TOPICS IN THE DESIGN OF AN AUCTION FOR LICENSES

By

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To my parents

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CHAPTER I

INTRODUCTION

Auctions have recently been used to sell a variety of items such as electricity, electromagnetic spectrum, treasury bills, and drilling rights. They have also been used to sell patents. In my dissertation, I consider a situation wherein a research lab has developed a process innovation and wants to sell licenses to members of an industry because the lab is not in a position to produce and sell the good that the process innovation concerns. This sale generates revenue for the lab and also influences firms in the industry. The literature has mostly considered the sale of exclusive licenses. I demonstrate, using both theoretical and experimental methods, that when firms sell differentiated products, there is a strong case for selling non-exclusive licenses. I also analyze the effect of product market characteristics on the optimal number of licenses.

How to Commercialize Technology Using Auctions

This essay considers the auction of licenses to a cost-reducing technology. Firms differ in their ability to reduce their marginal costs by using this technology and this ability is a firm's private information. In an auction of licenses, the acquisition of a license imposes a negative externality on the other firms. I determine the optimal number of licenses to sell when each firm is allowed to bid for one license. I derive the value of winning a license and how this value varies with the number of licenses auctioned. I then show that, given any fixed number of licenses, several well-known auctions are revenue-equivalent. The auction revenue, however, changes with the number of licenses and the lab can maximize its revenue by optimally choosing the number of licenses. I show that it may be optimal to auction multiple licenses when the level of product differentiation among the firms in the industry is high. Moreover, in the presence of negative externalities, the optimal reserve price can be zero.

A Laboratory Test of an Auction with Negative Externalities

We employ laboratory methods to test a model of bidding in an auction with externalities in which a firm can bid for one license for a cost-reducing technology. Since the winners impose a negative externality on the losers, bids must account for both the value of winning and the negative value of losing. Experimental treatments differ in terms of the severity of the negative externality (based on the substitutability of competitors' products), and the number of licenses being auctioned. We find that subjects underbid relative to theoretical benchmarks for auctions of one license, but overbid when two licenses are auctioned. In terms of mean revenues, the experimental revenues are consistent with the predicted revenues. However, there are some differences in the distribution of the experimental revenues and the predicted revenues and we propose a possible explanation rooted in a simple bidding heuristic for the difference.

When Should an Inventor Sell Licenses Exclusively? The Determinants of the Optimal Number of Licensees

A framework is proposed to analyze the sale of multiple licenses to use a cost-reducing technology. Firms differ in their ability to reduce their costs by purchasing a license. A purchaser imposes a negative externality on others. The payoff of each firm depends on the number and abilities of the licensees. The seller maximizes her revenue by optimally choosing the licensees. The optimal mechanism is determined both when each firm's ability to reduce its cost is publicly observable and when it is not. I also show the role of several product market factors in the determination of the optimal number of licensees.

CHAPTER II

HOW TO COMMERCIALIZE TECHNOLOGY USING AUCTIONS

The last century has witnessed the emergence of several new technologies. On many occasions, independent research labs (which includes university research groups) have been at the forefront of the development of new technologies. These labs often license their inventions to firms who can make use of the technology. According to Antonio Regaldo writing in The Wall Street Journal (June 18, 2004), "IBM already takes in more than $1 billion a year in licensing revenue from its vast portfolio of patents...". Given the fact that considerable uncertainties exist about the value of the licenses, a natural way to sell these licenses is by auction.[1] There are two issues related to the auction of a license. First, in an auction of licenses, the cost of making an additional blueprint of the technology is virtually cost less for the lab. Therefore, when designing the auction, the lab should take into account the optimal number of licenses to sell. The existing literature (for example, Jehiel and Moldovanu 2000 and Das Varma 2003) has mostly considered the sale of only one license. In this article, we extend the literature by letting the lab sell multiple licenses. Second, there is the issue of how much will a firm pay to the lab to ensure exclusivity. This question has been addressed in a subsequent paper of mine entitled "*A Framework for Analyzing a Seller's Revenue from the Sale of Multiple Licenses in the Presence of Negative Externalities.*" [2] In this article, we assume that each firm can bid for only one license. The presence of externalities reduces the willingness to pay of a potential buyer when the lab increases the number of licenses. However this may be offset by the fact that the lab extracts payments from more buyers when it increases the number of licenses. We analyze factors that determine which of the

[1]There are many recent instances of organizations selling intellectual property using auctions. For example, Tony Kontzer reports in Information Week (December 13, 2004) that in December 2004, Commerce One Inc. sold 39 patents using an auction.

[2]The paper is available online at http://www.people.vanderbilt.edu/~aniruddha.bagchi/chapter2.pdf

above two effects dominate when the lab increases the number of licenses it sells.[3]

The technological innovations that we consider are process innovations that reduce the costs of firms who have access to the new technology. The magnitude of these cost savings is firm specific.[4] Furthermore, we assume that the marginal cost of a firm that acquires a license is private information. We consider a number of auction formats and their corresponding direct mechanisms for allocating these licenses. We investigate the properties of the equilibria of these mechanisms both when the number of licenses is predetermined and when the number of licenses is chosen optimally to maximize the lab's revenue. In this model, a firm that wins a license imposes a negative externality on the other firms. Moreover, the magnitude of the externality may depend on the characteristics of the firm that wins a license, such as its marginal cost of production (the firm's "type"). We assume that during the auction, each firm knows its own type. Furthermore, it also knows that the types of its competitors are independently drawn from a distribution (which is common knowledge). Therefore, in our model, we have *interdependent values but independent types.*

In order to analyze the equilibrium of a license auction, we need to determine the value of a license to each type of firm. We show that the appropriate way to define the value of a license when multiple licenses are being allocated is a generalization of the *intrinsic value* proposed by Das Varma (2003) for the sale of one license. The intrinsic value of a license to a firm with marginal cost c is the value to this firm of winning a license rather than losing it to a competitor with the same marginal cost. The intrinsic value plays a similar role in our analysis as the exogenously determined value of an object does in the Milgrom and Weber (1982) framework.[5] By substituting our intrinsic value for the exogenously specified value in Milgrom and Weber's model, we can exploit their analysis when types are independent to derive some of our results.

[3]See the discussion preceding Proposition 2.4.

[4]This is because the ability of a firm to exploit a technology depends on many firm-specific characteristics that determine the quality of the match between the firm and the technology.

[5]The value of an object in Milgrom and Weber (1982) is the difference between a firm's payoff from winning the object instead of losing it.

We begin by considering the case in which there is no reserve price and the number of licenses is predetermined to be an integer less than the number of firms. We construct a direct mechanism that replicates the outcome of some well-known auction forms such as the uniform price auction (the multi-unit generalization of a second price auction) and the discriminatory auction (the pay-your-bid auction). In the symmetric equilibrium of an auction of k licenses that uses any of the above-mentioned formats, the winners are the firms with the lowest types. For each type of firm, we determine the expected payment in the truth-telling equilibrium of the direct mechanism. We then use this result to characterize the equilibrium bidding functions in the uniform price and discriminatory auctions. We also determine the ex ante expected revenue of the lab. The restriction to the above-mentioned auction formats when each firm can bid for only one license is without loss of generality, because it can be shown using standard techniques of mechanism design, that the allocations corresponding to the equilibrium of such auctions are the only allocations that occur in the truth-telling equilibrium of a direct mechanism in which each firm can win at most one license.

Next, we determine how the lab's revenues change when the number of licenses is allowed to vary. These revenues depend on the intrinsic values of the licenses, which in turn depend on the number of licenses sold as well as the types of the other firms who win a license. Therefore, when we compare auctions with different numbers of licenses, we have to keep in mind the fact that both the number of licenses and the intrinsic value of winning a license for a firm of a given type is different in the two auctions.

We then introduce reserve prices into the model. It is a well-known fact that when types are independently distributed, a reserve price enhances the auctioneer's revenue.[6] However, in our model, this is not necessarily the case because the reserve price, by excluding some firms, also reduces the cost of losing the auction, which in turn reduces the amount that firms bid. It is not possible to determine (as a general rule) which of the two effects dominate.

[6] In a discriminatory auction, a reserve price reduces the extent of bid shading, while in a uniform price auction, it insures the auctioneer from the possibility that the highest losing bid may be too low.

Indeed, as we shall show, it is possible for the optimal reserve price to be zero.

When the number of licenses is equal to the number of firms, the analysis differs in a number of respects. In this case, no firm can preclude another firm from winning a license and the best that the lab can do is to make a take-it-or-leave-it offer to every firm. We determine the lab's revenue with this mechanism.

Finally, we consider how many licenses the lab should offer to maximize its revenues. In order to address the issue, it is necessary to make specific assumptions about the structure of the product market, the magnitude of the externalities and the distribution of the types. By means of examples, we show that selling an exclusive license is optimal when the magnitude of the externalities is high or when firms are more likely to have high marginal costs should they win a license. However, when either of these conditions does not hold, selling multiple licenses may be optimal for the lab.

There are several related articles in the literature.[7] Katz and Shapiro (1986) and Hoppe, Jehiel, and Moldovanu (2004) have studied the problem of determining the optimal number of licenses when there is complete information. We, on the other hand, suppose that a firm's marginal cost of production after acquiring a license is private information.

Recently, there has been an emerging literature on auctions with externalities in which there is incomplete information. Several of these studies discuss auctions of licenses. Jehiel, Moldovanu and Stachetti (1999) have considered the problem of optimal auction design in the presence of externalities. However, they have assumed that the auctioneer sells only one license. Some other important articles that deal with an auction of a single object (not necessarily a license to a technology) in the presence of externalities are Jehiel, Moldovanu and Stachetti (1996), Jehiel and Moldovanu (2000) and Moldovanu and Sela (2003).

Jehiel and Moldovanu (2000) have studied the equilibrium in a second price sealed bid auction of a single license. However, in their article, because the innovation being licensed is not necessarily cost-reducing, the externality may not be negative. In particular, they have

[7]There has also been previous work on the efficacy of different mechanisms in selling licenses. This literature has been surveyed in Kamien (1992).

allowed for the possibility that the winner may impose a positive externality on the losers (a likely possibility in network industries). These authors have identified the separating equilibrium in the presence of negative externalities. However, in the presence of positive externalities, they have proved that a separating equilibrium may fail to exist. In our model, we deal with negative externalities only and, therefore, a separating equilibrium exists. Jehiel and Moldovanu (2000) assume that after the auction, but before the firms produce any output, the types become common knowledge. We retain this assumption here.

There is a related literature (for example, Das Varma (2003)) that does not assume that types will be revealed exogenously after the auction. In Das Varma's model, the winner of a license signals his type through his bid. Although with rational expectations, the other firms correctly infer his type from his bid, the equilibrium in this model is different from that of Jehiel and Moldovanu (2000) because the signalling motive modifies the equilibrium bidding function. Here, the focus is on analyzing multi-unit auctions and, therefore, for the sake of tractability, we assume that a firm's type is revealed exogenously after the auction. Consequently, the signalling motive will not influence the bidding function in our model.[8]

There has been some work on auctions of multiple licenses under incomplete information. Jehiel and Moldovanu (2001 and 2004) show the impossibility of implementing efficient outcomes when types are multi-dimensional. We assume that a firm's type is of one dimension. Moreover, in our model, when the lab sells multiple licenses, each firm is allowed to bid for only one license. Hence, the allocations we consider are not always the efficient allocations as defined in Jehiel and Moldovanu (2001). Dana and Spier (1994) consider auctions of production rights in an industry, but in their model, the seller (which is the government) maximizes social welfare. Moreover, the payoff of a firm that does not win a license is assumed to be zero. To the best of my knowledge, Schmitz (2002) is the only article that has analyzed revenue maximizing allocations from an auction of multiple licenses with incomplete information. He has shown that the optimal number is not always one. There

[8] Other articles with this feature are Goeree (2000), Haile (2000), Haile (2002) and Katzman and Rhodes-Kropf (2002).

are two key differences between our model and his. First, Schmitz assumes that there is some probability that a firm that wins a license may not be able to commercially exploit the technology, whereas, in our model, this is not the case. Second, Schmitz assumes that a firm's payoff is zero if it does not win a license or if one of the other firms does. We do not restrict the profit functions in this way.

It can be shown that if either of these assumptions is relaxed in Schmitz's model, the lab will only want to sell an exclusive license. In our model, we do not adopt either of Schmitz's assumptions. Nevertheless, we show that the lab will sometimes want to sell multiple licenses. Moreover, we show through an example that the optimal reserve price may be zero. If it is assumed that the payoff of a firm that does not win a license is zero (or a constant), then the optimal reserve price must be positive.

The plan of the rest of the article is as follows. In Section 2, we lay out the basic structure of the model. In Section 3, we assume that the lab chooses to sell a predetermined number of licenses (but not to every firm in the industry) and compute the expected revenue of the lab when the licenses are allocated using a direct mechanism. We then use this result to determine the bids firms will make in the uniform price and discriminatory auctions. In Section 4, we analyze the effect of a reserve price on the lab's revenue. In Section 5, we determine the revenue the lab obtains when it sells one license to each firm in the industry. Finally, in Section 6, we compute the optimal number of licenses in specific examples. Most of the proofs are presented in the Appendix.

Preliminaries

We consider an industry with n firms, $N = \{1, 2, \ldots, n\}$. There are no potential entrants. Below, we make an assumption that implies that no firm would want to leave the industry. Therefore, the number of firms is fixed in the analysis. Each firm i produces a single product at a constant marginal cost with no fixed cost. The gross profit (before paying any license fees) of any firm depends on its own marginal cost c_i as well as the marginal costs of the other firms in the industry. Moreover, we assume that the firms' profit functions are symmetric in the following senses:

(1) If the costs of firms i and j are permuted holding everything else constant, their profits are also permuted.

(2) Firm i's profit is invariant to any permutation of the marginal costs of its competitors.

Thus, we can write the (indirect) *profit function* of any firm i as $\pi\left(c_i; \hat{c}_{-i}\right)$ where $\hat{c}_{-i}$ is a non-decreasing permutation of the vector

$$c_{-i} = \left(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n\right).$$

The profit function is the level of profits that a firm earns given its costs, the costs of its competitors and the structure of the industry.[9] The function $\pi(\cdot)$ is assumed to be twice continuously differentiable in all of its arguments. Moreover, the following relations are assumed to hold:

$$\frac{\partial \pi(c_i; \hat{c}_{-i})}{\partial c_i} < 0 \ , \qquad \frac{\partial \pi(c_i; \hat{c}_{-i})}{\partial \hat{c}_j} > 0 \qquad j \neq i.$$

These assumptions are satisfied if for example there is a fixed demand function and the firms compete in quantities. They are also satisfied if the firms have differentiated products and compete in prices.[10] Also, a firm with type $\hat{c}_j$ is said to impose a *negative externality* on firm i with type c_i whenever $\frac{\partial \pi(c_i; \hat{c}_{-i})}{\partial \hat{c}_j} > 0$.

Initially, all firms have access to the best publicly available technology. The marginal cost of production with this technology is 1. Now suppose that there is a research laboratory that

[9]By the structure of the industry, we mean the nature of competition (that is, whether the firms compete in prices or in quantities), the demand function and the number of firms in the industry.

[10]These assumptions will not be satisfied if the firms compete among themselves in prices but have undifferentiated products. However, if the firms compete in prices using differentiated products, then these assumptions will be satisfied. For instance, given the parameters specified in Example 2.1, the equilibrium profit function of firm i given by

$$2\left(\frac{5 - 2c_i + \sum_{j \neq i} c_j}{5}\right)^2$$

satisfies these inequalities.

has invented a new cost-reducing technology (or process innovation) for this industry and has acquired the patent to this technology. A firm that has access to the new technology is able to produce at a marginal cost that does not exceed 1. It is assumed that $\pi(1; 0, 0, .., 0) \geq 0$. This assumption ensures that no firm will exit from the industry.

The research lab auctions licenses to its technology. The lab chooses the auction format and the number of licenses to sell so as to maximize its revenue subject to the constraint that no firm can obtain more than one license. Once the auction has been completed and the licenses have been transferred to the firms that have been successful in the auction, the firms produce and earn profits that depend on their post-auction marginal costs of production.

Firms differ in their ability to benefit from the new technology. We identify a firm's "type" with its marginal cost if it has access to the technology. At the time of the auction, each firm knows what its marginal cost will be if it wins a license. However, it only knows that the ex post marginal costs of all of the other firms who win a license are independently drawn from a distribution $G(\cdot)$ with support $[0, 1]$.[11] We assume that $G(\cdot)$ is continuously differentiable. Let $g(\cdot)$ be the corresponding density function, where $g(c) > 0$ $\forall c \in (0, 1)$. We also let $F_m^n(\cdot)$ denote the distribution function of $c_{(m)}^n$, where $c_{(m)}^n$ is the mth lowest cost realization among the n firms, and we let $f_m^n(\cdot)$ denote the associated density function. Let $F_{1k}^n \left(c_{(1)}, c_{(2)}, \ldots, c_{(k)}\right)$ denote the joint distribution of $c_{(1)}^n, c_{(2)}^n, \ldots, c_{(k)}^n$ and let $f_{1k}^n \left(c_{(1)}, c_{(2)}, \ldots, c_{(k)}\right)$ denote its corresponding density function.

If firm i wins a license, its marginal cost of production will be $c_i \leq 1$. Thus, for example, if k licenses are being auctioned (without a reserve price) and firm i wins one of them, then $(k - 1)$ of its competitors will also have a license. Therefore, when production occurs, at most k firms will have marginal costs less than 1 and at least $(n-k)$ firms will have marginal costs of production equal to 1.[12] To simplify the analysis, it is assumed that all marginal costs become common knowledge after the auction and before firms produce.[13]

[11] The ex post marginal cost of a firm will be its marginal cost if it wins a license and will be 1 otherwise.
[12] If there is a reserve price, then all k licenses may not be sold.
[13] As mentioned in the Introduction, we invoke this assumption to abstract from signalling issues.

In the subsequent analysis, we consider an auction of k licenses in which the licenses are awarded to firms with the k lowest ex post marginal costs. This allocation is a natural extension of the ones considered in Jehiel and Moldovanu (2000) and DasVarma (2003).[14] Such an allocation will occur in a symmetric equilibrium of a standard auction.[15] If k licenses are being auctioned and firm i wins one of them, then firm i 's continuation profit will be $\pi(c_i; c_{(1)}^{n-1}, c_{(2)}^{n-1}, \ldots, c_{(k-1)}^{n-1}, e_{n-k})$, where $c_{(j)}^{n-1}$ ($j = 1, 2, \ldots, (k-1)$) are the post auction marginal costs of the other licensees and e_{n-k} is a $(n-k)$ vector of ones. If firm i does not win a license, then its profit after production will be $\pi(1; c_{(1)}^{n-1}, c_{(2)}^{n-1}, \ldots, c_{(k)}^{n-1}, e_{n-k-1})$, where e_{n-k-1} is a $(n-k-1)$ vector of ones.

In the following discussion, we drop the superscripts $n-1$ or n from the order statistics $c_{(m)}^{n-1}$ and $c_{(m)}^{n}$ if it is clear from the context whether $n-1$ or n draws are being made from the distribution $G(\cdot)$. Furthermore, to simplify the notation, we also define the following:

$$\pi^W(c_i; c_{(1)}, c_{(2)}, \ldots, c_{(k-1)}) \equiv \pi(c_i; c_{(1)}, c_{(2)}, \ldots, c_{(k-1)}, e_{n-k}),$$

$$\pi^L(1; c_{(1)}, c_{(2)}, \ldots, c_{(k)}) \equiv \pi(1; c_{(1)}, c_{(2)}, \ldots, c_{(k)}, e_{n-k-1})$$

and

$$\pi^0 \equiv \pi(1; e_{n-1}).$$

Consider the situation in which one firm has marginal cost of c if it is awarded a license and the kth lowest of the ex post marginal costs of its competitors is also c. In this case, not all firms with marginal cost c can win a license. The *intrinsic value* $V_k(c)$ of a license to a firm with marginal cost c when $k < n$ licenses are being auctioned and each firm can bid for only one license is the value to this firm of winning a license rather than losing it to

[14] Another reason is that such allocations are the only implementable allocations when each firm can win at most one license.

[15] In a standard auction of k licenses, the firms with the k highest bids win a license each. We assume that ties will be broken randomly. However, in our analysis, the probability of a tie is 0.

a competitor with the same marginal cost. Formally,

$$V_k(c) \equiv E\left[\pi^W(c; c_{(1)}, \ldots, c_{(k-1)}) - \pi^L(1; c_{(1)}, \ldots, c_{(k-1)}, c_{(k)}) | c_{(k)} = c\right], \tag{1}$$

where the expectation is taken over $c_{(1)}, c_{(2)}, \ldots, c_{(k-1)}$ conditional on $c_{(k)} = c$. Equivalently,

$$\begin{aligned} V_k(c) &= \int_0^c \int_0^{c_{(k-1)}} \cdots \int_0^{c_{(2)}} [\pi^W(c; c_{(1)}, \ldots, c_{(k-1)}) - \pi^L(1; c_{(1)}, \ldots, c_{(k-1)}, c)] \\ &\quad \times \frac{f_{1k}^{n-1}\left(c_{(1)}, c_{(2)}, \ldots, c_{(k-1)}, c_{(k)} = c\right)}{f_k^{n-1}(c)} dc_{(1)} \cdots dc_{(k-1)}. \end{aligned} \tag{2}$$

Note that $V_k(1) = 0$.[16] Also, the intrinsic value of a license is not defined when n licenses are auctioned. The concept of the intrinsic value of a license was first introduced by Das Varma (2003) for the special case in which one license is being auctioned.[17] Below, we show that $V_k(c)$ is the price that a firm is willing to pay in a uniform price auction of $k < n$ licenses.[18]

It is assumed that the profit function and the distribution of costs are such that

$$V_k'(c) < 0. \tag{3}$$

This assumption is sufficient for the existence of a separating equilibrium in symmetric

[16] To understand the limits of the integral in (2), notice the following inequality:

$$0 \leq c_{(1)} \leq \cdots \leq c_{(k-1)} \leq c_{(k)} = c.$$

Hence, $c_{(k-1)}$ can take any value from 0 to c, and given any value of $c_{(k-1)}$ in this range, $c_{(k-2)}$ can take any value from 0 to $c_{(k-1)}$, and so on. This explains the specification of the integral limits.

[17] Similiar expressions have also been found in Milgrom and Weber (1982) and in Jehiel and Moldovanu (2000).

[18] The ex post value of a license to a firm with marginal cost c is

$$\pi^W(c; c_{(1)}, \ldots, c_{(k-1)}) - \pi^L(1; c_{(1)}, \ldots, c_{(k-1)}, c_{(k)}).$$

Therefore, in a symmetric equilibrium of a standard auction, the expected value of the ex post value of a license is

$$E\left[\pi^W(c; c_{(1)}, \ldots, c_{(k-1)}) | c_{(k)} \geq c\right] - E\left[\pi^L(1; c_{(1)}, \ldots, c_{(k-1)}, c_{(k)}) | c_{(k)} \leq c\right].$$

Note that this expression is different from the intrinsic value.

strategies in standard auctions.[19] In general, this inequality may not be satisfied because the inequality depends both on the profit function and on the distribution of costs. However, this assumption is satisfied in the examples considered in Section 6. Note that when $k = 1$, the assumptions on the profit functions are sufficient to guarantee that $V_1'(c) < 0$.

License Revenue with Incomplete Licensing

In this section, we consider a lab that uses a *standard auction* to sell a fixed number of licenses k, where $k \in \{1, 2, \ldots, (n-1)\}$. In a standard auction, the k highest bidders win a license each (Krishna 2002, p. 168). By the revelation principle, associated with any standard auction there is an equivalent direct mechanism whose truth-telling equilibrium has the same outcome as the standard auction. In a direct mechanism, the lab announces the probability that a firm wins a license (the *allocation rule*) and its expected payment (the *payment rule*) as a function of the firms' reports about their types. It then implements an outcome based on the reports, the allocation rule and the payment rule. An auction may have multiple equilibria. We only consider symmetric monotonic equilibria of standard auctions, that is, equilibria in which all firms follow the same bidding strategy and the bids are strictly decreasing in types. In terms of the corresponding direct mechanisms, we are therefore supposing that if two firms' types are permuted, their allocations and payments are permuted as well.

Analogous to the definition of a standard auction, we call a direct mechanism *standard* if the mechanism allocates one license each to the k firms with the lowest reported types. Standard direct mechanisms differ only in their payment rules. Moreover, we call a standard mechanism *symmetric* if the allocation and payment rules are symmetric. Note that the equivalent direct mechanism corresponding to the symmetric monotonic equilibria of a standard auction is symmetric.

[19] If the inequality is not satisfied, there does not exist any symmetric equlibrium in the uniform price auction.

The allocation rule that we consider is one in which the lab awards the licenses to the k firms with the lowest reported types. In this mechanism, in a truth-telling equilibrium, firm i wins the auction if its type c_i is less than the kth lowest of its competitors' types, that is, if $c_i < c_{(k)}$.[20] Therefore, if firm i wins a license, its expected profit in equilibrium is:

$$\int_{c_i}^{1}\int_{0}^{c_{(k)}}\cdots\int_{0}^{c_{(2)}}\pi^W(c_i;c_{(1)},\ldots,c_{(k-1)})f_{1k}^{n-1}\left(c_{(1)},\ldots,c_{(k)}\right)dc_{(1)}\cdots dc_{(k)}.$$

Similarly, if this firm does not win a license, its expected profit is:

$$\int_{0}^{c_i}\int_{0}^{c_{(k)}}\cdots\int_{0}^{c_{(2)}}\pi^L(1;c_{(1)},\ldots,c_{(k)})f_{1k}^{n-1}\left(c_{(1)},\ldots,c_{(k)}\right)dc_{(1)}\cdots dc_{(k)}.$$

We define $m_k\left(c_i\right)$ to be the expected payment of a firm that reports type c_i when k licenses are being auctioned and each firm can bid for only one license.

Let $P(\widetilde{c},c_i)$ denote the payoff of firm i when its true type is c_i but it reports its type as being $\widetilde{c}$. From the preceding discussion, it is apparent that $P(\widetilde{c},c_i)$ takes the following form:

$$\begin{aligned}P(\widetilde{c},c_i)=&\int_{\widetilde{c}}^{1}\int_{0}^{c_{(k)}}\cdots\int_{0}^{c_{(2)}}\pi^W(c_i;c_{(1)},\ldots,c_{(k-1)})f_{1k}^{n-1}\left(c_{(1)},\ldots,c_{(k)}\right)dc_{(1)}\cdots dc_{(k)}\\&+\int_{0}^{\widetilde{c}}\int_{0}^{c_{(k)}}\cdots\int_{0}^{c_{(2)}}\pi^L(1;c_{(1)},\ldots,c_{(k)})f_{1k}^{n-1}\left(c_{(1)},\ldots,c_{(k)}\right)dc_{(1)}\cdots dc_{(k)}-m_k\left(\widetilde{c}\right).\end{aligned}\tag{4}$$

The first term on the right hand side of (4) is the expected gross profit to firm i from winning a license when its true type is c_i and its reported type is $\widetilde{c}$. The second term is the expected gross profit to firm i from not winning the license. Finally, the third term is the expected payment of a firm with reported signal $\widetilde{c}$. Notice that the first two terms depend on the types of the winners, which in the case of the first term includes firm i. Expressions

[20] It is a zero probability event for c_i to equal $c_{(k)}$. Therefore, it is of no consequence to the seller who is awarded a license in such a situation.

analogous to the first term appear in other models of auctions with interdependent values, such as those of Schmitz (2002), Dana and Spier (1994), Jehiel and Moldovanu (2004) and Milgrom and Weber (1982).[21] The second term is what distinguishes our model from the above class of models. If firm i does not win a license, then its profit is higher, the higher the marginal cost of a winner. Because the winners have marginal costs below 1, the profit of firm i if it does not win a license is lower than its profit before the innovation. In other words, the winners impose a negative externality on firm i. This externality has implications for the revenue that an auction generates. Below we show that, because of this type-dependent negative externality, firms bid more aggressively relative to what would occur in a model in which the winner does not impose an externality on the loser. In many well-known models in the literature (including the ones cited above), the second term is assumed to be a constant. To distinguish these models from our own, the former models are said to have interdependencies and fixed externalities, whereas our model has interdependencies and type-dependent externalities.

When firm i reports its true type, its expected payoff is

$$U(c_i) \equiv P(c_i, c_i). \tag{5}$$

In the truth-telling equilibrium of a standard direct mechanism, $U(c_i)$ is the payoff of a firm with type c_i. In this equilibrium, the payment rule must be such that it is a best response for a firm to report its true type, given that the competitors also reveal their types. This is known as the *incentive compatibility constraint* and implies that the following inequalities must be satisfied:

$$U(c_i) \geq P(\widetilde{c}, c_i) \text{ for all } i \in N \text{ and all } c_i, \widetilde{c} \in [0,1]. \tag{6}$$

Moreover, because participation in the mechanism is voluntary, no firm should be made

[21] In Schmitz (2002), it is assumed that if a firm does not win a license, its payoff is 0.

worse off by participating in the mechanism. This is known as the *participation constraint* and implies that the following inequalities should be satisfied:

$$U(c_i) \geq \int_0^1 \int_0^{c_{(k)}} \cdots \int_0^{c_{(2)}} \pi^L(1; c_{(1)}, \ldots, c_{(k)}) f_{1k}^{n-1}\left(c_{(1)}, \ldots, c_{(k)}\right) dc_{(1)} \cdots dc_{(k)} \forall c_i \in [0,1]. \quad (7)$$

By substituting $c_i = 1$ in the left hand side of (7), we can easily check that the following inequality must be satisfied:

$$m_k(1) \leq 0.$$

When types are independent, it is a well-known fact that there is revenue equivalence for symmetric auctions that have the same allocation rule and that extract equal expected payments from the worst type of bidders. Proposition 1 is a version of the revenue equivalence theorem for the sale of k licenses by a standard direct mechanism.

Proposition 2.1. *In the truth-telling equilibrium of any symmetric standard direct mechanism for the sale of k $(< n)$ licenses, the expected payment of a firm with type c_i is given by*

$$m_k(c_i) = m_k(1) + \int_{c_i}^1 V_k(c) f_k^{n-1}(c)\, dc. \quad (8)$$

Proof. See Appendix A. □

We have argued above that $m_k(1)$ must be non-positive in order to satisfy the participation constraint and hence, we select

$$m_k(1) = 0 \quad (9)$$

to ensure that the revenue of the lab is maximized. Notice that it follows from (A.8) in the Appendix that

$$U'(c_i) < 0 \text{ for all } c_i \in [0,1) \quad (10)$$

and, hence, the equilibrium payoff is decreasing in type. Thus, it follows from (7) and (10) that in the truth-telling equilibrium, the participation constraints are satisfied for every type.

