ON THE CLASSIFICATION OF CLOSED FLAT FOUR-MANIFOLDS

By

Thomas Paul Lambert

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Approved:

Professor John G. Ratcliffe

Professor Bruce Hughes

Professor Guoliang Yu

Professor Steven T. Tschantz

Professor Thomas W. Kephart

To my family and friends, without whom this project would never be completed

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Chapter I

The Classification of Closed Flat 4-Manifolds: An Introduction

An n-dimensional *Euclidean space-form* is the orbit space of the action on $\mathbb{R}^n$ of a torsion-free discrete group of Euclidean isometries Γ. It is well known that if $X = \mathbb{R}^n/\Gamma$ is an n-dimensional Euclidean space-form, then the fundamental group $\pi_1(X) \cong \Gamma$. An important special case is given by the class of compact Euclidean space-forms: any n-dimensional compact Euclidean space-form is a closed flat Riemannian n-manifold, and conversely, any closed flat Riemannian n-manifold is isometric to a compact Euclidean space-form.

An n-dimensional *crystallographic group* is a discrete group of Euclidean isometries Γ whose orbit space $\mathbb{R}^n/\Gamma$ is compact. Crystallographic groups in general were first studied in the 3-dimensional case by physicists and chemists. The 3-dimensional crystallographic groups were classified independently by Federov in 1885, Schoenflies in 1891, and Barlow in 1894.

In 1900, David Hilbert proposed a set of famous problems; his eighteenth dealt with sphere-packings, crystallographic groups, and tilings of n-dimensional space for $n > 3$. Bieberbach's Theorems, stated below, provided an answer to part of Hilbert's 18th problem by showing that there are only finitely many essentially different n-dimensional crystallographic groups for each n.

Theorem I.1 (**Bieberbach's Theorems**)

(a) *If Γ is an n-dimensional crystallographic group, then the subgroup $\Gamma^\star$ of all the translations in Γ is a free abelian normal subgroup of rank n and of finite index in Γ. Moreover, $\Gamma^\star$ is maximal abelian in Γ.*

(b) *Two crystallographic groups Γ_1 and Γ_2 are isomorphic if and only if they are conjugate by an affine transformation.*

(c) *Up to affine equivalence, there are only finitely many n-dimensional crystallographic groups for each n.*

Corollary I.1 *Any two closed flat n-manifolds are homeomorphic if and only if they are affinely equivalent. Moreover, there are only finitely many pairwise affine inequivalent closed flat n-manifolds for each n.*

Remark: In dimensions $1, 2, 3, 4$ there are $1, 2, 10, 74$ affine equivalence classes of closed flat manifolds, respectively.

The problem of classifying the 4-dimensional compact Euclidean space-forms was first studied by Calabi in 1957; Calabi used a recursive approach involving the known 3-dimensional case. Others who have considered this subject include Charlap and Sah. The resulting lists were later found to be incomplete and incorrect; in 1970, in his Berkeley doctoral thesis, Levine [Lev70] published an improved list of computations and groups. This list was later shown to have a duplication. Other classifications include those of Brown *et al* [Bro78] and Hillman [Hil95]; they give complete classifications of the 4-dimensional crystallographic groups and of the flat 4-manifold groups, respectively. More historical details can be found in the introduction of Levine's thesis and in Schwarzenberger's book *N-Dimensional Crystallography* [Sch80].

The classification of the closed flat 4-manifolds can be found in Levine's Berkeley doctoral thesis [Lev70], Brown *et al* [Bro78], and a paper of Hillman [Hil95]. The usability and completeness of their lists bear special mention. The tables in Levine's thesis seem to be the most useful, because the first homology groups were computed; however, Levine's tables contain a duplication. Brown gives a complete classification; however, this description seems to be of more use to crystallographers than geometers and algebraists. Moreover, the tables are presented in notation that is very difficult to decipher. We used Brown's description mainly to obtain sets of generators for the groups. However, these generators are described as *affine* transformations, which are not ideal from a geometrical viewpoint. One of the tasks to address in that regard is to present the groups as *isometry* groups.

The classification in Hillman's paper is also complete, but emphasizes the algebra more than the geometry. Beyond a discussion of Betti numbers, geometry is not used. We used Hillman's results to reconcile Levine's classification with other classifications; through comparisons, we found the duplication in Levine's thesis. It is worth noting that the inconsistency only occurrs in the nonorientable case; the classification of the orientable manifolds appears to be universally agreed upon.

In this paper, we attempt to unify these different classifications of the closed flat 4-manifold groups. We use the tables of Levine and Brown as a starting point to describe the isometry groups. We then construct presentations for the groups and cast them in a form that is as geometrically transparent as possible; then, we realize each group as either a semidirect product or an amalgamated free product. The nature of these decompositions gives more insight into the structure of these closed flat 4-manifolds. In the process, we give a complete description of the fiber-bundle structures of the manifolds where appropriate. For the nonorientable manifolds, we also determine their orientable double-covers. Ultimately, we employ the nomenclature of Brown *et al* with respect to naming the groups, since Levine's list was found to contain 75 groups, with one duplicated nonorientable manifold. We also use a comparative labeling scheme for linking the lists of Levine and Brown wherever possible. Some of the algebraic ideas and inspirations used came from Hillman's paper, though his description was too algebraic in nature and did not explicitly describe the underlying geometry beyond a discussion of fiberings over the circle.

The main goal of this paper is to illustrate in fuller detail the connections between the group structure of the flat 4-manifold groups and the internal geometry of the corresponding closed flat 4-manifolds. In particular, we detail how the group structure as a semidirect product or amalgamated free product lets us distinguish different closed flat 4-manifolds with the same fiber and the same (first) homology. Before, this type of distinction would be very difficult to find. To this end, we give a complete description of the fiber-bundle structures of the closed flat 4-manifolds as well as the orientable double covers of the nonorientable manifolds. The latter information, which gives more insight into the manifold structures, appears to be absent in the current literature. As for the fiber-bundle descriptions, in the case where the Betti number equals 1, the fibering is unique; this gives us an invariant of the manifolds for that situation. Overall, a finer description of the geometry and group structure will let us more easily identify the manifolds under consideration.

The flat 4-manifold groups studied here are basic objects with applications to hyperbolic manifolds and cosmology. For instance, non-compact, finite-volume hyperbolic 5-manifolds have cusp cross-sections that are closed, flat 4-manifolds which are geometric and topological invariants of the hyperbolic manifolds. Applications to cosmology can be found in the study of constant-curvature gravitational instantons; the interested reader is referred to a 1998 paper of Ratcliffe and Tschantz [Rat98] for more details.

The next sections deal with well-known background material which is necessary in the exposition and development of this paper.

Covering Spaces

Let X be a path-connected topological space; then a continuous map $p : \tilde{X} \to X$ is said to be a *covering map* if, for every point $x \in X$, there is an open neighborhood U_x of x in X with $p^{-1}(U_x)$ a disjoint union of open sets (called *sheets*), each of which is mapped homeomorphically onto U_x via p. Any such neighborhood U_x is said to be *evenly covered* by p. For each $x \in X$, the set $p^{-1}(x)$ is called the *fiber* over x; it can be shown that the cardinality of the fiber is independent of x. Thus, it makes sense to speak of the number of sheets of the covering p. If this number is finite, we say that X is *finitely covered* by $\tilde{X}$.

Theorem I.2 *Any compact Euclidean space-form of dimension n is finitely covered by a flat n-torus $\mathbb{R}^n/T$, where T is generated by n linearly independent translations. In fact, if $M = \mathbb{R}^n/\Gamma$, then M is finitely covered by $\mathbb{R}^n/\Gamma^\star$, where $\Gamma^\star$ is the translation subgroup of Γ, which is n-dimensional by Theorem I.1.*

With the setup above, the *holonomy* of M is the quotient $F = \Gamma/\Gamma^\star$; by (a), F is a finite group. The *point group* of M is given as follows; this description can be found in Ratcliffe's book *Foundations of Hyperbolic Manifolds* [Rat2006]:

Any element $\gamma \in \Gamma$ can be written uniquely in the form $\gamma = a + A$, where $a \in \mathbb{R}^n$ and $A \in O(n)$, the group of all orthogonal $n \times n$ matrices over $\mathbb{R}$. There is a natural homomorphism $\eta : \Gamma \to O(n)$ sending $\gamma = a + A$ to A. The *point group* Π of Γ is then the image of η. Note that ker $\eta = \Gamma^\star$; thus, we have a short exact sequence of groups

$$0 \longrightarrow \Gamma^\star \hookrightarrow \Gamma \xrightarrow{\eta} \Pi \longrightarrow 1.$$

Note that the short exact sequence implies that $F = \Gamma/\Gamma^\star \cong \Pi$. If Π is a subgroup of $SO(n)$, the group of all orthogonal $n \times n$ real matrices with determinant 1, then M is said to be *orientable*. If Π contains a matrix with determinant -1, then M is said to be *nonorientable*.

For the cases where M is nonorientable, there is a unique orientable (compact) Euclidean space-form $\tilde{M}$ which double-covers M. Namely, we obtain a fundamental domain for $\tilde{M}$ by gluing two copies of a fundamental domain for M along their boundaries with a twist (to give a consistent orientation for the new space). The natural map $p : \tilde{M} \to M$ identifying the two copies is a 2-sheeted covering map. The n-manifold $\tilde{M}$ is called the *orientable double-cover* of M and is found by taking only the subgroup Γ_0 of orientation-preserving isometries of Γ; then $\tilde{M} = \mathbb{R}^n/\Gamma_0$. One of the noteworthy features of this paper is the complete classification of the orientable double-covers of the nonorientable closed flat 4-manifolds.

Fiber Bundles

The notion of fiber bundles is a natural generalization of covering spaces and vector bundles. Formally, the setup is the following:

Definition I.1 *Let E, B, F be topological spaces, and let $\pi : E \to B$ be a continuous surjective map such that the following local trivialization property holds:*

For any point $b \in B$, there exists an open neighborhood U_b of b in B such that $\pi^{-1}(U_b)$ is homeomorphic to the product $U_b \times F$, with the homeomorphism acting as π in the first component; i.e. if $\varphi_b : \pi^{-1}(U_b) \to U_b \times F$ is the homeomorphism and $p_1 : U_b \times F \to U_b$ is the natural projection, then the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_b) & \xrightarrow{\varphi_b} & U_b \times F \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle p_1} & \\ U_b & & \end{array}$$

Then E is said to be a fiber bundle over B with fiber F; the map π is called the bundle projection. Note that, for each $b \in B$, the preimage $\pi^{-1}(b)$, called the fiber over b, is homeomorphic to F.

The prototypical examples of fiber bundles include the following:

- The *trivial bundle*, with $E = B \times F$ and $\pi : E \to B$ given by projection onto B.
- Any covering space is a fiber bundle with discrete fiber.
- Any (real) vector bundle is a fiber bundle with a (real) vector space as fiber.
- If X is a path-connected space, and $h : X \to X$ is a self-homeomorphism, then the *mapping torus* $T_h(X)$ is the quotient space of $X \times I$ resulting from

identifying $(x,0)$ with $(h(x),1)$ for all $x \in X$. The map $\pi : T_h(X) \to S^1 = I/(0 \sim 1)$ given by $\pi([(x,t)]) = [t]$ is a bundle projection; therefore, $T_h(X)$ is a fiber bundle over S^1 with fiber X. The self-homeomorphism h is called the *monodromy* of the mapping torus. The canonical examples of mapping tori are the torus and the *Klein bottle*, with $X = S^1$ and $h : (S^1, 1) \to (S^1, 1)$ given by the identity and complex conjugation, respectively.

- Suppose X is an n-manifold; an $(n+1)$-manifold E is a *twisted I-bundle* over X if E is a fiber bundle over X with fiber I such that ∂E is connected. In particular, since $\partial(X \times I)$ is not connected, any twisted I-bundle is nontrivial. The canonical example of a twisted I-bundle is the *Möbius strip* E, with $X = S^1$.

An excellent, detailed resource on fiber bundles is Steenrod's book *The Topology of Fibre Bundles* [Ste51].

To understand the geometry of closed flat 4-manifolds better, we make use of the following result found in Hillman's paper [Hil95]:

Theorem I.3 *If M is a closed flat 4-manifold, then either M is a mapping torus $T_h(X)$ for some closed flat 3-manifold X, or M is the union of two twisted I-bundles over closed flat 3-manifolds, joined together along their (common) closed flat 3-manifold boundaries.*

This description is captured by the structure of the isometry groups Γ. If $M = \mathbb{R}^4/\Gamma$ is a mapping torus, then Γ splits as a semidirect product $\Gamma = \Gamma' \rtimes \mathbb{Z}$, where Γ' is a flat 3-manifold group. In fact, assume that $M = T_h(X)$, where X is a closed flat 3-manifold and where $h : (X, x_0) \to (X, x_0)$ is basepoint-preserving. Then a standard argument using Van Kampen's Theorem yields a presentation of $\Gamma = \pi_1(M)$:

$$\pi_1(M) = \langle \pi_1(X), t \mid t\gamma t^{-1} = h_*(\gamma),\ \gamma \in \pi_1(X) \rangle,$$

where $\pi_1(X) = \Gamma'$. Here, M fibers over S^1 with fiber X. In the case where M has Betti number 1, the fibering of M is unique. In other cases, M may fiber in two different ways; for those manifolds, we describe two different fiber-bundle structures in our tables and presentations in Chapters III and V.

If M is a union of twisted I-bundles, then Γ can be realized as an amalgamated free product $\Gamma = \Gamma_1 \star_G \Gamma_2$, where each Γ_i is a flat 3-manifold group containing an isomorphic copy of another orientable flat 3-manifold group G as a subgroup of index 2. If M_1 and M_2 are the corresponding flat 3-manifolds, then the group G corresponds to the common flat 3-manifold boundary of M_1, M_2 along which the bundles are joined. In the case where M has Betti number 0 (i.e. when Γ has finite abelianization), M does not fiber over S^1; Hillman gives several different amalgamated free product decompositions for each such group. These decompositions give rise to several different I-bundle decompositions for each corresponding manifold. In our tables and presentations, we list a single decomposition for each of the four flat 4-manifold groups with finite abelianization. It turns out that these are all nonorientable; therefore, it is of particular interest to examine closely the decompositions listed below.

Chapter II

Geometric Descriptions of the Flat 4-Manifold Groups: Constructing the Tables

In this chapter, we describe the techniques employed to convert the descriptions in Brown's book [Bro78] to ones which reflect the geometry more readily. This is necessary for those descriptions which use matrices which are not all orthogonal, so that the corresponding affine maps are not isometries.

For those manifolds with abelian holonomy, all generating matrices commute, and therefore there exists a basis for $\mathbb{R}^4$ with respect to which all the matrices are diagonal with entries ± 1. In particular, the resulting matrices are orthogonal. Assuming that the original matrices represent linear maps with respect to the standard ordered lattice basis (e_1, e_2, e_3, e_4), it is then straighforward to compute the new lattice basis using the change of basis which simultaneously diagonalizes the matrices.

It is clear that if A, B are simultaneously diagonalizable, then A, B must commute. The converse is harder, and the proof will give us a constructive method of computing a change of basis which does the job.

Suppose A and B commute and are diagonalizable. Let P be an invertible matrix such that $P^{-1}AP = D$, a diagonal matrix. Without loss of generality, assume that the eigenvalues of A are ordered so that all repeated eigenvalues are consecutive. Then if $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A, with multiplicities $m_1, \ldots, m_k$, respectively, we have

$$D = \begin{pmatrix} \lambda_1 I_{m_1} & & \\ & \ddots & \\ & & \lambda_k I_{m_k} \end{pmatrix}.$$

Now let $B' = P^{-1}BP$; then D and B' commute, since A and B commute. Hence

$$D_{ii}B'_{ij} = B'_{ij}D_{jj}$$

for all i, j. Thus, $B'_{ij}[D_{ii} - D_{jj}] = 0$. Therefore, $B'_{ij} = 0$ if $D_{ii} \neq D_{jj}$. Because of how D was constructed, if then follows that if D_{ii} and D_{jj} belong to different diagonal blocks (i.e. if they correspond to distinct eigenvalues), then $B'_{ij} = 0$. This means that B' is a block diagonal matrix of the form

$$B' = \begin{pmatrix} B'_1 & & \\ & \ddots & \\ & & B'_k \end{pmatrix},$$

where each B'_i is an $m_i \times m_i$ matrix. Since B is diagonalizable, so is B'; hence, each B'_i is diagonalizable, say by invertible matrices Q_i. Then the matrix

$$Q = \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_k \end{pmatrix}$$

will diagonalize B', yielding

$$Q^{-1}B'Q = \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_k \end{pmatrix},$$

where each E_i is diagonal. Notice that applying Q in this way to D yields a diagonal matrix; in fact,

$$Q^{-1}DQ = \begin{pmatrix} Q_1^{-1}(\lambda_1 I_{m_1})Q & & \\ & \ddots & \\ & & Q_k^{-1}(\lambda_k I_{m_k})Q_k \end{pmatrix} = \begin{pmatrix} \lambda_1 I_{m_1} & & \\ & \ddots & \\ & & \lambda_k I_{m_k} \end{pmatrix} = D.$$

Thus, the (invertible) matrix PQ will simultaneously diagonalize A and B.

Sometimes, simultaneous diagonalization (over the reals) is impossible; e.g. if the matrices do not commute, or if at least one has complex eigenvalues. In this case, we use a standard technique to realize the matrix as a block orthogonal matrix, as follows:

Suppose A is a real $n \times n$ matrix with at least one complex eigenvalue $\lambda = \mu + i\nu$. Let $v = u + iw$ be a corresponding eigenvector, where u, w are real. First, observe that $\bar{\lambda} = \mu - i\nu$ is also an eigenvalue of A, with corresponding eigenvector $\bar{v} = u - iw$, since A is a real matrix, and since any complex roots of a real polynomial (here, the characteristic polynomial of A) occur in complex conjugate pairs.

We now claim that the vectors u and w are linearly independent over $\mathbb{R}$. Indeed, any two eigenvectors associated to distinct eigenvalues are linearly independent. In particular, since $\bar{v} = u - iw$ corresponds to the eigenvalue $\bar{\lambda} = \mu - i\nu$, and since $\nu \neq 0$, the vectors $v, \bar{v}$ are linearly independent over $\mathbb{C}$.

Now suppose that $au + bw = 0$ for some real scalars a and b, not both zero. Let $c = (a - ib)/2$. We then have that $c \neq 0$; moreover,

$$cv + \bar{c}\bar{v} = 2Re(cv);$$

but

$$cv = \frac{1}{2}(a - ib)(u + iw) = \frac{1}{2}[(au + bw) + i(aw - bu)].$$

Thus,

$$cv + \bar{c}\bar{v} = 2Re(cv) = au + bw = 0,$$

which contradicts the linear independence (over $\mathbb{C}$) of v and $\bar{v}$. Hence, $a = b = 0$ and u, v are indeed linearly independent.

We now study the action of A on the 2-dimensional subspace spanned by u and v. To do this, expand the identity $Av = \lambda v$ to obtain

$$Av = \lambda v \Rightarrow Au + iAw = (\mu + i\nu)(u + iw) = (\mu u - \nu v) + i(\nu u + \mu w);$$

equating real and imaginary parts,

$$Au = \mu u - \nu v$$

and

$$Aw = \nu u + \mu w.$$

This means that the matrix of the restriction $A|_{\langle u,w\rangle}$ with respect to the ordered basis $\{u, w\}$ is:

$$A_{\langle u,w\rangle} = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix}.$$

If we start with a matrix A with all its eigenvalues having norm 1, then $\mu = \cos\theta$ and $\nu = \sin\theta$ for some $\theta \in [0, 2\pi)$. Hence, the restriction of A to $\langle u, w\rangle$ is simply a counterclockwise rotation of θ about the origin.

