ON THE AXISYMMETRIC SURFACE DIFFUSION FLOW

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To to my biggest supporters,

my beloved wife Kelsey,

and my parents,

Scott and Carol

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CHAPTER 1

INTRODUCTION

The central focus of this dissertation is the development of an analytic setting and rigorous results for the axisymmetric surface diffusion flow (ASD) with periodic boundary conditions. In particular, we establish well-posedness of ASD and we investigate geometric properties of solutions, including characterizing equilibria and investigating their stability, instability and bifurcation behavior. We begin with a motivation and derivation of the general surface diffusion flow, of which ASD is a special case, and we introduce the main features of the manuscript.

The mathematical equations modeling surface diffusion go back to a paper by Mullins [59] from the 1950s, who was in turned motivated by earlier work of Herring [44]. Both of these authors investigate phenomena witnessed in sintering processes, a method by which objects are created by heating powdered material to a high temperature, while remaining below the boiling point of the particular substance. When the applied temperature reaches a critical point, the atoms on the surfaces of individual particles will diffuse across to other particles, fusing the powder together into one solid object. In response to gradients of the chemical potential along the surface of this newly formed object, the surface atoms may undergo diffusive mass transport *on* the surface of the object, attempting to reduce the surface free energy. Given the right conditions – temperature, pressure, grain size, sample size, etc. – the mass flux due to this chemical potential will dominate the dynamics on the surface, and it is the resulting morphological evolution of the surface which the surface diffusion flow aims to model.[1] We also note that the surface diffusion flow has been used to model the motion of surfaces in other physical processes (e.g. growth of crystals and nano-structures), though the dynamics of the model are typically restricted to *small-scale* problems, where the chemical potential is the dominant force governing the dynamics of the surface mass flux and other forces, such as surface tension, are negligible.

Research into surface diffusion phenomena continues from both physical and mathematical perspectives with many contributions from a wide range of researchers. The models were studied

[1] The description provided for sintering phenomena is meant only for intuitive instruction. The study of powder metallurgy is an active and complex science of which the author claims only a passing understanding.

from a more general viewpoint, both mathematically and physically, by Davì and Gurtin [23] and Cahn and Taylor [16]. In fact, the article [16] contains the formulation of the model which we present in the following section, which is set in a more general framework than the original model developed by Mullins. Moreover, surface diffusion was shown, by Cahn, Elliot, and Novick–Cohen [15], to have a certain connection to solutions of the Cahn-Hilliard equation with concentration–dependent mobility. Meanwhile, various formulations of the model in two–dimensions, where the model tracks the evolution of curves via the aptly named *curve–diffusion flow*, have been studied by several authors, e.g. [13, 31, 33, 41, 62, 71], and have produced a wide range of analytic and numerical results. Most notably, the literature on the curve–diffusion flow contains a variety of analytic well-posedness and regularity results in addition to abundant numerical and analytic results regarding the general dynamic picture of the model, e.g. development of curvature singularities, self-intersection of initially embedded curves, loss of convexity of curves, etc. In contrast, the first results regarding well-posedness of the general surface diffusion flow (in arbitrary space dimensions, in fact) were given by Escher, Mayer and Simonett [36] and the first adequate framework for using numerical techniques to track solutions of the three–dimensional surface diffusion flow, without assuming additional symmetries of the problem, was developed by Mayer [56, 57] in the early 2000s. Meanwhile, analytic results regarding the general dynamic picture of the surface diffusion flow in three (or higher) dimensions are still lacking from the literature.

The techniques and results of this dissertation may be a first step in the process of filling that gap. This paper provides a rigorous analytic setting for studying general dynamic properties of ASD. In particular, we develop a theory with which we can establish and take full advantage of *maximal regularity* for the problem. Most notably, with maximal regularity we gain access to the implicit function theorem, a very powerful tool in dynamical systems theory. The work contained herein, which we will summarize in more detail below, demonstrates only a small portion of the analytic results that one can hope to prove with help from maximal regularity and the implicit function theorem, among other tools from dynamical systems theory and nonlinear functional analysis. With these results, the door has been opened for a wide range of analytic results to follow.

1.1 The Surface Diffusion Flow

From a mathematical perspective, the governing equation for motion via surface diffusion can be expressed for hypersurfaces in arbitrary space dimensions. In particular, let $\Gamma \subset \mathbb{R}^n$ be a closed, compact, immersed, oriented Riemmanian manifold with codimension 1. Then we denote by $\mathcal{H} = \mathcal{H}(\Gamma)$ the (normalized) mean curvature on Γ, which is simply the sum of the principle curvatures on the hypersurface, and Δ_Γ denotes the Laplace–Beltrami operator, or *surface Laplacian*, on Γ. These quantities are defined in terms of the Riemannian metric and the second fundamental form of Γ, c.f. [28], however, for the purpose of this dissertation, we will be interested in these quantities only for axisymmetric surfaces embedded in $\mathbb{R}^3$, for which we will express them explicitly in Section 1.2 below. The motion of the surface Γ by surface diffusion is then governed by the equation

$$V = \Delta_\Gamma \mathcal{H},$$

where V denotes the velocity of Γ in the normal direction to the surface. Notice that the motion of the surface is determined by the geometric properties of the surface itself, hence the surface diffusion flow is an example of a geometric evolution law. Moreover, we note that this equation leads to a nonlinear, fourth–order equation of parabolic type for which many of the techniques traditionally applied to second–order geometric evolution laws fail. Most notably, there does not appear to be a suitable maximum principle for the surface diffusion flow, an invaluable tool in the analysis of many second–order equations.

A solution to the surface diffusion problem on the interval $J \subset \mathbb{R}_+$, with $0 \in J$, is a family $\{\Gamma(t) : t \in J\}$ of closed, compact immersed hypersurfaces in $\mathbb{R}^n$ which satisfy the equation

$$\begin{cases} V(\Gamma(t)) = \Delta_{\Gamma(t)} \mathcal{H}(\Gamma(t)), \quad t \in \dot{J} := J \setminus \{0\}, \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{1.1}$$

for a given initial hypersurface Γ_0. It can be shown that solutions to (1.1) are volume–preserving, in the sense that the signed volume of the region Ω *enclosed* by the surface Γ is preserved along solutions. Additionally, (1.1) is surface–area–reducing, a feature that the surface diffusion flow shares with the mean curvature flow (a second–order, nonlinear geometric evolution law) among other well–known geometric evolution laws. In fact, many of the interesting questions that remain unanswered for the surface diffusion flow can be motivated by analogous questions which have been answered for the mean curvature flow and/or the *volume–constrained* mean curvature flow, a volume–preserving version of the mean curvature flow, c.f. [29, 45]. For instance, the existence of solutions to the mean curvature flow which develop finite–time pinch–off has been established analytically by Angenent in [7], see also [10]. Analogously, researchers on the surface diffusion flow have seen numerical evidence to justify the fact that some solutions will pinch–off in finite time, c.f. [18, 20, 14, 57]. However, no analytic methods have yet been developed to confirm these numerical results.

More broadly, as mentioned above, a general picture of the dynamic nature of solutions to the surface diffusion problem is still an open question. Analytic well–posedness was established by Escher, Mayer and Simonett in [36], where it was also shown that the $(n-1)$–dimensional spheres are asymptotically stable equilibria. Meanwhile, Mayer and Simonett [58] demonstrate the existence of initially embedded hypersurfaces which are driven to self–intersection under the surface diffusion flow. However, beyond these initial results, the literature lacks general analytic results regarding the behavior of solutions, e.g. occurrence of singularities and conditions under which solutions breakdown in finite time. An important feature of the surface diffusion flow that Escher, Mayer and Simonett exploit in order to obtain well-posedness results is the fact that the equation has a quasilinear structure and the linear part of the equation exhibits *maximal regularity* properties on appropriately chosen function spaces. These features will also play an important role in our analysis of ASD, for which we establish a robust theory for local and global well-posedness.

1.2 Axisymmetric Surface Diffusion (ASD)

For the remainder of the paper, we will focus our attention on a special case of the surface diffusion flow. Namely, we consider the case of $\Gamma \subset \mathbb{R}^3$ an embedded surface which is symmetric about an axis of rotation (which we take to be the x–axis, without loss of generality) and satisfies prescribed periodic boundary conditions on some interval L of periodicity (we take $L = [-\pi, \pi]$ and enforce 2π periodicity, without significant loss of generality). In particular, the axisymmetric surface Γ is characterized by the parametrization

$$\Gamma = \Big\{(x, r(x)\cos(\theta), r(x)\sin(\theta)) : \; x \in \mathbb{R}, \; \theta \in [-\pi, \pi]\Big\},$$

where the function $r : \mathbb{R} \to (0, \infty)$ is the *profile function* for the surface Γ. Conversely, a profile function $r : \mathbb{R} \to (0, \infty)$ generates an axisymmetric surface $\Gamma = \Gamma(r)$ via the parametrization given above. We will also enforce periodicity on the profile functions *up to the regularity of* r, which will be made more precise in Section 2.1

Utilizing the explicit parametrization for axisymmetric surfaces, we can recast the surface diffusion problem as an evolution equation for the profile functions $r = r(t)$. In particular, one can see that the surface $\Gamma(r)$ inherits the Riemannian metric

$$g = (1 + r_x^2)\, dx \wedge dx + r^2\, d\theta \wedge d\theta,$$

from the embedding $\Gamma \hookrightarrow \mathbb{R}^3$, with respect to the surface coordinates (x, θ); where the subscript $f_{x_i} := \partial_{x_i} f$ indicates the derivative of $f = f(x_1, \ldots, x_m)$ with respect to the variable x_i. It follows that the (normalized) mean curvature of the surface is $\mathcal{H}(r) = \kappa_1 + \kappa_2$, where

$$\kappa_1 = \frac{1}{r\sqrt{1 + r_x^2}} \quad \text{and} \quad \kappa_2 = \frac{-r_{xx}}{(1 + r_x^2)^{3/2}}$$

are the *azimuthal* and *axial* principle curvatures, respectively, on $\Gamma(r)$. Meanwhile, the Laplace–Beltrami operator on Γ and the normal velocity of $\Gamma = \Gamma(t)$ are

$$\begin{aligned} \Delta_{\Gamma(r)} &= \frac{1}{r\sqrt{1 + r_x^2}} \left(\partial_x \left[\frac{r}{\sqrt{1 + r_x^2}} \partial_x \right] + \partial_\theta \left[\frac{\sqrt{1 + r_x^2}}{r} \partial_\theta \right] \right), \\ V(t) &= \frac{r_t}{\sqrt{1 + r_x^2}}. \end{aligned}$$

Finally, plugging these terms into the equation (1.1) and simplifying, we arrive at the expression

$$\begin{cases} r_t = \dfrac{1}{r}\,\partial_x \left[\dfrac{r}{\sqrt{1+r_x^2}}\,\partial_x\left(\dfrac{1}{r\sqrt{1+r_x^2}} - \dfrac{r_{xx}}{(1+r_x^2)^{\frac{3}{2}}}\right)\right], & t>0,\ x\in\mathbb{R}, \\ r(t,x+2\pi) = r(t,x), & t\geq 0,\ x\in\mathbb{R}, \\ r(0,x) = r_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.2}$$

for the periodic axisymmetric surface diffusion problem.

The first investigations of evolution of an axisymmetric surface via surface diffusion can be traced back to the work of Nichols and Mullins [60, 61] in 1965, where one can already see some of the benefits of this special case of the surface diffusion problem. In particular, Mullins and Nichols are able to take advantage of the symmetry of the problem in order to develop an adequate scheme for numerical techniques. Recall that for the full surface diffusion problem this was not achieved until the work of Mayer [57] in 2001. Moreover, Mullins and Nichols are already predicting the finite–time pinch–off of tube like surfaces via surface diffusion flow, a feature similar to the mean curvature flow and a natural phenomenon to study in exactly this axisymmetric setting. Following this seminal work by Nichols and Mullins, there is a plethora of publications investigating ASD. Many researchers continued to study pinch–off behavior using numerical methods, c.f. [14, 18, 19, 20, 24, 52, 53], developing schemes for the continuation of solutions after the change of topology that occurs at the moment of pinch–off. Meanwhile, there has also been a lot of focus on the numerical investigation of the stability/instability and bifurcation behavior of cylinders, which are natural equilibria of ASD, under perturbations of various types, c.f. [14, 18, 20]. Though we again note that these investigations still lack a rigorous analytic development. The paper [14] by Bernoff, Bertozzi, and Witelski has served as a motivation for the breadth of results that have been shown via numerical techniques for ASD which we can hope to establish in a rigorous analytic framework.

In Chapter 3, we prove existence of solutions to (1.2), which, to the best of the author's knowledge, is the first analytic well–posedness result in the literature for the axisymmetric surface diffusion flow with periodic boundary conditions. In particular, we establish existence and uniqueness of maximal solutions which are analytic in time and space for positive time, with a prescribed singularity at time $t = 0$, for initial conditions which are $(2+\alpha)$–little–Hölder continuous in space.

Additionally, we establish conditions for global existence (solutions existing for all time) and regularity of the semiflow induced by (1.2). With these well–posedness results established, all of which depend heavily upon the theory developed in Chapter 2 and the well–posedness results for quasilinear equations with maximal regularity proved by Clément and Simonett [17], we then go on to investigate more general dynamic properties of solutions. First, we characterize the equilibria of ASD using a result of Delaunay [26] and Kenmotsu [47], which describes all of the constant mean curvature surfaces in the axisymmetric setting. We prove that all cylinders with radius $r_\star > 1$ are asymptotically, exponentially stable under a large class of nonlinear perturbations, which maintain the same axis of symmetry and satisfy the prescribed periodic boundary conditions. To prove this stability result, we linearize the equation for ASD and see that the spectrum of the linearized operator is contained in the *left half* of the complex plane, although the spectrum will always contain zero as an eigenvalue. Hence, we reduce the equation, by essentially eliminating non-volume-preserving perturbations, in order to eliminate the zero eigenvalue. Then we use results regarding maximal regularity on exponentially weighted function spaces to generate the desired exponential stability for the reduced equation, which is then transferred back to the (full) ASD problem via a *lifting* operator.

We also go on in Chapter 3 to establish results regarding the instability of cylinders with radius $0 < r_\star < 1$ and the existence of branches of bifurcating equilibria which intersect the family of cylinders at radii $r_\star = 1/\ell$, for every $\ell \in \mathbb{N} := \{1, 2, \ldots\}$. The instability result makes use of a contradiction technique reminiscent of results from the theory of ordinary differential equations, c.f. Prüss and Wilke [68]. More precisely, by isolating the linearization of the governing equation, one takes advantage of a *spectral gap* and associated spectral projections in order to derive necessary conditions for *stable* initial data, which in turn lead to a contradiction. We also refer to Prüss, Simonett and Zacher [67] and Prüss, Simonett and Wilke [64] for related results. Finally, for the bifurcation result we apply classic results of Crandall and Rabinowitz [21] regarding bifurcation from simple eigenvalues. However, even in the reduced setting mentioned above, developed while proving stability results, we find that the eigenvalues associated with the problem are not simple. We restrict our attention to surfaces which are even (symmetric about the surface $x = 0$) and satisfy prescribed regularity and periodicity, similar to a method used by Escher and Matioc [35]. In this setting, the problem does have simple eigenvalues, so we are able to derive bifurcation results and apply them back to the full ASD problem via a posteriori symmetries of equilibria. We note that, rather than using the restriction to even functions in order to apply the results of Crandall and Rabinowitz [21], we could also have chosen to apply more general bifurcation techniques, such as the methods contained in the manuscript of Kielhöfer [48, Section I.19], in order to generate (more general) bifurcation results. In fact, we do plan to make use of the results and methods of [48] in future research endeavors in order to further classify the type of bifurcations that arise and investigate the global and secondary bifurcation behavior of equilibria to ASD.

In Chapter 2, the contents of which are also contained in the publication [50], we develop an abstract theory and rigorous framework in order to generate maximal regularity results for a large class of abstract inhomogeneous equations with prescribed periodic boundary conditions. The results and framework developed in Chapter 2 are invaluable for the geometric analysis performed in Chapter 3, and will also be applicable to a large class of elliptic, inhomogeneous and quasilinear equations. In particular, we prove that every even–order differential operator with (little–Hölder regular and periodic) variable coefficients which satisfies certain uniform ellipticity conditions will satisfy (continuous) maximal regularity conditions in the setting of periodic little-Hölder spaces. Utilizing this result, as we demonstrate in Chapter 3, one gains access to the well–posedness results of Clément and Simonett [17] for quasilinear parabolic problems, and the results of Chapter 2 allow one to apply various techniques from nonlinear functional analysis, including the implicit function theorem, which opens the door to the analysis of a wide variety of dynamic properties of solutions.

1.3 Conventions and Notation

Throughout the paper, E and F will denote arbitrary Banach spaces over the field $\mathbb{K}$, which will either be $\mathbb{R}$ or $\mathbb{C}$, depending upon the context. The space $\mathcal{L}(E,F)$ consists of all bounded linear operators mapping E into F. We say that E is continuously embedded in F, denoted $E \hookrightarrow F$, if there exists an injective operator $i \in \mathcal{L}(E,F)$. Moreover, we say that E is densely embedded in F, denoted $E \overset{d}{\hookrightarrow} F$, if $i(E) \subset F$ is dense.

For $U \subset E$ an open set, we denote by $C^{\omega}(U,F)$ the collection of real analytic mappings $f : U \to F$. In particular, for every $x_0 \in U$ there exists a positive constant $r(x_0) > 0$ and a sequence $(T_k)_{k=0}^{\infty}$ with $T_k \in \mathcal{L}(E^k, F)$, i.e. T_k is a continuous, symmetric k-linear operator from $E \times \ldots \times E$

to F, such that f admits the representation

$$f(x) = f(x_0) + \sum_{k=0}^{\infty} T_k(x - x_0)^k, \qquad \text{(convergence in } F\text{)}$$

for every $x \in \mathbb{B}_E(x_0, r(x_0)) := \{x \in E : \|x - x_0\|_E < r(x_0)\}$.

CHAPTER 2

ELLIPTIC OPERATORS AND MAXIMAL REGULARITY

In this chapter we consider the following abstract periodic inhomogeneous equation

$$\begin{cases} \partial_t u(t,x) + \mathcal{A}(x,D)u(t,x) = f(t,x), & t > 0,\ x \in \mathbb{R} \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $\mathcal{A}(\cdot, D) := \sum_{k=0}^{2m} b_k(\cdot)D^k$ is a differential operator of order $2m$, with variable coefficients $b_k : \mathbb{R} \to \mathbb{C}$. Further, we enforce periodic boundary conditions on the problem by assuming the given functions $u_0, b_k, f(t,\cdot)$, for $t \geq 0$, $k = 0, \ldots, 2m$, are all 2π-periodic in $x \in \mathbb{R}$. Hence, we will be looking for solutions $u(t,\cdot)$ which also exhibit 2π-periodicity on $\mathbb{R}$, for $t > 0$. We will also consider the more general setting of vector-valued functions $u_0, f(t,\cdot), u(t,\cdot) : \mathbb{R} \to E$, and operator-valued coefficients $b_k : \mathbb{R} \to \mathcal{L}(E)$, for an arbitrary Banach space E over $\mathbb{C}$. This more general setting is discussed in Section 2.6.

Understanding the nature of solutions (i.e. existence, uniqueness and regularity) to inhomogeneous equations of this form is integral to the study of abstract quasilinear equations. In the quasilinear setting, we see that (2.1) takes the form

$$\partial_t u + \mathtt{A}(u, D)u = \mathtt{F}(u),$$

where the coefficients $b_k = b_k(u, u', \ldots, u^{(2m-1)})$, and subsequently the differential operator $\mathtt{A}$, may depend upon the solution u and its lower order derivatives $u^{(j)}$, $j \leq 2m-1$. Meanwhile, the inhomogeneity takes the form $\mathtt{F}(u) = \mathtt{F}(u, u', \ldots, u^{(2m-1)})$, for some nonlinear mapping $\mathtt{F}$. Several authors have studied abstract quasilinear equations, including [1, 8, 17, 49, 51, 63]. Among the techniques employed to study quasilinear problems, the notion of maximal regularity has proven to be a valuable tool in establishing both qualitative and quantitative results, c.f. [5, 8, 17, 36, 49, 51, 63, 65, 70]. The axisymmetric surface diffusion problem is one example of a problem to which one can apply the results of this chapter. Indeed, we utilize the results of this chapter to establish well-posedness of (1.2) and investigate dynamic properties of solutions in Chapter 3.

Given an elliptic differential operator with periodic coefficients, it is our goal in this chapter to show that one can establish (continuous) maximal regularity results in the setting of periodic little-Hölder spaces. Moreover, we assume only minimal regularity conditions on the coefficients b_k, lending our results to applications in periodic quasilinear problems. In order to establish maximal regularity, we make use of a result originally proved by DaPrato and Grisvard [22], which gives a construction of pairs of function spaces with the property of continuous maximal regularity for a given operator, under the assumption that the operator generates a strongly continuous analytic semigroup. Hence, we focus first on showing generation of an analytic semigroup.

In fact, we will show that elliptic operators with periodic coefficients generate analytic semigroups in the periodic Hölder and little-Hölder settings. However, we focus on the results in the little-Hölder setting, because we get strong continuity of the semigroups generated, due to density of embeddings in the little-Hölder scale, a necessary condition for applying the results of DaPrato and Grisvard. To the best of the author's awareness, the work contained herein constitutes the first systematic treatment of semigroup generation in the case of variable coefficients for elliptic operators with periodic boundary conditions. A related result for constant coefficients in the periodic setting was proved by Escher and Matioc [34], see also [55], where they considered a specific abstract operator of third order, in the periodic little-Hölder setting.