Consider two common forms of standard auctions—the uniform price auction and the discriminatory auction. In each of these auctions, the firms submit a bid and the firms with the k highest bids win a license each. In the uniform price auction, the winners pay the $(k+1)th$ highest bid, while in the discriminatory auction, they pay their own bids. Notice that the uniform price auction and the discriminatory auction are standard auctions in which the losers pay nothing. Therefore, the direct mechanisms corresponding to the symmetric equilibria of these auctions are standard symmetric direct mechanisms in which the expected payment of a firm with type 1 is 0, provided that the bids are decreasing in type. Hence, by Proposition 2.1, it follows that these two auctions generate the same amount of revenue for the lab.

We denote the bid function in the symmetric equilibrium in which bids are decreasing in types of a uniform price auction by $B_{kU}(\cdot)$ and that in the discriminatory auction by $B_{kP}(\cdot)$. By the revelation principle, Proposition 2.1 can be used to determine the functional forms of these bid functions.

Corollary 2.2. *In a uniform price auction of k ($< n$) licenses, the equilibrium bidding function in a symmetric equilibrium in which bids are decreasing in type is given by:*

$$B_{kU}(c_i) = V_k(c_i)\,;\ \forall\ c_i \in [0,1]\,. \tag{11}$$

In a discriminatory auction of k ($< n$) licenses, the equilibrium bidding function in a symmetric equilibrium in which bids are decreasing in type is:

$$B_{kP}(c_i) = \frac{\int_{c_i}^{1} V_k(c)\, f_k^{n-1}(c)dc}{1 - F_k^{n-1}(c_i)} = E\left[V_k\left(c_{(k)}^{n-1}\right) | V_k\left(c_{(k)}^{n-1}\right) \leq V_k(c_i)\right]\,;\ \forall\ c_i \in [0,1]\,. \tag{12}$$

Proof. Because, by assumption, bids are decreasing in type in the symmetric equilibrium, in a uniform price auction of k licenses,

$$m_k(c_i) = \int_{c_i}^{1} B_{kU}(c)\, f_k^{n-1}(c)\, dc. \tag{13}$$

Because this is true for all c_i, it follows from (8), (13) and the revelation principle that (11) holds. Similarly, in a discriminatory auction,

$$m_k(c_i) = \left[1 - F_k^{n-1}(c_i)\right] B_{kP}(c_i). \tag{14}$$

It then follows from (8), (14) and the revelation principle that (12) holds. □

In Corollary 2.2, the bid functions are determined indirectly using properties of the equivalent direct mechanism. However, one can directly check that the bid functions in Corollary 2.2 are indeed the symmetric equilibrium bids in the discriminatory and uniform price auctions, and assuming (3) is satisfied, that these bids are monotonically decreasing in type.

There is a substantial literature that analyzes bidding behavior when one license is being auctioned. For our purposes, the most relevant articles are Jehiel and Moldovanu (2000) and Das Varma (2003). Jehiel and Moldovanu considered the sale of one license using the uniform price auction. In this case, our model is identical to the one used by Jehiel and Moldovanu and, as a consequence, the equilibrium bid of firm i in their model is $V_1(c_i)$.[22] Das Varma considered the sale of one license using the discriminatory auction. In his model, firms signalled their types through their bids and his objective was to study the effect of signalling on the equilibrium bids. Das Varma was able to decompose the bidding function into two parts—the first part is associated with a firm's true type and the second part is the signalling component. In our model, firms cannot signal their types through their bids and hence the signalling component will be absent. One can easily check that when $k = 1$, the

[22] They described their auction as being a "second price auction," but it is identical to the uniform price auction for the sale of one license.

equilibrium bid in the discriminatory auction predicted by our model is the one derived by Das Varma without the signalling component.

The analysis of Milgrom and Weber (1982) can be used to derive the bidding functions for a number of well-known auction formats for selling a single object when there are interdependencies and fixed externalities; Weber (1983) is an extension of the analysis of Milgrom and Weber for the sale of multiple objects without externalities. For the discriminatory auction and the uniform price auction, the bidding functions in our model are identical to the ones derived by Weber except that the intrinsic value of a license replaces the "actual value of the object" (see Weber ((1983), pp. 168-169) in the expressions for the equilibrium bidding functions. Also note that, in our model, the intrinsic value changes as the number of licenses increases.

Milgrom and Weber have provided an intuitive explanation for the equilibrium bids observed in the uniform price (second price) auction for a single object (see Migrom and Weber (1982), p. 1101). We can use the same intuition to explain the bids observed in the uniform price auction of multiple licenses when there are externalities. Each firm bids so that it is indifferent between winning and losing, which in the case considered here is the value of the license to this firm conditional on the kth lowest competitor having the same marginal cost.

Notice that because there is a cost of losing the auction in our model, firms bid more aggressively than they would have in the absence of the type-dependent negative externality. This point is brought out very clearly in the uniform price auction. In the counterfactual case of $\frac{\partial}{\partial c_{(j)}}\pi^L\left(1; c_{(1)}, \ldots, c_{(k)}\right) = 0$ for $j = 1, 2, \ldots, k$, a firm earns a profit of π^0 if it does not win a license and it bids $\left[\pi^W\left(c; c_{(1)}, \ldots, c_{(k-1)}\right) - \pi^0\right]$ in the uniform price auction. When there are negative externalities (that is, when $\frac{\partial \pi(c_i; \hat{c}_{-i})}{\partial \hat{c}_j} > 0; \quad j \neq i$), the intrinsic value $V_k(c)$ is larger than $\left[\pi^W\left(c; c_{(1)}, \ldots, c_{(k-1)}\right) - \pi^0\right]$. Thus, the bidding is more aggressive in the presence of a type-dependent negative externality.

In Proposition 2.1, we have identified the expected payment of a firm in the truth-telling equilibrium of a symmetric standard direct mechanism. Using this expected payment, we can compute the ex ante expected revenue of the seller.

Proposition 2.3. *The ex ante expected revenue of the lab in a symmetric standard direct mechanism for the sale of $k < n$ licenses that satisfies the conditions of Proposition 2.1 is given by*

$$R_k = k\int_0^1 V_k\left(c\right) f_{k+1}^n\left(c\right) dc = kEV_k\left(c_{(k+1)}^n\right). \tag{15}$$

Proof. By definition,

$$R_k = n\int_0^1 m_k\left(c\right) g\left(c\right) dc. \tag{16}$$

Substituting (8) in (16) and integrating by parts, we obtain,

$$R_k = n\int_0^1 V_k\left(c\right) f_k^{n-1}(c) G\left(c\right) dc. \tag{17}$$

Note that

$$F_k^{n-1}\left(c\right) = 1 - \sum_{j=0}^{k-1} \begin{pmatrix} n-1 \\ j \end{pmatrix} \left[G\left(c\right)\right]^j \left[1 - G\left(c\right)\right]^{n-1-j}. \tag{18}$$

It can be shown that (18) implies

$$f_k^{n-1}\left(c\right) = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \left(n-k\right) \left[G\left(c\right)\right]^{k-1} \left[1 - G\left(c\right)\right]^{n-k-1} g\left(c\right), \tag{19}$$

from which it follows that

$$n f_k^{n-1}(c) G\left(c\right) = k f_{k+1}^n\left(c\right). \tag{20}$$

Substituting (20) in (17) yields (15). □

The expression for R_k in (15) can be interpreted easily. Notice that the uniform price auction satisfies the two conditions of Proposition 2.1 and, hence, the revenue from the uniform price auction must be equal to the revenue from any other auction that satisfies these two conditions. Given any profile of types, the price in the uniform price auction is $V_k\left(c^n_{(k+1)}\right)$ because each type bids $V_k(\cdot)$ and the winners pay the $(k+1)th$ highest bid.[23] Because k licenses are being sold, the expected revenue is therefore as stated in Proposition 2.3.

It is of interest to know whether it is optimal to sell multiple licenses when there are no reserve prices.[24] It is easiest to address this question in the context of a uniform price auction. Suppose the lab increases the number of licenses from 1 to k where $k = 2, 3, \ldots, n-1$. Then (without a reserve price), its revenue changes because of the following three effects:

(1) The number of licenses goes up, which increases the lab's revenue.

(2) When one license is being sold, the winners pay the second highest bid while in the case of k licenses, the winners pay the $(k+1)\,th$ highest bid, which reduces the lab's revenue.

(3) Firm i of type c_i bids $V_1(c_i)$ when one license is auctioned and $V_k(c_i)$ when k licenses are auctioned. If $V_1(c_i) \geq V_k(c_i)\ \forall\ c_i \in [0,1]$, each firm bids less aggressively when there are more licenses being auctioned, and this may reduce the lab's revenue. Similarly, if each firm bids more aggressively, the lab's revenue may increase.

The first two effects are present even in models without externalities.[25] However, in the presence of type-dependent negative externalities, the third factor also applies. We are interested in knowing if the three effects simply counteract each other; that is, is it always optimal to sell only one license? In order to address this question, in Proposition 2.4 we present sufficient conditions under which it is always optimal for the lab to sell one license

[23] Recall that $c^n_{(k+1)}$ is the $(k+1)th$ lowest cost from a sample of n independent draws.

[24] In most cases, setting no reserve price is not an optimal policy for the lab. However, the factors that are mentioned below affect the revenue of the lab even in the presence of a reserve price.

[25] In the absence of externalities, when k units are sold to n bidders whose types are uniformly distributed on the unit interval, the expected price per unit in the uniform price auction turns out to be $(n-k)/(n+1)$. Therefore, the seller's revenue from selling k units is $k[(n-k)/(n+1)]$. The first term in the product represents the first effect mentioned above, while the term in square brackets represents the second. It follows that the optimal number of units to sell in this case is $n/2$ (ignoring integer constraints).

rather than to sell k licenses, where $k = 2, \ldots, n-1$. In Section 6, we consider examples in which, for some distribution of costs, the optimal number of licenses (without a reserve price) is greater than one.

Proposition 2.4. *For any $k = 2, 3, \ldots, n-1$, if*

$$V_1(c) > kV_k(c) \text{ for all } c \in [0,1), \tag{21}$$

the lab earns more revenue by selling one license rather than by selling k licenses.

Proof. See Appendix A. □

Proposition 2.4 says that if the intrinsic value $V_1(c)$ sufficiently exceeds the intrinsic value $V_k(c)$ for all c, then the optimal number of licenses is one regardless of the distribution of costs. However, as we shall see, we can find cost distributions and profit functions that do not satisfy (21) for which selling multiple units is optimal even without a reserve price. Notice that if the payoff to a firm from winning a non-exclusive license is 0, as in Schmitz (2002), then $V_k(c) = 0$ for all c when $k > 1$ and, hence, it would always be optimal to sell only one license.

Another point of interest is to determine the lab's revenue when the types of the firms become common knowledge after the lab has announced the number of licenses $k < n$ that it sells, but before the actual sale occurs.[26] In this case, the lab can perfectly price discriminate among firms. If firm i with type c_i has one of the k lowest types (that is, if $c_i \leq c^n_{(k)}$), then the lab makes a take-it-or-leave-it offer to firm i of

$$\pi^W(c_i; c_{(1)}, \ldots, c_{(k-1)}) - \pi^L(1; c_{(1)}, \ldots, c_{(k)}). \tag{22}$$

[26]If the types are revealed before the lab commits to a fixed number of licenses, the lab might want to sell a different number of licenses depending on the realization of the types. Notice that under private information, we do not allow the lab to vary the number of licenses depending on the reports. Hence, to find out the exact role of private information in determining the lab's revenue, we consider the case when the type of a firm is revealed to the lab when the lab has already committed to sell a fixed number of licenses.

The first term is the profit of firm i when firm i and its competitors with the $(k-1)$ lowest costs win a license each. The second term is the profit of firm i when it does not win a license and its competitors with the k lowest costs win a license each. In effect, if firm i refuses to accept the lab's offer, the lab inflicts the maximum punishment on firm i by allocating the licenses in a manner that minimizes firm i's profits. A similar strategy of the lab has been used in Jehiel, Moldovanu, and Stacchetti (1996, p. 820). Notice that, it is an equilibrium for each firm to accept the lab's offer, provided the lab makes the offer. It follows immediately that the ex ante expected profit of the lab, in this case, is given by

$$\begin{aligned} \hat{R}_k &= E[\pi^W\left(c^n_{(1)}; c^n_{(2)}, \ldots, c^n_{(k-1)}\right) + \cdots + \pi^W\left(c^n_{(k)}; c^n_{(1)}, \ldots, c^n_{(k-1)}\right) \\ &\quad -\pi^L(1; c^n_{(2)}, \ldots, c^n_{(k+1)}) - \ldots - \pi^L(1; c^n_{(1)}, \ldots, c^n_{(k-1)}, c^n_{(k+1)})]. \end{aligned} \tag{23}$$

Effect of a Reserve Price

A well-known method of increasing revenues in an auction is through the introduction of a reserve price. In *a standard auction with a reserve price,* the lab allocates a license each to the k highest bidders, provided that these bidders bid at least as much as the reserve price. If some of the k highest bidders bid strictly less than the reserve price, the lab allocates a license to only those bidders who bid at least the reserve price.

In this section, we consider how the preceding analysis needs to be modified if the lab can set a reserve price. In order to address this question, we analyze an equivalent direct mechanism that we call a *standard direct mechanism with a critical type* c_r. Such a mechanism has the following features:

(1) The mechanism allocates a license to the firms with the k lowest reported types (marginal costs) provided that these types are less than c_r. If some of the k lowest reports are strictly greater than c_r, then only the firms that report types (weakly) less than c_r win a license.

(2) A firm with a reported type strictly greater than c_r never wins a license.

In addition, we say that a standard mechanism with a critical type c_r is *symmetric* if the allocation and payment rules are symmetric. As in the preceding section, we assume that in equilibrium, the types that never win a license pay 0 in expectation; that is,

$$m_{kr}(c_i) = 0 \text{ for all } c_i \in (c_r, 1]. \tag{24}$$

In the following proposition, we prove that all symmetric direct mechanisms with the same critical type c_r must be revenue equivalent if the expected payment of any firm with a type that exceeds c_r is 0.[27]

Proposition 2.5. *In the truth-telling equilibrium of any symmetric standard direct mechanism with a critical type c_r for the sale of k $(< n)$ licenses for which the expected payment of any firm with type $c_i > c_r$ is 0, the expected payment of a firm with type c_i is given by*

$$m_{kr}(c_i) = \begin{cases} \int_{c_i}^{c_r} V_k(c)\, f_k^{n-1}(c)\, dc + m_{kr}(c_r) & \text{if} \quad c_i \le c_r, \\ 0 & \text{if} \quad c_i > c_r, \end{cases} \tag{25}$$

where

$$\begin{aligned} m_{kr}(c_r) = & \int_{c_r}^{1} \int_{0}^{c_r} \int_{0}^{c_{(k-1)}} \cdots \int_{0}^{c_{(2)}} \left[\pi^W\left(c_r; c_{(1)}, \ldots, c_{(k-1)}\right) - \pi^L\left(1; c_{(1)}, \ldots, c_{(k-1)}\right)\right] \\ & \times f_{1k}^{n-1}\left(c_{(1)}, \ldots, c_{(k)}\right) dc_{(1)} \ldots dc_{(k)} \\ & + \int_{c_r}^{1} \int_{c_r}^{c_{(k)}} \int_{0}^{c_r} \cdots \int_{0}^{c_{(2)}} \left[\pi^W\left(c_r; c_{(1)}, \ldots, c_{(k-2)}\right) - \pi^L\left(1; c_{(1)}, \ldots, c_{(k-2)}\right)\right] \\ & \times f_{1k}^{n-1}\left(c_{(1)}, \ldots, c_{(k)}\right) dc_{(1)} \ldots dc_{(k)} \\ & \cdots + \int_{c_r}^{1} \int_{c_r}^{c_{(k)}} \cdots \int_{c_r}^{c_{(2)}} \left[\pi^W(c_r) - \pi^0\right] f_{1k}^{n-1}\left(c_{(1)}, \ldots, c_{(k)}\right) dc_{(1)} \ldots dc_{(k)}. \end{aligned} \tag{26}$$

Proof. See Appendix A. □

[27] As in the preceding section, the allocations we consider correspond to what would be implemented by the symmetric Bayesian Nash equilibrium of a standard auction with a reserve price.

As in the preceding section, incentive compatibility combined with the assumption that types that never win a license pay 0 in expectation ensures that the participation constraint is satisfied for each type.

We now analyze how to implement the outcome of a standard direct mechanism with a critical type c_r using a uniform price auction with a reserve price. In a uniform price auction of k licenses with a reserve price r_k, each winner pays a price equal to the maximum of the $(k+1)$th highest bid and r_k and each loser pays nothing.[28] We consider a particular symmetric equilibrium of the uniform price auction in which the bid function $B(\cdot)$ is decreasing in type for all $c \leq \overline{c}_r$, where

$$\overline{c}_r = \sup \{c \in [0,1] \,|B(c) \geq r_k\}.$$

A symmetric equilibrium of a uniform price auction with this property is said to be *decreasing*. By the revelation principle, there exists a truth-telling equilibrium in the corresponding direct mechanism that has the same outcome as the uniform price auction, where the outcome consists of the types that win a license and the expected payment of each type, for any given profile of types (see Krishna (2002, p. 62)).

We first show that, for any profile of types, a direct mechanism in which the types of the winners in the truth-telling equilibrium is the same as the types of the winners in a decreasing symmetric equilibrium of a uniform price auction, must be a symmetric standard direct mechanism with the critical type $\overline{c}_r$.

Lemma 2.6. *For any profile of types* $\left(c^n_{(1)}, \ldots, c^n_{(n)}\right)$, *in the truth-telling equilibrium of a symmetric standard direct mechanism with critical type* $\overline{c}_r$, *the types of winners are the same as the types of winners in a decreasing symmetric equilibrium of the uniform price auction with the reserve price* r_k. *Moreover, if a direct mechanism is either* (i) *a symmetric standard direct mechanism with a critical type* $\overline{c}'_r \neq \overline{c}_r$, *or* (ii) *a symmetric direct mechanism that*

[28] In this section, a uniform price auction refers to a uniform price auction of $k < n$ licenses with a reserve price r_k.

is not standard, then for some profile of types, the truth-telling equilibrium of the direct mechanism and any decreasing symmetric equilibrium of the uniform price auction allocate licenses differently.

Proof. Fix the profile of types $\left(c^n_{(1)}, \ldots, c^n_{(n)}\right)$. First, consider the case in which $c^n_{(k)} \leq \overline{c}_r$. Then in a decreasing symmetric equilibrium of the uniform price auction, the firms with types $c^n_{(1)}, \ldots,\ c^n_{(k)}$ win a license each (and the others do not win anything). Notice that, the same allocation of licenses is obtained in the truth-telling equilibrium of any symmetric standard direct mechanism with a critical type $\overline{c}_r$. An analogous argument can be used to show that the same conclusion holds when $c^n_{(k)} > \overline{c}_r$.

Now, consider a symmetric direct mechanism with a critical type $\overline{c}'_r$ for which $\overline{c}'_r > \overline{c}_r$. When the profile of types is such that $c^n_{(k)}$ lies between $\overline{c}'_r$ and $\overline{c}_r$, then in any decreasing symmetric equilibrium of the uniform price auction, the firm with type $c^n_{(k)}$ does not win a license, but in the truth-telling equilibrium of the symmetric standard direct mechanism, such a firm wins a license. Similar reasoning applies when $\overline{c}'_r < \overline{c}_r$.

Finally, consider a symmetric direct mechanism that is not standard. Suppose, contrary to the lemma, that in the truth-telling equilibrium, such a mechanism allocates licenses in the same way as the decreasing symmetric equilibrium of a uniform price auction. From the first part of the proof, we know that a symmetric standard direct mechanism with a critical type $\overline{c}_r$ also allocates licenses in the same way as a decreasing symmetric equilibrium of a uniform price auction. The allocation from the non-standard mechanism therefore coincides with the allocation from a symmetric standard direct mechanism for any profile of types, which is not possible. □

Hence, we know that a direct mechanism for which, for any profile of types, the outcome from the truth-telling equilibrium coincides with the outcome from the decreasing symmetric equilibrium of the uniform price auction with reserve price r_k must be a symmetric standard direct mechanism with a critical type $\overline{c}_r$. Henceforth, we use c_r to denote the critical type for both a uniform price auction and its equivalent symmetric standard direct mechanism.

Because types with $c > c_r$ make an expected payment of 0 in the uniform price auction, in the equivalent symmetric direct mechanism, these types must have an expected payment of 0 as well. Note that by Proposition 2.5, we already know the expected payment $m_{kr}(c_r)$ of the critical type c_r for a symmetric standard direct mechanism in which the types $c > c_r$ pay 0 in expectation. In addition, we know that in the uniform price auction with reserve price r_k, a firm of type c_r wins a license with probability $\left[1 - F_k^{n-1}(c_r)\right]$ in a decreasing symmetric equilibrium and, if it wins a license, then it pays the reserve price r_k. Hence, in the uniform price auction, the critical type c_r makes an expected payment of $\left[1 - F_k^{n-1}(c_r)\right] r_k$. By the revelation principle and Lemma 2.6, the expected payment of every type is the same in both of the mechanisms mentioned above, and hence, the reserve price r_k and the critical type c_r must be related as follows:

$$m_{kr}(c_r) = \left[1 - F_k^{n-1}(c_r)\right] r_k. \tag{27}$$

In other words, the reserve price must be

$$r_k = \frac{m_{kr}(c_r)}{\left[1 - F_k^{n-1}(c_r)\right]} \tag{28}$$

in order for the uniform price auction to implement the direct mechanism described above. In other standard auctions, the reserve price can similarly be chosen as a function of the critical type c_r.[29]

In view of the preceding discussion, we can use Proposition 2.5 to characterize the decreasing symmetric equilibrium bid function $B_{kUr}(\cdot)$ in the uniform price auction with the reserve price given in (28).

Corollary 2.7. *In a decreasing symmetric equilibrium of a uniform price auction for the*

[29] In the discriminatory auction, we have the same reserve price as in the uniform price auction because a firm has the same probability of winning in both the auctions.

sale of $k < n$ licenses with the reserve price r_k given by (28), the bid of a firm with type c_i is

$$B_{kUr}(c_i) = \begin{cases} V_k(c_i) & \text{if} \quad c_i \leq c_r; \\ 0 & \text{if} \quad c_i > c_r. \end{cases} \tag{29}$$

Proof. It has been argued above that a decreasing symmetric equilibrium of a uniform price auction with a reserve price r_k given by (28) and the truth-telling equilibrium of a symmetric standard direct mechanism with a critical type c_r have the same outcome and, hence, the expected payment of a type is the same with both of these mechanisms.

Consider a firm of type c_i with $c_i \leq c_r$. In a decreasing symmetric equilibrium of a uniform price auction with a reserve price r_k, this firm wins a license if and only if its type is less than $c_{(k)}^{n-1}$. Let $p_k\left(c_{(k)}^{n-1}\right)$ denote the price that this firm pays if it wins a license when $c_{(k)}^{n-1} \leq c_r$. If $c_{(k)}^{n-1} > c_r$, this firm pays r_k. Hence, the expected payment of such a firm is given by

$$m_{kr}(c_i) = \int_{c_i}^{c_r} p_k(c) f_k^{n-1}(c)\, dc + \left[1 - F_k^{n-1}(c_r)\right] r_k. \tag{30}$$

By comparing (30) and (25), it follows that $p_k(c) = V_k(c)$ for all $c \leq c_r$. Thus, a firm with type $c_i \leq c_r$ bids $V_k(c_i)$ in a decreasing symmetric equilibrium of a uniform price auction with the reserve price r_k.

Next, consider a firm of type c_i with $c_i > c_r$. With a bid of 0, the expected payment of such a firm is 0 in the uniform price auction with the reserve price r_k and in the symmetric standard direct mechanism with the critical type c_r. Hence, we have the result. □

Notice that, in the uniform price auction with a reserve price r_k, the critical type bids an amount strictly greater than the reserve price because $V_k(c_r) > r_k$.[30]

We now use Proposition 2.5 to determine the ex ante expected revenue of the lab in a symmetric standard direct mechanism with critical type c_r.

Proposition 2.8. *The ex ante expected revenue of the lab in a symmetric standard direct*

[30]In models without externalities, the critical type bids exactly the reserve price.

mechanism with a critical type c_r for the sale of $k < n$ licenses that satisfies the conditions of Proposition 2.5 is given by

$$R_{kr} = k \int_0^{c_r} V_k(c) f_{k+1}^n(c)\, dc + nG(c_r)\, m_{kr}(c_r). \tag{31}$$

Proof. The proof of Proposition 2.8 is similar to the proof of Proposition 2.3 and is therefore omitted. □

We can interpret the expression on the right-hand side of (31) by considering the corresponding uniform price auction. Observe that the first term on the right-hand side of (31) can be written as:[31]

$$\text{Prob}\left(c_{(k+1)}^n \leq c_r\right) kE\left(V_k\left(c_{(k+1)}^n\right) | c_{(k+1)}^n \leq c_r\right). \tag{32}$$

When $c_{(k+1)}^n$ is less than c_r, all k licenses are sold at an expected price of

$$E\left(V_k\left(c_{(k+1)}^n\right) | c_{(k+1)}^n \leq c_r\right).$$

Hence, the first term is the ex ante expected revenue of the lab when the $(k+1)th$ highest bid is greater than the reserve price. The second term in the right-hand side of (31), $nG(c_r)\, m_{kr}(c_r)$, is the ex ante expected revenue of the lab when the $(k+1)th$ highest bid is less than the reserve price. To understand this term, observe that the expected number of licenses sold conditional on $c_{(k+1)}^n > c_r$ is given by:

$$\frac{\sum_{s=0}^{k} s \begin{pmatrix} n \\ s \end{pmatrix} [G(c_r)]^s [1 - G(c_r)]^{n-s}}{1 - F_{k+1}^n(c_r)} = \frac{nG(c_r)\left[1 - F_k^{n-1}(c_r)\right]}{1 - F_{k+1}^n(c_r)} \tag{33}$$

[31]Recall that

$$E\left(V_k\left(c_{(k+1)}^n\right) | c_{(k+1)}^n \leq c_r\right) = \int_0^{c_r} V_k(c) \frac{f_{k+1}^n(c)}{F_{k+1}^n(c_r)} dc.$$

and for each license sold, the lab receives a price of r_k. Therefore, the second term is the product of (i) the probability that $c^n_{(k+1)} > c_r$, (ii) the expected number of licenses sold (conditional on this event) and (iii) the reserve price.

Comparing the expression in (15) for the lab's expected revenue R_k when there is no reserve price to the expression in (31) for the lab's expected revenue R_{kr} with a reserve price, we see that R_k is the same as the first term in the expression for R_{kr} except that the upper limit of the integral is 1 for R_k whereas it is c_r for R_{kr}. However, the presence of the second term in the expression for R_{kr} ensures that $R_{kr} \geq R_k$. The reason is that if the lab sets the reserve price equal to 0, then $c_r = 1$. Therefore, when the lab chooses the optimal reserve price, R_{kr} must be at least as great as R_k.

There is another interesting feature of the reserve price. In the presence of fixed externalities, but with independent "types", a well-known result is that the optimal reserve price is positive.[32] However, in the presence of externalities, the optimal reserve price may be 0.[33] By setting a positive reserve price, the lab reduces the extent of bid shading by excluding some "types." However, a positive reserve price also implies that there is a positive probability that not all of the licenses will be sold, which in turn reduces the cost of losing.[34] This has a dampening effect on the bids. It is not possible to say, as a general rule, which of the two effects dominate and, in particular cases, it may turn out that the optimal reserve price is 0. This is the case in Example 2.1 (see Section 6).[35]

The Lab's Revenue With Complete Licensing

When the number of licenses being auctioned equals the number of firms n, no firm can preempt a competitor from winning a license and similarly no competitor can preempt it from winning a license. If the lab does not specify a price for a license, it is a dominant

[32]This is known as the exclusion principle. For details, see Krishna (2002, p. 26).

[33]For an example with affiliated types and private values, see Levin and Smith (1996).

[34]One can check that the equilibrium payoff of type 1 given by $U(1)$ increases as c_r is reduced.

[35]In models in which the outside option is normalized to some constant (as in Schmitz (2002)) and the types are independent, the optimal reserve price is always positive by the exclusion principle.

strategy for each firm to pay 0, and hence, the lab's revenue in this case is 0. Therefore, when selling n licenses, the lab makes a take-it-or-leave-it offer to each firm. Moreover, because the type of a firm is its private information, the lab has to make the same offer to every firm. We call this a *posted price mechanism*.[36]

Suppose that at the posted price r_n, it is not the case that all firms prefer winning (resp. losing) a license to losing (resp. winning).[37] Let c_r be the critical type that is indifferent between winning and not winning a license at this price. Therefore, c_r must be the solution of the following equation:

$$E\pi^W(c_r; c_{(1)}, \ldots, c_{(n-1)}) - r_n = E\pi^L(1; c_{(1)}, \ldots, c_{(n-1)}), \tag{34}$$

where

$$\begin{aligned} E\pi^W(c_r; c_{(1)}, \ldots, c_{(n-1)}) \quad = \quad & \int_0^{c_r} \int_0^{c_{(n-1)}} \cdots \int_0^{c_{(2)}} \pi^W(c_r; c_{(1)}, \ldots, c_{(n-1)}) \\ & \times f_{1(n-1)}^{n-1}\left(c_{(1)}, \ldots, c_{(n-1)}\right) dc_{(1)} \ldots dc_{(n-1)} \\ & + \cdots + \int_{c_r}^1 \int_{c_r}^{c_{(n-1)}} \cdots \int_{c_r}^{c_{(2)}} \pi^W(c_r) \\ & \times f_{1(n-1)}^{n-1}\left(c_{(1)}, \ldots, c_{(n-1)}\right) dc_{(1)} \ldots dc_{(n-1)} \end{aligned} \tag{35}$$

is the profit of a firm of type c_r from winning a license. We can similarly define the profits of a firm of type c_r from not winning a license as

[36] There is an article by Wang (1993) that has compared auctions to a posted-price mechanism in a private value framework and has specified conditions under which an auction raises more revenue than a posted-price mechanism. A similar comparison in the presence of externalities is beyond the scope of this article and is left for future research.

[37] The subscript n in r_n represents the fact that n licenses are auctioned.

$$
\begin{aligned}
E\pi^L(1;c_{(1)},\ldots,c_{(n-1)}) \quad = \quad & \int_0^{c_r}\int_0^{c_{(n-1)}}\cdots\int_0^{c_{(2)}} \pi^L(1;c_{(1)},\ldots,c_{(n-1)}) \\
& \times f_{1(n-1)}^{n-1}\left(c_{(1)},\ldots,c_{(n-1)}\right) dc_{(1)}\ldots dc_{(n-1)} \\
& +\cdots+\int_{c_r}^{1}\int_{c_r}^{c_{(n-1)}}\cdots\int_{c_r}^{c_{(2)}} \pi^0 \\
& \times f_{1(n-1)}^{n-1}\left(c_{(1)},\ldots,c_{(n-1)}\right) dc_{(1)}\ldots dc_{(n-1)}. \qquad (36)
\end{aligned}
$$

Using (35) and (36), we can compute the ex ante expected payment of each firm and the lab's revenue in the posted price mechanism.

