POISSON BOUNDARIES OF FINITE VON NEUMANN ALGEBRAS

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To my parents,

for never ending support

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TABLE OF CONTENTS

Chapter 1

INTRODUCTION

The study of group actions (on spaces) and group representations occupies a central position in modern mathematics. This relates naturally to the study of von Neumann algebras, as we shall describe. A von Neumann algebra is a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ (bounded linear operators on a Hilbert space $\mathscr{H}$), containing I (the identity operator), and is closed under the strong operator topology, abbreviated SOT.(SOT can be thought as "topology of pointwise convergence": a net of bounded operators T_α converges in SOT to a bounded operator T if and only if $||(T_\alpha - T)\psi|| \to 0$ for all $\psi \in \mathscr{H}$). A von Neumann algebra M is called a *factor* if the only operators commuting with all of M are scalar multiples of identity. (i.e. $\mathscr{Z}(M) = M \cap M' = \mathbb{C}$. Here M' denotes the *commutant* of M inside $\mathscr{B}(\mathscr{H})$). We say M is a type II_1 factor if M is infinite dimensional, and has a faithful tracial state, denoted by τ. In general, if a von Neumann algebra admits a faithful tracial state τ, we say that M is a *finite* von Neumann algebra.

Some examples of von Neumann algebras are $L^\infty[0,1]$, $\mathbb{M}_n(\mathbb{C})$, or more generally $\mathscr{B}(\mathscr{H})$. Note that $L^\infty[0,1]$ is an abelian algebra, while the other two are factors. As it turns out, every abelian von Neumann algebra (on a separable Hilbert space) is isomorphic to $L^\infty(X,\mu)$ for some measure space (X,μ). Hence the study of von Neumann algebras can be thought of as a noncommutative measure theory. Other exciting examples of von Neumann algebras are constructed via group representations and actions, as described below.

The first examples of von Neumann algebras different from the above examples was the so called group von Neumann algebras of discrete groups. Given a countable, discrete group G, we define $vN(G)$ to be the SOT closure of the group algebra, $\mathbb{C}(G)$ inside $\mathscr{B}(\ell^2 G)$. This von Neumann algebra is a factor if and only if the group is i.c.c. (i.e. every non identity element has infinite conjugacy class. The i.c.c. conditions tells that the group is "highly noncommutative").

Also, whenever a group G acts a probability measure space (X,μ), we can construct a von

Neumann algebra, denoted by $L^\infty(X,\mu) \rtimes G$. If the G action on (X,μ) is measure preserving, free and ergodic, then we obtain a II_1 factor. Also, the algebra $L^\infty(X,\mu) \rtimes G$ provides a natural setting for studying the underlying group action. In fact, the dynamics of group actions has an exciting relation with studying this algebra. Thus many questions in ergodic theory and group representations can be studied naturally in the von Neumann algebra context. This mutual symbiosis has been a very active area of research for more than 50 years.

To study analytic properties of a group, like amenability and property (T), it's desirable to look for "nice" spaces on which the group acts. Poisson boundary of a group, $(\mathscr{B},\beta)$, provides a natural example of such a probability measure space, on which the group acts. In general, the actions is not measure preserving: in fact, in the case of non-amenable groups, the action on its Poisson boundary can never be measure preserving. (Nevertheless, $L^\infty(\mathscr{B},\beta) \rtimes G$ is a factor as soon as $vN(G)$ is a factor, i.e. G is i.c.c). As we shall see below, the study of Poisson boundaries of groups has been an exciting research area, and has led to a variety of deep results in the rigidity theory of lattices in Lie groups, including the celebrated Normal Subgroup Theorem of Margulis.

1.0.1 Poisson boundaries of groups

The notion for Poisson boundary of groups was introduced by Fustenberg to study Harmonic analysis over Lie groups. He was motivated by the Poisson transform in the classical complex analysis. In the classical case, any real valued bounded harmonic function h on the disc can be written in terms of a bounded function f on the boundary of $\mathbb{D}$. Notice that the group $G = PSL_2(\mathbb{R})$ acts on $\mathbb{D}$. Then, every bounded harmonic function h on $\mathbb{D}$ determines a bounded harmonic function (see definition below) $\hat{h}$ on G , by $\hat{h}(g) = h(g(0))$. In fact, every bounded harmonic function on G arises in this fashion. So, there's a one-one correspondence between bounded harmonic function on G and bounded functions on the unit circle. This was the main motivation for Furstenberg to study the notion of "boundary" of a Lie group, such that bounded harmonic functions on the group would be in one-one correspondence with bounded functions on this "boundary". As it turned out, the notion could be generalized to arbitrary locally compact groups, and the study of the Poisson boundary has been a beautiful area of research henceforth. In the treatment below, we shall focus on discrete groups and follow the perspective of Kaimanovich and Vershik [KV83]

Let G be a discrete group, and μ a symmetric probability measure of G. We say that $f \in \ell^\infty(G)$ is μ-harmonic if $P_\mu f = f$, where $P_\mu f(g) = \sum_{x\in G} f(gx)\mu(x)$.

Even though the space of μ-harmonic functions doesn't form an algebra, in general, it's possible to define a product on the space of harmonic functions to make it into a commutative von Neumann algebra. The space of harmonic functions with this new product then becomes isomorphic to $L^\infty(\mathscr{B},\beta)$, for some probability measure space (B,β)- which we call the Poisson boundary of the group G. There is a natural G action on this space, which preserves the β null sets, i.e. is quasi-invariant. The measure β is μ-stationary, (i.e. $\mu * \beta = \beta$) and the G action is amenable, (i.e. $L^\infty(\mathscr{B},\beta) \rtimes G$ is an amenable von Neumann algebra).

For the case of the free group on 2 generators, $\mathbb{F}_2 = \langle a,b\rangle$, with the uniform probability measure μ on a, a^{-1}, b, b^{-1}, the Poisson boundary can be identified with the space of one sided infinite reduced words, with a natural Borel probability measure. This example hints that the Poisson boundary can be thought of as an "exit boundary" for random walks on the Cayley graph of G. Indeed, such is the case, (see [KV83]). Therefore the study of Poisson boundaries has natural interactions with the study of random walks on groups as well.

One of the most striking application of Poisson boundaries was by Margulis' in his proof of the Normal subgroup theorem ([Mar78] [Mar79]). The Normal Subgroup Theorem states that: If H is a lattice in a center free higher rank semisimple Lie group G, then every nontrivial normal subgroup N of H has finite index. His strategy of showing the finiteness of the quotient H/N was to show that it's amenable, and has property (T). The amenability half of the result relied on looking at the action of G on its Poisson boundary (which is an amenable action). As we shall see, this was one of the key motivations to study Poisson boundary in the noncommutative setting.

1.0.2 Poisson boundary of finite von Neumann algebras

Based on the Mostow rigidity theorem [Mos73], Margulis' superrigidity theorem, and Zimmer's cocycle rigidity theorem [Zim80], Alain Connes' suggested in the early 80's that to study superrigidity type phenomenon in operator algebra setting, one should first study Poisson boundaries in the setting of operator algebras (see discussion in page 86 of [Jon00]).

In late 90's Izumi realized that the notion of Poisson boundaries for groups can be extended to the von Neumann algebra setting by studying fixed points of u.c.p. maps (defined below). Izumi introduced this notion in [Izu99], and extended the theory further in [Izu02] and [Izu12]. Later Peterson and Creutz studied the noncommutative boundary in depth in [CP13] and used it to prove the amenability half of the noncommutative analog of Margulis' Normal subgroup theorem.

Motivated by the above, in [DP17] we focus on the study of Poisson Boundary of finite von Neumann algebras arising from regular, symmetric, hyperstates. The contents of this Thesis is based on the joint work with Dr. Peterson in [DP17]

Chapter 2

PRELIMINARIES

In this section we record a few preliminaries needed for the rest of the paper.

2.0.1 von Neumann algebras, u.c.p. maps and amenability

A von Neumann algebra M is a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$, containing 1, and closed under the strong operator topology. Recall that $\mathscr{B}(\mathscr{H})$ carries three natural topologies, namely, the norm topology, the strong operator topology and the weak operator topology. A deep theorem of von Neumann states that if M is a von Neumann algebra, then M equals its double commutant, i.e. $M = M''$. Throughout this paper we shall assume that M acts on a separable Hilbert space.

A linear functional φ on M is said to be *normal* if there exists $\xi, \eta \in \mathscr{H} \otimes \ell^2(\mathbb{N})$ such that $\Phi(T) = \langle T \otimes I\xi, \eta \rangle$. A linear functional φ is said to be a state, if it is positive, and $\Phi(1) = 1$. A state τ on M is said to be tracial is $\tau(ab) = \tau(ba)$ for all $a, b \in M$.

We say that M is a finite von Neumann algebra, or a tracial von Neumann algebra, if there exists a faithful, normal tracial state τ on M. Given such a state we can perform the GNS construction, and get a Hilbert space, denoted by $L^2(M, \tau)$ or sometimes $L^2(M)$ if τ is fixed. A finite von Neumann algebra M is a factor if and only if it admits a unique tracial state τ.

An operator system $\mathscr{B}$ is a $*$-closed linear subspace of a unital C^* algebra $\mathscr{A}$, containing 1. We say a map $\Phi : \mathscr{B} \to \mathscr{A}$ is positive, if it takes positive elements of $\mathscr{B}$ to positive elements os $\mathscr{A}$. We say Φ is unital, if $\Phi(1) = 1$. Such a map Φ induces linear maps Φ_n from $\mathscr{B} \otimes \mathbb{M}_n(\mathbb{C}) \to \mathscr{B} \otimes \mathbb{M}_n(\mathbb{C})$, by acting on each matrix coordinate. We say that Φ is completely positive, if Φ_n is positive for all $n \in \mathbb{N}$. We say that Φ is normal, if $\varphi \circ \Phi$ is normal, for all normal linear functionals φ.

One example of normal u.c.p. map is given by unital homomorphisms between C^* algebras. In fact, Stinespring Dilation theorem says that, up to dilation, those are the only examples.

A very imporant class of examples of u.c.p. maps are *Conditional Expectations*. Let $\mathscr{B}$ be an operator system in $\mathscr{B}(\mathscr{H})$. A map $\mathscr{E}:\mathscr{B}(\mathscr{H})\to\mathscr{B}$ is said to be a conditional expectation, if it's u.c.p., and $\mathscr{E}\circ\mathscr{E}=\mathscr{E}$. In that case, $\mathscr{B}$ is said to be an injective operator system. A von Neumann algebra M is said to be amenable, if it is injective as an operator system. A deep theorem of Connes' asserts that M is injective, if and only if, M is *hyperfinite*.

Chapter 3

BOUNDARIES OF FINITE VON NEUMANN ALGEBRAS

In this chapter we record the main results and proofs of the thesis. The results here are from the joint work in [DP17], with Dr. Peterson.

3.1 Hyperstates and bimodular u.c.p. maps

Fix a tracial von Neumann algebra (M,τ), and suppose we have an embedding $M \subset \mathscr{A}$ where $\mathscr{A}$ is a C^*-algebra. Given a state $\varphi \in \mathscr{A}^*$ we will say that φ is a **τ-hyperstate** (or just a hyperstate if τ is fixed) if it extends τ. We denote by $\mathscr{S}_\tau(\mathscr{A})$ the convex set of all hyperstates on $\mathscr{A}$. To each a hyperstate φ we obtain a natural inclusion $L^2(M,\tau) \subset L^2(\mathscr{A},\varphi)$ induced from the map $x1_\tau \mapsto x1_\varphi$ for $x \in M$. We let $e_M \in \mathscr{B}(L^2(\mathscr{A},\varphi))$ denote the orthogonal projection onto $L^2(M,\tau)$. We may then consider the unital completely positive (u.c.p.) map $\mathscr{P}_\varphi : \mathscr{A} \to \mathscr{B}(L^2(M,\tau))$, defined by

$$\mathscr{P}_\varphi(T) = e_M T e_M, \qquad T \in \mathscr{A}. \tag{3.1}$$

Note that if $x \in M \subset \mathscr{A}$ then we have $\mathscr{P}_\varphi(x) = x$. We shall refer to the map $\mathscr{P}_\varphi$ as the **Poisson transform** (with respect to φ) of the inclusion $M \subset \mathscr{A}$.

The following proposition is well known, and implicit in [Con76b]. We include a proof for the benefit of the reader.

