A DETERMINATION OF THE EXISTENCE OF VARIOUS TYPES OF POSITIVE SYSTEMS IN L^p

By

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To my parents, Miriam and Jim Spaeth,

and my brother Hans Spaeth,

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and to my guinea pigs, Alfred, Walter, and Ernie,

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CHAPTER 1

INTRODUCTION

The main theme of this work is motivated by the following two observations:

Observation 1.0.1. *There exists a system* $\{f_n\}_{n=1}^{\infty}$*, which is complete in* $L^2(\mathbb{T})$*, with the property that for each n,* $f_n(x) \geq 0$ *for a.e.* $x \in \mathbb{T}$*.*

Namely, it is well known that taking characteristic functions of the dyadic subintervals of $\mathbb{T} = [0,1]$ provides one such example.

□

We shall refer to a system with the property that each function is a.e. non-negative, as a positive system.

Observation 1.0.2. *There does not exist an O.N.B.,* $\{f_n\}_{n=1}^{\infty}$ *for* $L^2(\mathbb{T})$ *which is a positive system.*

Proof.

Suppose that there does exist such a system. Then for $n \neq m$,

$$\int f_n f_m dx = 0,$$

but since the f_n are non-negative a.e., and do not have norm zero, we must conclude that for $n \neq m$, $\mu(supp(f_n) \cap supp(f_m)) = 0$, where μ denotes Lebesgue measure. To obtain a contradiction to completeness of the system, simply choose a set $G \subseteq supp(f_1)$, where $\mu(G) = \frac{1}{2}\mu(supp(f_1))$, and consider χ_G. This is a non-zero function which cannot be expressed in-terms of our O.N.B., contradicting completeness. □

Not only may one ask what happens for systems that lie "between" these observations, but the question can also be extended to general spaces $L^p(\mathbb{T})$. The chart below summarizes the results contained in this dissertation:

Table 1.1: Summary of Results

Positive System Type	$L^p(\mathbb{T})$	**Existence**
Unconditional Schauder Basis	$1 \leq p < \infty$	No
Monotone Basis	$1 < p < \infty$	No
Unconditional Quasibasis	$1 \leq p < \infty$	No
Conditional Quasibasis	$1 \leq p < \infty$	Yes
Conditional Pseudobasis	$1 \leq p < \infty$	Yes
Exact System with Exact Dual System	$1 < p < \infty$	Yes
Exact System	$1 \leq p < \infty$	Yes
Hamel Basis	$0 < p \leq \infty$	Yes
Frame	$p = 2$	No
Orthonormal Basis	$p = 2$	No
Riesz Basis	$p = 2$	No

In Chapter 2, we attempt to provide, within reason, all necessary definitions, and notations used throughout this work, as well as an explicit definition of the Rademacher, Walsh, and Haar systems on $\mathbb{T}$. There, we also inform the reader of some necessary background facts regarding spaces in which these systems are complete, or are bases, for example.

Chapter 3 is split into relevant subsections, addressing various types of generalized bases. In Section 3.1, we relate the generalized notion of perpendicularity to monotone bases, and use this to show that positive monotone bases do not exist in $L^P(\mathbb{T})$ for $1 < p < \infty$. In Section 3.2, we rely heavily upon Khinchine's Inequalities for the Rademacher system, as well as stability results for unconditional bases. Khinchine's inequalities help us establish more general inequalities, which we may apply to our work with other systems, including unconditional bases, Riesz bases, and frames.

Section 3.4 contains, in some sense, the most "difficult" and general results in this work, in which we show that the collection of dyadic characteristic functions forms a positive conditional quasibasis for the spaces $L^p(\mathbb{T})$, for $1 \leq p < \infty$, as well as the fact that any quasibasis of dyadic step functions with positive coefficients must be conditional in $L^p(\mathbb{T})$ for $1 \leq p < \infty$. The other main tool we develop in this section is a stability theorem for quasibases, which aids us in demonstrating the non-existence of positive unconditional quasibases.

In Section 3.5, the proof of the existence of positive Hamel bases for $0 < p \leq \infty$ is extremely different in flavor than the rest of the work, making use of Zorn's Lemma.

Chapter 4 contains some proofs of basic properties of the Walsh system, and some properties of windowed Walsh systems with deletions. There, we show the existence of positive exact systems with positive dual systems for $L^p(\mathbb{T})$, where $1 < p < \infty$. We also give a positive exact system in $L^1(\mathbb{T})$.

In Chapter 5, Section 5.1 contains results of the translation of completeness and minimality to product systems. Following, in Section 5.2, we prove similar results to those in Chapter 4, but for the product Walsh system defined on $\mathbb{T}^2$.

In Chapter 6, our results are not immediately related to the positive system question. However, results regarding windowed exponential systems on $\mathbb{T}^2$ are related in flavor to the explorations of the windowed Walsh system in Chapter 4.

As a final note for completeness, we address the issue of unconditional pseudobases. Though explicit details are not provided in either paper, it is claimed by Kazarian and Zink in [1] that the work of Ul'yanov in [2], demonstrates the existence of an unconditional positive pseudobasis for $L^p(\mathbb{T})$, specifically, the Schauder system.

CHAPTER 2

PRELIMINARIES AND NOTATIONS

2.1 Notations

- Generally when we write $:=$ as opposed to $=$, we are defining an equality of two items for the first time.
- In the following $\mu(A)$ will denote the Lebesgue measure of the set, $A \subseteq \mathbb{R}^d$, where d will be given in context.
- The notation $\overline{A}$ will denote the closure of a set, A.
- Given a Banach space, X, X^* will represent the dual space of all bounded linear functionals on X.
- Given x in a Banach space, X, and some $a \in X^*$, $\langle x, a \rangle$ will usually be written in place of $a(x)$. This is in an attempt to avoid confusion between writing $a(x)$, meaning that a is a constant, depending on x, and $a(x)$ interpreted as a functional, evaluated at x.
- It will be understood that $\mathbb{T} = [0,1]$, and that $\mathbb{T}^2 = [0,1] \times [0,1]$, in what follows.
- For $f \in L^p(E)$, $g \in L^q(E)$, with $\frac{1}{p} + \frac{1}{q} = 1$, we understand that

$$\langle f, g \rangle = \int_E f\overline{g}.$$

- Where confusion is possible, we denote the zero element of L^p by $[0]$.

- We will use $supp(f)$ to denote the support of f - that is, set of x for which $f(x) \neq 0$.
- Given a sequence $\{x_n\}_{n=1}^{\infty}$, in a vector space, $\text{span}(\{x_n\}_{n=1}^{\infty})$ will denote the set of all finite linear combinations of elements in $\{x_n\}_{n=1}^{\infty}$.

Definition 2.1.1 (Positive System)**.** We shall say that a sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^p$ is a positive system, if each f_n is almost everywhere non-negative.

Definition 2.1.2 (Dyadic Characteristic Function)**.** In the following, we will use the notation:

$$\chi_{k,N}(t) := \chi_{[k2^{-N},(k+1)2^{-N}]}(t),$$

and refer to $\chi_{k,N}$ as a dyadic characteristic function.

Definition 2.1.3 (Dyadic Step Function)**.** Letting $c_i \in \mathbb{R}$ or $\mathbb{C}$, we call a function of the form

$$D(t) = \sum_{i=1}^{M} c_i \chi_{k_i,N_i}(t),$$

a dyadic step function, or a dyadic simple function.

Definition 2.1.4 (Dyadic Characteristic Function in Two-Dimensions)**.** We define the two dimensional dyadic characteristic function as follows:

$$\chi_{j,k,M}(x,y) := \chi_{j,M}(x)\chi_{k,M}(y).$$

2.2 Generalized Bases

Definition 2.2.1 (Basis for a Banach Space)**.** We say that $\{x_n\}_{n=1}^{\infty}$ is a basis for a Banach space, X if for all $x \in X$, there exist unique scalars $a_n(x)$, such that

$$x = \sum_{n=1}^{\infty} a_n(x) x_n := \lim_{N\to\infty} \sum_{n=1}^{N} a_n(x) x_n.$$

It follows from the uniqueness of the scalars in the basis representation, that these constants define linear functionals on X, and that there is only one such sequence of linear functionals satisfying the above expression. This merits the following definition:

Definition 2.2.2 (Sequence of Coefficient Functionals)**.** Given a basis $\{x_n\}_{n=1}^{\infty}$ for a Banach space, X, where the $a_n(x)$ are as in Definition 2.2.1, we say that the sequence of linear functionals $\{a_n\}_{n=1}^{\infty}$ is the sequence of coefficient functionals associated with the sequence $\{x_n\}_{n=1}^{\infty}$.

Definition 2.2.3 (Schauder Basis)**.** Given a basis $\{x_n\}_{n=1}^{\infty}$ for a Banach space, X, we say that $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis for X if the coefficient functionals associated with $\{x_n\}_{n=1}^{\infty}$ are continuous.

It is a well established fact that every basis for a Banach space is a Schauder basis. See, for example Theorem 4.13 in, [3].

Definition 2.2.4 (Pseudobasis)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a system in a Banach space, X. If for each $x \in X$, there exists a sequence of scalars, $\{c_n\}_{n=1}^{\infty}$, such that:

$$x = \sum_{n=1}^{\infty} c_n x_n,$$

we say that $\{x_n\}_{n=1}^{\infty}$ is a pseudobasis.

Note that uniqueness of the scalars is not required here, as it was in the definition of basis.

Definition 2.2.5 (Quasibasis)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a Banach space, X. If there exists a dual sequence, or dual system, $\{a_n\}_{n=1}^{\infty} \subseteq X^*$, such that for all $x \in X$:

$$x = \sum_{n=1}^{\infty} \langle x, a_n \rangle x_n,$$

then we say that $\{x_n\}_{n=1}^{\infty}$ is a quasibasis.

Note that uniqueness of the dual system is not required, in opposition to Schauder bases, which have a unique dual system.

Definition 2.2.6 (Partial Sum Operators)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a basis for a Banach space, X, with associated coefficient functionals $\{a_n\}_{n=1}^{\infty}$. We define the partial sum operators, S_N associated with $\{x_n\}_{n=1}^{\infty}$ as follows:

$$S_N(x) := \sum_{n=1}^{N} a_n(x)x_n.$$

Notice that it follows immediately from the continuity of the a_n that each partial sum operator is also continuous, and in fact S_N is continuous for each N if and only if a_n is continuous for each n, [3].

Definition 2.2.7 (Basis Constant)**.** If $\{x_n\}_{n=1}^{\infty}$ is a basis for a Banach space, X, we call $\mathscr{C} = \sup_N \|S_N\|$ the basis constant of $\{x_n\}_{n=1}^{\infty}$.

It can be shown that $1 \leq \mathscr{C} = \sup_N \|S_N\| < \infty$, [3]. The finiteness of this supremum follows from the Uniform Boundedness Principle.

Definition 2.2.8 (Monotone Basis)**.** A basis $\{x_n\}_{n=1}^{\infty}$ with basis constant, $\mathscr{C} = 1$, is said to be a monotone basis.

Definition 2.2.9 (Unconditional Convergence)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a Banach space,

X. We say that

$$\sum_{n=1}^{\infty} x_n$$

is unconditionally convergent if

$$\sum_{\sigma(n)} x_n$$

converges for every permutation $\sigma(n)$ of $\mathbb{N}$. Alternatively, we may use the notation $\sum_{n\in\mathbb{N}} x_{\sigma(n)}$.

Definition 2.2.10 (Unconditional Basis)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a basis for a Banach space, X. We say that $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis for X if for each $x \in X$, the series

$$x = \sum_{n=1}^{\infty} a_n(x) x_n,$$

converges unconditionally.

Note for an unconditionally convergent series $\sum_{n=1}^{\infty} c_n x_n$ in a Banach space, X, given some permutation $\sigma(n)$ of $\mathbb{N}$, $\sum_{n=1}^{\infty} c_n x_n = \sum_{\sigma(n)} c_n x_n$. For a proof of this fact, see Corollary 3.11 in, [3].

Definition 2.2.11 (Unconditional Quasibasis)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a quasibasis for a Banach space, X. We say that $\{x_n\}_{n=1}^{\infty}$ is an unconditional quasibasis for X if there exists a dual system $\{a_n\}_{n=1}^{\infty} \subseteq X^*$, such that for each $x \in X$, the series

$$x = \sum_{n=1}^{\infty} \langle x, a_n \rangle x_n,$$

converges unconditionally.

Definition 2.2.12 (Frame)**.** Let H be a Hilbert space. We say that a sequence $\{f_n\}_{n=1}^{\infty}$ is a frame for H if there exist constants $A, B > 0$, such that for all $f \in H$:

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2.$$

Definition 2.2.13 (Topological Isomorphism)**.** Let X, and Y be normed linear spaces. Then $T : X \to Y$ is a topological isomorphism, if T is a bijection, and both T and T^{-1} are continouous.

Definition 2.2.14 (Riesz Basis)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a Hilbert space, H. Then $\{x_n\}_{n=1}^{\infty}$ is a Riesz basis if it is equivalent to some orthonormal basis, $\{e_n\}_{n=1}^{\infty}$ for H, that is, if there exists a topological isomorphism $T : H \to H$, such that $T(x_n) = e_n$ for all $n \in \mathbb{N}$.

Definition 2.2.15 (Hamel Basis)**.** Let V be a vector space. We say that $\{x_\gamma\}_{\gamma\in\Lambda}$ is a Hamel basis for V if:

1. V is equal to the finite linear span of $\{x_\gamma\}_{\gamma\in\Lambda}$.

2. $\{x_\gamma\}_{\gamma\in\Lambda}$ is finitely linearly independent.

Definition 2.2.16 (Perpendicularity)**.** Let X be a Banach space. We say that f is perpendicular to g, and write $f \perp g$, if for all scalars λ, $\|f\| \leq \|f + \lambda g\|$.

We note that we have taken this generalized notion from, [4]. We will later give an example that it is not true in general that $f \perp g \implies g \perp f$. We specifically give a counterexample to this fact for $L^1(\mathbb{T})$, in Lemma 3.1.5 .

Definition 2.2.17 (Adjoint)**.** Given Banach spaces, X and Y, and a bounded linear operator, $T : X \to Y$, the unique operator, $T^* : Y^* \to X^*$, which is a bounded linear operator, and satisfies:

$$\forall\, x \in X,\ \forall\, y^* \in Y^*,\ \langle Tx, y^*\rangle = \langle x, T^*y^*\rangle\,,$$

is called the adjoint of T.

That this definition makes sense follows from Exercise 2.9, [3]. Additionally, $\|T^*\| = \|T\|$, which is a consequence of the Hahn-Banach Theorem, see page 62, [3].

2.3 Complete and Minimal Systems

Definition 2.3.1 (Complete System)**.** A sequence $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$ in a Banach space, X is said to be complete if $\overline{\text{span}}(\mathbf{x}) = X$.

Note that complete systems and pseudobases are not equivalent notions. Each pseudobasis is a complete system, but the converse is not true. We refer to Example 1.29 in, [3]. Heil considers the Banach space, $C[a,b]$, of continuous functions on the compact interval $[a,b]$, under the norm $\|f\| := \sup_{t\in[a,b]} |f(t)|$. Continuous functions can be approximated by polynomials under the sup norm, and so are in the closure of the finite linear span of the set of polynomials. However, not every $f \in C[a,b]$ has a power series which converges on $[a,b]$.

Definition 2.3.2 (Minimal System)**.** A system $\{x_n\}_{n=1}^{\infty}$ in a Banach space, X, is said to be minimal if for all $m \in \mathbb{N}$,

$$x_m \notin \overline{\text{span}}\{x_n\}_{n\neq m}.$$

Definition 2.3.3 (Biorthogonal System)**.** Given a system $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$ in a Banach space X, we say that $\{a_n\}_{n=1}^{\infty} \subseteq X^*$ is a biorthogonal system to $\mathbf{x}$, if $\langle x_n, a_m\rangle = \delta_{m,n}$, where $\delta_{m,n} = 0$ if $n \neq m$, and $\delta_{m,n} = 1$ if $m = n$.

Definition 2.3.4 (Exact System)**.** A system $\{x_n\}_{n=1}^{\infty}$ in a Banach space, X, is an exact system if it is both complete and minimal.

The following lemmas give equivalent statements to completeness, minimality, and exactness, which we will use freely in this work. While proofs of the following facts are given in,

[3], either explicitly, or as exercises, they are important enough to merit their inclusion here.

Lemma 2.3.5 (Equivalent Notion to Completeness)**.** *A system $\{x_n\}_{n=1}^{\infty}$ in a Banach space, X is complete if and only if, given $x^* \in X^*$, if $\langle x_n, x^* \rangle = 0$ for all $n \in \mathbb{N}$, then $x^* = 0$.*

Proof.

First suppose that $\{x_n\}_{n=1}^{\infty}$ is complete in X. Let $x^* \in X^*$, and suppose that $\langle x_n, x^* \rangle = 0$ for all $n \in \mathbb{N}$. Because x^* is linear, given $y = \sum_{j=1}^{J} c_j x_{n_j} \in \operatorname{span}\{x_n\}_{n=1}^{\infty}$,

$$\langle y, x^* \rangle = \left\langle \sum_{j=1}^{J} c_j x_{n_j}, x^* \right\rangle = \sum_{j=1}^{J} c_j \left\langle x_{n_j}, x^* \right\rangle = 0.$$

Given $x \in X$, $x \in \overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty}$, and so there is a sequence $\{y_k\}_{k=1}^{\infty} \subseteq \operatorname{span}\{x_n\}_{n=1}^{\infty}$, converging to x. Since x^* is continuous, and $\langle y_k, x^* \rangle = 0$ for all $k \in \mathbb{N}$, it must be that $\lim_{k\to\infty} \langle y_k, x^* \rangle = \langle \lim_{k\to\infty} y_k, x^* \rangle = \langle x, x^* \rangle = 0$, hence $x^* = 0$.

Suppose that if $x^* \in X^*$, and $\langle x_n, x^* \rangle = 0$, for all $n \in \mathbb{N}$, then $x^* = 0$. Proving by contradiction, suppose that $\{x_n\}_{n=1}^{\infty}$ is not complete, so that $\overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty} \neq X$. Therefore, there exists some $x_0 \in X$, where $x_0 \notin \overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty}$. By the Hahn-Banach theorem, there exists some $y^* \in X^*$, such that $\langle x_0, y^* \rangle = 1$, and for all $x \in \overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty}$, $\langle x, y^* \rangle = 0$. But then, in particular, $\langle x_n, y^* \rangle = 0$ for all $n \in \mathbb{N}$. By hypothesis, it must be that $y^* = 0$, contradicting the fact that $\langle x_0, y \rangle = 1$. Hence, $\{x_n\}_{n=1}^{\infty}$ must be complete in X.

□

Lemma 2.3.6 (Equivalent Notion to Biorthogonality)**.** *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a Banach space, X. Then $\{x_n\}_{n=1}^{\infty}$ is minimal if and only if it has a biorthogonal system, $\{a_n\}_{n=1}^{\infty} \subseteq X^*$.*

Proof.

First suppose that $\{x_n\}_{n=1}^{\infty}$ is minimal. Fix $x_{n_0} \in \{x_n\}_{n=1}^{\infty}$. Then by definition of minimal, x_{n_0} is not in the closed subspace, $\overline{\operatorname{span}}\{x_n\}_{n=1, n\neq n_0}^{\infty}$. Employing the Hahn-Banach theorem, there exists some $a_{n_0} \in X^*$, such that $\langle x_{n_0}, a_{n_0} \rangle = 1$, and for all $x \in \overline{\operatorname{span}}\{x_n\}_{n=1, n\neq n_0}^{\infty}$, $\langle x, a_{n_0} \rangle = 0$.

Hence, the sequence, $\{a_n\}_{n=1}^{\infty}$, of elements of X^* found in this way form a biorthogonal system to $\{x_n\}_{n=1}^{\infty}$.

Now suppose that $\{x_n\}_{n=1}^{\infty}$ has a biorthogonal system $\{a_n\}_{n=1}^{\infty} \subseteq X^*$. Fix $n_0 \in \mathbb{N}$. Let $y \in \operatorname{span}\{x_n\}_{n=1,n\neq n_0}^{\infty}$, so for some constants, c_j:

$$\langle y, a_{n_0}\rangle = \left\langle \sum_{j=1}^{J} c_j x_{n_j}, a_{n_0} \right\rangle = \sum_{j=1}^{J} c_j \left\langle x_{n_j}, a_{n_0}\right\rangle = 0,$$

since each $n_j \in \mathbb{N} \setminus \{n_0\}$, for $j = 1, \cdots, J$. Hence, $a_{n_0} \equiv 0$ on $\operatorname{span}\{x_n\}_{n=1,n\neq n_0}^{\infty}$. Because a_{n_0} is continuous, it follows that $\langle y, a_{n_0}\rangle = 0$ for all $y \in \overline{\operatorname{span}}\{x_n\}_{n=1,n\neq n_0}^{\infty}$. Since $\langle x_{n_0}, a_{n_0}\rangle = 1$, it must be the case then that $x_{n_0} \notin \overline{\operatorname{span}}\{x_n\}_{n=1,n\neq n_0}^{\infty}$, and so $\{x_n\}_{n=1}^{\infty}$ is minimal.

$\square$

Lemma 2.3.7 (Equivalent Notion to Exactness)**.** *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a Banach space, X. Then $\{x_n\}_{n=1}^{\infty}$ is exact if and only if it has a unique biorthogonal system $\{a_n\}_{n=1}^{\infty} \subseteq X^*$.*

Proof.

First, suppose that $\{x_n\}_{n=1}^{\infty}$ is exact. Then $\{x_n\}_{n=1}^{\infty}$ is complete and minimal. By Lemma 2.3.6, $\{x_n\}_{n=1}^{\infty}$ has a biorthogonal system, $\{a_n\}_{n=1}^{\infty} \subseteq X^*$. Suppose that $\{b_n\}_{n=1}^{\infty} \subseteq X^*$ is another biorthogonal system. It is clear that $a_m = b_m$ on $\operatorname{span}\{x_n\}_{n=1}^{\infty}$, for all $m \in \mathbb{N}$. Now, let $y \in X$. Since $\{x_n\}_{n=1}^{\infty}$ is complete, $y \in \overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty}$. Hence, there exists a sequence $\{y_j\}_{j=1}^{\infty} \subseteq \operatorname{span}\{x_n\}_{n=1}^{\infty}$, converging to y. But then:

$$\langle y, a_m - b_m\rangle = \lim_{j\to\infty} \left\langle y_j, a_m - b_m\right\rangle = 0,$$

and so $a_m(y) = b_m(y)$. Hence, $a_m \equiv b_m$ on $\overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty} = X$. Since this holds for all $m \in \mathbb{N}$, $\{a_n\}_{n=1}^{\infty} = \{b_n\}_{n=1}^{\infty}$, and the biorthogonal system must be unique.

Now suppose that $\{x_n\}_{n=1}^{\infty}$ has a unique biorthogonal system. In order to show that $\{x_n\}_{n=1}^{\infty}$

is exact, since $\{x_n\}_{n=1}^{\infty}$ is minimal, it remains to show completeness in X. Proving by contradiction, suppose that $\{x_n\}_{n=1}^{\infty}$ is not complete in X. Then there is some $y \in X$, where $y \notin \overline{\text{span}}\{x_n\}_{n=1}^{\infty}$. Since $\overline{\text{span}}\{x_n\}_{n=1}^{\infty}$ is a closed subspace of X, of which y is not an element, by the Hahn-Banach theorem, there is some $a \in X^*$ such that $\langle y, a\rangle = 1$, and $\langle x, a\rangle = 0$ for all $x \in \overline{\text{span}}\{x_n\}_{n=1}^{\infty}$. Let $b_n = a_n + a$ for $n \in \mathbb{N}$, noting that $b_n \in X^*$. Note also that $a_n \neq a_n + a$, and that given $n, m \in \mathbb{N}$:

$$\langle x_n, b_m\rangle = \langle x_n, a_m\rangle + \langle x_n, a\rangle = \delta_{m,n} + 0 = \delta_{m,n}.$$

But then $\{b_n\}_{n=1}^{\infty}$ is a biorthogonal system to $\{x_n\}_{n=1}^{\infty}$, not equal to $\{a_n\}_{n=1}^{\infty}$, contradicting the uniqueness of the biorthogonal system. Thus, it must be that $\{x_n\}_{n=1}^{\infty}$ is complete.

$\square$

It is interesting to note that given a Schauder basis, B, for a Banach space, X, the sequence of associated coefficient functionals forms a biorthogonal sequence for B, since by definition, the coefficient functionals of a Schauder basis are bounded. Hence, every Schauder basis is minimal. Clearly, a Schauder basis is complete, and so we have that every Schauder basis is an exact system.

To clarify the necessity of the discussion of exact systems, we note that one might be tempted to conjecture that an exact system whose dual system is also exact must be a Schauder basis. In fact, this is not the case, even in the particularly nice situation of Hilbert spaces, which are reflexive. We provide one such counterexample to that conjecture, as presented in [5] and [3]. Let $e_n(x) := e^{2\pi i n x}$. Note that $\{e_n\}_{n\in\mathbb{Z}}$ is a Schauder basis for $L^2(\mathbb{T})$, since it is an orthonormal basis.

Theorem 2.3.8. *The sequence $\{xe_n(x)\}_{n\in\mathbb{Z}, n\neq 0}$ is not a Schauder basis, but it is an exact system in $L^2(\mathbb{T})$,with biorthogonal sequence $\{\tilde{e}_n\}_{n\in\mathbb{Z},n\neq 0} := \left\{\frac{e_n - 1}{x}\right\}_{n\in\mathbb{Z},n\neq 0}$, which is also exact in $L^2(\mathbb{T})$.*

Proof.

First we demonstrate that $\tilde{e}_n \in L^2(\mathbb{T})$, for $n \neq 0$:

$$\|\tilde{e}_n\|_2 = \int_0^1 \left| \frac{e^{2\pi inx} - 1}{x} \right|^2 dx = \int_0^1 \frac{2 - 2\cos(2\pi nx)}{x^2} dx,$$

using integration by parts, and simple substitution,

$$= 4\pi n \int_0^{2\pi n} \frac{\sin(u)}{u} du < \infty. \tag{2.1}$$

Hence, $\tilde{e}_n \in L^2(\mathbb{T})$. Computing for $n, m \in \mathbb{Z} \setminus \{0\}$:

$$\langle xe_n, \tilde{e}_m \rangle = \langle e_n, e_m \rangle - \langle e_n, 1 \rangle = \delta_{n,m},$$

which demonstrates the biorthogonality of $\{\tilde{e}_n\}_{n \in \mathbb{Z}, n \neq 0}$, and thus the minimality of both systems, since we are working in $L^2(\mathbb{T})$.

To show completeness of $\{xe_n\}_{n \in \mathbb{Z}, n \neq 0}$, let $f \in L^2(\mathbb{T})$, and suppose that $\langle xe_n, f \rangle = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $0 = \langle xe_n, f \rangle = \langle e_n, xf \rangle$, where $xf \in L^2(\mathbb{T})$, since x is bounded on $\mathbb{T}$. These inner products are the Fourier coefficients of xf, and so $xf = c$, for some constant, c. But then, $f = \frac{c}{x} \in L^2(\mathbb{T})$, and so $c = 0$. Hence, $f = 0$. Therefore the system $\{xe_n\}_{n \in \mathbb{Z}, n \neq 0}$ is complete, and we have shown that it is exact.

We now show that $\{\tilde{e}_n\}_{n \in \mathbb{Z}, n \neq 0}$ is complete. Suppose that $h \in L^2(\mathbb{T})$, and $\langle \tilde{e}_n, h \rangle = 0$ for all $n \in \mathbb{Z}, n \neq 0$. Define $\tilde{h}(x) := h(x) \cdot \frac{e_1 - 1}{x}$, and note that $\tilde{h}(x) \in L^2(\mathbb{T})$, since by Hölder's inequality:

$$\int_0^1 \left| h \frac{e_1 - 1}{x} \right|^2 dx \leq \|h^2\|_1 \left\| \left(\frac{e_1 - 1}{x} \right)^2 \right\|_\infty = \|h\|_2^{\frac{1}{2}} \left\| \left(\frac{e_1 - 1}{x} \right)^2 \right\|_\infty,$$

where the infinity norm is finite, since $\frac{e_1 - 1}{x}$ has a finite limit as $x \to 0$, and is continuous elsewhere on $\mathbb{T}$. Defining $g_0 := \frac{e^{2\pi i \cdot 0} - 1}{x} = 0$, we have that $\langle g_0, h \rangle = 0$.

Computing, for $m \in \mathbb{Z}$,

$$\left\langle e_m, \tilde{h} \right\rangle = \left\langle \frac{e_{-1}e_m - e_m}{x}, h \right\rangle = \left\langle \frac{e_{m-1} - 1 + 1 - e_m}{x}, h \right\rangle = \langle \tilde{e}_{m-1}, h \rangle - \langle \tilde{e}_m, h \rangle = 0.$$

Since all the Fourier coefficients of $\tilde{h}$ are zero, $\tilde{h} = 0$, which yields $h = 0$. Hence, the system $\{\tilde{e}_n\}_{n \in \mathbb{Z}, n \neq 0}$ is complete, and since it is minimal, it is also exact.

It remains to demonstrate that $\{xe_n\}_{n \in \mathbb{X}, n \neq 0}$ is not a Schauder basis. To do so, we first examine the value of $\|\tilde{e}_n\|_2^2$ for $n > 0$. Notice that for $j = 1, 2 \cdots, n$, $\int_{2\pi(j-1)}^{2\pi\left(j-\frac{1}{2}\right)} \frac{\sin(u)}{u} du > 0$, and in fact there, $\frac{\sin(u)}{u} > 0$. Also, $\int_{2\pi\left(j-\frac{1}{2}\right)}^{2\pi j} \frac{\sin(u)}{u} du < 0$, and in fact there, $\frac{\sin(u)}{u} < 0$. Finally, it is easy to see that for $j \geq 1$:

$$\int_{2\pi(j-1)}^{2\pi\left(j-\frac{1}{2}\right)} \left| \frac{\sin(u)}{u} \right| dx > \int_{2\pi\left(j-\frac{1}{2}\right)}^{2\pi j} \left| \frac{\sin(u)}{u} \right| du.$$

Therefore, we can see that the integrals $\int_0^{2\pi n} \frac{\sin(u)}{u} du$ are strictly increasing with n, and so, using 2.1,

$$\lim_{n \to \infty} \|\tilde{e}_n\|_2 = \lim_{n \to \infty} \left(4\pi n \int_0^{2\pi n} \frac{\sin(u)}{u} du \right) = \infty \tag{2.2}$$

since the norms have the value given above in (2.1). However, the original system, $\{xe_n\}_{n \in \mathbb{Z}, n \neq 0}$ is bounded above in norm. Were $\{xe_n\}_{n \in \mathbb{Z}, n \neq 0}$ to be a Schauder Basis for some ordering, then it would have a finite basis constant, $\mathscr{C}$, and the biorthogonal system would act as the system of coefficient functionals. But then, letting S_n denote partial sums, where we now index our sequences by $\mathbb{N}$, for $g \in L^2(\mathbb{T})$, and $n \geq 2$:

$$\langle g, \tilde{e}_n \rangle xe_n = S_n g - S_{n-1} g,$$

and so

$$|\langle g, \tilde{e}_n \rangle| \, \|xe_n\|_2 = \| \langle g, \tilde{e}_n \rangle \tilde{e}_n \|_2 \leq \|S_n g\| + \|S_{n-1} g\| \leq 2\mathscr{C} \|g\|_2.$$

For $n = 1$:

$$|\langle g, \tilde{e}_1 \rangle| \, xe_1 = S_1 g,$$

and so

$$|\langle g, \tilde{e}_1 \rangle| \, \|xe_1\|_2 = \|S_1 g\| \leq 2\mathscr{C} \|g\|_2.$$

Since $\|xe_n\|_2 = \frac{1}{3}$ for all n, this implies that for all $\tilde{e}_n$ in our system:

$$|\langle g, \tilde{e}_n \rangle| \leq 6\mathscr{C} \|g\|_2.$$

But this yields that $\tilde{e}_n$ must be uniformly bounded in norm, which contradicts (2.2). Hence, we can see by a similar argument, it must be that $\{xe_n\}_{n \in \mathbb{Z}, n \neq 0}$ is not a Schauder basis for any ordering.

