PROPERTIES OF HYPERBOLIC GROUPS: FREE NORMAL SUBGROUPS, QUASICONVEX SUBGROUPS AND ACTIONS OF MAXIMAL GROWTH

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To my parents

and Aliya

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CHAPTER I

INTRODUCTION

In the founding paper [**Gro**] M. Gromov defined the notion of hyperbolic groups and outlined a number of research directions in this (now well established) area. As a subfield of geometric group theory, the hyperbolic groups are defined using the analogy between an algebraic objects – groups, and hyperbolic metric spaces and manifolds. The analogy between fundamental groups of compact surfaces and tessellations of a hyperbolic plane was surely known to Max Dehn, back in the wake of 20th century. Some of the early group theory methods such as small cancellation and Bass-Serre theory of groups acting on trees may be viewed now as early chapters of geometric (hyperbolic) group theory.

One of the most astounding facts about hyperbolic groups is often referred by saying that " hyperbolic groups are generic". By that one means the hypothesis of Gromov, stating that almost every group is hyperbolic (one can find both formalization and proof of such statement in [**Olsh1**]). Another excellent example of richness of the class of hyperbolic groups is that every (non-elementary group) G is SQ-universal, i.e. every countable group can be embedded into the quotient of G. One of the results of this paper is a slight strengthening of this SQ-universality property (see Corollary I.1.5).

At the same time, one can also say that hyperbolic groups are "easy to deal with": the word problem (i.e. the algorithmic problem, asking if a word, encoding an element of the group, actually represents the identity element) is solvable in linear time.

In this dissertation we work with two very different classes of subgroups in a hyperbolic group G: normal subgroups and quasiconvex subgroups. The first class of subgroups – normal– are embedded "nicely" in G in the classical group theoretic sense, while the quasiconvex subgroups are embedded "hyperbolically" in G as geometric objects. It is well known that a subgroup in G, which is both normal and quasiconvex, is either finite or of finite index (see [**Arzh**]) and hence, not very interesting. In one way, the two types of subgroups are "completing" each other in a hyperbolic group G: if $\mathscr{H}$ is a quasiconvex subgroup, then there is a normal complement $\mathscr{N}$ intersecting $\mathscr{H}$ trivially ([**Min**], Lemma 3.8 or [**GSS**]).

I.1 Main results

In Chapter II we give basic definitions and properties which we will use later.

Chapter III is devoted to small cancellation constructions of the normal subgroups. In the first two sections we present the small cancellation techniques by T. Delzant and A. Olshanskiy. One finds the following Statement 5.3E in [**Gro**]:

There exists a constant $m = m(k,\delta)$ such that for every k hyperbolic elements $x_1,\dots,x_k$ in a word δ-hyperbolic group G the normal subgroup generated by $x_1^{m_1},\dots,x_k^{m_k}$ is free for all $m_i \geq m$.

Although not correct in full generality (as a counter-example in the appendix to [**Delz**] shows) the following theorems are true:

Theorem I.1.1. *(Delzant [**Delz**], Theoremé I) Let G be a non-elementary δ-hyperbolic group. There* *exists an integer N such that for any elements $f_1,\dots,f_n$ such that $[[f_i]] = [[f_j]] \geq 1000\delta$ (where $[[f]] = lim_{n\to\infty}\frac{|f^n|}{n}$), the normal subgroup $\mathscr{N}(f_1^{kN},\dots,f_n^{kN})$ is free for every k. Moreover, (for every k) the group $G/\mathscr{N}(f_1^{kN},\dots,f_n^{kN})$ is hyperbolic.*

The Theorem I.1.1 is obtained in [**Delz**] from Theorem I.1.2 by arguing that for sufficiently large N (independent of choice f_i) the system of elements $f_1^N,\dots,f_n^N$ can be completed to that satisfying small cancellation $C'(\mu)$ (see definition III.1.4).

Theorem I.1.2. *(Delzant [**Delz**], Theorem é II) Let $\mathscr{R}$ be a finite set of elements satisfying the the small cancellation condition $C'(\mu)$. A normal subgroup $\mathscr{N}(\mathscr{R})$ generated by $\mathscr{R}$ is free. The quotient $G/\mathscr{N}(\mathscr{R})$ is hyperbolic.*

However we think that the proof of Theorem I.1.2 requires some additional arguments. To be more precise, the proof of the Theorem 2.1 (iii) [**Delz**] pp 677-678 (stating that if a (finite) system $\mathscr{R}$ satisfies condition $C'(\mu),\ \mu < 1/8$ then the normal subgroup $\mathscr{N}(\mathscr{R})$ generated by $\mathscr{R}$ is free) is incomplete. We provide a proof of essentially the same fact in somewhat different setting (in particular, the set $\mathscr{R}$ can be infinite) using both techniques of Delzant (such as Lemmas III.1.6, III.1.10) and diagram techniques of A. Olshanskiy from [**Olsh**], [**Olsh93**]. We would like to note that the Lemma III.4.10 of this paper provides justification for the formula on top of page 678 of [**Delz**]. One may replace Theorem I.1.2 with the following statement:

Theorem I.1.3. *There exists $\mu_0 > 0$ such that for any $\mu < \mu_0$ there are ε and ρ such that if $\mathscr{R}$ is a set of geodesic words satisfying $\tilde{C}(\varepsilon,\mu,\rho)$-condition (see Definition III.2.9) in the hyperbolic group G then:*

(i) the normal subgroup $\mathscr{N} = \mathscr{N}(\mathscr{R})$ is free;

(ii) if G is non-elementary and $\mathscr{R}$ is finite then $G/\mathscr{N}(\mathscr{R})$ is non-elementary hyperbolic.

As a corollary we get:

Theorem I.1.4. *Let G be a non-elementary hyperbolic group. For any finite set of elements $x_1,\dots,x_m$ there exists an integer N such that the normal closure $\mathscr{N} = \mathscr{N}(x_1^{s_1N},\dots,x_m^{s_mN})$ in G of elements $x_1^{s_1N},\dots,x_m^{s_mN}$ is free for any integer $s_i > 0$ and the quotient $G/\mathscr{N}$ is non-elementary hyperbolic.*

A stronger and more general version of the above statement, using the language of rotating families, appeared recently (but after [**Cha**]) in the paper [**DGO**], theorem 2.13.

Let us note that in our result I.1.4, the choice of constant N depends on the elements $x_1,\dots,x_m$ rather then being an absolute constant as in Theorem I.1.1. On the other hand we do not assume any significant restrictions on the set of elements $x_1,\dots,x_m$.

The following Corollary somewhat strengthens the Theorem proved by T. Delzant and A. Olshanskiy independently (see [**Delz**], [**Olsh95**]) stating that every non-elementary hyperbolic group is SQ-universal.

Corollary I.1.5. *Let G be a non-elementary hyperbolic group. Then:*

(i) there exists a free normal subgroup $\mathscr{N}$ of G of rank greater than 1;

(ii) for any free normal subgroup $\mathscr{N}$ of rank greater than 1 and any countable group H there exists a free subgroup $M < \mathscr{N}$, $M \triangleleft G$ such that H embeds in quotient G/M.

As an application of Theorem I.1.3 we also obtain:

Corollary I.1.6. *Let G be a non-elementary hyperbolic group. Then there exist free normal subgroups of infinite index A, B such that $AB = G$ if and only if $Z(G) = E(G)$.*

We would like to mention the following

Open problem *([**Kour**], 15.69): Does every hyperbolic group G have a free normal subgroup N such that the quotient G/N is a torsion group of bounded exponent?*

The above problem is motivated by the result of Ivanov and Olshanskiy [**IvOl**] stating that for every non-elementary hyperbolic group G there is a number $n = n(G)$ such that the quotient group G/G^n is infinite.

In Chapter IV we discuss the growth of highly transitive actions of hyperbolic groups. We generalize some results of [**BO**], where the authors discuss the notion of growth of actions of a group (monoid, ring) on a set (module).

Let us denote the growth function of a transitive action of a group G generated by a finite set S on a set X with respect to some base point $o \in X$ by $g_{o,S}(n) = \#\{o' = og| \ |g| \leq n\}$ (see section IV.1). A distinguished class of actions defined and studied in [**BO**] is that of actions of maximal growth. We observe that in case of a non-amenable group G the growth of action of G on X is *maximal* if there exists $c_1 > 0$ such that

$$c_1 f(n) \leq g_{o,S}(n)$$

for every natural n, where $f(n)$ is the growth of the group G itself (see remark IV.1.4). In general, the notion of maximal growth depends on the choice of the generating set S of the group, thus throughout the Chapter IV we assume that S is fixed. The author tends to consider the maximal growth action as a new characterization for the finite radical of the hyperbolic group. One observes in remark IV.1.11 that the kernel of arbitrary 2-transitive action of maximal growth by a hyperbolic group G is exactly the finite radical $E(G)$.

In [**BO**] the authors construct some examples of actions by the free group of maximal growth and satisfying additional properties, see for example corollary IV.1.9. The first result of our paper is the following broadening of the aforementioned corollary:

Theorem I.1.7. *Let G be a non-elementary hyperbolic group. Then there exist a set X and a transitive action of G on X such that the growth of this action is maximal, each orbit of action by every element $g \in G$ is finite and the stabilizer of every element $x \in X$ is a free group.*

One can observe that the above result follows from Theorem IV.1.10 in this paper.

The Theorem IV.1.10 stems from the technical result IV.2.8, which also allows us to generalize and strengthen the result of Arzhantseva [**Arzh**] conjectured by M. Gromov [**Gro**].

The following Theorem and corollary generalize Theorem 1 in [**Arzh**] by removing the requirement on the hyperbolic group to be torsion-free and formulating the necessary and sufficient conditions. We recall the notation $E(g)$ – a unique maximal elementary subgroup of hyperbolic group G containing g (it exists whenever g is of infinite order, see section II.1). Recall also that there exists a unique finite maximal normal subgroup $E(G)$ in every non-elementary hyperbolic group G. We will call $E(G)$ *the finite radical*[1] of G.

[1]the term proposed by A. Olshanskiy.

Theorem I.1.8. *Let G be a non-elementary hyperbolic group and $\mathscr{H}$ be a quasiconvex infinite index subgroup of G.*

(i) Consider an element x in G of infinite order. Then the following are equivalent:

(a) there exists a natural number $t > 0$ such that the subgroup $\langle \mathscr{H}, x^t \rangle$ (i.e. generated by $\mathscr{H}$ and x^t) is isomorphic to $\mathscr{H} * \langle x^t \rangle$;

(b) $E(x) \cap \mathscr{H} = \{e\}$.

(ii) [2] *An element x, satisfying part (i), exists if and only if $\mathscr{H} \cap E(G) = \{e\}$.*

(iii) If $\mathscr{H} \cap E(G) = \{e\}$ then for x and t described in part (i) the subgroup $\langle \mathscr{H}, x^t \rangle$ is quasiconvex of infinite index and the intersection $E(G) \cap \langle \mathscr{H}, x^t \rangle$ is trivial.

Part (i) of Theorem I.1.8 follows also from a more general statement in [**M-P**](Corollary 1.12) and a particular case when $E(x) = E^+(x)$ appears in Theorem 5 [**Min**]. We also formulate the following (somewhat more general) result concerning arbitrary quasiconvex subgroups of infinite index.

Corollary I.1.9. *Let G be a non-elementary hyperbolic group and $\mathscr{H}$ be a quasiconvex subgroup of infinite index in G. Then there exists $g \in G$ of infinite order such that $\langle \mathscr{H} \cdot E(G), g \rangle \cong \mathscr{H} \cdot E(G) *_{E(G)} \langle g, E(G) \rangle$. Moreover $\langle \mathscr{H} \cdot E(G), g \rangle$ is a quasiconvex subgroup of infinite index.*

The main results of this chapter concern the highly transitive actions of finitely generated groups (i.e. the actions which are k-transitive for every $k \in \mathbb{N}$) on infinite countable sets. The first result of this sort, known to the author, is that finitely generated non-abelian free groups admit faithful highly transitive actions, [**McD**]. In recent years, similar results has been proved for several classes of groups, we summarized thm in the following:

Theorem I.1.10. *([**GaGl**], [**MoSt**], [**Kit**]) The following groups admit a highly transitive faithful action on infinite countable sets:*

(i) the fundamental group of a closed orientable surface of genus > 1;

(ii) free products $G * H$ where at least G or H is not isomorphic to $\mathbb{Z}_2$;

(iii) The group of outer automorphisms of the free group $Out(F_n)$ for $n > 3$.

Theorem I.1.11 is our main result of the chapter, it includes the case of fundamental groups of the closed orientable surfaces of genus > 1 as well as some of the free products $G * H$, described by I.1.10(ii). Moreover, in some statistical sense (see [**Olsh1**]), almost every finitely presented group is hyperbolic. Thus, almost every group satisfies conditions (i)–(iv) of the Theorem below.

Theorem I.1.11. *Let G be a non-elementary δ-hyperbolic group. There exist a set X and an action of maximal growth of G on X such that:*

(i) each orbit of action by every element $g \in G$ is finite;

(ii) the action is highly transitive;

(iii) the stabilizer of every point $x \in X$ is an extension of $E(G)$ by a free group;

(iv) the kernel of the action is the finite radical $E(G)$ of G.

[2] While preparing this result for publication the author learned that a version of this statement has been presented by F. Dudkin and K. Sviridov in a Group Theory seminar at IM SORAN (Novosibirsk) in November, 2011.

We would also like to stress that if we assume (in the Theorem I.1.11) that the finite radical $E(G)$ of the group G is trivial, then the action is faithful and the stabilizer of every $x \in X$ is a maximal free subgroup of G (see also corollary IV.4.7).

CHAPTER II

HYPERBOLIC SPACES AND GROUPS

II.1 Hyperbolic spaces

We recall some definitions and properties from the founding article of Gromov [**Gro**] (see also [**Ghys**]). Let $(X, |\ |)$ be a metric space. We sometimes denote the distance $|x-y|$ between $x, y \in X$ by $d(x,y)$. We assume that X is geodesic, i.e. every two points can be connected by a geodesic line. We refer to a geodesic between some point x, y of X as $[x,y]$. For convenience we denote by $|x|$ the distance $|x-y_0|$ to some fixed point y_0 (usually the identity element of the group).

For a path γ in X we denote the initial (terminal) vertex of γ by γ_- (γ_+), denote by $\|\gamma\|$ the length of path γ and by $|\gamma|$ the distance $|\gamma_+ - \gamma_-|$. Recall that if $0 < \lambda \leq 1$ and $c \geq 0$ then a path γ in X is called (λ, c)-*quasigeodesic* if for every subpath γ_1 of γ the following inequality is satisfied:

$\|\gamma_1\| \leq \frac{1}{\lambda}|\gamma_1| + c$.

We call the path γ *geodesic up to c*, if it is $(1,c)$-quasigeodesic.

Define a scalar (Gromov) product of x, y with respect to z by formula

$$(x,y)_z = \frac{1}{2}(|x-z| + |y-z| - |x-y|).$$

An (equivalent) implicit definition of the Gromov product illustrates its geometric significance:

$$(x,y)_z + (x,z)_y = |z-y|; \tag{II.1}$$

$$(x,y)_z + (y,z)_x = |z-x|;$$

$$(y,z)_x + (z,x)_y = |x-y|.$$

A space X is called δ-hyperbolic if there exists a non-negative integer δ such that the following inequality holds:

$$(\mathbf{H1})\ \ \forall x,y,z,t \in X,\ \ (x,y)_z \geq min\{(x,t)_z, (y,t)_z\} - \delta.$$

The condition (H1) implies (and in fact is equivalent up to constant) the following:

(H2) *For every triple of points x, y, z in X every geodesic $[x,y]$ is within the (closed) 4δ-neighborhood of the union $[x,z] \cup [y,z]$.*

(H3) *For every four points x,y,z,t in X we have $|x-y| + |z-t| \leq max\{|x-z| + |y-t|, |x-t| + |y-z|\} +$* 2δ.

We will need a few properties of hyperbolic spaces and Gromov products:

Lemma II.1.1. *([**Delz**], Lemma 1.3.3) Let K be a nonnegative real number, $[x,y]$ and $[x',y']$ – two segments in a δ-hyperbolic space of length at least $2K + 20\delta$ and suppose that $|x-x'| \leq K$, $|y-y'| \leq K$. Choose* *points u and v on $[x,y]$ at distance $K+2\delta$ from x and y respectively. Then every point P on $[u,v]$ is in the 6δ-neighborhood of the segment $[x',y']$.*

Lemma II.1.2. *([**Ghys**], Chapter 3, §17) For any three points x,y,z in a δ-hyperbolic space X, we have $d(x,[y,z])-\delta \le \langle y,z\rangle_x \le d(x,[y,z])$.*

We will use the following easy remark.

Remark II.1.3. Let X be a hyperbolic space. Then:

(i) In the notations of Lemma II.1.1 it is immediate that the segment $[x,y]$ is within $K+2\delta+6\delta$-neighborhood of $[x',y']$.

(ii) Suppose γ is a path, geodesic up to some $c \ge 0$ in X, and o is an arbitrary point on γ. Then

$$(\gamma_-,\gamma_+)_o \le c/2. \tag{II.2}$$

Combining the previous inequality with Lemma II.1.2 we get that:

$$d(o,[\gamma_-,\gamma_+])-\delta \le c/2. \tag{II.3}$$

We recall the notion of the *metric tree T* ([**Ghys**], Chapter 2, §1). Let T' be a tree (i.e. graph without cycles), we construct the geometric realization T in the following way. For every edge a of T' we choose a real positive number $l(a)$. Then there exists a unique (up to isometry) metric d on T maximal with respect to the following condition: edge a is isometric to interval $[0,l(a)]$ on the real line. Then T with the metric d is a metric tree.

Various versions of the following Gromov's Theorem provide an approximation of a finite set of geodesics in hyperbolic space by metric trees:

Theorem II.1.4. *([**Ghys**], Chapter 2, Theorem 12) Let F be a δ-hyperbolic metric space. Suppose that $F=\cup_{i=1}^n F_i$, where each $F_i=[w,w_i]$ is a geodesic and $n \le 2^k$.*

Then there exists a metric tree T and function $\Phi : F \to T$ such that

(i)$|[\Phi(x),\Phi(w)]| = |[x,w]|$, $\forall x \in F$;

(ii)$|x-y|-2(k+1)\delta \le |\Phi(x)-\Phi(y)| \le |x-y|$ for all $x,y \in F$.

It is clear that if x is some vertex in a metric graph T in the Theorem above then either

(i) there exist some indexes i,j such that the images of F_i and F_j under Φ depart at x: $\Phi([w,w_i]) \cap \Phi([w,w_j]) = [\Phi(w),x]$ (in this case we call vertex x a branching point), or

(ii) there exists some index i such that $\Phi(w_i)=x$ or $\Phi(w)=x$. In this case we call x a leaf (because it is adjacent to a single vertex).

When we talk about an approximation tree for a set of vertices $w,w_1,\dots,w_n$ in the hyperbolic space X, we mean an approximation of the set $F=\cup_{i=1}^n F_i$ in the sense of the previous Theorem.

By a tripod we mean a metric tree with one branching point (center o) and three edges (pods).

Remark II.1.5. ([**Ghys**] Chapter 2, §1) Let x,y,z be some points in a δ-hyperbolic space X, and o_1 be a point on $[x,y]$ at distance $s \le (y,z)_x$ from x, o_2 be on $[x,z]$ at distance s from x. Then there exists a tripod T and a map $\Phi : [x,y]\cup[x,z] \longrightarrow T$ such that:

(i) a restriction of the map Φ on each segment $[x,y],[x,z]$ is an isometry which sends x,y,z to different ends of pods of T and $\Phi(o_1)=\Phi(o_2)$;

(ii) Φ, T satisfies the previous Theorem.

Lemma II.1.6. *([**Gro**], [**Ghys**] p. 87) There exists a constant $H=H(\delta,\lambda,c)$ such that for any (λ,c)-quasigeodesic path p in a δ-hyperbolic space and any geodesic path q with conditions $q_-=p_-$ and $q_+=p_+$, the paths p and q are within (closed) H-neighborhoods of each other.*

II.2 Hyperbolic Groups

Let G be a finitely presented group with presentation $gp(S|\mathscr{D})$. We assume that no generator in S is equal to e in G. We consider G as a metric space with respect to the distance function $|g-h|=|g^{-1}h|$ for every g and h. We denote by $|g|$ the length of a minimal (geodesic) word with respect to the generators S equal to g. The notation (g,h) is the Gromov product $(g,h)_e$ with respect to the identity vertex e.

We denote the (right) Cayley graph of the group by $Cay(G)$. The graph $Cay(G)$ has a set of vertices G, and a pair of vertices g_1,g_2 is connected by an edge of length 1 labeled by s if and only if $g_1^{-1}g_2=s$ in G for some $s\in S^{\pm1}$. It is clear that $Cay(G)$ may be considered as a geodesic space: one identifies every edge of $Cay(G)$ with interval $[0,1]$ and chooses the maximal metric d which agrees with metric on every edge. Define a label function on paths in $Cay(G)$. From now on, by a path in $Cay(G)$ we mean a path $p=p_1...p_n$, where p_i is an edge in $Cay(G)$ between some group elements g_i, g_{i+1} for every $1\le i\le n$. A label $lab(p)$ function is defined on any path p by equality $lab(p)=lab(p_1)...lab(p_n)$, i.e. $lab(p)$ is a word in alphabet $S^{\pm1}$.

Hence a unique word $lab(p)$ is assigned to a path p in $Cay(G)$. On the other hand for every word w in alphabet $S^{\pm1}$ there exists a unique path p in $Cay(G)$ starting from the identity vertex with label w. Hence there is a one-to-one correspondence between paths with initial vertex e (the identity vertex in G) and words in alphabet $S^{\pm1}$, so we will not distinguish between a word in the alphabet $S^{\pm1}$ and it's image in $Cay(G)$, i.e. a path starting from the identity vertex. Thus, when considering some words X,Y,Z in the alphabet $S^{\pm1}$, we can talk about the path $\gamma=XYZ$ in the Cayley graph of G originating in the identity vertex e. To distinguish a path Y with initial vertex e from the subpath of γ with label Y we denote the latter as ${}_\gamma Y$. We will talk about values $|X|,\|X\|$ for a word X in alphabet $S^{\pm1}$ meaning these values on the corresponding paths in $Cay(G)$. Given elements $x_1,...,x_k$ in G we may write $lab(p)=x_1^{t_1}...x_k^{t_k}$ for some path p in $Cay(G)$, $t_i\in\mathbb{Z}$ if for some geodesic words $X_1,...,X_k$ representing elements $x_1,...,x_k$ we have $lab(p)=X_1^{t_1}...X_k^{t_k}$.

For a point x in a metric space X and $r\ge0$ we denote by $B_r(x)$ a metric ball of radius r around x. For a set $D\subset X$ we denote by $B_r(D)$ a (closed) r-neighborhood of D in X (i.e. $B_r(D)=\cup_{x\in D}B_r(x)$). We denote the ball $B_R(e)$ in the Cayley graph $Cay(G)$ by B_R. Given a set $D\in Cay(G)$ we denote by $\#\{D\}$ a number of vertices in D.

A group G is called *δ-hyperbolic* for some $\delta\ge0$, if it's Cayley graph is δ-hyperbolic. It is well known that hyperbolicity does not depend on choice of a finite presentation of the group G (while δ does depend on presentation).

We recall that a (sub)group is called elementary if it contains a cyclic group of finite index. For any element $g \in G$ of infinite order in a hyperbolic group, there exists a unique maximal elementary subgroup $E(g)$ containing g (see [**Gro**], [**Olsh93**] Lemma 1.16). It is well known that for a hyperbolic group G

$$E(g) = \{x \in G | \exists n \neq 0 \text{ such that } xg^n x^{-1} = g^{\pm n} \text{ in } G\},$$

and if a is an element in $E(g)$ of infinite order then $E(g) = E(a)$. We recall also that if G is a non-elementary hyperbolic then the subgroup $E(G) = \cap_g \{E(g) | g \in G, \text{ order of } g \text{ is infinite}\}$ is a unique maximal finite normal subgroup ([**Olsh93**], Prop.1). As agreed in the introduction, we will call $E(G)$ the *finite radical* of a non-elementary group G.

Definition II.2.1. A subset A is called K-quasiconvex in the metric space X if for any pair of points $a, b \in A$ every geodesic connecting a and b (in X) is within (closed) K-neighborhood of A. A subgroup $\mathscr{H}$ of a hyperbolic group G is K-quasiconvex if it forms a K-quasiconvex subset in the graph $Cay(G)$.

It is said that $\mathscr{H}$ is quasiconvex if it is K-quasiconvex for some $K \geq 0$. Note also that the left multiplication $g \to ag$ induces an isometry of G and hence, for a K-quasiconvex subgroup $\mathscr{H}$, the right coset $a\mathscr{H}$ is K-quasiconvex for any a in G.

Lemma II.2.2. *([**GMRS**], Lemma 1.2) Let H be a K-quasiconvex subgroup of a δ-hyperbolic group G. If the shortest representative of a double coset HgH has length greater than $2K+2\delta$, then the intersection $H \cap g^{-1}Hg$ consists of elements shorter than $2K+8\delta+2$ and, hence, is finite.*

Proposition II.2.3. *([**Arzh**], Prop.1) Let G be a word hyperbolic group and H a quasiconvex subgroup of G of infinite index. Then the number of double cosets of G modulo H is infinite.*

Another important property that we are going to use is:

Lemma II.2.4. *([**Min**] Lemma 3.8) Assume H, K are subgroups of a δ-hyperbolic group G, H is quasiconvex, K is non-elementary and $|K : (K \cap H^g)| = \infty$ for every $g \in G$. Then there exists an element $y \in K$ such that $E(y) = E(K) \times \langle y \rangle$ and $\langle y \rangle^G \cap H = e$, where $\langle y \rangle^G$ is the set of conjugates of all elements of $\langle y \rangle$ and $E(K)$ is a unique maximal finite subgroup normalized by K.*

We quote the following:

Theorem II.2.5. *([**Mack**], Theorem 6.4) Let G be a hyperbolic group and H be a quasiconvex subgroup of infinite index in G. Then there exist $C > 0$ and a set-theoretic section $s : G/H \to G$ such that:*

(i) the section s maps each coset gH to an element $g' \in gH$ with $|g'|$ minimal among all representatives in gH;

(ii) the group G is within C-neighborhood of $s(G/H)$.

The following Lemma summarize some properties of elementary subgroups of hyperbolic groups ([**Gro**]; [**Ghys**] p.150, p.154; [**CDP**] Pr. 4.2, Ch.10; [**Olsh93**] Lemma 2.2).

Lemma II.2.6. *Let G be a hyperbolic group.*

(i) For any word W of infinite order in the hyperbolic group G there exist constants $0 < \lambda \leq 1$ and $c \geq 0$ such that any path with label W^m in $Cay(G)$ is (λ, c)-quasigeodesic for any m.

(ii) Let E be an infinite elementary subgroup in G. Then there exists a constant $K = K(E) \geq 0$ such that the subgroup E is K-quasiconvex.

(iii) If W is a geodesic word and p is a path with label W^n then there exists K (independent of n) such that the path p and the geodesic $[p_-, p_+]$ are within K-neighborhoods of each other.

(iv) Let g, h be elements of infinite order such that $E(g) \neq E(h)$. Then the Gromov products (g^m, h^n), (g^u, g^v), (h^u, h^v) are bounded by some constant C depending on g, h only provided $uv < 0$.

Following [**Olsh93**], we call a pair of elements x, y of infinite order in G non-commensurable if x^k is not conjugate to y^s for any non-zero integers k, s.

Lemma II.2.7. *([**Olsh93**], Lemmas 3.4, 3.8) There exist infinitely many pairwise non-commensurable elements $g_1, g_2, \ldots$ in a non-elementary hyperbolic group G such that $E(g_i) = \langle g_i \rangle \times E(G)$ for every i.*

We will say (adopting the terminology from [**Min2006**]) that the element $g \in G$ is G-suitable if $E(g) = E(G) \times < g >$.

Let W be a word, and let us fix some factorization $W \equiv W_1^{i_1} W_2^{i_2} \ldots W_k^{i_k}$ for some words $W_1, \ldots, W_k$. Consider a path q with label W in $Cay(G)$.

Consider all vertices o_i which are the terminal vertices of initial subpaths p_i of q such that $lab(p_i) = W_1^{i_1} \ldots W_{m-1}^{i_{m-1}} W_m^s$, where $m \leq k$ and $s = 0, \ldots, i_m$. Following [**Olsh93**], we call vertices $\{o_i\}$ *phase vertices* of q relative to factorization $W_1^{i_1} W_2^{i_2} \ldots W_k^{i_k}$ of the $lab(q)$. We enumerate distinct phase vertices along the path q starting from $o_0 = q_-$; the total number of such vertices is $(|i_1| + \ldots + |i_k| + 1)$.

Assume we have a pair of paths $q, \bar{q}$ in $Cay(G)$ with phase vertices o_i and $\bar{o}_j$ where $i = 1, \ldots, l$, $j = 1, \ldots, m$ for some positive integers l, m. We call a shortest path between a phase vertex o_i and some phase vertex $\bar{o}_j$ of $\bar{q}$ a *phase path* with initial vertex o_i. We may also talk about phase vertices of subpaths p of q meaning these vertices o_i which belong to p.

Definition II.2.8. [**Olsh93**] Let the words $W_1, \ldots, W_l$ represent some elements of infinite order in G. Fix some $A \geq 0$ and an integer m to define a set $S_m = S(W_1, \ldots, W_l, A, m)$ of words

$$W = X_0 W_1^{m_1} X_1 W_2^{m_2} \ldots W_l^{m_l} X_l \text{ where } |m_2|, \ldots, |m_{l-1}| \geq m,$$

such that $\|X_i\| \leq A$ for $i = 0, \ldots, l$ and $X_i^{-1} W_i X_i \notin E(W_{i+1})$ in G for $i = 1, \ldots, l-1$. If $l = 1$ we assume that $|m_1| \geq m$.

Lemma II.2.9. *([**Olsh93**], Lemma 2.4) There exist $\lambda > 0$, $c \geq 0$ and $m > 0$ (depending on $K, W_1, W_2, \ldots, W_l$) such that any word $W \in S_m$ is (λ, c)-quasigeodesic. If $W_i \equiv W_j$ for all i, j then the constant λ does not depend on A, l.*

Consider a closed path $p_1 q_1 p_2 q_2$ in $Cay(G)$. Let $q_1 = x_1 t_1 x_2 t_2 \ldots x_l t_l$ where $lab(x_i) = X_i$ and $lab(t_i) = W_i^{m_i}$ for some $W = X_0 W_1^{m_1} X_1 W_2^{m_2} \ldots W_l^{m_l} X_l \in S_m$. Similarly, we let $q_2^{-1} = \bar{x}_1 \bar{t}_1 \ldots \bar{x}_l \bar{t}_l$ where $lab(\bar{x}_i) = \overline{X}_i$ and $lab(\bar{t}_i) = \overline{W}_i^{\overline{m}_i}$ for some $\overline{W} = \overline{X}_0 \overline{W}_1^{\overline{m}_1} \overline{X}_1 \overline{W}_2^{\overline{m}_2} \ldots \overline{W}_l^{\overline{m}_l} \overline{X}_l \in \overline{S}_m$. Define phase vertices o_i and $\overline{o}_j$ on q_1 and q_2

relative to factorizations $X_0W_1^{m_1}X_1W_2^{m_2}...W_l^{m_l}X_l$ and $\overline{X}_0\overline{W}_1^{\overline{m}_1}\overline{X}_1\overline{W}_2^{\overline{m}_2}...\overline{W}_l^{\overline{m}_l}\overline{X}_l$. As in [**Olsh93**], We say that paths t_i and $\bar{t}_j$ are compatible if there exists a phase path v_i with $lab(v_i) = V_i$ between a phase vertex of t_i and $\bar{t}_j$ such that there exist natural numbers a, b satisfying $(V_i\overline{W}_jV_i^{-1})^a = W_i^b$.

