra viewpoint, the connection between subfactor theory and geometric group theory was made through Jones' construction of representations of Thompson group F and T [Jon14]. The Thompson group F is viewed as the direct limit of sets graded by planar binary trees [Jon16]. Each set consists of planar binary trees which have the same number of leaves as the grading tree. Therefore one can construct many actions of Thompson groups. In particular, there are interesting subgroups of Thompson groups constructed by considering the stabilizer of some vectors from the representations constructed, such as $\vec{F}$ and the 3-colorable subgroup $\mathscr{F}$. For the spaces of binary trees with the number of leaves being fixed, vertical isotopy is sufficient to distinguish different elements. Therefore, we prove that the topological relations deduced from this vertical isotopy completely determine the algebraic relations of the subgroups. Specifically, we determine the subgroup $\vec{F}$ [Jon14][GS17] and the 3-colorable subgroup uniformly and can apply the strategy to other possible algebraic structures associated with the Thompson group.

Theorem 1.0.3. *The Jones subgroup $\vec{F}$ is isomorphic to F_3 and the 3-colorable subgroup $\mathscr{F}$ is isomorphic to F_4.*

Chapter 2

Preliminary

We refer the readers to [JS97][Jon] for the definition of subfactors and planar algebras. Here we briefly recall some basic results.

2.1 Subfactors

A von Neumann algebra is $*$-subalgebra of the bounded operators $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ which is closed under the strong operator topology (SOT). A factor M of type II_1 is an infinite dimensional von Neumann algebra with trivial center and admitting a unique faithful normal tracial linear functional τ on M, i.e, $\tau(xy) = \tau(yx)$, $\forall x, y \in M$ and $\tau(1) = 1$. Let $L^2(M, \tau)$ be the standard form of the factor (M, τ), i.e, the GNS construction of M with respect to τ. $L^2(M, \tau)$. We define a M-module $\mathscr{H}_\infty$ as $L^2(M) \otimes \ell^2$.

Theorem 2.1.1 ([JS97]). *If $\mathscr{H}$ is any separable M-module, then there exists a projection $p \in M_\infty(M)$ such that $\mathscr{H} \equiv M_\infty(M)p$, and such a projection p is determined uniquely up to Murray-von Neumann equivalence.*

Definition 2.1.2. *Let $\mathscr{H}$ be an arbitrary separable module over a II_1 factor M with separable pre-dual. Then define*

$$\dim_M \mathscr{H} = Tr(p)$$

where $p \in M_\infty(M)$ is any projection such that $\mathscr{H}_\infty p$ is isomorphic to $\mathscr{H}$ as an M-module.

A subfactor $N \subset M$ is a unital inclusion of II_1 factors. The standard form $L^2(M)$ is viewed as an N-module where the action of N is defined by left multiplication.

Definition 2.1.3. *Let $N \subset M$ is a subfactor of type II_1. We define the index of $N \subset M$ as*

$$[M : N] = \dim_N(L^2(M)).$$

In [Jon83], Jones proved the index theorem as we introduced in §1.

Theorem 2.1.4 (Jones)**.** *Suppose $N \subset M$ is a subfactor of type II_1, then we have*

$$[M:N] \in \{4\cos^2(\frac{\pi}{n}) : n \geq 3\} \cup [4, +\infty]$$

Furthermore, each possible value can be realized as the index of a subfactor.

One of the main structures to reveal the nature of subfactors is obtained by the basic construction as follows: Let e_1 be the orthogonal projection from $L^2(M)$ onto the subspace $L^2(N)$. We construct a von-Neumann algebra $M_1 \subset \mathscr{B}(L^2(M))$ generated by M and e_1. The inclusion $N \subset M \subset M_1$ is called the basic construction of $N \subset M$. One can prove that M_1 is a II_1 factor when the index of $N \subset M$ is finite. Furthermore, we have $[M:N] = [M_1:M]$. Therefore, a sequence of II_1 factors is obtained by iterating the basic construction, called the Jones tower:

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

With the Jones tower, one can construct the standard invariant.

Definition 2.1.5 (The standard invariant)**.** *Let $N \subset M$ be a subfactor of type II_1 with the following Jones tower:*

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

The standard invariant is the relative commutants of the Jones tower with respect to N and M, defined as

$$\begin{array}{ccccccc} N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & \cdots \\ & & \cup & & \cup & & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & \cdots \end{array}$$

The standard invariant has a natural correspondence from the bimodule viewpoint which was fully discussed in [Bis97]. Let X be the N-M-bimodule $L^2(M)$. The homomorphism space of $X \otimes \overline{X} \otimes X \cdots \otimes X$, $X \otimes \overline{X} \otimes X \cdots \otimes \overline{X}$, $\overline{X} \otimes X \otimes \overline{X} \cdots \otimes X$, $\overline{X} \otimes X \otimes \overline{X} \cdots \otimes \overline{X}$ corresponds to $N' \cap M_{2k}$, $N' \cap M_{2k+1}$, $M' \cap M_{2k}$, $M' \cap M_{2k+1}$. This leads to another invariant which describes the structure of the inclusions of the standard invariant called the principal graphs:

Definition 2.1.6 (Principal graphs)**.** *The principal graph of a subfactor $N \subset M$ is a bipartite graph. The vertices are given by the irreducible N-N-bimodules as submodules of $X \otimes \overline{X} \otimes X \cdots X$ and N-M submodules as submodules of $X \otimes \overline{X} \otimes \cdots \overline{X}$. An N-N bimodule Y and an N-M-bimodule are connected in the principal graph if Z is a submodule of $Y \otimes X$ and the number of edges is the multiplicity of Z as submodules in $Y \otimes X$. Similarly, we have the definition of the dual principal graph with respect to $\overline{X} \otimes X \otimes \overline{X} \cdots X$ and $\overline{X} \otimes X \otimes \overline{X} \cdots \overline{X}$.*

2.2 Planar algebras

Planar algebras were introduced by Jones [Jon] as an axiomatization of standard invariants.

Definition 2.2.1 (Planar tangles)**.** *A planar tangle T is determined by the following data:*

(1) A smooth disc D_T in $\mathbb{R}^2$ as the "output disc";

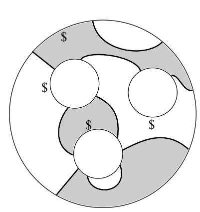
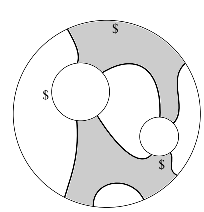
(2) A finite set $\mathfrak{D}_T$ consists of disjoint discs in the interior of D_T as "input discs";

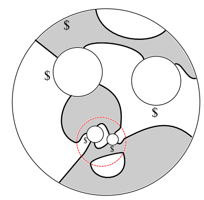
(3) Finitely many disjoint smooth curves which do not meet the interior of input discs inside the output disc. Those strands meet the boundary of discs transversally. The connected components of the output disc by taking out the input discs are called regions.

(4) Each boundary is divided into several intervals (possibly one). There exists a distinguished interval which we put a \$ near it to indicate the position.

Definition 2.2.2 (Composition of tangles)**.** *Suppose T and S are two planar tangles and the boundary condition of one input disc D of T coincides the boundary condition of the output disc of S. Then we define the composition of T and S as the planar tangle constructed by replacing the input disc D of T with the planar tangle S.*

Example 2.2.3.

$T =$  , $S =$  .

$T \circ S =$  .

Definition 2.2.4. *A (shaded) planar algebra is a family of $\mathbb{Z}_2$-graded vector spaces $\mathscr{P}_{n,\pm}$, $n \in \mathbb{N}$, with multilinear maps determined by planar tangles. For a planar tangle T, we have the multilinear map:*

$$Z_T : \otimes_{D \in \mathfrak{D}_T} \mathscr{P}_{\partial D} \to \mathscr{P}_{\partial D_T}$$

For a disc D, $\mathscr{P}_{\partial D}$ is $\mathscr{P}_{n,+}$ where n is the half of number of the boundary points of D and the \$ is in a unshaded region; $\mathscr{P}_{n,-}$ where n is the half of number of the boundary points of D and the \$ is in a shaded region. Furthermore, if two planar tangles are isotopy equivalent, they correspond to the same multilinear map and the multilinear maps are compatible with the composition of planar tangles.

Example 2.2.5. *We illustrate the compatibility with the tangles in Example 2.2.3:*

$$Z_T : \mathscr{P}_{2,+} \otimes \mathscr{P}_{1,+} \otimes \mathscr{P}_{3,-} \to \mathscr{P}_{3,-}$$

$$Z_S : \mathscr{P}_{2,+} \otimes \mathscr{P}_{2,-} \to \mathscr{P}_{3,-}$$

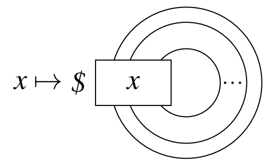
Then we have

$$Z_{T \circ S} = Z_T(1_{\mathscr{P}_{2,+}} \otimes 1_{\mathscr{P}_{1,+}} \otimes Z_S).$$

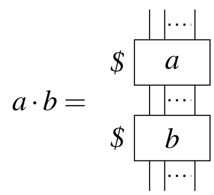
Definition 2.2.6. *A subfactor planar algebra $\mathscr{P}_\bullet$ is a planar algebra satisfying the following properties:*

- $\dim(\mathscr{P}_{0,\pm}) = 1$; $\dim(\mathscr{P}_{n,\pm}) < \infty$;
- *There is an involution $*$ on each vector space $\mathscr{P}_{n,\pm}$ which is compatible with the involution on planar tangles defined by vertical reflection.*

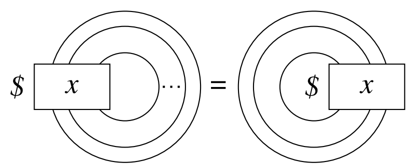
- *Each vector space* $\mathscr{P}_{n,\pm}$ *admits a positive-definite trace as*



Remark . *For each* $n \in \mathbb{N}$, $\mathscr{P}_{n,\pm}$ *is a* C^**-algebra with the algebra structure given by*



Remark . *The planar algebra is said to be spherical if for each* $x \in \mathscr{P}_{n,\pm}$ *we have*



Theorem 2.2.7 (Jones)**.** *The standard invariant of an extremal subfactor of finite index is a spherical subfactor planar algebra. Furthermore, the converse is true.*

Chapter 3

Classification

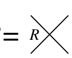
3.1 One-way Yang-Baxter planar algebras

3.1.1 Introduction

The classification of Yang-Baxter planar algebras has been given by a series of papers [BJ97][BJ03] [DBL17][Liu15]. Here we recall some basic results.

Let $\mathscr{P}_\bullet$ be a subfactor planar algebra generated by a single element in $\mathscr{P}_{2,+}$ whose 3-box space is 15 dimensional. This condition implies that the 2-box space has the basis $\{id, e, R\}$, where id, e are the identity and the Jones projection which are Temperley-Lieb diagrams, and R is an element which is linearly independent from $\{id, e\}$. Every element in $\mathscr{P}_\bullet$ can be written as a linear combination of diagrams labelled by R.

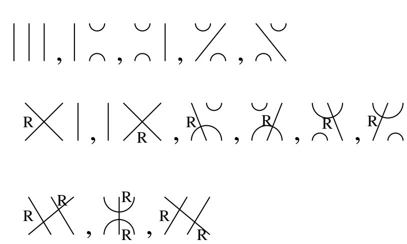
Notation 3.1.1. *For the generator $R \in \mathscr{P}_{2,+}$, we use the following notation*

$R=$ 

where R indicates the position of the \$.

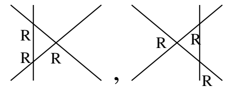
We first explore the basis of the 3-box space and consider the number of generators as the measurement of complexity of diagrams.

We can list the first 14 diagrams:

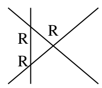
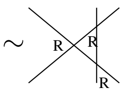


It has been proved in [BJ03] that if these 14 diagrams are linearly dependent, the planar algebra admits the exchange relation [Bis94][Lan02] and such planar algebras have been classified in [BJ03].

The next two diagrams with 3 generators are



Since the dimension of the 3-box space is 15, the above 16 diagrams must be linearly dependent. The Yang-Baxter relation is

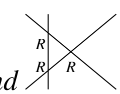
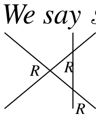
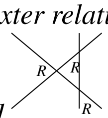
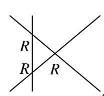
 $\sim$ ,

where $\sim$ means the diagram P on the left hand side is equivalent to the diagram Q on the right hand side modulo lower terms, i.e., P is a linear combination of Q and diagrams with fewer generators. Moreover, the coefficient of Q is nonzero. The Yang-Baxter planar algebras have been completely classified in [BJ97][BJ03][DBL17][Liu15]. Note that it is not sufficient to force the Yang-Baxter relation to hold with the assumption that the 3-box space is 15 dimensional. Therefore, we have the question that whether the classification can be extended to the subfactor planar algebras only with the above dimension conditions. Because of the classification of Yang-Baxter planar algebras, we are left with the so-called One-way Yang-Baxter case.

Definition 3.1.2 (One-Way Yang-Baxter relation)**.** *Suppose $\mathscr{P}_\bullet$ is a planar algebra with the following conditions:*

(1) $\mathscr{P}_\bullet$ is generated by its 2-box space;

(2) $\dim \mathscr{P}_2 = 3$*;* $\dim \mathscr{P}_3 = 15$*;*

We say $\mathscr{P}_\bullet$ admits one-way Yang-Baxter relation if the above 14 diagrams and  *(respectively,* *) forms a basis of $\mathscr{P}_{3,+}$ and*  *(respectively,* *) is a linear combination of the above 14 diagrams.*

Question 1. *Doe there exist a subfactor planar algebra satisfying the One-way Yang-Baxter relation without having Yang-Baxter relation?*

3.1.2 Subgroup subfactors and Spin models

Let M be a II_1 factor and a finite group G with an outer action on M. Let H be a subgroup of G and thus H has the restricted outer action on M. Therefore we have the so-called subgroup subfactor $M \rtimes H \subset M \rtimes G$. We refer to [JS97] for details about subgroup subfactors. The planar algebra of subgroup subfactors has been studied in [Gup08]. However, it is not clear about skein theory for the subgroup subfactor planar algebras. Here, we first realize the subgroup subfactor as a fixed point subalgebra of a spin model planar algebra. This has been introduced in [Jon][Cur03].

Definition 3.1.3 (Spin model planar algebra, [Jon])**.** *Suppose V is a vector space of dimension Q with a basis $\{e_1, e_2, \cdots, e_Q\}$. Let $\mathscr{P}_{0,+}$ be the ground field $\mathbb{C}$ and $\mathscr{P}_{0,-}$ be $End(V)$. For each odd n, let $\mathscr{P}_{n,\pm}$ be subalgebra of $End(V^{\otimes \frac{n+1}{2}})$ given by $End(V^{\otimes \frac{n-1}{2}}) \otimes \Delta$ where Δ is the subalgebra $End(V)$ which consists of linear maps diagonal with respect to the basis. For every n, each element in $\mathscr{P}_{n,\pm}$ is identified with a linear functional on $V^{\otimes n}$. For each even $n > 0$, let $\mathscr{P}_{n,\pm}$ be the algebra $End(V^{\otimes \frac{n}{2}})$. Suppose T is a planar tangle with input discs $D_1, D_2, \cdots, D_m$ and an output disc D_{out}. For a disc with $2k$ boundary points, we number the segments of the boundary attached to black regions cyclically $1, 2, 3, \cdots, k$ starting from the position of the \$ and going clockwisely. Let σ be a function from the black regions to the basis $\{e_1, e_2, \cdots, e_Q\}$. For a disc D with $2k$ boundary points, let σ_D be the vector $e_{\sigma(B_1)} \otimes e_{\sigma(B_2)} \otimes \cdots \otimes e_{\sigma(B_k)}$ for k even; the vector $e_{\sigma(B_1)} \otimes e_{\sigma(B_2)} \otimes \cdots \otimes e_{\sigma(B_{\frac{k-1}{2}})} \otimes e_{\sigma(B_{\frac{k+1}{2}})} \otimes e_{\sigma(B_{\frac{k+1}{2}})} \otimes e_{\sigma(B_{\frac{k+3}{2}})} \otimes \cdots \otimes e_{\sigma(B_k)}$ for k odd, where B_i is the black region which the i-th segment of D is attached to. Thus T gives a multilinear map Z_T from $\otimes \mathscr{P}_{\partial D_k} \to \mathscr{P}_{\partial D}$ by: Given $f_1 \otimes f_2 \otimes \cdots \otimes f_m \in \otimes \mathscr{P}_{\partial D_k}$ and a basic tensor vector $v \in \mathscr{P}_{\partial D}$,*

$$Z_T(f_1 \otimes f_2 \otimes \cdots \otimes f_m)(v) = \sum_{\sigma : \sigma_D = v} \prod f_i(\sigma_{D_i})$$

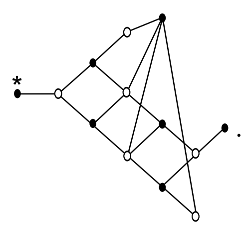
It follows from the definition that the dimension of $\mathscr{P}_{0,-}$ is Q. Let G be a finite group with a transitive action on the basis $\{e_1, e_2, \cdots, e_Q\}$ of V, and thus we obtain an action of G on $\mathscr{P}_\bullet$ since each box space is identified as functionals on $V^{\otimes n}$ for some $n \in \mathbb{N}$. Therefore, we consider the following planar algebra as fixed point algebra of the spin model planar algebra under the group action.

Definition 3.1.4. *Let V be a vector space of dimension Q with a basis $\{e_1, e_2, \cdots, e_Q\}$ and $\mathscr{P}_\bullet$ be* *the spin model planar algebra associated to V. Suppose G is a group with a transitive action α on V and the extended action of G on $\mathscr{P}_\bullet$ is still denoted by α. Let $\mathscr{P}^G_{n,\pm}$ be the fixed point subalgebra of $\mathscr{P}_{n,\pm}$ under the action α. Given a planar tangle T provided with the same planar tangle action as in Definition 3.1.3, we verify that Z_T commutes with the group action α as follows: Given $g \in G$, $f_1 \otimes f_2 \otimes \cdots f_m \in \otimes \mathscr{P}_{\partial D_k}$ and a basic tensor vector $v \in \mathscr{P}_{\partial D}$,*

$$\begin{aligned}
Z_T(f_1 \otimes f_2 \otimes \cdots \otimes f_m)(gv) &= \sum_{\sigma:\sigma_D = gv} \prod f_i(\sigma_{D_i}) \\
&= \sum_{\sigma:\sigma_D = v} \prod f_i(g^{-1}\sigma_{D_i}) \\
&== \sum_{\sigma:\sigma_D = v} \prod f_i(\sigma_{D_i}) \\
&= Z_T(f_1 \otimes f_2 \otimes \cdots \otimes f_m)(v)
\end{aligned}$$

Remark . *The planar algebra $\mathscr{P}^G_\bullet$ satisfies that* $\dim(\mathscr{P}_{0,\pm}) = 1$ *and has a positive-definite trace. It is the group subgroup subfactor planar algebra for $G_v \subset G$, where v is an arbitrary element of V and G_v is the stabilizer of v.*

The candidate of an example of Question 1 is the subgroup subfactor $S_2 \times S_3 \subset S_5$ with the following principal graph:

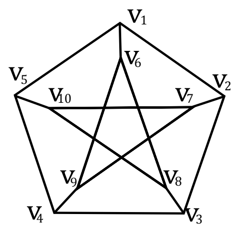


It is a straightforward observation that the dimension of the 2-box space is 3 and the 3-box space is 15, but it is difficult to prove the planar algebra is generated by its 2-box space. First we introduce the subgroup subfactor $S_2 \times S_3 \subset S_5$ in an different way as follows:

Let V be the set $\{u \subset \{1,2,3,4,5\} : \ |u| = 2\}$ and we define an action of the symmetric group S_5 on the set V by for a permutation $\sigma \in S_5$,

$$\sigma(u) = \{\sigma(x) : x \in u\}.$$

It follows directly that S_5 acts on V transitively and for $u = \{1,2\} \in V$, the stablizer is generated by the permutations $(12),(34),(45),(35)$ which is isomorphic to $S_2 \times S_3$. Therefore, the subfactor planar algebra $\mathscr{P}_\bullet$ is the fixed point subalgebra of the spin model planar algebra of V under the action of S_5. We define a graph Γ by: The vertex set is V and two vertices u,v are connected by an edge if and only if $u \cap v = \emptyset$. It is easy to see that the graph Γ is isomorphic to the Petersen graph as follows:



Where the vertices are labelled by $\{V_1, V_2, \cdots, V_{10}\}$ as above.