$\square$

2.4 The Rademacher System

Definition 2.4.1 (Rademacher System)**.** Define the Rademacher system, $\{R_n\}_{n=0}^{\infty}$ on $\mathbb{T}$ by:

$$R_n(t) := sign(\sin(2^n \pi t)),$$

where we take $sign(0) = 0$.

An alternative way to define the Rademacher System is as follows.

For $n = 0$:

$$R_0(t) := \begin{cases} 1, & 0 < t < 1 \\ 0, & t \in \{0,1\} \end{cases}.$$

For $n \in \mathbb{N}$, and $k \in \mathbb{N}$ with $1 \leq k \leq 2^{n-1}$:

$$R_n(t) = \begin{cases} 1, & t \in \left(\frac{2(k-1)}{2^n}, \frac{2k-1}{2^n}\right) \\ -1, & t \in \left(\frac{2k-1}{2^n}, \frac{2k}{2^n}\right) \\ 0, & t \in \{\frac{j}{2^n} \,|\, 0 \leq j \leq 2^n\} \end{cases}.$$

2.5 The Walsh System

Definition 2.5.1 (The Walsh System)**.** We define the Walsh system, $\{w_n\}_{n=1}^{\infty}$ on $\mathbb{T}$ in-terms of the Rademacher system as follows:

$$w_1(x) \equiv 1$$

For $k = 1, 2, \cdots$, we define:

$$w_{k+1}(x) := R_{n_1+1}(x) \cdot R_{n_2+1}(x) \cdot \cdots \cdot R_{n_\nu+1}(x),$$

where $k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_\nu}$, and $n_1 > n_2 > \cdots > n_\nu \geq 0$.

Notice that the entire Rademacher system itself is contained as a subset of the Walsh system, and it is easy to show that both systems are orthonormal systems in $L^2(\mathbb{T})$, see for instance, [6]. Singer shows completeness of the Walsh system for $1 \leq p < \infty$, using a dimension argument, through the expression of Walsh functions as finite linear combinations of Haar functions, which are complete in those spaces - see pages 399, and 405 in, [6]. In particular, the Walsh system is a complete, orthonormal system in $L^2(\mathbb{T})$, and so it is an O.N.B. there. Finally, Lemma 4.1.2 gives an explicit way that we may write dyadic characteristic functions in terms finitely many Walsh functions. For some further interesting properties of the Walsh system, and Walsh product system on $\mathbb{T}^2$, beyond the scope of what is necessary in this work, see for instance [7], [8], [9], and [10].

2.6 The Haar System

We define the Haar System as follows:

Definition 2.6.1 (The Haar System)**.** For $k = 0, 1, \cdots$, and $j = 1, 2, \cdots, 2^k$, define:

$$h_{k,j}(t) = \begin{cases} 2^{\frac{k}{2}}, & \text{if } \frac{2j-2}{2^{k+1}} < t < \frac{2j-1}{2^{k+1}} \\ -2^{\frac{k}{2}} & \text{if } \frac{2j-1}{2^{k+1}} < t < \frac{2j}{2^{k+1}} \\ 0, & \text{elsewhere on } [0,1] \end{cases}.$$

For $n = 1$, define:

$$h_1(t) \equiv 1.$$

Now, for $n = 2^k + j$, where $k = 0, 1, \cdots$,and $j = 1, 2, \cdots, 2^k$, define the n^{th} Haar function to be: $h_n(t) := h_{k,j}(t)$. Hence, we denote the Haar system by $\{h_n\}_{n=1}^{\infty}$.

An intuitive proof of the fact that the Haar system forms a basis for $1 \leq p < \infty$ (and in fact that it is a monotone basis there) is given on page 168 in, [3]. Though we will not need its unconditionality, it is a well established fact that $h_n(t)$ is an unconditional Schauder basis for $L^p(\mathbb{T})$ for $1 < p < \infty$, [6]. Alternatively, that the Haar system is a monotone basis for $L^p(\mathbb{T})$, $1 < p < \infty$, implies that it is an unconditional basis there, by the work of Dor and Odell, who show that monotone bases must be unconditional bases in the spaces with $1 < p < \infty$, [11].

CHAPTER 3

GENERALIZED BASES

3.1 Positive Monotone Bases and Perpendicularity

In this section, we first show in Lemma 3.1.1 that there is an equivalent, and perhaps more intuitive condition, to monotonicity for a basis, which is motivated by Exercise 5.1.1 in, [3], though the perpendicularity portion is not contained there. Additionally we demonstrate a generalized version of perpendicularity between certain elements of a monotone basis. The result following will demonstrate that non-negative a.e. perpendicular elements must be a.e. disjointly supported. Finally, using these results, a similar method to that in Observation 1.0.2, will show that there cannot exist a positive, monotone basis for $L^p(\mathbb{T})$, $1 < p < \infty$.

Lemma 3.1.1 (An Equivalent Condition to Monotonicity)**.** *Let the system $\{x_n\}_{n=1}^{\infty}$ be a basis in a Banach Space X. Then, $\{x_n\}_{n=1}^{\infty}$ is a monotone basis for X if and only if given $x \in X$, and $N \leq M$, $\|S_N(x)\| \leq \|S_M(x)\|$. Moreover, if $\{x_n\}_{n=1}^{\infty}$ is monotone, for $m, n \in \mathbb{N}$, with $m < n$, $x_m \perp x_n$.*

Proof.

First suppose that $\{x_n\}_{n=1}^{\infty}$ is a monotone basis for X, that is, let $\sup_N \|S_N\| = 1$. Then:

$$\|S_N(x)\| = \|S_N(S_{N+1}(x))\| \leq \|S_N\| \cdot \|S_{N+1}(x)\| \leq 1 \cdot \|S_{N+1}(x)\|,$$

where we have used the uniqueness of the coefficient functionals to yield $S_N(x) = S_N(S_{N+1}(x))$. The result clearly follows, then, for all $N, M \in \mathbb{N}$, with $N \leq M$.

Now, suppose that the basis $\{x_n\}_{n=1}^{\infty}$ has the property that given $x \in X$, and $N \leq M$, $\|S_N(x)\| \leq \|S_M(x)\|$. Then:

$$\lim_{N\to\infty} \|S_N(x)\| \leq \lim_{N\to\infty} \|S_N(x) - x\| + \|x\| = \|x\|.$$

Since the sequence $\{\|S_N(x)\|\}_{N=1}^{\infty}$ is a monotone, increasing sequence of numbers, it must hold that for all N, $\|S_N(x)\| \leq \|x\|$. So $\|S_N\| \leq 1$ for all $N \in \mathbb{N}$. Now, $\|S_1(x_1)\| = \|1 \cdot x_1\| = \|x_1\|$, using the uniqueness of coefficients, and so $\sup_N \|S_N\| = 1$. Hence, by definition, $\{x_n\}_{n=1}^{\infty}$ is a monotone basis.

It remains to show the perpendicularity property for monotone bases. Let $m < n$, and $y := \sum_{k=1}^{n} (\delta_{k,m} + \lambda \delta_{k,n}) x_k$. Hence, $S_m(y) = x_m$, and $S_n(y) = x_m + \lambda x_n$. Therefore, $\|x_m\| = \|S_m(y)\| \leq \|S_n(y)\| = \|x_m + \lambda x_n\|$. Hence, by definition, $x_m \perp x_n$.

$\square$

Lemma 3.1.2. *Let $f, g \in L^p(\mathbb{T})$, $1 < p < \infty$, and suppose that non-zero functions, $f, g \geq 0$ a.e. in $\mathbb{T}$. Then if $f \perp g$,*

$$\mu(supp(f) \cap supp(g)) = 0.$$

Proof.

Without loss of generality, suppose that $\|f\| = 1$, and $\|g\| = 1$. Suppose by contradiction that $\mu(supp f \cap supp g) > 0$. Now, define

$$T(\lambda) := \int_0^1 |f + \lambda g|^p = \|f + \lambda g\|_p^p. \tag{3.1}$$

We will contradict $f \perp g$, by showing that T has positive derivative at $\lambda = 0$.

Consider a sequence, $\{\lambda_n\}_{n=1}^{\infty}$ with $0 < |\lambda_n| < 1$, and so that either $\lambda_n > 0$ for all n, or $\lambda_n < 0$ for all n, and $\lim_{n\to\infty} \lambda_n = 0$. Directly computing the right or left hand derivative of T at 0:

$$\lim_{n\to\infty}\frac{T(\lambda_n)-T(0)}{\lambda_n}=\lim_{n\to\infty}\frac{\int_0^1|f+\lambda_n g|^p-\int_0^1|f|^p}{\lambda_n}=\lim_{n\to\infty}\int_0^1\frac{|f+\lambda_n g|^p-|f|^p}{\lambda_n}.$$

Define:

$$A_n=\{x\,|\,|\lambda_n|g\geq f>0,\text{ and }|f+\lambda_n g|^p\geq|f|^p\},$$

$$B_n=\{x\,|\,|\lambda_n|g\geq f>0,\text{ and }|f+\lambda_n g|^p<|f|^p\},$$

and

$$E_n=\{x\,|\,|\lambda_n|g\geq f>0\},$$

noting that $E_n=A_n\cup B_n$. Also, let

$$C=\{x|f=0\text{ or }g=0\},\ D=\{x|f<0\text{ or }g<0\},$$

and

$$G_n=\{x\,|\,0<|\lambda_n|g<f\}. \tag{3.2}$$

Define:

$$\phi_n:=\frac{|f+\lambda_n g|^p-|f|^p}{\lambda_n}. \tag{3.3}$$

We first determine a bounding function for $\phi_n(x)$, where $x\in A_n$ which does not depend upon n:

$$|\phi_n| = \left|\frac{|f+\lambda_n g|^p - |f|^p}{\lambda_n}\right| = \frac{|f+\lambda_n g|^p - |f|^p}{|\lambda_n|}$$

$$\leq \frac{|f+\lambda_n g|^p}{|\lambda_n|} \leq 2^p|\lambda_n|^{p-1}g^p \leq 2^p g^p \in L^1(\mathbb{T}). \tag{3.4}$$

We now find a bounding function for $\phi_n(x)$, where $x \in B_n$, which does not depend upon n:

$$|\phi_n| = \left|\frac{|f+\lambda_n g|^p - |f|^p}{\lambda_n}\right| = \frac{|f|^p - |f+\lambda_n g|^p}{|\lambda_n|}$$

$$\leq \frac{|f|^p}{|\lambda_n|} \leq \frac{|\lambda_n|^p g^p}{|\lambda_n|} \leq g^p \in L^1(\mathbb{T}). \tag{3.5}$$

Specifically, for $x \in E_n$, $\phi_n(x) \leq 2^p g^p(x)$, by (3.4) and (3.5). It is easy to see that $\lim\limits_{n\to\infty} \mu(E_n) = 0$, since the E_n decrease to a set of measure zero, and all have finite measure.

Hence by the Lebesgue dominated convergence theorem:

$$\lim_{n\to\infty}\int_{E_n}\left|\frac{|f+\lambda_n g|^p - |f|^p}{\lambda_n}\right| \leq \lim_{n\to\infty}\int_{E_n} 2^p|g|^p$$

$$= \int \lim_{n\to\infty} 2^p|g|^p \chi_{E_n} = 0. \tag{3.6}$$

For $x \in C$,

$$\left|\frac{|f+\lambda_n g|^p - |f|^p}{\lambda_n}\right| = |\lambda_n|^{p-1}|g|^p \leq |g|^p \in L^1(\mathbb{T}),$$

and so by the Lebesgue dominated convergence theorem:

$$\lim_{n\to\infty}\int_C \phi_n = \int_C \lim_{n\to\infty} |\lambda_n|^{p-1}|g|^p = 0. \tag{3.7}$$

Since $\mu(D)=0$, it is clear that $\int_D \phi_n = 0$ for all n, and so

$$\lim_{n\to\infty}\int_D \phi_n = 0. \tag{3.8}$$

Now, we find an $L^1(\mathbb{T})$ bound for all the $\phi_n \chi_{G_n}$ on

$$G=\bigcup_{n=0}^{\infty} G_n, \tag{3.9}$$

which does not depend upon n.

First note that for $0\leq y<1$:

$$||1+y|^p-1| = (1+y)^p-1 \leq (1+y)^{\lceil p\rceil}-1 = \sum_{k=0}^{\lceil p\rceil}\binom{\lceil p\rceil}{k}y^k - 1 = y\sum_{k=1}^{\lceil p\rceil}\binom{\lceil p\rceil}{k}y^{k-1}$$

$$\leq y\sum_{k=1}^{\lceil p\rceil}\binom{\lceil p\rceil}{k} \leq |y|2^{\lceil p\rceil}. \tag{3.10}$$

Similarly, for $-1<y<0$:

$$||1+y|^p-1| = 1-(1+y)^p \leq 1-(1+y)^{\lceil p\rceil} = -\sum_{k=1}^{\lceil p\rceil}\binom{\lceil p\rceil}{k}y^k$$

$$\leq |y|\sum_{k=1}^{\lceil p\rceil}\binom{\lceil p\rceil}{k} \leq |y|2^{\lceil p\rceil}. \tag{3.11}$$

Employing (3.10) and (3.11) for all n, and $x\in G_n$:

$$||f(x)+\lambda_n g(x)|^p - |f(x)|^p| = |f(x)|^p\left|\left|1+\frac{\lambda_n g(x)}{f(x)}\right|^p - 1\right|$$

$$\leq 2^{\lceil p\rceil}|f(x)|^p\left|\frac{\lambda_n g(x)}{f(x)}\right| = 2^{\lceil p\rceil}f(x)^{p-1}g(x)|\lambda_n|.$$

Then, for all n, and $x \in G$, recalling that ϕ_n has the form given in (3.3):

$$\phi_n(x)\chi_{G_n} \leq 2^{\lceil p \rceil} f(x)^{p-1} g(x)\chi_G.$$

It is also the case that $2^{\lceil p \rceil} f(x)^{p-1} g(x)\chi_G \in L^1(\mathbb{T})$ by Hölder's inequality.

Therefore, by the Lebesgue dominated convergence theorem:

$$\lim_{n\to\infty} \int_{G_n} \phi_n = \lim_{n\to\infty} \int_0^1 \phi_n \chi_{G_n} = \int_0^1 \lim_{n\to\infty} \phi_n \chi_{G_n}$$

$$= \int_0^1 pg|f|^{p-1}\chi_G = p\int_G g f^{p-1} < \infty, \tag{3.12}$$

where we have used the fact that $\lim_{n\to\infty} \phi_n$ is the derivative with respect to λ, at $\lambda = 0$, of the function $|f + \lambda g|^p$, as well as the non-negativity of f. Thus for either the left-hand or right-hand limit, combining (3.7), (3.8), (3.6), and (3.12) :

$$T'(0) = \lim_{n\to\infty} \int_0^1 \frac{|f + \lambda_n g|^p - |f|^p}{\lambda_n} = \lim_{n\to\infty} \int_0^1 \phi_n$$

$$= \lim_{n\to\infty} \int_{E_n} \phi_n + \lim_{n\to\infty} \int_C \phi_n + \lim_{n\to\infty} \int_D \phi_n + \lim_{n\to\infty} \int_{G_n} \phi_n$$

$$= 0 + 0 + 0 + p\int_G g f^{p-1} \leq p\|f\|_p^{p-1}\|g\|_p < \infty,$$

by Hölder's inequality. Hence, $T'(0) = p\int_G g f^{p-1}$. Notice that $G = supp(f) \cap supp(g)$, by how G_n and G are defined in (3.2) and (3.9), respectively. We supposed $\mu(G) > 0$, which implies that $T'(0) > 0$, using the non-negativity of f and g in the following:

$$T'(0) = p\int_G g f^{p-1} = p\int_{supp(f)\cap supp(g)} g f^{p-1} > 0.$$

Hence, there is some $\lambda_0 < 0$ such that:

$$T(\lambda_0) = \|f+\lambda_0 g\|_p^p < T(0) = \|f\|_p^p,$$

which, taking p^{th} roots yields a contradiction to perpendicularity. Hence it must be that:

$$\mu(supp(f)\cap supp(g)) = 0.$$

$\square$

Corollary 3.1.3. *Let $f,g \in L^p(\mathbb{T})$, for $1 < p < \infty$, and suppose that $f,g \geq 0$ a.e. If $f \perp g$, then $g \perp f$.*

Proof.

In Lemma 3.1.2 we showed that having $f,g \geq 0$ a.e., and $f \perp g$ resulted in $\mu(supp(f)\cap supp(g)) = 0$. It follows immediately from this fact that $g \perp f$.

$\square$

Theorem 3.1.4 (Non-Existence of Positive Monotone Bases)**.** *If $\{f_n\}_{n=1}^\infty$ is a basis for $L^p(\mathbb{T})$ with $1 < p < \infty$, and for all n, $f_n \geq 0$ a.e. on $\mathbb{T}$, then $\{f_n\}_{n=1}^\infty$ is not monotone.*

Proof.

By way of contradiction, assume that $\{f_n\}_{n=1}^\infty$ is monotone, and let f_{n_1} and f_{n_2} be two basis elements, where $n_2 > n_1$. Then by Lemma 3.1.1, $f_{n_1} \perp f_{n_2}$. Hence, by Lemma 3.1.2, $\mu(supp(f_{n_1})\cap supp(f_{n_2})) = 0$. But then the supports of all basis elements must be pairwise disjoint, except possibly on sets of measure zero. To obtain a contradiction now, choose one basis element, f_{n_0} and let $E_0 \subset supp(f_{n_0})$ be a set with $\mu(E_0) = \frac{1}{2}\mu(supp(f_{n_0}))$. Now, χ_{E_0} is a non-zero function, which is clearly outside $\overline{\text{span}}\{f_n\}_{n=1}^{\infty}$, contradicting the fact that $\{f_n\}_{n=1}^{\infty}$ is a basis for $L^p(\mathbb{T})$.

$\square$

The following lemma demonstrates an important difference between perpendicularity in $L^p(\mathbb{T})$ for $p=1$ and $1<p<\infty$. In particular, the example given demonstrates that $f \perp g$ in $L^1(\mathbb{T})$implies neither that $\mu(supp(f)\cap supp(g))=0$, nor that $g \perp f$, in contrast to $1<p<\infty$, motivating the exclusion of the case of $p=1$ from the above arguments.

Lemma 3.1.5. *There exist functions $f,g \in L^1(\mathbb{T})$, with $f,g \geq 0$ a.e. and $f \perp g$, but $g \not\perp f$, and so that $\mu(supp(f)\cap supp(g))>0$.*

Proof.

Let $f=\chi_{[0,\frac{1}{2}]}$, and $g=\chi_{[\frac{1}{8},1]}$. We first demonstrate that $f \perp g$:

$$\|f+\lambda g\|_1=\int_0^{\frac{1}{8}} 1+\int_{\frac{1}{8}}^{\frac{1}{2}} |1+\lambda|+\int_{\frac{1}{2}}^{1} |\lambda|. \tag{3.13}$$

For $\lambda \geq -1$:

$$(3.13)=\frac{1}{8}+\frac{3}{8}(1+\lambda)+\frac{1}{2}|\lambda| \geq \frac{1}{2}=\|f\|_1.$$

For $\lambda < -1$:

$$(3.13)=\frac{1}{8}+\frac{3}{8}(|\lambda|-1)+\frac{1}{2}|\lambda| \geq \frac{1}{8}+\frac{1}{2} \geq \frac{1}{2}=\|f\|_1.$$

Now we demonstrate that $g \not\perp f$:

Let $\lambda=-1$.

$$\|g+\lambda f\|_1=\int_0^{\frac{1}{8}} 1+\int_{\frac{1}{2}}^{1} 1=\frac{5}{8}<\|g\|_1=\frac{7}{8}.$$

Lastly, $\mu(supp(f) \cap supp(g)) = \frac{3}{8} > 0$.

$\square$

3.2 Unconditional Bases, and the Rademacher system

In this section, we introduce some well known properties of unconditional bases which will be used in the following section. See [3] for proofs of theorems which are omitted in this section. Khinchine's Inequalities for the Rademacher system prove to be particularly useful tools in this section.

Khinchine's Inequalities:

For each $1 \leq p < \infty$ there exist constants $k_p, K_p > 0$ such that for every $N \in \mathbb{N}$, and real scalars, $c_1, \cdots, c_N$,

$$k_p \left(\sum_{n=1}^{N} c_n^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{n=1}^{N} c_n R_n \right\|_{L^p(\mathbb{T})} \leq K_p \left(\sum_{n=1}^{N} c_n^2 \right)^{\frac{1}{2}}. \tag{3.14}$$

Note, that we only use that the inequality in the following theorem is implied by a sequence being an unconditional basis, though we state the equivalence for the sake of completeness.

Theorem 3.2.1. *Let $\{x_n\}_{n=1}^{\infty}$ be a complete sequence in a Banach space, X, such that $x_n \neq 0$ for every $n \in \mathbb{N}$. Then $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis for a Banach space, X, if and only if there exists $C \geq 1$ (independent of b_i, a_i, and N) such that if $|b_n| \leq |c_n|$ for $n = 1, \cdots, N$, then:*

$$\left\| \sum_{n=1}^{N} b_n x_n \right\| \leq C \left\| \sum_{n=1}^{N} c_n x_n \right\|.$$

Proof. Refer to Theorem 6.7 in, [3].

$\square$

Lemma 3.2.2. *Given $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$,*

$$\lim_{N\to\infty} \int_0^1 fR_N = 0.$$

Proof.

We first consider the case that $1 \leq p < \infty$. Let $\varepsilon > 0$. Since the set of dyadic step functions is dense in $L^p(\mathbb{T})$, take:

$$D(t) = \sum_{n=1}^{M} b_n \chi_{k_n,N_n}(t),$$

so that $\|f - D\|_p < \varepsilon$. Let $N_0 = \max\{N_n \,|\, n = 1, \cdots, M\}$. Fix $\tilde{N} > N_0 + 1$. Then for $n = 1, \cdots, M$,

$$\int_0^1 b_n \chi_{[k_n 2^{-N_n}, (k_n+1)2^{-N_n}]}(t) R_{\tilde{N}}(t)dt = 0.$$

This follows from how the Rademacher system is defined, in Definition 2.4.1, since $R_{\tilde{N}}(t)$ is -1 on exactly half of the measure of $\left[k_n 2^{-N_n}, (k_n+1)2^{-N_n}\right]$, and $+1$ on the other half of its measure, because $\tilde{N} > N_0 + 1 \geq N_n + 1$. Thus:

$$\int_0^1 D(t) R_{\tilde{N}}(t)dt = \sum_{n=1}^{M} b_n \int_0^1 \chi_{k_n,N_n} R_{\tilde{N}}(t)dt = 0.$$

Now, for $N = 0, 1, \cdots$:

$$\left|\int_0^1 f(t)R_N(t)dt\right| \leq \left|\int_0^1 f(t)R_N(t)dt - \int_0^1 D(t)R_N(t)dt\right| + \left|\int_0^1 D(t)R_N(t)dt\right|$$

$$= \left|\int_0^1 (f(t) - D(t))R_N(t)dt\right| + \left|\int_0^1 D(t)R_N(t)dt\right| \leq \int_0^1 |f(t) - D(t)|dt + \left|\int_0^1 D(t)R_N(t)dt\right|$$

$$\leq \|f(t)-D(t)\|_p + \left|\int_0^1 D(t)R_N(t)\right|, \tag{3.15}$$

where we have employed that $\|\cdot\|_{L^1(\mathbb{T})} \leq \|\cdot\|_{L^p(\mathbb{T})}$ for $1 \leq p < \infty$, which follows using $\mu(\mathbb{T}) = 1$, and Hölder's inequality.

So, using (3.15),

$$\limsup_{N\to\infty} \left|\int_0^1 f(t)R_N(t)dt\right| \leq \|f(t)-D(t)\|_p + \lim_{N\to\infty}\left|\int_0^1 D(t)R_N(t)dt\right| = \varepsilon + 0,$$

and since $\varepsilon > 0$ is arbitrary, we must have that $\lim\limits_{N\to\infty}\int_0^1 fR_N(t)dt = 0.$

For the $p = \infty$ case, note that since $\mu(\mathbb{T}) = 1$ (more generally, since $\mathbb{T}$ does not contain sets of arbitrarily large measure), $L^\infty(\mathbb{T}) \subseteq L^1(\mathbb{T})$. Hence, the result must hold also for $p = \infty$.

□

The following theorem provides two inequalities that must be violated by at least one function in the dual space of $L^p(\mathbb{T})$, $1 < p < \infty$, beginning with any positive system. This is a particularly useful tool in obtaining a contradiction to the existence of positive frames, positive Riesz bases, and positive unconditional bases.

Theorem 3.2.3.

Let $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$, and let $\{f_n\}_{n=0}^\infty$ be a sequence in $L^p(\mathbb{T})$ with the property that $f_n \geq 0$ a.e. on $\mathbb{T}$. Then for all systems $\{g_n\}_{n=0}^\infty$ of measurable functions, and constants $0 < J, K < \infty$, there exists some real-valued $h \in L^q(\mathbb{T})$ such that the following inequality does not hold:

$$J\|h\|_q \leq \left(\int_0^1 \left(\sum_{n=0}^\infty |\langle h, f_n\rangle|^2 |g_n(t)|^2\right)^{\frac{q}{2}} dt\right)^{\frac{1}{q}} \leq K\|h\|_q. \tag{3.16}$$

Proof.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^p(\mathbb{T})$, $1 < p < \infty$, with $f_n \geq 0$ a.e. on $\mathbb{T}$ for all n. If each g_n is equivalent to zero, it is clear that the lower inequality is violated for any $J > 0$, and $h \in L^q(\mathbb{T})$, which is non-zero. Suppose then by contradiction, that $\{g_n\}_{n=0}^{\infty}$ is a sequence of measurable functions, not all equivalent to the zero function, such that there exist constants $0 < J, K < \infty$, for which (3.16) holds for all real-valued $h \in L^q(\mathbb{T})$.

As before, let R_N denote the N^{th} Rademacher function, as given in Definition 2.4.1, which is real-valued, and note that $R_N \in L^q(\mathbb{T})$.

Now, for all $N > 0$, employing the a.e. non-negativity of the f_n, and the fact that $R_0 = 1$, it is easy to see that:

$$|\langle R_N, f_n\rangle|^2 \leq |\langle R_0, f_n\rangle|^2 . \tag{3.17}$$

Applying (3.16), and noting that K is independent of N:

$$0 < J = J\|R_N\|_q \leq \left(\int_0^1 \left(\sum_{n=0}^{\infty} |\langle R_N, f_n\rangle|^2 |g_n(t)|^2\right)^{\frac{q}{2}} dt\right)^{\frac{1}{q}} \leq K\|R_N\|_q = K < \infty. \tag{3.18}$$

Employing (3.17), for all $N \geq 0$, and $t \in \mathbb{T}$:

$$\left(\sum_{n=0}^{\infty} |\langle R_N, f_n\rangle|^2 |g_n(t)|^2\right)^{\frac{q}{2}} \leq \left(\sum_{n=0}^{\infty} |\langle R_0, f_n\rangle|^2 |g_n(t)|^2\right)^{\frac{q}{2}} \in L^1(\mathbb{T}),$$

by the upper bound in (3.18).

By the Lebesgue Dominated Convergence Theorem, formally, using (3.18):

$$0 < J \leq \lim_{N\to\infty} \left(\int_0^1 \left(\sum_{n=0}^{\infty} |\langle R_N, f_n\rangle|^2 |g_n(t)|^2\right)^{\frac{q}{2}} dt\right)^{\frac{1}{q}}$$

$$= \left(\int_0^1 \left(\lim_{N\to\infty} \sum_{n=0}^{\infty} |\langle R_N, f_n\rangle|^2 \, |g_n(t)|^2 \right)^{\frac{q}{2}} dt \right)^{\frac{1}{q}}. \tag{3.19}$$

Now, define $\Phi_N(n)(t) := |\langle R_N, f_n\rangle|^2 \, |g_n(t)|^2$, and $\Phi_N(t) := \{\Phi_N(n)(t)\}_{n=0}^{\infty}$.

Using (3.17):

$$\|\Phi_N(t)\|_{\ell^1} \le \|\Phi_0(t)\|_{\ell^1} < \infty,$$

for a.e. t, since

$$\|\Phi_0(t)\|_{\ell^1}^{\frac{q}{2}} = \left(\sum_{n=0}^{\infty} |\langle R_0, f_n\rangle|^2 \, |g_n(t)|^2 \right)^{\frac{q}{2}} \in L^1(\mathbb{T}).$$

Then, for a.e. t, by the Lebesgue dominated convergence theorem for series:

$$\lim_{N\to\infty} \sum_{n=0}^{\infty} |\langle R_N, f_n\rangle|^2 \, |g_n(t)|^2 = \sum_{n=0}^{\infty} \lim_{N\to\infty} |\langle R_N, f_n\rangle|^2 \, |g_n(t)|^2 = 0,$$

by Lemma 3.2.2.

Returning to (3.19), then:

$$0 < J \le \left(\int_0^1 \left(\lim_{N\to\infty} \sum_{n=0}^{\infty} |\langle R_N, f_n\rangle|^2 \, |g_n(t)|^2 \right)^{\frac{q}{2}} dt \right)^{\frac{1}{q}}$$

$$= \left(\int_0^1 \left(\sum_{n=0}^{\infty} \lim_{N\to\infty} |\langle R_N, f_n\rangle|^2 \, |g_n(t)|^2 \right)^{\frac{q}{2}} dt \right)^{\frac{1}{q}} = 0,$$

which is a contradiction. Hence there must not exist a system $\{g_n\}_{n=0}^{\infty}$, and constants $0 < J, K < \infty$, where (3.16) holds for all real-valued functions, $h \in L^q(\mathbb{T})$.

$\square$

Before we use Theorem 3.2.3 to assist us in showing the non-existence of non-negative a.e. frames, and unconditional bases, we require a few lemmas.

Lemma 3.2.4. *Let $\{x_n\}_{n=1}^{\infty}$ be an unconditional basis of real-valued functions in $L^p(\mathbb{T})$ for $1 \leq p < \infty$. Then there exists some $K > 0$, such that for all real scalars, $\{b_n\}_{n=1}^{\infty}$ for which $\sum_{n=1}^{\infty} b_n x_n$ converges in $L^p(\mathbb{T})$:*

$$K^{-1} \left\| \sum_{n=1}^{\infty} b_n x_n \right\|_p \leq \left(\int_0^1 \left(\sum_{n=1}^{\infty} b_n^2 x_n^2(t) \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} \leq K \left\| \sum_{n=1}^{\infty} b_n x_n \right\|_p .$$

Proof.