Proposition 2.9. *The ex ante expected payment of a firm in the posted price mechanism when the posted price is* r_n *is*

$$M_n = r_n G(c_r) \qquad (37)$$

and the ex ante expected revenue of the lab is

$$R_{nr} = nG\left(c_r\right) r_n. \qquad (38)$$

Proof. The conclusions of the proposition follow from the fact that each firm's probability of winning is $G(c_r)$ and each winning firm pays r_n. □

When the types become common knowledge after the lab has committed to sell n licenses, the lab makes a take-it-or leave it offer to firm i at a price of

$$\pi^W(c_i;c_{(1)},\ldots,c_{(n)}) - \pi^L(1;c_{(1)},\ldots,c_{(n)}). \qquad (39)$$

Notice that, in this case, if a firm refuses to accept the lab's offer, the lab cannot inflict any punishment because the lab is selling to all the firms anyway. Hence, the lab offers to sell a license to firm i at a price equal to firm i's gain from accepting the offer. The revenue of the lab in this case is,

$$\hat{R}_n = E[\pi^W\left(c^n_{(1)}; c^n_{(2)}, \ldots, c^n_{(n)}\right) + \cdots + \pi^W\left(c^n_{(n)}; c^n_{(1)}, \ldots, c^n_{(n-1)}\right) - \pi^L(1; c^n_{(2)}, \ldots, c^n_{(n)}) - \ldots - \pi^L(1; c^n_{(1)}, \ldots, c^n_{(n-1)})]. \tag{40}$$

The Optimal Number Of Licenses

So far, we have derived the expressions for the lab's revenue as a function of the number of licenses. However, in order to use our analysis to find the optimal number of licenses, we need to make assumptions about the product market. In this section, we investigate the lab's choice of the optimal number of licenses to sell both when a firm's type is its private information and when a firm's type is revealed to the lab after the lab commits to sell a fixed number of licenses. Moreover, in the case in which there is a reserve price under private information about firms' types, we also determine the optimal reserve price and optimal critical value by considering a series of examples. The examples are distinguished by the nature of competition in the product market, the magnitude of the externalities firms impose on each other and the distribution of costs. Each of these factors affect the optimal number of licenses through its effect on the intrinsic values. We show that the optimal number of licenses may or may not be one. Hence, it is restrictive to assume, as in Jehiel and Moldovanu (2000) and Das Varma (2003), that the lab only sells one license. We also compare the optimal number of licenses when the types of the firms are revealed to the lab after it commits to sell a fixed number of licenses, with the optimal number of licenses when the type of a firm is its private information.

In each of our examples, there are three firms. Another common feature of all our examples is that the distribution of types (marginal costs) is given by the beta distribution with parameters λ_1 and λ_2. That is,

$$g(c) = \frac{1}{\beta(\lambda_1, \lambda_2)} c^{\lambda_1 - 1} (1 - c)^{\lambda_2 - 1}; \quad \forall\, c \in [0, 1].$$

For concreteness, (λ_1, λ_2) is chosen to be either $(1, 1)$ or $(1, 2)$. When $\lambda_1 = \lambda_2 = 1$, the distribution is uniform. When $\lambda_1 = 1$ and $\lambda_2 = 2$, the distribution is first-order stochastically dominated by the uniform distribution. The density functions for the order statistics can be computed from the density function for the distribution of types $g(c)$ (see David (1969, p. 9) and Reiss (1989, pp. 27-32)).

Competition in Quantities

First, we consider the case in which the firms are Cournot competitors in the product market. Let the inverse demand take the following form:

$$p_i = 3 - q_i - \mu \sum_{j \neq i} q_j, \text{ where } \quad \mu \in [0, 1].$$

The demand function has been chosen to ensure that all the assumptions on the profit function made in Section 2 hold. In the above (inverse) demand function, the parameter μ measures the magnitude of the externalities that competitors impose on each other. In the following three examples, we consider several specifications for μ and λ_2 and, in each case, determine the optimal number of licenses.

Example 1.1

We first let $\mu = 1$, $\lambda_1 = 1$ and $\lambda_2 = 1$. We first analyze the lab's revenue when a firm's type is its private information. In this case, the intrinsic value functions for the sale of one and two licenses are

$$V_1(c) = \frac{1}{2}(3 - c)(1 - c)$$

and

$$V_2(c) = \frac{1}{4}(4-c)(1-c).$$

Recall that when $k = n$, the intrinsic value is not defined.

We now compute the expected revenue of the lab, both with and without a reserve price (assuming that the reserve price is chosen optimally). The results are tabulated below (significant to two digits). In the table, k denotes the number of licenses, $\hat{R}_k$ denotes the lab's revenue when the types are revealed after the lab commits to sell a fixed number of licenses, while R_k denotes the lab's revenue under private information without a reserve price, and R_{kr} denotes the lab's revenue under private information with the optimal reserve price r_k and the corresponding optimal critical type c_r.

Table 1: Quantity Competition When The Externality Parameter is 1 And The Types are Drawn From The Uniform Distribution

k	$\hat{R}_k$	R_k	R_{kr}	c_r	r_k
1	1	0.65	0.66	0.55	0.45
2	1.21	0.43	0.64	0.40	0.55
3	1.13	0	0.61	0.36	0.56

Note that without a reserve price, the optimal number of licenses is 1. Moreover, with the optimal reserve price, the lab's optimal strategy is still to auction one license (with a reserve price of 0.45). However, the optimal number of licenses when the types are revealed after the lab commits to sell a fixed number of licenses, is 2.

Example 1.2

For the parameters characterizing the type distribution in Example 1.1, when the externality parameter is changed to $\mu = 0.8$, that is, when the firms sell more differentiated products compared to the previous example, the optimal number of licenses (under private information) depends on whether there is a reserve price or not. Table 2 is the analogue to Table 1 for the new value of μ.

Table 2: Quantity Competition When The Externality Parameter is 0.8 And The Types are Drawn From The Beta(1,2) Distribution

k	$\hat{R}_k$	R_k	R_{kr}	c_r	r_k
1	0.88	0.58	0.59	0.56	0.40
2	1.16	0.43	0.61	0.43	0.48
3	1.14	0	0.59	0.40	0.50

Note that compared to Example 1.1, the revenue of the lab under private information is lower in this example for any number of licenses. Because we have a smaller value of the externality parameter, the cost of losing the auction is also smaller for the firms, and this reduces the revenues of the lab. In this example, under private information, the optimal number of licenses without a reserve price is 1, but it changes to 2 when we use the optimal reserve price. Observe that, the optimal number of licenses when the types are revealed after the lab commits to sell a fixed number of licenses, is still 2.

Example 1.3

We now consider another interesting specification of the parameter values. Suppose that $\mu = 0.7$, $\lambda_1 = 1$ and $\lambda_2 = 2$. Compared to Example 1.2, the magnitude of the externalities that firms impose on each other has been reduced (which tends to reduce the lab's revenue), but at the same time, the probability that a firm has a better (lower) draw from the cost distribution increases (which tends to increase the lab's revenue). In this example, the optimal number of licenses is 3, if the lab can observe the firms' types after it has committed to sell a fixed number of licenses.

Table 3: Quantity Competition When The Externality Parameter is 0.7 And The Types are Drawn From The Beta(1,2) Distribution

k	$\hat{R}_k$	R_k	R_{kr}	c_r	r_k
1	1.01	0.79	0.79	0.62	0.33
2	1.46	0.82	0.91	0.39	0.50
3	1.48	0	0.87	0.33	0.53

Compared to Example 1.2, the lab's revenues increase in this example under private information. Furthermore, the optimal number of licenses with or without a reserve price is 2. This example shows that, under private information, the optimal number of licenses without a reserve price need not be 1.

Competition in Prices

We now suppose that the three firms compete in prices in a differentiated product market. Let the demand function for firm i be given by

$$q_i = 2 - 2p_i + \mu \sum_{j \neq i} p_j.$$

As in Section 6.1, μ measures the magnitude of the externalities that firms impose on their competitors.

Example 2.1

First, consider the benchmark example with $\mu = 1$, $\lambda_1 = 1$ and $\lambda_2 = 1$. In this example, the intrinsic value functions are

$$V_1(c) = \frac{6}{25}(11 - c)(1 - c)$$

and

$$V_2(c) = \frac{54}{25}(1-c).$$

One can easily check that for all c in $[0,1)$, we have $V_1(c) > V_2(c)$. We present our findings in Table 4.

Table 4: Price Competition When The Externality Parameter is 1 And The Types are Drawn From The Beta(1,2) Distribution

k	$\hat{R}_k$	R_k	R_{kr}	c_r	$\mathbf{r}_k$
1	1.77	1.27	1.27	1	0
2	2.36	1.08	1.24	0.54	0.74
3	2.24	0	1.15	0.45	0.86

In this example, under private information, the optimal number of licenses with or without a reserve price is 1. However, the most interesting conclusion to be drawn from Table 4 is that the optimal reserve price when one license is being auctioned is 0.[38]

Example 2.2

We now change the externality parameter to $\mu = 0.75$, keeping the parameters of the type distribution as in the preceding example. Our findings are summarized in Table 5.

Because the externality parameter is smaller than in Example 2.1, the revenues of the lab are smaller as well. In this example, under private information, the lab's optimal strategy

[38] See the discussion at the end of Section 4 to see why this may be the case.

Table 5: Price Competition When The Externality Parameter is 0.75 And The Types are Drawn From The Beta(1,2) Distribution

k	$\hat{R}_k$	R_k	R_{kr}	c_r	$\mathbf{r}_k$
1	1.43	0.81	0.82	0.6	0.5
2	2.13	0.67	0.88	0.47	0.63
3	2.18	0	0.86	0.43	0.13

is to auction two licenses with a reserve price of 0.63. However, without a reserve price, the optimal number of licenses is 1. Therefore, under private information, the optimal number of licenses with and without a reserve price may be different.

Note that in all of our examples, under private information, setting the reserve price optimally rather than having no reserve price has a much more significant impact on the lab's revenues when two licenses are being sold instead of one. To understand this phenomenon, consider a uniform price auction. In our examples, we have $V_1(c) \geq V_2(c)$ for all c. Moreover, the price when one license is being sold is the second highest bid, whereas the price when two licenses are being sold is the third highest bid. Both of the above mentioned factors combine to ensure that in the absence of a reserve price, the price of a license tends to be lower when two licenses are being auctioned instead of one. A reserve price bounds from below the price of a license and, hence, it tends to increase the price of a license more when two licenses are being auctioned instead of one. Because this is true for each license sold, this effect gets magnified when more licenses are being auctioned. Hence, the optimal reserve price tends to have a much more significant impact on the revenue of the lab when it auctions two licenses rather than one.

Another regularity in our examples is that the optimal number of licenses under private information is never higher than the optimal number of licenses when each firm's type is revealed to a lab after it has committed to sell a fixed number of licenses. This is in contrast to a finding in Schmitz (2002) that the optimal number of licenses under private information can be higher, than the optimal number of licenses when the type of a firm is revealed to the lab. Schmitz assumes that if two firms win a license each, the profit of each of these firms is 0. We obtain different results because we do not restrict the profit function in this way.

Concluding Remarks

In this article, we have analyzed an auction of licenses to a process innovation. Firms who win a license impose a negative externality on their competitors. More generally, this article can be viewed as extending the analysis of Milgrom and Weber (1982) to allow for the possibility that the acquisition of the object generates a negative externality to the other bidders. We have shown that the value of a license to a firm of a given type is its intrinsic value; that is, this firm's value of winning a license rather than having a competitor with the same marginal cost win it instead. We have found that the amount that each type will bid in a uniform price auction is its intrinsic value, whereas, in a discriminatory auction, the amount a firm bids is the expectation of the intrinsic value for a license, conditional on winning.

Our two most striking conclusions are (*i*) it is not always optimal to sell an exclusive license, whether there is a reserve price or not, and (*ii*) the optimal reserve price may be zero. The first of these conclusions demonstrates the restrictiveness of the assumption used in the previous literature that the lab only auctions one license. The second of these conclusions demonstrates that the standard result that optimal reserve prices with independent types are positive may be overturned in the presence of type-dependent negative externalities.

Because of the possibility that a winner of a license imposes a negative externality on other firms, each firm tries to preempt its competitors from winning a license by raising its bid. Failure to recognize this preemption motive may lead the lab to sell a nonoptimal number of licenses. In our model, we have assumed that each firm can bid for only one license and, therefore, it can prevent at most one competitor from winning a license. It may be more realistic to suppose that a firm would try to prevent as many competitors as possible from winning licenses. To capture the full force of the bid-raising effect described above, we need to relax the assumption that firms can only bid for one license. This issue is the subject of ongoing research.

CHAPTER III

A LABORATORY TEST OF AN AUCTION WITH NEGATIVE EXTERNALITIES

Consider several competing firms with similar cost structures. A newly invented process innovation promises to reduce the firms' marginal costs, though the firms differ in the extent of this reduction. If the inventor elects to auction licenses to this technology, how do industry structure and auction design factors influence the auctioneer's revenue?

While several authors forward theoretical predictions about such auction markets (e.g., Jehiel, Moldovanu, and Stachetti, 1996 and 1999; Jehiel and Moldovanu, 2000; Das Varma, 2002 and 2003; Bagchi, 2005b), we provide theoretical benchmarks and examine experimentally two specific questions. First, as the industry becomes more competitive, perhaps through less differentiation of products, what happens to auction revenue? More intense competition depresses overall industry profitability but also implies that losers of the auction suffer a greater externality from winners' lower costs, perhaps increasing bids. Second, how does auction revenue change with the decision to offer multiple licenses for sale? Selling more licenses implies that more firms pay the auctioneer but also diminishes the competitive advantage of any single winner since others also enjoy reduced costs, potentially reducing bids. In all cases, participants' bids depend not only on the extent of cost savings they may anticipate from the new technology but also on their beliefs about the efficiency of rivals.

We develop a model of an inventor selling licenses for a cost-reducing technology using a uniform price auction. Each firm bids for at most one license; if k licenses are auctioned, the k firms with the highest bids win a license and pay an amount equal to the $(k+1)^{th}$ highest bid.[39] Prior to the auction, firms have identical costs though they differ in their

[39] Acquiring one license gives a firm user rights to the cost-reducing technology. If a single firm was able to acquire additional licenses, these would not have any direct effect on the firm's cost structure, but would have a preemptive motive, foreclosing on the use of the license by a competitor. This is not our focus. For a theoretical treatment of such cases, see Bagchi (2005a).

abilities to implement the process improvement. Firms receive private signals as to the degree of cost savings they can realize by winning a license. In our experiment, three firms participate in an auction for either one or two licenses. Payoffs from the auction are derived from Cournot competition in a differentiated-goods industry with costs determined by the allocation of licenses resulting from the auction. We vary the level of product differentiation in the Cournot industry which alters the extent to which one firm's cost savings impacts the profitability of its rivals.

In the symmetric equilibrium of our model, each firm bids its "intrinsic value" (Das Varma, 2003; Bagchi, 2005b), the expected difference between its profit from winning a license and its profit from losing the license to a competing firm with the same signal. Thus, the equilibrium bid incorporates, in an additive fashion, the expected change in profit both from winning and from losing the auction. Relative to theoretical predictions, subjects undervalue the profit from winning though possibly compensate for this by adding a positive constant to bids. In auctions of two licenses, subjects also overemphasize the profit from losing. That is, they overreact to the possibility of profit losses from not obtaining a license. The net effect of these factors is overbidding in auctions of two licenses and underbidding in auctions of one license. Nevertheless, mean revenues are mostly in line with theoretical predictions except in the cases of multiple licenses being sold in the presence of moderate externalities. Therefore, we conclude that when the products are moderately differentiated, an auction of two licenses performs better than what the model suggests; in the other cases, the observed revenues are close enough to the predicted revenues.

Although revenues are mostly in line with theoretical predictions, there are systematic departures between observed and predicted *distributions* of revenues. In the sale of one license, observed revenues show less dispersion than predicted, while in the sale of two licenses, the observed revenue distributions generally lie below the theoretically predicted distribution. We show that a simple bidding heuristic can explain both anomalies. When bidding, a subject does not know the signals of other firms, which represent their level of cost savings upon acquiring a license. In the absence of this information, it is possible that subjects do not undertake the rather arduous task of calculating equilibrium expectations conditional on their own signals but instead simply assume (or act as if they assume) that the winners' private information is equal to the unconditional expected value of the signal. Predicted revenues for the auctioneer when each firm bids according to this simple heuristic are consistent with our data. On the whole, subjects appear to react to externalities in the manner that we predict and discrepancies between the experimental data and our predictions can be explained by the fact that subjects do not correctly estimate the signals of competitors who win a license. In real world auctions, firms are likely to form better estimates of competitors' signals than do our subjects, leading revenues to approximate our predictions more closely. However, for auctions in which bidders are not likely to show great sophistication, our heuristic provides a simple revenue benchmark. We show that both bidding frameworks – in line with the equilibrium or the heuristic – generally lead to the same guidance as to the optimal number of licenses to auction.

Markets for a single license with private information have recently received considerable attention (Jehiel, Moldovanu, and Stachetti, 1996 and 1999; Moldovanu and Sela, 2003; Katzman and Rhodes-Kropf, 2002; Goeree, 2003), including the analysis by Jehiel and Moldovanu (2000) of a second price auction and Das Varma (2003) of a first price auction. Several papers have considered the sale of multiple licenses. Katz and Shapiro (1986) and Hoppe, Jehiel, and Moldovanu (forthcoming) assume that the signal of a firm is publicly observable. Jehiel and Moldovanu (2001, 2004) demonstrate the impossibility of implementing efficient allocations when signals are multi-dimensional. We consider private, unidimensional signals representing the realized cost savings if a license is acquired, allowing for equilibrium characterization. Dana (1994) and Schmitz (2002) also consider the problem of auctioning production rights but assume that a firm that does not acquire a license earns zero profits, while we, following Bagchi (2005b), allow a losing firm's profits to decrease as a result of the cost savings realized by license-acquiring competitors.

Several papers examine experimentally auctions with interdependent valuations. Kirchkamp and Moldovanu (2004) consider auctions in which the winner's payoff depends on the private information of another, specific bidder. Only one object is sold, however, and winners do not impose an externality on losing bidders. Goeree, Offerman, and Sloof (2004) examine bidding when new entrants impose a negative externality on existing market participants. In multi-unit auctions, the authors find that the existence of externalities does not eliminate strategic demand reduction (Alsemgeest, Noussair, and Olson, 1998; List and Lucking-Reiley, 2000) when subjects can bid for multiple licenses. To our knowledge, Goeree, Offerman, and Sloof (2004) is the only other manuscript to consider auctions in which negative externalities impact both winners and losers.

Theoretical Considerations

Model

The model follows Bagchi (2005b) which extends the previous literature on single-object auctions with externalities to the case of multiple licenses. Notable papers on license auctions are Jehiel and Moldovanu (2000) that analyzes a second price auction and Das Varma (2003) that analyzes a first price auction of licenses when a winner signals her type through her bid. Consider an industry with n competing firms. The profit of firm i, $\pi(c_i; c_{-i}, \xi)$, depends on its own marginal cost of production, c_i, the vector of rivals' costs, c_{-i}, and a parameter, ξ, representing the the strength of externalities or the degree to which one firm's cost savings imposes an externality on its rivals. It may be convenient to interpret ξ as the degree of product substitutability. When ξ is large, firms are selling similar products leading to more intense competition, while a low value of ξ implies that competition among rivals is low. Firm i's profit is (i) decreasing in its own costs ($\frac{\partial \pi(c_i; c_{-i}, \xi)}{\partial c_i} < 0$), (ii) increasing in others' marginal cost ($\frac{\partial \pi(c_i;c_{-i},\xi)}{\partial c_j} > 0, j \neq i$), and (iii) the effect of a change in a competitor's cost on i's profits is increasing in ξ, the externality parameter ($\frac{\partial^2 \pi(c_i;c_{-i},\xi)}{\partial \xi \partial c_j} > 0, j \neq i$). Assumption (ii) suggests that a decrease in costs imposes a negative externality on one's competitors. Assumption (iii) allows us to vary the strength of this externality.

Initially, all firms have identical marginal costs, $\overline{c}$, perhaps a result of access to an existing publicly available technology or convergence of industry practice. An independent inventor develops a new cost-reducing technology which benefits firms differentially. In particular, each firm i receives a private signal $\theta_i \in [0, \overline{c}]$ drawn independently from the distribution function $F(\theta)$. These signals represent the cost savings realized by the firm if it were to employ the new technology, reducing its marginal cost to $c_i = \overline{c} - \theta_i$. Let $\theta_{(k)}^{n-1}$ be the k^{th} highest signal among firm i's competitors, so that

$$\theta_{(1)}^{n-1} \geq \theta_{(2)}^{n-1} \geq \ldots \geq \theta_{(n-1)}^{n-1}. \tag{41}$$

It will be convenient to express profit as a function of the firms' signals. Suppose that the k firms who could achieve the greatest cost savings (have the highest values of θ_i) obtain licenses to the new production technology through an auction. If firm i is among the winners, its profit is given by:

$$\Pi\left(\theta_i; \theta_{(1)}^{n-1}, \ldots, \theta_{(k-1)}^{n-1}, \xi\right) \equiv \pi\left(\overline{c} - \theta_i; \overline{c} - \theta_{(1)}^{n-1}, \ldots, \overline{c} - \theta_{(k-1)}^{n-1}, \overline{c}_{n-k}, \xi\right) \tag{42}$$

where $\overline{c}_{n-k}$ is a vector of dimension $n-k$ whose components equal $\overline{c}$.

Analogously, when firm i is not among the k firms with the highest values of θ, the firm continues to employ the old technology and earns profits of:

$$\Pi\left(0; \theta_{(1)}^{n-1}, \ldots, \theta_{(k)}^{n-1}, \xi\right) \equiv \pi\left(\overline{c}; \overline{c} - \theta_{(1)}^{n-1}, \ldots, \overline{c} - \theta_{(k)}^{n-1}, \overline{c}_{n-1-k}, \xi\right). \tag{43}$$

Lastly, we consider the profit prior to any firm acquiring a license. Each firm has a cost of

$\overline{c}$, or equivalently, a cost saving of 0, with profit given by

$$\Pi(0;\xi) \equiv \pi(\overline{c};\overline{c}_{n-1},\xi). \tag{44}$$

The inventor of the technology auctions $k < n$ licenses, with firms submitting bids for a single license. Though the model may be generalized to most common auction formats, we consider a uniform-price auction in which each of the k highest bidders wins a license and pays an amount equal to the $(k+1)^{th}$ highest bid. We restrict our attention to the increasing symmetric equilibrium, which implies that the firms with the greatest cost savings (signals) win the auction.

Equilibrium

Firms that win a license enjoy increased profits while losing firms suffer decreased profits when competitors reduce costs. In our experiment, we study the role of each of these effects – the profit from winning and from losing – in determining subjects' bids. Define the *winning value*, the change in profit accruing to a winner of the auction, as

$$W_k(\theta_i,\xi) = E\left[\Pi\left(\theta_i;\theta_{(1)}^{n-1},\ldots,\theta_{(k-1)}^{n-1},\xi\right)|\theta_{(k)}^{n-1}=\theta_i\right] - \Pi(0;\xi) \tag{45}$$

The winning value is the expected increase in profit from obtaining a license. It is the difference between firm i's profit when winning the auction and its profit when every firm uses the old technology. Analogously, the *losing value* is given by

$$L_k(\theta_i,\xi) = E\left[\Pi\left(0;\theta_{(1)}^{n-1},\ldots,\theta_{(k)}^{n-1},\xi\right)|\theta_{(k)}^{n-1}=\theta_i\right] - \Pi(0;\xi) \tag{46}$$

and is the difference between firm i's profits from losing the auction when the marginal winner has the same signal as firm i and firm i's profits when all firms use the old technology.

For concreteness, consider the bidder's problem in the absence of externalities, as would occur if each firm were a monopolist in its market. The winning value would simply reflect the profit gain from lowering costs. The losing value would equal zero since other firms' costs do not enter into a monopolist's profit function. A private value auction with *i.i.d* valuations would ensue. In the presence of externalities, bidders must account for the reduction in profit that results from losing the auction and the impact of other winners if more than one license is sold. We define the *intrinsic value*, $V_k(\theta_i, \xi)$, as the difference between the winning value and the losing value for firm i when k licenses are auctioned.[40]

$$\begin{aligned} V_k(\theta_i, \xi) &\equiv W_k(\theta_i, \xi) - L_k(\theta_i, \xi). \\ &= E\left[\Pi\left(\theta_i; \theta_{(1)}^{n-1}, \ldots, \theta_{(k-1)}^{n-1}, \xi\right) - \Pi\left(0; \theta_{(1)}^{n-1}, \ldots, \theta_{(k)}^{n-1}, \xi\right) | \theta_{(k)}^{n-1} = \theta_i\right]. \end{aligned} \tag{47}$$

The intrinsic value represents the expected difference in a firm's profit between winning a license and losing a license when the marginal winner's signal is θ_i. The following proposition characterizes the equilibrium of the auction.

Proposition 3.1. *In the unique increasing symmetric equilibrium of a uniform price auction of $k < n$ licenses, the bid of firm i is given by $b_k(\theta_i; \xi) = V_k(\theta_i, \xi)$.*

Proof. See Appendix B. □

Since bids equal intrinsic values in equilibrium, expected revenue is characterized by the auctioneer collecting k payments each equal to the $(k+1)^{th}$ order statistic of intrinsic values.

Corollary 3.2. *The ex-ante expected revenue for the auctioneer of $k < n$ licenses in a uniform price auction is given by*

$$R_k(\xi) = kE\left[V_k\left(\theta_{(k+1)}^{n}, \xi\right)\right] \tag{48}$$

[40] Das Varma (2003) analyzes a first price auction of a single license in which firms signal their types through their bids. One component of the bidding function in his model is a function of the intrinsic value. Bagchi (2005b) generalizes the intrinsic value for auctions of multiple licenses. The expression shares similarities with the equilibrium bid for a common value auction derived in Milgrom and Weber (1982).

Very little insight on the revenue effects of these auctions can be gleaned directly from this equation. First, increasing the externality parameter, ξ, may increase or decrease revenues, depending on the profit function. Greater competition generally decreases the value from winning the auction but also decreases a loser's profit, leading to ambiguous effects in the aggregate. Second, an increase in the number of licenses auctioned likewise may increase or decrease auction revenue. The effect depends on the relative changes in winning value and losing value caused by offering more units and on the properties of the order statistics of signals induced by the distribution $F(\theta)$. Lastly, revenue effects need not be monotonic in either the number of licenses sold, k, or the externality parameter, ξ. Thus, for some profit specifications, selling to a very competitive industry may result in higher profits than selling to a less competitive one, while the reverse can hold for other specifications. The experimental treatments we consider exhibit several of these features.

Experimental Design

We now describe the specific form of the model we use in the experiments. Three firms with differentiated products compete in quantities *a la* Cournot. The inverse demand function for firm i is given by

$$p_i = 300 - q_i - \xi \sum_{j \neq i} q_j \ , \tag{49}$$

where $\xi \in [0, 1]$ captures the level of product differentiation. We consider three cases:

((i)) $\xi = 0$ (monopoly) in which competitors' quantities do not influence own price,

((ii)) $\xi = 1/2$ (differentiation) in which firms produce imperfect substitutes, and

((iii)) $\xi = 1$ (homogeneity) which is the case of identical products.

Prior to the auction, each firm has a marginal cost of $\overline{c} = 100$. Auctions are either for one license ($k = 1$) or two licenses ($k = 2$). Each firm receives a private signal, θ_i, distributed *i.i.d.* uniform on $[0, 100]$, representing the cost savings resulting from winning a license.

For any outcome of the auction, the resulting change in profits for subject i is given by:

$$\Delta\Pi\left(\theta_i;\theta_j,\theta_k\right)=\left[\frac{200\left(2-\xi\right)+\left(2+\xi\right)I_i\theta_i-\xi\left(I_j\theta_j+I_k\theta_k\right)}{2\left(1+\xi\right)\left(2-\xi\right)}\right]^2-\left[\frac{100}{1+\xi}\right]^2 \tag{50}$$

where the first term is the post-auction profit, the second term is the pre-auction profit when each firm has a cost of 100, and I_l is an indicator function taking the value of 1 if player l is a winner of the auction and zero otherwise. Next, we derive predictions for equilibrium bids and revenues.

Proposition 3.3. *Equilibrium bids and resulting expected revenues for $k \in \{1,2\}$ are given by:*

$$b_1\left(\theta_i;\xi\right)=V_1\left(\theta_i;\xi\right) = \frac{\left[200\left(2-\xi\right)+\theta_i\right]\theta_i}{\left(1+\xi\right)\left(2-\xi\right)^2} \tag{51}$$

$$b_2\left(\theta_i;\xi\right)=V_2\left(\theta_i;\xi\right) = \frac{\left[100\left(8-5\xi\right)+\left(2-\xi\right)\theta_i\right]\theta_i}{2\left(1+\xi\right)\left(2-\xi\right)^2} \tag{52}$$

$$R_1(\xi) = \frac{10000\left(2.3-\xi\right)}{(1+\xi)(2-\xi)^2} \tag{53}$$

$$R_2(\xi) = \frac{10000\left(2.2-1.35\xi\right)}{(1+\xi)(2-\xi)^2} \tag{54}$$

In keeping with our intuition about Cournot competition, it can be verified that total industry profits decline as products become more substitutable (as ξ increases) for any profile of costs. However, declining industry profitability need not imply that the auctioneer's profit is decreasing. This specification of profits has four revenue implications of interest, reflected in Table 6. First, when selling a single license ($k = 1$), a perfectly homogeneous product market yields greater profit for the auctioneer than a market of monopolists. Second, the opposite result obtains for the sale of two licenses ($k = 2$). Third, in the single license case ($k = 1$), the intermediate case of $\xi = 0.5$ results in lower revenue than either polar case of

$\xi = 0$ or $\xi = 1$, implying that revenues need not be monotonic in ξ. Finally, for the chosen parameters, auctioning one license generates more revenue than auctioning two licenses. We wish to examine how closely these revenue predictions match experimental data.