Lemma 3.1.5. *Suppose σ is a function from the vertices of the Petersen graph to the set V and $\sigma(V_i) \cap \sigma(V_j) = \emptyset$ for any (V_i, V_j) is an edge of the Petersen graph, then σ is a bijection.*

Proof. Note σ is a map between finite sets of the same size and therefore we only need to show that σ is an injection. Suppose σ is not injective, and without loss of generality we assume $\sigma(V_1) = \sigma(V_7) = \{1,2\}$. Note that $(V_1,V_5),(V_5,V_{10}),(V_{10},V_7)$ are edges of the Petersen graph, we have

$$\sigma(V_5) \sqcup \sigma(V_{10}) \sqcup \{1,2\} \subset \{1,2,3,4,5\}$$

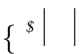
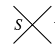
This is a contradiction and therefore σ is a bijection. $\square$

Let W be the vector space with a basis $\{e_u : u \in V\}$, and thus there exists an action of S_5 on W induced by the above action of S_5 on V. Let $\mathscr{P}_\bullet$ be the fixed point algebra of the spin model of W under S_5. By previous discussions, we know that $\mathscr{P}_\bullet$ is the subgroup subfactor planar algebra $S_2 \times S_3 \subset S_5$.

3.1.2.1 Generators

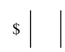
In this section, we give a choice of generators of $\mathscr{P}_\bullet$, which is the subgroup subfactor planar algebra $S_2 \times S_3 \subset S_5$ introduced in the end of §3.1.2. To simplify the notation, we use u to represent the basis vector e_u for $u \in V$ in the following sections.

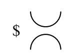
Proposition 3.1.6. *The* 2*-box space* $\mathscr{P}_{2,+}$ *has the following basis:*

$\{$  ,  ,  $\}$,

where S is the characteristic function of the orbit $[u \otimes v]$ *such that* (u, v) *is an edge of the Petersen graph.*

Proof. Note that $\mathscr{P}_{2,+}$ has the basis of $\{\chi_{[v\otimes v]}, \chi_{[v\otimes u]}, \chi_{[v\otimes w]:u,v\in V} : v \neq u, v \cap u = \emptyset, v \cap w \neq \emptyset\}$ and

 $= \chi_{[v\otimes v]}$,

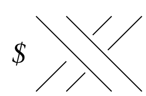
 $= \chi_{[v\otimes v]+[v\otimes u]+[v\otimes w]}$,

 $= \chi_{[v\otimes u]}$.

Therefore, it follows that the proposition is proved. $\square$

Next we introduce an important symmetric brading S in the planar algebra $\mathscr{P}_\bullet$ and $\{S, R\}$ is a generating set of the planar algebra $\mathscr{P}_\bullet$.

Definition 3.1.7. *The element* $R = \chi_{[v\otimes w\otimes w\otimes v]} \in \mathscr{P}_{4,+}$ *is represented as the following diagram*



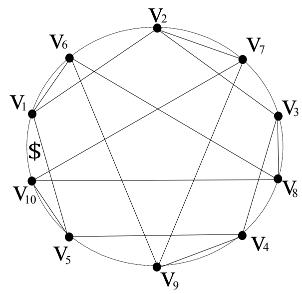
Furthermore, R is a symmetric brading viewed as a linear map from $V \otimes V \to V \otimes V$.

The following theorem has been proved independently by Jones and Curtin [Cur03] for an arbitrary spin model planar algebra. We give another proof of the theorem for the planar algebra $\mathscr{P}_\bullet$ which helps to provide a generating set and a new skein theory as in §3.1.2.2

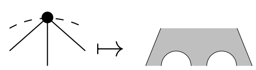
Theorem 3.1.8. *The planar algebra* $\mathscr{P}_\bullet$ *is generated by* $\{S, R\}$.

Proof. Let $\mathscr{A}_\bullet$ be the planar subalgebra generated by $\{S, R\}$.

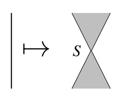
We first draw the Petersen graph in the unit disc of $\mathbb{R}^2$ such that that the vertices are put on the unit circle with equal distances. The unit circle is divided into several intervals by these vertices and we put a \$ near the leftmost interval as illustrated in the following:



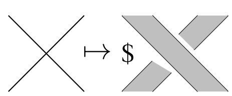
Next we construct an element in $\mathscr{P}_{10,+}$ by the following procedure: Note that the vertices of Petersen graph are put on the unit circle with equal distances. We put two boundary points near the position of each vertex of the Petersen graph. Since the degree of each vertex is 3, we applying the following operation in the neighbourhood of each vertex:



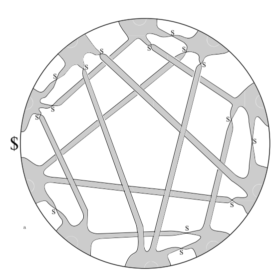
Suppose two vertices are connected by an edge in the Petersen graph, we apply the following operation to the edge:



If two edges of the Petersen graph come across each other, we apply the following operation to the crossing:



Therefore, we have the following element X in $\mathscr{P}_{10,+}$:

$X =$ 

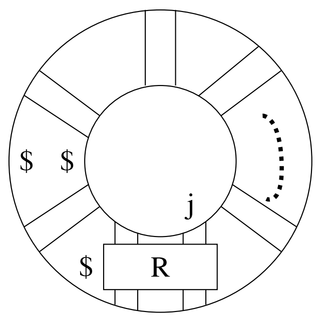
By the construction, the black regions of X which touch the output disc correspond to the vertices of the Petersen graph. Let D_i be the black region which corresponds to the vertex v_i and σ be a function from the black regions of X to the set V. Let σ be a function from the black regions of X to the set V. It follows from the definition of R that σ is uniquely determined by the value $\sigma(D_i)$, $1 \leq i \leq 10$. Therefore, the value $X(\sigma(D_1) \otimes \sigma(D_6) \otimes \sigma(D_2) \otimes \sigma(D_7) \otimes \sigma(D_3) \otimes \sigma(D_8) \otimes \sigma(D_9) \otimes \sigma(D_4) \otimes \sigma(D_9) \otimes \sigma(D_5) \otimes \sigma(D_{10}))$ is nonzero if and only $\sigma(D_i) \cap \sigma(D_j) = \emptyset$ for (v_i, v_j) is an edge of the Petersen graph. By Lemma 3.1.5, we know that σ is an bijection between the black regions D_i,$1 \leq i \leq 10$ and the set V. Therefore, we have

$$X = \chi_{[v_1 \otimes v_6 \otimes v_2 \otimes v_7 \otimes v_3 \otimes v_8 \otimes v_4 \otimes v_9 \otimes v_5 \otimes v_{10}]} \in \mathscr{A}_\bullet,$$

where $v_i \in V$ and $v_i \neq v_j$ for $i \neq j$.

Before we prove the theorem, we first introduce the following annular tangle Φ_j^m in $\mathscr{A}_bullet$:

$$\Phi_j^m =$$

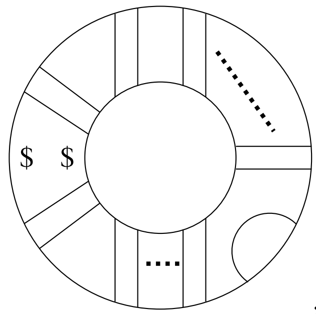


It follows that $\Phi_j^m(\chi_{[v_{i_1} \otimes v_{i_2} \otimes \cdots v_{i_j} \otimes v_{i_{j+1}} \cdots \otimes v_{i_m}]}) = \chi_{[v_{i_1} \otimes v_{i_2} \otimes \cdots v_{i_{j+1}} \otimes v_{i_j} \cdots \otimes v_{i_m}]}$. Therefore, we obtain an action of S_m on $\mathscr{P}_{m,+}$ induced by the planar tangles Φ_j^m, $1 \leq j \leq m$. For a permutation $\sigma \in S_m$, let Φ_σ^m be the corresponding planar tangle.

Let $\sigma \in S_{10}$ be the permutation $(268953)(47)$ and thus we have

$$\chi_[v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_{10}] = \Phi_\sigma^{10}(X) \in \mathscr{A}_\bullet.$$

Now, let $\chi_[v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_m}]$ be the characteristic function of an arbitrary orbit. Since we have an action of S_m on $\mathscr{P}_{m,+}$, one can assume that $j_1 = j_2 = \cdots = j_{k_1} = t_1; j_{k_1+1} = j_{k_1+2} = \cdots = j_{k_2} = t_2; \cdots j_{k_{l-1}+1} = \cdots = j_m = t_l$ and $t_1 < t_2 < \cdots < t_l$ without loss of generality. Next we consider the following annular tangle $\mathscr{E}_j^m$ in $\mathscr{A}_\bullet$:

$\mathscr{E}_j^m =$ 

,

where the j-th boundary point on the output disc is connected to the $(j+1)$-th boundary point. It follows that $\chi_{[v_{j_1}\otimes v_{j_2}\otimes v_{j_3}\otimes\cdots\otimes v_{j_m}]} = \mathscr{E}_2^m(\chi_{[v_{j_1}\otimes v_{j_3}\otimes\cdots\otimes v_{j_m}]})$. To show $\chi_{[v_{j_1}\otimes v_{j_2}\otimes v_{j_3}\cdots\otimes v_{j_m}]} \in \mathscr{A}_\bullet$, we only need to show $\chi_{[v_{j_1}\otimes v_{j_3}\cdots\otimes v_{j_m}]} \in \mathscr{A}_\bullet$. By repeating this procedure, we only need to show

$$\chi_{[j_{t_1}\otimes\cdots\otimes j_{t_l}]} \in \mathscr{A}_\bullet$$

Let $\Psi_{t_1,t_2,\cdots,t_l}$ be an $(2l,20)$-tangle such that the $(2n-1)$-th boundary point of the input disc is connected to the $2n$-th boundary point for $n \neq t_k,\ 1 \leq k \leq l$ and other boundary points are connected to the corresponding boundary point of the output disc. It follows that

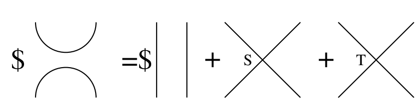
$$\chi_{[j_{t_1}\otimes\cdots\otimes j_{t_l}]} = \Psi_{t_1,t_2,\cdots,t_l}(\chi_{[v_1\otimes v_2\otimes\cdots\otimes v_{10}]}) \in \mathscr{A}_\bullet.$$

Therefore, the planar algebra $\mathscr{P}_\bullet$ equals to the planar subalgebra $\mathscr{A}_\bullet$ which is generated by $\{S,R\}$, i.e., the planar algebra $\mathscr{P}_\bullet$ is generated by $\{S,R\}$. $\square$

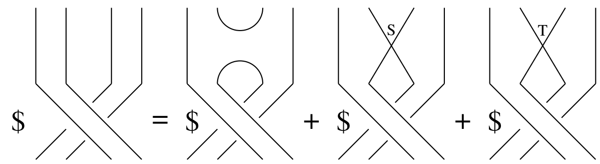
To show that $\mathscr{P}_\bullet$ is generated by its 2-box space, we only need to show that R can be generated by its 2-box space. Let $\mathscr{Q}_\bullet$ be the planar subalgebra of $\mathscr{P}_\bullet$ generated by $\mathscr{P}_{2,+}$.

Theorem 3.1.9. *The subgroup subfactor planar algebra $S_2 \times S_3 \subset S_5$ is generated by its 2-box space.*

Proof. Let $T = \chi_{[v\otimes u]}$, where $v \neq u$ and (v,u) is not an edge of the Petersen graph and thus,

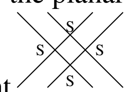


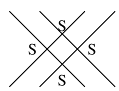
Therefore, we have



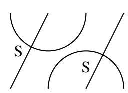
$$ \tag{3.1} $$

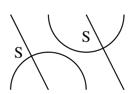
It follows from the definition that the first term of the right hand side of Equation (3.1) is a Temperley-Lieb diagram and hence, it belongs to the planar subalgebra $\mathscr{Q}_\bullet$. Now we first prove that the second term belongs to the planar subalgebra $\mathscr{Q}_\bullet$.

We consider the element  $\in \mathscr{Q}_{4,+}$. Let $[v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}]$ be a non-zero state. Therefore, $(v_{i_j}, v_{i_{j+1}})$ is an edge in the Petersen graph for any $j \in \{1,2,3,4\}$ modulo 4. However, there is no subgraph of the Petersen graph isomorphic to a square. Therefore, we have either $v_{i_1} = v_{i_3}$ or $v_{i_2} = v_{i_4}$, i.e.,

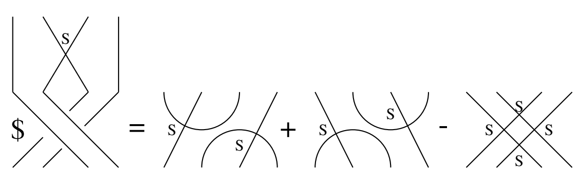
 $= \chi_{[v_1 \otimes v_3 \otimes v_1 \otimes v_3]} + \chi_{[v_1 \otimes v_3 \otimes v_1 \otimes v_9]} + \chi_{[v_3 \otimes v_1 \otimes v_9 \otimes v_1]}.$

Similarly, we have

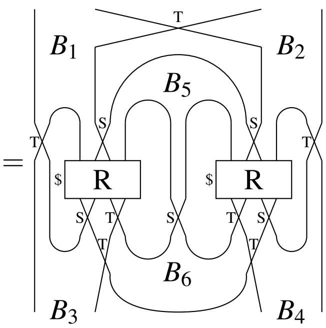
 $= \chi_{[v_1 \otimes v_3 \otimes v_1 \otimes v_3]} + \chi_{[v_3 \otimes v_1 \otimes v_9 \otimes v_1]}$

 $= \chi_{[v_1 \otimes v_3 \otimes v_1 \otimes v_3]} + \chi_{[v_1 \otimes v_3 \otimes v_1 \otimes v_9]}$

Since the second term of the right hand side of Equation (3.1) equals to $\chi_{[v_1 \otimes v_3 \otimes v_1 \otimes v_3]}$, we have

 $\in \mathscr{Q}_\bullet$

Next we will show the third term of the right hand side of Equation (3.1) belongs to the planar subalgebra $\mathscr{Q}$. Since we have shown that the second term of Equation (3.1) is in the planar subalgebra $\mathscr{Q}_\bullet$, we have the following element D also belongs to $\mathscr{Q}_\bullet$.

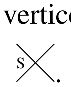
$D =$ 

Let σ be a function from the black regions to the set V. Without loss of generality, we assume that $\sigma(B_1) = \{1,2\}$ and $\sigma(B_2) = \{1,3\}$. By the defintion of S, we know that $\sigma(B_3)$ is disjoint with $\sigma(B_1)$ and $\sigma(B_2)$ and thus $\sigma(B_3) = \{4,5\}$. Similarly, we know that $\sigma(B_5) = \{2,3\}$. Therefore, $\sigma(B_3)$ is disjoint with $\sigma(B_5)$ and different from $\sigma(B_1)$ and $\sigma(B_5)$ and thus it has to be $\{1,3\}$ and symmetrically we obtain that $\sigma(B_4)$ must be $\{1,2\}$. Hence, we have that $\chi_{[v_1 \otimes v_7 \otimes v_1 \otimes v_7]} = D \in \mathscr{Q}_\bullet$.

From Equation (3.1), we know that the element R belongs to the planar subalgebra $\mathscr{Q}_\bullet$. By Theorem 3.1.8, we have the subgroup subfactor planar algebra $S_2 \times S_3 \subset S_5$ is generated by its 2-box space. $\square$

3.1.2.2 An evaluable skein theory

Since the subgroup subfactor $S_2 \times S_3 \subset S_5$ is realized as the fixed point subalgebra of the spin model of the Petersen graph, the definition provides us with a global skein theory:

For a closed diagram D, we construct a planar graph Γ_D: the vertices are the shaded regions of D, and two vertices are connected by an edge if and only if the corresponding shaded regions sharing a crossing 

. A non-zero state σ of the diagram D gives a Petersen coloring of the planar graph Γ_D, which is an injective graph homomorphism from Γ_D to the Petersen graph. The evaluation of D is defined as the number of the Petersen coloring of the planar graph Γ_D. This explanation gives an equivalent description of Petersen coloring conjecture of Jaeger in the language of planar algebras.

Conjecture (F. Jaeger)**.** *Every single diagram in $\mathscr{P}_\bullet$ labelled by R and S is non-zero.*

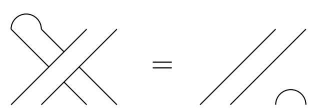
However, it is difficult to compute the number of the Petersen coloring of a planar graph in general. Therefore, we introduce a new skein theory for the planar algebra $\mathscr{P}_\bullet$.

Proposition 3.1.10. *The planar algebra $\mathscr{P}_\bullet$ is generated by $\{R,X\}$.*

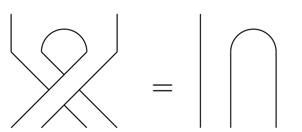
Proof. By Theorem 3.1.8, we know that the planar algebra is generated by $\{S,R\}$. Therefore, we only need to show that S belongs to the planar subalgebra generated by $\{R,X\}$. Let Ψ be the $(4,20)$-annular tangle whose $(2n-1)$-th and $2n$-th boundary point of the input disc are connected a string for every $3 \leq n \leq 10$ and other boundary points of the input disc is connected to the corresponding boundary points of the output disc with through strings. It follows that $S = \Psi(X)$ and thus $\mathscr{P}_\bullet$ is generated by $\{R,X\}$. $\square$

Proposition 3.1.11. *The generators R and X of the planar algebra $\mathscr{P}_\bullet$ satisfy the following relations:*

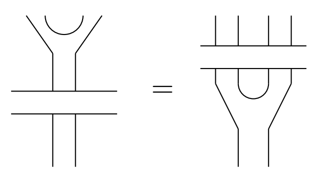
(1) R is a symmetric brading and satisfies the following:



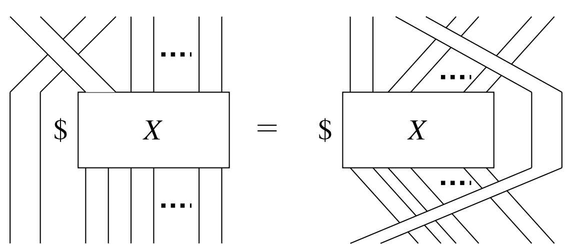
(3.2)



(3.3)

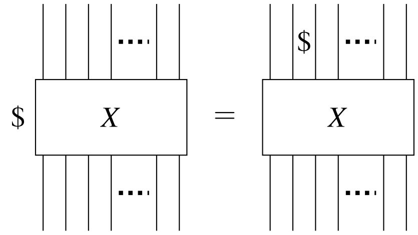


(3.4)

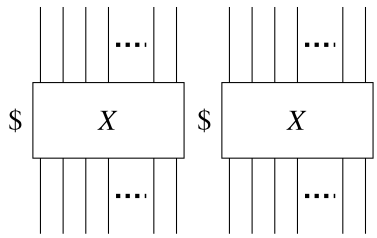


(3.5)

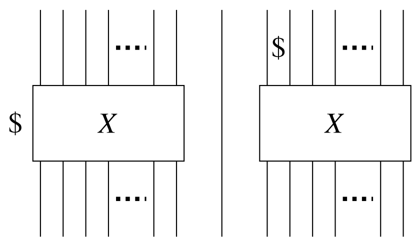
(2) X is invariant under 4*-click rotation, i.e.,*



(3) X satisfies the following relations:

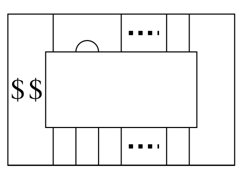


=*lower terms* (3.6)



=*lower terms* (3.7)

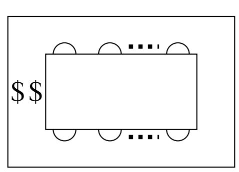
(4) Let Δ_1 *be the annular tangle following tangle:*



(3.8)

Then $\Delta_1 \circ \Phi_\sigma^{10}(X) = 0$ *for any* $\sigma \in S_{10}$*.*

(5) Let Δ_2 *be the following annular tangle:*



(3.9)

Then $\Delta_2 \circ \Phi^{10}_{\sigma}(X) \in \mathbb{C}$ *for any* $\sigma \in S_{10}$.