Lemma II.2.10. *([**Olsh93**], Lemma 2.5) Provided the conditions for q_1 and q_2 hold, and $|p_1|, |p_2| < C$ for some C, there exists an integer m and an integer k, where $|k| \leq 1$ such that t_i and $\bar{t}_{i+k}$ are compatible for any $i = 2, ..., l-1$ provided that $|m_2|, ..., |m_{l-1}|, |\overline{m}_2|, ..., |\overline{m}_{l-1}| \geq m$ and for $i = 1$ (resp. $i = l$) if $|m_1| \geq m$, (resp. $|m_l| \geq m$). Moreover t_i is not compatible with $\bar{t}_j$ if $j \neq i+k$.*

CHAPTER III

ON THE GENERATORS OF THE KERNELS OF HYPERBOLIC GROUP PRESENTATIONS

III.1 T. Delzant's small cancellation

In this section we recall some definitions and Lemmas from [**Delz**], but with certain modifications. We would like to formulate all the statements in the language of (geodesic) and cyclically reduced words rather then group elements and cyclically reduced group elements (element g of the group G is called *a cyclically reduced element* if g has a minimal length in it's conjugacy class in G). The proofs of these Lemmas can be repeated while changing the terminology.

We first recall the following Lemmas:

Lemma III.1.1. *([**Delz**], Lemma 1.2.1) Let V,W be geodesic words in G; their scalar product is an integer or $\frac{1}{2}$ times integer. If $V \equiv AB$ such that $|A| = [\langle V,W\rangle_1]$ and C is defined by equality $AC = W$ in G then the path AC is geodesic up to constant 2δ (we denote by $[x]$ a maximal integer smaller or equal to x).*

Lemma III.1.2. *([**Delz**] Lemma 1.5.1) Let V be a geodesic word in G which is shortest in it's conjugacy class and of length no less than 20δ. Assume that W is conjugate to V. Then there exists a geodesic word U and a cyclic conjugate V' of V such that $W = UV'U^{-1}$ and the path $UV'U^{-1}$ is geodesic up to 10δ.*

Let us mention the following property of metric trees with finite number of vertices. If a metric tree T is a union of n segments $\cup_{i=1}^{n}[l_0,l_i]$ originating from a fixed vertex w_0, it is easy to see that an addition of a new segment $[l_0,l_{n+1}]$ to T can increase the number of edges by at most 2. To be more precise we can prove by induction on n that $|E(T)| \leq 2n-1$, where $E(T)$ is a set of edges in T.

The Proposition below provides a "pull-back" of the tree approximation T for the set F in the situation of Theorem II.1.4 in the original hyperbolic space X. It will be formulated for hyperbolic groups. In order to formulate this Proposition we need to add some edges of zero length to $E(T)$. The reason for this adjustment is that a trivial edge in T may correspond to a nontrivial group word ("edge in the pullback tree") in the Proposition III.1.3. For every $k \leq n$ we consider a subtree $T_k = \Phi(\cup_{s=1}^{k}[w_0,w_s])$. For every $i \leq n$, if $\Phi(w_i) \in T_{i-1}$, then we add to the set of edges $E(T)$ a new edge of zero length $[\Phi(w_i),\Phi(w_i)]$. The inequality $|E(T)| \leq 2n-1$ still holds if we take into account edges of zero length. We choose an (arbitrary) orientation on every edge $\alpha \in E(T)$. When we consider a segment $[\Phi(w_i),\Phi(w_j)] = \alpha_{s_1}^{\varepsilon_1}\dots\alpha_{s_m}^{\varepsilon_m}$ ($\alpha_{s_i} \in E(T)$) in Proposition III.1.3 such that a zero length edge was defined for i (for j), we assume that α_{s_1} is the edge $[\Phi(w_i),\Phi(w_i)]$ (respectively, α_{s_m} is the edge $[\Phi(w_j),\Phi(w_j)]$). After described conventions, we may formulate the following:

Proposition III.1.3. *([**Delz**] Lemma 1.3.2) Let $g_0,g_1,\dots,g_n$ be elements in G, $n \leq 2^k$ and let Φ, T be the corresponding approximation tree and function provided by Theorem II.1.4. Denote by $E(T) = \{\alpha_1,\dots,\alpha_{2n-1}\}$ the set of edges of T. Let W be a geodesic word such that $W = g_0^{-1}g_1$ in G. Then there exist geodesic words $A_1,\dots,A_{2n-1}$ in G satisfying the following properties:*

(i) $||\alpha_i| - |A_i|| \leq 2\delta(k+1)+2$.

(ii) If the geodesic $[\Phi(g_i),\Phi(g_j)]$ is a path $\alpha_{s_1}^{\varepsilon_1}\dots\alpha_{s_m}^{\varepsilon_m}$ in the tree T, then $g_i^{-1}g_j = A_{s_1}^{\varepsilon_1}\dots A_{s_m}^{\varepsilon_m}$ in G , $\varepsilon_i = \pm 1$ and $A_{s_1}^{\varepsilon_1}\dots A_{s_m}^{\varepsilon_m}$ is geodesic up to $n(2\delta(k+1)+2)$.

(iii) The word $A_{s_1}^{\varepsilon_1}\dots A_{s_m}^{\varepsilon_m}$ defined in (ii) for $g_0^{-1}g_1$ is geodesic and $W \equiv A_{s_1}^{\varepsilon_1}\dots A_{s_m}^{\varepsilon_m}$.

Small Cancellation Properties on the Cayley Graph of Hyperbolic Groups

The following definitions can be found in [**LSch**]. We call the set of words $\mathscr{R}$ *symmetrized* if it is a set of freely cyclically reduced words in alphabet $S^{\pm 1}$, i.e.

(i) $R \in \mathscr{R} \implies R^{-1} \in \mathscr{R}$,

(ii)$R \in \mathscr{R}, R \equiv R_1R_2 \implies R_2R_1 \in \mathscr{R}$.

We will sometimes talk about cyclic word R meaning R or one of it's cyclic conjugates. Denote by G_1 the factor group $G/\mathscr{N}(\mathscr{R})$ of G by the normal closure (in G) of the set $\mathscr{R}$. For a pair of words X,Y in the alphabet $S^{\pm 1}$ let us denote by $X \equiv Y$ a letter-by-letter equality of X and Y.

Definition III.1.4. Let $\mathscr{R}$ be a symmetrized set of geodesic words in the δ-hyperbolic group G and $\mu < 1/8$. Assume furthermore that every $R \in \mathscr{R}$ is a cyclically reduced element of G. The family $\mathscr{R}$ satisfies a small cancellation condition $C'(\mu)$ if:

(i) For every words A,B in G , $|A|,|B| \leq 100\delta$, $\forall R_1,R_2 \in \mathscr{R}$, if $\langle AR_1B,R_2\rangle > \mu min(|R_1|,|R_2|)$, then $R_2 = AR_1A^{-1}$ in G;

(ii) $min_{R\in\mathscr{R}}(|R|) \geq 5000\delta/(1-8\mu)$.

The previous definition is essentially the same as that in [**Delz**], 2.1 up to some adjustment of constants (the difference between them is that $b = 1$ in [**Delz**]).

Definition III.1.5. [**Delz**] We say that a geodesic word U of G contains more then half of a relation if there exists $R \equiv r_1r_2$ from $\mathscr{R}$ such that

(i) $R \equiv r_1r_2$ is geodesic, $|r_1| \geq |r_2| + 60\delta$ and

(ii) U equals to the word $U_1r_1U_2$ in G, which is geodesic up to 50δ.

We denote the set of all geodesic words U which do not contain more then half of a relation by $\mathscr{U}$.

Lemma III.1.6. *([**Delz**], Lemma 2.2) Consider the set $\mathscr{X}$ of words URU^{-1} geodesic up to 10δ in G such that U does not contain more then half of a relation from $\mathscr{R}$. Then every element g in the normal closure $\mathscr{N}(\mathscr{R})$ is a product of words from $\mathscr{X}$.*

The proof of the Lemma III.1.6 follows immediately from the remark below.

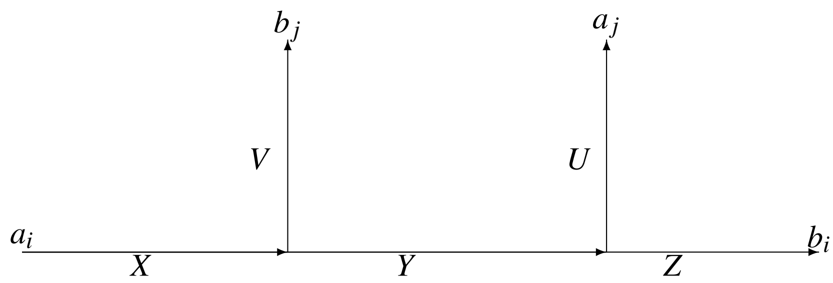
Remark III.1.7. (i) Suppose that a geodesic word U contains more then half of a relation (i.e. $U = U_1r_1U_2$ for some geodesic words U_1,U_2,r_1 satisfying Definition III.1.5). Then

$URU^{-1} = (U_1r_1r_2U_1^{-1})[(U_1r_2^{-1}U_2)R(U_1r_2^{-1}U_2)^{-1}](U_1r_1r_2U_1^{-1})^{-1}$ in G and, evidently,

$\left|U_1r_2^{-1}U_2^{-1}\right|, |U_1| < |U|$.

(ii) Suppose that $R \in \mathscr{R}$, and URU^{-1} is not geodesic up to 10δ. Then by Lemma III.1.2 there exists $R' \in \mathscr{R}$ (so $|R| = |R'|$) and a geodesic word V such that $URU^{-1} = VR'V^{-1}$ in G and $VR'V^{-1}$ is geodesic up to 10δ.

Figure III.1: Cancellation Tree



We introduce some notation and conventions. Let g be an element in the normal closure of $\mathscr{R}$, choose n minimal such that

$$g = U_1R_1U_1^{-1}\dots U_nR_nU_n^{-1} \text{ with } U_iR_iU_i^{-1} \in \mathscr{X}.$$

Then we denote: $g_0 = 1,\ g_1 = U_1R_1U_1^{-1}, \dots, g_n = g$. Also we set $a_i = g_{i-1}U_i$ and $b_i = a_iR_i = g_iU_i$.

Assume that for some indices $i < j$ the approximation tree T for vertices a_i, b_i, a_j, b_j is of shape on the Figure III.1 (T is provided by Gromov's Theorem II.1.4 where $w = a_i, k = 2,\ n = 3$). For convenience we label vertices of the tree on Figure III.1 by corresponding group elements. Proposition III.1.3 provides us with with five geodesic words X, Y, Z, U, V such that $R_i = XYZ$, where XYZ is geodesic and $R_j = U^{-1}Y^{-1}V$, where $U^{-1}Y^{-1}V$ is geodesic up to $3(2\cdot 3\delta + 2) = 18\delta + 6$. We label edges of the tree T with X, Y, Z, U, V for convenience of the reader. Note that Φ and T determine the exponents of X, Y, Z, U, V in equalities for R_i, R_j uniquely.

The following Lemma is an application of the small cancellation, we provide a proof of it (following [**Delz**]) for future references.

Lemma III.1.8. *([**Delz**], Lemma 2.3) Suppose that a fixed element g is equal to a word $W = U_1R_1U_1^{-1}\dots U_nR_nU_n^{-1}$ in G and that for some indices $i < j$ the tree approximation of vertices a_i, b_i, a_j, b_j in $Cay(G)$ (with geodesic words X, Y, Z, U, V provided by Proposition III.1.3) has the shape on Figure III.1.*

(i) Assume that n is a minimal possible number among all words W equal to g. Then the following inequality holds:

$$|Y| \leq \mu min(|R_i|; |R_j|) + 10\delta + 3. \tag{III.1}$$

(ii) If the equality (III.1) is violated then n is not minimal and the following equality holds in G:

$$U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_{j-1}R_{j-1}U_{j-1}^{-1} = U_iR_iU_i^{-1}\dots U_jR_jU_j^{-1}. \tag{III.2}$$

Proof Assume that the inequality (III.1) does not hold. In notations used in Figure III.1 we have $R_i = XYZ$ and XYZ is geodesic, $R_j = U^{-1}Y^{-1}V$, where the right-hand side is geodesic up to $3(2\cdot 3\delta + 2) = 18\delta + 6$. We consider the conjugate $R_i' = YZX$ of R_i, which is also geodesic: $|R_i'| \geq |R_i|$ (since R_i is a cyclically reduced geodesic word), but on the other hand $|R_i'| \leq |Y| + |Z| + |X| = |R_i|$. Consider also the conjugate $R_j' = YUV^{-1}$ of R_j^{-1} which is geodesic up to $3(2\cdot 3\delta + 2) = 18\delta + 6$ (we have $|R_j| \leq |R_j'| \leq |Y| + |U| + |V| \leq |R_j| + 18\delta + 6$).

By Lemma III.1.2, there exists a geodesic word $R'' = AR'_jA^{-1}$ cyclically conjugate to R_j such that $2|A| + |R''| \leq \left|R'_j\right| + 10\delta$ and $|R''| = \left|R_j\right|$. Now the computation

$$2|A| + \left|R''\right| \leq \left|R'_j\right| + 10\delta \leq \left|R_j\right| + 28\delta + 6$$

implies that $|A| \leq 14\delta + 3$. We also have that $R'' \in \mathscr{R}$: it is a cyclic conjugate of R_j.

By definition of hyperbolicity, we have that

$$(R'_i, R'_j) \geq min((Y, R'_j), (R'_i, Y)) - \delta.$$

Both Gromov products on the right side of the last equation are not greater than $|Y|$ and the second is actually equal to $|Y|$ because $R'_i = YZX$ is geodesic. So $(R'_i, R'_j) \geq (Y, R'_j) - \delta = |Y| - \delta - (1, R'_j)_Y$, where the last equality follows from $(Y, R'_j)_1 + (1, R'_j)_Y = |Y|$. Since $R'_j = YUV^{-1}$ is geodesic up to $18\delta + 6$ we have by inequality (II.2) that $\langle 1, R'_j \rangle_Y \leq 9\delta + 3$ and finally

$$(R'_i, R'_j) \geq |Y| - 10\delta - 3.$$

We hence obtained that $(AR''A^{-1}, R'_i) \geq \mu min(|R_i|; \left|R_j\right|)$ and by the condition $C'(\mu)$ we get that $A^{-1}R''A = YUV^{-1} = R'_i = YZX$. Thus $UV^{-1} = ZX$, hence $Z^{-1}U = XV$ and so ${b_i}^{-1}a_j = a_i^{-1}b_j$, which in turn is equivalent to $U_i^{-1}g_i^{-1}g_{j-1}U_j = U_i^{-1}g_{j-1}^{-1}g_jU_j$ and hence $g_i^{-1}g_{j-1} = g_{i-1}^{-1}g_j$. Rewriting the last equality in the explicit form, we get precisely equation (III.2).

The left-hand side of the last equality contains fewer elements of $\mathscr{X}$ contrary to the minimality of number n for g. Contradiction. □

The following definition utilizes the Lemma

Definition III.1.9. [**Delz**] A word (or, equivalently, a path in $Cay(G)$) $U_1R_1U_1^{-1} \dots U_nR_nU_n^{-1}$ is called *reduced* if for every pair of indices $i < j$ such that the approximating tree for a_i, b_i, a_j, b_j is of shape on Figure III.1, the inequality (III.1) holds. If for a pair of indexes $i < j$ the tree approximation is of shape on Figure III.1, the inequality (III.1) is violated, then we call $i < j$ a *reducible pair of indexes*.

Note that if we switch the labels a_j and b_j on Figure III.1, the pair $i < j$ will no longer be a reducible pair. The following corollary summarizes [**Delz**] Lemma 2.4.

Lemma III.1.10. *Suppose G is hyperbolic and $\mathscr{R}$ satisfies $C'(\mu)$, $\mu \leq 1/8$. Let $\gamma = \prod_{i=1}^{n} U_iR_iU_i^{-1}$ be a reduced path in $Cay(G)$, $U_iR_iU_i^{-1} \in \mathscr{X}$ and denote by $\bar{\gamma}$ some geodesic between γ_-, γ_+. Then there exist an index $1 \leq i_0 \leq n$, a subsegment x of geodesic segment ${}_\gamma R_{i_0}$ such that x is in 30δ-neighborhood of $\bar{\gamma}$ and $|x| \geq (1 - 3\mu)|R_{i_0}| - 1500\delta$.* □

III.2 Diagrams and small cancellation

Suppose we are given a hyperbolic group G with a combinatorial presentation $G = gp(S|\mathscr{D})$. For technical purposes we assume that $\mathscr{D}$ contains all relations of the group G.

For $\varepsilon \geq 0$ a subword U is called an *ε-piece* of a word R in a symmetrized set $\mathscr{R}$ with respect to G if there exists a word $R' \in \mathscr{R}$ such that

(i) $R \equiv UV,\ R' \equiv U'V'$ for some U', V', V;

(ii) $U' = YUZ$ in G for some words Y, Z where $\|Y\|, \|Z\| \leq \varepsilon$;

(iii) $YRY^{-1} \neq R'$ in the group G.

We say that the system $\mathscr{R}$ satisfies the $C(\varepsilon, \mu, \rho)$-condition (with respect to G) for some $\varepsilon \geq 0,\ \mu \geq 0,\ \rho \geq 0$ if

(i)$\|R\| \geq \rho$ for any $R \in \mathscr{R}$;

(ii) any word $R \in \mathscr{R}$ is geodesic;

(iii) for any ε-piece of any word $R \in \mathscr{R}$ the inequalities $\|U\|, \|U'\| < \mu \|R\|$ hold (using notations of the definition of the ε-piece).

Definition III.2.1. Consider a finite, two dimensional complex Δ with directed edges such that:

(i) The underlying topological space of complex M is a disc with a boundary P.

(ii) For any path in Δ there defined a label function $\phi(*)$. If x is an edge in Δ, $\phi(x) \in S \cup S^{-1} \cup 1$ and $\phi(x^{-1}) = \phi(x)^{-1}$. For a path q in Δ, $q = q_1 ... q_n$, where q_i is an edge for every i, we define $\phi(q) = \phi(q_1)...\phi(q_n)$. If q is a simple closed path we choose a base vertex o and read off the labels of edges in the clockwise direction.

(iii) A boundary label of any 2-cell of M is either an element of $\mathscr{R}$ (then we call it an *$\mathscr{R}$-face*) or has a label D where $D = 1$ in the hyperbolic group G (*$\mathscr{D}$-face*).

We call the triple $(M, \phi(*), P)$ *a* (disc) diagram Δ with respect to $gp(S|\mathscr{D} \cup \mathscr{R})$ with a boundary path P.

Similarly we may define notions of *annular or spherical diagrams*.

For convenience we often fix a base point o of the diagram Δ – a vertex on one of the boundary components of Δ. We may also choose a base point o_1 on the boundary of a face Π and write $\partial_{o_1}\Pi = r$ where r is a simple closed boundary path of Π with a initial (terminal) vertex o_1.

Consider a path γ in Δ as a path in the underlying topological space M. We say that γ is a *simple path* in Δ if for every open set U in M containing γ there exists a homotopy (in U) from γ to a simple curve $\gamma' = \gamma'(U)$. A simple closed path γ in Δ bounds a subdiagram Δ_1 with boundary $\partial\Delta_1 = \gamma$ consisting of all edges, vertices and faces which are inside the simple closed curve $\gamma' = \gamma'(U)$ for every open set U containing γ. Subdiagrams Δ_1, Δ_2 are called disjoint if for every neighborhood of $\partial\Delta_1 \cup \partial\Delta_2$ (in the underlying space for Δ) there exists a homotopy inside U of $\partial\Delta_1$ to a simple γ_1 such that $\Delta_2 \cap \gamma_1 = \emptyset$.

The following operations (and their inverses) are referred to as *elementary transformations* of diagram Δ over G_1:

1. Let Π_1, Π_2 be $\mathscr{D}$-faces in Δ with a common boundary subpath p. Then we can erase p making Π_1, Π_2 into a single $\mathscr{D}$-face.

2. Let p be a simple path in Δ. Then we cut the diagram Δ along p (i.e. consider the path pp^{-1} as a new boundary component) and glue in a $\mathscr{D}$-face labeled by $\phi(p)\phi(p)^{-1}$.

It is clear that elementary transformations define an equivalence relation on the set of all reduced diagrams over G_1. We say that Δ is equivalent to Δ' if there exists a finite sequence of elementary transformations starting from Δ and ending with Δ'.

Definition III.2.2. ([**Olsh93**]) Let Π_1, Π_2 be different $\mathscr{R}$-faces of a diagram Δ having boundary labels R_1, R_2 reading in a clockwise direction, starting from vertices o_1, o_2 respectively. Suppose also that there exists a simple path t in Δ such that $t_- = o_1, t_+ = o_2$. Call Π_1, Π_2 *opposite* (with respect to the path t) if the following equality holds:

$$\phi(t)^{-1} R_1 \phi(t) R_2 = 1 \text{ in } G. \tag{III.3}$$

If a diagram Δ contains no opposite faces then we call it *reduced*.

Lemma III.2.3. *(van Kampen, see [**Olsh93**]) Let w_0 be an nonempty word in the alphabet S. Then $w_0 = 1$ in G_1 if and only if there exists a reduced disc diagram over $gp(S|\mathscr{D} \cup \mathscr{R})$ with boundary label equal to w_0.*

Let p be a path in Δ over G, define $\|p\| = \|\phi(p)\|$ and $|p| = |\phi(p)|$. We call a path p geodesic if $\|p\| = |p|$ (recall that $|p|$ equals the distance $|p_+ - p_-|$ in G).

One can define a map ϕ' (see [**Olsh93**], §5) from a disc diagram Δ over G with the base point o to Caley graph $Cay(G)$. Set $\phi'(o) = 1$, where 1 is the identity vertex of $Cay(G)$. For an arbitrary vertex a in Δ we define $\phi'(a)$ to be the vertex of $Cay(G)$ labeled by the geodesic word $\phi(p)$ where p is a path in Δ connecting o and a (it follows from the van Kampen Lemma that $\phi'(a)$ does not depend on the choice of p). If p is an edge in Δ labeled by $s \in S^{\pm 1}$, then define $\phi'(p)$ to be the edge labeled by s in Cayley graph $Cay(G)$ with vertices $\phi'(p_-), \phi'(p_+)$. If $\phi(p) \equiv 1$ for an edge p of Δ then $\phi'(p) = \phi'(p_-) = \phi'(p_+)$. One can verify that $|p| = |\phi'(p)|$, $\|p\| = \|\phi'(p)\|$ for any path p in diagram Δ over G ([**Olsh93**], Lemma 5.1).

When Δ is a diagram over G_1 we still use functions $\|p\|, |p|$, where p is a path in Δ.

In the following remark we translate some hyperbolic properties of $Cay(G)$ into the context of diagrams over G.

Remark III.2.4. (i) Suppose Δ is a reduced diagram over G, p_1 and p_2 are disjoint paths in Δ, vertices $(p_i)_\pm$ are on the boundary $\partial\Delta$. Then there exists a diagram Δ' equivalent to Δ, such that $\partial\Delta' = \partial\Delta$, vertices $(p_i)_\pm$ are connected by a *geodesic* path p'_i for $i = 1, 2$, and paths p'_1, p'_2 are disjoint. Furthermore, a point x of the path p'_i is on $\partial\Delta'$ if and only if it is an initial or terminal vertex of p'_i.

(ii) Suppose Γ is a diagram over G, $\partial\Gamma = p_1 q_1 p_2 q_2$, where q_i are geodesic in G and $\|p_i\| \le K$, $|q_i| \ge 2K + 20\delta$ for $i = 1, 2$ and some $K \ge 0$. Then (after elementary transformations) there exists a subdiagram Γ' in Γ with boundary $\partial\Gamma' = p'_1 q'_1 p'_2 q'_2$ such that $\|p'_i\| \le 6\delta$, q'_i are geodesic subpaths of q_i and $|(q_1)_+ - (q'_1)_+| = |(q_1)_- - (q'_1)_-| = K + 2\delta$. In particular,

$$|q'_1| = |q_1| - 2K - 4\delta.$$

(iii) If a subdiagram Γ satisfies the conditions of part (ii), then every vertex x of q_1 is at distance not greater than $K + 8\delta$ from q_2 (i.e. there exists a vertex y on q_2 such that $|x - y| \le K + 8\delta$).

Proof (i) Consider the map ϕ' from diagram Δ to $Cay(G)$. For $i = 1, 2$ we pick a geodesic in $Cay(G)$ with label P'_i between vertices $\phi'(p_{i\pm})$ in $Cay(G)$. We apply an elementary transformation of type (ii) to p_i: cut Δ along p_i to get a new boundary component $p_i \tilde{p}_i$, $\phi(\tilde{p}_i) = \phi(p_i)^{-1}$ in G and glue inside a $\mathscr{D}$-face Π_i with boundary $p_i \tilde{p}_i$. Then apply the inverse type (ii) to Π_i: replace it with a pair of faces Π_{i1}, Π_{i2} with common subpath p'_i labeled by P'_i ($\partial\Pi_{i1} = p_i {p'}_i^{-1}$, $\partial\Pi_{i2} = p'_i \tilde{p}_i$). We have constructed the desired diagram

Δ'. It remains to notice that no vertex belongs to both closed paths $p_1\tilde{p}_1$ and $p_2\tilde{p}_2$ since $p_i, \tilde{p}_i$ are copies of disjoint paths p_i in Δ. Also, all vertices of p'_i except for $p'_{i\pm}$ are interior in a subdiagram bounded by $p_i\tilde{p}_i$, and the remark is proved completely.

(ii) We consider $\phi'(\Gamma)$, and apply Lemma II.1.1 to the pair of geodesic paths $\phi'(q_1)$, $\phi'(q_2)$ in $Cay(G)$ to find the subpath q''_1 of $\phi'(q_1)$ such that $|(q''_1)_\pm - \phi'((q'_1)_\pm)| = K + 2\delta$ and vertices $(q'_1)_\pm$ are in 6δ-neighborhood of geodesic $\phi'(q_2)$. Define a subpath q''_2 of $\phi'(q'_1)$ so that the inequality $|(q''_1)_\pm - (q''_2)_\pm| \leq 6\delta$ holds. It remains to choose a subpath q'_i on q_i satisfying equality $\phi'(q'_i) = q''_i$. Now apply part (i) to two pairs of points $(q'_2)_+, (q'_1)_-$ and $(q'_1)_+, (q'_2)_-$ in Γ which provides paths p'_i and observe that the path $p'_1q'_1p'_2q'_2$ bounds the desired diagram Γ'.

(iii) Follows from remark II.1.3 and properties of the mapping ϕ'. □

We will need the following:

Lemma III.2.5. *Suppose we have a diagram Δ consisting of cells Π_1, Π_2, a simple path t between them such that Π_1, Π_2 is pair of opposite cells with respect to a path t. Then, for any vertices o_1, o_2 on $\partial\Pi_1, \partial\Pi_2$ respectively, there exists a path s_1ts_2 such that $\phi(s_1ts_2) = P\phi(a)$ in G, where $|a| \leq \frac{1}{2}|\partial\Pi_2|$, P is a geodesic word and $|P| \leq |t| + 8\delta$, s_i is a subpath of $\partial\Pi_i$ ($i = 1, 2$), a is a subpath of $\partial\Pi_2$ and $s_{1-} = o_1, s_{2+} = o_2$. Moreover, the following equality holds in G:*

$$(P\phi(a))^{-1}\phi(\partial_{o_1}\Pi_1)(P\phi(a))\phi(\partial_{o_2}\Pi_2) = 1 \text{ in } G. \tag{III.4}$$

Proof. We denote r_1 to be the boundary path $\partial_{t-}\Pi_1$, r_2 to be the boundary path $\partial_{t+}\Pi_2$. By definition of an opposite pair (bounded by $r_1tr_2t^{-1}$) and the van-Kampen Lemma, there exists a diagram Γ over G with boundary $r_1tr_2t_1^{-1}$, where $\phi(t_1) = \phi(t)$. Since each path r_i is geodesic, by Remark III.2.4 (iii) the distance between a vertex on r_1 and r_2 is not greater than $|t| + 8\delta$, hence there exists a vertex o'_1 on r_2 such that $|o_1 - o'_1| \leq |t| + 8\delta$.

Consider a subpath of the form $s_1t's'_2$ on $\partial\Gamma$, where s_1 is a subpath of $r_1^{\pm 1}$, s'_2 is a subpath of $r_2^{\pm 1}$, $(s_1)_- = o_1$, $(s'_2)_+ = o'_1$, t' is either t or t_1.

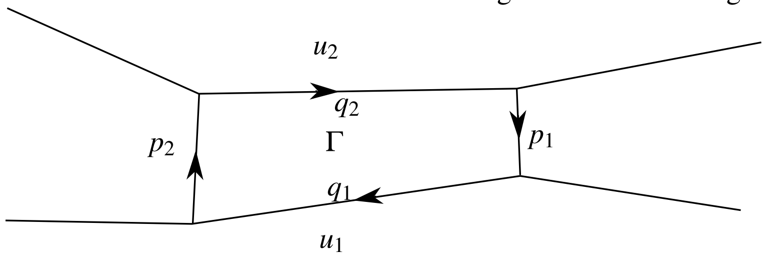
Let P be a geodesic word equal in G to the label of the path $s_1t's'_2$, so $|P| \leq |t| + 8\delta$. Now we consider $s_1t's'_2$ as a subpath of boundary $\partial\Delta$, so t' is t. We choose a path a on $\partial\Pi_2$ between o'_1 and o_2 satisfying inequality $|a| \leq \frac{1}{2}|\partial\Pi_2|$. Define the path s_2 to be s'_2a after elimination of returns, hence $\phi(s'_2a) = \phi(s_2)$ in a free group generated by S. Since the boundary labels of Δ and Γ are the same, we may consider the path $s_1t's'_2$ as a path $s_1ts'_2$ in Δ. We have that $\phi(s_1ts'_2) = P$ in G, and so the following first two equalities hold in the free group generated by S while the last one holds in G:

$$\phi(s_1ts_2) = \phi(s_1t's'_2s) = \phi(s_1ts'_2)\phi(a) = P\phi(a).$$

To establish (III.4), we observe that the path $(s_1^{-1}\partial_{o_1}\Pi_1s_1)t(s_2\partial_{o_2}\Pi_2s_2^{-1})t^{-1}$ coincide with $(\partial_{t_-}\Pi_1)t(\partial_{t_+}\Pi_2)t^{-1}$ after the elimination of returns. Thus

$$\phi((s_1^{-1}\partial_{o_1}\Pi_1s_1)t(s_2\partial_{o_2}\Pi_2s_2^{-1})t^{-1}) = (\partial_{t_-}\Pi_1)t(\partial_{t_+}\Pi_2)t^{-1} = 1 \text{ in } G,$$

Figure III.2: $\varepsilon-$Contiguity



which after conjugation provides $\phi^{-1}(s_1ts_2)\phi(\partial_{o_1}\Pi_1)\phi(s_1ts_2)\phi(\partial_{o_2}\Pi_2)=1$ in G providing (III.4).□

The following notion of ε-contiguity subdiagram will be used extensively. Let Δ be a diagram over G_1. Let u_1 and u_2 be a pair of paths in Δ with subpaths q_1 and q_2 respectively, such that there exists a pair of simple paths p_1,p_2, $|p_1|,|p_2|\leq\varepsilon$ and suppose that a path $p_1q_1p_2q_2$ bounds a disc diagram Γ which does not contain any $\mathscr{R}$-faces (see Figure IV.1). Then we call Γ an *ε-contiguity subdiagram* between paths u_1 and u_2. When we talk about the contiguity subdiagram Γ between u_1 and u_2 we use the formula $\partial(u_1,\Gamma,u_2)=p_1q_1p_2q_2$ to define notation for arcs of Γ. In this case q_1,q_2 are referred to as *contiguity arcs* and p_1,p_2 as *side arcs* of the ε-contiguity subdiagram Γ. We usually consider contiguity subdiagrams between a pair of $\mathscr{R}$-faces or between an $\mathscr{R}$-face and a boundary path (i.e. u_1 is the boundary path of $\mathscr{R}$-face Π_1 and u_2 is the boundary path of $\mathscr{R}$-face Π_2 or is a subpath of the boundary of Δ). If u_1 is the boundary of an $\mathscr{R}$-face Π_1, u_2 is a path of a boundary of an $\mathscr{R}$-face Π_2 with ε-contiguity diagram Γ described above then we define the degree of contiguity of Π_1 to Π_2 to be $(\Pi_1,\Gamma,\Pi_2)=\frac{\|q_1\|}{\|\Pi_1\|}$ (or, if u_2 is a boundary subpath of Δ, the degree of contiguity of Π_1 to the boundary subpath u_2 to be $(\Pi_1,\Gamma,u_2)=\frac{\|q_1\|}{\|\Pi_1\|}$).