Table 6: Predicted Revenue In Experimental Treatments

	$\xi = 0$ monopoly (no externalities)	$\xi = 0.5$ differentiation (weak externalities)	$\xi = 1$ homogeneity (strong externalities)
$k = 1$	5750	5333	6500
$k = 2$	5500	4519	4250

Experimental Method

Our subject population was comprised of 78 students at Vanderbilt University. Because the focus of the present study is on how subjects react to externalities, traditional overbidding observed in experiments of uniform-price auctions (e.g., Kagel, Harstad, and Levin, 1987; Kagel and Levin, 1993) would confound data interpretation. To avoid this, most subjects (92%) were M.B.A. students who had completed introductory lectures on the theory of auctions. Specifically, subjects had learned of the dominant strategy in second price auctions to bid one's value in private value settings, and had participated in several high-stakes experiments (for exemption from a final exam) as part of a course in game theory. No treatment of externalities was included in course materials. While there is evidence that experience in previous economics experiments tends to improve bidding behavior (Harstad, 2000), the results we describe in our monopoly treatments – which are equivalent to private value auctions – suggest that formal instruction is also likely to improve bidding behavior (McCabe and Smith, 2000).

Each subject participated in a series of either second price auctions for one license ($k = 1$) or third price auctions for two licenses ($k = 2$). Subjects bid in a total of 15 auctions, five at each of three values of the substitution parameter, $\xi \in \{0, 0.5, 1\}$. Subjects were told that in each auction, they would be randomly matched with two other participants whose identities would not be revealed.

Prior to each series of five auctions (for a specific value of ξ), subjects were shown the relevant payoff function (Equation 50 for the specific value of k and ξ) and were provided with tables that provided numerical values of this profit under various scenarios. In every auction, each subject received a signal from the uniform distribution on $[0, 100]$. For a specific signal in a particular auction, subjects were presented with tables of payoffs accruing to both a winner and loser of the auction for various signals of their competitors. Prior to bidding in each series of five auctions, subjects worked through an example and had to correctly calculate the resulting profits in two scenarios (winning and losing) to proceed. The majority of subjects successfully completed each of these tests on the first try. The average number of attempts on each test was 1.39. Subjects did not observe the outcome of any auction until the conclusion of the experiment, which limits intra-game learning and path dependency of wealth effects.

Subjects received a participation fee of $20 to which winnings were added and losses deducted. Payoffs were denoted in points with 1000 points convertible into one dollar. These payoffs were governed by Equation 50. A winner of an auction earns the change in profit due to her decreased cost and pays the resulting price (second or third highest bid). A subject who loses an auction suffers lower profits due to others' cost reductions (except in the no-externality, monopoly case, when $\xi = 0$). Three subjects went broke, losing the entire participation fee during the experiment. These subjects were dropped from the sample for the purpose of the analysis.[41] The treatments and number of observations in each is reported in Table 7.

The experiment required an average of 29 minutes to complete and subjects earned an

[41] All three subjects participated in the $k = 1$ treatment, and generally bid the same (unusually large) amounts in each auction regardless of their signals. Their inclusion in the data analysis increases parameter values in the $k = 1$ treatment.

average of \$33. In actuality, all subjects were paid a minimum of \$15 even if their net earnings were lower than this amount, though they were not informed of this prior to the experiment.[42]

Table 7: Experimental Treatments

	monopoly ($\xi = 0$) 5 bids per subject	differentiation ($\xi = 0.5$) 5 bids per subject	homogeneity ($\xi = 1$) 5 bids per subject
one license ($k = 1$) $N = 39$	180 bids 60 auctions	180 bids 60 auctions	180 bids 60 auctions
two licenses ($k = 2$) $N = 39$	195 bids 65 auctions	195 bids 65 auctions	195 bids 65 auctions

While subjects were free to bid any amount, we compute the maximum value a license could possibly hold for a participant. As is common in experiments with second price auctions, several subjects bid unreasonably high amounts which would significantly skew the analysis. For the purpose of analysis, subjects' bids were censored from above at the maximum possible difference in value between winning and losing the auction.[43] Any bid above this value is weakly dominated by bidding this value. Specifically, the most value one can derive from winning an auction occurs when the subject has the maximum signal of 100 and, if $k = 2$, when the other winner has a signal of 0. The worst loss can occur when a subject loses the auction and each winner has the maximum signal of 100. We censor at the difference between the best gain and worse loss. Thus, for $k = 1$, bids above 12,500, 11,852, and 15,000 were censored for $\xi = 0$, 0.5, and 1. Similarly, for $k = 2$, bids were censored at 12,500, 10,371, and 10,000. This censoring affected 2.4% of bids in the $\xi = 0$ condition, 9.1%

[42] Keeping MBA subjects happy is a secondary, though institutionally-imposed, concern.

[43] Methods of dealing with severe overbidding above any reasonably obtainable value include prohibiting subjects from bidding above the maximum value (e.g., Mares and Shor, 2004), cautioning subjects not to do so (e.g., Kagel and Levin, 2001), or simply censoring such bids by setting them equal to the maximum possible valuation prior to data analysis (e.g., Kagel and Levin, 2004). Since the maximum intrinsic value is a more complicated object than just the maximum draw, censoring after the experiment seems the easiest approach as it requires no further explanation to subjects.

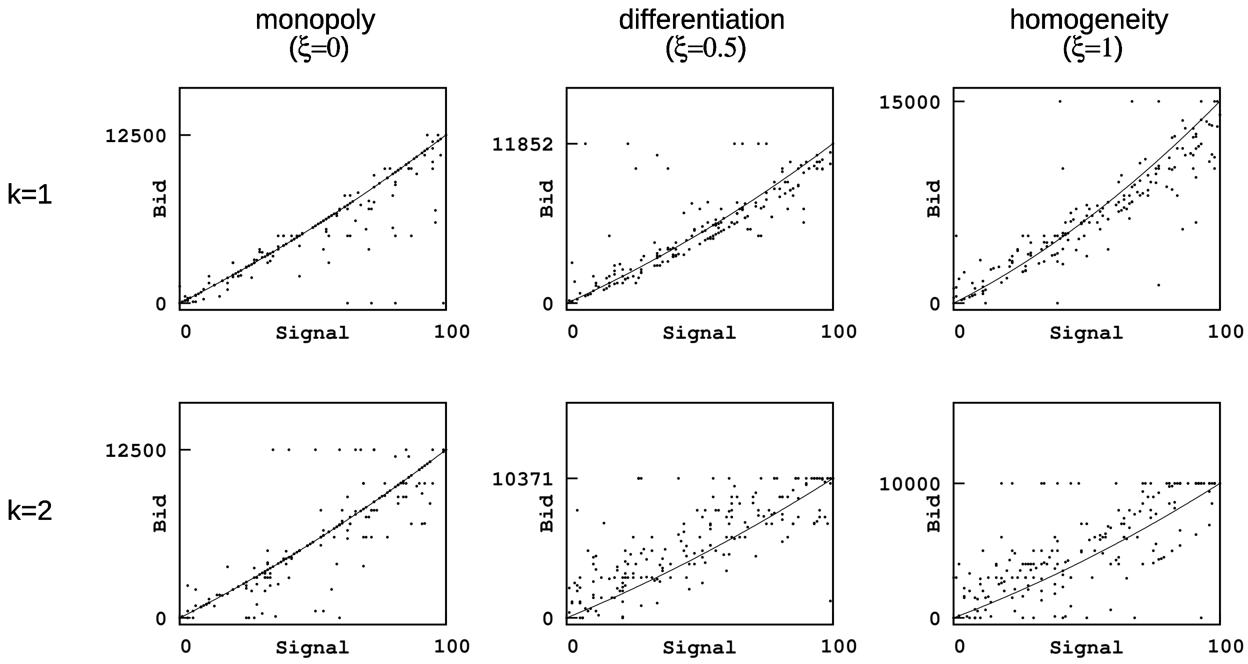

Figure 1: Observed versus predicted bids

when $\xi = 0.5$, and 8.3% when $\xi = 1$.

Results

Summary Statistics of Bids

We begin by examining raw data on bids. Plots suggest a strong co-movement between observed and equilibrium bids (Figure 1). In both monopoly cases, in which no externalities exist, most bids are at or slightly below the intrinsic value. When externalities are introduced, bids appear more dispersed and exhibit a higher incidence of values at the censored upper bound of permissible bids.

We do not find any evidence of overbidding in the monopoly cases. In the aggregate, subjects underbid relative to equilibrium predictions in an auction of one license and conform

Table 8: Mean and Standard Deviation of the Bids

		Observed (std. dev.)	Predicted† (std. dev)	p-value ‡ $H_1 : obs \neq pred$
k=1	monopoly	5333	5876	< 0.01
$N = 180$	($\xi = 0$)	(3550)	(3676)	
	differentiation	5619	5558	0.66
	($\xi = 0.5$)	(3170)	(3237)	
	homogeneity	6558	6975	0.01
	($\xi = 1$)	(3997)	(4392)	
k=2	monopoly	5947	6090	0.28
$N = 195$	($\xi = 0$)	(3740)	(3388)	
	differentiation	5646	4678	< 0.01
	($\xi = 0.5$)	(3218)	(3151)	
	homogeneity	5172	4434	< 0.01
	($\xi = 1$)	(3329)	(2940)	

†Mean and standard deviation of intrinsic values evaluated at subjects' signals.
‡Results of matched pair t-test determining whether the distribution of differences between bids and corresponding intrinsic values is significant.

to equilibrium predictions in an auction of two licenses (Table 8). This departure from previous experiments lends support for our use of "sophisticated" bidders with familiarity of uniform-price auctions and allows us to conclude with some confidence that any departures from equilibrium in the presence of externalities (especially overbidding) are due to the externalities themselves. In the presence of externalities, we find evidence of overbidding only for auctions of two licenses.

Remark 1. *Bids exceed theoretical predictions when two licenses are auctioned in the presence of externalities. In all other cases, subjects either underbid or bid in line with theoretical predictions.*

In the sale of one license, we expect bids to first decrease and then increase as the negative externalities become more prominent. Comparing bidding in the monopoly case ($\xi = 0$, no externalities) with the differentiated products case ($\xi = 0.5$), average bids actually increase, though not significantly ($p = .21$ one-tailed). As ξ increases from 0.5 to 1, bids increase ($p < .01$) in accord with theoretical predictions. When two licences are sold, consistent with theoretical predictions, subjects bid less when the level of competition increases. However, on average, this decline is not as dramatic as theoretically predicted. For our data, the change in mean bid is not significant ($p = .20$, one-tailed) when ξ increases from 0 to 0.5 and only mildly significant ($p = .08$) when ξ increases from 0.5 to 1.

Remark 2. *Overall, bids appear to increase with the level of competition in the case of one license and to decrease (though by less than predicted) in the case of two licenses.*

In the presence of externalities, our subjects do not exceed theoretically predicted bids in an auction of one license but do overbid in auctions of two licenses. Before examining the implication of these behaviors on revenue, the next two subsections consider factors that may have contributed to the bidding patterns. We examine whether subjects properly account for both the value of winning and the value of losing a license in their bids. As a preview of the results, bids are on the whole in line with theoretical predictions. However, a simple heuristic model that does not require subjects to calculate conditional expectations about rivals' signals also conforms to observed bids.

Auctions of One License

We analyze whether subjects bid in accordance with theoretical predictions. Because we observe multiple bids for each subject, these 15 observations may exhibit similar subject-specific idiosyncracies and may not be considered independent. We also must account for the fact that bids are censored from above, and the points at which they are censored differ across treatments. We estimate a series of fixed-effects censored normal regressions of the following form:

$$bid_{ij} = \alpha_i + \beta X_{ij} + \epsilon_{ij},$$

where i indexes the subject and j indexes each of the 15 bids placed by that subject. The variable α_i captures person-specific fixed effects and X_{ij} is the matrix of independent variables of interest. As an initial test of theoretical predictions, we regress subjects' bids on

Table 9: Estimation Of Bids In The Sale of One License

	Censored regression models			
	1	2	3	4
Intrinsic Value	0.843***	0.846***		
	(0.024)	(0.024)		
Winning Value			0.818***	0.910***
			(0.020)	(0.018)
Losing Value			−1.091***	
			(0.119)	
Lose 50				−1.187***
				(0.136)
$\xi = 0.5$ (differentiation)		584***		
		(149)		
$\xi = 1.0$ (homogeneity)		297***		
		(150)		
Average fixed effect	679	371	669	104
% of fixed effects significant at < 0.05	55.6%	38.9%	55.6%	27.8%

Note: dependent variable is a subject's bid. Standard errors are shown in parentheses.
*** All coefficients are significant at 0.01. $N = 585$.

intrinsic values, which are the equilibrium bids (Table 9, Model 1). The coefficient on intrinsic value is highly significant though less than 1 ($p < .01$). However, contrary to theoretical predictions, the fixed effects are significant in a majority of cases and generally positive.

To understand why subjects deviate from equilibrium predictions, Models 2 and 3 incorporate additional potential variables. First, we include dummy variables for the two types of auctions with externalities: $\xi = 0.5$ (differentiation) and $\xi = 1.0$ (homogeneity). The results (Model 2) suggest that bids are higher relative to equilibrium predictions when externalities are present. Additionally, deviations from equilibrium are similar in both auctions with externalities as the coefficients on the two dummy variables are not highly significantly different from each other ($p = .06$).

Since the equilibrium bid additively incorporates a subject's profit from winning and from losing the auction, we explore how each of these components contributes to a subject's bid. The winning value (Equation 45) is equal to the added profit from winning the auction and decreasing one's costs. In the case of a single license offered for sale, this profit improvement is always positive. The losing value (Equation 46) is the change in profit when another bidder who is presumed to have the same signal wins the auction. This losing value is equal to zero when $\xi = 0$ since rival firms have no impact on profit, and is negative in the other cases. To understand the relative weight subjects place on winning and losing, we regress bids on winning value and losing value separately (Model 3). Since the equilibrium bid is the difference between the winning and losing values, we predict that the coefficients on winning value and losing value are equal to 1 and -1. Subjects appear to place a weight less than 1 on the winning value but they place a weight not significantly different from -1 on the losing value ($p = .44$).

Again in Model 3 we note that the fixed effects are significant for a majority of the subjects, implying that bids are not derived solely from the values from winning and losing the auction. We consider another specification that might provide an explanation. While determining the profit from winning an auction for a single license is straightforward – it is the profit given one's signal – determining the expected profit from losing requires assumptions about what signal the winner is likely to have. In equilibrium, the appropriate "assumption" is that the winner has an identical signal. That is, conditional on the information revealed by the fact that I lose the auction, in equilibrium, I bid as if the winner's signal is equal to mine. Yet, we cannot necessarily expect this level of sophistication from bidders. Instead, it is quite possible that subjects formulate bids by assuming that the winner has some signal that is independent of their own. A mean (or, in this context, equivalently, the median) may serve as a reasonable focal point, so that a subject may think "If I lose, I don't know what signal the winner has, but it is likely to be 50 since that's the average signal." We construct a variable, *lose 50*, representing the reduction in profit if the subject loses the auction and the winner has the mean signal (50). This variable takes on only one of three values corresponding to the three possible values of ξ and is equal to 0 in the monopoly condition ($\xi = 0$).

Model 4 shows the results with the simple heuristic *lose 50* instead of the losing value. The coefficient on *lose 50* is not significantly different from -1 ($p = .17$). Additionally, the average of the fixed effects for this specification is quite low relative to other models and insignificant for nearly three out of four subjects. These data suggest that subjects bid *as if* they (almost) properly account for winning value and assume that, if they lose, the winner possesses the mean signal.

Remark 3. *In auctions of one license, subjects incorporate the winning value and the losing value somewhat in accord with theoretical predictions. However, the data are also consistent with a model in which subjects assume that a competitor who wins a license has the mean signal.*

The main departure from theoretical predictions comes from insufficient weight being placed on the winning value. While, in all estimations, this parameter is numerically close to one, it is also, in all cases, statistically significantly less than one, leading generally to underbidding in the sale of one license.

Auctions of Two Licenses

The sale of two licenses presents subjects with the need to estimate not only winners' private information if they lose, but also to predict what the *other* winner's signal might be if they win. We investigate next whether bidder behavior changes in this environment compared to the sale of one license. Our estimation parallels that for the sale of one license (Table 10).

Table 10: Estimation Of Bids In The Sale Of Two Licenses

	Censored regression models			
	1	2	3	4
Intrinsic Value	0.897***	0.934***		
	(0.024)	(0.024)		
Winning Value			0.886***	
			(0.023)	
Losing Value			−1.336***	
			(0.080)	
Win 50				0.908***
				(0.024)
Lose 50				−1.101***
				(0.100)
$\xi = 0.5$ (differentiation)		1205***		
		(179)		
$\xi = 1.0$ (homogeneity)		923***		
		(180)		
Average fixed effect	1219	328	744	623
% of fixed effects significant at < 0.05	64.1%	46.2%	53.8%	48.7%

Note: dependent variable is a subject's bid. Standard errors are shown in parentheses.
*** All coefficients are significant at 0.01. $N = 585$.

Again we find (in Model 1) that the coefficient on intrinsic value is significant but statistically less than the theoretical prediction of 1 ($p < .01$). However, when compared to the auctions of one license (where the coefficient on intrinsic value is 0.843), there seems to be a stronger co-movement between the bid and the corresponding intrinsic values in the auction for two licenses. Incorporating dummy variables for the auctions with externalities, Model 2 suggests overbidding in the presence of externalities, and this overbidding is greater than in the sale of one license. When considering the components of the intrinsic value separately

(Model 3), the coefficient on winning value is again significantly closer to the theoretical prediction of 1 than in the sale of one license. However, the coefficient on losing value also grows in magnitude (in absolute value).

We again consider bidding behavior when subjects substitute the average signal for unknown signals of their competitors (Model 4). This is more complex than in the sale of one license since this not only is incorporated in losing value (assuming both winners each have a signal of 50), but also requires making an assumption about the other winner if the subject wins. The variable *win 50* is a subject's value from winning if the other winner has the average signal. Thus, *win 50* substitutes for winning value when subjects utilize a simple heuristic for determining the profit from winning the auction. Analogous to the sale of one license, we find that this model yields parameters on *lose 50* not significantly different from -1 ($p = .32$), though positive fixed effects persist, and fixed effects are significant for nearly half of our subjects.

Remark 4. *In auctions of two licenses, subjects appear to incorporate the winning value in a manner close to theoretical predictions while they overemphasize the role of losing value. The data are also consistent with a model in which subjects assume that competitors who win a license have the mean signal.*

Revenues

Previously, we found evidence of overbidding for auctions of two licenses and underbidding in auctions of one license. This need not imply a similar result for revenue since revenue also depends on the distribution of bids. We now turn to exploring the mean and distribution of observed revenues.

Table 11: Estimated And Predicted Revenues

	Predicted Revenue	*Observed Revenue*	*Recombinant Revenue*
k=1			
monopoly	5750	5350	5197
($\xi = 0$) $N = 180$		(0.28)	(0.08)
differentiation	5333	5483	5513
($\xi = 0.5$) $N = 180$		(0.64)	(0.50)
homogeneity	6500	6333	6402
($\xi = 1$) $N = 180$		(0.77)	(0.78)
k=2			
monopoly	5500	5298	5499
($\xi = 0$) $N = 195$		(0.51)	(0.98)
differentiation	4519	5657	5685
($\xi = 0.5$) $N = 195$		(0.00)	(0.02)
homogeneity	4250	4965	4690
($\xi = 1$) $N = 195$		(0.01)	(0.32)

In parentheses are p-values for estimated revenues compared to predicted revenues, two sided.

Revenues from the experiment are reported in Table 11. Observed revenues do not differ from predictions in the sale of one license. In the sale of two licenses ($k = 2$), we observe significantly higher revenues than predicted in the presence of externalities. This result is potentially sensitive to process of matching groups of subjects. Observed revenues represent the outcome of a single matching of subjects' bids into groups of three for each auction. An auctioneer interested in expected revenue may wish to know what revenue would occur over many such matchings. For robustness, we also estimate revenues using the recombinant estimator first proposed by Mullin and Reiley (2006), similar to a bootstrap procedure which accounts for the correlations across resamplings due to the same subject appearing in multiple samples. Results with the recombinant estimator cast doubt on the significance of the two license $\xi = 1$ case though confirm that, when $\xi = 0.5$, revenues exceed theoretical predictions. Both observed revenues and the recombinant revenue estimates indicate that revenues are broadly consistent with corresponding predicted revenues; there is, however, weak evidence that experimental revenues exceed predicted revenues for auctions of two licenses in the presence of externalities.[44] This implies that an inventor interested in using theoretical benchmarks to predict revenue or to decide how to structure the auction may wish to adjust predicted revenue upwards for an auction of two licenses.

Remark 5. *Obtained revenues are broadly consistent with theoretical predictions. There is weak evidence that the model slightly understates revenue for auctions of two licenses when externalities are present.*

Revenues exceeding theoretical benchmarks need not change auction design guidance unless the departures from equilibrium are systematic and severe enough to change the revenue ranking of different auctions. Given the parameters of our model, the predicted optimal number of licenses to auction is 1 for any value of the externality parameter, ξ. However, this result does not hold in our experiment. The recombinant revenue and the empirically observed expected revenue are often higher for an auction of two licenses relative to an auction of one license. This is because the obtained revenues are much higher than predicted revenues for auctions of two licenses relative to auctions of one license. If such an effect holds generally, then it may be optimal for the seller to sell one license only for a subset of the parameter values for which the model predicts that the optimal number of licenses is one.

Remark 6. *In the presence of externalities, theoretical predictions understate the relative advantage of selling two licenses.*

Comparison to a Model of Heuristic Bidding

In the previous section, we conclude that *mean* auction revenues in our experiment correspond quite closely to predictions. In this section, we show that the *distribution* of revenues

[44] As an additional robustness check, we also considered directly the empirical distribution of bids in each treatment. We determine the distribution of the k^{th} highest order statistic from $N = 3$ draws from the distribution where $k = 2$ or 3. The expected value and standard deviation of this distribution implies results similar to those above.

departs systematically from what we should observe in theory and demonstrate that both this observation and the slight overbidding in the sale of two licenses are consistent with the simple heuristic bidding model.

In Figure 2, we compare the distribution of auction revenues if subjects bid according to theoretical predictions with the simulated distribution of revenues from the experiment.[45] In both monopoly cases, the distributions mostly coincide. We observe several differences in the distributions in the presence of externalities: (i) for auctions of one license, the distribution of observed revenues display an s-shape, lying below predictions for lower revenues and above predictions for higher revenues; (ii) for auctions of two licenses, the distribution function of observed revenues lies below theoretical revenues, except for a small region in the lower end of the distribution. Below, we propose a possible explanation for such differences.

In the model with rational bidders, firm i's bid is the intrinsic value which is the difference between its profit from winning and its profit from losing, when the marginal winner has the same signal as firm i. As suggested by the regression results in the previous sections, it is plausible that instead of bidding the intrinsic value as the model predicts, each bidder adopts a simpler heuristic and assumes that a winner of the auction has the mean signal. Then, each bidder bids

$$\tilde{b}_k(\theta_i;\xi) = \Pi\left(\theta_i; \theta_{(1)}^{n-1} = 50, \ldots, \theta_{(k-1)}^{n-1} = 50, \xi\right) - \Pi\left(0; \theta_{(1)}^{n-1} = 50, \ldots, \theta_{(k)}^{n-1} = 50, \xi\right) \quad (55)$$

which is the difference between her profit from winning and her profit from losing when each winning competitor has the expected value of the signal. In the experiment, the signal of each bidder is independently uniformly distributed between 0 and 100; hence, the expected value of the signal of each bidder is 50. We call $\tilde{b}_k$ of equation (55) the *heuristic value.*

Figure 2 also contains plots of predicted revenues if subjects bid according to their heuristic values. Heuristic bids are equal to equilibrium bids in the monopoly cases, since competi-

[45]The simulated distributions are obtained from 10,000 samples for every treatment, each composed of the bids of three subjects.

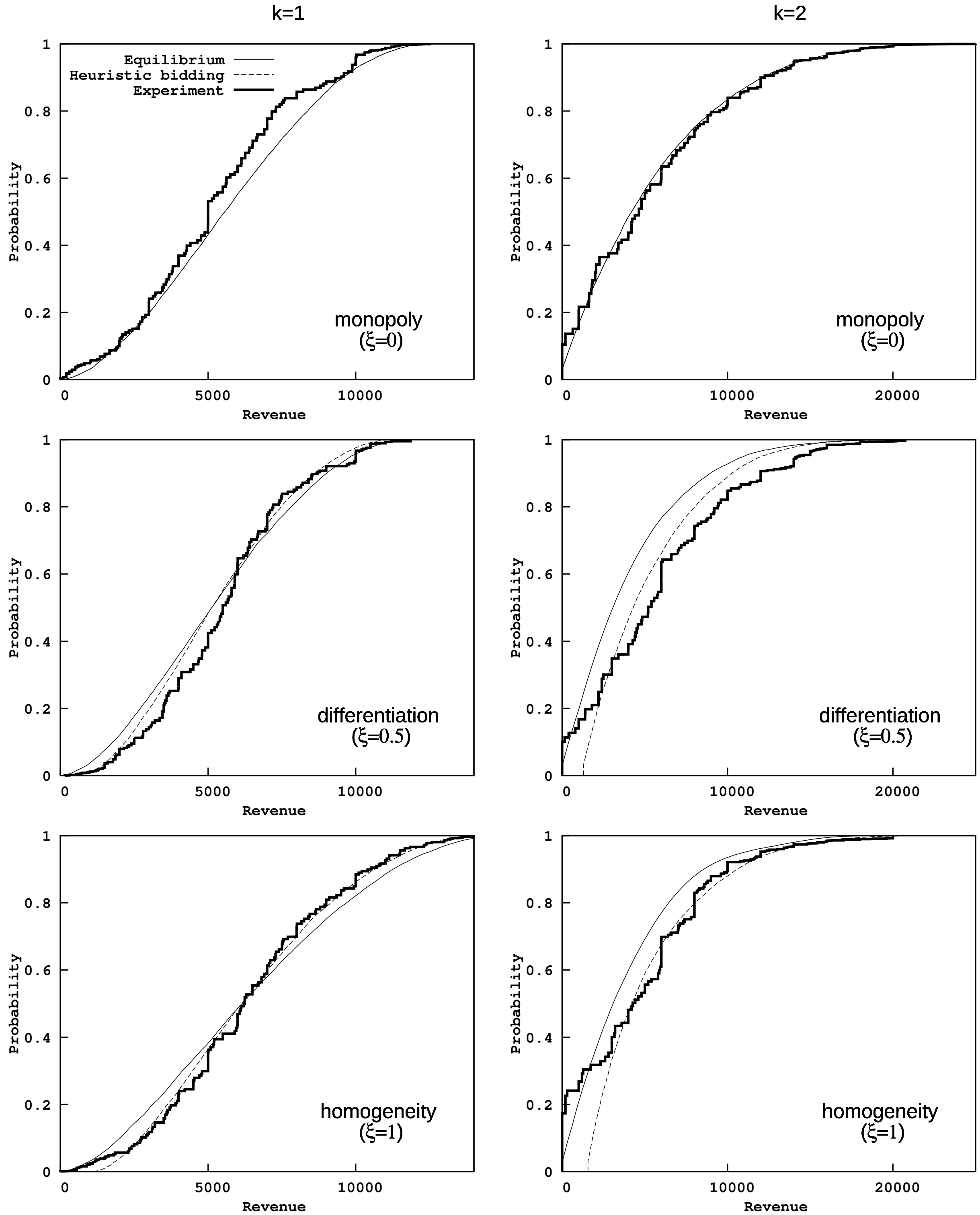

Figure 2: Simulation Distribution of Revenues from Equilibrium bids Intrinsic Values), heuristic bids and observed bids.

Table 12: Estimated and Predicted Heuristic Revenues

	Heuristic Predicted Revenue	*Observed Revenue*	*Recombinant Revenue*
k=1			
monopoly	5750	5350	5197
($\xi = 0$) $N = 180$		(0.28)	(0.08)
differentiation	5340	5483	5513
($\xi = 0.5$) $N = 180$		(0.65)	(0.51)
homogeneity	6531	6333	6402
($\xi = 1$) $N = 180$		(0.56)	(0.70)
k=2			
monopoly	5500	5298	5499
($\xi = 0$) $N = 195$		(0.51)	(0.98)
differentiation	5309	5657	5685
($\xi = 0.5$) $N = 195$		(0.28)	(0.42)
homogeneity	5500	4965	4690
($\xi = 1$) $N = 195$		(0.06)	(0.07)

In parentheses are two-sided p-values for estimated revenues compared to heuristic predicted revenues.

tors' signals do not enter one's bidding function. In the presence of externalities, heuristic bidding reflects both the s-shape of observed bids in the sale of one license[46] and higher bids (a lower curve) in the case of two licenses.[47]

In Table 12, we present predicted revenues if each bidder bids her heuristic value. Notably, experimental revenues (and their recombinant estimates) are never significantly different from those predicted by heuristic bidding at 5%. In the sale of one license with externalities, heuristic bidding marginally increases predicted revenue (by less than 1%). In the sale of two licenses, this heuristic implies significantly greater bids (17% and 29% for $\xi = 0.5, 1$). These observations are qualitatively consistent with our finding that the theoretical predictions are

[46]For an auction of one license, the profit from winning a license is the same for the intrinsic value and the heuristic value. To calculate the profit from losing, the equilibrium assumption that the winner has the same signal as the bidder is replaced with the heuristic assumption that the winner always has a signal of 50, leading to higher bids for lower signals and lower bids for higher signals than in equilibrium.

[47]For an auction of two licenses, the profit from winning a license is always higher under the heuristic assumption (the other winner has a signal of 50) than in equilibrium (the other winner has a signal higher than mine). The profit from losing is again ambiguous but given the parameters of the model, the intrinsic value is always lower than the heuristic value.

in line with revenues for the sale of one license but may underestimate revenue in the sale of two licenses.

Discussion

A challenge for auctions models is that people, be they experimental subjects or decision-makers in the "real world," rarely exhibit the level of sophistication required for equilibrium calculations. We find evidence that our subjects adopt simple heuristics, acting as if winners' signals are equal to the mean of the distribution of signals. Fortunately, we find that this need not imply significant departures from equilibrium revenues.

In the sale of one license, both equilibrium and heuristic bids imply nearly identical revenues, and these are observed in our experiments. For auctions of two licenses, there is weak evidence that the model slightly underpredicts revenue. The observed overbidding is, however, in line with the heuristic bidding model. How does an auctioneer choose whether to sell one or many licenses if he is unaware of whether participants bid rationally? To examine this question, we replace the assumption in the experiment that the distribution of cost savings is uniform with a special case of the beta distribution; let θ_i be distributed on $[0, 100]$ according to $F(\theta) = (\frac{\theta}{100})^{\alpha}$. When $\alpha = 1$, we recover the uniform distribution. Higher values of alpha imply a greater likelihood of larger cost savings. Again let market demand be given by $p_i = 300 - q_i - \xi \sum_{j \neq i} q_j$ with each firm having a pre-auction marginal cost of $\bar{c} = 100$. Figure 3 compares the revenue predictions when $\xi = 1/2$ under both heuristic and equilibrium bidding. Revenues in the sale of one license are nearly identical under both bidding rules. In the sale of two licenses, heuristic bidders lead to greater revenue.