Proof. It follows from the proof of Theorem 3.1.8. □

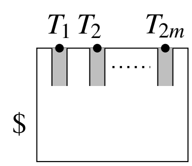
We will give an evaluation algorithm for the planar algebra $\mathscr{P}_\bullet$ using the relations in Proposition 3.1.11 as skein relations. Moreover, the skein theory is evaluable, i.e., one can deduce that $\mathscr{P}_\bullet$ by the skein relations.

For a planar diagram in $\mathscr{P}_\bullet$, we use the number of X's as the complexity of the diagram. First we study the simplest diagrams with repsect to the complexity, i.e., the diagrams labelled only by R.

Lemma 3.1.12. *Let* $y \in \mathscr{P}_{2m,+}$ *be an diagram labelled only by* R *for some* $m \in \mathbb{N}$, *then there exists a permutation* $\alpha \in S_{2m}$ *and an Temperley-Lieb element* $z \in \mathscr{P}_{2m,+}$, *such that*

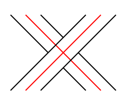
$$y = \Phi^m_\alpha(z).$$

Proof. We consider the diagram y in the following form with the output disc being $[0,1] \times [0,1] \subset \mathbb{R}^2$:



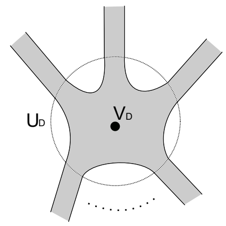
where $T_1, T_2, \cdots, T_{2m}$ be the midpoint of the corresponding intervals on the output disc.

For each label R, we draw two red lines inside the label as follows:



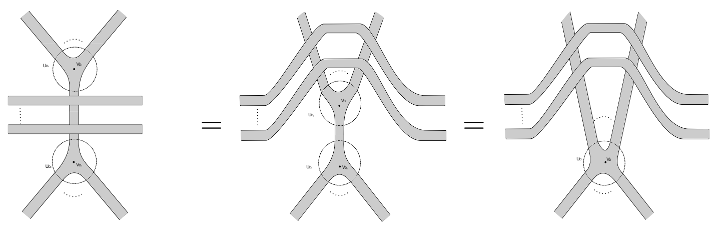
A curve in $\mathbb{R}^2$ is called admissible if every interior point of the curve is either in the interior of some black region of the planar diagram or on the red line as above. We define an equivalence relation on $\{T_1, T_2, \cdots, T_{2m}\}$ as: for $1 \leq i, j \leq 2m$, we say $T_i \sim T_j$ if T_i and T_j can be connected by an admissible curve.

A black region D is said to be effective if the boundary of D has only one or at least three connected segments on the boundary of input discs and output disc. For each effecitve black region D, we pick an arbitrary point v_D in the interior and a neighbourhood U_D of v_D in the interior of the output disc such that the neighbourhoods are disjoint. We reorganize the strands of the diagram of y by isotopy and removing closed strands such that inside each U_D, we have



and there is no  outside the union of U_D's. Note that a diagram with a closed strand equals to a multiple of the diagram after removing the closed strand. Therefore, the operation of removing closed strands does not affect the lemma.

Let D_1 and D_2 be two effective black regions such that v_{D_1} and v_{D_2} are connected by an admissible curve which has only one connected component in some lable R. By Relation (3.4), we have the following:

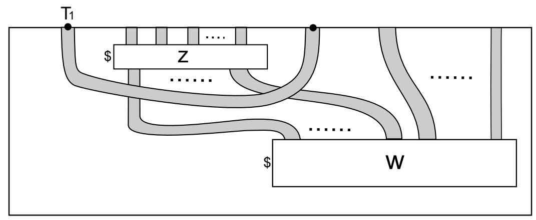


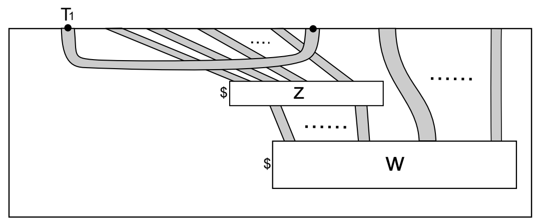
(3.10)

Therefore, we can join D_1 and D_2 to obtain only one effective black region. Note that in this process, the number of effecitve black regions are strictly decrease. Let E be an equivalence class in $\{T_1, T_2, \cdots, T_{2m}\}$ and D_1, D_2, D_k be all the effective black regions such that v_{D_j} can be connected to the mindpoint of some element $T_E \in E$ by an admissible curve. For any $1 \leq i, j \leq k$, we know v_{D_i} and v_{D_j} can be connected by an admissible curve. Therefore we can apply the operation (3.10) such that there is at most one effective black region D_E for the equivalence classe E.

Let E_1 be the equivalence class of T_1. We have three different cases:

(1) There is no effective black region for E_1. Then we must have the following:

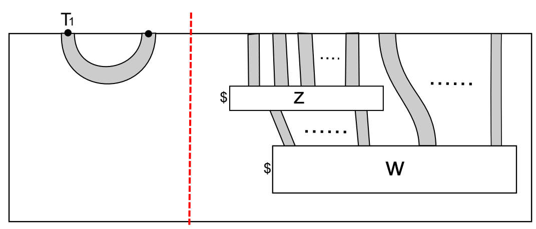
$y = \$$ 

$= \$$ 

.

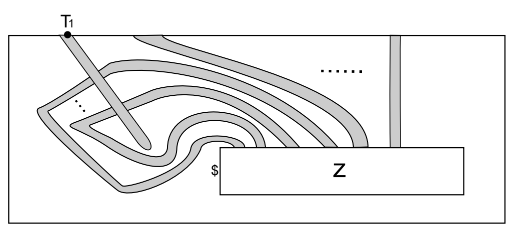
where z and w are planar diagrams labelled only by R.

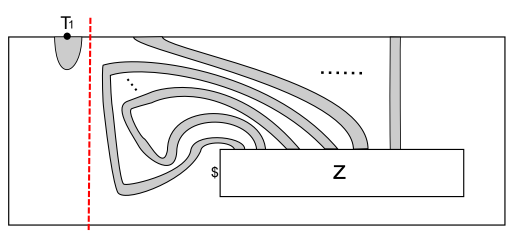
Note that E_1 contains only $\{T_1, T_k\}$ for some k. Let σ be the pemutation $(234\cdots k)$, and then we have

$\Phi_\sigma^{2m}(y) = \$$ 

.

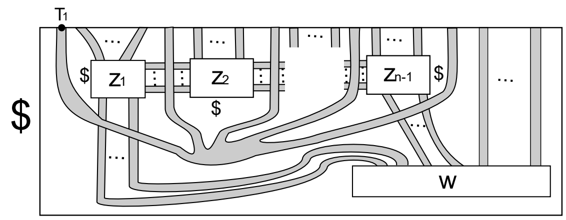
(2) The boundary of the effective black region for E_1 has only one connected component on the boundary of the input discs and output disc. Then we must have the following:

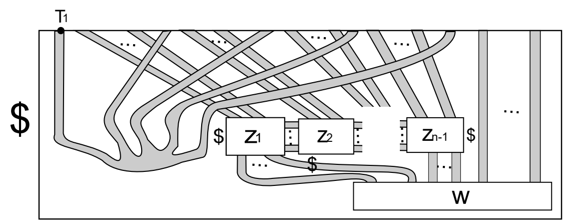
$y = \$$ 

$= \$$ 

where z is a planar diagram labelled only by R.

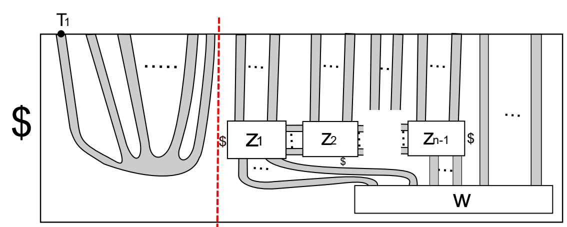
(3) The boundary of the effective black region for E_1 has at least three connected component on the boundary of the input discs and output disc. Then we must have the following:

$y =$ 

$=$  .

where $z_1, z_2, \cdots, z_{n-1}$ and w are planar diagrams labelled only by R.

Suppose $E_1 = \{T_1, T_{k_2}, \cdots, T_{k_n}\}$, let σ be the permuation such that σ is increasing as a function from $\{1,2,\cdots,2m\}/\{1,k_2,\cdots,k_n\}$ to $\{n+1,\cdots,2m\}$ and $\sigma(k_i) = i$ for $2 \leq i \leq n$. Then we have

$\Phi_\sigma^{2m}(y) =$  .

In all cases, there exists a $\sigma \in S_{2m}$ such that the diagram $\Phi_\sigma^{2m}(y)$ can be sepreated into two parts by a vertical dashed red line such that the left part is a Temperley-Lieb diagram and the right part is a diagram only labelled by R in $\mathscr{P}_{2n,+}$ for some $n < m$. Therefore we can repeat the above procedure for another equivalence class in the right part until exhaustion. The resulting diagram is a Temperley-Lieb diagram, denoted by z. Note that in the procedure, we have only applied annular tangles Φ_σ^{2m}, $\sigma \in S_{2m}$ to the diagram y. Therefore, let α be the the inverse of the product of all the permuations and we have

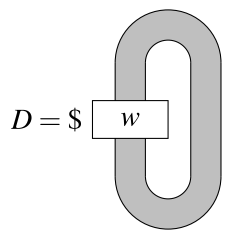
$$y = \Phi_\alpha^{2m}(z).$$

$\square$

Remark . *The lemma follows directly for the planar algebra $\mathscr{P}_\bullet$ from Lemma 2 in [Jon94]. However, to provide a skein theory we need to prove the lemma using only the skein relations in Proposition 3.1.11.*

Now we give the evulation algorithm for the planar algebra $\mathscr{P}_\bullet$: Let D be a closed diagram in $\mathscr{P}_\bullet$. First suppose $D \in \mathscr{P}_{0,+}$.

Step (1): Suppose D has no X, then there exists an element $w \in \mathscr{P}_{2,+}$ such that

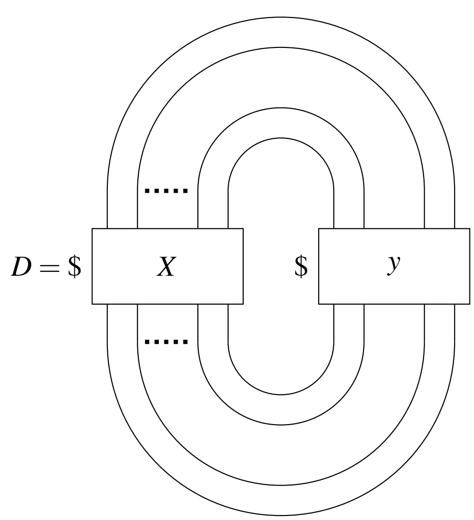


By Lemma 3.1.12, we know that there exists a Temperley-Lieb diagram $z \in \mathscr{P}_{2+}$ and a permutation $\sigma \in S_2$ such that

$$w = \Phi_\sigma^2(z).$$

If $\sigma = (1) \in S_2$, D is a closed Temperley-Lieb diagram; If $\sigma = (12) \in S_2$, we can reduce the diagram D to a closed Temperley-Lieb diagram by applying Type (1) move for the symmetric braiding of R. Therefore, we obtain the evaluation of D.

Step (2): Suppose D has only one X. We can pull the label X to the leftmost by applying Relation (3.5) to obtain D as in the form:

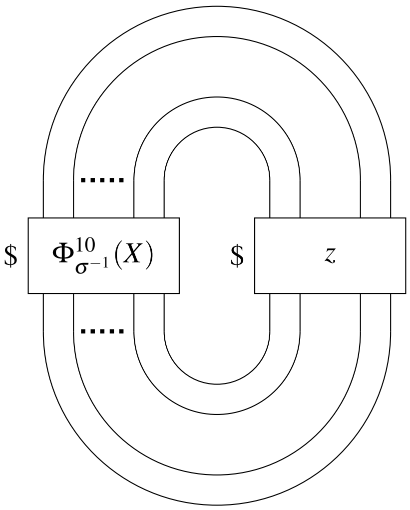


where $y \in \mathscr{P}_{10,+}$ is a diagram labelled only by R.

By Lemma 3.1.12, we know that there exists a permutation $\sigma \in S_{10}$ and a Temperley-Lieb diagram $z \in \mathscr{P}_{10,+}$ such that

$$y = \Phi_{\sigma}^{10}(z).$$

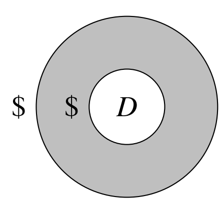
Therefore, we have

$D =$ 

Since z is a Temperley-Lieb diagram, D can be evaluated by Relation (3.8) or (3.9).

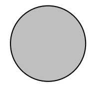
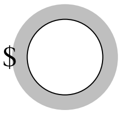
Step (3) Suppose D has more thant one X's. we put two labels of X on the same horizontal level such that they are seperated by straight strands. By applying Relation (3.5), the two labels can be pulled close such that they are either next to each other as in Relation (3.6) or sperated by a single strand as in Relation (3.7). Therefore, the evaluation of D can be reduced to the evaluation of lower terms. By induction on the complexity, we obtain the evaluation of D.

Suppose $D \in \mathscr{P}_{0,-}$, we have



$\in \mathscr{P}_{0,+}$

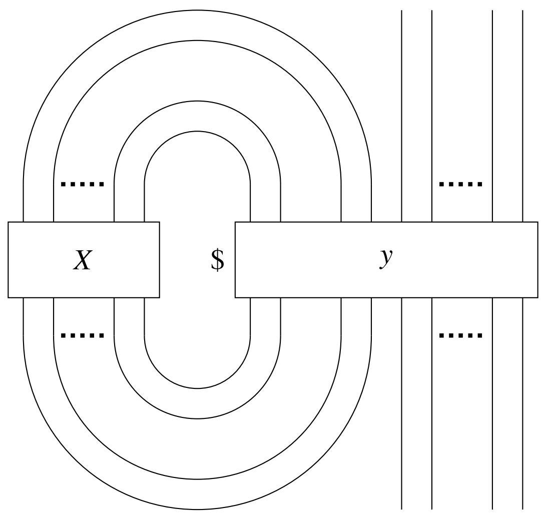
We set the evaluation of D to be the evaluation of the above diagram divided by δ, wehre δ is the circle parameter, i.e.,

$\delta = \$$  $= \$$ 

Proposition 3.1.13. *Suppose $\mathscr{P}_\bullet$ is a planar algebra with above generators and relations, we have that for every $n \in \mathbb{N}$,*

$$\dim(\mathscr{P}_{n,\pm}) < \infty.$$

Proof. First note that by Step (3) of the evaluation algorithm, the space $\mathscr{P}_{2n,+}$ is spanned by diagrams with at most one X. For a diagram $D \in \mathscr{P}_{n,+}$, we can apply Relation (3.5) to obtain D as in the form:

$D = \$$ 

where $y \in \mathscr{P}_{2n+10,+}$ is a diagram labelled only by R.

By Lemma 3.1.12, we know there are only finitely many diagrams lablled only by R in $\mathscr{P}_{2n+10,+}$. Therefore, we have that $\mathscr{P}_{2n,+}$ is finite dimensional for every $n \in \mathbb{N}$. Note that $\mathscr{P}_{2n-1,+}$ and $\mathscr{P}_{n,-}$ can be embeded into $\mathscr{P}_{2n,+}$ for every $n \in \mathbb{N}$ and thereby, the proposition is proved. $\square$

3.2 Thurston-relation planar algebras

3.2.1 Introduction

In this section, we study planar algebras generated by a single 3-box. There is a known example which is a two-parameter family $\{\mathscr{P}_\bullet^H(q,r)\}$ related to quantum $SU(N)$, which has been introduced as Example 2.5 in [Jon]. We determine all q,r for which $\mathscr{P}_\bullet^H(q,r)$ has a positive-definite partition function, (see theorem 3.2.24). Its semisimple quotient is a subfactor planar algebra. The corresponding subfactors are known as Jones-Wenzl subfactors [Jon87, Wen88]. The subfactor planar algebras were constructed by Xu [Xu98]. The skein theory of $\mathscr{P}_\bullet^H(q,r)$ is inherited from the HOMFLY-PT skein relations [FYH$^+$85, PT88]. Thurston provided an intrinsic skein theory which was designed for 6-valent planar graphs [Thu17], which are called Thurston relation.

In this paper, we classify subfactor planar algebras with Thurston relation generated by a non-trivial 3-box, namely singly generated Thurston-relation planar algebras:

Theorem 3.2.1 (Main Theorem)**.** *Any singly generated Thurston-relation planar algebra is either E_6 or (the semisimple quotient of) $\mathscr{P}_\bullet^H(q,r)$. Moreover, $r=q^N$ for some $N\in\mathbb{N}$, $N\geq 3$, and $q=e^{\frac{i\pi}{N+l}}$ for some $l\in\mathbb{N}$, $l\geq 3$, or $q\geq 1$.*

We remark that the appearance of E_6 in our classification is exceptional, since it is not of the form $\mathscr{P}_\bullet^H(q,r)$. It would be interesting to find another skein theory for a single 3-box which gives a family of subfactor planar algebras involving E_6.

The paper is organized as follows. In §3.2.2 , we recall planar algebras, HOMFLY-PT skein theory, and Thurston's skein theory. The m-box space of a singly generated Thurston-relation planar algebra $\mathscr{P}_\bullet$ generically has dimension $m!$.

In §3.2.3 , we classify the generic case, namely $\dim\mathscr{P}_{4,\pm}=24$. We set up five formal variables for Thurston relation using 3-box relations. We prove that only two variables survive after considering 4-box relations (Theorems 3.2.15, 3.2.16). Then we identify the two-parameter family with $\mathscr{P}_\bullet^H(q,r)$ (Theorem 3.2.18). Technically we simplify the computation by working on the reduced planar algebra with respect to the second Jones-Wenzl idempotent f_2. The reduced planar algebra has smaller m-box spaces and their dimensions are $1,0,1,2,9,\cdots$. [1]

[1] One may consider the planar algebra to be generated by two trivalent vertices, but the relations become much more

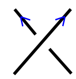
In §3.2.4 , we classify the reduced case, namely $\dim \mathscr{P}_{4,\pm} \leq 23$.

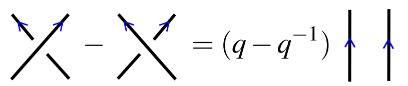
In §3.2.5, we classify all q, r for which $\mathscr{P}^H_\bullet(q,r)$ has a positive Markov trace. In this case, $\mathscr{P}^H_\bullet(q,r)$ has a unique involution $*$, so that its semi-simple quotient is a subfactor planar algebra. Then we complete our classification, Theorem 3.2.1.

3.2.2 Preliminaries

We refer the readers to [Jon] for definition, properties, examples and skein theory of planar algebras.

3.2.2.1 HOMFLY-PT planar algebras

The HOMFLY-PT polynomial is a link invariant given by a braid  satisfying Reidemeister moves I, II, III and the Hecke relation.

Hecke relation:  (3.11)

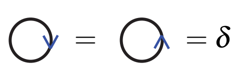
Reidemeister moves I:  (3.12)

 (3.13)

Reidemeister moves II:  (3.14)

 (3.15)

Reidemeister moves III:  (3.16)

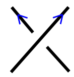
circle parameter:  (3.17)

$$r - r^{-1} = \delta(q - q^{-1}) \tag{3.18}$$

Remark . *When $q = \pm 1$, we have $r = \pm 1$. Then the skein relation is determined by the circle*

complicated. The H-I relation of the trivalent vertices is known as the 6-j symbol. However, the 6-j symbol is only known for the first couple of objects in terms of the formal variables. Our planar algebra may contain infinitely many simple objects. Thus the H-I relation in the reduced planar algebra does not give an evaluation algorithm. Thereby the H-I relation for the other shading (given by Thurston relation) is necessary and important.

parameter δ. When $q \neq \pm 1$, $\delta = \frac{r-r^{-1}}{q-q^{-1}}$.