In the process of establishing semigroup generation results, we consider the parameter-dependent elliptic equation

$$(\lambda - \mathcal{A}(\cdot, D))u = f, \qquad \lambda \in \mathbb{C},$$

for which we show invertibility in the periodic Hölder and little-Hölder settings, provided $\operatorname{Re}\lambda$ is sufficiently large. Additionally, we establish parameter-dependent estimates on the resolvent of an elliptic operator under minimal regularity assumptions on the cefficients. With invertibility and resolvent estimates, semigroup generation follows from a standard result in semigroup theory, [2] and [40]. One will note that semigroup generation results are sufficient to derive well-posedness for the inhomogeneous problem (2.1) by classic semigroup techniques. However, as stated, we focus on establishing maximal regularity results, for which the little-Hölder setting is desirable.

The chapter is organized as follows. In the first section, we express regularity conditions for periodic functions on $\mathbb{R}$, exploiting a connection with functions defined over the one-dimensional torus $\mathbb{T}$, and establish necessary results regarding these spaces. In the second section we state and prove a Marcinkiewicz-type Fourier multiplier result, which is a slight generalization of a result in [11]. In the third section we prove that a simplified operator $-\mathcal{A}_b$, with highest-order terms and constant coefficients, generates a (strongly continuous) analytic semigroup on periodic little-Hölder spaces. In the fourth section, we extend this result to the principal part $-\mathcal{A}_p$, with highest-order terms and variable coefficients, using a partition technique as seen in [4, 6]. In the fifth section, we present a generation result for the full operator, discuss maximal regularity and solutions to the linear problem (2.1). We conclude the paper by discussing the case of vector-valued functions and necessary modifications to our methods for results to carry over to this setting.

2.1 Periodic Functions Over $\mathbb{R}$

Given a 2π-periodic function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ with some known regularity, we can restrict $\tilde{f}$ to an interval of periodicity (the interval $[-\pi, \pi]$, for instance) and the full function can still be recovered, i.e. the restricted function $f := \tilde{f}|_{[-\pi,\pi]}$ can be extended periodically to all of $\mathbb{R}$ and this extension will coincide exactly with $\tilde{f}$. Reversing this process, we want to start with a function $f : [-\pi, \pi] \to \mathbb{C}$ and prescribe minimal conditions on f so that the periodic extension exhibits desired regularity on $\mathbb{R}$. In this section, we characterize several regularity classes for periodic functions with respect to their properties on the interval $[-\pi, \pi]$.

Let $\mathbb{T} := [-\pi, \pi]$, where the points π and $-\pi$ are identified; we denote this point by $\{\pi, -\pi\}$. Endow $\mathbb{T}$ with the metric topology τ generated by the metric

$$d_{\mathbb{T}} : \mathbb{T} \times \mathbb{T} \to \mathbb{R} \qquad d_{\mathbb{T}}(x, y) := |x - y| \wedge (2\pi - |x - y|), \quad \text{where } a \wedge b := \min\{a, b\}.$$

(For computational purposes, we follow the convention $\{\pi, -\pi\} = \pi$ so that $\{\pi, -\pi\} \geq x$ for all $x \in \mathbb{T}$ and $d_{\mathbb{T}}(x, \{\pi, -\pi\}) = |x - \pi| \wedge (2\pi - |x - \pi|)$.) Notice that $(\mathbb{T}, \tau)$ is a topological group which is isomorphic to the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ endowed with the quotient topology. We see, moreover, that $\mathbb{T}$ is a complete, compact, metric space and we denote open balls of $\mathbb{T}$ by

$$\mathbb{B}_{\mathbb{T}}(x, \varepsilon) := \{y \in \mathbb{T} : d_{\mathbb{T}}(x, y) < \varepsilon\} \qquad \varepsilon > 0,\ x \in \mathbb{T}.$$

We will demonstrate how using this metric gives intrinsic regularity conditions for functions defined on $\mathbb{T}$, which can be naturally extended periodically to $\mathbb{R}$.

2.1.1 Regularity on $\mathbb{T}$

Given a function $f : \mathbb{T} \to \mathbb{C}$, we define its periodic extension

$$\tilde{f}(x) := f(x - 2\pi k) \qquad \text{for} \quad x \in [\pi(2k-1), \pi(2k+1)], \quad k \in \mathbb{Z}.$$

Denote by ϕ the periodic extension operator taking f defined on $\mathbb{T}$ to $\tilde{f}$ defined on $\mathbb{R}$. One can immediately see that ϕ is bijective from $\mathbb{C}^{\mathbb{T}}$ to $\left(\mathbb{C}^{\mathbb{R}}\right)_{per}$, the space of 2π-periodic functions on $\mathbb{R}$. Now, we define spaces of regular periodic functions over $\mathbb{R}$ into which we want ϕ to map.

Denote by $C_{per}(\mathbb{R})$ and $C^k_{per}(\mathbb{R})$ the spaces of 2π-periodic functions over $\mathbb{R}$ which are continuous and k-times continuously differentiable, respectively, for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$; we take $C^0_{per}(\mathbb{R}) = C_{per}(\mathbb{R})$ by convention. Each is a closed subspace of the corresponding non-periodic spaces and are Banach spaces when equipped with the following norms

$$\|f\|_{C(\mathbb{R})} := \sup_{x \in \mathbb{R}} |f(x)|, \qquad \|f\|_{C^k(\mathbb{R})} := \sum_{j=0}^{k} \|f^{(j)}\|_{C(\mathbb{R})}. \tag{2.2}$$

Moreover, for $\alpha \in (0, 1)$ and $k \in \mathbb{N}_0$, we define the space of Hölder continuous functions $C^{k+\alpha}_{per}(\mathbb{R})$ to be those functions $f \in C^k_{per}(\mathbb{R})$ such that

$$\left[f^{(k)}\right]_{\alpha,\mathbb{R}} := \sup_{\substack{x,y \in \mathbb{R} \\ x \neq y}} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^{\alpha}} < \infty.$$

We call $[\cdot]_{\alpha,\mathbb{R}}$ the α-Hölder seminorm over $\mathbb{R}$ and one can see that $C^{k+\alpha}_{per}(\mathbb{R})$ is a Banach space with the norm

$$\|f\|_{C^{k+\alpha}(\mathbb{R})} := \|f\|_{C^k(\mathbb{R})} + \left[f^{(k)}\right]_{\alpha,\mathbb{R}}. \tag{2.3}$$

For simplicity of notation, given $\theta \in \mathbb{R}_+$, we define $C^{\theta}_{per}(\mathbb{R}) := C^{\lfloor\theta\rfloor + \{\theta\}}_{per}(\mathbb{R})$, where $\lfloor\theta\rfloor$ denotes the largest integer not exceeding θ and $\{\theta\} := \theta - \lfloor\theta\rfloor$.

With the periodic spaces over $\mathbb{R}$ established we define the spaces over $\mathbb{T}$ as follows. For $\theta \in \mathbb{R}_+$

let

$$C^\theta(\mathbb{T}) := \left\{ f \in \mathbb{C}^\mathbb{T} : \phi(f) \in C^\theta_{per}(\mathbb{R}) \right\} \qquad \text{with} \quad \|f\|_{C^\theta(\mathbb{T})} := \|\phi(f)\|_{C^\theta(\mathbb{R})}. \tag{2.4}$$

It follows immediately that $C^\theta(\mathbb{T})$ is a Banach space and ϕ is a linear isometric isomorphism from $C^\theta(\mathbb{T})$ to $C^\theta_{per}(\mathbb{R})$. Further, if $\theta \geq 1$ and $f \in C^\theta(\mathbb{T})$, we define the derivative $f' \in \mathbb{C}^\mathbb{T}$ by $f' := \phi^{-1}(\phi(f)') = \left(\frac{d}{dx}\tilde{f}\right)|_\mathbb{T}$.

It is interesting to note that continuity, differentiability and Hölder continuity can all be defined intrinsically on $\mathbb{T}$, making use of an ordered adaptation of the metric $d_\mathbb{T}$, such that ϕ remains a linear isomorphism. Intrinsic definitions of regularity provide a different perspective for functions over the periodic domain $\mathbb{T}$ and setting for regularity independent of periodic extensions. Although the connection between functions over $\mathbb{T}$ and periodic functions over $\mathbb{R}$ has been used widely in the literature, c.f. [12, 34, 69], little attention has been paid to the local conditions and geometry on $\mathbb{T}$, which are important to the partition argument that we use in Section 2.4. We will state some of the results regarding this intrinsic viewpoint that will be of use later in the paper, in particular we state equivalent definitions for (Hölder) continuity over $\mathbb{T}$ and an application of the Mean Value theorem. For simplicity of notation, we denote by $d^\alpha_\mathbb{T}(\cdot,\cdot)$ the quantity $d_\mathbb{T}(\cdot,\cdot)^\alpha$.

Proposition 2.1.1. *Let $f \in \mathbb{C}^\mathbb{T}$, then*

a) $f \in C(\mathbb{T})$ if and only if f is continuous in the metric topology τ.

b) for $\alpha \in (0,1)$, $f \in C^\alpha(\mathbb{T})$ if and only if $[f]_{\alpha,\mathbb{T}} := \sup\limits_{\substack{x,y\in\mathbb{T} \\ x\neq y}} \dfrac{|f(x)-f(y)|}{d^\alpha_\mathbb{T}(x,y)} < \infty.$

Moreover, $[f]_{\alpha,\mathbb{T}} = [\tilde{f}]_{\alpha,\mathbb{R}}$ in this case.

c) if $f \in C^1(\mathbb{T})$ and $x, y \in \mathbb{T}$, then $|f(x) - f(y)| \leq \|f'\|_{C(\mathbb{T})} d_\mathbb{T}(x,y)$.

Proof. a) Follows from direct computation.

b) First, assume $f \in C^\alpha(\mathbb{T})$ and let $x, y \in \mathbb{T}$ such that $x \neq y$, without loss of generality assume $x < y$ (recalling the convention that $\{\pi, -\pi\} \geq x$ for all $x \in \mathbb{T}$). By definition of the metric $d_\mathbb{T}$, we see that $d_\mathbb{T}(x,y)$ is either equal to $|x-y|$ or $|(x+2\pi)-y|$. Now, we examine both cases

- if $d_\mathbb{T}(x,y) = |x-y|$, then $\dfrac{|f(x)-f(y)|}{d^\alpha_\mathbb{T}(x,y)} = \dfrac{|\tilde{f}(x)-\tilde{f}(y)|}{|x-y|^\alpha} \leq [\tilde{f}]_{\alpha,\mathbb{R}}$,

- if $d_{\mathbb{T}}(x,y) = |(x+2\pi)-y|$, then, by periodicity of $\tilde{f}$,

$$\frac{|f(x)-f(y)|}{d^{\alpha}_{\mathbb{T}}(x,y)} = \frac{|\tilde{f}(x)-\tilde{f}(y)|}{|(x+2\pi)-y|^{\alpha}} = \frac{|\tilde{f}(x+2\pi)-\tilde{f}(y)|}{|(x+2\pi)-y|^{\alpha}} \leq [\tilde{f}]_{\alpha,\mathbb{R}}.$$

Hence, $[f]_{\alpha,\mathbb{T}} \leq [\tilde{f}]_{\alpha,\mathbb{R}} < \infty$. Conversely, assume that $[f]_{\alpha,\mathbb{T}} < \infty$ and consider $x, y \in \mathbb{R}$, $x < y$. Here we consider the following three cases

- if $|x-y| \leq \pi$ and there exists $k \in \mathbb{Z}$ so that $x, y \in [\pi(2k-1), \pi(2k+1)]$. Then we have $(x-2\pi k), (y-2\pi k) \in \mathbb{T}$, $|x-y| = d_{\mathbb{T}}((x-2\pi k),(y-2\pi k))$ and

$$\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|x-y|^{\alpha}} = \frac{|f(x-2\pi k)-f(y-2\pi k)|}{d^{\alpha}_{\mathbb{T}}((x-2\pi k),(y-2\pi k))} \leq [f]_{\alpha,\mathbb{T}}.$$

- if $|x-y| \leq \pi$ and there exists $k \in \mathbb{Z}$ so that $x \in [\pi(2k-1), \pi(2k+1)]$ and $y \in (\pi(2k+1), \pi(2k+3)]$. Then we have that $(x-2\pi k), (y-2\pi(k+1)) \in \mathbb{T}$, $|x-y| = d_{\mathbb{T}}((x-2\pi k),(y-2\pi(k+1)))$ and

$$\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|x-y|^{\alpha}} = \frac{|f(x-2\pi k)-f(y-2\pi(k+1))|}{d^{\alpha}_{\mathbb{T}}((x-2\pi k),(y-2\pi(k+1)))} \leq [f]_{\alpha,\mathbb{T}}.$$

- if $|x-y| > \pi$, then we can find $l \in \mathbb{Z}$ so that $|(x+2\pi l)-y| \leq \pi$. Then we have, taking advantage of the periodicity of $\tilde{f}$ and the fact that $|x-y| \geq |(x+2\pi l)-y|$, that

$$\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|x-y|^{\alpha}} \leq \frac{|\tilde{f}(x+2\pi l)-\tilde{f}(y)|}{|(x+2\pi l)-y|^{\alpha}} \leq [f]_{\alpha,\mathbb{T}},$$

  where the last inequality follows from the previous two cases.

Therefore, we can see that $[\tilde{f}]_{\alpha,\mathbb{R}} \leq [f]_{\alpha,\mathbb{T}} < \infty$, so $f \in C^{\alpha}(\mathbb{T})$ and the claim follows. Moreover, we see that $[f]_{\alpha,\mathbb{T}} = [\tilde{f}]_{\alpha,\mathbb{R}}$.

c) Fix $f \in C^1(\mathbb{T})$, $x, y \in \mathbb{T}$ and assume, without loss of generality, that $x \leq y$. As before, it follows that $d_{\mathbb{T}}(x,y)$ equals either $|x-y|$ or $|(x+2\pi)-y|$. We consider these two cases separately and see that the claim holds;

- if $d_{\mathbb{T}}(x,y) = |x-y|$, then
  $|f(x)-f(y)| = |\tilde{f}(y)-\tilde{f}(x)| = \left|\int_x^y \tilde{f}'(t)dt\right| \leq \|\tilde{f}'\|_{C(\mathbb{R})}|y-x| = \|f'\|_{C(\mathbb{T})} d_{\mathbb{T}}(x,y).$

- if $d_{\mathbb{T}}(x,y) = |(x+2\pi) - y|$, then
$$|f(x)-f(y)| = |\tilde{f}(x+2\pi)-\tilde{f}(y)| = \left|\int_y^{x+2\pi} \tilde{f}'(t)dt\right| \le \|\tilde{f}'\|_{C(\mathbb{R})}|(x+2\pi)-y| = \|f'\|_{C(\mathbb{T})} d_{\mathbb{T}}(x,y).$$

□

Finally, we define the so-called little-Hölder spaces over $\mathbb{R}$ and $\mathbb{T}$. We discuss equivalent characterizations and results on little-Hölder spaces, important for maximal regularity and generation of analytic semigroups. For $\theta \in \mathbb{R}_+ \setminus \mathbb{Z}$ define the periodic little-Hölder spaces over $\mathbb{R}$ as

$$h^\theta_{per}(\mathbb{R}) := \left\{ f \in C^\theta_{per}(\mathbb{R}) : \lim_{\delta \to 0} \sup_{\substack{x,y\in\mathbb{R} \\ 0<|x-y|<\delta}} \frac{|f^{\lfloor\theta\rfloor}(x) - f^{\lfloor\theta\rfloor}(y)|}{|x-y|^{\{\theta\}}} = 0 \right\}.$$

Then, $h^\theta_{per}(\mathbb{R})$ is a closed subspace of $C^\theta_{per}(\mathbb{R})$ and likewise a Banach space with the inherited norm $\|\cdot\|_{C^\theta(\mathbb{R})}$, defined by (2.3). Moreover, it follows that the little-Hölder spaces are, in fact, Banach algebras, in both the periodic and non-periodic settings. Now, we define $h^\theta(\mathbb{T}) := \{f \in \mathbb{C}^{\mathbb{T}} : \phi(f) \in h^\theta_{per}(\mathbb{R})\}$, for $\theta \in \mathbb{R}_+ \setminus \mathbb{Z}$. Following Proposition 2.1.1, one easily verifies that an equivalent definition is

$$h^\theta(\mathbb{T}) := \left\{ f \in C^\theta(\mathbb{T}) : \lim_{\delta \to 0} \sup_{\substack{x,y\in\mathbb{T} \\ 0<d_{\mathbb{T}}(x,y)<\delta}} \frac{|f^{\lfloor\theta\rfloor}(x) - f^{\lfloor\theta\rfloor}(y)|}{d^{\{\theta\}}_{\mathbb{T}}(x,y)} = 0 \right\}. \tag{2.5}$$

Little-Hölder spaces have been studied by several authors in context with analytic semigroups and maximal regularity, c.f. [17, 34, 36, 51]. The proposition that follows demonstrates two properties of little-Hölder spaces which make them a natural choice for maximal regularity results; it is the periodic analog of well-known results on little-Hölder spaces over $\mathbb{R}$, c.f. [51]. Here we let $(\cdot,\cdot)_\eta := (\cdot,\cdot)^0_{\eta,\infty}$ denote the continuous interpolation functor of Da Prato and Grisvard, with exponent $\eta \in (0,1)$, see [2, 51] for reference.

Proposition 2.1.2.

a) For $\theta \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $\sigma \in (\theta, \infty]$, $h^\theta(\mathbb{T})$ is the closure of $C^\sigma(\mathbb{T})$ in $(C^\theta(\mathbb{T}), \|\cdot\|_{C^\theta(\mathbb{T})})$. Hence, $h^\sigma(\mathbb{T}) \overset{d}{\hookrightarrow} h^\theta(\mathbb{T})$ for $\sigma \in (\theta,\infty) \setminus \mathbb{Z}$.

b) For $\theta_1, \theta_2 \in \mathbb{R}_+ \setminus \mathbb{Z}$ with $\theta_2 \ge \theta_1$, it follows that $(h^{\theta_1}(\mathbb{T}), h^{\theta_2}(\mathbb{T}))_\eta = h^{\eta\theta_2 + (1-\eta)\theta_1}(\mathbb{T})$, provided $(\eta\theta_2 + (1-\eta)\theta_1) \notin \mathbb{Z}$.

Remarks on Proof: a) The proof of this statement is identical to the non-periodic case and can be found in Lunardi, [51, Proposition 0.2.1]. We remark that the approximating functions from $C^\infty(\mathbb{R})$ established in Lunardi's proof, which are convolutions with smooth approximations of the identity, are in $C^\infty_{per}(\mathbb{R})$ in the periodic case. This fact follows from a property of convolutions involving periodic functions. Namely, given a function φ and a 2π-periodic function f such that the convolution $f * \varphi$ is well-defined, then the convolution is periodic, as

$$(f * \varphi)(x + 2\pi) = \int_{\mathbb{R}} f((x + 2\pi) - y)\varphi(y)dy = \int_{\mathbb{R}} f(x - y)\varphi(y)dy = (f * \varphi)(x).$$

b) The proof of this statement is identical to the non-periodic case, as demonstrated in [51, Theorem 1.2.17]. Again, this method applies to the periodic case because we consider convolutions of smoothing kernels φ_t with periodic functions f over $\mathbb{R}$. Hence, the resulting convolutions are contained in $C^\infty_{per}(\mathbb{R})$. □

2.1.2 Periodic Besov Spaces

In order to state the Fourier multiplier theorem upon which our generation results heavily rely, we must first introduce the scale of Sobolev and Besov spaces. We present here a definition of periodic Besov spaces with respect to dyadic-type decompositions, similar to the development in [12], for more details on these spaces, and equivalent definitions, see Triebel and Schmeisser [69, Section 3.5].

Following the notation of Arendt and Bu [12], let $\mathcal{D}(\mathbb{T})$ denote the space $C^\infty(\mathbb{T})$ equipped with the locally convex topology generated by the family of semi-norms $\|f\|_k := \|f^{(k)}\|_{C(\mathbb{T})}$, for $k \in \mathbb{N}_0$. We define the space of periodic distributions $\mathcal{D}'(\mathbb{T}) := (\mathcal{D}(\mathbb{T}))^*$, the set of all bounded linear functionals on $\mathcal{D}(\mathbb{T})$, and we equip $\mathcal{D}'(\mathbb{T})$ with the weak-star topology over $\mathcal{D}(\mathbb{T})$. Now we will investigate how the Fourier transform interacts with these spaces.

Denote by e_k the function $[x \mapsto e^{ikx}] : \mathbb{T} \to \mathbb{C}$, then $e_k \in \mathcal{D}(\mathbb{T})$ for $k \in \mathbb{Z}$. For $T \in \mathcal{D}'(\mathbb{T})$, we define the *Fourier coefficients* $\hat{T}(k) := \langle T, e_{-k}\rangle$, where $\langle \cdot, \cdot \rangle : \mathcal{D}'(\mathbb{T}) \times \mathcal{D}(\mathbb{T}) \to \mathbb{C}$ denotes the duality pairing. Notice that every test function $\varphi \in \mathcal{D}(\mathbb{T})$ can be identified with the induced distribution $T_\varphi \in \mathcal{D}'(\mathbb{T})$ defined by $\langle T_\varphi, \psi\rangle := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \varphi(x)\psi(x)dx$, $\psi \in \mathcal{D}(\mathbb{T})$. Then the Fourier coefficients of

T_φ coincide with the usual Fourier coefficients for $\varphi \in \mathcal{D}(\mathbb{T})$, namely

$$\hat{T}_\varphi(k) = \hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \varphi(x) e^{-ikx} dx.$$

When no confusion is likely, we will denote by φ both the function and its induced distribution. Moreover, by [30, Theorem 12.5.3], we have the Fourier series representation

$$f = \sum_{k\in\mathbb{Z}} \hat{f}(k) e_k \qquad \text{for} \quad f \in \mathcal{D}'(\mathbb{T}) \quad (\text{convergence in } \mathcal{D}'(\mathbb{T})).$$

To define Besov spaces over $\mathbb{T}$, let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$ and $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions on $\mathbb{R}$. Further, let $\Phi(\mathbb{R})$ denote the collection of all systems $(\varphi_j)_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ satisfying the properties:

- $\operatorname{supp} \varphi_0 \subset [-2, 2], \qquad \operatorname{supp} \varphi_j \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}], \quad j \geq 1,$
- $\displaystyle\sum_{j\in\mathbb{N}_0} \varphi_j(x) = 1, \quad x \in \mathbb{R},$
- $\forall\, l \in \mathbb{N}_0$, $\exists\, C_l > 0$ so that $\displaystyle\sup_{j\in\mathbb{N}_0} 2^{lj} \|\varphi_j^{(l)}\|_{C(\mathbb{R})} \leq C_l$.