Proposition 3.1.1. *The correspondence $\varphi \mapsto \mathscr{P}_\varphi$ defined by (3.1) gives a bijective correspondence between hyperstates on M, and u.c.p., M-bimodular maps from $\mathscr{A}$ to $\mathscr{B}(L^2(M,\tau))$. Moreover, if $\mathscr{A}$ is a von Neumann algebra, then $\mathscr{P}_\varphi$ is normal if and only if φ is normal.*

Also, this corresondence is a homeomorphism where the space of hyperstates is endowed with the weak*-topology, and the space of u.c.p., M-bimodular maps with the topology of pointwise weak operator topology convergence.

Proof. First note that if φ is a hyperstate on $\mathscr{A}$, then for all $T \in \mathscr{A}$ we have

$$\varphi(T) = \langle T, \hat{1} \rangle_\varphi = \langle \mathscr{P}_\varphi(T)\hat{1}, \hat{1} \rangle_\tau.$$

From this it follows that the correspondence $\varphi \mapsto \mathscr{P}_\varphi$ is one-to-one. To see that it is onto, suppose that $\mathscr{P} : \mathscr{A} \to \mathscr{B}(L^2(M,\tau))$ is u.c.p. and M-bimodular. We define a state φ on $\mathscr{A}$ by $\varphi(T) = \langle \mathscr{P}(T)\hat{1}, \hat{1} \rangle_\tau$. For all $y \in M$ we then have $\varphi(y) = \langle \mathscr{P}(y)\hat{1}, \hat{1} \rangle_\tau = \tau(y)$, hence φ is a hyperstate. Moreover, if $y, z \in M$, and $T \in \mathscr{A}$ then we have

$$\begin{aligned} \langle \mathscr{P}_\varphi(T)y, z \rangle_\tau &= \langle \mathscr{P}_\varphi(z^*Ty)\hat{1}, \hat{1} \rangle \\ &= \varphi(z^*Ty) = \langle \mathscr{P}(T)y, z \rangle_\tau, \end{aligned} \tag{3.2}$$

hence, $\mathscr{P}_\varphi = \mathscr{P}$.

It is also easy to check that $\mathscr{P}_\varphi$ is normal if and only if φ is.

To see that this correspondence is a homeomorphism when given the topologies above, suppose that φ is a hyperstate, and φ_α is a net of hyperstates. From (3.2) and the fact that u.c.p. maps are contractions in norm we see that $\mathscr{P}_{\varphi_\alpha}$ converges in the pointwise weak operator topology to $\mathscr{P}_\varphi$ if φ_α converges weak* to φ. Conversely, setting $y = z = 1$ in (3.2) shows that if $\mathscr{P}_{\varphi_\alpha}$ converges in the pointwise weak operator topology to $\mathscr{P}_\varphi$ then φ_α converges weak* to φ. $\square$

Considering the case $\mathscr{A} = \mathscr{B}(L^2(M,\tau))$ we see that to each hyperstate on $\mathscr{B}(L^2(M,\tau))$ we obtain a u.c.p. M-bimodular map on $\mathscr{B}(L^2(M,\tau))$. In particular, composing such maps gives a convolution operation on the space of hyperstates. More generally, if $\mathscr{A}$ is a C^*-algebra, with $M \subset \mathscr{A}$, then for hyperstates $\psi \in \mathscr{A}^*$, and $\varphi \in \mathscr{B}(L^2(M,\tau))^*$ we define the **convolution** $\varphi * \psi$ to be the unique hyperstate on $\mathscr{A}$ such that

$$\mathscr{P}_{\varphi * \psi} = \mathscr{P}_\varphi \circ \mathscr{P}_\psi. \tag{3.3}$$

We say that ψ is **φ-stationary** if we have $\varphi * \psi = \psi$, or equivalently, if $\mathscr{P}_\psi$ maps into the space of $\mathscr{P}_\varphi$**-harmonic operators**

$$\mathrm{Har}(\mathscr{B}(L^2(M,\tau)),\mathscr{P}_\varphi) = \{T \in \mathscr{B}(L^2(M,\tau)) \mid \mathscr{P}_\varphi(T) = T\}.$$

Lemma 3.1.2. *For a fixed $\psi \in \mathscr{S}_\tau(\mathscr{A})$ the mapping*

$$\mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)) \ni \varphi \mapsto \varphi * \psi \in \mathscr{S}_\tau(\mathscr{A})$$

is continuous in the weak*-topology.

Moreover, if $\varphi \in \mathscr{B}(L^2(N,\tau))_*$ is a fixed normal hyperstate, then the mapping

$$\mathscr{S}_\tau(\mathscr{A}) \ni \psi \mapsto \varphi * \psi \in \mathscr{S}_\tau(\mathscr{A})$$

is also weak*-continuous.

Proof. By Proposition 3.1.1 the correspondence $\varphi \mapsto \mathscr{P}_\varphi$ is a homeomorphism from the weak*-topology to the topology of pointwise weak operator topology convergence, this lemma then follows easily from (3.3). □

3.1.1 Poisson boundaries of II_1 factors

If $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau))$ is a hyperstate then we define the **Poisson boundary** B_φ of M with respect to φ to be the noncommutative Poisson boundary of the u.c.p. map $\mathscr{P}_\varphi$ as defined by Izumi [Izu02] (see the next section for an explicit construction).

The Poisson boundary contains M as a subalgebra and the inclusion $(M \subset B_\varphi)$ is determined up to isomorphism by the property that there exists a completely positive isometric isomorphism $\mathscr{P} : B_\varphi \to \mathrm{Har}(\mathscr{B}(L^2(M,\tau)),\mathscr{P}_\varphi)$ which restricts to the identity map on M. We will always assume that $\mathscr{P}$ is fixed and we also call $\mathscr{P}$ the **Poisson transform**.

Given any initial hyperstate $\varphi_0 \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$ we may consider the hyperstate given by $\varphi_0 \circ \mathscr{P}$ on B_φ. Of particular interest is the state η on B_φ arising from the initial state $x \mapsto \langle x\hat{1},\hat{1}\rangle$, which we call the **stationary state** on B_φ. In this case it is easy to see that we have $\mathscr{P}_\eta = \mathscr{P}$, and hence $\varphi * \eta = \eta$.

Proposition 3.1.3. *Let (M, τ) be a tracial von Neumann algebra and take $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$, then the Poisson boundary B_φ is injective.*

Proof. If we take any cluster point E of in the topology of pointwise weak operator topology convergence, then $E : \mathscr{B}(L^2(M,\tau)) \to \mathrm{Har}(\mathscr{B}(L^2(M,\tau)), \mathscr{P}_\varphi)$ gives a conditional expectation. As B_φ is isomorphic to $\mathrm{Har}(\mathscr{B}(L^2(M,\tau)), \mathscr{P}_\varphi)$ as an operator system it then follows that B_φ is injective. □

The trivial case is when $\varphi_e(x) = \langle x1, 1\rangle_\tau$ in which case we have that $\mathscr{P}_{\varphi_e} = \mathrm{id}$, and the Poisson boundary is nothing but $\mathscr{B}(L^2(M,\tau))$. Note that φ_e gives an identity with respect to convolution. Also note that if $\varphi \in \mathscr{B}(L^2(M,\tau))^*$ is a hyperstate, then we have a description of the space of harmonic operators as:

$$\mathrm{Har}(\mathscr{B}(L^2(M,\tau)), \mathscr{P}_\varphi) = \{T \in \mathscr{B}(L^2(M,\tau)) \mid \varphi(aTb) = \varphi_e(aTb) \text{ for all } a, b \in M\}.$$

Since $\mathscr{P}_\varphi$ is M-bimodular it follows that $\mathscr{P}_\varphi(M') \subset M'$. We say that φ is **regular** if the restriction of $\mathscr{P}_\varphi$ to M' preserves the canonical trace on M', and we say that φ is **generating** if M is the largest $*$-subalgebra of $\mathscr{B}(L^2(M,\tau))$ which is contained in $\mathrm{Har}(\mathscr{B}(L^2(M,\tau)), \mathscr{P}_\varphi)$. If φ is regular, then the **conjugate** of φ is given by $\varphi^*(T) = \varphi(JT^*J)$, which is again a hyperstate. We'll say that φ is **symmetric** if it is regular and we have $\varphi^* = \varphi$.

Regular, generating, symmetric hyperstates are easy to find. Suppose (M, τ) is a separable finite von Neumann algebra with a faithful normal trace τ. We consider the unit ball $(M)_1$ of M as a Polish space endowed with the weak operator topology, and suppose we have a σ-finite measure μ on $(M)_1$ such that $\int x^* x \, d\mu(x) = 1$. We obtain a normal hyperstate as

$$\varphi(T) = \int \langle T\widehat{x^*}, \widehat{x^*}\rangle \, d\mu(x) \tag{3.4}$$

and using (3.2) we may explicitly compute the Poisson transform $\mathscr{P}_\varphi$ on $\mathscr{B}(L^2(M,\tau))$ as

$$\mathscr{P}_\varphi(T) = \int (Jx^*J)T(JxJ)\,d\mu(x).$$

Proposition 3.1.4. *Consider φ as given by (3.4), then*

1. *φ is generating if and only if the support of μ generates M as a weakly closed subalgebra of $\mathscr{B}(L^2(M,\tau))$ containing the identity.*

2. *φ is regular if and only if $\int xx^*\,d\mu(x) = 1$.*

3. *If φ is regular then $\mathscr{P}_{\varphi^*}(T) = \int (JxJ)T(Jx^*J)\,d\mu(x)$ and φ is symmetric if $J_*\mu = \mu$, where J is the adjoint operation.*

Proof. If the support of μ generates a weakly closed subalgebra M_0 containing the identity such that $M_0 \neq M$, then we have $[Jx^*J, e_{M_0}] = 0$ for each x in the support of μ. Hence, $\mathscr{P}_\varphi(T) = \int (JxJ)T(Jx^*J)\,d\mu(x) = T$, for each T in the $*$-algebra generated by M and e_{M_0}. Therefore, φ is not generating. On the other hand, if $T \in \mathrm{Har}(\mathscr{B}(L^2(M,\tau)), \mathscr{P}_\varphi)$ is such that we also have $T^*T \in \mathrm{Har}(\mathscr{B}(L^2(M,\tau)), \mathscr{P}_\varphi)$ then for each $a \in M$ we have

$$\begin{aligned}&\int \|((JxJ)T - T(JxJ))\hat{a}\|_2^2\,d\mu(x) \\ &\qquad = \langle (T^*\mathscr{P}_\varphi(1)T - \mathscr{P}_\varphi(T^*)T - T^*\mathscr{P}_\varphi(T) + \mathscr{P}_\varphi(T^*T))\hat{a}, \hat{a}\rangle = 0\end{aligned}$$

Hence, $[JxJ, T] = 0$ for μ-almost every $x \in (M)_1$. Therefore, if the support of μ generates M as a weakly dense subalgebra containing the identity then we then have that $T \in JMJ' = M$, showing that φ is generating.

If $y \in M$ then we have $\mathscr{P}_\varphi(JyJ) = \int Jx^*yxJ\,d\mu(x)$. Hence, we see that φ is regular if and only if for all $y \in M$ we have $\tau(y) = \int \tau(x^*yx)\,d\mu(x) = \int \tau(xx^*y)\,d\mu(x)$, which is if and only if $\int xx^*\,d\mu(x) = 1$.

If φ is regular then

$$\begin{aligned}\varphi^*(T) = \varphi(JT^*J) &= \int \langle JT^*J\widehat{x^*}, \widehat{x^*}\rangle \, d\mu(x) \\ &= \int \langle \hat{x}, T^*\hat{x}\rangle \, d\mu(x) = \int \langle T\widehat{x^*}, \widehat{x^*}\rangle \, dJ_*\mu(x).\end{aligned}$$

Therefore, if $J_*\mu = \mu$ then φ is symmetric. $\square$

The following lemma is well known, see, e.g., [FNW94], or Lemma 3.4 in [BJKW00]. We include a proof for the convenience of the reader.