Let $R_n(y)$ denote the n^{th} Rademacher function, as given in Definition 2.4.1. Using a technique of Heil from Lemma 3.26 in [3], for a.e. $y \in \mathbb{T}$, employing Khinchine's Inequalities given in (3.14), which we may do since the x_n are real-valued:

$$\int_0^1 \left(\sum_{n=1}^{N} b_n^2 x_n^2(t) \right)^{\frac{p}{2}} dt \leq k_p^{-p} \int_0^1 \left\| \sum_{n=1}^{N} b_n x_n(t) R_n(y) \right\|_p^p dt$$

$$= k_p^{-p} \int_0^1 \int_0^1 \left| \sum_{n=1}^{N} b_n x_n(t) R_n(y) \right|^p dy dt = k_p^{-p} \int_0^1 \int_0^1 \left| \sum_{n=1}^{N} b_n x_n(t) R_n(y) \right|^p dt dy, \tag{3.20}$$

by Tonelli's theorem since the integrand is non-negative, [12].

Now, letting $f(t) := \sum_{n=1}^{\infty} b_n x_n(t)$, and $f_y(t) := \sum_{n=1}^{\infty} R_n(y) b_n x_n(t)$, note that by Theorem 3.2.1, there exists some constant $C \geq 1$ such that:

$\|S_N f_y\|_p \leq C \|S_N f\|_p$, since $|R_i(y) b_i| \leq |b_i|$ for every $y \in \mathbb{T}$. Hence:

$$(3.20) = k_p^{-p} \int_0^1 \|S_N f_y\|_p^p dy \leq k_p^{-p} \int_0^1 C^p \|S_N f\|_p^p dy = \frac{C^p}{k_p^p} \|S_N f\|_p^p.$$

Thus, employing the monotone convergence theorem,

$$\int_0^1 \left(\sum_{n=1}^{\infty} b_n^2 x_n^2(t)\right)^{\frac{p}{2}} dt = \lim_{N\to\infty} \int_0^1 \left(\sum_{n=1}^{N} b_n^2 x_n^2(t)\right)^{\frac{p}{2}} dt$$

$$\leq \lim_{N\to\infty} \frac{C^p}{k_p^p} \|S_N f\|_p^p = \frac{C^p}{k_p^p} \|f\|_p^p = \frac{C^p}{k_p^p} \left\|\sum_{n=1}^{\infty} b_n x_n\right\|_p^p .$$

Taking p^{th} roots yields:

$$\left(\int_0^1 \left(\sum_{n=1}^{\infty} b_n^2 x_n^2(t)\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}} \leq \frac{C}{k_p} \left\|\sum_{n=1}^{\infty} b_n x_n\right\|_p . \tag{3.21}$$

For the lower inequality we omit some steps, since we proceed as before:

$$\int_0^1 \left(\sum_{n=1}^{N} b_n^2 x_n^2(t)\right)^{\frac{p}{2}} dt \geq K_p^{-p} \int_0^1 \left\|\sum_{n=1}^{N} b_n x_n(t) R_n(y)\right\|_p^p dt$$

$$= K_p^{-p} \int_0^1 \int_0^1 \left|\sum_{n=1}^{N} b_n x_n(t) R_n(y)\right|^p dt dy = K_p^{-p} \int_0^1 \|S_N f_y\|_p^p dy. \tag{3.22}$$

In this case, $|b_i| \leq |R_i(y) b_i|$ for a.e. $y \in \mathbb{T}$. Hence, by Theorem 3.2.1, for a.e. $y \in \mathbb{T}$, $\|S_N f\| \leq C\|S_N f_y\|$, where C is as above so:

$$(3.22) \geq \frac{1}{C^p K_p^p} \int_0^1 \|S_N f\|_p^p dy.$$

Arguing as before:

$$\left(\int_0^1 \left(\sum_{n=1}^{\infty} b_n^2 x_n^2(t)\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}} \geq \frac{1}{CK_p} \left\|\sum_{n=1}^{\infty} b_n x_n\right\|_p . \tag{3.23}$$

Taking K to be the maximum of CK_p and $\frac{C}{k_p}$ yields the result.

$\square$

The following lemma is given as an exercise in [3]. We use similar techniques to those used in Heil's proof that given a basis in a reflexive Banach space, X, the dual system forms a basis for X^*.

Lemma 3.2.5. *Given an unconditional basis $\{x_n\}_{n=1}^{\infty}$ in a reflexive Banach space, X, the associated system of coefficient functionals $\{a_n\}_{n=1}^{\infty}$ is an unconditional basis for X^*.*

Proof.

Since $\{x_n\}_{n=1}^{\infty}$ is a basis in a reflexive Banach space, X, the associated system of coefficient functionals, $\{a_n\}_{n=1}^{\infty}$ is a basis for X^*, see Corollary 5.22 in, [3], . Let $\sigma(n) = \{n_1, n_2, \cdots\}$ be a permutation of $\mathbb{N}$. It suffices to show that $\{a_{n_j}\}_{j=1}^{\infty}$ is a basis for X^* as well, using the uniqueness property of coefficients of bases. For $x \in X$, and $x^* \in X^*$, define $\pi(x)(x^*) := x^*(x) = \langle x, x^* \rangle$, where this denotes the point-evaluation operator, which is the standard embedding of X into X^{**}, and note that $\pi(x) \in X^{**}$. Then: $\left\langle a_{n_k}, \pi(x_{n_j}) \right\rangle = \left\langle x_{n_j}, a_{n_k} \right\rangle = \delta_{n_k, n_j}$, and so $\{\pi(x_{n_j})\}_{j=1}^{\infty}$ is a biorthogonal sequence to $\{a_{n_j}\}_{j=1}^{\infty}$. For $x^* \in X^*$, define the partial sum operator:

$$T_J(x^*) := \sum_{j=1}^{J} \left\langle x^*, \pi(x_{n_j}) \right\rangle a_{n_j}.$$

We use the fact that $\{a_{n_j}\}_{j=1}^{\infty}$ is a basis for X^* if and only if it is exact and $\sup_J \|T_J\| < \infty$, see [3], Theorem 5.12 (e). Exactness of $\{a_{n_j}\}_{j=1}^{\infty}$ follows from the biorthogonality of $\{\pi(x_{n_j})\}_{j=1}^{\infty}$, and the fact that the system is complete, since $\{a_n\}_{n=1}^{\infty}$ is complete. Now, we demonstrate $\sup_J \|T_J\| < \infty$. For $x \in X$, and $x^* \in X^*$, and letting S_J denote the J^{th} partial sum operator for $\{x_{n_j}\}_{j=1}^{\infty}$, and S_J^* denote the adjoint operator to S_J, as given in Definition 2.2.17:

$$\langle x, S_J^*(x^*) \rangle = \langle S_J(x), x^* \rangle = \left\langle \sum_{j=1}^{J} \left\langle x, a_{n_j} \right\rangle x_{n_j}, x^* \right\rangle = \sum_{j=1}^{J} \left\langle x, a_{n_j} \right\rangle \left\langle x_{n_j}, x^* \right\rangle$$

$$= \left\langle x, \sum_{j=1}^{J} \langle x_{n_j}, x^* \rangle a_{n_j} \right\rangle = \left\langle x, \sum_{j=1}^{J} \langle x^*, \pi(x_{n_j}) \rangle a_{n_j} \right\rangle = \langle x, T_J(x^*) \rangle .$$

Hence, $T_J = S_J^*$, and so $\|T_J\| = \|S_J^*\| = \|S_J\|$. Since $\{x_{n_j}\}_{n=1}^{\infty}$ is also a basis for X due to the unconditionality of $\{x_n\}_{n=1}^{\infty}$, $\sup_J \|S_J\| < \infty$. Hence, $\sup_J \|T_J\| < \infty$, which concludes the proof.

□

Theorem 3.2.6 (Non-Existence of Positive Unconditional Bases)**.** *If $\{f_n\}_{n=1}^{\infty}$ is a basis for $L^p(\mathbb{T})$ with $1 < p < \infty$, and for all n, $f_n \geq 0$ a.e. on $\mathbb{T}$, then $\{f_n\}_{n=1}^{\infty}$ is not an unconditional basis.*

Proof.

Let $\{g_n\}_{n=1}^{\infty} \subseteq L^q(\mathbb{T})$, where $\frac{1}{p} + \frac{1}{q} = 1$, denote the dual system to $\{f_n\}_{n=1}^{\infty}$. Note that by Lemma 3.2.5, $\{g_n\}_{n=1}^{\infty}$ is an unconditional basis for $L^q(\mathbb{T})$. Let $h \in L^q(\mathbb{T})$ be a real-valued function, so:

$$h(t) = \sum_{n=1}^{\infty} \langle h, f_n \rangle g_n,$$

where each coefficient, $\langle h, f_n \rangle$, is real-valued, since both h and f_n are.

In order to apply Lemma 3.2.4, to the $\{g_n\}_{n=1}^{\infty}$, we must show that each g_n is real-valued. This follows readily from the following, using the fact that the f_n are real-valued:

$$1 = \langle f_n, g_n \rangle = \int_0^1 f_n \overline{g}_n = \int_0^1 f_n Re(g_n) - i \int_0^1 f_n Im(g_n) = \int_0^1 f_n Re(g_n),$$

and for $m \neq n$:

$$0 = \langle f_m, g_n \rangle = \int_0^1 f_m \overline{g}_n = \int_0^1 f_m Re(g_n) - i \int_0^1 f_m Im(g_n) = \int_0^1 f_m Re(g_n).$$

Hence, for all $m \in \mathbb{N}$, $\langle f_m, Im(g_n) \rangle = 0$. Since $\{f_n\}_{n \in \mathbb{N}}$ is complete, it must be that $Im(g_n) \in [0]$.

Then by Lemma 3.2.4, there exists some $K > 0$, such that:

$$K^{-1}\|h\|_q \leq \left(\int_0^1 \left(\sum_{n=1}^{\infty} |\langle h, f_n \rangle|^2 \, |g_n|^2 \right)^{\frac{q}{2}} dt \right)^{\frac{1}{q}} \leq K\|h\|_q.$$

But this yields a contradiction according to Theorem 3.2.3, since h was an arbitrary real-valued function in $L^q(\mathbb{T})$. Hence, $\{f_n\}_{n=1}^{\infty}$ cannot be a positive unconditional basis for $L^p(\mathbb{T})$.

$\square$

It is interesting to note that Dor and Odell have shown that for $1 < p < \infty$, every monotone basis is an unconditional basis, [11]. Hence, we could obtain the non-existence of a positive monotone basis for $1 < p < \infty$, also as a corollary to Theorem 3.2.6, though we have earlier proved that result independently in Theorem 3.1.4.

3.3 Riesz Bases and Frames

Corollary 3.3.1 (Non-Existence of Positive Riesz Bases)**.** *There does not exist a positive Riesz basis for $L^2(\mathbb{T})$.*

Proof.

By Theorem 7.11 in, [3], every Riesz basis for $L^2(\mathbb{T})$ must be an unconditional basis for $L^2(\mathbb{T})$. Hence, it follows directly from Theorem 3.2.6, that there cannot be a positive Riesz basis for $L^2(\mathbb{T})$.

$\square$

Theorem 3.3.2 (Non-Existence of Positive Frames)**.** *There does not exist a frame, $\{f_n\}_{n=1}^{\infty}$ for $L^2(\mathbb{T})$ with the property that for all n, $f_n \geq 0$ a.e. on $\mathbb{T}$.*

Proof.

Consider the system $\{g_j\}_{j=1}^{\infty} \subseteq L^2(\mathbb{T})$ where $g_j(t) \equiv 1$ on $\mathbb{T}$. Then, for all real-valued $h \in L^2(\mathbb{T})$:

$$\sum_{n=1}^{\infty} |\langle h, f_n \rangle|^2 = \int_0^1 \sum_{n=1}^{\infty} |\langle h, f_n \rangle|^2 \left|g_j(t)\right|^2 dt$$

By the definition of frame, then there exist constants $A, B > 0$ such that:

$$A\|h\|_2^2 \leq \int_0^1 \sum_{n=1}^{\infty} |\langle h, f_n \rangle|^2 \left|g_j(t)\right|^2 dt \leq B\|h\|_2^2$$

Taking square roots:

$$A^{\frac{1}{2}}\|h\|_2 \leq \left(\int_0^1 \sum_{n=1}^{\infty} |\langle h, f_n \rangle|^2 \left|g_j(t)\right|^2 dt \right)^{\frac{1}{2}} \leq B^{\frac{1}{2}}\|h\|_2,$$

for all $h \in L^2(\mathbb{T})$, where $0 < A, B < \infty$. More specifically, this holds for all real-valued functions, $h \in L^2(\mathbb{T})$. This contradicts Theorem 3.2.3.

□

3.4 Quasibases and Conditional Pseudobases

In this section, some basic properties of quasibases will be demonstrated, with the goal of addressing the existence question for positive conditional, and positive unconditional quasibases in $L^p(\mathbb{T})$.

In Theorem 3.4.8, we demonstrate that the set of dyadic characteristic functions forms a positive conditional quasibasis for $L^p(\mathbb{T})$, $1 \leq p < \infty$, regardless of the dual system. The proof of the conditionality of the quasibasis of dyadic characteristic functions uses techniques employed by Kazarian and Zink in [1], where they consider the Schauder system. The proof of Theorem 3.4.10 shares many techniques with Theorem 3.4.8, however it requires more cumbersome notation, due to the more general nature of the dyadic step functions which we deal with there. Finally, we prove both a stability theorem for quasibases, and a stability theorem for unconditional quasibases.

In summary, to show non-existence of positive unconditional quasibases, we approximate the elements of a supposed general positive unconditional quasibasis with dyadic step functions with positive coefficients. Applying the stability theorem for unconditional quasibases, we obtain a contradiction to the conclusion of Theorem 3.4.8, which says that any such quasibasis of dyadic step functions must be conditional.

Note that since each positive unconditional Schauder basis is an positive unconditional Quasibasis in its space, the non-existence of positive unconditional quasibases in $L^p(\mathbb{T})$ for $1 \le p < \infty$ yields that there can be no positive unconditional Schauder basis there either. Up to this point, we only explicitly demonstrated the non-existence of positive, unconditional Schauder bases in $L^p(\mathbb{T})$ for $1 < p < \infty$ - the quasibasis result extends non-existence of positive unconditional Schauder bases to $p = 1$. Though it is a well-known fact that there cannot exist *any* unconditional Schauder basis for $L^1(\mathbb{T})$, it is interesting to note that we have shown non-existence of positive unconditional Schauder bases, independent of this result.

Definition 3.4.1 (Partial Sums for Quasibases)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a quasibasis for a Banach space, X, with some dual system $A = \{a_n\}_{n=1}^{\infty} \subseteq X^*$. Then, we define the partial sum operator:

$$S_N^A(x) := \sum_{n=1}^{N} \langle x, a_n \rangle x_n.$$

We will use the notation $S_N(x) = S_N^A(x)$, where there is no ambiguity regarding which dual system we are using to expand x.

Lemma 3.4.2 (An Equivalent Notion to Being a Quasibasis)**.** *Let X be a Banach space, and sup-*

pose that $\{x_n\}_{n=1}^{\infty} \subseteq X$*, and let* $\{a_n\}_{n=1}^{\infty} \subseteq X^*$*. Then* $\{x_n\}_{n=1}^{\infty}$ *is a quasibasis for* $\overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$ *if and only if* $\sup_N \|S_N\| < \infty$*, where the* S_N *are restricted to* $\overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$*, and partial sums are taken with respect to the dual system under consideration, and for all* $n \in \mathbb{N}$*, it holds that*

$$\lim_{N\to\infty} \|S_N(x_n) - x_n\| = 0.$$

Proof.

First suppose that $\{x_n\}_{n=1}^{\infty}$ is a quasibasis for $\overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$ with a dual system $\{a_n\}_{n=1}^{\infty}$. Letting $x \in \overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$, $\{S_N(x)\}_{N=1}^{\infty}$ is a convergent sequence, and hence is bounded. That is, $\sup_N \|S_N(x)\| < \infty$, for all $x \in \overline{\text{span}}\left(\{x_n\}_{n=1}^{\infty}\right)$. Hence, by the uniform boundedness principle [3], it follows that $\sup_N \|S_N\| < \infty$, where the S_N are restricted to $\overline{\text{span}}\left(\{x_n\}_{n=1}^{\infty}\right)$. By the definition of quasibasis, since $x_n \in \overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$, it holds that $\lim_{N\to\infty} \|S_N(x_n) - x_n\| = 0$ for $n \in \mathbb{N}$.

To prove the other implication suppose that $C := \sup_N \|S_N\| < \infty$, and for all $n \in \mathbb{N}$, it holds that $\lim_{N\to\infty} \|S_N(x_n) - x_n\| = 0$. Let $x \in \overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$, and let $\varepsilon > 0$. Then there exists a non-zero $y = \sum_{n=1}^{M} c_n x_n$ such that:

$$\|x - y\| < \frac{\varepsilon}{2(1+C)}$$

Now, choose $\tilde{N}$ large enough so that for all $m = 1, 2, \cdots, M$:

$$\|S_{\tilde{N}}(x_m) - x_m\| < \frac{\varepsilon}{2M \max_{m=1,\cdots,M}\{|c_m|\}}.$$

Therefore:

$$\|S_{\tilde{N}}(x) - x\| \leq \|x - y\| + \|y - S_{\tilde{N}}(y)\| + \|S_{\tilde{N}}(y) - S_{\tilde{N}}(x)\|$$

$$\leq \|x-y\| + \left\| \sum_{n=1}^{M} c_n x_n - S_{\tilde{N}} \left(\sum_{n=1}^{M} c_n x_n \right) \right\| + \|S_{\tilde{N}}\| \, \|x-y\|$$

$$\leq (1+C)\|x-y\| + \sum_{n=1}^{M} \|c_n(x_n - S_{\tilde{N}}(x_n))\|$$

$$\leq \frac{\varepsilon}{2} + \sum_{n=1}^{M} \max_{m=1,\cdots M} \{|c_m|\} \, \|S_{\tilde{N}}(x_n) - x_n\| < \varepsilon.$$

Hence, for $x \in \overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$,

$$\lim_{N\to\infty} \|S_N(x) - x\| = 0,$$

and so $\{x_n\}_{n=1}^{\infty}$ is a quasibasis for $\overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$.

$\square$

Definition 3.4.3. Let X be a Banach space with a quasibasis, $\{x_n\}_{n=1}^{\infty}$, and some associated set of coefficient functionals $A := \{a_n\}_{n=1}^{\infty}$, and let $F \subseteq \mathbb{N}$ be finite. Define the following partial sum functional:

$$S_F^A(x) = \sum_{n \in F} \langle x, a_n \rangle x_n,$$

and define $\Lambda^A(x) := \sup_F \left\| S_F^A(x) \right\|$.

Definition 3.4.4. Let X be a Banach space with a quasibasis, $\{x_n\}_{n=1}^{\infty}$, and some associated set of coefficient functionals $A := \{a_n\}_{n=1}^{\infty}$, and let $F \subseteq \mathbb{N}$ be finite. Also let $\mathscr{E} := \{\varepsilon_n\}_{n=1}^{\infty}$, be a sequence, where $\varepsilon_n \in \{-1, 0, 1\}$. Define the following partial sum functional:

$$S_{F,\mathscr{E}}^A(x) = \sum_{n \in F} \varepsilon_n \langle x, a_n \rangle x_n,$$

and define $\Lambda^A_{\mathscr{E}}(x) := \sup_{F,\mathscr{E}} \left\| S^A_{F,\mathscr{E}}(x) \right\|$, and $\Lambda^A_{\mathscr{E}} := \sup_{F,\mathscr{E}} \left\| S_{F,\mathscr{E}} \right\|$.

Lemma 3.4.5. *Let X be a Banach space, and suppose that $\{x_n\}_{n=1}^{\infty}$ is an unconditional quasibasis with an associated sequence of coefficient functionals $A = \{a_n\}_{n=1}^{\infty}$, for which, for all $x \in X$, $\sum_{n=1}^{\infty} \langle x, a_n \rangle x_n$ converges unconditionally. As defined previously, $\Lambda^A(x) < \infty$ for each $x \in X$.*

Proof.

This follows directly from Theorem 3.15 in, [3].

□

Lemma 3.4.6. *Let X be a Banach space, and suppose that $\{x_n\}_{n=1}^{\infty}$ is an unconditional quasibasis with an associated sequence of coefficient functionals $A = \{a_n\}_{n=1}^{\infty}$, for which for all $x \in X$, $\sum_{n=1}^{\infty} \langle x, a_n \rangle x_n$ converges unconditionally, and let $\mathscr{E} = \{\varepsilon_n\}_{n=1}^{\infty}$, where $\varepsilon_n \in \{-1, 0, 1\}$. As defined previously, $\Lambda^A_{\mathscr{E}}(x) < \infty$. Moreover:*

$$\Lambda^A_{\mathscr{E}} = \sup_{F,\mathscr{E}} \left\| S^A_{F,\mathscr{E}} \right\| < \infty.$$

Proof.

Let $F \subseteq \mathbb{N}$ be finite, and define $F^+ := \{n \in F \mid \varepsilon_n > 0\}$ and $F^- := \{n \in F \mid \varepsilon_n < 0\}$. Then:

$$\left\| S^A_{F,\mathscr{E}}(x) \right\| = \left\| \sum_{n \in F} \varepsilon_n \langle x, a_n \rangle x_n \right\| = \left\| \sum_{n \in F^+} \langle x, a_n \rangle x_n - \sum_{n \in F^-} \langle x, a_n \rangle x_n \right\|$$

$$\leq \left\| \sum_{n \in F^+} \langle x, a_n \rangle x_n \right\| + \left\| \sum_{n \in F^-} \langle x, a_n \rangle x_n \right\| \leq 2\Lambda^A(x) < \infty,$$

by Lemma 3.4.5. Hence, $\Lambda^A_{\mathscr{E}}(x) < \infty$.

Now, since for each $x \in X$, $\Lambda^A_{\mathscr{E}}(x) = \sup_{F,\mathscr{E}} \left\| S^A_{F,\mathscr{E}}(x) \right\| < \infty$, the uniform boundedness principle, [3], yields that:

$$\Lambda_{\mathscr{E}}^{A} = \sup_{F,\mathscr{E}} \left\| S_{F,\mathscr{E}}^{A} \right\| < \infty.$$

□

Theorem 3.4.7. *Let $\{f_n\}_{n=1}^{\infty}$ be a real-valued basis for $L^p(\mathbb{T})$, $1 \le p < \infty$, with dual system $\{g_n\}_{n=1}^{\infty} \subseteq L^q(\mathbb{T})$, and $\frac{1}{p} + \frac{1}{q} = 1$. Let $A_n = \{x \mid f_n(x) > 0\}$, $B_n = \{x \mid f_n(x) < 0\}$, and define $a_n(t) := |f_n(t)|\chi_{A_n}(t)$, and $b_n(t) := |f_n(t)|\chi_{B_n}(t)$. Then, the system $\{\tilde{f}_n\}_{n=1}^{\infty} := \{a_1, b_1, \cdots, a_j, b_j, \cdots\}$ is a quasibasis for $L^p(\mathbb{T})$ with dual system, $\{\tilde{g}_n\}_{n=1}^{\infty} := \{g_1, -g_1, g_2, -g_2, \cdots, g_j, -g_j, \cdots\}$ in $L^q(\mathbb{T})$.*

Proof.

Notice that $a_j(t) - b_j(t) = f_j(t)$.

Letting $h \in L^p(\mathbb{T})$:

$$h(t) = \sum_{j=1}^{\infty} \langle h, g_j \rangle f_j(t) = \sum_{j=1}^{\infty} \langle h, g_j \rangle (a_j(t) - b_j(t)) = \sum_{j=1}^{\infty} \left(\langle h, g_j \rangle a_j(t) + \langle h, -g_j \rangle b_j(t) \right),$$

reindexing, formally,

$$= \sum_{n=1}^{\infty} \langle h, \tilde{g}_n \rangle \tilde{f}_n, \tag{3.24}$$

where it remains to show that this sum converges after reindexing.

Considering partial sums $\sum_{n=1}^{N} \langle h, \tilde{g}_n \rangle \tilde{f}_n$ of (3.24) there are two possibilities. When N is even, $\tilde{f}_N = b_{\frac{N}{2}}$, and:

$$\sum_{n=1}^{N} \langle h, \tilde{g}_n \rangle \tilde{f}_n = \sum_{j=1}^{\frac{N}{2}} \langle h, g_j \rangle f_j(t).$$

If N is odd, then $\tilde{f}_N = a_{\frac{N-1}{2}+1}$, and:

$$\sum_{n=1}^{N} \langle h, \tilde{g}_n \rangle \tilde{f}_n = \sum_{j=1}^{\frac{N-1}{2}} \langle h, g_j \rangle f_j(t) + \left\langle h, g_{\frac{N-1}{2}+1} \right\rangle a_{\frac{N-1}{2}+1}.$$

Before proceeding, we compute the following limit:

$$\lim_{N\to\infty} \|\langle h, g_N \rangle a_N\|_p \le \lim_{N\to\infty} \left(\|\langle h, g_N \rangle a_N\|_p + \|\langle h, -g_N \rangle b_N\|_p \right)$$

$$= \lim_{N\to\infty} \|\langle h, g_N \rangle a_N + \langle h, -g_N \rangle b_N\|_p = \lim_{N\to\infty} \|\langle h, g_N \rangle (a_N - b_N)\|_p = \lim_{N\to\infty} \|\langle h, g_N \rangle f_N\|_p = 0,$$

where we have employed the disjointness of the supports of a_N and b_N, and we have used the fact that the partial sums of $\sum_{j=1}^{\infty} \langle h, g_j \rangle f_j$ form a Cauchy sequence to yield convergence to 0.

Let $\varepsilon > 0$, and take N large enough that for all $M \ge N$, where M is even,

$$\left\| h - \sum_{n=1}^{M} \langle h, \tilde{g}_n \rangle \tilde{f}_n \right\|_p = \left\| h - \sum_{j=1}^{\frac{M}{2}} \langle h, g_j \rangle f_j \right\|_p < \frac{\varepsilon}{2}, \tag{3.25}$$

and large enough so that if $M \ge N$, and M is odd,

$$\left\| \left\langle h, g_{\frac{M-1}{2}+1} \right\rangle a_{\frac{M-1}{2}+1} \right\|_p < \frac{\varepsilon}{2}.$$

If $M \ge N+1$ is odd, then:

$$\left\| h - \sum_{n=1}^{M} \langle h, \tilde{g}_n \rangle \tilde{f}_n \right\|_p \le \left\| h - \sum_{n=1}^{M-1} \langle h, \tilde{g}_n \rangle \tilde{f}_n \right\|_p + \left\| \left\langle h, g_{\frac{M-1}{2}+1} \right\rangle a_{\frac{M-1}{2}+1} \right\|_p < \varepsilon.$$

If $M \ge N$ is even, we refer to (3.25).

Thus, $h = \sum_{n=1}^{\infty} \langle h, \tilde{g}_n \rangle \tilde{f}_n$, and so $\{a_1, b_1, \cdots, a_j, b_j, \cdots\}$ is a quasibasis of $L^p(\mathbb{T})$ with dual system $\{g_1, -g_1, \cdots, g_j, -g_j, \cdots\}$ in $L^q(\mathbb{T})$.

$\square$

We note that while the following proof demonstrates the existence of a positive conditional quasibasis, it is not the first theorem to do so. In [1], using similar methods to those demonstrated here, the existence of a positive conditional quasibasis is shown. Specifically, Kazarian and Zink show that the Schauder (or Faber-Schauder) system, which is a positive system, is a quasibasis which is conditional for $1 < p < \infty$. In their proof, Kazarian and Zink note that the proof of non-existence of any unconditional Schauder basis for $L^1(\mathbb{T})$ can be modified to demonstrate non-existence of any unconditional quasibasis for $L^1(\mathbb{T})$, though they do not provide details of this proof. Here we provide a proof of the non-existence of a positive unconditional quasibasis in $L^1(\mathbb{T})$, independent of the more general result claimed in [1].

Theorem 3.4.8 (Existence of Positive Conditional Quasibases)**.** *The set of dyadic characteristic functions is a quasibasis for $L^p(\mathbb{T})$ for $1 \leq p < \infty$, that is not unconditional.*

Proof.

We start with the Haar system, $\{h_n\}_{n=1}^{\infty}$, as given in Definition 2.6.1, and recall that the Haar system forms a basis for $L^p(\mathbb{T})$, $1 \leq p < \infty$, [13].

Let $f \in L^p(\mathbb{T})$. Then:

$$f(t) = \sum_{n=1}^{\infty} \langle f, h_n \rangle h_n(t) = \sum_{n=1}^{\infty} \langle f, h_n \rangle (c_n a_n(t) - c_n b_n(t)),$$

where for the n^{th} Haar function, $h_n(t) = h_{j,k}(t)$, with $n \geq 2$,

$$a_n(t) = \chi_{\left[\frac{2j-2}{2^{k+1}}, \frac{2j-1}{2^{k+1}}\right]}(t),\ b_n(t) = \chi_{\left[\frac{2j-1}{2^{k+1}}, \frac{2j}{2^{k+1}}\right]}, \tag{3.26}$$

and $c_n = 2^{\frac{k}{2}}$. For $n = 1$, $h_1(t) \equiv 1$, and so we take

$$a_1(t) \equiv 1,\ b_1(t) \equiv 0, \tag{3.27}$$

and $c_1 = 1$. Since a_n and b_n are as in the statement of Theorem 3.4.7, in relation to the Haar basis for $L^p(\mathbb{T})$, $1 \leq p < \infty$, the following ordering of the scaled dyadic characteristic functions on $\mathbb{T}$ is a quasibasis for $L^p(\mathbb{T})$ with $1 \leq p < \infty$, $\{c_1a_1, c_1b_1, \cdots, c_na_n, c_nb_n, \cdots\} := \{\tilde{x}_n\}_{n=1}^{\infty}$ and has a dual system defined by $\{h_1, -h_1, \cdots, h_n, -h_n, \cdots\}$. It is easy to see that the system

$$C := \{a_1, b_1, \cdots, a_n, b_n, \cdots\},$$

is also a quasibasis, this time with dual system,

$$\{c_1h_1, -c_1h_1, \cdots, c_nh_n, -c_nh_n, \cdots\}.$$

For simplicity in the following, we remove b_1, since it is 0, and we are left with

$$\{a_1, a_2, b_2, a_3, b_3, \cdots\} := \{x_n\}_{n=1}^{\infty}, \tag{3.28}$$

as our quasibasis.

Proving by contradiction, suppose that $C = \{x_n\}_{n=1}^{\infty}$ is an unconditional quasibasis for $L^p(\mathbb{T})$, and let $G = \{g_n\}_{n=1}^{\infty}$ be some corresponding dual system in $L^q(\mathbb{T})$, where $\frac{1}{p} + \frac{1}{q} = 1$, and so that it holds for each $f \in L^p(\mathbb{T})$ that the following expression converges unconditionally to f in L^p norm:

$$f = \sum_{n=1}^{\infty} \langle f, g_n \rangle x_n, \quad \text{where} \quad \langle f, g_n \rangle = \int_0^1 f\overline{g}_n dt.$$

Where R_n denotes the n^{th} Rademacher function, as given in Definition 2.4.1, let

$$c_k^n := \int_0^1 R_n\overline{g}_k dt = \langle R_n, g_k \rangle, \tag{3.29}$$

so that for all n:

$$R_n = \sum_{k=1}^{\infty} c_k^n x_k, \tag{3.30}$$

where the sum converges to R_n in $L^p(\mathbb{T})$, and convergence is unconditional by assumption.