Now, let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ be fixed parameters and $\varphi = (\varphi_j) \in \Phi(\mathbb{R})$. For $f \in \mathcal{D}'(\mathbb{T})$, $j \in \mathbb{N}_0$, the series $\sum_{k\in\mathbb{Z}} \varphi_j(k) \hat{f}(k) e_k$ has only finitely many nonzero terms, by compactness of the support of φ_j (we refer to finite series of this form as *trigonometric polynomials*), and it follows that $\sum_{k\in\mathbb{Z}} \varphi_j(k) \hat{f}(k) e_k \in L_p(\mathbb{T})$. The norm on $L_p(\mathbb{T})$ is given by

$$\|g\|_p := \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{T}} |g(x)|^p dx \right)^{1/p} & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{T}} |g(x)| & p = \infty. \end{cases}$$

Now we define the periodic Besov space

$$B^{s,\varphi}_{p,q}(\mathbb{T}) := \left\{ f \in \mathcal{D}'(\mathbb{T}) : \left(2^{sj} \left\| \sum_{k\in\mathbb{Z}} \varphi_j(k) \hat{f}(k) e_k \right\|_p \right)_{j\in\mathbb{N}_0} \in \ell^q(\mathbb{N}_0) \right\}. \tag{2.6}$$

Then $B^{s,\varphi}_{p,q}(\mathbb{T})$ is a Banach space when equipped with the norm

$$\|f\|_{B^{s,\varphi}_{p,q}} := \begin{cases} \left(\sum_{j\in\mathbb{N}_0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k \right\|_p^q \right)^{1/q} & \text{for } q < \infty, \\ \sup_{j\in\mathbb{N}_0} 2^{sj} \left\| \sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k \right\|_p & \text{for } q = \infty. \end{cases} \tag{2.7}$$

Although the definition of a periodic Besov space depends explicitly upon the choice of system $\varphi \in \Phi(\mathbb{R})$, it can be shown that $(B^{s,\varphi}_{p,q}(\mathbb{T}), \|\cdot\|_{B^{s,\varphi}_{p,q}})$ is equivalent to $(B^{s,\psi}_{p,q}(\mathbb{T}), \|\cdot\|_{B^{s,\psi}_{p,q}})$, for two systems $\varphi, \psi \in \Phi(\mathbb{R})$, c.f. [69, Theorem 3.5.1(i)]. Hence, we drop reference to particular systems $\varphi \in \Phi(\mathbb{R})$ and simply refer to Besov spaces parametrized by $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. See [3, 4, 12, 69] for more information on Besov spaces and their properties. We mention one property that comes up in the sequel, c.f. [12, Theorem 3.1 (ii)] or [69, Theorem 3.5.4 (i)].

Proposition 2.1.3. *For $s \in \mathbb{R}_+ \setminus \mathbb{Z}$, it holds that $B^s_{\infty,\infty}(\mathbb{T}) = C^s(\mathbb{T})$.*

2.2 A Fourier Multiplier Theorem

The Fourier multiplier result that we will need is a slight modification of the result [12, Theorem 4.5 (ii)], which gives sufficient conditions on the symbol of a Fourier multiplier so that the associated operator is continuous from $B^s_{p,q}(\mathbb{T})$ to itself. We modify the result to get sufficient conditions for continuity from $B^s_{p,q}(\mathbb{T})$ to $B^r_{p,q}(\mathbb{T})$ for distinct values of r and s. The modification we apply is the same technique used by B.V. Matioc [55] in altering the result [12, Theorem 4.5 (i)].

For $1 \le p \le \infty$, we define the Sobolev space

$$W^1_p(\mathbb{T}) := \{f \in L_p(\mathbb{T}) : f \text{ is weakly differentiable and } f' \in L_p(\mathbb{T})\}$$

with the norm $\|f\|_{W^1_p} := \|f\|_p + \|f'\|_p$.

Theorem 2.2.1. *Let $r, s \in \mathbb{R}_+$ and $1 \le p, q \le \infty$. Suppose that $(M_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is a sequence such that*

$$s_1 := \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s}|M_k| < \infty, \quad \text{and} \quad s_2 := \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+1}|M_{k+1} - M_k| < \infty. \tag{2.8}$$

Then the Fourier multiplier with symbol $(M_k)_{k\in\mathbb{Z}}$ is a continuous mapping from $B^s_{p,q}(\mathbb{T})$ to $B^r_{p,q}(\mathbb{T})$, *namely*

$$T : \left[\sum_{k \in \mathbb{Z}} \hat{f}(k) e_k \longmapsto \sum_{k \in \mathbb{Z}} M_k \hat{f}(k) e_k \right] \in \mathcal{L}\left(B^s_{p,q}(\mathbb{T}), B^r_{p,q}(\mathbb{T})\right).$$

Moreover, the operator norm of T as a mapping from $B^s_{p,q}(\mathbb{T})$ to $B^r_{p,q}(\mathbb{T})$ can be bounded with respect to s_1 and s_2.

The proof of this result relies upon the following Lemma, which is a simple version of [12, Lemma 4.4]. Here we only consider $\mathbb{C}$-valued functions over $\mathbb{R}$, so that the spaces involved are of Fourier type 2 and the statement is simplified as follows.

Lemma 2.2.2. *Let $1 \le p \le \infty$ and let $m \in C_c(\mathbb{R}, \mathbb{C}) \cap \mathcal{F}L_1(\mathbb{R}, \mathbb{C})$. Then*

$$\left\| \sum_{k \in \mathbb{Z}} m(k) \hat{f}(k) e_k \right\|_p \le C_p \eta_2(m) \| \sum_{k \in \mathbb{Z}} \hat{f}(k) e_k \|_p \tag{2.9}$$

holds whenever $f \in L_p(\mathbb{T})$ is a trigonometric polynomial, where C_p is a constant depending only on p, and $\eta_2(m) := \inf\{\|m(a \cdot)\|_{W^1_2} : a > 0\}$.

Proof of Theorem 2.2.1: We provide the proof here for the reader's convenience and reference [12, Theorem 4.5(ii)] and [55, Theorem 2.2.1]. Fix $(M_k)_{k \in \mathbb{Z}} \subset \mathbb{C}$ satisfying (2.8) and parameters $s, r \in \mathbb{R}$, $1 \le p, q \le \infty$ and $\varphi := \{\varphi_j\}_{j \ge 0} \in \Phi(\mathbb{R})$. We follow the same method as Arendt and Bu, with modifications to account for the (possibly nonzero) difference $|r - s|$, which is zero in the case considered in [12]. To see that T is a bounded operator from $B^s_{p,q}(\mathbb{T})$ to $B^r_{p,q}(\mathbb{T})$ as stated, it will suffice to show that there exists some constant $C > 0$ such that the bound

$$\left\| \sum_{k \in \mathbb{Z}} \left(2^{(r-s)j} M_k \right) \varphi_j(k) \hat{f}(k) e_k \right\|_p \le C \left\| \sum_{k \in \mathbb{Z}} \varphi_j(k) \hat{f}(k) e_k \right\|_p ,$$

holds uniformly for $f \in B^s_{p,q}(\mathbb{T})$ and $j \ge 0$. To demonstrate this bound, we define an appropriate sequence of compactly supported continuous functions and take advantage of Lemma 2.2.2.

For $j \ge 1$, define $m_j : \mathbb{R} \to \mathbb{C}$ by $m_j(x) = 0$ if $|x| \ge 2^{j+2}$ or $|x| \le 2^{j-2}$, $m_j(k) = 2^{(r-s)j} M_k$ for $k \in \mathbb{Z}$ with $2^{j-1} \le |k| \le 2^{j+1}$, and m_j is affine on $[k, k+1]$ for all $k \in \mathbb{Z}$. We define m_0 in a similar manner, where $m_0(x) = 0$ if $|x| \ge 2$, $m_0(k) = M_k$ for $-1 \le k \le 1$, and m_0 is affine on every interval $[k, k+1]$, $k \in \mathbb{Z}$.

One can see that $m_j \in C_c(\mathbb{R}) \cap \mathcal{F}L^1(\mathbb{R})$ and, by compactness of $\operatorname{supp} \varphi_j$, $\sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k$ is a trigonometric polynomial, for $j \geq 0$. Hence, we can apply Lemma 2.2.2 to see that, for $j \geq 1$, the following bounds hold.

$$\begin{aligned}\left\| \sum_{k\in\mathbb{Z}} \left(2^{(r-s)j} M_k\right) \varphi_j(k)\hat{f}(k)e_k \right\|_p &= \left\| \sum_{2^{j-1}\leq |k| \leq 2^{j+1}} m_j(k)\varphi_j(k)\hat{f}(k)e_k \right\|_p \\ \leq C_p\, \eta_2(m_j) \left\| \sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k \right\|_p &\leq C_p \, \|m_j(2^j \cdot)\|_{W_2^1} \left\| \sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k \right\|_p .\end{aligned}$$

Hence, it suffices to show that $\{\|m_j(2^j\cdot)\|_{W_2^1}\}_{j\geq 1}$ is uniformly bounded. From direct computation, one can see that this bound follows from the property $\operatorname{supp} m_j \subset [\frac{1}{4}, 4]$ and it holds that

$$\sup_{x\in\mathbb{R}} |m_j(2^j x)| \leq \sup_{2^{j-1}\leq |k| \leq 2^{j+1}} 2^{(r-s)j}|M_k| \leq \sup_{2^{j-1}\leq |k| \leq 2^{j+1}} \left(\frac{2^{(r-s)j}}{|k|^{r-s}}\right) s_1 \leq 2^{|r-s|} s_1,$$

and

$$\sup_{2^{j-1}\leq |p| \leq 2^{j+1}} 2^{(r-s+1)j}|M_{p+1} - M_p| \leq \sup_{2^{j-1}\leq |p| \leq 2^{j+1}} \left(\frac{2^{(r-s+1)j}}{|p|^{(r-s+1)}}\right) s_2 \leq 2^{|r-s+1|} s_2.$$

Then, the $W_2^1(\mathbb{T})$ norms can be bounded explicitly, for all $j \geq 0$, and it follows that the operator norm of T can be bounded with respect to the constants s_1 and s_2 alone. $\square$

2.3 Ellipticity and Generation of Analytic Semigroups

Having established a setting within which we will look for solutions to the inhomogeneous Cauchy problem (2.1) in Section 2.1, we turn our attention back to the differential operator $\mathcal{A} = \mathcal{A}(\cdot, D)$. First, we define ellipticity conditions on a differential operator of order $2m$ and then we demonstrate our first result regarding generation of analytic semigroups on periodic little-Hölder spaces.

Denote by $D := i\frac{d}{dx}$ the elementary differential operator over $\mathbb{T}$ and let $m \in \mathbb{N}$ be an arbitrary positive integer. Now, fix a collection $\{b_k : k = 0, \ldots, 2m\} \subset h^\alpha(\mathbb{T})$ of coefficient functions and consider the differential operator $\mathcal{A}$, acting on $u \in h^{2m+\alpha}(\mathbb{T})$, defined by

$$\mathcal{A}u(x) := \mathcal{A}(x, D)u(x) := \sum_{k=0}^{2m} b_k(x)\,(D^k u)(x) = \sum_{k=0}^{2m} i^k\, b_k(x)\, u^{(k)}(x), \qquad x \in \mathbb{T}.$$

By the embedding property Proposition 2.1.2(a) and the fact that $h^\alpha(\mathbb{T})$ is a Banach algebra, it follows immediately that $\mathcal{A}$ maps $h^{2m+\alpha}(\mathbb{T})$ into $h^\alpha(\mathbb{T})$. Now, denote by $\sigma\mathcal{A} : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ the *principal symbol* of $\mathcal{A}$, defined by $\sigma\mathcal{A}(x,\xi) := b_{2m}(x)\xi^{2m}$. Then we say that $\mathcal{A}$ is a *uniformly elliptic* operator on $\mathbb{T}$ if there exists a constant $c_1 > 0$ such that

$$\operatorname{Re}\big(\sigma\mathcal{A}(x,\xi)\big) \geq c_1 \qquad \text{for all} \quad x \in \mathbb{T},\ |\xi| = 1. \tag{2.10}$$

In case b_{2m} is simply a $\mathbb{R}$-valued function, we see that uniform ellipticity is equivalent to the condition $b_{2m}(x) \geq c_1$ for all $x \in \mathbb{T}$. Meanwhile, when b_{2m} takes values in $\mathbb{C}\backslash\mathbb{R}$, uniform ellipticity is equivalent to the more general condition $b_{2m}(\mathbb{T}) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq c_1\}$. Also notice, by assumption we have b_{2m} continuous on $\mathbb{T}$, so that there always exists some constant $c_2 > 0$ for which $b_{2m}(\mathbb{T}) \subset \{z \in \mathbb{C} : |z| \leq c_2\}$. Indeed, we can always take $c_2 = \|b_{2m}\|_{C(\mathbb{T})}$.

Definition 2.3.1. Following the notation of Amann [2], given Banach spaces E_0 and E_1 with $E_1 \overset{d}{\hookrightarrow} E_0$, we denote by $\mathcal{H}(E_1, E_0)$ the collection of $A \in \mathcal{L}(E_1, E_0)$ such that $-A$ is the infinitesimal generator of an analytic semigroup on E_0, with domain $D(A) = E_1$. Moreover, given parameters $\kappa \geq 1$, $\omega > 0$, we denote by $\mathcal{H}(E_1, E_0, \kappa, \omega)$ the set of linear operators $A : E_1 \to E_0$, closed in E_0, such that $\omega + A \in \mathcal{L}_{isom}(E_1, E_0)$ and

$$\kappa^{-1} \leq \frac{\|(\lambda + A)x\|_0}{|\lambda|\|x\|_0 + \|x\|_1} \leq \kappa, \qquad x \in E_1 \setminus \{0\},\ \operatorname{Re}\lambda \geq \omega. \tag{2.11}$$

Then, it follows, c.f. [2, Theorem 1.2.2], that $\mathcal{H}(E_1, E_0) = \bigcup\limits_{\substack{\kappa \geq 1 \\ \omega > 0}} \mathcal{H}(E_1, E_0, \kappa, \omega)$.

Theorem 2.3.2. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$ and consider the differential operator $\mathcal{A}_b := b\, D^{2m}$ with constant coefficient $b \in \mathbb{C}$. If $\mathcal{A}_b$ is uniformly elliptic, with constant $c_1 > 0$, and $c_2 \geq c_1 > 0$ is chosen so that $|b| \leq c_2$, then $-\mathcal{A}_b$ generates a (strongly continuous) analytic semigroup on $h^\alpha(\mathbb{T})$ with domain $h^{2m+\alpha}(\mathbb{T})$. Moreover, for any $\omega > 0$, there exists $\kappa = \kappa(\omega, c_1, c_2, m)$ such that*

$$\mathcal{A}_b \in \mathcal{H}\big(h^{2m+\alpha}, h^\alpha, \kappa(\omega, c_1, c_2, m), \omega\big).$$

The method for proving this theorem is inspired by an argument presented by Escher and Matioc in [34], where they demonstrated that a particular third order operator, associated with Stokesian

Hele-Shaw flow, generates an analytic semigroup on periodic little-Hölder spaces. Before we present the proof, we need to state a result which helps establish a connection between little-Hölder spaces and Fourier multiplier results, stated earlier in the scale of Besov spaces. In particular, notice if we can apply Theorem 2.2.1 for the case $p = q = \infty$, then the identification in Proposition 2.1.3 gives results in $C^s(\mathbb{T})$ which we then need to connect with the little-Hölder spaces $h^s(\mathbb{T})$. The following Lemma provides this necessary connection.

Lemma 2.3.3. *Suppose $T \in \mathcal{L}(C^{k+\alpha}(\mathbb{T}), C^{l+\alpha}(\mathbb{T}))$ such that $T(C^{k+r}(\mathbb{T})) \subset C^{l+r}(\mathbb{T})$, for $k, l \in \mathbb{N}_0$, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $r > \alpha$. Then $T \in \mathcal{L}(h^{k+\alpha}(\mathbb{T}), h^{l+\alpha}(\mathbb{T}))$.*

Proof. This result is a straight forward consequence of the dense embedding $C^{l+r}(\mathbb{T}) \overset{d}{\hookrightarrow} h^{l+\alpha}(\mathbb{T})$, c.f. Proposition 2.1.2(a), we present the proof here for the readers convenience. First, notice that for $T \in \mathcal{L}(C^{k+\alpha}(\mathbb{T}), C^{l+\alpha}(\mathbb{T}))$, it follows that $T \in \mathcal{L}(h^{k+\alpha}(\mathbb{T}), C^{l+\alpha}(\mathbb{T}))$. Hence, it suffices to show that $T(h^{k+\alpha}(\mathbb{T})) \subset h^{l+\alpha}(\mathbb{T})$. Let $f \in h^{k+\alpha}(\mathbb{T})$ and we can find $(f_j)_j \subset C^{k+r}(\mathbb{T})$ such that $f_j \to f$ in $\|\cdot\|_{C^{k+\alpha}}$. Then $Tf_j \to Tf$ in $\|\cdot\|_{C^{l+\alpha}}$, by $T \in \mathcal{L}(C^{k+\alpha}(\mathbb{T}), C^{l+\alpha}(\mathbb{T}))$, and $Tf_j \in C^{l+r}(\mathbb{T})$ for $j \in \mathbb{N}$, by assumption. Therefore, we have $Tf \in \overline{C^{l+r}(\mathbb{T})}^{\|\cdot\|_{C^{l+\alpha}}} = h^{l+\alpha}(\mathbb{T})$ and the lemma is proved. $\square$

Proof of Theorem 2.3.2. Fix $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$, $\omega > 0$ and $b \in \mathbb{C}$ as indicated, in particular we assume that $b \in \Sigma(c_1, c_2) := \{z \in \mathbb{C} : \operatorname{Re} z \geq c_1\} \cap \{z \in \mathbb{C} : |z| \leq c_2\}$. First, we realize the operator $-\mathcal{A}_b$ as a Fourier multiplier. Since

$$\mathcal{A}_b\left(\sum_{k\in\mathbb{Z}} a_k e_k\right) = \sum_{k\in\mathbb{Z}} b(i)^{2m} a_k (ik)^{2m} e_k = \sum_{k\in\mathbb{Z}} b k^{2m} a_k e_k,$$

we see that $-\mathcal{A}_b$ is associated with the multiplier symbol $(M_k)_k := (-bk^{2m})_k$.

Claim 1: *$(\lambda + \mathcal{A}_b) \in \mathcal{L}_{isom}(h^{2m+\alpha}(\mathbb{T}), h^{\alpha}(\mathbb{T}))$ for $Re\,\lambda \geq \omega$, i.e. $\rho(-\mathcal{A}_b) \supset \{\lambda \in \mathbb{C} : Re\,\lambda \geq \omega\}$. Moreover, the set $\{\|(\lambda + \mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^\alpha, h^{2m+\alpha})} : Re\,\lambda \geq \omega\}$ is uniformly bounded by some $M_1 = M_1(\omega, c_1, c_2, m) < \infty$.*

First notice that $(\lambda + \mathcal{A}_b) \in \mathcal{L}(C^{2m+\sigma}(\mathbb{T}), C^{\sigma}(\mathbb{T}))$ is a natural consequence of the embedding $C^{2m+\sigma}(\mathbb{T}) \hookrightarrow C^{\sigma}(\mathbb{T})$, for arbitrary $\sigma \in \mathbb{R}_+$. In particular, we see that

$$\|(\lambda + \mathcal{A}_b)f\|_{C^\sigma} \leq |\lambda|\|f\|_{C^\sigma} + |b|\,\|f^{(2m)}\|_{C^\sigma} \leq (c(\sigma)\,|\lambda| + c_2)\|f\|_{C^{2m+\sigma}},$$

where $c(\sigma) > 0$ is the embedding constant, i.e. $\|f\|_{C^\sigma} \leq c(\sigma)\,\|f\|_{C^{2m+\sigma}}$ for all $f \in C^{2m+\sigma}(\mathbb{T})$. Now, we focus on showing continuous invertibility of the operator $(\lambda + \mathcal{A}_b)$. We will demonstrate invertibility in the classic Hölder spaces, then apply Lemma 2.3.3 to get the stated result.