The next two Lemmas provide the basic connection between the notions of small cancellation and diagrams over hyperbolic groups.

Lemma III.2.6. *(i)([**Olsh93**], Lemma 5.2) If the symmetized system $\mathscr{R}$ satisfies the $C(\varepsilon,\mu,\rho)$-condition, then for any reduced diagram Δ and any ε-contiguity subdiagram Γ of a face Π_1 to another face Π_2 the following inequalities hold:*

$$\|q_1\|<\mu\,\|\partial\Pi_1\|,\ \|q_2\|<\mu\,\|\partial\Pi_2\|,$$

where $\partial(\Pi_1,\Gamma,\Pi_2)=p_1q_1p_2q_2$ for any reduced diagram Δ over G_1.

(ii) Suppose a diagram Δ has a pair of $\mathscr{R}$-faces Π_1,Π_2 and an ε-contiguity subdiagram Γ ($\partial\Gamma=p_1q_1p_2q_2$) such that

$$max\{(\Pi_1,\Gamma,\Pi_2),(\Pi_2,\Gamma,\Pi_1)\}\geq\mu.$$

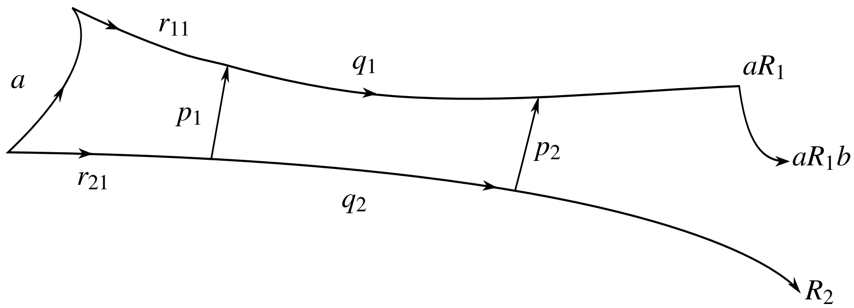
Then Π_1,Π_2 are opposite with respect to each of the paths p_1,p_2.

Note that part 2 of the above Lemma is an immediate corollary of small cancellation property □.

Lemma III.2.7. *([**OlOsSa**], Lemma 4.6) For any hyperbolic group G there exists $\mu_0>0$ such that for any $0<\mu\leq\mu_0$ there are $\varepsilon\geq0$ and ρ (it is suffice to choose $\rho>\frac{10^6\varepsilon}{\mu}$) with the following property:*

Let the symmetized system $\mathscr{R}$ satisfy the $C(\varepsilon,\mu,\rho)$-condition and furthermore let Δ be a reduced disc diagram over G_1 whose boundary $\partial\Delta$ is decomposed into geodesic sections $q^1,\dots,q^r$, where $1\leq r\leq12$.

Figure III.3: $C(\varepsilon,\mu,\rho)\ =>\ C'(2\mu)$



Then, provided Δ has an $\mathscr{R}$-face, there exists a reduced diagram Δ' equivalent to Δ, an $\mathscr{R}$-face Π in Δ and disjoint ε-contiguity subdiagrams $\Gamma_1,\dots,\Gamma_r$ (some of them can be absent) of Π to $q^1,\dots,q^r$ respectively such that

$$(\Pi,\Gamma_1,q_1)+\cdots+(\Pi,\Gamma_r,q_r)>1-23\mu.$$

The following Lemma is a special case of that in [**Olsh93**]:

Lemma III.2.8. *([**Olsh93**], Lemmas 6.7, 7.4) Let G be a non-elementary hyperbolic group. There exists $\mu_0>0$ such that for any $0<\mu\le\mu_0$ there exists $\varepsilon\ge 0$ such that for every $N>0$ there exists $\rho>0$ with the following property:*

if $\mathscr{R}$ is finite and satisfies $C(\varepsilon,\mu,\rho)$ then G_1 is a non-elementary hyperbolic group and $W=1$ in G_1 if and only if $W=1$ in G for every word W with $\|W\|\le N$.

Definition III.2.9. We say that a system $\mathscr{R}$ of geodesic words satisfies the $\tilde{C}(\varepsilon,\mu,\rho)$-condition if $\mathscr{R}$ is symmetrized, satisfies $C(\varepsilon,\mu,\rho)$-condition and consists of words which represent cyclically reduced elements in G.

III.3 Condition $C'(\mu)$ and connection to $C(\varepsilon,\mu,\rho)$-condition

Remark III.3.1. Suppose the system of geodesic words $\mathscr{R}$ satisfies $\tilde{C}(\varepsilon,\mu,\rho)$-condition, $\mu<1/100$, $\varepsilon\ge\varepsilon_0\ge 6\delta$, $\rho>\frac{500\delta}{\mu(1-8\mu)}$. Then $\mathscr{R}$ satisfies $C'(2\mu)$.

Proof Take arbitrary words $R_1,R_2\in\mathscr{R}$. We denote by M the minimum $min(|R_1|,|R_2|)$. To check the condition $C'(2\mu)$ we assume that $\langle aR_1b,R_2\rangle>2\mu M$ for some $a,b\in G$ such that $|a|,|b|\le 100\delta$.

We denote by W a geodesic equal to aR_1b, by ν a path R_2 and by γ a path aR_1b in the Cayley graph $Cay(G)$.

Consider vertices o_2 on ν and o_3 on the geodesic W at distance $[2\mu M]$ from identity vertex 1. By Remark II.1.5 (part 1), we have that $\Phi(o_2)=\Phi(o_3)$ and (by part 2) $|o_2-o_3|\le 4\delta$. Now we may apply Lemma II.1.1 (for $K=100\delta$) to segments ${}_\gamma R_1,W$ and hence there exists a subsegment $[u,v]$ of W such that $|u-e|\le 102\delta,\ |v-\gamma_+|\le 102\delta$ and $[u,v]$ is within 6δ-neighborhood of ${}_\gamma R_1$. Vertex o_3 lies on $[u,v]$ because on one hand $|o_3-e|=[2\mu M]>2K+20\delta$ and on the other hand

$$|o_3-\gamma_+|\ge|R_1|-|a|-|b|-[2\mu M]\ge(1-3\mu)M>2K+20\delta.$$

We get that o_3 is within 6δ-neighborhood of some vertex o_1 on path ${}_\gamma R_1$.

We consider two subsegments $[e,o_2]$ and $[({}_\gamma a)_+,o_1]$ of ν and ${}_\gamma R_1$ respectively and apply Lemma II.1.1 to get that there exists a subsegment q_2 of R_2 between e and o_2 such that

$$|q_2| \geq [2\mu M] - 200\delta - 4\delta > \frac{3}{2}\mu M + 20\delta$$

which is within 6δ-neighborhood from ${}_\gamma R_1$. Now define q_1 to be a subsegment of ${}_\gamma R_1$ with $|q_{1-} - q_{2-}|, |q_{1+} - q_{2+}| \leq 6\delta$.

We have that

$$|q_i| > \frac{3}{2}\mu min(|R_1|,|R_2|) \text{ for i=1,2.} \tag{III.5}$$

Define p_1 (p_2) to be a geodesic path between q_{2-}, q_{1-} (q_{1+}, q_{2+}), see Figure III.3. To justify the Figure III.3, we must show that $\left|({}_\gamma a)_+ - (q_1)_-\right| < \left|({}_\gamma a)_+ - (q_1)_+\right|$ (this inequality follows from [**Olsh93**] Lemma 1.10, but we include the argument here). By triangle inequality and definition of q_1, we have that

$$\left|({}_\gamma a)_+ - (q_1)_-\right| \leq |a| + |p_1| + |e - (q_1)_-| \leq 100\delta + 6\delta + 102\delta = 208\delta;$$

on the other hand,

$$\left|({}_\gamma a)_+ - (q_1)_+\right| \geq |e - (q_2)_+| - |p_2| - |a| = |e - (q_2)_-| + |q_2| - |p_2| - |a| \geq$$

$$102\delta + \frac{3}{2}\mu M + 20\delta - 100\delta - 6\delta > \mu M \geq 500\delta$$

and hence we got $\left|({}_\gamma a)_+ - (q_1)_-\right| < \left|({}_\gamma a)_+ - (q_1)_+\right|$, as desired.

We denote labels of q_i and p_i as Q_i and P_i respectively. Define four subpaths r_{ij}, $i,j \in \{1,2\}$ by equalities ${}_\gamma R_1 = r_{11}r_{12}$, $\nu = r_{21}r_{22}$ and $(r_{11})_+ = (p_1)_+$, $(r_{21})_+ = (p_1)_-$. Define words R_{ij}, Q', Q'' by equalities $lab(r_{ij}) = R_{ij}$, $R_{12}R_{11} \equiv Q_1Q'$, $R_{22}R_{21} \equiv Q_2Q''$. We have that $Q_2 = P_1Q_1P_2^{-1}$, $\|P_i\| \leq 6\delta$, and taking into account the inequality (III.5) we conclude by $\tilde{C}(\varepsilon,\mu,\rho)$-condition that $P_1R_{12}R_{11}P^{-1} = R_{22}R_{21}$, which in turn is equivalent to $(R_{21}P_1R_{11}^{-1})(R_{11}R_{12})(R_{11}P_1^{-1}R_{21}^{-1}) = R_{21}R_{22}$. It remains to observe that $a = (R_{21}P_1R_{11}^{-1})$ and so $aR_1a^{-1} = R_2$.□

Corollary III.3.2. *Suppose $\mathscr{R}$ satisfies $\tilde{C}(\varepsilon,\mu,\rho)$-condition and $n \geq 1$,*

$$\prod_{k=1}^{n} U_kR_kU_k^{-1} = 1 \textit{ in } G, \textit{ where } U_kR_kU_k^{-1} \in \mathscr{X}. \tag{III.6}$$

Then (i) There exists a reducible pair $i < j$ in the sense of Definition III.1.9 and

$$U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_{j-1}R_{j-1}U_{j-1}^{-1} = U_iR_iU_i^{-1}\dots U_jR_jU_j^{-1} \textit{in } G. \tag{III.7}$$

(ii) For every reducible pair $i < j$ in (III.6), there exists a van-Kampen diagram Δ' over G with the boundary γ' labeled by the word
$U_1R_1U_1^{-1}\dots U_nR_nU_n^{-1}$ and a subdiagram Γ in Δ' with boundary $p_1q_1p_2q_2$ such that q_1 is a subpath of ${}_{\gamma'}R_i$,

q_2 is a subpath of ${}_{\gamma}R_j$, $|p_i| \leq 11\delta+3$ and max$(\frac{|q_1|}{|R_i|}, \frac{|q_2|}{|R_j|}) \geq 2\mu - \frac{10\delta+3}{\rho}$. The only vertices of paths p_i that are on the boundary of Δ are initial and terminal vertices $p_{i\pm}$.

(iii) Consider the diagram Δ' from part (ii) and let ν' be any of the four paths given by the formula $\nu' = {}_{\gamma'}(U_i^{\pm 1}) s_1 p_1^{-1} s_2 {}_{\gamma'}(U_j^{\pm 1})$, where s_1 is a subpath of ${}_{\gamma}R_i$, s_2 is a subpath of ${}_{\gamma}R_j$. Then

$$\phi(({}_{\gamma}U_i)^{\pm 1} s_1 p_1^{-1} s_2 ({}_{\gamma}U_j^{\pm 1})) = \prod_{k=i+d}^{j-c} U_k R_k U_k^{-1} \text{ in } G,$$

where c,d take values 0 or 1 depending on the path ν' and $(c,d) \neq (0,0)$. Moreover, depending on values c and d, the word $H \equiv \prod_{k=i+c}^{j-d} U_k R_k U_k^{-1}$ conjugates $U_i R_i U_i^{-1}$ to $U_j R_j^{\pm 1} U_j^{-1}$, namely:

$$H^{-1} U_i R_i U_i^{-1} H = U_j R_j^e U_j^{-1}, \text{ where } e \in \{\pm 1\}.$$

Proof By Remark III.3.1, $\tilde{C}(\varepsilon,\mu,\rho)$-condition implies the condition $C'(2\mu)$. The product $\prod_{k=1}^{n} U_k R_k U_k^{-1}$ equals to identity in G so by Lemma III.1.10 it is not reduced in the sense of Definition III.1.9. Hence there exists a reducible pair $i<j$ (in particular, we have that $|R_i| = |R_j|$) such that the approximation tree for a_i, b_i, a_j, b_j is of shape on Figure III.1 and by Lemma III.1.8 the corresponding geodesic word Y satisfies:

$$|Y| \geq 2\mu M + 10\delta + 3, \text{ where } M = |R_i|. \tag{III.8}$$

Lemma III.1.8 also provides the equation (III.2) and thus (i) is proved.

Diagram Δ' over G with boundary γ' labeled by $\prod_{k=1}^{n} U_k R_k U_k^{-1}$ exists by van-Kampen Lemma. Consider the map $\phi' : \Delta' \longmapsto Cay(G)$. We denote $\phi'(\gamma')$ as γ'' (a path in $Cay(G)$ with label $\prod_{k=1}^{n} U_k R_k U_k^{-1}$). We adopt notations from the definition of a reducible pair $i<j$ and Figure III.1. Consider a geodesic path α in $Cay(G)$ starting from a_i with label XYZ (hence it ends at b_i) and a geodesic up to $18\delta+6$ path β in $Cay(G)$ starting from a_j with label $U^{-1}Y^{-1}V$ (it ends at b_j). By definition of X,Y,Z,U,V, we have $({}_{\alpha}Y)^{-1} =_{\beta} Y^{-1}$. From the fact that XYZ is geodesic, it follows from Remark II.1.3 (ii) that there exists a subpath q_1' of ${}_{\gamma''}R_i$ such that:

$$\left|{}_{\alpha}Y_- - q'_{1-}\right|, \left|{}_{\alpha}Y_+ - q'_{1+}\right| \leq \delta, \tag{III.9}$$

which implies that:

$$\left|q_1'\right| \geq |Y| - 2\delta. \tag{III.10}$$

Similarly, we consider the path β geodesic up to $18\delta+6$ and apply again Remark II.1.3 (ii) to obtain that there exists a subpath q_2' of ${}_{\gamma''}R_j$ such that:

$$\left|{}_{\alpha}Y_- - q'_{2+}\right|, \left|{}_{\alpha}Y_+ - q'_{2-}\right| \leq (9\delta+3)+\delta, \tag{III.11}$$

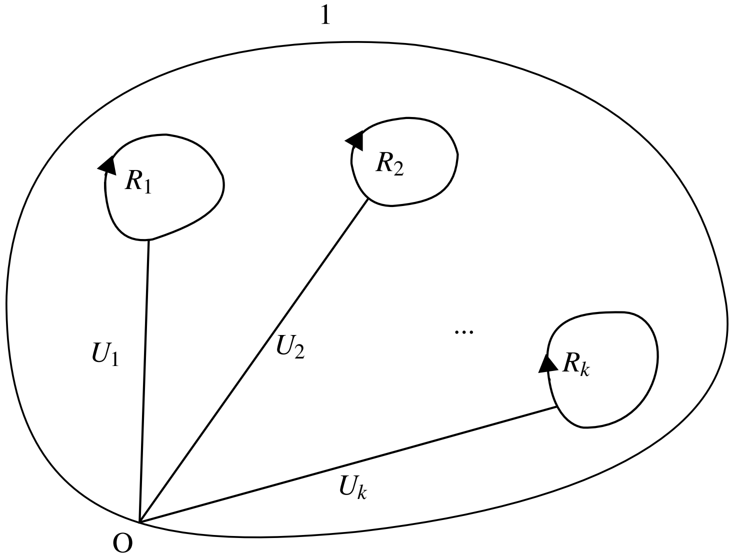
and hence :

$$\left|q_2'\right| \geq |Y| - 20\delta - 6. \tag{III.12}$$

The inequalities (III.9), (III.11) imply also that $\left|q'_{1-} - q'_{2+}\right|, \left|q'_{1+} - q'_{2-}\right| \leq 11\delta+3$.

Consider subpaths q_1 of ${}_{\gamma}R_i$ and q_2 of ${}_{\gamma}R_j$ in the boundary $\partial\Delta'$ such that $\phi'(q_{i-}) = q'_{i-}$, $\phi'(q_{i+}) = q'_{i+}$.

Figure III.4: Standard Diagram



The Remark III.2.4 implies that (after some elementary transformations) there exists a subdiagram Γ in Δ' with boundary $p_1q_1p_2q_2$, vertices of p_i are interior except for initial and terminal ones and $|p_i| \leq 11\delta+3$. Equations (III.10), (III.12), (III.8) provide that:

$max(\frac{|q_1|}{|R_i|}, \frac{|q_2|}{|R_j|}) \geq \frac{|Y|-20\delta-6}{M} \geq \frac{2\mu M+10\delta+3-20\delta-6}{M} \geq 2\mu - \frac{10\delta+3}{M}$. Part (ii) is proved.

To justify part (iii) we look at each of the 4 options for the path ν'. For example, if $\nu' = ({}_{\gamma'}U_i)s_1p_1^{-1}s_2({}_{\gamma'}U_j^{-1})$ then ϕ' maps the vertex $\nu_- = ({}_{\gamma'}U_i)_-$ of Δ' to the vertex $g_{i-1} = \prod_{k=1}^{i-1} U_kR_kU_k^{-1}$ in $Cay(G)$, $\nu'_+ = ({}_{\gamma'}U_j^{-1})_+$ to the vertex $g_j = \prod_{k=1}^{j} U_kR_kU_k^{-1}$ in $Cay(G)$. Hence $lab(\phi'(\nu')) = g_{i-1}^{-1}g_j = \prod_{k=i}^{j} U_kR_kU_k^{-1}$.

A direct computation using the relation (III.7) yields that for every possible value of c and d the word H conjugates $U_iR_iU_i^{-1}$ to $U_jR_j^{\pm1}U_j^{-1}$. For example, $H \equiv U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1}$ conjugates $U_iR_iU_i^{-1}$ to $U_jR_j^{-1}U_j^{-1}$:

$$U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1}U_jR_j^{-1}U_j^{-1}(U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1})^{-1} =$$

$$U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_{j-1}R_{j-1}U_{j-1}^{-1}(U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1})^{-1} =$$

$$U_iR_iU_i^{-1}\dots U_jR_jU_j^{-1}(U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1})^{-1} = U_iR_iU_i^{-1},$$

where the last inequality holds by (III.7). It remains to notice that by relation (III.7), in the word H the parameters $c = d = 0$ may be replaced by $c = d = 1$. □

Definition III.3.3. For every reducible pair $i < j$ consider the diagram Δ' from Corollary III.3.2, identify each edge of ${}_{\gamma'}U_s$ with corresponding edge of ${}_{\gamma'}U_s^{-1}$ and fill in the $\mathscr{R}$-faces Π_s to get a van-Kampen diagram Δ over G_1 which has a $(11\delta+3)$-contiguity subdiagram Γ such that $max\{(\Pi_i,\Pi_j),(\Pi_j,\Pi_i)\} \geq 2\mu - \frac{10\delta+3}{\rho}$. We will refer to a described diagram Δ as *a* standard diagram for relation (III.6). We denote the image of γ' in Δ by γ.

By definition, the standard diagram is a spherical diagram, but for convenience we draw it on Figure III.4 as a disc diagram with boundary label 1.

Remark III.3.4. According to the identifications made in the definition of the standard diagram Δ, any of the four paths v' in Δ' corresponds to a closed path in Δ with label $v = ({}_\gamma U_i) r_1 p_1 r_2 ({}_\gamma U_j^{-1})$, where r_i correspond to s_i. One can observe that different paths v' have different images in Δ, but we will not use this fact later. Note that the subpaths $({}_\gamma U_i)^{\pm 1}$ and $({}_\gamma U_j)^{\pm 1}$ of v' in Δ' correspond respectively to subpaths ${}_\gamma U_i$ and ${}_\gamma U_j$ of v.

III.4 Generators of a free normal subgroup in G

In this section we assume that the set $\mathscr{R}$ satisfies $\tilde{C}(\varepsilon,\mu,\rho)$-condition, where the parameters ε,μ,ρ are chosen according to Lemma III.2.7 and satisfy inequalities $\varepsilon > \varepsilon_0 = 19\delta + 3$, $\mu < 1/100$, $\rho > \frac{500\varepsilon}{6\mu(1-8\mu)}$.

It is well known (see [**Gro**]2.2A) that a hyperbolic group contains only finitely many conjugacy classes of torsion elements. So, given a group G, we may choose the constant ρ to be larger then the length of shortest representative in each conjugacy class of torsion elements. Thus we will assume in the sequel that for values of ρ large enough:

Remark III.4.1. The set $\mathscr{R}$ consists of elements of infinite order.

Definition III.4.2. We call a (reduced) diagram Δ *an octagon diagram* if $\partial\Delta = l_1 j_1 \dots l_4 j_4$, where l_i are geodesic in G, and $\|j_i\| \le \varepsilon$.

Definition III.4.3. Consider an octagon reduced diagram Δ with boundary $\partial\Delta = l_1 j_1 \dots l_4 j_4$ and pick a number $0 < \kappa < 1$. We say that an arc l_i satisfies the condition $\mathscr{U}_\Delta(\kappa)$ if for every diagram Δ' equivalent to Δ and every $\mathscr{R}$-face Π in Δ' such that there is a contiguity subdiagram Γ between Π and l_i, we have the inequality $(\Pi,\Gamma,l_i) < \kappa$.

It is clear that if l_i has a subpath l which is a boundary arc of some subdiagram Δ_1 of Δ then l satisfies $\mathscr{U}_{\Delta_1}(\kappa)$ as well.

Lemma III.4.4. *Let Δ be an arbitrary octagon diagram and $\phi(l_1) = U \in \mathscr{U}$, then (in notations of Definition III.4.2) l_1 satisfies $\mathscr{U}_\Delta(\frac{1}{2}+\frac{1}{5}\mu)$.*

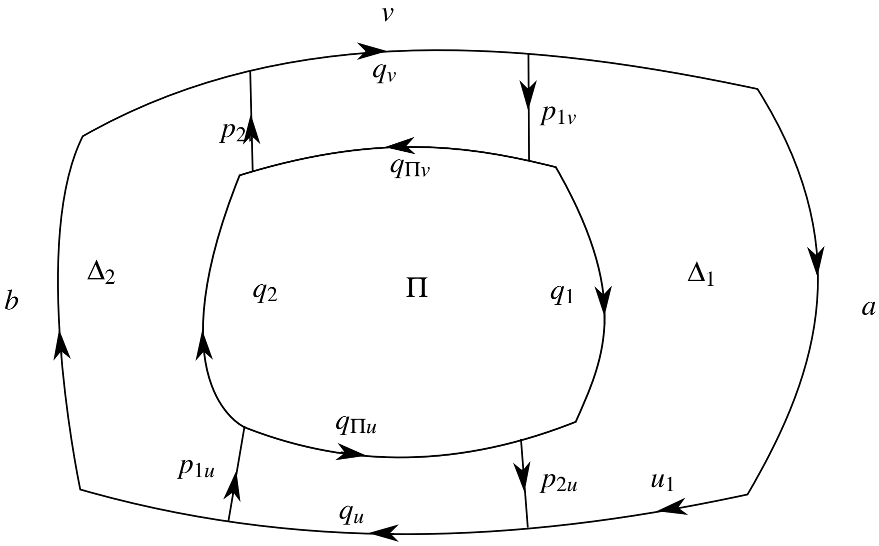
Proof Note that by definition of ρ we have that $\frac{2\varepsilon+34\delta}{\rho} < \frac{1}{5}\mu$. We suppose that there exists an octagon diagram Δ, with boundary arc l_1, $\phi(l_1) = U \in \mathscr{U}$. Assume that (after elementary transformations) there exists an $\mathscr{R}$-face Π in Δ and a corresponding subdiagram Γ between Π and l_1 with boundary $\partial(\Pi,\Gamma,l_1) = p_1 q_1 p_2 q_2$ such that $(\Pi,\Gamma,l_1) \ge \frac{1}{2} + \frac{2\varepsilon+34\delta}{\rho}$.

Now we may apply Remark III.2.4(ii) to the diagram Γ and conclude that (after elementary transformations) there exists a subdiagram Γ' of Γ with boundary $p_1' q_1' p_2' q_2'$ such that q_i' are subpaths of q_i and:

$$\left|p_i'\right| \le 6\delta,\ \left|q_1'\right| = |q_1| - 2\varepsilon - 4\delta. \tag{III.13}$$

By definition of q_1', we have $|q_1'| = |q_1| - 2\varepsilon - 4\delta \ge \frac{1}{2}|\partial\Pi| + 30\delta$ and it's complement q_3' ($\partial\Pi = q_1' q_3'$) satisfies $|q_3'| \le \frac{1}{2}|\partial\Pi| - 30\delta$. Thus the condition (i) of definition III.1.5 is satisfied.

Figure III.5: Bond Between u and v



We define paths l', l'' such that $l_1 = l'q_2'l''$. The equality $U = \phi(l_1) = \phi(l'p_1'q_1'p_2'l'')$ holds in G, moreover, by inequalities (III.13), we have:

$$|l'| + |p_1'| + |q_1'| + |p_2'| + |l''| \leq |l'| + 2|p_1'| + |q_2'| + 2|p_2'| + |l''| \leq |l_1| + 4 \cdot 6\delta.$$

Hence the condition (ii) of definition III.1.5 is checked for the factorization
$\phi(l'p_1')\phi(q_1')\phi(p_2'l'')$ of the word U.

By Definition III.1.5, the word U does contain more then half of a relation and thus $U \notin \mathscr{U}$ contrary to our assumption. $\square$

Definition III.4.5. Consider a reduced octagon diagram Δ with boundary $l_1j_1\dots l_4j_4$. Denote for simplicity of notation $u = l_1$ and $v^{-1} = l_3$, $a = j_3l_4j_4$, $b = j_1l_2j_2$ and define the base point of Δ to be $o = (l_1)_-$. Consider an $\mathscr{R}$-face Π and disjoint contiguity subdiagrams Γ_u, Γ_v of Π to boundary arcs u, v, define boundary arcs of Γ_u, Γ_v by $\partial(\Pi, \Gamma_u, u) = p_{1u}q_{\Pi u}p_{2u}q_u$, $\partial(\Pi, \Gamma_v, v) = p_{1v}q_{\Pi v}p_{2v}q_v$ and define q_1, q_2 by equality $\partial\Pi = q_{\Pi v}^{-1}q_1q_{\Pi u}^{-1}q_2$ (see Figure III.5). We say that a subdiagram $\Delta_0 = \Delta_0(\Delta, \Pi)$ with a boundary path $p_{2u}q_up_{1u}q_2p_{2v}q_vp_{1v}q_1$ (u,v)*-bond (through* Π*)* if both values $(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v)$ are greater than μ. We define subdiagrams $\Delta_1 = \Delta_1(\Delta, \Pi)$, $\Delta_2 = \Delta_2(\Delta, \Pi)$ of Δ with boundaries $u_1p_{2u}^{-1}q_1p_{1v}^{-1}v_1^{-1}a$ and $u_2bv_2^{-1}p_{2v}^{-1}q_2^{-1}p_{1u}^{-1}$ respectively, where u_1 (v_1) is an initial subpath of u (v) and v_2 (u_2) is a terminal subpath of v (u) (recall that the orientation of the boundary is clockwise).

For an arbitrary reduced octagon diagram Δ, $\partial\Delta = l_1j_1\dots l_4j_4$, where l_i are geodesic in G, $\|j_i\| \leq \varepsilon$, there exist a pair of (possibly empty) sets $V = \{\Pi_1, \dots \Pi_m\}$ of $\mathscr{R}$-faces and $\Sigma(\Delta) = \{\Gamma_{1,u}, \Gamma_{1,v}, \dots, \Gamma_{m,u}, \Gamma_{m,v}\}$ of disjoint ε-contiguity subdiagrams, where $\Gamma_{i,u}, \Gamma_{i,v}$ are contiguity subdiagrams such that $\Delta_0(\Pi_i) = \Pi_i \cup \Gamma_{iu} \cup \Gamma_{iv}$ is a (u,v)-bond. We call a pair $(V, \Sigma(\Delta))$ *a system of bonds between u and v*.

Remark III.4.6. (i) It is clear that in a non-empty system of (u,v)-bonds $(V, \Sigma(\Delta))$ for a reduced diagram Δ there exists a unique face Π in V such that the associated (see definition III.4.5) paths u_1 and v_1 are the longest. Moreover, any other face $\Pi' \in V$ belongs to $\Delta_1(\Pi)$.

(ii) For every face Π in V we have that

$$|u_1| \leq |u| - (\Pi, \Gamma_u, u)\,|\partial \Pi| + 2\varepsilon, \;\; |v_1| \leq |v| - (\Pi, \Gamma_v, v)\,|\partial \Pi| + 2\varepsilon. \tag{III.14}$$

The following remark will allow us to extend systems of bonds of subdiagrams Δ_i to the diagram Δ.

Remark III.4.7. Consider a reduced octagon diagram Δ over G_1 and assume that there is a (u,v)-bond $\Delta_0(\Pi) = \Pi \cup \Gamma_u \cup \Gamma_v$ in Δ satisfying $(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v) \geq \mu$ and two systems of (u_i, v_i)-bonds $(V_i, \Sigma(\Delta_i))$ in $\Delta_i = \Delta_i(\Pi, \Delta)$, $i = 1, 2$. Then the sets $V = V_1 \cup V_2 \cup \{\Pi\}$ and $\Sigma(\Delta) = \Sigma(\Delta_1) \cup \Sigma(\Delta_2) \cup \{\Gamma_u, \Gamma_v\}$ comprise the system of (u,v)-bonds $(V, \Sigma(\Delta))$ in Δ. □

Lemma III.4.8. *Let Δ be a reduced octagon diagram with at least one $\mathscr{R}$-face with boundary $\partial\Delta = a j_1 u j_2 b j_3 v^{-1} j_4$, where u, v, a satisfy the condition $\mathscr{U}_\Delta(\frac{1}{2} + \frac{\mu}{5})$, b satisfies $\mathscr{U}_\Delta(\mu)$ and $|j_k| \leq \varepsilon$ for every k.*

(i) Then Δ has a non-empty system of (u,a)-, (v,a)- or (u,v)-bonds.

(ii) Assume in addition that Δ does not have (u,a)- or (v,a)-bonds. Then, for the set V consisting of all $\mathscr{R}$-faces, there exists a system of (u,v)-bonds $(V, \Sigma(\Delta))$ such that for every $\mathscr{R}$-face Π in Δ there exist subdiagrams $\Gamma_u, \Gamma_v \in \Sigma(\Delta)$ satisfying:

$$(\Pi, \Gamma_u, u) + (\Pi, \Gamma_v, v) > 1 - 26\mu; \tag{III.15}$$

$$max[(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v)] > \frac{1}{2} - 13\mu; \tag{III.16}$$

$$min[(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v)] > \frac{1}{2} - 27\mu. \tag{III.17}$$

Proof (i) On the one hand we may consider an $\mathscr{R}$-face Π satisfying Lemma III.2.7 such that $(\Pi, \Gamma_a, a) + (\Pi, \Gamma_b, b) + (\Pi, \Gamma_u, u) + (\Pi, \Gamma_v, v) > (1 - 23\mu) - \frac{4\cdot 3\varepsilon}{|\partial\Pi|}$ (note that $(\Pi, \Gamma_{j_i}, j_i)\,|\partial\Pi| \leq 3\varepsilon$ because $|j_i| \leq \varepsilon$). Together with condition on b it means that

$$(\Pi, \Gamma_a, a) + (\Pi, \Gamma_u, u) + (\Pi, \Gamma_v, v) > (1 - 24\mu) - \frac{4\cdot 3\varepsilon}{|\partial\Pi|} \tag{III.18}$$

On the other hand each summand on the left-hand side of (III.19) is smaller than $\frac{1}{2} + \frac{\mu}{5}$. Hence at least two of them are larger than 12μ.