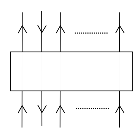
Let σ_i, $i \geq 1$, be the diagram obtained by adding $i-1$ oriented (from bottom to top) through-strings on the left of  . The Hecke algebra of type A is a (unital) filtered algebra $H_\bullet$. The algebra H_n is generated by σ_i, $1 \leq i \leq n-1$ and H_n is identified as a subalgebra of H_{n+1} by adding an oriented through string on the right. Over the field $\mathbb{C}(q,r)$, the equivalence classes of minimal idempotents of H_n are indexed by Young diagrams with n cells. The trace formula is given by

Theorem 3.2.2 ([? ?])**.** *Let λ be a Young diagram and m_λ the minimal idempotent corresponding to λ, then*

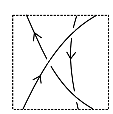
$$Tr(m_\lambda) = \prod_{i,j} \frac{rq^{c(i,j)} - r^{-1}q^{-c(i,j)}}{q^{h_{(i,j)}} - q^{-h_{(i,j)}}} \tag{3.19}$$

where $c(i,j) = j - i$ is the content of the cell (i,j) in λ and $h_{(i,j)}$ is its hook length.

Jones studied the planar algebras $\mathscr{P}^H_\bullet(q,r)$ associated with HOMFLY-PT skein relation [Jon]. Its n-box space consists of HOMFLY-PT diagrams which have $2n$ boundary points and alternating orientation on the boundary as follows:

 .

Moreover, he proved that this planar algebra is generated by a 3-box:

Theorem 3.2.3 (Jones)**.** *The planar algebra $\mathscr{P}^H_\bullet(q,r)$ is generated by*  .

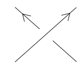
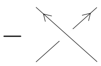
When $r = q^N$ for some $N \in \mathbb{N}$ and $q = e^{\frac{i\pi}{N+l}}$ for some $l \in \mathbb{N}$ or $q \geq 1$, $\mathscr{P}^H_\bullet(q,r)$ admits an involution * such that the Markov trace is positive semidefinite. Therefore, the semisimple quotient of $\mathscr{P}^H_\bullet(q,r)$ is a subfactor planar algebra, which can be constructed from the representation theory of quantum SU(N) [Xu98]. (When $q = 1$, we have $r = 1$ and $\delta = N$.)

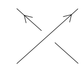
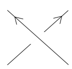
Remark . *When $q = \pm 1, r = \pm 1$, the planar algebra in Theorem 3.2.3 is determined by the circle parameter δ. Therefore, we use the notation $\mathscr{P}^H_\bullet(1,1,\delta)$ for the planar algebra.*

We prove that these are the only possibilities such that $\mathscr{P}^H_\bullet(q,r)$ has positivity in Theorem 3.2.24. First, we identify the isomorphism classes of $\mathscr{P}^H_\bullet(q,r)$:

Proposition 3.2.4. *The four planar algebras $\mathscr{P}^H_\bullet(q,r)$, $\mathscr{P}^H_\bullet(q,-r^{-1})$, $\mathscr{P}^H_\bullet(-q^{-1},r)$, $\mathscr{P}^H_\bullet(-q^{-1},-r^{-1})$, $\mathscr{P}^H_\bullet(-q,r^{-1})$, $\mathscr{P}^H_\bullet(-q,-r)$, $\mathscr{P}^H_\bullet(q^{-1},r^{-1})$, $\mathscr{P}^H_\bullet(q^{-1},-r)$ are isomorphic.*

Proof. Note that (q,r) and $(-q^{-1},r)$ define the same skein relations, hence the planar algebras are isomorphic.

The isomorphism between $\mathscr{P}^H_\bullet(q,r)$ and $\mathscr{P}^H_\bullet(q,-r^{-1})$ is induced by sending  to $-$  .

The isomorphism between $\mathscr{P}^H_\bullet(q,r)$ and $\mathscr{P}^H_\bullet(q^{-1},r^{-1})$ is induced by sending  to  .

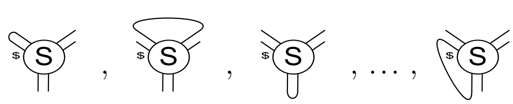
From the isomorphisms, we can show the 8 planar algebras are isomorphic. ☐

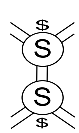
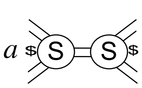
3.2.2.2 Thurston's skein relations

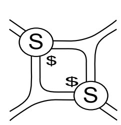
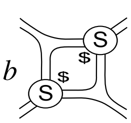
As emphasized in the introduction, skein theory provides an important perspective from which to understand a planar algebra for many reasons. Skein theories are important starting points for the construction and classification of planar algebras. In this paper we will study subfactor planar algebras generated by a 3-box.

Recall that $\mathscr{P}^H_\bullet(q,r)$ is generated by a 3-box. Although it has a skein theory derived from the HOMFLY-PT skein relation, Thurston provides a skein theory intrinsic to the 3-box generator of $\mathscr{P}^H_\bullet(q,r)$ [Thu17]:

Definition 3.2.5 (Thurston relation [Thu17])**.** *We say a 3-box S satisfies Thurston relation if*

$1 \to 0$ *Move:*  $= (lower\ terms)$; (3.20)

Unshaded $2 \leftrightarrow 2$ move:  $= a$  $+ (lower\ terms),\ a \neq 0$; (3.21)

Shaded $2 \leftrightarrow 2$ move:  $= b$  $+ (lower\ terms),\ b \neq 0$, (3.22)

where lower terms are a linear combination of diagrams with less generators.

Remark . *One can generalize Thurston relation for multiple 3-box generators.*

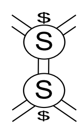
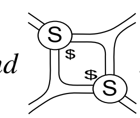
He proved that the above relations are evaluable, in the sense that there exists an algorithm for evaluating every closed planar diagram to a scalar. Moreover, the standard forms in the sense of Thurston [Thu17] give a basis of the n-box space when their dimension achieves the maximum of $n!$.

Theorem 3.2.6 (Thurston, [Thu17])**.** *Suppose $\mathscr{P}_\bullet$ admits Thurston relation. Then $\mathscr{P}_{n,\pm}$ is spanned by the standard forms and* $\dim \mathscr{P}_{n,\pm} \leq n!$.

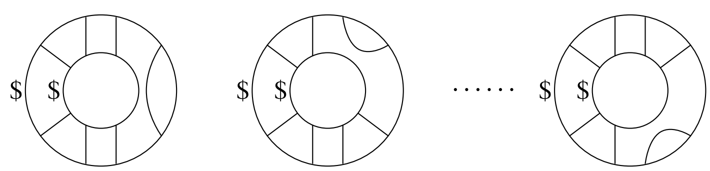
Corollary 3.2.7 (Thurston, [Thu17])**.** *In the generic case, namely $\mathscr{P}_{4,\pm} = 24$, we have the basis of the 4-box space given by the standard form as follows:*

- *14 Temperley-Lieb diagrams;*

- *8 diagrams in the annular consequences, which we denote by AC;*

- *2 diagrams with two generators:*  *and*  *. Moreover, one can replace them by the other diagrams with two generators using Thurston relation* (3.21)*,* (3.22).

Remark . *The annular consequences of the generator S is the 8 diagrams obtained by the applying the annular tangles:*



Thurston relation looks similar to the 6j symbol in a monoidal category, but it provides a much better evaluation algorithm. The 6j symbols in a monoidal category are the coefficients of the change of basis matrix in the hom spaces, and a monoidal category is determined by the 6j symbols up to monoidal equivalence, however it seems hopeless to determine the 6j symbols in general. Actually for the example $\mathscr{P}_\bullet^H(q,r)$ appearing in our classification, it is difficult to compute the 6j symbols. Thurston relation only requires partial 6j symbols for two objects, together with the data for the dual planar algebra. This combination, rather than considering only one side, appears to be powerful, and determines the planar algebra completely.

The main purpose of this paper is to classify all subfactor planar algebras generated by a 3-box S satisfying Thurston relation. We give the classification for the generic case in §3.2.3 and for the reduced case in §3.2.4. Since the subfactor with index at most 4 is classified, we only need to consider the case $\delta^2 > 4$. Actually, E_6 and extended E_6 subfactor planar algebras are generated by a 3-box with Thurston relation. The dimension of their 4-box spaces are 21 and 22 respectively.

3.2.3 Generic case

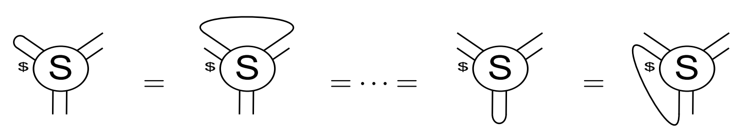
In this section, we classify subfactor planar algebras $\mathscr{P}_\bullet$ generated by a 3-box S with Thurston relation, such that $\mathscr{P}_{4,\pm} = 24$. In this case, we have $\delta > 2$ and $\mathscr{P}_{3,\pm} = 6$. Most results in this section also work for case $\mathscr{P}_{4,\pm} = 23$.

3.2.3.1 Generators

Notation 3.2.8. *Suppose $\mathscr{P}_\bullet$ is a spherical planar algebra. We use the following notations: e_n is the n^{th} Jones projection; f_n is the n^{th} Jones-Wenzl idempotent; and $\mathscr{I}_{n,\pm}$ is the basic construction ideal in $\mathscr{P}_{n,\pm}$.*

Since $\dim(\mathscr{P}_{3,+}) = 6$, there exists two minimal idempotents P and Q in $\mathscr{P}_{3,+}/\mathscr{I}_{3,+}$ and $P+Q = f_3$. Since $tr(f_3) \neq 0$, we assume that $tr(Q) \neq 0$. We take $S = \gamma Q - P$, where $\gamma = \frac{tr(P)}{tr(Q)}$, as the generator for $\mathscr{P}_\bullet$. Then S has the following relations:

1. S is totally uncappable, i.e,

$0 =$  . (3.23)

2. S is an eigenvector of the (2-click) rotation ρ. i.e

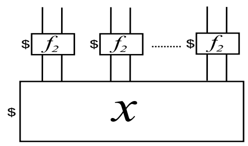
$$\rho(S) = \omega S, \text{ and } \omega^3 = 1. \tag{3.24}$$

3. S satisfies a quadratic relation:

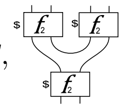
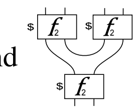
$$S^2 = (\gamma - 1)S + \gamma f_3. \tag{3.25}$$

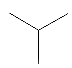
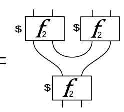
Note that Property (1) is $(1 \to 0)$ move in Thurston relation.

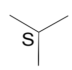
Now we focus on the f_2-cutdown of $\mathscr{P}_{n,+}$ instead of the entire subfactor planar algebra. Technically, this reduces the dimension of the n-box space and simplify the computation. Elements in the n-box cutdown space will be elements $x \in \mathscr{P}_{n,+}$ of the form

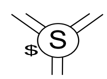


.

Then the f_2-cutdown of $\mathscr{P}_{3,+}$ has a basis $\{S,$  $\}$. We can consider the elements in the f_2-cutdown as morphisms in the $N-N$ bimodule category (the even part of the subfactor planar algebra). In terms of $N-N$ bimodule maps, we rewrite f_2 as a single string labelled by f_2, and we ignore the label if there is no confusion. In this setting, we express S and  as $N-N$ bimodule maps as follows:

 $:=$  , (3.26)

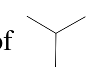
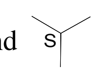


$:=$ 

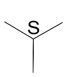
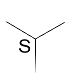
. (3.27)

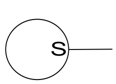
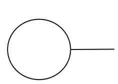
where the position of S indicates the position of the \$ sign.

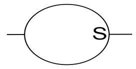
3.2.3.2 Relations in 3-boxes

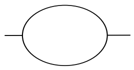
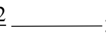
Now let us set up and simplify the formal variables for the relations of  and  in $\mathscr{P}_{3,+}$

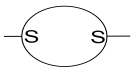
Lemma 3.2.9. *We have the following skein relations in the f_2 cut-down in terms of δ, ω, γ and one new parameter ε:*

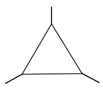
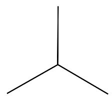
(i)  $= \omega$  ;

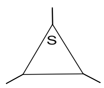
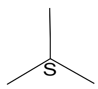
(ii)  $=$  $= 0$;

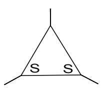
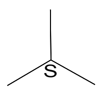
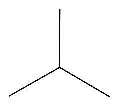
(iii)  $= 0;$

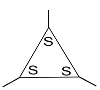
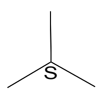
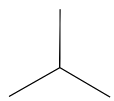
(iv)  $= \frac{\delta^2-2}{\delta}$ ;

(v)  $= \gamma\frac{\delta^3-2\delta}{\delta^2-1}$;

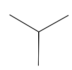
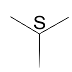
(vi)  $= \frac{\delta^2-3}{\delta}$  ;

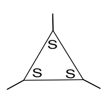
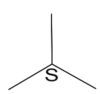
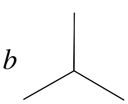
(vii)  $= -\frac{1}{\delta}$  ;

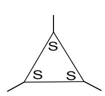
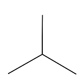
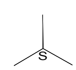
(viii)  $= (\gamma-1)$  $-\gamma\frac{\delta}{\delta^2-1}$  ;

(ix)  $= \delta_{\omega,1}\varepsilon$  $+\gamma(\gamma-1)\frac{\delta^2}{\delta^2-1}$  *, where $\varepsilon \in \mathbb{C}$ is an undetermined variable.*

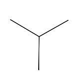
Proof. The relations (i)-(viii) follow from Relations 3.23–3.25 if one consider the relations in the whole planar algebra $\mathscr{P}_\bullet$.

For the relation (ix), Since  and  form an orthogonal basis of the f_2-cutdown of $\mathscr{P}_{3,+}$, we assume that

 $= a$  $+ b$  .

First note that both  and  are invariant under 1-click rotation, while  is invariant under 1-click rotation if and only if $\omega = 1$, thus we can assume that

$$a = \delta_{\omega,1}\varepsilon.$$

For the coefficient b, we multiply  from the bottom of both sides of the equation. Since the f_2-cutdown of $\mathscr{P}_{1,+}$ is one-dimensional, we obtain the following by evaluate both sides with the relation (viii),

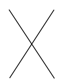
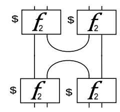
$$b = \gamma(\gamma - 1)\frac{\delta^2}{\delta^2 - 1}$$

□

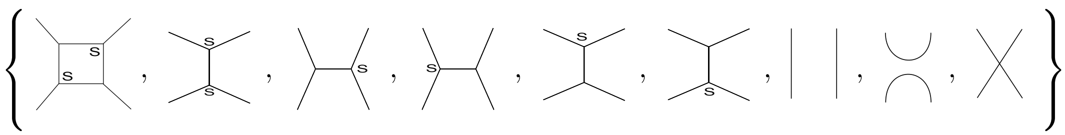
3.2.3.3 Relations in 4-boxes

We proceed to discuss the relations in the f_2-cutdown of $\mathscr{P}_{4,+}$.

Notation 3.2.10. *We define*

 $:=$  .

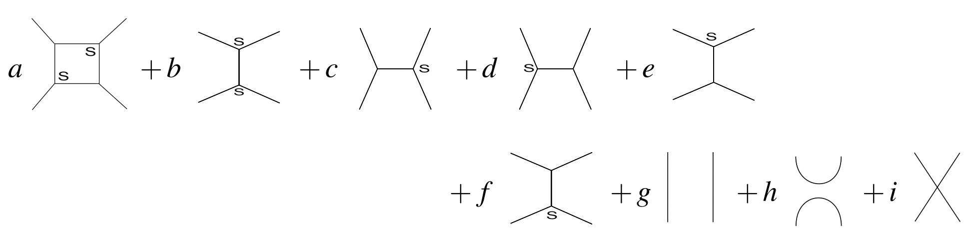
Lemma 3.2.11. *The set B defined as*

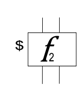
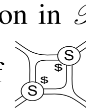
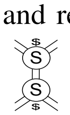
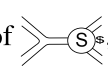
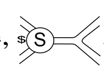
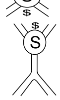
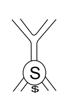
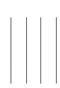
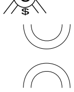
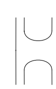


forms a basis of the f_2-cutdown of $\mathscr{P}_{4,+}$.

Proof. Note that the f_2-cutdown of the 24 diagrams in the basis of $\mathscr{P}_{4,+}$ in Corollary 3.2.7 is in the linear span of B, thus f_2-cutdown of $\mathscr{P}_{4,+}$ is spanned by B.

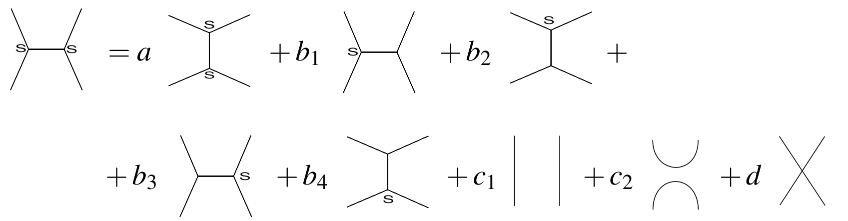
Suppose

 $= 0, \quad (3.28)$

for some $a,b,c,d,e,f,g,h,i \in \mathbb{C}$. Recall that each string in Equation (3.28) represents  in $\mathscr{P}_\bullet$. We consider it as an equation in $\mathscr{P}_{4,+}$ and rewrite it in terms of the basis of $\mathscr{P}_{4,+}$ in Corollary 3.2.7. The coefficients of  and  are a and b respectively, thus $a = b = 0$. Furthermore, the coefficients of , ,  and  are c, d, e and f respectively, thus $c = d = e = f = 0$. Finally, the coefficients of ,  and  are g, h, and i

respectively, thus $g = h = i = 0$. Therefore B is linearly independent, and B is a basis. $\square$

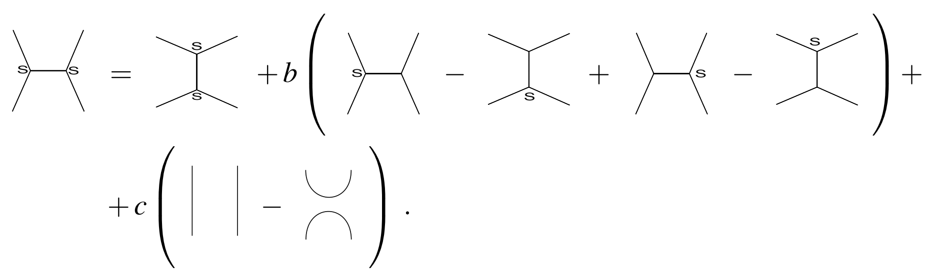
Since B is a basis, we have the unshaded $2 \leftrightarrow 2$ move (3.21) for S in terms of formal variables:



(3.29)

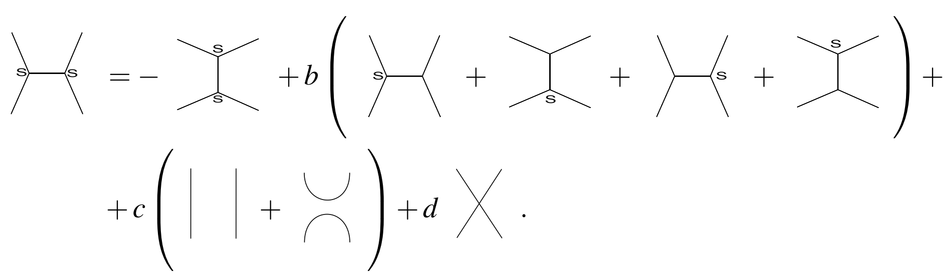
Using the rotational symmetry, we can immediately simplify the formal variables.