We use Theorem 2.2.1 and the identification $B^\sigma_{\infty,\infty}(\mathbb{T}) = C^\sigma(\mathbb{T})$, for $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. In particular, let $\operatorname{Re}\lambda \geq \omega$ and consider the symbol $\left(\tilde{M}_k(\lambda)\right)_k := \left(\frac{1}{\lambda + bk^{2m}}\right)_k$, which we will show satisfies (2.8), with $r = 2m + \sigma$ and $s = \sigma$. Then $r - s = 2m$ and we have,

$$\begin{aligned} |k|^{2m}|\tilde{M}_k(\lambda)| = \frac{k^{2m}}{|\lambda + bk^{2m}|} \leq \frac{k^{2m}}{\operatorname{Re} b\, k^{2m}} \leq \frac{1}{\operatorname{Re} b} \quad \text{for } k \in \mathbb{Z}\setminus\{0\} \\ \implies \qquad s_1 := \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s}|\tilde{M}_k(\lambda)| \leq \frac{1}{c_1} < \infty, \end{aligned}$$

and

$$\begin{aligned} |k|^{2m+1}|\tilde{M}_{k+1}(\lambda) - \tilde{M}_k(\lambda)| &= |k|^{2m+1}\left|\frac{1}{\lambda + b(k+1)^{2m}} - \frac{1}{\lambda + bk^{2m}}\right| \\ &= \frac{|k|^{2m}}{|\lambda + b(k+1)^{2m}|}\,\frac{|k|^{2m}}{|\lambda + bk^{2m}|}\,\frac{|b||(k+1)^{2m} - k^{2m}|}{|k|^{2m-1}} \\ &\leq \frac{|k|^{2m}}{|\lambda + b(k+1)^{2m}|}\,\frac{|b|}{\operatorname{Re} b}\,\frac{|(k+1)^{2m} - k^{2m}|}{|k|^{2m-1}} \end{aligned}$$

If $k = -1$, then this last term is equal $1/|\lambda|$, which is majorized by $1/\omega$. For all other $k \in \mathbb{Z}\setminus\{0\}$, we eliminate dependence on λ, as in the bound for s_1, so that we have

$$\begin{aligned} s_2 &:= \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+1}\left|\tilde{M}_{k+1}(\lambda) - \tilde{M}_k(\lambda)\right| \\ &\leq \left(\frac{1}{\omega} \vee \frac{c_2}{(c_1)^2}\right) \sup_{k\in\mathbb{Z}\setminus\{-1\}} \left(\frac{|k|^{2m}}{|k+1|^{2m}} \sum_{j=0}^{2m-1} \binom{2m}{j} |k|^{j-2m+1}\right) < \infty. \end{aligned}$$

Hence, by Theorem 2.2.1 we have $R(\lambda) \in \mathcal{L}(B^r_{p,q}(\mathbb{T}), B^{r+2m}_{p,q}(\mathbb{T}))$ for any $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}_+$, where $R(\lambda)$ is the operator associated with the symbol $\left(\tilde{M}_k(\lambda)\right)_k$. Taking $p = q = \infty$ and $r = \sigma$, we see $R(\lambda) \in \mathcal{L}(C^\sigma(\mathbb{T}), C^{2m+\sigma}(\mathbb{T}))$. Meanwhile, it holds that

$$R(\lambda)(\lambda + \mathcal{A}_b)f = f \quad \text{and} \quad (\lambda + \mathcal{A}_b)R(\lambda)g = g, \quad \text{for} \quad f \in C^{2m+\sigma}(\mathbb{T}),\ g \in C^\sigma(\mathbb{T}),$$

which demonstrates that $R(\lambda) = (\lambda + \mathcal{A}_b)^{-1}$ and

$$(\lambda + \mathcal{A}_b) \in \mathcal{L}_{isom}(C^{2m+\sigma}(\mathbb{T}), C^{\sigma}(\mathbb{T})) \qquad \text{for} \quad \operatorname{Re}\lambda \geq \omega,\ \sigma \in \mathbb{R}_+ \setminus \mathbb{Z}. \tag{2.12}$$

Now, it is clear that Claim 1 follows from Lemma 2.3.3.

Meanwhile, for any $\operatorname{Re}\lambda \geq \omega$, notice that s_1, s_2 and so, by Theorem 2.2.1, the operator norm $\|(\lambda+\mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^\alpha, h^{2m+\alpha})}$ can be bounded by terms depending only on the constants ω, c_1, c_2 and m. In particular, there exists some $M_1 = M_1(\omega, c_1, c_2, m) < \infty$ such that $\|(\lambda+\mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^\alpha, h^{2m+\alpha})} \leq M_1$ for all $\operatorname{Re}\lambda \geq \omega$ and for all $b \in \Sigma\,(c_1, c_2)$.

Claim 2: *$\lambda(\lambda + \mathcal{A}_b)^{-1} \in \mathcal{L}(h^\alpha(\mathbb{T}))$ for $Re\,\lambda \geq \omega$. Moreover, there is an upper bound $M_2 = M_2(\omega, c_1, c_2, m) < \infty$ for the set $\big\{|\lambda|\|(\lambda + \mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^\alpha(\mathbb{T}))} : Re\,\lambda \geq \omega\big\}$.*

Fix $\operatorname{Re}\lambda \geq \omega$ and notice that the operator $\lambda(\lambda + \mathcal{A}_b)^{-1}$ has the associated multiplier symbol $\left(\frac{\lambda}{\lambda+bk^{2m}}\right)_k$. We established in Claim 1 that $(\lambda + \mathcal{A}_b)^{-1}$ is a well-defined operator mapping $h^\alpha(\mathbb{T})$ into $h^{2m+\alpha}(\mathbb{T})$. Now, by the embedding property, Theorem 2.1.2(a), we can also consider the mapping properties of $(\lambda + \mathcal{A}_b)^{-1}$ as an operator from $h^\alpha(\mathbb{T})$ into itself. Again, we make use of Lemma 2.3.3 and Theorem 2.2.1, where now we are taking $r = s = \sigma$ and $p = q = \infty$. Moreover, we show that s_1 and s_2 can be bounded independent of $\operatorname{Re}\lambda \geq \omega$.

Notice that we can find $\vartheta = \vartheta(c_1, c_2) \in (0, \frac{\pi}{2})$ such that $\Sigma\,(c_1, c_2) := \{z \in \mathbb{C} : \operatorname{Re} z \geq c_1\} \cap \{z \in \mathbb{C} : |z| \leq c_2\} \subset S_\vartheta := \{z \in \mathbb{C} : |\arg z| < \vartheta\}$. Moreover, there exists a constant $C(\vartheta)$ such that $|\lambda + z| \geq |\lambda|/C(\vartheta)$ for all $z \in S_\vartheta \cup \{0\}$, $\operatorname{Re}\lambda > 0$, since $\vartheta < \frac{\pi}{2}$. In particular, we have

$$s_1 = \sup_{k \in \mathbb{Z}\setminus\{0\}} \frac{|\lambda|}{|\lambda + b\,k^{2m}|} \leq C(\vartheta) \qquad \text{for all} \quad \operatorname{Re}\lambda \geq \omega.$$

Now, considering s_2, we have the bound

$$\begin{aligned} |k|\left|\frac{|\lambda|}{\lambda + b(k+1)^{2m}} - \frac{|\lambda|}{\lambda + bk^{2m}}\right| &= \frac{|\lambda|}{|\lambda + b(k+1)^{2m}|}\,\frac{k^{2m}}{|\lambda + bk^{2m}|}\,\frac{|b||(k+1)^{2m} - k^{2m}|}{|k|^{2m-1}} \\ &\leq C(\vartheta)\,\frac{k^{2m}}{\operatorname{Re} b\,k^{2m}}\,\frac{|b|((k+1)^{2m} - k^{2m})}{|k|^{2m-1}} \leq C(\vartheta)\,\frac{c_2}{c_1}\,\frac{(k+1)^{2m} - k^{2m}}{|k|^{2m-1}}, \end{aligned}$$

for $k \in \mathbb{Z} \setminus \{0\}$. Hence,

$$s_2 \leq \left(C(\vartheta)\, \frac{c_2}{c_1} \right) \sup_{k \in \mathbb{N}} \left(\sum_{j=0}^{2m-1} \binom{2m}{j} k^{j-2m+1} \right) < \infty,$$

again uniformly in $\operatorname{Re}\lambda \geq \omega$. Now we see that $\lambda(\lambda + \mathcal{A}_b)^{-1} \in \mathcal{L}(C^\sigma(\mathbb{T}), C^{2m+\sigma}(\mathbb{T}))$ holds by application of Theorem 2.2.1, for $\lambda \geq \omega$, $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. Hence, the claim holds by Lemma 2.3.3 and we fix a constant $M_2 = M_2(\omega, c_1, c_2, m) < \infty$ such that $\|(\lambda + \mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^\alpha)} \leq M_2/|\lambda|$ holds uniformly for $\operatorname{Re}\lambda \geq \omega$ and $b \in \Sigma\,(c_1, c_2)$.

By Claims 1 and 2, we see that $-\mathcal{A}_b$ satisfies the conditions necessary to generate an analytic semigroup, c.f. Amann [2, Theorem 1.2.2]. Moreover, if we choose

$$\kappa = \kappa(c_1, c_2, \omega) \geq 2\big(M_1 \vee M_2\big) \vee \big(1 \vee c_2\big),$$

it holds that

$$\kappa^{-1} \leq \frac{\|(\lambda + \mathcal{A}_b)f\|_{h^\alpha(\mathbb{T})}}{|\lambda|\|f\|_{h^\alpha(\mathbb{T})} + \|f\|_{h^{2m+\alpha}(\mathbb{T})}} \leq \kappa, \qquad f \in h^{2m+\alpha}(\mathbb{T}) \setminus \{0\},\ \operatorname{Re}\lambda \geq \omega.$$

Hence, we see that $\mathcal{A}_b \in \mathcal{H}(h^\alpha(\mathbb{T}), h^{2m+\alpha}(\mathbb{T}), \kappa, \omega)$, as claimed. □

2.4 Partition and Generation Result

Now that we have a generation result for the operator with constant coefficients, we can extend the result to variable coefficients through the following partition and perturbation argument. Here we consider the operator

$$\mathcal{A}_p := \mathcal{A}_p(\cdot, D) := b(\cdot) D^{2m}, \qquad \text{for} \quad b \in \mathbb{C}^{\mathbb{T}}, \tag{2.13}$$

and we assume that $\mathcal{A}_p$ satisfies the conditions of uniform ellipticity (2.10). We will show that, under minimal regularity assumptions on the coefficient function b, $\mathcal{A}_p$ generates an analytic semigroup on $h^\alpha(\mathbb{T})$ with domain $h^{2m+\alpha}(\mathbb{T})$.

For the following localization argument, we make use of the fact that $\mathbb{T}$ is isomorphic to the (additive) quotient group $\mathbb{R}/2\pi\mathbb{Z}$. In particular, for $x \in \mathbb{T}$, we consider the associated coset $[x] \in \mathbb{R}/2\pi\mathbb{Z}$, $[x] := \{x + 2\pi k : k \in \mathbb{Z}\}$. Note that the element $x \in \mathbb{T}$ is the unique member of the coset $[x]$ contained in the interval $[-\pi, \pi]$; except in the notable case $x = \{\pi, -\pi\}$, where the points π and $-\pi$ are both members of the coset $[\pi]$ and they are identified in $\mathbb{T}$. Moreover, for $x \in \mathbb{T}$ we see that the inverse element $-[x] \in \mathbb{R}/2\pi\mathbb{Z}$ corresponds to $-x \in \mathbb{T}$. Then, for $z \in \mathbb{T}$, define the translation operator $T_z(y) := y - z$, where $y - z \in \mathbb{T}$ is the unique element in $\mathbb{T}$ associated with the coset $[y - z] \in \mathbb{R}/2\pi\mathbb{Z}$. Note that the metric $d_{\mathbb{T}}$ is invariant under translations on $\mathbb{T}$, i.e. $d_{\mathbb{T}}(T_z(x), T_z(y)) = d_{\mathbb{T}}(x, y)$ for any $x, y, z \in \mathbb{T}$.

2.4.1 Localized Coefficient

We begin by localizing the function b to open sets of the form $\mathbb{B}_{\mathbb{T}}(z, \varepsilon)$, for $z \in \mathbb{T}$ and $\varepsilon \in (0, 1/2)$. We define cut-off functions and 'local retractions' which work together to accomplish this goal. For the cut-off functions, choose $X \in C^1(\mathbb{T})$ such that

$$\operatorname{supp} X \subset (-1, 1) \quad \text{and} \quad X|_{[-\frac{1}{2}, \frac{1}{2}]} \equiv 1.$$

Then, define $X_z := X \circ T_z$ (the cut-off function centered at $z \in \mathbb{T}$) and notice that $X_z \in C^1(\mathbb{T})$ with $\operatorname{supp}(X_z) \subset \mathbb{B}_{\mathbb{T}}(z, 1)$ for every $z \in \mathbb{T}$.

For our 'local retractions' we define $r_\varepsilon : [-1, 1] \to [-\varepsilon, \varepsilon]$, for $\varepsilon \in (0, 1/2)$, as

$$r_\varepsilon(x) := \begin{cases} x & \text{if } x \in [-\varepsilon, \varepsilon], \\ \varepsilon & \text{if } x \in (\varepsilon, 1], \\ -\varepsilon & \text{if } x \in [-1, -\varepsilon). \end{cases} \tag{2.14}$$

Then, for $z \in \mathbb{T}$ arbitrary, we define $r_{z,\varepsilon} := T_{-z} \circ r_\varepsilon \circ T_z$, the local retraction centered at z, which maps the closed neighborhood $\overline{\mathbb{B}}_{\mathbb{T}}(z, 1)$ to $\overline{\mathbb{B}}_{\mathbb{T}}(z, \varepsilon)$.

Proposition 2.4.1. *For $\varepsilon \in (0, 1/2)$, r_ε is Lipschitz continuous from $[-1, 1]$ to $[-\varepsilon, \varepsilon]$, with Lipschitz constant 1. Consequently, $r_{z,\varepsilon}$ is Lipschitz continuous (with respect to the metric $d_{\mathbb{T}}$) from $\overline{\mathbb{B}}_{\mathbb{T}}(z, 1)$ to $\overline{\mathbb{B}}_{\mathbb{T}}(z, \varepsilon)$ for all $z \in \mathbb{T}$, $\varepsilon \in (0, 1/2)$.*

Proof. By considering cases for points $x, y \in \mathbb{T}$, the first claim is easily verified. Furthermore, notice that $d_{\mathbb{T}}(x,y) = |x-y|$ for $x, y \in [-1,1]$, so that r_ε is Lipschitz in the metric $d_{\mathbb{T}}$ on $[-1,1]$. Then the second claim follows from invariance of the metric $d_{\mathbb{T}}$ under translations T_z. $\square$

Now, given a function $b \in \mathbb{C}^{\mathbb{T}}$, we combine these 'local retractions' and cut-off functions to define the parameter-dependent functions

$$b_{z,\varepsilon}(x) := \begin{cases} X_z(x)\,[b \circ r_{z,\varepsilon}(x) - b(z)] & \text{if } x \in \mathbb{B}_{\mathbb{T}}(z,1), \\ 0 & \text{otherwise,} \end{cases} \qquad z \in \mathbb{T},\ \varepsilon \in (0,1/2),$$

which essentially compare the local behavior of b against a fixed value $b(z)$. Before we make use of these 'localized coefficients', we establish the following results regarding their regularity.

Lemma 2.4.2. *Let $b \in h^\alpha(\mathbb{T})$ for $\alpha \in (0,1)$. Then the following results hold:*

a) $b_{z,\varepsilon} \in h^\alpha(\mathbb{T})$ *for* $\varepsilon \in (0,1/2),\ z \in \mathbb{T}$,

b) $\lim\limits_{\varepsilon \to 0^+} \sup\limits_{z \in \mathbb{T}} \|b_{z,\varepsilon}\|_{h^\alpha} = 0.$

Proof. First notice, since $b \in h^\alpha(\mathbb{T})$, it follows from the intrinsic characterization of little-Hölder spaces (2.5) that for $\varepsilon \in (0,1/2)$, there exists $C(\varepsilon) > 0$ such that

$$\sup_{z \in \mathbb{T}} [b\,]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)} = C(\varepsilon) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0^+. \tag{2.15}$$

Now, let $z \in \mathbb{T}$ be a fixed sample point and $\varepsilon \in (0,1/2)$. To see that $b_{z,\varepsilon}$ has the necessary regularity, we make use of Proposition 2.1.1(c) and Proposition 2.4.1. In particular, let $x, y \in \mathbb{T}$ and consider the following cases:

- $x, y \in \mathbb{B}_{\mathbb{T}}(z,1)$: Then $r_{z,\varepsilon}(x), r_{z,\varepsilon}(y) \in \overline{\mathbb{B}}_{\mathbb{T}}(z,\varepsilon)$ and

$$\begin{aligned} |b_{z,\varepsilon}(x) - b_{z,\varepsilon}(y)| &= \big|X_z(x)\big(b(r_{z,\varepsilon}(x)) - b(z)\big) - X_z(y)\big(b(r_{z,\varepsilon}(y)) - b(z)\big)\big| \\ &\le |X_z(x)||b(r_{z,\varepsilon}(x)) - b(r_{z,\varepsilon}(y))| + |X_z(x) - X_z(y)||b(r_{z,\varepsilon}(y)) - b(z)| \\ &\le \Big(\|X_z\|_{C(\mathbb{T})} d^\alpha_{\mathbb{T}}(r_{z,\varepsilon}(x), r_{z,\varepsilon}(y)) + \|X'_z\|_{C(\mathbb{T})}\, d_{\mathbb{T}}(x,y)\, d^\alpha_{\mathbb{T}}(r_{z,\varepsilon}(y), z)\Big) [b\,]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)} \\ &\le \Big(\|X_z\|_{C(\mathbb{T})} + \|X'_z\|_{C(\mathbb{T})}\, d^{1-\alpha}_{\mathbb{T}}(x,y)\, \varepsilon^\alpha\Big) [b\,]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)} d^\alpha_{\mathbb{T}}(x,y) \end{aligned} \tag{2.16}$$

- $x \in \mathbb{B}_{\mathbb{T}}(z,1)$, $y \in \mathbb{T} \setminus \mathbb{B}_{\mathbb{T}}(z,1)$: Then $X_z(y) = 0$ and

$$\begin{aligned}
|b_{z,\varepsilon}(x) - b_{z,\varepsilon}(y)| = |b_{z,\varepsilon}(x)| &= |X_z(x)\big(b(r_{z,\varepsilon}(x)) - b(z)\big)| \\
&\leq |X_z(x) - X_z(y)|\ d_{\mathbb{T}}^{\alpha}(r_{z,\varepsilon}(x), z)\ [b\,]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)} \\
&\leq \|X_z'\|_{C(\mathbb{T})}\ d_{\mathbb{T}}(x,y)\ \varepsilon^{\alpha}\ [b\,]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)} \\
&\leq \Big(\|X_z'\|_{C(\mathbb{T})}\ d_{\mathbb{T}}^{1-\alpha}(x,y)\ \varepsilon^{\alpha}\Big)\ [b\,]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)}\ d_{\mathbb{T}}^{\alpha}(x,y) \qquad (2.17)
\end{aligned}$$

Together with the trivial case $x, y \in \mathbb{T} \setminus \mathbb{B}_{\mathbb{T}}(z,1)$ – where $X_z(x) = X_z(y) = 0$ – this is enough to see that $b_{z,\varepsilon} \in C^{\alpha}(\mathbb{T})$ with the α-Hölder norm of $b_{z,\varepsilon}$ bounded as

$$[b_{z,\varepsilon}]_{\alpha,\mathbb{T}} \leq \big(\|X_z\|_{C(\mathbb{T})} + \varepsilon^{\alpha}\ \pi^{1-\alpha}\ \|X_z'\|_{C(\mathbb{T})}\big)\ [b\,]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)}.$$

Furthermore, we can see that $\|b_{z,\varepsilon}\|_{C(\mathbb{T})} \leq \varepsilon^{\alpha}\ \|X_z\|_{C(\mathbb{T})}\ [b\,]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)}$ so that the C^{α}-norm of $b_{z,\varepsilon}$ is bounded as

$$\|b_{z,\varepsilon}\|_{C^{\alpha}} \leq \big(\,(1+\varepsilon^{\alpha})\,\|X_z\|_{C(\mathbb{T})} + \varepsilon^{\alpha}\,\pi^{1-\alpha}\,\|X_z'\|_{C(\mathbb{T})}\big)[b]_{\alpha,\overline{\mathbb{B}}(z,\varepsilon)}. \qquad (2.18)$$

Hence, by the property (2.15) and the inequalities (2.16) and (2.17), we see that

$$\lim_{\delta\to 0^+} \sup_{\substack{x,y\in\mathbb{T} \\ 0<d_{\mathbb{T}}(x,y)<\delta}} \frac{|b_{z,\varepsilon}(x) - b_{z,\varepsilon}(y)|}{d_{\mathbb{T}}^{\alpha}(x,y)} = 0,$$

which demonstrates $b_{z,\varepsilon} \in h^{\alpha}(\mathbb{T})$ as claimed in (a). Now the second claim follows from (2.15) and (2.18).

□

2.4.2 Partition and Generation Result

For $\varepsilon \in (0, 1/2)$, let $n(\varepsilon) := \lceil \frac{2\pi}{\varepsilon} \rceil$, where $\lceil a \rceil$ denotes the smallest integer n such that $n \geq a$, $a \in \mathbb{R}$. Now, let $\{x_{\varepsilon,j} : j = 1, \ldots, n(\varepsilon)\} \subset \mathbb{T}$ be a collection of sample points from $\mathbb{T}$ so that $x_{\varepsilon,1} = -\pi$ and $x_{\varepsilon,j} = x_{\varepsilon,(j-1)} + \varepsilon$, $j = 2, \ldots, n(\varepsilon)$. Further, define $\Omega_{\varepsilon} := \{\mathbb{B}_{\mathbb{T}}(x_{\varepsilon,j}, \varepsilon) : j = 1, \ldots, n(\varepsilon)\}$, which is a finite open cover for $\mathbb{T}$, and let $\Pi_{\varepsilon} := \{\pi_{\varepsilon,j}^2\} \subset C^{\infty}(\mathbb{T})$ be a resolution of unity subordinate to Ω_{ε}.

In particular, Π_ε is a collection of infinitely differentiable functions such that

$$\mathrm{supp}(\pi_{\varepsilon,j}) \subset \mathbb{B}_{\mathbb{T}}(x_{\varepsilon,j},\varepsilon),\ j=1,\ldots,n(\varepsilon), \qquad \text{and} \qquad \sum_{j=1}^{n(\varepsilon)} \pi_{\varepsilon,j}^2(x) = 1,\ x \in \mathbb{T}.$$

Now we are prepared to prove the following result, which is a generalization of Theorem 2.3.2 to the case of non-constant coefficients. The method of the proof is motivated by results in [4, 6].