Lemma 3.1.5. *Suppose A is a unital C^*-algebra with a faithful state φ. If $\mathscr{P} : A \to A$ is a u.c.p. map such that $\varphi \circ \mathscr{P} = \varphi$, then* $\mathrm{Har}(A, \mathscr{P}) \subset A$ *is a C^*-subalgebra.*

Proof. $\mathrm{Har}(A, \mathscr{P})$ is clearly a self-adjoint closed subspace, thus we must show that $\mathrm{Har}(A, \mathscr{P})$ is an algebra. By the polarization identity it is enough to show that $x^*x \in \mathrm{Har}(A, \mathscr{P})$ whenever $x \in \mathrm{Har}(A, \mathscr{P})$. Suppose $x \in \mathrm{Har}(A, \mathscr{P})$. By Kadison's indequality we have $\mathscr{P}(x^*x) - x^*x = \mathscr{P}(x^*x) - \mathscr{P}(x^*)\mathscr{P}(x) \geq 0$. Also, $\varphi(\mathscr{P}(x^*x) - x^*x) = 0$ so that by faithfulness of φ we have $\mathscr{P}(x^*x) = x^*x$. $\square$

Proposition 3.1.6. *Let M be a finite von Neumann algebra with a normal faithful trace τ. Let $\varphi \in \mathscr{B}(L^2(M, \tau))^*$ be a regular generating hyperstate, and let B_φ be the corresponding Poisson boundary, then $M' \cap B_\varphi = \mathscr{Z}(M)$. In particular, when M is a factor then so is B_φ.*

Proof. Let $\mathscr{P} : B_\varphi \to \mathrm{Har}(\mathscr{B}(L^2(M, \tau)), \mathscr{P}_\varphi)$ denote the Poisson transform. If $x \in M' \cap B_\varphi$, then $\mathscr{P}(x) \in M' \cap \mathscr{B}(L^2(M, \tau)) = JMJ$. Since φ is regular, $\mathscr{P}_\varphi$ preserves the trace when restricted to JMJ. Thus, $\mathrm{Har}(\mathscr{P}_\varphi, JMJ)$ is a von Neumann subalgebra of JMJ by Lemma 3.1.5, which must be $\mathscr{Z}(M)$ since φ is generating. Therefore, $\mathscr{P}(x) \in \mathrm{Har}(\mathscr{P}_\varphi, JMJ) = \mathscr{Z}(M)$,and hence $x \in \mathscr{Z}(M)$ since $\mathscr{P}$ is injective. $\square$

If φ is a normal hyperstate in $\mathscr{S}_\tau(\mathscr{B}(L^2(M, \tau)))$, then $\mathscr{P}_\varphi : \mathscr{B}(L^2(M, \tau)) \to \mathscr{B}(L^2(M, \tau))$ is a normal map, and hence the dual map $\mathscr{P}_\varphi^*$ preserves the predual of $\mathscr{B}(L^2(M, \tau))$ which we identify with the space of trace class operators.

We let $A_\varphi \in \mathscr{B}(L^2(M,\tau))$ denote the density operator associated with φ, i.e., A_φ is the unique trace class operator so that $\varphi(T) = \mathrm{Tr}(A_\varphi T)$ for all $T \in \mathscr{B}(L^2(M,\tau))$. Since φ is positive we have that A_φ is a positive operator. If $P_{\hat{1}}$ denotes the rank one orthogonal projection onto $\mathbb{C}\hat{1}$, then we have $\varphi(T) = \langle \mathscr{P}_\varphi(T)\hat{1}, \hat{1}\rangle = \mathrm{Tr}(\mathscr{P}_\varphi(T)P_{\hat{1}})$, and hence we see that $A_\varphi = \mathscr{P}_\varphi^*(P_{\hat{1}})$. In particular we have that $A_{\varphi^{*n}} = (\mathscr{P}_\varphi^n)^*(P_{\hat{1}})$ for $n \geq 1$.

Proposition 3.1.7. *Let $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$ be a normal hyperstate, then there exists a τ-orthogonal family $\{z_n\}_n$ which gives a partition of the identity as $1 = \sum_n z_n^* z_n$ so that*

$$\mathscr{P}_\varphi(T) = \sum_n (Jz_n^*J)T(Jz_nJ)$$

for all $T \in \mathscr{B}(L^2(M,\tau))$.

Moreover, if $\{\tilde{z}_m\}_m$ is a τ-orthogonal family which gives a partition of the identity as $1 = \sum_n \tilde{z}_n^*\tilde{z}_n$, then the map $\sum_n (J\tilde{z}_n^*J)T(J\tilde{z}_nJ)$ agrees with $\mathscr{P}_\varphi$ if and only if for each $t > 0$ we have

$$\mathrm{sp}\{z_n \mid \|z_n\|_2 = t\} = \mathrm{sp}\{\tilde{z}_n \mid \|\tilde{z}_n\|_2 = t\}.$$

Proof. Since A_φ is a positive trace class operator we may write $A_\varphi = \sum_n a_n P_{y_n}$ where $a_1, a_2, \ldots$ are positive and $\{y_n\}_n$ is an orthonormal family with P_{y_n} denoting the rank one projection onto $\mathbb{C}y_n$. For $T \in \mathscr{B}(L^2(M,\tau))$ we then have

$$\mathrm{Tr}(TA_\varphi) = \sum_n a_n \langle Ty_n, y_n\rangle.$$

Taking $T = x^*x \in M$ we have $a_n\|xy_n\|_2^2 \leq \mathrm{Tr}(x^*xA_\varphi) = \|x\|_2^2$, so that $y_n \in M \subset L^2(M,\tau)$ for each n. Hence, for $T \in \mathscr{B}(L^2(M,\tau))$ we have

$$\begin{aligned}\mathrm{Tr}(\mathscr{P}_\varphi(T)P_{\hat{1}}) = \mathrm{Tr}(TA_\varphi) &= \left\langle \sum_n a_n (Jy_nJ)T(Jy_n^*J)\hat{1}, \hat{1}\right\rangle \\ &= \mathrm{Tr}\left(\left(\sum_n a_n(Jy_nJ)T(Jy_n^*J)\right)P_{\hat{1}}\right).\end{aligned}$$

Since $\mathscr{P}_\varphi$ is M-bimodular and since $Jy_nJ \in M'$ it follows that for all $x, y \in M$ we have

$$\mathrm{Tr}(\mathscr{P}_\varphi(T)xP_{\hat{1}}y) = \mathrm{Tr}\left(\left(\sum_n a_n(Jy_nJ)T(Jy_n^*J)\right)xP_{\hat{1}}y\right).$$

In particular, setting $T = y = 1$ we have

$$\tau(x) = \sum_n a_n \tau(y_n^* y_n x),$$

which shows that $\sum_n a_n y_n^* y_n = 1$.

Since the span of operators of the form $xP_{\hat{1}}y$ is dense in the space of trace class operators it then follows that $\mathscr{P}_\varphi(T) = \sum_n a_n(Jy_nJ)T(Jy_n^*J)$ for all $T \in \mathscr{B}(L^2(M,\tau))$. Setting $z_n = \sqrt{a_n}y_n^*$ then finishes the existence part of the proposition.

Suppose now that $\{\tilde{z}_m\}_m$ is a τ-orthogonal family which gives a partition of the identity $1 = \sum_n \tilde{z}_n^* \tilde{z}_n$, and set $\tilde{\varphi}(T) = \mathrm{Tr}((\sum_n (J\tilde{z}_n^*J)T(J\tilde{z}_nJ))P_{\hat{1}})$. Then, the density matrix corresponding to $\tilde{\varphi}$ is $\sum_n \tilde{z}_n^* P_{\hat{1}} \tilde{z}_n$. Since $\{\tilde{z}_n\}_n$ forms a τ-orthogonal family it then follows easily that $\tilde{z}_n^*$ is an eigenvector for $A_{\tilde{\varphi}}$, and the corresponding eigenvalue is $\|z_n^*\|_2^2 = \|z_n\|_2^2$. Since $\{z_n\}_n$ above was constructed using any orthonormal basis of eigenvectors from A_φ the rest of the proposition then follows easily.

$\square$

We say that the form $\mathscr{P}_\varphi(T) = \sum_n (Jz_n^*J)T(Jz_nJ)$ (resp. $\varphi(T) = \sum_n \langle T\widehat{z_n^*}, \widehat{z_n^*}\rangle$) is a **standard form** for $\mathscr{P}_\varphi$ (resp. φ).

Remark: It follows from 3.1.4 that φ is generating if and only if the weakly closed subalgebra generated by $\{z_n\}$ is M

Proposition 3.1.8. *If φ is generating then the stationary state ζ on B_φ is faithful.*

Proof. By considering the Poisson transform, it suffices to show that φ is faithful on $\mathrm{Har}(\mathscr{P}_\varphi)$. Let $T \in \mathrm{Har}(\mathscr{P}_\varphi)$, with $T \geq 0$ and $\varphi(T) = 0$. Let $\varphi(S) = \sum_n \langle S\hat{z_n^*}, \hat{z_n^*}\rangle$ be the standard form of φ. Then $\varphi(T) = 0$ implies that $T\hat{z_n^*} = 0$ for all n. Since $T \in \mathrm{Har}(\mathscr{P}_\varphi)$, we have that $\mathscr{P}_\varphi^k(T) = \mathscr{P}_\varphi(T)(= T)$. So, we have that $\varphi^{*k}(T) = 0$ for all $k \geq 1$. As φ is generating, we get that $T\hat{m} = 0$ for all $m \in M$, and hence $T = 0$. This shows that φ is faithful on $\mathrm{Har}(\mathscr{P}_\varphi)$, and thus the stationary state ζ on B_φ is faithful. $\square$

3.1.2 Construction of the boundary

In this section, we give an explicit construction of the boundary, based on the Bhat's dilation theorem [Bha99]. The author first learned about this construction from Dr. Peterson.

Poisson boundaries of completely positive maps were first defined by Izumi in [Izu02] using the Choi-Effros product from [CE77]. Izumi further developed the theory in [Izu04], and in [Izu12] he credits Arveson with the description of Poisson boundaries as the fixed point algebra of Bhat's dilation, and this is the perspective we take here.

If $A \subset \mathscr{B}(\mathscr{H})$ is a unital C^*-algebra, and $\phi : A \to A$ a unital completely positive map, then a projection $p \in A$ is said to be **coinvariant**, if $\{\phi^n(p)\}$ defines an increasing sequence of projections which strongly converge to 1 in $\mathscr{B}(\mathscr{H})$, and such that for $y \in \mathscr{B}(\mathscr{H})$ we have $y \in A$ if and only if $\phi^n(p)y\phi^n(p) \in A$ for all $n \geq 0$. Note that for $n \geq 0$, $\phi^n(p)$ is in the multiplicative domain for ϕ, and is again coinvariant. We define $\phi_p : pAp \to pAp$ to be the map $\phi_p(x) = p\phi(x)p$, then ϕ_p is normal unital completely positive. Moreover, we have that $\phi_p^k(x) = p\phi^k(x)p$ for all $x \in pAp$, which can be seen by induction from

$$p\phi^k(x)p = p\phi^{k-1}(p)\phi^k(x)\phi^{k-1}(p)p = p\phi^{k-1}(\phi_p(x))p.$$

Theorem 3.1.9 (Prunaru [Pru09])**.** *Let $A \subset \mathscr{B}(\mathscr{H})$ be a unital C^*-algebra, $\phi : A \to A$ a unital completely positive map, and $p \in A$ a coinvariant projection. Then the map $\theta : \mathrm{Har}(A,\phi) \to \mathrm{Har}(pAp,\phi_p)$ given by $\theta(x) = pxp$ defines a completely positive isometric surjection, between $\mathrm{Har}(A,\phi)$ and $\mathrm{Har}(pAp,\phi_p)$.*

Moreover, if A is a von Neumann algebra and ϕ is normal then θ is also normal.

Proof. First note that θ is well-defined since if $x \in \mathrm{Har}(A,\phi)$ we have $\phi_p(pxp) = p\phi(p)x\phi(p)p = pxp$. Clearly θ is completely positive (and normal in the case when A is a von Neumann algebra and ϕ is normal).

To see that it is surjective, if $x \in \mathrm{Har}(pAp, \phi_p)$ then consider the sequence $\phi^n(x)$. For each $m, n \geq 0$, we have

$$\phi^m(p)\phi^{m+n}(x)\phi^m(p) = \phi^m(p\phi^n(x)p) = \phi^m(\phi_p^n(x)) = \phi^m(x).$$

It follows that $\{\phi^n(x)\}$ converges in the strong operator topology to an element $y \in \mathscr{B}(\mathscr{H})$ such that $\phi^m(p)y\phi^m(p) = \phi^m(x)$ for each $m \geq 0$, consequently we have $y \in A$.