Now suppose that A is diagonalizable over the complex numbers; that is, $\mathbb{C}^n$ has a basis $\{v_1, \ldots, v_n\}$ of (complex) eigenvectors of A. Without loss of generality, suppose that $v_1, \ldots, v_k$ are real and $v_{k+1}, \ldots, v_n$ are complex. As any complex roots of a real polynomial occur in complex conjugate pairs, it follows that $n - k$ must be even, say $n - k = 2l$. Write the eigenvectors as

$$v_1, \ldots, v_k, v_{k+1} = u_1 + iw_1, v_{k+2} = u_1 - iw_1, \ldots, v_{n-1} = u_l + iw_l, v_n = u_l - iw_l,$$

with corresponding eigenvalues

$$\lambda_1, \ldots, \lambda_k, \lambda_{k+1} = \mu_1 + i\nu_1, \lambda_{k+2} = \mu_1 - i\nu_1, \ldots, \lambda_{n-1} = \mu_l + i\nu_l, \lambda_n = \mu_l - i\nu_l.$$

We claim that the vectors $v_1, \ldots, v_k, u_1, w_1, \ldots, u_l, w_l$ are linearly independent over $\mathbb{R}$. Indeed, suppose that

$$(a_1 v_1 + \cdots a_k v_k) + (r_1 u_1 + s_1 w_1) + \cdots + (r_l u_l + s_l w_l) = 0$$

for some real scalars a_i, r_j, s_j. Letting $c_j = (r_j - is_j)/2$ as before, we have

$$(a_1 v_1 + \cdots a_k v_k) + (c_1 v_{k+1} + \bar{c}_1 v_{k+2}) + \cdots + (c_l v_{n-1} + \bar{c}_l v_n) = 0.$$

Since $\{v_1, \ldots, v_n\}$ is a basis, we have $a_i = 0$ for $i = 1, \ldots, k$ and $c_j = 0$ for $j = 1, \ldots, l$. In turn, we have $r_j = 0$ and $s_j = 0$ for $j = 1, \ldots, l$. This shows that $v_1, \ldots, v_k, u_1, w_1, \ldots, u_l, w_l$ are indeed linearly independent over $\mathbb{R}$ and therefore form a basis for $\mathbb{R}^n$, since there are $k + l + l = n$ vectors given here.

By considering the restrictions of A to each subspace $\langle u_i, w_i \rangle$, for $i = 1, \ldots, l$, we see that the matrix of A with respect to the basis $\{v_1, \ldots, v_k, u_1, w_1, \ldots, u_l, w_l\}$ is block diagonal:

$$A_{\langle v_1,\ldots,v_k,u_1,w_1,\ldots,u_l,w_l\rangle} = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_k & & & & & \\ & & & \mu_1 & -\nu_1 & & & \\ & & & \nu_1 & \mu_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \mu_l & -\nu_l \\ & & & & & & \nu_l & \mu_l \end{pmatrix}.$$

Again, if we start with a matrix A whose eigenvalues all have norm 1, we have $\mu_k = \cos\theta_k$ and $\nu_k = \sin\theta_k$ for some angles $\theta_k \in [0, 2\pi)$, for $k = 1, \ldots, l$. Denoting the corresponding rotation matrices by

$$R_k = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix},$$

we have that

$$A_{\langle v_1,\ldots,v_k,u_1,w_1,\ldots,u_l,w_l\rangle} = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & R_1 & & \\ & & & & \ddots & \\ & & & & & R_l \end{pmatrix}.$$

We employ the ideas of this technique using *Mathematica* to carry out most of the computations needed to find a change of basis which makes all matrices orthogonal. To this end, we use the well-known fact that the full affine (isometry) group of $\mathbb{R}^n$

can be realized as the matrix group of all $(n+1)\times(n+1)$ matrices of the form

$$\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix},$$

where $A \in GL_n(\mathbb{R})$ $(A \in O(n))$ and a is a vector in $\mathbb{R}^n$. Then all the computations can be carried out using matrix multiplication, which makes the technique more efficient.

Once we perform a change of basis, say M, which makes every matrix which generates the holonomy orthogonal, we let $\Lambda = \mathrm{M}^{-1}$. Then we construct a linear isomorphism $\Lambda' = 0 + \Lambda$ and put $\Gamma' = \Lambda'\Gamma(\Lambda')^{-1}$. Since conjugation is an isomorphism of groups, and since the new tranformations are all isometries, we obtain a genuine isometry group which is isomorphic to Γ. In addition, we show that Γ' has the same representation, with respect to Brown, as the original group Γ.

In Brown's description, the generators for the various crystallographic groups are notated as follows: the map $a + A$ is written as

$$A : [x_1, x_2, x_3, x_4]/r \text{ if } a = (x_1, x_2, x_3, x_4)/r.$$

Here, the basis is arbitrary; it will be convenient to begin with the standard basis $\{e_1, e_2, e_3, e_4\}$. Then any change of basis M effects a transformation $A' = \Lambda A \Lambda^{-1}$ which satisfies

$$A'(\Lambda(e_i)) = \Lambda(A(e_i)) = \Lambda(\sum_j A_{ij} e_j) = \sum_j A_{ij} \cdot \Lambda(e_j);$$

this shows that the new lattice basis will be $\{\Lambda(e_i) : i = 1, 2, 3, 4\}$. Also, the fixed-point-free groups are given special notation, so we can easily identify the flat manifold groups.

For brevity, put $a_i = \Lambda(e_i), i = 1, 2, 3, 4$. Suppose $\alpha = a + A$ with respect to the standard basis; then $a = \sum_j \frac{x_j}{r} e_j$. In the new basis, an easy calculation shows that

$\alpha' = \Lambda(a) + A'$. But

$$\Lambda(a) = \sum_j \frac{x_j}{r}\Lambda(e_j) = \sum_j \frac{x_j}{r}a_j.$$

So we can follow the notation of Brown and denote the new map α' by

$$A' : [x_1, x_2, x_3, x_4]/r$$

with respect to the new basis.

We now proceed to show that the relations for Γ are preserved, up to relabeling as above, by conjugation by Λ'. First, we deal with translations and powers of affine maps. As each of the groups we are considering has finite point group, for each $\alpha = a + A$ there is an integer $k \geq 0$ with $\alpha^k = T$, where $T = t + I$ is a pure translation. The next proposition shows that this relationship is preserved under conjugation by Λ':

Proposition II.1 *(a) If $T = t + I$ is a translation, then so is $T' = (\Lambda')T(\Lambda')^{-1}$.*

(b) If $\alpha^k = T$, where T is a translation, then $(\alpha')^k = T'$, and so $(\alpha')^k$ is also a translation.

Proof This is a simple calculation; for part (a), note that $T' = \Lambda(t) + I'$ and $I' = \Lambda' I(\Lambda')^{-1} = I$. For part (b), consider the equality $\alpha^k = T$ and conjugate both sides by Λ':

$$\Lambda'\alpha^k(\Lambda')^{-1} = T' \Rightarrow [\Lambda'\alpha(\Lambda')^{-1}]^k = T' \Rightarrow (\alpha')^k = T'.$$

This proves (b). ♦

Moreover, any relation of the form $\alpha\beta\alpha^{-1} = \gamma$ will be preserved under the change of basis:

Proposition II.2 *Suppose $\alpha = a + A$ and $\beta = b + B$ are affine maps, where A, B are invertible $n \times n$ matrices. If*

$$\alpha\beta\alpha^{-1} = \gamma$$

for some affine map $\gamma = c + C$ (C invertible), then, in the notation above,

$$\alpha'\beta'(\alpha')^{-1} = \gamma'.$$

Proof This turns out to be another straightforward calculation, based on the identity

$$\begin{aligned}(a + A)(b + B)(a + A)^{-1} &= (a + A)(b + B)(-A^{-1}(a) + A^{-1}) \\ &= (a + A)(b - BA^{-1}(a) + BA^{-1}) \\ &= a + A(b) - ABA^{-1}(a) + ABA^{-1} \\ &= [(I - ABA^{-1})(a) + A(b)] + ABA^{-1}.\end{aligned}$$

Suppose $\alpha\beta\alpha^{-1} = \gamma$; then $\gamma = c + C$, where $C = ABA^{-1}$ and

$$c = (I - C)(a) + A(b).$$

Now consider the transformed maps α', β', γ'. Then

$$\alpha'\beta'(\alpha')^{-1} = (\Lambda(a) + A')(\Lambda(b) + B')(\Lambda(a) + A')^{-1}$$

and

$$\gamma' = \Lambda(c) + C';$$

thus,

$$\alpha'\beta'(\alpha')^{-1} = [(I - A'B'(A')^{-1})(\Lambda(a)) + A'(\Lambda(b))] + A'B'(A')^{-1}.$$

But

$$\begin{aligned}\Lambda(c) &= \Lambda(I - C)(a) + \Lambda(A(b)) \\ &= (\Lambda - \Lambda C)(a) + \Lambda A(b) \\ &= (\Lambda - C'\Lambda)(a) + A'\Lambda(b) = (I - C')(\Lambda(a)) - A'(\Lambda(b));\end{aligned}$$

moreover,

$$C' = \Lambda C \Lambda^{-1} = \Lambda(ABA^{-1})\Lambda^{-1} = (\Lambda A \Lambda^{-1})(\Lambda B \Lambda^{-1})(\Lambda A \Lambda^{-1})^{-1} = A'B'A'^{-1}.$$

Therefore,

$$\begin{aligned} \alpha'\beta'(\alpha')^{-1} &= [(I - A'B'(A')^{-1})(\Lambda(a)) + A'(\Lambda(b))] + A'B'(A')^{-1} \\ &= [(I - C')(\Lambda(a)) - A'(\Lambda(b))] + C' = \Lambda(c) + C' = \gamma', \end{aligned}$$

which was what we wanted to prove. ♦

In summary, suppose Γ is an n-dimensional crystallographic group generated by several affine maps $\alpha_j = a_j + A_j, j = 1, \ldots, r$ and basic translations $t_k = e_k + I, k = 1, \ldots, n$. For each i, j, denote by x_{ij} the translation $\alpha_i(t_j) = A_i(e_j) + I$. Also, suppose Γ is presented by

$$\Gamma = \langle \alpha_j, t_k \mid [t_i, t_j] = 1, 1 \le i < j \le n,\ \alpha_j^{s_j} = T_j, j = 1, \ldots, r,$$
$$\alpha_i \alpha_j \alpha_i^{-1} = \beta_j, 1 \le i < j \le r,\ \alpha_i t_j \alpha_i^{-1} = x_{ij}, i = 1, \ldots, r, j = 1, \ldots, n\rangle.$$

In the notation of this chapter, $t'_k = \Lambda(e_k) + I = a_k + I$; hence, $[t'_i, t'_j] = 1$. Also, each T_j is given by

$$T_j = (\sum_k u_k e_k) + I,$$

and so

$$T'_j = (\sum_k u_k a_k) + I.$$

Moreover, by Propositions II.1 and II.2, $(\alpha'_j)^{s_j} = T'_j$ for all j and $\alpha'_i \alpha'_j (\alpha'_i)^{-1} = \beta'_j$ for all i, j. Also,

$$x'_{ij} = \Lambda(A_i(e_j)) + I = A'_i(\Lambda(e_j)) + I = A'_i(a_j) + I.$$

Therefore, Γ' is presented by

$$\Gamma' = \langle \alpha'_j, t'_k \mid [t'_i, t'_j] = 1, 1 \le i < j \le n,\ (\alpha'_j)^{s_j} = T'_j, j = 1, \ldots, r,$$
$$\alpha'_i \alpha'_j (\alpha'_i)^{-1} = \beta'_j, 1 \le i < j \le r,\ \alpha'_i t'_j (\alpha'_i)^{-1} = x'_{ij}, i = 1, \ldots, r, j = 1, \ldots, n\rangle.$$

That is, up to a change of indices, the new group Γ', which is now a true group of Euclidean isometries (since the matrices A'_j are orthogonal by construction), can be described exactly as the old group Γ was depicted.

Chapter III

Tables of Flat 3- and 4-Manifold Groups: Algebraic Descriptions

In this chapter are presented lists of all 10 (closed) flat 3-manifold groups and all 74 flat 4-manifold groups, with their fiberings (if applicable), homology, holonomy, and product structures of the groups. Using these tables, along with the notation of this section, one can reproduce an abstract presentation for each group.

For geometric descriptions of the generators and relations, one can consult [Wol84] or [Lev70]; the tables for the 4-manifold groups follow loosely the tables in [Lev70].

For ease of presentation, we will label the generators of the flat 3-manifold groups (except the Hantzsche-Wendt group O^3_6) by x, y, and z, where $\langle x, y\rangle = \mathbb{Z} \oplus \mathbb{Z}$ and $\langle z\rangle = \mathbb{Z}$. Thus, we may express O^3_2, for example, as

$$O^3_2 = \langle x, y, z \mid xy = yx,\ zxz^{-1} = x^{-1},\ zyz^{-1} = y^{-1}\rangle.$$

Also, for semidirect products $G \rtimes_T \mathbb{Z} = \langle x, y\rangle \rtimes_T \langle z\rangle$, we identify the homomorphism $T : \mathbb{Z} \to \mathrm{Aut}(G)$ with its image $T(z)$ as a automorphism of G. Furthermore, if $G = \mathbb{Z}^2$, we will identify $T(z)$ with its corresponding matrix in $GL_2(\mathbb{Z})$. In all cases, we will describe T by its action on the generators of G; namely, $\mathbb{Z} = \langle z\rangle$ acts on G via conjugation. For instance, in the case of O^3_2, T would be described by

$$z \leftrightarrow c_z : x \mapsto x^{-1}, y \mapsto y^{-1},$$

where $c_z : g \mapsto zgz^{-1}$ denotes conjugation by z.

For the group O^3_6, we will use the presentation

$$O^3_6 = \langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2}\rangle.$$

In the case of the flat 4-manifold groups, we follow a similar notation; for semidirect products

$$G \rtimes_T \mathbb{Z} = \langle x, y, z \rangle \rtimes_T \langle w \rangle,$$

identify the homomorphism $T : \mathbb{Z} \rightarrow \text{Aut}(G)$ with the image $T(w) \in \text{Aut}(G)$. Here, the group $\mathbb{Z} = \langle w \rangle$ acts on G by conjugation, and T is described similarly. We will describe the corresponding fibering as $M \rtimes S^1$, where M is the appropriate closed flat 2- or 3-manifold. By abuse of notation, we will often denote the flat manifold and its group by the same label. For each group, the automorphism T is determined up to its outer automorphism class in $\text{Aut}(G)$: if two automorphisms $T_1, T_2 \in \text{Aut}(G)$ are conjugate in $\text{Aut}(G)$ up to inversion, then the groups $G \rtimes_{T_1} \mathbb{Z}$ and $G \rtimes_{T_2} \mathbb{Z}$ are isomorphic and therefore represent the same flat manifold. For an analysis of the outer automorphism groups, see [Hil95].

When the manifold has $\beta_1 = 0$, it does not possess a decomposition as a fiber bundle over S^1. In this case, Γ decomposes as an amalgamated free product $\Gamma_1 \star_G \Gamma_2$ by [Hil95]. This decomposition of Γ corresponds to a decomposition of the flat manifold as a union of two twisted I-bundles joined along their common boundaries. The only such 3-dimensional example is the Hantzsche-Wendt manifold; this manifold is orientable. In the 4-dimensional case, there are 4 such manifolds, all of which are nonorientable.

Notation:

(1) D_n will denote the dihedral group of order $2n$.

(2) Δ will denote the *tetrahedral group*, the group of symmetries of a regular tetrahedron (which is isomorphic to the alternating group A_4 on four letters).

(3) π_1 denotes the fundamental group of the manifold, H_1 its first homology group, and β_1 its first Betti number.

(4) For the flat 3-manifold groups, K^2 will denote the *Klein bottle group*, with presentation

$$K^2 = \langle x, y \mid xyx^{-1} = y^{-1} \rangle.$$

(5) In the tables of the nonorientable flat 3- and 4-manifold groups, we list the orientable double-covers (abbreviated *ODC*) along with the algebraic presentations.

The Flat 3-Manifold Groups

Manifold	Fibering	π_1	H_1	β_1	$T : z \leftrightarrow c_z$	Holonomy
O^3_1	$T^2 \times S^1$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\{1\}$
O^3_2	$T^2 \rtimes S^1$	$\mathbb{Z}^2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mathbb{Z}_2$
O^3_3	$T^2 \rtimes S^1$	$\mathbb{Z}^2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_3$	1	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\mathbb{Z}_3$
O^3_4	$T^2 \rtimes S^1$	$\mathbb{Z}^2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\mathbb{Z}_4$
O^3_5	$T^2 \rtimes S^1$	$\mathbb{Z}^2 \rtimes_T \mathbb{Z}$	$\mathbb{Z}$	1	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\mathbb{Z}_6$

Table 1: Orientable Flat 3-Manifold Groups with Infinite Abelianization

Manifold	π_1	H_1	β_1	G	Holonomy
O^3_6	$K^2 \star_G K^2$	$(\mathbb{Z}_4)^2$	0	$\mathbb{Z}^2$	$(\mathbb{Z}_2)^2$

Table 2: Flat 3-Manifold Groups with Finite Abelianization

Manifold	Fibering	π_1	H_1	β_1	$T : z \leftrightarrow c_z$	Holonomy	ODC
N_1^3	$K^2 \times S^1$ $T^2 \rtimes S^1$	$K^2 \times \mathbb{Z}$ $\mathbb{Z}^2 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$\begin{matrix} x \mapsto x \\ y \mapsto y \end{matrix}$; $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mathbb{Z}_2$	O_1^3
N_2^3	$K^2 \rtimes S^1$ $T^2 \rtimes S^1$	$K^2 \rtimes_T \mathbb{Z}$ $\mathbb{Z}^2 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2$	2	$\begin{matrix} x \mapsto xy^{-1} \\ y \mapsto \ \ y \end{matrix}$; $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\mathbb{Z}_2$	O_1^3
N_3^3	$K^2 \rtimes S^1$	$K^2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$\begin{matrix} x \mapsto x^{-1} \\ y \mapsto y^{-1} \end{matrix}$	$(\mathbb{Z}_2)^2$	O_2^3
N_4^3	$K^2 \rtimes S^1$	$K^2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4$	1	$\begin{matrix} x \mapsto x^{-1}y \\ y \mapsto y^{-1} \end{matrix}$	$(\mathbb{Z}_2)^2$	O_2^3

Table 3: Nonorientable Flat 3-Manifold Groups

Notes:

- O_i^3 fibers over S^1 with fiber the flat 2-torus T^2 for $i = 1, \ldots, 5$.
- The Hantzsche-Wendt manifold O_6^3 is a union of two twisted I-bundles over K^2 joined along their common boundaries (in this case, the common boundary is T^2).
- N_1^3 and N_2^3 both fiber over S^1 in two ways (with fiber either the flat 2-torus T^2 or the Klein bottle K^2).
- N_3^3 and N_4^3 fiber uniquely over S^1 (with fiber K^2).