Lemma 2.4.3. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$ and consider the differential operator $\mathcal{A}_p := \mathcal{A}_p(\cdot, D) := b(\cdot)\, D^{2m}$ with coefficient $b \in h^\alpha(\mathbb{T})$. If $\mathcal{A}_p$ is uniformly elliptic, then $-\mathcal{A}_p$ generates a (strongly continuous) analytic semigroup on $h^\alpha(\mathbb{T})$ with domain $h^{2m+\alpha}(\mathbb{T})$. i.e. $\mathcal{A}_p \in \mathcal{H}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}))$.*

Proof. Fix $\omega > 0$ and $b \in h^\alpha(\mathbb{T})$. By assumption, there exist constants c_1 and c_2, with $c_2 \geq c_1 > 0$ such that $b(\mathbb{T}) \subset \{z \in \mathbb{C} : \mathrm{Re}\, z \geq c_1\} \cap \{z \in \mathbb{C} : |z| \leq c_2\}$.

(i) First we demonstrate that it suffices to prove the result for $\alpha \in (0,1)$. Suppose that the claim holds for $\alpha \in (0,1)$ and let $\beta := \alpha + 1$. In particular, we assume $b \in h^\beta(\mathbb{T})$ and $\mathcal{A}_p = b(\cdot)D^{2m} \in \mathcal{H}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}))$. It follows that $(\lambda + \mathcal{A}_p) : h^{2m+\alpha}(\mathbb{T}) \to h^\alpha(\mathbb{T})$ is invertible for $\mathrm{Re}\,\lambda \geq \omega$ and we have the resolvent estimates

$$\|(\lambda + \mathcal{A}_p)^{-1}\|_{\mathcal{L}(h^\alpha, h^{2m+\alpha})} \leq \kappa, \qquad |\lambda| \|(\lambda + \mathcal{A}_p)^{-1}\|_{\mathcal{L}(h^\alpha)} \leq \kappa, \tag{2.19}$$

for $\mathrm{Re}\,\lambda \geq \omega$, for some $\omega > 0$ and $\kappa \geq 1$. Now, fix $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \omega$ and consider $f \in h^\beta(\mathbb{T})$. Then $f \in h^\alpha(\mathbb{T})$, by Proposition 2.1.2(a), and we define $u := (\lambda + \mathcal{A}_p)^{-1} f \in h^{2m+\alpha}(\mathbb{T})$. Then u satisfies the equation $(\lambda + \mathcal{A}_p)u = f$ and, differentiating this equation, we see that

$$(\lambda + \mathcal{A}_p)u' = f' - b'\, u^{(2m)},$$

where, a priori, we know that $u' \in h^{2m-1+\alpha}(\mathbb{T})$. However, notice that
$f', b', u^{(2m)} \in h^\alpha(\mathbb{T})$ and $b'\, u^{(2m)} \in h^\alpha(\mathbb{T})$, since $h^\alpha(\mathbb{T})$ is a Banach algebra, so that

$u' = (\lambda + \mathcal{A}_p)^{-1}\big(f' - b'\, u^{(2m)}\big) \in h^{2m+\alpha}(\mathbb{T})$. Hence, we see that $u \in h^{2m+\beta}(\mathbb{T})$ and

$$\begin{aligned}
\|u\|_{h^{2m+\beta}(\mathbb{T})} &= \sum_{k=0}^{2m+1} \|u^{(k)}\|_{C(\mathbb{T})} + [u^{(2m+1)}\,]_{\alpha,\mathbb{T}} = \|u'\|_{h^{2m+\alpha}(\mathbb{T})} + \|u\|_{C(\mathbb{T})} \\
&\le \kappa\big(\|f' - b'\, u^{(2m)}\|_{h^\alpha(\mathbb{T})} + \|u\|_{C(\mathbb{T})}\big) \\
&\le \kappa\big(\|f'\|_{h^\alpha(\mathbb{T})} + (1 \vee \|b'\|_{h^\alpha(\mathbb{T})})\|u\|_{h^{2m+\alpha}(\mathbb{T})}\big) \\
&\le \kappa\big(\|f'\|_{h^\alpha(\mathbb{T})} + \kappa(1 \vee \|b'\|_{h^\alpha(\mathbb{T})})\|f\|_{h^\alpha(\mathbb{T})}\big) \\
&\le K(\kappa, b)\big(\|f'\|_{C(\mathbb{T})} + [f'\,]_{\alpha,\mathbb{T}} + \|f\|_{C(\mathbb{T})} + [f\,]_{\alpha,\mathbb{T}}\big) \\
&\le K(\kappa, b)\big((1 + \pi^{1-\alpha})\|f'\|_{C(\mathbb{T})} + \|f\|_{C(\mathbb{T})} + [f'\,]_{\alpha,\mathbb{T}}\big) \le \tilde{K}\|f\|_{h^\beta(\mathbb{T})}.
\end{aligned}$$

Then $\|(\lambda + \mathcal{A}_p)^{-1}\|_{\mathcal{L}(h^\beta, h^{2m+\beta})} \le \tilde{K}$ for $\operatorname{Re}\lambda \ge \omega$. Meanwhile, in a similar fashion, we see that

$$|\lambda|\|u\|_{h^\beta(\mathbb{T})} = |\lambda|\big(\|u\|_{C(\mathbb{T})} + \|u'\|_{C(\mathbb{T})} + [u'\,]_{\alpha,\mathbb{T}}\big) \le \tilde{K}\|f\|_{h^\beta(\mathbb{T})},$$

holds for $\operatorname{Re}\lambda \ge \omega$. Hence, it follows that $|\lambda|\|(\lambda + \mathcal{A}_p)^{-1}\|_{\mathcal{L}(h^\beta)} \le \tilde{K}$ for $\operatorname{Re}\lambda \ge \omega$ and so the claim holds for $\beta = \alpha + 1$. Then, we extend the result to any $\beta > 1$, $\beta \notin \mathbb{Z}$, by induction on α.

(ii) Now we demonstrate the claim for $\alpha \in (0, 1)$. By uniform ellipticity of $\mathcal{A}_p$, it follows from Theorem 2.3.2 that there exists some constant $\kappa = \kappa(\omega, c_1, c_2) \ge 1$ such that

$$\mathcal{A}_p(x_0) := b(x_0)D^{2m} \in \mathcal{H}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}), \kappa, \omega)$$

for any fixed $x_0 \in \mathbb{T}$. Fix η so that $0 < \eta < 1/\kappa$. By Lemma 2.4.2(b), there exists $\varepsilon_0 > 0$ with associated sampling set $\{x_j\} := \{x_{\varepsilon_0,j}\}$ and partition $\Omega := \Omega_{\varepsilon_0} = \{\mathbb{B}_{\mathbb{T}}(x_j, \varepsilon_0)\}$, $j = 1, \ldots, n := n(\varepsilon_0)$, such that

$$\sup_{j=1,\ldots,n} \|b_j\|_{h^\alpha} < \eta, \qquad \text{where} \quad b_j := b_{x_j,\varepsilon_0}\,. \tag{2.20}$$

Moreover, by Lemma 2.4.2(a) and the fact that $h^\alpha(\mathbb{T})$ is a Banach Algebra, the operator $b_j(\cdot)D^{2m}$ is in $\mathcal{L}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}))$ with $\|b_j(\cdot)D^{2m}\|_{\mathcal{L}(h^{2m+\alpha}, h^\alpha)} \le \|b_j\|_{h^\alpha}$, for $j = 1, \ldots, n$. Hence, by [2, Theorem 1.3.1(i)] and (2.20) we can see that perturbations of $\mathcal{A}_p(x_j)$ remain in the class $\mathcal{H}$, namely

$$\mathcal{A}_j := [b(x_j) + b_j(\cdot)]\, D^{2m} \in \mathcal{H}\left(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}), \frac{\kappa}{1 - \kappa\eta}, \omega\right), \qquad j = 1, \ldots, n.$$

In particular, this implies that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \omega\} \subset \rho(-\mathcal{A}_j)$ and the resolvent estimates

$$|\lambda| \|(\lambda + \mathcal{A}_j)^{-1}\|_{\mathcal{L}(h^\alpha)} \leq \frac{\kappa}{1-\kappa\eta} \quad \text{and} \quad \|(\lambda + \mathcal{A}_j)^{-1}\|_{\mathcal{L}(h^\alpha, h^{2m+\alpha})} \leq \frac{\kappa}{1-\kappa\eta}, \tag{2.21}$$

hold uniformly for $\operatorname{Re}\lambda \geq \omega$ and $j = 1, \dots, n$.

Let $\Pi := \Pi_{\varepsilon_0} = \{\pi_j^2\}$ be a resolution of unity subordinate to Ω, where we also insist that $\|\pi_j\|_{h^\alpha}, \|\pi_j\|_{h^{2m+\alpha}} \leq M$ uniformly in j, for some $M = M(\varepsilon_0) \geq 1$. Now define the composite little-Hölder spaces

$$(h^\sigma(\mathbb{T}))^n := \{(f_j)_{j\in\mathbb{N}} \in \ell^\infty(h^\sigma(\mathbb{T})) : f_j = 0 \text{ for } j \geq n+1\}, \qquad \sigma \in \mathbb{R}_+ \setminus \mathbb{Z}.$$

Then, it is easy to see that $(h^\sigma(\mathbb{T}))^n$ is a Banach space, with the norm topology inherited from $\ell^\infty(h^\sigma(\mathbb{T}))$. Moreover, we have the following retraction and coretraction

$$R : (h^\sigma(\mathbb{T}))^n \to h^\sigma(\mathbb{T}) \qquad \text{where } R((f_j)_j) := \sum_{j=1}^n \pi_j f_j \tag{2.22}$$

$$R^C : h^\sigma(\mathbb{T}) \to (h^\sigma(\mathbb{T}))^n \qquad \text{where } R^C(u) := (\pi_j u)_j\,. \tag{2.23}$$

With finiteness of the partition Ω and the properties of the resolution of unity Π, we easily see that $R \in \mathcal{L}((h^\sigma(\mathbb{T}))^n, h^\sigma(\mathbb{T}))$ and $R^C \in \mathcal{L}(h^\sigma(\mathbb{T}), (h^\sigma(\mathbb{T}))^n)$ with $R \circ R^C = id_{h^\sigma(\mathbb{T})}$ and

$$\|R\|_{\mathcal{L}((h^\sigma(\mathbb{T}))^n, h^\sigma(\mathbb{T}))} \leq nM, \qquad \|R^C\|_{\mathcal{L}(h^\sigma(\mathbb{T}), (h^\sigma(\mathbb{T}))^n)} \leq M, \qquad \sigma \in \{\alpha, 2m+\alpha\}. \tag{2.24}$$

We will make use of R and R^C together with the spaces $(h^\sigma(\mathbb{T}))^n$ to construct a left and right inverse for $(\lambda + \mathcal{A}_p)$, for $\operatorname{Re}\lambda \geq \omega_0 \geq \omega$ sufficiently large. Toward this goal, we define the following operators:

- $\Lambda : \left(h^{2m+\alpha}(\mathbb{T})\right)^n \to (h^\alpha(\mathbb{T}))^n$ defined by $\Lambda(f_j)_j := (\mathcal{A}_j f_j)_j$. Then $\Lambda \in \mathcal{L}\left(\left(h^{2m+\alpha}(\mathbb{T})\right)^n, (h^\alpha(\mathbb{T}))^n\right)$ with $\|\Lambda\| \leq \sup\limits_{j=1,\dots,n} \|\mathcal{A}_j\|_{\mathcal{L}(h^{2m+\alpha}, h^\alpha)}$.
- $B_j := \pi_j \mathcal{A}_j - \mathcal{A}_j \pi_j = [\pi_j, \mathcal{A}_p]$ the commutator of π_j and $\mathcal{A}_p$, $j = 1, \dots, n$. The second expression for B_j follows from the fact that $\operatorname{supp}(\pi_j) \subset \mathbb{B}_\mathbb{T}(x_j, \varepsilon_0)$ and $b_j(x) = b(x) - b(x_j)$ for $x \in \mathbb{B}_\mathbb{T}(x_j, \varepsilon_0)$, so that $\mathcal{A}_p$ and $\mathcal{A}_j$ coincide on $\operatorname{supp}(\pi_j)$. Moreover, the highest order

terms are eliminated in B_j so that we have $B_j \in \mathcal{L}\left(h^{(2m-1)+\alpha}(\mathbb{T}), h^{\alpha}(\mathbb{T})\right)$ with $\|B_j\| \leq C(m)\|\pi_j\|_{h^{2m+\alpha}} \leq C(m)M$.

- $\mathcal{B} : h^{(2m-1)+\alpha}(\mathbb{T}) \to (h^{\alpha}(\mathbb{T}))^n$ defined by $\mathcal{B}f := (B_j f)_j$. Then $\mathcal{B} \in \mathcal{L}\left(h^{(2m-1)+\alpha}(\mathbb{T}), (h^{\alpha}(\mathbb{T}))^n\right)$ with $\|\mathcal{B}\| \leq \sup_{j=1,\dots,n} \|B_j\|_{\mathcal{L}\left(h^{(2m-1)+\alpha}, h^{\alpha}\right)} \leq C(m)M$.

- $\mathcal{D} : (h^{(2m-1)+\alpha}(\mathbb{T}))^n \to h^{\alpha}(\mathbb{T})$ defined by $\mathcal{D}(f_j)_j := \sum_{j=1}^{n} B_j f_j$. Then $\mathcal{D} \in \mathcal{L}((h^{(2m-1)+\alpha}(\mathbb{T}))^n, h^{\alpha}(\mathbb{T}))$ with $\|\mathcal{D}\| \leq n\, C(m)M$.

- $C_{j,k}(\lambda) := B_j \circ \pi_k \circ (\lambda + \mathcal{A}_k)^{-1}$, $j, k = 1, \dots, n$, $\operatorname{Re}\lambda \geq \omega$. We easily see that $C_{j,k} \in \mathcal{L}(h^{\alpha}(\mathbb{T}))$. Moreover, since B_j maps $h^{(2m-1)+\alpha}(\mathbb{T})$ to $h^{\alpha}(\mathbb{T})$ we can consider the mapping $(\lambda + \mathcal{A}_k)^{-1}$ from $h^{\alpha}(\mathbb{T})$ to $h^{(2m-1)+\alpha}(\mathbb{T})$. In this way, we take advantage of the interpolation result for little-Hölder spaces, Proposition 2.1.2(b), in conjunction with the resolvent estimates (2.21) on $\mathcal{A}_k$, to see that

$$\begin{aligned}
\|(\lambda + \mathcal{A}_k)^{-1}\|_{\mathcal{L}(h^{\alpha}, h^{(2m-1)+\alpha})} &\leq \|(\lambda + \mathcal{A}_k)^{-1}\|^{1-1/2m}_{\mathcal{L}(h^{\alpha}, h^{2m+\alpha})} \|(\lambda + \mathcal{A}_k)^{-1}\|^{1/2m}_{\mathcal{L}(h^{\alpha})} \\
&\leq \left(\frac{\kappa}{1-\kappa\eta}\right)^{1-1/2m} \left(\frac{\kappa}{1-\kappa\eta}\right)^{1/2m} |\lambda|^{-1/2m} \\
&\leq \left(\frac{\kappa}{1-\kappa\eta}\right) |\lambda|^{-1/2m} = \tilde{c}|\lambda|^{-1/2m}.
\end{aligned} \tag{2.25}$$

Here, we take advantage of the fact that the continuous interpolation method used in Propostion 2.1.2(b) is exact. Hence, the $C_{j,k}$ operator norms are bounded as

$$\|C_{j,k}\|_{\mathcal{L}(h^{\alpha}(\mathbb{T}))} \leq \tilde{c}\, C(m) M^2 |\lambda|^{-1/2m} \qquad j, k = 1, \dots, n,\ \operatorname{Re}\lambda \geq \omega. \tag{2.26}$$

- $\mathcal{C}(\lambda) : (h^{\alpha}(\mathbb{T}))^n \to (h^{\alpha}(\mathbb{T}))^n$ defined $\mathcal{C}(f_j)_j := \left(B_j \sum_{k=1}^{n} \pi_k (\lambda + \mathcal{A}_k)^{-1} f_k\right)_j$, for $\operatorname{Re}\lambda \geq \omega$. Notice that $\operatorname{supp}\pi_k \subset \mathbb{B}_{\mathbb{T}}(x_k, \varepsilon_0)$ and $\operatorname{supp} B_j \subset \mathbb{B}_{\mathbb{T}}(x_j, \varepsilon_0)$ for $j, k = 1, \dots, n$, so $C_{j,k}(\lambda) = 0$ for $1 < |j-k| < n-1$. Hence, by (2.26), we can choose $\omega_1 > 0$ large enough to ensure that $\|\mathcal{C}(\lambda)\|_{\mathcal{L}((h^{\alpha}(\mathbb{T}))^n)} \leq 1/2$ for $\operatorname{Re}\lambda \geq \omega_1$.

Claim 1: *For $Re\,\lambda \geq \omega_1$, $(\lambda + \Lambda + \mathcal{B}R) : \left(h^{2m+\alpha}(\mathbb{T})\right)^n \to (h^{\alpha}(\mathbb{T}))^n$ is invertible and $L(\lambda) := R\,(\lambda + \Lambda + \mathcal{B}R)^{-1} R^C$ is a left inverse for $(\lambda + \mathcal{A}_p)$.*

From the definition and discussion of $\mathcal{C}(\lambda)$ above, we can choose $\omega_1 \geq \omega$ large enough so that $\|\mathcal{C}(\lambda)\|_{\mathcal{L}((h^\alpha(\mathbb{T}))^n)} \leq 1/2$ for $\operatorname{Re}\lambda \geq \omega_1$. Hence, by the Neumann series, we see that $\big(id_{(h^\alpha(\mathbb{T}))^n} + \mathcal{C}(\lambda)\big)$ is invertible on $(h^\alpha(\mathbb{T}))^n$ for $\operatorname{Re}\lambda \geq \omega_1$ and $\|(id_{(h^\alpha(\mathbb{T}))^n} + \mathcal{C}(\lambda))^{-1}\|_{\mathcal{L}((h^\alpha(\mathbb{T}))^n)} \leq 2$. For any $(f_j)_j \in \big(h^{2m+\alpha}(\mathbb{T})\big)^n$ and $\operatorname{Re}\lambda \geq \omega_1$, we have

$$\begin{aligned}
\mathcal{B}R(f_j)_j &= \mathcal{B}\Big(\sum_{k=1}^n \pi_k f_k\Big) = \Big(B_j \sum_{k=1}^n \pi_k f_k\Big)_j \\
&= \Big(B_j \sum_{k=1}^n \pi_k (\lambda + \mathcal{A}_k)^{-1}(\lambda + \mathcal{A}_k) f_k\Big)_j = \mathcal{C}(\lambda)\Big((\lambda + \mathcal{A}_j) f_j\Big)_j \\
\implies (\lambda + \Lambda + \mathcal{B}R)(f_j)_j &= (\lambda + \Lambda)(f_j)_j + \mathcal{C}(\lambda)\Big((\lambda + \mathcal{A}_j) f_j\Big)_j \\
&= \Big(id_{(h^\alpha(\mathbb{T}))^n} + \mathcal{C}(\lambda)\Big)(\lambda + \Lambda)(f_j)_j.
\end{aligned}$$

Hence, invertibility of $(\lambda + \Lambda + \mathcal{B}R)$ follows from invertibility of $(id_{(h^\alpha(\mathbb{T}))^n} + \mathcal{C}(\lambda))$ and invertibility of $(\lambda + \Lambda)$, both of which hold if $\operatorname{Re}\lambda \geq \omega_1 \geq \omega$. Furthermore, we see that $(\lambda + \Lambda + \mathcal{B}R)^{-1} = (\lambda + \Lambda)^{-1}(id_{(h^\alpha(\mathbb{T}))^n} + \mathcal{C}(\lambda))^{-1}$, $\operatorname{Re}\lambda \geq \omega_1$.

Now, we apply (2.21) to see that

$$\|(\lambda + \Lambda)^{-1}\|_{\mathcal{L}((h^\alpha(\mathbb{T}))^n)} \leq \left(\frac{\kappa}{1 - \kappa\eta}\right) |\lambda|^{-1},$$

and so, with (2.24), we get the bound

$$\|L(\lambda)\|_{\mathcal{L}(h^\alpha)} = \|R(\lambda + \Lambda + \mathcal{B}R)^{-1} R^C\| \leq \left(\frac{2\kappa}{1 - \kappa\eta}\right) n M^2 |\lambda|^{-1}. \tag{2.27}$$

Finally, to see that $L(\lambda)$ is indeed a left inverse for $(\lambda + \mathcal{A}_p)$, when $\operatorname{Re}\lambda \geq \omega_1$. Let $u \in h^{2m+\alpha}(\mathbb{T})$ and $\pi_j \in \Pi$, then we see that

$$\pi_j(\lambda + \mathcal{A}_p)u = (\lambda + \mathcal{A}_j)\pi_j u + B_j u = (\lambda + \mathcal{A}_j)\pi_j u + B_j R R^C u.$$

Hence, it follows, by exhibiting all components for $j = 1, \dots, n$, that

$$R^C(\lambda + \mathcal{A}_p)u = (\lambda + \Lambda + \mathcal{B}R)R^C u \qquad \text{and so} \qquad L(\lambda)(\lambda + \mathcal{A}_p) = id_{h^{2m+\alpha}(\mathbb{T})}.$$

Claim 2: $\left(\lambda+\Lambda-R^C\mathcal{D}\right) : \left(h^{2m+\alpha}(\mathbb{T})\right)^n \to (h^\alpha(\mathbb{T}))^n$ *is invertible for* $Re\,\lambda \geq \omega_2 \geq \omega$ *with* ω_2 *sufficiently large. Moreover,* $R(\lambda) := R\left(\lambda+\Lambda-R^C\mathcal{D}\right)^{-1}R^C$ *is a right inverse for* $(\lambda+\mathcal{A}_p)$ *and* $L(\lambda) = R(\lambda) = (\lambda+\mathcal{A}_p)^{-1}$.