Fix $n \geq 0$, and let

$$I_j = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right], \text{ where } j \in \{1, \cdots, 2^n\}. \tag{3.31}$$

Note that on such an interval, R_{n+1} "oscillates" once, since its period is $\frac{1}{2^n}$, and so is 1 on half of the measure of the interval, and -1 on the other half.

Recall that the supports of the x_i are dyadic intervals of the form $\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right]$, since they are each some a_i, or b_i as given in (3.26), or (3.27). Now, if $i \leq 2^{n+1} - 1$, then it is easy to see that $\mu(supp(x_i)) \geq \frac{1}{2^n}$. This follows from the fact that there are $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ of the x_i with support intervals of length greater than or equal to $\frac{1}{2^n}$, and from the fact that the measures of the supports of the x_i are non-increasing with respect to i. Hence for $i \leq 2^{n+1} - 1$, by the construction of the x_i, either $x_i(t) \equiv 1$ on I_j, or $x_i \equiv 0$ on I_j. Let

$$\Lambda_j := \{i | i \leq 2^{n+1} - 1, \text{ and } x_i = 1 \text{ on } I_j\}. \tag{3.32}$$

Then for a.e. $t \in I_j$, using (3.30):

$$\sum_{k=2^{n+1}}^{\infty} c_k^{n+1} x_k(t) = R_{n+1}(t) - \sum_{k=1}^{2^{n+1}-1} c_k^{n+1} x_k(t) = R_{n+1}(t) - \sum_{k \in \Lambda_j} c_k^{n+1}. \tag{3.33}$$

Now, let

$$\Delta_j = \{i > 2^{n+1} - 1 \,|\, supp(x_i) \subseteq I_j\}. \tag{3.34}$$

Note that if $i > 2^{n+1} - 1$ and $i \notin \Delta_j$, then $supp(x_i) \cap I_j = \emptyset$. Therefore, for a.e. $t \in I_j$, using

(3.33):

$$R_{n+1}(t) - \sum_{k\in\Lambda_j} c_k^{n+1} = \sum_{k=2^{n+1}}^{\infty} c_k^{n+1} x_k(t) = \sum_{k\in\Delta_j} c_k^{n+1} x_k(t). \tag{3.35}$$

Hence, for a.e. $t \in I_j$, by (3.35):

$$\sum_{k\in\Delta_j} \left|c_k^{n+1}\right| x_k(t) \geq \left|\sum_{k\in\Delta_j} c_k^{n+1} x_k(t)\right| = \left|R_{n+1}(t) - \sum_{k\in\Lambda_j} c_k^{n+1}\right|. \tag{3.36}$$

Now we will obtain a lower bound on the following integral, where I_j' denotes the half of I_j on which $R_{n+1} = 1$, and I_j'' denotes the half of I_j on which $R_{n+1} = -1$:

$$\int_{I_j} \left|R_{n+1}(t) - \sum_{k\in\Lambda_j} c_k^{n+1}\right| dt = \int_{I_j'} \left|1 - \sum_{k\in\Lambda_j} c_k^{n+1}\right| dt + \int_{I_j''} \left|-1 - \sum_{k\in\Lambda_j} c_k^{n+1}\right| dt. \tag{3.37}$$

We must consider a few cases. If $\sum_{k\in\Lambda_j} c_k^{n+1} > 1$ on I_j:

$$(3.37) = \frac{\mu(I_j)}{2}\left(-1 + \sum_{k\in\Lambda_j} c_k^{n+1} + 1 + \sum_{k\in\Lambda_j} c_k^{n+1}\right) \geq \mu(I_j).$$

If $1 \geq \sum_{k\in\Lambda_j} c_k^{n+1} \geq -1$ on I_j:

$$(3.37) = \frac{\mu(I_j)}{2}\left(1 - \sum_{k\in\Lambda_j} c_k^{n+1} + 1 + \sum_{k\in\Lambda_j} c_k^{n+1}\right) \geq \mu(I_j).$$

If $-1 > \sum_{k\in\Lambda_j} c_k^{n+1}$ on I_j:

$$(3.37) = \frac{\mu(I_j)}{2}\left(1 - \sum_{k\in\Lambda_j} c_k^{n+1} - 1 - \sum_{k\in\Lambda_j} c_k^{n+1}\right) \geq \mu(I_j).$$

Hence, it holds that

$$\int_{I_j} \left| R_{n+1}(t) - \sum_{k \in \Lambda_j} c_k^{n+1} \right| dt \geq \mu(I_j). \tag{3.38}$$

Thus, employing (3.35), and (3.38):

$$\int_{I_j} \sum_{k \in \Delta_j} \left| c_k^{n+1} \right| x_k(t) dt \geq \int_{I_j} \left| \sum_{k \in \Delta_j} c_k^{n+1} x_k(t) \right| dt$$

$$= \int_{I_j} \left| R_{n+1}(t) - \sum_{k \in \Lambda_j} c_k^{n+1} \right| dt \geq \mu(I_j) = \frac{1}{2^n}.$$

Then using (3.35), and the fact that the I_j are disjoint dyadic intervals defined in (3.31):

$$\int_0^1 \sum_{k=2^{n+1}}^{\infty} \left| c_k^{n+1} \right| x_k(t) dt = \sum_{j=1}^{2^n} \int_{I_j} \sum_{k=2^{n+1}}^{\infty} \left| c_k^{n+1} \right| x_k(t) dt$$

$$= \sum_{j=1}^{2^n} \int_{I_j} \sum_{k \in \Delta_j} \left| c_k^{n+1} \right| x_k(t) dt \geq \sum_{j=1}^{2^n} \frac{1}{2^n} = 1.$$

In summary:

$$\int_0^1 \sum_{k=2^{n+1}}^{\infty} \left| c_k^{n+1} \right| x_k(t) dt \geq 1, \tag{3.39}$$

for all $n \in \mathbb{N}$.

Now, let $\varepsilon_k := sign(c_k^{n+1})$, and $\mathscr{E} := \{\varepsilon_k\}_{k=1}^{\infty}$. Define,

$$S_{N,\mathscr{E}}^G(R_{n+1}) := \sum_{k=1}^{N} \varepsilon_k c_k^{n+1} x_k(t). \tag{3.40}$$

Using the non-negativity of the x_k, we can see that the partial sums $\sum_{k=1}^N \left| c_k^{n+1} \right| x_k$ increase with N to $\sum_{k=1}^\infty \left| c_k^{n+1} \right| x_k$ pointwise. Hence, the Monotone Convergence Theorem yields:

$$\int_0^1 \sum_{k=1}^{\infty} \left|c_k^{n+1}\right| x_k(t)dt = \lim_{N\to\infty} \int_0^1 \sum_{k=1}^{N} \left|c_k^{n+1}\right| x_k(t)dt = \lim_{N\to\infty} \int_0^1 \left| \sum_{k=1}^{N} \varepsilon_k c_k^{n+1} x_k(t) \right| dt$$

$$\leq \lim_{N\to\infty} \left(\int_0^1 \left| \sum_{k=1}^{N} \varepsilon_k c_k^{n+1} x_k(t) \right|^p dt \right)^{\frac{1}{p}} = \lim_{N\to\infty} \|S_{N,\mathscr{E}}^G(R_{n+1})\|_p$$

$$\leq \lim_{N\to\infty} \left\| S_{N,\mathscr{E}}^G \right\| \|R_{n+1}\|_p \leq \Lambda_{\mathscr{E}}^G \cdot 1 < \infty,$$

by Lemma 3.4.6 and Hölder's inequality.

Hence:

$$\int_0^1 \sum_{k=1}^{\infty} \left|c_k^{n+1}\right| x_k(t)dt < \infty, \tag{3.41}$$

from which it follows readily that given any $\varepsilon > 0$, we can find some N large enough that

$$\left\| \sum_{k=N+1}^{\infty} \left|c_k^{n+1}\right| x_k \right\|_1 < \varepsilon. \tag{3.42}$$

Now, let $n(0) = 1$, and using (3.39), choose $m(1) > n(0)$ so that,

$$\int_0^1 \sum_{k=2^{n(0)+1}}^{2^{m(1)}} \left| c_k^{n(0)+1} \right| x_k(t)dt \geq \frac{2}{3}.$$

Since $g_k \in L^q(\mathbb{T})$ with $1 < q \leq \infty$, Lemma 3.2.2 yields that $\lim_{j\to\infty} \left|c_k^j\right| = \left|\langle R_j, g_k \rangle\right| = 0$. Using this and (3.42), we may find some $n(1) > m(1)$ satisfying :

$$\int_0^1 \sum_{k=2^{n(1)+1}}^{\infty} \left| c_k^{n(0)+1} \right| x_k(t)dt < \frac{1}{2^4},$$

and

$$\left|c_k^{n(1)+1}\right| < \frac{1}{2^3}\left|c_k^{n(0)+1}\right|,$$

for all $k \in A_0 := \left\{k \,|\, 2^{n(0)+1} \leq k \leq 2^{m(1)}, \text{ and } \left|c_k^{n(0)+1}\right| > 0\right\}$.

Similarly to the previous step, using (3.39) there exists some $m(2) \in \mathbb{N}$, where $m(2) > n(1)$, such that

$$\int_0^1 \sum_{k=2^{n(1)+1}}^{2^{m(2)}} \left|c_k^{n(1)+1}\right| x_k dt \geq \frac{2}{3},$$

and using 3.42, there exists some $n(2) \in \mathbb{N}$, where $n(2) > m(2)$, such that:

$$\int_0^1 \sum_{k=2^{n(2)+1}}^{\infty} \left|c_k^{n(0)+1}\right| x_k dt < \frac{1}{2^5},$$

$$\int_0^1 \sum_{k=2^{n(2)+1}}^{\infty} \left|c_k^{n(1)+1}\right| x_k dt < \frac{1}{2^5},$$

$$\left|c_k^{n(2)+1}\right| < \frac{1}{2^4}\left|c_k^{n(0)+1}\right|,$$

for all $k \in A_0 = \left\{k \,|\, 2^{n(0)+1} \leq k \leq 2^{m(1)}, \text{ and } \left|c_k^{n(0)+1}\right| > 0\right\}$, and

$$\left|c_k^{n(2)+1}\right| < \frac{1}{2^4}\left|c_k^{n(1)+1}\right|,$$

for all $k \in A_1 := \left\{k \,|\, 2^{n(1)+1} \leq k \leq 2^{m(2)}, \text{ and } \left|c_k^{n(1)+1}\right| > 0\right\}$.

We proceed inductively in this way, constructing sequences $\{n(j)\}_{j=0}^{\infty}$, and $\{m(j)\}_{j=1}^{\infty}$ such that:

$$1 = n(0) < m(1) < n(1) < m(2) < \cdots m(j) < n(j) < \cdots,$$

and for every $j \in \mathbb{N}$:

$$\int_0^1 \sum_{k=2^{n(j)+1}}^{2^{m(j+1)}} \left| c_k^{n(j)+1} \right| x_k dt \geq \frac{2}{3}, \tag{3.43}$$

$$\int_0^1 \sum_{k=2^{n(j+1)+1}}^{\infty} \left| c_k^{n(\ell)+1} \right| x_k dt < \frac{1}{2^{j+4}}, \tag{3.44}$$

for all $\ell = 0, 1, \cdots, j$, and

$$\left| c_k^{n(j+1)+1} \right| < \frac{1}{2^{j+3}} \left| c_k^{n(\ell)+1} \right|, \tag{3.45}$$

for all $\ell = 0, 1, \cdots, j$, and $k \in A_\ell := \left\{ k \,|\, 2^{n(\ell)+1} \leq k \leq 2^{m(\ell+1)}, \text{ and } \left| c_k^{n(\ell)+1} \right| > 0 \right\}$.

Now, consider the series $\sum_{j=1}^{\infty} \frac{1}{j} R_{n(j)+1}$. Applying Khinchine's inequality from (3.14), there is a positive constant, C, depending only upon p, such that for all $M, N \in \mathbb{N}$, where $M \leq N$:

$$\left\| \sum_{j=M}^{N} \frac{1}{j} R_{n(j)+1} \right\|_p \leq C \left(\sum_{j=M}^{N} \frac{1}{j^2} \right)^{\frac{1}{2}}.$$

Hence, letting $M, N \to \infty$, we see the sequence of partial sums of $\sum_{j=1}^{\infty} \frac{1}{j} R_{n(j)+1}$ is Cauchy in $L^p(\mathbb{T})$, and so it must be the case that for some $f \in L^p(\mathbb{T})$:

$$f = \sum_{j=1}^{\infty} \frac{1}{j} R_{n(j)+1} = \sum_{k=1}^{\infty} \langle f, g_k \rangle x_k, \tag{3.46}$$

where equality is in $L^p(\mathbb{T})$, and the second series converges unconditionally in $L^p(\mathbb{T})$ by assumption.

Notice that:

$$\left|\langle f,g_k\rangle-\sum_{i=1}^{j}\frac{1}{i}c_k^{n(i)+1}\right|=\left|\int_0^1 f\overline{g}_k dt-\sum_{i=1}^{j}\frac{1}{i}\int_0^1 R_{n(i)+1}\overline{g}_k dt\right|$$

$$=\left|\int_0^1\left(f-\sum_{i=1}^{j}\frac{1}{i}R_{n(i)+1}\right)\overline{g}_k dt\right|\leq\int_0^1\left|\left(f-\sum_{i=1}^{j}\frac{1}{i}R_{n(i)+1}\right)\overline{g}_k\right|dt,$$

applying Hölder's inequality,

$$\leq\left\|f-\sum_{i=1}^{j}\frac{1}{i}R_{n(i)+1}\right\|_p\|g_k\|_q.$$

Letting j go to infinity, we see that from (3.46):

$$\lim_{j\to\infty}\left|\langle f,g_k\rangle-\sum_{i=1}^{j}\frac{1}{i}c_k^{n(i)+1}\right|\leq\lim_{j\to\infty}\left\|f-\sum_{i=1}^{j}\frac{1}{i}R_{n(i)+1}\right\|_p\|g_k\|_q=0.$$

Hence, for each k:

$$\langle f,g_k\rangle=\sum_{i=1}^{\infty}\frac{1}{i}c_k^{n(i)+1}, \tag{3.47}$$

That is:

$$f=\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}\frac{1}{i}c_k^{n(i)+1}\right)x_k. \tag{3.48}$$

Therefore, for all $k\in A_j=\left\{k\,|\,2^{n(j)+1}\leq k\leq 2^{m(j+1)},\text{ and }\left|c_k^{n(j)+1}\right|>0\right\}$, and for every $j\geq 4$, using (3.47):

$$|\langle f,g_k\rangle|=\left|\sum_{i=1}^{\infty}\frac{1}{i}c_k^{n(i)+1}\right|\geq\frac{1}{j}\left|c_k^{n(j)+1}\right|-\left|\sum_{i\neq j}\frac{1}{i}c_k^{n(i)+1}\right|$$

$$\geq \frac{1}{j}\left|c_k^{n(j)+1}\right| - \sum_{i\neq j}\frac{1}{i}\left|c_k^{n(i)+1}\right| = \frac{1}{j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right| - \sum_{i=j+1}^{\infty}\left|\frac{1}{i}c_k^{n(i)+1}\right|$$

$$\geq \frac{1}{j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right| - \sum_{i=j+1}^{\infty}\left|\frac{1}{i}\frac{1}{2^{i+2}}c_k^{n(j)+1}\right|, \tag{3.49}$$

where the last inequality follows from the construction of the $c_k^{n(i)+1}$. More specifically, (3.45) says :

$$\left|c_k^{n(i)+1}\right| = \left|c_k^{n((i-1)+1)+1}\right| < \frac{1}{2^{i-1+3}}\left|c_k^{n(\ell)+1}\right| = \frac{1}{2^{i+2}}\left|c_k^{n(\ell)+1}\right|,$$

for all $k \in A_\ell$, and $\ell = 0, \cdots, i-1$. Since when $i \geq j+1$, $i-1 \geq j$, the inequality certainly holds for $\ell = j$.

Hence,

$$(3.49) \geq \frac{1}{j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right| - \left|\frac{1}{j}c_k^{n(j)+1}\right|\sum_{i=j+1}^{\infty}\frac{1}{2^{i+2}}$$

$$\geq \frac{1}{j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right| - \left|\frac{1}{j}c_k^{n(j)+1}\right|\frac{1}{4}\sum_{i=2}^{\infty}\frac{1}{2^i} = \frac{7}{8j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right|.$$

Summarizing,

$$|\langle f, g_k\rangle| \geq \frac{7}{8j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right|. \tag{3.50}$$

Then for all $j \geq 4$, and employing the positivity of the x_n, the definition of A_j in (3.45), and (3.50) :

$$\int_0^1 \sum_{k=2^{n(j)+1}}^{2^{m(j+1)}} |\langle f,g_k\rangle| x_k dt \geq \int_0^1 \sum_{k\in A_j} |\langle f,g_k\rangle| x_k dt \geq \frac{7}{8j}\int_0^1 \sum_{k\in A_j} \left|c_k^{n(j)+1}\right| x_k dt - \sum_{i=1}^{j-1}\frac{1}{i}\int_0^1 \sum_{k\in A_j}\left|c_k^{n(i)+1}\right| x_k dt$$

$$= \frac{7}{8j}\int_0^1 \sum_{k=2^{n(j)+1}}^{2^{m(j+1)}} \left|c_k^{n(j)+1}\right| x_k dt - \sum_{i=1}^{j-1}\frac{1}{i}\int_0^1 \sum_{k\in A_j}\left|c_k^{n(i)+1}\right| x_k dt$$

$$\geq \frac{7}{8j}\int_0^1 \sum_{k=2^{n(j)+1}}^{2^{m(j+1)}} \left|c_k^{n(j)+1}\right| x_k dt - \sum_{i=1}^{j-1}\frac{1}{i}\int_0^1 \sum_{k=2^{n(j)+1}}^{2^{m(j+1)}}\left|c_k^{n(i)+1}\right| x_k dt$$

$$\geq \frac{7}{12j} - \sum_{i=1}^{j-1}\frac{1}{i}\int_0^1 \sum_{k=2^{n(j)+1}}^{\infty}\left|c_k^{n(i)+1}\right| x_k dt \geq \frac{7}{12j} - \sum_{i=1}^{j-1}\frac{1}{i}\frac{1}{2^{j+3}},$$

by (3.44),

$$\geq \frac{7}{12j} - \frac{1}{8j^2}\sum_{i=1}^{j-1}\frac{1}{i} \geq \frac{7}{12j} - \frac{j-1}{8j^2} \geq \frac{7}{12j} - \frac{1}{8j}\frac{j-1}{j} \geq \frac{11}{24j}. \tag{3.51}$$

Now, define the sequence $E := \{\tilde{\varepsilon}_k\}_{k=1}^{\infty}$ as follows:

$$\begin{cases} sign(\langle f,g_k\rangle) & \text{if } 2^{n(j)+1} \leq k \leq 2^{m(j+1)}, j = 0,1,\cdots; \\ 0, & \text{otherwise;} \end{cases}$$

and define

$$f_\varepsilon(t) := \sum_{k=1}^{\infty} \tilde{\varepsilon}_k \langle f,g_k\rangle x_k(t) = \sum_{j=0}^{\infty}\sum_{k=2^{n(j)+1}}^{2^{m(j+1)}} |\langle f,g_k\rangle| x_k(t). \tag{3.52}$$

Employing the Monotone Convergence Theorem as earlier, since the x_k are non-negative, as well as Hölder's inequality:

$$\|f_\varepsilon\|_p \geq \|f_\varepsilon\|_1 = \int_0^1 \sum_{j=0}^{\infty} \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |\langle f, g_k\rangle| x_k(t) dt = \int_0^1 \lim_{J\to\infty} \sum_{j=0}^{J} \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |\langle f, g_k\rangle| x_k(t) dt$$

$$\geq \int_0^1 \lim_{J\to\infty} \sum_{j=4}^{J} \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |\langle f, g_k\rangle| x_k(t) dt = \lim_{J\to\infty} \int_0^1 \sum_{j=4}^{J} \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |\langle f, g_k\rangle| x_k(t) dt$$

$$= \lim_{J\to\infty} \sum_{j=4}^{J} \int_0^1 \sum_{k=2^{n(j)}+1}^{2^{m(j+1)}} |\langle f, g_k\rangle| x_k(t) dt \geq \lim_{J\to\infty} \sum_{j=4}^{J} \frac{11}{24j} dt = \infty.$$

However, it is also true, using (3.52), and employing the Monotone Convergence Theorem with partial sums as denoted in Definition 3.4.4, and Lemma 3.4.6:

$$\|f_\varepsilon\|_p = \left\| \sum_{k=1}^{\infty} \tilde{\varepsilon}_k \langle f, g_k\rangle x_k(t) \right\|_p = \left\| \lim_{J\to\infty} \sum_{k=1}^{J} \tilde{\varepsilon}_k \langle f, g_k\rangle x_k(t) \right\|_p =$$

$$= \lim_{J\to\infty} \|S_{J,E}^G(f)\|_p \leq \lim_{J\to\infty} \left\| S_{J,E}^G \right\| \|f\|_p \leq \Lambda_E^G \|f\|_p < \infty,$$

which is a contradiction. Thus, it must be that the system $\{x_k\}_{k=1}^{\infty}$ is not an unconditional quasibasis for $L^p(\mathbb{T})$, since $\{g_k\}_{k=1}^{\infty}$ was an arbitrary associated dual system. □

Corollary 3.4.9 (Existence of Positive Conditional Pseudobases)**.** *There exists a positive conditional pseudobasis for $L^p(\mathbb{T})$, where $1 \leq p < \infty$.*

Proof.

This follows immediately from Theorem 3.4.8 since the dyadic characteristic functions as constructed there must also form a conditional pseudobasis, since they form a conditional quasibasis.

□

We will use similar techniques to those used in Theorem 3.4.8 to prove the following result.

Theorem 3.4.10. *Let* $\{x_n\}_{n=1}^{\infty} := \left\{a_1, a_2, b_2, a_3, b_3, \cdots, a_j, b_j, \cdots\right\}$*, where* $a_n(t) = \chi_{\left[\frac{2j-2}{2^{k+1}}, \frac{2j-1}{2^{k+1}}\right]}(t)$*,* $b_n(t) = \chi_{\left[\frac{2j-1}{2^{k+1}}, \frac{2j}{2^{k+1}}\right]}$*, for* $n \geq 2$*, as given in (3.26), and* $a_1 \equiv 1$*, as in (3.27). Moreover suppose that* $\{\phi_n(t)\}_{n=1}^{\infty}$ *is a quasibasis in* $L^p(\mathbb{T})$*, with* $1 \leq p < \infty$*, where for each* $n \in \mathbb{N}$*, there is a sequence of non-negative constants,* $\{\gamma_j^n\}_{j=1}^{N_n}$*, where* N_n *depends upon* n*, such that* $\gamma_{N_n}^n \neq 0$*, and* $\phi_n(t)$ *can be written as:*

$$\phi_n(t) = \sum_{j=1}^{N_n} \gamma_j^n x_j(t),$$

then $\{\phi_n\}_{n=1}^{\infty}$ *is not an unconditional quasibasis.*

Proof.

First, we demonstrate that we may make further assumptions regarding the expression of $\phi_n(t)$ in terms of the $x_j(t)$. By the ordering of the x_j, the lengths of the supports of the $x_j(t)$ are non-increasing with respect to j. Considering ϕ_1, the support of $x_{N_1}(t)$ has the minimum length of all the supports of x_j, $j = 1, \cdots, N_1$, say, $supp(x_{N_1}(t)) = \left[\frac{j-1}{2^{\ell_1}}, \frac{j}{2^{\ell_1}}\right]$. Now, subdivide $[0,1]$ into intervals of the form $\left[\frac{i-1}{2^{\ell_1}}, \frac{i}{2^{\ell_1}}\right]$, $i = 1, \cdots, 2^{\ell_1}$. Now, let $k_1 := \ell_1$, and notice that $\phi_1(t)$ is constant and non-negative on each interval of this form. Since there are $2^{k_1} - 1$ dyadic sub-intervals of $[0,1]$ of length strictly greater than $\frac{1}{2^{k_1}}$, we see that we may in fact write:

$$\phi_1(t) = \sum_{j=2^{k_1}}^{2^{k_1+1}-1} d_j^1 x_j(t),$$

for the appropriate non-negative constants, d_j^1.

Continuing in this fashion, we define ℓ_n analogously to how we defined ℓ_1, and we let

$k_n = \max\{k_{n-1}+1, \ell_n\}$, so that the sequence $\{k_n\}_{n=1}^{\infty}$ is strictly increasing, and:

$$\phi_n(t) = \sum_{j=2^{k_n}}^{2^{k_n+1}-1} d_j^n x_j(t). \tag{3.53}$$

Notice that the indices over which any two sums of the form in (3.53), representing $\phi_n(t)$ and $\phi_m(t)$, where $n \neq m$, respectively, are non-overlapping. In the following arguments in this proof, we will assume that $\phi_n(t)$ has the form in (3.53).

Let $G = \{g_n(t)\}_{n=1}^{\infty} \subseteq L^q(\mathbb{T})$ be some dual system to $\{\phi_n\}_{n=1}^{\infty}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Proving by contradiction, suppose that for each $f \in L^p(\mathbb{T})$ the following expression converges unconditionally to f in L^p norm:

$$f = \sum_{n=1}^{\infty} \langle f, g_n \rangle \phi_n, \quad \text{where} \quad \langle f, g_n \rangle = \int_0^1 f \overline{g}_n dt.$$

Where R_n denotes the n^{th} Rademacher function, let

$$c_k^n := \int_0^1 R_n \overline{g}_k dt = \langle R_n, g_k \rangle, \tag{3.54}$$

so that for all $n = 0, 1, \cdots$:

$$R_n = \sum_{k=1}^{\infty} c_k^n \phi_k, \tag{3.55}$$

where the sum converges to R_n in $L^p(\mathbb{T})$, and convergence is unconditional by assumption.

Fix n, and let,

$$I_\alpha = \left(\frac{\alpha - 1}{2^{k_n}}, \frac{\alpha}{2^{k_n}} \right), \tag{3.56}$$

where $\alpha \in \{1, \cdots, 2^{k_n}\}$. Note that on such an interval, R_{k_n+1} "oscillates" once, and so is 1 on half of the measure of the interval, and -1 on the other half. Recall that the sequence k_n is a strictly increasing sequence, and so by our representation given in (3.53) for the ϕ_n, for all

$m = 1, \cdots, n$, $\phi_m(t)$ is constant on dyadic intervals of the form I_α - moreover, each x_j, where $j = 2^{k_m}, \cdots, 2^{k_m+1} - 1$ is constant on dyadic intervals of the form I_α, for $m = 1, \cdots, n$. Let

$$\Gamma_\alpha := \{m | 1 \le m \le n, \text{ and } \phi_m > 0 \text{ on } I_\alpha\}, \tag{3.57}$$

and

$$\Lambda_\alpha := \{\ell | \, m \in \Gamma_\alpha, \, 2^{k_m} \le \ell \le 2^{k_m+1} - 1, \text{ and } x_\ell > 0 \text{ on } I_\alpha\}. \tag{3.58}$$

Then for a.e. $t \in I_\alpha$, using the non-negativity of the ϕ_k, and the fact that the ϕ_k and x_j are constant on I_α in the range in the summation below, and equations (3.53), and (3.55):

$$\sum_{j=n+1}^{\infty} c_j^{k_n+1} \phi_j(t) = R_{k_n+1}(t) - \sum_{j=0}^{n} c_j^{k_n+1} \phi_j(t) = R_{k_n+1}(t) - \sum_{j=1}^{n} \left(c_j^{k_n+1} \sum_{\beta=2^{k_j}}^{2^{k_j+1}-1} d_\beta^j x_\beta(t) \right)$$

$$= R_{k_n+1}(t) - \sum_{m \in \Gamma_\alpha} \left(c_m^{k_n+1} \sum_{\ell \in \Lambda_\alpha} d_\ell^m \right), \tag{3.59}$$

for a.e. $t \in I_\alpha$.

Now, let $\Delta_\alpha = \{i > n \,|\, \mu(supp(\phi_i) \cap I_\alpha) \neq 0\}$. Therefore, for a.e. $t \in I_\alpha$, using (3.59):

$$R_{k_n+1}(t) - \sum_{j=0}^{n} c_j^{k_n+1} \phi_j(t) = R_{k_n+1}(t) - \sum_{m \in \Gamma_\alpha} c_m^{k_n+1} \sum_{\ell \in \Lambda_\alpha} d_\ell^m$$

$$= \sum_{j=n+1}^{\infty} c_j^{k_n+1} \phi_j(t) = \sum_{j \in \Delta_\alpha} c_j^{k_n+1} \phi_j(t). \tag{3.60}$$

Hence, for a.e. $t \in I_\alpha$, by (3.59), and (3.60):

$$\sum_{j\in\Delta_\alpha}\left|c_j^{k_n+1}\right|\phi_j(t)\geq\left|\sum_{j\in\Delta_\alpha}c_j^{k_n+1}\phi_j(t)\right|=\left|\sum_{j=n+1}^{\infty}c_j^{k_n+1}\phi_j(t)\right|=\left|R_{k_n+1}(t)-\sum_{j=0}^{n}c_j^{k_n+1}\phi_j(t)\right|$$

$$=\left|R_{k_n+1}(t)-\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m\right|. \tag{3.61}$$

Now we will obtain a lower bound on the following integral, where I'_α denotes the half of I_α, from (3.56), on which $R_{k_n+1}=1$, and I''_α denotes the half of I_α on which $R_{k_n+1}=-1$:

$$\int_{I_\alpha}\left|R_{k_n+1}(t)-\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m\right|dt \tag{3.62}$$

$$=\int_{I'_\alpha}\left|1-\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m\right|dt+\int_{I''_\alpha}\left|-1-\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m\right|dt$$

We must consider a few cases. If $\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m>1$ on I_α:

$$(3.62)=\frac{\mu(I_\alpha)}{2}\left(-1+\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m+1+\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m\right)\geq\mu(I_\alpha).$$

If $1\geq\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m\geq-1$ on I_α:

$$(3.62)=\frac{\mu(I_\alpha)}{2}\left(1-\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m+1+\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m\right)\geq\mu(I_\alpha).$$

If $-1>\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m$ on I_α:

$$(3.62)=\frac{\mu(I_\alpha)}{2}\left(1-\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m-1-\sum_{m\in\Gamma_\alpha}c_m^{k_n+1}\sum_{\ell\in\Lambda_\alpha}d_\ell^m\right)\geq\mu(I_\alpha).$$

Hence, it holds that $(3.62) \geq \mu(I_\alpha)$. Thus, employing (3.60):

$$\int_{I_\alpha} \sum_{j\in\Delta_\alpha} \left|c_j^{k_n+1}\right| \phi_j(t)dt \geq \int_{I_\alpha} \left| \sum_{j\in\Delta_\alpha} c_j^{k_n+1}\phi_j(t)\right| dt$$

$$= \int_{I_\alpha} \left| R_{k_n+1}(t) - \sum_{m\in\Gamma_\alpha} c_m^{k_n+1} \sum_{\ell\in\Lambda_\alpha} d_\ell^m \right| dt \geq \mu(I_\alpha) = \frac{1}{2^{k_n}}.$$

Then using that the I_α from (3.56) are disjoint, and (3.60):

$$\int_0^1 \sum_{j=n+1}^{\infty} \left|c_j^{k_n+1}\right| \phi_j(t)dt = \sum_{\alpha=1}^{2^{k_n}} \int_{I_\alpha} \sum_{j=n+1}^{\infty} \left|c_j^{k_n+1}\right| \phi_j(t)dt$$

$$\geq \sum_{\alpha=1}^{2^{k_n}} \int_{I_\alpha} \sum_{j\in\Delta_\alpha} \left|c_j^{k_n+1}\right| \phi_j(t)dt \geq \sum_{j=1}^{2^{k_n}} \frac{1}{2^{k_n}} = 1.$$

That is:

$$\int_0^1 \sum_{j=n+1}^{\infty} \left|c_j^{k_n+1}\right| \phi_j(t)dt \geq 1, \tag{3.63}$$

for each $n = 1, 2, \cdots$.