The Orientable Flat 4-Manifold Groups

Manifold	Fibering	π_1	H_1	β_1	$T : w \leftrightarrow c_w$	Holonomy
O^4_1	$O^3_1 \times S^1$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	4	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\{1\}$
O^4_2	$O^3_2 \times S^1$ $O^3_1 \rtimes S^1$	$O^3_2 \times \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2$	2	$\begin{matrix} x \mapsto x \\ y \mapsto y \\ z \mapsto z \end{matrix}$; $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbb{Z}_2$
O^4_3	$O^3_2 \rtimes S^1$ $O^3_1 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$\begin{matrix} x \mapsto x \\ y \mapsto y \\ z \mapsto y^{-1}z \end{matrix}$; $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\mathbb{Z}_2$
O^4_4	$O^3_3 \times S^1$ $O^3_1 \rtimes S^1$	$O^3_3 \times \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3$	2	$\begin{matrix} x \mapsto x \\ y \mapsto y \\ z \mapsto z \end{matrix}$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$	$\mathbb{Z}_3$
O^4_5	$O^3_3 \rtimes S^1$ $O^3_1 \rtimes S^1$	$O^3_3 \rtimes_T \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2$	2	$\begin{matrix} x \mapsto x \\ y \mapsto y \\ z \mapsto y^{-1}z \end{matrix}$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}$	$\mathbb{Z}_3$
O^4_6	$O^3_4 \times S^1$ $O^3_1 \rtimes S^1$	$O^3_4 \times \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$\begin{matrix} x \mapsto x \\ y \mapsto y \\ z \mapsto z \end{matrix}$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\mathbb{Z}_4$
O^4_7	$O^3_4 \rtimes S^1$ $O^3_1 \rtimes S^1$	$O^3_4 \rtimes_T \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2$	2	$\begin{matrix} x \mapsto x \\ y \mapsto y \\ z \mapsto x^{-1}z \end{matrix}$; $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\mathbb{Z}_4$
O^4_8	$O^3_5 \times S^1$ $O^3_1 \rtimes S^1$	$O^3_5 \times \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2$	2	$\begin{matrix} x \mapsto x \\ y \mapsto y \\ z \mapsto z \end{matrix}$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$	$\mathbb{Z}_6$
O^4_9	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	1	$\begin{matrix} x \mapsto x^{-1} \\ y \mapsto y \\ z \mapsto z^{-1} \end{matrix}$	$(\mathbb{Z}_2)^2$
O^4_{10}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	1	$\begin{matrix} x \mapsto x^{-1} \\ y \mapsto y \\ z \mapsto xz^{-1} \end{matrix}$	$(\mathbb{Z}_2)^2$

Table 4: Orientable Flat 4-Manifold Groups

Manifold	Fibering	π_1	H_1	β_1	$T: w \leftrightarrow c_w$	Holonomy
O^4_{11}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	1	$x \mapsto x^{-1}$ $y \mapsto y$ $z \mapsto xyz^{-1}$	$(\mathbb{Z}_2)^2$
O^4_{12}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto y$ $y \mapsto x$ $z \mapsto z^{-1}$	$(\mathbb{Z}_2)^2$
O^4_{13}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4$	1	$x \mapsto y$ $y \mapsto x$ $z \mapsto xz^{-1}$	$(\mathbb{Z}_2)^2$
O^4_{14}	$O^3_6 \times S^1$	$O^3_6 \times \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_4)^2$	1	$x \mapsto x$ $y \mapsto y$	$(\mathbb{Z}_2)^2$
O^4_{15}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto x^{-1}$ $y \mapsto y^{-1}$	$(\mathbb{Z}_2)^2$
O^4_{16}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto x^{-1}$ $y \mapsto x^2y^{-1}$	$(\mathbb{Z}_2)^2$
O^4_{17}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	1	$x \mapsto xy^2$ $y \mapsto y$	$(\mathbb{Z}_2)^2$
O^4_{18}	$O^3_3 \rtimes S^1$	$O^3_3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_6$	1	$x \mapsto y$ $y \mapsto x$ $z \mapsto z^{-1}$	D_3
O^4_{19}	$O^3_3 \rtimes S^1$	$O^3_3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$x \mapsto y^{-1}$ $y \mapsto x^{-1}$ $z \mapsto z^{-1}$	D_3
O^4_{20}	$O^3_3 \rtimes S^1$	$O^3_3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$x \mapsto y^{-1}$ $y \mapsto x^{-1}$ $z \mapsto xz^{-1}$	D_3
O^4_{21}	$O^3_4 \rtimes S^1$	$O^3_4 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto y$ $y \mapsto x$ $z \mapsto z^{-1}$	D_4
O^4_{22}	$O^3_4 \rtimes S^1$	$O^3_4 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4$	1	$x \mapsto y$ $y \mapsto x$ $z \mapsto xz^{-1}$	D_4
O^4_{23}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4$	1	$x \mapsto y$ $y \mapsto x$	D_4
O^4_{24}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$x \mapsto x^2y$ $y \mapsto x$	D_4
O^4_{25}	$O^3_5 \rtimes S^1$	$O^3_5 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$x \mapsto y^{-1}$ $y \mapsto x^{-1}$ $z \mapsto z^{-1}$	D_6
O^4_{26}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z}$	1	$x \mapsto x^{-1}y$ $y \mapsto x^{-1}$	Δ
O^4_{27}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z}$	1	$x \mapsto y^{-1}$ $y \mapsto xy$	Δ

The Nonorientable Flat 4-Manifold Groups

Manifold	Fibering	π_1	H_1	β_1	$T : w \leftrightarrow c_w$	Holonomy	ODC
N_1^4	$N_1^3 \times S^1$ $O_1^3 \rtimes S^1$	$N_1^3 \times \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2$	3	$\begin{array}{l} x \mapsto x \\ y \mapsto y \\ z \mapsto z \end{array} ; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\mathbb{Z}_2$	O_1^4
N_2^4	$N_2^3 \times S^1$ $O_1^3 \rtimes S^1$	$N_2^3 \times \mathbb{Z}$ $\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^3$	3	$\begin{array}{l} x \mapsto x \\ y \mapsto y \\ z \mapsto z \end{array} ; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\mathbb{Z}_2$	O_1^4
N_3^4	$N_1^3 \rtimes S^1$ $O_2^3 \rtimes S^1$	$N_1^3 \rtimes_T \mathbb{Z}$ $O_2^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2$	2	$\begin{array}{l} x \mapsto x^{-1} \\ y \mapsto y^{-1} \\ z \mapsto z \end{array} ; \begin{array}{l} x \mapsto x \\ y \mapsto y^{-1} \\ z \mapsto z \end{array}$	$(\mathbb{Z}_2)^2$	O_2^4
N_4^4	$N_1^3 \rtimes S^1$ $O_2^3 \rtimes S^1$	$N_1^3 \rtimes_T \mathbb{Z}$ $O_2^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$\begin{array}{l} x \mapsto x^{-1} \\ y \mapsto y^{-1} \\ z \mapsto yz \end{array} ; \begin{array}{l} x \mapsto x \\ y \mapsto y^{-1} \\ z \mapsto y^{-1}z \end{array}$	$(\mathbb{Z}_2)^2$	O_2^4
N_5^4	$N_1^3 \rtimes S^1$ $O_2^3 \rtimes S^1$	$N_1^3 \rtimes_T \mathbb{Z}$ $O_2^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$\begin{array}{l} x \mapsto x^{-1} \\ y \mapsto y^{-1} \\ z \mapsto xyz \end{array} ; \begin{array}{l} x \mapsto x^{-1} \\ y \mapsto y \\ z \mapsto xyz \end{array}$	$(\mathbb{Z}_2)^2$	O_2^4
N_6^4	$N_2^3 \rtimes S^1$ $O_2^3 \rtimes S^1$	$N_2^3 \rtimes_T \mathbb{Z}$ $O_2^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$\begin{array}{l} x \mapsto x^{-1} \\ y \mapsto y^{-1} \\ z \mapsto z \end{array} ; \begin{array}{l} x \mapsto x \\ y \mapsto xy^{-1} \\ z \mapsto z \end{array}$	$(\mathbb{Z}_2)^2$	O_2^4
N_7^4	$N_2^3 \rtimes S^1$ $O_2^3 \rtimes S^1$	$N_2^3 \rtimes_T \mathbb{Z}$ $O_2^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2$	2	$\begin{array}{l} x \mapsto y \\ y \mapsto x \\ z \mapsto yz^{-1} \end{array} ; \begin{array}{l} x \mapsto xy \\ y \mapsto y^{-1} \\ z \mapsto xz \end{array}$	$(\mathbb{Z}_2)^2$	O_3^4
N_8^4	$N_3^3 \times S^1$ $N_1^3 \rtimes S^1$	$N_3^3 \times \mathbb{Z}$ $N_1^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2$	2	$\begin{array}{l} x \mapsto x \\ y \mapsto y \\ z \mapsto z \end{array} ; \begin{array}{l} x \mapsto x \\ y \mapsto y^{-1} \\ z \mapsto z^{-1} \end{array}$	$(\mathbb{Z}_2)^2$	O_2^4
N_9^4	$N_4^3 \times S^1$ $N_1^3 \rtimes S^1$	$N_4^3 \times \mathbb{Z}$ $N_1^3 \rtimes \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4$	2	$\begin{array}{l} x \mapsto x \\ y \mapsto y \\ z \mapsto z \end{array} ; \begin{array}{l} x \mapsto x \\ y \mapsto y^{-1} \\ z \mapsto yz^{-1} \end{array}$	$(\mathbb{Z}_2)^2$	O_2^4
N_{10}^4	$N_1^3 \rtimes S^1$ $N_3^3 \rtimes S^1$	$N_1^3 \rtimes_T \mathbb{Z}$ $N_3^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2$	2	$\begin{array}{l} x \mapsto x^{-1}z^2 \\ y \mapsto y^{-1} \\ z \mapsto z \end{array} ; \begin{array}{l} x \mapsto x \\ y \mapsto y^{-1} \\ z \mapsto z \end{array}$	$(\mathbb{Z}_2)^2$	O_3^4

Table 5: Nonorientable Flat 4-Manifold Groups with Infinite Abelianization

Manifold	Fibering	π_1	H_1	β_1	$T : w \leftrightarrow c_w$	Holonomy	ODC
N^4_{11}	$N^3_2 \rtimes S^1$ $N^3_3 \rtimes S^1$	$N^3_2 \rtimes_T \mathbb{Z}$ $N^3_3 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$x \mapsto y$, $y \mapsto x$, $z \mapsto z^{-1}$; $x \mapsto xy^{-1}$, $y \mapsto y$, $z \mapsto yz$	$(\mathbb{Z}_2)^2$	O^4_3
N^4_{12}	$N^3_1 \rtimes S^1$ $N^3_4 \rtimes S^1$	$N^3_1 \rtimes_T \mathbb{Z}$ $N^3_4 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$x \mapsto x^{-1}z^2$, $y \mapsto y^{-1}$, $z \mapsto yz$; $x \mapsto x$, $y \mapsto y^{-1}$, $z \mapsto yz$	$(\mathbb{Z}_2)^2$	O^4_3
N^4_{13}	$N^3_2 \rtimes S^1$ $N^3_4 \rtimes S^1$	$N^3_2 \rtimes_T \mathbb{Z}$ $N^3_4 \rtimes_T \mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	2	$x \mapsto x^{-1}z^2$, $y \mapsto y^{-1}z^2$, $z \mapsto z$; $x \mapsto xy$, $y \mapsto y^{-1}$, $z \mapsto z$	$(\mathbb{Z}_2)^2$	O^4_3
N^4_{14}	$O^3_1 \rtimes S^1$	$\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	1	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\mathbb{Z}_2$	O^4_1
N^4_{15}	$O^3_1 \rtimes S^1$	$\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\mathbb{Z}_4$	O^4_2
N^4_{16}	$O^3_1 \rtimes S^1$	$\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4$	1	$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_4$	O^4_3
N^4_{17}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto y$, $y \mapsto x^{-1}$, $z \mapsto z^{-1}$	$\mathbb{Z}_4$	O^4_2
N^4_{18}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4$	1	$x \mapsto y$, $y \mapsto x^{-1}$, $z \mapsto xz^{-1}$	$\mathbb{Z}_4$	O^4_3
N^4_{19}	$O^3_1 \rtimes S^1$	$\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_6$	1	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\mathbb{Z}_6$	O^4_4
N^4_{20}	$O^3_1 \rtimes S^1$	$\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	$\mathbb{Z}_6$	O^4_4

Manifold	Fibering	π_1	H_1	β_1	$T: w \leftrightarrow c_w$	Holonomy	ODC
N^4_{21}	$O^3_1 \rtimes S^1$	$\mathbb{Z}^3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_6$	O^4_5
N^4_{22}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	1	$x \mapsto x^{-1}$ $y \mapsto y^{-1}$ $z \mapsto z^{-1}$	$(\mathbb{Z}_2)^2$	O^4_2
N^4_{23}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	1	$x \mapsto x^{-1}$ $y \mapsto y^{-1}$ $z \mapsto yz^{-1}$	$(\mathbb{Z}_2)^2$	O^4_2
N^4_{24}	$N^3_1 \rtimes S^1$	$N^3_1 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	1	$x \mapsto x^{-1}$ $y \mapsto y$ $z \mapsto z^{-1}$	$(\mathbb{Z}_2)^2$	O^4_2
N^4_{25}	$N^3_1 \rtimes S^1$	$N^3_1 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	1	$x \mapsto x^{-1}$ $y \mapsto y$ $z \mapsto yz^{-1}$	$(\mathbb{Z}_2)^2$	O^4_2
N^4_{26}	$N^3_2 \rtimes S^1$	$N^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto y^{-1}$ $y \mapsto x^{-1}$ $z \mapsto z^{-1}$	$(\mathbb{Z}_2)^2$	O^4_3
N^4_{27}	$N^3_1 \rtimes S^1$	$N^3_1 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto x^{-1}z^2$ $y \mapsto y$ $z \mapsto x^{-1}z$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	O^4_6
N^4_{28}	$N^3_1 \rtimes S^1$	$N^3_1 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto x^{-1}z^2$ $y \mapsto y$ $z \mapsto x^{-1}yz$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	O^4_6
N^4_{29}	$N^3_2 \rtimes S^1$	$N^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$x \mapsto xz^{-2}$ $y \mapsto yz^{-2}$ $z \mapsto yz^{-1}$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	O^4_7
N^4_{30}	$N^3_3 \rtimes S^1$	$N^3_3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	1	$x \mapsto x$ $y \mapsto y^{-1}$ $z \mapsto z^{-1}$	$(\mathbb{Z}_2)^3$	O^4_9
N^4_{31}	$N^3_3 \rtimes S^1$	$N^3_3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^3$	1	$x \mapsto x^{-1}$ $y \mapsto y$ $z \mapsto x^2z^{-1}$	$(\mathbb{Z}_2)^3$	O^4_{10}
N^4_{32}	$N^3_3 \rtimes S^1$	$N^3_3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto xy^{-1}$ $y \mapsto y^{-1}$ $z \mapsto yz^{-1}$	$(\mathbb{Z}_2)^3$	O^4_{10}
N^4_{33}	$N^3_3 \rtimes S^1$	$N^3_3 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto xy^{-1}$ $y \mapsto y^{-1}$ $z \mapsto x^2yz^{-1}$	$(\mathbb{Z}_2)^3$	O^4_{11}
N^4_{34}	$N^3_4 \rtimes S^1$	$N^3_4 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto xy$ $y \mapsto y^{-1}$ $z \mapsto z^{-1}$	$(\mathbb{Z}_2)^3$	O^4_9

Manifold	Fibering	π_1	H_1	β_1	$T : w \leftrightarrow c_w$	Holonomy	ODC
N^4_{35}	$N^3_4 \rtimes S^1$	$N^3_4 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto x^{-1}$ $y \mapsto y$ $z \mapsto x^2 z^{-1}$	$(\mathbb{Z}_2)^3$	O^4_{10}
N^4_{36}	$N^3_4 \rtimes S^1$	$N^3_4 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	1	$x \mapsto x$ $y \mapsto y^{-1}$ $z \mapsto yz^{-1}$	$(\mathbb{Z}_2)^3$	O^4_{10}
N^4_{37}	$N^3_4 \rtimes S^1$	$N^3_4 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	1	$x \mapsto x^{-1}y^{-1}$ $y \mapsto y$ $z \mapsto x^2 y z^{-1}$	$(\mathbb{Z}_2)^3$	O^4_{11}
N^4_{38}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	1	$x \mapsto x$ $y \mapsto y^{-1}$	$(\mathbb{Z}_2)^3$	O^4_{14}
N^4_{39}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}_2)^2$	1	$x \mapsto xy^{-2}$ $y \mapsto x^2 y^{-1}$	$(\mathbb{Z}_2)^3$	O^4_{14}
N^4_{40}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$x \mapsto y^{-1}$ $y \mapsto x$	D_4	O^4_{15}
N^4_{41}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_4$	1	$x \mapsto x^{-1}$ $y \mapsto xy^{-1}$	D_4	O^4_{17}
N^4_{42}	$O^3_2 \rtimes S^1$	$O^3_2 \rtimes_T \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	1	$x \mapsto y$ $y \mapsto x^{-1}y^{-1}$ $z \mapsto z^{-1}$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	O^4_8
N^4_{43}	$O^3_6 \rtimes S^1$	$O^3_6 \rtimes_T \mathbb{Z}$	$\mathbb{Z}$	1	$x \mapsto x^{-1}y$ $y \mapsto x$	$\Delta \times \mathbb{Z}_2$	O^4_{26}

Manifold	π_1	H_1	β_1	G	Holonomy	ODC
N^4_{44}	$O^3_2 \star_G N^3_1$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2$	0	$\mathbb{Z}^3$	$(\mathbb{Z}_2)^2$	O^4_2
N^4_{45}	$O^3_6 \star_G N^3_3$	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$	0	O^3_2	$(\mathbb{Z}_2)^3$	O^4_9
N^4_{46}	$O^3_6 \star_G N^3_4$	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$	0	O^3_2	$(\mathbb{Z}_2)^3$	O^4_{10}
N^4_{47}	$O^3_2 \star_G N^3_2$	$(\mathbb{Z}_4)^2$	0	$\mathbb{Z}^3$	D_4	O^4_{12}

Table 6: Nonorientable Flat 4-Manifold Groups with Finite Abelianization

Notes:

- The two fiberings for the manifold N^4_9 were listed separately in [Lev70]; in Proposition III.1, we show that the corresponding groups are *isomorphic*. This accounts for the extra nonorientable group with $\beta_1 = 2$ in [Lev70].
- As in the flat 3-manifold case, each of the flat 4-manifold groups with $\beta_1 = 0$ are unions of twisted I-bundles over flat 3-manifolds, joined together along their

common boundaries (for N^4_{44} and N^4_{47}, the boundary is the flat 3-torus T^3; for N^4_{45} and N^4_{46}, the boundary is O^3_2).

Proposition III.1 *The two fiberings listed for N^4_9 are decompositions of the same closed flat 4-manifold.*

Proof It suffices to prove that the corresponding groups are isomorphic. Put $G_1 = N^3_4 \times \mathbb{Z}$ and $G_2 = N^3_1 \rtimes \mathbb{Z}$. From the table entry for N^4_9, we have the following presentations for the two groups:

$$G_1 = \langle x, y, z, w \mid xyx^{-1} = y^{-1},\ zxz^{-1} = x^{-1}y,\ zyz^{-1} = y^{-1},\ wx = xw,\ wy = yw,\ wz = zw\rangle$$

and

$$G_2 = \langle x, y, z, w \mid xy = yx,\ zx = xz,\ zyz^{-1} = y^{-1},\ wx = xw,\ wyw^{-1} = y^{-1},\ wzw^{-1} = yz^{-1}\rangle.$$

To show that these groups are isomorphic, we will exhibit a finite sequence of Tietze transformations which carries the first presentation into the second.