We make use of the same observations that led to the invertibility of $(\lambda+\Lambda+\mathcal{B}R)$ in the previous claim. Notice that

$$(\lambda+\Lambda-R^C\mathcal{D}) = (id_{(h^\alpha(\mathbb{T}))^n} - R^C\mathcal{D}(\lambda+\Lambda)^{-1})(\lambda+\Lambda), \qquad \operatorname{Re}\lambda \geq \omega,$$

so it suffices to show that $(id_{(h^\alpha(\mathbb{T}))^n} + R^C\mathcal{D}(\lambda+\Lambda)^{-1})$ is invertible in $\mathcal{L}((h^\alpha(\mathbb{T}))^n)$, for $\operatorname{Re}\lambda$ sufficiently large. However, this follows by the Neumann series, taking into account the fact that $\mathcal{D} \in \mathcal{L}((h^{(2m-1)+\alpha}(\mathbb{T}))^n, h^\alpha(\mathbb{T}))$, so that (2.25) implies

$$\|R^C\mathcal{D}(\lambda+\Lambda)^{-1}\|_{\mathcal{L}((h^\alpha(\mathbb{T}))^n)} \leq \tilde{c}\,nC(m)M^2|\lambda|^{-1/2m}.$$

Hence, we can choose $\omega_2 \geq \omega$ large enough that $\|R^C\mathcal{D}(\lambda+\Lambda)^{-1}\|_{\mathcal{L}((h^\alpha(\mathbb{T}))^n)} \leq 1/2$ for $\operatorname{Re}\lambda \geq \omega_2$, which implies invertibility of $(\lambda+\Lambda-R^C\mathcal{D})$. Furthermore, to see that $R(\lambda)$ is a right inverse for $(\lambda+\mathcal{A}_p)$, let $(f_j)_j \in (h^{2m+\alpha}(\mathbb{T}))^n$ and notice that

$$\begin{aligned}(\lambda+\mathcal{A}_p)R(f_j)_j &= \sum_{j=1}^n(\lambda+\mathcal{A}_p)\pi_j f_j = \sum_{j=1}^n\left(\pi_j(\lambda+\mathcal{A}_j)f_j - B_jf_j\right)\\ &= \sum_{j=1}^n \pi_j(\lambda+\mathcal{A}_j)f_j - \sum_{j=1}^n B_jf_j = R(\lambda+\Lambda)(f_j)_j - \mathcal{D}(f_j)_j\\ &= R(\lambda+\Lambda)(f_j)_j - RR^C\mathcal{D}(f_j)_j = R(\lambda+\Lambda-R^C\mathcal{D})(f_j)_j\,.\end{aligned}$$

Hence, $(\lambda+\mathcal{A}_p)R(\lambda) = id_{h^\alpha(\mathbb{T})}$ and $R(\lambda)$ is a right inverse for $(\lambda+\mathcal{A}_p)$.

Finally, let $\omega_0 = \omega_1 \vee \omega_2$, so that $L(\lambda)$ and $R(\lambda)$ are both defined for $\operatorname{Re}\lambda \geq \omega_0$. Then $L(\lambda)f = L(\lambda)\left[(\lambda+\mathcal{A}_p)R(\lambda)\right]f = \left[L(\lambda)(\lambda+\mathcal{A}_p)\right]R(\lambda)f = R(\lambda)f$, for $f \in h^\alpha(\mathbb{T})$. Hence, $(\lambda+\mathcal{A}_p)$ is invertible for $\operatorname{Re}\lambda \geq \omega_0$ and $\mathcal{A}_p \in \mathcal{H}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}))$ follows from the resolvent estimate (2.27).

$\square$

With this generation result for the principal operator $\mathcal{A}_p$ established, we return to the full elliptic operator $\mathcal{A}$. Making use of perturbation results for generators of analytic semigroups, we prove that -$\mathcal{A}$ generates an analytic semigroup in the scale of little-Hölder spaces.

Theorem 2.4.4. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}, b_k \in h^\alpha(\mathbb{T})$, for $k = 0, \dots, 2m$, and suppose the operator $\mathcal{A} := \mathcal{A}(\cdot, D) := \sum_{k=0}^{2m} b_k(\cdot) D^k$ is uniformly elliptic. Then $\mathcal{A} \in \mathcal{H}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}))$.*

Proof. By Lemma 2.4.3 and [2, Theorem I.1.2.2] we can find $\omega > 0$ and $\kappa \geq 1$ such that $\mathcal{A}_p := \mathcal{A}_p(\cdot, D) := b_{2m}(\cdot) D^{2m} \in \mathcal{H}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}), \kappa, \omega)$. Fix η so that $0 < \eta < 1/\kappa$ and consider the operator $B_1 := B_1(\cdot, D) := b_{2m-1}(\cdot) D^{2m-1}$. For any $f \in h^{2m+a}(\mathbb{T})$, we make use of the interpolation inequality, c.f. [2, Proposition I.2.2.1], and Young's inequality to see that

$$\begin{aligned}
\|B_1 f\|_{h^\alpha(\mathbb{T})} &\leq \|b_{2m-1}\|_{h^\alpha(\mathbb{T})} \, \|f^{2m-1}\|_{h^\alpha(\mathbb{T})} \leq \|b_{2m-1}\|_{h^\alpha(\mathbb{T})} \, \|f\|_{h^{2m-1+\alpha}(\mathbb{T})} \\
&\leq c \|b_{2m-1}\|_{h^\alpha(\mathbb{T})} \left(\|f\|_{h^\alpha(\mathbb{T})}^{\frac{1}{2m}} \, \|f\|_{h^{2m+\alpha}(\mathbb{T})}^{\frac{2m-1}{2m}} \right) \\
&\leq c \|b_{2m-1}\|_{h^\alpha(\mathbb{T})} \left(\varepsilon \, \|f\|_{h^{2m+\alpha}(\mathbb{T})} + \tilde{c} \, \varepsilon^{1-2m} \, \|f\|_{h^\alpha(\mathbb{T})} \right),
\end{aligned}$$

which holds for arbitrary $\varepsilon > 0$. If we choose $\varepsilon > 0$ such that $\tilde{\varepsilon} := c\varepsilon \|b_{2m-1}\|_{h^\alpha(\mathbb{T})} < \eta$, it follows from [2, Theorem 1.3.1(ii)] that

$$\mathcal{A}_p + B_1 \in \mathcal{H}\left(h^{2m+\alpha}(\mathbb{T}),\ h^\alpha(\mathbb{T}),\ \frac{\kappa}{1 - \kappa \tilde{\varepsilon}},\ \omega \ \vee \ \frac{c \tilde{c} \, \varepsilon^{1-2m} \ \|b_{2m-1}\|_{h^\alpha(\mathbb{T})}}{\tilde{\varepsilon}} \right).$$

Now the theorem follows by repeating this argument for the remaining lower-order terms of the operator $\mathcal{A}$. □

Remarks 2.4.5. Notice that the results of Theorem 2.4.4 also hold in the setting of classic Hölder spaces $C^\sigma(\mathbb{T})$, though one must still take coefficients from the little-Hölder spaces to preserve smallness of localized coefficients, c.f. Lemma 2.4.2(b). One notable difference when considering these analogous results in the classic Hölder setting is that the semigroups generated are no longer strongly continuous, due to a lack of dense embeddings in this setting. For the methods leading to maximal regularity that follow, strong continuity of semigroups is necessary, so the results in the little-Hölder setting are required for our purposes.

2.5 Maximal Regularity and The Inhomogeneous Problem

We return to the task of finding solutions to the inhomogeneous problem

$$\begin{cases} \partial_t u(t,x) + \mathcal{A}(x,D)u(t,x) = f(t,x), & t > 0,\, x \in \mathbb{R} \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.28}$$

Given an interval $J := [0,T]$ and $\dot{J} := J \setminus \{0\}$, we say that $u : [t \mapsto u(t,\cdot)]$ is a *classical solution* to (2.28) if

$$u \in C^1(\dot{J}, C(\mathbb{T})) \cap C(\dot{J}, C^{2m}(\mathbb{T})) \cap C(J, C(\mathbb{T})),$$

and u satisfies (2.28). Following results of DaPrato, Grisvard and Angenent, we will show how the analytic semigroup generation result of Theorem 2.4.4 leads to existence and uniqueness of solutions to (2.28), with maximal regularity of solutions. We begin by defining function spaces which define the temporal regularity of solutions (i.e. mapping the interval J into the little-Hölder spaces), then we define a class of maximal regularity and use properties of maximal regularity to acquire existence and uniqueness of solutions.

2.5.1 Function Spaces and Maximal Regularity

Addressing temporal regularity of solutions, let $\mu \in (0,1]$, $J := [0,T]$, for some $T > 0$, and let E be a Banach space. Following the notation of [17], we define spaces of functions defined on $\dot{J} := J \setminus \{0\}$ with prescribed singularity at 0. Namely, for $\mu \in (0,1)$, define

$$\begin{aligned} &BUC_{1-\mu}(J,E) := \left\{ u \in C(\dot{J},E) : [t \mapsto t^{1-\mu}u(t)] \in BUC(\dot{J},E) \text{ and } \lim_{t\to 0^+} t^{1-\mu}\|u(t)\|_E = 0 \right\} \\ &\|u\|_{C_{1-\mu}} := \sup_{t\in J} t^{1-\mu}\|u(t)\|_E, \end{aligned} \tag{2.29}$$

where BUC denotes the space consisting of bounded, uniformly continuous functions. It is easy to verify that $BUC_{1-\mu}(J,E)$ is a Banach space when equipped with the norm $\|\cdot\|_{C_{1-\mu}}$. Moreover, we define the subspace

$$BUC^1_{1-\mu}(J,E) := \left\{ u \in C^1(\dot{J},E) : u, \dot{u} \in BUC_{1-\mu}(J,E) \right\}, \quad \mu \in (0,1)$$

and we set

$$BUC_0(J,E) := BUC(J,E) \qquad BUC_0^1(J,E) := BUC^1(J,E).$$

Now, fix $\mu \in (0,1]$ and consider the spaces

$$\begin{aligned}
\mathbb{E}_0(J) &:= BUC_{1-\mu}(J, h^\alpha(\mathbb{T})), \qquad \alpha \in \mathbb{R}_+ \setminus \mathbb{Z} \\
\mathbb{E}_1(J) &:= BUC^1_{1-\mu}(J, h^\alpha(\mathbb{T})) \cap BUC_{1-\mu}(J, h^{2m+\alpha}(\mathbb{T})),
\end{aligned}$$

where $\mathbb{E}_1(J)$ is a Banach space with the norm

$$\|u\|_{\mathbb{E}_1(J)} := \sup_{t \in \dot{J}} t^{1-\mu}\Big(\|\dot{u}(t)\|_{h^\alpha} + \|u(t)\|_{h^{2m+\alpha}}\Big).$$

It follows that the trace operator $\gamma : \mathbb{E}_1(J) \to h^\alpha(\mathbb{T})$, defined by $\gamma v := v(0)$, is well-defined and we denote by $\gamma\mathbb{E}_1$ the image of γ in $h^\alpha(\mathbb{T})$, which is a Banach space when equipped with the norm

$$\|f\|_{\gamma\mathbb{E}_1} := \inf\Big\{\|v\|_{\mathbb{E}_1(J)} : v \in \mathbb{E}_1(J) \text{ and } \gamma v = f\Big\}.$$

By [2, Theorem III.2.3.1] and Proposition 2.1.2(b) we see that

$$\begin{aligned}
\gamma\mathbb{E}_1 = (h^\alpha(\mathbb{T}), h^{2m+\alpha}(\mathbb{T}))_\mu = h^{2m\mu+\alpha}(\mathbb{T}), &\qquad \mu \in (0,1) \\
\gamma\mathbb{E}_1 := h^{2m+\alpha}(\mathbb{T}) &\qquad \mu = 1,
\end{aligned}$$

where $(\cdot,\cdot)_\eta$ denotes the continuous interpolation functor of DaPrato and Grisvard, c.f. [2, 51], and the interpolation space characterization holds (up to equivalent norms) when $2m\mu + \alpha \notin \mathbb{Z}$.

For $B \in \mathcal{L}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}))$, closed on $h^\alpha(\mathbb{T})$, we say that $\big(\mathbb{E}_0(J), \mathbb{E}_1(J)\big)$ is a *pair of (continuous) maximal regularity* for B, and write $B \in \mathcal{M}_\mu(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}))$, if

$$\left(\frac{d}{dt} + B,\, \gamma\right) \in \mathcal{L}_{isom}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times \gamma\mathbb{E}_1),$$

$\mu \in (0,1]$, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $J = [0,T]$ for some $T > 0$. In particular, we see that $\big(\mathbb{E}_0(J), \mathbb{E}_1(J)\big)$ is a pair of maximal regularity for B if and only if for every $(f, u_0) \in \mathbb{E}_0(J) \times \gamma\mathbb{E}_1$, there exists a unique solution $u \in \mathbb{E}_1(J)$ to the inhomogeneous Cauchy problem with operator B.

2.5.2 Maximal Regularity and Generation of Analytic Semigroups

Our goal is to show that $\big(\mathbb{E}_0(J), \mathbb{E}_1(J)\big)$ is a pair of maximal regularity for $\mathcal{A}$ for arbitrary $\alpha \in \mathbb{R}_+ \backslash \mathbb{Z}$ and $J = [0, T]$, given minimal regularity assumptions on the coefficients b_k. In particular, fix $m \in \mathbb{N}$ and coefficients $b_k \in h^\alpha(\mathbb{T})$, $k = 0, \ldots, 2m$ such that $\mathcal{A} := \mathcal{A}(\cdot, D) := \sum_{k \le 2m} b_k(\cdot) D^k$ satisfies the *uniform ellipticity* conditions (2.10). The tool we are going to use to prove this maximal regularity result is the following theorem of DaPrato, Grisvard and Angenent, which was originally proved by DaPrato and Grisvard [22] in the case $\mu = 1$ and then generalized to $\mu \in (0, 1)$ by Angenent [8].

Theorem 2.5.1 (DaPrato-Grisvard-Angenent)**.** *Fix $\eta \in (0, 1)$, $\mu \in (0, 1]$ and $J := [0, T]$ for $T > 0$. Suppose that (E_0, E_1) is a pair of densely embedded Banach spaces and consider an operator $A \in \mathcal{H}(E_1, E_0)$. Now, set*

$$E_2 := E_2(A) := (D(A^2), \|\cdot\|_2) \quad \textit{equipped with the norm} \quad \|\cdot\|_2 := \|A\cdot\|_1 + \|\cdot\|_1,$$

$$E_\eta := (E_0, E_1)_\eta, \qquad\qquad E_{1+\eta} := E_{1+\eta}(A) := (E_1, E_2(A))_\eta,$$

$$A_\eta := \textit{ the maximal } E_\eta\textit{-realization of } A.$$

It follows that

$$\big(\mathbb{E}_\eta(J), \mathbb{E}_{1+\eta}(J)\big) := \Big(BUC_{1-\mu}(J, E_\eta), BUC^1_{1-\mu}(J, E_\eta) \cap BUC_{1-\mu}(J, E_{1+\eta})\Big),$$

is a pair of maximal regularity for A_η.

It is also a well-known result that $A_\eta \in \mathcal{H}(E_{1+\eta}, E_\eta)$, c.f. [2, Section III.3.2].

Due to the continuous interpolation spaces constructed in the theorem, we see that we cannot directly derive maximal regularity results for $\mathcal{A}$ in $h^\alpha(\mathbb{T})$. In particular, when applying Theorem 2.5.1, the derived maximal regularity results are necessarily in a little-Hölder space with slightly larger exponent than where we assume analytic semigroup generation results. Moreover, it is in general quite difficult to characterize the operator-dependent space $E_2(\mathcal{A})$, which is in turn dependent upon the regularity conditions imposed on the coefficients b_k. However, we are able to take advantage of flexibility in Theorem 2.4.4, with respect to the regularity exponents, in order to work around these difficulties and prove the following result.

Theorem 2.5.2. *Fix $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$, $\mu \in (0,1]$ and $J = [0,T]$, for $T > 0$ arbitrary. Suppose the operator $\mathcal{A} := \mathcal{A}(\cdot, D) = \sum_{k \le 2m} b_k(\cdot) D^k$, with coefficients $b_k \in h^\alpha(\mathbb{T})$ is uniformly elliptic, as in* (2.10). *Then*

$$\left(\frac{d}{dt} + \mathcal{A}, \, \gamma\right) \in \mathcal{L}_{isom}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times \gamma\mathbb{E}_1).$$

In particular, given any pair $(f, u_0) \in BUC_{1-\mu}(J, h^\alpha(\mathbb{T})) \times \gamma\mathbb{E}_1$, there exists a unique solution $u \in BUC^1_{1-\mu}(J, h^\alpha(\mathbb{T})) \cap BUC_{1-\mu}(J, h^{2m+\alpha}(\mathbb{T}))$ to the inhomogeneous Cauchy problem (2.28).

Proof. Fix $\beta \in \mathbb{R}_+ \setminus \mathbb{Z}$ such that $\beta < \alpha < 2m + \beta$ and fix $\eta := \frac{\alpha - \beta}{2m}$, then we see that $\eta \in (0,1)$ and $2m\eta + \beta = \alpha$. $\mathcal{A}$ is trivially realized as an operator from $h^{2m+\beta}(\mathbb{T})$ to $h^\beta(\mathbb{T})$ by Proposition 2.1.2(a), so that, by Theorem 2.4.4, we know $\mathcal{A} \in \mathcal{H}(h^{2m+\beta}(\mathbb{T}), h^\beta(\mathbb{T}))$. Now we construct the spaces E_2, E_η and $E_{1+\eta}$ as in Theorem 2.5.1, and we apply Proposition 2.1.2(b) when possible. Namely, we set

$$E_2 := \left\{ f \in h^\beta(\mathbb{T}) : \mathcal{A}f \in h^{2m+\beta}(\mathbb{T}) \right\},$$

equipped with the graph norm $\|\cdot\|_2 := \|\mathcal{A}\cdot\|_{h^{2m+\beta}} + \|\cdot\|_{h^{2m+\beta}}$,

$$E_\eta := (h^\beta(\mathbb{T}), h^{2m+\beta}(\mathbb{T}))_\eta = h^\alpha(\mathbb{T}) \quad \text{and} \quad E_{1+\eta} := (h^{2m+\beta}(\mathbb{T}), E_2)_\eta;$$

notice that, a priori, we cannot conclude $E_{1+\eta}$ coincides with a little-Hölder space without a proper characterization of E_2. However, by uniform ellipticity of $\mathcal{A}$, with coefficients b_k in $h^\alpha(\mathbb{T})$, we know that $\mathcal{A} \in \mathcal{H}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T}))$, by Theorem 2.4.4 again. Meanwhile, by the remark following Theorem 2.5.1, we see that $\mathcal{A} \in \mathcal{H}(E_{1+\eta}, h^\alpha(\mathbb{T}))$. Hence, we can find $\omega > 0$ sufficiently large so that

$$(\omega + \mathcal{A}) \in \mathcal{L}_{isom}(h^{2m+\alpha}(\mathbb{T}), h^\alpha(\mathbb{T})) \cap \mathcal{L}_{isom}(E_{1+\eta}, h^\alpha(\mathbb{T})).$$

However, it follows that $(\omega + \mathcal{A})^{-1} \circ (\omega + \mathcal{A}) : h^{2m+\alpha}(\mathbb{T}) \to E_{1+\eta}$ is an isometric isomorphism, by commutativity. So, $h^{2m+\alpha}(\mathbb{T})$ and $E_{1+\eta}$ coincide (up to equivalent norms) and it follows that $\big(\mathbb{E}_0(J), \mathbb{E}_1(J)\big)$ is a pair of (continuous) maximal regularity for $\mathcal{A}$, by Theorem 2.5.1. $\square$

2.6 Vector-Valued Setting

For the remainder of this chapter, let $E = (E, |\cdot|)$ denote an arbitrary (non-trivial) Banach space over $\mathbb{C}$. Again, consider the inhomogeneous problem (2.1) with periodicity enforced. However, suppose that one is given vector-valued functions, $u_0, f(t,\cdot) : \mathbb{T} \to E$, and operator-valued coefficients, $b_k : \mathbb{T} \to \mathcal{L}(E)$. It turns out that, with only minor modifications and appropriate alterations to definitions, the preceding results continue to hold in this more general setting. In this section, we highlight the necessary changes to the preceding theory and state results in this vector-valued setting.

2.6.1 Vector-Valued Function Spaces

Following common conventions, we denote by $C(\mathbb{T}, E)$, $C^\theta(\mathbb{T}, E)$, and $h^\theta(\mathbb{T}, E)$, the classes of regular E-valued functions analogous to the scalar-valued cases defined in Section 2.1.1, the definitions of which remain essentially unchanged. Moreover, one will note that Proposition 2.1.2 is a simplified version of [51, Proposition 0.2.1 and Theorem 1.2.17], which were already stated in the vector-valued setting, so there is no trouble in getting these same results for E-valued functions. In order to give an adequate definition of E-valued Besov spaces, however, one will need the concept of vector-valued distributions.

Taking $\mathcal{D}(\mathbb{T})$ to be the smooth $\mathbb{C}$-valued functions over $\mathbb{T}$, as before, we define the space of E-valued distributions $\mathcal{D}'(\mathbb{T}, E) := \mathcal{L}(\mathcal{D}(\mathbb{T}), E)$ and we equip $\mathcal{D}'(\mathbb{T}, E)$ with the weak-star topology over $\mathcal{D}(\mathbb{T})$. One can see that the same definitions of Fourier coefficients and results on Fourier series representations continue to hold, c.f. [12]. In particular, for every $f \in \mathcal{D}'(\mathbb{T}, E)$, it holds that

$$f = \sum_{k\in\mathbb{Z}} e_k \otimes \hat{f}(k) \qquad \text{(convergence in } \mathcal{D}'(\mathbb{T}, E)),$$

where $e_k \in \mathcal{D}(\mathbb{T})$ has the same definition as before and $e_k \otimes y$ denotes the function $[x \mapsto e^{ikx} y] : \mathbb{T} \to E$ for $y \in E$ given. Then, we define the E-valued periodic Besov spaces $B^s_{p,q}(\mathbb{T}, E)$ as before, by making use of collections of dyadic decompositions $\Phi(\mathbb{R})$, and we derive analogous results to those discussed in the scalar setting.