Of particular import is the point at which selling two licenses dominates the sale of one license. Under the heuristic bidding rule, the sale of two licenses is better whenever $\alpha > 1.02$ and under equilibrium bidding, the sale of two licenses is better when $\alpha > 1.45$, confirming our earlier intuition that the sale of two licenses is optimal under a broader sense of parameters when subjects bid according to the simple heuristic. Figure 4 summarizes

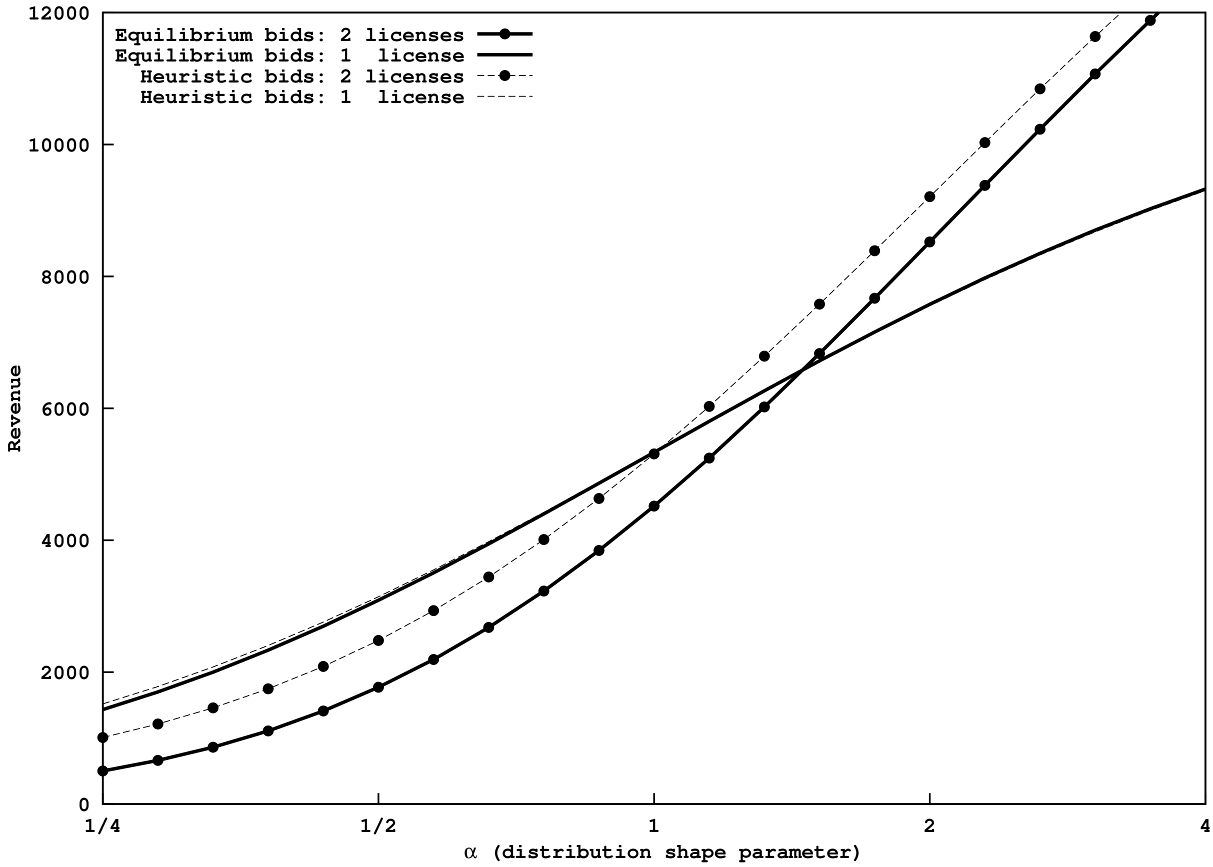

Figure 3: Expected revenues for the sale of one and two licenses under equilibrium and heuristic bidding for the differentiation case.

the predictions of the two bidding models for a range of distributions (α) and externality parameters (ξ). In short, the optimal number of licenses to auction is the same under both models (Region I and Region III) except in a narrow band (Region II) in which the sale of two licenses is preferred under heuristic bidding but one license is optimal under equilibrium bidding.

Our results imply that theory is a useful predictor of revenues in the sale of one license, but a model in which subjects assume that each competing winner has the mean signal can act as a good predictor of auction revenues in multi-license auctions. Because in an auction of two licenses, the heuristic model predicts higher revenues than in equilibrium, it may be optimal to sell one license for a smaller set of parameter values than what we predict theoretically.

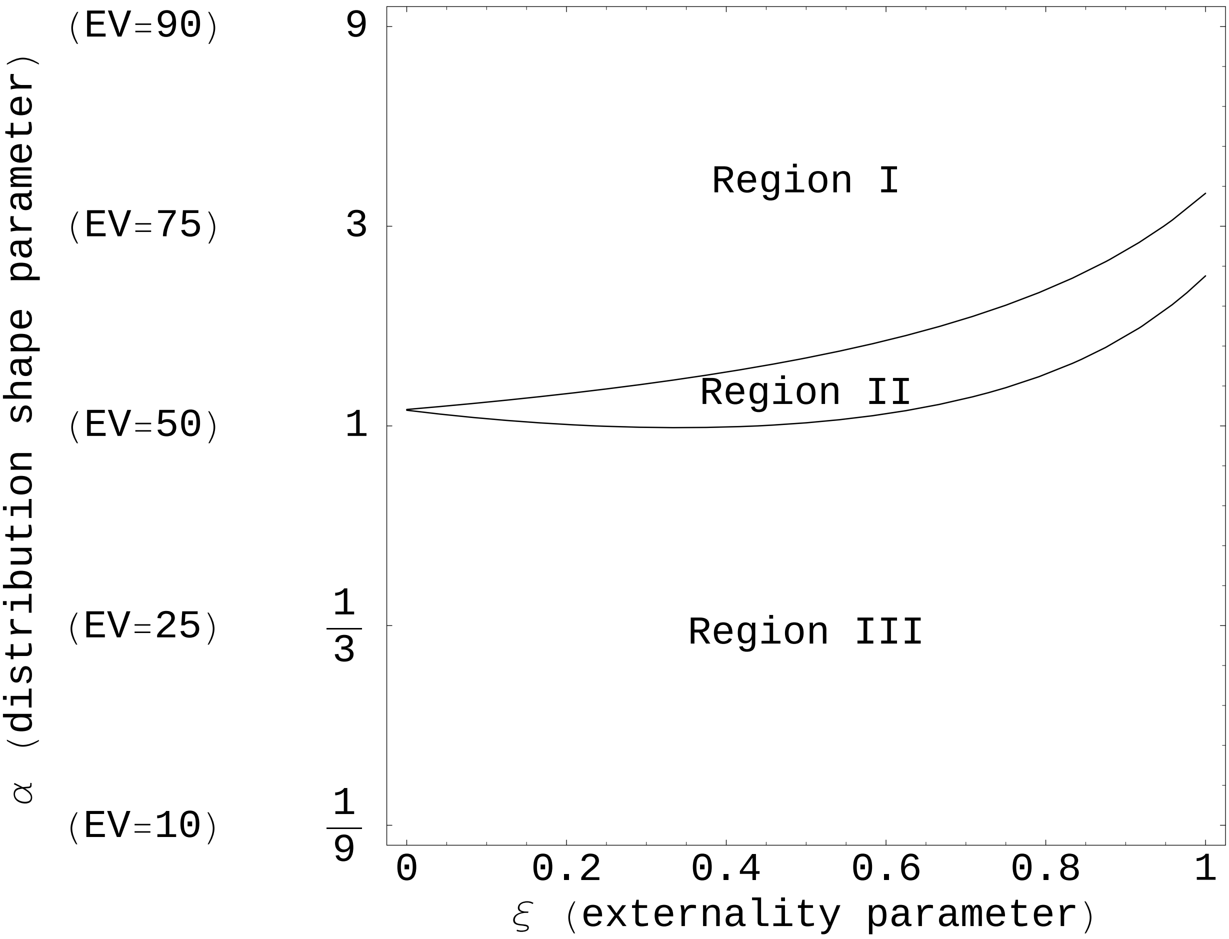

Figure 4: Optimal number of licenses to auction under equilibrium and heuristic bidding. In Region I, both models favor the sale of two licenses. In Region II, only the heuristic bidding model favors the sale of two licenses. In Region III, both models favor the sale of one license.

CHAPTER IV

WHEN SHOULD AN INVENTOR SELL LICENSES EXCLUSIVELY? THE DETERMINANTS OF THE OPTIMAL NUMBER OF LICENSEES

This paper considers the problem of a seller who has developed a cost-reducing technology and wants to earn revenue by selling multiple licenses to use the technology. The licenses are identical in the sense that they endow the licensee with the right to use the same technology. The buyers of the licenses are the firms in the relevant industry that compete in the product market.

The payoff of each firm depends on the marginal costs of production of every firm in the industry; there is no fixed cost of production. Each firm has access to an existing publicly available technology that allows it to produce at a marginal cost of one. The marginal cost of production of any firm that uses the new technology is at most one. However, it is assumed that different firms in the industry have different abilities to exploit the technology and, consequently, there is heterogeneity in the marginal costs of the firms that use the new technology. Each firm receives a one-dimensional signal that is a measure of the cost reduction that firm can achieve by using the new technology. A firm that receives a higher signal than another firm achieves a lower marginal cost than the other firm and hence has a higher payoff. If a firm cannot purchase a license to the new technology, it uses the existing technology and achieves a marginal cost of one. Furthermore, the payoff of a firm also depends negatively on the signal of any other licensee of a license. This has the implication that even though a firm obtains the user rights to the technology by purchasing just one license, it still has a positive value for the second license because, by purchasing the second license, it prevents a competitor from acquiring a license. Moreover, the payoff of a firm depends on the signals of the firms that purchase a license, not on how many licenses each of these firms has. Hence, given a profile of signals, the seller maximizes her revenue by optimally choosing the number of licensees. I provide a framework for the analysis of the optimal number of licensees. Because the payoff of a licensee depends on the nature of the product market, I also analyze the role of different product market factors in the determination of the optimal number of licensees.[48]

In the paper, I consider the simplified problem of determining the seller's revenue when the seller sells two identical licenses and there are three firms. The analysis can be extended to the sale of k licenses to n firms with its attendant combinatorial complexity. The payoffs of the firms are taken to have the same properties as the payoffs of firms that compete in quantities or prices using differentiated products. First, I determine the seller's revenue when the signal of each firm is publicly observable. I find that in the optimal selling mechanism when the signal of each firm is publicly observable, the seller selects the allocation rule that maximizes the industry (gross) payoff. The optimal mechanism of the seller for the case in which all licensees have the same signal was solved in Kamien ((1992), pp. 348-352) and in Kamien, Oren, and Tauman (1992). I extend their analysis to the case in which licensees can have different signals.

Next, I analyze the case wherein the signal of each firm is its private information. In order to analyze the problem, I define an allocation rule as a specification of the licensees, and consider the allocation in a truth-telling equilibrium of a direct mechanism. The seller's problem is to choose the allocation rule that maximizes her revenue from the sale of two licenses, provided the allocation rule satisfies the incentive compatibility and individual rationality constraints. Therefore, I first derive the necessary and sufficient conditions for incentive compatibility. I then show that some well-known allocation rules satisfy the necessary and sufficient conditions for incentive compatibility. Because of the assumption of independence of signals, I prove that two selling procedures that have the same allocation rule and the same payoff of a firm with the worst possible signal yield the same revenue to

[48] Other situations in which special cases of the model apply are the sale of production rights in an industry (as in Dana and Spier (1994)), the sale of airport takeoff and landing rights (as in Gale (1994)), and the sale of franchise rights (as in Degraba and Postlewaite (1992)).

the seller. This is known as the *revenue equivalence theorem,* first established in a simpler context by Riley and Samuelson (1981).

I then determine the seller's revenue in an incentive compatible mechanism (that is, in a mechanism in which the allocation rule satisfies the necessary and sufficient conditions for incentive compatibility). In order to determine the seller's revenue, I define the concept of *industry virtual payoff*, which is the analog, for the situation in which there are externalities, of the concept of virtual value (Myerson (1981), Bulow and Roberts (1989), and Bulow and Klemperer (1996)). The industry virtual payoff is the industry profit, less the information rents of the licensees; hence, the industry virtual payoff, given any profile of signals, depends on the allocation rule. The revenue of the seller, given any allocation rule, is the expected industry virtual payoff, less the product of the payoff of a firm with the lowest possible signal and the number of firms in the industry. In models of sales with externalities, the payoff of the firm with the lowest possible signal depends on the mechanism (Kamien (1992), Jehiel, Moldovanu, and Stachetti (1996) and (1999)). If the seller can credibly commit to punish a firm that refuses to participate in the mechanism by allocating a license to the other firms, then the payoff of the firm with the lowest possible signal is minimized and is independent of the equilibrium allocation. Hence, in the optimal mechanism, the seller chooses the allocation mechanism that maximizes the industry virtual payoff for any arbitrary profile of signals, and threatens to punish a firm that refuses to participate in the mechanism by allocating a license to the other firms. I determine the number of licensees in the optimal mechanism, both when the signal of each firm is publicly observable, and when the signal of each firm is its private information.

I then illustrate the role of several product market factors in the determination of the optimal number of licensees. First, I show that the presence of significant externalities along with private information may cause the seller to select a fewer number of licensees compared to the situation in which the signal of each firm was publicly observable. Next, I show that an increase in the magnitude of externalities may lead to a decrease in the expected number of licensees. Finally, I show that when the firms are likely to be more efficient users of the technology, then the expected number of licensees increases.

Other Related Literature

This paper is closely related to the literature on sales with externalities. One of the earliest analyses of sales of licenses in the presence of externalities is Katz and Shapiro (1986). Their analysis assumes that the signal of a firm is publicly observable, and that each firm can purchase at most one license. Another article in the same spirit is Hoppe, Jehiel, and Moldovanu (2004). I relax both of the assumptions mentioned above. There has been work on sale with externalities, in which the signal of each firm is its private information. Jehiel, Moldovanu, and Stachetti ((1996) and (1999)) show that the payoff of the firm with the worst signal is endogenous to the mechanism, when the licensee of the license imposes an externality on the others. I find such a result in my model. Other examples of articles that analyze sales with externalities are Jehiel and Moldovanu (2000), Moldovanu and Sela (2003), DasVarma (2003), Katzman and Rhodes-Kropf (2002) and Goeree (2003). However, unlike my paper, none of these papers consider multiple licenses.

There has also been some work on sales of multiple licenses in the presence of externalities. Jehiel and Moldovanu (2001 and 2004) show the impossibility of implementing efficient allocations when the signals are multi-dimensional. In my paper, the signals are unidimensional. My paper is also closely related to Dana and Spier (1994). In their article, Dana and Spier consider the problem of auctioning production rights to firms in an industry. Dana and Spier assume that "a firm earns zero profits if it is not awarded a production right" (Dana and Spier (1994), p. 129) while I assume that if a firm does not purchase a license, its payoff is lower (and depends on the signals of the other licensees) compared to its payoff before the sale. In Dana and Spier, the seller (which is the government) maximizes social welfare which is a function of the revenue from the sale, profits of firms, and consumer surplus, while in my model, the seller maximizes her revenue from the sale.

Schmitz (2002) has analyzed revenue-maximizing allocations from a sale of multiple licenses when the signal of each firm is its private information. He has shown that the optimal number of licensees under private information can be two even when the optimal number of licensees under complete information is one. There are two major difference between Schmitz's model and mine. First, in Schmitz's model, each firm can win at most one license but I impose no such restriction. Second, Schmitz assumes that with positive probability a licensee is not able to commercially exploit the technology whereas, in my model, this is not the case. It can be shown that if this assumption is relaxed in Schmitz's model, it is always optimal for the seller to sell both licenses to one firm. In contrast, I do not make such an assumption but show that it can be optimal for the seller to choose multiple licensees. Brocas (2005) also analyzes a model of sale of k licenses in which each firm can win at most one license but her payoff functions are not motivated by standard models of market competition. In simultaneous but independent work, Figueroa and Skreta (2005) have considered a general model of sale of multiple objects in the presence of externalities whereas my model deals only with the sale of licenses. They derive the optimal mechanism both when the non-participation payoffs are own-type dependent and when they are own-type independent; I consider the optimal mechanism when the non-participation payoffs are own-type independent. In my model, the payoff function of firms have the properties of payoff functions that arise in equilibrium when firms compete in quantities or in prices using differentiated products; in Figueroa and Skreta the payoff functions are not derived from an oligopoly model. Because I model the nature of competition in the product market more explicitly, I show the relationship between the level of product differentiation and the optimal number of licensees. Moreover, because I assume that the marginal payoff of a firm is decreasing in a competitor's signal (as in many oligopoly models), I derive a different regularity condition from the one in Figueroa and Skreta. Further, I analyze the problem both when the signal of every firm is publicly observable and when they are private information. This allows me to highlight the effect of negative externalities alone and the effect of both negative externalities and private information.

There is another related literature that analyzes auctions of heterogeneous objects. Palfrey (1983), Armstrong (2000) and Avery and Hendershott (2000) analyze auctions of heterogeneous objects when buyers have an exogenously specified private value for each object. In contrast, in my model, the licenses are identical and the value of winning a license is determined only after the sale. In Palfrey's model, the seller, who is the owner of many heterogeneous objects, decides to partition the objects into separate bundles and sell each bundle separately using auctions. In comparison, in my model, the seller makes the bundling decision *ex post*. Palfrey finds that the desirability of selling all the objects as a bundle depends on the number of bidders. Armstrong (2000) and Avery and Hendershott (2000) extend Palfrey's analysis to determine the revenue-maximizing auction, under different assumptions about the buyers.[49]

Seller's Revenue under Complete Information

Model

Consider a seller who wants to maximize revenue by selling two licenses to use a cost-reducing technology (process innovation). There are three potential buyers for the licenses, labelled firms 1, 2, and 3. Initially, the firms produce with a marginal cost of 1; there are no fixed costs of production. Before the sale occurs, each firm i ($i = 1, 2, 3$) receives a signal $s_i \in [0, 1]$ that determines its marginal cost of production c_i if it purchases a license to the technology, in the following way:

$$c_i = 1 - s_i. \tag{56}$$

[49] In Armstrong (2000), all buyers gain a positive payoff from winning any object. In contrast, in Avery and Hendershott (2000), only some buyers gain a positive payoff from winning any object, while the others gain a positive payoff from winning only some of the objects.

Otherwise, the firm continues to produce at a marginal cost of 1. It is assumed that no firm exits the industry after the sale of licenses to the new technology.

The signals are assumed to be identically and independently distributed across firms, with $G(s)$ (resp., $g(s)$) as the distribution function (resp., density function). In this section, it is assumed that the signal of each firm is publicly observable. Let $s^3_{(k)}$ be the kth highest statistic from a sample of size 3 where the sample includes all the potential buyers for the licenses and assume that $s^3_{(k)}$ is distributed as $F^3_k(\cdot)$ with the associated density function $f^3_k(\cdot)$. A typical firm has two competitors and the expected payoff of a firm depends on the signals of the two competitors. Therefore, when a sample refers to a firm's competitors, I work with a sample size of 2 and in these cases, I replace 3 with 2 in the superscripts of the expressions above. In such a case, the sample consists of all the competitors of firm i. I denote the joint density of $s^j_{(1)}, \ldots, s^j_{(k)}$ by $f^j_{1\ldots k}\left(s_{(1)}, \ldots, s_{(k)}\right)$ where $j, k = 1, 2, 3$ and $k \leq j$.

A firm's payoff depends upon its own signal as well as on the signals drawn by the other firms. If firm i purchases a license and, if the firm with signal $s^2_{(p)}$ also purchases a license, then firm i 's payoff is given by $\pi(s_i; s^2_{(p)}, 0)$ where p is either 1 or 2. If firm i does not purchase a license and, if the firms with signals $s^2_{(1)}$ and $s^2_{(2)}$ purchase one license each, then firm i' s payoff is $\pi(0; s^2_{(1)}, s^2_{(2)})$. Finally, if firm i is not a licensee and the firm with signal $s^2_{(p)}$ purchases both licenses, then the payoff of firm i is $\pi(0; s^2_{(p)}, 0)$; $p = 1, 2$. The payoff function $\pi(\cdot;\cdot,\cdot)$ is also assumed to be symmetric across firms; that is, if the signals amongst any two firms are permuted, their payoffs are permuted as well.

The exact specification of the payoff function $\pi(\cdot;\cdot,\cdot)$ depends on the nature of competition among the firms and other market parameters such as the demand function. However, regardless of the functional form, I assume that the payoff function $\pi(\cdot;\cdot,\cdot)$ is twice continuously differentiable in all its arguments and has the following properties:

$$\pi_1(s_i;\cdot,\cdot) > 0\ ,\quad \pi_{11}(s_i;\cdot,\cdot) \geq 0, \tag{57}$$

$$\pi_j(\cdot;\cdot,\cdot) < 0,\ j = 2, 3, \tag{58}$$

$$\pi_{12}(\cdot;\cdot,\cdot) < 0,\quad \pi_{13}(\cdot;\cdot,\cdot) < 0,\ \ \pi_1(\cdot;\cdot,\cdot) > -\pi_2(\cdot;\cdot,\cdot)\,, \tag{59}$$

where $\pi_j(\cdot;\cdot,\cdot)$ refers to the partial derivative of $\pi(\cdot;\cdot,\cdot)$ with respect to the jth argument. The inequalities in (57) imply that the payoff of firm i, when it purchases a license, is increasing and convex in its own signal. The inequality in (58) captures the effect of negative externalities in this model because it implies that when a competitor of firm i purchases a license, then the payoff of firm i is strictly decreasing in that competitor's signal. The first two inequalities in (59) imply that the marginal payoff of a firm's signal is decreasing in another firm's signal, while the third inequality in (59) implies that the payoff of a firm is more sensitive to its own signal than to another firm's signal.

Below, I illustrate the payoff function for different specifications about the nature of competition and show that the payoff function in each of these examples satisfies the properties described in (57), (58), and (59). In the first example, I consider a market in which the firms compete in quantities.

Example 4.1. (Sale of Licenses to use a Process Innovation): Suppose an independent research lab wants to sell two licenses to use a cost-reducing technology. There are three potential buyers, who are firms that compete in quantites producing differentiated products. The inverse demand function for firm i is given by:

$$p_i = \tau - q_i - \mu q^2_{(1)} - \mu q^2_{(2)};\ \ \mu \in [0, 1]\,.$$

In this demand function, $q^2_{(j)}$ is the output of the firm with cost $1 - s^2_{(j)}$.[50] Also, μ is the

[50]Recall that $s^2_{(1)} \geq s^2_{(2)}$.

externality parameter that captures the effect of the other firms' decisions on firm i's payoff.[51] Given $\mu \in [0,1]$, the payoff function is given by:

$$\pi\left(s_i; s_{(1)}^2, s_{(2)}^2\right) = \left[\frac{(\tau-1)(2-\mu)+(2+\mu)\,s_i-\mu\left(s_{(1)}^2+s_{(2)}^2\right)}{2(1+\mu)(2-\mu)}\right]^2. \tag{60}$$

Because I am considering the problem of allocating two licenses, at least one of s_i, $s_{(1)}^2$, or $s_{(2)}^2$ is 0. It can be verified that the payoff function (which is the reduced form equilibrium profit function) satisfies (57), (58), and (59). □

Next, I consider another example in which the firms compete in prices instead of in quantities.

Example 4.2. This example is similar to the example above, except for the nature of competition. Suppose the three firms compete in prices selling differentiated products, and let the demand function be given by

$$q_i = \tau - p_i + \mu p_{(1)}^2 + \mu p_{(2)}^2; \;\; \mu \in [0, 0.5].$$

Analogous to the example above, $p_{(j)}^2$ is the price of the firm with cost $1 - s_{(j)}^2$ and μ is the externality parameter. If all the firms were able to use the new technology, the payoff function of firm i would be given by

$$\pi\left(s_i; s_{(1)}^2, s_{(2)}^2\right) = \left[\frac{(\tau+2\mu-1)(2+\mu)+(2-\mu-2\mu^2)\,s_i-\mu\left(s_{(1)}^2+s_{(2)}^2\right)}{2(1+\mu)(2+\mu)}\right]^2.$$

It can be verified that the payoff function (which is the reduced form equilibrium profit function) satisfies (57), (58), and (59). *square*

Dana and Spier (1994) consider the sale of production rights in an industry and their problem has a similar flavor to my paper. One important way in which their paper is different

[51] Another interpretation of μ is that it captures the level of product differentiation in the industry.

is the nature of the payoff function of firm i when firm i does not win a license. Dana and Spier assume that

$$\pi_2(0;\cdot,\cdot) = \pi_3(0;\cdot,\cdot) = 0, \tag{61}$$

while I assume that

$$\pi_2(0;\cdot,\cdot) < 0 \text{ and } \pi_3(0;\cdot,\cdot) < 0. \tag{62}$$

A payoff function that satisfies (61) is said to have *fixed externalities,* while a payoff function that satisfies (62) is said to have *signal-dependent externalities.* The nature of the externality depends on the context of the problem. In Dana and Spier, a firm that does not win a production right cannot enter the industry and has a payoff of 0, thereby implying that the payoff function has fixed externalities. In my model, I assume that the number of firms in the industry is fixed and the seller can only determine the number of firms that can use its technology. Hence, in my model, I have signal-dependent externalities.

It is also important to note the relationship between firm i's payoff from purchasing a license when it has a signal of 0, and its payoff from not purchasing a license. First, consider the case that only one of firm i's competitors—the firm with signal $s^2_{(p)}$—purchases a license. Then firm i's payoff is $\pi(0; s^2_{(p)}, 0)$ when either it purchases a license or when the firm with signal $s^2_{(p)}$ purchases both the licenses. Second, in the case when both of firm i's competitors with signals $s^2_{(1)}$ and $s^2_{(2)}$ purchase a license, firm i's payoff is $\pi(0; s^2_{(1)}, s^2_{(2)})$. However, if firm i instead purchases a license by displacing one of its rivals (say the firm with signal $s^2_{(1)}$), then its payoff is $\pi(0; s^2_{(2)}, 0)$ which is different from $\pi(0; s^2_{(1)}, s^2_{(2)})$. It follows from (58) that firm i with signal 0 may be better off by purchasing a license if it displaces a competitor; even though it cannot obtain a reduction in its own marginal cost, it can prevent a rival from doing so.

Revenue

In the rest of this section, I determine the revenue of the seller in the optimal mechanism, when the signal of each firm is publicly observable.[52] In order to do so, I fix a profile of signals $\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$. Each allocation is denoted by the vector $a = [a1, a2, a3]$ where ai is the number of licenses that the firm with signal $s^3_{(i)}$ purchases in equilibrium; $i = 1, 2, 3$.[53] Suppose the seller commits to allocate the two licenses according to a in exchange for a payment of $m^c_i(a)$ from firm i; $i = 1, 2, 3$. If firm i does not accept the seller's offer (that is, if firm i does not participate in the mechanism), the payoff of firm i is denoted by $\underline{\pi}^c_i$. Let $I^c(ai)$ be an indicator variable that takes a value 1 if $ai > 0$ and 0 otherwise. Therefore, the seller's problem is the following:

$$\max_a \sum_{i=1}^{3} m^c_i(a)$$

$$\text{s.t. } \pi\left(I^c(ai)\, s_i; \cdot, \cdot\right) - m^c_i(a) \geq \underline{\pi}^c_i; \quad i = 1, 2, 3. \tag{63}$$

Observe that the seller can extract a higher payment from firm i if she can reduce the payoff of firm i from not participating in the mechanism. In the lemma below, $\underline{\pi}^c_i$ has been determined for every firm i in the optimal mechanism.

Lemma 4.3. *In the optimal mechanism,*

$$\underline{\pi}^c_i = \pi(0; s^2_{(1)}, s^2_{(2)}) \text{ for } i = 1, 2, 3, \tag{64}$$

where $s^2_{(1)}$ *and* $s^2_{(2)}$ *are the signals of firm i's competitors such that* $s^2_{(1)} \geq s^2_{(2)}$.

Proof. See Appendix C. □

If a firm does not participate in the mechanism, then its payoff depends on how the licenses are allocated in such an eventuality. The maximum punishment that the seller can credibly threaten to inflict on a non-participating firm is to allocate the licenses to both of

[52] The mechanism is an extension of the "chutzpah mechanism" (described in Kamien ((1992), pp. 348-352) or in Kamien, Oren, and Tauman (1992)) to the case in which licensees achieve different marginal costs of production with the new technology.

[53] In any allocation, $a1$, $a2$ and $a3$ are integers that must sum to 2, because the seller sells two licenses.

its competitors, and the payoff of firm i in such a case is given by $\pi(0; s^2_{(1)}, s^2_{(2)})$; note that $\pi(0; s^2_{(1)}, s^2_{(2)})$ is less than $\pi(0; 0, 0)$ which is the profit a firm makes before the innovation is introduced. In the optimal mechanism, the seller makes each firm indifferent between participating and non-participating and hence we have the above lemma.

Under complete information, the seller can extract the entire surplus of the three firms and hence, in the optimal mechanism, all the constraints in (63) must be tight. Therefore, the seller's problem can be re-stated as

$$\max_a \left\{ \sum_{i=1}^{3} \pi \left(I^c \left(ai \right) s_i; \cdot, \cdot \right) - \sum_{i=1}^{3} \underline{\pi}_i^c \right\}, \tag{65}$$

and it follows from (64) that

$$\sum_{i=1}^{3} \underline{\pi}_i^c = \pi(0; s^3_{(2)}, s^3_{(3)}) + \pi(0; s^3_{(1)}, s^3_{(3)}) + \pi(0; s^3_{(1)}, s^3_{(2)}). \tag{66}$$

Notice that $\sum_{i=1}^{3} \pi \left(I^c \left(ai \right) s_i; \cdot, \cdot \right)$ is the industry gross payoff from the mechanism. Hence, the seller's optimal allocation a^* maximizes the industry gross payoff. This is stated formally in the following proposition.

Proposition 4.4. *Suppose the signal of each firm is publicly observable. Then, in the optimal allocation a^*, the seller maximizes the industry gross payoff, that is,*

$$a^* \in \arg\max \sum_{i=1}^{3} \pi \left(I^c \left(ai \right) s_i; \cdot, \cdot \right),$$

and the seller's revenue in the optimal mechanism is

$$\sum_{i=1}^{3} \pi \left(I^c \left(ai^* \right) s_i; \cdot, \cdot \right) - \sum_{i=1}^{3} \underline{\pi}_i^c,$$

where $\sum_{i=1}^{3} \underline{\pi}_i^c$ is given by (66).

Proof. Follows from the discussion above. □

In later sections, I show that when the signal of each firm is its private information, then the seller cannot extract all the surplus of the firms and hence the optimal allocation in such a case may not be the allocation that maximizes the industry gross payoff. In the presence of private information, the seller sometimes sells licenses to a smaller number of licensees compared to what she would have done if the signals were publicly observable. In order to determine the seller's revenue when the signal of each firm is its private information, in the section below I describe the seller's problem when the signal of each firm is its private information.