In particular, for $m = 0$ we have $pyp = x$. To see that $y \in \mathrm{Har}(A, \phi)$ we use that for all $z \in A$ we have the strong operator topology limit

$$\lim_{n\to\infty} \phi(\phi^n(p)z\phi^n(p)) = \phi^{n+1}(p)\phi(z)\phi^{n+1}(p) = \phi(z),$$

and hence

$$\phi(y) = \lim_{m\to\infty} \phi(\phi^m(p)y\phi^m(p)) = \lim_{m\to\infty} \phi^{m+1}(x) = y.$$

Thus θ is surjective, and since $\phi^n(p)$ converges strongly to 1, and each $\phi^n(p)$ is in the multiplicative domain of ϕ, it follows that if $x \in \mathrm{Har}(A, \phi)$ then $\phi^n(pxp)$ converges strongly to x and hence

$$\|x\| = \lim_{n\to\infty} \|\phi^n(pxp)\| \leq \|pxp\| \leq \|x\|.$$

Thus, θ is also isometric. □

Corollary 3.1.10 (Izumi [Izu02]). *Let A be a unital C^*-algebra, and $\phi : A \to A$ a unital completely positive map. Then there exists a C^*-algebra B and a completely positive isometric surjection $\theta : B \to \mathrm{Har}(A, \phi)$.*

Moreover B and θ are unique in the sense that if $\tilde{B}$ is another C^*-algebra, and $\tilde{\theta} : \tilde{B} \to \mathrm{Har}(A, \phi)$ is a completely positive isometric surjection, then $\theta^{-1} \circ \tilde{\theta}$ is an isomorphism.

Also, if A is a von Neumann algebra and ϕ is normal, then B is also a von Neumann algebra and θ is normal.

Proof. Note that we may assume $A \subset \mathscr{B}(\mathscr{H})$. Existence then follows by applying the previous theorem to Bhat's dilation. Uniqueness follows from Theorem **??**. □

We refer to the C^*-algebra B from the previous corollary as the **Poisson boundary of** ϕ, and we refer to the map θ as the **Poisson transform**.

Corollary 3.1.11 (Choi-Effros [CE77])**.** *Let A be a unital C^*-algebra and $F \subset A$ an operator system. If $E : A \to F$ is a completely positive map such that $E_{|F} = \mathrm{id}$, then F has a unique C^*-algebraic structure which is given by $x \cdot y = E(xy)$. Moreover, if A is a von Neumann algebra and F is weakly closed then this gives a von Neumann algebraic structure on F.*

Proof. When A is a C^*-algebra this follows from Corollary 3.1.10 since $\mathrm{Har}(A,E) = F$. Also note that since $E^n = E$ it follows from the proof of Theorem 3.1.9 that the product structure coming from the Poisson boundary is given by $x \cdot y = E(xy)$.

If A is a von Neumann algebra and F is weakly closed then F has a predual $F_\perp = \{\varphi \in A_* \mid \varphi(x) = 0, \text{ for all } x \in F\}$ and hence A is isomorphic to a von Neumann algebraic by Sakai's theorem. □

Note that if A is a C^*-algebra, $F \subset A$ an operator system, and $E : A \to F$ completely positive with $E_{|F} = \mathrm{id}$, then we still have a form of bimodularity for E when we endow F with the Choi-Effros product from Corollary 3.1.11. In this case though the bimodularity is with respect to two different product structures, i.e., we have

$$E(xay) = x \cdot E(a) \cdot y,$$

for all $x, y \in F$, $a \in A$.

3.2 Bi-harmonic operators

If $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$ is regular and normal then we define $\mathscr{P}^o_\varphi$ to be the u.c.p. map given by $\mathscr{P}^o_\varphi = \mathrm{Ad}(J) \circ \mathscr{P}_{\varphi^*} \circ \mathrm{Ad}(J)$. Note that $\mathscr{P}^o_\varphi$ and $\mathscr{P}_\eta$ commute for any normal hyperstate η.

Indeed, if we have standard forms $\mathscr{P}_\varphi(T) = \sum_n (Jz_n^*J)T(Jz_nJ)$ and $\mathscr{P}_\eta(T) = \sum_m (Jy_m^*J)T(Jy_mJ)$ then by Proposition 3.1.4 we have $\mathscr{P}_\varphi^o(T) = \sum_n z_n T z_n^*$ and hence

$$\mathscr{P}_\varphi^o \circ \mathscr{P}_\eta(T) = \mathscr{P}_\eta \circ \mathscr{P}_\varphi^o(T) = \sum_{n,m} z_n (Jy_m^*J)T(Jy_mJ)z_n^*.$$

The following is a noncommutative analogue of double ergodicity which was established in [Kai03].

Theorem 3.2.1. *Let $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$ be a normal regular generating hyperstate, then*

$$\mathrm{Har}(\mathscr{B}(L^2(M,\tau)), \mathscr{P}_\varphi) \cap \mathrm{Har}(\mathscr{B}(L^2(M,\tau)), \mathscr{P}_\varphi^o) = \mathscr{Z}(M).$$

Proof. We fix a standard form $\mathscr{P}_\varphi(T) = \sum_n (Jz_n^*J)T(Jz_nJ)$, so that we also have $\mathscr{P}_\varphi^o(T) = \sum_m z_m T z_m^*$. We identify the Poisson boundary B_φ with $\mathrm{Har}(\mathscr{B}(L^2(M,\tau)))$, and let ζ denote the stationary state on B_φ, which is faithful by Proposition 3.1.8. For $T \in B_\varphi$ we have

$$\zeta(\mathscr{P}_\varphi^o(T)) = \langle \mathscr{P}_\varphi^o(T)\hat{1}, \hat{1}\rangle = \langle \mathscr{P}_\varphi(T)\hat{1}, \hat{1}\rangle = \zeta(\mathscr{P}_\varphi(T)) = \zeta(T).$$

By Lemma 3.1.5 we then have that $B_0 = \mathrm{Har}(B_\varphi, \mathscr{P}^o_{|B_\varphi})$ is a von Neumann subalgebra of B_φ. If $p \in B_0$ is a projection and $\xi \in L^2(B_\varphi, \zeta)$ then

$$\sum_n \|pz_n^*p^\perp \xi\|_2^2 = \sum_n \langle z_n p z_n^* p^\perp \zeta, p^\perp \xi\rangle = 0.$$

We must therefore have $\|pz_n^*p^\perp\xi\|_2 = 0$ for each n, and hence $pz_n^* = pz_n^*p$, for each n. Repeating this argument with p and $p^\perp$ reversed shows that $z_n^*p = pz_n^*p$, so that $p \in M' \cap B_\varphi$. Since p was an arbitrary projection we then have $B_0 \subset M' \cap B_\varphi$ and by Proposition 3.1.6 we have $B_0 = \mathscr{Z}(M)$. $\square$

The previous result allows us to obtain an analogue of the Choquet-Deny theorem.

Corollary 3.2.2 (The Choquet-Deny theorem)**.** *Suppose M is abelian and $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$*

is a normal regular generating hyperstate, then

$$\mathrm{Har}(\mathscr{B}(L^2(M,\tau)),\mathscr{P}_\varphi)=\mathscr{Z}(M)=M.$$

Theorem 3.2.3. *Let $\varphi\in\mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$ be a normal regular generating hyperstate, suppose $C\subset B_\varphi$ is a weakly closed M-bimodule. If $\delta:M\to C$ is a norm continuous derivation then there exists $c\in C$ so that $\delta(x)=[x,c]$ for $x\in M$. Moreover, c may be chosen so that $\|c\|\leq\|\delta\|$.*

Proof. We first view δ as a derivation from M into $\mathscr{P}(C)\subset\mathrm{Har}(\mathscr{P}_\varphi)\subset\mathscr{B}(L^2M,\tau)$. Henceforth, we shall identify C with $\mathscr{P}(C)$. Since $L^2(M,\tau)$ has a cyclic vector for M, $\delta(m)=mT-Tm$ for some $T\in\mathscr{B}(L^2M,\tau)$. (Theorem 5.3, [Chr78]). Taking the conditional expectation onto $\mathrm{Har}(\mathscr{P}_\varphi)$, we may assume $T\in\mathrm{Har}(\mathscr{P}_\varphi)$. Note that by our hypothesis, $z_m\delta(z_m^*)\in C$. So we get:

$$\sum_m z_m\delta(z_m^*)=\sum_m z_mz_m^*T-\sum_m z_mTz_m^*=T-\mathscr{P}_\varphi^o(T)$$

The left hand side of the above equation defines an element of C. So, $T-\mathscr{P}_\varphi^o(T)\in C$. As $\mathscr{P}_\varphi^o$ leaves C invariant (as C is an M-bimodule), by induction we get that $T-(\mathscr{P}_\varphi^o)^n(T)\in C$ for all $n\geq 1$. So, we get:

$$T-\frac{1}{N}\sum_{n=1}^{N}(\mathscr{P}_\varphi^o)^n(T)\in C$$

But, $\frac{1}{N}\sum_{n=1}^{N}(\mathscr{P}_\varphi^o)^n(T)$ has a weak operator topology limit point $z\in\mathrm{Har}(\mathscr{P}_\varphi^o)$. As $z\in\mathrm{Har}(\mathscr{P}_\varphi)$ as well, we get by theorem 3.2.1, that $z\in\mathscr{Z}(M)$. So, $T=(T-z)+z\in C$ □

Theorem 3.2.4. *Let (M,τ) be a tracial von Neumann algebra, and suppose $T\in\mathscr{B}(L^2(M,\tau))$, then*

$$\mathrm{dist}(T,\mathscr{Z}(M))\leq\|\delta_{T|M'}\|+\|\delta_{T|M}\|.$$

Proof. Let $\mathscr{P}_\varphi(x)=\sum_i\mu_iJu_i^*JxJu_iJ$, where $u_i's\in\mathscr{U}(M),\mu_i>0,\sum_i\mu_i=1$, and $\mathrm{span}\{u_i1_\tau\}$ form a dense set in $L^2(M,\tau)$. Then, $\mathscr{P}_\varphi$ is a normal u.c.p. map, which corresponds to a regular, symmetric hyperstate φ.

Note that

$$\|T - Ju_i^* JTJu_iJ\| = \|[Ju_iJ, T]\| \le \|\delta_{T|M'}\|$$

So, we get:

$$\|T - \frac{1}{N}\sum_{n=1}^{N} \mathscr{P}_\varphi^n(T)\| \le \|\delta_{T|M'}\|$$

But, $\sum_{n=1}^{N} \mathscr{P}_\varphi^n(T)$ has a weak operator topology limit point $S \in \mathrm{Har}(\mathscr{P}_\varphi)$. By same techniques as above, we get:

$$\|S - \frac{1}{N}\sum_{n=1}^{N} (\mathscr{P}_\varphi^o)^n(S)\| \le \|\delta_{S|M}\|$$

It's easy to see that $\|\delta_{S|M}\| \le \|\delta_{T|M}\|$. Now, let R be a weak operator topology limit point of $\frac{1}{N}\sum_{n=1}^{N} (\mathscr{P}_\varphi^o)^n(S)$. Note that $R \in \mathrm{Har}(\mathscr{P}_\varphi) \cap \mathrm{Har}(\mathscr{P}_\varphi^o) = \mathscr{Z}(M)$. So,

$$\mathrm{dist}(T, \mathscr{Z}(M)) \le \|T - R\| \le \|T - S\| + \|S - R\| \le \|\delta_{T|M'}\| + \|\delta_{T|M}\|$$

□

Corollary 3.2.5. *If M is an injective II_1 factor, then* $\mathrm{dist}(T, M') \le \|\delta_{T|M}\|$

Proof. Since $M' = JMJ$ is isomorphic to $L(S_\infty)$, by [KV83], there exists a regular, symmetric, generating hyperstate φ such that $\mathrm{Har}(\mathscr{P}_\varphi) = M'$. Then, by same techniques as above, we can find $R \in \mathrm{Har}(\mathscr{P}_\varphi) = M'$ with $\|T - R\| \le \|\delta_{T|M}\|$. Hence, $\mathrm{dist}(T, M') \le \|\delta_{T|M}\|$. □

Remark: $M \subseteq B_\varphi$ has the weak relative Dixmier property, in the sense of [Pop00]. The proof follows from theorem 3.2.1.