First, we will add the following generators and relations to the presentation for G_1:

$$G_1 = \langle x, x', y, y', z, z', w, w' \mid x' = w,\ y' = y^{-1},\ z' = x,\ w' = z,\ xyx^{-1} = y^{-1},$$
$$zxz^{-1} = x^{-1}y,\ zyz^{-1} = y^{-1},\ wx = xw,\ wy = yw,\ wz = zw\rangle.$$

Then delete the generators x, y, z, w, which are now redundant, and rewrite the relations in terms of the remaining generators:

$$G_1 = \langle x', y', z', w' \mid z'y'z'^{-1} = y'^{-1},\ w'z'w'^{-1} = y'z'^{-1},\ w'y'w'^{-1} = y'^{-1},$$
$$x'z' = z'x',\ x'y' = y'x',\ x'w' = w'x'\rangle.$$

The presentation now becomes

$$G_1 = \langle x', y', z', w' \mid x'y' = y'x',\ z'x' = x'z',\ z'y'z'^{-1} = y'^{-1},\ w'x' = x'w',$$
$$w'y'w'^{-1} = y'^{-1},\ w'z'w'^{-1} = y'z'^{-1}\rangle,$$

which is (with relabeling) the given presentation for G_2. This shows that G_1 and G_2 are indeed isomorphic. ♦

Flat 4-Manifold Groups and Their Correspondences

In this section, we give a detailed correspondence among the classifications of all flat 4-manifold groups given in [Lev70], [Hil95], and [Bro78]. We refer to the groups in Levine's table in the numerical order that they appear there.

Brown	Lambert	Hillman	Levine
01/01/01/001	O_1^4	$\mathbb{Z}^3 \times \mathbb{Z}$	1
03/01/01/002	O_2^4	$O_2^3 \times \mathbb{Z}$	2
03/01/02/002	O_3^4	$O_2^3 \rtimes \mathbb{Z}$	3
05/01/02/007	O_9^4	$O_2^3 \rtimes \mathbb{Z}$	5
05/01/02/008	O_{10}^4	$O_2^3 \rtimes \mathbb{Z}$	6
05/01/02/009	O_{14}^4	$O_6^3 \times \mathbb{Z}$	4
05/01/02/010	O_{11}^4	$O_2^3 \rtimes \mathbb{Z}$	7
05/01/03/006	O_{12}^4	$O_2^3 \rtimes \mathbb{Z}$	11
05/01/04/006	O_{17}^4	$O_6^3 \rtimes \mathbb{Z}$	8
05/01/06/006	O_{16}^4	$O_6^3 \rtimes \mathbb{Z}$	9
05/01/07/004	O_{15}^4	$O_6^3 \rtimes \mathbb{Z}$	10
05/01/10/004	O_{13}^4	$O_2^3 \rtimes \mathbb{Z}$	12
07/02/01/002	O_6^4	$O_4^3 \times \mathbb{Z}$	13
07/02/02/002	O_7^4	$O_4^3 \rtimes \mathbb{Z}$	14
08/01/01/002	O_5^4	$O_3^3 \rtimes \mathbb{Z}$	20
08/01/02/002	O_4^4	$O_3^3 \times \mathbb{Z}$	19
09/01/01/002	O_8^4	$O_5^3 \times \mathbb{Z}$	21
13/04/01/014	O_{21}^4	$O_4^3 \rtimes \mathbb{Z}$	15
13/04/01/020	O_{22}^4	$O_4^3 \rtimes \mathbb{Z}$	17
13/04/01/023	O_{23}^4	$O_6^3 \rtimes \mathbb{Z}$	16
13/04/04/011	O_{24}^4	$O_6^3 \rtimes \mathbb{Z}$	18
14/03/01/004	O_{20}^4	$O_3^3 \rtimes \mathbb{Z}$	24
14/03/05/004	O_{18}^4	$O_3^3 \rtimes \mathbb{Z}$	22
14/03/06/004	O_{19}^4	$O_3^3 \rtimes \mathbb{Z}$	23
15/04/01/010	O_{25}^4	$O_5^3 \rtimes \mathbb{Z}$	25
24/01/02/004	O_{26}^4	$O_6^3 \rtimes \mathbb{Z}$	26
24/01/04/004	O_{27}^4	$O_6^3 \rtimes \mathbb{Z}$	27

Table 7: The Orientable Flat 4-Manifold Groups: Correspondences

Brown	Lambert	Hillman	Levine
02/01/01/002	N_1^4	$N_1^3 \times \mathbb{Z}$	1
02/01/02/002	N_2^4	$N_2^3 \times \mathbb{Z}$	2
02/02/01/002	N_{14}^4	$\mathbb{Z}^3 \rtimes \mathbb{Z}$	3
04/01/01/006	N_8^4	$N_3^3 \times \mathbb{Z}$	4
04/01/01/007	N_9^4	$N_4^3 \times \mathbb{Z}$	5,8
04/01/01/010	N_3^4	$N_1^3 \rtimes \mathbb{Z}$	6
04/01/01/011	N_4^4	$N_1^3 \rtimes \mathbb{Z}$	7
04/01/01/013	N_5^4	$N_1^3 \rtimes \mathbb{Z}$	9
04/01/02/004	N_6^4	$N_2^3 \rtimes \mathbb{Z}$	15
04/01/03/004	N_{11}^4	$N_2^3 \rtimes \mathbb{Z}$	10
04/01/03/011	N_{10}^4	$N_1^3 \rtimes \mathbb{Z}$	11
04/01/03/012	N_{12}^4	$N_1^3 \rtimes \mathbb{Z}$	12
04/01/04/005	N_{13}^4	$N_2^3 \rtimes \mathbb{Z}$	13
04/01/06/004	N_7^4	$N_2^3 \rtimes \mathbb{Z}$	14
04/02/01/008	N_{22}^4	$O_2^3 \rtimes \mathbb{Z}$	16
04/02/01/011	N_{24}^4	$N_1^3 \rtimes \mathbb{Z}$	18
04/02/01/012	N_{25}^4	$N_1^3 \rtimes \mathbb{Z}$	19
04/02/01/016	N_{23}^4	$O_2^3 \rtimes \mathbb{Z}$	17
04/02/03/004	N_{26}^4	$N_2^3 \rtimes \mathbb{Z}$	20
04/03/01/006	N_{44}^4	$O_2^3 \star N_1^3$	21
06/01/01/041	N_{31}^4	$N_3^3 \rtimes \mathbb{Z}$	23
06/01/01/045	N_{37}^4	$N_4^3 \rtimes \mathbb{Z}$	24
06/01/01/049	N_{38}^4	$O_6^3 \rtimes \mathbb{Z}$	22
06/01/01/063	N_{30}^4	$N_3^3 \rtimes \mathbb{Z}$	26

Brown	Lambert	Hillman	Levine
06/01/01/064	N_{32}^4	$N_3^3 \rtimes \mathbb{Z}$	31
06/01/01/066	N_{33}^4	$N_3^3 \rtimes \mathbb{Z}$	29
06/01/01/081	N_{36}^4	$N_4^3 \rtimes \mathbb{Z}$	27
06/01/01/082	N_{34}^4	$N_4^3 \rtimes \mathbb{Z}$	30
06/01/01/083	N_{35}^4	$N_4^3 \rtimes \mathbb{Z}$	28
06/01/01/092	N_{39}^4	$O_6^3 \rtimes \mathbb{Z}$	25
06/02/01/027	N_{46}^4	$O_6^3 \star N_4^3$	33
06/02/01/050	N_{45}^4	$O_6^3 \star N_3^3$	32
12/01/02/002	N_{15}^4	$\mathbb{Z}^3 \rtimes \mathbb{Z}$	34
12/01/03/002	N_{17}^4	$O_2^3 \rtimes \mathbb{Z}$	35
12/01/04/002	N_{16}^4	$\mathbb{Z}^3 \rtimes \mathbb{Z}$	37
12/01/06/002	N_{18}^4	$O_2^3 \rtimes \mathbb{Z}$	36
12/03/04/006	N_{41}^4	$O_6^3 \rtimes \mathbb{Z}$	41
12/03/10/005	N_{40}^4	$O_6^3 \rtimes \mathbb{Z}$	42
12/04/03/011	N_{47}^4	$O_2^3 \star N_2^3$	43
13/01/01/008	N_{27}^4	$N_1^3 \rtimes \mathbb{Z}$	38
13/01/01/011	N_{28}^4	$N_1^3 \rtimes \mathbb{Z}$	39
13/01/03/008	N_{29}^4	$N_2^3 \rtimes \mathbb{Z}$	40
14/01/01/002	N_{21}^4	$\mathbb{Z}^3 \rtimes \mathbb{Z}$	45
14/01/03/002	N_{20}^4	$\mathbb{Z}^3 \rtimes \mathbb{Z}$	46
14/02/03/002	N_{19}^4	$\mathbb{Z}^3 \rtimes \mathbb{Z}$	44
15/01/01/010	N_{42}^4	$O_2^3 \rtimes \mathbb{Z}$	47
25/01/01/010	N_{43}^4	$O_6^3 \rtimes \mathbb{Z}$	48

Table 8: The Nonorientable Flat 4-Manifold Groups: Correspondences

Chapter IV

The Flat 3-Manifold Groups: Geometric Descriptions

In this chapter, we give a geometric description of the (closed) flat 3-manifold groups as a companion to the abstract presentations in Chapter III. Wherever possible, we will give generators which correspond to the abstract generators given there. In the cases where the matrices given in [Wol84] are not all orthogonal, we perform a change of basis which simultaneously diagonalizes them wherever possible (i.e. when the matrices commute); if this is not possible, we employ complex diagonalization to obtain a standard orthogonal matrix of the form

$$\begin{pmatrix} \pm 1 & \\ & B \end{pmatrix},$$

where B is a 2×2 orthogonal matrix. This procedure is outlined in Chapter II. As a byproduct of this technique, it was shown that the defining relations are preserved, so that we can continue to use the notation in [Wol84, Bro78] without change.

The notation in these tables is adapted from [Wol84]: We will list the orthogonal matrices as generators, and the basic translations will always be included. The other generating isometries will be listed in the format of [Bro78]; e.g. $A : [1, 0, 0, 0]/2$ means $\frac{1}{2}a_1$ is added to the rotation matrix A, yielding the isometry $\frac{1}{2}a_1 + A$. The orientable groups will be denoted by O_k^3, and the nonorientable groups will be labeled N_l^3, as in the abstract tables in Chapter III.

Other Notation: We shall denote by e_i $(i = 1, 2, 3)$ the standard basis vectors of $\mathbb{R}^3$; i.e. e_i is the vector whose jth component is δ_{ij}. For each group, we will list the corresponding lattice basis as an ordered basis; e.g. $(a_1, a_2, a_3) = (e_1, e_2, e_3)$ if the matrices in [Wol84] were already orthogonal. The corresponding translations will be denoted by $t_i = a_i + I$. In the following presentations, we will take for granted that the t_i's commute rather than repeatedly write the appropriate relations.

The Orientable Flat 3-Manifold Groups

O_1^3: (Flat 3-torus)

Rotation matrices: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Holonomy: $\{1\}$
First Homology: $\mathbb{Z}^3$
Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, e_3)$
$\Gamma = \langle t_1, t_2, t_3 \rangle = \mathbb{Z}^3$

O_2^3:

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
Holonomy: $\langle A \rangle = \mathbb{Z}_2$
Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, e_3)$
Generators: $\alpha = A : [1, 0, 0]/2$
First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\Gamma = \langle t_1, t_2, t_3, \alpha \mid \alpha^2 = t_1,\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 2, 3 \rangle = \langle t_2, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^2 \rtimes \mathbb{Z}$

O_3^3:

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
Holonomy: $\langle A \rangle = \mathbb{Z}_3$
Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, -\frac{1}{2} e_2 + \frac{\sqrt{3}}{2} e_3)$
Generators: $\alpha = A : [1, 0, 0]/3$
First Homology: $\mathbb{Z} \oplus \mathbb{Z}_3$
$\Gamma = \langle t_1, t_2, t_3, \alpha \mid \alpha^3 = t_1,\ \alpha t_2 \alpha^{-1} = t_3,\ \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1} \rangle = \langle t_2, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^2 \rtimes \mathbb{Z}$

O^3_4:

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
Holonomy: $\langle A \rangle = \mathbb{Z}_4$
Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, e_3)$
Generators: $\alpha = A : [1, 0, 0]/4$
First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$
$\Gamma = \langle t_1, t_2, t_3, \alpha \mid \alpha^4 = t_1,\ \alpha t_2 \alpha^{-1} = t_3,\ \alpha t_3 \alpha^{-1} = t_2^{-1} \rangle = \langle t_2, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^2 \rtimes \mathbb{Z}$

O^3_5:

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
Holonomy: $\langle A \rangle = \mathbb{Z}_6$
Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, \frac{1}{2} e_2 + \frac{\sqrt{3}}{2} e_3)$
Generators: $\alpha = A : [1, 0, 0]/6$
First Homology: $\mathbb{Z}$
$\Gamma = \langle t_1, t_2, t_3, \alpha \mid \alpha^6 = t_1,\ \alpha t_2 \alpha^{-1} = t_3,\ \alpha t_3 \alpha^{-1} = t_2^{-1} t_3 \rangle = \langle t_2, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^2 \rtimes \mathbb{Z}$

O^3_6:

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$
Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, e_3)$
Generators: $\alpha = A : [1, 0, 0]/2,\ \beta = B : [0, 1, 1]/2$
First Homology: $\mathbb{Z}_4 \oplus \mathbb{Z}_4$
$\Gamma = \langle t_1, t_2, t_3, \alpha, \beta \mid \alpha^2 = t_1,\ \beta^2 = t_2,\ \beta \alpha \beta^{-1} = t_2 t_3 \alpha^{-1},\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 2, 3,$
$\beta t_i \beta^{-1} = t_i^{-1}, i = 1, 3 \rangle$
$= \langle \alpha, t_2 \rangle \star_{\langle t_1, t_2 \rangle} \langle \beta, t_1 \rangle = K^2 \star_{\mathbb{Z}^2} K^2$

The Nonorientable Flat 3-Manifold Groups

N_1^3:

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_2$
Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, e_3)$
Generators: $\alpha = A : [1, 0, 0]/2$
First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$
$\Gamma = \langle t_1, t_2, t_3, \alpha \mid \alpha^2 = t_1,\ \alpha t_2 \alpha^{-1} = t_2,\ \alpha t_3 \alpha^{-1} = t_3^{-1} \rangle = \langle \alpha, t_3 \rangle \times \langle t_2 \rangle = K^2 \times \mathbb{Z}$
$\Gamma = \langle t_2, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^2 \rtimes \mathbb{Z}$

N_2^3:

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_2$
Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, \frac{1}{2}(e_1 + e_2 + e_3))$
Generators: $\alpha = A : [1, 0, 0]/2$
First Homology: $\mathbb{Z} \oplus \mathbb{Z}$
$\Gamma = \langle t_1, t_2, t_3, \alpha \mid \alpha^2 = t_1,\ \alpha t_2 \alpha^{-1} = t_2,\ \alpha t_3 \alpha^{-1} = t_1 t_2 t_3^{-1} \rangle$
$= \langle \alpha, t_1 t_2 t_3^{-2} \rangle \rtimes \langle t_1 t_2 t_3^{-1} \rangle = K^2 \rtimes \mathbb{Z}$
$\Gamma = \langle t_3, t_1 t_2 t_3^{-1} \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^2 \rtimes \mathbb{Z}$

N^3_3:

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, e_3)$

Generators: $\alpha = A : [1,0,0]/2,\ \beta = B : [0,1,0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle t_1, t_2, t_3, \alpha, \beta \mid{} & \alpha^2 = t_1,\ \beta^2 = t_2,\ \alpha\beta\alpha^{-1} = \beta^{-1},\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 2, 3, \\ & \beta t_1 \beta^{-1} = t_1,\ \beta t_3 \beta^{-1} = t_3^{-1}\rangle \\ = \langle \beta, t_3\rangle \rtimes \langle\alpha\rangle = K^2 \rtimes \mathbb{Z} &\end{aligned}$$

N^3_4:

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3) = (e_1, e_2, e_3)$

Generators: $\alpha = A : [1,0,0]/2,\ \beta = B : [0,1,1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_4$

$$\begin{aligned}\Gamma = \langle t_1, t_2, t_3, \alpha, \beta \mid{} & \alpha^2 = t_1,\ \beta^2 = t_2,\ \alpha\beta\alpha^{-1} = \beta^{-1} t_3,\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 2, 3, \\ & \beta t_1 \beta^{-1} = t_1,\ \beta t_3 \beta^{-1} = t_3^{-1}\rangle \\ = \langle \beta, t_3\rangle \rtimes \langle\alpha\rangle = K^2 \rtimes \mathbb{Z} &\end{aligned}$$

Chapter V

The Flat 4-Manifold Groups: Geometric Descriptions

We now give a geometric description of the flat 4-manifold groups as a companion to the abstract presentations in Chapter III. Wherever possible, we will give generators which correspond to the abstract generators given there. In the cases where the matrices given in [Bro78] are not all orthogonal, we perform a change of basis which simultaneously diagonalizes them wherever possible (i.e. when the matrices commute); if this is not possible, we employ complex diagonalization to obtain a standard orthogonal matrix of the form

$$\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & B_3 & \\ & & & B_4 \end{pmatrix},$$

where each B_i is either ± 1 or a 2×2 orthogonal matrix. This procedure is outlined in Chapter II. As a byproduct of this technique, it was shown that the defining relations are preserved, so that we can continue to use Brown's notation without change.

The notation in these tables is adapted from the tables in [Lev70] and [Bro78]: We will list the orthogonal matrices as generators, and the basic translations will always be included. The other generating isometries will be listed in the format of Brown; e.g. $A : [1, 0, 0, 0]/2$ means $\frac{1}{2}a_1$ is added to the rotation matrix A, yielding the isometry $\frac{1}{2}a_1 + A$. For reference, we will also list the standard designation of [Bro78] along with our labeling.

Other Notation: We shall denote by e_i $(i = 1, 2, 3, 4)$ the standard basis vectors of $\mathbb{R}^4$; i.e. e_i is the vector whose jth component is δ_{ij}. Also, $t_i = a_i + I$ will denote the basic translations of $\mathbb{R}^4$. For each group, we will list the corresponding lattice basis as an ordered basis; e.g. $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$ if the matrices in [Bro78] were already orthogonal. In the following presentations, we will take for granted that the t_i's commute rather than repeatedly write the appropriate relations.