The Mechanism under Incomplete Information

From now on, unless otherwise mentioned, the signal of each firm is assumed to be its private information and the seller is assumed to have the power to credibly commit to a mechanism for the sale of two licenses. By the revelation principle, there is no loss of generality in restricting our attention to direct mechanisms. In a direct mechanism, the seller asks each firm to give a report about its signal, and implements some outcome depending on the profile of reports.

Direct Mechanism

Before I proceed to set up the direct mechanism, I define its building blocks. The first building block is the ***signal space*** of the firms. I denote it by Ω. Because the signal of each firm is distributed on the unit interval, $\Omega = [0, 1] \times [0, 1] \times [0, 1]$.

The second building block of a direct mechanism is the payment rule and it is defined below:

A ***payment rule*** *is a mapping that specifies the payments of the firms as a function of the profile of reports. Let* $r = \{r_1, r_2, r_3\} \in \Omega$ be the reports of the three firms about their signals. Then, the payment rule $M(r)$ *is specified by the following vector:*

$$M(r) = (M_1(r),\ M_2(r),\ M_3(r)),$$

where $M_j(r)$ *is the payment of firm* j *when the profile of reports is* r. The third building block of a direct mechanism is the allocation rule which is specified below:

An (ex post) ***allocation rule*** *is a mapping that prescribes the distribution of licenses among the firms as a function of the firms' reports* $r = \{r_1, r_2, r_3\} \in \Omega$. *Let* $q_j(r)$ be the number of licenses firm j obtains when the reported profile is r, where $j = 1, 2$, or 3. *The allocation rule* $Q(r)$ *is then defined to be the vector* $Q(r) = (q_1(r), q_2(r), q_3(r))$. *Because the seller commits to sell two licenses, the following relations must also hold:*

$$\text{For all } r \in \Omega,\ q_j(r) \in \{0, 1, 2\} \text{ and } \sum_{j=1}^{3} q_j(r) = 2; \qquad j = 1, 2, 3.$$

I restrict consideration to allocation rules that are symmetric and with one exception, ex post deterministic.[54] The exception is when $r_1 = r_2 = r_3$; in this case, because there are only two licenses to be allocated, the seller can allocate them in any random way that does not depend on the identites of the would-be licensees. The probability of such an event occuring in a truth-telling equilibrium is zero. For all other report profiles, the allocation rule is deterministic and, if r_i and r_j are permuted in the report profile for $i \neq j$, then $q_i(\cdot)$ and $q_j(\cdot)$ are permuted as well.[55]

I now define the direct mechanism as follows:

A ***direct mechanism*** *is the set* $\Gamma = \{\ \Omega,\ Q(\cdot),\ M(\cdot)\}$.

I denote the *ex ante* expected revenue of the seller when she commits to implement an allocation rule $Q(\cdot)$ by R_Q. The seller's problem is to choose the allocation rule that

[54] In other words, given any profile of reports, the allocation rule specifies the number of licenses each firm obtains, with certainty, in all but one case.

[55] It is possible to analyze the problem with any arbitrary allocation rule. However, for the sake of exposition, I have considered only symmetric allocation rules throughout the paper.

maximizes her expected revenue. In a symmetric allocation rule, all that matters for the *ex post* allocation is the signal of a firm and not its identity. Hence, from now on, only the non-decreasing permutation of the reports $\hat{r} \equiv \left(r^3_{(1)}, r^3_{(2)}, r^3_{(3)}\right)$ is considered, where $r^3_{(1)} \geq r^3_{(2)} \geq r^3_{(3)}$.

Expected Payoff of a Firm

A revenue-maximizing seller extracts the maximum possible amount from each firm, under the condition that each firm's signal is its private information and the seller has two licenses to sell. The amount that the seller extracts from each firm depends on the expected payoff of each firm, which in turn depends on the seller's allocation rule. In the rest of this section, I show the relationship between the seller's allocation rule and the expected payoff of a firm.

In order to do so, first notice that given any symmetric allocation rule Q, the signal space can be partitioned into six (mutually exclusive and exhaustive) sets

$$A\left(\hat{r}|Q\right) \equiv \left\{A1\left(\hat{r}|Q\right), A2\left(\hat{r}|Q\right), \ldots, A6\left(\hat{r}|Q\right)\right\}$$

such that each set maps an ordered profile of reports $\hat{r}$ to a particular *ex post* allocation of licenses. The allocation corresponding to each set is presented in the table below. As an example, suppose that under an allocation rule Q, a particular ordered profile of reports $\hat{r}$ belongs to the set $A6\left(\hat{r}|Q\right)$. Then, the firm reporting $r^3_{(1)}$ purchases two licenses and the others purchase nothing. Similarly, depending on other profile of reports, a different *ex post* allocation of licenses is obtained. *Also notice that any allocation rule is associated with a unique partition of the signal space and vice versa.* It is assumed that the seller can commit to allocate the licenses according to the allocation rule Q.

Next, in order to relate the allocation rule to firm i's expected payoff, I determine the

Table 13: Partition Of The Report Space Induced By An Allocation Rule

Set	Allocation of $r^3_{(1)}$	Allocation of $r^3_{(2)}$	Allocation of $r^3_{(3)}$
$A1\,(\hat{r}\vert Q)$	0	2	0
$A2\,(\hat{r}\vert Q)$	0	0	2
$A3\,(\hat{r}\vert Q)$	0	1	1
$A4\,(\hat{r}\vert Q)$	1	1	0
$A5\,(\hat{r}\vert Q)$	1	0	1
$A6\,(\hat{r}\vert Q)$	2	0	0

ex post distribution of licenses from firm i's perspective, when the seller commits to the allocation rule Q, firm i reports r_i, and the other firms report truthfully. Let $s^2_{(1)}$ (resp., $s^2_{(2)}$) be the highest (resp., lowest) of the signals of firm i's competitors and denote

$$\hat{s}_{-i} = \left(s^2_{(1)}, s^2_{(2)}\right)$$

as the non-decreasing permutation of the signals of firm i's competitors. Then, given the profile of reports $(r_i,\ \hat{s}_{-i})$ and the partition $A\,(\hat{r}|Q)$, there also exists a partition

$$B\,(r_i) \equiv \{B1\,(r_i)\,, \ldots, B6\,(r_i)\}$$

of the signal space of firm i's competitors, as a function of firm i's report r_i. Each subset of the partition $B\,(r_i)$, given r_i, is the set of ordered signals of firm i's competitors $\hat{s}_{-i}$ that result in the same allocation. In the table below, I list the allocation corresponding to each subset of the partition $B\,(r_i)$. For example, it follows from the table that $B3\,(r_i)$ is defined as follows:

$$B3\,(r_i) = \{\hat{s}_{-i}|\text{firm } i \text{ is not a licensee and its competitors purchase one license each}\}\,.$$

Notice that there are six subsets of $B\,(r_i)$ because there are six ways of allocating the two licenses.

Table 14: Allocation of Licenses As A Function Of Firm i's Report

	Number of licenses allocated to each firm		
Set	Firm i	Firm with signal $s^2_{(1)}$	Firm with signal $s^2_{(2)}$
$B1(r_i)$	0	2	0
$B2(r_i)$	0	0	2
$B3(r_i)$	0	1	1
$B4(r_i)$	1	1	0
$B5(r_i)$	1	0	1
$B6(r_i)$	2	0	0

The relationship of the partition $B(r_i)$ and the partition $A(\hat{r}|Q)$ is explained in the Appendix. The partition $B(r_i)$ can be used to determine the probability $\Phi_k(r_i)$ that firm i obtains k licenses $(k = 0, 1, 2)$ and its expected payoff conditional on obtaining k licenses $\Pi(r_i, s_i|k)$ when firm i reports r_i and the others report truthfully. Hence, I can determine the expected payoff of firm i from reporting r_i when its competitors report truthfully, and this is discussed below.

Let $\phi_{Bk}(r_i)$ be the probability that the ordered profile of signals of firm i's competitors belong to the set $Bk(r_i)$ when firm i reports its signal as r_i, the rivals report truthfully, and the seller commits to the allocation rule Q, that is,

$$\phi_{Bk}(r_i) \equiv P\{\hat{s}_{-i} \in Bk(r_i)\}; \quad k = 1, \ldots, 6.$$

The expected payment of firm i, denoted by $m_i(r_i)$, can be similarly defined when firm i reports r_i, the other firms report truthfully, and the seller commits to the allocation rule Q.[56] Therefore,

$$m_i(r_i) = \int_{\hat{s}_{-i}} M_i(r_i, \hat{s}_{-i}) f^2_{12}(\cdot)\, d\hat{s}_{-i}.$$

[56] Note that, while I consider only symmetric allocation rules, I allow the payment rule to be asymmetric across firms.

Using the definitions of $\phi_{Bk}(r_i)$ and $m_i(r_i)$, the interim expected payoff of firm i, denoted by $V_{iQ}(r_i, s_i)$, is defined below, when firm i with a signal of s_i reports r_i, the others report truthfully, and the seller chooses Q.

$$\begin{aligned} V_{iQ}(r_i, s_i) = {} & \phi_{B1}(r_i) E\left\{\pi\left(0; s_{(1)}^2, 0\right) | \hat{s}_{-i} \in B1(r_i)\right\} \\ & + \phi_{B2}(r_i) E\left\{\pi\left(0; s_{(2)}^2, 0\right) | \hat{s}_{-i} \in B2(r_i)\right\} \\ & + \phi_{B3}(r_i) E\left\{\pi\left(0; s_{(1)}^2, s_{(2)}^2\right) | \hat{s}_{-i} \in B3(r_i)\right\} \\ & + \phi_{B4}(r_i) E\left\{\pi\left(s_i; s_{(1)}^2, 0\right) | \hat{s}_{-i} \in B4(r_i)\right\} \\ & + \phi_{B5}(r_i) E\left\{\pi\left(s_i; s_{(2)}^2, 0\right) | \hat{s}_{-i} \in B5(r_i)\right\} \\ & + \phi_{B6}(r_i) \pi(s_i; 0, 0) - m_i(r_i). \end{aligned} \tag{67}$$

As mentioned above, let $\Phi_k(r_i)$ be the probability that firm i purchases k licenses when it reports r_i, the other firms report truthfully, and the seller commits to the allocation rule Q. In particular, the probabilities $\Phi_k(r_i)$ for $k = 1, 2, 3$ are related to the probabilities $\phi_{Bj}(r_i)$ for $j = 1, 2, \ldots, 6$ as follows:

$$\begin{aligned} \Phi_0(r_i) & \equiv \phi_{B1}(r_i) + \phi_{B2}(r_i) + \phi_{B3}(r_i), \\ \Phi_1(r_i) & \equiv \phi_{B4}(r_i) + \phi_{B5}(r_i) \\ \text{and } \Phi_2(r_i) & \equiv \phi_{B6}(r_i). \end{aligned} \tag{68}$$

Observe that the probability of purchasing k licenses depends on firm i's reported signal r_i and not on its true signal s_i. Moreover, the expected gross payoff of firm i from purchasing k licenses (when its true signal is s_i and its reported signal is r_i) denoted by $\Pi(r_i, s_i|k)$ is described below for $k = 0, 1,$and 2.

$$\Pi(r_i, s_i|0) \equiv \frac{\phi_{B1}(r_i)}{\Phi_0(r_i)} E\left\{\pi\left(0; s_{(1)}^2, 0\right) |\hat{s}_{-i} \in B1(r_i)\right\} + \frac{\phi_{B2}(r_i)}{\Phi_0(r_i)} E\left\{\pi\left(0; s_{(2)}^2, 0\right) |\hat{s}_{-i} \in B2(r_i)\right\} + \frac{\phi_{B3}(r_i)}{\Phi_0(r_i)} E\left\{\pi\left(0; s_{(1)}^2, s_{(2)}^2\right) |\hat{s}_{-i} \in B3(r_i)\right\}, \tag{69}$$

$$\Pi(r_i, s_i|1) \equiv \frac{\phi_{B4}(r_i)}{\Phi_1(r_i)} E\left\{\pi\left(s_i; s_{(1)}^2, 0\right) |\hat{s}_{-i} \in B4(r_i)\right\} + \frac{\phi_{B5}(r_i)}{\Phi_1(r_i)} E\left\{\pi\left(s_i; s_{(2)}^2, 0\right) |\hat{s}_{-i} \in B5(r_i)\right\}, \tag{70}$$

and

$$\Pi(r_i, s_i|2) \equiv \pi(s_i; 0, 0). \tag{71}$$

Furthermore, $\Pi_j(r_i, s_i|k)$ denotes the partial derivative of $\Pi(r_i, s_i|k)$ with respect to the jth argument where j is either 1 or 2.

Remark: It is interesting to observe that, when the payoff function $\pi(\cdot;\cdot,\cdot)$ exhibits fixed externalities, the expected gross payoff of firm i, given that it has not won any license (denoted by $\Pi(r_i, s_i|0)$), is a constant and does not depend on either its reported signal or its true signal. However, in the presence of signal-dependent externalities, $\Pi(r_i, s_i|0)$ depends only on firm i's reported signal r_i and not on its true signal s_i. Moreover, firm i's expected gross payoff from purchasing one license (denoted by $\Pi(r_i, s_i|1)$), depends both on its reported signal and on its true signal. Finally, firm i's expected gross payoff from purchasing both licenses (denoted by $\Pi(r_i, s_i|2)$), depends only on its true signal s_i.

The Seller's Problem

I now state the seller's problem formally. In order to do so, I use (69), (70) and (71) to rewrite the interim expected payoff of firm i as given in (67) as follows:

$$V_{iQ}\left(r_i, s_i\right) = \Phi_0\left(r_i\right)\Pi\left(r_i, s_i|0\right) + \Phi_1\left(r_i\right)\Pi\left(r_i, s_i|1\right) + \Phi_2\left(r_i\right)\Pi\left(r_i, s_i|2\right) - m_i\left(r_i\right). \tag{72}$$

In the truth-telling equilibrium, it is a best response of firm i to report its signal truthfully, given that other firms report truthfully. This is known as *Bayesian Incentive Compatibility* and is defined formally below:[57]

The allocation and payment rule satisfy ***Bayesian Incentive Compatibility*** *(henceforth BIC) if, for every firm i,*

$$V_{iQ}\left(s_i, s_i\right) \geq V_{iQ}\left(r_i, s_i\right) \text{ for all } r_i \text{ and } s_i \in [0,1]. \tag{73}$$

Moreover, no firm can be forced to participate in the mechanism. This can be ensured if no firm becomes worse off participating in the mechanism than by staying out. This is known as *Individual Rationality.* Suppose that if firm i stays out of the mechanism, it gets a payoff of $\underline{\pi}_i$. Then, the Individual Rationality constraint is formally as follows.

The allocation and payment rule satisfy ***Individual Rationality*** *(henceforth IR) if, for every firm i,*

$$V_{iQ}\left(s_i, s_i\right) \geq \underline{\pi}_i. \tag{74}$$

I now define the seller's problem as follows:

$$\begin{aligned} &\text{Select } \left\{B1\left(\cdot\right), \ldots, B6\left(\cdot\right), m_1\left(\cdot\right), m_2\left(\cdot\right), m_3\left(\cdot\right)\right\} \text{ to} \\ &Max \sum_{i=1}^{3} \int_0^1 m_i\left(s_i\right) g\left(s_i\right) ds_i \text{ s.t. } BIC \text{ and } IR. \end{aligned} \tag{75}$$

The optimal mechanism solves the problem for all possible $\{Q, M_i\}$. In principle, this problem can be solved in two steps. First, fix the allocation rule $Q\left(\cdot\right)$ arbitrarily.[58] Then,

[57] The term "Bayesian" has been used because the equilibrium concept used is Bayes-Nash. See Krishna (2002, p. 280).

[58] Recall that with a Q, we can associate a unique partition A and hence a resulting partition $B\left(\cdot\right)$.

determine $m_i(\cdot)$ to satisfy BIC and IR and use this function to obtain R_Q where

$$R_Q = \sum_{i=1}^{3} \int_0^1 m_i(s_i)\, g(s_i)\, ds_i. \tag{76}$$

In the second step, select $Q(\cdot)$ to maximize R_Q.

In the next section, I use incentive compatibility to narrow down the search to a subset of all possible allocation and payment rules. Then, I show that the expected payment function $m_i(\cdot)$, in an incentive compatible direct mechanism, is a function of the allocation rule and the equilibrium payoff of the firm with signal 0, given by $V_{iQ}(0,0)$. I then discuss how $V_{iQ}(0,0)$ is determined in the optimal mechanism.

Seller's Revenue under Incomplete Information

In this section, I determine the seller's revenue in an incentive compatible direct mechanism, as a function of the allocation rule. In order to do so, I first provide an alternative characterization of incentive compatibility in the following proposition:

Proposition 4.5. *The direct mechanism is incentive compatible if and only if*

$$V_{iQ}(s_i, s_i) = V_{iQ}(0,0) + \int_0^{s_i} \left[\Phi_1(s)\, \Pi_2(s, s|1) + \Phi_2(s)\, \Pi_2(s, s|2)\right] ds, \tag{77}$$

and

$$\Phi_1(r_i)\, \Pi_2(r_i, s_i|1) + \Phi_2(r_i)\, \Pi_2(r_i, s_i|2) \text{ is non-decreasing in } r_i \text{ for all } r_i \in [0,1]. \tag{78}$$

Proof. See Appendix C. □

From (77) it follows that the marginal change in the equilibrium payoff $V_{iQ}(s_i, s_i)$ with respect to the signal s_i is given by

$$\Phi_1(s_i)\,\Pi_2(s_i, s_i|1) + \Phi_2(s_i)\,\Pi_2(s_i, s_i|2)\,. \tag{79}$$

Notice that under the assumptions of the model, the expression in (79) is positive. In addition, I also show that, under the assumptions of the model, the expression in (79) is non-decreasing in the signal s_i.

Corollary 4.6. *Suppose the payoff of a firm is strictly convex in its signal s_i.Then (78) implies that the marginal change in the equilibrium payoff (given in (79)) with respect to s_i is non-decreasing in s_i.*

Proof. See Appendix C. □

The above corollary and (79) implies that, in my model, the equilibrium payoff function $V_{iQ}(s_i, s_i)$ is positively sloped and convex in s_i. Hence, incentive compatibility ensures that if the IR constraint is satisfied for a firm with signal 0, then it is satisfied for a firm with any arbitrary signal.

Incentive Compatible Allocations

It is instructive at this point to consider the kind of allocations that satisfy (78) and hence are implementable. First, I consider allocations in which the number of licensees is known with certainty when the firms report their types. There are six possible allocation rules with such a feature and they are the following: *(i)* The firm with the highest report purchases both the licenses, *(ii)* the firm with the second highest report purchases both the licenses, *(iii)* the firm with the third highest report purchases both the licenses, *(iv)* the firms with the highest and second highest reports purchase a license each, *(v)* the firms with the second highest and third highest reports purchase a license each, and *(vi)* the firms with the highest and third highest reports purchase a license each. Suppose the seller commits to sell both the licenses to the firm with the highest report, that is, it commits to implement the first allocation rule mentioned above. In this case,

$$\begin{aligned} & \Phi_1(r_i)\,\Pi_2(r_i, s_i|1) + \Phi_2(r_i)\,\Pi_2(r_i, s_i|2) \\ = & \int_0^{r_i} \pi_1(s_i; 0, 0)\, f_1^2\left(s_{(1)}^2\right) ds_{(1)}^2 \end{aligned}$$

and the above expression is non-decreasing in the report r_i. Hence, the allocation in which the firm with the highest report purchases both licenses is implementable. One can check that the allocations mentioned in (ii) and (iii) are not implementable. Next consider (iv), that is, the allocation in which the seller commits to sell one license each to the firms with the highest two reports. In this case,

$$\begin{aligned} & \Phi_1(r_i)\,\Pi_2(r_i, s_i|1) + \Phi_2(r_i)\,\Pi_2(r_i, s_i|2) \\ = & \int_0^{r_i}\int_{s_{(2)}^2}^{1} \pi_1\left(s_i; s_{(1)}^2, 0\right) f_1^2\left(s_{(1)}^2|s_{(2)}^2\right) f_2^2\left(s_{(2)}^2\right) ds_{(1)}^2 ds_{(2)}^2 \end{aligned}$$

and this is also non-decreasing in r_i. Hence, the allocation in which the two firms with the highest signals purchase one license each is implementable. One can check that the allocations in (v) and (vi) are not implementable.

Next, I consider allocations that have the feature that the number of licensees is uncertain before the sale. There are potentially many allocation rules that have this feature. Below, I consider a particular class of such allocation rules. I call each rule in this class the *Non-decreasing cutoff (NDC) rule.* These allocation rules have the feature that if the reports $r_{(1)}^3$ and $r_{(2)}^3$ are "close" to each other, then the firms with the reports $r_{(1)}^3$ and $r_{(2)}^3$ purchase one license each; else, the firm with the report $r_{(1)}^3$ purchases both licenses. In particular, corresponding to each allocation Q, there is a non-decreasing function $\underline{s}(r; Q) \leq r$, such that if $r_{(2)}^3 \geq \underline{s}\left(r_{(1)}^3; Q\right)$, then $\left(r_{(1)}^3, r_{(2)}^3, r_{(3)}^3\right) \in A4(\hat{r}|Q)$ and the firms with the highest and second highest reports purchase one license each, while if $r_{(2)}^3 < \underline{s}\left(r_{(1)}^3; Q\right)$, then

$\left(r^3_{(1)}, r^3_{(2)}, r^3_{(3)}\right) \in A6\left(\hat{r}|Q\right)$ and the firm with the highest report purchases both licenses.[59] Formally,

$$\left(r^3_{(1)}, r^3_{(2)}, r^3_{(3)}\right) \in \begin{cases} A4\left(\hat{r}|Q\right) \text{ if } 0 \leq \max\left\{r^3_{(3)}, \underline{s}\left(r^3_{(1)}; Q\right)\right\} \leq r^3_{(2)} \leq r^3_{(1)} \leq 1, \\ A6\left(\hat{r}|Q\right) \text{ if } 0 \leq r^3_{(3)} \leq r^3_{(2)} < \underline{s}\left(r^3_{(1)}; Q\right) \leq r^3_{(1)} \leq 1. \end{cases} \tag{80}$$

I now determine the expected payoff of firm i under the NDC rule, when it reports r_i. It follows from (80) that under any allocation rule that belongs to the NDC class, if $r^2_{(1)} < \underline{s}\left(r_i; Q\right)$, then firm i purchases both licenses, and if $\underline{s}\left(r_i; Q\right) \leq r^2_{(1)} < r_i$, firm i purchases one license.[60] In order to specify firm i's allocation under the NDC rule in the case that $r^2_{(1)}$ is greater than r_i, define

$$\overline{s}(r_i; Q) \equiv \max\left\{r^2_{(1)} | \underline{s}\left(r^2_{(1)}; Q\right) \leq r_i\right\}.$$

Given firm i's report r_i, $\overline{s}(r_i; Q)$ is, by construction, the maximum value of $r^2_{(1)}$ such that firm i can purchase a license. The above statement implies that if $r^2_{(2)} < r_i < r^2_{(1)} \leq \overline{s}(r_i; Q)$, then firm i can purchase exactly one license while, if either $r^2_{(2)} > r_i$ or if $r^2_{(1)} > \overline{s}(r_i; Q)$, then firm i cannot purchase any license. Observe that $\overline{s}(r_i; Q)$ is non-decreasing in r_i because of the fact that $\underline{s}\left(r_i; Q\right)$ is non-decreasing in r_i.

Suppose firm i reports r_i and the other firms report truthfully. Under an allocation rule that belongs to the NDC class described above,

$$\Phi_1\left(r_i\right) \Pi\left(r_i, s_i|1\right) = \int_{\underline{s}(r_i;Q)}^{\overline{s}(r_i;Q)} \int_0^{\min\left\{r_i, s^2_{(1)}\right\}} \pi\left(s_i; s^2_{(1)}, 0\right) f^2_{12}\left(s^2_{(1)}, s^2_{(2)}\right) ds^2_{(2)} ds^2_{(1)} \tag{81}$$

and

$$\Phi_2\left(r_i\right) \Pi\left(r_i, s_i|2\right) = \pi\left(s_i; 0, 0\right) \int_0^{\underline{s}(r_i;Q)} \int_0^{s^2_{(1)}} f^2_{12}\left(s^2_{(1)}, s^2_{(2)}\right) ds^2_{(2)} ds^2_{(1)}. \tag{82}$$

[59] The optimal allocation rule for fixed externalities belongs to this class. See Dana and Spier (1994) and Schmitz (2002).

[60] Observe that in this case firm i has the highest of the three signals.

I now check whether this class of allocations satisfy (78). Notice that, from (81) and (82),

$$\begin{aligned} & \frac{\partial}{\partial r_i}\left\{\Phi_1(r_i)\,\Pi_2(r_i,s_i|1)+\Phi_2(r_i)\,\Pi_2(r_i,s_i|2)\right\} \\ = & \ \overline{s}'(r_i;Q)\,\pi_1(s_i;\overline{s}(r_i;Q),0)\,F_2^2\left(r_i|s_{(1)}^2=\overline{s}(r_i;Q)\right)f_1^2(\overline{s}(r_i;Q)) \\ & +\underline{s}'(r_i;Q)\left\{\int_0^{\underline{s}(r_i;Q)}-\pi_{12}(s_i;s,0)\,ds\right\}f_1^2(\underline{s}(r_i;Q)). \end{aligned} \tag{83}$$

It follows that, given (57) and (59), an allocation rule in this class satisfies (78), and is hence implementable.

Revenue

I now determine the expected payment of a firm in an incentive compatible direct mechanism. Below, I prove a version of the revenue equivalence theorem by showing that the expected payments of firm i with signal s_i in two mechanisms that have the same allocation rule[61] and the same net payoff for a firm with signal 0 are equal.

I define $\alpha_k(r_i)$ for $k=1,2,\ldots,6$ as follows:

$$\alpha_k(r_i)=\begin{cases} E\left\{\pi(\cdot;\cdot,\cdot)\right\} \text{ if } \hat{s}_{-i}\in Bk(r_i), \\ 0 \text{ if } \hat{s}_{-i}\notin Bk(r_i). \end{cases}$$

In particular, $\alpha_k(r_i)$ is the expected gross payoff to firm i when it reports a signal r_i, the others report truthfully and the profile of reports belong to the set $Bk(r_i)$ for $k=1,2,\ldots,6$. Observe that by construction, the following equality must hold:

$$V_{iQ}(s_i,s_i)=\sum_{k=1}^{6}\alpha_k(s_i)-m_i(s_i). \tag{84}$$

[61] Recall that an allocation rule can be associated with one and only one partition $\{B1(r_i),\ldots,B6(r_i)\}$ of the signal space of buyer i's competitors.

Analogously, I define $\beta_k(r_i, s_i)$ for $k = 4, 5$,or 6 as follows:

$$\beta_k(r_i, s_i) = \begin{cases} E\{\pi_1(\cdot;\cdot,\cdot)\} \text{ if } \hat{s}_{-i} \in Bk(r_i), \\ 0 \text{ if } \hat{s}_{-i} \notin Bk(r_i). \end{cases}$$

It follows from the above definition that $\beta_k(r_i, s_i)$ is the expected value of the marginal gross payoff to firm i when it reports r_i, the others report truthfully and the profile of reports belong to the set $Bk(r_i)$ for $k = 1, 2, \ldots, 6$. The explicit forms of $\alpha_k(r_i)$ and $\beta_k(r_i, s_i)$ are presented in the Appendix. The expected payment of a firm in an incentive compatible direct mechanism is presented in the following proposition.

Proposition 4.7. *In the truth-telling equilibrium of an incentive compatible and individually rational direct mechanism in which a firm with signal* 0 *obtains a net payoff of* $V_Q(0,0)$, *the expected payment of firm* i *with signal* s_i *is given by*

$$m(s_i) = \sum_{k=1}^{6}\left[\alpha_k(s_i) - \int_0^{s_i} \beta_k(s,s)\,ds\right] - V_Q(0,0) \text{ provided } V_Q(0,0) \geq \underline{\pi}. \tag{85}$$

Proof. See Appendix C. □

The above proposition shows that the expected payment of a firm in two mechanisms are the same whenever these mechanisms have the same partition $\{Bk(r_i)\}_{k=1}^{6}$, and the same net payoff of the firm with signal 0. Also, notice that, under the condition that a firm with signal 0 earns a payoff of $V_Q(0,0)$, the equilibrium payment function $m_i(\cdot)$ is the same for all the firms. Consequently, the subscript i from the expected payment function $m_i(s_i)$ has been dropped.

I now use Proposition 4.7 to determine the revenue of the seller in the truth-telling equilibrium of an incentive compatible direct mechanism. Below, I define the *industry virtual payoff* and show that the *ex ante* expected revenue of the seller is the expected value of the industry virtual payoff. The industry virtual payoff depends on the *ex post* allocation rule and hence, the seller maximizes her *ex ante* expected revenue by choosing the allocation rule that maximizes the expected industry virtual payoff.

Definition: Given the profile of signals $\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$ and the allocation rule Q, such that

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \in Ak\left(\hat{r}|Q\right); \; k = 1, \ldots, 6,$$

the industry virtual payoff, denoted by $\lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$, is given by

$$\begin{aligned} & \lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \\ = & \sum_{j=0}^{3}\{\pi\left(\cdot;\cdot,\cdot\right) - I_w\left(s^3_{(j)}\right)\frac{1-G\left(s^3_{(j)}\right)}{g\left(s^3_{(j)}\right)}\pi_1\left(\cdot;\cdot,\cdot\right) | \left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \in Ak\left(\hat{r}|Q\right)\}; \\ k = & 1, \ldots, 6, \end{aligned} \tag{86}$$

where $I_w\left(s^3_{(j)}\right)$ is an indicator variable that takes value 1 if the firm with type $s^3_{(j)}$ purchases a license.

As an example, suppose the profile of signals $\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$ belongs to the set $A4\left(\hat{r}|Q\right)$ under the seller's allocation rule. Then,

$$\begin{aligned} \lambda_{A4(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) &= \pi\left(s^3_{(1)}; s^3_{(2)}, 0\right) + \pi\left(s^3_{(2)}; s^3_{(1)}, 0\right) + \pi\left(0; s^3_{(1)}, s^3_{(2)}\right) \\ & - \frac{1-G\left(s^3_{(1)}\right)}{g\left(s^3_{(1)}\right)}\pi_1\left(s^3_{(1)}; s^3_{(2)}, 0\right) - \frac{1-G\left(s^3_{(2)}\right)}{g\left(s^3_{(2)}\right)}\pi_1\left(s^3_{(2)}; s^3_{(1)}, 0\right). \end{aligned}$$

The industry virtual payoff is the gross industry payoff less the information rents of the licensees. The information rent depends on the distribution function of the signals and the signals of the licensees. It also follows from (59) that $\pi_{12}\left(\cdot;\cdot,\cdot\right) < 0$ and hence, the information rent of a licensee is non-increasing in the signal of another licensee. Below, I use (85) and (86) to determine the revenue of the seller in the truth-telling equilibrium of any incentive compatible direct mechanism.