Lemma 3.2.12. *Either $a = 1$, and*



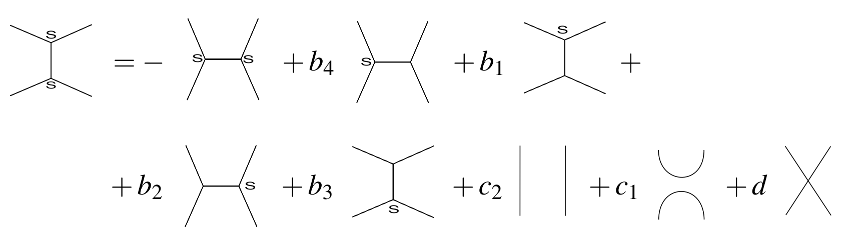
(3.30)

Or $a = -1$, and



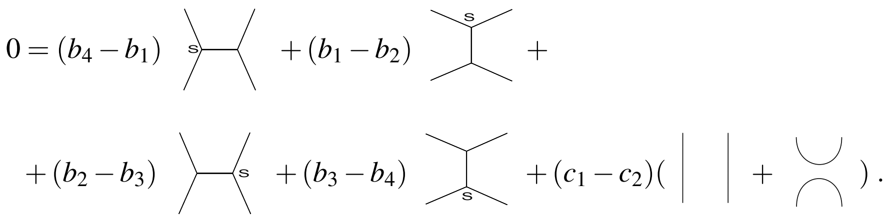
(3.31)

Proof. If $a = -1$, then we apply rotations by $\frac{\pi}{2}$ to Equation (3.29) and obtain



(3.32)

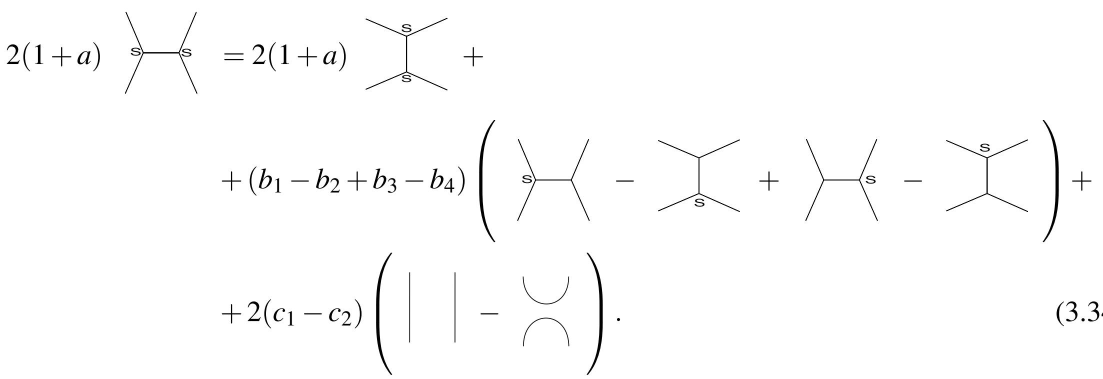
Subtracting Equations (3.29) from (3.32) yields



(3.33)

Thus $b_1 = b_2 = b_3 = b_4$ and $c_1 = c_2$ and we obtain Equation (3.31).

If $a \neq -1$, then we apply rotation by $\frac{k\pi}{2}, k = 0, 1, 2, 3$, to Equation (3.29) and take the alternating sum of the four resultant equations. We have



(3.34)

Taking the quotient by $2(1+a)$, we obtain Equation (3.30). $\square$

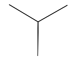
Furthermore, we can determine the coefficients b, c, d in Lemma 3.2.12 in terms of δ, γ, ω.

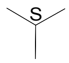
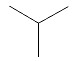
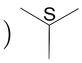
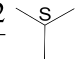
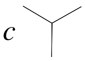
Lemma 3.2.13. *If* $a = 1$*, then*

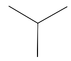
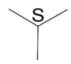
$$\begin{cases} b = -(\gamma - 1)\dfrac{\delta}{\delta^2 - 2 + \omega + \omega^{-1}}\,, \\ c = -\gamma\dfrac{\delta}{\delta^2 - 1}\,. \end{cases} \tag{3.35}$$

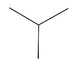
If $a = -1$*, then*

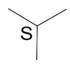
$$\begin{cases} b = \dfrac{(\gamma - 1)\delta}{\delta^2 - 2 - \omega - \omega^{-1}}\,, \\ c = \gamma\delta(2\dfrac{\delta^2 - 2}{\delta^4 - 3\delta^2 + 1} - \dfrac{1}{\delta^2 - 1})\,, \\ d = -2\gamma\dfrac{\delta^2}{\delta^4 - 3\delta^2 + 1}\,. \end{cases} \tag{3.36}$$

Proof. When $a = 1$, applying  to the bottom on both sides of Equation (3.30), we obtain

$(\gamma - 1)$  $-\gamma\dfrac{\delta}{\delta^2 - 1}$  $= b\Big((\omega + \omega^{-1})(-\dfrac{1}{\delta})$  $-\dfrac{\delta^2 - 2}{\delta}$  $\Big) + c$  .

By comparing the coefficient in  and , we solve for b, c as in Eqution (3.35).

When $a = -1$, we apply  to the bottom on both sides of Equation (3.31). By a similar computation, we solve for b, c, d as in Equation (3.36). □

We remark that the variable ε in Lemma 3.2.9 does not appear in the solutions 3.35, 3.36. We can solve for ε in terms of δ, γ, ω by applying  to the bottom of both sides of Equations (3.30), (3.31). We will determine ε later.

Theorem 3.2.14. *Suppose $\mathscr{P}_\bullet$ is a planar algebra generated by a non-trivial 3-box satisfying Thurston relations, then $\mathscr{P}_\bullet$ is determined by $(\delta, \gamma, \omega, a, a')$, where δ^2 is the index, γ the ratio of the trace of the two orthogonal minimal idempotents, ω is the rotation eigenvalue and a, a' are the signs in the unshaded and shaded $(2 \leftrightarrow 2)$ moves.*

Proof. Suppose $\mathscr{P}_\bullet$ and $\mathscr{P}'_\bullet$ are two planar algebras and there exist orthogonal minimal idempotents $P, Q \in \mathscr{P}_\bullet$ and $P', Q' \in \mathscr{P}'_\bullet$. Then we can construct uncappable rotation eigenvectors S and S' as in section 3 satisfying the same quadratic relation. Since the planar algebra is generated by a 3-box, we define a map $\phi : \mathscr{P}_3 \to \mathscr{P}'_3$ by sending S to S'. Since all the coefficients in the skein theory are determined by (δ, γ, ω), ϕ extends to a planar algebra isomorphism. □

With the structure of the Thurston-relation coefficients determined, we are ready to turn to the classification of Thurston-relation planar algebras.

3.2.3.4 Classification

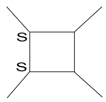
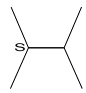
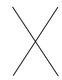
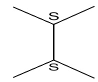
In this section, we prove our classification result (Theorem 3.2.1) for the generic case.

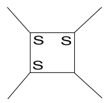
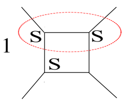
In Theorem 3.2.14, we show that $\mathscr{P}_\bullet$ is determined by $\delta > 2, \gamma, \omega, a$ and a' for the unshaded and shaded $2 \leftrightarrow 2$ moves. First we prove that $\omega = 1$ and $a, a' = 1$ in the $2 \leftrightarrow 2$ move. Thus $\mathscr{P}_\bullet$ is determined by δ, γ. Then we identify $\mathscr{P}_\bullet$ with $\mathscr{P}^H_\bullet(q, r)$.

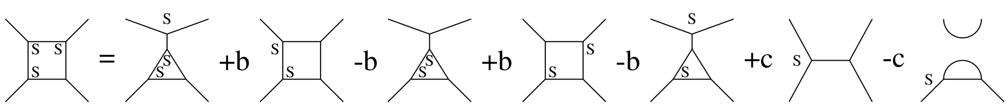
Lemma 3.2.15. *If $\mathscr{P}_\bullet$ is a planar algebra generated by a non-trivial 3-box satisfying Thurston relation with parameters $(\delta, \gamma, \omega, a, a')$, $\delta > 2$, then the rotation eigenvalue $\omega = 1$.*

Proof. Suppose $\mathscr{P}_\bullet$ is a planar algebra with above assumption with $\omega \neq 1$.

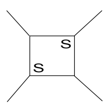
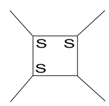
A direct computation using Lemma 3.2.9 shows that in general,

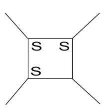
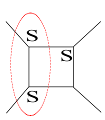
 $= (\gamma - 1)$  $+\gamma$ $- \dfrac{\gamma\delta}{\delta^2 - 1}$  $- \dfrac{1}{\delta}$  (3.37)

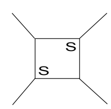
Suppose $a = 1$, notice that  $= \omega\omega^{-1}$ , and apply the $(2 \leftrightarrow 2)$ move in the dotted circle, we can rewrite with respect to the basis in Corollary 3.2.7.



The right hand side can be expressed in terms of the basis by using Lemma 3.2.9 and Relation (3.37).

The coefficient of  in the linear representation of  with respect to this basis is $b\omega^{-1}$.

Similarly  $= \omega\omega^{-1}$  ,

so applying the $(2 \leftrightarrow 2)$ move in the dotted circle, we see that the coefficient of  is $b\omega$.

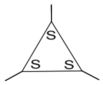
Equating these coefficients from the two different computations yields

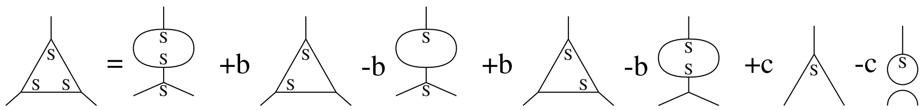
$$b(\omega - \omega^{-1}) = 0.$$

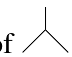
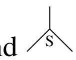
Since $\omega \neq 1$, $\omega - \omega^{-1} \neq 0$, and so

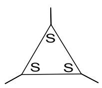
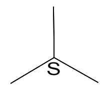
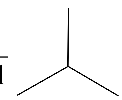
$$b = -\frac{(\gamma - 1)\delta}{\delta^2 - 3} = 0,$$

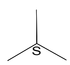
hence $\gamma = 1$.

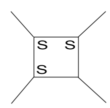
Note that by  is an element in the f_2-cutdown of $\mathscr{P}_{3,+}$, we apply the $(2 \leftrightarrow 2)$ move of Thurston relation and thus obtain

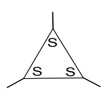


Each term on the right hand side can be expressed in terms of  and  by applying the relations in Lemma 3.2.9 and thus obtain


$$= \left(\frac{\gamma\delta}{\delta^2 - 1}(\delta^2 - 3) - 2(\gamma - 1)^2 \frac{\delta}{\delta^2 - 3} \right)$$

$$+ \gamma(\gamma - 1)\frac{\delta}{\delta^2 - 1}$$


Since the coefficient before  is 0 with $\omega \neq 1$, we obtain $\frac{\delta(\delta^2-3)}{\delta^2-1} = 0$, which contradicts the fact that $\delta > 2$.

Now suppose $a = -1$. Again, we can evaluate  as a sum of basis diagrams in two different ways as above, which will again give us that $\gamma = 1$.

Similarly by checking , we obtain

$$\frac{\delta^2 - 2}{\delta^2 - 1} = 2\frac{\delta^2 - 2}{\delta^4 - 3\delta^2 + 1} - \frac{1}{\delta^2 - 1},$$

which yields the equation

$$1 = 2\frac{\delta^2 - 2}{\delta^4 - 3\delta^2 + 1}.$$

This has no real solutions with $\delta > 2$.

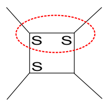
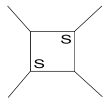
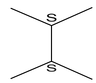
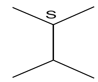
This concludes the proof of the theorem.

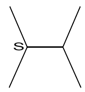
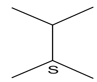
□

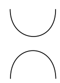
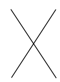
Theorem 3.2.16. *If $\mathscr{P}_\bullet$ is a planar algebra generated by a non-trivial 3-box satisfying Thurston* *relation with parameters* $(\delta,\gamma,1,a,a')$, $\delta>2$, *then* $a=a'=1$.

Proof. Suppose $a=-1$. First note that by 3.2.15, we have $\omega=1$.

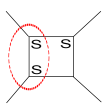
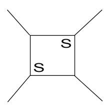
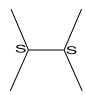
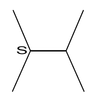
Applying $(2\leftrightarrow 2)$ move in the dotted circle, we obtain the relation

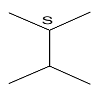
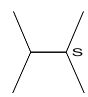
 $=b$  $+\left(-(\gamma-1)+b(-\frac{2}{\delta})\right)$  $+\frac{\gamma\delta}{\delta^2-1}$ 

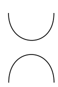
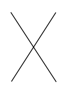
$+(b(\gamma-1)+c)$  $+\left(b(\gamma-1)+d(-\frac{1}{\delta})\right)$ 

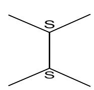
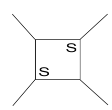
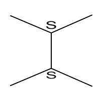
$+b\gamma$ $+\frac{b\gamma}{\delta^2-1}$  $+2b(-\gamma\frac{\delta}{\delta^2-1})$ 

Applying $(2\leftrightarrow 2)$ move in the dotted circle, we obtain

 $=b$  $+\left(-(\gamma-1)+b(-\frac{2}{\delta})\right)$  $+\frac{\gamma\delta}{\delta^2-1}$ 

$+(b(\gamma-1)+c)$  $+\left(b(\gamma-1)+d(-\frac{1}{\delta})\right)$ 

$+b\gamma$  $+\frac{b\gamma}{\delta^2-1}$ $+2b(-\gamma\frac{\delta}{\delta^2-1})$  .

Let λ be the coefficient of  in the linear expansion of  with respect to basis. (Note that if $\dim\mathscr{P}_{4,+}=24$, then $\lambda=0$. We do not use this assumption, so that the proof also work for the case $\dim\mathscr{P}_{4,+}=23$.) Equating the coefficients of  in the two expressions yields

$$b\lambda-(\gamma-1)+b(-\frac{2}{\delta})=b\lambda+(\gamma-1)+b(\frac{2}{\delta}),$$

thus

$$-(\gamma-1)+b(-\frac{2}{\delta})=0.$$

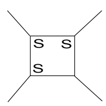
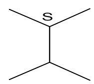
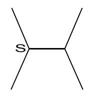
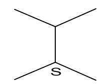
Note that $b = \frac{(\gamma-1)\delta}{\delta^2-4}$, when $\omega = 1$, so

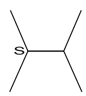
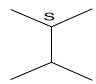
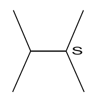
$$(\gamma-1)\left(-1-\frac{\delta}{\delta^2-4}\frac{2}{\delta}\right)=0$$

hence

$$(\gamma-1)\left(\frac{\delta^2-2}{\delta^2-4}\right)=0$$

$\delta > 2$ implies that $\gamma = 1$ and thus $b = 0$. Therefore,

 $= \frac{\delta}{\delta^2-1}$  $+c$  $+d(-\frac{1}{\delta})$ 

$= \frac{\delta}{\delta^2-1}$  $+c$  $+d(-\frac{1}{\delta})$ 

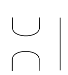
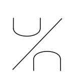
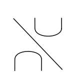
Hence, $d = -2\gamma\frac{\delta^2}{\delta^4-3\delta^2+1} = 0$. Note that this contradicts $\gamma = 1$. □

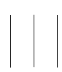
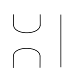
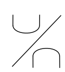
These two theorems show that a singly genrated Thurston-relation planar algebra with parameters $(\delta,\gamma,\omega,a,a')$ satisfies $\omega = a = a' = 1$. Thus the planar algebra is parameterized by (δ,γ). With the following lemma and theorem, we will identify any Thurston-relation planar algebra as $\mathscr{P}^H_\bullet(q,r)$ for some (q,r).

Lemma 3.2.17. *There exist two idempotents $P,Q \in \mathscr{P}^H(q,r)_{3,+}$ with $P+Q = f_3$ and*

$$tr(P) = \frac{(r-r^{-1})(rq-r^{-1}q^{-1})(rq^{-2}-r^{-1}q^2)}{(q+q^{-1})(q-q^{-1})^3} \tag{3.38}$$

$$tr(Q) = \frac{(r-r^{-1})(rq^2-r^{-1}q^{-2})(rq^{-1}-r^{-1}q)}{(q+q^{-1})(q-q^{-1})^3} \tag{3.39}$$

Proof. Consider the basis $\{$ , ,  $\}$ of $\mathscr{P}^H(q,r)_{3,+}$.

First set $Q = aR + b$  $+c\Big($  $+$  $\Big) + e$  $+f$  . Since we want $Q \in \mathscr{I}(\mathscr{P}^H(q,r)_{3,+})$, comparing the coefficients with respect to $\mathscr{B}$ of $e_kQ = Qe_k = 0$ for $k = 1,2$ we obtain the following equations:

$$\begin{cases} ar + c + e\dfrac{r-r^{-1}}{q-q^{-1}} = 0 \\ b + c\dfrac{r-r^{-1}}{q-q^{-1}} + f = 0 \\ ar^{-1} + c + f\dfrac{r-r^{-1}}{q-q^{-1}} = 0 \\ a(q-q^{-1}) + b + c\dfrac{r-r^{-1}}{q-q^{-1}} + e = 0 \end{cases} \tag{3.40}$$

Which have solutions in terms of a, b, r and q:

$$\begin{cases} c = a\dfrac{r^{-1}(q-q^{-1})^2}{r^2+r^{-2}-q^2-q^{-2}} - b\dfrac{(r-r^{-1})(q-q^{-1})}{r^2+r^{-2}-q^2-q^{-2}} \\ e = -\dfrac{q-q^{-1}}{r-r^{-1}}\left(ar + ar^{-1}\dfrac{(q-q^{-1})^2}{r^2+r^{-2}-q^2-q^{-2}} - b\dfrac{(r-r^{-1})(q-q^{-1})}{r^2+r^{-2}-q^2-q^{-2}}\right) \\ f = -\dfrac{q-q^{-1}}{r-r^{-1}}\left(ar^{-1}\dfrac{(r-r^{-1})^2}{r^2+r^{-2}-q^2-q^{-2}} + b\dfrac{(r-r^{-1})(q-q^{-1})}{r^2+r^{-2}-q^2-q^{-2}}\right) \end{cases} \tag{3.41}$$

We want Q to be an idempotent, so we set up equations by comparing the coefficients with respect $\mathscr{B}$ of Q^2 and Q and we find two possible solutions:

$$\begin{cases} a = \dfrac{1}{q+q^{-1}},\ b = \dfrac{q^{-1}}{q+q^{-1}} \quad (a) \\ a = -\dfrac{1}{q+q^{-1}},\ b = \dfrac{q}{q+q^{-1}} \quad (b) \end{cases} \tag{3.42}$$

Taking (a), we have

$$
\begin{aligned}
tr(Q) =& ar(\frac{r-r^{-1}}{q-q^{-1}})^2 + b(\frac{r-r^{-1}}{q-q^{-1}})^3 + c(2(\frac{r-r^{-1}}{q-q^{-1}})^2) + (e+f)\frac{r-r^{-1}}{q-q^{-1}} \\
=& \frac{r}{q+q^{-1}}(\frac{r-r^{-1}}{q-q^{-1}})^2 + \frac{q^{-1}}{q+q^{-1}}(\frac{r-r^{-1}}{q-q^{-1}})^3 \\
&+ \frac{q-q^{-1}}{q+q^{-1}}\frac{r^{-1}q-r^{-1}q}{r^2+r^{-2}-q^2-q^{-2}}(2(\frac{r-r^{-1}}{q-q^{-1}})^2) \\
&+ (-\frac{r+r^{-1}}{q+q^{-1}} - 2\frac{q-q^{-1}}{q+q^{-1}}\frac{r^{-1}q-r^{-1}q}{r^2+r^{-2}-q^2-q^{-2}}) \\
=& \frac{r}{q+q^{-1}}(\frac{r-r^{-1}}{q-q^{-1}})^2 + \frac{q^{-1}}{q+q^{-1}}(\frac{r-r^{-1}}{q-q^{-1}})^3 \\
&- \frac{r+r^{-1}}{q+q^{-1}} + 2\frac{q-q^{-1}}{q+q^{-1}}\frac{r^{-1}q-r^{-1}q}{r^2+r^{-2}-q^2-q^{-2}}((\frac{r-r^{-1}}{q-q^{-1}})^2-1) \\
=& \frac{1}{(q+q^{-1})(q-q^{-1})^3}(r(r-r^{-1})^2(q-q^{-1}) + q^{-1}(q-q^{-1})^3 \\
&- (r+r^{-1})(q-q^{-1})^3 + 2(r^{-1}q-rq^{-1})(q-q^{-1})^2) \\
=& \frac{(r-r^{-1})(r^2q+r^{-2}q^{-1}-q^3-q^{-3})}{(q^2-q^{-2})(q-q^{-1})} \\
=& \frac{(r-r^{-1})(rq^2-r^{-1}q^{-2})(rq^{-1}-r^{-1}q)}{(q+q^{-1})(q-q^{-1})^3}
\end{aligned}
\tag{3.43}
$$

Applying a similar computation, we see that $P = f_3 - Q$ is an idempotent with

$$
tr(P) = \frac{(r-r^{-1})(rq-r^{-1}q^{-1})(rq^{-2}-r^{-1}q^2)}{(q+q^{-1})(q-q^{-1})^3}
$$

We see that taking the second case (b) simply switches the role of P and Q. □

Remark . *Another strategy to show the above trace formula is to use the trace formula* (3.19) *for q a root of unity, and proving that the trace formula for idempotents are rational functions on q, r.*

We are now ready for the following theorem:

Theorem 3.2.18. *Suppose $\mathscr{P}_\bullet$ is a singly generated Thurston-relation planar algebra and* $\dim \mathscr{P}_{4,\pm} = 24$. *Then it is isomorphic to the semisimple quotient of $\mathscr{P}_\bullet^H(q,r)$ for some (q,r).*

Proof. Suppose $\mathscr{P}_\bullet$ is a singly generated Thurston-relation planar algebra with parameter $(\delta > 2, \gamma)$. We find (q,r) such that $\mathscr{P}_\bullet^H(q,r)$ has the same parameter (δ, γ).