3.3 Entropy

3.3.1 Asymptotic entropy

We let M be a tracial von Neumann algebra with a faithful normal tracial state τ. For a normal hyperstate $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M, \tau)))$ we define the **entropy of** φ, denoted by $H(\varphi)$, to be the von

Neumann entropy of the corresponding density matrix A_φ:

$$H(\varphi) = -\mathrm{Tr}(A_\varphi \log(A_\varphi)).$$

If we have a standard form $\varphi(T) = \sum_n \langle T\widehat{z_n^*}, \widehat{z_n^*}\rangle$ then we may compute this explicitly as

$$H(\varphi) = -\sum_n \|z_n\|_2^2 \log(\|z_n\|_2^2).$$

Theorem 3.3.1. *If φ and ψ are two normal hyperstates with ψ regular, then*

$$H(\varphi * \psi) \leq H(\varphi) + H(\psi)$$

Proof. Let A_φ and A_ψ be the corresponding density matrices and $\mathscr{P}_\varphi$ and $\mathscr{P}_\psi$ be the corresponding u.c.p. M-bimodular maps. Let $\{a_i\}_{i\in I}$ and $\{c_j\}_{j\in J}$ be τ orthogonal families, as in proof of proposition 3.1.7, such that $A_\varphi = \sum\limits_i \mu_i P_{\hat{a}_i}$ and $A_\psi = \sum\limits_j \nu_j P_{\hat{c}_j}$. Let $b_i = Ja_iJ$ and $d_i = Jc_iJ$. Then $b_i, d_i \in M'$. We have that:

$$\mathscr{P}_\varphi(T) = \sum_i \mu_i b_i T b_i^* \text{ and } \mathscr{P}_\psi(T) = \sum_j \nu_j d_j T d_j^*.$$

Since ψ is a regular hyperstate we have that $\sum\limits_i \nu_i d_i^* d_i = \sum\limits_i \nu_i d_i d_i^* = 1$. Since φ is a hyperstate, we have that $\sum\limits_i \mu_i b_i b_i^* = 1$. Now,

$$H(A_{\varphi*\psi}) = -\sum_{i,j} Tr[\mu_i \nu_j b_i^* d_j^* P_{\hat{1}} d_j b_i \log(A_{\varphi*\psi})].$$

But

$$b_i^* d_j^* P_{\hat{1}} d_j b_i = \tau(b_i b_i^* d_j^* d_j) P_{\widehat{b_i^* d_j^*}}.$$

So,

$$A_{\varphi*\psi} = \sum_{i,j} \mu_i \nu_j b_i^* d_j^* P_{\hat{1}} d_j b_i \geq \mu_i \nu_j \tau(b_i b_i^* d_j^* d_j) P_{\widehat{b_i^* d_j^*}}$$

for each i, j.

Thus

$$-\log(A_{\varphi*\psi}) = -\log(\sum_{i,j}\mu_i\nu_j b_i^* d_j^* P_{\hat{1}} d_j b_i) \leq -\log((\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j))P_{\widehat{b_i^*d_j^*}})$$

for each i, j, as log is operator monotone.

So,

$$\begin{aligned}
H(A_{\varphi*\psi}) &\leq -\sum_{i,j} Tr[\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)P_{\widehat{b_i^*d_j^*}}\log(\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)P_{\widehat{b_i^*d_j^*}})] \\
&= -\sum_{i,j} Tr[\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)P_{\widehat{b_i^*d_j^*}}\log(\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j) - \sum_{i,j} Tr[\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)P_{\widehat{b_i^*d_j^*}}\log(P_{\widehat{b_i^*d_j^*}})] \\
&= -\sum_{i,j}\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)\log(\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j),
\end{aligned}$$

as the second term vanishes, and $Tr(P_{\widehat{b_i^*d_j^*}}) = 1$. Now define m on $I \times J$ by $m(i, j) = \mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)$. Note that $\sum_i m(i, j) = \nu_j\tau(\sum_i \mu_i b_ib_i^*d_j^*d_j) = \nu_j\tau(d_j^*d_j) = \nu_j$ and $\sum_j m(i, j) = \mu_i\tau(\sum_i \nu_j b_ib_i^*d_j^*d_j) = \mu_i\tau(b_ib_i^*) = \mu_i$. We claim that

$$H(m) = -\sum_{i,j} m(i, j)\log(m(i, j)) \leq H(\mu) + H(\nu).$$

Proof of claim:

$$\begin{aligned}
H(m) &= -\sum_{i,j} m(i, j)\log(m(i, j)) \\
&= -\sum_{i,j}\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)\log(\mu_i\tau(b_ib_i^*d_j^*d_j)) - \sum_{i,j}\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)\log(\nu_j) \\
&= -\sum_{i,j}\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)\log(\mu_i) - \sum_{i,j}\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)\log(\nu_j) - \sum_{i,j}\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)\log(\tau(b_ib_i^*d_j^*d_j)).
\end{aligned}$$

Note that in the above sum, the first term is $H(\mu)$ (summing over j we get $-\sum_i \mu_i\tau(b_ib_i^*)\log(\mu_i) = -\sum_i \mu_i\log(\mu_i)$), and the second term is $H(\nu)$. So, the inequality will be satisfied if we can show: $\sum_{i,j}\mu_i\nu_j\tau(b_ib_i^*d_j^*d_j)\log(\tau(b_ib_i^*d_j^*d_j)) \geq 0$.

Let $\eta(x) = -x\log(x)$ for $x \in [0,1]$. Note that η is concave, and so $\eta(\sum_i \alpha_i x_i) \geq \sum_i \alpha_i \eta(x_i)$ whenever $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$.

So,

$$\begin{aligned}
-\sum_{i,j} \mu_i \nu_j \tau(b_i b_i^* d_j^* d_j) \log(\tau(b_i b_i^* d_j^* d_j)) &= \sum_{i,j} \mu_i \nu_j \eta(\tau(b_i b_i^* d_j^* d_j)) \\
&= \sum_i \mu_i (\sum_j \nu_j \eta(\tau(b_i b_i^* d_j^* d_j))) \\
&\leq \sum_i \mu_i \eta(\sum_j \nu_j \tau(b_i b_i^* d_j^* d_j)) \\
&= \sum_i \mu_i \eta(\tau(b_i b_i^*)) = 0
\end{aligned}$$

□

Corollary 3.3.2. *If φ is a normal, regular hyperstate, then the limit $\lim_{n\to\infty} \frac{H(\varphi^{*n})}{n}$ exits.*

Proof. The sequence $\{H(\varphi^{*n})\}$ is subadditive by Theorem 3.3.1 □

The **asymptotic entropy** $h(\varphi)$ of a normal regular hyperstate φ is defined to be the limit

$$h(\varphi) = \lim_{n\to\infty} \frac{H(\varphi^{*n})}{n}.$$

3.3.2 A Furstenberg type entropy

Suppose G is a Polish group and $\mu \in \mathrm{Prob}(G)$. Given a quasi-invariant action $G \curvearrowright^a (X,\nu)$ the corresponding Furstenberg entropy (or μ-entropy) is defined [Fur63, Section 8] to be

$$h_\mu(a,\nu) = -\iint \log\left(\frac{dg^{-1}\nu}{d\nu}(x)\right) d\nu(x) d\mu(g).$$

If we consider the measure space $(G\times X, \nu\times\mu)$ then we have a non-singular map $\pi : G\times X \to$

$G \times X$ given by $\pi(g,x) = (g, g^{-1}x)$, whose Radon-Nikodym derivative is given by

$$\frac{d\pi(\mu \times \nu)}{d(\mu \times \nu)}(x,g) = \frac{dg^{-1}\nu}{d\nu}(x).$$

We may thus rewrite the μ-entropy as a relative entropy

$$h_\mu(a,\nu) = -\iint \log\left(\frac{d\pi(\nu \times \mu)}{d(\nu \times \mu)}(g,x)\right) d(\nu \times \mu) = S((\nu \times \mu)|\pi(\nu \times \mu)).$$

Let (M,τ) be a tracial von Neumann algebra, φ a normal hyperstate for M, and $\mathscr{A}$ a C^*-algebra, such that $M \subseteq \mathscr{A}$. Let $\zeta \in \mathscr{S}_\tau(\mathscr{A})$ be a faithful hyperstate. Let $\Delta_\zeta : L^2(\mathscr{A},\zeta) \to L^2(\mathscr{A},\zeta)$ be the modular operator corresponding to ζ, and consider the spectral decomposition $\Delta_\zeta = \int_0^\infty \lambda\, dE(\lambda)$.

Since $\zeta|_M = \tau$, we have a natural inclusion of $L^2(M,\tau)$ in $L^2(\mathscr{A},\zeta)$. Let e denote the orthogonal projection from $L^2(\mathscr{A},\zeta)$ to $L^2(M,\tau)$. The entropy of the inclusion $(M,\tau) \subset (\mathscr{A},\zeta)$ with respect to φ is defined to be

$$h_\varphi(M \subset \mathscr{A},\zeta) = -\int \log(\lambda)\, d\varphi(eE(\lambda)e).$$

Example 3.3.3. *If Γ is a discrete group, $\mu \in \mathrm{Prob}(\Gamma)$ and $\Gamma \curvearrowright^a (X,\nu)$ is a quasi-invariant action, then we may consider the state φ on $\mathscr{B}(\ell^2\Gamma)$ given by $\varphi(T) = \int \langle T\delta_\gamma, \delta_\gamma\rangle\, d\mu(\gamma)$, and we may consider the state ζ on $L^\infty(X,\nu) \rtimes \Gamma \subset \mathscr{B}(\ell^2\Gamma \overline{\otimes} L^2(X,\nu))$ given by $\zeta\left(\sum_{\gamma\in\Gamma} a_\gamma u_\gamma\right) = \int a_e\, d\nu$. Note that we have that in this case we may compute $\varphi * \zeta\left(\sum_{\gamma\in\Gamma} a_\gamma u_\gamma\right) = \int a_e\, d\mu * \nu$. The modular operator Δ_ζ is then affiliated to the von Neumann algebra $\ell^\infty\Gamma \overline{\otimes} L^\infty(X,\nu)$, and we may compute this directly as*

$$\Delta_\zeta(\gamma,x) = \frac{d\gamma^{-1}\nu}{d\nu}(x).$$

We also have that the projection e from $\ell^2\Gamma \overline{\otimes} L^2(X,\nu) \to \ell^2\Gamma$ is given by $\mathrm{id} \otimes \int$. Thus, it follows that the measure $d\varphi(eE(\lambda)e)$ agrees with $d\alpha_*(\mu \times \nu)$, where $\alpha : \Gamma \times X \to \mathbb{R}_{>0}$ is the Radon-Nikodym cocycle, $\alpha(\gamma,x) = \frac{d\gamma^{-1}\nu}{d\nu}(x)$.

In this case we then have

$$h_\varphi(L\Gamma \subset L^\infty(X,\nu) \rtimes \Gamma, \zeta) = -\int \log(\lambda) d\varphi(eE(\lambda)e)$$
$$= -\iint \log\left(\frac{d\gamma^{-1}\nu}{d\nu}(x)\right) d(\nu \times \mu) = h_\mu(a,\nu).$$

Lemma 3.3.4. *Let $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$ be a normal hyperstate and write φ in a standard form $\varphi(T) = \sum_n \langle T\widehat{z_n^*}, \widehat{z_n^*}\rangle$. Suppose $\mathscr{A}$ is a C*-algebra with $M \subset \mathscr{A}$ and $\zeta \in \mathscr{S}_\tau(\mathscr{A})$ is a hyperstate. Then if $h_\varphi(M \subset \mathscr{A}, \zeta) < \infty$ we have that $z_n^* 1_\zeta \in D(\log \Delta_\zeta)$ for each n and*

$$h_\varphi(M \subset \mathscr{A}, \zeta) = \sum_n \langle \log \Delta_\zeta z_n^* 1_\zeta, z_n^* 1_\zeta \rangle = i \lim_{t\to 0} \frac{1}{t} \sum_n (\zeta(z_n \sigma_t^\zeta(z_n^*)) - 1).$$

Proof. As $\mathscr{A}1_\zeta$ forms a core for S_ζ we get that $z_n^* 1_\zeta \in D(\log(\Delta_\zeta))$. Also, we know that $\lim_{t\to 0} \frac{\Delta_\zeta^{it} - 1}{t}\xi = i\log(\Delta_\zeta)\xi$, for all $\xi \in D(\Delta_\zeta)$. So, we have that

$$\begin{aligned} h_\varphi(M \subset \mathscr{A}, \zeta) &= -\varphi(e \log(\Delta_\zeta) e) \\ &= \sum_n \langle \log \Delta_\zeta z_n^* 1_\zeta, z_n^* 1_\zeta \rangle \\ &= i \sum_n \langle z_n \lim_{t\to 0} \frac{\Delta_\zeta^{it} - 1}{t} z_n^* 1_\zeta, 1_\zeta \rangle = i \lim_{t\to 0} \frac{1}{t} \sum_n (\zeta(z_n \sigma_t^\zeta(z_n^*)) - 1). \end{aligned}$$

□

Example 3.3.5. *Fix two normal hyperstates $\varphi, \zeta \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$ such that φ is regular, and ζ is faithful, and consider the case $\mathscr{A} = \mathscr{B}(L^2(M,\tau))$. Then the density operator A_ζ is injective with dense range and the modular operator on $L^2(\mathscr{B}(L^2(M,\tau)),\zeta)$ is given by $\Delta_\zeta(T1_\zeta) = A_\zeta T A_\zeta^{-1} 1_\zeta$, for $T \in \mathscr{B}(L^2(M,\tau))$ such that $T1_\zeta \in D(\Delta_\zeta)$, so that $\log(\Delta_\zeta)(T1_\zeta) = (\mathrm{Ad}(\log A_\zeta)T)1_\zeta$, where* $\mathrm{Ad}(\log A_\zeta)T = (\log A_\zeta)T - T(\log A_\zeta)$.