Proposition 4.8. *The ex ante expected revenue of the seller in the truth-telling equilibrium* *of an incentive compatible direct mechanism is*

$$R_Q = \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda_{Ak(\hat{r}|Q)} \left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}\left(\cdot\right) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} - 3V_Q\left(0,0\right), \tag{87}$$

where

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \in Ak\left(\hat{r}|Q\right); \; k = 1, \ldots, 6.$$

Proof. See Appendix C. □

This proposition states that the revenue of the seller in the truth-telling equilibrium of a direct mechanism is the expected industry virtual payoff, less the product of the number of firms and the payoff of the firm with signal zero. Notice that the industry virtual payoff in (87) given any profile of signals depends on the seller's allocation rule.

Optimal Allocation

The allocation rule that maximizes (87) is defined to be the optimal allocation rule and the associated mechanism is said to be the optimal mechanism. Below, I use Proposition 4.8 to determine the optimal mechanism when the payoff function exhibits signal-dependent externalities. It follows from (87) that in the optimal mechanism, the expected industry virtual payoff is maximized and the payoff of the firm with signal zero is minimized. In order to describe the allocation that maximizes the expected industry virtual payoff, I define

$$\lambda^*\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) = \max\left\{\lambda_{A1(\hat{r}|Q)}\left(\cdot\right), \ldots, \lambda_{A6(\hat{r}|Q)}\left(\cdot\right)\right\}.$$

Further, one has to check that the allocation that maximizes the expected industry virtual payoff satisfies the conditions for incentive compatibility given by (77) and (78). Notice that the expression in (87) relies on (77) only and therefore, there is no guarantee that the optimal allocation derived by maximizing the expression in (87) satisfies (78). In models of sales without externalities, this problem is usually solved by assuming a regularity condition—in such cases, the regularity condition states that the virtual value is an increasing function of the signal.

In my model, I can solve the problem using an appropriate regularity condition and this is described below. I define a problem to be *regular* if the following conditions are true: *(i)* Given any profile of signals $\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$, either $\lambda^*(\cdot) = \lambda_{A6(\hat{r}|Q)}(\cdot)$ or $\lambda^*(\cdot) = \lambda_{A4(\hat{r}|Q)}(\cdot)$; hence, in the optimal allocation, the seller either sells both the licenses to the firm with the highest signal (when $\lambda^*(\cdot) = \lambda_{A6(\hat{r}|Q)}(\cdot)$) or to the two firms with the two highest signals (when $\lambda^*(\cdot) = \lambda_{A4(\hat{r}|Q)}(\cdot)$), *(ii)* The allocation induced by $\lambda^*(\cdot)$ belongs to the NDC class. Notice that if the design problem is regular, then the allocation that maximizes the industry virtual payoff is incentive compatible, because allocations that belong to the NDC class satisfy (78).[62]

In the proposition below, it has been assumed that if one of the firms decides to stay out of the mechanism, then the seller can credibly commit to allocate a license to each of the firms who participate in the mechanism.[63] Under such a commitment, the payoff to a firm that decides not to participate in the mechanism is

$$\underline{\pi} = \int_0^1 \int_0^{s^2_{(1)}} \pi\left(0; s^2_{(1)}, s^2_{(2)}\right) f^2_{12}(\cdot)\, ds^2_{(2)} ds^2_{(1)}. \tag{88}$$

The value of $\underline{\pi}$ in (88) is used in the description of the optimal mechanism in Proposition 4.9.

Proposition 4.9. *Suppose that the design problem is regular, and if a firm does not participate in the mechanism, the seller can credibly commit to allocate one license each to the two other firms. Moreover, suppose that the payoff function exhibits signal-dependent externali-*

[62]Figueroa and Skreta (2005) have a different regularity condition because they do not assume that $\pi_{12}(\cdot;\cdot,\cdot) < 0$.

[63]Such a commitment is similar in spirit to Jehiel, Moldovanu, and Stachetti ((1996), p. 820) and in Kamien, Oren, and Tauman (1992). A description of the mechanism by Kamien et al. is also available at Kamien ((1992), p. 348).

ties. Then the revenue of the seller in the truth-telling equilibrium of the optimal mechanism is given by

$$R_Q^* = \int_0^1 \int_0^{s_{(1)}^3} \int_0^{s_{(2)}^3} \lambda^* \left(s_{(1)}^3, s_{(2)}^3, s_{(3)}^3\right) f_{123}^3 (\cdot)\, ds_{(3)}^3 ds_{(2)}^3 ds_{(1)}^3 - 3\underline{\pi} \tag{89}$$

where $\underline{\pi}$ is defined by (88).

Proof. See Appendix C. □

The above proposition states that in a regular problem, the seller can achieve her optimal revenue by choosing the allocation rule that maximizes the industry virtual payoff for every profile of reports. It follows from the definition of the industry virtual payoff that the following three factors affect the seller's revenue when she selects one licensee (the firm with the highest signal) instead of two licensees (the firms with the two highest signals):

((*i*)) The industry gross profit may increase or decrease, and hence, the effect of this factor on the seller's revenue is ambiguous. In the examples below, the industry gross profit is always maximized by choosing two licensees.

((*ii*)) If the seller selects only one licensee, then the firm with the second highest signal cannot earn any information rent and this increases the revenue of the seller. Notice that this factor will be present even in a model with no externalities; however, the magnitude of this effect depends on the externality parameter. This factor reduces the number of licensees.

((*iii*)) If the seller selects only one licensee, then the information rent of the firm with the highest signal increases because $\pi_{12}(\cdot;\cdot,\cdot) < 0$. This factor therefore reduces the revenue of the seller. Notice that this factor emerges only in the presence of externalities.

The choice of the optimal number of licensees therefore depends on the relative strength of each of the above-mentioned factors which in turn depends on the nature of the product market. Below, I show the role of several channels through which the product market influences the optimal number of licensees.

Role of the product market in determining the optimal allocation

In order to show the role of several product market factors in determining the optimal number of licensees, I need to make more explicit assumptions about the product market. For the purpose of the discussion below, I consider the scenario described in Example 1 when $\tau = 4$. In the example, firms compete in quantites and a winner imposes a higher level of externalities on the others when the level of product differentiation is low.

Remark 7. *Significant externalities may cause the seller to select a fewer number of licensees in the presence of private information compared to what she would have done if firms had no private information.*

Suppose the signals follow the uniform distribution. In the table below, I present the optimal number of licensees when the profile of signals is

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) = (0.9, 0.8, 0.3)\,.$$

Table 15: The Optimal Number Of Licensees When The Profile Of Signals Is 0.9, 0.8, and 0.3, As A Function Of The Externality Parameter

	Externality Parameter		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1$
Signals are Publicly Observable	2	2	2
The Signal of a Firm is its Private Information	2	2	1

Notice that the optimal number of licensees is always two when the signal of each firm is publicly observable. Furthermore, the optimal number of licensees remains at two even when the signal of each firm is its private information, as long as the externality parameter is 0 or 0.5. However, when the externality parameter is 1, the seller selects only one licensee in the optimal allocation when the signal of each firm is its private information. In this example, private information by itself does not induce the seller to select a fewer number of licensees. Instead, private information combined with the presence of significant externalities induces the seller to sell licenses to a fewer number of firms.

Remark 8. *When firms have private information, a higher level of externality leads to a decrease in the expected number of licensees.*

Consider the scenario described in the previous remark. For each possible profile of signals, I compute the optimal number of licensees and use this to compute the expected number of licensees. In the table below, I show how the expected number of licensees vary with the externality parameter (level of product differentiation in the industry).

Table 16: The Expected Number Of Licensees As A Function Of The Externality Parameter

	Externality Parameter		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1$
Expected number of licensees	1.47	1.27	1.1

Observe that an increase in the externality parameter (or, a reduction in the level of product differentiation) leads to a decrease in the expected number of licensees.

In the model, I assume that the signals of firms are independenly drawn from a common distribution $G(s)$. Below, I show that the nature of the distribution function $G(s)$ is an important determinant of the optimal number of licensees.

Remark 9. *Suppose $G_A(s)$ and $G_B(s)$ are two different distributions of signals such that*

$G_A(s)$ first-order stochastically dominates $G_B(s)$. Then, the expected number of licensees is higher under $G_A(s)$.

To interpret the above remark, notice that when the signals are drawn from $G_A(s)$ instead of $G_B(s)$, each firm has a greater likelihood of drawing a higher signal. In other words, when signals are drawn from $G_A(s)$ instead of $G_B(s)$, each firm has a greater likelihood of achieving a lower cost of production. Therefore, when firms are likely to be more efficient users of the technology, the expected number of licensees goes up.

For an illustration of the above remark, consider the Beta (2,1) distribution and the Uniform distribution; the Beta (2,1) distribution first-order stochastically dominates the Uniform distribution. In the table below, I present the expected number of licensees under each of these distributions.

Table 17: The Expected Number Of Licensees Under The Uniform And The Beta(2,1) distributions

Distribution	Externality Parameter		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1$
Uniform	1.47	1.27	1.1
Beta (2,1)	1.71	1.50	1.25

Notice that the expected number of licensees is higher under the Beta (2,1) distribution than under the Uniform distribution, for every value of the externality parameter and the question is why is this the case. It follows from the discussion above that when the seller increases the number of licensees from one to two, her revenue changes due to change in three factors- the change in industry profit and the change in information rents of the firms with the highest and second highest signals. However, in this example, when the seller increases the number of licensees from one to two, the changes in the information rents are not substantially different across the two distributions. In contrast, when the seller selects two licensees instead of one, the industry profit increases more substantially when the signals are drawn from the Beta (2,1) distribution and this explains why the expected number of licensees is higher when the signals are drawn from the Beta (2,1) distribution instead of the Uniform distribution.

Conclusion

This paper analyzes the seller's revenue from the sale of two identical licenses, both when the signal of each firm is publicly observable, and when the signal of each firm is its private information. It is assumed that if a firm refuses to participate in the mechanism, then the seller can credibly commit to allocate a license to its competitors. When the signal of each firm is publicly observable, the seller's optimal allocation is the one that maximizes the industry payoff. In contrast, when the signal of each firm is its private information, the seller selects the allocation rule that maximizes the industry virtual payoff. Such an allocation may or may not be different from the allocation that maximizes the industry payoff. We find that the presence of private information leads the seller to sometimes sell licenses to a smaller number of firms.

An important assumption in this article is that the private information of each firm is uni-dimensional. It follows from Jehiel and Moldovanu (2001) that an efficient allocation cannot be implemented when the signals are multi-dimensional. An interesting extension of this paper is to analyze revenue-maximizing allocations when the signals are multi-dimensional.

CHAPTER V

EPILOGUE

In my dissertation, I have studied the problem of how to sell intellectual property. In my first chapter, I have written a model to analyze an auction of licenses to a technology; in my second chapter, I have tested some of the predictions in my first chapter using an experiment and in my third chapter, I have derived the maximum possible revenue of a seller who wants to sell licenses to a technology.

There are recent examples where auctions have been used to sell intellectual property. For example, Commerce One Inc. sold 39 e-commerce patents using an auction in December 2004. I would like to point out three salient features of an auction of licenses to a technology:

((*i*)) The payoff of any firm in the industry depends on the outcome of the auction as well as on the market structure. Hence, while designing an auction of licenses, one has to take into account the nature of the product market.

((*ii*)) The winner of such an auction imposes a negative externality on the competitors.

((*iii*)) It is virtually costless for the seller to produce an additional license. This implies that while analyzing an auction of licenses, the number of licenses that the seller sells is endogenous.

In my work, I highlight the role of product market factors in the design of an auction of licenses where one of the elements of the design is the number of licenses. In the paper *How to Commercialize Technology Using Auctions*, I consider a situation where an independent private lab has developed a cost-reducing technology and announces an auction of licenses to the technology. Firms can achieve different marginal costs of production with the new technology and this is private information to each firm. This paper assumes that each firm can bid for only one license; in another paper, I have relaxed this assumption. The major findings of the paper are as follows:

((*i*)) One might want to know whether for licenses, any auction format will work better than the others. I show that the choice of the auction format does not matter, that is, for a fixed number of licenses, certain well known auctions will yield the same revenue. This explains why in practice we observe different kinds of auction formats or the sale of licenses. However, the number of licenses does matter for revenue.

((*ii*)) Second, does the number of licenses matter for revenue? I show that the seller can influence revenue by the choice of the number of licenses.

((*iii*)) Third, how does market structure affect the optimal number of licenses? I show that if firms sell differentiated products, it may be optimal to sell licenses to multiple firms.

((*iv*)) Finally, in the design of an auction, we often think about a reserve price. What is the optimal reserve price in this context? I demonstrate that it can be 0.

Next, I test the performance of this model (with Mike Shor) in a controlled experiment. The results of this study have been described in the paper *A Laboratory Test of an Auction with Negative Externalities.* The major findings are as follows:

((*i*)) In terms of average revenues, the observed revenues were statistically equal to the predicted revenues.

((*ii*)) The simulated distribution of observed revenues seemed to have some difference with the distribution of predicted revenues. A possible reason for the discrepancy was that subjects seemed to be using a rule of thumb to estimate the signals of other competitors.

In the next paper entitled *When Should an Inventor Sell Licenses Exclusively? The Determinants of the Optimal Number of Licensees,* I have extended my previous work to the case where firms can potentially win multiple licenses and thereby prevent a competitor from winning a license.

((i)) I characterize the optimal mechanism when there is no private information. In the optimal mechanism, the seller ensures participation by threatening to award a license to each competitor of a non-participant. In the optimal mechanism under complete information, the seller chooses the allocation that maximizes the industry profit.

((ii)) I show that mechanisms with the same ex-post allocation rule and the same structure of the product market are revenue equivalent.

((iii)) I characterize the optimal mechanism under private information. The optimal mechanism has two parts. First, if any firm refuses to participate in the mechanism, the seller threatens to punish the firm by allocating a license to each the participating competitors. Second, given any profile of types, the seller chooses the allocation that maximizes the industry virtual payoff. The industry virtual payoff is the total industry profit less the information rents of the firms that win a license.

((iv)) I then determine factors that determine the licensees in the optimal mechanism. I find that if the lab selects multiple winners instead of one, three factors affect the lab's revenue. First, industry profits change. Second, when the lab sells to multiple firms, more firms earn an information rent which favors selling to one firm. Third, when the lab sells to multiple firms, then the information rent of the firm with the lowest cost goes down which favors selling to multiple firms.

((v)) Finally, I demonstrate the role of several product market factors, such as the level of product differentiation in the industry in determining the optimal number of licensees.

Proof of Proposition 2.1

For the sake of exposition, we present the proof for $k = 2$. The other cases can be proved analogously, or can be obtained on request from the author. When $k = 2$,

$$U(c_i) = \int_{c_i}^{1}\int_{0}^{c_{(2)}} \pi^W\left(c_i; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)}, c_{(2)}\right) dc_{(1)}dc_{(2)}$$
$$+ \int_{0}^{c_i}\int_{0}^{c_{(2)}} \pi^L\left(1; c_{(1)}, c_{(2)}\right) f_{12}^{n-1}\left(c_{(1)}, c_{(2)}\right) dc_{(1)}dc_{(2)} - m_2(c_i). \tag{A.1}$$

We also know that:

$$P(\widetilde{c}, c_i) = \int_{\widetilde{c}}^{1}\int_{0}^{c_{(2)}} \pi^W\left(c_i; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)}, c_{(2)}\right) dc_{(1)}dc_{(2)}$$
$$+ \int_{0}^{\widetilde{c}}\int_{0}^{c_{(2)}} \pi^L\left(1; c_{(1)}, c_{(2)}\right) f_{12}^{n-1}\left(c_{(1)}, c_{(2)}\right) dc_{(1)}dc_{(2)} - m_2(\widetilde{c}). \tag{A.2}$$

The expressions for $U(\widetilde{c})$ and $P(c_i, \widetilde{c})$ are obtained from (A.1) and (A.2) by permuting c_i and $\widetilde{c}$. From incentive compatibility, we have:

$$U(c_i) \geq P(\widetilde{c}, c_i), \tag{A.3}$$

and

$$U(\widetilde{c}) \geq P(c_i, \widetilde{c}). \tag{A.4}$$

Furthermore,

$$P(\widetilde{c}, c_i) = U(\widetilde{c}) + \int_{\widetilde{c}}^{1}\int_{0}^{c_{(2)}} \left[\pi^W\left(c_i; c_{(1)}\right) - \pi^W\left(\widetilde{c}; c_{(1)}\right)\right] f_{12}^{n-1}\left(c_{(1)}, c_{(2)}\right) dc_{(1)}dc_{(2)} \tag{A.5}$$

and

$$P\left(c_i,\widetilde{c}\right)=U\left(c_i\right)+\int_{c_i}^{1}\int_{0}^{c_{(2)}}\left[\pi^W\left(\widetilde{c};c_{(1)}\right)-\pi^W\left(c_i;c_{(1)}\right)\right]f_{12}^{n-1}\left(c_{(1)},c_{(2)}\right)dc_{(1)}dc_{(2)}. \tag{A.6}$$

Therefore, from (A.3), (A.4), (A.5) and (A.6), we obtain the following inequality:

$$\begin{aligned}\int_{\widetilde{c}}^{1}\int_{0}^{c_{(2)}}&\left[\pi^W\left(c_i;c_{(1)}\right)-\pi^W\left(\widetilde{c};c_{(1)}\right)\right]f_{12}^{n-1}\left(c_{(1)},c_{(2)}\right)dc_{(1)}dc_{(2)}\\ &\leq U\left(c_i\right)-U\left(\widetilde{c}\right)\\ &\leq\int_{c_i}^{1}\int_{0}^{c_{(2)}}\left[\pi^W\left(c_i;c_{(1)}\right)-\pi^W\left(\widetilde{c};c_{(1)}\right)\right]f_{12}^{n-1}\left(c_{(1)},c_{(2)}\right)dc_{(1)}dc_{(2)}.\end{aligned} \tag{A.7}$$

By letting $\widetilde{c}\rightarrow c_i$, it follows from (A.7) that

$$U'\left(c_i\right)=\int_{c_i}^{1}\int_{0}^{c_{(2)}}\frac{\partial\pi^W\left(c_i;c_{(1)}\right)}{\partial c_i}f_{12}^{n-1}\left(c_{(1)},c_{(2)}\right)dc_{(1)}dc_{(2)}. \tag{A.8}$$

Thus,

$$\begin{aligned}U\left(1\right)-U\left(c_i\right)&=\int_{c_i}^{1}\int_{c}^{1}\int_{0}^{c_{(2)}}\frac{\partial\pi^W\left(c;c_{(1)}\right)}{\partial c}f_{12}^{n-1}\left(c_{(1)},c_{(2)}\right)dc_{(1)}dc_{(2)}dc\\ &=\int_{c_i}^{1}\int_{0}^{c_{(2)}}\int_{c_i}^{c_{(2)}}\frac{\partial\pi^W\left(c;c_{(1)}\right)}{\partial c}f_{12}^{n-1}\left(c_{(1)},c_{(2)}\right)dc\,dc_{(1)}dc_{(2)}\\ &=\int_{c_i}^{1}\int_{0}^{c_{(2)}}\left[\pi^W\left(c_{(2)};c_{(1)}\right)-\pi^W\left(c_i;c_{(1)}\right)\right]f_{12}^{n-1}\left(c_{(1)},c_{(2)}\right)dc_{(1)}dc_{(2)}.\end{aligned} \tag{A.9}$$

Notice that, (4) and (5) imply that:

$$U\left(1\right)=\int_{0}^{1}\int_{0}^{c_{(2)}}\pi^L\left(1;c_{(1)},c_{(2)}\right)f_{12}^{n-1}\left(c_{(1)},c_{(2)}\right)dc_{(1)}dc_{(2)}-m_2\left(1\right). \tag{A.10}$$

Hence,

$$U(c_i) = \int_0^1 \int_0^{c_{(2)}} \pi^L\left(1; c_{(1)}, c_{(2)}\right) f_{12}^{n-1}\left(c_{(1)}, c_{(2)}\right) dc_{(1)} dc_{(2)}$$
$$- \int_{c_i}^1 \int_0^{c_{(2)}} \left[\pi^W\left(c_{(2)}; c_{(1)}\right) - \pi^W\left(c_i; c_{(1)}\right)\right] f_{12}^{n-1}\left(c_{(1)}, c_{(2)}\right) dc_{(1)} dc_{(2)} - m_2(1). \quad \text{(A.11)}$$

Combining (A.1) and (A.11) and rearranging, we obtain:

$$\begin{aligned} m_2(c_i) &= m_2(1) + \int_{c_i}^1 \int_0^{c_{(2)}} \left[\pi^W\left(c_{(2)}; c_{(1)}\right) - \pi^L\left(1; c_{(1)}, c_{(2)}\right)\right] f_{12}^{n-1}\left(c_{(1)}, c_{(2)}\right) dc_{(1)} dc_{(2)} \\ &= m_2(1) + \int_{c_i}^1 \left\{ \int_0^c \left[\pi^W\left(c; c_{(1)}\right) - \pi^L\left(1; c_{(1)}, c\right)\right] \frac{f_{12}^{n-1}\left(c_{(1)}, c_{(2)} = c\right)}{f_2^{n-1}(c)} dc_{(1)} \right\} f_2^{n-1}(c)\, dc \\ &= m_2(1) + \int_{c_i}^1 V_2(c) f_2^{n-1}(c)\, dc. \end{aligned} \quad (90)$$

where the last equality follows from (2).

Proof of Proposition 2.4

Because $V_k(1) = 0$ for all k, substituting (21) in (15), we obtain:

$$R_k = k \int_0^1 V_k(c) f_{k+1}^n(c)\, dc < \int_0^1 V_1(c) f_{k+1}^n(c)\, dc. \quad \text{(A.13)}$$

Observe that $F_{k+1}^n(c)$ first order stochastically dominates $F_2^n(c)$ and both $V_1(c)$ and $V_k(c)$ are monotonically decreasing. Therefore, we obtain that

$$R_k < \int_0^1 V_1(c) f_{k+1}^n(c)\, dc \leq \int_0^1 V_1(c) f_2^n(c)\, dc = R_1. \quad \text{(A.14)}$$

Proof of Proposition 2.5

For the sake of exposition, we present the proof for $k = 2$. The other cases can be proved analogously, or can be obtained on request from the author. Suppose that $c_i \leq c_r$. Then,

$$\begin{aligned} U(c_i) = & \int_{c_i}^{c_r} \int_0^{c_{(2)}} \pi^W\left(c_i; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)} dc_{(2)} \\ & + \int_{c_r}^{1} \int_0^{c_r} \pi^W\left(c_i; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)} dc_{(2)} \\ & + \int_{c_r}^{1} \int_{c_r}^{c_{(2)}} \pi^W\left(c_i\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)} dc_{(2)} \\ & + \int_0^{c_i} \int_0^{c_{(2)}} \pi^L\left(1; c_{(1)}, c_{(2)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)} dc_{(2)} - m_{2r}(c_i). \end{aligned} \tag{A.15}$$

The expected profit of firm i is given by the first four terms on the right hand side of (A.15). Because $c_i \leq c_r$, firm i wins (resp. does not win) a license if $c_i < c_{(2)}^{n-1}$ (resp. $c_i > c_{(2)}^{n-1}$). Thus, in the first three terms, firm i wins a license, whereas in the fourth term it does not. In the third term, $c_{(1)}^{n-1} > c_r$, so only firm i wins a license. In the first two terms, both firm i and the firm of type $c_{(1)}^{n-1}$ win licenses. In the first term, the firm of type $c_{(2)}^{n-1}$ has a type below the critical value c_r, whereas in the second term, it does not.

Following the same steps as in Proposition 2.1, we can show that:

$$\begin{aligned} m_{2r}(c_i) = & -U(c_r) + \int_{c_i}^{c_r} \int_0^{c_{(2)}} \pi^W\left(c_{(2)}; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)} dc_{(2)} \\ & + \int_{c_r}^{1} \int_0^{c_r} \pi^W\left(c_r; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)} dc_{(2)} \\ & + \int_{c_r}^{1} \int_{c_r}^{c_{(2)}} \pi^W\left(c_r\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)} dc_{(2)} \\ & + \int_0^{c_i} \int_0^{c_{(2)}} \pi^L\left(1; c_{(1)}, c_{(2)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)} dc_{(2)}. \end{aligned} \tag{A.16}$$

Moreover, from (A.15), we obtain:

$$\begin{aligned}U(c_r) = &\int_{c_r}^{1}\int_{0}^{c_r} \pi^W\left(c_r; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} \\ &+ \int_{c_r}^{1}\int_{c_r}^{c_{(2)}} \pi^W(c_r)\, f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} \\ &+ \int_{0}^{c_r}\int_{0}^{c_{(2)}} \pi^L\left(1; c_{(1)}, c_{(2)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} - m_{2r}(c_r). \qquad \text{(A.17)}\end{aligned}$$

Substituting (A.17) in (A.16), we obtain

$$\begin{aligned}m_{2r}(c_i) &= \int_{c_i}^{c_r}\left\{\int_{0}^{c}\left[\pi^W\left(c; c_{(1)}\right) - \pi^L\left(1; c_{(1)}, c\right)\right]\frac{f_{12}^{n-1}\left(c_{(1)}, c\right)}{f_2^{n-1}(c)}dc_{(1)}\right\} f_2^{n-1}(c)\,dc + m_{2r}(c_r) \\ &= \int_{c_i}^{c_r} V_2(c)\, f_2^{n-1}(c)\,dc + m_{2r}(c_r), \qquad \text{(A.18)}\end{aligned}$$

where the last equality follows from (2).

Now consider any type $c_i > c_r$. The equilibrium payoff of such a type is given by

$$\begin{aligned}U(c_i) = &\int_{c_r}^{1}\int_{0}^{c_r} \pi^L\left(1; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} \\ &+ \int_{c_r}^{1}\int_{c_r}^{c_{(2)}} \pi^0 f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} \\ &+ \int_{0}^{c_r}\int_{0}^{c_{(2)}} \pi^L\left(1; c_{(1)}, c_{(2)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)}. \qquad \text{(A.19)}\end{aligned}$$

Observe that because a firm of type $c_i > c_r$ never wins a license, the equilibrium payoff $U(c_i)$ is the same for all such types. The first term in the right-hand side of (A.19) is the profit of a firm of type c_i when only the firm of type $c_{(1)}^{n-1}$ wins a license. In the second term, no firm wins a license. In the third term, the firms of types $c_{(1)}^{n-1}$ and $c_{(2)}^{n-1}$ win a license each. Note that, by assumption, $m_{2r}(c_i) = 0$ for all $c_i > c_r$.

Next, we show that $m_{2r}(c_r)$ must equate $U(c_r)$ to $U(c_i)$ when $c_i > c_r$. Suppose to the contrary that $m_{2r}(c_r)$ is such that $U(c_r) \neq U(c_i)$. First, consider the case in which $U(c_r) > U(c_i)$. In equilibrium, a firm of type c_i earns a payoff of $U(c_i)$, where $U(c_i)$ is given in (A.19). If type c_i deviates from its truth-telling strategy and instead reports c_r as its type, then this firm's payoff is given by

$$\begin{aligned} P(c_r, c_i) = & \int_{c_r}^{1}\int_{0}^{c_r} \pi^W\left(c_i; c_{(1)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} \\ & + \int_{c_r}^{1}\int_{c_r}^{c_{(2)}} \pi^W(c_i)\, f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} \\ & + \int_{0}^{c_r}\int_{0}^{c_{(2)}} \pi^L\left(1; c_{(1)}, c_{(2)}\right) f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} - m_{2r}(c_r) \end{aligned} \tag{A.20}$$

By the continuity of the profit function, it follows from (A.17) and (A.20) that for any $\epsilon > 0$, by choosing c_i sufficiently close to c_r with $c_i > c_r$, $|P(c_r, c_i) - U(c_i)| < \epsilon$. Hence, for such a c_i, we have $P(c_r, c_i) > U(c_i)$, which is impossible in the truth-telling equilibrium. Next, we consider the case in which $U(c_r) < U(c_i)$. In this case, if a firm of type c_r falsely reports its type as c_i with $c_i > c_r$, it earns a payoff of $P(c_i, c_r)$. In this case, this firm never wins a license and its expected payment is 0. Therefore, from (A.19) we know that $P(c_i, c_r) = U(c_i)$. Hence, $P(c_i, c_r) > U(c_r)$, which is not possible in a truth-telling equilibrium. Therefore, we conclude that $U(c_r) = U(c_i)$ when $c_i > c_r$.