(ii) We continue the considerations in the proof of part (i). We cannot have $(\Pi, \Gamma_a, a) \geq \mu$ because at least one of the other summands in (III.18) is larger thn 12μ and we would get a (u,a)- or (v,a)-bond involving a which is impossible. Hence we get that

$$(\Pi, \Gamma_u, u) + (\Pi, \Gamma_v, v) > (1 - 25\mu) - \frac{4\cdot 3\varepsilon}{|\partial\Pi|} \tag{III.19}$$

and so the inequality (III.15) holds for Π. The inequality

$$max[(\Pi,\Gamma_u,u),(\Pi,\Gamma_v,v)] > \frac{1}{2}-\frac{25}{2}\mu-\frac{2\cdot 3\varepsilon}{|\partial\Pi|}$$

follows immediately since $\mu < 1/100$, while for

$$min[(\Pi,\Gamma_u,u),(\Pi,\Gamma_v,v)] > \frac{1}{2}-26\frac{1}{5}\mu$$

it is enough to recall that $|q_u|,|q_v| < (\frac{1}{2}+\frac{1}{5}\mu)\,|\partial\Pi|$. We have proved the formulas (III.15)–(III.17) for the face Π satisfying Lemma III.2.7, taking into account that (by definition of ρ): $\frac{4\cdot 3\varepsilon}{|\partial\Pi|} \leq \frac{4\cdot 3\varepsilon}{\rho} < \frac{1}{5}\mu$.

When $n=1$, the diagram Δ has a single $\mathscr{R}$-face Π and we are done by the argument above.

We induct on a number n of $\mathscr{R}$-faces in the octagon diagram Δ with base $n=1$. If $n>1$ we consider subdiagrams $\Delta_i=\Delta_i(\Delta,\Pi)$ for the face Π (we follow notations of Definition III.4.5 here). It is clear that diagrams Δ_i satisfy the induction assumption. Each has a number of $\mathscr{R}$-faces strictly less than n because neither contains the face Π, the arcs p_{iu},p_{iv} on the boundary of Δ_i are not longer than ε. The boundary arcs q_i of Δ_i satisfy the condition $\mathscr{U}_{\Delta_i}(\mu)$ by Lemma III.2.6 because they are boundary arcs of the $\mathscr{R}$-face Π in the reduced diagram Δ. As we mentioned before the proof of the Lemma, conditions $\mathscr{U}_{\Delta_i}(\mu)$ for q_i imply that there are no bonds involving q_i in Δ_i. The induction assumption is now checked for Δ_i, hence there exist systems of (u_i,v_i)-bonds $(V_i,\Sigma(\Delta_i))$ in Δ_i satisfying the conclusion of the Lemma. Finally we are in position to apply the Lemma III.4.7 to Δ relative to the bond $\Delta_0(\Pi)$: we obtain a system of (u,v)-bonds $(V,\Sigma(\Delta))$ such that V contains all $\mathscr{R}$-faces and the set $\Sigma(\Delta)$ is comprised of $\Sigma(\Delta_i)$ for $i=1,2$ and Γ_u,Γ_v. The inequalities (III.15)–(III.17) hold for every $\mathscr{R}$-face in Δ except for the face Π by induction assumption, and for the face Π we have obtained them above.□

We denote words URU^{-1} by $A_{R,U}$. If u is a path in some diagram Δ, we write $A_{R,u}$ for $A_{R,\phi(u)}$.

Definition III.4.9. Define a weight of a word $A_{R,U}$ by $\psi(A_{R,U})=|R|+4\,|U|$.

Lemma III.4.10. *Let Δ be a reduced diagram over the group G_1 with boundary $uj_1aj_2v^{-1}$, where u,v,a satisfy the condition $\mathscr{U}_\Delta(\frac{1}{2}+\frac{\mu}{5})$, $|j_i|\leq\varepsilon$ for $i=1,2$ and there are no (u,a)– or (v,a)-bonds. Then $\phi(uj_1aj_2v^{-1})=\prod_{i=1}^{n}A_{R_j,U'_j}$ in G, where $max_{1\leq j\leq n}\psi(A_{R_j,U'_j})<4max(|u|,|v|)$.*

Proof We proceed by induction on the number n of $\mathscr{R}$-faces in Δ. The conclusion of the Lemma holds for $k=0$ because $\phi(uj_1aj_2v^{-1})=1$ in G and there are no $A_{R,U}$'s.

Assume that the Lemma is true for $n-1$. Consider a face Π satisfying the Remark III.4.6. By Lemma III.4.8(ii), the $\mathscr{R}$-face Π of Δ is in the set V for some system of (u,v)-bonds $(V,\Sigma(\Delta))$, and inequalities (III.15)–(III.17) hold for Π. We recall the inequality (III.16) and assume that

$$(\Pi,\Gamma_u,u) > (\frac{1}{2}-13\mu), \tag{III.20}$$

in the other case proof is the same.

By the choice of Π, we have that every other $\mathscr{R}$-face of Δ is in the subdiagram Δ_1 ($\Delta_i=\Delta_i(\Delta,\Pi)$) and the subdiagram Δ_2 is a diagram over G (we are using notations from Definition III.4.5 and the reader can refer to Figure III.5 in the sequel of the proof). We consider a system of (u,v)-bonds provided by Lemma

III.4.8. Denote a subdiagram of Δ consisting of Δ_2, Δ_0 by Δ'. It contains a single $\mathscr{R}$-face Π, so we get the following equations in the group G:

$$\phi(\partial_{u_{1+}}\Delta') = \phi(\partial_{u_{1+}}\Delta_0) = \phi(p_{2u}^{-1}(\partial_{p_{2u-}}\Pi)p_{2u}). \tag{III.21}$$

Now notice that paths $\partial_{u_-}\Delta$ and $u_1(\partial_{u_{1+}}\Delta')u_1^{-1}(\partial_{u_-}\Delta_1)$ coincide after the elimination of returns in the latter path, so their labels are equal in the free group generated by S. We get that

$$\phi(\partial_{u_-}\Delta) = \phi(u_1(\partial_{u_{1+}}\Delta')u_1^{-1}(\partial_{u_-}\Delta_1)) = \phi(u_1(\partial_{u_{1+}}\Delta')u_1^{-1})\phi(\partial_{u_-}\Delta_1), \tag{III.22}$$

and taking into account (III.21),

$$\phi(u_1p_{2u}^{-1})\phi(\partial_{(p_{2u})_-}\Pi)\phi(u_1p_{2u}^{-1})^{-1}\phi(\partial_{u_-}\Delta_1) = 1 \text{ in } G_1,$$

where the number of faces in the diagram Δ_1, bounded by the path $u_1p_{2u}^{-1}q_1^{-1}p_{1v}^{-1}v_1^{-1}$, is $n-1$. For convenience we denote $\phi(\partial_{(p_{2u})_-}\Pi)$ by R_1. By induction assumption, we have the following equality in G for the boundary of Δ':

$$\phi(u_1p_{2u}^{-1}q_1^{-1}p_{1v}v_1^{-1}) = \prod_{i=2}^{n} A_{R_j,u_j},$$

where for every $1 < j \leq n$ we have $\psi(A_{R_j,u_j}) < 4max(|u_1|,|v_1|)$.

By Remark III.4.6 part (ii), we have that $max(|u_1|,|v_1|) < max(|u|,|v|)$. By inequalities (III.14) and (III.20), we have $\left|u_1p_{2u}^{-1}\right| \leq |u| - (\Pi,\Gamma_u,u)\,|\partial\Pi| + 2\varepsilon + \varepsilon < |u| - (\frac{1}{2} - 13\mu)\,|\partial\Pi| + 3\varepsilon < |u| - \frac{1}{4}\,|\partial\Pi|$, hence

$$\psi(A_{R_1,u_1p_{2u}^{-1}}) = |\partial\Pi| + 4\left|u_1p_{2u}^{-1}\right| < |\partial\Pi| + 4|u| - |\partial\Pi| = 4|u|.\square$$

Remark III.4.11. Let Δ be a reduced octagon diagram with boundary $\partial\Delta = l_1j_1\ldots l_4j_4$. Assume that $\phi(l_1)$ is a subword of some $R \in \mathscr{R}$ and $|l_1| \leq \frac{1}{2}|R|$. Then l_1 satisfies $U_\Delta(\frac{1}{2} + \frac{\mu}{5})$.

Proof Suppose on the contrary, there exists an $\mathscr{R}$-face Π and a contiguity subdiagram Γ such that $(\Pi,\Gamma,l_1) \geq \frac{1}{2} + \frac{\mu}{5}$, $\partial(\Pi,\Gamma,l_1) = p_1q_1p_2q_2$. Then, by $\tilde{C}(\varepsilon,\mu,\rho)$-condition, R and $\phi(\partial\Pi)$ are conjugate so $|\partial\Pi| = |R|$. Hence we get

$$\frac{1}{2}\,|\partial\Pi| \geq |l_1| \geq |q_1| - 2\varepsilon \geq (\frac{1}{2} + \frac{\mu}{5})\,|\partial\Pi|,$$

which is a contradiction. $\square$

For technical reasons we introduce a notation

$$N_{R,U} = gp\langle A_{R',U'} \mid \psi(A_{R',U'}) < \psi(A_{R,U})\rangle.$$

We say that $A_{R',U'}$ *is equivalent* ($\approx$) *to* $A_{R,U}$ if and only if $\psi(A_{R',U'}) = \psi(A_{R,U})$ and there exists a word H in $N_{R,U}$ such that $HA_{R',U'}H^{-1} = A_{R,U}$ in G. To prove that the relation $\approx$ is a correctly defined equivalence it is enough to notice that $N_{R,U} = N_{R',U'}$ whenever $\psi(A_{R',U'}) = \psi(A_{R,U})$. It is clear that equivalence classes with respect to $\approx$ are finite.

Definition III.4.12. Let $\mathscr{A}$ be a maximal set of words $A_{R,U}$ where $R \in \mathscr{R}$, $U \in \mathscr{U}$ such that

(i) $A_{R,U} \notin N_{R,U}$;

(ii) if $A_{R',U'} \approx A^{\pm 1}_{R,U}$, then at most one of them belongs to $\mathscr{A}$.

Lemma III.4.13. *(i) Suppose that some geodesic word U contains more then half of a relation, then for every $R \in \mathscr{R}$ we have that $A_{R,U} \in \mathscr{N}_{R,U}$.*

(ii) If URU^{-1} is not geodesic up to 10δ then there exists a geodesic up to 10δ word $VR'V^{-1}$ such that $A_{R,U} = A_{R',V}$ in G and $\psi(A_{R,U}) > \psi(A_{R',V})$.

(iii) $\mathscr{A}$ is a subset of $\mathscr{X}$ from Lemma III.1.6.

(iv) $\mathscr{A}$ generates $\mathscr{N}(\mathscr{R})$, moreover every $A_{R,U}$ is a product of elements of $\mathscr{A}^{\pm 1}$ with weights not larger then $\psi(A_{R,U})$.

Proof Pick some word $A_{R,U}$.

(i) Assume that U contains more then half of a relation, then (using notations and statement of Remark III.1.7(i)) we have

$$A_{R,U} = A_{r_1 r_2, U_1} A_{R, U_1 r_2^{-1} U_2} A^{-1}_{r_1 r_2, U_1}, \text{ where } U = U_1 r_1 U_2,\ r_1 r_2 \in \mathscr{R}, \tag{III.23}$$

and the following inequalities hold:

$$|r_1| + |U_1| + |U_2| \le |U| + 50\delta,\ \ |r_1| \ge |r_2| + 60\delta. \tag{III.24}$$

It follows from III.1.7(i) that $\psi(A_{R,U_1 r_2^{-1} U_2}) < \psi(A_{R,U})$. Now we use inequalities (III.24) to estimate:

$$\psi(A_{r_1 r_2, U_1}) = |r_1 r_2| + 4|U_1| = |r_1| + |r_2| + 4|U_1| \le$$

$$2|r_1| + 4|U_1| = 2(|r_1| + |U_1|) + 2|U_1| \le$$

$$\le 2(|U| + 50\delta) + 2(|U| + 50\delta - |r_1|) \le 4|U| + 200\delta - \rho < 4|U|.$$

Hence $A_{R,U}$ is equal to the product (III.23) such that both $\psi(A_{r_1 r_2, U_1})$ and $\psi(A_{R,U_1 r_2^{-1} U_2})$ are strictly less then $\psi(A_{R,U})$ and we conclude that $A_{R,U} \in \mathscr{N}_{R,U}$. Contradiction with Definition III.4.12. Hence, if $A_{R,U} \in \mathscr{A}$ then U does not contain more then half of a relation.

(ii) Suppose that $A_{R,U}$ is not geodesic up to 10δ. The Remark III.1.7 (ii) implies that then there exists $R' \in \mathscr{R}$ and a geodesic word V such that $URU^{-1} = VR'V^{-1}$ in G. By the same remark, the word $VR'V^{-1}$ is geodesic up to 10δ and $|R| = |R'|$ and so $|U| > |V|$. Thus we have got inequality $\psi(A_{R,U}) > \psi(A_{R',V})$ contradicting the choice $A_{R,U} \in \mathscr{A}$ again.

(iii) Follows from (i) and (ii) by definition of $\mathscr{X}$ in Lemma III.1.6.

(iv) By Lemma III.1.6, if $g \in \mathscr{N}$ then $g = \prod_{s=1}^{n} U_s R_s U_s^{-1}$ for some $U_s R_s U_s^{-1} \in \mathscr{X}$. Hence it is enough to show that every $A_{R,U} \in \mathscr{X}$ is equal to a product of elements of $\mathscr{A}$. We proceed by induction on possible values of $k = \psi(*)$ on the set $\mathscr{X}$.

If $A_{R_0,1} \in \mathscr{X}$ has minimal weight $\psi(A_{R_0,1})$, we have that $\mathscr{N}_{R_0,1} = \{1\}$ and so $A_{R_0,1} \notin \mathscr{N}_{R_0,1}$. By maximality of the set $\mathscr{A}$, the exists a word $A_{R',U'} \in \mathscr{A}$ such that $A_{R_0,1} \approx A^{\pm 1}_{R',U'}$ which implies that $A_{R_0,1} = A^{\pm 1}_{R',U'}$

in G.

Now pick $A_{R,U} \in \mathscr{X}$ such that $\psi(A_{R,U}) = k$. There are two cases.

CASE 1. $A_{R,U} \in \mathscr{N}_{R,U}$. In this case $A_{R,U}$ is a product of words $A_{R',U'}$ such that $\psi(A_{R',U'}) < \psi(A_{R,U})$ and we are done by the induction assumption.

CASE 2. $A_{R,U} \notin \mathscr{N}_{R,U}$. Consider all words $A_{R',U'}$ such that $A_{R',U'} \approx A_{R,U}$. Clearly, $A_{R',U'} \notin \mathscr{N}_{R,U} = \mathscr{N}_{R',U'}$. By maximality of the set $\mathscr{A}$, there exists a word $A_{R',U'} \in \mathscr{A}$ and by Corollary III.3.2 (iii) we have that there exists $H \in N_{R,U}$ such that $HA^{\pm 1}_{R',U'}H^{-1} = A_{R,U}$ in G. By induction assumption, H is a product of elements of $\mathscr{A}$ with weights smaller then $\psi(A_{R,U})$, while $\psi(A_{R,U}) = \psi(A_{R',U'})$. $\square$

Lemma III.4.14. *Let Δ be a reduced diagram over the group G_1 with boundary $upav^{-1}$ where $|p| \leq \varepsilon$, $\phi(u), \phi(v) \in \mathscr{U}$, $\phi(a)^{-1}A' \equiv R \in \mathscr{R}$ for some word A' and $|\phi(a)| \leq \frac{1}{2}|R|$.*

(i) Suppose that there exist an $\mathscr{R}$-face Π and contiguity subdiagrams Γ_a, Γ_v such that $(\Pi, \Gamma_a, a), (\Pi, \Gamma_v, v) \geq \mu$. Then $A_{R,v} \notin \mathscr{A}^{\pm 1}$.

(ii) Suppose that there exist an $\mathscr{R}$-face Π and disjoint contiguity subdiagrams Γ_a, Γ_u such that $(\Pi, \Gamma_a, a), (\Pi, \Gamma_u, u) \geq \mu$. In addition assume that $\phi(p)A'\phi(a)^{-1}\phi(p)^{-1} = R'$ in G for some $R' \in \mathscr{R}$. Then $A_{R',u} \notin \mathscr{A}^{\pm 1}$.

Proof (i) We define arcs of Γ_a, Γ_v by equalities $\partial(\Pi, \Gamma_v, v) = p_{1v}q_{\Pi v}p_{2v}q_v$, $\partial(\Pi, \Gamma_a, a) = p_{1a}q_{\Pi a}p_{2a}q_a$ and define q_1, q_2 by equality $\partial\Pi = q_{\Pi v}^{-1}q_1q_{\Pi a}^{-1}q_2$. We also define v_1, v_2 by equality $v = v_1q_v^{-1}v_2$ (see Figure III.6).

Consider a subdiagram Δ' with boundary $p_{2v}^{-1}q_2^{-1}p_{1a}^{-1}a_2v_2^{-1}$. Observe that q_2 satisfies $\mathscr{U}_{\Delta'}(\mu)$ by Lemma III.2.6 (because it is a boundary subpath of the $\mathscr{R}$-face Π in the reduced diagram Δ), $|p_{1v}|, |p_{1a}| \leq \varepsilon$ and a_2, v_2 satisfy $\mathscr{U}_{\Delta'}(\frac{1}{2} + \frac{\mu}{5})$ (they are subpaths of a, v and a satisfies $\mathscr{U}_{\Delta}(\frac{1}{2} + \frac{\mu}{5})$ by Lemma III.4.11). Choose $(a_2)_-$ as a base point of Δ. By Lemma III.4.8, there exists a system of (a,v)-bonds $(V, \Sigma(\Delta'))$ such that V contains all $\mathscr{R}$-faces of Δ' and (assuming there are $\mathscr{R}$-faces in Δ'), by Remark III.4.6, there exists a face Π' such that the diagram $\Delta_2(\Pi', \Delta')$ does not have $\mathscr{R}$-faces. The face Π' is in V so in order to simplify the notation we assume that $\Pi' = \Pi$ and Δ' itself is a diagram over G (i.e. it does not contain $\mathscr{R}$-faces).

Consider an $\mathscr{R}$-face $\bar{\Pi}$ disjoint from Δ and glue $\bar{\Pi}$ and Δ together along a. Define $\partial\bar{\Pi} = a^{-1}a'$ so that $\phi(a^{-1}a') \equiv R$. Since $(\Pi, \Gamma_a, a) \geq \mu$ we have that $\Pi, \bar{\Pi}$ comprise a pair of opposite faces with respect to p_{1a} hence

$$\phi((\partial_{(p_{1a})_+}\Pi)p_{1a}^{-1}(\partial_{(q_a)_+}\bar{\Pi})p_{1a}) = 1 \text{ in } G. \tag{III.25}$$

Now notice that $\phi(p_{1a}) = \phi(a_2v_2^{-1}q_vp_{1v}q_{\Pi v}q_2^{-1})$ in the group G because it bounds the diagrams Δ' and Γ_v over G. We plug in the latter expression into the equation (IV.2) and then conjugate by $\phi(p_{1v}q_{\Pi v}q_2^{-1})$ to obtain

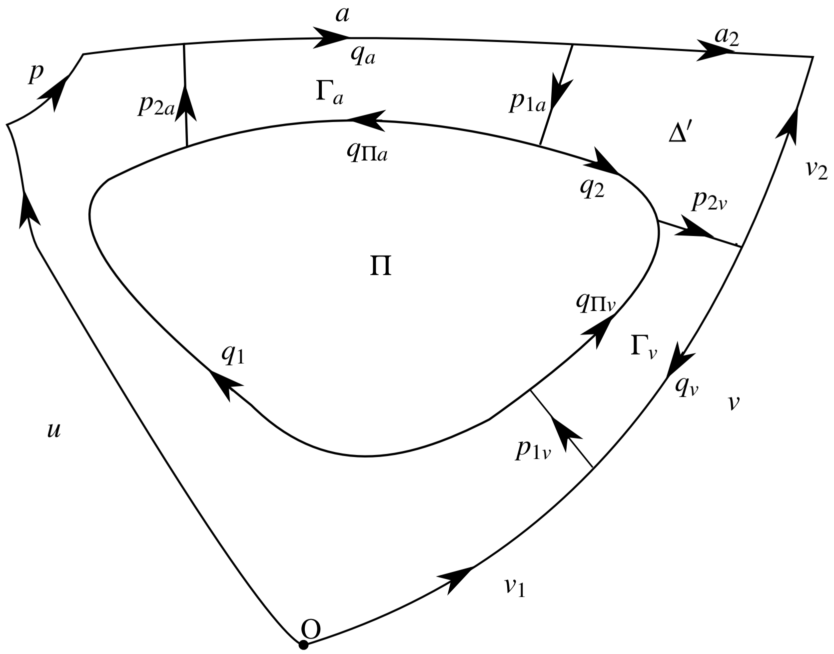
$$\phi(p_{1v}[q_{\Pi v}q_2^{-1}(\partial_{(p_{2a})_+}\Pi)q_2q_{\Pi v}^{-1}]p_{1v}^{-1}q_v^{-1}v_2[a_2^{-1}(\partial_{(q_a)_+}\bar{\Pi})a_2]v_2^{-1}q_v) = 1 \text{ in } G.$$

The paths in the square brackets are equal after elimination of returns to $\partial_{(p_{1v})_+}\Pi$ and $\partial_{v_+}\bar{\Pi}$ respectively. Denote $R' = \phi(\partial_{(p_{1v})_+}\Pi)$, recall that $R = \phi(\partial_{v_+}\bar{\Pi})$. Thus we have obtained that $A_{R',p_{1v}}A_{R,q_{\Pi}^{-1}v_2} = 1$ in G and, conjugating by v_1, we get:

$$A_{R',v_1p_{1v}}A_{R,v} = 1 \text{ in } G. \tag{III.26}$$

But on the other hand we have that $|R| = |R'|$ (because they are labels of opposite $\mathscr{R}$-faces in Δ) and,

Figure III.6: To the Proof of Lemma III.4.14



using inequality (III.14),

$$|v_1 p_{1v}| \leq |v_1| + |p_{1v}| = |v| - |q_v| - |v_2| + |p_{1v}| \leq$$

$$\leq |v| - ((\Pi, \Gamma_v, v)\, |\partial \Pi| - 2\varepsilon) + \varepsilon < |v|\,.$$

Hence we get $\psi(A_{R', v_1 p_{1v}}) < \psi(A_{R,v})$ and so $A_{R,v} \notin \mathscr{A}^{\pm 1}$.

Proof of part (ii) repeats part (i) with obvious changes in notation. □

Recall that in the beginning of section 5 we chose constants ε, μ, ρ according to Lemmas III.2.7, III.2.8. Hence part (ii) of Theorem I.1.3 follows immediately from aforementioned Lemmas (and is due to Olshanskiy [**Olsh93**]). We prove part (i) below:

Theorem III.4.15. *The subgroup $\mathscr{N} = \mathscr{N}(\mathscr{R})$ is freely generated by the set $\mathscr{A}$.*

Proof $\mathscr{A}$ generates $\mathscr{N}$ by Lemma III.4.13(iv).

We have to show that the set $\mathscr{A}$ generates $\mathscr{N}$ freely. We define a partial short-lex ordering on all words in alphabet $\mathscr{A}^{\pm 1}$. Let $W = A^{\varepsilon_1}_{R_1,U_1} \dots A^{\varepsilon_k}_{R_k,U_k}$ ($\varepsilon_i \in \pm 1$), $W' = \tilde{A}^{\varepsilon'_1}_{R'_1,U'_1} \dots \tilde{A}^{\varepsilon_{k'}}_{R'_{k'},U'_{k'}}$, we say that $W \succ W'$ if either

(i) $k > k'$ or

(ii) length of W is equal to length of W' ($k = k'$) and there exists $m_0 \leq k$ such that $\psi(A_{R_m,U_m}) = \psi(\tilde{A}_{R'_m,U'_m})$ for any $m < m_0$ and $\psi(A_{R_{m_0},U_{m_0}}) > \psi(\tilde{A}_{R'_{m_0},U'_{m_0}})$.

Let $W(\mathscr{A}) \equiv A^{\varepsilon_1}_{R_1,U_1} \dots A^{\varepsilon_n}_{R_n,U_n}$ be a nontrivial freely reduced word (in alphabet $\mathscr{A}$) such that $W = 1$ in G, assume that it is minimal with respect to the above ordering $\succ$. We are in position to apply Corollary III.3.2 and consider the corresponding standard diagram Δ for the word W, a reducible pair of indexes $i < j$, the standard contiguity subdiagram Γ between Π_i and Π_j with $|p_1| < 11\delta + 3$. We apply Lemma III.2.5

to faces Π_i, Π_j, path p_1 and vertices $o_1 = (_\gamma U_i)_+, o_2 = (_\gamma U_j)_+$. It provides the path $s_1 p_1 s_2$ in Δ such that $\phi(s_1 p_1 s_2) = P\phi(a)$ in G with $|P| \leq 11\delta + 3 + 8\delta$, $|a| \leq \frac{1}{2}\left|\partial \Pi_j\right|$, a is a subpath of $\partial \Pi_j$ and (using formula (III.4)) provides the equality $(P\phi(a))^{-1} R_i^{\varepsilon_i} (P\phi(a)) R_j^{\varepsilon_j} = 1$ in G or, equivalently,

$$P^{-1} R_i^{\varepsilon_i} P[\phi(a) R_j^{\varepsilon_j} \phi(a)^{-1}] = 1 \text{ in } G, \tag{III.27}$$

where the the word $[\phi(a) R_j^{\varepsilon_j} \phi^{-1}(a)]$ is a cyclic conjugation of $R_j^{\varepsilon_j}$ so $R_j^{\varepsilon_j} \equiv \phi^{-1}(a) A'$ for some A'.

We have that the path $_\gamma U_i s_1 p_1 s_2 (_\gamma U_j)^{-1}$ is closed in the standard diagram Δ by Remark III.3.4 and we have chosen $s_1 p_1 s_2$ so that

$$\phi(_\gamma U_i s_1 p_1 s_2 {_\gamma U_j^{-1}}) = U_i P \phi(a) U_j^{-1}. \tag{III.28}$$

Consider a reduced diagram $\tilde{\Delta}$ with boundary upa_1v^{-1} such that $\phi(u) = U_i$, $\phi(p) = P$, $\phi(a_1) = A$, where $A = \phi(a)$, $a \in \Delta$, $\phi(v) = U_j$. We will show that in fact it satisfies conditions of Lemma III.4.10. We first check conditions of Lemma III.4.14: we have that paths u, v are in $\mathscr{U}$, thus they satisfy condition $\mathscr{U}_{\tilde{\Delta}}(\frac{1}{2} + \frac{\mu}{5})$ by Lemma III.4.4 and so does the path a_1 by Lemma III.4.11. We also have that $\phi(v) R_j^{\varepsilon_j} \phi^{-1}(v) \in \mathscr{A}^{\pm 1}$ by definition of v and $R_i^{\varepsilon_i} = PA'\phi^{-1}(a_1)P^{-1}$ by equation (III.27), so Lemma III.4.14 provides us that there are no (u, a_1)- or (v, a_1)-bonds in $\tilde{\Delta}$. We have just checked the conditions of Lemma III.4.10 for the diagram $\tilde{\Delta}$ and conclude that:

$$\phi(ua_1 s v^{-1}) = \prod_{m=1}^{k} A_{R'_m, U'_m} \text{ in } G,$$

where $max_{1 \leq m \leq k} \psi(A_{R'_m, U'_m}) < 4 max(|u|, |v|)$.

The last relation together with (III.28) implies that $\phi(_\gamma U_i s_1 p_1 s_2 {_\gamma U_j^{-1}})$ belongs to at least one of the groups $\mathscr{N}_{R_i, U_i}, \mathscr{N}_{R_j, U_j}$. By Corollary III.3.2(iii), we have that $\phi(_\gamma U_i s_1 p_1 s_2 {_\gamma U_j^{-1}}) = H$ in G (where $H \equiv \prod_{k=i+d}^{j-c} A_{R_k, U_k}^{\varepsilon_k}$, $(c, d) \neq (0, 0)$, $c, d \in \{0, 1\}$) and that

$$H^{-1} A_{R_i, U_i}^{\varepsilon_i} H = A_{R_j, U_j}^{e} \text{ in } G \text{ for some } e \in \{\pm 1\}. \tag{III.29}$$

Suppose that $A_{R_i, U_i} \succ A_{R_j, U_j}$, then both words H and A_{R_j, U_j} belong to $\mathscr{N}_{R_i, U_i}$. Hence $A_{R_i, U_i} \in \mathscr{N}_{R_i, U_i}$, contradiction.

It remains consider the case when $\psi(A_{R_i, U_i}) = \psi(A_{R_j, U_j})$. By equation (III.29), $A_{R_i, U_i} \approx A_{R_j, U_j}^{e}$ and since they are both in $\mathscr{A}$ we have that $U_i \equiv U_j$, $R_i \equiv R_j$. Thus we can glue together the paths u and v of the boundary of $\tilde{\Delta}$ and obtain a diagram with boundary pa_1 (we will also call it $\tilde{\Delta}$). For every $\mathscr{R}$-face Π in $\tilde{\Delta}$ we now have that $(\Pi, \Gamma_p, p) \leq 3\varepsilon$ because $|p| \leq \varepsilon$ and $(\Pi, \Gamma_{a_1}, a_1) \leq \frac{1}{2} + \frac{1}{5}\mu$ thus

$$(\Pi, \Gamma_{a_1}, a_1) + (\Pi, \Gamma_p, p) \leq \frac{1}{2} + \frac{\mu}{5} + 3\varepsilon < 1 - 23\mu,$$

which contradicts Lemma III.2.7. Hence there are no $\mathscr{R}$-faces in $\tilde{\Delta}$ and $H = \phi(pa_1) = 1$ in G. But the word $H \equiv \prod_{k=i+d}^{j-c} A_{R_k, U_k}$ is a subword of W which is strictly shorter then W so $W \succ H$ and $H = 1$ in G. By minimality of W, we have equality $H \equiv 1$ which can only happen if $i + 1 = j$ so $A_{R_i, U_i}^{\varepsilon_i} A_{R_{i+1}, U_{i+1}}^{\varepsilon_{i+1}}$ is a subword of W, $U_i \equiv U_j$, $R_i \equiv R_j$ and by the relation (III.7) in G:

$$U_i R_i^{\varepsilon_i} U_i^{-1} U_{i+1} R_{i+1}^{\varepsilon_{i+1}} U_{i+1}^{-1} \equiv U_i R_i^{\varepsilon_i} U_i^{-1} U_i R_i^{\varepsilon_{i+1}} U_i^{-1} = 1,$$

which is equivalent to $R_i^{\varepsilon_i+\varepsilon_{i+1}} = 1$ in G and, taking into account the Remark III.4.1, we have that $\varepsilon_i + \varepsilon_{i+1} = 0$. Hence $A_{R_i,U_i}^{\varepsilon_i} A_{R_{i+1},U_{i+1}}^{\varepsilon_{i+1}} \equiv A_{R_i,U_i}^{\varepsilon_i} A_{R_i,U_i}^{-\varepsilon_i}$ is a subword of W. Contradiction with choice of W. □

In order to deduce Theorem I.1.4 we will use the following remark.