We also have that the projection $e : L^2(\mathscr{B}(L^2(M,\tau)),\zeta) \to L^2(M,\tau)$ is given by $e(T1_\zeta) =$

$\mathscr{P}_\zeta(T)1_\tau$. *Therefore,* $e\log\Delta_\zeta ex1_\tau = \mathscr{P}_\zeta(\mathrm{Ad}(\log A_\zeta)x)1_\tau = \mathscr{P}_\zeta(\mathrm{Ad}(\log A_\zeta))x1_\tau$. *Hence,*

$$\begin{aligned} h_\varphi(M \subset \mathscr{B}(L^2(M,\tau)),\zeta) &= \varphi(\mathscr{P}_\zeta(\mathrm{Ad}(\log A_\zeta))) \\ &= \mathrm{Tr}(A_{\varphi*\zeta}\mathrm{Ad}(\log A_\zeta)) \\ &= \mathrm{Tr}(A_{\varphi*\zeta}\log A_\zeta) - \langle \log A_\zeta 1_\tau, 1_\tau\rangle. \end{aligned}$$

Where the last equality follows since φ *is regular.*

We recall the following two lemmas from works of D.Petz

Lemma 3.3.6 (D.Petz: Properties of relative entropy of states of von Neumann algebras)**.** *Let* Δ_j *be positive, self adjoint operators on* $\mathscr{H}_j, j=1,2$. *If* $T : \mathscr{H}_1 \to \mathscr{H}_2$ *is a bounded operator such that:*

- $T(\mathscr{D}(\Delta_1)) \subseteq \mathscr{D}(\Delta_2)$
- $||\Delta_2 T\xi|| \leq ||T|| \cdot ||\Delta_1\xi||$ *(*$\xi \in \mathscr{D}(\Delta_1)$*),*

then we have for each $t \in [0,1]$, *and* $\xi \in \mathscr{D}(\Delta_1^t)$,

$$||\Delta_2^t T\xi|| \leq ||T|| \cdot ||\Delta_1^t\xi||$$

Lemma 3.3.7 (D Petz)**.** *Let* Δ *be a positive self adjoint operator and* $\xi \in \mathscr{D}(\Delta)$. *Then:*

$$\lim_{t\to 0+} \frac{||\Delta^{t/2}\xi||^2 - ||\xi||^2}{t}$$

exists. It's finite or $-\infty$ and equals $\int_0^\infty \log\lambda\, d\langle E_\lambda\xi,\xi\rangle$ where $\int_0^\infty \log\lambda\, dE_\lambda$ is the spectral resolution of Δ.

Corollary 3.3.8. $h_\varphi(M \subset \mathscr{A},\zeta) = -\lim_{t\to 0+} \frac{\sum_{k=1}^\infty ||\Delta_\varphi^{t/2} e z_n^* 1_\tau||^2 - ||e z_n^* 1_\tau||^2}{t}$

Lemma 3.3.9. $h_\varphi(M \subset \mathscr{A},\zeta) \geq 0$

Proof. Let $\mathscr{P}_\zeta(T) = eTe$ for $T \in \mathscr{A}$.

$h_\varphi(M \subset \mathscr{A}, \zeta) = \lim_{n\to\infty} \varphi(-e\log\Delta_n e) = -\lim_{n\to\infty} \langle \mathscr{P}_\varphi \circ \mathscr{P}_\zeta(\log\Delta_n)1_\tau, 1_\tau\rangle \geq \lim_{n\to\infty} -\langle \log(\mathscr{P}_\varphi \circ \mathscr{P}_\zeta(\Delta_n))1_\tau, 1_\tau\rangle$ (using the operator Jensen's inequality; recall that log is operator concave).

Now, $e\Delta_n e \leq e\Delta e = I$. So, $\mathscr{P}_\varphi \circ \mathscr{P}_\zeta(\Delta_n) \leq I$. As log is operator monotone, we get that $\log(\mathscr{P}_\varphi \circ \mathscr{P}_\zeta(\Delta_n)) \leq \log(I) = 0$. Hence we are done. □

Theorem 3.3.10. *Let $\varphi, \psi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M,\tau)))$ be two normal hyperstates such that ψ is regular, and suppose $\mathscr{A}$ is a C^*-algebra with $M \subset \mathscr{A}$, and $\zeta \in S_\tau(\mathscr{A})$ is a faithful hyperstate, then*

$$h_{\varphi*\psi}(M \subset \mathscr{A}, \zeta) = h_\varphi(M \subset \mathscr{A}, \psi * \zeta) + h_\psi(M \subset \mathscr{A}, \zeta).$$

Proof. Let $\mathscr{P}_\varphi$ and $\mathscr{P}_\psi$ be the corresponding u.c.p. maps. Let $\mathscr{P}_\varphi(T) = \sum_k \mu_k J a_k^* J T J a_k J$ and $\mathscr{P}_\psi(T) = \sum_l \nu_l J b_l^* J T J b_l J$. We shall denote the projection from $L^2(\mathscr{A}, \zeta)$ to $L^2(M, \tau)$ by e and Δ_ζ by Δ. We then have:

$$\begin{aligned} h_\varphi(M \subset \mathscr{A}, \zeta) &= i\lim_{t\to 0} \varphi\Big(\frac{e\Delta^{it}e - 1}{t}\Big) = i\lim_{t\to 0} \frac{1}{t}\varphi(e\Delta^{it}e - 1) \\ &= i\lim_{t\to 0}\frac{1}{t}\Big(\sum_k \mu_k \langle (\Delta^{it} - 1)a_k^* 1_\zeta, a_k^* 1_\zeta\rangle\Big) \end{aligned}$$

Similarly,

$$h_\psi(M \subset \mathscr{A}, \zeta) = i\lim_{t\to 0}\frac{1}{t}\Big(\sum_l \nu_l \langle (\Delta^{it} - 1)b_l^* 1_\zeta, b_l^* 1_\zeta\rangle\Big)$$

and,

$$\begin{aligned} h_{\varphi*\psi}(M \subset \mathscr{A}, \zeta) &= i\lim_{t\to 0}\frac{1}{t}\Big(\sum_{k,l} \mu_k \nu_l \langle (\Delta^{it} - 1)a_k^* b_l^* 1_\zeta, a_k^* b_l^* 1_\zeta\rangle\Big) \\ &= i\lim_{t\to 0}\frac{1}{t}\Big(\sum_{k,l} \mu_k \nu_l \langle (b_l a_k \sigma_t(a_k^* b_l^*)1_\zeta, 1_\zeta\rangle - 1\Big) \end{aligned}$$

We shall now show: $\lim_{t\to 0}\frac{1}{t}\big(\sum_{k,l} \mu_k \nu_l \langle b_l a_k \sigma_t(a_k^* b_l^*)1_\zeta, 1_\zeta\rangle - \sum_{k,l} \mu_k \nu_l \langle b_l \sigma_t(b_l^*)\sigma_t(a_k^*)1_\zeta, 1_\zeta\rangle\big) = 0.$

Let $y_t = a_k \sigma(a_k^*)$. Note that $y_t \to a_k a_k^*$ as $t \to 0$, in SOT. We have:

$$\begin{aligned} y_t \sigma_t(b_l^*) - \sigma_t(b_l^*) y_t &= y_t \sigma_t(b_l^*) - y_t b_l^* + y_t b_l^* - \sigma_t(b_l^*) y_t \\ &= y_t(\sigma_t(b_l^*) - b_l^*) + (y_t b_l^* - b_l^* y_t) + (b_l^* - \sigma_t(b_l^*)) y_t \end{aligned}$$

Now,

$$\begin{aligned} \frac{1}{t}(\sum_{k,l} \mu_k \nu_l \langle (y_t b_l^* - b_l^* y_t) 1_\zeta, b_l^* 1_\zeta \rangle &= \frac{1}{t}(\sum_{k,l} \mu_k \nu_l \langle b_l y_t b_l^* 1_\zeta, 1_\zeta \rangle - \frac{1}{t}(\sum_{k,l} \mu_k \nu_l \langle y_t 1_\zeta, b_l b_l^* 1_\zeta \rangle \\ &= \frac{1}{t} \sum_k \mu_k \langle (\sum_l \nu_l b_l y_t b_l^*) 1_\zeta, 1_\zeta \rangle - \frac{1}{t} \sum_k \langle y_t 1_\zeta, 1_\zeta \rangle \\ &= \frac{1}{t} \sum_k \langle y_t 1_\zeta, 1_\zeta \rangle - \frac{1}{t} \sum_k \langle y_t 1_\zeta, 1_\zeta \rangle = 0 (Stationarity). \end{aligned}$$

Also, $\lim_{t \to 0} \frac{1}{t}(y_t(\sigma_t(b_l^*) - b_l^*))$ exists, and hence $\lim_{t \to 0} \frac{1}{t}(\sum_{k,l} \mu_k \nu_l \langle b_l a_k \sigma_t(a_k^* b_l^*) 1_\zeta, 1_\zeta \rangle - \sum_{k,l} \mu_k \nu_l \langle b_l \sigma_t(b_l^*) \sigma_t(a_k^*) 1_\zeta, 1$
0. So, we get that:

$$\begin{aligned} h_{\varphi * \psi}(M \subset \mathscr{A}, \zeta) &= i \lim_{t \to 0} \frac{1}{t}(\sum_{k,l} \mu_k \nu_l \langle (b_l \sigma_t(b_l^*) a_k \sigma_t(a_k^*) - 1) 1_\zeta, 1_\zeta \rangle \\ &= i \lim_{t \to 0} \frac{1}{t}(\sum_{k,l} \mu_k \nu_l [\langle (b_l \sigma_t(b_l^*) - 1) 1_\zeta, 1_\zeta \rangle \\ &+ \langle (a_k \sigma_t(a_k^*) - 1) 1_\zeta, 1_\zeta \rangle \\ &+ \langle (a_k \sigma_t(a_k^*) - 1) 1_\zeta, (b_l \sigma_t(b_l^*) - 1)^* 1_\zeta \rangle] \end{aligned}$$

The first term equals $h_\varphi(M \subset \mathscr{A}, \zeta)$, while second term equals $h_\psi(M \subset \mathscr{A}, \zeta)$, and the third term equals zero, as $\lim_{t \to 0} \frac{1}{t}(a_k \sigma_t(a_k^*) - 1) 1_\zeta$ exists, while $\lim_{t \to 0} \sum_l \nu_l (b_l \sigma_t(b_l^*) - 1)^* 1_\zeta = 0$ □

Corollary 3.3.11. *Let $\varphi \in \mathscr{S}_\tau(\mathscr{B}(L^2(M, \tau)))$ be a regular normal hyperstate and suppose $\mathscr{A}$ is a C^*-algebra with $M \subset \mathscr{A}$, and $\zeta \in S_\tau(\mathscr{A})$ is a faithful hyperstate φ-stationary hyperstate, then for $n \geq 1$ we have*

$$h_{\varphi^{*n}}(M \subset \mathscr{A}, \zeta) = n h_\varphi(M \subset \mathscr{A}, \zeta).$$