The Orientable Flat 4-Manifold Groups

O^4_1 : 01/01/01/001 (Flat 4-torus)

Rotation matrices: $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: 1

First Homology: $\mathbb{Z}^4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

$\Gamma = \langle t_1, t_2, t_3, t_4 \rangle = \mathbb{Z}^4$

O^4_2 : 03/01/01/002

Rotation matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^2 = t_4,\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 1, 2,\ \alpha t_3 \alpha^{-1} = t_3 \rangle$
$\quad = \langle t_1, t_2, \alpha \rangle \times \langle t_3 \rangle = O^3_2 \times \mathbb{Z}$

$\Gamma = \langle t_1, t_2, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$O_3^4 : 03/01/02/002$

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}(e_1 + e_4))$

Generators: $\alpha = A : [0, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^2 = t_2,\ \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_3^{-1},\ \alpha t_4 \alpha^{-1} = t_1 t_4^{-1} \rangle$

$= \langle t_3, t_1 t_4^{-2}, \alpha \rangle \rtimes \langle t_4 \rangle = O_2^3 \rtimes \mathbb{Z}$

$\Gamma = \langle t_4, t_1 t_4^{-2}, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$O_4^4 : 08/01/02/002$

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_3$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/3$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^3 = t_2,\ \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_3^{-1} t_4, \alpha t_4 \alpha^{-1} = t_3^{-1} \rangle$

$= \langle t_4^{-1}, t_3, \alpha \rangle \times \langle t_1 \rangle = O_3^3 \times \mathbb{Z}$

$\Gamma = \langle t_1, t_4^{-1}, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$O_5^4 : 08/01/01/002$

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_3$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, -\frac{1}{3}e_2 + \frac{2\sqrt{3}}{3}e_3, \frac{1}{3}e_2 + \frac{\sqrt{3}}{3}e_3 + e_4)$

Generators: $\alpha = A : [1, 0, 0, 0]/3$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^3 = t_1,\ \alpha t_2 \alpha^{-1} = t_2,\ \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1} t_4, \alpha t_4 \alpha^{-1} = t_3^{-1} \rangle$

$= \langle t_3 t_4, t_2^{-1} t_3^{-2} t_4, \alpha \rangle \rtimes \langle t_3 \rangle = O_3^3 \rtimes \mathbb{Z}$

$\Gamma = \langle t_3, t_3 t_4, t_2^{-1} t_3^{-2} t_4 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

O_6^4 : 07/02/01/002

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/4$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^4 = t_2,\ \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_4, \alpha t_4 \alpha^{-1} = t_3^{-1} \rangle$

$\quad = \langle t_3, t_4, \alpha \rangle \times \langle t_1 \rangle = O_4^3 \times \mathbb{Z}$

$\Gamma = \langle t_1, t_3, t_4 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

O_7^4 : 07/02/02/002

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, \frac{1}{2}e_1 + \frac{1}{2}e_2 + e_3, \frac{1}{2}e_1 + \frac{1}{2}e_2 + e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/4$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^4 = t_2,\ \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_4, \alpha t_4 \alpha^{-1} = t_1 t_2 t_3^{-1} \rangle$

$\quad = \langle t_1 t_2 t_3^{-1} t_4^{-1}, t_3 t_4^{-1}, \alpha \rangle \rtimes \langle t_4 \rangle = O_4^3 \rtimes \mathbb{Z}$

$\Gamma = \langle t_4, t_1 t_2 t_3^{-1} t_4^{-1}, t_3 t_4^{-1} \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

O_8^4 : 09/01/01/002

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_6$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/6$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^6 = t_2,\ \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_4, \alpha t_4 \alpha^{-1} = t_3^{-1} t_4 \rangle$

$\quad = \langle t_3, t_4, \alpha \rangle \times \langle t_1 \rangle = O_5^3 \times \mathbb{Z}$

$\Gamma = \langle t_1, t_3, t_4 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$O^4_9 : 05/01/02/007$

Rotation matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2,\ \beta = B : [0, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_2,\ \beta^2 = t_4,\ \beta\alpha\beta^{-1} = \alpha^{-1}, \alpha t_i \alpha^{-1} = t_i^{-1}, i = 1, 3,$
$\quad \alpha t_4 \alpha^{-1} = t_4, \beta t_i \beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_3 \beta^{-1} = t_3\rangle$
$= \langle t_1, t_3, \alpha\rangle \rtimes \langle\beta\rangle = O^3_2 \rtimes \mathbb{Z}$

$O^4_{10} : 05/01/02/008$

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1, 1, 0, 0]/2,\ \beta = B : [0, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_1,\ \beta^2 = t_4,\ \beta\alpha\beta^{-1} = t_2^{-1}\alpha^{-1}, \alpha t_i \alpha^{-1} = t_i^{-1}, i = 2, 3,$
$\quad \alpha t_4 \alpha^{-1} = t_4,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_3 \beta^{-1} = t_3\rangle$
$= \langle t_2^{-1}, t_3, \alpha\rangle \rtimes \langle\beta\rangle = O^3_2 \rtimes \mathbb{Z}$

$O^4_{11} : 05/01/02/010$

Rotation matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2,\ \beta = B : [1, 1, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_1 t_4,\ \beta\alpha\beta^{-1} = t_1 t_2 \alpha^{-1}, \alpha t_i \alpha^{-1} = t_i^{-1}, i = 1, 2,$
$\quad \alpha t_4 \alpha^{-1} = t_4,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 2, 3,\ \beta t_i \beta^{-1} = t_i, i = 1, 4\rangle$
$= \langle t_2, t_1, \alpha\rangle \rtimes \langle\beta\rangle = O^3_2 \rtimes \mathbb{Z}$

O^4_{12} : 05/01/03/006

Rotation matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2,\ \beta = B : [0, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle & t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_4,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 1, 2, \\ & \alpha t_4 \alpha^{-1} = t_4,\ \beta t_1 \beta^{-1} = t_2,\ \beta t_2 \beta^{-1} = t_1,\ \beta t_3 \beta^{-1} = t_3^{-1}\rangle \\ = & \langle t_1, t_2, \alpha\rangle \rtimes \langle\beta\rangle = O^3_2 \rtimes \mathbb{Z}\end{aligned}$$

O^4_{13} : 05/01/10/004

Rotation matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (\frac{1}{2}(e_2 - e_3), \frac{1}{2}(e_2 + e_3), \frac{1}{2}(e_1 - e_2), e_4)$

Generators: $\alpha = A : [0, 1, 1, 1]/2,\ \beta = B : [0, 0, 1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_4$

$$\begin{aligned}\Gamma = \langle & t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_4,\ \beta^2 = t_3,\ \beta\alpha\beta^{-1} = t_1 t_3 \alpha^{-1},\ \alpha t_1 \alpha^{-1} = t_2, \\ & \alpha t_2 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_1^{-1} t_2^{-1} t_3^{-1},\ \beta t_1 \beta^{-1} = t_2,\ \beta t_2 \beta^{-1} = t_1,\ \beta t_4 \beta^{-1} = t_4^{-1}\rangle \\ = & \langle t_1 t_3, t_2 t_3, \alpha\rangle \rtimes \langle\beta\rangle = O^3_2 \rtimes \mathbb{Z}\end{aligned}$$

O^4_{14} : 05/01/02/009

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1, 1, 0, 0]/2,\ \beta = B : [-1, 1, -1, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$

$$\begin{aligned}\Gamma = \langle & t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_1,\ \beta^2 = t_2,\ \alpha\beta\alpha^{-1} = t_1 t_3 \beta^{-1}, \\ & \alpha t_i \alpha^{-1} = t_i^{-1}, i = 2, 3,\ \alpha t_4 \alpha^{-1} = t_4,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 1, 3,\ \beta t_4 \beta^{-1} = t_4\rangle \\ = & \langle\alpha, \beta\rangle \times \langle t_4\rangle = O^3_6 \times \mathbb{Z}\end{aligned}$$

O^4_{15} : 05/01/07/004

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}(e_1 + e_2 + e_3 + e_4))$

Generators: $\alpha = A : [0, 0, 1, 0]/2,\ \beta = B : [1, 0, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_1,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_1\alpha^{-1} = t_1,\ \alpha t_2\alpha^{-1} = t_2^{-1},$
$\quad \alpha t_4\alpha^{-1} = t_1t_3t_4^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,\ \beta t_4\beta^{-1} = t_2^{-1}t_3^{-1}t_4\rangle$
$\quad = \langle \alpha\beta t_4^{-1}, \alpha\rangle \rtimes \langle t_3\beta\rangle = O^3_6 \rtimes \mathbb{Z}$

O^4_{16} : 05/01/06/006

Rotation matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (\frac{1}{2}(e_1 + e_2 - e_3), \frac{1}{2}(-e_1 + e_2 + e_3), \frac{1}{2}(e_1 - e_2 + e_3), e_4)$

Generators: $\alpha = A : [-2, -1, -1, 1]/2,\ \beta = B : [1, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_4,\ \beta^2 = t_1t_2,\ \alpha\beta\alpha^{-1} = t_1^{-1}t_3^{-1}t_4\beta^{-1},$
$\quad \alpha t_1\alpha^{-1} = t_1^{-1}t_2^{-1}t_3^{-1},\ \alpha t_2\alpha^{-1} = t_3,\ \alpha t_3\alpha^{-1} = t_2, \beta t_1\beta^{-1} = t_3^{-1},\ \beta t_2\beta^{-1} = t_1t_2t_3,$
$\quad \beta t_3\beta^{-1} = t_1^{-1},\ \beta t_4\beta^{-1} = t_4^{-1}\rangle$
$\quad = \langle \alpha, \beta\rangle \rtimes \langle \alpha\beta t_3\rangle = O^3_6 \rtimes \mathbb{Z}$

O^4_{17} : 05/01/04/006

Rotation matrices: $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1, 1, 1, 0]/2,\ \beta = B : [-1, -1, -1, -1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_1^{-1}t_2^{-1},\ \alpha\beta\alpha^{-1} = t_3t_4\beta^{-1},\ \alpha t_1\alpha^{-1} = t_2^{-1},$
$\quad \alpha t_2\alpha^{-1} = t_1^{-1},\ \alpha t_4\alpha^{-1} = t_4^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 3, 4,\ \beta t_i\beta^{-1} = t_i, i = 1, 2\rangle$
$\quad = \langle \alpha, \beta\rangle \rtimes \langle t_1\rangle = O^3_6 \rtimes \mathbb{Z}$

O^4_{18} : 14/03/05/004

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Holonomy: $\langle A, B \mid A^3 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_3$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/3,\ \beta = B : [3, 2, 0, 0]/6$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_6$

$$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^3 = t_2,\ \beta^2 = t_1,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_4^{-1},$$
$$\alpha t_4 \alpha^{-1} = t_3 t_4^{-1},\ \beta t_2 \beta^{-1} = t_2^{-1},\ \beta t_3 \beta^{-1} = t_4^{-1},\ \beta t_4 \beta^{-1} = t_3^{-1} \rangle$$
$$= \langle t_3, t_4^{-1}, \alpha \rangle \rtimes \langle \beta \rangle = O^3_3 \rtimes \mathbb{Z}$$

O^4_{19} : 14/03/06/004

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Holonomy: $\langle A, B \mid A^3 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_3$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4)$

Generators: $\alpha = A : [1, 0, 0, 0]/3,\ \beta = B : [2, 3, 0, 0]/6$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^3 = t_1,\ \beta^2 = t_2,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_2 \alpha^{-1} = t_2,\ \alpha t_3 \alpha^{-1} = t_4^{-1},$$
$$\alpha t_4 \alpha^{-1} = t_3 t_4^{-1},\ \beta t_1 \beta^{-1} = t_1^{-1},\ \beta t_3 \beta^{-1} = t_4,\ \beta t_4 \beta^{-1} = t_3 \rangle$$
$$= \langle t_3, t_4^{-1}, \alpha \rangle \rtimes \langle \beta \rangle = O^3_3 \rtimes \mathbb{Z}$$

O^4_{20} : 14/03/01/004

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B \mid A^3 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_3$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [2, 3, -3, 0]/6,\ \beta = B : [2, -3, 0, 6]/6$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^3 = t_1,\ \beta^2 = t_2^{-1} t_3 t_4,\ \beta\alpha\beta^{-1} = t_2^{-1} t_3 \alpha^{-1},$$
$$\alpha t_1 \alpha^{-1} = t_1,\ \alpha t_2 \alpha^{-1} = t_4,\ \alpha t_3 \alpha^{-1} = t_2,\ \alpha t_4 \alpha^{-1} = t_3,\ \beta t_1 \beta^{-1} = t_1^{-1},$$
$$\beta t_2 \beta^{-1} = t_2,\ \beta t_3 \beta^{-1} = t_4,\ \beta t_4 \beta^{-1} = t_3 \rangle$$
$$= \langle t_2^{-1} t_3, t_2 t_4^{-1}, \alpha \rangle \rtimes \langle \beta \rangle = O^3_3 \rtimes \mathbb{Z}$$

O^4_{21} : 13/04/01/014

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B \mid A^4 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1, 0, 0, 0]/4,\ \beta = B : [0, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus Z_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^4 = t_1,\ \beta^2 = t_2,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_2 \alpha^{-1} = t_2,\ \alpha t_3 \alpha^{-1} = t_4^{-1},$
$\alpha t_4 \alpha^{-1} = t_3,\ \beta t_1 \beta^{-1} = t_1^{-1},\ \beta t_3 \beta^{-1} = t_4,\ \beta t_4 \beta^{-1} = t_3 \rangle$
$= \langle t_4, t_3, \alpha \rangle \rtimes \langle \beta \rangle = O^3_4 \rtimes \mathbb{Z}$

O^4_{22} : 13/04/01/020

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B \mid A^4 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1, 0, 2, 0]/4,\ \beta = B : [0, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^4 = t_1,\ \beta^2 = t_2,\ \beta\alpha\beta^{-1} = t_4\alpha^{-1},\ \alpha t_2 \alpha^{-1} = t_2,$
$\alpha t_3 \alpha^{-1} = t_4^{-1},\ \alpha t_4 \alpha^{-1} = t_3,\ \beta t_1 \beta^{-1} = t_1^{-1},\ \beta t_3 \beta^{-1} = t_4,\ \beta t_4 \beta^{-1} = t_3 \rangle$
$= \langle t_4, t_3, \alpha \rangle \rtimes \langle \beta \rangle = O^3_4 \rtimes \mathbb{Z}$

O^4_{23} : 13/04/01/023

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B \mid A^4 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1, 2, 2, 0]/4,\ \beta = B : [0, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^4 = t_1 t_2^2,\ \beta^2 = t_2,\ \beta\alpha\beta^{-1} = t_2 t_4 \alpha^{-1},$
$\alpha t_i \alpha^{-1} = t_i, i = 1, 2,\ \alpha t_3 \alpha^{-1} = t_4^{-1},\ \alpha t_4 \alpha^{-1} = t_3,\ \beta t_1 \beta^{-1} = t_1^{-1},$
$\beta t_3 \beta^{-1} = t_4,\ \beta t_4 \beta^{-1} = t_3 \rangle$
$= \langle \beta^{-1}\alpha, \alpha\beta^{-1} \rangle \rtimes \langle \beta \rangle = O^3_6 \rtimes \mathbb{Z}$

O^4_{24} : 13/04/04/011

Rotation matrices: $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B \mid A^4 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (\frac{1}{2}(-e_1 - e_2 + e_3), \frac{1}{2}(e_1 - e_2 - e_3), \frac{1}{2}(e_1 + e_2 + e_3), e_4)$

Generators: $\alpha = A : [0, 0, 2, 1]/4,\ \beta = B : [1, 0, 1, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^4 = t_1t_3t_4,\ \beta^2 = t_1t_3,\ \beta\alpha\beta^{-1} = t_1t_2t_3\alpha^{-1},$
$\alpha t_1\alpha^{-1} = t_1t_2t_3,\ \alpha t_2\alpha^{-1} = t_1^{-1},\ \alpha t_3\alpha^{-1} = t_2^{-1},\ \alpha t_4\alpha^{-1} = t_4,\ \beta t_1\beta^{-1} = t_2^{-1},$
$\beta t_2\beta^{-1} = t_1^{-1},\ \beta t_3\beta^{-1} = t_1t_2t_3,\ \beta t_4\beta^{-1} = t_4^{-1} \rangle$
$= \langle t_3^{-1}\beta, \beta\alpha^{-2} \rangle \rtimes \langle \beta\alpha^{-1} \rangle = O^3_6 \rtimes \mathbb{Z}$

O^4_{25} : 15/04/01/010

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B \mid A^6 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_6$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/6,\ \beta = B : [1, -1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^6 = t_2,\ \beta^2 = t_1,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_1\alpha^{-1} = t_1,\ \alpha t_3\alpha^{-1} = t_4,$
$\alpha t_4\alpha^{-1} = t_3^{-1}t_4,\ \beta t_2\beta^{-1} = t_2^{-1},\ \beta t_3\beta^{-1} = t_3,\ \beta t_4\beta^{-1} = t_3t_4^{-1} \rangle$
$= \langle t_4, t_3^{-1}t_4, \alpha \rangle \rtimes \langle \beta \rangle = O^3_5 \rtimes \mathbb{Z}$

O^4_{26} : 24/01/02/004

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B, C \mid A^2 = B^2 = C^3 = 1, AB = BA, CAC^{-1} = A^{-1}B,$
$CBC^{-1} = A^{-1} \rangle = \Delta$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, -1, 0, -1]/2,\ \beta = B : [0, 0, 1, 1]/2,\ \gamma = C : [2, 3, 0, -3]/6$

First Homology: $\mathbb{Z}$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_2^{-1},\ \beta^2 = t_4,\ \gamma^3 = t_1,\ \beta\alpha\beta^{-1} = t_3t_4\alpha^{-1},$
$\gamma\alpha\gamma^{-1} = \alpha^{-1}\beta,\ \gamma\beta\gamma^{-1} = \alpha^{-1},\ \alpha t_1\alpha^{-1} = t_1,\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 3, 4,$
$\beta t_1\beta^{-1} = t_1,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,\ \gamma t_2\gamma^{-1} = t_3,\ \gamma t_3\gamma^{-1} = t_4,\ \gamma t_4\gamma^{-1} = t_2 \rangle$
$= \langle \alpha, \beta \rangle \rtimes \langle \gamma \rangle = O^3_6 \rtimes \mathbb{Z}$

O^4_{27} : 24/01/04/004

Rotation matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Holonomy: $\langle A, B, C \mid A^2 = B^2 = C^3 = 1, AB = BA, CAC^{-1} = A^{-1}B, CBC^{-1} = A^{-1} \rangle = \Delta$

Lattice Basis:

$$(a_1, a_2, a_3, a_4) = (e_1, \frac{1}{2}(-e_1 + e_2 - e_3 + e_4), \frac{1}{2}(-e_1 + e_2 + e_3 - e_4), \frac{1}{2}(e_1 - e_2 + e_3 + e_4))$$

Generators: $\alpha = A : [0, 1, 1, 2]/2,\ \beta = B : [-1, 0, -1, 1]/2,\ \gamma = C : [1, 0, 3, 3]/6$

First Homology: $\mathbb{Z}$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_3 t_4,\ \beta^2 = t_2 t_4,\ \gamma^3 = t_2^{-1}, \beta\alpha\beta^{-1} = t_1^{-1} t_3^{-1} t_4 \alpha^{-1}, \gamma\alpha\gamma^{-1} = \beta^{-1}\alpha^{-1},\ \gamma\beta\gamma^{-1} = t_2^{-1} t_4^{-1} \alpha^{-1},\ \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1}, \alpha t_3 \alpha^{-1} = t_1^{-1} t_4,\ \alpha t_4 \alpha^{-1} = t_1 t_3,\ \beta t_1 \beta^{-1} = t_1,\ \beta t_2 \beta^{-1} = t_1^{-1} t_4,\ \beta t_3 \beta^{-1} = t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1}, \beta t_4 \beta^{-1} = t_1 t_2,\ \gamma t_1 \gamma^{-1} = t_1,\ \gamma t_3 \gamma^{-1} = t_1^{-1} t_4,\ \gamma t_4 \gamma^{-1} = t_2^{-1} t_3^{-1} t_4^{-1} \rangle$
$= \langle \alpha, \alpha\beta \rangle \rtimes \langle \gamma \rangle = O^3_6 \rtimes \mathbb{Z}$

The Nonorientable Flat 4-Manifold Groups

We now give a geometric description of the nonorientable flat 4-manifold groups as a companion to the abstract presentations in Chapter III. Wherever possible, we will give generators which correspond to the abstract generators given there.