Remark III.4.16. (i) ([**Swe**] Theorem 13) For every element x in a hyperbolic group G there exists $n > 0$ and a *straight word* Y_x (i.e. a word Y_x such that Y_x^s is geodesic for every s) such that Y_x is conjugate to x^n.

(ii) Given a set of geodesic words words $X_1, \ldots, X_m$ we will denote by $\mathscr{R}_n = \mathscr{R}(X_1^{s_1}, \ldots, X_m^{s_m}, n)$ a system of all cyclic permutations of $R_i^{\pm 1}$ where $R_i \equiv X_i^{s_i n}$. If $X_1, \ldots, X_m$ are straight pairwise non-commensurable words in G, then for every $\mu > 0$, $\varepsilon \geq \varepsilon_0$ and $\rho > 0$ there exists a number $n > 0$ such that $\mathscr{R}_n$ satisfies $C(\varepsilon, \mu, \rho)$-condition independent of a choice of non-zero integers $s_1, \ldots, s_m$.

(iii) If Y is a straight word in G then for every integer m the word Y^m has a minimal length in it's conjugacy class.

Proof of part (ii) up to minor modifications repeats the proof of Lemma 4.1 in [**Olsh93**] which states the same property for $m = 1$.

Part (iii). Assume that $Y^s = TZT^{-1}$ for some T and that $|Z| \leq |Y^s| - 1$ then for every k we have that

$$k|Z| + k \leq k(|Y^s| - 1) + k = k(|Y^s|) = \left|Y^{sk}\right| \leq 2|T| + \left|Z^k\right| \leq 2|T| + k|Z|,$$

which implies that $k \leq 2|T|$. Contradiction.□

Proof of Theorem I.1.4. Let us first consider a set of pairwise non-commensurable elements $x_1, \ldots, x_m$ of infinite order. By remark III.4.16 (i), for each x_i there exists a straight word $\bar{Y}_{x_i}$ conjugate to $x_i^{n_i}$ for some $n_i > 0$. Define $n_0 = \prod_{1 \leq i \leq m} n_i$. Clearly words $Y_{x_1} \equiv \bar{Y}_{x_1}^{n_0}, \ldots, Y_{x_m} \equiv \bar{Y}_{x_m}^{n_0}$ are pairwise non-commensurable and, by parts (ii) and (iii) remark III.4.16, there exists an integer $K > 0$ such that the system $\mathscr{R}_K = \mathscr{R}(Y_1^{s_1}, \ldots, Y_m^{s_m}, K)$ satisfies $\tilde{C}(\varepsilon, \mu, \rho)$-condition for any choice of positive $s_1, \ldots, s_m$. By Theorem I.1.3, the group $\mathscr{N}(\mathscr{R}_K)$ is free and the quotient $G/\mathscr{N}(\mathscr{R}_K)$ is non-elementary hyperbolic.

Now consider an arbitrary set of elements $x_1, \ldots, x_m$ in G. If some of the elements x_i have finite orders $n_{i_1}, \ldots, n_{i_q}$ we define $n_0 = n_{i_1} \ldots n_{i_q}$ and replace the set $x_1, \ldots, x_m$ with $x_1^{n_0}, \ldots, x_m^{n_0}$ (which after deletion of identity elements contains only the elements of infinite order). Hence we can assume that all elements $x_1, \ldots, x_m$ are of infinite order. For every pair x_i, x_j $(i < j)$ define a pair of nonzero integers k_{ij}, k_{ji} such that $x_i^{k_{ij}}$ is conjugate to $x_j^{k_{ji}}$ if x_i, x_j are commensurable and let $k_{ij} = k_{ji} = 1$ if the pair x_i, x_j is not commensurable. Define $K_0 = \prod_{1 \leq i,j \leq m} k_{ij}$ and let $K_0 = 1$ if $m = 1$. We show by induction on m that

there exists an integer N such that $\mathscr{N} = \mathscr{N}(x_1^{s_1 K_0 N}, \ldots, x_m^{s_m K_0 N})$ is free for any choice of integers $s_1, \ldots, s_m$.

We have showed that the statement holds if the elements $x_1, \ldots, x_m$ are pairwise non-commensurable and in particular if $m = 1$. Hence, in order to prove the induction step, we may assume that (after reenumeration of x_i's) x_1 is commensurable to x_2. Using the normality of $\mathscr{N}$ and the fact that for every $x \in G$ a subgroup generated by x^a, x^b is the equal to the one generated by $x^{gcd(a,b)}$ we get that

$$\mathscr{N}(x_1^{s_1K_0N}, x_2^{s_2K_0N}, \ldots, x_m^{s_mK_0N}) = \mathscr{N}(x_1^{k_{12}s_1\frac{K_0}{k_{12}}N}, x_2^{s_2K_0N}, \ldots) =$$

$$\mathscr{N}(x_2^{k_{21}s_1\frac{K_0}{k_{12}}N}, x_2^{s_2K_0N}, \ldots) = \mathscr{N}(x_2^{gcd(k_{21}s_1\frac{K_0}{k_{12}}, s_2K_0)N}, x_3^{s_3K_0N}, \ldots, x_m^{s_mK_0N}).$$

Thus $\mathscr{N}$ is generated by $m-1$ elements and we may apply the induction assumption completing the proof of Theorem I.1.4.□

We recall the notions of an SQ-universal group and a CEP-subgroup. A group G is said to be *SQ-universal* if every countable group K embeds in a quotient of G. Let H be a subgroup of G, then H is said to have a *congruence extension property* (CEP) if for every subgroup K, $K \lhd H$ there exists a subgroup K_1, $K_1 \lhd G$, such that $K_1 \cap H = K$. It is easy to see that if the group G has a free infinitely generated CEP-subgroup then G is SQ-universal (see, for example, Proposition [**Olsh95**]).

Proof of Corollary I.1.5 (i) If G is non-elementary, there exists a pair of non-commensurable straight words X_1, X_2 in G (see for example [**Olsh93**], Lemma 1.14). By Remark III.4.16, there exists a number n such that $\mathscr{R} = \mathscr{R}(X_1, X_2, n)$ satisfies the small cancellation property $\tilde{C}(\varepsilon, \mu, \rho)$-condition for sufficiently small μ and hence $\mathscr{N}(\mathscr{R})$ is a free group by Theorem I.1.4. The rank $\mathscr{N}(\mathscr{R})$ is greater than 1 because X_1, X_2 are non-commensurable.

(ii) It is a result of Olshanskiy [**Olsh95**] that

(*) inside every non-elementary subgroup of G there exists a free countably generated CEP-subgroup in G (Theorem 4, [**Olsh95**]);

Consider a free normal subgroup $\mathscr{N}$ in G of rank greater than 1. There exists a free infinite rank CEP-subgroup N_1 in G, $N_1 < \mathscr{N}$ by (*). Hence for every countable group H there exists $M_1 \lhd N_1$ such that $H \cong N_1/M_1$. By congruence extension property, the (normal in G) subgroup $M = M_1^G$ satisfies $M \cap H = M_1$, so H embeds in G/M. Clearly $M = M_1^G$ is free (being a subgroup of a free group $\mathscr{N}$), and thus (ii) is proved.
□

III.5 Proof of corollary I.1.6

We will often deal with paths which are geodesic up to 10δ, so we set, using Lemma II.1.6, the constant $h = H(\delta, 1, 10\delta)$.

Lemma III.5.1. *Let $X_1\bar{R}_1X_1^{-1} = R_1$ in G, where R_1 is a (λ, c)-quasigeodesic in G and the path $X_1\bar{R}_1X_1^{-1}$ is geodesic up to 10δ. We choose the constant $H = H(\delta, \lambda, c)$ according to Lemma II.1.6. Consider an arbitrary factorization $\bar{R}_1 \equiv UV$ for some words U, V. Then there exists a cyclic conjugate R_2 of R_1 such that $X(VU)X^{-1} = R_2$ in G and $|X| \le H + h$.*

If we assume that $\bar{R}_1$ is a shortest element in the conjugacy class of R_1, then $|\bar{R}_1| \ge \lambda \|R_1\| - c - 2(H+h)$.

Proof We consider the paths $X_1\bar{R}_1X_1^{-1}$ and R_1 in $Cay(G)$. Combining together Lemma II.1.6 and definition of h before this Lemma we get that the paths $X_1\bar{R}_1X_1^{-1}$ and R_1 are within $(H+h)$-neighborhoods of each other and hence we may consider a geodesic ν between a vertex of the path R_1 and the vertex with label equal to X_1U such that $lab(\nu) = X$, $|X| \leq H+h$. Vertex ν_- provides the factorization $R_1 \equiv U_1V_1$ of R_1 and we have that $X(VU)X^{-1} = V_1U_1$ in G and thus define $R_2 \equiv V_1U_1$, it is an element of $\mathscr{R}$.

If we assume now that $\bar{R}_1$ is shortest in the conjugacy class of R_1, we get that $|\bar{R}_1| = |UV| = |VU| \geq |V_1U_1| - 2|X| \geq \lambda \|V_1U_1\| - c - 2|X| = \lambda \|R_1\| - c - 2|X| \geq \lambda \|R_1\| - c - 2(H+h)$, which proves the last assertion of the Lemma. $\square$

Lemma III.5.2. *Let $\mathscr{R}$ be a system of words satisfying the $C(\varepsilon,\mu,\lambda,c,\rho)$-condition, choose $H = H(\delta,\lambda,c)$ according to Lemma II.1.6. Assume that $\varepsilon = \varepsilon_0 + 2(H+h)$ for some $\varepsilon_0 > 0$ and that $\lambda\rho > c+4H+2h+16\delta$. Then there exists a system $\bar{\mathscr{R}}$ such that $\mathscr{N}(\bar{\mathscr{R}}) = \mathscr{N}(\mathscr{R})$ and for any positive number k, satisfying $\lambda\rho \geq \frac{k+1}{k}(c+4H+2h+16\delta)$, the condition $\tilde{C}(\varepsilon_0,\mu_0,\rho_0)$ holds for $\bar{\mathscr{R}}$, where $\mu_0 = \frac{\mu(1+k)}{\lambda}$ and $\rho_0 = \lambda\rho - c - 2(H+h)$.*

Proof We define an equivalence relation $\approx$ on the set of freely reduced words over $S^{\pm 1}$: words W_1 and W_2 are equivalent if and only if W_1 is conjugate to W_2 or W_2^{-1}. Then we consider a set $\{R_1,\ldots,R_m,\ldots\}$ of representatives of equivalence classes of $\mathscr{R}$ with respect to $\approx$. By Lemma III.1.2, there exist geodesic words $\bar{R}_i, X_i$ such that $X_i\bar{R}_iX_i^{-1} = R_i$, where $\bar{R}_i$ is geodesic up to 10δ and is shortest in the conjugacy class of R_i. Finally we define $\bar{\mathscr{R}}$ to be the closure of $\{\bar{R}_1,\ldots,\bar{R}_m,\ldots\}$ with respect to cyclic conjugates and inverses. Clearly $\mathscr{N}(\mathscr{R}) = \mathscr{N}(\bar{\mathscr{R}})$.

It remains to check that $\bar{\mathscr{R}}$ satisfies $\tilde{C}(\varepsilon_0,\mu_0,\rho_0)$-condition. Let $\bar{R}_i \equiv U_iV_i$ for $i = 1,2$, $U_2 = YU_1Z$ for some $|Y|,|Z| \leq \varepsilon_0$. Assume that $|U_1| \geq \mu_0 min(\bar{R}_1,\bar{R}_2)$.

By Lemma III.5.1 and the definition of $\bar{\mathscr{R}}$, there exist $R_i \in \mathscr{R}$ and (geodesic) words X_i such that $X_i\bar{R}_iX_i^{-1} = R_i$ and $|X_i| \leq H+h$. By Remark II.1.3, the path $p_i = X_i\bar{R}_iX_i^{-1}$ in the $Cay(G)$ is within the $H+h+8\delta$ neighborhood of a geodesic q_i joining the ends of p_i and, by Lemma II.1.6, q_i is in the H-neighborhood of R_i. Hence we can define geodesic words Y_i,Z_i, $i = 1,2$, of lengths not greater than $2H+h+8\delta$ and factorizations $R_i \equiv C_iD_iE_i$ such that the following equalities hold in G: $X_1Y_1 = C_1$, $X_2Y_2^{-1} = C_2$, $C_1D_1Z_1 = X_1U_1$ and $X_2U_2Z_2 = C_2D_2$. We have obtained that $(Y_2YY_1)D_1(Z_1ZZ_2) = D_2$. Now we estimate the length of D_1 using quasigeodesicity of R_1: $|D_1| \geq |U_1| - |Y_1| - |Z_1| \geq |U_1| - 2(2H+h+8\delta) \geq \mu_0|\bar{R}_i| - (4H+2h+16\delta) \geq \mu_0(\lambda\|R_i\| - c - (4H+2h+16\delta)) \geq \frac{\mu(1+k)}{\lambda}\lambda\|R_i\| - \frac{\mu(1+k)}{\lambda}(c+4H+2h+16\delta) = \mu\|R_i\| + \mu[k\|R_i\| - \frac{(1+k)}{\lambda}(c+4H+2h+16\delta)] \geq \mu\|R_i\|$.

By the small cancellation condition on $\mathscr{R}$, we conclude that $(Y_2YY_1)(D_1E_1C_1)(Y_2YY_1)^{-1} = D_2E_2C_2$ and so $Y(Y_1D_1E_1C_1Y_1^{-1})Y^{-1} = Y_2^{-1}D_2E_2C_2Y_2$ in G implying that $Y\bar{R}_1Y^{-1} = \bar{R}_2$. $\square$

We say that a set of elements $\{x_1,\ldots,x_n\}$ satisfy the *condition (I)* relative to some element g of infinite order if

(a) $x_i \notin E(g)$ for every i;

(b) x_i is in the centralizer of $E(G)$ in G: $x_i \in C_G(E(G))$;

(c) if an equality $ax_i = x_ib$ holds in G for some $a,b \in E(g)$, then $a = b \in E(G)$;

(d) if $a,b \in E(g)$ and $ax_i = x_jb$, then $i = j$.

Lemma III.5.3. *([**Olsh93**], Lemma 3.7)*[1] *Let g be an element such that $E(g) = \langle g \rangle \times E(G)$. Then for any integer $l \geq 1$, one can find elements $x_1,\ldots,x_l \in G$ satisfying the condition I relative to the element g.*

[1]Our formulation is less general then that in [**Olsh93**]. One can observe that in our case the condition (I)(c) follows from (I)(a)-(b).

Let us consider a set of geodesic words $W_1,\ldots,W_k$ which represent respectively elements $g_1,\ldots,g_k$, satisfying Lemma II.2.7. For every $1 \le i \le k$ we can choose a sequence of geodesic words $X_{0i},\ldots,X_{li}$ satisfying Lemma III.5.3 relative to W_i. Define a set $\mathscr{R}_{l,m}$ as a closure with respect to taking inverses and cyclic conjugates of words $R_1,...,R_k$, where $R_i \equiv X_{0i}W_i^m \ldots X_{li}W_i^m$ for $i = 1,\ldots,k$.

Lemma III.5.4. *([**Olsh93**], Lemma 4.2) There exists $\lambda > 0$ such that for every $\mu > 0$ there exists $l > 0$ and $c \ge 0$ such that for any $\varepsilon \ge 0$, $\rho > 0$ there are $m_0 > 0$ such that the system $\mathscr{R}_{l,m}$ satisfies the $C(\varepsilon,\mu,\lambda,c,\rho)$-condition if $m \ge m_0$.*

We will denote the center $C_G(G)$ of the group G by $Z(G)$.

Proposition III.5.5. *Assume that G is a hyperbolic group and $Z(G) = E(G)$. Let g be an element in G satisfying Lemma II.2.7 and consider an element y such that $y \notin E(g)$. Then for any integer $l \ge 1$, one can find elements $x_1,\ldots,x_l \in G$ such that each set $\{x_1,\ldots,x_l\}$ and $\{yx_1,x_2,\ldots,x_l\}$ satisfies the condition (I) relative to the element g.*

Proof We take an element g satisfying Lemma II.2.7 and a set $\mathscr{X} = \{x_1,\ldots,x_{l+1}\}$ satisfying Lemma III.5.3 relative to g. We only need to check the conditions of property (I) involving yx_1.

(a) Assume that $yx_1 \in E(g)$, then $yx_2 \notin E(g)$. Indeed, if $yx_1, yx_2 \in E(g)$, then $x_2^{-1}x_1 \in E(g)$. Thus $x_1 = x_2 b$ for some $b \in E(g)$, which contradicts the condition (d) of (I) for $\mathscr{X}$.

Hence, if $yx_1 \in E(g)$, we can redefine $\mathscr{X}$ to be $\{x_2,x_1,x_3\ldots,x_{l+1}\}$ and thus each set $\{x_2,x_1,x_3\ldots,x_{l+1}\}$ and $\{yx_2,x_1,x_3\ldots,x_{l+1}\}$ satisfies (I)(a).

Condition (b) holds because $C_G(E(G)) = G$.

(c) Assume that the equality $ayx_1 = yx_1b$ holds for some $a,b \in E(g)$. If the element a is of finite order, then $a \in E(G) = Z(G)$ by the choice of g and thus $a = b$. If a is of infinite order, then $a^{yx_1} \notin E(a) = E(g)$ because $yx_1 \notin E(a) = E(g)$, which contradicts the choice of $b \in E(g)$.

(d) We can observe that an equality $ayx_1 = xb$, where $x \in \{x_2,\ldots,x_{l+1}\}$ can hold for at most one element x_i. Assume on the contrary, that both equalities

$$a_1yx_1 = x_2b_1, a_2yx_1 = x_ib_2$$

hold for some $a_j,b_j \in E(g)$. Then we have $a_1^{-1}x_2b_1 = a_2^{-1}x_ib_2$ and the equality $x_2b_1b_2^{-1} = a_1a_2^{-1}x_i$ contradicts property (I) for $\mathscr{X}$.

Hence we may choose a set $x_1',\ldots,x_l'$ satisfying the Lemma by eliminating at most one element from $\mathscr{X}$. □

For each k, we can choose elements $W_1,\ldots,W_k$ satisfying Lemma II.2.7, Proposition III.5.5 allows us to pick sequences of words $\{X_{0i},\ldots,X_{li}\}$ for every $1 \le i \le k$ and any $l > 0$ such that both $\{X_{0i},\ldots,X_{li}\}$ and $\{s_iX_{0i},\ldots,X_{li}\}$ satisfy the condition (I) relative to W_i. In addition to the set $\mathscr{R}_{l,m}$, we consider a set of relations $S\mathscr{R}_{l,m}$ which is a closure of the relations s_iR_i for $1 \le i \le k$ under inversion and cyclic conjugation. Clearly, $S\mathscr{R}_{l,m}$ also satisfies Lemma III.5.4. We will need the following, slightly stronger, Corollary:

Corollary III.5.6. *([**Olsh93**], Lemmas 4.2) There exists $\lambda > 0$ such that for every $\mu > 0$ there exists $l > 0$*

and $c \geq 0$ such that for any $\varepsilon \geq 0$, $\rho > 0$ there are $m_0 > 0$ and $\{X_{0i}, \dots, X_{li}\}$ for every $1 \leq i \leq k$ such that each system $\mathscr{R}_{l,m}$ and $S\mathscr{R}_{l,m}$ satisfies the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition if $m \geq m_0$.

Proof repeats that of Lemma III.5.4 [**Olsh93**] with minor modifications. First, as in Lemma III.5.4, we may choose $\lambda > 0$ depending only on $W_1, \dots, W_k$. Then we choose l (after μ) so that

$$\lambda \mu (l+1) \frac{min_i \|W_i\|}{max_i \|W_i\|} > 6.$$

Similarly to III.5.4, there exists c, depending on $W_i, X_{ji}, s_i X_{0i}$ such that $\mathscr{R}_{l,m}$ and $S\mathscr{R}_{l,m}$ consist of (λ, c)-quasigeodesics for all sufficiently large m(c does not depend on m).

Finally, one chooses m_0 (after ε, ρ) such that

$$\lambda \mu (l+1) m_0 min_i \|W_i\| - c - 2\varepsilon \geq 6 m_0 max_i \|W_i\|$$

and such that $\mathscr{R}_{l,m}$, $S\mathscr{R}_{l,m}$ consist of (λ, c)-quasigeodesics. □

Remark III.5.7. Let G be a hyperbolic group. Then:

(i) if s is an element of finite order then there exists an element $t \in G$ such that both t and st^m are of infinite order for any $m \geq m_0$ for some number m_0.

(ii) the group G is generated by elements of infinite order.

Proof (i) Let t be an element satisfying Lemma II.2.7. Assume that for every $m > 0$ the element (st^m) is of finite order. Since a hyperbolic group can have only finite number of conjugacy classes of torsion elements, we can take $K > 0$, independent on m, such that $(st^m)^K = 1$ in G for every m. We have that values $(t^{-m}s^{-1}, st^m)$ are unbounded as $m \to \infty$ and hence the values of (t^{-m}, st^m) are also unbounded which implies that $s \in E(t)$. But if s is in $E(t)$ then st is of infinite order. Contradiction; the elements (st^m) have infinite order for all sufficiently large m.

Let $s_1, \dots, s_k$ be a generating set for the group G, consider an element t satisfying Lemma II.2.7. By part (i), there exists $m > 0$ such that $s_i t^m$ is of infinite order for every generator s_i of finite order. If we replace $s_i \in S$ with a pair $t' = t^m$, $s_i' = s_i t^m$ whenever s_i is of finite order and keep $s_i \in S$ otherwise. Then the generating set S satisfies (ii). □

Proof of Corollary I.1.6 Assume first that $Z(G) = E(G)$ and consider the generators $s_1, \dots, s_k$ of G. By Remark III.5.7 we may assume that the orders of all generators are infinite. We consider a set of pairwise non-commensurable geodesic words $W_1, \dots, W_k$ such that $E(W_i) = E(G) \times < W_i >$ in G (see conventions before the Lemma III.5.4).

For every $l > 0$ and $1 \leq i \leq k$, we choose a sequence of geodesic words $\{X_{0i}, \dots, X_{li}\}$ satisfying Proposition III.5.5 relative to W_i. We apply Lemma III.5.6 to the sets $\mathscr{R}_{l,m}$, $S\mathscr{R}_{l,m}$. Hence we may choose consequently $\lambda > 0$, $\mu < \frac{\lambda}{200}$, $l > 0$, $c > 0$, $\varepsilon = \varepsilon_0 + 2(2H(\lambda, c) + h)$ where $\varepsilon_0 \geq 19\delta + 3$ and ρ such that $\rho > \frac{1}{\lambda}(c + 2H + 2h + \frac{500\varepsilon_0}{6\frac{2\mu}{\lambda}(1 - 8\frac{2\mu}{\lambda})})$ such that both $\mathscr{R}_{l,m}$ and $S\mathscr{R}_{l,m}$ satisfy $C(\varepsilon, \mu, \lambda, c, \rho)$ for some m. By Lemma III.5.2 there are sets $\overline{\mathscr{R}_{l,m}}$ and $\overline{S\mathscr{R}_{l,m}}$ which generate the same normal subgroups as $\mathscr{R}_{l,m}$, $S\mathscr{R}_{l,m}$ respectively and satisfy $\tilde{C}(\varepsilon_0, \mu_0, \rho_0)$ for $\rho_0 = \lambda\rho - c - 2H - 2h$ and $\mu_0 = \frac{2\mu}{\lambda}$. By the choice of constants above we have that $\varepsilon_0 \geq 19\delta + 3$, $\mu_0 < 1/100$, and $\rho_0 \geq \frac{500\varepsilon_0}{6\frac{2\mu}{\lambda}(1 - 8\frac{2\mu}{\lambda})}$ and hence we may apply Theorem III.4.15 to conclude that both normal subgroups $A = \mathscr{N}(\mathscr{R}_{l,m}) = \mathscr{N}(\overline{\mathscr{R}_{l,m}})$ and $B = \mathscr{N}(S\mathscr{R}_{l,m}) = \mathscr{N}(\overline{S\mathscr{R}_{l,m}})$ are free. By Lemma III.2.8 each subgroup A and B has an infinite index in G.

Clearly $AB = BA \ni s_i = (s_i R_i)(R_i)^{-1}$ for every $1 \leq i \leq k$, hence $AB = G$.

Now assume that $AB = G$ for some free normal subgroups A, B in G. For every normal subgroup N in G we have that $[N, E(G)] \leq N \cap E(G)$, so $[A, E(G)] = 1$ and $[B, E(G)] = 1$ because the intersection of free subgroup with torsion subgroup is trivial. Hence $E(G) \leq C_G(A) \cap C_G(B) = C_G(AB) = Z(G)$. The inclusion $Z(G) \leq E(G)$ is immediate. □

CHAPTER IV

HIGHLY TRANSITIVE AND PERIODIC ACTIONS OF MAXIMAL GROWTH OF HYPERBOLIC GROUPS

IV.1 Actions of Maximal Growth

Let G be a group generated by a finite set S and suppose that G acts on a set X from the right:

$$xe = x,\ \ (xg_1)g_2 = x(g_1g_2) \text{ in } G \text{ for all } x \in X;\ g_1, g_2 \in G.$$

We assume that the action is transitive (i.e. $X = oG$, where o is some element from X). Consider the set $B_n(o)$ of elements $og \in X$ such that $g \in G$ and $|g| \leq n$. Then *the growth function of the right action* of G on X is $f_{o,S}(n) = \#\{B_n(o)\}$. Let $o' = og_0 \in G$ and denote $|g_0|$ by C. It is clear that $B_n(o') \subset B_{n+C}(o)$ and hence

$$f_{o,S}(n+C) \geq f_{o',S}(n).$$

Recall that the action of G on X is called k-transitive for some natural number k, if for any two ordered k-tuples $(x_1,...,x_k)$, $(y_1,...,y_k)$ of distinct elements in X there exists an element $g \in G$ such that $x_ig = y_i$ for every i, $1 \leq i \leq k$. The action is said to be *highly transitive* if it is k-transitive for every natural k.

Consider a set $\mathscr{F}$ of functions from $\mathbb{N}_0$ to $\mathbb{N}_0$. A pair $f, g \in \mathscr{F}$ is said (see [**BO**], §1.4) to satisfy the relation $f \prec g$ if there exists a non-negative integer C such that $f(n) \leq g(n+C)$ for every $n \in \mathbb{N}_0$. Clearly the relation $\prec$ is transitive and reflexive. Functions $f, g \in \mathscr{F}$ are said to be *equivalent* ([**BO**], §1.4) if $f \prec g$ and $g \prec f$. According to the discussion above, growth functions of transitive action of G on X with respect to different base points o, o' are equivalent.

One can observe that the notion of equivalence of functions, which we discuss here, is stronger then that used for study of asymptotic invariants. It is easy to see that the pairs $2n$ and $3n$ are not equivalent in the sense of definition above and neither are the functions 2^n, 3^n. We would also like to note that there exists a transitive action of a finitely generated free group which has maximal growth with respect to one finite set of generators and does not have maximal growth with respect to the other.

If the group G acts from the right on $X = G$, we get the usual growth function and denote it by $f(n)$; clearly the growth of any action of G is bounded by the usual growth function of G: $f_{o,S}(n) \leq f(n)$ for any $o \in X$.

If H is a stabilizer of o, then every element $x \in X$ is in one-to-one correspondence with a coset Hg in G such that $x = og$ and the right actions of G on X and on $H\backslash G$ are isomorphic.

Definition IV.1.1. ([**BO**], §2) Let $f(n)$ be a growth function of G relative to a finite generating set S and consider a transitive action of G on a set X. Then the growth of the action is called maximal if the function $f_{o,S}(n)$ is equivalent to $f(n)$.

In this paper we discuss the growth of actions of hyperbolic groups which are known to be non-amenable (see remark IV.1.2). We recall that a group G is called amenable if there exists a finitely additive left invariant probability measure on G (see [**Gre**]).

Remark IV.1.2. Every non-elementary hyperbolic group is non-amenable.

Proof If G is a non-elementary hyperbolic group, then it contains a free non-cyclic subgroup F_2 ([**Gro**], [**Ghys**], p. 157). But a free group of rank greater then one is non-amenable (see [**Gre**] 1.2.8). On the other hand, a subgroup of an amenable group is amenable ([**Gre**], Theorem 1.2.5). Hence G cannot be amenable.$\square$

The famous Fölner amenability criterion ([**Gre**]) yields the following:

Corollary IV.1.3. *For every non-elementary hyperbolic group G there exists $\varepsilon > 0$ (depending on G only) such that $\#\{B_{R+1}\} \geq (1+\varepsilon)\#\{B_R\}$ for any R.*

Remark IV.1.4. Let G be a non-amenable group with growth function $f(x)$ relative to a finite generating set S. Assume G acts on X with respect to some base point $o \in X$; denote the growth function of this action by $g_{o,S}(x)$. Then there exists $c_1 > 0$ such that the inequality $g_{o,S}(n) \geq c_1 f(n)$ holds for all natural n if and only if the action has maximal growth.

Proof We first show the "only if" part. By corollary IV.1.3, there exists $\varepsilon > 0$ such that the recursive formula $f(n+1) \geq (1+\varepsilon)f(n)$ holds for every n. We choose a natural C satisfying $c_1(1+\varepsilon)^C \geq 1$. Then, applying the recursive formula C times we get:

$$g_{o,S}(n+C) \geq c_1 f(n+C) \geq c_1(1+\varepsilon)^C f(n) \geq f(n).$$

Now assume that the action has maximal growth, i.e. $g_{o,S}(n+C) \geq f(n)$ for some natural number $C \geq 0$ and every natural n. It is clear from definition of $g_{o,S}$ that $g_{o,S}(n+C) \leq (2\#\{S\})^C \times g_{o,S}(n)$ and hence for $c_1 = (2\#\{S\})^{-C}$ the inequality $g_{o,S}(n) \geq c_1 f(n)$ holds.$\square$

We recall the notion of exponential growth rate of a group G with respect to the set of generators S: $\lambda(G,S) = lim_{n\to\infty} \sqrt[n]{f(n)}$, where $f(n)$ is a growth function of G.

Let S be a finite generating set in G and let N be an infinite normal subgroup of G. We denote the image of S under the canonical homomorphism $G \to G/N$ by $\overline{S}$. The following Theorem is often summarized by saying that the hyperbolic groups are "growth tight":

Theorem IV.1.5. *[**AL**] Let G be a non-elementary hyperbolic group and S any finite set of generators for G. Then for any infinite normal subgroup N of G we have $\lambda(G,S) > \lambda(G/N, \overline{S})$.*

The next corollary restates the above Theorem in terms of maximal growth.