Using (A.17) and (A.19), we find that

$$\begin{aligned} m_{2r}(c_r) = & \int_{c_r}^{1}\int_{0}^{c_r} \left[\pi^W\left(c_r; c_{(1)}\right) - \pi^L\left(1; c_{(1)}\right)\right] f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)} \\ & + \int_{c_r}^{1}\int_{c_r}^{c_{(2)}} \left[\pi^W(c_r) - \pi^0\right] f_{12}^{n-1}\left(c_{(1)},\ c_{(2)}\right) dc_{(1)}dc_{(2)}. \end{aligned}$$

Proof of Proposition 3.1

Suppose the firms other than firm i bid according to $b_k(\cdot;\xi)$ and suppose firm i with signal θ_i bids b_i. In an increasing symmetric equilibrium, $b_k(\theta;\xi)$ is increasing in θ, and hence, firm i wins a license if and only if

$$b_i > b_k\left(\theta_{(k)}^{n-1};\xi\right). \tag{B.1}$$

Notice that the inequality in (B.1) is equivalent to the following condition:

$$\theta_{(k)}^{n-1} < b_k^{-1}\left(b_i;\xi\right).$$

Hence, the payoff of firm i by bidding b_i, is given by

$$\begin{aligned}&\int_0^{b_k^{-1}(b_i;\xi)}\int_{\theta_{(k)}^{n-1}}^{\overline{c}}\int_{\theta_{(k-1)}^{n-1}}^{\overline{c}}\cdots\int_{\theta_{(2)}^{n-1}}^{\overline{c}}\left[\Pi\left(\theta_i;\theta_{(1)}^{n-1},\ldots,\theta_{(k-1)}^{n-1},\xi\right)-b_k\left(\theta_{(k)}^{n-1}\right)\right]\\&\times f_{1.k}^{n-1}\left(\theta_{(1)}^{n-1},\ldots,\theta_{(k)}^{n-1}\right)d\theta_{(1)}^{n-1}d\theta_{(2)}^{n-1}\ldots d\theta_{(k)}^{n-1}\\&+\int_{b_k^{-1}(b_i;\xi)}^{\overline{c}}\int_{\theta_{(k)}^{n-1}}^{\overline{c}}\int_{\theta_{(k-1)}^{n-1}}^{\overline{c}}\cdots\int_{\theta_{(2)}^{n-1}}^{\overline{c}}\Pi\left(0;\theta_{(1)}^{n-1},\ldots,\theta_{(k)}^{n-1},\xi\right)f_{1.k}^{n-1}\left(\theta_{(k)}^{n-1}\right)d\theta_{(1)}^{n-1}d\theta_{(2)}^{n-1}\ldots d\theta_{(k)}^{n-1}.\end{aligned}$$

where $f_{1.k}^{n-1}\left(\theta_{(1)}^{n-1},\ldots,\theta_{(k)}^{n-1}\right)$ is the joint density function of $\theta_{(1)}^{n-1},\ldots,\theta_{(k)}^{n-1}$. From the first-order condition, we obtain the following:

$$\begin{aligned}
&\int_{b_k^{-1}(b_i;\xi)}^{\bar{c}} \int_{\theta_{(k-1)}^{n-1}}^{\bar{c}} \cdots \int_{\theta_{(2)}^{n-1}}^{\bar{c}} \left[\begin{array}{c} \Pi\left(\theta_i;\theta_{(1)}^{n-1},\ldots,\theta_{(k-1)}^{n-1},\xi\right) - b_k\left(b_k^{-1}\left(b_i;\xi\right)\right) \\ -\Pi\left(0;\theta_{(1)}^{n-1},\ldots,b_k^{-1}\left(b_i;\xi\right)\right) \end{array} \right] \\
&\times f_{1.k}^{n-1}\left(\theta_{(1)}^{n-1},\ldots,\theta_{(k-1)}^{n-1},b_k^{-1}\left(b_i;\xi\right)\right) d\theta_{(1)}^{n-1} d\theta_{(2)}^{n-1}\ldots d\theta_{(k-1)}^{n-1} \\
&= E\left[\begin{array}{c} \Pi\left(\theta_i;\theta_{(1)}^{n-1},\ldots,\theta_{(k-1)}^{n-1},\xi\right) - b_k\left(b_k^{-1}\left(b_i;\xi\right)\right) - \\ \Pi\left(0;\theta_{(1)}^{n-1},\ldots,b_k^{-1}\left(b_i;\xi\right)\right) | \theta_{(k)}^{n-1} = b_k^{-1}\left(b_i;\xi\right) \end{array} \right] \\
&= 0. \qquad \text{(B.2)}
\end{aligned}$$

In a symmetric equilibrium, firm i has the same bidding strategy as its competitors, and hence,

$$b_i = b_k\left(\theta_i;\xi\right). \qquad \text{(B.3)}$$

Notice that the condition above is equivalent to the following condition:

$$b_k^{-1}\left(b_i;\xi\right) = \theta_i. \qquad \text{(B.4)}$$

Substituting (B.3) and (B.4) in (B.2), we obtain that

$$b_k\left(\theta_i;\xi\right) = E\left[\Pi\left(\theta_i;\theta_{(1)}^{n-1},\ldots,\theta_{(k-1)}^{n-1},\xi\right) - \Pi\left(0;\theta_{(1)}^{n-1},\ldots,\theta_{(k)}^{n-1},\xi\right) | \theta_{(k)}^{n-1} = \theta_i\right] = V_k\left(\theta_i,\xi\right). \qquad \text{(B.5)}$$

APPENDIX TO CHAPTER IV

Proof of Lemma 4.3

If firm i does not participate in the mechanism, then its payoff $\underline{\pi}_i^c$ depends on how the seller allocates the licenses in such an eventuality. Notice that, if firm i does not participate in the mechanism, then the seller can choose to allocate a license to each of its competitors, or a license to one of its competitors, or to not sell any license. Hence, firm i's payoff if it does not participate in the mechanism is either $\pi(0; s^2_{(1)}, s^2_{(2)})$, $\pi(0; s^2_{(1)}, 0)$, $\pi(0; s^2_{(2)}, 0)$, or $\pi(0; 0, 0)$. Next, notice that it follows from (57)–(59) that

$$\pi(0; s^2_{(1)}, s^2_{(2)}) = \min\left\{\pi(0; s^2_{(1)}, s^2_{(2)}), \pi(0; s^2_{(1)}, 0), \pi(0; s^2_{(2)}, 0), \pi(0; 0, 0)\right\}. \tag{C.1}$$

For example,

$$\begin{aligned} & \pi(0; s^2_{(1)}, s^2_{(2)}) - \pi(0; s^2_{(1)}, 0) \\ = & \int_0^{s^2_{(2)}} \pi_3\left(0; s^2_{(1)}, s\right) ds \\ \leq & \ 0 \text{ because } \pi_3(\cdot; \cdot, \cdot) < 0. \end{aligned}$$

Notice that (C.1) holds regardless of the seller's choice of the allocation rule a in the mechanism. It also follows from (63) that in the optimal mechanism, the payoff of every non-participating firm has to be minimized. Therefore, we obtain the result.

Relationship between the Partition $A(\hat{r}|Q)$ and the Partition $B(r_i)$

Given a profile of reports, let a *rank-set pair* be the pair whose first element is the rank of firm i's report in the profile of reports, and whose second element is the subset of $A(\hat{r}|Q)$

that contains the profile of reports. For example, if firm i's report is the second highest among the three reports (that is, if $r_i = r^3_{(2)}$), and if the profile of reports $\left(r^3_{(1)}, r^3_{(2)}, r^3_{(3)}\right) = \left(s^2_{(1)}, r_i, s^2_{(2)}\right) \in A3\left(\hat{r}|Q\right)$, then the corresponding rank-set pair is $(2, A3\left(\hat{r}|Q\right))$. Notice that, given a partition $A\left(\hat{r}|Q\right) = \{A1\left(\hat{r}|Q\right), \ldots, A6\left(\hat{r}|Q\right)\}$ and a profile of reports $\hat{r}$, each rank-set pair can be associated with a unique set $Bk\left(r_i\right)$; $k = 1, \ldots, 6$. However, each set $Bk\left(r_i\right)$; $k = 1, \ldots, 6$, can result from several rank-set pairs. For example, one can check that the set $B1\left(r_i\right)$ can be generated by three rank-set pairs $(1, A1\left(\hat{r}|Q\right))$, $(2, A6\left(\hat{r}|Q\right))$ and $(3, A6\left(\hat{r}|Q\right))$. In the following table, I associate each set $Bk\left(r_i\right)$; $k = 1, \ldots, 6$, with the rank-set pairs that generate $Bk\left(r_i\right)$.

Table 18: Derivation of the partition $B(r_i)$ from the partition A

Subsets	Corresponding Rank-Set Pairs
$B1\left(r_i\right)$	$(1, A1\left(\hat{r}\|Q\right))$ or $(2, A6\left(\hat{r}\|Q\right))$ or $(3, A6\left(\hat{r}\|Q\right))$
$B2\left(r_i\right)$	$(1, A2\left(\hat{r}\|Q\right))$ or $(2, A2\left(\hat{r}\|Q\right))$ or $(3, A1\left(\hat{r}\|Q\right))$
$B3\left(r_i\right)$	$(1, A3\left(\hat{r}\|Q\right))$ or $(2, A5\left(\hat{r}\|Q\right))$ or $(3, A4\left(\hat{r}\|Q\right))$
$B4\left(r_i\right)$	$(1, A4\left(\hat{r}\|Q\right))$ or $(2, A4\left(\hat{r}\|Q\right))$ or $(3, A5\left(\hat{r}\|Q\right))$
$B5\left(r_i\right)$	$(1, A5\left(\hat{r}\|Q\right))$ or $(2, A3\left(\hat{r}\|Q\right))$ or $(3, A3\left(\hat{r}\|Q\right))$
$B6\left(r_i\right)$	$(1, A6\left(\hat{r}\|Q\right))$ or $(2, A1\left(\hat{r}\|Q\right))$ or $(3, A2\left(\hat{r}\|Q\right))$

The probability of the realization of the set $Bk\left(r_i\right)$; $k = 1, 2, \ldots, 6$

Let the indicator function $I_{Bk}\left(r_i, \hat{s}_{-i}\right)$ take the value 1 if the profile of reports is an element of $Bk\left(r_i\right)$, and let $I_{Bk}\left(r_i, \hat{s}_{-i}\right)$ be 0 otherwise. Formally,

$$I_{Bk}\left(r_i, \hat{s}_{-i}\right) = \begin{cases} 1 & \text{if} \quad \hat{s}_{-i} \in Bk\left(r_i\right); \quad k = 1, 2, \ldots, 6, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, $\phi_{Bk}(r_i)$ is the probability that the ordered profile of signals of firm i's competitors belong to the set $Bk(r_i)$ when firm i reports its signal as r_i, the rivals report truthfully, and the seller commits to the allocation rule Q. The function $\phi_{Bk}(r_i)$ is related to the indicator function $I_{Bk}(r_i, \hat{s}_{-i})$ as follows:

$$\begin{aligned}\phi_{Bk}(r_i) &\equiv P\{\hat{s}_{-i} \in Bk(r_i)\}; \quad k = 1, \ldots, 6 \\ &= \int_{\hat{s}_{-i}} I_{Bk}(r_i, \hat{s}_{-i}) f_{12}^2(\cdot)\, d\hat{s}_{-i}.\end{aligned}$$

Proof of Proposition 4.5

I denote the payoff of firm i in the truth-telling equilibrium of the direct mechanism as follows:

$$\Psi_{iQ}(s_i) = V_{iQ}(s_i, s_i).$$

First, I prove the necessity part. Note that incentive compatibility implies that

$$\Psi_{iQ}(s_i) \geq V_{iQ}(r_i, s_i) \text{ for all } r_i,\ s_i \in [0,1]. \tag{C.2}$$

Moreover, I can re-write $V_{iQ}(r_i, s_i)$ as follows:

$$\begin{aligned}V_{iQ}(r_i, s_i) &= \Psi_{iQ}(r_i) + \sum_{k=1}^{2} \Phi_k(r_i)\left[\Pi(r_i, s_i|k) - \Pi(r_i, r_i|k)\right] &\text{(C.3)}\\ &= \Psi_{iQ}(r_i) + \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i)\, \Pi_2(r_i, s|k)\, ds. &\text{(C.4)}\end{aligned}$$

Therefore, from (C.2) and (C.4), I find that incentive compatibility implies the following condition:

$$\Psi_{iQ}(s_i) - \Psi_{iQ}(r_i) \geq \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i)\, \Pi_2(r_i, s|k)\, ds, \tag{C.5}$$

and, by interchanging the variables, I find that,

$$\Psi_{iQ}(r_i) - \Psi_{iQ}(s_i) \geq \int_{s_i}^{r_i} \sum_{k=1}^{2} \Phi_k(s_i)\,\Pi_2(s_i, s|k)\,ds. \tag{C.6}$$

Combining (C.5) and (C.6), I obtain the following inequality:

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(s_i)\,\Pi_2(s_i, s|k)\,ds \geq \Psi_{iQ}(s_i) - \Psi_{iQ}(r_i) \geq \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i)\,\Pi_2(r_i, s|k)\,ds. \tag{C.7}$$

Notice that, the above inequality implies (78). Next, I divide all the terms in (C.7) and let $r_i \to s_i$ to obtain the result that

$$\Psi'_{iQ}(s_i) = \sum_{k=1}^{2} \Phi_k(s_i)\,\Pi_2(s_i, s_i|k) \tag{C.8}$$

and hence,

$$\Psi_{iQ}(s_i) = \Psi_{iQ}(0) + \int_{0}^{s_i} \sum_{k=1}^{2} \Phi_k(s)\,\Pi_2(s, s|k)\,ds, \tag{C.9}$$

which is (77).

Next, I prove the sufficiency part. Suppose, (77) and (78) are satisfied, but the mechanism is not incentive compatible. Then, there exists s_i and r_i such that

$$V_{iQ}(r_i, s_i) > \Psi_{iQ}(s_i), \tag{C.10}$$

and substituting (C.4) in (C.10), I obtain that,

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i)\,\Pi_2(r_i, s|k)\,ds > \Psi_{iQ}(s_i) - \Psi_{iQ}(r_i). \tag{C.11}$$

Furthermore, using (77) in (C.11), I obtain

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i)\,\Pi_2(r_i, s|k)\,ds > \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(s_i)\,\Pi_2(s_i, s|k)\,ds. \tag{C.12}$$

Notice that (C.12) contradicts (78).

Proof of Corollary 4.6

Pick any two values of s_i, say s_i' and s_i''. Without loss of generality, let $s_i' < s_i''$. Using (78), I obtain that

$$\Phi_1(s_i')\,\Pi_2(s_i', s_i'|1) + \Phi_2(s_i')\,\Pi_2(s_i', s_i'|2) \le \Phi_1(s_i'')\,\Pi_2(s_i'', s_i'|1) + \Phi_2(s_i'')\,\Pi_2(s_i'', s_i'|2)\,. \quad \text{(C.13)}$$

Further, because the payoffs are convex in the signal s, therefore,

$$\Pi_2(s_i'', s_i'|1) \le \Pi_2(s_i'', s_i''|1) \quad \text{(C.14)}$$

and

$$\Pi_2(s_i'', s_i'|2) \le \Pi_2(s_i'', s_i''|2)\,. \quad \text{(C.15)}$$

Combining (C.13), (C.14) and (C.15), I obtain the result.

Proof of the claim that the allocation in which the firm with the second highest report purchases both the licenses is not incentive compatible

Notice that for the allocation in which the firm with the second highest report purchases both the licenses,

$$\begin{aligned}&\Phi_1(r_i)\,\Pi_2(r_i, s_i|1) + \Phi_2(r_i)\,\Pi_2(r_i, s_i|2)\\&= \pi_1(s_i; 0, 0)\int_{r_i}^{1}\int_{0}^{r_i} f_{12}^2(\cdot)\, ds_{(2)}^2 ds_{(1)}^2.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\frac{\partial}{\partial r_i}\left\{\Phi_1(r_i)\,\Pi_2(r_i,s_i|1)+\Phi_2(r_i)\,\Pi_2(r_i,s_i|2)\right\}\\
&=\pi_1(s_i;0,0)\\
&\times\left[-\int_0^{r_i} f_{12}^2\left(r_i,s_{(2)}^2\right)ds_{(2)}^2+\int_{r_i}^1 f_{12}^2\left(s_{(1)}^2,r_i\right)ds_{(1)}^2\right] \qquad \text{(C.16)}
\end{aligned}
$$

Notice that the expression in (C.16) is negative for $r_i = 1$. Moreover, this expression is also continuous in r_i. Hence, the expression in (C.16) is negative in the neighborhood of 1, and violates the incentive compatibility condition (78).

Definitions of $\alpha_k(s_i)$ and $\beta_k(r_i, s_i)$

The functions $\alpha_k(s_i)$; $k = 1, \ldots, 6$, are defined as follows:

$$\alpha_1(r_i) = \int_{\hat{s}_{-i}\,\in\,B1(r_i)} \pi\left(0; s_{(1)}^2, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_2(r_i) = \int_{\hat{s}_{-i}\,\in\,B2(r_i)} \pi\left(0; s_{(2)}^2, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_3(r_i) = \int_{\hat{s}_{-i}\,\in\,B3(r_i)} \pi\left(0; s_{(1)}^2, s_{(2)}^2\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_4(r_i) = \int_{\hat{s}_{-i}\,\in\,B4(r_i)} \pi\left(s_i; s_{(1)}^2, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_5(r_i) = \int_{\hat{s}_{-i}\,\in\,B5(r_i)} \pi\left(s_i; s_{(2)}^2, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_6(r_i) = \int_{\hat{s}_{-i}\,\in\,B6(r_i)} \pi\left(s_i; 0, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i}.$$

Analogously, the functions $\beta_k(r_i, s_i)$; $k = 1, \ldots, 6$, are defined as follows:

$$\beta_1(r_i, s_i) = \beta_2(r_i, s_i) = \beta_3(r_i, s_i) = 0,$$

$$\beta_4(r_i, s_i) = \int_{\hat{s}_{-i} \in B4(r_i)} \pi_1\left(s_i; s^2_{(1)}, 0\right) f^2_{12}(\cdot)\, d\hat{s}_{-i}\ ,$$

$$\beta_5(r_i, s_i) = \int_{\hat{s}_{-i} \in B5(r_i)} \pi_1\left(s_i; s^2_{(2)}, 0\right) f^2_{12}(\cdot)\, d\hat{s}_{-i},$$

$$\beta_6(r_i, s_i) = \int_{\hat{s}_{-i} \in B6(r_i)} \pi_1(s_i; 0, 0)\, f^2_{12}(\cdot)\, d\hat{s}_{-i}.$$

Proof of Proposition 4.7

From (77), it follows that:

$$V_{iQ}(s_i, s_i) = V_{iQ}(0,0) + \int_0^{s_i} \sum_{k=1}^{6} \beta_k(s, s)\, ds. \tag{C.17}$$

Combining (84) and (C.17), I obtain:

$$m_i(s_i) = \sum_{k=1}^{6} \left[\alpha_k(s_i) - \int_0^{s_i} \beta_k(s, s)\, ds\right] - V_{iQ}(0,0)\,.$$

Proof of Proposition 4.8

Substituting (85) in (76), I obtain the following relation:

$$\begin{aligned} R_Q &= 3\int_0^1 \sum_{k=1}^{6} \left[\alpha_k(s_i) - \int_0^{s_i} \beta_k(s, s)\, ds\right] g(s_i)\, ds_i - 3V_Q(0,0) \\ &= 3\sum_{k=1}^{6} \left[\int_0^1 \alpha_k(s_i)\, g(s_i)\, ds_i - \int_0^1 \int_0^{s_i} \beta_k(s, s)\, g(s_i)\, ds ds_i\right] - 3V_Q(0,0)\,. \end{aligned} \tag{C.18}$$

Furthermore, I integrate by parts the second expression in the right hand side of (C.18), and obtain the following expression for the seller's revenue:

$$R_Q = 3\sum_{k=1}^{6}\int_0^1 \gamma_k(s_i)\, g(s_i)\, ds_i \;\; - 3V_Q(0,0) \tag{C.19}$$

$$\text{where } \gamma_k(s) \equiv \alpha_k(s) - \beta_k(s,s)\frac{1-G(s)}{g(s)}; \quad k = 1,2,\ldots,6.$$

I now simplify the expression in (C.19). For the sake of exposition, I only present the simplification for the case in which the seller commits to sell both the licenses to the firm with the highest signal for all possible profile of reports; one can use the same technique for any arbitrary allocation rule. Under such a commitment,

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \in A6\left(\hat{r}|Q\right) \text{ for all possible values of } \left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right).$$

Pick any arbitrary $s_i \in [0,1]$. Notice that if firm i with signal s_i has the highest signal among the three firms, then the set $B6(s_i)$ is realized. Moreover, if $s_i = s^3_{(2)}$ or $s^3_{(3)}$, then the set $B1(s_i)$ is realized. From (C.19), it follows that

$$R_Q = 3\sum_{k=1}^{6}\int_0^1 \gamma_k(s_i)\, g(s_i)\, ds_i \;\; - 3V_Q(0,0). \tag{C.20}$$

Therefore, I first evaluate the expression

$$\sum_{k=1}^{6} \gamma_k(s_i) = \gamma_6(s_i) + \gamma_1(s_i) + \gamma_1(s_i). \tag{C.21}$$

By expanding the expression in the right hand side of (C.20), I obtain the following:

$$\sum_{k=1}^{6} \gamma_k (s_i) = \int_0^{s_i} \int_0^{s^2_{(1)}} \left\{ \pi (s_i; 0.0) - \frac{1 - G(s_i)}{g(s_i)} \frac{\partial \pi (s_i; 0.0)}{\partial s_i} \right\} f^2_{12} (\cdot)\, ds^2_{(2)} ds^2_{(1)} + \int_{s_i}^{1} \int_0^{s_i} \pi \left(0; s^2_{(1)}, 0\right) f^2_{12} (\cdot)\, ds^2_{(2)} ds^2_{(1)} + \int_{s_i}^{1} \int_{s_i}^{s^2_{(1)}} \pi \left(0; s^2_{(1)}, 0\right) f^2_{12} (\cdot)\, ds^2_{(2)} ds^2_{(1)}. \tag{C.22}$$

Notice that, in the first term on the right hand side of (C.22), $s_i = s^3_{(1)}$, and hence,

$$s^2_{(1)} = s^3_{(2)} \text{ and } s^2_{(2)} = s^3_{(3)}.$$

Similarly, in the second term $s_i = s^3_{(2)}$, and in the third term, $s_i = s^3_{(3)}$. Also, the following relation holds:

$$3g(s_i) f^2_{12} \left(s_{(1)}, s_{(2)}\right) = f^3_{123} \left(s_i, s_{(1)}, s_{(2)}\right).$$

Therefore, I can re-write the first term (on the right hand side) of (C.22) as follows:

$$\begin{aligned} & 3 \int_0^1 \gamma_6 (s_i)\, g(s_i)\, ds_i \\ &= 3 \int_0^1 \int_0^{s_i} \int_0^{s^2_{(1)}} \left\{ \pi (s_i; 0.0) - \frac{1 - G(s_i)}{g(s_i)} \frac{\partial \pi (s_i; 0.0)}{\partial s_i} \right\} g(s_i) f^2_{12} (\cdot)\, ds^2_{(2)} ds^2_{(1)} ds_i \\ &= 3 \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \left\{ \pi \left(s^3_{(1)}; 0.0\right) - \frac{1 - G\left(s^3_{(1)}\right)}{g\left(s^3_{(1)}\right)} \frac{\partial \pi \left(s^3_{(1)}; 0.0\right)}{\partial s^3_{(1)}} \right\} g\left(s^3_{(1)}\right) f^2_{12} (\cdot)\, ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} \\ &= \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \left\{ \pi \left(s^3_{(1)}; 0.0\right) - \frac{1 - G\left(s^3_{(1)}\right)}{g\left(s^3_{(1)}\right)} \frac{\partial \pi \left(s^3_{(1)}; 0.0\right)}{\partial s^3_{(1)}} \right\} f^3_{123} (\cdot)\, ds^3_{(3)} ds^3_{(2)} ds^3_{(1)}. \end{aligned}$$

Similarly, I can expand the other terms to obtain the seller's revenue as follows:

$$
\begin{aligned}
R_Q &= \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \left\{ \pi\left(s^3_{(1)};0,0\right) + 2\pi\left(0;s^3_{(1)},0\right) - \frac{1-G\left(s^3_{(1)}\right)}{g\left(s^3_{(1)}\right)} \frac{\partial \pi\left(s^3_{(1)};0.0\right)}{\partial s^3_{(1)}} \right\} \\
&\quad \times f^3_{123}\left(\cdot\right) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} - 3V_Q\left(0,0\right) \\
&= \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}\left(\cdot\right) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} - 3V_Q\left(0,0\right).
\end{aligned}
$$

Analogously, the revenue of the seller under any arbitrary allocation rule can be determined.

Proof of Proposition 4.9

It follows from inspection of (87) that in the optimal mechanism, the payoff of a firm with signal 0, given by $V_Q\left(0,0\right)$, has to be minimized, subject to

$$V_Q\left(0,0\right) \geq \underline{\pi}.$$

Hence, in the optimal mechanism, I must have

$$V_Q\left(0,0\right) = \underline{\pi}$$

where $\underline{\pi}$ is defined in (88). Moreover, by construction, the maximum value of

$$\int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}\left(\cdot\right) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} \tag{C.23}$$

is given by

$$\int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda^*\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}\left(\cdot\right) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)}. \tag{C.24}$$

Notice that maximizing the expression in (C.23) and minimizing $V_Q\left(0,0\right)$ are independent of each other. Hence, I have the result.

BIBLIOGRAPHY

Alsemgeest, P., C. Noussair, and M. Olson (1998). Experimental comparisons of auctions under single and multi-unit demand. Economic Inquiry 36 (1), 87—97.

Armstrong, M. (2000). Optimal multi-object auctions. Review of Economic Studies 67 (3), 455—481.

Avery, C. and T. Hendershott (2000). Bundling and optimal auctions of multiple products. Review of Economic Studies 67 (3), 483—497.

Bagchi, A. (2005a). When should an inventor sell licenses exclusively? The determinants of the optimal number of licensees. Working Paper, Vanderbilt University.

Bagchi, A. (2005b, June). How to commercialize technology using auctions. Vanderbilt University.

Brocas, I. Designing auctions in R and D: Optimal licensing of an innovation. 2005.

Bulow, J. and P. Klemperer (1996). Auctions versus negotiations. American Economic Review 86 (1), 180—194.

Bulow, J. and J. Roberts (1989). The simple economics of optimal auctions. Journal of Political Economy 97 (5), 1060—1090.

Dana, J. D. and K. E. Spier (1994). Designing a private industry: Government auctions with endogenous market structure. Journal of Public Economics 53 (1), 127—147.

Das Varma, G. (2002). Standard auctions with identity-dependent externalities. RAND Journal of Economics 33 (4), 689—708.

DasVarma, G. (2003). Bidding for a process innovation under alternate modes of competition. International Journal of Industrial Organization 21 (1), 15—37.

David, H. (1969). Order Statistics. New York: Wiley.

DeGraba, P. and A. Postlewaite (1992). Exclusivity clauses and best price policies in input markets. Journal of Economics and Management Strategy 1, 423—454.

Figueroa, N. and V. Skreta. Optimal multi-unit auctions: Who gets what matters. 2005.

Gale, I. (1994). Competition for scarce inputs: The case for airport takeoff and landing slots. Economic Review Federal Reserve Bank of Cleveland 30 (2), 18—25.

Goeree, J. K. (2003). Bidding for the future: Signalling in auctions with an aftermarket. Journal of Economic Theory 108, 345—364.

Goeree, J. K., T. Offerman, and R. Sloof (2004). Demand reduction and preemptive bidding in multiunit license auctions. Working Paper, California Institute of Technology.

Haile, P. A. (2000). Partial pooling at the reserve price in auctions with resale opportunities. Games and Economic Behavior 33 (2), 231—248.

Haile, P. A. (2003). Auctions with private uncertainty and resale opportunities. Journal of Economic Theory 108 (1), 72—110.

Harstad, R. M. (2000). Dominant strategy adoption and bidders' experience with pricing rules. Experimental Economics 3 (3), 261—280.

Hoppe, H., P. Jehiel, and B. Moldovanu (2004). License auctions and market structure. forthcoming in Journal of Economics and Management Strategy.

Hoppe, H., P. Jehiel, and B. Moldovanu (forthcoming). License auctions and market structure. Journal of Economics and Managment Strategy.

Jehiel, P. and B. Moldovanu (2000). Auctions with downstream interaction among buyers. RAND Journal of Economics 31 (4), 768—791.

Jehiel, P. and B. Moldovanu (2001). Efficient design with interdependent valuations. Econometrica 69 (5), 1237—1259.

Jehiel, P. and B. Moldovanu (2004). The design of an efficient private industry. Journal of the European Economic Association 2 (2), 516—525.

Jehiel, P., B. Moldovanu, and E. Stacchetti (1996). How (not) to sell nuclear weapons. American Economic Review 86 (4), 814—829.

Jehiel, P., B. Moldovanu, and E. Stacchetti (1999). Multidimensional mechanism design for auctions with externalities. Journal of Economic Theory 85 (2), 258—293.

Kagel, J. H., R. M. Harstad, and D. Levin (1987). Information impact and allocation rules in auctions with affiliated private values: A laboratory study. Econometrica 55 (6), 1275—1304.

Kagel, J. H. and D. Levin (1993). Independent private value auctions: Bidder behavior in first, second, and third price auctions with varying numbers of bidders. Economic Journal 103 (419), 868—879.

Kagel, J. H. and D. Levin (2001). Behavior in multi-unit demand auctions: Experiments with uniform price and dynamic vickrey auctions. Econometrica 69 (2), 413—454.

Kagel, J. H. and D. Levin (2004). Multi-unit demand auctions with synergies: Behavior in sealed bid vs ascending bid uniform price auctions. Games and Economic Behavior forthcoming.

Kamien, M. (1992). Patent licensing. In R. J. Aumann and S. Hart (Eds.), Handbook of Game Theory, Chapter 11, pp. 332—354. Amsterdam: Elsevier.

Kamien, M. I., S. S. Oren, and Y. Tauman (1992). Optimal licensing of cost-reducing innovation. Journal of Mathematical Economics 21 (5), 483—508.

Katz, M. L. and C. Shapiro (1986). How to license intangible property. Quarterly Journal of Economics 101 (3), 567—589.

Katzman, B. and M. Rhodes-Kropf (2002). The consequences of information revealed in auctions. Working Paper, Columbia University.

Kirchkamp, O. and B. Moldovanu (2004). An experimental analysis of auctions with interdependent valuations. Games and Economic Behavior 48 (1), 54—85.

Kontzer, T. (2004, December 13). Mystery company buys ecommerce patents. Information Week.

Krishna, V. (2002). Auction Theory. San Diego: Academic Press.

Levin, D. and J. L. Smith (1996). Optimal reservation prices in auctions. Economic Journal 106 (438), 1271—1283.

List, J. A. and D. Lucking Reiley (2000). Demand reduction in a multi-unit auction: Evidence from a sportscard field experiment. American Economic Review 90 (4), 961— 972.

Mares, V. and M. Shor (2004). Industry concentration in common value auctions. Working Paper, Washington University in St. Louis.

McCabe, K. A. and V. L. Smith (2000). A comparison of naive and sophisticated subject behavior with game theoretic predictions. Proceedings of the National Academy of Sciences 97 (7), 3777—3781.

Milgrom, P. R. and R. J. Weber (1982). A theory of auctions and competitive bidding. Econometrica 50 (5), 1089—1122.

Moldovanu, B. and A. Sela (2003). Patent licensing to bertrand competitors. International Journal of Industrial Organization 21 (1), 1—13.

Mullin, C. H. and D. H. Reiley (2006). Recombinant estimation for normal form games, with applications to auctions and bargaining. Games and Economic Behavior 54 (1), 159—182.

Myerson, R. (1981). Optimal auction design. Mathematics of Operations Research 6 (1), 58—73.

Palfrey, T. R. (1983). Bundling decisions by a multiproduct monopolist with incomplete information. Econometrica 51 (2), 463—484.

Regaldo, A. (2004, June 18). Nanotechnology patents surge as companies vie to stake claim. The Wall Street Journal.

Reiss, R. D. (1989). Approximate Distributions of Order Statistics. New York: SpringerVerlag.

Riley, J. G. and W. F. Samuelson (1981). Optimal auctions. American Economic Review 71 (3), 381—392.

Schmitz, P. W. (2002). On monopolistic licensing strategies under asymmetric information. Journal of Economic Theory 106 (1), 177—189.

Wang, R. (1993). Auctions versus posted-price selling. American Economic Review 83 (4), 838—851.

Weber, R. J. (1983). Multiple-object auctions. In Auctions, Bidding and Contracting: Uses and Theory. New York University Press.