Case 1: $\gamma = \dfrac{(\delta+2)(\delta-1)}{(\delta-2)(\delta+1)}$.

In this case, we show that that $\mathscr{P}^H_\bullet(1,1)$ with a circle parameter δ is a solution. First note that this planar algebra has the desired δ. Therefore, we only need to show that the ratio of two traces of the two minimal idempotents in $\mathscr{P}^H_3(1,1)$ equals to γ. By Lemma 3.2.17, we know that $\mathscr{P}^H_3(1,1)$ with circle parameter δ has two minimal idempotents P,Q with

$$tr(P) = \frac{\delta(\delta+2)(\delta-1)}{2} \tag{3.44}$$

$$tr(Q) = \frac{\delta(\delta-2)(\delta+1)}{2} \tag{3.45}$$

The ratio equals to $\frac{(\delta+2)(\delta-1)}{(\delta-2)(\delta+1)}$. Therefore, $\mathscr{P}^H_\bullet(1,1)$ with circle parameter δ gives a solution.

Case 2: $\gamma \neq \frac{(\delta+2)(\delta-1)}{(\delta-2)(\delta+1)}$. Let (q,r) be the solution of

$$q^2+q^{-2} = 4(\delta^2-1)/\left(\delta^2-(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1})^2\right)-2 \tag{3.46}$$

$$r-r^{-1} = \delta(q-q^{-1}) \tag{3.47}$$

(Note that the assumption $\gamma \neq \frac{(\delta+2)(\delta-1)}{(\delta-2)(\delta+1)}$ implies that $q \neq \pm 1$.) From Equation (3.46), we see that

$$\delta^2(q^2+q^{-2}-2)+2 = \left(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}(q+q^{-1})\right)^2-2 \tag{3.48}$$

$$\delta^2(q-q^{-1})^2+4 = \left(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}(q+q^{-1})\right)^2 \tag{3.49}$$

$$(r-r^{-1})^2+4 = \left(\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}(q+q^{-1})\right)^2 \tag{3.50}$$

From Equation (3.50), we obtain

$$r+r^{-1} = \pm\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1}(q+q^{-1}) \tag{3.51}$$

Then we have the following equations:

$$r^2 - r^{-2} = \pm(\delta^2 - 2)\frac{\gamma - 1}{\gamma + 1}(q^2 - q^{-2}) \tag{3.52}$$

$$r^2 + r^{-2} = \delta^2(q^2 + q^{-2} - 2) + 2 \tag{3.53}$$

Recall that $q \neq \pm 1$, we let A, B to be defined by the formulas in Lemma 3.2.17

$$A = \frac{(r - r^{-1})(rq - r^{-1}q^{-1})(rq^{-2} - r^{-1}q^2)}{(q + q^{-1})(q - q^{-1})^3} \tag{3.54}$$

$$B = \frac{(r - r^{-1})(rq^2 - r^{-1}q^{-2})(rq^{-1} - r^{-1}q)}{(q + q^{-1})(q - q^{-1})^3} \tag{3.55}$$

Then we have

$$A + B = \frac{(r - r^{-1})((r^2 + r^{-2})(q + q^{-1}) + 2(q^3 + q^{-3}))}{(q + q^{-1})(q - q^{-1})^3} \tag{3.56}$$

$$A - B = \frac{(r - r^{-1})(r^2 - r^{-2})(q - q^{-1})}{(q + q^{-1})(q - q^{-1})^3} \tag{3.57}$$

Combined with the Equation (3.47),(3.52) and Equation (3.53), we see that either

$$A = (\delta^3 - 2\delta)\frac{1}{1 + \gamma} \tag{3.58}$$

$$B = (\delta^3 - 2\delta)\frac{\gamma}{1 + \gamma} \tag{3.59}$$

or

$$A = (\delta^3 - 2\delta)\frac{\gamma}{1 + \gamma} \tag{3.60}$$

$$B = (\delta^3 - 2\delta)\frac{1}{1 + \gamma} \tag{3.61}$$

In both cases, the singly generated Thurston-relation planar algebra with (δ, γ) and $\mathscr{P}_\bullet^H(q, r)$ have the same skein theory. Therefore, the planar algebras are isomorphic.

$\square$

Remark . *There are 8 solutions (q, r) in the above theorem. Proposition 3.2.4 shows that the 8* *corresponding planar algebras are isomorphic.*

3.2.4 Reduced case

In this section, we will classify subfactor planar algebras $\mathscr{P}_\bullet$ generated by a 3-box with Thurston relation for the reduced case, namely $\mathscr{P}_{4,\pm} \leq 23$.

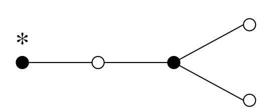
3.2.4.1 The case for at most 22 dimensional 4-box space

Proposition 3.2.19. *If $\mathscr{P}_\bullet$ is 2-supertransitive, then $\delta \leq 2$ if and only if $\dim \mathscr{P}_{4,\pm} \leq 22$. In this case, $\mathscr{P}_\bullet$ is either the E_6 or $E_6^{(1)}$ subfactor planar algebra.*

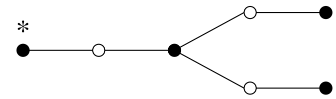
Thus the main Theorem 3.2.1 holds for the reduced case $\dim \mathscr{P}_{4,\pm} \leq 22$ by noting that $E_6^{(1)}$ is corresponding the $SU(3)_3$.

Proof. If $\delta \leq 2$, then $\mathscr{P}_\bullet$ is either the E_6 or $E_6^{(1)}$ by Popa's classification of subfactor standard invariants up to index 4 [Pop94]. Moreover, $\dim \mathscr{P}_{4,\pm} \leq 22$.

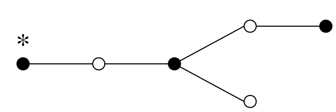
In general, we know that if $\dim \mathscr{P}_{4,\pm} \leq 22$, the principal graph begins with



One can compute that the dimension of the two-sided ideal generated by Jones projections in $\mathscr{P}_{4,+}$ is 20. Therefore, there are at most two vertices in the next step connected to the previous vertices with multiplicity 1. If $\delta > 2$, then we know that the set of annular consequences of the generator S together with the Temperley-Lieb diagrams form a linearly independent set, and since there are 14 Temperley-Lieb diagrams and 8 annular consequences [JR06], these must form a basis for $\mathscr{P}_{4,+}$. Thus the annular multiplicity sequences begins $0^3 10$, and in particular the principal graph must start with



when $\dim \mathscr{P}_{4,+} = 22$;

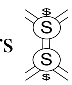
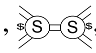
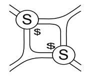
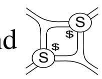


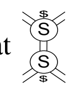
when $\dim \mathscr{P}_{4,+} = 21$.

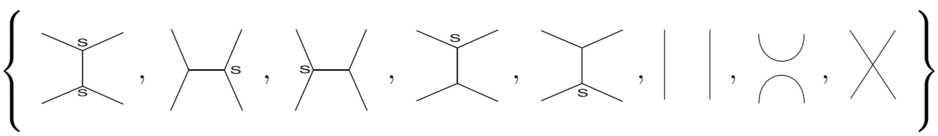
But by Ocneanu's triple point obstruction [Ocn88], we obtain $\delta \leq 2$, contradicting our assumption.

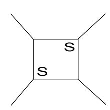
□

3.2.4.2 The case for 23 dimensional 4-box space

In this section, we classify subfactor planar algebras $\mathscr{P}_\bullet$ generated by a 3-box S with Thurston relation, such that $\dim(\mathscr{P}_{4,\pm}) = 23$. In this case, we have $\delta > 2$ and $\mathscr{P}_{3,+} = 6$. By the result of Jones [Jon01], the 14 Temperley-Lieb diagrams and the 8 diagrams in the annular consequence are linearly independent. Then one of the diagram with two generators , ,  and  is linearly independent with these 22 diagrams. Otherwise $\mathscr{P}_{4,\pm} = 22$.

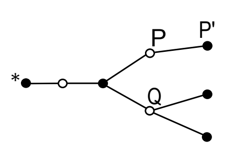
Up to rotation and the duality of the shading, we can assume that  and the other 22 diagrams form a basis. Similarly to Lemma 3.2.11, the basis for the f_2-cutdown of $\mathscr{P}_{4,+}$ is given by

$B' =$  .

Note that all results in §3.2.3 work for the case $\mathscr{P}_{4,\pm} = 23$, except Theorem 3.2.15. There we used the fact that  is linearly independent with B', which is no longer true for the reduced case. Now we give a different proof of $\omega = 1$ for the reduced case. Consequently the main Theorem 3.2.1 holds for the reduced case $\dim \mathscr{P}_{4,\pm} \leq 23$.

Theorem 3.2.20. *If $\mathscr{P}_\bullet$ is a singly genrated Thurston-relation subfactor planar algebra with parameters $(\delta, \gamma, \omega, a, a')$, then $\omega = 1$.*

Proof. One can deduce that the principal graph starts with the graph



where P and Q correspond to 2 minimal projections in $\mathscr{P}_{3,+}/\mathscr{I}_{3,+}$ described above. Now, we can apply Penney's triple point obstruction [**?**]:

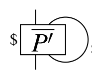
$$(\gamma-1)-\frac{\sigma+\sigma^{-1}}{[3]}=-(1+\gamma)\frac{[4]}{[3]}\left(Coef_{\in\cap_{n+1}\bar{P'}}(S)\right) \tag{3.62}$$

Where $\sigma^2=\omega$ and $[n]=\frac{q^n-q^{-n}}{q-q^{-1}}$ is the n^{th} quantum number (where q is defined so that $\delta=[2]=q+q^{-1}$ and $q>1$), P' is the projection labelled in our picture, and $\overline{P}'$ is the dual projection.

Under the assumption $\omega=e^{i\frac{1}{3}\pi}$ or $e^{i\frac{2}{3}\pi}$, $\sigma+\sigma^{-1}=\pm 1$. This implies that

$$(\gamma-1)\pm\frac{1}{[3]}=-(1+\gamma)\frac{[4]}{[3]}\left(Coef_{\in\cap_{n+1}\bar{P'}}(S)\right) \tag{3.63}$$

We now consider 2 subcases:

Suppose $P^{'}$ is self-dual, i.e.,  $=P$. In this case, we have

$$Coef_{\in\cap_{n+1}\bar{P'}}(S)=\frac{tr(P')}{tr(P)}\frac{1}{\gamma+1}=\frac{[2]tr(P)-[3]}{tr(P)}\frac{1}{\gamma+1}=([2]-\frac{[3]}{[4]}(\gamma+1))\frac{1}{\gamma+1} \tag{3.64}$$

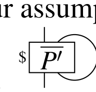
Therefore,

$$\gamma-1\pm\frac{1}{[3]}=-\frac{[4][2]}{[3]}+1+\gamma, \tag{3.65}$$

which implies

$$\delta^4-4\delta^2+1=0,$$

hence $\delta^2=2+\sqrt{3}<4$, contradicting our assumption that $\delta>2$.

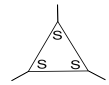
Now suppose that P' is not self-dual, i.e.,  $=Q$. In this case, we have

$$Coef_{\in\cap_{n+1}\bar{P'}}(S)=\frac{tr(P')}{tr(Q)}\frac{1}{\gamma+1}=-\frac{[2]tr(P)-[3]}{tr(Q)}\frac{1}{\gamma+1}=-([2]\frac{1}{r}-\frac{[3]}{[4]}(\frac{1}{\gamma}+1))\frac{1}{\gamma+1} \tag{3.66}$$

Therefore,

$$\gamma-1\pm\frac{1}{[3]}=-\frac{[4]}{[3]}([2]\frac{1}{\gamma}-\frac{[3]}{[4]}(1+\frac{1}{\gamma})),$$

hence $[3]\gamma^2\pm\gamma-[5]=0$.

Now suppose $a = 1$, then by considering evaluating 

, we have the following equations,

$$\frac{\gamma\delta}{\delta^2 - 1}(\delta^2 - 3) = 2(\gamma - 1)^2 \frac{\delta}{\delta^2 - 3}\,, \tag{3.67}$$

$$[3]\gamma^2 \pm \gamma - [5] = 0\,. \tag{3.68}$$

One can solove Equation (3.68) for γ in terms of δ and plug it in back to (3.67). We use Mathematica to solve numerical solutions for δ. They are far below 2 hence there is no solution when $\delta > 2$.

If $a = -1$, we obtain two equations following simialr arguments,

$$-\gamma\frac{\delta^3 - 2\delta}{\delta^2 - 1} + b(2(\gamma - 1)) + c = 0\,, \tag{3.69}$$

$$[3]\gamma^2 \pm \gamma - [5] = 0\,, \tag{3.70}$$

where b, c are given in Equation (3.36).

One can show there is no solution with $\delta > 2$ in a similar way to the a=1 case.

We conclude that $\omega = 1$ for the dimension 23 case. □

Theorem 3.2.21. *Suppose $\mathscr{P}_\bullet$ is a singly generated Thurston-relation planar algebra and* $\dim \mathscr{P}_{4,\pm} =$ 23, *then it is isomorphic to the semisimple quotient of $\mathscr{P}^H_\bullet(q,r)$ for some (q,r).*

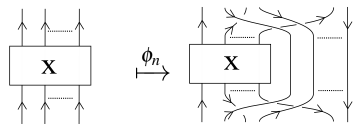
Proof. It follows from Theorem 3.2.20, 3.2.16 and the proof of Theorem 3.2.18. □

Remark . *We notice that in the case of 24 dimensional 4-box spaces, our proof did not need the full strength of the assumption that we had a subfactor planar algebra. In particular, we did not need that the canonical inner product on the box spaces induces a positive definite inner product on the hom spaces in any essential way, and it would have been sufficient in principal to assume that this inner product was simply non-degenerate. For the cases in this section, when the 4-box space has dimension less than 24, we appealed to triple point obstructions and classification of subfactors, which are theorems for subfactor planar algebras, and thus we used the subfactor assumption in an essential way.*

3.2.5 Positivity

In this section, we will determine the positivity of $\mathscr{P}^H_\bullet(q,r)$.

Consider the map $\phi_n : H_n(q,r) \to \mathscr{P}^H_n(q,r)$, for $x \in H_n(q,r)$ defined $\phi_n(x)$ as follows:



.

Proposition 3.2.22. *The map ϕ_n is an algebra homomorphism from $H_n(q,r)$ to $\mathscr{P}^H_{2n}(q,r)$ preserving the normalized Markov trace. Furthermore, $\phi_{n+1}|_{H_n(q,r)} = \phi_n$.*

Proof. This follows from the HOMFLY-PT skein relations. □

Notation 3.2.23. *We define ϕ on $H_\bullet(q,r)$ as $\lim_{n\to\infty} \phi_n$.*

Recall that when $r = q^N$ for some $N \in \mathbb{N}$ and $q = e^{\frac{i\pi}{N+l}}$ for some $l \in \mathbb{N}$ or $q \geq 1$, $\mathscr{P}^H_\bullet(q,r)$ admits an involution $*$ such that the Markov trace is semipositive-definite. We show that they are the only values of (q,r) and involution such that positivity holds.

Theorem 3.2.24. *The planar algebra $\mathscr{P}^H_\bullet(q,r)$ has positivity if and only if $r = q^N$ for some $N \in \mathbb{N}$, and $q = e^{\frac{i\pi}{N+l}}$ for some $l \in \mathbb{N}$ or $q \geq 1$.*

Proof. If $\mathscr{P}^H_\bullet(q,r)$ has positivity, then the idempotents P and Q in Lemma 3.2.17 are projections, therefore the generator S is self-adjoint. Since $\mathscr{P}^H_\bullet(q,r)$ is generated by S, the involution $*$ is uniquely determined.

In Theorem 3.2.18 and 3.2.21, we show that a singly generated Thurston-relation planar algebra with parameters (δ,γ) and $\delta > 2$ is isomorphic to $\mathscr{P}^H_\bullet(q,r)$ for some (q,r). By Lemma 3.2.4, we can assume that $\Re q \geq 0$, $\Im q \geq 0$. Note that $\delta > 2$ and $\gamma > 0$, then by Equation (3.46), we have

$$q + q^{-1} = \sqrt{2(\delta^2-2)/(\delta^2 - (\frac{\delta^2-2}{\delta}\frac{\gamma-1}{\gamma+1})^2)}. \tag{3.71}$$

(One can check the term in the square root is positive.) We have $q = e^{i\theta}$ with $0 \leq \theta \leq \pi/2$ or $q \geq 1$.

Let $[n]$ denote the Young diagram with 1 row and n columns and $[1^n]$ denote the Young diagram with n rows and 1 column.

Case 1: $q>1$. By Lemma 3.2.4, we can assume that $\Re r\geq 0$. By $\dfrac{r-r^{-1}}{q-q^{-1}}=\delta>2$, we have that $r>1$. If $r=q^N$, for some $N\in\mathbb{N}$, then we know that $\mathscr{P}^H_\bullet(q,r)$ has positivity. Otherwise, $q^N<r<q^{N+1}$. Then the idempotent $m_{[N+2]}$ is well-defined and $Tr(m_{[N+2]})<0$. By Proposition 3.2.22, $\mathscr{P}^H_\bullet(q,r)$ does not have positivity.

Case 2: $q=e^{i\theta}$, and $q\neq 1$: By Lemma 3.2.4, we can assume that $\Im r\geq 0$. By $\dfrac{r-r^{-1}}{q-q^{-1}}=\delta>2$, we have $r=e^{i\alpha}$, for some $\theta<\alpha<\pi-\theta$.

Subcase 1: If $N\theta<\alpha<(N+1)\theta$, for some $N\in\mathbb{N}$, then the idempotent $m_{[N+2]}$ is well-defined and $Tr(m_{[N+2]})<0$. By Proposition 3.2.22, $\mathscr{P}^H_\bullet(q,r)$ does not have positivity.

Subcase 2: If $\alpha=N\theta$ and $\frac{\pi}{N+l+1}<\theta<\frac{\pi}{N+l}$, for some $N,l\in\mathbb{N}$, then the idempotent $m_{[1^{l+1}]}$ is well-defined and $Tr(m_{1_{[l+1]}})<0$. By Proposition 3.2.22, $\mathscr{P}^H_\bullet(q,r)$ does not have positivity.

Subcase 3: If $\alpha=N\theta$ and $\theta=\frac{\pi}{N+l}$, for some $N,l\in\mathbb{N}$, then we know that $\mathscr{P}^H_\bullet(q,r)$ has positivity.

Case 3: $q=1$: By $r-r^{-1}=\delta(q-q^{-1})$, we have $r=1$. By a similar argument in Case 1, one can show that $\delta=N$, for some $N\in\mathbb{N}$. In this case, we know that $\mathscr{P}^H_\bullet(q,r)$ has positivity.

$\square$

Therefore we obtain our classification result, Theorem 3.2.1. The ones with positivity all come from representations of the quantum groups $U_q(SU(N))$ or E_6.

Chapter 4

Applications of Skein theory

4.1 Introduction

As mentioned in §1, a skein theory of a planar algebra plays the role as a presentation of a group. In [Jon14][Jon16], Jones gave a general construction of actions and unitary representations of Thompson groups. Let us first recall the basic results.

Let $\mathscr{F}$ be a category with the following fraction properties:

(i) (Unit) There exists an object 1 such that $Mor(1,a) \neq \emptyset$, $\forall a \in Obj(\mathscr{F})$.

(ii) (Stabilisation) Let $\mathscr{D} = \bigcup_{a \in Obj(\mathscr{F})} Mor(1,a)$. For each $f, g \in \mathscr{D}$, there exits morphisms p, q such that $pf = qg$.