Corollary IV.1.6. *Assume G is a hyperbolic group acting on some set X from the right with maximal growth. Then the kernel of this action is a finite normal subgroup.*

Proof Let N be the kernel of the action on X. For any point $o \in X$ we have that $oNg = og$ for all $g \in G$ and hence the growth function of the action $g_{o,S}$ satisfies:

$$g_{o,S}(n) \leq f_{G/N}(n) \text{ for every } n \in \mathbb{N}, \tag{IV.1}$$

where $\overline{f}(n)$ is the growth function of G/N with respect to images $\overline{S}$ of generators S of G. If $f(n)$ is the growth function of G with respect to S and the growth of the action is maximal, then there exists $c_1 > 0$ such that $g_{o,S}(n) \geq c_1 f(n)$ for every $n \in \mathbb{N}$. Hence, using (IV.1) and the last inequality, we have:

$$\lambda(G/N, \overline{S}) = lim_{n\to\infty} \sqrt[n]{\overline{f}(n)} \geq lim_{n\to\infty} \sqrt[n]{g_{o,S}(n)} \geq lim_{n\to\infty} \sqrt[n]{c_1 f(n)} = \lambda(G,S),$$

which, by Theorem IV.1.5, can only hold when N is finite. $\square$

Throughout this paper we will mainly discuss properties of left cosets. The connection between the right and left cosets is established by the following observation:

Remark IV.1.7. The right coset Hg intersects the ball B_R in G if and only if the left coset $g^{-1}H$ intersects B_R.$\square$

The abundance of examples of actions of maximal growth is evident from the following:

Proposition IV.1.8. *Let G be a hyperbolic group and H be a quasiconvex subgroup of infinite index in G. Then the natural right action of G on $H\backslash G$ has maximal growth.*

Proof We first consider left cosets G/H. By Theorem II.2.5, there exists $C > 0$ and the section s such that the group G is in $B_C(s(G/H))$. Hence for every $g \in B_R$ there exists $\overline{g} \in s(G/H)$ such that $|g - \overline{g}| \leq C$. By definition of s, $|\overline{g}| \leq |g|$ and thus $\overline{g} \in B_R$. We get that $B_R \subset \cup_{g \in B_R \cap s(G/H)} B_C(g)$, which implies:

$$f(R) = \#\{B_R\} \leq \#\{B_C\} \times \#\{s(G/H) \cap B_R\}. \tag{IV.2}$$

If $g_1, g_2 \in s(G/H) \cap B_R$ then (because the map s is a section) $g_1 H \neq g_2 H$. We get that $\#\{s(G/H) \cap B_R\} \leq \#\{gH | gH \cap B_R \neq \emptyset\}$ and the remark IV.1.7 provides $\#\{gH | gH \cap B_R \neq \emptyset\} = \#\{Hg^{-1} | Hg^{-1} \cap B_R \neq \emptyset\}$, thus

$$\#\{s(G/H) \cap B_R\} \leq \#\{Hg^{-1} | Hg^{-1} \cap B_R \neq \emptyset\}. \tag{IV.3}$$

Evidently the sets $\{Hg^{-1} | Hg^{-1} \cap B_R \neq \emptyset\}$ and $\{Hg | \exists g_1 \ : \ Hg_1 = Hg \ \& \ |g_1| \leq R\}$ contain the same cosets, and, by definition of the growth function $f_{H,G/H}(R)$ of natural right action of G on G/H: $\#\{Hg | \exists g_1 : Hg_1 = Hg \& |g_1| \leq R\} = f_{H,G/H}(R)$. Using inequalities (IV.2), (IV.3) and the last equality we get:

$$f(R) \leq \#\{B_C\} \times \#\{s(G/H) \cap B_R\} \leq \#\{B_C\} \times \#\{Hg | \exists g_1 : Hg_1 = Hg \ \& |g_1| \leq R\}$$

$$\leq \#\{B_C\} \times f_{H,G/H}(R).$$

By remark IV.1.4, the action has maximal growth.$\square$

In [**BO**] the authors provide examples of maximal growth actions of free groups satisfying some additional properties.

Recall that in [**Sta**] a subgroup H of a group G is said to satisfy the Burnside condition if for any $a \in G$ there exists a natural number $n \neq 0$ such that a^n is in H. One of the main results of the aforementioned paper is the following:

Corollary IV.1.9. *([**BO**], corollary 6) Any finitely generated subgroup H of infinite index in the free group F of rank greater than 1 is contained as a free factor in a free subgroup K satisfying the Burnside condition. One can choose K with maximal growth of action of F on $K\backslash F$. It follows that there exists a transitive action of F, with maximal growth and with finite orbits for each element $g \in F$, which factors through the action of F on $H\backslash F$.*

The following Theorem generalizes the corollary 6 of [**BO**] from free groups to non-elementary hyperbolic ones:

Theorem IV.1.10. *Let G be a non-elementary hyperbolic group with growth function $f(n)$. Then for any $0 < q < 1$ there exists a free subgroup H in G satisfying the Burnside condition and such that the growth $f_{H\backslash G}(n)$ of the (transitive) right action of G on $H\backslash G$ satisfies $f_{H\backslash G}(n) \geq qf(n)$. In particular, the growth of such action is maximal.*

Consequently, for every non-elementary hyperbolic group G there exists a transitive action of G with maximal growth such that the orbit of action of any element $g \in G$ is finite.

Remark IV.1.11. Let G be an infinite group acting 2-transitively on some infinite set X.

(i) If $\mathscr{N}$ is a finite normal subgroup in G, then $\mathscr{N}$ is in the kernel of the action.

(ii) Assume that G is a non-elementary hyperbolic group and that the growth of action on X is maximal. Then the kernel of this action is the finite radical $E(G)$.

Proof (i) The action is 2-transitive and thus is imprimitive, i.e. there are no non-trivial block partitions (see [**Rot**], pp. 256-258). On the other hand, one easily checks that for any $x \in X$ the set $\mathscr{N}x$ is a block: if

$$g \in G \text{ and } g\mathscr{N}x \cap \mathscr{N}x \neq \emptyset,$$

then $\mathscr{N}(g\mathscr{N}x \cap \mathscr{N}x) = \mathscr{N}x$ and, because $\#\{g\mathscr{N}x\} = \#\{\mathscr{N}x\}$, we have that $g\mathscr{N}x = \mathscr{N}x$. Using the finiteness of $\mathscr{N}x$ again, we have that $\mathscr{N}x \neq X$ and thus $\mathscr{N}x = \{x\}$ for every x.

(ii) The kernel $\mathscr{N}$ of action is a normal subgroup, thus we use the condition that the growth of action is maximal and Proposition IV.1.8 to obtain that $\mathscr{N}$ is finite. By definition of finite radical, we have $\mathscr{N} \leq E(G)$. It remains to apply part (i).□

IV.2 Proof of Theorem IV.1.10

Throughout this paragraph we assume that the group G is non-elementary hyperbolic. The following Lemma summarizes some geometric properties that we will need later.

Lemma IV.2.1. *Let a,b,c,d be points in a δ-hyperbolic space X.*

(i)Assume that $(a,c)_b,(b,d)_c \leq M$. If we take Q such that $(a,d)_b \leq Q$ then $|a-d| \geq |a-b|+|b-c|+|c-d|-2M-2Q$. Moreover, if $|b-c| > 2M+\delta$, then we can choose $Q = M+\delta$.

Assume that the point d is on the segment $[a,b]$ and

(ii) $d \in B_{M_1}([a,c])$ for some $M_1 \geq 0$. Then

$$|d-b| \geq (a,c)_b - \delta - M_1 \text{ and} \tag{IV.4}$$

$$|a-c| \geq |a-b|+|b-c|-2\,|d-b|-2\delta-2M_1. \qquad \text{(IV.5)}$$

(iii) that $(b,c)_a-5\delta > |a-d|$. Then $d \in B_{4\delta}([a,c])$.
(iv) the vertex d is at least $D>0$ away from each point a,b. Then

$$|d-c| \leq max\{|a-c|\,,|b-c|\}+2\delta-D.$$

Proof (i) By definition of Gromov product and conditions of part (i), we get that

$$|a-d| = |a-b|+|b-d|-2(a,d)_b = |a-b|+(|b-c|+\ |c-d|-2(b,d)_c-2(a,d)_b) \geq$$

$$\geq |a-b|+|b-c|+|c-d|-2M-2Q.$$

It remains to show the second claim in part (i). If $|b-c|>2M+\delta$, then by (II.1):

$$(c,d)_b = |b-c|-(b,d)_c > 2M+\delta-M. \qquad \text{(IV.6)}$$

By inequality (IV.6) and definition (H1) of δ-hyperbolic space, we have

$$(c,d)_b > M+\delta \geq (a,c)_b+\delta \geq min\{(a,d)_b,(c,d)_b\},$$

which implies that $(a,d)_b \leq M+\delta$.

(ii) Let d' be a point on $[a,c]$ at distance at most M_1 from d. Then, by (H1) and definitions of d,d':

$$(d,c)_b = \frac{1}{2}(|b-d|+|b-c|-|c-d|) = \frac{1}{2}(|a-b|-|a-d|+|b-c|-|c-d|) =$$

$$= \frac{1}{2}(|a-b|+|b-c|-|a-c|)+\frac{1}{2}(|a-c|-|a-d|-|c-d|) \geq$$

$$\geq (a,c)_b+\frac{1}{2}(|a-c|-[\left|a-d'\right|+\left|d-d'\right|]-[\left|d-d'\right|+\left|c-d'\right|]) = (a,c)_b-\left|d-d'\right|.$$

We obtain the first claim of part (ii) using definition (H1) and the expression for $(d,c)_b$ above:

$$|d-b| = (d,a)_b \geq min\{(d,c)_b,(a,c)_b\}-\delta \geq (a,c)_b-\delta-\left|d-d'\right|,$$

and apply it to obtain the second claim:

$$|a-c| = |a-b|+|b-c|-2(a,c)_b \geq |a-b|+|b-c|-2(|d-b|+\left|d-d'\right|+\delta).$$

(iii) Assume $d \notin B_{4\delta}([a,c])$, then $d \in B_{4\delta}([b,c])$ by (H2). We apply part (ii) to the points b,a,c,d with $M_1 = 4\delta$ and obtain that $|a-d| \geq (b,c)_a-\delta-4\delta$. Contradiction.

(iv) By definition (H3), we have that

$$|d-c|+|b-a| \leq max\{|a-c|+|b-d|\,,|b-c|+|a-d|\}+2\delta, \text{ hence}$$

$$|d-c| \leq max\{|a-c|+(|b-d|-|b-a|), |b-c|+(|a-d|-|b-a|)\}+2\delta \leq$$

$$\leq max\{|a-c|, |b-c|\}+2\delta+max\{|b-d|-|b-a|, |a-d|-|b-a|\} \leq$$

$$\leq max\{|a-c|, |b-c|\}+2\delta-D.\square$$

Lemma IV.2.2. *Consider subgroups $\mathscr{H}_1$ and $\mathscr{H}_2$ in a finitely generated group G. If there exists $M \geq 0$ such that $\#\{B_M(\mathscr{H}_1) \cap B_M(\mathscr{H}_2)\} = \infty$ then $\#\{\mathscr{H}_1 \cap \mathscr{H}_2\} = \infty$.*

The Lemma is equivalent to the statement: if $\#\{\mathscr{H}_1 \cap \mathscr{H}_2\} < \infty$ then the set $B_M(\mathscr{H}_1) \cap B_M(\mathscr{H}_2)$ is finite for every non-negative M.

Proof Assume that $\#\{B_M(\mathscr{H}_1) \cap B_M(\mathscr{H}_2)\} = \infty$ for some $M \geq 0$, then there exist infinite sequences of elements $\{h_{1i}\} \subset \mathscr{H}_1$ and $\{h_{2i}\} \subset \mathscr{H}_2$ such that $\left|h_{1i}^{-1}h_{2i}\right| = |h_{1i}-h_{2i}| \leq 2M$ for every $i \in \mathbb{N}$. We denote the element $h_{1i}^{-1}h_{2i}$ by l_i. Since $|l_i| \leq 2M$ and the geometry of $Cay(G)$ is proper, there exists an element $l \in G$ and a subsequence $\{i_j\}$, $j \in \mathbb{N}$ such that $l_{i_{j_1}} = l_{i_{j_2}} = l$ in G for any $j_1, j_2 \in \mathbb{N}$ and thus $h_{1s}^{-1}h_{2s} = h_{1k}^{-1}h_{2k}$ in G for every $s, k \in \{i_j\}$. We obtained that $h_{1k}h_{1s}^{-1} = h_{2k}h_{2s}^{-1}$ belongs to $\mathscr{H}_1 \cap \mathscr{H}_2$ for every $s, k \in \{i_j\}$. For every fixed k, $lim_{s\to\infty}\left|h_{1k}h_{1s}^{-1}\right| \geq (lim_{s\to\infty}\left|h_{1s}^{-1}\right|) - |h_{1k}| = \infty$ which implies that the intersection $\mathscr{H}_1 \cap \mathscr{H}_2$ does not belong to a ball B_R for any $R \geq 0$ and thus is infinite. $\square$

Corollary IV.2.3. *Let $\mathscr{H}$ be a K-quasiconvex subgroup and assume that for infinitely many distinct natural numbers s_i and some $z_1, z_2 \in G$ the elements $z_1x^{s_i}z_2$ belong to $\mathscr{H}$. Then there exists $n \neq 0$ such that $z_1x^nz_1^{-1} \in \mathscr{H}$.*

Proof We denote $|z_1|+|z_2|$ by C. We have that $z_1x^{s_i}z_1^{-1}(z_1z_2) \in \mathscr{H}$ and hence the distance $d(z_1x^{s_i}z_1^{-1}, \mathscr{H})$ is bounded by C for infinitely many distinct s_i. In other words the intersection $\langle z_1xz_1^{-1}\rangle \cap B_C(\mathscr{H})$ is infinite and by Lemma IV.2.2 the intersection $\langle z_1xz_1^{-1}\rangle \cap \mathscr{H}$ is also infinite.$\square$

Lemma IV.2.4. *Let $\mathscr{H}$ be a K-quasiconvex subgroup in a δ-hyperbolic group G and let E be an infinite elementary subgroup in G. Then the following assertions are equivalent:*

(i) for any number $M > 0$ we have $E \not\subset B_M(\mathscr{H})$;

(ii)$\#\{E \cap \mathscr{H}\} < \infty$;

(iii) There exists $M > 0$ (depending on E and $\mathscr{H}$ only) such that $(x,h) < M$ for any $x \in E$, $h \in \mathscr{H}$.

Proof We first show that (ii) implies (i). If the intersection $E \cap \mathscr{H}$ is finite then, by Lemma IV.2.2, we have $\#\{B_M(E) \cap B_M(\mathscr{H})\} < \infty$ for any $M \geq 0$. In particular, $\#\{E \cap B_M(\mathscr{H})\} < \infty$ and hence $E \not\subset B_M(\mathscr{H})$ for any $M \geq 0$.

Now we show that (iii) implies (ii). Let x be an element of E and $|x| > 2M$. We have

$$|x-h| = |x|+|h|-2(x,h) \geq |x|-2M > 0,$$

hence $x \notin \mathscr{H}$ and so the intersection $E \cap \mathscr{H}$ belongs to the ball $B_{2M}(e)$ which is a finite set.

It remains to show that (i) implies (iii). Since E is infinite virtually cyclic we can choose an element x in E of infinite order and thus E is of finite index in $E(x)$. Hence

$$\text{there exists a constant } M_0 \text{ such that } E(x) \subset B_{M_0}(\langle x\rangle). \tag{IV.7}$$

Let $K_x = K(\langle x \rangle)$ be a constant provided by Lemma II.2.6(iii).

Assume (iii) does not hold, i.e. for every $M \geq 0$ there exist $y \in E$, $h \in \mathscr{H}$, satisfying $(y,h) > M + M_0$. Then, by (IV.7), $y = x^t a$ for some $a \in E, |a| \leq M_0$, $t \neq 0$ and

$$(x^t, h) \geq (x^t a, h) - M_0 > M + M_0 - M_0 = M. \tag{IV.8}$$

Hence for every $M > 0$ there exists an integer t and an element $h \in \mathscr{H}$ such that $(x^t, h) > M$.

Now we fix an arbitrary t and choose M so that $|x^t| < M - K_x - 5\delta$. We may assume without loss of generality that $t \geq 0$. Then by (IV.8), there exist $t' \geq t$ and $h \in \mathscr{H}$ such that $(x^{t'}, h) > M$. By Lemma II.2.6(ii), vertices x^m are within K_x-neighborhood of $[e, x^{t'}]$ for any $0 \leq m \leq t'$. In particular, there exists a vertex $b \in [e, x^{t'}]$ such that $|x^t - b| \leq K_x$ and thus $|b| \leq M - 5\delta < (x^{t'}, h) - 5\delta$. By Lemma IV.2.1(iii), we have that $b \in B_{4\delta}([e,h])$ and, because $\mathscr{H}$ is K-quasiconvex, $b \in B_{4\delta+K}(\mathscr{H})$. Finally, we get that x^t belongs to $B_{4\delta+K+K_x}(\mathscr{H})$ for every t contrary to (i). $\square$

In Lemma IV.2.5 and Theorem IV.2.8 we follow in part the line of argument from [**Arzh**] (in particular we apply Lemma 13[**Arzh**]).

Lemma IV.2.5. *Let x be an element of infinite order in G and choose a constant $M_1 \geq 0$. Then there exist a natural number m and a number $M_2 \geq 0$ such that for any element h in G satisfying conditions $|h| < 2M_1$ and $h \notin E(x)$ and any $|t|, |s| \geq m$ the following inequality holds:*

$$\left|x^t h x^s\right| \geq \left|x^t\right| + |h| + |x^s| - M_2.$$

Proof For a pair of integers s,t we consider a closed path $p_1 q_1 p_2 q_2$ in $Cay(G)$, where the path p_1 starts from e and $lab(p_1) = x^{-t}$, the path q_1 is geodesic and ends at vertex hx^s, the path p_2 satisfies $lab(p_2) = x^{-s}$ and q_2 is geodesic with $lab(q_2) = h^{-1}$. We define phase vertices a_i on p_1 and phase vertices b_j on p_2^{-1} relative to the natural factorizations x^{-t} and x^s respectively ($i = 0, \ldots, -t$ and $j = 0, \ldots, s$).

Step 1. We take constants $\lambda, c, K_x = K(\langle x \rangle)$ provided by Lemma II.2.6 for the cyclic group $\langle x \rangle$ and define

$$C = max\{2K_x + \frac{1}{2}|x|, K_x + 2M_1\} + 8\delta.$$

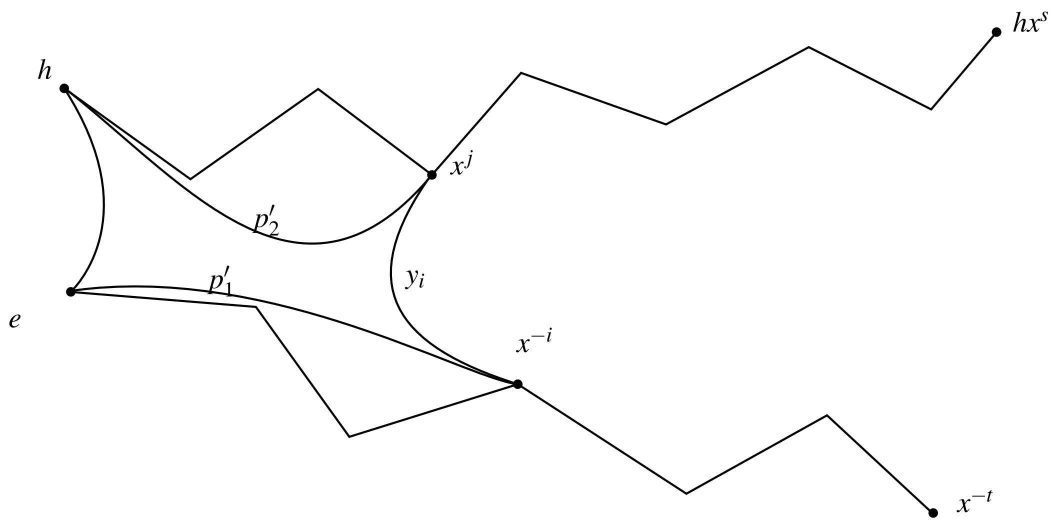
Let us denote by y_i a phase path connecting vertex a_i with some phase vertex b_j of p_2. Assume that $|y_i| \leq C$ for some i. We define subpaths p_1', p_2' of paths p_1, p_2, where the path p_1' connects a_0 to a_i and p_2' connects b_j to b_0. Considering the closed path $p_1' y_i p_2' q_2$, we have

$$|j|\,|x| \geq \left|h - hx^j\right| \geq \left|x^i\right| - |h| - |y_i| \geq \lambda\,|x|\,|i| - c - 2M_1 - C,$$

which implies that $|j| \geq \frac{\lambda |i| |x| - c - C - 2M_1}{|x|} \geq \lambda\,|i| - c_1$, where $c_1 = \frac{c + C + 2M_1}{|x|}$.

Since we have fixed the constant C, we may apply Lemma II.2.10 to the closed path $p_1' y_i p_2' q_2$ to obtain an integer m_0 such that if we choose a number i_0 satisfying $\lambda\,|i_0| - c_1 \geq m_0$ and hence $|i_0| \geq m_0$ then there exists a phase path $y_{i'}$, $|i'| \leq i_0$ such that $lab(y_{i'}) \in E(x)$. If the vertex $b_{j'}$ is the end vertex of $y_{i'}$, we get that $x^{-i'} lab(y_{i'}) x^{-j'} h = e$ in G and hence $h \in E(x)$, contradiction. We obtained that there exists i_0 depending on

Figure IV.1: Step I of IV.2.5



x and C, such that

$$|y_{i_0}| > C. \tag{IV.9}$$

Step 2. We show now that $a_{i_0} \in B_{8\delta+K_x}(q_1)$. By Lemma II.2.6, $a_i \in B_{K_x}([e, x^{-t}])$ and using twice the condition (H2), we get that a_{i_0} belongs to $B_{8\delta+K_x}(q_2 \cup [h, hx^s] \cup q_1)$.

Clearly $a_{i_0} \notin B_{8\delta+K_x}(q_2)$: since $|y_{i_0}|$ is minimal, i.e. $|y_{i_0}| \le \left|a_{i_0} - b_{j'}\right|$ for every $j = 0, \ldots, s$, we get for $j' = 0$ that

$$|y_{i_0}| \le |a_{i_0} - b_0| = |a_{i_0} - h| \le d(a_{i_0}, q_2) + |q_2| \le 8\delta + K_x + |h| < C$$

contrary to (IV.9).

Similarly, $a_{i_0} \notin B_{8\delta+K_x}([h, hx^s])$. Otherwise we may consider a vertex z on $[h, hx^s]$ at distance at most $8\delta + K_x$ from a_{i_0} and choose a vertex z' on p_2 at distance no more than K_x from z. Finally, there exists a phase vertex $b_{j'}$ on p_2 such that $\left|b_{j'} - z'\right| \le |x|/2$. Using the minimality of $|y_{i_0}|$ we obtain the estimate for the length of phase path:

$$|y_{i_0}| \le |a_{i_0} - z| + \left|z - z'\right| + \left|z' - b_{j'}\right| \le (K_x + 8\delta) + K_x + |x|/2 \le C,$$

which again contradicts (IV.9). The claim of Step 2 is proved.

Step 3. Let us choose some $|t|, |s| \ge |i_0|$. We choose z on $[e, x^{-t}]$ such that

$$|a_{i_0} - z| \le K_x. \tag{IV.10}$$

By Step 2, the vertex a_{i_0} is in the set $B_{8\delta+K_x}(q_1)$ and hence $z \in B_{8\delta+2K_x}(q_1)$. Applying Lemma IV.2.1 (ii) to vertices e, x^{-t}, hx^s, z we get using (IV.5), $|h| < 2M_1$:

$$|q_1| \ge \left|x^t\right| + |hx^s| - 2|z| - 2\delta - 2(8\delta + 2K_x) \ge \left|x^t\right| + (|h| + |x^s| - 4M_1) - 2|z| - 2\delta - 2(8\delta + 2K_x).$$

Inequality (IV.10) implies that $\left|x^{i_0}\right| + K_x \geq |z|$ and we conclude that

$$|q_1| \geq \left|x^t\right| + |h| + |x^s| - 2(\left|x^{i_0}\right| + K_x) + 2(9\delta + 2K_x) = \left|x^t\right| + |h| + |x^s| - (4M_1 + 2\left|x^{i_0}\right| + 6K_x + 18\delta).$$

It remains to define the constant M_2 (depending only on $\langle x\rangle$, M_1, $\mathscr{H}$) to be $4M_1 + 2\left|x^{i_0}\right| + 4K_x + 18\delta$ and define $m = |i_0|$. $\square$

Lemma IV.2.6. *([**Arzh**], Lemma 13) Let $n \geq 1$, $r_0 \geq 48\delta$ and elements $h_i, g_i \in G$ ($1 \leq i \leq n$) satisfy :*

$$|g_i| > 15r_0,\ (1 \leq i \leq n),\ |h_1 g_1| \geq |h_1| + |g_1| - r_0, \tag{IV.11}$$

$$|g_{i-1} h_i g_i| \geq |g_{i-1}| + |h_i| + |g_i| - 2r_0 (1 < i \leq n). \tag{IV.12}$$

Then the following assertions are true:

(i) One has

$$|h_1 g_1 ... h_n g_n| \geq |h_1 g_1 ... h_{n-1} g_{n-1}| + |h_n| + |g_n| - 5r_0.$$

In particular one has (by induction) $h_1 g_1 ... h_n g_n \neq e$ in G.

(ii) Let p be a path in $Cay(G)$ labeled by $h_1 g_1 ... h_n g_n$. Then the path p and any geodesic $[p_-, p_+]$ are contained within $4r_0$-neighborhood of each other.

For technical reasons we introduce a the following corollary (related to the Lemma quoted above).

Corollary IV.2.7. *Let $\mathscr{H}$ be a quasiconvex subgroup of infinite index and $r_0 \geq 0$ be a constant. Assume that for every $r \geq r_0$ there exists a finite set $\mathscr{F} = \mathscr{F}(r)$ of elements of G such that the number $\#\{\mathscr{F}\}$ is independent of r and:*

$$(A0) \quad \mathscr{F} \cap \mathscr{F}^{-1} = \emptyset.$$

Moreover, for every $f_1, f_2 \in \mathscr{F}^{\pm 1}$ and $h \in \mathscr{H}$:

$$(A1) \quad |f_1| > 15r;$$

$$(A2) \quad |hf_1| \geq |h| + |f_1| - r_0;$$

$$(A3) \quad |f_1 h f_2| \geq |f_1| + |h| + |f_2| - 2r_0,\ \textit{unless}\ h = e\ \textit{and}\ f_1 = f_2^{-1}.$$

Then (i) the conclusion of previous Lemma holds for every path labeled by $h_1 f_1 ... h_n f_n$ unless there exists i such that $h_i = e$ and $f_{i-1} = f_i^{-1}$, where $f_1, ..., f_n \in \mathscr{F}^{\pm 1}$;

(ii) the group $H_1 = \langle \mathscr{H}, \mathscr{F}\rangle$ is isomorphic to $\mathscr{H} * (*_{g\in\mathscr{F}} < g >)$ and is $(4\delta + 4r + max\{K, max_{g\in\mathscr{F}} |g|/2\})$-quasiconvex in G.

Proof (i) Follows immediately from the above Lemma IV.2.6.

(ii) If the equality $h_1 f_1 ... h_n f_n = e$ holds in G (where $f_i \in \mathscr{F}$, $n \geq i \geq 1$) then there exists an index $i \leq n$ such that $h_i = e$ for some and $f_{i-1} = f_i^{-1}$ and exactly one of the elements f_{i-1}, f_i^{-1} belongs to $\mathscr{F}$. Thus the group generated by $\mathscr{H}$ and $f_1, ..., f_n$ is isomorphic to $\mathscr{H} * \langle f_1\rangle * ... * \langle f_n\rangle$.

Consider an element $h = h_1 f_1 ... h_n f_n h_{n+1}$ in G, where $f_1, ..., f_n \in \mathscr{F}$ and if $h_i = e$ for $i \le n$ then $f_{i-1} f_i \neq e$ (h_{n+1} can be the identity e). Define a path pp' in $Cay(G)$ where the subpath p starts from e and is labelled by $h_1 f_1 ... h_n f_n$ in $Cay(G)$ and the path p' labeled by h_{n+1}. By (H2) and Lemma IV.2.6(ii), we have that $[e, p'_+] \subset B_{4\delta+4r_0}(p \cup p')$. In turn, every vertex v of $p \cup p'$ is either within K-neighborhood of $\mathscr{H}_1 = \langle \mathscr{H}, f_1, ..., f_n \rangle$ (if v is a vertex of a subpath labeled by h_i) or at most $max_{g \in \mathscr{F}(r)} |g|/2$ away from $\langle \mathscr{H}, f_1, ..., f_n \rangle$ (if v is a vertex of the subpath labeled by f_i). We conclude that every vertex $z \in [e, p'_+]$ is within $4\delta + 4r + max\{K, max\{K, max_{g \in \mathscr{F}(r)} |g|/2\}$ from a vertex in $\mathscr{H}_1$. Hence $\mathscr{H}_1$ is $(4\delta + 4r + max\{K, max\{K, max_{g \in \mathscr{F}(r)} |g|/2\})$-quasiconvex. $\Box$

Theorem IV.2.8. *Let G be a non-elementary hyperbolic group and $\mathscr{H}$ be a K-quasiconvex subgroup of G. Consider an element x in G of infinite order such that $E(x) \cap \mathscr{H} = \{e\}$. Then there exists a number r_0 (depending on $\mathscr{H}$ and x only) such that*

(i) $(x^s, h) < \frac{r_0}{2}$ for any $h \in H$ and $s \in \mathbb{Z}$ and

(ii) for any $r \ge r_0$ there exists $t' > 0$ such that for every $t \ge t'$ and $g = x^t$ the subgroup $\mathscr{H}_1 = \langle g, \mathscr{H} \rangle$ is $(4\delta + 4r + max\{K, |g|/2\})$-quasiconvex, of infinite index and canonically isomorphic to $\langle g \rangle * \mathscr{H}$. Moreover, the inequalities of Lemma IV.2.6 hold for r; $f_i \in \langle g \rangle \setminus \{e\}$ and $h_i \in \mathscr{H}$.

Proof By Lemma IV.2.4(iii), there exists $M > 0$ such that $(x^s, h) < M$ for any $h \in H$ and $s \in \mathbb{Z}$, hence

$$|hx^s| = |h| + |x^s| - 2(x^s, h) \ge |h| + |x^s| - 2M. \tag{IV.13}$$

Now we consider an arbitrary element $x^{m_1} h x^{m_2}$. If $|h| > 2M + \delta$, then apply Lemma IV.2.1(i) to vertices e, x^{m_1}, $x^{m_1} h$, $x^{m_1} h x^{m_2}$ in $Cay(G)$:

$$|x^{m_1} h x^{m_2}| \ge |x^{m_1}| + |h| + |x^{m_2}| - 4M - 2\delta. \tag{IV.14}$$

Lemma II.2.6(iv) provides a constant $M' \ge 0$ such that the following inequality holds provided $m_1 m_2 \ge 0$:

$$|x^{m_1} x^{m_2}| \ge |x^{m_1}| + |x^{m_2}| - 2M'. \tag{IV.15}$$

By Lemma IV.2.5, there exist a natural number m and a non-negative constant M_2 such that for any $|m_1|, |m_2| \ge m$ the following inequality holds:

$$|x^{m_1} h x^{m_2}| \ge |x^{m_1}| + |h| + |x^{m_2}| - M_2 \tag{IV.16}$$

for every $h \neq 1, |h| < 2M + 2\delta$. Now we choose

$$r_0 = max\{2M + \delta, M_2/2, 48\delta, M'\}, \tag{IV.17}$$

and then for any $r \ge r_0$ we choose t' satisfying $|t'| \ge m$ so that the inequality

$$\left|x^t\right| > 15r, \text{ holds for every } t, |t| \ge \left|t'\right|. \tag{IV.18}$$

Note that the choice of r_0 in (IV.17) and inequality (IV.13) proves part (i) of our Theorem.

We can observe now that the corollary IV.2.7 holds for $\mathscr{H}, r_0, r, \mathscr{F}(r) = \{x^t\}$. The condition $(A0)$ is immediate, condition $(A1)$ holds by the choice of t, $(A2)$ is satisfied by (IV.13) and $(A3)$ is justified because (IV.14)–(IV.16) hold.