In addition to the information given in the section on orientable flat 4-manifold groups, we will also describe the orientable double covers of each manifold. Recall that, for every n-dimensional (closed) flat manifold group Γ, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \hookrightarrow \Gamma \xrightarrow{\eta} \Pi \longrightarrow 1,$$

giving Γ as an extension of $\mathbb{Z}^n = \langle t_i : i = 1, \ldots, n \rangle$ by a finite group Π, called the *point group*. In our case, Π is isomorphic to the holonomy group of the manifold. Note that here Π is a subgroup of $O(n)$, the group of all $n \times n$ real orthogonal matrices. In the case where Γ is nonorientable, Π will contain an orientation-reversing transformation

A with $\det A = -1$. Let $\Pi_0 \leq \Pi$ be the (normal) subgroup of all orientation-preserving elements of Π; i.e. Π_0 consists of all matrices $A \in \Pi$ with $\det A = 1$. Then the preimage Γ_0 of Π_0 under this extension gives us the orientable double cover of the manifold to which Γ corresponds. As Γ_0 is of index 2 in Γ, this indeed gives us a 2-sheeted covering of Γ.

N_1^4 : 02/01/01/002 ($K^2 \times T^2$)

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1, 0, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^2 = t_1,\ \alpha t_i \alpha^{-1} = t_i, i = 2, 3,\ \alpha t_4 \alpha^{-1} = t_4^{-1} \rangle$

$\quad = \langle t_3, t_4, \alpha \rangle \times \langle t_2 \rangle = N_1^3 \times \mathbb{Z}$

$\Gamma = \langle t_2, t_3, t_4 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha^2 \rangle = \langle t_1, t_2, t_3, t_4 \rangle = \mathbb{Z}^4 = O_1^4$

N_2^4 : 02/01/02/002

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}(e_3 + e_4))$

Generators: $\alpha = A : [1, 0, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^2 = t_1,\ \alpha t_i \alpha^{-1} = t_i, i = 2, 3,\ \alpha t_4 \alpha^{-1} = t_3 t_4^{-1} \rangle$

$\quad = \langle t_4, t_3 t_4^{-1}, \alpha \rangle \times \langle t_2 \rangle = N_2^3 \times \mathbb{Z}$

$\Gamma = \langle t_2, t_4, t_3 t_4^{-1} \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha^2 \rangle = \langle t_1, t_2, t_3, t_4 \rangle = \mathbb{Z}^4 = O_1^4$

$N_3^4 : 04/01/01/010\ (K^2 \times K^2)$

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0,1,0,0]/2,\ \beta = B : [1,1,0,0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_2,\ \beta^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = \alpha,\ \alpha t_i \alpha^{-1} = t_i, i = 1,3,$
$\quad \alpha t_4 \alpha^{-1} = t_4^{-1},\ \beta t_i \beta^{-1} = t_i, i = 1,2,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3,4 \rangle$
$\quad = \langle t_3, t_4, \alpha \rangle \rtimes \langle \beta \rangle = N_1^3 \rtimes \mathbb{Z}$

$\Gamma = \langle t_3, t_4, \beta \rangle \rtimes \langle \alpha \rangle = O_2^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_1 t_2,\ \beta t_i \beta^{-1} = t_i, i = 1,2,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3,4 \rangle$
$\quad = \langle t_3, t_4, \beta \rangle \times \langle t_1 \rangle = O_2^3 \times \mathbb{Z} = O_2^4$

$N_4^4 : 04/01/01/011$

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0,1,0,0]/2,\ \beta = B : [1,1,0,1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_2,\ \beta^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = t_4\alpha,\ \alpha t_i \alpha^{-1} = t_i, i = 1,3,$
$\quad \alpha t_4 \alpha^{-1} = t_4^{-1},\ \beta t_i \beta^{-1} = t_i, i = 1,2, \beta t_i \beta^{-1} = t_i^{-1}, i = 3,4 \rangle$
$\quad = \langle t_3, t_4, \alpha \rangle \rtimes \langle \beta \rangle = N_1^3 \rtimes \mathbb{Z}$

$\Gamma = \langle t_3, t_4, \beta \rangle \rtimes \langle \alpha \rangle = O_2^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_1 t_2,\ \beta t_i \beta^{-1} = t_i, i = 1,2,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3,4 \rangle$
$\quad = \langle t_3, t_4, \beta \rangle \times \langle t_1 \rangle = O_2^3 \times \mathbb{Z} = O_2^4$

$N_5^4 : 04/01/01/013$

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1,0,0,1]/2,\ \beta = B : [1,1,1,-1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_1 t_4,\ \beta^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = t_3 t_4^{-1}\alpha,\ \alpha t_i \alpha^{-1} = t_i, i = 1,2,4,$
$\quad \alpha t_3 \alpha^{-1} = t_3^{-1},\ \beta t_i \beta^{-1} = t_i, i = 1,2,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3,4 \rangle$
$\quad = \langle t_4^{-1}, t_3, \alpha \rangle \rtimes \langle \beta \rangle = N_1^3 \rtimes \mathbb{Z}$

$\Gamma = \langle t_3^{-1}, t_4, \beta \rangle \rtimes \langle \alpha \rangle = O_2^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_1 t_2,\ \beta t_i \beta^{-1} = t_i, i = 1,2,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3,4 \rangle$
$\quad = \langle t_3, t_4, \beta \rangle \times \langle t_1 \rangle = O_2^3 \times \mathbb{Z} = O_2^4$

$N_6^4 : 04/01/02/004$

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}(e_3 + e_4))$

Generators: $\alpha = A : [0, 1, 0, 0]/2,\ \beta = B : [1, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_2,\ \beta^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = \alpha,\ \alpha t_i \alpha^{-1} = t_i, i = 1, 3,$$
$$\alpha t_4 \alpha^{-1} = t_3 t_4^{-1},\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3, 4\rangle$$
$$= \langle t_4, t_3 t_4^{-1}, \alpha \rangle \rtimes \langle \beta \rangle = N_2^3 \rtimes \mathbb{Z}$$
$$\Gamma = \langle t_3, t_4, \beta \rangle \rtimes \langle \alpha \rangle = O_2^3 \rtimes \mathbb{Z}$$
$$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_1 t_2,\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3, 4\rangle$$
$$= \langle t_3, t_4, \beta \rangle \times \langle t_1 \rangle = O_2^3 \times \mathbb{Z} = O_2^4$$

$N_7^4 : 04/01/06/004$

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, \frac{1}{2}(e_1 + e_3), \frac{1}{2}(e_1 + e_4))$

Generators: $\alpha = A : [0, 0, 1, 0]/2,\ \beta = B : [1, 1, -1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}$

$$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = t_4 \alpha^{-1},\ \alpha t_i \alpha^{-1} = t_i, i = 1, 2,$$
$$\alpha t_4 \alpha^{-1} = t_1 t_4^{-1},\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_1 t_3^{-1},\ \beta t_4 \beta^{-1} = t_1 t_4^{-1}\rangle$$
$$= \langle t_1 t_4^{-1}, t_4, \alpha \rangle \rtimes \langle \beta \rangle = N_2^3 \rtimes \mathbb{Z}$$
$$\Gamma = \langle t_3 t_4^{-1}, t_1^{-1} t_4^2, \beta \rangle \rtimes \langle \alpha \rangle = O_2^3 \rtimes \mathbb{Z}$$
$$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_1 t_2,\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_1 t_3^{-1},\ \beta t_4 \beta^{-1} = t_1 t_4^{-1}\rangle$$
$$= \langle t_3 t_4^{-1}, t_1 t_4^{-2}, \beta \rangle \rtimes \langle t_4 \rangle = O_2^3 \rtimes \mathbb{Z} = O_3^4$$

$N_8^4 : 04/01/01/006$

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2$, $\beta = B : [0, 1, -1, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_2,\ \beta\alpha\beta^{-1} = \alpha^{-1}, \alpha t_i \alpha^{-1} = t_i, i = 1, 2,$
$\quad \alpha t_4 \alpha^{-1} = t_4^{-1}, \beta t_1 \beta^{-1} = t_1,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3, 4\rangle$
$\quad = \langle \alpha, t_4, \beta\rangle \times \langle t_1\rangle = N_3^3 \times \mathbb{Z}$

$\Gamma = \langle t_1, t_4, \alpha\rangle \rtimes \langle \beta\rangle = N_1^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_2,\ \beta t_1 \beta^{-1} = t_1,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3, 4\rangle$
$\quad = \langle t_3, t_4, \beta\rangle \times \langle t_1\rangle = O_2^3 \times \mathbb{Z} = O_2^4$

$N_9^4 : 04/01/01/007$

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2$, $\beta = B : [0, 1, -1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_2,\ \beta\alpha\beta^{-1} = t_4\alpha^{-1},\ \alpha t_i \alpha^{-1} = t_i, i = 1, 2,$
$\quad \alpha t_4 \alpha^{-1} = t_4^{-1},\ \beta t_1 \beta^{-1} = t_1,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3, 4\rangle$
$\quad = \langle \alpha, t_4^{-1}, \beta\rangle \times \langle t_1\rangle = N_4^3 \times \mathbb{Z}$

$\Gamma = \langle t_1, t_4, \alpha\rangle \rtimes \langle \beta\rangle = N_1^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_2,\ \beta t_1 \beta^{-1} = t_1,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3, 4\rangle$
$\quad = \langle t_3, t_4, \beta\rangle \times \langle t_1\rangle = O_2^3 \times \mathbb{Z} = O_2^4$

$N_{10}^4 : 04/01/03/011$

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, \frac{1}{2}(e_1 + e_3), e_4)$

Generators: $\alpha = A : [1, 0, 0, 0]/2$, $\beta = B : [1, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_1,\ \beta^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = \alpha,\ \alpha t_i \alpha^{-1} = t_i, i = 2, 3,$
$\quad \alpha t_4 \alpha^{-1} = t_4^{-1},\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_1 t_3^{-1},\ \beta t_4 \beta^{-1} = t_4^{-1}\rangle$
$\quad = \langle t_3, t_4, \alpha\rangle \rtimes \langle \beta\rangle = N_1^3 \rtimes \mathbb{Z}$

$\Gamma = \langle \alpha^{-1} t_3, t_4, \beta\rangle \rtimes \langle \alpha\rangle = N_3^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_1 t_2,\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_1 t_3^{-1},\ \beta t_4 \beta^{-1} = t_4^{-1}\rangle$
$\quad = \langle t_4, t_1 t_3^{-2}, \beta\rangle \rtimes \langle t_3\rangle = O_2^3 \rtimes \mathbb{Z} = O_3^4$

N_{11}^4 : 04/01/03/004

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, \frac{1}{2}(e_1 + e_3), e_4)$

Generators: $\alpha = A : [0, 0, 0, 1]/2,\ \beta = B : [0, 1, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_4,\ \beta^2 = t_2,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_i \alpha^{-1} = t_i, i = 1, 2,$
$\quad \alpha t_3 \alpha^{-1} = t_1 t_3^{-1},\ \beta t_1 \beta^{-1} = t_1,\ \beta t_3 \beta^{-1} = t_1 t_3^{-1},\ \beta t_4 \beta^{-1} = t_4^{-1}\rangle$
$= \langle t_3, t_1 t_3^{-1}, \alpha\rangle \rtimes \langle\beta\rangle = N_2^3 \rtimes \mathbb{Z}$

$\Gamma = \langle \alpha, t_1^{-1} t_3^2, \beta\rangle \rtimes \langle t_3\rangle = N_3^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_2,\ \beta t_1 \beta^{-1} = t_1,\ \beta t_3 \beta^{-1} = t_1 t_3^{-1},\ \beta t_4 \beta^{-1} = t_4^{-1}\rangle$
$= \langle t_4, t_1 t_3^{-2}, \beta\rangle \rtimes \langle t_3\rangle = O_2^3 \rtimes \mathbb{Z} = O_3^4$

N_{12}^4 : 04/01/03/012

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, \frac{1}{2}(e_1 + e_3), e_4)$

Generators: $\alpha = A : [1, 0, 0, 0]/2,\ \beta = B : [1, 1, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_1,\ \beta^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = t_4\alpha,\ \alpha t_i \alpha^{-1} = t_i, i = 2, 3,$
$\quad \alpha t_4 \alpha^{-1} = t_4^{-1},\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_1 t_3^{-1},\ \beta t_4 \beta^{-1} = t_4^{-1}\rangle$
$= \langle t_3, t_4, \alpha\rangle \rtimes \langle\beta\rangle = N_1^3 \rtimes \mathbb{Z}$

$\Gamma = \langle \alpha^{-1} t_3, t_4^{-1}, \beta\rangle \rtimes \langle\alpha\rangle = N_4^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_1 t_2,\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_1 t_3^{-1},\ \beta t_4 \beta^{-1} = t_4^{-1}\rangle$
$= \langle t_4, t_1 t_3^{-2}, \beta\rangle \rtimes \langle t_3\rangle = O_2^3 \rtimes \mathbb{Z} = O_3^4$

N^4_{13} : 04/01/04/005

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}(-e_1 + e_3 + e_4))$

Generators: $\alpha = A : [-1, 0, 0, 0]/2,\ \beta = B : [1, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_1^{-1},\ \beta^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = \alpha,\ \alpha t_2 \alpha^{-1} = t_2,\ \alpha t_3 \alpha^{-1} = t_3^{-1},$
$\alpha t_4 \alpha^{-1} = t_3^{-1} t_4,\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_3^{-1},\ \beta t_4 \beta^{-1} = t_1^{-1} t_4^{-1} \rangle$
$= \langle t_4, t_3^{-1} t_4, \alpha \rangle \rtimes \langle \beta \rangle = N^3_2 \rtimes \mathbb{Z}$

$\Gamma = \langle \alpha^{-1} t_4, t_3^{-1}, \beta \rangle \rtimes \langle \alpha \rangle = N^3_4 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_1 t_2,\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_3^{-1},\ \beta t_4 \beta^{-1} = t_1^{-1} t_4^{-1} \rangle$
$= \langle t_3, t_1^{-1} t_4^{-2}, \beta \rangle \rtimes \langle t_4 \rangle = O^3_2 \rtimes \mathbb{Z} = O^4_3$

N^4_{14} : 02/02/01/002

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^2 = t_4,\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 1, 2, 3 \rangle$
$= \langle t_1, t_2, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha^2 \rangle = \langle t_1, t_2, t_3, t_4 \rangle = \mathbb{Z}^4 = O^4_1$

N^4_{15} : 12/01/02/002

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/4$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^4 = t_2,\ \alpha t_1 \alpha^{-1} = t_1^{-1},\ \alpha t_3 \alpha^{-1} = t_4,\ \alpha t_4 \alpha^{-1} = t_3^{-1} \rangle$
$= \langle t_1, t_3, t_4 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha^2 \mid (\alpha^2)^2 = t_2,\ \alpha^2 t_1 (\alpha^2)^{-1} = t_1,\ \alpha^2 t_i (\alpha^2)^{-1} = t_1^{-1}, i = 3, 4 \rangle$
$= \langle t_3, t_4, \alpha^2 \rangle \times \langle t_1 \rangle = O^3_2 \times \mathbb{Z} = O^4_2$

N^4_{16} : 12/01/04/002

Orthogonal matrices: $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (\frac{1}{2}(e_1 + e_2 + e_3), \frac{1}{2}(-e_1 + e_2 - e_3), \frac{1}{2}(-e_1 - e_2 + e_3), e_4)$

Generators: $\alpha = A : [0, 0, 0, 1]/4$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^4 = t_4,\ \alpha t_1 \alpha^{-1} = t_1^{-1} t_2^{-1} t_3^{-1},\ \alpha t_2 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_2 \rangle$
$\quad = \langle t_1, t_2, t_3 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha^2 \mid (\alpha^2)^2 = t_4,\ \alpha^2 t_1 (\alpha^2)^{-1} = t_3,\ \alpha^2 t_2 (\alpha^2)^{-1} = t_1^{-1} t_2^{-1} t_3^{-1},$
$\qquad \alpha^2 t_3 (\alpha^2)^{-1} = t_1 \rangle$
$\quad = \langle t_2 t_3, t_1 t_3^{-1}, \alpha^2 \rangle \rtimes \langle t_3 \rangle = O^3_2 \rtimes \mathbb{Z} = O^4_3$

N^4_{17} : 12/01/03/002

Orthogonal matrices: $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B | AB = BA, A^2 = B \rangle = \mathbb{Z}_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2,\ \beta = B : [-1, 1, 0, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^4 = t_1 t_2,\ \beta = \alpha^2 t_1^{-1},\ \alpha \beta \alpha^{-1} = \beta^{-1},\ \alpha t_1 \alpha^{-1} = t_2,$
$\qquad \alpha t_2 \alpha^{-1} = t_1,\ \alpha t_3 \alpha^{-1} = t_4,\ \alpha t_4 \alpha^{-1} = t_3^{-1},\ \beta t_i \beta^{-1} = t_i, i = 1, 2,$
$\qquad \beta t_i \beta^{-1} = t_i^{-1}, i = 3, 4 \rangle$
$\quad = \langle t_3, t_4, \beta \rangle \rtimes \langle \alpha \rangle = O^3_2 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha^2, \beta \mid (\alpha^2)^2 = t_1 t_2,\ \beta = \alpha^2 t_1^{-1}, \alpha^2 \beta (\alpha^2)^{-1} = \beta,$
$\qquad \alpha^2 t_i (\alpha^2)^{-1} = t_i, i = 1, 2,\ \alpha^2 t_i (\alpha^2)^{-1} = t_i^{-1}, i = 3, 4,$
$\qquad \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 3, 4 \rangle$
$\quad = \langle t_3, t_4, \beta \rangle \times \langle t_1 \rangle = O^3_2 \times \mathbb{Z} = O^4_2$

N^4_{18} : 12/01/06/002

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B | AB = BA, A^2 = B \rangle = \mathbb{Z}_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, \frac{1}{2}(e_1 + e_2 + 2e_3), \frac{1}{2}(e_1 + e_2 + 2e_4))$