Now, let $\varepsilon_j := sign(c_j^{k_n+1})$, and $\mathscr{E} := \{\varepsilon_j\}_{j=1}^{\infty}$. Define, $S_{N,\mathscr{E}}^G(R_{k_n+1}) := \sum_{j=1}^N \varepsilon_j c_j^{k_n+1}\phi_j(t)$. Using the positivity of the ϕ_j, we can see that the partial sums $\sum_{j=1}^N \left|c_k^{k_n+1}\right| \phi_j$ increase with N to $\sum_{j=1}^{\infty} \left|c_j^{k_n+1}\right| \phi_j$ pointwise. Hence, the Monotone Convergence Theorem yields:

$$\int_0^1 \sum_{j=1}^{\infty} \left|c_j^{k_n+1}\right| \phi_j(t)dt = \lim_{N\to\infty} \int_0^1 \sum_{j=1}^{N} \left|c_j^{k_n+1}\right| \phi_j(t)dt = \lim_{N\to\infty} \int_0^1 \left| \sum_{j=1}^{N} \varepsilon_j c_j^{k_n+1}\phi_j(t)\right| dt$$

$$\leq \lim_{N\to\infty} \left(\int_0^1 \left| \sum_{j=1}^{N} \varepsilon_j c_j^{k_n+1}\phi_j(t)\right|^p dt \right)^{\frac{1}{p}} = \lim_{N\to\infty} \|S_{N,\mathscr{E}}^G(R_{k_n+1})\|_p$$

$$\leq \lim_{N\to\infty} \left\| S^G_{N\mathscr{E}} \right\| \|R_{k_n+1}\|_p \leq \Lambda^G_{\mathscr{E}} \cdot 1 < \infty,$$

by Lemma 3.4.6, and Hölder's inequality. Hence,

$$\int_0^1 \sum_{j=1}^{\infty} \left| c_j^{k_n+1} \right| \phi_j(t) dt < \infty. \tag{3.64}$$

It follows readily that given any $\varepsilon > 0$, we can find some N large enough that

$$\left\| \sum_{j=N+1}^{\infty} \left| c_j^{k_n+1} \right| \phi_j \right\|_1 < \varepsilon. \tag{3.65}$$

We now let $n(0) = k_1$, and say that $n(0) = k_{r(0)}$ (i.e. set $r(0) = 1$). Using (3.63), choose $m(1) > r(0) + 1$ so that,

$$\int_0^1 \sum_{j=r(0)+1}^{m(1)} \left| c_j^{n(0)+1} \right| \phi_j(t) dt \geq \frac{2}{3}.$$

Since $g_k \in L^q(\mathbb{T})$ with $1 < q \leq \infty$, Lemma 3.2.2 yields that $\lim_{n\to\infty} \left| c_j^n \right| = \left| \langle R_n, g_j \rangle \right| = 0$, for each j. Using this, and (3.65) we may find $n(1)$, and $r(1)$, where $r(1) > m(1)$, $n(1) \in \{k_i\}_{i=r(0)+1}^{\infty}$, and $n(1) := k_{r(1)}$, so that:

$$\int_0^1 \sum_{j=r(1)+1}^{\infty} \left| c_j^{n(0)+1} \right| \phi_j(t) dt < \frac{1}{2^4},$$

and

$$\left| c_j^{n(1)+1} \right| < \frac{1}{2^3} \left| c_j^{n(0)+1} \right|,$$

for all $j \in A_0 := \left\{ j \,|\, r(0) + 1 \leq j \leq m(1), \text{ and } \left| c_j^{n(0)+1} \right| > 0 \right\}$.

In a similar fashion to the previous step, using (3.63) there exists some $m(2) \in \mathbb{N}$, where $m(2) > r(1) + 1$, such that

$$\int_0^1 \sum_{j=r(1)+1}^{m(2)} \left|c_j^{n(1)+1}\right| \phi_j dt \geq \frac{2}{3},$$

and there exist $r(2)$, and $n(2)$, where $r(2) > m(2)$, and $n(2) \in \{k_n\}_{n=r(1)+1}^{\infty}$, and we say that $n(2) := k_{r(2)}$, such that:

$$\int_0^1 \sum_{j=r(2)+1}^{\infty} \left|c_j^{n(0)+1}\right| \phi_j dt < \frac{1}{2^5},$$

$$\int_0^1 \sum_{j=r(2)+1}^{\infty} \left|c_j^{n(1)+1}\right| \phi_j dt < \frac{1}{2^5},$$

$$\left|c_j^{n(2)+1}\right| < \frac{1}{2^4}\left|c_j^{n(0)+1}\right|,$$

for all $j \in A_0 = \left\{ j \,|\, r(0)+1 \leq j \leq m(1), \text{ and } \left|c_j^{n(0)+1}\right| > 0 \right\}$, and

$$\left|c_j^{n(2)+1}\right| < \frac{1}{2^4}\left|c_j^{n(1)+1}\right|,$$

for all $j \in A_1 := \left\{ j \,|\, r(1)+1 \leq j \leq m(2), \text{ and } \left|c_j^{n(1)+1}\right| > 0 \right\}$. We proceed inductively in this way, constructing sequences, $\{r(i)\}_{i=0}^{\infty}$, $\{n(i)\}_{i=0}^{\infty}$, and $\{m(i)\}_{i=1}^{\infty}$, where $n(i) = k_{r(i)}$, such that:

$$1 = r(0) < m(1) < r(1) < m(2) < \cdots m(i) < r(i) < \cdots,$$

$$k_1 = n(1) < n(2) < \cdots < n(i) < \cdots$$

and for every $i \in \mathbb{N}$:

$$\int_0^1 \sum_{j=r(i)+1}^{m(i+1)} \left|c_j^{n(i)+1}\right| \phi_j dt \geq \frac{2}{3}, \tag{3.66}$$

$$\int_0^1 \sum_{j=r(i+1)+1}^{\infty} \left| c_j^{n(\ell)+1} \right| \phi_j dt < \frac{1}{2^{i+4}}, \tag{3.67}$$

for all $\ell = 0, 1, \cdots, i$, and

$$\left| c_j^{n(i+1)+1} \right| < \frac{1}{2^{i+3}} \left| c_j^{n(\ell)+1} \right|, \tag{3.68}$$

for all $\ell = 0, 1, \cdots, i$, and $j \in A_\ell := \left\{ j \,|\, r(\ell) + 1 \leq j \leq m(\ell+1), \text{ and } \left| c_j^{n(\ell)+1} \right| > 0 \right\}$.

Now, consider the series $\sum_{j=1}^{\infty} \frac{1}{j} R_{n(j)+1}$. Applying Khinchine's inequality, from (3.14), there is a positive constant, C, depending only upon p, such that for all $M, N \in \mathbb{N}$, where $M \leq N$:

$$\left\| \sum_{j=M}^{N} \frac{1}{j} R_{n(j)+1} \right\|_p \leq C \left(\sum_{j=M}^{N} \frac{1}{j^2} \right)^{\frac{1}{2}}.$$

Hence, letting $M, N \to \infty$, we see that the sequence of partial sums of $\sum_{j=1}^{\infty} \frac{1}{j} R_{n(j)+1}$ is Cauchy, and so it must be the case that for some $f \in L^p(\mathbb{T})$:

$$f = \sum_{j=1}^{\infty} \frac{1}{j} R_{n(j)+1} = \sum_{k=0}^{\infty} \langle f, g_k \rangle \phi_k, \tag{3.69}$$

where equality is in $L^p(\mathbb{T})$, and the second series converges unconditionally in $L^p(\mathbb{T})$ by assumption.

Now, note that:

$$\left| \langle f, g_k \rangle - \sum_{i=1}^{j} \frac{1}{i} c_k^{n(i)+1} \right| = \left| \int_0^1 f \overline{g}_k dt - \sum_{i=1}^{j} \frac{1}{i} \int_0^1 R_{n(i)+1} \overline{g}_k dt \right|$$

$$= \left| \int_0^1 \left(f - \sum_{i=1}^{j} \frac{1}{i} R_{n(i)+1} \right) \overline{g}_k dt \right| \leq \int_0^1 \left| \left(f - \sum_{i=1}^{j} \frac{1}{i} R_{n(i)+1} \right) \overline{g}_k \right| dt$$

applying Hölder's inequality,

$$\leq \left\| f - \sum_{i=1}^{j} \frac{1}{i} R_{n(i)+1} \right\|_p \|g_k\|_q .$$

Letting j go to infinity, and applying (3.69):

$$\lim_{j\to\infty} \left| \langle f, g_k \rangle - \sum_{i=1}^{j} \frac{1}{i} c_k^{n(i)+1} \right| \leq \lim_{j\to\infty} \left\| f - \sum_{i=1}^{j} \frac{1}{i} R_{n(i)+1} \right\|_p \|g_k\|_q = 0.$$

Hence, for each $k \in \mathbb{N}$,

$$\langle f, g_k \rangle = \sum_{i=1}^{\infty} \frac{1}{i} c_k^{n(i)+1}. \tag{3.70}$$

Therefore, for all $k \in A_j$, and for every $j \geq 4$ using (3.70):

$$|\langle f, g_k \rangle| = \left| \sum_{i=1}^{\infty} \frac{1}{i} c_k^{n(i)+1} \right| \geq \frac{1}{j} \left| c_k^{n(j)+1} \right| - \left| \sum_{i\neq j} \frac{1}{i} c_k^{n(i)+1} \right|$$

$$\geq \frac{1}{j} \left| c_k^{n(j)+1} \right| - \sum_{i\neq j} \frac{1}{i} \left| c_k^{n(i)+1} \right| = \frac{1}{j} \left| c_k^{n(j)+1} \right| - \sum_{i=1}^{j-1} \left| \frac{1}{i} c_k^{n(i)+1} \right| - \sum_{i=j+1}^{\infty} \left| \frac{1}{i} c_k^{n(i)+1} \right|$$

$$\geq \frac{1}{j} \left| c_k^{n(j)+1} \right| - \sum_{i=1}^{j-1} \left| \frac{1}{i} c_k^{n(i)+1} \right| - \sum_{i=j+1}^{\infty} \left| \frac{1}{i} \frac{1}{2^{i+2}} c_k^{n(j)+1} \right|, \tag{3.71}$$

where the last inequality follows from the construction of the $c_k^{n(i)+1}$. More specifically from (3.68):

$$\left| c_k^{n(i)+1} \right| = \left| c_k^{n((i-1)+1)} \right| < \frac{1}{2^{i-1+3}} \left| c_k^{n(\ell)+1} \right| = \frac{1}{2^{i+2}} \left| c_k^{n(\ell)+1} \right|,$$

for all $k \in A_\ell$, and $\ell = 0, \cdots, i-1$. Since when $i \geq j+1$, $i-1 > j$, this inequality certainly holds when $\ell = j$. So,

$$(3.71) \geq \frac{1}{j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right| - \left|\frac{1}{j}c_k^{n(j)+1}\right|\sum_{i=j+1}^{\infty}\frac{1}{2^{i+2}}$$

$$\geq \frac{1}{j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right| - \left|\frac{1}{j}c_k^{n(j)+1}\right|\frac{1}{4}\sum_{i=2}^{\infty}\frac{1}{2^{i}}$$

$$= \frac{7}{8j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right|.$$

In summary, for all $k \in A_j$:

$$|\langle f, g_k\rangle| \geq \frac{7}{8j}\left|c_k^{n(j)+1}\right| - \sum_{i=1}^{j-1}\left|\frac{1}{i}c_k^{n(i)+1}\right|. \tag{3.72}$$

Then for all $j \geq 4$, employing the positivity of the ϕ_n, and (3.68):

$$\int_0^1 \sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k\rangle|\,\phi_k dt \geq \int_0^1 \sum_{k\in A_j} |\langle f, g_k\rangle|\,\phi_k dt$$

$$\geq \frac{7}{8j}\int_0^1 \sum_{k\in A_j}\left|c_k^{n(j)+1}\right|\phi_k dt - \sum_{i=1}^{j-1}\frac{1}{i}\int_0^1 \sum_{k\in A_j}\left|c_k^{n(i)+1}\right|\phi_k dt$$

$$= \frac{7}{8j}\int_0^1 \sum_{k=r(j)+1}^{m(j+1)}\left|c_k^{n(j)+1}\right|\phi_k dt - \sum_{i=1}^{j-1}\frac{1}{i}\int_0^1 \sum_{k\in A_j}\left|c_k^{n(i)+1}\right|\phi_k dt$$

$$\geq \frac{7}{8j}\int_0^1 \sum_{k=r(j)+1}^{m(j+1)}\left|c_k^{n(j)+1}\right|\phi_k dt - \sum_{i=1}^{j-1}\frac{1}{i}\int_0^1 \sum_{k=r(j)+1}^{m(j+1)}\left|c_k^{n(i)+1}\right|\phi_k dt$$

$$\geq \frac{7}{12j} - \sum_{i=1}^{j-1}\frac{1}{i}\int_0^1 \sum_{k=r(j)+1}^{\infty}\left|c_k^{n(i)+1}\right|\phi_k dt \geq \frac{7}{12j} - \sum_{i=1}^{j-1}\frac{1}{i}\frac{1}{2^{j+3}},$$

where we have used (3.67),

$$\geq \frac{7}{12j} - \frac{1}{8j^2}\sum_{i=1}^{j-1}\frac{1}{i} \geq \frac{7}{12j} - \frac{j-1}{8j^2} \geq \frac{7}{12j} - \frac{1}{8j}\frac{j-1}{j} \geq \frac{11}{24j}.$$

Now, define the sequence $E := \{\tilde{\varepsilon}_k\}_{k=1}^{\infty}$ as follows:

$$\begin{cases} sign(\langle f, g_k\rangle) & \text{if } r(j)+1 \leq k \leq m(j+1), j = 0, 1, \cdots; \\ 0, & \text{otherwise;} \end{cases}$$

and define

$$f_\varepsilon(t) := \sum_{k=0}^{\infty} \tilde{\varepsilon}_k \langle f, g_k\rangle \phi_k(t) = \sum_{j=0}^{\infty}\sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k\rangle| \phi_k(t). \tag{3.73}$$

Employing the Monotone Convergence Theorem as earlier, since the ϕ_k are non-negative, as well as Hölder's inequality:

$$\|f_\varepsilon\|_p \geq \|f_\varepsilon\|_1 = \int_0^1 \sum_{j=0}^{\infty}\sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k\rangle| \phi_k(t)dt = \int_0^1 \lim_{J\to\infty}\sum_{j=0}^{J}\sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k\rangle| \phi_k(t)dt$$

$$\geq \int_0^1 \lim_{J\to\infty}\sum_{j=4}^{J}\sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k\rangle| \phi_k(t)dt = \lim_{J\to\infty}\int_0^1 \sum_{j=4}^{J}\sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k\rangle| \phi_k(t)dt$$

$$= \lim_{J\to\infty}\sum_{j=4}^{J}\int_0^1 \sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k\rangle| \phi_k(t)dt \geq \lim_{J\to\infty}\sum_{j=4}^{J}\frac{11}{24j} = \infty,$$

by (3.51). However, it is also true, using (3.73), and employing the Monotone Convergence Theorem with partial sums denoted as in Definition 3.4.4, and Lemma 3.4.6:

$$\|f_\varepsilon\|_p = \left\|\sum_{k=0}^{\infty} \tilde{\varepsilon}_k \langle f, g_k\rangle \phi_k(t)\right\|_p = \left\|\lim_{J\to\infty}\sum_{k=0}^{J} \tilde{\varepsilon}_k \langle f, g_k\rangle \phi_k(t)\right\|_p = \lim_{J\to\infty}\left\|\sum_{k=0}^{J} \tilde{\varepsilon}_k \langle f, g_k\rangle \phi_k(t)\right\|_p$$

$$= \lim_{J\to\infty} \|S^G_{J,E}(f)\|_p \leq \lim_{J\to\infty} \left\|S^G_{J,E}\right\| \|f\|_p \leq \Lambda^G_E \|f\|_p < \infty,$$

which is a contradiction. Thus, it must be that the system $\{\phi_k\}_{k=1}^{\infty}$ is not an unconditional quasibasis for $L^p(\mathbb{T})$.

□

Theorem 3.4.11 (Stability of Quasibases)**.** *Let $\{x_n\}_{n=1}^{\infty}$ be a quasibasis for a Banach space, X, with dual system $\{a_n\}_{n=1}^{\infty}$ in X^*. Let $1 > \varepsilon > 0$. Suppose that $\{y_n\}_{n=1}^{\infty}$ is a system in X with the property that,*

$$\|x_i - y_i\| \leq \frac{\varepsilon}{2^{i+1}\|a_i\|}.$$

Then $\{y_i\}_{i=1}^{\infty}$ is also a quasibasis in X.

Proof.

Formally, define

$$S(x) := \sum_{i=1}^{\infty} \langle x, a_i\rangle (x_i - y_i).$$

We show first that this sum converges, by demonstrating that the sequence of partial sums is Cauchy. Suppose $N > M$:

$$\left\|\sum_{i=1}^{N} \langle x, a_i\rangle (x_i - y_i) - \sum_{i=1}^{M} \langle x, a_i\rangle (x_i - y_i)\right\| = \left\|\sum_{i=M}^{N} \langle x, a_i\rangle (x_i - y_i)\right\| \leq \sum_{i=M}^{N} |\langle x, a_i\rangle| \|x_i - y_i\|$$

$$\leq \sum_{n=M}^{N} \|x\|\|a_i\|\|x_i - y_i\| \leq \|x\|\varepsilon \sum_{i=N}^{M} \frac{1}{2^{i+1}},$$

which goes to 0 as $N, M \to \infty$.

Using a similar computation, we show that S is bounded:

$$\|S(x)\| = \left\| \sum_{i=1}^{\infty} \langle x, a_i \rangle (x_i - y_i) \right\| = \left\| \lim_{N\to\infty} \sum_{i=1}^{N} \langle x, a_i \rangle (x_i - y_i) \right\|. \tag{3.74}$$

Since the partial sums converge to $S(x)$,

$$(3.74) = \lim_{N\to\infty} \left\| \sum_{n=1}^{N} \langle x, a_i \rangle (x_i - y_i) \right\| \leq \lim_{N\to\infty} \sum_{i=1}^{N} |\langle x, a_i \rangle| \, \|x_i - y_i\|$$

$$\leq \lim_{N\to\infty} \sum_{i=1}^{N} \|x\| \, \|a_i\| \, \frac{\varepsilon}{2^{i+1} \|a_i\|} = \frac{\varepsilon}{2} \|x\|.$$

Therefore, $\|S\| < 1$.

Now, define $T(x) := (I - S)(x) = x - S(x)$. Note that T is invertible, and has bounded inverse, since $\|S\| < 1$, by exercise 2.40 in, [3]. We now show that $(I - S)(x) = \sum_{i=1}^{\infty} \langle x, a_i \rangle y_i$.

First note that the sequence of partial sums of $\sum_{i=1}^{\infty} \langle x, a_i \rangle y_i$ is Cauchy, and hence the series converges, since:

$$\left\| \sum_{n=M}^{N} \langle x, a_n \rangle y_n \right\| \leq \left\| \sum_{n=M}^{N} \langle x, a_n \rangle (x_n - y_n) \right\| + \left\| \sum_{n=M}^{N} \langle x, a_n \rangle x_n \right\|,$$

where both of the sums are from Cauchy sequences of partial sums from $S(x)$, and the representations of x with respect to the quasibasis $\{x_i\}_{i=1}^{\infty}$, respectively.

Now, consider the following:

$$\left\| \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n - T(x) \right\| = \left\| \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n - (I - S)(x) \right\|$$

$$= \left\| \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n - \sum_{n=1}^{\infty} \langle x, a_n \rangle x_n + \sum_{n=1}^{\infty} \langle x, a_n \rangle (x_n - y_n) \right\|,$$

using the fact that the second and third sums converge,

$$= \left\| \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n + \lim_{N\to\infty} \left(-\sum_{n=1}^{N} \langle x, a_n \rangle x_n + \sum_{n=1}^{N} \langle x, a_n \rangle (x_n - y_n) \right) \right\|$$

$$= \left\| \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n - \lim_{N\to\infty} \left(\sum_{n=1}^{N} \langle x, a_n \rangle y_n \right) \right\| = \left\| \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n - \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n \right\| = 0.$$

Hence,

$$T(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n.$$

Given $x \in X$:

$$x = T(T^{-1}(x)) = \sum_{i=1}^{\infty} \left\langle T^{-1}(x), a_i \right\rangle y_i = \sum_{i=1}^{\infty} \left\langle x, (T^{-1})^* a_i \right\rangle y_i. \tag{3.75}$$

Let $\{b_i\}_{i=1}^{\infty} = \{(T^{-1})^* a_i\}_{i=1}^{\infty}$. Since T^{-1} is bounded, and $\|T^{-1}\| = \|(T^{-1})^*\|$, it follows that $(T^{-1})^* a_i \in X^*$. Hence, we can see (3.75) yields that $\{y_i\}_{i=1}^{\infty}$ is a quasibasis with dual system $\{b_i\}_{i=1}^{\infty} \subseteq X^*$.

□

Theorem 3.4.12 (Stability of Unconditional Quasibases)**.** *Let $\{x_n\}_{n=1}^{\infty}$ be an unconditional quasibasis for a Banach space, X, with dual system $\{a_n\}_{n=1}^{\infty} \subseteq X^*$, in terms of which expansions of elements of X are unconditional. Let $1 > \varepsilon > 0$. Suppose that $\{y_n\}_{n=1}^{\infty}$ is a system in X with the property that,*

$$\|x_i - y_i\| \le \frac{\varepsilon}{2^{i+1} \|a_i\|}.$$

Then $\{y_i\}_{i=1}^{\infty}$ is an unconditional quasibasis in X.

Proof.

Let $S(x) := \sum_{n=1}^{\infty} \langle x, a_n \rangle (x_n - y_n)$, and $T(x) = (I-S)(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n$, as in Theorem 3.4.11, which then yields that $\{y_n\}_{n=1}^{\infty}$ is a quasibasis with a dual system in X^*, given by $\{(T^{-1})^* a_n\}_{n=1}^{\infty}$. It remains to show that,

$$x = \sum_{n=1}^{\infty} \left\langle x, \left(T^{-1}\right)^* a_n \right\rangle y_n,$$

converges unconditionally. Let $\sigma(n) := \{i(1), i(2), \cdots\}$ be some permutation of $\mathbb{N}$.

Then, computing the norm of the difference between S_N and S_M for $N > M$:

$$\left\| \sum_{n=M}^{N} \left\langle x, (T^{-1})^* a_{i(n)} \right\rangle y_{i(n)} \right\| \leq \left\| \sum_{n=M}^{N} \left\langle x, (T^{-1})^* a_{i(n)} \right\rangle \left(x_{i(n)} - y_{i(n)} \right) \right\| + \left\| \sum_{n=M}^{N} \left\langle x, (T^{-1})^* a_{i(n)} \right\rangle x_{i(n)} \right\|$$

$$= \left\| \sum_{n=M}^{N} \left\langle T^{-1}(x), a_{i(n)} \right\rangle \left(x_{i(n)} - y_{i(n)} \right) \right\| + \left\| \sum_{n=M}^{N} \left\langle T^{-1}(x), a_{i(n)} \right\rangle x_{i(n)} \right\|$$

$$\leq \sum_{n=M}^{N} \left\| T^{-1}(x) \right\| \left\| a_{i(n)} \right\| \left\| x_{i(n)} - y_{i(n)} \right\| + \left\| \sum_{n=M}^{N} \left\langle T^{-1}(x), a_{i(n)} \right\rangle x_{i(n)} \right\|$$

$$\leq \left\| T^{-1}(x) \right\| \varepsilon \sum_{n=M}^{N} \frac{1}{2^{i(n)+1}} + \left\| \sum_{n=M}^{N} \left\langle T^{-1}(x), a_{i(n)} \right\rangle x_{i(n)} \right\|.$$

Taking the limit as $N, M \to \infty$ of both sides of the previous inequality, and using the unconditionality of $\{x_n\}_{n=1}^{\infty}$:

$$\lim_{M \to \infty} \left\| \sum_{n=M}^{N} \left\langle x, \left(T^{-1}\right)^* a_{i(n)} \right\rangle y_{i(n)} \right\| = 0.$$

Hence, the sequence of partial sums of

$$\sum_{n=1}^{\infty}\left\langle x,\left(T^{-1}\right)^{*}a_{i(n)}\right\rangle y_{i(n)},$$

is Cauchy, and so it must also converge. Though it is not required to show that $\{y_i\}_{i=1}^{\infty}$ is an unconditional quasibasis, note that convergence to x, regardless of the permutation $\sigma(n)$, is shown in Corollary 3.11 of, [3]. Hence $\{y_i\}_{i=1}^{\infty}$ is an unconditional quasibasis.

$\square$

Theorem 3.4.13 (Non-Existence of Positive Unconditional Quasibases)**.** *There does not exist a positive unconditional quasibasis for $L^p(\mathbb{T})$, where $1 \le p < \infty$.*

Proof.

Proving by contradiction, suppose that $\{y_n\}_{n=1}^{\infty}$ is a positive unconditional quasibasis for $L^p(\mathbb{T})$, where $1 \le p < \infty$, with some dual system $\{a_n\}_{n=1}^{\infty} \subseteq L^q(\mathbb{T})$, in terms of which expansions of elements of X are unconditional, and where $\frac{1}{p}+\frac{1}{q}=1$. Let $\varepsilon > 0$. Since the dyadic characteristic functions are complete in $L^p(\mathbb{T})$, for each $n \in \mathbb{N}$, there exists a dyadic step function x_j^n, and strictly positive constants, γ_j^n such that

$$\left\|y_n-\sum_{j=1}^{N_n}\gamma_j^n x_j^n\right\| < \frac{\varepsilon}{2^{n+1}\|a_n\|}.$$

Then, by Theorem 3.4.11, $\left\{\sum_{j=1}^{N_n}\gamma_j^n x_j^n\right\}_{n=1}^{\infty}$ is also an unconditional quasibasis. Noting that $\gamma_j^{N_n} > 0$, we may apply Theorem 3.4.10 to yield that

$$\left\{\sum_{j=1}^{N_n}\gamma_j^n x_j^n\right\}_{n=1}^{\infty},$$

must be a conditional quasibasis. Hence, we have obtained the desired contradiction, and it must be that $\{y_n\}_{n=1}^{\infty}$ is a conditional quasibasis.

$\square$

Corollary 3.4.14 (Non-Existence of Positive Unconditional Bases for $L^1(\mathbb{T})$)**.** *There does not exist a positive unconditional basis for* $L^1(\mathbb{T})$.

Proof.

This follows directly from Theorem 3.4.13, since any positive unconditional Schauder basis is a positive unconditional quasibasis for the same space.

□

3.5 Hamel Bases

Theorem 3.5.1. *Let L be a linear space of functions $f : E \to \mathbb{F}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, with the property that given any real-valued $f \in L$, the functions defined by:*

$$f_+(x) := \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

$$f_-(x) := \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases},$$

are both in L. Note that in this case, $f = f_+ + (-f_-)$. Suppose also that given a complex valued, $f \in L$, where $f = g + ih$, both g, and h are in L.

Then, there exists a Hamel Basis M for L, such that each $f \in M$ has the property that $f(x) \geq 0$ for all $x \in E$.

Proof.

Let P be the family of all finitely linearly independent subsets of L, each of which contains only functions that are non-negative. Clearly P is non-empty, since $\emptyset \in P$. Consider the partial ordering on P given by subset inclusion. We will first show that every chain, or totally ordered non-empty subset of P, has an upper bound, with the goal of applying Zorn's Lemma to obtain a maximal element in P, [14].

Let B be a chain in P. Let $\tilde{B}$ be the union of the sets in B. Clearly, $\tilde{B} \subseteq L$, and each $f \in \tilde{B}$ is non-negative. We must show that $\tilde{B}$ is finitely linearly independent. Let $\{f_n\}_{n=1}^N \subseteq \tilde{B}$. For each $n = 1, \cdots, N$, there is some set $B_n \in B$ such that $f_n \in B_n$. Since B is well ordered, there is some n_0, where $1 \leq n_0 \leq N$, such that $B_n \subseteq B_{n_0}$ for all $n = 1, \cdots, N$. Hence, $\{f_n\}_{n=1}^N \subseteq B_{n_0}$, and so $\{f_n\}_{n=1}^N$ is a linearly independent set, since $B_n \in B$. Hence, $\tilde{B}$ is finitely linearly independent, and so $\tilde{B} \in P$.

Because every chain in P has an upper bound in P, we may apply Zorn's lemma to obtain a maximal element, M, in P, [14] . Since $M \in P$, M is finitely linearly independent.

We now show that M spans L. Let $f \in L$, and suppose that $f(x) > 0$ for all $x \in E$. If $M \cup \{f\}$ were finitely linearly independent, the maximality of M would be contradicted. Hence, f can be written as a finite linear combination of elements of M. Now, let g be an arbitrary element of L. Then $g = Re(g)_+ - Re(g)_- + iIm(g)_+ - iIm(g)_-$, where $Re(g)_+, Re(g)_-, Im(g)_+$, and $Im(g)_-$ are all non-negative. But then by what was argued earlier in this paragraph, it must be that $Re(g)_+, Re(g)_-, Im(g)_+$, and $Im(g)_-$ are finite linear combinations of elements of M. Therefore, g must be expressible as a finite linear combination of elements of M. Hence, M forms a Hamel Basis for L whose elements are non-negative. $\square$

Corollary 3.5.2 (Existence of Positive Hamel Bases)**.** *There exists a Hamel basis for $L^p(E)$, $0 < p \leq \infty$, where $E = \mathbb{R}^d$ or $\mathbb{T}^d$, where $d \in \mathbb{N}$, with the property that each element of the basis is pointwise a.e. non-negative.*

Proof.

This follows from Theorem 3.5.1, since $L^p(E)$ is a linear space, and given $f \in L^p(E)$, where $0 < p \leq \infty$, it is the case that $Re(f) \in L^p(E)$, $Im(f) \in L^p(E)$, and for any real-valued f, f_+, and f_-, as defined in Theorem 3.5.1, are both in $L^p(E)$. $\square$

CHAPTER 4

THE WINDOWED WALSH SYSTEM

4.1 Properties of The Walsh System

First, we establish some basic background properties of the Walsh system, as given in Definition 2.5.1.

Lemma 4.1.1. $\forall\, k \in \{0,1,\cdots,2^N-1\}$, $\chi_{k,N} \in \{\text{span}(w_n) : 1 \leq n \leq 2^N\}$. *Moreover,*

$$\chi_{k,N}(x) = \sum_{m=1}^{2^N} \left\langle \chi_{k,N}, w_m \right\rangle w_m(x),$$

for all $x \in [0,1] \setminus \{\frac{k}{2^N} \,|\, k = 0,1,\cdots,2^N\}$.

Proof.