Thus the group generated by $\mathscr{H}$ and g is isomorphic to $\mathscr{H} * \langle g \rangle$ and the group $\mathscr{H}_1 = \mathscr{H} * \langle g \rangle$ is $(4\delta + 4r + max\{K, |g|/2\})$-quasiconvex. All conclusions of the Theorem are checked for $\mathscr{H}_1$ except the infiniteness of index. It only remains to observe that the subgroup $\mathscr{H}_2 = \langle \mathscr{H}, g^2 \rangle$ has infinite index in $\langle \mathscr{H}, g \rangle$ and hence in G. It satisfies the all of the conditions of the Theorem and hence the conclusion holds for the same constant r and $g = x^{2t}$.□

Let us consider a path p in $Cay(G)$ starting at vertex a and ending with b with label $h_1 f_1 ... h_n f_n h_n$, i.e. $a h_1 f_1 ... h_n f_n h_n = b$ in G, where $f_i \in \mathscr{F}^{\pm 1}(r)$, $h_i \in \mathscr{H}$ and if $h_i = e$ for $1 < i \leq n$ then $f_{i-1} f_i \neq e$. We shall denote:

$$\begin{aligned} a_0 &= a, & b_1 &= a h_1, \\ a_i &= a h_1 ... h_i f_i \text{ for } 1 < i \leq n, & b_i &= a h_1 f_1 ... h_i, \text{ for } 1 < i \leq n. \end{aligned} \tag{IV.19}$$

Lemma IV.2.9. *Assume that $\mathscr{H}$, r_0 and a set $\mathscr{F}(r)$ (where $r \geq r_0$) satisfy the corollary IV.2.7. In the notations (IV.19) we have that $(a, b_{i+1})_{a_i} \leq \frac{r_0}{2} + \delta$ for any $i \geq 1$.*

Proof The definition (H3) for a, b_i, a_i, b_{i+1} reads:

$$|a_i - a| + |f_i h_{i+1}| \leq max\{|b_i - a| + |h_{i+1}|, |b_{i+1} - a| + |f_i|\} + 2\delta. \tag{IV.20}$$

The corollary IV.2.7 permits us to apply the inequalities of Lemma IV.2.6(i) and $(A2)$ to the left side of (IV.20):

$$|a_i - a| + |f_i h_{i+1}| \geq (|a_{i-1} - a| + |h_i| + |f_i| - 5r_0) + (|f_i| + |h_{i+1}| - r_0),$$

applying $(A1)$ we obtain

$$|a_i - a| + |f_i h_{i+1}| \geq (|a_{i-1} - a| + |h_i|) + |h_{i+1}| + (2|f_i| - 6r_0) > |b_i - a| + |h_{i+1}| + 24r_0.$$

By$(A2)$ and the conditions on r_0 in Lemma IV.2.6:

$$|a_i - a| + |f_i h_{i+1}| > |b_i - a| + |h_{i+1}| + 24r_0 > |b_i - a| + |h_{i+1}| + 2\delta. \tag{IV.21}$$

Hence we may rewrite (IV.21) as

$$|a_i - a| + |f_i h_{i+1}| \leq |b_{i+1} - a| + |f_i| + 2\delta$$

and thus (using $(A2)$ again):

$$|b_{i+1} - a| \geq |a_i - a| + |f_i h_{i+1}| - |f_i| - 2\delta \geq |a_i - a| + |f_i| + |h_{i+1}| - r_0 - |f_i| - 2\delta =$$

$= |a_i - a| + |h_{i+1}| - 2\delta - r_0$, which, by (H1), implies that $(a, b_{i+1})_{a_i} \leq \frac{r_0}{2} + \delta$.□

Lemma IV.2.10. *Assume that $\mathscr{H}$, r_0 and a set $\mathscr{F}(r)$ (where $r \geq r_0$) satisfy the corollary IV.2.7. In the conventions above (see (IV.19)), assume that*

(i) $a_i \in B_R$ for some $i > 1$. Then a_1 belongs to $B_{R+2\delta-2r}$.

(ii) $b_{i+1} \in B_R$ for some $i > 1$. Then a_1 belongs to B_R.

Proof (i) By Lemma IV.2.6(ii), there exists $b' \in [a, a_i]$ such that

$$\left|b' - a_1\right| \leq 4r_0. \tag{IV.22}$$

Using inequalities $(A2)$ and $(A1)$ of corollary IV.2.7, we have

$$\left|b' - a\right| \geq |a - a_1| - \left|b' - a_1\right| \geq |f_1| + |h_1| - r_0 - 4r_0 \geq 10r. \tag{IV.23}$$

Similarly, we may inductively apply Lemma IV.2.6(i) to the subpath of p connecting a_j, a_i for $j < i$:

$$\left|a_j - a_i\right| \geq \left|a_j - a_{i-1}\right| + |h_i| + |f_i| - 5r_0 > \left|a_j - a_{i-1}\right| + 15r - 5r_0 \geq 10r(i-j), \tag{IV.24}$$

and apply it in order to estimate (for $i > 1$):

$$\left|b' - a_i\right| \geq |a_1 - a_i| - \left|a_1 - b'\right| \geq 10r(i-1) - 4r_0 \geq 6r.$$

The inequalities (IV.23) and (IV.24) allow to apply Lemma IV.2.1(iv) to a, a_i, e, b' (with $D = 6r$) and get that $|b' - e| \leq max\{|a-e|, |a_i - e|\} + 2\delta - 6r$. Since $a, a_i \in B_R$ we have that $|b' - e| \leq R + 2\delta - 6r$. We use the previous inequality together with (IV.22) to conclude that: $|a_1 - e| \leq |a_1 - b'| + |b' - e| \leq 4r_0 + R - 6r + 2\delta \leq R + 2\delta - 2r$.

(ii) The inequality (IV.24) provides that $|a_i - a| \geq 10r > \frac{r_0}{2} + 6\delta + 1$; on the other hand Lemma IV.2.9 implies that $\frac{r_0}{2} + 6\delta + 1 \geq (a, b_{i+1})_{a_i} + 5\delta + 1$. Thus we can choose a vertex d on a geodesic $[a, a_i]$ satisfying inequalities:

$$(a, b_{i+1})_{a_i} + 5\delta + 1 \geq |d - a_i| \geq (a, b_{i+1})_{a_i} + 5\delta.$$

Then, by Lemma IV.2.1(iii), d belongs to $B_{4\delta}([a, b_{i+1}])$ and using Lemma IV.2.9 we get

$$d(a_i, [a, b_{i+1}]) \leq |d - b| + 4\delta \leq (a, b_{i+1})_{a_i} + 5\delta + 1 + 4\delta \leq \frac{r_0}{2} + 10\delta + 1.$$

By (H2), segment $[a, b_{i+1}]$ belongs to the 4δ-neighborhood of union $[e, a] \cup [e, b_{i+1}]$ which is a subset of B_R because $a, b_{i+1} \in B_R$. Hence

$$|a_i - e| \leq d(a_i, [a, b_{i+1}]) + 4\delta + R \leq R + \frac{r_0}{2} + 14\delta + 1$$

and, by part (i) of this Lemma, we conclude that a_1 belongs to $B_{R-r+16\delta+1} \subset B_R$.□

Lemma IV.2.11. *Let $\mathscr{H}$ be a K-quasiconvex subgroup in a hyperbolic group G. Take some $a \in B_R$ and $h \in \mathscr{H}$ such that $ah \notin B_R$. Then either $(a^{-1}, h) \leq 13\delta + K$ or there exists $b_1 \in a\mathscr{H} \cap B_R$ such that $b_1 h_1 = ah$ and $|h_1| < |h|$ for some $h_1 \in \mathscr{H}$.*

Proof Assume that $(a^{-1},h) > 13\delta + K$. We choose a vertex d on the segment $[a, ah]$ such that $|d-a| = K+8\delta$. By Lemma IV.2.1(iii), $d \in B_{4\delta}([e,a])$ and we can choose $d' \in [e,a]$ to satisfy the inequality $|d-d'| \le 4\delta$. Then we have

$$|d-e| \le \left|e-d'\right| + \left|d-d'\right| \le (|e-a| - \left|a-d'\right|) + 4\delta \le R - \left|a-d'\right| + 4\delta \le$$

$$\le R - |a-d| + 4\delta + 4\delta \le R - K.$$

By quasiconvexity of $\mathscr{H}$, there exists $b_1 \in a\mathscr{H}$, $|b_1 - d| \le K$ and hence $b_1 \in B_R$. By the choice of b_1, we have that $b_1^{-1}ah = h_1 \in \mathscr{H}$ and

$$|b_1 - ah| \le |b_1 - d| + |d - ah| \le |b_1 - d| + (|a-ah| - |d-a|) \le K + (|h| - K - 8\delta) < |h|.\square$$

Lemma IV.2.12. *Assume that $\mathscr{H}$, r_0 and a set $\mathscr{F}(r)$ (where $r \ge r_0$) satisfy the corollary IV.2.7. We adopt notations (IV.19) and let a, b be vertices in B_R and $ah_1gh_2 = b$ in G for some $h_1, h_2 \in \mathscr{H}$, $g \in \mathscr{F}^{\pm 1}(r)$. Assume furthermore that $(a^{-1}, h_1) \le 13\delta + K$ and that $b_1 \notin B_R$. Then*

$$|h_1| \le K + \frac{r_0}{2} + 15\delta.$$

Proof Definition (H1) and $(A2)$ yield:

$$\frac{r_0}{2} \ge (a_1, a)_{b_1} \ge min\{(a,b)_{b_1}, (a_1,b)_{b_1}\} - \delta \ge min\{(e,a)_{b_1}, (e,b)_{b_1}, (a_1,b)_{b_1}\} - 2\delta. \qquad \text{(IV.25)}$$

Consider the last two Gromov products on the right-hand side of (IV.25). We have:

$$(e,b)_{b_1} = \frac{1}{2}(|b_1| + |b-b_1| - |b|) = \frac{1}{2}(|b_1| - |b|) + \frac{1}{2}|b - b_1|,$$

by the conditions of this Lemma, $|b_1| > R \ge |b|$ and using $(A1)$ and $(A2)$ we conclude

$$(e,b)_{b_1} \ge 0 + \frac{1}{2}(|g| + |h_2| - r_0) \ge \frac{1}{2}|g| - \frac{r_0}{2} > 7r \ge 7r_0.$$

Similarly,

$$(a_1,b)_{b_1} = \frac{1}{2}(|g| + |b-b_1| - |h_2|) \ge \frac{1}{2}(|g| + (|g| + |h_2| - r_0) - |h_2|) \ge |g| - \frac{r_0}{2} \ge 14\frac{1}{2}r_0.$$

Now we may rewrite (IV.25) as $\frac{r_0}{2} \ge (e,a)_{b_1} - 2\delta$. Note that $(e,b_1)_a = (a^{-1},h_1) \le 13\delta + K$ and thus, by definition of the Gromov product (II.1):

$$|h_1| = (e,a)_{b_1} + (b_1,e)_a \le (\frac{r_0}{2} + 2\delta) + (K + 13\delta).\square$$

In the following Lemma, we denote $\langle \mathscr{H}, \mathscr{F}(r)\rangle$ by $\mathscr{H}_1$. In order to estimate the number of $\mathscr{H}_1$-cosets in the ball B_R from below, we define $M_R = \{a\mathscr{H} \mid a\mathscr{H} \cap B_R \ne \emptyset\}$ and $Q_R = \{a\mathscr{H} \in M_R | \exists b \in B_R,\ b\mathscr{H} \ne$

$a\mathscr{H}$ & $b\mathscr{H}_1 = a\mathscr{H}_1\}$.

Lemma IV.2.13. *Assume that $\mathscr{H}$, r_0 and a set $\mathscr{F}(r)$ (where $r \geq r_0$) satisfy the corollary IV.2.7. Then for any $k \in \mathbb{N}$ and all sufficiently large r the following inequality holds: $\frac{\#\{Q_R\}}{\#\{B_R\}} \leq \frac{1}{2^k}$ for any $R > 0$.*

Proof Recall that the number $\#\{\mathscr{F}\} = \#\{\mathscr{F}(r)\}$ is a constant independent of r. We use corollary IV.1.3 to choose c so that that $2^{k+1}\#\{\mathscr{F}\}(\#\{B_{K+r_0+15\delta}\}^2\#\{B_{R-c}\}) \leq \#\{B_R\}$ and r so that:

$$r \geq max\{c/7, 2(K+\frac{r_0}{2}+17\delta)\}. \tag{IV.26}$$

Let $a\mathscr{H}$ belong to Q_R, then there exist $b \in B_R$, $b \notin a\mathscr{H}$, and elements $h_i \in \mathscr{H}$ $(i = 1,...,k)$ such that $ah_1f_1...f_kh_{k+1} = b$ in G, where $f_i \in \mathscr{F}^{\pm 1}$. By Lemma IV.2.10, we have that either $b_1 = ah_1$ or $a_1 = ah_1f_1$ or $ah_1f_1h_2$ belongs to B_R. Hence we can assume that $b = ah_1gh_2$, where $h_1, h_2 \in \mathscr{H}$, $g \in \mathscr{F}^{\pm 1}$. Clearly $a\mathscr{H} \neq b\mathscr{H}$, otherwise $ah_1gh_2 = ah$ and $g \in \mathscr{H}$, contradiction.

We may also assume that a, h_1 are chosen so that $|h_1|$ is minimal with respect to all factorizations $a'h' = ah_1$ in G where $a' \in a\mathscr{H} \cap B_R$. Similarly, we may assume that b, h_2 are chosen so that for any $b' \in b\mathscr{H} \cap B_R$ and $h' \in \mathscr{H}$ the equality $b'h'^{-1} = bh_2^{-1}$ implies that $|h'| \geq |h|$. According to the choice we made, if $h_1 \neq e$ ($h_2 \neq e$) then $b_1 = ah_1 \notin B_R$ (respectively $a_1 = ah_1g \notin B_R$). Now we are in position to apply Lemma IV.2.11 to the pairs a, ah_1 and b, bh_2^{-1}, which provides that $(a^{-1}, h_1) \leq K + 13\delta$, $(b^{-1}, h_2^{-1}) \leq K + 13\delta$, and then, by Lemma IV.2.12, we conclude that

$$|h_i| \leq K + \frac{r_0}{2} + 15\delta, \text{ for } i = 1, 2. \tag{IV.27}$$

If $h_i = e$ for $i = 1$ or 2 then the corresponding inequality in (IV.27) holds trivially.

We have that $b_1, a_1 \in B_{R+K+\frac{r_0}{2}+15\delta}$. Since $|g| > 15r$, we can fix a factorization $g = g_1g_2$ in G for every $g \in \mathscr{F}$ such that $|g_1| + |g_2| = |g|$ and $|g_1|, |g_2| \geq \frac{15r}{2}$. Let $b' = ah_1g_1$ if $g \in \mathscr{F}$ and $b' = ah_1g_2^{-1}$ if $g^{-1} \in \mathscr{F}$, we will call b' a middle point of the path p starting at a with label h_1gh_2.

Applying Lemma IV.2.1(iv) to vertices b_1, a_1, b', we obtain that $|b' - e| \leq (R + K + \frac{r_0}{2} + 15\delta) + 2\delta - \frac{15r}{2}$. As we choose r according to (IV.26) we obtain:

$$b' \in B_{R-7r}. \tag{IV.28}$$

We have obtained that if a coset $a'\mathscr{H}$ belongs to Q_R then there exist $a \in a'\mathscr{H} \cap B_R$, $b \in B_R$, $h_1, h_2 \in \mathscr{H}$ and $g \in \mathscr{F}^{\pm 1}$ such that the equation $ah_1gh_2 = b$ holds in G together with conditions (IV.27) and (IV.28). Hence the number of elements in Q_R is not greater then the number of paths with label h_1gh_2 in $Cay(G)$ such that the middle point b' of each path satisfies (IV.28):

$$\#\{Q_R\} \leq \#\{\text{of } h_1gh_2 \text{ satisfying (IV.27)}\} \times \#\{B_{R-7r}\} \leq 2\#\{\mathscr{F}\}\#\{B_{K+\frac{r_0}{2}+15\delta}\}^2 \times \#\{B_{R-7r}\}. \tag{IV.29}$$

Due to our choice of r in (IV.26) we finally get

$$\#\{Q_R\} \leq 2\#\{\mathscr{F}\}\#\{B_{K+\frac{r_0}{2}+15\delta}\}^2\#\{B_{R-7r}\} \leq \frac{1}{2^k}\#\{B_R\}. \square$$

Corollary IV.2.14. *Let $\mathscr{H}$ be a free K-quasiconvex subgroup in G. Then for any $k \in \mathbb{N}$ and any $x \in G$ of infinite order either:*

(i) there exists $t \neq 0$ such that $x^t \in \mathscr{H}$, or

(ii)there exists t such that for $g = x^t$ the group $\langle \mathscr{H}, g \rangle$ is quasiconvex and canonically isomorphic to the free product $\mathscr{H} * \langle g \rangle$. Moreover $\frac{\#\{Q_R\}}{\#\{B_R\}} \leq \frac{1}{2^k}$ for any $R > 0$.

Proof Assume that (i) does not hold, so $x^t \in \mathscr{H}$ implies $t = 0$. By Lemma IV.2.2, we have that for every $M \geq 0$ the number of vertices in $B_M(\langle x \rangle) \cap B_M(\mathscr{H})$ is finite. There exists $M_0 \geq 0$ such that $E(x)$ is in M_0-neighborhood of $\langle x \rangle$, hence $B_M(E(x)) \cap B_M(\mathscr{H}) \subset B_{M+M_0}(\langle x \rangle) \cap B_{M+M_0}(\mathscr{H})$ and hence $\#\{B_M(E(x)) \cap B_M(\mathscr{H})\}$ is finite thus $\#\{E(x) \cap \mathscr{H}\} < \infty$. Since $\mathscr{H}$ is free, the last inequality means that $E(x) \cap \mathscr{H} = \{e\}$. It remains to define $\mathscr{F}(r) = \{x^t\}$ so that $|x^t| \geq 15r$ and apply the previous Lemma.□

Remark IV.2.15. (i)Let $\mathscr{H}$ be an infinite quasiconvex subgroup of G of infinite index. Then there exists an element $x \in G$ of infinite order such that it is non-commensurable with any element of $\mathscr{H}$.

(ii)Part (i) implies that no infinite index subgroup satisfying the Burnside condition in a non-elementary hyperbolic group is quasiconvex.

Proof (i) Follows from A. Minasyan's Lemma II.2.4 for $K = G$. □

Theorem IV.2.16. *For every non-elementary δ-hyperbolic group G and any $0 < q < 1$ there exists a free subgroup H satisfying the Burnside condition and such that $\frac{\#\{aH|\ aH \cap B_R \neq \emptyset\}}{\#\{B_R\}} \geq q$.*

Proof We choose a sequence $\{k_i\}_{i \in \mathbb{N}}$ such that

$$\Sigma_{i=1}^{\infty} \frac{1}{2^{k_i}} < 1 - q. \tag{IV.30}$$

Let $\{x_j\}$, $j \in \mathbb{N}$ be a list of all elements of infinite order in G. We fix notations $\mathscr{H}_i = \langle x_1^{t_1}, ..., x_i^{t_i} \rangle$ for some positive numbers $t_i \in \mathbb{N}$ which we will determine later. We define $H = \cup_{i=1}^{\infty} \mathscr{H}_i$, it clearly satisfies the Burnside condition. Then we denote $M_R^i = \{a\mathscr{H}_i |\ a\mathscr{H}_i \cap B_R \neq \emptyset\}$ and $Q_R^i = \{a\mathscr{H}_i |\ \exists b \in B_R$ such that $a\mathscr{H}_i \neq b\mathscr{H}_i$ & $a\mathscr{H}_{i+1} = b\mathscr{H}_{i+1}\}$.

We set $\mathscr{H}_0 = \{e\}$ and thus $M_R^0 = B_R$. Lemma IV.2.14 (applied to $\mathscr{H}_0$, x_1 and k_1) provides that there exists $t_1 > 0$ such that $\mathscr{H}_1 = \langle f_1 \rangle$ (where $f_1 = x_1^{t_1}$) is cyclic, quasiconvex and $\frac{\#\{Q_R^0\}}{\#\{B_R\}} \leq \frac{1}{2^{k_1}}$ for any $R > 0$. It provides the following estimate for M_R^1:

$$\#\{M_R^1\} \geq \#\{M_R^0\} - \#\{Q_R^0\} \geq (1 - \frac{1}{2^{k_1}})\#\{B_R\}.$$

Now we assume by induction that a free quasiconvex subgroup $\mathscr{H}_i = \langle x_1^{t_1}, ..., x_i^{t_i} \rangle$ has been constructed by repeated application of Lemma IV.2.14 and M_R^i satisfies inequality

$$\#\{M_R^i\} \geq (1 - \frac{1}{2^{k_1}} - \frac{1}{2^{k_2}} - ... - \frac{1}{2^{k_i}})\#\{B_R\}. \tag{IV.31}$$

If $\langle x_{i+1} \rangle \subset B_M(\mathscr{H}_i)$ for some $M \geq 0$ then, by Lemma IV.2.4, there exists $t_{i+1} > 0$ such that $x_{i+1}^{t_{i+1}} \in \mathscr{H}_i$ and we can set $\mathscr{H}_{i+1} = \mathscr{H}_i$, finishing the induction step ($M_R^{i+1} = M_R^i$).

Assume now that $\langle x_{i+1}\rangle \not\subset B_M(\mathscr{H}_i)$ for any non-negative M, then, by Lemma IV.2.4, we have $\#\{E(x)\cap \mathscr{H}_i\} < \infty$ and hence (because $\mathscr{H}_i$ is free) $E(x)\cap\mathscr{H}_i = \{e\}$. We choose t_{i+1} appplying Lemma IV.2.14 to $\mathscr{H}_i, x_{i+1}, k_{i+1}$ and using the induction assumption (IV.31):

$$\#\{M_R^{i+1}\} \geq \#\{M_R^i\} - \#\{Q_R^i\} \geq (1 - \frac{1}{2^{k_1}} - \frac{1}{2^{k_2}} - \dots - \frac{1}{2^{k_i}})\#\{B_R\} - \#\{Q_R^i\} \geq$$

$$\geq (1 - \frac{1}{2^{k_1}} - \frac{1}{2^{k_2}} - \dots - \frac{1}{2^{k_{i+1}}})\#\{B_R\},$$

and, by (IV.50):

$$\#\{M_R\} \geq (1 - \Sigma_{i=1}^{\infty}\frac{1}{2^{k_i}})\#\{B_R\} > q\#\{B_R\}.\square$$

Proof of Theorem IV.1.10 By the remark IV.1.7, the number of left cosets intersecting B_n is equal to the number $f_{H\backslash G}(n)$ of right ones. We can now fix some $0 < q < 1$ and using Theorem IV.2.16 find a group H such that $f_{H\backslash G}(n) \geq qf(n)$. Thus, by remark IV.1.4, the growth of action of G on $H\backslash G$ is maximal. $\square$

IV.3 Proof of Theorem I.1.8 and corollary I.1.9

The proof of the claim that the element g is G-suitable (i.e. $E(g) = \langle g\rangle \times E(G)$) in the Lemma below is similar to that in [**Olsh93**] and to that in Lemma 3.8 in [**Min2005**], which we will apply. However, we write up the proof completely because the the entire statement does not follow from these considerations.

Lemma IV.3.1. *Let G be a non-elementary hyperbolic group and $\mathscr{H}$ be a quasiconvex subgroup of infinite index in G. Assume there exists an element $y \in \mathscr{H}$ such that $E(y) = \langle y'\rangle \times E(G)$ for some $y' \in G$ and take an element $x \in G$ (it exists by remark IV.2.15) of infinite order which is non-commensurable with any element in $\mathscr{H}$. Then*

(i) there exist $k_1, k_2 > 0$ such that for every $s > 0$ the element $g = y^{k_1 s}x^{k_2 s}$ is G-suitable and non-commensurable with any element of $\mathscr{H}$;

(ii) for every $C_0 \geq 0$ there exists $s_0 > 0$ such that for every $s \geq s_0$, every $a, b \in B_{C_0}$ and every $t \neq 0$:

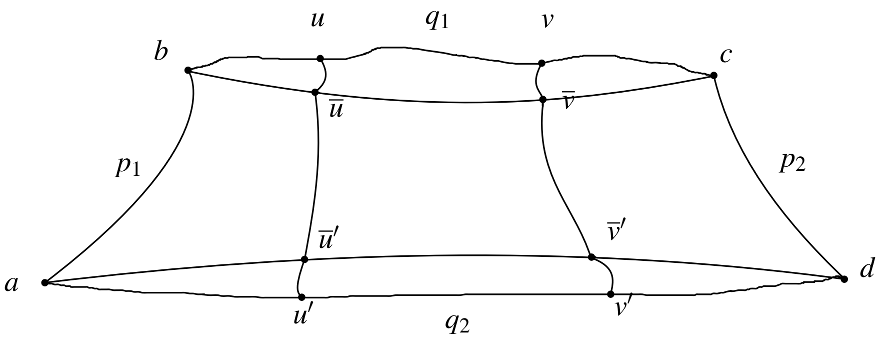
$$a(y^{k_1 s}x^{k_2 s})^t b \notin \mathscr{H}.$$

Proof (i) We define a subgroup $\mathscr{H}_1 = \langle y\rangle$, which is quasiconvex by Lemma II.2.6(ii). Since $\mathscr{H}_1 \cap E(x) = \{e\}$, there exists a constant $r_0 \geq 0$ such that $(x^t, y^s) < \frac{r_0}{2}$ for all $t, s \in \mathbb{Z}$ by Theorem IV.2.8. By part (ii) of the same Theorem and $r = r_0$, there exists $t' > 0$ such that $\langle x^{t'}, y\rangle \cong \langle x^{t'}\rangle * \langle y\rangle$. We denote $x_1 = x^{t'}$ so the subgroup $\langle x_1, y\rangle$ is free quasiconvex and inequalities of Lemma IV.2.6 hold for $\mathscr{H}_1, x_1$ and $r = r_0$. In particular, for every reduced word w in $\langle x_1, y\rangle$, the corresponding path in $Cay(G)$ with label w is within $4r_0$-neighborhood of a geodesic connecting its ends.

By Lemma IV.2.4, there exists $M \geq 0$ such that $(x^s, h) < M$ for every $s \in \mathbb{Z}$, $h \in \mathscr{H}$. Choose $t > 0$ such that $|x_1^s| > 4r_0 + K + 2M$ for every $|s| \geq t$ and denote $x_2 = x_1^t$. We have that for any non-zero m:

$$d(x_2^m, h) \geq |x_2^m| + |h| - 2M \geq |x_2^m| - 2M > 4r_0 + K. \qquad \text{(IV.32)}$$

Figure IV.2: Lemma IV.3.1



Let element $w = y^{s_0}x_2^{t_1}y^{s_1}...x_2^{t_n}y^{s_n}$ satisfy $s_i, t_i, t_n \neq 0$ for $i = 1, ..., n-1$ and assume that $w \in \mathscr{H}$. Then every phase vertex of w is in $(4r_0 + K)$-neighborhood of $\mathscr{H}$, which contradicts inequality (IV.32) because $d(y^{s_0}x_2^{t_1}, y^{s_0}) = d(x_2^{t_1}, e) > 4r_0 + K$. We conclude that an element w of the free group $\mathscr{H}_2 = \langle x_2, y\rangle$ is commensurable with an element of $\mathscr{H}$ if and only if $w = y^s$ for some integer s. We have proved the first part of (i): for sufficiently large k_1, k_2 and every $s \neq 0$ the element $y^{k_1 s}x^{k_2 s}$ is not commensurable with any element of $\mathscr{H}$.

Now we consider an element $y^k x_2^k$ for sufficiently large k and will show that the group $E(y^k x_2^k)$ is equal to $\langle y^k x_2^k\rangle \times E(G)$. Let z be an element of $E(y^k x_2^k)$, i.e. the equality $z(y^k x_2^k)^m z^{-1} = (y^k x_2^k)^{m'}$ holds in G for some $m = \pm m' \neq 0$. We choose a constant

$$M_0 > 2|z| + 8r_0 + 26\delta + (3k+1)(max\{|y|, |x_2|\})$$

and a natural number s divisible by m such that $\left|(y^k x_2^k)^s\right| \geq M_0$. We consider a closed path $p_1 q_1 p_2 q_2$ in $Cay(G)$ such that $lab(p_1) = lab(p_2) = z$, $lab(q_1) = (y^k x_2^k)^s$, and $lab(q_2^{-1}) = (y^k x_2^k)^{s'}$, where $s' = \pm s$. For convenience we denote the initial vertices of p_1, q_1, p_2, q_2 by a, b, c, d respectively. We choose a vertex $\overline{u}$ on $[b, c]$ at distance $|z| + 5\delta$ in $Cay(G)$ from the vertex b. Then, using (H2), $\overline{u}$ is in 4δ-neighborhood of some u_1 on $[a, c] \cup [a, b]$ and, by the choice of $\overline{u}$, is actually on $[a, c]$. Using (H2) again, and taking into account that

$$|\overline{u_1} - c| \geq |b - c| - |z| - 5\delta - 4\delta \geq M_0 - |z| - 9\delta > |z| + 5\delta$$

we obtain that there exists $\overline{u}'$ on $[a, d]$ satisfying $|\overline{u}' - u_1| \leq 4\delta$. Hence

$$\left|\overline{u}' - \overline{u}\right| \leq 8\delta. \tag{IV.33}$$

Similarly we can choose $\overline{v}$ on $[b, c]$ at distance $|z| + 5\delta$ from the vertex c and a vertex $\overline{v}'$ on $[a, d]$ such that

$$\left|\overline{v}' - \overline{v}\right| \leq 8\delta. \tag{IV.34}$$

Since q_1 and $[b, c]$ are within $4r_0$-neighborhood of each other, we find phase vertices u, v on q_1 relative to the factorization $y^k x_2^k ... y^k x_2^k$ of $lab(q_1)$ such that

$$|u-\overline{u}|, |v-\overline{v}| \leq 4r_0 + \frac{1}{2}max\{|y|, |x_2|\}. \tag{IV.35}$$

Similarly we find phase vertices u', v' on q_2 such that

$$|u'-\overline{u}'|, |v'-\overline{v}'| \leq 4r_0 + \frac{1}{2}max\{|y|, |x_2|\}. \tag{IV.36}$$

Now we consider a closed path $p_1' q_1' p_2' q_2'$, where q_1', q_2' are subpaths of q_1 and q_2 respectively and $p_1' = [u', u]$, $p_2' = [v', v]$. According to inequalities (IV.33)–(IV.36) above:

$$|p_i| \leq 8r_0 + max\{|y|, |x_2|\} + 8\delta, \text{ where } i = 1, 2. \tag{IV.37}$$

Note that

$$|q_1'| = |u-v| \geq |\overline{u}-\overline{v}| - |u-\overline{u}| - |v-\overline{v}| \geq |q_1| - |\overline{u}-c| - |\overline{v}-c| - |u-\overline{u}| - |v-\overline{v}|,$$

and using the definitions of $\overline{u}, \overline{v}$ and (IV.35) we get:

$$|q_1'| \geq M_0 - 2|z| - 10\delta - 8r_0 - max\{|y|, |x_2|\} > 3k\, max\{|y|, |x_2|\}. \tag{IV.38}$$

We consider $q_1' = t_1 ... t_n$, where either $lab(t_{2i-1}) = y^k$, $lab(t_{2i}) = x_2^k$ for every $1 < 2i \leq n-1$ or $lab(t_{2i-1}) = x_2^k$, $lab(t_{2i}) = y^k$ for every $1 < 2i \leq n-1$. By the estimate on q_1' above, we have that $n \geq 4$.

Now we use (IV.33), (IV.34) and (IV.38) to obtain:

$$|q_2'| = |u'-v'| \geq |\overline{u}'-\overline{v}'| - |u'-\overline{u}| - |v'-\overline{v}| \geq |\overline{u}-\overline{v}| - |\overline{u}'-\overline{u}| - |\overline{v}'-\overline{v}| - |u'-\overline{u}| - |v'-\overline{v}|$$

$$\geq M_0 - 16\delta - 2|z| - 10\delta - 8r_0 - max\{|y|, |x_2|\} > 3k\, max\{|y|, |x_2|\}.$$

We consider $(q_2')^{-1} = t_1' ... t_{n'}'$, where $lab(t_{2i}) = y^{k'}, lab(t_{2i+1}) = x_2^{k'}$ for every $1 < 2i < n'-1$ or $lab(t_{2i}) = x_2^{k'}, lab(t_{2i+1}) = y^{k'}$ for every $1 < 2i < |n'|-1$ where $k = \pm k'$. By the estimate on q_1' above, $n' \geq 4$.