Generators: $\alpha = A : [0, 0, 1, 0]/2,\ \beta = B : [1, 2, 1, -3]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_4$

$$\begin{aligned}\Gamma = \langle & t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^4 = t_1,\ \beta = t_2 t_4^{-1} \alpha^2,\ \alpha\beta\alpha^{-1} = t_3^{-1} t_4 \beta^{-1},\ \alpha t_2 \alpha^{-1} = t_2^{-1}, \\ & \alpha t_3 \alpha^{-1} = t_1 t_4^{-1},\ \alpha t_4 \alpha^{-1} = t_2^{-1} t_3,\ \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_1 t_2 t_3^{-1}, \\ & \beta t_4 \beta^{-1} = t_1 t_2 t_4^{-1} \rangle \\ = \langle & t_3^{-1} t_4, t_1^{-1} t_2^{-1} t_3 t_4, \beta \rangle \rtimes \langle \alpha \rangle = O_2^3 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 = \langle & t_1, t_2, t_3, t_4, \alpha^2, \beta \mid (\alpha^2)^2 = t_1,\ \beta = t_2 t_4^{-1} \alpha^2, \alpha^2 \beta (\alpha^2)^{-1} = t_1^{-1} t_4^2 \beta^{-1}, \\ & \alpha^2 t_2 (\alpha^2)^{-1} = t_2,\ \alpha^2 t_3 (\alpha^2)^{-1} = t_1 t_2 t_3^{-1},\ \alpha^2 t_4 (\alpha^2)^{-1} = t_1 t_2 t_4^{-1}, \\ & \beta t_i \beta^{-1} = t_i, i = 1, 2,\ \beta t_3 \beta^{-1} = t_1 t_2 t_3^{-1},\ \beta t_4 \beta^{-1} = t_1 t_2 t_4^{-1} \rangle \\ = \langle & t_3 t_4^{-1}, t_1 t_2 t_4^{-2}, \alpha^2 \rangle \rtimes \langle t_4 \rangle = O_2^3 \rtimes \mathbb{Z} = O_3^4\end{aligned}$$

N^4_{19} : 14/02/03/002

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_6$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2} e_3 + \frac{\sqrt{3}}{2} e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/6$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_6$

$$\begin{aligned}\Gamma &= \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^6 = t_2,\ \alpha t_1 \alpha^{-1} = t_1^{-1},\ \alpha t_3 \alpha^{-1} = t_3^{-1} t_4,\ \alpha t_4 \alpha^{-1} = t_3^{-1} \rangle \\ &= \langle t_1, t_3, t_4 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 &= \langle t_1, t_2, t_3, t_4, \alpha^2 \mid (\alpha^2)^3 = t_2,\ \alpha^2 t_1 (\alpha^2)^{-1} = t_1,\ \alpha^2 t_3 (\alpha^2)^{-1} = t_4^{-1},\ \alpha^2 t_4 (\alpha^2)^{-1} = t_3 t_4^{-1} \rangle \\ &= \langle t_3, t_4^{-1}, \alpha^2 \rangle \times \langle t_1 \rangle = O_3^3 \times \mathbb{Z} = O_4^4\end{aligned}$$

N^4_{20} : 14/01/03/002

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_6$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/6$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^6 = t_2,\ \alpha t_1 \alpha^{-1} = t_1^{-1},\ \alpha t_3 \alpha^{-1} = t_3 t_4^{-1},\ \alpha t_4 \alpha^{-1} = t_3 \rangle$
$\quad = \langle t_1, t_3, t_4 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha^2 \mid (\alpha^2)^3 = t_2,\ \alpha^2 t_1 (\alpha^2)^{-1} = t_1,\ \alpha^2 t_3 (\alpha^2)^{-1} = t_4^{-1},\ \alpha^2 t_4 (\alpha^2)^{-1} = t_3 t_4^{-1} \rangle$
$\quad = \langle t_3, t_4^{-1}, \alpha^2 \rangle \times \langle t_1 \rangle = O^3_3 \times \mathbb{Z} = O^4_4$

N^4_{21} : 14/01/01/002

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

Holonomy: $\langle A \rangle = \mathbb{Z}_6$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [1, 0, 0, 0]/6$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^6 = t_1,\ \alpha t_2 \alpha^{-1} = t_4^{-1},\ \alpha t_3 \alpha^{-1} = t_2^{-1},\ \alpha t_4 \alpha^{-1} = t_3^{-1} \rangle$
$\quad = \langle t_2, t_3, t_4 \rangle \rtimes \langle \alpha \rangle = \mathbb{Z}^3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha^2 \mid (\alpha^2)^3 = t_1,\ \alpha^2 t_2 (\alpha^2)^{-1} = t_3,\ \alpha^2 t_3 (\alpha^2)^{-1} = t_4,\ \alpha^2 t_4 (\alpha^2)^{-1} = t_2 \rangle$
$\quad = \langle t_2^{-1} t_3, t_3^{-1} t_4, \alpha^2 \rangle \rtimes \langle t_3 \rangle = O^3_3 \rtimes \mathbb{Z} = O^4_5$

N^4_{22} : 04/02/01/008

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2$, $\beta = B : [0, 0, 1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_4,\ \beta \alpha \beta^{-1} = \alpha^{-1},\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 1, 2,$
$\qquad \alpha t_4 \alpha^{-1} = t_4,\ \beta t_i \beta^{-1} = t_i^{-1}, i = 1, 2, 3 \rangle$
$\quad = \langle t_1, t_2, \alpha \rangle \rtimes \langle \beta \rangle = O^3_2 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^2 = t_3,\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 1, 2,\ \alpha t_4 \alpha^{-1} = t_4 \rangle$
$\quad = \langle t_1, t_2, \alpha \rangle \times \langle t_4 \rangle = O^3_2 \times \mathbb{Z} = O^4_2$

N^4_{23} : 04/02/01/016

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, -1, 1, 0]/2$, $\beta = B : [0, 0, 1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_3,\ \beta^2 = t_4,\ \beta\alpha\beta^{-1} = t_2\alpha^{-1},\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 1, 2,$
$\quad \alpha t_4\alpha^{-1} = t_4,\ \beta t_i\beta^{-1} = t_i, i = 1, 2, 3\rangle$
$\quad = \langle t_1, t_2, \alpha\rangle \rtimes \langle\beta\rangle = O^3_2 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^2 = t_3,\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 1, 2,\ \alpha t_4\alpha^{-1} = t_4\rangle$
$\quad = \langle t_1, t_2, \alpha\rangle \times \langle t_4\rangle = O^3_2 \times \mathbb{Z} = O^4_2$

N^4_{24} : 04/02/01/011

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2$, $\beta = B : [0, 1, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_2,\ \beta^2 = t_4,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 1, 4,$
$\quad \alpha t_3\alpha^{-1} = t_3^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_3\beta^{-1} = t_3\rangle$
$\quad = \langle t_1, t_3, \alpha\rangle \rtimes \langle\beta\rangle = N^3_1 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_3\beta^{-1} = t_3\rangle$
$\quad = \langle t_1, t_2, \beta\rangle \times \langle t_3\rangle = O^3_2 \times \mathbb{Z} = O^4_2$

N^4_{25} : 04/02/01/012

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2$, $\beta = B : [0, 1, -1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_2,\ \beta^2 = t_3^{-1}t_4,\ \beta\alpha\beta^{-1} = t_3^{-1}\alpha^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 1, 4,$
$\quad \alpha t_3\alpha^{-1} = t_3^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_i\beta^{-1} = t_i, i = 3, 4\rangle$
$\quad = \langle t_1, t_3^{-1}, \alpha\rangle \rtimes \langle\beta\rangle = N^3_1 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_3^{-1}t_4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_i\beta^{-1} = t_i, i = 3, 4\rangle$
$\quad = \langle t_1, t_2, \beta\rangle \times \langle t_3\rangle = O^3_2 \times \mathbb{Z} = O^4_2$

N^4_{26} : 04/02/03/004

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, \frac{1}{2}(e_1 + e_3), e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2,\ \beta = B : [0, 1, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid\ & \alpha^2 = t_2,\ \beta^2 = t_4,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 1, 4,\\ & \alpha t_3\alpha^{-1} = t_1t_3^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_3\beta^{-1} = t_1^{-1}t_3\rangle\\ = \langle t_3, t_1t_3^{-1}, \alpha\rangle & \rtimes \langle\beta\rangle = N^3_2 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 &= \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_3\beta^{-1} = t_1^{-1}t_3\rangle\\ &= \langle t_2, t_1^{-1}, \beta\rangle \rtimes \langle t_3\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_3\end{aligned}$$

N^4_{27} : 13/01/01/008

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_4 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, -2, -2]/4,\ \beta = B : [0, 0, 1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid\ & \alpha^4 = t_2,\ \beta^2 = t_3t_4,\ \alpha\beta\alpha^{-1} = t_3^{-1}\beta,\ \alpha t_1\alpha^{-1} = t_1,\\ & \alpha t_3\alpha^{-1} = t_4,\ \alpha t_4\alpha^{-1} = t_3^{-1},\ \beta t_1\beta^{-1} = t_1^{-1},\ \beta t_i\beta^{-1} = t_i, i = 2, 3, 4\rangle\\ = \langle t_3, t_1, \beta\rangle & \rtimes \langle\alpha\rangle = N^3_1 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 &= \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^4 = t_2,\ \alpha t_1\alpha^{-1} = t_1,\ \alpha t_3\alpha^{-1} = t_4,\ \alpha t_4\alpha^{-1} = t_3^{-1}\rangle\\ &= \langle t_3, t_4, \alpha\rangle \times \langle t_1\rangle = O^3_4 \times \mathbb{Z} = O^4_6\end{aligned}$$

N^4_{28} : 13/01/01/011

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_4 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [-2, 1, 2, 2]/4,\ \beta = B : [1, 0, 1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid\ & \alpha^4 = t_1^{-2}t_2,\ \beta^2 = t_3t_4,\ \alpha\beta\alpha^{-1} = t_1^{-1}t_3^{-1}\beta,\ \alpha t_1\alpha^{-1} = t_1,\\ & \alpha t_3\alpha^{-1} = t_4,\ \alpha t_4\alpha^{-1} = t_3^{-1},\ \beta t_1\beta^{-1} = t_1^{-1},\ \beta t_i\beta^{-1} = t_i, i = 2, 3, 4\rangle\\ = \langle t_3, t_1^{-1}, \beta\rangle & \rtimes \langle\alpha\rangle = N^3_1 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 &= \langle t_1, t_2, t_3, t_4, \alpha \mid \alpha^4 = t_1^{-2}t_2,\ \alpha t_1\alpha^{-1} = t_1,\ \alpha t_3\alpha^{-1} = t_4,\ \alpha t_4\alpha^{-1} = t_3^{-1}\rangle\\ &= \langle t_3, t_4, \alpha\rangle \times \langle t_1\rangle = O^3_4 \times \mathbb{Z} = O^4_6\end{aligned}$$

N^4_{29} : 13/01/03/008

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (\frac{1}{2}(e_1 - e_2 - e_3), \frac{1}{2}(-e_1 + e_2 - e_3), \frac{1}{2}(e_1 + e_2 + e_3), e_4)$

Generators: $\alpha = A : [1, 1, 0, 0]/2,\ \beta = B : [0, 0, -2, 1]/4$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle &t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_1t_2,\ \beta^4 = t_1^{-1}t_3^{-1}t_4,\ \beta\alpha\beta^{-1} = t_3^{-1}\alpha^{-1},\ \alpha t_1\alpha^{-1} = t_3^{-1},\\ &\alpha t_2\alpha^{-1} = t_1t_2t_3,\ \alpha t_3\alpha^{-1} = t_1^{-1},\ \alpha t_4\alpha^{-1} = t_4,\ \beta t_1\beta^{-1} = t_2^{-1},\ \beta t_2\beta^{-1} = t_3^{-1},\\ &\beta t_3\beta^{-1} = t_1t_2t_3,\ \beta t_4\beta^{-1} = t_4\rangle\\ = \langle &t_1, t_3^{-1}, \alpha\rangle \rtimes \langle\beta\rangle = N^3_2 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 = \langle &t_1, t_2, t_3, t_4, \beta \mid \beta^4 = t_1^{-1}t_3^{-1}t_4,\ \beta t_1\beta^{-1} = t_2^{-1},\ \beta t_2\beta^{-1} = t_3^{-1},\ \beta t_3\beta^{-1} = t_1t_2t_3,\\ &\beta t_4\beta^{-1} = t_4\rangle\\ = \langle &t_1t_2, t_2^{-1}t_3^{-1}, \beta\rangle \rtimes \langle t_3\rangle = O^3_4 \rtimes \mathbb{Z} = O^4_7\end{aligned}$$

N^4_{30} : 06/01/01/063

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2,\ \beta = B : [0, 1, -1, 0]/2,\ \gamma = C : [0, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle &t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_3,\ \beta^2 = t_2,\ \gamma^2 = t_4,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \gamma\alpha\gamma^{-1} = \alpha,\\ &\gamma\beta\gamma^{-1} = \beta^{-1},\ \alpha t_1\alpha^{-1} = t_1^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 2, 4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3,\\ &\beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle\\ = \langle &\alpha, t_1, \beta\rangle \rtimes \langle\gamma\rangle = N^3_3 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 = \langle &t_1, t_2, t_3, t_4, \beta, \gamma \mid \beta^2 = t_2,\ \gamma^2 = t_4,\ \gamma\beta\gamma^{-1} = \beta^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3,\\ &\beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle\\ = \langle &t_1, t_3, \beta\rangle \rtimes \langle\gamma\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_9\end{aligned}$$

N^4_{31} : 06/01/01/041

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2,\ \beta = B : [1, -1, 0, 0]/2,\ \gamma = C : [1, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_2,\ \beta^2 = t_1,\ \gamma^2 = t_4,\ \beta\alpha\beta^{-1} = \alpha^{-1},$
$\gamma\alpha\gamma^{-1} = \alpha^{-1},\ \gamma\beta\gamma^{-1} = t_2\beta^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 1, 4,\ \alpha t_3\alpha^{-1} = t_3^{-1},$
$\beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,\ \beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle$
$= \langle \alpha, t_3, \beta\rangle \rtimes \langle\gamma\rangle = N^3_3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid \beta^2 = t_1,\ \gamma^2 = t_4,\ \gamma\beta\gamma^{-1} = t_2\beta^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,$
$\beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle$
$= \langle t_2, t_3, \beta\rangle \rtimes \langle\gamma\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_{10}$

N^4_{32} : 06/01/01/064

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2,\ \beta = B : [0, 1, -1, 0]/2,\ \gamma = C : [1, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_3,\ \beta^2 = t_2,\ \gamma^2 = t_4,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \gamma\alpha\gamma^{-1} = t_1\alpha,$
$\gamma\beta\gamma^{-1} = t_1\beta^{-1},\ \alpha t_1\alpha^{-1} = t_1^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 2, 4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3,$
$\beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle$
$= \langle \alpha, t_1, \beta\rangle \rtimes \langle\gamma\rangle = N^3_3 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid \beta^2 = t_2,\ \gamma^2 = t_4,\ \gamma\beta\gamma^{-1} = t_1\beta^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3,$
$\beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle$
$= \langle t_1, t_3, \beta\rangle \rtimes \langle\gamma\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_{10}$

N^4_{33} : 06/01/01/066

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2$, $\beta = B : [0, 1, -1, 0]/2$, $\gamma = C : [1, 0, 1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid\ & \alpha^2 = t_3,\ \beta^2 = t_2,\ \gamma^2 = t_3t_4,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \gamma\alpha\gamma^{-1} = t_1\alpha, \\ & \gamma\beta\gamma^{-1} = t_1t_3\beta^{-1},\ \alpha t_1\alpha^{-1} = t_1^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 2, 4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3, \\ & \beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_i\gamma^{-1} = t_i, i = 3, 4\rangle \\ = \langle \alpha, t_1, \beta\rangle \rtimes \langle\gamma\rangle = N^3_3 \rtimes \mathbb{Z} &\end{aligned}$$

$$\begin{aligned}\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid\ & \beta^2 = t_2,\ \gamma^2 = t_3t_4,\ \gamma\beta\gamma^{-1} = t_1t_3\beta^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3, \\ & \beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_i\gamma^{-1} = t_i, i = 3, 4\rangle \\ = \langle t_1, t_3, \beta\rangle \rtimes \langle\gamma\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_{11} &\end{aligned}$$

N^4_{34} : 06/01/01/082

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2$, $\beta = B : [1, 1, 1, 0]/2$, $\gamma = C : [1, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid\ & \alpha^2 = t_3,\ \beta^2 = t_2,\ \gamma^2 = t_4,\ \beta\alpha\beta^{-1} = t_1\alpha^{-1},\ \gamma\alpha\gamma^{-1} = t_1\alpha, \\ & \gamma\beta\gamma^{-1} = \beta^{-1},\ \alpha t_1\alpha^{-1} = t_1^{-1}\ \alpha t_i\alpha^{-1} = t_i, i = 2, 4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3, \\ & \beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle \\ = \langle \alpha, t_1^{-1}, \beta\rangle \rtimes \langle\gamma\rangle = N^3_4 \rtimes \mathbb{Z} &\end{aligned}$$

$$\begin{aligned}\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid\ & \beta^2 = t_2,\ \gamma^2 = t_4,\ \gamma\beta\gamma^{-1} = \beta^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3, \\ & \beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle \\ = \langle t_1, t_3, \beta\rangle \rtimes \langle\gamma\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_9 &\end{aligned}$$

N^4_{35} : 06/01/01/083

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2$, $\beta = B : [1, 1, -1, 0]/2$, $\gamma = C : [0, -1, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_3,\ \beta^2 = t_2,\ \gamma^2 = t_4,\ \beta\alpha\beta^{-1} = t_1\alpha^{-1},\ \gamma\alpha\gamma^{-1} = \alpha^{-1},$
$\gamma\beta\gamma^{-1} = t_3\beta^{-1},\ \alpha t_1\alpha^{-1} = t_1^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 2, 4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3,$
$\beta t_4\beta^{-1} = t_4,\ \gamma t_1\gamma^{-1} = t_1,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 2, 3\rangle$
$= \langle \alpha, t_1^{-1}, \beta\rangle \rtimes \langle\gamma\rangle = N^3_4 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid \beta^2 = t_2,\ \gamma^2 = t_4,\ \gamma\beta\gamma^{-1} = t_3\beta^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3,$
$\beta t_4\beta^{-1} = t_4,\ \gamma t_1\gamma^{-1} = t_1,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 2, 3\rangle$
$= \langle t_3, t_1, \beta\rangle \rtimes \langle\gamma\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_{10}$

N^4_{36} : 06/01/01/081

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 0, 1, 0]/2$, $\beta = B : [-1, 1, -1, 0]/2$, $\gamma = C : [0, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_3,\ \beta^2 = t_2,\ \gamma^2 = t_4,\ \beta\alpha\beta^{-1} = t_1^{-1}\alpha^{-1},\ \gamma\alpha\gamma^{-1} = \alpha,$
$\gamma\beta\gamma^{-1} = t_1\beta^{-1},\ \alpha t_1\alpha^{-1} = t_1^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 2, 4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3,$
$\beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle$
$= \langle \alpha, t_1, \beta\rangle \rtimes \langle\gamma\rangle = N^3_4 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid \beta^2 = t_2,\ \gamma^2 = t_4,\ \gamma\beta\gamma^{-1} = t_1\beta^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 3,$
$\beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle$
$= \langle t_1, t_3, \beta\rangle \rtimes \langle\gamma\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_{10}$

N^4_{37} : 06/01/01/045

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2,\ \beta = B : [1, -1, -1, 0]/2,\ \gamma = C : [1, 0, 1, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_2,\ \beta^2 = t_1,\ \gamma^2 = t_3t_4,\ \beta\alpha\beta^{-1} = t_3^{-1}\alpha^{-1},$
$\gamma\alpha\gamma^{-1} = t_3\alpha^{-1},\ \gamma\beta\gamma^{-1} = t_2t_3\beta^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 1, 4,\ \alpha t_3\alpha^{-1} = t_3^{-1},$
$\beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,\ \beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle$
$= \langle \alpha, t_3, \beta\rangle \rtimes \langle\gamma\rangle = N^3_4 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid \beta^2 = t_1,\ \gamma^2 = t_3t_4,\ \gamma\beta\gamma^{-1} = t_2t_3\beta^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,$
$\beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 2,\ \gamma t_3\gamma^{-1} = t_3\rangle$
$= \langle t_2, t_3, \beta\rangle \rtimes \langle\gamma\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_{11}$