2.6.2 Operator-Valued Fourier Multipliers

Now, with vector-valued Besov spaces established, we consider Fourier multiplier results in this setting. As discussed in [12], the *Fourier type* of the underlying Banach space E will affect the statement of the Fourier multiplier result. To be clear regarding Fourier multipliers in this setting, we are given a sequence $(M_k)_k \subset \mathcal{L}(E)$ and consider the associated (formal) operator

$$T : \sum_{k\in\mathbb{Z}} e_k \otimes \hat{f}(k) \longmapsto \sum_{k\in\mathbb{Z}} e_k \otimes M_k \hat{f}(k).$$

The following multiplier theorem will work for the general case where E is a Banach space with arbitrary Fourier type. We note that the analogue to Lemma 2.2.2 does not hold in this general case.

Theorem 2.6.1. *Let $r, s \in \mathbb{R}_+$ and $1 \leq p, q \leq \infty$. Suppose that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(E)$ is a sequence such that*

$$\begin{gathered} s_1 := \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s}\|M_k\| < \infty, \qquad s_2 := \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+1}\|M_{k+1} - M_k\| < \infty, \\ s_3 := \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+2}\|M_{k+1} - 2M_k + M_{k-1}\| < \infty. \end{gathered} \tag{2.30}$$

Then the Fourier multiplier with symbol $(M_k)_{k\in\mathbb{Z}}$ is a continuous mapping from $B^s_{p,q}(\mathbb{T}, E)$ to $B^r_{p,q}(\mathbb{T}, E)$, namely

$$T : \left[\sum_{k\in\mathbb{Z}} e_k \otimes \hat{f}(k) \longmapsto \sum_{k\in\mathbb{Z}} e_k \otimes M_k \hat{f}(k)\right] \in \mathcal{L}\left(B^s_{p,q}(\mathbb{T}, E), B^r_{p,q}(\mathbb{T}, E)\right).$$

A proof of Theorem 2.6.1 follows from [55, Theorem 2.2.1], by restating the proof in the E-valued setting. On the other hand, if we find that the Fourier type of E is in the interval $(1, 2]$, then Lemma 2.2.2 is known to hold and we have the analogous statement to Theorem 2.2.1, without the necessity of checking the term s_3. Note that this sharper case includes the situation E a Hilbert space, where the Fourier type is exactly 2.

2.6.3 Ellipticity With Operator-Valued Coefficients

Now, fix a collection $\{b_k : k = 0, \ldots, 2m\} \subset h^{\alpha}(\mathbb{T}, \mathcal{L}(E))$ of operator-valued coefficient functions and consider the differential operator $\mathcal{A}$, acting on $h^{2m+\alpha}(\mathbb{T}, E)$, defined by

$$\mathcal{A}u(x) := \mathcal{A}(x, D)u(x) := \sum_{k=0}^{2m} b_k(x)\,(D^k u)(x) = \sum_{k=0}^{2m} i^k\, b_k(x)\, u^{(k)}(x), \qquad x \in \mathbb{T}.$$

By the embedding property Proposition 2.1.2(a) and the regularity assumptions on b_k and u, it follows immediately that $\mathcal{A}$ maps $h^{2m+\alpha}(\mathbb{T})$ into $h^{\alpha}(\mathbb{T})$. Now, denote by $\sigma\mathcal{A} : \mathbb{T} \times \mathbb{R} \to \mathcal{L}(E)$ the *principal symbol* of $\mathcal{A}$, defined by $\sigma\mathcal{A}(x, \xi) := \xi^{2m} b_{2m}(x)$. We say that $\mathcal{A}$ is a *normally elliptic* operator on $\mathbb{T}$ if there exist constants $c_1 \geq 1$ and $\theta \in (\pi/2, \pi)$ so that the properties

$$\begin{gathered} \rho(-\sigma\mathcal{A}(x, \xi)) \supset \Sigma_\theta := \{z \in \mathbb{C} : |\arg z| \leq \theta\} \cup \{0\} \\ (1 + |\lambda|)\|(\lambda + \sigma\mathcal{A}(x, \xi))^{-1}\| \leq c_1, \quad \lambda \in \Sigma_\theta, \end{gathered} \tag{2.31}$$

hold for all $x \in \mathbb{T}$ and $|\xi| = 1$. This definition coincides with the definition of normally elliptic operators presented in [4, Section 3] and one will note that this definition generalizes the notion of uniform ellipticity, as in (2.10). Moreover, as mentioned by Amann in [4], in the case that E is finite-dimensional, this definition of normal ellipticity is equivalent to the condition that there exist $0 < r < R$ such that

$$\sigma(\sigma\mathcal{A}(x, \xi)) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq r\} \cap \{z \in \mathbb{C} : |z| \leq R\}, \qquad \text{for } x \in \mathbb{T},\ |\xi| = 1.$$

Theorem 2.6.2. *Let E be a Banach space, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$ and consider the differential operator $\mathcal{A}_b := b\, D^{2m}$ with constant coefficient $b \in \mathcal{L}(E)$. If $\mathcal{A}_b$ is normally elliptic, with constant $c_1 > 0$, and $c_2 \geq c_1 > 0$ is chosen so that $\|b\| \leq c_2$, then $-\mathcal{A}_b$ generates a (strongly continuous) analytic semigroup on $h^{\alpha}(\mathbb{T}, E)$ with domain $h^{2m+\alpha}(\mathbb{T}, E)$. Moreover, for any $\omega > 0$, there exists $\kappa = \kappa(\omega, c_1, c_2, m)$ such that*

$$\mathcal{A}_b \in \mathcal{H}\big(h^{2m+\alpha}, h^{\alpha}, \kappa(\omega, c_1, c_2, m), \omega\big).$$

Proof. The proof of this result follows the same method used to prove Theorem 2.3.2, however, in this vector-valued setting, we must derive bounds for the term s_3 before applying our Fourier multiplier result, Theorem 2.6.1 in this case. Fix $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$, $\omega > 0$ and $b \in \mathcal{L}(E)$ as indicated. Notice that $-\mathcal{A}_b$ is now associated with the operator-valued multiplier symbol $(M_k)_k := \left(-k^{2m}b\right)_k \subset \mathcal{L}(E)$. Now, we can make formally identical claims to those stated in the scalar-valued setting.

Claim 1: $(\lambda + \mathcal{A}_b) \in \mathcal{L}_{isom}(h^{2m+\alpha}(\mathbb{T}, E), h^\alpha(\mathbb{T}, E))$ *for* $Re\,\lambda \geq \omega$, *i.e.* $\rho(-\mathcal{A}_b) \supset \{\lambda \in \mathbb{C} : Re\,\lambda \geq \omega\}$. *Moreover, the set* $\{\|(\lambda + \mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^\alpha, h^{2m+\alpha})} : Re\,\lambda \geq \omega\}$ *is uniformly bounded by some* $M_1 = M_1(\omega, c_1, c_2, m) < \infty$.

Let λ be fixed with $\mathrm{Re}\,\lambda \geq \omega$. The fact that $\mathcal{A}_b \in \mathcal{L}(C^{2m+\sigma}(\mathbb{T}), C^\sigma(\mathbb{T}))$ follows in the same way as the scalar case with $\|\lambda + \mathcal{A}_b\| \leq (c(\sigma)|\lambda| + c_2)$. Consider the symbol $\left(\tilde{M}_k(\lambda)\right)_k := \left((\lambda + k^{2m}b)^{-1}\right)_k$, where the condition of normal ellipticity guarantees that $\mathrm{Re}\,\lambda \geq 0$ is sufficient to see that $\lambda \in \rho(\sigma\mathcal{A}_b(x,\xi))$. Moreover, in the constant coefficient case, it follows that $\sigma\mathcal{A}_b(x,\xi) \equiv b$ for $|\xi| = 1$. Now, the second condition of normal ellipticity, (2.31), gives adequate flexibility to see that $\lambda/k^{2m} \in \rho(b)$ and we conclude that $\tilde{M}_k(\lambda) \in \mathcal{L}(E)$ is well-defined, for $k \in \mathbb{Z}$. Further, notice that

$$\tilde{M}_k(\lambda) := \left(\lambda + k^{2m}b\right)^{-1} = k^{-2m}\left(\frac{\lambda}{k^{2m}} + b\right)^{-1} \qquad k \neq 0,$$

which we make use of for verifying the conditions of the Fourier multiplier theorem.

Using the resolvent bounds given in the normal ellipticity definition, we see that, concerning the symbol $\left(\tilde{M}_k(\lambda)\right)_k$, we have

$$s_1 \leq c_1 < \infty, \qquad s_2 \leq \left(\frac{c_1c_2}{\omega} \vee c_2\right) \sup_{k\in\mathbb{Z}\setminus\{-1\}} \left(\frac{|k||(k+1)^{2m} - k^{2m}|}{|k+1|^{2m}}\right) < \infty.$$

Meanwhile, notice that, for $k \neq \pm 1$,

$$
\begin{aligned}
&|k|^{2m+2} \left\|(\lambda + (k+1)^{2m}b)^{-1} - 2(\lambda + k^{2m}b)^{-1} + (\lambda + (k-1)^{-1}b)^{-1}\right\| \\
&\leq \left\| \left(\frac{\lambda}{(k+1)^{2m}} + b \right)^{-1} \right\| \left\| \left(\frac{\lambda}{k^{2m}} + b \right)^{-1} \right\| \left\| \left(\frac{\lambda}{(k-1)^{2m}} + b \right)^{-1} \right\| \\
&\quad \left(\frac{|k|^2}{|k+1|^{2m}|k-1|^{2m}} \right) \Big[\left\| -\lambda \left((k+1)^{2m} - 2k^{2m} + (k-1)^{2m} \right) b \right\| \\
&\qquad + \left\| \left((k+1)^{2m}(k^{2m} - (k-1)^{2m}) + (k-1)^{2m}((k+1)^{2m} - k^{2m}) \right) b^2 \right\| \Big] \\
&\leq \frac{c_1^3 c_2}{(1+|\lambda|)^3} \Big(|\lambda| + c_2 \Big) \mathcal{K}_1(k) \leq c_1^3 c_2 (1 + c_2) \mathcal{K}_1(k),
\end{aligned}
$$

where $\mathcal{K}_1$ is a bounded function in k. Similarly, in case $k = \pm 1$, we see that

$$
\left\|(\lambda)^{-1} - 2(\lambda + b)^{-1} + (\lambda + 2^{2m}b)^{-1}\right\| \leq 2^{2m} c_1^2 c_2 (1 + c_2).
$$

Hence, it follows that $s_3 < \infty$, and bounded by terms which only depend upon ω, m, c_1, and c_2. Hence, we can apply Theorem 2.6.1 to prove the claim. We again see that the operator $R(\lambda)$ associated with the symbol $\left(\tilde{M}_k(\lambda)\right)_k$ coincides with the inverse of $(\lambda + \mathcal{A}_b)$.

Claim 2: *$\lambda(\lambda + \mathcal{A}_b)^{-1} \in \mathcal{L}(h^\alpha(\mathbb{T}, E))$ for $Re\,\lambda \geq \omega$, and there is an upper bound $M_2 = M_2(\omega, c_1, c_2, m) < \infty$ for the set $\{|\lambda| \|(\lambda + \mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^\alpha)} : Re\,\lambda \geq \omega\}$.*

This claim is verified by applying the same techniques as above to the symbol $\left(\lambda(\lambda + k^{2m}b)^{-1}\right)_k$. Working through the details, one verifies that the s_i terms, $i = 1, 2, 3$, are bounded exactly the same as in Claim 1 above. Hence, the desired result holds, and the proof of the theorem proceeds exactly as in the scalar-valued setting.

$\square$

2.6.4 Semigroup Generation and Maximal Regularity

We conclude this chapter with statements of the main results in the setting of vector-valued functions. Their proofs are obtained by direct application of the methods employed in the scalar-valued setting, with only minor changes of notation and definitions, which have already been addressed in the preceding parts of this section.

Theorem 2.6.3. *Let E be a Banach space, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}_+ \backslash \mathbb{Z}, b_k \in h^\alpha(\mathbb{T}, \mathcal{L}(E))$, for $k = 0, \ldots, 2m$, and suppose the operator $\mathcal{A} := \mathcal{A}(\cdot, D) := \sum_{k=0}^{2m} b_k(\cdot) D^k$ is normally elliptic. Then*

$$\mathcal{A} \in \mathcal{H}(h^{2m+\alpha}(\mathbb{T}, E), h^\alpha(\mathbb{T}, E)).$$

Fix $\mu \in (0, 1]$ and define the spaces

$$\mathbb{E}_0(J) := BUC_{1-\mu}(J, h^\alpha(\mathbb{T}, E)), \qquad \alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$$
$$\mathbb{E}_1(J) := BUC^1_{1-\mu}(J, h^\alpha(\mathbb{T}, E)) \cap BUC_{1-\mu}(J, h^{2m+\alpha}(\mathbb{T}, E)).$$

Then we get the maximal regularity result.

Theorem 2.6.4. *Fix a Banach space E, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$, $\mu \in (0, 1]$ and $J = [0, T]$, for $T > 0$ arbitrary. Suppose the operator $\mathcal{A} := \mathcal{A}(\cdot, D) = \sum_{k \le 2m} b_k(\cdot) D^k$, with coefficients $b_k \in h^\alpha(\mathbb{T}, \mathcal{L}(E))$ is normally elliptic, as in* (2.31)*. Then*

$$\left(\frac{d}{dt} + \mathcal{A}, \gamma\right) \in \mathcal{L}_{isom}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times \gamma\mathbb{E}_1).$$

In particular, given any pair $(f, u_0) \in BUC_{1-\mu}(J, h^\alpha(\mathbb{T}, E)) \times \gamma\mathbb{E}_1$, there exists a unique solution $u \in BUC^1_{1-\mu}(J, h^\alpha(\mathbb{T}, E)) \cap BUC_{1-\mu}(J, h^{2m+\alpha}(\mathbb{T}, E))$ to the inhomogeneous Cauchy problem (2.1) *on J.*

CHAPTER 3

WELL POSEDNESS AND GEOMETRIC PROPERTIES OF SOLUTIONS

3.1 Well-Posedness of ASD

Recall the expression for the axisymmetric surface diffusion problem

$$\begin{cases} r_t = \dfrac{1}{r}\,\partial_x \left[\dfrac{r}{\sqrt{1+r_x^2}}\,\partial_x \left(\dfrac{1}{r\sqrt{1+r_x^2}} - \dfrac{r_{xx}}{(1+r_x^2)^{\frac{3}{2}}} \right) \right], & t>0,\ x\in\mathbb{R}, \\ r(t,x+2\pi) = r(t,x), & t\geq 0,\ x\in\mathbb{R}, \\ r(0,x) = r_0(x), & x\in\mathbb{R}. \end{cases} \tag{3.1}$$

To simplify notation, we define the operator

$$G(r) := \frac{1}{r}\,\partial_x \left[\frac{r}{\sqrt{1+r_x^2}}\,\partial_x \mathcal{H}(r) \right], \tag{3.2}$$

which is formally equivalent to the right hand side of the governing equation for (3.1).

In this chapter, we will establish well-posedness and dynamic properties of solutions to (3.1) in the setting of periodic little–Hölder spaces. Shifting slightly from the setting considered in the previous chapter, we will only be considering $\mathbb{R}$–valued functions over $\mathbb{T}$ which satisfy the regularity conditions defined in Section 2.1.1. Although this setting differs slightly, we will demonstrate that only minor difficulties arise in applying the results of Chapter 2 to the setting considered here.

In order to make explicit the quasilinear structure of (3.1), we reformulate the problem. By expanding the governing equation we arrive at the formally equivalent problem

$$\begin{cases} \partial_t r(t,x) + \mathcal{A}(r(t,x))r(t,x) = f(r(t,x)), & t>0,\ x\in\mathbb{T}, \\ r(0,x) = r_0(x), & x\in\mathbb{T}, \end{cases} \tag{3.3}$$

where, for appropriately chosen functions ρ,

$$\mathcal{A}(\rho) := \frac{1}{(1+\rho_x^2)^2}\,\partial_x^4 + \frac{2\rho_x\big(1+\rho_x^2 - 3\rho\rho_{xx}\big)}{\rho\big(1+\rho_x^2\big)^3}\,\partial_x^3 \tag{3.4}$$

is a fourth-order differential operator with variable coefficients over $\mathbb{T}$ and

$$f(\rho) := \frac{\rho_x^2 - 1}{\rho^2(1+\rho_x^2)^2}\,\rho_{xx} + \frac{6\rho_x^2 - 1}{\rho(1+\rho_x^2)^3}\,\rho_{xx}^2 + \frac{3 - 15\rho_x^2}{(1+\rho_x^2)^4}\,\rho_{xx}^3 + \frac{\rho_x^2}{\rho^3(1+\rho_x^2)} \tag{3.5}$$

is a $\mathbb{R}$-valued function over $\mathbb{T}$. Here we use the notation $\mathcal{A}(r(t,x))$ to denote the evaluation of the function $\mathcal{A}(r(t))$ at the value $x \in \mathbb{T}$. Looking at these formal expressions, one can deduce several properties that the functions ρ must satisfy in order to get good mapping properties for f and $\mathcal{A}$. In particular, we want to choose ρ such that $\rho(x) \neq 0$ for all $x \in \mathbb{T}$, also we want that the spacial derivatives ρ_x and ρ_{xx} make sense and the products ρ^2, ρ^3, $\rho\rho_x^2$, etc. have desired regularity properties. With these conditions in mind, we proceed with our well-posedness result.

3.1.1 Existence and Uniqueness of Solutions

Fix $\alpha \in (0,1)$ and define the spaces of $\mathbb{R}$-valued little–Hölder continuous functions

$$E_0 := h^\alpha(\mathbb{T},\mathbb{R}), \quad E_1 := h^{4+\alpha}(\mathbb{T},\mathbb{R}), \quad \text{and} \quad E_\mu := (E_0, E_1)^0_{\mu,\infty}, \quad \mu \in (0,1), \tag{3.6}$$

where $(\cdot,\cdot)^0_{\mu,\infty}$ denotes the continuous interpolation functor of Da Prato and Grisvard, c.f. [22] or [2]. Recall from Theorem 2.1.2 that the little-Hölder spaces are stable under this interpolation method, in particular we know that $E_\mu = h^{4\mu+\alpha}(\mathbb{T},\mathbb{R})$ whenever $4\mu + \alpha \notin \mathbb{Z}$ (up to equivalent norms). Also recall from Section 2.5 that $\mathcal{MR}_\mu(E_1, E_0)$ denotes the space of all operators $B \in \mathcal{L}(E_1, E_0)$ such that

$$\Big(BUC_{1-\mu}(J, E_0), BUC^1_{1-\mu}(J, E_0) \cap BUC_{1-\mu}(J, E_1)\Big)$$

is a pair of maximal regularity for B, for some interval $J := [0, T]$. Further, let V be the set of functions $r : \mathbb{T} \to \mathbb{R}$ such that $r(x) > 0$ for all $x \in \mathbb{T}$ and define $V_\mu := V \cap E_\mu$ for $\mu \in [0,1]$. We note that V_μ is an open subset of E_μ for all $\mu \in [0,1]$.

Lemma 3.1.1. *Let $\mu \in [1/2, 1]$ such that $4\mu + \alpha \notin \mathbb{Z}$. Then*

$$(\mathcal{A}, f) \in C^\omega\Big(V_\mu,\ \mathcal{MR}_\nu(E_1, E_0) \times E_0\Big), \qquad \textit{for} \quad \nu \in (0,1],$$

where C^ω denotes the space of real analytic mappings between Banach spaces, as discussed in

Proof. Fix $\mu \in [1/2, 1]$ as indicated. This result relies on the fact that the little-Hölder spaces $h^\sigma(\mathbb{T})$ are Banach algebras with pointwise multiplication of functions.

CLAIM 1: $\mathcal{A}(\rho) \in \mathcal{M}R_\nu(E_1, E_0)$ for $\rho \in V_\mu\,,\ \nu \in (0,1]$. This claim will follow from Theorem 2.5.2, though the current setting differs slightly from the statement of the theorem and warrants some discussion. First, for $\rho \in V_\mu$ define the coefficients

$$b_4(\rho) := \frac{1}{(1+\rho_x^2)^2} \quad \text{and} \quad b_3(\rho) := \frac{2\rho_x\big(1+\rho_x^2 - 3\rho\rho_{xx}\big)}{\rho\big(1+\rho_x^2\big)^3},$$

so that $\mathcal{A}(\rho) = b_4(\rho)\,\partial_x^4 + b_3(\rho)\,\partial_x^3$. By our choice of μ, it follows that $V_\mu \subset h^{2+\alpha}(\mathbb{T}, \mathbb{R})$, so that $b_4, b_3 \in E_0$ and $\mathcal{A}(\rho)$ is a uniformly elliptic differential operator. By Theorem 2.5.2 we conclude that

$$\mathcal{A}(\rho) \in \mathcal{M}R_\nu\big(h^{4+\alpha}(\mathbb{T}, \mathbb{C}), h^\alpha(\mathbb{T}, \mathbb{C})\big), \qquad \nu \in (0,1],$$

where we utilize the notation $h^{k+\alpha}(\mathbb{T}, \mathbb{C})$ to be clear that the space consists of $\mathbb{C}$–valued functions over $\mathbb{T}$, and does not coincide with the spaces E_μ defined in this chapter. However, $h^{k+\alpha}(\mathbb{T}, \mathbb{C})$ does coincide with the complexification of $h^{k+\alpha}(\mathbb{T}, \mathbb{R})$ (up to equivalent norms) and it is a straightforward exercise to see that the property of maximal regularity continues to hold under restriction to the subspaces $h^\sigma(\mathbb{T}, \mathbb{R})$.