Let $k \in \{0,1,\cdots,2^N-1\}$, and recall that $supp(\chi_{k,N}) = [k2^{-N},(k+1)2^{-N}]$. To determine $\left\langle \chi_{k,N}, w_m \right\rangle$, where $m > 2^N$, it suffices to consider the behavior of w_m on $[k2^{-N},(k+1)2^{-N}]$. Suppose that $m-1 = 2^{n_1}+2^{n_2}+\cdots+2^{n_\nu}$, where $n_1 > n_2 > \cdots > n_\nu \geq 0$. Then by how it is defined, $sign(w_m(x))$ changes exactly once on $[k2^{-n_1},(k+1)2^{-n_1}]$, at its midpoint. So $w_m(x)$ is 1 on half of the interval, $[k2^{-n_1},(k+1)2^{-n_1}]$, and -1 on the other half. Since $m-1 > 2^N-1$, we know that $[k2^{-N},(k+1)2^{-N}]$ is a non-overlapping union of an even number of intervals of the form $[k2^{-n_1},(k+1)2^{-n_1}]$, leading us to conclude that $w_m(x)$ is 1 on exactly half of the measure of $[k2^{-N},(k+1)2^{-N}]$, and -1 on the other half of the measure. Hence, $\left\langle \chi_{k,N}, w_m \right\rangle = 0$ for $m > 2^N$. Since the Walsh system is an orthonormal basis for $L^2(\mathbb{T})$ (see explanation given with Definition 2.5.1):

$$\chi_{k,N}(x) = \sum_{m=1}^{\infty} \langle \chi_{k,N}, w_m \rangle w_m(x) = \sum_{m=1}^{2^N} \langle \chi_{k,N}, w_m \rangle w_m(x),$$

where the only x for which the equality may not hold are in the set: $\{\frac{k}{2^N} | k = 0, 1, \cdots 2^N\}$.

$\square$

Lemma 4.1.2.

$$\chi_{k,N}(x) = \sum_{n=1, n\neq m}^{2^N} \langle \chi_{k,N}, w_n \rangle (w_n - w_m)$$

for $k = 1, 2, \cdots 2^N - 1$*, and* $m \in \{1, \cdots, 2^N\}$.

Proof.

Using Lemma 4.1.1, and the fact that for $1 \leq n \leq 2^N$, $w_n(x) = 1$ for $x \in [0, 2^N]$:

$$\chi_{0,N} = \sum_{n=1}^{2^N} \langle \chi_{0,N}, w_n \rangle w_n = 2^{-N} \sum_{n=1}^{2^N} w_n.$$

So:

$$\sum_{n=1, n\neq m}^{2^N} w_n = 2^N \chi_{0,N} - w_m.$$

Using this fact, we compute for $1 \leq k \leq 2^N - 1$:

$$\sum_{n=1, n\neq m}^{2^N} \langle \chi_{k,N}, w_n \rangle (w_n(x) - w_m(x)) = \sum_{n=1, n\neq m}^{2^N} \langle \chi_{k,N}, w_n \rangle w_n - \left\langle \chi_{k,N}, \sum_{n=1, n\neq m}^{2^N} w_n \right\rangle w_m$$

$$= \sum_{n=1, n\neq m}^{2^N} \langle \chi_{k,N}, w_n \rangle w_n - \langle \chi_{k,N}, 2^N \chi_{0,N} - w_m \rangle w_m$$

$$= \sum_{n=1, n\neq m}^{2^N} \left\langle \chi_{k,N}, w_n \right\rangle w_n - \left\langle \chi_{k,N}, 2^N \chi_{0,N} \right\rangle w_m + \left\langle \chi_{k,N}, w_m \right\rangle w_m.$$

$$= \sum_{n=1}^{2^N} \left\langle \chi_{k,N}, w_n \right\rangle w_n = \chi_{k,N},$$

using Lemma 4.1.1.

□

Lemma 4.1.3. $\chi_{j,k,M}(x,y)$ *can be written as a finite linear combination of functions of the form:*

$$w_i(x)w_\ell(y) - w_a(x)w_b(y),$$

where $a, b \leq 2^M$*, and* $j = 1, 2, \cdots, 2^M - 1$*,* $k = 1, 2, \cdots, 2^M - 1$*. Specifically:*

$$\chi_{j,k,M}(x,y) = \sum_{n=1, n\neq b}^{2^M} \sum_{p=1, p\neq a}^{2^M} \left\langle \chi_{j,M}, w_p \right\rangle \left\langle \chi_{k,M}, w_n \right\rangle$$

$$\cdot \left([w_p(x)w_n(y) - w_a(x)w_b(y)] - [w_a(x)w_n(y) - w_a(x)w_b(y)] - [w_p(x)w_b(y) - w_a(x)w_b(y)] \right).$$

Proof.

Employing Lemma 4.1.2, where $1 \leq a, b \leq 2^M$, $j = 1, 2, \cdots, 2^M - 1$, and $k = 1, 2, \cdots, 2^M - 1$:

$$\chi_{j,k,M}(x,y) = \chi_{j,M}(x) \cdot \chi_{k,M}(y),$$

$$= \left(\sum_{p=1, p\neq a}^{2^M} \left\langle \chi_{j,M}, w_p \right\rangle (w_p(x) - w_a(x)) \right) \left(\sum_{n=1, n\neq b}^{2^M} \left\langle \chi_{k,M}, w_n \right\rangle (w_n(y) - w_b(y)) \right)$$

$$= \sum_{n=1, n \neq b}^{2^M} \sum_{p=1, p \neq a}^{2^M} \langle \chi_{j,M}, w_p \rangle \langle \chi_{k,M}, w_n \rangle ((w_p(x) - w_a(x))(w_n(y) - w_b(y)))$$

$$= \sum_{n=1, n \neq b}^{2^M} \sum_{p=1, p \neq a}^{2^M} \langle \chi_{j,M}, w_p \rangle \langle \chi_{k,M}, w_n \rangle$$

$$\cdot ([w_p(x)w_n(y) - w_a(x)w_b(y)] - [w_a(x)w_n(y) - w_a(x)w_b(y)] - [w_p(x)w_b(y) - w_a(x)w_b(y)]) .$$

□

The idea for the main technique used the following lemma, which is a standard fact regarding the Walsh system, comes partially from a proof of Josef Dick, though he has not formally published an argument of this particular type.

Lemma 4.1.4. *Let $f \in L^1(\mathbb{T})$, and suppose that:*

$$\langle w_n, f \rangle = 0,$$

for all $n \in \mathbb{N}$. Then $f(x) = 0$ for a.e. $x \in \mathbb{T}$.

Proof.

First, notice that we may reduce this to answering the question for real-valued functions, f, since $\langle w_n, f \rangle = 0$, implies that $\langle w_n, Re(f) \rangle = \langle w_n, Im(f) \rangle = 0$. Hence, we make this simplifying assumption. Now, define:

$$f^*(x) := \int_0^x f(t)dt.$$

Then $f^*(x)$ is absolutely continuous and for a.e. $x \in \mathbb{T}$, $(f^*)'(x) = f(x)$, since $f \in L^1(\mathbb{T})$, [12]. We will use induction to show that $f^*(x) = 0$ for all $x \in \mathbb{T}$ of the form $\frac{k}{2^N}$, where $k \in 0, 1, \cdots, 2^N$, and $N = 0, 1, \cdots$. Fix N. Computing for the endpoints $x = 0$, and $x = 1$:

$$f^*\left(\frac{0}{2^N}\right) = f^*(0) = \int_0^0 f(t)dt = 0,$$

and

$$f^*(1) = \int_0^1 f(t)dt = \int_0^1 f(t)w_1(t)dt = \langle w_1, f\rangle = 0.$$

Now, suppose that $f^*\left(\frac{k}{2^N}\right) = 0$ for all $k = 0, 1, \cdots k_0 < 2^N - 1$. Computing, using Lemma 4.1.1:

$$\left\langle \chi_{k_0,N}, f\right\rangle = \left\langle \sum_{n=1}^{2^N} \left\langle \chi_{k_0,N}, w_n\right\rangle (w_n), f \right\rangle = \sum_{n=1}^{2^N} \left\langle \chi_{k_0,N}, w_n\right\rangle \langle w_n, f\rangle = 0. \tag{4.1}$$

But also,

$$\left\langle \chi_{k_0,N}, f\right\rangle = \int_{\frac{k_0}{2^N}}^{\frac{k_0+1}{2^N}} f(t)dt = \int_0^{\frac{k_0+1}{2^N}} f(t)dt - \int_0^{\frac{k_0}{2^N}} f(t)dt = f^*\left(\frac{k_0+1}{2^N}\right) - f^*\left(\frac{k_0}{2^N}\right).$$

Combining this with (4.1), yields $f^*\left(\frac{k+1}{2^N}\right) = f^*\left(\frac{k}{2^N}\right) = 0$. Since N was arbitrary, we have now shown that f^* is equal to zero for all dyadic rationals in $\mathbb{T}$. Since f^* is continuous, and the dyadic rationals form a dense subset of $\mathbb{T}$, we must then have that $f^*(x) = 0$ for all $x \in \mathbb{T}$. Hence, $(f^*)'(x) = 0$ for all $x \in \mathbb{T}$. Since $f(x) = (f^*)'(x)$ a.e. in $\mathbb{T}$, we have finally obtained the result that $f(x) = 0$ for a.e. $x \in \mathbb{T}$.

□

4.2 The Windowed Walsh System and Positive Exact Systems

Let $\{w_n\}_{n=1}^{\infty}$ denote the Walsh System as given in Definition 2.5.1, which forms an orthonormal basis for $L^2(\mathbb{T})$, as described directly following the aforementioned definition. Recall that the Walsh functions form a complete system in $L^p(\mathbb{T})$ for $1 \leq p < \infty$ as a result of Lemma 4.1.2, by virtue of the fact that the dyadic characteristic functions are complete in those spaces.

Results in this section include the classification of windows which yield a complete system when windowing the Walsh System with one deleted element, and some properties of the set of such window functions, as well as a demonstration of the completeness, and thus uniqueness of the biorthogonal system for the given windowed Walsh System with one deletion. It appears that Kazarian may have completed some similar related work in [15], which is presented in Russian. Here we provide explicit and sometimes constructive proofs of our results in particular cases.

In this section, let $G := \{x \,|\, g(x) = 0\}$.

Lemma 4.2.1. *For any $m = 1, 2, \cdots$, and $g \in L^p(\mathbb{T}) \setminus \{0\}$ with $1 \leq p < \infty$, it is the case that $w_n(x)g(x) \in L^p(\mathbb{T})$.*

Proof.

$$\left(\int_0^1 |w_n(x)g(x)|^p dx\right)^{\frac{1}{p}} = \left(\int_0^1 |g(x)|^p dx\right)^{\frac{1}{p}} = \|g\|_{L^p(\mathbb{T})} < \infty.$$

□

Lemma 4.2.2. *If $\frac{1}{g} \in L^q([a,1])$ for all $0 < a \leq 1$, $1 < q < \infty$, then for all $m, n \in \mathbb{N}$, with $m \neq n$, $\frac{w_n(x) - w_m(x)}{\overline{g}(x)} \in L^q(\mathbb{T})$.*

Proof.

We begin by noting that $w_n(x) = w_m(x) = 1$ a.e. on any interval $[0,2^{-N}]$, if $2^N > m$ and $2^N > n$. Hence, assuming $2^N > m,n$:

$$\int_0^1 \left|\frac{w_n(x)-w_m(x)}{\overline{g}(x)}\right|^q dx = \int_{2^{-N}}^1 \left|\frac{w_n(x)-w_m(x)}{\overline{g}(x)}\right|^q dx \leq 2^q \int_{2^{-N}}^1 \frac{1}{|g(x)|^q} dx < \infty,$$

since $\frac{1}{g} \in L^q([2^{-N},1])$ by assumption.

$\square$

Lemma 4.2.3. *Given $1 < p,q < \infty$, with $\frac{1}{p}+\frac{1}{q} = 1$, $g \in L^p(\mathbb{T}) \setminus \{0\}$ and $\frac{1}{g} \in L^q([a,1])$ for all $0 < a \leq 1$, we have that for any $m \in \mathbb{N}$, $\{g(x)w_n(x)\}_{n=1,n\neq m}^{\infty}$, has a biorthogonal system $\{\frac{w_n(x)-w_m(x)}{\overline{g}(x)}\}_{n=1,n\neq m}^{\infty}$ in $L^q(\mathbb{T})$. Thus, both sequences are minimal.*

Proof.

First note that the systems $\{w_n(x)g(x)\}_{n=1,n\neq m}^{\infty}$, and $\{\frac{w_n(x)-w_m(x)}{\overline{g}(x)}\}_{n=1,n\neq m}^{\infty}$ are in $L^p(\mathbb{T})$ and $L^q(\mathbb{T})$ respectively, by Lemma 4.2.1, and Lemma 4.2.2 respectively. For $n \neq m$:

$$\left\langle w_n(x)g(x), \frac{w_n(x)-w_m(x)}{\overline{g}(x)}\right\rangle = \int_0^1 (w_n(x)g(x))\left(\frac{w_n(x)-w_m(x)}{g(x)}\right) dx$$
$$= \int_0^1 w_n(x)^2 dx - \int_0^1 w_n(x)w_m(x)dx = 1-0 = 1,$$

where we have used the orthonormality of the Walsh system in $L^2(\mathbb{T})$.

For $k \neq n$, and $n,k \neq m$:

$$\left\langle w_n(x)g(x), \frac{w_k(x)-w_m(x)}{\overline{g}(x)}\right\rangle = \int_0^1 (w_n(x)g(x))\left(\frac{w_k(x)-w_m(x)}{g(x)}\right) dx$$

$$= \int_0^1 w_n(x)w_k(x)dx - \int_0^1 w_n(x)w_m(x)dx = 0-0 = 0,$$

again using the orthonormality of the Walsh system. Similarly, in order to show that $\{\frac{w_n(x)-w_m(x)}{\overline{g}(x)}\}_{n=1,n\neq m}^{\infty}$ has biorthogonal system $\{g(x)w_n(x)\}_{n=1,n\neq m}^{\infty}$ we note that:

$$\left\langle \frac{w_k(x)-w_m(x)}{\overline{g}(x)}, w_n(x)g(x) \right\rangle = \delta_{k,n},$$

for $k \neq n$, and $n,k \neq m$.

$\square$

Theorem 4.2.4. *The system $\{g(x)w_n(x)\}_{n=1,n\neq m}^{\infty}$ is complete in $L^p(\mathbb{T})$, $1<p<\infty$, if and only if $g(x) \in L^p(\mathbb{T}) \setminus \{0\}$, $\mu(G)=0$, and $\frac{1}{g} \notin L^q(\mathbb{T})$, where $\frac{1}{p}+\frac{1}{q}=1$.*

Proof.

First suppose $g(x) \in L^p(\mathbb{T}) \setminus \{0\}$, $\frac{1}{g} \notin L^q(\mathbb{T})$, and $\mu(G)=0$, recalling that $G=\{x \,|\, g(x)=0\}$. By way of contradiction, suppose that $\{gw_n\}_{n=1,n\neq m}^{\infty}$ is not complete. Then there exists some $f(x) \in L^q(\mathbb{T})$, $\|f\|_q \neq 0$, where $\langle gw_n, f\rangle = 0$ for all $n \in \mathbb{N}$, $n \neq m$. Since $f \in L^p(\mathbb{T})$, and $g \in L^q(\mathbb{T})$, Hölder's inequality yields that $f\overline{g} \in L^1(\mathbb{T})$. Hence, it makes sense to write $\langle w_n, f\overline{g}\rangle = 0$ for all $n \neq m$, in light of Lemma 4.1.4 .

Now, consider the function $f\overline{g} - \langle w_m, f\overline{g}\rangle\, w_m \in L^1(\mathbb{T})$. Computing for $n \neq m$:

$$\langle w_n, f\overline{g} - \langle w_m, f\overline{g}\rangle\, w_m\rangle = \langle w_n, f\overline{g}\rangle - \langle w_m, f\overline{g}\rangle \langle w_n, w_m\rangle = 0,$$

and,

$$\langle w_m, f\overline{g} - \langle w_m, f\overline{g}\rangle\, w_m\rangle = \langle w_m, f\overline{g}\rangle - \langle w_m, f\overline{g}\rangle \langle w_m, w_m\rangle = \langle w_m, f\overline{g}\rangle - \langle w_m, f\overline{g}\rangle = 0,$$

by the orthonormality of the Walsh system in $L^2(\mathbb{T})$.

Therefore, $f\overline{g} - \langle w_m, f\overline{g}\rangle\, w_m \in L^1(\mathbb{T})$, and $\langle w_n, f\overline{g} - \langle w_m, f\overline{g}\rangle\, w_m\rangle = 0$ for all $n \in \mathbb{N}$. Applying Lemma 4.1.4, we see that $f\overline{g} - \langle w_m, f\overline{g}\rangle\, w_m = 0$ a.e. on $\mathbb{T}$, and so $f\overline{g} = \langle w_m, f\overline{g}\rangle\, w_m$ a.e. on

$\mathbb{T}$.

Now, since $\|f\|_q \neq 0$, and $\mu(G) = 0$ yields $\|f\overline{g}\|_1 \neq 0$, it must be that $\langle w_m, f\overline{g}\rangle \neq 0$, because $f\overline{g} = \langle w_m, f\overline{g}\rangle\, w_m$. Then for a.e. x, $\frac{\langle w_m, f\overline{g}\rangle w_m}{\overline{g}} = f \in L^q(\mathbb{T})$, which implies:

$$\|f\|_q = \left\|\frac{\langle w_m, f\overline{g}\rangle\, w_m}{\overline{g}}\right\|_q = |\,\langle w_m, f\overline{g}\rangle\,|\left\|\frac{1}{g}\right\|_q = \infty,$$

since $\langle w_m, f\overline{g}\rangle \neq 0$. This contradicts $f \in L^q(\mathbb{T})$. Hence the system $\{g(x)w_n(x)\}_{n=1,n\neq m}^{\infty}$ is complete.

Now we assume that the system $\{g(x)w_n(x)\}_{n=1,n\neq m}^{\infty}$ is complete in $L^p(\mathbb{T})$, and assume first by way of contradiction that $g(x) \notin L^p(\mathbb{T}) \setminus \{0\}$. It is clear that if $g(x) = 0$, that we obtain a contradiction, since we only have functions equal to zero a.e. in the system. So we suppose $g(x) \notin L^p(\mathbb{T})$. But then $\|gw_n\|_p = \|g\|_p = \infty$, so the system is not in $L^p(\mathbb{T})$, which is a contradiction as well.

Recall that $G := \{x\,|\,g(x) = 0\}$, and suppose that $\mu(G) > 0$. Then $\chi_G \notin [0]$, (that is χ_G is not in the equivalence class of 0 in $L^p(\mathbb{T})$), $\chi_G \in L^q(\mathbb{T})$, and it is the case that for all $n = 1, \cdots, \infty$, $n \neq m$,

$$\langle gw_n, \chi_G\rangle = 0,$$

contradicting the completeness of $\{g(x)w_n(x)\}_{n=1,n\neq m}^{\infty}$ in $L^p(\mathbb{T})$.

It is left to show that we obtain a contradiction if we assume $\{g(x)w_n(x)\}_{n=1,n\neq m}^{\infty}$ is complete in $L^p(\mathbb{T})$, and $\frac{1}{g} \in L^q(\mathbb{T})$. In this case,

$$0 < \left\|\frac{w_m(x)}{\overline{g}(x)}\right\|_q = \left\|\frac{1}{g(x)}\right\|_q < \infty.$$

Also, for all $n = 1, \cdots, \infty$, $n \neq m$,

$$\left\langle g w_n, \frac{w_m}{\overline{g}} \right\rangle = \langle w_n, w_m \rangle = 0,$$

which contradicts the completeness of the system.

□

Lemma 4.2.5. *The set, W of window functions $g(x)$ for which $\{g(x)w_n(x)\}_{n=1,n\neq m}^{\infty}$ is complete in $L^p(\mathbb{T})$, is not open, not closed, and not convex in $L^p(\mathbb{T})$, for $1 < p < \infty$.*

Proof.

We first show that the set of window functions is not closed. Let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Take $f_n(x) := x^{\frac{n}{q}}$, for $n = 1, 2, \cdots$. Then $\{f_n\}_{n=1}^{\infty} \subseteq W$, since $\{f_n\}_{n=1}^{\infty} \subseteq L^p(\mathbb{T})$, $\mu(\{x | x^{\frac{n}{q}} = 0\}) = 0$, and $\frac{1}{x^{\frac{n}{q}}} \notin L^q(\mathbb{T})$. However, $x^{\frac{n}{q}} \to 0$ in $L^p(\mathbb{T})$ by the Lebesgue dominated convergence theorem, and $0 \notin W$. Hence, W is not closed.

Now we show that W is not open. Let $f_n(x) = 0$ on $[0, 2^{-n}]$, and $f_n(x) = x$ on $(2^{-n}, 1]$ for $n = 1, 2, \cdots$. Then $f_n \notin W$, because $\mu(\{x \,|\, f_n(x) = 0\}) = 2^{-n} \neq 0$. However, $f_n(x) \to x$ in $L^p(\mathbb{T})$ by the Lebesgue dominated convergence theorem, and $x \in W$.

We demonstrate that W is not convex: Note that $x \in W$, and $-x \in W$. Let

$G(t) := tx + (1-t)(-x)$, we see that, $G\left(\frac{1}{2}\right) = 0$, and $0 \notin W$.

□

In the following theorem, to obtain completeness of the biorthogonal system to $\{gw_n\}_{n \neq a}$, notice that we strengthen the condition on $\frac{1}{g}$ from being in $L^q([a,1])$ for $0 < a \leq 1$ to being continuous on these intervals.

Theorem 4.2.6. *Let $\frac{1}{\bar{g}}$ be continuous on $[\gamma, 1]$, for all $0 < \gamma \leq 1$, $m \in \mathbb{N}$, and $\mu(G) = 0$. Then the system $\left\{ \frac{w_n(x) - w_a(x)}{\bar{g}(x)} \right\}_{n=1, n \neq a}^{\infty}$ is complete in $L^q(\mathbb{T})$ for $1 < q < \infty$.*

Proof.

For the proof, it suffices to show that dyadic characteristic functions, $\chi_{k,N}$, with $k \neq 0$, can be approximated by functions of the given system. Fix $\varepsilon > 0$, and $N \in \mathbb{N}$, so we have fixed $\left[0, 2^{-N}\right]$. Define:

$$\varepsilon_N = \frac{\varepsilon^{\frac{1}{q}}}{\sup_{x \in [2^{-N}, 1]}(|\bar{g}(x)|)}. \tag{4.2}$$

By the continuity of $\frac{1}{\bar{g}}$ on $\left[2^{-N}, 1\right]$, for some $\delta > 0$, for all $x_1, x_2 \in \left[2^{-N}, 1\right]$ with $|x_1 - x_2| < \delta$, $|\frac{1}{\bar{g}(x_1)} - \frac{1}{\bar{g}(x_2)}| < \varepsilon_N$. Choose $M \in \mathbb{N}$ large enough that $2^{-M} < \delta$, $2^M > a$, and $M > N$. Define:

$$A_{j,M} = \left[j2^{-M}, (j+1)2^{-M}\right],$$

$$c_{j,M} = \sup_{x \in A_{j,M}} \bar{g}(x),$$

and notice that $\chi_{j,M}(x) = \chi_{A_{j,M}}(x)$, and since $\mu(G) = 0$, $c_{j,M} \neq 0$. Using the continuity of $\frac{1}{\bar{g}}$, we find a bound for the following integral, where $j \geq 2^{M-N}$, so that $\frac{j}{2^M} \geq \frac{1}{2^N}$:

$$\int_0^1 \left| \frac{1}{c_{j,M}} \chi_{j,M}(x) - \frac{1}{\bar{g}(x)} \chi_{j,M}(x) \right|^q dx = \int_{A_{j,M}} \left| \frac{1}{c_{j,M}} - \frac{1}{\bar{g}(x)} \right|^q dx \leq \varepsilon_N^q \int_{A_{j,M}} 1 dx = \varepsilon_N^q 2^{-M}. \tag{4.3}$$

Now we will show the approximation of $\chi_{\alpha,N}$ where $\alpha \in \mathbb{N}$, and $\alpha \leq 2^N - 1$ by functions in our system. First, with M and N as above, define $J := \{j \in \mathbb{N} | A_{j,M} \subseteq A_{\alpha,N}\}$. Then the following equality holds:

$$\chi_{\alpha,N} = \sum_{j\in J} \chi_{j,M}. \tag{4.4}$$

For each $j \in J$, by Lemma 4.1.2, $\chi_{j,M}(x)$ is a linear combination of functions of the form $w_n(x) - w_a(x)$, where $n \neq a$, since $2^M > a$, and the definition of J implies that each $j \leq 2^M - 1$. Therefore:

$$\sum_{j\in J} \frac{c_{j,M}}{\overline{g}(x)} \chi_{j,M} = \sum_{k=1}^{K} d_k \frac{w_{n_k}(x) - w_a(x)}{\overline{g}(x)},$$

for some constants d_k, and $n_k \neq a$, and some $K \in \mathbb{N}$.

Hence:

$$\int_0^1 \left| \chi_{\alpha,N}(x) - \sum_{k=1}^{K} d_k \frac{w_{n_k}(x) - w_a(x)}{\overline{g}(x)} \right|^q dx = \int_0^1 \left| \chi_{\alpha,N}(x) - \sum_{j\in J} \frac{c_{j,M}}{\overline{g}(x)} \chi_{j,M}(x) \right|^q dx,$$

using (4.4),

$$= \int_{A_{\alpha,N}} \left| \sum_{j\in J} \frac{c_{j,M}}{c_{j,M}} \chi_{j,M}(x) - \sum_{j\in J} \frac{c_{j,M}}{\overline{g}(x)} \chi_{j,M}(x) \right|^q dx,$$

since the support of the second sum is simply $A_{\alpha,N}$, and the supports of the $\chi_{j,M}$ are disjoint,

$$= \sum_{j\in J} |c_{j,M}|^q \int_{A_{j,M}} \left| \frac{1}{c_{j,M}} \chi_{j,M}(x) - \frac{1}{\overline{g}(x)} \chi_{j,M}(x) \right|^q dx = \sum_{j\in J} |c_{j,M}|^q \int_{A_{j,M}} \left| \frac{1}{c_{j,M}} - \frac{1}{\overline{g}(x)} \right|^q dx$$

$$\leq \left[\sup_{x\in[2^{-N},1]} (|\overline{g}(x)|) \right]^q \sum_{j\in J} \varepsilon_N^q \cdot 2^{-M},$$

since $j \in J$ implies that $\frac{j}{2^M} \geq \frac{1}{2^N}$, because $\alpha \geq 1$, and employing (4.3),

$$\leq \left[\sup_{x \in [2^{-N},1]} (|\overline{g}(x)|) \right]^q \frac{2^M}{2^N} \varepsilon_N^q 2^{-M} \leq \frac{\varepsilon}{2^N} < \varepsilon,$$

using (4.2).

$\square$

Theorem 4.2.7. *Let μ denote Lebesgue measure, and $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Let $g(x) \in L^p(\mathbb{T}) \setminus \{0\}$, let $\frac{1}{\overline{g}(x)}$ be continuous on $[\gamma, 1]$ for all $0 < \gamma \leq 1$, $\frac{1}{g} \notin L^q(\mathbb{T})$, and $\mu(G) = 0$. Then $\{g(x)w_n(x)\}_{n=1, n\neq a}^{\infty}$ and $\left\{ \frac{w_n(x) - w_a(x)}{\overline{g}(x)} \right\}_{n=1, n\neq a}^{\infty}$ are exact systems in $L^p(\mathbb{T})$, and $L^q(\mathbb{T})$. Moreover, they are the unique biorthogonal systems to one-another.*

Proof.

The biorthogonality of the systems follows from Lemma 4.2.3. The completeness of the systems $\{g(x)w_n(x)\}_{n=1, n\neq a}^{\infty}$ and $\left\{ \frac{w_n(x) - w_a(x)}{\overline{g}(x)} \right\}_{n=1, n\neq a}^{\infty}$ in $L^p(\mathbb{T})$ and $L^q(\mathbb{T})$ respectively, follows from Theorem 4.2.4 and Theorem 4.2.6, respectively. The uniqueness of the two biorthogonal systems follows from Lemma 2.3.7, since both systems are exact.

$\square$

Theorem 4.2.8 (Existence of Positive Exact System with Exact Dual for $1 < p < \infty$)**.** *There exists an exact system $\{f_k\}_{k=1}^{\infty} \subseteq L^q(\mathbb{T})$ for each q with $1 < q < \infty$, where for all k, $f_k \geq 0$ a.e., with an exact dual system.*

Proof.

Suppose $\frac{1}{q} + \frac{1}{p} = 1$. Let $g(x) = -(x^{\frac{1}{q}})$. It clear that $g \in L^p(\mathbb{T})$, $\frac{1}{g}$ is continuous on $[\gamma, 1]$ for all $0 < \gamma \leq 1$, and that $\frac{1}{g} \notin L^q(\mathbb{T})$. Then the system

$$\left\{ \frac{w_n(x) - w_1(x)}{-(x^{\frac{1}{q}})} \right\}_{n=2}^{\infty}$$

is exact, and has an exact dual system, as given by Theorem 4.2.7 . Note that $w_1(x) \equiv 1$, and that for all $x \in \mathbb{T}$ and all $n = 2, \cdots$, we have $w_n(x) = \pm 1$, a.e. Hence:

$$\frac{w_n(x) - w_1(x)}{-(x^{\frac{1}{q}})} = \frac{1 - w_n(x)}{x^{\frac{1}{q}}} \geq 0,$$

for all $x \in \mathbb{T}$, and $n = 2, \cdots$.

$\square$

In the following theorem, where we take the original exact system to be in $L^1(\mathbb{T})$, notice that we do not claim that the dual system is exact. In fact, we have no hope of finding an an exact dual system, since it would need to be a complete sequence in $L^\infty(\mathbb{T})$. No such sequence can exist since $L^\infty(\mathbb{T})$ is not a separable space.

Theorem 4.2.9 (Existence of Positve Exact Systems for $L^1(\mathbb{T})$)**.** *There exists a positive exact system for $L^1(\mathbb{T})$.*

Proof.

Consider the system $\{f_n\}_{n=2}^{\infty}$, where $f_n(t) := w_1(t) - w_n(t)$. First, $f_n \in L^1(\mathbb{T})$, since the Walsh system is a subset of $L^1(\mathbb{T})$. Also, because $w_1(t) \equiv 1$ for $t \in \mathbb{T}$, and $w_n(t)$ takes on only the values of $1, 0$, or -1, $f_n(t) \geq 0$ for all $t \in \mathbb{T}$.

We now demonstrate that $\{f_n\}_{n=2}^{\infty}$ is minimal by showing that $\{w_n(t)\}_{n=2}^{\infty}$ is a biorthogonal system. Note that $\{w_n\}_{n=2}^{\infty} \subseteq L^\infty(\mathbb{T})$. For $n, m \in \{2, 3, \cdots\}$, with $n \neq m$:

$$\langle f_n, w_m \rangle = \langle w_n - w_1, w_m \rangle = \langle w_n, w_m \rangle - \langle w_1, w_m \rangle = 0,$$

using the orthonormality of the Walsh system in $L^2(\mathbb{T})$.

For $n = m \in \{2, 3, \cdots\}$:

$$\langle f_n, w_n \rangle = \langle w_n - w_1, w_n \rangle = \langle w_n, w_n \rangle - \langle w_1, w_m \rangle = 1,$$

again using the orthonormality of the Walsh system in $L^2(\mathbb{T})$.