Lemma 3.3.12. $h_\varphi(M \subset \mathscr{A}, \zeta) \leq H(\varphi)$

Proof. Let $\mathscr{P}_\varphi(T) = \sum_k \mu_k b_k T b_k^*$. Let $a_k = J b_k J \in M$. It follows from 3.3.7 that

$$H(\varphi) = -\lim_{t\to 0+} \frac{\sum_{k=1}^{\infty} \mu_k ||A_\varphi^{t/2} a_k^* 1_\tau||^2 - ||a_k^* 1_\tau||^2}{t}.$$

So by corollary 3.3.8 it's enough to show that

$$\lim_{t\to 0+} \frac{\sum_{k=1}^{\infty} \mu_k ||A_\varphi^{t/2} a_k^* 1_\tau||^2 - ||a_k^* 1_\tau||^2}{t} \leq \lim_{t\to 0+} \frac{\sum_{k=1}^{\infty} \mu_k ||\Delta_\varphi^{t/2} e a_k^* 1_\tau||^2 - ||e a_k^* 1_\tau||^2}{t}.$$

—

So, it's enough to show that

$$||A_\varphi^{t/2} a_k 1_\tau||^2 \leq ||\Delta_\zeta^{t/2} a_k 1_\zeta||^2$$

Define $T : L^2(\mathscr{A}, \zeta) \to L^2(M, \tau)$ by $T(a 1_\zeta) = \mathscr{P}_\zeta(a) 1_\tau$. Then $||T|| = 1$, as $||T(1_\zeta)|| = 1$ and $||\mathscr{P}_\zeta|| \leq 1$. T takes $\mathscr{D}(\Delta_\zeta)$ into $\mathscr{D}(A_\varphi) = L^2(M, \tau)$. By lemma 3.3.6 it's enough to show:

$$||A_\varphi^{1/2} T\xi|| \leq ||\Delta^{1/2}\xi|| \text{for all } \xi \in \mathscr{D}(\Delta).$$

In fact it's enough to show the above for all vectors in a core for $\mathscr{D}(\Delta)$. Recall that $\mathscr{A} 1_\zeta$ forms a core for $\mathscr{D}(\Delta)$. So, we only need to show

$$||A_\varphi^{1/2} T a 1_\zeta|| \leq ||\Delta^{1/2} a 1_\zeta||$$

Now we have:

$$
\begin{aligned}
||\Delta^{1/2}a1_\zeta||^2 &= \langle \Delta^{1/2}a1_\zeta, \Delta^{1/2}a1_\zeta\rangle = \langle JSa1_\zeta, JSa1_\zeta\rangle \\
&= \langle Ja^*1_\zeta, Ja^*1_\zeta\rangle = \langle a^*1_\zeta, a^*1_\zeta\rangle = \zeta(aa^*) \\
&= \langle \mathscr{P}_\zeta(aa^*)1_\tau, 1_\tau\rangle
\end{aligned}
$$

We also have $\mathscr{P}_\varphi \circ \mathscr{P}_\zeta = \mathscr{P}_\zeta \implies \varphi \circ \mathscr{P}_\zeta = \zeta$. Now:

$$
\begin{aligned}
||A_\varphi^{1/2}Ta1_\zeta||^2 &= \langle A_\varphi^{1/2}\mathscr{P}_\zeta(a)1_\tau, A_\varphi^{1/2}\mathscr{P}_\zeta(a)1_\tau\rangle = \langle A_\varphi \mathscr{P}_\zeta(a)1_\tau, \mathscr{P}_\zeta(a)1_\tau\rangle \\
&= \langle \mathscr{P}_\zeta(a)^* A_\varphi \mathscr{P}_\zeta(a)1_\tau, 1_\tau\rangle \leq Tr(\mathscr{P}_\zeta(a)^* A_\varphi \mathscr{P}_\zeta) \\
&= Tr(A_\varphi \mathscr{P}_\zeta(a)\mathscr{P}_\zeta(a^*)) \leq Tr(A_\varphi \mathscr{P}_\zeta(aa^*) \\
&= \langle \Phi \circ \mathscr{P}_\zeta(aa^*)1_\tau, 1_\tau\rangle = \langle \mathscr{P}_\zeta(aa^*)1_\tau, 1_\tau\rangle \\
&= \zeta(aa^*) = ||\Delta^{1/2}a1_\zeta||^2.
\end{aligned}
$$

Hence we are done. □

Corollary 3.3.13. $h_\varphi(M \subset \mathscr{A}, \zeta) \leq h(\varphi)$

Proof. By lemma 3.3.12, we have that $h_{\varphi^{*n}}(M \subset \mathscr{A}, \zeta) \leq H(\varphi^{*n})$. By corollary 3.3.11 we have that $h_{\varphi^{*n}}(M \subset \mathscr{A}, \zeta) = nh_\varphi(M \subset \mathscr{A}, \zeta)$. So we get,

$$
h_\varphi(M \subset \mathscr{A}, \zeta) \leq \frac{H(\varphi^{*n})}{n} \to h(\varphi).
$$

□

Lemma 3.3.14. $h_\varphi(M \subset \mathscr{A}, \zeta) = 0$ *if and only if there exists a* ζ *preserving conditional expectation from* $\mathscr{A}$ *to* M.

Proof. Let $\mathscr{E} : \mathscr{A} \to M$ be a ζ preserving conditional expectation. Then, we know that $\sigma_t^\zeta(m) = m$

for all $m \in M$. Hence,

$$\begin{aligned} h_{\varphi}(M \subset \mathscr{A}, \zeta) &= i \lim_{t \to 0} \frac{1}{t} \sum_k \langle (\Delta^{it} - 1) a_k^* 1_\zeta, a_k^* 1_\zeta \rangle \\ &= i \lim_{t \to 0} \frac{1}{t} \sum_k \langle \sigma_t(a_k^*) 1_\zeta, a_k^* 1_\zeta \rangle - 1 = 0 \end{aligned}$$

Conversely, suppose $h_{\varphi}(M \subset \mathscr{A}, \zeta) = 0$. Let $\Delta_\zeta = \Delta$ and let $\Delta = \int_0^\infty \lambda d\lambda$ be it's spectral resolution. Let $\Delta_n = \int_{1/n}^n \lambda d\lambda$, $n \geq 1$ be the truncations. We know that Δ_n converges to Δ in the resolvent sense. As usual, we denote by e the projection from $L^2(\mathscr{A}, \zeta)$ to $L^2(M, \tau)$. We have that $I = e\Delta e \geq e\Delta_n e$ for all n. So, $(I+t)^{-1} \leq (e\Delta_n e + t)^{-1} \leq e(\Delta_n + t)^{-1} e$ for all n and for all $t > 0$. Taking limits as $n \to \infty$, we get $(I+t)^{-1} \leq e(\Delta + t)^{-1} e$. Now we shall use the following integral representation of log:

$$\log(x) = \int_0^\infty [(1+t)^{-1} - (x+t)^{-1}] dt$$

So that :

$$h_{\varphi}(M \subset \mathscr{A}, \zeta) = -\int_0^\infty \sum_k \langle e[(I+t)^{-1} - (\Delta+t)^{-1}] e a_k^* 1_\tau, a_k^* 1_\tau \rangle.$$

From $h_{\varphi}(M \subset \mathscr{A}, \zeta) = 0$ and above discussion, we deduce that:

$$(I+t)^{-1} a_k^* 1_\zeta = (\Delta + t)^{-1} a_k^* 1_\zeta$$

for almost all $t > 0$, and hence by continuity, for all $t > 0$. This implies that $\Delta^{it} a_k^* 1_\zeta = a_k^* 1_\zeta$, which implies that $\sigma_t^\zeta(m) = m$ for all $m \in M$, as φ is generating. But this implies the existence of a ζ preserving conditional expectation from $\mathscr{A}$ to M. $\square$

Corollary 3.3.15. $\mathrm{Har}(\mathscr{B}(L^2M, \tau), \mathscr{P}_\varphi) = M$ *if and only if* $h_\varphi(M \subset \mathscr{B}_\varphi, \zeta) = 0$*, where* $\mathscr{B}_\varphi$ *denotes the Poisson boundary with respect to* φ*.*

Proof. If $h_\varphi(M \subset \mathscr{B}, \zeta) = 0$, then by lemma 3.3.14 there exists a conditional expectation $\mathscr{E} : \mathscr{B} \to$

M, given by $\mathscr{E}(b) = ebe = \mathscr{P}(b)$. So,

$$\mathrm{Har}(\mathscr{P}_\varphi) = \mathscr{P}(\mathscr{B}_\varphi) = M.$$

Conversely, if $\mathrm{Har}(\mathscr{B}(L^2M,\tau),\mathscr{P}_\varphi) = M$ then $\Delta_\zeta = I$ and hence $h_\varphi(M \subset \mathscr{B}_\varphi, \zeta) = 0$ □

Corollary 3.3.16. $\mathrm{Har}(\mathscr{P}_\varphi) = M$ *if* $h(\varphi) = 0$

Proof. Since $0 \leq h_\varphi(M \subset \mathscr{B}_\varphi, \zeta) \leq h(\varphi)$, this result follows from the previous corollary 3.3.15. □

3.4 An entropy gap for property (T) factors

If (M,τ) is a tracial von Neumann algebra, then a Hilbert M-bimodule consists of a Hilbert space $\mathscr{H}$, together with commuting normal representations $L : M \to \mathscr{B}(\mathscr{H})$, $R : M^{\mathrm{op}} \to \mathscr{B}(\mathscr{H})$. We will sometimes simplify notation by writing $x\xi y$ for the vector $L(x)R(y^{\mathrm{op}})\xi$. A vector $\xi \in \mathscr{H}$ is left (resp. right) tracial if $\langle x\xi,\xi\rangle = \tau(x)$ (resp. $\langle \xi x,\xi\rangle = \tau(x)$) for all $x \in M$. A vector is bi-tracial if it is both left and right tracial. A vector ξ is left (resp. right) bounded if there exists $C > 0$ so that $\|x\xi\| \leq c\|x\|_2$ (resp. $\|\xi x\| \leq C\|x\|_2$) for all $x \in M$. We denote by ${}^o\mathscr{H}$ (resp. $\mathscr{H}^o$) the subspace of all left (resp. right) bounded vectors, we let ${}^o\mathscr{H}^o =^o \mathscr{H} \cap \mathscr{H}^o$ be the (dense [Pop86, Theorem...]) subspace of left and right bounded vectors. A vector $\xi \in \mathscr{H}$ is central if $x\xi = \xi x$ for all $x \in M$. Note that if ξ is a unit central vector then $x \mapsto \langle x\xi,\xi\rangle$ gives a normal trace on M.

Suppose $M \subset \mathscr{A}$ is an inclusion of von Neumann algebras and $\varphi \in \mathscr{A}_*$ is a hyperstate. We may then consider the Hilbert space $L^2(\mathscr{A},\varphi)$ which is naturally a Hilbert M-bimodule where the left action is given by left multiplication $L(x)\hat{a} = \widehat{xa}$, and the right action is given by $R(x^{op}) = Jx^*J$. In this case the vector $\hat{1}$ is clearly left tracial, and we also have $Jx^*J\hat{1} = \Delta^{1/2}x\hat{1}$ from which it follows easily that $\hat{1}$ is also right tracial. If $\xi_0 \in L^2(\mathscr{A},\varphi)$ is a unit central vector, then $\tau_0(x) = \langle x\xi_0,\xi_0\rangle$ defines a normal trace on M. We let $s \in \mathscr{Z}(M)$ denote the support of τ_0.

The von Neumann algebra M has property (T) if for any sequence of Hilbert bimodules $\mathscr{H}_n$, and $\xi_n \in \mathscr{H}_n$ bi-triacial vectors, such that $\|x\xi_n - \xi_n x\| \to 0$ for all $x \in M$, then we have $\|\xi_n - P_0(\xi_n)\| \to$

0, where P_0 is the projection onto the space of central vectors. This is independent of the normal faithful trace τ [Pop06, Proposition 4.1]. Property (T) was first introduced in the factor case by Connes and Jones [CJ85] where they showed that for an ICC group Γ, the group von Neumann algebra $L\Gamma$ has property (T) if and only if Γ has Kazhdan's property (T) [Kaž67]. Their proof works equally well in the general case when Γ is not necessarily ICC.