We can now apply Lemma II.2.10 to the closed path $p_1' q_1' p_2' q_2'$ with upper bound on $|p_i'|$ provided by (IV.37) and obtain a constant m_0 such that for every $k \geq m_0$ the paths t_2 and t_3 are compatible with t_i' and t_{i+1}' respectively (for some unique i). Let us denote for convenience $lab(t_2) = W_2^k$, $lab(t_3) = W_3^k$, $lab(t_i') = \overline{W}_i^{k'}$, $lab(t_3) = \overline{W}_{i+1}^{k'}$, where the sets $\{W_2, W_3\}$, $\{\overline{W}_i, \overline{W}_{i+1}\}$ and $\{x, y\}$ are all equal. Lemma II.2.10 also provides that there exist compatibility paths v_2 and v_3 with labels V_2, V_3 such that:

$$V_2^{-1} W_2^r V_2 = \overline{W}_i^s,\ V_3^{-1} W_3^{r'} V_3 = \overline{W}_{i+1}^{s'}, \tag{IV.39}$$

for some $r, s, r', s' > 0$. Because x_2 and y are non-commensurable, the equalities (IV.39) are only possible if $W_2 \equiv \overline{W}_i^{\pm 1}$ and $W_3 \equiv \overline{W}_{i+1}^{\pm 1}$. Moreover, one of the exponents is positive because y is not conjugate with y^{-1} and thus $lab(q_2)^{-1} = (y^k x_2^k)^m$ and $W_2 \equiv \overline{W}_i$ and $W_3 \equiv \overline{W}_{i+1}$. Now by definition of compatible paths, we have that $V_2 \in E(W_2)$ and $V_3 \in E(W_3)$. Consider a path v connecting the terminal vertex of t_2 with the terminal vertex of t_i'. We also consider a pair of paths $\overline{q_1}v_2\overline{q_2}$ and $\overline{q_3}v_3\overline{q_4}$ each of which has the same initial and terminal vertices as the path v. Reading off their labels provides the following inequalities in G:

$$lab(v) = W_2^{s_1}V_2W_2^{s_2} = W_3^{s_3}V_3W_3^{s_4}$$

for some exponents $s_i \in \mathbb{Z}$. Hence $lab(v) \in E(W_2) \cap E(W_3) = E(x_2) \cap E(y) = E(G)$. We obtained that $z = lab(p_1)$ is equal to either $(y^kx_2^k)^{s'}lab(v)^{-1}(y^kx_2^k)^{-s''}$ or $(y^kx_2^k)^{s'}y^klab(v)^{-1}((y^kx_2^k)^{s''}y^k)^{-1} = (y^kx_2^k)^{s'}lab(v)^{-1}(y^kx_2^k)^{-s''}$ for some non-negative numbers s', s''. In both cases $z \in E(G) \times \langle (y^kx_2^k) \rangle$. We computed the maximal elementary subgroup $E(y^kx_2^k)$ for all sufficiently large k: $E(y^kx_2^k) = E(G) \times \langle (y^kx_2^k) \rangle$. Since x_2 is a power of x we conclude that there are $k_1 = k$ and $k_2 > 0$ and divisible by k such that for every $s \neq 0$ we have $E(y^{k_1s}x_2^{k_2s}) = E(G) \times \langle (y^{k_1s}x_2^{k_2s}) \rangle$. We have shown the second part of (i).

(ii) By the way of contradiction we assume that the inclusions $a_i(y^{k_1s_i}x^{k_2s_i})^{t_i}b_i = h_i \in \mathscr{H}$ hold for infinitely many distinct numbers $s_i \in \mathbb{N}$, some $t_i > 0$ and $a_i, b_i \in B_{C_0}$. Because B_{C_0} is finite there are $a, b \in B_{C_0}$ such that the equality $a(y^{k_1s_i}x^{k_2s_i})^{t_i}b = h_i$ holds for infinitely many natural numbers s_i and $t_i \neq 0$.

By the choice of x_1 in part (i) of the Lemma, the path q_i with label $(y^{k_1s_i}x^{k_2s_i})^{t_i}$ starting at a is within $4r_0$-neighborhood of the geodesic $[a, a(y^{k_1s_i}x^{k_2s_i})^{t_i}]$ which in turn is in the $C_0 + 8\delta$-neighborhood of the geodesic $q_i' = [e, a(y^{k_1s_i}x^{k_2s_i})^{t_i}b]$. Hence vertices $ay^{k_1s_i}$ and $ay^{k_1s_i}x^{k_2s_i}$ are within $4r_0 + C_0 + 8\delta$-neighborhood of q_i'. Because $\mathscr{H}$ is K-quasiconvex and $lab(q_i') = h_i \in \mathscr{H}$ we have

$$d(ay^{k_1s_i}, \mathscr{H}), d(ay^{k_1s_i}x^{k_2s_i}, \mathscr{H}) \leq 4r_0 + C_0 + 8\delta + K,$$

hence a subpath $\bar{q}_i$ of q_i with label $x^{k_2s_i}$ is within $C_1 = 4r_0 + C_0 + 8\delta + K$-neighborhood of $\mathscr{H}$ in $Cay(G)$. Thus there exist elements $z_{1i}, z_{2i} \in B_{C_1}$ such that $z_{1i}x^{k_2s_i}z_{2i} \in \mathscr{H}$ for infinitely many distinct natural numbers s_i. Because the balls in $Cay(G)$ are finite, there are $z_1, z_2 \in B_{C_1}$ such that $z_1x^{k_2s_i}z_2 \in \mathscr{H}$ hold for infinitely many s_i.

It follows from remark IV.2.3 that $z_1x^nz_1^{-1} \in \mathscr{H}$ for some $n \neq 0$. Contradiction with the choice of x in (i).□

Lemma IV.3.2. *Let G be a non-elementary hyperbolic group and $\mathscr{H}$ be a quasiconvex subgroup of infinite index in G such that $E(G) \cap \mathscr{H} = \{e\}$, then there exist a G-suitable element $g \in G$ and a number $t_0 > 0$ such that for every $t \geq t_0$ the group $\langle \mathscr{H}, g^t \rangle$ is quasiconvex of infinite index and is canonically isomorphic to $\mathscr{H} * \langle g^t \rangle$.*

Proof By Lemma II.2.7, there exists a G-suitable element $y \in G$. We have either

(I) $\#\{E(y) \cap \mathscr{H}\} < \infty$,

or (II) $\#\{E(y) \cap \mathscr{H}\}$ is infinite.

Take an element $y^ka \in E(g)$ and assume $y^ka \in \mathscr{H}$ for some $k \in \mathbb{Z}$ and $a \in E(G)$, then we have $(y^ka)^n = y^{kn}a^n$ for every n (because a commutes with y). Note that for a non-zero k the equality $y^{kn_1}a^{n_1} = y^{kn_2}a^{n_2}$ holds in G if and only if $n_1 = n_2$.

Hence in case (I) we have that $k = 0$ and thus $E(y) \cap \mathscr{H} \subset E(G) \cap \mathscr{H} = \{e\}$ and we may apply Theorem IV.2.8 to find $t > 0$ and obtain the canonical isomorphism $\langle \mathscr{H}, y^t \rangle \cong \mathscr{H} * \langle y^t \rangle$.

Thus we only need to consider case (II) when $y^k a \in \mathscr{H}$ for some non-zero k. Replacing y with $y^k a$ we may assume that y is in $\mathscr{H}$. By remark IV.2.15, there exists an element x of infinite order such that x is non-commensurable with any element of $\mathscr{H}$. Replacing x with its non-zero power if necessary, we may assume that x commutes with $E(G)$.

We can choose a (sufficiently large) k so that the conclusion of the Lemma IV.3.1 holds for $\mathscr{H}$ and $g = y^k x_2^k$. We have that $E(g) \cap \mathscr{H} = \{e\}$ for $g = y^k x_2^k$, and now the result follows from Theorem IV.2.8.□

Proof of Theorem I.1.8 (i) The sufficiency is provided by Theorem IV.2.8. Assume that there exists an element x of infinite order and $t \neq 0$ such that $\langle \mathscr{H}, x^t \rangle \cong \mathscr{H} * \langle x^t \rangle$. Take $h \in E(x) \cap \mathscr{H}$, then there exist $n \neq 0$ and $n' = \pm n$ such that $h^{-1} x^n h x^{n'} = e$ in G. Thus $h^{-1} x^{tn} h x^{tn'} = e$ in G which imlpies $h = e$.

(ii) The sufficiency follows from Lemma IV.3.2. To show the necessity it is enough to notice that if the element x satisfying part (i) exists then $E(G) \cap \mathscr{H} \leq E(x) \cap \mathscr{H} = \{e\}$.

(iii) Denote the subgroup $E(G) \cap (\mathscr{H} * \langle x^t \rangle)$ by K, it is a finite subgroup in $\mathscr{H} * \langle x^t \rangle$. By Kurosh subgroup Theorem, K is conjugate to a subgroup in $\mathscr{H}$. On the other hand K is normal in $\mathscr{H} * \langle x^t \rangle$ and thus $K < \mathscr{H}$, i.e. $E(G) \cap \mathscr{H} * \langle x^t \rangle \leq E(G) \cap \mathscr{H} = \{e\}$□.

Proof of corollary I.1.9

Consider a canonical homomorphism $\phi : G \to \overline{G} = G/E(G)$. It is clear that $E(\overline{G}) = \{e\}$: the subgroup $E(\overline{G})$ is finite normal, hence the subgroup $\phi^{-1}(E(\overline{G}))$ is finite normal and thus $\phi^{-1}(E(\overline{G})) \leq E(G)$. Homomorphism ϕ is a quasi-isometry because $E(G)$ is finite. Thus $\overline{\mathscr{H}} = \phi(\mathscr{H})$ is quasiconvex of infinite index in $\overline{G}$. We can apply Lemma IV.3.2 to $\overline{\mathscr{H}}, \overline{G}$ and find some $\bar{y}$ such that $E(\bar{y})$ is infinite cyclic and obtain the isomorphism $\langle \mathscr{H}, y \rangle \cong \overline{\mathscr{H}} * \langle \bar{y} \rangle$. Consider some preimage y of $\bar{y}$. Clearly,

$$\phi^{-1}(\langle \overline{\mathscr{H}}, \bar{y} \rangle) = \langle \mathscr{H} \cdot E(G), y \rangle \cong \mathscr{H} \cdot E(G) *_{E(G)} \langle y, E(G) \rangle.\square$$

IV.4 Maximal growth highly transitive actions of hyperbolic groups

Consider a pair of ordered sets $\mathscr{U} = \{u_1, ..., u_k\}$ and $\mathscr{V} = \{v_1, ..., v_k\}$ such that $u_i \mathscr{H} = u_j \mathscr{H}$ if and only if $i = j$ and $v_i \mathscr{H} = v_j \mathscr{H}$ if and only if $i = j$. We take a constant C_0 such that $\mathscr{U}, \mathscr{V} \subset B_{C_0}$. Having fixed $\mathscr{U}, \mathscr{V}$, some auxiliary element $g \in G$ and a natural number s_0, we fix some $t \neq 0$ and denote $f_i = v_i^{-1} g^{s_0 t} u_i$ for $i = 1, ..., k$, $u_i \in \mathscr{U}, v_i \in \mathscr{V}$.

Lemma IV.4.1. *Let $\mathscr{H}$ be a K-quasiconvex subgroup of infinite index in a hyperbolic group G with the trivial finite radical $E(G) = \{e\}$. For any $k > 0$ and any pair of ordered sets $\mathscr{U}, \mathscr{V}$ there exist an element $g \in G$,a number $r_0 \geq 0$, a natural number s_0 such that for any $r \geq r_0$ there exist a natural number t and the set $\mathscr{F}(r) = \{f_1, ..., f_k\}$ satisfying the conditions $(A0)$–$(A3)$ and hence corollary IV.2.7.*

Proof We apply Lemma IV.3.2 to $\mathscr{H}$: there exists an element y such that $E(y)$ is infinite cyclic and the group $\mathscr{H}_1 = \langle \mathscr{H}, y \rangle$ is quasiconvex of infinite index and is isomorphic to $\mathscr{H} * \langle y \rangle$ in G. Then we take an element x in G which is non-commensurable with any element of $\mathscr{H}_1$ (see remark IV.2.15). By Lemma IV.3.1, for sufficiently large s there exists an element $g = y^{k_1 s} x^{k_2 s}$ which is non-commensurable with elements of $\mathscr{H}$, $E(g) = \langle g \rangle$ and such that

$$\text{for every } t_1 \neq 0 \text{ and all } a, b \in B_{C_0} \text{ we have } a g^{t_1} b \notin \mathscr{H}. \qquad \text{(IV.40)}$$

We will define r_0, s_0 and t so that the system $\mathscr{F}(r)$ satisfies the conditions $(A0)-(A3)$.

Take an element $a \in B_{C_0}$. Element g is non-commensurable with any element of $\mathscr{H}$ and hence $ag^m a^{-1} \notin \mathscr{H}$ for any $m \neq 0$. We obtained that $\mathscr{H} \cap E(aga^{-1}) = \mathscr{H} \cap \langle aga^{-1} \rangle = \{e\}$ and thus we may apply Theorem IV.2.8 and get $r_0(a) \geq 0$ and $s_0(a) > 0$ such that there exist numbers $r_0(a) \geq 0$ and $s_0(a) \geq 0$:

$$\left| h(aga^{-1})^{s_0(a)\cdot t} \right| \geq |h| + \left| (aga^{-1})^{s_0(a)\cdot t} \right| - r_0(a) \text{ for any } t \in \mathbb{Z}. \tag{IV.41}$$

We now fix constants $r_1 \geq r_0(a)$ for every $a \in B_{C_0}$ and $s_1 = \prod_{a \in B_{C_0}} s_0(a)$. Hence, for every $b \in B_{C_0}$ and every $t \in \mathbb{Z}$ we have:

$$\left| hag^{s_1 t} b \right| \geq \left| hag^{s_1 t} a^{-1} \right| - 2C_0 \geq |h| + \left| ag^{s_1 t} a^{-1} \right| - 2C_0 - r_1 \geq |h| + \left| ag^{s_1 t} b \right| - 4C_0 - r_1.$$

One obtains that the following inequality holds:

$$\left| hag^{s_1 t} b \right| \geq |h| + \left| ag^{s_1 t} b \right| - 4C_0 - r_1. \tag{IV.42}$$

Now notice that if s_1 divides s_0 and $f_i = v_i^{-1} g^{s_0 t} u_i$ then for $\varepsilon \in \{\pm 1\}$, $h \in \mathscr{H}$ and arbitrary $t \neq 0$ we have that

$$(f_i^{\varepsilon}, h) \leq r_1/2 + 2C_0. \tag{IV.43}$$

We consider the expression $f_i^{\varepsilon_1} h f_j^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.

Case I. Assume that $|h| \geq 2r_1 + 4C_0 + \delta$. Then we have, by Lemma IV.2.1(i):

$$\left| f_i^{\varepsilon_1} h f_j^{\varepsilon_2} \right| \geq \left| f_i^{\varepsilon_1} \right| + |h| + \left| f_j^{\varepsilon_2} \right| - 2(r_1/2 + 2C_0) - 2(r_1/2 + 2C_0 + \delta). \tag{IV.44}$$

Case II. Now let $|h| < r_1 + 4C_0 + \delta$. By definition of elements f_i, f_j, we can rewrite $f_i^{\varepsilon_1} h f_j^{\varepsilon_2}$ as $\beta_1 g^{s_0 t \varepsilon_1} \alpha_1 h \alpha_2 g^{s_0 t \varepsilon_2} \beta_2$ for some $\alpha_1, \beta_1 \in \{u_i^{\pm 1}, v_i^{\pm 1}\}$ and $\alpha_2, \beta_2 \in \{u_j^{\pm 1}, v_j^{\pm 1}\}$. Using our assumption on h we get that $|\alpha_1 h \alpha_2| \leq r_1 + 6C_0 + \delta$.

Assume that $\alpha_1 h \alpha_2 \notin E(g)$. Then by Lemma IV.2.5, there exist M_1 and a natural number m_0 (both depending on $r_1 + 6C_0 + \delta$, $\mathscr{H}$ and g only) such that if $s_0 \geq m_0$ then for all $t \in \mathbb{Z}$:

$$\left| g^{s_0 t \varepsilon_1} \alpha_1 h \alpha_2 g^{s_0 t \varepsilon_2} \right| \geq \left| g^{s_0 t \varepsilon_1} \right| + |\alpha_1 h \alpha_2| + \left| g^{s_0 t \varepsilon_2} \right| - M_1. \tag{IV.45}$$

Thus

$$\left| f_i^{\varepsilon_1} h f_j^{\varepsilon_2} \right| \geq \left| g^{s_0 t \varepsilon_1} \alpha_1 h \alpha_2 g^{s_0 t \varepsilon_2} \right| - 2C_0 \geq \left| g^{s_0 t \varepsilon_1} \right| + |\alpha_1 h \alpha_2| + \left| g^{s_0 t \varepsilon_2} \right| - M_1 - 2C_0 \geq$$

$$\geq \left| f_i^{\varepsilon_1} \right| + |h| + \left| f_j^{\varepsilon_2} \right| - 8C_0 - M_1;$$

and we get the estimate:

$$\left| f_i^{\varepsilon_1} h f_j^{\varepsilon_2} \right| \geq \left| f_i^{\varepsilon_1} \right| + |h| + \left| f_j^{\varepsilon_2} \right| - 8C_0 - M_1. \tag{IV.46}$$

It remains to consider the subcase when $\alpha_1 h \alpha_2 \in E(g) = \langle g \rangle$. By the choice of element g, the equality

$\alpha_1^{-1} g^{t_0} \alpha_2^{-1} = h \in \mathscr{H}$ may hold if and only if $t_0 = 0$ and thus

$$\alpha_1 h \alpha_2 = e \text{ in } G. \tag{IV.47}$$

If $\varepsilon_1 \varepsilon_2 = 1$ we choose a constant $M(g)$ according to Lemma II.2.6(iv) and obtain that

$$\left| f_i^{\varepsilon_1} h f_j^{\varepsilon_2} \right| = \left| \beta_1 g^{2s_0t} \beta_2 \right| \geq \left| g^{2s_0t} \right| - 2C_0 \geq \left| g^{s_0t} \right| + \left| g^{s_0t} \right| - M(g) - 2C_0;$$

hence:

$$\left| f_i^{\varepsilon_1} h f_j^{\varepsilon_2} \right| \geq \left| f_i^{\varepsilon_1} \right| + |h| + \left| f_j^{\varepsilon_2} \right| - 8C_0 - M_1. \tag{IV.48}$$

If $\varepsilon_1 \varepsilon_2 = -1$, then the expression $f_i^{\varepsilon_1} h f_j^{\varepsilon_2}$ is either $v_i^{-1} g^{s_1t} u_i h u_j^{-1} g^{-s_1t} v_j$ or $u_i^{-1} g^{-s_1t} v_i h v_j^{-1} g^{s_1t} u_j$ and thus $\alpha_1 = u_i$ and $\alpha_2 = u_j^{-1}$ or $\alpha_1 = v_i$ and $\alpha_2 = v_j^{-1}$. From equality (IV.47) we get that $u_i h = u_j$ or, respectively, $v_i h = v_j$. Hence $i = j$ and thus $f_i = f_j$, $h = e$ and the expression $f_i^{\varepsilon_1} h f_j^{\varepsilon_2}$ is trivial.

We can now choose the constant $r_0 = max\{r_1 + \delta, \frac{M_1}{2}, \frac{M(g)}{2}\} + 4C_0$. Then choose a natural number s_0 so that it is divisible by s_1 and satisfy $s_0 \geq m_0$.

Finally, we choose the number t so that the condition $(A1)$ is satisfied for $\mathscr{F}(r)$.

The condition $(A0)$ holds because if $f_i = f_j^{-1}$ then $g^{s_0t} u_j v_i^{-1} g^{s_0t} = v_j u_i^{-1}$. It is a contradiction with the length estimate of the left-hand side of the last equality by means of (IV.45):

$$\left| g^{s_0t} u_j v_i^{-1} g^{s_0t} \right| \geq \left| g^{s_0t} \right| + \left| u_j v_i^{-1} \right| + \left| g^{s_0t} \right| - M_1 \geq 30 r_0 + 2C_0 - M_1 > 2C_0.$$

The condition $(A2)$ holds because the formula (IV.42) holds for all $a, b \in B_{C_0}$. The condition $(A3)$ holds by formula (IV.44) if $|h| \geq 2r_1 + 4C_0 + \delta$, and by (IV.46) and (IV.48) otherwise.□

Theorem IV.4.2. *Let G be a non-elementary hyperbolic group with trivial finite radical $E(G) = \{e\}$ and let $\mathscr{H}$ be a quasiconvex subgroup $\mathscr{H}$ of infinite index in G. If $\{u_1\mathscr{H}, ..., u_k\mathscr{H}\}$ and $\{v_1\mathscr{H}, ..., v_k\mathscr{H}\}$ is a pair of ordered k-tuples of pairwise distinct cosets, then for every $l \in \mathbb{N}$ there exists a quasiconvex subgroup of infinite index $\mathscr{H}_1$ such that:*

(i) $\mathscr{H}_1 \geq \mathscr{H}$ and $\mathscr{H}_1$ is of infinite index in G;

(ii) there exists an element g_1 such that $g_1 u_i \mathscr{H}_1 = v_i \mathscr{H}_1$ for every $i = 1, ..., k$;

(iii) $\frac{\#\{Q_R\}}{\#\{B_R\}} \leq \frac{1}{2^l}$ for any $R > 0$, where $Q_R = \{a\mathscr{H} |\ \exists b \in B_R$ such that $a\mathscr{H} \neq b\mathscr{H}$ & $a\mathscr{H}_1 = b\mathscr{H}_1\}$.

Proof Because the index of $\mathscr{H}$ is infinite in G we can choose some auxiliary elements u_0, v_0 such that $u_0\mathscr{H} \neq u_i\mathscr{H}$, $v_0\mathscr{H} \neq v_i\mathscr{H}$ for every $i = 1, ..., k$. We apply Lemma IV.4.1 to $\mathscr{U} = \{u_0, ..., u_k\}$, $\mathscr{V} = \{v_0, ..., v_k\}$. There exist an element $g \in G$, number $r_0 \geq 0$, number s_0 such that for any $r \geq r_0$ there exist $t = t(r)$ and the set $\mathscr{F}_0(r) = \{v_0^{-1} g^{s_0t} u_0, ..., v_k^{-1} g^{s_0t} u_k\}$ and hence (by corollary IV.2.7) the group $\mathscr{H}_0$ is quasiconvex and the following isomorphism is canonical:

$$\mathscr{H}_0 \equiv \mathscr{H} * (*_{g \in \mathscr{F}_0(r)} < g >). \tag{IV.49}$$

Consider a group $\mathscr{H}_1 \leq \mathscr{H}_0$ defined by $\mathscr{H}_1 = \langle \mathscr{H}, \mathscr{F}(r) \rangle$, where $\mathscr{F}(r) = \{v_1^{-1} g^{s_0t} u_1, ..., v_k^{-1} g^{s_0t} u_k\}$. Because of canonical isomorphism (IV.49), subgroup $\mathscr{H}_1$ is of infinite index in $\mathscr{H}_0$ and hence in G. Conditions

$(A0)-(A3)$ hold for $\mathscr{F}(r)$ because $\mathscr{F}(r)\subset\mathscr{F}_0(r)$. Thus, by corollary IV.2.7, $\mathscr{H}_1$ is quasiconvex of infinite index and (i) is checked for all sufficiently large *r*.

The following computation shows that the condition (ii) holds for $g_1 = g^{s_0t}$:

$$v_i\mathscr{H}_1 = v_i(v_i^{-1}g^{s_0t}u_i\mathscr{H}_1) = g^{s_0t}u_i\mathscr{H}_1$$

To conclude the proof we observe that Lemma IV.2.13 holds for all sufficiently large *r*, showing (iii). □

We are ready to generalize the following statements of Bahturin and Ol'shanskii.

Corollary IV.4.3. *[**BO**] Any finitely generated subgroup H of infinite index in the free group F of rank $r>1$, is contained as a free factor in a subgroup K of infinite index in F such that the right action of F on F/K is highly transitive. In particular, K is a maximal subgroup in F. One can choose K in such a way that the growth of the action of F on F/K is maximal.*

Corollary IV.4.4. *[**BO**] Any finitely generated subgroup H of infinite index in a free group F of rank $r>1$, is a free factor in a Burnside subgroup K of infinite index such that the action of F on F/K is highly transitive. One can choose K so that the growth of the action of F on F/K is maximal.*

It is a natural next step to generalize these corollaries to the class of hyperbolic groups. In particular, this question was asked by Z. Sela after the presentation of [**BO**] by A. Olshanskiy ("Models and Groups", Istambul 2009).

By remark IV.1.7, the growth functions of action of left action of G on G/H is equal to the growth function of right action of G on $H\backslash G$. By Proposition IV.1.8, for every quasiconvex subgroup $\mathscr{H}$ of infinite index there exists $c_1>0$ such that

$$\#\{M_R^0\}\geq c_1\#\{B_R\},\ \text{for all } R\geq 0,$$

where $M_R^0=\{a\mathscr{H} \mid a\mathscr{H}\cap B_R\neq\emptyset\}$.

Proposition IV.4.5. *Let G be a non-elementary δ-hyperbolic group with trivial finite radical $E(G)=\{e\}$ and let $\mathscr{H}$ be a quasiconvex subgroup $\mathscr{H}$ of infinite index in G. Pick a number $0<q<1$. There exists a subgroup $H\geq\mathscr{H}$ such that the natural left action of G on G/H is faithful, highly transitive and* $\frac{\#\{aH \mid aH\cap B_R\neq\emptyset\}}{\#\{B_R\}}\geq c_1 q$.

The proof is analogous to that in IV.2.16.

Proof We choose a sequence $\{l_i\}_{i\in\mathbb{N}}$ such that

$$\Sigma_{i=1}^{\infty}\frac{1}{2^{l_i}} < c_1(1-q). \tag{IV.50}$$

We will define an increasing sequence of quasiconvex subgroups $\mathscr{H}_0=\mathscr{H}\leq\mathscr{H}_1\leq\ldots$ and set $H=\cup_{i=1}^{\infty}\mathscr{H}_i$.

As in Theorem IV.2.16, we denote $M_R^i = \{a\mathscr{H}_i |\ a\mathscr{H}_i \cap B_R \neq \emptyset\}$ and $Q_R^i = \{a\mathscr{H}_i |\ \exists b \in B_R$ such that $a\mathscr{H}_i \neq b\mathscr{H}_i\ \&\ a\mathscr{H}_{i+1} = b\mathscr{H}_{i+1}\}$. Hence we have:

$$\#\{M_R^{i+1}\} \geq \#\{M_R^i\} - \#\{Q_R^i\}.$$

Assume that $\mathscr{H}_i$ has been defined and

$$\#\{M_R^i\} \geq (c_1 - \frac{1}{2^{l_1}} - \frac{1}{2^{l_2}} - ... - \frac{1}{2^{l_i}})\#\{B_R\}. \tag{IV.51}$$

Note that the inequality above holds trivially for $i = 0$. If $u_m\mathscr{H}_i = u_n\mathscr{H}_i$ (or $v_m\mathscr{H}_i = v_n\mathscr{H}_i$) for $1 \leq m < n \leq k$, then we set $\mathscr{H}_i = \mathscr{H}_{i+1}$ and the equation IV.52 holds trivially. If $u_m\mathscr{H}_i \neq u_n\mathscr{H}_i$ and $v_m\mathscr{H}_i \neq v_n\mathscr{H}_i$ for every m,n such that $1 \leq m < n \leq k$, then, by Theorem IV.4.2, there exist a quasiconvex subgroup $\mathscr{H}_{i+1}$ of infinite index and an element g_i such that $\mathscr{H}_{i+1} \geq \mathscr{H}_i$, for every $m = 1,...,k$ the equality $g_iu_m\mathscr{H}_{i+1} = v_m\mathscr{H}_{i+1}$ holds and $\frac{\#\{Q_R^i\}}{\#\{B_R\}} \leq \frac{1}{2^{l_i}}$ for any $R > 0$. Hence we have:

$$\#\{M_R^{i+1}\} \geq \#\{M_R^i\} - \#\{Q_R^i\} \geq (c_1 - \frac{1}{2^{l_1}} - \frac{1}{2^{l_2}} - ... - \frac{1}{2^{l_{i+1}}})\#\{B_R\} \tag{IV.52}$$

and thus

$$\frac{\#\{aH|\ aH \cap B_R \neq \emptyset\}}{\#\{B_R\}} \geq lim_{i\to\infty}\#\{M_R^{i+1}\} \geq (c_1 - \Sigma_{i=1}^{\infty}\frac{1}{2^{l_i}})\#\{B_R\} > qc_1\#\{B_R\}.$$

It is clear that the action is highly transitive. Choose a pair of k-tuples $(u_1H,...,u_kH)$ and $(v_1H,...,v_kH)$ such that $u_mH \neq u_nH$, $v_mH \neq v_nH$ for all $1 \leq m < n \leq k$. The k-tuples $(u_1,...,u_k)$ and $(v_1,...,v_k)$ were enumerated by some integer i and, because $\mathscr{H}_i \leq H$, we have $u_m\mathscr{H}_i \neq u_n\mathscr{H}_i$ and $v_m\mathscr{H}_i \neq v_n\mathscr{H}_i$. By the choice of $\mathscr{H}_{i+1}$, there exists g_i such that $g_iu_m\mathscr{H}_{i+1} = v_m\mathscr{H}_{i+1}$ holds for every $m \leq k$ and thus $g_iu_mH = v_mH$, as desired. □

The following Theorem can now be deduced from Proposition IV.4.5.

Theorem IV.4.6. *Let G be a non-elementary δ-hyperbolic group and $\mathscr{H}$ be a quasiconvex subgroup of infinite index in G. Then there exists a subgroup H containing $\mathscr{H}$ such that the natural right action of G on $H\backslash G$ is highly transitive and has maximal growth. Moreover, the kernel of any action of maximal growth is equal to the finite radical $E(G)$.*

Proof We consider the quotient $\phi : \overline{G} \to G/E(G)$ and the image $\overline{\mathscr{H}}$ of the subgroup $\mathscr{H}$ under this quotient. The corollary above implies that there exists a subgroup $\overline{H}$ of $\overline{G}$ such that the action of $\overline{G}$ on $\overline{H}\backslash\overline{G}$ is faithful, highly transitive and has maximal growth. One can define the subgroup $H = \phi^{-1}(\overline{H})$ in G and the action of G on $H\backslash G$. □

Corollary IV.4.7. *Let G be a non-elementary δ-hyperbolic group with trivial finite radical $E(G) = \{e\}$. For any number q, $0 < q < 1$ there exists a subgroup H such that:*

(i) H satisfies the Burnside condition;

(ii) the natural left action of G on G/H is faithful, highly transitive and for every $R \geq 0$:
$\frac{\#\{aH|\ aH\cap B_R \neq \emptyset\}}{\#\{B_R\}} \geq q$.

(iii) the subgroup H is a free and maximal subgroup in G.

The Theorem I.1.11 follows from the above corollary in the same way as did Theorem IV.4.6 from Proposition IV.4.5.

The **Proof** of (i)-(ii) is similar to that in IV.2.16 and IV.4.5, we only need to alternate applications of Theorem IV.4.2 and corollary IV.2.14. The action is faithful by remark IV.1.11(ii): it is of maximal growth and $E(G) = \{e\}$.

(iii) Because the stabilizer of any point of a 2-transitive group action is a maximal subgroup in the group G (see [**Rot**], pp. 256-258), we have that H is maximal. When constructing H, we start from a trivial subgroup, hence on every step of the inductive construction we have that the quasiconvex subgroup $\mathscr{H}_i$ is free. □

The following corollary follows immediately from that above.

Corollary IV.4.8. *Let G be a non-elementary δ-hyperbolic group with trivial finite radical $E(G) = \{e\}$. Then G has a free maximal subgroup (of infinite index).*

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