It remains to show that $\{f_n\}_{n=2}^{\infty}$ is complete in $L^1(\mathbb{T})$. Let $g \in L^{\infty}(\mathbb{T})$, and suppose that $\langle f_n, g\rangle = 0$ for all $n \in \{2,3,\cdots\}$. Then for $n \in \{2,3,\cdots\}$:

$$0 = \langle f_n, g\rangle = \langle w_n, g\rangle - \langle w_1, g\rangle\,,$$

and so for all $n \in \{2,3,\cdots\}$:

$$\langle w_n, g\rangle = \langle w_1, g\rangle\,. \tag{4.5}$$

Now, because $g \in L^{\infty}(\mathbb{T}) \subseteq L^2(\mathbb{T})$, and the Walsh system is orthonormal, Plancherel's Equality gives, combined with (4.5), [3]:

$$\infty > \|g\|_2^2 = \sum_{n=1}^{\infty} |\langle w_n, g\rangle|^2 = \sum_{n=1}^{\infty} |\langle w_1, g\rangle|^2\,,$$

which is infinite unless $0 = \langle w_1, g\rangle$. Hence, $\langle w_n, g\rangle = 0$ for all $n \in \mathbb{N}$. By the completeness of the Walsh system in $L^2(\mathbb{T})$, we must have $g = 0$ a.e. on $\mathbb{T}$. Hence, $\{f_n\}_{n=2}^{\infty}$ is complete.

Since $\{f_n\}_{n=2}^{\infty}$ is complete and minimal, it is exact.

$\square$

CHAPTER 5

PRODUCT SYSTEMS

5.1 Properties of Product Systems

Theorem 5.1.1. *Let $F := \{f_n\}_{n\in\mathbb{N}}$, and $G := \{g_m\}_{m\in\mathbb{N}}$ be systems in $L^p(\mathbb{T})$, $1 < p < \infty$, not containing $[0]$. Then F and G are complete in $L^p(\mathbb{T})$ if and only if $F \circ G$ is complete in $L^p(\mathbb{T}^2)$, where $F \circ G := \{f_n(x)g_m(y)\}_{(n,m)\in\mathbb{N}^2}$.*

Proof.

Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. In the proof, we will use $\|\cdot\|_p$ to denote both $L^p(\mathbb{T})$, and $L^p(\mathbb{T}^2)$ norms for conciseness, as context will allow the reader to differentiate between the two meanings. First, suppose that $F \circ G$ is complete in $L^p(\mathbb{T}^2)$. We will show that F is complete in $L^p(\mathbb{T})$, and note that the argument for G being complete follows analogously. Proving by contradiction, let $f \in L^q(\mathbb{T})$ be such that $\|f\|_q \neq 0$, and suppose that $\langle f_n, f\rangle = 0$ for all $n \in \mathbb{N}$. Then,

$$\left\langle f_n(x)g_m(y), f(x)\chi_{[0,1]}(y)\right\rangle = \int_0^1\int_0^1 f_n(x)g_m(y)\overline{f}(x)\overline{\chi}_{[0,1]}(y)dxdy$$

$$= \int_0^1 f_n(x)\overline{f}(x)dx\int_0^1 g_m(y)\overline{\chi}_{[0,1]}(y)dy = \langle f_n, f\rangle\left\langle g_m, \chi_{[0,1]}\right\rangle = 0,$$

for all $(m,n) \in \mathbb{N}^2$. Then, it must be the case that $\|f(x)\chi_{[0,1]}(y)\|_q = 0$ since $F \circ G$ is complete in $L^p(\mathbb{R}^2)$. Hence:

$$0 = \left(\int_0^1 \int_0^1 \left| f(x)\chi_{[0,1]}(y)\right|^q dxdy\right)^{\frac{1}{q}} = \left(\int_0^1 |f(x)|^q dx\right)^{\frac{1}{q}} \left(\int_0^1 \left|\chi_{[0,1]}(y)\right|^q dy\right)^{\frac{1}{q}} = \|f\|_q,$$

contradicting $\|f\|_q \neq 0$. Thus, F is complete. Similarly, we may prove that G is complete.

Now, suppose that F and G are complete in $L^p(\mathbb{T})$. Let $h(x,y) \in L^p(\mathbb{T}^2)$, and let $\varepsilon > 0$. We know that we may approximate $h(x,y)$ by products of characteristic functions, $\chi_A(x)\chi_B(y)$, where A and B are dyadic intervals, so we choose a dyadic step function so that:

$$\left\| h(x,y) - \sum_{n=1}^{N} \gamma_n \chi_{I_{in}}(x)\chi_{I_{jn}}(y) \right\|_p < \frac{\varepsilon}{3}.$$

Since G is complete in $L^p(\mathbb{R})$, we can choose $d_k^{(n)} g_k^{(n)}$ so that:

$$\left\| \chi_{I_{jn}}(y) - \sum_{k=1}^{K_n} d_k^{(n)} g_k^{(n)}(y) \right\|_p < \frac{\varepsilon}{3N\left\|\gamma_n \chi_{I_{in}}(x)\right\|_p}.$$

We subsequently choose $c_l^{(n)} f_l^{(n)}(x)$ such that :

$$\left\| \gamma_n \chi_{I_{in}}(x) - \sum_{l=1}^{L_n} c_l^{(n)} f_l^{(n)}(x) \right\|_p < \frac{\varepsilon}{3N\left\|\sum_{k=1}^{K_n} d_k^{(n)} g_k^{(n)}(y)\right\|_p}.$$

Noting that

$$\sum_{n=1}^{N}\left[\left(\sum_{l=1}^{L_n} c_l^{(n)} f_l^{(n)}(x)\right) \cdot \left(\sum_{k=1}^{K_n} d_k^{(n)} g_k^{(n)}(y)\right)\right] \in \operatorname{span}(F \circ G),$$

we compute:

$$\left\|h(x,y)-\sum_{n=1}^{N}\left[\left(\sum_{l=1}^{L_n}c_l^{(n)}f_l^{(n)}(x)\right)\cdot\left(\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right)\right]\right\|_p$$

$$\leq\left\|h(x,y)-\sum_{n=1}^{N}\left[\gamma_n\chi_{I_{i_n}}(x)\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right]\right\|_p$$

$$+\left\|\sum_{n=1}^{N}\left[\gamma_n\chi_{I_{i_n}}(x)\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right]-\sum_{n=1}^{N}\left[\left(\sum_{l=1}^{L_n}c_l^{(n)}f_l^{(n)}(x)\right)\cdot\left(\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right)\right]\right\|_p$$

$$\leq\left\|h(x,y)-\sum_{n=1}^{N}\gamma_n\chi_{I_{i_n}}(x)\chi_{I_{j_n}}(y)\right\|_p+\left\|\sum_{n=1}^{N}\gamma_n\chi_{I_{i_n}}(x)\chi_{I_{j_n}}(y)-\sum_{n=1}^{N}\left[\gamma_n\chi_{I_{i_n}}(x)\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right]\right\|_p$$

$$+\left\|\sum_{n=1}^{N}\left[\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\left(\gamma_n\chi_{I_{i_n}}(x)-\sum_{l=1}^{L_n}c_l^{(n)}f_l^{(n)}(x)\right)\right]\right\|_p$$

$$<\frac{\varepsilon}{3}+\sum_{n=1}^{N}\left\|\gamma_n\chi_{I_{i_n}}(x)\right\|_p\left\|\chi_{I_{j_n}}(y)-\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right\|_p$$

$$+\sum_{n=1}^{N}\left\|\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right\|_p\left\|\gamma_n\chi_{I_{i_n}}(x)-\sum_{l=1}^{L_n}c_l^{(n)}f_l^{(n)}(x)\right\|_p$$

$$<\frac{\varepsilon}{3}+\sum_{n=1}^{N}\left\|\gamma_n\chi_{I_{i_n}}(x)\right\|_p\frac{\varepsilon}{3N\left\|\gamma_n\chi_{I_{i_n}}(x)\right\|_p}+\sum_{n=1}^{N}\left\|\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right\|_p\frac{\varepsilon}{3N\left\|\sum_{k=1}^{K_n}d_k^{(n)}g_k^{(n)}(y)\right\|_p}=\varepsilon.$$

Hence, $F\circ G$ is complete in $L^p(\mathbb{T}^2)$. $\square$

Theorem 5.1.2. *Let $F := \{f_n\}_{n\in\mathbb{N}}$, and $G := \{g_m\}_{m\in\mathbb{N}}$ be systems in $L^p(\mathbb{T})$, $1 < p < \infty$, not containing $[0]$. Then F and G are minimal in $L^p(\mathbb{T}^2)$ if and only if $F \circ G$ is minimal in $L^p(\mathbb{T}^2)$, where we define $F \circ G := \{f_n(x)g_m(y)\}_{(n,m)\in\mathbb{N}^2}$.*

Proof.

First suppose that F and G are minimal in $L^p(\mathbb{T})$. Then, there exist biorthogonal systems $\tilde{F} := \{\tilde{f}_n\}_{n\in\mathbb{N}}$, and $\tilde{G} := \{\tilde{g}_m\}_{m\in\mathbb{N}}$ to F and G in $L^q(\mathbb{T})$ respectively, where $\frac{1}{p} + \frac{1}{q} = 1$. Consider the system $\tilde{F} \circ \tilde{G} := \{\tilde{f}_n(x)\tilde{g}_m(y)\}_{(n,m)\in\mathbb{N}^2}$ in $L^q(\mathbb{T}^2)$. We have:

$$\left\langle f_n(x)g_m(y), \tilde{f}_j(x)\tilde{g}_k(y)\right\rangle = \int_0^1 \int_0^1 f_n(x)g_m(y)\overline{\tilde{f}}_j(x)\overline{\tilde{g}}_k(y)dxdy$$

$$\int_0^1 f_n(x)\overline{\tilde{f}}_j(x)dx \int_0^1 g_m(y)\overline{\tilde{g}}_k(y)dy = \left\langle f_n, \tilde{f}_j\right\rangle \langle g_m, \tilde{g}_k\rangle = \delta_{(n,m),(j,k)},$$

and so $F \circ G$ has the biorthogonal system $\tilde{F} \circ \tilde{G} \subseteq L^q(\mathbb{T}^2)$, and is thus minimal in $L^p(\mathbb{T}^2)$.

Now suppose that $F \circ G$ is minimal in $L^p(\mathbb{T}^2)$, with dual system $\{\tilde{H}_{n,m}(x,y)\}_{(n,m)\in\mathbb{N}^2} \subseteq L^q(\mathbb{T}^2)$. We prove by contradiction, assuming that F is not minimal (in the case that we suppose G to not be minimal, the argument follows analogously). Since F is not minimal, it must be that for some k, $f_k \in \overline{\text{span}}\{f_n\}_{n\neq k, n\in\mathbb{N}}$. Hence, for some $\{y_n\}_{n=1}^{\infty} \subseteq \text{span}\{f_n\}_{n\neq k, n\in\mathbb{N}}$:

$$\lim_{n\to\infty} \|f_k - y_n\| = 0.$$

Defining $H_n(x,y) := y_n(x)g_m(y)$, where $m \in \mathbb{N}$, we compute:

$$\lim_{n\to\infty}\left(\int_0^1 \int_0^1 |H_n(x,y) - f_k(x)g_m(y)|^p \, dxdy\right)^{\frac{1}{p}}$$

$$= \lim_{n\to\infty}\left(\int_0^1 |g_m(y)|^p \, dy\right)^{\frac{1}{p}} \left(\int_0^1 |y_n(x) - f_k(x)|^p \, dx\right)^{\frac{1}{p}}$$

$$= \|g_m\|_p \cdot \left(\lim_{N\to\infty} \|y_n - f_k\|_p \right) = 0,$$

using $g_m \in L^p(\mathbb{T})$. Hence, $\lim_{n\to\infty} H_n(x,y) = f_k(x)g_m(y)$ in $L^p(\mathbb{T}^2)$.

Since $\tilde{H}_{(k,m)}(x,y)$ is continuous:

$$\lim_{n\to\infty} \left\langle H_n(x,y), \tilde{H}_{(k,m)}(x,y) \right\rangle = \left\langle f_k(x)g_m(y), \tilde{H}_{(k,m)}(x,y) \right\rangle = 1,$$

by biorthogonality. Also:

$$\lim_{n\to\infty} \left\langle H_n(x,y), \tilde{H}_{(k,m)}(x,y) \right\rangle = \lim_{n\to\infty} \left\langle y_n(x)g_m(y), \tilde{H}_{(k,m)}(x,y) \right\rangle = 0,$$

since each y_n is a finite linear combination of f_n, where $n \neq k$. This contracts the fact above that we found this limit to be equal to 1. Hence, F must be minimal. A similar argument shows that G must be minimal as well.

□

5.2 The Windowed Walsh Product System

Since the Walsh system is complete in $L^2(\mathbb{T})$, as described following Definition 2.5.1, Theorem 5.1.1 tells us that multiplying the Walsh system by itself to form $\{w_n(x)w_m(y)\}_{(n,m)\in\mathbb{N}^2}$ results in a complete system in $L^2(\mathbb{T}^2)$, where $\mathbb{N}^2 := \{(m,n) \,|\, m,n \in \mathbb{N}\}$. In fact, the Walsh product system forms a complete orthonormal system, and so is a basis for $L^2(\mathbb{T}^2)$. Results given here include the classification of windows which yield a complete system when windowing the Walsh product system with one deletion, as well as a demonstration of the complete, and thus unique biorthogonal system for the Walsh product system with one deletion. In this section, let $F = \mathbb{N}^2 \setminus \{(a,b)\}$, and let $G := \{(x,y) \,|\, g(x,y) = 0\}$.

We first prove the two-dimensional analogy of Lemma 4.1.4.

Lemma 5.2.1. *Let $f \in L^1(\mathbb{T}^2)$, and suppose that:*

$$\langle w_n(x)w_m(y), f(x,y)\rangle = 0$$

for all $(m,n) \in \mathbb{N}^2$. Then $f(x,y) = 0$ for a.e. $(x,y) \in \mathbb{T}^2$.

Proof.

Let $f(x,y) \in L^1(\mathbb{T}^2)$, and suppose that $\langle w_n(x)w_m(y), f(x,y)\rangle = 0$ for all $(n,m) \in \mathbb{N}^2$. Then for $y_0 \in \mathbb{T} \setminus Y$, where $Y := \{\frac{k}{2^N} \,|k \in \{0,1,\cdots,2^N\}, N \in \mathbb{N}\}$:

$$0 = \langle w_n(x)w_m(y_0), f(x,y_0)\rangle = \langle w_n(x), f(x,y_0)\rangle\,,$$

or

$$0 = \langle w_n(x)w_m(y_0), f(x,y_0)\rangle = \langle -w_n(x), f(x,y_0)\rangle\,.$$

Note that $\mu(Y) = 0$, so that for a.e. $y_0 \in \mathbb{T}$,

$$0 = \langle w_n(x), f(x,y_0)\rangle\,,$$

for each $n \in \mathbb{N}$. Applying Lemma 4.1.4, we see that $f(x,y_0) = 0$ for a.e. $x \in \mathbb{T}$, if $y_0 \in \mathbb{T} \setminus Y$.

Now, let $A = \{(x,y) \subseteq \mathbb{T}^2 \,|\, f(x,y) \neq 0\} \setminus (\mathbb{T} \times Y)$, and notice that $A \cup (\mathbb{T} \setminus Y) = \{(x,y) \in \mathbb{T}^2 \,|\, f(x,y) \neq 0\}$. We wish to show that $\mu(A \cup (\mathbb{T} \times Y)) = 0$, where we understand that μ may denote one or two-dimensional Lebesgue measure, depending upon context. Now:

$$\mu(A \cup (\mathbb{T} \times Y)) = \iint \chi_{A\cup(\mathbb{T}\times Y)}dxdy = \iint \chi_A dxdy + \iint \chi_{\mathbb{T}\times Y}dxdy,$$

since A and $\mathbb{T} \times Y$ are disjoint. By Fubini's theorem [12],

$$\iint \chi_{\mathbb{T}\times Y} dxdy = \int_{\mathbb{T}} \left(\int_Y 1dy \right) dx = \int_{\mathbb{T}} 0dx = 0.$$

Again, by Fubini's Theorem:

$$\iint \chi_A dxdy = \int_0^1 \left(\int_0^1 \chi_A dx \right) dy = 0,$$

since $\chi_A(x,y) = 0$ for a.e. x given any $y \in \mathbb{T}$. Hence, $\mu(A \cup (\mathbb{T} \times Y)) = 0$.

Therefore $f(x,y) = 0$ for a.e. $(x,y) \in \mathbb{T}^2$.

$\square$

Lemma 5.2.2. *For any* $(m,n) \in \mathbb{N}^2$, *and* $g(x,y) \in L^p(\mathbb{T}^2) \setminus \{0\}$, $w_n(x)w_m(y)g(x,y) \in L^p(\mathbb{T})$.

Proof.

$$\left(\int_0^1 \int_0^1 |w_n(x)w_m(y)g(x,y)|^p dxdy \right)^{\frac{1}{p}} = \left(\int_0^1 \int_0^1 |g(x,y)|^p dxdy \right)^{\frac{1}{p}} < \infty.$$

$\square$

Lemma 5.2.3. *Let* $1 < q < \infty$. *If* $\frac{1}{g} \in L^q(\mathbb{T}^2 \setminus [0,\gamma) \times [0,\gamma))$ *for all* $0 < \gamma \leq 1$, *then for all* $(m,n),(j,k) \in \mathbb{N}^2$, *with* $(m,n) \neq (j,k)$, $\frac{w_n(x)w_m(y) - w_j(x)w_k(y)}{\overline{g}(x,y)} \in L^q(\mathbb{T}^2)$.

Proof.

Suppose that 2^N is strictly larger than each of m,n,j,k. Then by how they are defined, each of w_n, w_m, w_j, and w_k is equal to 1 a.e. on $\left[0,2^{-N},\right]$. Thus, a.e. on $\left[0,2^{-N}\right] \times \left[0,2^{-N}\right]$, $w_n(x)w_m(y) - w_j(x)w_k(y) = 0$. Hence:

$$\iint_{\mathbb{T}^2} \left| \frac{w_n(x)w_m(y) - w_j(x)w_k(y)}{g(x,y)} \right|^q dxdy \leq 2^q \int_{2^{-N}}^1 \int_{2^{-N}}^1 \left| \frac{1}{g(x,y)} \right|^q dxdy < \infty.$$

$\square$

Lemma 5.2.4. *Let $1 < p < \infty$. The system $\{g(x,y)w_n(x)w_m(y)\}_{(n,m)\in F}$, where $F = \mathbb{N}^2 \setminus \{(a,b)\}$, is complete in $L^p(\mathbb{T}^2)$ if and only if $g(x,y) \in L^q(\mathbb{T}^2) \setminus \{0\}$, $\mu(G) = 0$, and $\frac{1}{\overline{g}(x,y)} \notin L^q(\mathbb{T}^2)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof.

First suppose that $g(x,y) \in L^p(\mathbb{T}^2) \setminus \{0\}$, $\mu(G) = 0$, and that $\frac{1}{\overline{g}(x,y)} \notin L^q(\mathbb{T}^2)$. By way of contradiction to completeness of $\{g(x,y)w_n(x)w_m(y)\}_{(n,m)\in F}$, let $f(x,y) \in L^q(\mathbb{T}^2)$, and $\|f\|_q \neq 0$, and suppose that $\langle g(x,y)w_n(x)w_m(y), f(x,y)\rangle = 0$ for all $(n,m) \in F$. Since $f \in L^p(\mathbb{T}^2)$, and $g \in L^q(\mathbb{T}^2)$, $f\overline{g} \in L^1(\mathbb{T}^2)$. Hence, it makes sense to write:

$$\langle w_n(x)w_m(y), f(x,y)\overline{g}(x,y)\rangle = 0,$$

for all $(n,m) \neq (a,b)$, in light of Lemma 5.2.1 .

Now consider the function $h(x,y) := f(x,y)\overline{g}(x,y) - \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle\, w_a(x)w_b(y) \in L^1(\mathbb{T}^2)$. Computing for $(n,m) \neq (a,b)$, and using the orthonormality of the Walsh system:

$$\langle w_n(x)w_m(y), f(x,y)\overline{g}(x,y) - \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle\, w_a(x)w_b(y)\rangle$$

$$= \langle w_n(x)w_m(y), f(x,y)\overline{g}(x,y)\rangle - \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle\, \langle w_n(x)w_m(y), w_a(x)w_b(y)\rangle$$

$$= 0 - \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle\, \langle w_n(x), w_a(x)\rangle\, \langle w_m(y), w_b(y)\rangle = 0.$$

For $(n,m) = (a,b)$:

$$\langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y) - \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle\, w_a(x)w_b(y)\rangle$$

$$= \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle - \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle \langle w_a(x)w_b(y), w_a(x)w_b(y)\rangle$$

$$\langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle - \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle \langle w_a(x), w_a(x)\rangle \langle w_b(y), w_b(y)\rangle$$

$$= \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle - \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle \cdot 1 \cdot 1 = 0.$$

Then $\langle w_n(x)w_m(y), h(x,y)\rangle = 0$ for all $(n,m) \in \mathbb{N}^2$, and so by Lemma 4.1.4, $h(x,y) = 0$ for a.e. $(x,y) \in \mathbb{T}^2$. Therefore, $f(x,y)\overline{g}(x,y) = \langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle\, w_a(x)w_b(y)$ for a.e. $(x,y) \in \mathbb{T}^2$. It follows that since $\|f\|_q \neq 0$, and $\mu(G) = 0$ yields $\|f\overline{g}\|_1 \neq 0$, we have $\langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle \neq 0$. Then for a.e. $(x,y) \in \mathbb{T}^2$,

$$f(x,y) = \frac{\langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle\, w_a(x)w_b(y)}{\overline{g}(x,y)},$$

which yields,

$$\|f(x,y)\|_q = \left\| \frac{\langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle\, w_a(x)w_b(y)}{\overline{g}(x,y)} \right\|_q = |\langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle| \left\| \frac{1}{\overline{g}} \right\|_q = \infty,$$

since $|\langle w_a(x)w_b(y), f(x,y)\overline{g}(x,y)\rangle| \neq 0$. This contradicts $f(x,y) \in L^q(\mathbb{T}^2)$. Hence the system $\{g(x,y)w_n(x)w_m(y)\}_{(n,m)\in F}$ is complete.

Now we assume that the system $\{g(x,y)w_n(x)w_m(y)\}_{(n,m)\in G}$ is complete in $L^p(\mathbb{T}^2)$, and assume first by way of contradiction that $g(x) \notin L^p(\mathbb{T}^2) \setminus \{0\}$. It is clear that if $g(x,y) = 0$, that we obtain a contradiction, since we only have functions equal to zero a.e. in the system. So we suppose $g(x,y) \notin L^p(\mathbb{T}^2)$. But then $\|g(x,y)w_n(x)w_m(y)\|_p = \|g(x,y)\|_p = \infty$, so the system is not in $L^p(\mathbb{T})$, which is a contradiction as well.

Now, suppose $\mu(G) \neq 0$. Then $\chi_G \notin [0]$, and for all $(n,m) \in F$,

$$\langle g(x,y)w_n(x)w_m(y), \chi_G \rangle = 0,$$

contradicting the completeness of $\{g(x,y)w_n(x)w_m(y)\}_{(n,m)\in F}$.

It is left to show that we obtain a contradiction if we assume $\{g(x,y)w_n(x)w_m(y)\}_{(n,m)\in F}$ is complete in $L^p(\mathbb{T}^2)$, and $\frac{1}{\overline{g}} \in L^q(\mathbb{T}^2)$. In this case,

$$0 < \left\| \frac{w_a(x)w_b(y)}{\overline{g}(x)} \right\|_q = \left\| \frac{1}{\overline{g}(x)} \right\|_q < \infty.$$

Also, for all $(n,m) \in F$,

$$\left\langle g(x,y)w_m(x)w_n(y), \frac{w_a(x)w_b(y)}{\overline{g}(x,y)} \right\rangle = \langle w_n(x)w_m(y), w_a(x)w_b(y) \rangle = 0,$$

since $(m,n) \neq (a,b)$, by the orthonormality of the Walsh system, which contradicts the completeness of the system.

$\square$

Lemma 5.2.5. *Suppose* $1 < p < \infty$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $g(x,y) \in L^p(\mathbb{T}^2) \setminus \{0\}$, $\mu(G) = 0$, $\frac{1}{\overline{g}(x,y)} \in L^q(\mathbb{T}^2 \setminus [0,\gamma) \times [0,\gamma))$ *for all* $0 < \gamma \leq 1$. *Then* $\{g(x,y)w_n(x)w_m(y)\}_{(m,n)\in F}$ *has a biorthogonal system* $\left\{ \frac{w_n(x)w_m(x) - w_a(x)w_b(y)}{\overline{g}(x,y)} \right\}_{(m,n)\in F}$ *in* $L^q(\mathbb{T}^2)$. *Thus, both sequences are minimal.*

Proof.

That $\{g(x,y)w_n(x)w_m(y)\}_{(m,n)\in F} \subseteq L^p(\mathbb{T}^2)$ follows from Lemma 5.2.2, and that

$\left\{\frac{w_n(x)w_m(x)-w_a(x)w_b(y)}{\overline{g}(x,y)}\right\}_{(m,n)\in F} \subseteq L^q(\mathbb{T}^2)$ follows from Lemma 5.2.3.

Computing:

$$\left\langle g(x,y)w_n(x)w_m(y), \frac{w_j(x)w_k(y)-w_a(x)w_b(y)}{\overline{g}(x,y)} \right\rangle$$

$$= \left\langle w_n(x)w_m(y), w_j(x)w_k(y) \right\rangle - \left\langle w_n(x)w_m(y), w_a(x)w_b(y) \right\rangle = \delta_{(n,m),(j,k)} + 0 = \delta_{(n,m),(j,k)},$$

using the orthonormality of the Walsh system. Thus $\{g(x,y)w_n(x)w_m(y)\}_{(m,n)\in F}$ is minimal. Minimality of its biorthogonal system follows from the reflexivity of L^p.

$\square$

Lemma 5.2.6. *Let $1 < q < \infty$, and $g(x,y) \in L^q(\mathbb{T}^2) \setminus \{0\}$. Suppose that $\frac{1}{g(x,y)}$ is continuous on $\mathbb{T}^2 \setminus [0,\gamma) \times [0,\gamma)$ for all $0 < \gamma \leq 1$. Then the system $\left\{\frac{w_n(x)w_m(y)-w_a(x)w_b(y)}{g(x,y)}\right\}_{(m,n)\in F}$ is complete in $L^q(\mathbb{T}^2)$.*

Proof.

For the proof, it suffices to show that functions of the form $\chi_{j,k,N}$, whose support does not include $(0,0)$, can be approximated by functions of the given system. First, fix $\varepsilon > 0$. Fix $N \in \mathbb{N}$, and so we have fixed $[0,2^{-N}) \times [0,2^{-N})$. Let $\varepsilon_N = \frac{\varepsilon^{\frac{1}{q}}}{4\sup_{(x,y)\in\mathbb{T}^2\setminus[0,2^{-N})\times[0,2^{-N})}(|g(x,y)|)}$. Then for some $\delta > 0$, for all $(x_1,y_1),(x_2,y_2) \in \mathbb{T}^2 \setminus [0,2^{-N}) \times [0,2^{-N})$ with $|(x_1,y_1)-(x_2,y_2)| < \delta$, $|g(x_1,y_1)-g(x_2,y_2)| < \varepsilon_N$. Choose $M \in \mathbb{N}$ large enough so that the distance between any two points in

$[j2^{-M},(j+1)2^{-M}] \times [k2^{-M},(k+1)2^{-M}]$ is less than δ, and so that $2^M > a$, $2^M > b$, and $M > N$. Define

$$A^M_{j,k} := [j2^{-M},(j+1)2^{-M}] \times [k2^{-M},(k+1)2^{-M}],$$

and

$$c^M_{j,k} := \sup_{(x,y)\in A^M_{j,k}} g(x,y),$$

and note that

$\chi_{j,k,M}(x,y) = \chi_{A^M_{j,k}}(x,y)$. We find a bound for the following integral, where $j,k \in \mathbb{N}$, and $j \geq 2^{M-N}$, $k \geq 2^{M-N}$, so that $\frac{j}{2^M} \geq \frac{1}{2^N}$, and $\frac{k}{2^M} \geq \frac{1}{2^N}$:

$$\int\int_{\mathbb{T}^2} \left| \frac{1}{c^M_{j,k}} \chi_{j,k,M}(x,y) - \frac{1}{g(x,y)} \chi_{j,k,M}(x,y) \right|^q dxdy$$

$$= \int\int_{A^M_{j,k}} \left| \frac{1}{c^M_{j,k}} - \frac{1}{g(x,y)} \right|^q dxdy \leq \varepsilon^q_N 2^{-2M}. \tag{5.1}$$

Now we will show the approximation of $\chi_{\alpha,\beta,N}$ by functions in our system, when $\alpha,\beta \in \mathbb{N}$, and $\alpha,\beta \leq 2^N - 1$, noting that this gives us $\alpha,\beta \geq 1$.

First we see that the following equality holds since $M > N$:

$$\chi_{\alpha,\beta,N} = \sum_{k\in K} \sum_{j\in J} \chi_{j,k,M}, \tag{5.2}$$

where

$$J := \{ j \,|\, \alpha 2^{-N} \leq j 2^{-M}, \text{ and } (j+1)2^{-M} \leq (\alpha+1)2^{-N} \},$$

and

$$K := \{ k \,|\, \beta 2^{-N} \leq k 2^{-M} \text{ and } (k+1)2^{-M} \leq (\beta+1)2^{-N} \}.$$

We compute the following approximation to $\chi_{\alpha,\beta,N}(x,y)$ by functions in our system, noting that each $\chi_{j,k,M}(x,y)$ is a finite linear combination of functions of the form,

$$w_n(x)w_m(y) - w_a(x)w_b(y),$$

by Lemma 4.1.3, since $a,b < 2^M$, and since by definition of J, $(j+1)2^{-M} \leq (\alpha+1)2^{-N} \leq 1$ yields $j \leq 2^M - 1$. Similarly for the k index, $k \leq 2^M - 1$. Hence where $(n,m) \neq (a,b)$ it must be that,

$$\sum_{k\in K}\sum_{j\in J}\frac{c^M_{j,k}}{g(x,y)}\chi_{j,k,M}(x,y),$$

is a linear combination of functions in the system,

$$\left\{\frac{w_n(x)w_m(y) - w_a(x)w_b(y)}{g(x,y)}\right\}_{(m,n)\in F}.$$

Computing:

$$\int\int_{\mathbb{T}^2}\left|\chi_{\alpha,\beta,N}(x,y) - \sum_{k\in K}\sum_{j\in J}\frac{c^M_{j,k}}{g(x,y)}\chi_{j,k,M}(x,y)\right|^q dxdy$$

$$= \int\int_{A^N_{\alpha,\beta}}\left|\sum_{k\in K}\sum_{j\in J}\frac{c^M_{j,k}}{c^M_{j,k}}\chi_{j,k,M}(x,y) - \sum_{k\in K}\sum_{j\in J}\frac{c^M_{j,k}}{g(x,y)}\chi_{j,k,M}(x,y)\right|^q dxdy,$$

since the support of the second sum is simply $A^N_{\alpha,\beta}$,

$$= \sum_{k\in K}\sum_{j\in J}|c^M_{j,k}|^q\int\int_{A^M_{j,k}}\left|\frac{1}{c^M_{j,k}}\chi_{j,k,M}(x,y) - \frac{1}{g(x,y)}\chi_{j,k,M}(x,y)\right|^q dxdy,$$

since the supports of the $A_{j,k,M}$ are disjoint,

$$\leq \left[\sup_{(x,y)\in[2^{-N},1]\times[2^{-N},1]}(|g(x,y)|)\right]^q\sum_{k\in K}\sum_{j\in J}\int\int_{A^M_{j,k}}\left|\frac{1}{c^M_{j,k}} - \frac{1}{g(x,y)}\right|^q dxdy$$

$$\leq \left[\sup_{(x,y)\in[2^{-N},1]\times[2^{-N},1]} (|g(x,y)|) \right]^q \sum_{k\in K} \sum_{j\in J} \varepsilon_N^q 2^{-2M},$$

by (5.1),

$$= \left[\sup_{(x,y)\in[2^{-N},1]\times[2^{-N},1]} (|g(x,y)|) \right]^q \left(\frac{2^M}{2^N} \right)^2 \varepsilon_N^q 2^{-2M} = \frac{\varepsilon}{2^{2N}} < \varepsilon.$$

□

Theorem 5.2.7. *Let $g(x,y) \in L^p(\mathbb{T}^2) \setminus \{0\}$, let $\frac{1}{\overline{g}(x,y)}$ be continuous on $\mathbb{T}^2 \setminus [0,\gamma) \times [0,\gamma)$ for all $0 < \gamma \leq 1$, $\frac{1}{\overline{g}(x,y)} \notin L^q(\mathbb{T}^2)$, and $\mu(G) = 0$. Then the systems $\{g(x,y)w_n(x)w_m(y)\}_{(m,n)\in F}$ and $\left\{ \frac{w_n(x)w_m(y) - w_a(x)w_b(y)}{\overline{g}(x,y)} \right\}_{(m,n)\in F}$ are both exact systems in $L^p(\mathbb{T}^2)$, and $L^q(\mathbb{T}^2)$, respectively. Moreover, they are the unique biorthogonal systems to one-another.*

Proof.