(iii) (Cancellation) If $pf = pg$, then $f = g$.

We first define a partial order on the set $\mathscr{D}$ by: $f \leq g \Leftrightarrow \exists\ p,\ g = pf$, where p is a morphism in $\mathscr{F}$. Then we choose an arbitrary category $\mathscr{C}$ and a functor $\Phi : \mathscr{F} \to \mathscr{C}$. Together with the partial order on $\mathscr{D}$, we construct the following direct system graded by $\mathscr{D}$: For each $f \in \mathscr{D}$,

$$A(\Phi)_f = Mor(\Phi(1), \Phi(target(f)))$$

together with the inclusion $l_f^g : A(\Phi)_f \to A(\Phi)_g$ when $f \leq g$ and $g = pf$ given by

$$l_f^g(v) = \Phi(p)v\ \forall \quad v \in A(\Phi)_f$$

Thus, we take the direct limit $A(\Phi) = \lim\limits_{\rightarrow} A(\Phi)_f$, which is defined as the disjoint union of $A_f,\ f \in \mathscr{D}$ modulo the equivalence relation induced by

$$x \in A_f \sim y \in A_g \Leftrightarrow \exists k \geq f, g\ l_f^k(x) = l_g^k(y)$$

For the convenience, we introduce the following notation to study the direct limit $A(\Phi)$.

Notation 4.1.1. *Given a category $\mathscr{F}$ satisfying the fraction properties with another category $\mathscr{C}$ and* *a functor Φ from $\mathscr{F}$ to $\mathscr{C}$. The pair (f,x) represents the element x in the direct limit $A(\Phi)$, where $x \in A(\Phi)_f$ and $f \in \mathscr{D}$.*

Now we put a group structure on $A(\Phi)$ when Φ is the identity functor $\mathscr{I}$. Given two elements $(f_1,g_1),(f_2,g_2) \in A(\mathscr{I})$. By the stabilisation property, there exist morphisms p,q such that $pg_1 = qf_2$. Thus we define

$$(f_1,g_1)\circ(f_2,g_2) = (pf_1,qg_2).$$

Proposition 4.1.2 ([Jon16])**.** *The above definition is independent of the choice p,q in the direct limit $A(\mathscr{I})$ and thus, $A(\mathscr{I})$ forms a group with the identity to be $[(f,f)]$ for some $f \in \mathscr{D}$ and the inverse of $[(f,g)]$ equals to $[(g,f)]$.*

The choice of the category $\mathscr{C}$ and the functor Φ gives an action of the group $A(\mathscr{I})$ on the set $A(\Phi)$. Given $(f_1,g_1) \in A(\mathscr{I})$ and $(f_2,g_2) \in A(\Phi)$, we can choose morphisms p,q by the stabilisation property such that $pg_1 = qf_2$. Then define the action as

$$(f_1,g_1)(f_2,g_2) = (pf_1,\Phi(q)g_2).$$

Proposition 4.1.3 ([Jon16])**.** *The above definition is independent of the choice p,q in the direct limit $A(\mathscr{I})$ and thus, this gives an action of $A(\mathscr{I})$ on $A(\Phi)$.*

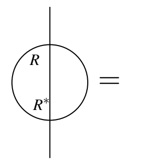
Now we consider the category $\mathscr{F}$ to be the category of binary forests, i.e., the objects are positive integers and the morphism space from n to m is the set of binary forests with n roots and m leaves when $n \leq m$ and is the empty set when $n > m$.

Proposition 4.1.4 ([Jon16])**.** *Let $\mathscr{F}$ be the category of binary forests, then $A(\mathscr{I})$ is isomorphic to the Thompson group F.*

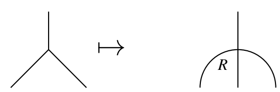
From this viewpoint, there are many actions of the Thompson group F with various choices of the category $\mathscr{C}$ and the functor Φ. One of the most natural choices of the target category is the subfactor planar algebras. In this case, the direct system $\{A(\Phi)_f, f \in \mathscr{D}\}$ consists of Hilbert spaces. With a nice choice of the functor, one can construct unitary representations of the Thompson group F. Furthermore, these constructions can be generalized to the Thompson group T when taking the category $\mathscr{F}$ to be the category of annular binary forests.

The Jones' subgroup $\vec{F}$ and $\vec{T}$ were introduced as stabilizer of the vacuum vector in the above representations in [Jon14]. The definition of the subgroups needs a slight modification to the construction. Here we first introduce the modified construction.

We start with $\mathscr{P}_\bullet$ an irreducible subfactor planar algebra with a normalised 2-box R, i.e.,



For each $f \in \mathscr{D}$, we consider the set $A_f = \mathscr{P}_{n_f}$, where $n_f = \frac{1+target(f)}{2}$. For each morphism p in the category $\mathscr{F}$, we define a map Φ by mapping it to an element in the planar algebra determined by

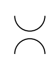


Therefore by applying the construction of the direct limit, we obtain an action of the Thompson group F on the direct limit $A(\Phi)$. However, each A_f is a finite-dimensional Hilbert space where the inner product is given in the planar algebra $\mathscr{P}_\bullet$. The normalisation condition of R guarantees that each inclusion map is an isometry. Hence, the direct limit $A(\Phi)$ is a pre-Hilbert space. By taking the completion of $A(\Phi)$, we obtain a Hilbert space $\mathscr{H}_R$ and a map π_R from the Thompson group F to the bounded operators of the Hilbert space $\mathscr{H}_R$. From the theory of subfactors, one can show that $(\pi, \mathscr{H}_R)$ is a unitary representation of the Thompson group F. Naturally we obtain a subgroup by taking the stabilizer of the vacuum vector $\xi = |$.

Definition 4.1.5. *The subgroup $F_{\xi,R}$ is defined as*

$$F_{\xi,R} = \{g \in F | \pi(g)\xi = \xi\}$$

Definition 4.1.6 (Jones' subgroup)**.** *Let $\mathscr{P}_\bullet$ be the Temperley-Lieb planar algebra with index 2 and the 2-box R a multiple of the second Jones-Wenzl idempotent, i.e,*

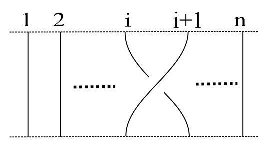
$R = 2^{1/4}$ $-2^{-1/4}$ 

Then the Jones subgroup $\vec{F}$ is defined as the subgroup $F_{\xi,R}$.

Remark . *The subgroup vecF were shown to be isomorphic to the Thompson group F_3 by Golan and Sapir [GS17]. This definition can be generalized to the Thompson group T to obtain the Jones' subgroup $\vec{T}$.*

4.2 Singly generated subgroups

In this section, we introduce a class of subgroups of Thompson group F which are motivated by the braid groups $B_n, n \in \mathbb{N}$. The braid group B_n is the group formed by appropriate isotopy classes of braids with obvious concatenation operation. A preferred set of generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ is given by the following pictures:



One can easily to verify the following relations:

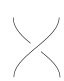
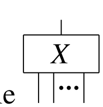
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ i = 1, 2, \cdots, n-1 \tag{4.1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2 \tag{4.2}$$

E. Artin proved that relations (4.1) and (4.2) give a presentation of B_n. It follows that B_n can be embedded into B_{n+1} by adding a through string on the right hand side. Therefore, one can consider the braid group obtained by taking the inductive limit of $B_n, n \in \mathbb{N}$. In this case, every element can be interpreted as a diagram in B_n for some $n \in \mathbb{N}$ with infinitely many through strings on its right hand side.

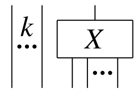
Definition 4.2.1. *The braid group B_∞ is defined as:*

$$B_\infty \cong \langle \sigma_1, \sigma_2, \cdots | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, i = 1, 2, \cdots; \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2\rangle$$

One can think of B_∞ as a group generated by a $(2,2)$-tangle  . In this section, we consider groups in this form for an arbitrary $(1,N)$-tangle 

which arise as subgroups of the Thompson group F.

Definition 4.2.2 (Shifts of X)**.** *Let X_k, called the k-shift of X, be an $(k+1, k+N)$-tangle defined as follows:*

$X_k =$ 

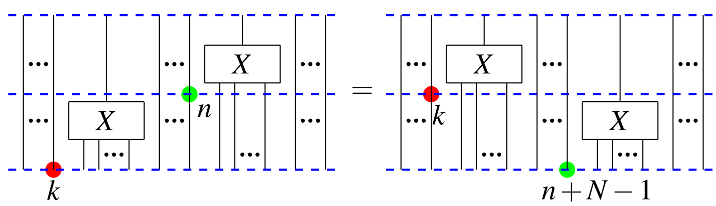
where k stands for k through strings on the left.

Remark . *For each X_k, we identify it as the same diagram with infinitely many strings on the right. Therefore, we define multiplication $\cdot$ of X_m and X_n as stacking the tangle from top to bottom as the multiplication in $\mathscr{P}_\bullet$. We denote this set as $Alg(X)$ and call it singly generated by X. Let $Alg(X)_n$ to be set of all $(1,n)$-tangles in $Alg(X)$.*

Proposition 4.2.3. *For $k, n \in \mathbb{N}$ with $k < n$, we have*

$$X_n \cdot X_k = X_k \cdot X_{n+N-1} \tag{4.3}$$

Proof. Relation (4.3) follows from



□

Motivated by the pair of binary trees representation of Thompson group F, we consider the group consisting of the pairs of elements of a certain type from $Alg(X)$.

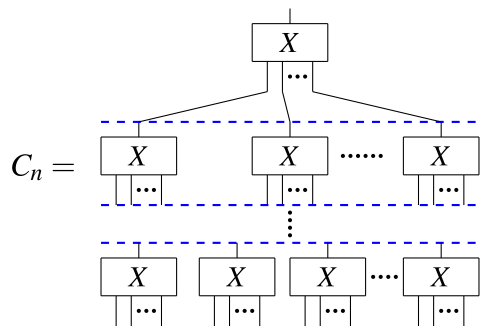
Definition 4.2.4. *Let $\widehat{G_X} = \{(T_+, T_-) : T_\pm \in Alg(X)_n \ \forall n \in \mathbb{N}\}$. We define a relation on $\widehat{G_X}$ by*

$$(T_+, T_-) \sim (S_+, S_-) \Leftrightarrow$$

$$\exists R \in Alg(X) \text{ such that } T_\pm = S_\pm \cdot R \text{ or } S_\pm = T_\pm \cdot R.$$

Proposition 4.2.5. *Suppose $T \in Alg(X)_n, S \in Alg(X)_m$ for some $n, m \in \mathbb{N}$. There exists $P, Q \in Alg(X)$ such that $T \cdot P = S \cdot Q$.*

Proof. Set $R_k = \prod_{j=0}^{k} X_{jN}$ to be an $(k+1, (k+1)N)$-tangle. We consider the elements $C_n = \prod_{j=0}^{n} R_j \in Alg(X)_{(n+1)N}$ illustrated as follows:



Suppose $T \in Alg(X)_n$. There exists $k_1 \in \mathbb{N}$ such that T is a sub diagram of C_{k_1}, i.e, there exists $P_1 \in Alg(X)$ such that $T \cdot P_1 = C_{k_1}$. Similarly we obtain such $k_2 \in \mathbb{N}$ and $Q_1 \in Alg(X)$ for $S \in Alg(X)_m$.

Set $k = \max(k_1, k_2)$. If $k_1 = k_2$, then $T \cdot P_1 = S \cdot Q_1$. If $k_1 \neq k_2$, then without loss of generality we may assume $k_1 > k_2$. Set $Q_2 = \prod_{j=k_2+1}^{k_1} R_j$. Then

$$T \cdot P_1 = C_{k_1}$$

$$S \cdot Q_1 \cdot Q_2 = C_{k_2} \cdot \prod_{j=k_2+1}^{k_1} R_j = C_{k_1}$$

Therefore, $P = P_1$ and $Q = Q_1 \cdot Q_2$ satisfies the requirement of the proposition. $\square$

Corollary 4.2.6. *Suppose $(T_+, T_-), (S_+, S_-) \in \widehat{G_X}$. There exists $(\widetilde{T_+}, \widetilde{T_-}), (\widetilde{S_+}, \widetilde{S_-}) \in \widehat{G_X}$ such that*

$$(\widetilde{T_+}, \widetilde{T_-}) \sim (T_+, T_-) \tag{4.4}$$

$$(\widetilde{S_+}, \widetilde{S_-}) \sim (S_+, S_-) \tag{4.5}$$

$$\widetilde{T_-} = \widetilde{S_+} \tag{4.6}$$

Proof. By Proposition 4.2.5, there exists $P, Q \in Alg(X)$ such that $T_- \cdot P = S_+ \cdot Q$. We define $\widetilde{T_\pm} = T_\pm \cdot P$ and $\widetilde{S_\pm} = S_\pm \cdot Q$. It follows from definitions that $(\widetilde{T_+}, \widetilde{T_-}), (\widetilde{S_+}, \widetilde{S_-}) \in \widehat{G_X}$ satisfy Relations

(4.4), (4.5) and (4.6). □

Let $G_X = \{(T_+, T_-) : T_\pm \in Alg(X)_n \; \forall n \in \mathbb{N}\} / \sim$, where $\sim$ is the equivalence relation in Definition 4.2.4. Suppose $g, h \in G_X$. Then there exists $(T,S), (S,R) \in \widehat{G_X}$ such that $g = [(T,S)], h = [(S,R)]$ by Corollary 4.2.6. Thus we define a binary operation $\circ$ as

$$g \circ h = [(T,R)]$$

Theorem 4.2.7. *The binary operation $\circ$ is well defined on G_X and G_X is a group with the binary operation $\circ$.*

Proof. Suppose $(T,S) \sim (\widetilde{T}, \widetilde{S})$ and $(S,R) \sim (\widetilde{S}, \widetilde{R})$. Then there exists $P \in Alg(X)$ such that

$$\widetilde{T} = T \cdot P$$
$$\widetilde{S} = S \cdot P$$
$$\widetilde{R} = R \cdot P$$

Therefore $(T,R) \sim (\widetilde{T}, \widetilde{R})$, i.e, $\circ$ is well defined on G_X.

Suppose $T, S \in Alg(X)_n$ for some $n \in \mathbb{N}$. It follows that $[(T,T)]$ is the identity element with respect to $\circ$. For the element $[(T,S)]$, $[(S,T)]$ is the inverse element. By definition, G_X is closed under the binary operation $\circ$. Therefore G_X is a group with $\circ$.

□

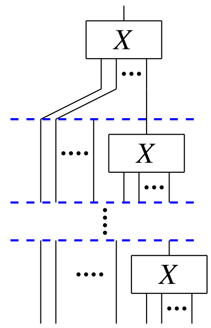
Remark . *We omit the $\circ$ when there is no confusion and call such a group G_X a singly generated subgroup by X. Furthermore, $[(T,R)]$ equals to the identity if and only if T is isotopically equivalent of R for $T, R \in Alg(X)_n$ with $n \in \mathbb{N}$.*

Notation 4.2.8. *In the following sections, we denote (T,R) for $T, R \in Alg(X)_n$ for the equivalence class of $[(T,R)]$ for elements in G_X.*

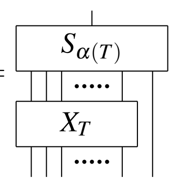
4.3 The Classical presentation

In this section we discuss the structure of G_X and its classical presentation derived from the vertical isotopy.

Definition 4.3.1. *Suppose X is an $(1,N)$-tangle. Set $S_n = X_0 \cdot X_{N-1} \cdot X_{2(N-1)} \cdots X_{n(N-1)}$. We call these S_n's basic forms and illustrate them as following:*



Proposition 4.3.2. *Suppose $T \in Alg(X)_n$, there exists $\alpha(T) \in \mathbb{N}$ and $X_T \in Alg(X)$ which is $(\alpha(T)(N-1)+1, n)$-tangle such that*



Proof. Since $T \in Alg(X)_n$, the first word in T is X_0. If there exists an X such that it is attached to the rightmost string on the bottom of X_0, then apply the vertical isotopy to obtain a diagram starting with $X_0 \cdot X_{N-1}$. Repeating this procedure and letting $\alpha(T)$ to be the number of the steps, we get that $T = S_{\alpha(T)} \cdot X_T$, where X_T is the rest of the word. $\square$

Lemma 4.3.3. *Let $P_X = \{(T, S_n) : T \in Alg(X)_n,\ \forall n \in \mathbb{N}\}$. Then P_X is a semigroup under $\circ$. Furthermore, it generates the group G_X.*

Proof. Suppose $g, h \in P_X$ and $g = (T, S_n), h = (R, S_m)$.

$$
\begin{aligned}
gh =& (S_{\alpha(T)} \cdot X_T, S_n) \circ (S_{\alpha(R)} \cdot X_R, S_m) \\
=& (S_{\alpha(T)+\max(n,\alpha(R))-n} \cdot X_T, S_{\max(n,\alpha(R))}) \circ \\
& (S_{\max(n,\alpha(R))} \cdot X_R, S_{m+\max(n,\alpha(R))-\alpha(R)}) \\
=& (S_{\alpha(T)+\max(n,\alpha(R))-n} \cdot X_T \cdot X_R, S_{m+\max(n,\alpha(R))-\alpha(R)}) \in P_X
\end{aligned}
$$

Therefore P_X is a semigroup under the binary operation $\circ$.

Note that for $(T,R) \in G_X$, $(T,R) = (T,S_n)(R,S_n)^{-1}$ for some $n \in \mathbb{N}$. Hence P_X is a generating set for G_X. $\square$

Now we give a description of classical generators for the group G_X.

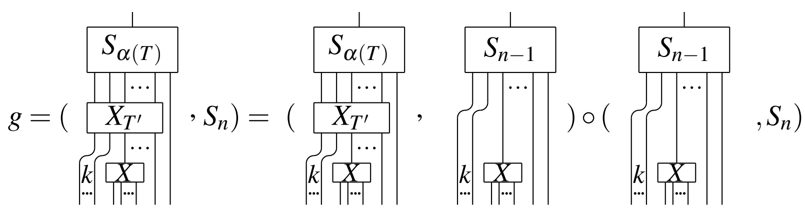
Definition 4.3.4. *Let $n \in \mathbb{N}$. There exists $a(n), b(n) \in \mathbb{N}$ such that $a(n)$ is the largest integer satisfying $(N-1)a(n) + b(n) = n$ with $0 \le b(n) < N$. We define*

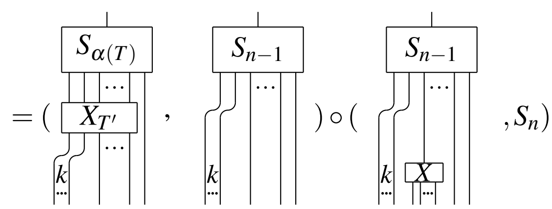
$$x_n = (S_{a(n)} \cdot X_n, S_{a(n)+1})$$

Lemma 4.3.5. *The set $\{x_n : n \in \mathbb{N}\}$ is a generating set for G_X and satisfies the relation*

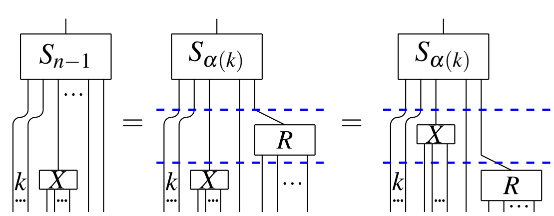
$$x_k^{-1} x_n x_k = x_{n+N-1} \tag{4.7}$$

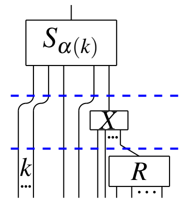
Proof. By Lemma 4.3.3, we only need to show every element $g \in P_X$ belongs to the subgroup generated by $\{x_n : n \in \mathbb{N}\}$. Suppose $g = (T, S_n) \in P_X$. By Proposition 4.3.2, $X_T = X_{T'} \cdot X_k$ for some $k \in \mathbb{N}$.



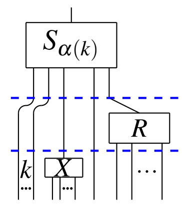
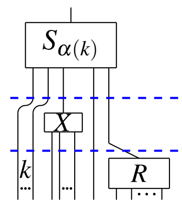
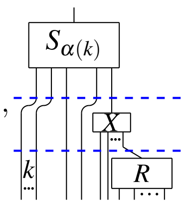


By Definition 4.3.1, $S_n = X_0 \cdot X_{N-1} \cdot X_{2(N-1)} \cdots X_{n(N-1)}$. Let $R = X_{(m+1)(N-1)} \cdot X_{n(N-1)}$. By Definition 4.3.4, $a(k)$ is the smallest integer such that $(N-1)a(k) + b(k) = k$ with $0 \le b(k) < N$.