We now fix a normal hyperstate φ on $\mathscr{B}(L^2(M,\tau))$ and we take $\{a_k\}_k^n \subset M$ with $n \in \mathbb{N} \cup \{\infty\}$ so that $\sum_{k=1}^n a_k^* a_k = 1$ and $\varphi(T) = \sum_{k=1}^n \langle T\widehat{a_k^*}, \widehat{a_k^*}\rangle$ for $T \in \mathscr{B}(L^2(M,\tau))$. For a fixed Hilbert bimodule $\mathscr{H}$ we define $\nabla_L, \nabla_R : \mathscr{H}^o \to \mathscr{H}^{\oplus n}$ by

$$\nabla_L(\xi) = \oplus a_k \xi$$

$$\nabla_R(\xi) = \oplus \xi a_k.$$

Note that we have

$$\|\nabla_L(\xi)\|^2 = \sum_{k=1}^n \|a_k\xi\|^2 = \left\langle \sum_{k=1}^n a_k^* a_k \xi, \xi \right\rangle = \|\xi\|^2.$$

Also, if $\xi \in \mathscr{H}^o$ is such that $\|\xi x\| \leq C\|x\|_2$, then we similarly have that $\|\nabla_R(\xi)\|^2 \leq C^2$.

Definition 3.4.1. *A tracial von Neumann algebra M together with a hyperstate φ is said to have an entropy gap, if there exists a constant $\varepsilon = \varepsilon(M,\varphi)$ such that for any stationary space $(\mathscr{A},\zeta)$ the Furstenberg entropy is atleast ε.*

We shall show that if M has property (T) in the sense of Connes and Jones, then (M,φ) has an entropy gap for any regular, symmetric hyperstate.

Lemma 3.4.2. *Let $(\mathscr{A},\varphi_0)$ be a stationary space. Define $T : L^2(\mathscr{A},\zeta) \to L^2(\mathscr{A},\zeta)$ by :*

$$T(\xi) = (\sum_k L_{a_k^*} R_{a_k})(\xi)$$

Then $h_\varphi(\mathscr{A},\zeta) \geq -2\log\langle T1_\zeta, 1_\zeta\rangle$.

Proof. Note that, $L_{a_k^*}R_{a_k}1_\zeta = a_k^*\Delta^{1/2}a_k 1_\zeta$. Now,

$$\begin{aligned}
-2\log\langle T1_\zeta, 1_\zeta\rangle &= -2\log(\sum_k \langle L_{a_k^*}R_{a_k}\hat{1},\hat{1}\rangle) \\
&= -2\log(\sum_k \langle a_k^*\Delta^{1/2}a_k\hat{1},\hat{1}\rangle) = -2\log(\lim_{n\to\infty}\sum_k \langle a_k^*\Delta_n^{1/2}a_k\hat{1},\hat{1}\rangle) \\
&= -2\lim_{n\to\infty}\log(\sum_k \langle a_k^*\Delta_n^{1/2}a_k\hat{1},\hat{1}\rangle) \\
&\leq \lim_{n\to\infty}\sum_k \langle a_k^*\log(\Delta_n)a_k\hat{1},\hat{1}\rangle = h_\varphi(\mathscr{A},\zeta),
\end{aligned}$$

where the last inequality follows from Jensen's operator inequality.

□

Lemma 3.4.3. $\langle T1_\zeta, 1_\zeta\rangle \leq ||T|| \leq 1$.

Proof. Let ∇_L and ∇_R be defined as before, for the Hilbert bimodule $L^2(\mathscr{A},\zeta)$. Then:

$$\begin{aligned}
\langle \nabla_R^*\nabla_L(\psi),\eta\rangle &= \sum_k \langle a_k\psi, \eta a_k\rangle \\
&= \sum_k \langle a_k\psi a_k^*, \eta\rangle = \langle T(\psi),\eta\rangle.
\end{aligned}$$

We also have:

$$\begin{aligned}
\langle \nabla_L^*\nabla_L(\psi),\eta\rangle &= \sum_k \langle a_k\psi, a_k\eta\rangle \\
&= \sum_k \langle a_k^*a_k\psi, \eta\rangle = \langle \sum_k a_k^*a_k\psi, \eta\rangle = \langle \psi,\eta\rangle,
\end{aligned}$$

and:

$$\begin{aligned}
\langle \nabla_R^*\nabla_R(\psi),\eta\rangle &= \sum_k \langle \psi a_k, \eta a_k\rangle \\
&= \sum_k \langle \psi a_k a_k^*, \eta\rangle = \langle \sum_k \psi a_k a_k^*, \eta\rangle = \langle \psi,\eta\rangle.
\end{aligned}$$

So, $||T|| \leq ||\nabla_L|| \cdot ||\nabla_R|| \leq 1$. $\square$

Remark: The operator T is unambiguously defined.

Lemma 3.4.4. *If M has property (T) then $\langle T 1_\zeta, 1_\zeta \rangle \leq c < 1$ where c is independent of $(\mathscr{A}, \zeta)$*

Proof. We already showed that $\langle T 1_\zeta, 1_\zeta \rangle \leq 1$. Suppose that there exists (stationary) spaces $(\mathscr{A}_\setminus, \zeta_n)$, such that $\langle T 1_{\zeta_n}, 1_{\zeta_n} \rangle \to 1$. Since 1_{ζ_n} are bi-tracial vectors, satisfying $||a_k 1_{\zeta_n} - 1_{\zeta_n} a_k|| \to 0$ (by convexity of Hilbert spaces), we get that there exists a central vector ψ (since a_k's generate M and M has property (T).)The rest follows easily. $\square$

3.5 Rigidity for u.c.p. maps on the boundary

Theorem 3.5.1. *Suppose we have an intermediate von Neumann algebra $M \subset \mathscr{C} \subset \mathscr{B}$, and $\Psi : \mathscr{C} \to \mathscr{B}$ is a normal unital completely positive map such that $\Psi_{|M} = \mathrm{id}$, then $\Psi = \mathrm{id}$.*

Proof. By identifying $\mathscr{C}$ with $\mathscr{P}(\mathscr{C})$ we may assume that $\phi : \mathscr{C} \to \mathrm{Har}(\mathscr{P}_\varphi)$ is a normal unital completely positive map such that $\Psi_{|M} = \mathrm{id}$. Under this identification, $\mathscr{C}$ is a weakly closed M-sub bimodule of $\mathrm{Har}(\mathscr{P}_\varphi)$. Note that for $T \in \mathscr{C}$ we have,

$$\begin{aligned}\langle \Psi(T) 1_\tau, 1_\tau \rangle = \langle \mathscr{P}_\varphi(\Psi(T)) 1_\tau, 1_\tau \rangle = \langle \mathscr{P}^o_\varphi(\Psi(T)) 1_\tau, 1_\tau \rangle \\ = \sum_n \langle z_n \Psi(T) z_n^* 1_\tau, 1_\tau \rangle = \langle \Psi(\mathscr{P}^o_\varphi(T)) 1_\tau, 1_\tau \rangle.\end{aligned}$$

Where the last equality follows from the fact that Ψ is normal and M-bimodular. So we get that $\langle \Psi(\mathscr{P}^o_\varphi(T)) 1_\tau, 1_\tau \rangle = \langle \Psi(T) 1_\tau, 1_\tau \rangle$ for all $T \in C$, which immediately implies that

$$\langle \Psi(\frac{1}{N} \sum_{n=1}^N (\mathscr{P}^o_\varphi)^n(T)) 1_\tau, 1_\tau \rangle = \langle \Psi(T) 1_\tau, 1_\tau \rangle \text{for all } T \in C.$$

Let z be a weak operator topology limit point of $\frac{1}{N} \sum_{n=1}^N (\mathscr{P}^o_\varphi)^n(T)$. Then, $z \in \mathscr{Z}(M)$ by theorem 3.2.1. So, $\Psi(z) = z$, and we get that

$$\langle \Psi(T) 1_\tau, 1_\tau \rangle = \langle z 1_\tau, 1_\tau \rangle = \langle T 1_\tau, 1_\tau \rangle$$

where the last equality follows because z is independent of Ψ. Now, let $a, b \in M$, and $T \in \mathscr{C}$. Then, we have that $b^*Ta \in \mathscr{C}$, and hence by above computation, we get:

$$\langle \Psi(T)a1_\tau, b1_\tau \rangle = \langle \Psi(b^*Ta)1_\tau, 1_\tau \rangle = \langle b^*Ta1_\tau, 1_\tau \rangle = \langle Ta1_\tau, b1_\tau \rangle$$

and hence $\Psi(T) = T$ $\square$

Theorem 3.5.1 should be compared with Theorem 3.2 in [CP13] where this is established for crossed products of commutative Poisson boundaries.

Corollary 3.5.2. *Suppose M_i is a II_1 factor for $i \in \{1, 2\}$, and φ_i be regular, generating hyperstates (for M_i on $B(L^2M_i)$). Then,*

$$\mathrm{Har}(\mathscr{P}_{\varphi_1} \otimes \mathscr{P}_{\varphi_2}) = \mathrm{Har}(\mathscr{P}_{\varphi_1}) \overline{\otimes} \mathrm{Har}(\mathscr{P}_{\varphi_2}).$$

Proof. We clearly have $\mathrm{Har}(\mathscr{P}_{\varphi_1}) \overline{\otimes} \mathrm{Har}(\mathscr{P}_{\varphi_2}) \subset \mathrm{Har}(\mathscr{P}_{\varphi_1} \otimes \mathscr{P}_{\varphi_2})$ so we only need to show the reverse inclusion. Note that

$$(\mathscr{P}_{\varphi_1} \otimes \mathrm{id}) \circ (\mathscr{P}_{\varphi_1} \otimes \mathscr{P}_{\varphi_2}) = (\mathscr{P}_{\varphi_1} \otimes \mathscr{P}_{\varphi_2}) \circ (\mathscr{P}_{\varphi_1} \otimes \mathrm{id}),$$

hence $(\mathscr{P}_{\varphi_1} \otimes \mathrm{id})_{|\mathrm{Har}(\mathscr{P}_{\varphi_1} \otimes \mathscr{P}_{\varphi_2})}$ gives a normal ucp map which restricts to the identity on $M_1 \overline{\otimes} M_2$. By Theorem 3.5.1 we have that $(\mathscr{P}_{\varphi_1} \otimes \mathrm{id})_{|\mathrm{Har}(\mathscr{P}_{\varphi_1} \otimes \mathscr{P}_{\varphi_2})}$ is the identity map and hence

$$\mathrm{Har}(\mathscr{P}_{\varphi_1} \otimes \mathscr{P}_{\varphi_2}) \subset \mathrm{Har}(\mathscr{P}_{\varphi_1} \otimes \mathrm{id}) = \mathrm{Har}(\mathscr{P}_{\varphi_1}) \overline{\otimes} \mathscr{B}(L^2M_2).$$

We similarly have

$$\mathrm{Har}(\mathscr{P}_{\varphi_1} \otimes \mathscr{P}_{\varphi_2}) \subset \mathscr{B}(L^2M_1) \overline{\otimes} \mathrm{Har}(\mathscr{P}_{\varphi_2}).$$

Since $\mathrm{Har}(\mathscr{P}_{\varphi_1})$ is injective it is semidiscrete [Con76a], and hence has property S_σ of Kraus

[Kra91, Theorem 1.9]. We then have

$$\mathrm{Har}(\phi_1 \otimes \phi_2) \subset (\mathrm{Har}(\mathscr{P}_{\varphi_1})\overline{\otimes}\mathscr{B}(L^2M_2)) \cap (\mathscr{B}(L^2M_1)\overline{\otimes}\mathrm{Har}(\mathscr{P}_{\varphi_2})) \subset \mathrm{Har}(\mathscr{P}_{\varphi_1})\overline{\otimes}\mathrm{Har}(\mathscr{P}_{\varphi_2}).$$

□

Corollary 3.5.3. *Let M be a II_1 factor, and let φ be a regular, generating hyperstate. Then, M is a maximal type II_1 factor inside $\mathscr{B}_\varphi$.*

Proof. Suppose $N \subset \mathscr{B}_\varphi$ is a type II_1 factor containing M. Then there exists a normal conditional expectation $E : N \to M$. Hence, by theorem 3.5.1, $E(n) = n$ for all $n \in N$, and hence $N = M$. □

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