This is a corollary of Lemmas 5.2.4, 5.2.5, and 5.2.6, and 2.3.7.

□

CHAPTER 6

THE WINDOWED EXPONENTIAL SYSTEM ON $\mathbb{T}^2$

In [5], Heil and Yoon discuss the relationship between completeness and minimality of windowed exponential systems of the form $\{g(x)e^{2\pi inx}\}_{n\in\mathbb{Z}\setminus F} \subseteq L^2(\mathbb{T})$, where F is a set of varying finite cardinality. They show that the properties of the system are greatly dependent upon the cardinality of the set F, and the structure of the zeros of the window function, $g(x)$. In particular, they consider specific window functions of the form $g(x) = x^N$. In light of these results, it is natural to ask whether such questions can be answered for similar systems in $L^2(\mathbb{T}^2)$. Here, the question is addressed for the system $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F}$, with $\alpha, \beta, p > 0$, where $|F| = 1$ - that is, for which α, β, and p is this windowed system complete and/or minimal?

Lemma 6.0.8. *Let $F \subseteq \mathbb{Z}^2$, $|F| = 1$, and assume $\alpha, \beta, p > 0$.*

(a) If $\min(\alpha p, \beta p) \geq 1$, then the system

$$\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F},$$

is complete in $L^2(\mathbb{T}^2)$.

(b) If the system

$$\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F},$$

is complete, then $\max(\alpha p, \beta p) \geq 1$.

Proof.

To demonstrate (a) suppose that $\min(\alpha p, \beta p) \geq 1$. Let $f \in L^2(\mathbb{T}^2)$, and suppose that $\left\langle f, (x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)} \right\rangle = 0$ for all $(j,k) \in \mathbb{Z}^2$, except (a,b). Since $\left\langle f, (x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)} \right\rangle = \left\langle (x^\alpha + y^\beta)^p f, e^{2\pi i(jx+ky)} \right\rangle$, this is simply the $(j,k)^{th}$ Fourier coefficient of $f(x,y)(x^\alpha + y^\beta)^p$, which is in $L^2(\mathbb{T}^2)$ since $(x^\alpha + y^\beta)^{2p}$ is bounded on $\mathbb{T}^2$. Thus:

$$f(x,y)(x^\alpha + y^\beta)^p = ce^{2\pi i(ax+by)},$$

for some constant c. Hence:

$$f(x,y) = \frac{ce^{2\pi i(ax+by)}}{(x^\alpha + y^\beta)^p} \in L^2(\mathbb{T}^2).$$

Computing:

$$\infty > \|f\|^2_{L^2(\mathbb{T}^2)} = \int_0^1 \int_0^1 \left| \frac{ce^{2\pi i(ax+by)}}{(x^\alpha + y^\beta)^p} \right|^2 dxdy = \int_0^1 \int_0^1 \frac{|c|^2}{(x^\alpha + y^\beta)^{2p}} dxdy$$

$$\geq \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \frac{|c|^2}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} rdrd\theta \geq \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \frac{|c|^2}{(2r^{\min(\alpha,\beta)})^{2p}} rdrd\theta$$

$$= \frac{1}{2^{2p}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \frac{|c|^2}{r^{2p\min(\alpha,\beta)-1}} drd\theta, \tag{6.1}$$

where we have made a change to polar coordinates in the standard way.

But since $f(x,y) \in L^2(\mathbb{T}^2)$, we must have then have that $c = 0$, or (6.1) will be infinite since $2p\min(\alpha,\beta) - 1 \geq 1$. Hence, $f(x,y) = 0$. Thus the system $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F}$ is complete.

Now we show (b), assuming that the system is complete. We prove by contradiction, supposing that $\max(\alpha p, \beta p) < 1$. Assume $F = \{(a,b)\}$, where $a, b \in \mathbb{Z}$. Consider the function

$$g(x,y) := \frac{e^{2\pi i(ax+by)}}{(x^\alpha + y^\beta)^p}.$$

We first show that $g \in L^2(\mathbb{T}^2)$, converting to polar coordinates, and noting that g is continuous for $r > \frac{1}{2}$, so that the integral, I, over the remaining subset of $\mathbb{T}^2$ is finite:

$$\|g\|_2^2 = \int_0^1 \int_0^1 \left| \frac{e^{2\pi i(ax+by)}}{(x^\alpha + y^\beta)^p} \right|^2 dxdy \le \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \frac{1}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} rdrd\theta + I$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{2}} \frac{1}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} rdrd\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \frac{1}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} rdrd\theta + I$$

$$\le \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{2}} \frac{1}{(r^\alpha \cos^\alpha(\theta))^{2p}} rdrd\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \frac{1}{(r^\beta \sin^\beta(\theta))^{2p}} rdrd\theta + I$$

$$\le \left(\frac{2}{\sqrt{2}}\right)^{2p\alpha} \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{2}} \frac{1}{r^{2p\alpha-1}} drd\theta + \left(\frac{2}{\sqrt{2}}\right)^{2p\beta} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{1}{2}} \frac{1}{r^{2p\beta-1}} drd\theta + I < \infty,$$

since $\max(\alpha p, \beta p) < 1$ gives $2p\alpha - 1 < 1$, and $2p\beta - 1 < 1$.

We have shown that $g(x,y) \in L^2(\mathbb{T}^2)$, and it is clear that $\|g(x,y)\|_{L^2(\mathbb{T}^2)} = \left\| \frac{e^{2\pi i(ax+by)}}{(x^\alpha+y^\beta)^p} \right\|_{L^2(\mathbb{T}^2)} \neq 0$. However, for $(j,k) \neq (a,b)$:

$$\left\langle g(x,y), (x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)} \right\rangle = \left\langle e^{2\pi i(ax+by)}, e^{2\pi i(jx+ky)} \right\rangle = 0.$$

Therefore $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F}$ is not complete. This is a contradiction.

$\square$

Lemma 6.0.9. *Let $F \subseteq \mathbb{Z}^2$, $|F| = 1$, and assume $\alpha, \beta, p > 0$. If $\max(\alpha p, \beta p) < 2$, then the system $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F}$ is minimal in $L^2(\mathbb{T}^2)$, and has a biorthogonal system, $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^2\setminus F}$, of the following form, where $F = \{(a,b)\}$:*

$$g_{m,n}(x,y) = \frac{e^{2\pi i(mx+ny)} - e^{2\pi i(ax+by)}}{(x^\alpha + y^\beta)^p}.$$

Proof.

Without loss of generality, suppose $\max(\alpha p, \beta p) = \alpha p$. We will show that $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^2\setminus F}$ is a biorthogonal system to $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F}$. First we demonstrate that $g_{m,n} \in L^2(\mathbb{T}^2)$, where $(m,n) \neq (a,b)$:

$$||g_{m,n}||^2_{L^2(\mathbb{T}^2)} = \int_0^1 \int_0^1 \left| \frac{e^{2\pi i(mx+ny)} - e^{2\pi i(ax+by)}}{(x^\alpha + y^\beta)^p} \right|^2 dxdy$$

$$= \int_0^1 \int_0^1 \frac{(e^{2\pi i(mx+ny)} - e^{2\pi i(ax+by)})(e^{-2\pi i(mx+ny)} - e^{-2\pi i(ax+by)})}{(x^\alpha + y^\beta)^{2p}} dxdy$$

$$= \int_0^1 \int_0^1 \frac{2 - 2\cos(2\pi((m-a)x + (n-b)y))}{(x^\alpha + y^\beta)^{2p}} dxdy$$

$$\leq \int_0^1 \int_0^1 \left| \frac{2 - 2\cos(2\pi((m-a)x + (n-b)y))}{(x^\alpha + y^\beta)^{2p}} \right| dxdy,$$

converting to polar coordinates, and choosing $R_1, R_2 < \frac{1}{2}$,

$$= 2\int_0^{\frac{\pi}{4}} \int_0^{R_1} \left| \frac{1 - \cos(2\pi\left[(m-a)r\cos(\theta) + (n-b)r\sin(\theta)\right])}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} \right| rdrd\theta$$

$$+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{R_2} \left| \frac{1 - \cos(2\pi\left[(m-a)r\cos(\theta) + (n-b)r\sin(\theta)\right])}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} \right| rdrd\theta + I, \qquad (6.2)$$

where I is the integral over the remaining area of $\mathbb{T}^2$, which does not include $(0,0)$ since

$R_1, R_2 > 0$. Since the integrand of I is continuous away from zero, and the integral is taken over a compact set, $I < \infty$. Now:

$$(6.2) = 2\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi r\left[(m-a)\cos(\theta)+(n-b)\sin(\theta)\right])}{(r^\alpha\cos^\alpha(\theta)+r^\beta\sin^\beta(\theta))^{2p}}\right| r dr d\theta$$

$$+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi r\left[(m-a)\cos(\theta)+(n-b)\sin(\theta)\right])}{(r^\alpha\cos^\alpha(\theta)+r^\beta\sin^\beta(\theta))^{2p}}\right| r dr d\theta + I. \tag{6.3}$$

Let $l = l(\theta) = [(m-a)\cos(\theta)+(n-b)\sin(\theta)]$ for $\theta \in \left[0, \frac{\pi}{2}\right]$. Note that on $[0, \frac{\pi}{2}]$, $l(\theta)$ is continuous, so there is some $0 < M < \infty$ such that:

$$|l(\theta)| \leq M < \infty. \tag{6.4}$$

Substituting:

$$(6.3) = 2\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi r l)}{(r^\alpha\cos^\alpha(\theta)+r^\beta\sin^\beta(\theta))^{2p}}\right| r dr d\theta$$

$$+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi r l)}{(r^\alpha\cos^\alpha(\theta)+r^\beta\sin^\beta(\theta))^{2p}}\right| r dr d\theta + I. \tag{6.5}$$

Note that for $0 \leq \theta \leq \frac{\pi}{4}$, $r^\alpha\cos^\alpha(\theta)+r^\beta\sin^\beta(\theta) \geq r^\alpha\left(\frac{\sqrt{2}}{2}\right)^\alpha$, and for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$,
$r^\alpha\cos^\alpha(\theta)+r^\beta\sin^\beta(\theta) \geq r^\beta\left(\frac{\sqrt{2}}{2}\right)^\beta$. Hence:

$$(6.5) \leq 2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi r l)}{r^{(2p\alpha-1)}}\right| dr d\theta$$

$$+2\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi r l)}{r^{(2p\beta-1)}}\right| dr d\theta + I. \tag{6.6}$$

Our goal is to show that the sum of these integrals is finite. We make a note that in the case that

$p\beta < 1$, we have $2p\beta - 1 < 1$, and so :

$$2\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\beta-1)}}\right|drd\theta \le 4\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1}{r^{(2p\beta-1)}}\right|drd\theta < \infty.$$

We may argue similarly that if $p\alpha < 1$,

$$2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\alpha-1)}}\right|drd\theta < \infty.$$

Hence, it remains to show that in (6.6), the first integral is finite if $p\alpha \ge 1$, and the second integral is finite if $p\beta \ge 1$. Considering the integrals simultaneously, we let $u = 2\pi rl$, so that

$$2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\alpha-1)}}\right|drd\theta$$

$$= 2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}(2\pi|l|)^{(2p\alpha-2)}\cdot\int_0^{2\pi R_1|l|}\left|\frac{1-\cos(u)}{|u|^{(2p\alpha-1)}}\right|dud\theta$$

$$\le 2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}(2\pi M)^{(2p\alpha-2)}\int_0^{2\pi R_1 M}\left|\frac{1-\cos(u)}{|u|^{(2p\alpha-1)}}\right|dud\theta, \tag{6.7}$$

where we have used (6.4), and the fact that $2p\alpha - 2 \ge 0$. Similarly,

$$2\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\beta-1)}}\right|drd\theta$$

$$\le 2\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\pi M)^{(2p\beta-2)}\int_0^{2\pi R_2 M}\left|\frac{1-\cos(u)}{|u|^{(2p\beta-1)}}\right|dud\theta, \tag{6.8}$$

since $2\beta p - 2 \ge 0$ also.

In (6.7), $-2\pi R_1 M \le u = 2\pi rl \le 2\pi R_1 M$, so $\lim_{R_1\to 0} u = 0$, regardless of the value of θ.

Similarly, in (6.8), $-2\pi R_2 M \leq u = 2\pi rl \leq 2\pi R_2 M$, so $\lim_{R_2\to 0} u = 0$, independent of θ. It is easy to show using Taylor series, that for u sufficiently small, $\frac{1-\cos(u)}{|u|^{2p\alpha-1}} \leq \frac{\cos(u)}{|u|^{2p\alpha-3}}$, and so, choosing R_1 sufficiently small:

$$(6.7) \leq 2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha} \int_0^{\frac{\pi}{4}} (2\pi M)^{(2p\alpha-2)} \int_0^{2\pi R_1 M} \left|\frac{\cos(u)}{|u|^{(2p\alpha-3)}}\right| du d\theta < \infty,$$

since $2p\alpha - 3 < 1$ by hypothesis. We argue analogously to show that (6.8) is finite in the case that $p\beta \geq 1$.

Thus, choosing R_1, R_2 sufficiently small, we see that both (6.7) and (6.8) finite in the respective cases of $1 \leq p\alpha < 2$, and $1 \leq p\beta < 2$. Hence, we showed that in all cases, (6.6) is finite. Therefore, when $\max\{\alpha p, \beta p\} < 2$, $g_{m,n}(x,y) \in L^2(\mathbb{T}^2)$. It is clear that $\|g_{m,n}(x,y)\|_{L^2(\mathbb{T}^2)} \neq 0$. Now we compute, first for for $(j,k) \notin \{(a,b),(m,n)\}$,

$$\left\langle g_{m,n}(x,y), (x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)} \right\rangle = \left\langle e^{2\pi i(mx+ny)} - e^{2\pi i(ax+by)}, e^{2\pi i(jx+ky)} \right\rangle = 0,$$

using the orthogonality of the double exponentials. For $(j,k) = (m,n)$,

$$\left\langle g_{m,n}(x), (x^\alpha + y^\beta)^p e^{2\pi i(mx+ny)} \right\rangle = \left\langle e^{2\pi i(mx+ny)} - e^{2\pi i(ax+by)}, e^{2\pi i(mx+ny)} \right\rangle = 1,$$

using the orthonormality of the double exponentials. Hence, the sequence $\{g_{j,k}\}_{\mathbb{Z}^2\setminus F}$ is biorthogonal to $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{\mathbb{Z}^2\setminus F}$, proving that this sequence is minimal.

$\square$

Theorem 6.0.10. *Lef $F \subseteq \mathbb{Z}^2$, $|F| = 1$, and suppose $\alpha, \beta, p > 0$. If the system $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{\mathbb{Z}^2\setminus F}$ is minimal in $L^2(\mathbb{T}^2)$, then* $\min(\alpha p, \beta p) < 2$.

Proof.

We assume that $\beta p = \min(\alpha p, \beta p)$ without loss of generality, and suppose by way of contradiction that $\beta p \geq 2$. Since the system is minimal, there is a biorthogonal sequence $\{\sigma_{j,k}\}_{\mathbb{Z}^2 \setminus F} \subseteq L^2(\mathbb{T}^2)$. We let $F = \{(a,b)\}$, where $a, b \in \mathbb{Z}$.

Hence,

$$0 = \left\langle (x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}, \sigma_{a,b+1} \right\rangle = \left\langle e^{2\pi i(jx+ky)}, (x^\alpha + y^\beta)^p \sigma_{a,b+1} \right\rangle, \tag{6.9}$$

for $(j,k) \in \mathbb{Z}^2 \setminus F$, and $(j,k) \neq (a, b+1)$. Now, $\sigma_{a,b+1}(x^\alpha + y^\beta)^p \in L^2(\mathbb{T}^2)$, since $(x^\alpha + y^\beta)^p$ is bounded on $\mathbb{T}^2$ and $\sigma_{a,b+1} \in L^2(\mathbb{T}^2)$. Hence, (6.9) implies that the (j,k) Fourier coefficients of $\sigma_{a,b+1}(x^\alpha + y^\beta)^p$ are zero, except for the (a,b) and $(a,b+1)$ terms. Thus for constants c and d:

$$\sigma_{a,b+1}(x^\alpha + y^\beta)^p = de^{2\pi i(ax+(b+1)y)} + ce^{2\pi i(ax+by)}.$$

Note also that,

$$\left\langle \sigma_{a,b+1}, (x^\alpha + y^\beta)^p e^{2\pi i(ax+(b+1)y)} \right\rangle = \left\langle \sigma_{a,b+1}(x^\alpha + y^\beta)^p, e^{2\pi i(ax+(b+1)y)} \right\rangle = 1,$$

forces that $d = 1$. Therefore:

$$\sigma_{a,b+1} = \frac{e^{2\pi i(ax+(b+1)y)} + ce^{2\pi i(ax+by)}}{(x^\alpha + y^\beta)^p} \in L^2(\mathbb{T}^2).$$

We compute the L^2 norm of $\sigma_{a,b+1}$:

$$\|\sigma_{a,b+1}\|_2^2 = \int_0^1 \int_0^1 \frac{\left| e^{2\pi i(ax+(b+1)y)} + ce^{2\pi i(ax+by)} \right|^2}{(x^\alpha + y^\beta)^{2p}} dxdy$$

$$= \int_0^1 \int_0^1 \frac{1+|c|^2+2\operatorname{Re}(c)\cos(2\pi y)}{(x^\alpha+y^\beta)^{2p}} dxdy$$

$$\geq \int_0^1 \int_0^y \frac{1+|c|^2+2\operatorname{Re}(c)\cos(2\pi y)}{(x^\beta+y^\beta)^{2p}} dxdy$$

$$\geq \int_0^1 \int_0^y \frac{1+|c|^2+2\operatorname{Re}(c)\cos(2\pi y)}{(y^\beta+y^\beta)^{2p}} dxdy$$

$$= \frac{1}{2^{2p}} \int_0^1 \frac{1+|c|^2+2\operatorname{Re}(c)\cos(2\pi y)}{y^{2\beta p}} ydy$$

$$= \frac{1}{2^{2p}} \int_0^1 \frac{1+|c|^2+2\operatorname{Re}(c)\cos(2\pi y)}{y^{2\beta p-1}} dy. \tag{6.10}$$

If $1+|c|^2 \neq -2\operatorname{Re}(c)$, then this integral will be unbounded, since $2\beta p - 1 \geq 3$, and the numerator is non-negative. So, assume $1+|c|^2 = -2\operatorname{Re}(c)$. Supposing that $c = v + wi$, where $v, w \in \mathbb{R}$, we must have that $v^2 + 2v + (1+w^2) = 0$. Applying the quadratic formula yields $v = -1 \pm iw$. Hence, $w = 0$, and $v = -1$. Using this and the fact that $2\beta p - 1 \geq 3$:

$$(6.10) = \frac{1}{2^{2p-1}} \int_0^1 \frac{1-\cos(2\pi y)}{y^{2\beta p-1}} dy \geq \frac{1}{2^{2p-1}} \int_0^1 \frac{1-\cos(2\pi y)}{y^3} dy$$

$$= \frac{1}{2^{2p-1}} \int_0^1 \frac{1-\cos(2\pi y)}{y^2} \cdot \frac{1}{y} dy. \tag{6.11}$$

Since $\lim_{y\to 0} \frac{1-\cos(2\pi y)}{y^2} = 2\pi^2$, choose $\delta > 0$ small enough that $\frac{1-\cos(2\pi y)}{y^2} > m$, where m is some finite, positive real number. Hence:

$$(6.11) \geq \frac{1}{2^{2p-1}} \int_0^\delta \frac{1-\cos(2\pi y)}{y^2} \cdot \frac{1}{y} dy \geq \frac{m}{2^{2p-1}} \int_0^\delta \frac{1}{y} dy = \infty.$$

That is, $\left\|\sigma_{a,b+1}\right\|_2 = \infty$, contradicting $\sigma_{a,b+1} \in L^2(\mathbb{T}^2)$. Therefore, it must be that $\min(\alpha p, \beta p) < 2$.

$\square$

Corollary 6.0.11. *Let $F \subseteq \mathbb{Z}^2$, $|F| = 1$, and assume $\alpha, \beta, p > 0$. If*

$$1 \leq \min(\alpha p, \beta p) \text{ and } \max(\alpha p, \beta p) < 2,$$

then the system,

$$\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F},$$

is exact in $L^2(\mathbb{T}^2)$.

Proof.

This follows directly from Lemma 6.0.8 and Lemma 6.0.9.

$\square$

Corollary 6.0.12. *Let $F \subseteq \mathbb{Z}^2$, $|F| = 1$, and assume $\alpha, \beta, p > 0$. If the system*

$$\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2\setminus F},$$

is exact in $L^2(\mathbb{T}^2)$, then:

$$1 \leq \max(\alpha p, \beta p) \text{ and } \min(\alpha p, \beta p) < 2$$

Proof.

This follows directly from Lemma 6.0.8, and Lemma 6.0.10.

$\square$

We note to the reader, that the following proof is virtually identical to portions of the proof of Theorem 6.0.9.

Theorem 6.0.13. *Let $F \subseteq \mathbb{Z}^2$, $|F| \geq 2$. If $\max(\alpha p, \beta p) < 2$, then the system $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{\mathbb{Z}^2 \setminus F}$ is not complete in $L^2(\mathbb{T}^2)$.*

Proof.

Let $\max(\alpha p, \beta p) < 2$, and suppose that $F = \{(a,b),(m,n)\}$. Let $g(x) = \frac{e^{2\pi i(ax+by)} - e^{2\pi i(mx+ny)}}{(x^\alpha + y^\beta)^p}$. We will demonstrate first that $g \in L^2(\mathbb{T}^2)$:

$$||g||_2^2 = \int_0^1 \int_0^1 \left| \frac{e^{2\pi i(ax+by)} - e^{2\pi i(mx+ny)}}{(x^\alpha + y^\beta)^p} \right|^2 dxdy$$

$$= \int_0^1 \int_0^1 \frac{(e^{2\pi i(ax+by)} - e^{2\pi i(mx+ny)})(e^{-2\pi i(ax+by)} - e^{-2\pi i(mx+ny)})}{(x^\alpha + y^\beta)^{2p}} dxdy$$

$$= \int_0^1 \int_0^1 \frac{2 - 2\cos(2\pi((a-m)x + (b-n)y))}{(x^\alpha + y^\beta)^{2p}} dxdy$$

$$\leq \int_0^1 \int_0^1 \left| \frac{2 - 2\cos(2\pi((a-m)x + (b-n)y))}{(x^\alpha + y^\beta)^{2p}} \right| dxdy, \tag{6.12}$$

converting to polar coordinates, and choosing $R_1, R_2 < \frac{1}{2}$:

$$= 2\int_0^{\frac{\pi}{4}} \int_0^{R_1} \left| \frac{1 - \cos(2\pi[(a-m)r\cos(\theta) + (b-n)r\sin(\theta)])}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} \right| rdrd\theta$$

$$+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{R_2} \left| \frac{1 - \cos(2\pi[(a-m)r\cos(\theta) + (b-n)r\sin(\theta)])}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} \right| rdrd\theta + I, \tag{6.13}$$

where I is the integral over the remaining area of $\mathbb{T}^2$, which does not include $(0,0)$ since $R_1, R_2 > 0$. Since the integrand of I is continuous away from zero, and the integral is taken over a compact set, $I < \infty$. Now:

$$(6.13) = 2\int_0^{\frac{\pi}{4}} \int_0^{R_1} \left| \frac{1 - \cos(2\pi r[(a-m)\cos(\theta) + (b-n)\sin(\theta)])}{(r^\alpha \cos^\alpha(\theta) + r^\beta \sin^\beta(\theta))^{2p}} \right| rdrd\theta$$

$$+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi r\left[(a-m)\cos(\theta)+(b-n)\sin(\theta)\right])}{(r^{\alpha}\cos^{\alpha}(\theta)+r^{\beta}\sin^{\beta}(\theta))^{2p}}\right| rdrd\theta+I. \tag{6.14}$$

Let $l=l(\theta)=[(a-m)\cos(\theta)+(b-n)\sin(\theta)]$ for $\theta\in\left[0,\frac{\pi}{2}\right]$. Note that on $[0,\frac{\pi}{2}]$, $l(\theta)$ is continuous, so there is some $0<M<\infty$, such that

$$|l(\theta)|\leq M<\infty. \tag{6.15}$$

Substituting:

$$(6.14)=2\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi rl)}{(r^{\alpha}\cos^{\alpha}(\theta)+r^{\beta}\sin^{\beta}(\theta))^{2p}}\right| rdrd\theta$$

$$+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi rl)}{(r^{\alpha}\cos^{\alpha}(\theta)+r^{\beta}\sin^{\beta}(\theta))^{2p}}\right| rdrd\theta+I. \tag{6.16}$$

Note that for $0\leq\theta\leq\frac{\pi}{4}$, $r^{\alpha}\cos^{\alpha}(\theta)+r^{\beta}\sin^{\beta}(\theta)\geq r^{\alpha}\left(\frac{\sqrt{2}}{2}\right)^{\alpha}$, and for $\frac{\pi}{4}\leq\theta\leq\frac{\pi}{2}$, $r^{\alpha}\cos^{\alpha}(\theta)+r^{\beta}\sin^{\beta}(\theta)\geq r^{\beta}\left(\frac{\sqrt{2}}{2}\right)^{\beta}$. Hence:

$$(6.16)\leq 2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\alpha-1)}}\right| drd\theta$$

$$+2\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\beta-1)}}\right| drd\theta+I. \tag{6.17}$$

Our goal is to show that the sum of these integrals is finite. We make a note that in the case that $p\alpha<1$, we have $2p\alpha-1<1$, and so:

$$2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\alpha-1)}}\right| drd\theta$$

$$\leq 4\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}\int_0^{R_1}\frac{1}{r^{(2p\alpha-1)}}drd\theta < \infty.$$

We may argue similarly that if $p\beta < 1$,

$$2\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\beta-1)}}\right|drd\theta < \infty.$$

Hence, it remains to show that in (6.17), the first integral is finite if $p\alpha \geq 1$, and the second integral is finite if $p\beta \geq 1$. Considering the integrals simultaneously, we let $u = 2\pi rl$, so that

$$2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}\int_0^{R_1}\left|\frac{1-\cos(2\pi rl)}{2^{(2p\alpha-1)}}\right|drd\theta$$

$$= 2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}(2\pi|l|)^{(2p\alpha-2)}\int_0^{2\pi R_1|l|}\left|\frac{1-\cos(u)}{|u|^{(2p\alpha-1)}}\right|dud\theta$$

$$\leq 2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha}\int_0^{\frac{\pi}{4}}(2\pi M)^{(2p\alpha-2)}\int_0^{2\pi R_1 M}\left|\frac{1-\cos(u)}{|u|^{(2p\alpha-1)}}\right|dud\theta, \tag{6.18}$$

where we have used (6.15), and the fact that $2p\alpha - 2 \geq 0$. Similarly,

$$2\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{R_2}\left|\frac{1-\cos(2\pi rl)}{r^{(2p\beta-1)}}\right|drd\theta$$

$$\leq 2\left(\frac{2}{\sqrt{2}}\right)^{2p\beta}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\pi M)^{(2p\beta-2)}\int_0^{2\pi R_2 M}\left|\frac{1-\cos(u)}{u^{(2p\beta-1)}}\right|dud\theta, \tag{6.19}$$

since $2\beta p - 2 \geq 0$ also.

In (6.18), $-2\pi R_1 M \leq u = 2\pi rl \leq 2\pi R_1 M$, so $\lim_{R_1\to 0} u = 0$, regardless of the value of θ. Similarly, in (6.19), $-2\pi R_2 M \leq u = 2\pi rl \leq 2\pi R_2 M$, so $\lim_{R_2\to 0} u = 0$, independent of θ.

It is easy to show using Taylor series, that for u sufficiently small, $\frac{1-\cos(u)}{|u|^{2p\alpha-1}} \leq \frac{\cos(u)}{|u|^{2p\alpha-3}}$, and so, choosing R_1 sufficiently small:

$$(6.18) \leq 2\left(\frac{2}{\sqrt{2}}\right)^{2p\alpha} \int_0^{\frac{\pi}{4}} (2\pi M)^{(2p\alpha-2)} \int_0^{2\pi R_1 M} \left|\frac{\cos(u)}{|u|^{(2p\alpha-3)}}\right| dud\theta < \infty,$$

since $2p\alpha - 3 < 1$ by hypothesis. We argue analogously to show that (6.19) is finite in the case that $p\beta \geq 1$.

Thus, choosing R_1, R_2 sufficiently small, we see that both (6.18) and (6.19) finite in the respective cases of $1 \leq p\alpha < 2$, and $1 \leq p\beta < 2$. Hence, we showed that in all cases, (6.16) is finite. Therefore, when $\max(\alpha p, \beta p) < 2$, $g(x,y) \in L^2(\mathbb{T}^2)$. It is clear that $\|g(x,y)\|_2 \neq 0$. Also:

$$\left\langle g(x), (x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)} \right\rangle = \left\langle e^{2\pi i(ax+by)} - e^{2\pi i(mx+ny)}, e^{2\pi i(jx+ky)} \right\rangle = 0,$$

for $(j,k) \in \mathbb{Z}^2 \setminus F$. Thus, removing more than one element from $\{(x^\alpha + y^\beta)^p e^{2\pi i(jx+ky)}\}_{(j,k)\in\mathbb{Z}^2}$ results in an incomplete system.

$\square$

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