$S_n =$ 

Therefore, by Definition 4.2.4,

( $, S_n) = ($  ,  $) = x_k$

Hence g is written as the product of a word in P_X with smaller length and x_k for some $k \in \mathbb{N}$. Then by induction on the length, G_X is generated by $\{x_n : n \in \mathbb{N}\}$.

Now we prove Relation (4.7). Suppose $k, n \in \mathbb{N}$ with $k < n$.

$$\begin{aligned}
x_n x_k &= (S_{a(n)} \cdot X_n, S_{a(n)+1}) \circ (S_{a(k)} \cdot X_k, S_{a(k)+1}) \\
&= (S_{a(n)} \cdot X_n, S_{a(n)+1}) \circ (S_{a(n)+1} \cdot X_k, S_{a(n)+2}) \\
&= (S_{a(n)} \cdot X_n \cdot X_k, S_{a(n)+1} \cdot X_k) \circ (S_{a(n+N)} \cdot X_k, S_{a(n)+2}) \\
&= ((S_{a(n)} \cdot X_n \cdot X_k, S_{a(n)+2})
\end{aligned}$$

By Definition 4.3.4, we know that $a(n+N-1) = a(n)+1$. Therefore,

$$\begin{aligned}
x_k x_{n+N-1} &= (S_{a(k)} \cdot X_k, S_{a(k)+1}) \circ (S_{a(n+N-1)} \cdot X_{n+N-1}, S_{a(n+N-1)+1}) \\
&= (S_{a(n)} \cdot X_k, S_{a(n)+1}) \circ (S_{a(n)+1} \cdot X_{n+N-1}, S_{a(n)+2}) \\
&= (S_{a(n)} \cdot X_k \cdot X_{n+N-1}, S_{a(n)+1} \cdot X_{n+N-1}) \circ (S_{a(n)+1} \cdot X_n, S_{a(n)+2}) \\
&= (S_{a(n)} \cdot X_k \cdot X_{n+N-1}, S_{(a(n)+2)})
\end{aligned}$$

Note that we have the following from Proposition 4.2.3:

$$X_n \cdot X_k = X_k \cdot X_{n+N-1}$$

Hence $x_n x_k = x_k x_{n+N-1}$. $\square$

From the proof of Lemma 4.3.5, we have the following corollary,

Corollary 4.3.6. *Suppose $g \in P_X$. Then there exists $n \in \mathbb{N}; i_1, i_2, \cdots, i_n \in \mathbb{N}; k_1, k_2, \cdots, k_n \in \mathbb{N}$ such that*

$$g = x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}.$$

Furthermore,

$$g = 1 \Leftrightarrow k_1 = k_2 \cdots = k_n = 0.$$

Theorem 4.3.7. *The group G_X has a classical presentation*

$$G_X \cong \langle t_n, n \in \mathbb{N} | t_k^{-1} t_n t_k = t_{n+N-1},\ \forall k < N \rangle. \tag{4.8}$$

i.e, the group G_X is isomorphic to F_N.

Proof. Let M denote the normal subgroup generated by $\{t_k^{-1} t_n^{-1} t_k t_{n+N-1},\ \forall k < n\}$. Define the map Φ by sending t_k to x_k. It follows that Φ extends to a surjective group homomorphism from $\langle t_n, n \in \mathbb{N} | t_k^{-1} t_n t_k = t_{n+N-1},\ \forall k < N \rangle$ to G_X. Thus we only need to show Φ is injective. Note that for $k, n \in \mathbb{N},\ k < n$,

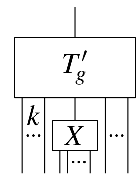
$$t_k^{-1} t_n = t_{n+N-1} t_k^{-1}$$
$$t_n^{-1} t_k = t_k t_{n+N-1}^{-1}$$

Therefore, for every element $g \in \langle t_n, n \in \mathbb{N} | t_k^{-1} t_n t_k = t_{n+N-1},\ \forall k < N \rangle$, there exists g_+, g_- such that $g = g_+ g_-^{-1}$ where $g_\pm$ is a word on $\{t_1, t_2, \cdots\}$ with positive powers. These are called positive elements of Thompson group F_N. Thus to show Φ is injective, we only need to show that for every two positive elements $g, h \in F_N$,

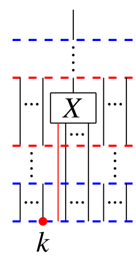
$$\Phi(g) = \Phi(h) \Leftrightarrow g = h$$

Since g,h are positive elements and Φ is a group homomorphism, $\Phi(g),\Phi(h) \in P_X$. Hence there exists $T_g, T_h \in Alg(X)_{n(N-1)+1}$ such that $\Phi(g) = (T_g, S_n)$ and $\Phi(h) = (T_h, S_n)$ where T_g and T_h are isotopically equivalent.

Assume $T_g = T_g' \cdot X_k$, i.e,

$T_g =$ 

Since T_g and T_h are isotopically equivalent, T_h must be of the form:



where the red string is the $(k+1)$th string from the left. Therefore, $T_h = T_{h_1} \cdot X_n \cdot T_{h_2}$, where X_n corresponds to the red layer. From the proof of Lemma 4.3.5, we know that there exits h_1, h_2 such that

$$h = h_1 t_n h_2.$$

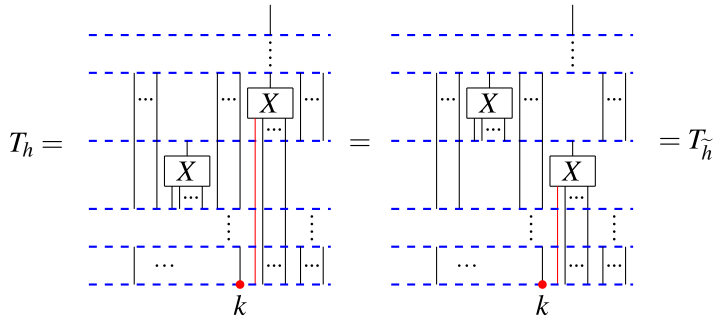
Suppose $h_2 = 1$, then $n = k$ and $g = g' t_k$, $h = h' t_k$ for some g', h' positive elements. Note that

$$g = h \Leftrightarrow g' = h'.$$

Suppose $h_2 \neq 1$, let t_m be the first word in h_2, i.e, $h = h_1 t_n t_m h_3$.

By the structure of T_h, we know that either $m < n$ or $m > n+N-1$. If $m < n$, then $t_n t_m = t_m t_{n+N-1}$; if $m > n+N-1$, then $t_n t_m = t_{m-N+1} t_n$ by Relation (4.7). By symmetry we just discuss the first case.

Let $\widetilde{h} = h_1 t_m t_{n+N-1} h_3$.



Note that $h^{-1}\widetilde{h} = h_3^{-1} t_m^{-1} t_n^{-1} t_m t_{n+N-1} h_3 \in R$. Therefore to show $g^{-1}h \in R$, we only need to show $g^{-1}\widetilde{h} \in R$, i.e, to show that $\Phi(g) = \Phi(\widetilde{h}) \Leftrightarrow g = \widetilde{h}$. By repeating this procedure, we obtain a positive element $\widehat{h}$ with $\widehat{h} = \widehat{h}' t_k$. We only need to show

$$\begin{aligned} \Phi(g) = \Phi(\widehat{h}) &\Leftrightarrow g = \widehat{h} \\ &\Leftrightarrow g' = \widehat{h}' \end{aligned}$$

To show $g = h$, the computation in both cases is reduced to show the equality of two positive elements with smaller length. Eventually it reduces to the case that for $i_1, \cdot, i_m; n_1, \cdot, n_m \in \mathbb{N}$

$$\begin{aligned} t_{i_1}^{n_1} t_{i_2}^{n_2} \cdots t_{i_m}^{n_m} &= 1 \\ \Rightarrow x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_m}^{n_m} &= 1 \end{aligned}$$

Therefore $n_1 = n_2 = \cdots n_m = 1$, i.e, Φ is injective. $\square$

4.4 Examples

In this section, we mainly introduce two examples studied by Jones.

4.4.1 The Jones subgroup $\vec{F}$

Jones introduced the Jones subgroup $\vec{F}$ [Jon14], and it is shown that $\vec{F} \cong F_3$ by Golan and Sapir [GS17]. We first recall the definition of $\vec{F}$.

Let $\mathscr{P}_\bullet$ be the subfactor planar algebra, $\mathscr{TL}(2)$. We construct a unitary representation π with Approach 1 in §**??** by taking the 2-box R to be $2^{1/4}({}_{s}\Big|\ \Big| - \frac{1}{\sqrt{2}}\, {}_{s}\genfrac{}{}{0pt}{}{\cup}{\cap})$, a multiple of the 2nd Jones-Wenzl idempotent.

For every element $g \in F$, the matrix coefficient $\langle \pi(g)\xi, \xi \rangle$ has a specific interpretation:

Suppose g has a pair of binary tree representation as (T_+, T_-). We arrange the pair of trees in $\mathbb{R}^2$ such that the leaves of $T_\pm$ are the points $(1/2,0), (3/2,0), \cdots ((2n-1)/2, 0)$ with all the edges being the straight line segments sloping either up from left to right or down from left to right. T_+ is in the upper half plane and T_- is in the lower half plane. We construct a simply-laced planar graph $\Gamma(T_+, T_-)$ from the given trees. Let the vertices be $\{(0,0), (1,0), ..., (n,0)\}$ contained in each region between the edges of each tree. The vertices $(k,0)$ and $(j,0)$ is connected by an edge if and only if the corresponding regions are separated by an edge sloping up in T_+ or down in T_- from left to right.

Proposition 4.4.1 (Jones, [Jon14])**.** *$\Gamma(T_+, T_-)$ defined as above consists of a pair of trees, $\Gamma(T_+)$ in the upper half plane and $\Gamma(T_-)$ in the lower half plane with following properties:*

- *The vertices are $0, 1, 2, \cdots, n$.*
- *Each vertex other than 0 is connected to exactly one vertex to its left*
- *Each edge can be parameterised as $(x(t), y(t))$ for $0 \leq t \leq 1$ such that $x'(t) > 0$ and $y(t) > 0$ on $(0,1)$ or $y(t) < 0$ on $(0,1)$.*

Proposition 4.4.2 (Jones, [Jon14])**.** *We have the following for the matrix coefficient,*

$$\langle \pi(g)\xi, \xi \rangle = \frac{1}{2} Chr_{\Gamma(T_+,T_-)}(2),$$

where $Chr_{\Gamma(T_+,T_-)}(2)$ is the value of the chromatic polynomial for $\Gamma(T_+, T_-)$ at 2, i.e, the number of 2-coloring of $\Gamma(T_+, T_-)$.

Remark . *By definition of the chromatic polynomial, we have*

$$Chr_{\Gamma(T_+,T_-)}(2) = \begin{cases} 2, & \text{if } \Gamma(T_+,T_-) \text{ is bipartite} \\ 0, & \text{if } \Gamma(T_+,T_-) \text{ is not bipartite} \end{cases}$$

Definition 4.4.3 (the Jones subgroup)**.** *We denote the Jones subgroup $\vec{F}$ as the stabilizer of the* *vacuum vector, i.e,*

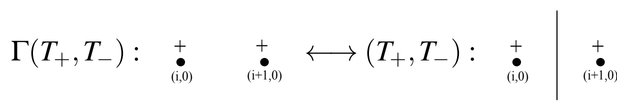
$$\vec{F} = \{g \in F | \pi(g)\xi = \xi\}$$

Remark . *Suppose g has a pair of trees representation as (T_+, T_-). By the remark following Proposition 4.4.2, we have that $g \in \vec{F}$ if and only if $\Gamma(T_+, T_-)$ is a bipartite planar graph.*

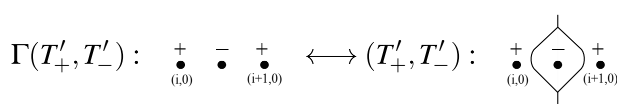
Now we show that $\vec{F}$ is isomorphic to F_3.

Lemma 4.4.4. *Suppose $g \in \vec{F}$, then g has a pair of trees representation as (T_+, T_-) such that the coloring of the vertices of $\Gamma(T_+, T_-)$ is $\pm \mp \pm \mp \cdots \pm$ from the left to right.*

Proof. Suppose (T_+, T_-) is a pair of trees representation of $g \in \vec{F}$. Since $g \in \vec{F}$, $\Gamma(T_+, T_-)$ is a bipartite graph, i.e, there exists a coloring of the vertices. Suppose there exists a vertex $(i, 0)$ with $i \in \mathbb{N}$ and the coloring of $(i, 0)$ and $(i+1, 0)$ are different, i.e,

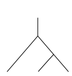


Let (T'_+, T'_-) be the pair of trees such that $T'_\pm$ is the tree obtained by adding a caret on the endpoint $(i+1/2, 0)$ of $T_\pm$. Thus we have

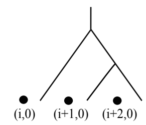


Note that $(T_+, T_-) \sim (T'_+, T'_-)$ and thus by repeating this procedure we obtain a pair of trees representation of g, thereby proving the lemma. ☐

Theorem 4.4.5. *The Jones subgroup $\vec{F}$ is isomorphic to F_3.*

Proof. Let $X =$  . Suppose $g \in \vec{F}$ is a non-trivial element with a pair of trees representation (T_+, T_-) as in Lemma 4.4.4 and T_+ has $2n-1$ leaves.

Claim. *There exists $i \in \mathbb{N}$ such that in T_+*



Proof of the claim. By Proposition 4.4.1, we consider the upper half part of $\Gamma(T_+, T_-)$, denoted by $\Gamma(T_+)$. The claim is equivalent to that there exists $i \in \mathbb{N}$ such that $(i+2,0)$ is connected to $(i+1,0)$ and $(i+1,0)$ is connected to $(i,0)$ by edges in $\Gamma(T_+)$. We prove this statement by induction on the number of vertices. The base case is trivially true since the coloring is $\pm\mp\pm$.

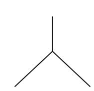
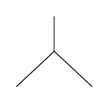
Now suppose the statement holds for $2n-1$ vertices and $\Gamma(T_+)$ has $2n+1$ vertices. Since every vertex in $\Gamma(T_+)$ is connected to exactly one vertex to its left, there exist $m \in \mathbb{N}$ such that $(m,0)$ is connected to $(m-1,0)$ by induction. Let $(k,0)$ be the only vertex that $(m-1,0)$ is connected to on its left. If $k = m-2$, then we find such vertex required by the claim. If $k < m-2$, we apply the induction on the induced sub graph of $\Gamma(T_+)$ on vertices $\{(k,0),(k+1,0),\cdots,(m.0)\}$. Therefore we conclude the claim. $\square$

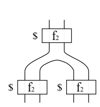
From the claim we obtain that $T_+ \in Alg(X)_{2n-1}$. Therefore $\vec{F} \cong F_3$ by Theorem 4.3.7. $\square$

Remark . *Jones defined a family of subgroups of F which start with $\vec{F}$ using the R-matrix in spin models [Jon14]. These subgroups have been redefined by Golan and Sapir denoted by $\vec{F}_n, n \geq 2$ and shown to be isomorphic to F_{n+1} [GS17]. These results can be obtained by using the same idea in the proof of Theorem 4.4.5.*

4.4.2 The 3-colorable subgroup

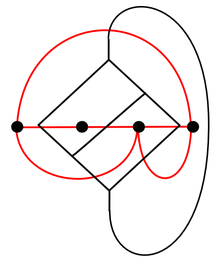
We now recall the definition of the 3-colorable subgroup $\mathscr{F}$ introduced by Jones.

Let $\mathscr{P}_\bullet$ is the subfactor planar algebra $\mathscr{TL}(2)$. We set S to be $3^{\frac{1}{4}}$  , where  =

 .

We construct a unitary representation π using S following Approach 2 in §**??**. Similar to the case in §4.4.1, the matrix coefficient with respect to the vacuum vector has a specific chromatic meaning:

Suppose $g \in F$ with (T_+, T_-) a pair of trees representation. We arrange the pair of trees as in §4.4.1 to obtain a cubic planar graph. Let $\Gamma(T_+, T_-)$ be the dual graph of the cubic graph with vertices $\{(0,0), (1,0), \cdots, (n,0)\}$. For instance,



(4.9)

Proposition 4.4.6. *The matrix coefficient satisfies the following:*

$$\langle \pi(g)\xi, \xi \rangle = \begin{cases} 1, & \Gamma(T_+, T_-) \text{ is 3-colorable} \\ 0, & \Gamma(T_+, T_-) \text{ is not 3-colorable} \end{cases}$$

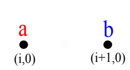
Remark . *We use $\{a, b, c\}$ as the coloring of $\Gamma(T_+, T_-)$ for g having (T_+, T_-) as a pair of representation with $\langle \pi(g)\xi, \xi \rangle = 1$.*

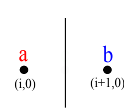
Definition 4.4.7 (Jones)**.** *We define the 3-colorable subgroup $\mathscr{F}$ as the stabilizer of the vacuum vector*

$$\mathscr{F} = \{g \in F | \pi(g)\xi = \xi\}$$

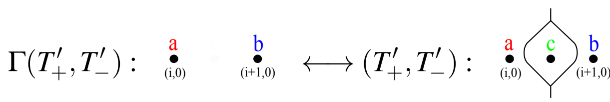
Lemma 4.4.8. *Suppose $g \in \mathscr{F}$, then g has a pair of trees representation (T_+, T_-) such that the coloring of the vertices of $\Gamma(T_+, T_-)$ is $acbacb \cdots ac$.*

Proof. First note that this lemma should hold for the coloring after applying any permutation of $\{a, b, c\}$ to $acbacb \cdots ac$. We will prove the lemma by induction. Suppose (T_+, T_-) is a pair of trees representation of $g \in \mathscr{F}$ and there exists $i \in \mathbb{N}$ with coloring of the vertex $(i, 0)$ is a and the vertex $(i+1, 0)$ is b. Then we apply the same technique as in Lemma 4.4.4

$\Gamma(T_+, T_-)$: 

$\longleftrightarrow$ (T_+, T_-) : 

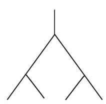
Let (T'_+, T'_-) be the pair of trees such that $T'_\pm$ is the tree obtained by adding a caret on the endpoint $(i+1/2, 0)$ of $T_\pm$. Thus we have



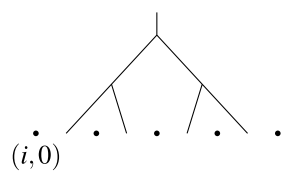
Note that $(T_+, T_-) \sim (T'_+, T'_-)$ and thus by repeating this procedure we obtain a pair of trees representation of g, thereby proving the lemma.

$\square$

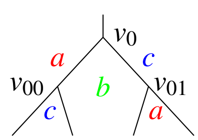
Theorem 4.4.9. *The 3-colorable subgroup $\mathscr{F}$ is isomorphic to F_4.*

Proof. Let $X =$  . Suppose $g \in \mathscr{F}$ having (T_+, T_-) as a pair of trees representation satisfying Lemma 4.4.8 where $T_\pm$ has $3n+1$ leaves.

Claim. *There exits $i \in \mathbb{N}$ such that in T_+,*



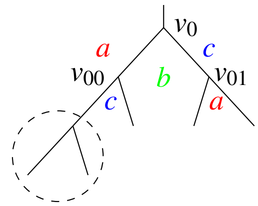
Proof of Claim. We prove the claim by induction. The base case is trivially true since it corresponds to the coloring *acbac*. Suppose this claim holds for binary trees with $3n-2$ leaves. T_+ must start with the form



If there are no more other vertices connected to v_{00} or v_{01}, then the claim holds.

If there are more vertices on the left edge of v_{00}, then the dotted circle part is a tree with

$acbacb\cdots ac$. By induction, we know that the claim holds.



The same argument holds for the cases that there are more vertices on the right edge of v_{00} or the left (or right) edge of v_{01}. $\square$

From the claim we obtain that $T_+ \in Alg(X)_{3n+1}$. Therefore, $\mathscr{F} \cong F_4$ by Theorem 4.3.7 $\square$

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