N^4_{38} : 06/01/01/049

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2,\ \beta = B : [-1, 1, -1, 0]/2,\ \gamma = C : [0, 0, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_2,\ \beta^2 = t_1^{-1},\ \gamma^2 = t_4,\ \beta\alpha\beta^{-1} = t_1^{-1}t_3^{-1}\alpha^{-1},$
$\gamma\alpha\gamma^{-1} = \alpha,\ \gamma\beta\gamma^{-1} = \beta^{-1},\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 1, 3,\ \alpha t_4\alpha^{-1} = t_4,$
$\beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,\ \beta t_4\beta^{-1} = t_4,\ \gamma t_1\gamma^{-1} = t_1^{-1},\ \gamma t_i\gamma^{-1} = t_i, i = 2, 3\rangle$
$= \langle \alpha, \beta\rangle \rtimes \langle\gamma\rangle = O^3_6 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_2,\ \beta^2 = t_1^{-1},\ \beta\alpha\beta^{-1} = t_1^{-1}t_3^{-1}\alpha^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 1, 3,$
$\alpha t_4\alpha^{-1} = t_4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,\ \beta t_4\beta^{-1} = t_4\rangle$
$= \langle \alpha, \beta\rangle \times \langle t_4\rangle = O^3_6 \times \mathbb{Z} = O^4_{14}$

N^4_{39} : 06/01/01/092

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, -1, 0]/2,\ \beta = B : [1, 0, 1, 0]/2,\ \gamma = C : [0, 1, 0, 1]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_2,\ \beta^2 = t_3,\ \gamma^2 = t_2t_4,\ \beta\alpha\beta^{-1} = t_1t_3\alpha^{-1},$
$\gamma\alpha\gamma^{-1} = t_3\alpha,\ \gamma\beta\gamma^{-1} = t_2\beta^{-1},\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 1, 3,\ \alpha t_4\alpha^{-1} = t_4,$
$\beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_4\beta^{-1} = t_4,\ \gamma t_i\gamma^{-1} = t_i, i = 1, 2, 4,\ \gamma t_3\gamma^{-1} = t_3^{-1}\rangle$
$= \langle\alpha, \beta\rangle \rtimes \langle\gamma\rangle = O^3_6 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_2,\ \beta^2 = t_3,\ \beta\alpha\beta^{-1} = t_1t_3\alpha^{-1},\ \alpha t_i\alpha^{-1} = t_i, i = 1, 3,$
$\alpha t_4\alpha^{-1} = t_4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_4\beta^{-1} = t_4\rangle$
$= \langle\alpha, \beta\rangle \times \langle t_4\rangle = O^3_6 \times \mathbb{Z} = O^4_{14}$

N^4_{40} : 12/03/10/005

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B, C \mid B = C^2A, A^2 = C^4 = 1, ACA^{-1} = C^{-1}\rangle = D_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, \frac{1}{2}(e_1 + e_2 + e_3 - e_4), \frac{1}{2}(e_1 + e_2 + e_3 + e_4))$

Generators: $\alpha = A : [0, 0, -1, 1]/2,\ \beta = B : [1, 2, 0, -2]/2,\ \gamma = C : [0, 0, 1, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_3^{-1}t_4,\ \beta^2 = t_1t_2t_3^{-1}t_4^{-1},\ \gamma^2 = t_1t_2t_4^{-1}\alpha\beta,\ \gamma^4 = t_1,$
$\alpha\beta\alpha^{-1} = t_2^{-1}t_3^{-1}t_4\beta^{-1},\ \gamma\alpha\gamma^{-1} = \beta^{-1},\ \gamma\beta\gamma^{-1} = \alpha,\ \alpha t_1\alpha^{-1} = t_1,$
$\alpha t_2\alpha^{-1} = t_2^{-1},\ \alpha t_3\alpha^{-1} = t_1t_4^{-1},\ \alpha t_4\alpha^{-1} = t_1t_3^{-1},$
$\beta t_1\beta^{-1} = t_1,\ \beta t_2\beta^{-1} = t_2^{-1},\ \beta t_3\beta^{-1} = t_2^{-1}t_4,\ \beta t_4\beta^{-1} = t_2^{-1}t_3,$
$\gamma t_1\gamma^{-1} = t_1,\ \gamma t_2\gamma^{-1} = t_2^{-1},\ \gamma t_3\gamma^{-1} = t_1t_4^{-1},\ \gamma t_4\gamma^{-1} = t_2^{-1}t_3\rangle$
$= \langle\alpha, \beta\rangle \rtimes \langle\gamma\rangle = O^3_6 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma^2 \mid \alpha^2 = t_3^{-1}t_4,\ \beta^2 = t_1t_2t_3^{-1}t_4^{-1},\ \gamma^2 = t_1t_2t_4^{-1}\alpha\beta,\ (\gamma^2)^2 = t_1,$
$\alpha\beta\alpha^{-1} = t_2^{-1}t_3^{-1}t_4\beta^{-1},\ (\gamma^2)\alpha(\gamma^2)^{-1} = \alpha^{-1},\ (\gamma^2)\beta(\gamma^2)^{-1} = \beta^{-1},\ \alpha t_1\alpha^{-1} = t_1,$
$\alpha t_2\alpha^{-1} = t_2^{-1},\ \alpha t_3\alpha^{-1} = t_1t_4^{-1},\ \alpha t_4\alpha^{-1} = t_1t_3^{-1},\ \beta t_1\beta^{-1} = t_1,\ \beta t_2\beta^{-1} = t_2^{-1},$
$\beta t_3\beta^{-1} = t_2^{-1}t_4,\ \beta t_4\beta^{-1} = t_2^{-1}t_3,\ \gamma^2t_1(\gamma^2)^{-1} = t_1,\ \gamma^2t_2(\gamma^2)^{-1} = t_2,$
$\gamma^2t_3(\gamma^2)^{-1} = t_1t_2t_3^{-1},\ \gamma^2t_4(\gamma^2)^{-1} = t_1t_2t_4^{-1}\rangle$
$= \langle\alpha, \beta\rangle \rtimes \langle\gamma^2\rangle = O^3_6 \rtimes \mathbb{Z} = O^4_{15}$

N^4_{41} : 12/03/04/006

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B, C \mid A = C^2, B^2 = C^4 = 1, BCB^{-1} = C^{-1} \rangle = D_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [-1, 1, 1, -1]/2,\ \beta = B : [0, 0, 0, 1]/2,\ \gamma = C : [0, 1, 1, 0]/2$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_4$

$$\begin{aligned}\Gamma = \langle & t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha = \gamma^2 t_1^{-1},\ \beta^2 = t_4,\ \gamma^4 = t_1 t_2,\ \beta\alpha\beta^{-1} = t_3^{-1} t_4 \alpha^{-1}, \\ & \gamma\alpha\gamma^{-1} = \alpha^{-1},\ \gamma\beta\gamma^{-1} = \alpha\beta^{-1},\ \alpha t_i \alpha^{-1} = t_i, i = 1, 2,\ \alpha t_i \alpha^{-1} = t_i^{-1}, i = 3, 4, \\ & \beta t_1 \beta^{-1} = t_2,\ \beta t_2 \beta^{-1} = t_1,\ \beta t_3 \beta^{-1} = t_3^{-1},\ \gamma t_1 \gamma^{-1} = t_2,\ \gamma t_2 \gamma^{-1} = t_1, \\ & \gamma t_3 \beta^{-1} = t_4^{-1},\ \gamma t_4 \gamma^{-1} = t_3 \rangle \\ = \langle & \alpha, \beta \rangle \rtimes \langle \gamma \rangle = O^3_6 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 = \langle & t_1, t_2, t_3, t_4, \alpha, \beta, \gamma^2 \mid \alpha = \gamma^2 t_1^{-1},\ \beta^2 = t_4,\ (\gamma^2)^2 = t_1 t_2,\ \beta\alpha\beta^{-1} = t_3^{-1} t_4 \alpha^{-1}, \\ & (\gamma^2)\alpha(\gamma^2)^{-1} = \alpha,\ (\gamma^2)\beta(\gamma^2)^{-1} = t_3\beta^{-1},\ \alpha t_i \alpha^{-1} = t_i, i = 1, 2, \\ & \alpha t_i \alpha^{-1} = t_i^{-1}, i = 3, 4,\ \beta t_1 \beta^{-1} = t_2,\ \beta t_2 \beta^{-1} = t_1,\ \beta t_3 \beta^{-1} = t_3^{-1}, \\ & \gamma^2 t_i (\gamma^2)^{-1} = t_i, i = 1, 2,\ \gamma^2 t_i (\gamma^2)^{-1} = t_i^{-1}, i = 3, 4 \rangle \\ = \langle & \beta, \alpha^{-1} \rangle \rtimes \langle t_2 \rangle = O^3_6 \rtimes \mathbb{Z} = O^4_{17}\end{aligned}$$

N^4_{42} : 15/01/01/010

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Holonomy: $\langle A, B \rangle = \mathbb{Z}_2 \times \mathbb{Z}_6$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, \frac{1}{2} e_3 + \frac{\sqrt{3}}{2} e_4)$

Generators: $\alpha = A : [1, 0, 0, 0]/2,\ \beta = B : [0, 1, 0, 0]/6$

First Homology: $\mathbb{Z} \oplus \mathbb{Z}_2$

$$\begin{aligned}\Gamma = \langle & t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_1,\ \beta^6 = t_2,\ \beta\alpha\beta^{-1} = \alpha^{-1},\ \alpha t_2 \alpha^{-1} = t_2, \\ & \alpha t_i \alpha^{-1} = t_i^{-1}, i = 3, 4,\ \beta t_1 \beta^{-1} = t_1^{-1},\ \beta t_3 \beta^{-1} = t_4^{-1},\ \beta t_4 \beta^{-1} = t_3 t_4^{-1} \rangle \\ = \langle & t_3^{-1}, t_4, \alpha \rangle \rtimes \langle \beta \rangle = O^3_2 \rtimes \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_0 = \langle & t_1, t_2, t_3, t_4, \alpha, \beta^2 \mid \alpha^2 = t_1,\ (\beta^2)^3 = t_2,\ \beta^2\alpha(\beta^2)^{-1} = \alpha,\ \alpha t_2 \alpha^{-1} = t_2, \\ & \alpha t_i \alpha^{-1} = t_i^{-1}, i = 3, 4,\ \beta^2 t_1 (\beta^2)^{-1} = t_1,\ \beta^2 t_3 (\beta^2)^{-1} = t_3^{-1} t_4,\ \beta^2 t_4 (\beta^2)^{-1} = t_3^{-1} \rangle \\ = \langle & t_3, t_3 t_4^{-1}, \alpha\beta^2 \rangle \times \langle t_1 t_2 \rangle = O^3_5 \times \mathbb{Z} = O^4_8\end{aligned}$$

N^4_{43} : 25/01/01/010

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B, C \mid A^2 = B^2 = C^6 = 1, [A, B] = 1, CAC^{-1} = B^{-1}A,$
$CBC^{-1} = A^{-1}\rangle = \Delta \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 1, 0]/2,\ \beta = B : [0, 1, 0, -1]/2,\ \gamma = C : [1, 0, 0, 0]/6$

First Homology: $\mathbb{Z}$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_2,\ \beta^2 = t_4^{-1},\ \gamma^6 = t_1,\ \beta\alpha\beta^{-1} = t_3^{-1}t_4^{-1}\alpha^{-1},$
$\gamma\alpha\gamma^{-1} = \alpha^{-1}\beta,\ \gamma\beta\gamma^{-1} = \alpha,\ \alpha t_1\alpha^{-1} = t_1,\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 3, 4,$
$\beta t_1\beta^{-1} = t_1,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,\ \gamma t_2\gamma^{-1} = t_3,\ \gamma t_3\gamma^{-1} = t_4,\ \gamma t_4\gamma^{-1} = t_2^{-1}\rangle$
$= \langle \alpha, \beta\rangle \rtimes \langle\gamma\rangle = O^3_6 \rtimes \mathbb{Z}$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma^2 \mid \alpha^2 = t_2,\ \beta^2 = t_4^{-1},\ (\gamma^2)^3 = t_1,\ \beta\alpha\beta^{-1} = t_3^{-1}t_4^{-1}\alpha^{-1},$
$(\gamma^2)\alpha(\gamma^2)^{-1} = \beta^{-1},\ \gamma^2\beta(\gamma^2)^{-1} = \alpha^{-1}\beta,\ \alpha t_1\alpha^{-1} = t_1,\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 3, 4,$
$\beta t_1\beta^{-1} = t_1,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 2, 3,\ \gamma^2 t_2(\gamma^2)^{-1} = t_4,\ \gamma t_3\gamma^{-1} = t_2^{-1},\ \gamma t_4\gamma^{-1} = t_3^{-1}\rangle$
$= \langle \alpha^{-1}\beta, \beta^{-1}\rangle \rtimes \langle\gamma^2\rangle = O^3_6 \rtimes \mathbb{Z} = O^4_{26}$

N^4_{44} : 04/03/01/006

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Holonomy: $\langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 1]/2,\ \beta = B : [0, 1, -1, -1]/2$

First Homology: $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^2 = t_4,\ \beta^2 = t_2,\ \beta\alpha\beta^{-1} = t_2t_3^{-1}\alpha^{-1},\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 1, 2, 3,$
$\beta t_1\beta^{-1} = t_1,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 3, 4\rangle$
$= \langle t_1, t_2, \alpha\rangle \star_{\langle t_1, t_2, t_4\rangle} \langle t_1, t_4, \beta\rangle = O^3_2 \star_{\mathbb{Z}^3} N^3_1$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta \mid \beta^2 = t_2,\ \beta t_1\beta^{-1} = t_1,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 3, 4\rangle$
$= \langle t_3, t_4, \beta\rangle \times \langle t_1\rangle = O^3_2 \times \mathbb{Z} = O^4_2$

N^4_{45} : 06/02/01/050

Orthogonal matrices: $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [-1, -1, 0, 0]/2,\ \beta = B : [1, 0, 0, 1]/2,\ \gamma = C : [-1, 0, -1, 0]/2$

First Homology: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_1^{-1},\ \beta^2 = t_4,\ \gamma^2 = t_1^{-1},\ \alpha\beta\alpha^{-1} = t_1^{-1}t_2^{-1}\beta^{-1},$
$\alpha\gamma\alpha^{-1} = t_2^{-1}t_3\gamma,\ \beta\gamma\beta^{-1} = \gamma^{-1},\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 2, 3, 4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,$
$\beta t_3\beta^{-1} = t_3,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 2, 3,\ \gamma t_4\gamma^{-1} = t_4\rangle$
$= \langle \alpha, \beta\rangle \star_{\langle \alpha^2, t_2^{-1}, \beta\rangle} \langle \gamma, t_2^{-1}, \beta\rangle = O^3_6 \star_{O^3_2} N^3_3$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid \beta^2 = t_4,\ \gamma^2 = t_1^{-1},\ \beta\gamma\beta^{-1} = \gamma^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,$
$\beta t_3\beta^{-1} = t_3,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 2, 3,\ \gamma t_4\gamma^{-1} = t_4\rangle$
$= \langle t_2, t_3, \gamma\rangle \rtimes \langle \beta\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_9$

N^4_{46} : 06/02/01/027

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Holonomy: $\langle A, B, C\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 0]/2,\ \beta = B : [1, 0, 0, 1]/2,\ \gamma = C : [0, 1, -1, 0]/2$

First Homology: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$\Gamma = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma \mid \alpha^2 = t_2,\ \beta^2 = t_4,\ \gamma^2 = t_2,\ \beta\alpha\beta^{-1} = t_1t_4\alpha^{-1},\ \gamma\alpha\gamma^{-1} = t_3^{-1}\alpha,$
$\beta\gamma\beta^{-1} = t_1\gamma^{-1},\ \alpha t_i\alpha^{-1} = t_i^{-1}, i = 1, 3, 4,\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,\ \beta t_3\beta^{-1} = t_3,$
$\gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 3,\ \gamma t_4\gamma^{-1} = t_4\rangle$
$= \langle \alpha, \beta\rangle \star_{\langle \alpha^2, t_1^{-1}, \beta\rangle} \langle \gamma, t_1^{-1}, \beta\rangle = O^3_6 \star_{O^3_2} N^3_4$

$\Gamma_0 = \langle t_1, t_2, t_3, t_4, \beta, \gamma \mid \beta^2 = t_4,\ \gamma^2 = t_2,\ \beta\gamma\beta^{-1} = t_1\gamma^{-1},\ \beta t_i\beta^{-1} = t_i^{-1}, i = 1, 2,$
$\beta t_3\beta^{-1} = t_3,\ \gamma t_i\gamma^{-1} = t_i^{-1}, i = 1, 3,\ \gamma t_4\gamma^{-1} = t_4\rangle$
$= \langle t_1, t_3, \gamma\rangle \rtimes \langle \beta\rangle = O^3_2 \rtimes \mathbb{Z} = O^4_{10}$

N^4_{47} : 12/04/03/011

Orthogonal matrices: $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

Holonomy: $\langle A, B \mid A^4 = B^2 = 1, BAB^{-1} = A^{-1} \rangle = D_4$

Lattice Basis: $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$

Generators: $\alpha = A : [0, 1, 0, 2]/4$, $\beta = B : [1, 0, 0, 0]/2$

First Homology: $\mathbb{Z}_4 \oplus \mathbb{Z}_4$

$$\begin{aligned}\Gamma = \langle & t_1, t_2, t_3, t_4, \alpha, \beta \mid \alpha^4 = t_2,\ \beta^2 = t_1,\ \beta\alpha\beta^{-1} = t_1 t_3^{-1} \alpha^{-1},\ \alpha t_1 \alpha^{-1} = t_1^{-1}, \\ & \alpha t_3 \alpha^{-1} = t_4^{-1},\ \alpha t_4 \alpha^{-1} = t_3,\ \beta t_2 \beta^{-1} = t_2^{-1},\ \beta t_3 \beta^{-1} = t_4^{-1},\ \beta t_4 \beta^{-1} = t_3^{-1} \rangle \\ = & \langle t_1, t_4^{-1}, \alpha^{-1}\beta \rangle \star_{\langle t_1, t_3, t_4^{-1} \rangle} \langle t_3, t_4^{-1}, \beta \rangle = O^3_2 \star_{\mathbb{Z}^3} N^3_2\end{aligned}$$

$$\begin{aligned}\Gamma_0 = \langle & t_1, t_2, t_3, t_4, \alpha^2, \beta \mid (\alpha^2)^4 = t_2,\ \beta^2 = t_1,\ \beta\alpha^2\beta^{-1} = t_3^{-1} t_4 (\alpha^2)^{-1},\ \alpha^2 t_1 (\alpha^2)^{-1} = t_1, \\ & \alpha^2 t_i (\alpha^2)^{-1} = t_i^{-1}, i = 3, 4,\ \beta t_2 \beta^{-1} = t_2^{-1},\ \beta t_3 \beta^{-1} = t_4^{-1},\ \beta t_4 \beta^{-1} = t_3^{-1} \rangle \\ = & \langle t_3, t_4^{-1}, t_3^{-1}\alpha^2 \rangle \rtimes \langle \beta \rangle = O^3_2 \rtimes \mathbb{Z} = O^4_{12}\end{aligned}$$

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