CLAIM 2: The operation of inversion is real analytic from V_0 into E_0, i.e.

$$T_i : [r \mapsto 1/r] \in C^\omega(V_0, E_0).$$

Fix $r_0 \in V_0$ and choose $a > 0$ so that $r_0(x) > a$ for all $x \in \mathbb{T}$. If $r \in E_0$ is chosen so that $\|r - r_0\|_{C(\mathbb{T})} < a$, then the representation

$$\frac{1}{r(x)} = \frac{1}{r_0(x)\left(1 + \frac{r(x)-r_0(x)}{r_0(x)}\right)} = \frac{1}{r_0(x)} \sum_{n=0}^\infty \left(\frac{r_0(x) - r(x)}{r_0(x)}\right)^n,$$

holds for $x \in \mathbb{T}$ arbitrary, where the last equality follows by an elementary geometric series argument. Hence, the given power series represents the function $1/r$ point-wise for $x \in \mathbb{T}$. Moreover, it follows from the algebraic structure of E_0 that

$$S_k := \sum_{n=0}^{k} \frac{(r_0 - r)^n}{r_0^{n+1}} \in E_0, \qquad \text{for} \quad k \in \mathbb{N}.$$

Finally, if $\|r_0 - r\|_{E_0} \leq a/2$, then

$$\sum_{n=0}^{\infty} \left\| \frac{(r_0 - r)^n}{r_0^{n+1}} \right\|_{E_0} \leq \sum_{n=0}^{\infty} \frac{\|r_0 - r\|_{E_0}^n}{\|r_0\|_{E_0}^{n+1}} \leq \sum_{n=0}^{\infty} \frac{(a/2)^n}{a^{n+1}} = 2/a,$$

which demonstrates the power series converges absolutely in the topology of E_0 and the claim follows.

CLAIM 3: The operations of differentiation and multiplication are real analytic in the setting of little-Hölder spaces, i.e.

$$\begin{aligned}
&\partial_x : [r \mapsto r_x] \in C^\omega(h^{\sigma+1}(\mathbb{T}), h^\sigma(\mathbb{T})), \qquad \sigma \in \mathbb{R}_+ \setminus \mathbb{Z}, \\
&T_m : [(r, s) \mapsto rs] \in C^\omega(E_0 \times E_0, E_0).
\end{aligned}$$

This claim follows since ∂_x and T_m are bounded, linear and bilinear (respectively) on the indicated spaces.

The remainder of the lemma follows from the fact that $\mathcal{A} : V_\mu \to \mathcal{L}(E_1, E_0)$ inherits the regularity of the coefficients $b_1, b_2 : V_\mu \to E_0$ and the fact that $\mathcal{M}R_\mu(E_1, E_0)$ is an open subset of $\mathcal{L}(E_0, E_1)$, c.f. [17, Lemma 2.5(a)]. $\square$

It follows from Lemma 3.1.1 that we can take full advantage of the well-posedness results for quasilinear parabolic equations presented in the article [17] of Clément and Simonett. In particular, we conclude results regarding local existence and uniqueness of solutions with continuous dependence on initial data, as well as maximal solutions and conditions for global existence. However, we give only a limited presentation, focusing on those results which will be of most direct use to us in the sequel, and we refer the interested reader to [17] for further details on well-posedness of (3.3).

Before we can properly state a result on maximal solutions, we need to introduce one more space of functions from an interval $J \subset \mathbb{R}_+$ to a Banach space E, with prescribed singularity at zero. Namely, if $J = [0, a)$ for $a > 0$, i.e. J is a right-open interval containing 0, then we set

$$C_{1-\mu}(J, E) := \{u \in C(\dot{J}, E) : u \in BUC_{1-\mu}([0, T], E), \quad T < \sup J\},$$
$$C^1_{1-\mu}(J, E) := \{u \in C^1(\dot{J}, E) : u, \dot{u} \in C_{1-\mu}(J, E)\}, \qquad \mu \in (0, 1]$$

and equip these spaces with the natural Fréchet topologies induced by $BUC_{1-\mu}([0, T], E)$ and $BUC^1_{1-\mu}([0, T], E)$, respectively.

Theorem 3.1.1 (Existence and Uniqueness)**.** *Fix $\alpha \in (0, 1)$ and take $\mu \in [1/2, 1]$ so that $4\mu+\alpha \notin \mathbb{Z}$. For each initial value $r_0 \in V_\mu$, there exists a unique maximal solution*

$$r(\cdot, r_0) \in C^1_{1-\mu}(J(r_0), E_0) \cap C_{1-\mu}(J(r_0), E_1),$$

where $J(r_0) = [0, t^+(r_0)) \subseteq \mathbb{R}_+$ denotes the maximal interval of existence for initial data r_0. Further, it follows that

$$\mathcal{D} := \bigcup_{r_0 \in V_\mu} J(r_0) \times \{r_0\}$$

is open in $\mathbb{R}_+ \times V_\mu$ and $\varphi : [(t, r_0) \mapsto r(t, r_0)]$ is an analytic semiflow on V_μ, i.e. using the notation $\varphi^t(r_0) := \varphi(t, r_0)$, the mapping φ satisfies the conditions

$$\begin{aligned}
&\bullet\ \varphi \in C\big(\mathcal{D}, V_\mu\big) \\
&\bullet\ \varphi^0 = id_{V_\mu} \\
&\bullet\ \varphi^{s+t}(r_0) = \varphi^t \circ \varphi^s(r_0) \quad \text{for } 0 \le s < t^+(r_0) \text{ and } 0 \le t < t^+(\varphi^s(r_0)) \\
&\bullet\ \varphi(t, \cdot) \in C^\omega(\mathcal{D}_t, V_\mu) \quad \text{for } t \in \mathbb{R}_+ \text{ with } \mathcal{D}_t := \{r \in V_\mu : (t, r) \in \mathcal{D}\} \neq \emptyset.
\end{aligned} \tag{3.7}$$

Proof. In case $\mu \in [1/2, 1)$, the result follows from Lemma 3.1.1 and [17, Theorems 4.1, 5.1 and 6.1]. When $\mu = 1$ we note that the existence and uniqueness of a maximal solution

$$r(\cdot, r_0) \in C^1(J(r_0), E_0) \cap C(J(r_0), E_1)$$

follows from [17, Theorem 4.1(b)]. However, for the semiflow properties, we will consider (3.1) as a fully nonlinear equation, and apply results of Angenent [8]. In particular, for $r \in V_1$ we use the representation $G(r) = -\mathcal{A}(r)r + f(r)$ and (3.4)–(3.5) to see that the Fréchet derivative DG has the structure

$$DG(r) = -\frac{1}{(1+r_x^2)^2}\,\partial_x^4 + \sum_{k=0}^{3} B_k(r)\,\partial_x^k,$$

where the coefficients $B_k(r) \in E_0$ for every $r \in V_1$, $k = 0, \ldots, 3$. From this computation it follows that $DG(r)$ is a uniformly elliptic operator from E_1 to E_0 and so, using the results of [50] as in Claim 1 of Lemma 3.1.1 above, we see that $DG(r) \in \mathcal{M}R_1(E_1, E_0)$ for all $r \in V_1$. Now the fact that (3.1) generates an analytic semiflow on V_1 follows from [8, Corollary 2.9]. $\square$

The results contained in [17] also give the following conditions for global solutions. We have separated this result from the previous existence result because breakdown of solutions to (3.3), in particular an analytic investigation of *pinch-off* behavior of certain solutions, is an open and interesting topic.

Theorem 3.1.2 (Global Solutions)**.** *Let $r_0 \in V_\mu$ for $\mu \in (1/2, 1]$ and suppose there exist $\varepsilon > 0$ and $0 < M < \infty$ so that, for all $t \in J(r_0)$*

- $r(t, r_0)(x) \geq \varepsilon, \ \forall x \in \mathbb{T}, \qquad$ *and*
- $\|r(t, r_0)\|_{E_\mu} \leq M,$

then it must hold that $t^+(r_0) = \infty$, so that $r(\cdot, r_0)$ is a global solution. Conversely, if $r_0 \in V_\mu$ and $t^+(r_0) < \infty$, i.e. the solution breaks down in finite–time, then one, or both, of the conditions stated must fail to hold.

We can also state the following result regarding analyticity of the maximal solutions $r(\cdot, r_0)$ in both space and time.

Theorem 3.1.3 (Regularity of Solutions)**.** *Under the same assumptions as in Theorem 3.1.1, it follows that*

$$r(\cdot, r_0) \in C^\omega((0, t^+(r_0)) \times \mathbb{T}) \qquad \text{for all} \quad r_0 \in V_\mu, \quad \mu \in [1/2, 1]. \tag{3.8}$$

Proof. Here we rely on an idea that goes back to Masuda [54] and Angenent [8, 9] to introduce parameters and use the implicit function theorem to obtain regularity results for solutions, see also [38, 39, 37].

First, for $a \in \mathbb{R}$ let $T_a : \mathbb{T} \to \mathbb{T}$ be the translation operator, where $T_a(x)$ denotes the unique element in $\mathbb{T}$ that is in the coset $[x+a] \in \mathbb{R}/2\pi\mathbb{Z}$ of $(x+a)$. T_a naturally acts on functions $u \in C(\mathbb{T},\mathbb{R})$ by virtue of $(T_a u)(x) := u(T_a(x))$. As in [38] one shows that, for $a \in \mathbb{R}$, the family of translations $\{T_{ta} : t \in \mathbb{R}\}$ induces a strongly continuous group of contractions on any of the spaces E_μ, with infinitesimal generator A_a given by

$$D(A_a) = h^{1+4\mu+\alpha}(\mathbb{T},\mathbb{R}), \qquad A_a = a\partial_x.$$

Let $r_0 \in V_\mu$ be fixed, and let

$$r = r(\cdot, r_0) \in C^1_{1-\mu}(J(r_0), E_0) \cap C_{1-\mu}(J(r_0), E_1)$$

be the unique solution to (3.3) on the maximal interval of existence $J(r_0) = [0, t^+(r_0))$. Let $t_1 \in (0, t^+(r_0))$ be fixed and set $I := [0, t_1]$. Then there exists $\delta > 0$ such that $(1+\lambda)t \in J(r_0)$ for all $(t,\lambda) \in I \times (-\delta,\delta)$. Finally, for $(\lambda, a) \in W := (-\delta,\delta)^2$ we set

$$r_{\lambda,a}(t) := T_{ta} r((1+\lambda)t), \qquad t \in I;$$

i.e. $r_{\lambda,a}(t,x) = r((1+\lambda)t, x+ta)$ for $(t,x) \in I \times \mathbb{T}$. One verifies that

$$r_{\lambda,a} \in \mathbb{E}_1(I) := BUC^1_{1-\mu}(I, E_0) \cap BUC_{1-\mu}(I, E_1).$$

Moreover, since the nonlinear mapping $[r \mapsto G(r)]$ is equivariant with respect to translations, i.e. $T_b G(r) = G(T_b r)$ for any $b \in \mathbb{R}$, we obtain that $r_{\lambda,a}$ is a solution of the parameter-dependent equation

$$\begin{cases} \partial_t v = (1+\lambda)G(v) + a\partial_x v, \quad t > 0 \\ v(0) = r_0, \end{cases} \tag{3.9}$$

on the time interval I.

Now, for $\mathbb{U}(I) := \mathbb{E}_1(I) \cap C(I, V)$ we define

$$\Phi : \mathbb{U}(I) \times W \to \mathbb{E}_0(I) \times E_\mu, \quad \Phi(v,(\lambda,a)) = \big(\partial_t v - (1+\lambda)G(v) - a\partial_x v,\, \gamma v - r_0\big),$$

where $\mathbb{E}_0(I) := BUC_{1-\mu}(I, E_0)$, and we note that $\Phi(r_{\lambda,a}, (\lambda, a)) = (0, 0)$. Moreover,

$$\Phi \in C^\omega\Big(\mathbb{U}(I) \times W, \mathbb{E}_0(I) \times E_\mu\Big), \quad D_1\Phi(r, (0,0)) = \left(\frac{d}{dt} - DG(r), \gamma\right),$$

where we use the same notation for $r = r(\cdot, r_0)$ and its restriction to the time interval I. Exactly as in the proof of [17, Theorem 6.1] one shows that

$$D_1\Phi(r, (0,0)) \in \mathcal{L}_{isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_\mu).$$

Finally, according to the implicit function theorem, c.f. [25, Theorem 15.3] or [27, (10.2.1)], there exist a neighborhood of r in $\mathbb{E}_1(I)$ and a neighborhood of $(0,0)$ in $\mathbb{R}^2$, which we will again denote by $\mathbb{U}(I)$ and W, respectively, and a mapping $g \in C^\omega(W, \mathbb{E}_1(I))$ such that

$$\Phi(v, (\lambda, a)) = (0, 0) \quad \text{if and only if} \quad v = g(\lambda, a)$$

whenever $(v, (\lambda, a)) \in \mathbb{U}(I) \times W$. We conclude that $g(\lambda, a) = r_{\lambda,a}$ and

$$[(\lambda, a) \mapsto r_{\lambda,a}] \in C^\omega(W, \mathbb{U}(I)). \tag{3.10}$$

For $t_0 \in (0, t_1)$ and $x_0 \in \mathbb{T}$ fixed, we see that

$$[(\lambda, a) \mapsto r((1+\lambda)t_0, x_0 + t_0 a)] \in C^\omega(W, \mathbb{R}), \tag{3.11}$$

and the assertion follows since (t_0, x_0) can be chosen arbitrarily. □

3.2 Characterizing The Equilibria of ASD

With well-posedness of (3.1) established, we move on to investigate geometric properties of solutions. We begin our analysis of the long-time behavior of solutions by characterizing and describing the equilibria of (3.1). For this characterization, we make use of a well-known, strict Lyapunov functional for the surface diffusion flow, namely the surface area functional, and a characterization of surfaces of revolution with prescribed mean curvature, as presented by Kenmotsu [47].

Recalling the operator G, as expressed by (3.2) and taking it to be defined on $V_1 \subset h^{4+\alpha}(\mathbb{T})$, one will see that the set of equilibria of (3.1) coincides with the null set of G. Although, from the well-posedness results of the previous section, we know that we can consider (3.1) with initial conditions in $h^{2+\alpha}(\mathbb{T})$, upon which the operator G is not defined, one immediately sees that all equilibria must be in $h^{4+\alpha}(\mathbb{T})$ (in fact, by Theorem 3.1.3, we can even conclude that equilibria are in $C^\infty(\mathbb{T})$). More specifically, if we define equilibria to be those elements $\bar{r} \in V_{1/2} = V \cap h^{2+\alpha}(\mathbb{T})$, such that the maximal solution $r(\cdot, \bar{r})$ satisfies

$$r(t, \bar{r}) = \bar{r}, \qquad t > 0,$$

then it follows immediately that $\bar{r} \in h^{4+\alpha}(\mathbb{T})$ and $G(\bar{r}) = 0$. Now, we proceed by characterizing the elements of the null set of G.

Consider the functional

$$S(r) := \int_{\mathbb{T}} r(x)\sqrt{1 + r_x^2(x)}dx,$$

which corresponds to the surface area of the generated surface $\Gamma(r)$ and is a strict Lyapunov functional for (3.14). Indeed, if $r = r(\cdot, r_0)$ is a solution to (3.1) on the interval $J(r_0)$, then (suppressing the variable of integration)

$$\begin{aligned}
\partial_t S(r(t)) &= \int_{\mathbb{T}} \left[\sqrt{1 + r_x^2(t)} + \frac{r(t) r_x(t)}{\sqrt{1 + r_x^2(t)}}\, \partial_x \right] G(r(t))\, dx \\
&= \int_{\mathbb{T}} \partial_x \left(\frac{r(t)}{\sqrt{1 + r_x^2(t)}}\, \partial_x \mathcal{H}(r(t)) \right) \mathcal{H}(r(t))\, dx \\
&= -\int_{\mathbb{T}} \frac{r(t)}{\sqrt{1 + r_x^2(t)}} \left(\partial_x \mathcal{H}(r(t)) \right)^2\, dx, \qquad\qquad t \in J(r_0) \setminus \{0\},
\end{aligned}$$

where we use integration by parts twice and eliminate boundary terms because of periodicity. Notice that the expression is non-positive for all times $t \in J(r_0) \setminus \{0\}$. Moreover, if $\bar{r}$ is an equilibrium of (3.1) it follows that $\partial_x \mathcal{H}(\bar{r})$ is identically zero on $\mathbb{T}$. Meanwhile, notice by the definition of the operator G that $G(\bar{r}) = 0$ whenever $\partial_x \mathcal{H}(\bar{r}) = 0$. Hence, we conclude that $S(r)$ is a strict Lyapunov functional for (3.1), as claimed, and we also see that the equilibria of (3.1) are exactly those functions $\bar{r} \in h^{4+\alpha}(\mathbb{T})$ for which the mean curvature function $\mathcal{H}(\bar{r})$ is constant on $\mathbb{T}$.

The axisymmetric surfaces with constant mean curvature have been characterized explicitly by

Kenmotsu in [47]. In particular, we see that all equilibria of (3.1) are so-called *undulary* curves, and the *unduloid* surfaces, which are generated by the undulary curves by revolution about the axis of symmetry, are stationary solutions of the original surface diffusion problem (1.1).

Theorem 3.2.1 (Delaunay [26] and Kenmotsu [47])**.** *Any complete surface of revolution with constant mean curvature $\mathcal{H}$ is either a sphere, a catenoid, or a surface whose profile curve is given by the parametric expression, parametrized by the arc-length parameter $s \in \mathbb{R}$,*

$$R(s; \mathcal{H}, B) := \left(\int_0^s \frac{1 + B\sin(\mathcal{H}t)}{\sqrt{1 + B^2 + 2B\sin(\mathcal{H}t)}}\, dt\,, \frac{\sqrt{1 + B^2 + 2B\sin(\mathcal{H}s)}}{|\mathcal{H}|} \right). \tag{3.12}$$

Remarks 3.2.1. We can immediately draw several conclusions from Theorem 3.2.1 and characterize the equilibria of (3.1). We use the notation $R(\mathcal{H}, B)$ to denote the graph in $\mathbb{R}^2$ of the parametric expression $R(\cdot\,; \mathcal{H}, B)$.

a) Although the curves $R(\mathcal{H}, B)$ are well-defined for arbitrary values $B \in \mathbb{R}$ and $\mathcal{H} \neq 0$, it is not difficult to see that, without loss of generality, we may restrict our attention to values $\mathcal{H} > 0$ and $B \geq 0$, c.f. [47, Section 2]

b) When $B = 1$, $R(\mathcal{H}, 1)$ is a family of spheres controlled by the parameter $\mathcal{H}$. The spheres are a well-known family of stable equilibria for the surface diffusion flow, c.f. [36], however they are outside of our current setting because they fail to be continuously differentiable functions on all of $\mathbb{T}$. Moreover, we should note that the spheres represented by $R(\mathcal{H}, 1)$ are in fact a *connected* family of spheres, or a *chain of pearls*, for which even general techniques for (1.1) break down, as the mean curvature is not well-defined at the points of intersection, c.f. Figure 3.1[1]. These families of connected spheres may be interesting objects to investigate in a weaker formulation of ASD, but they fall outside of the current setting.

c) Catenoids, or more precisely the generating catenary curves (which are essentially just the hyperbolic cosine, up to scaling), do not fall into the current setting because they fail to satisfy the periodic boundary conditions, c.f. Figure 3.1.

[1]All of the figures contained herein were generated with the program GNU Octave, version 3.4.3, copyright 2011 John W. Eaton, and GNUPLOT, version 4.4 patchlevel 3, copyright 2010 Thomas Williams, Colin Kelley

d) In case $B > 1$, the curve $R(\mathcal{H}, B)$ is called a *nodary*, which cannot be realized as the graph of a function over the x-axis and hence falls outside the current setting, c.f. Figure 3.2[2].

e) For values $0 \leq B < 1$, $R(\mathcal{H}, B)$ is a family of *undulary* curves, which generate the *unduloid* surfaces. The undulary curves are representable as graphs of functions over the x-axis, which are strictly positive for B in the given range, c.f. Figure 3.3. In fact, the case $B = 0$ corresponds to the cylinder of radius $1/\mathcal{H}$. Hence, by Theorem 3.2.1 above, we conclude that **all equilibria of** (3.1) **fall into the family of undulary curves.**

f) Notice that the curve $R(\mathcal{H}, B)$ is always periodic in both the parameter s and the spacial variable x. In order to ensure that the curve satisfies the 2π-periodic boundary conditions enforced in (3.1) (which we emphasize is a condition regarding periodicity over the variable x and not the arc-length parameter s), we must impose further conditions on the parameters $\mathcal{H} > 0$ and $B > 0$; here we avoid $B = 0$ because the curve $R(\mathcal{H}, 0)$ already satisfies periodic boundary conditions. One will see that if $\mathcal{H}$ and B satisfy the relationship

$$\frac{2\pi\,\mathcal{H}}{k} = \int_0^{2\pi} \frac{1 + B\sin t}{\sqrt{1 + B^2 + 2B\sin t}}\, dt\,, \tag{3.13}$$

then the curve $R(\mathcal{H}, B)$ is $2\pi/k$ periodic in the x variable, for $k \in \mathbb{N}$.

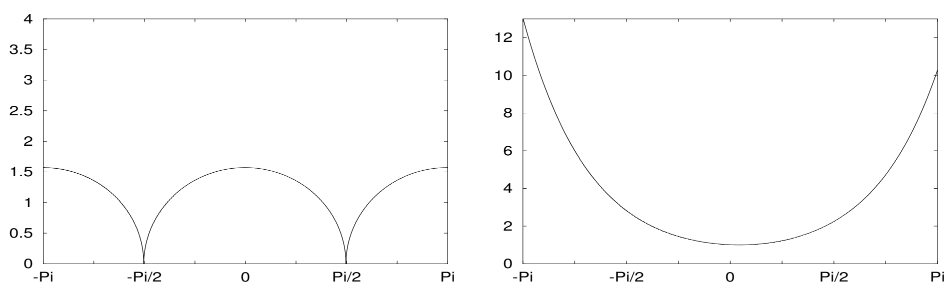

Figure 3.1: Profile curves for a family of spheres and a catenoid, respectively.

[2]The profile functions displayed are actually shifted versions of the curves generated by $R(\mathcal{H}, B)$, where the curve has been shifted along the x–axis to achieve an even representation of the profile curve, c.f. Proposition 3.5.1 below.

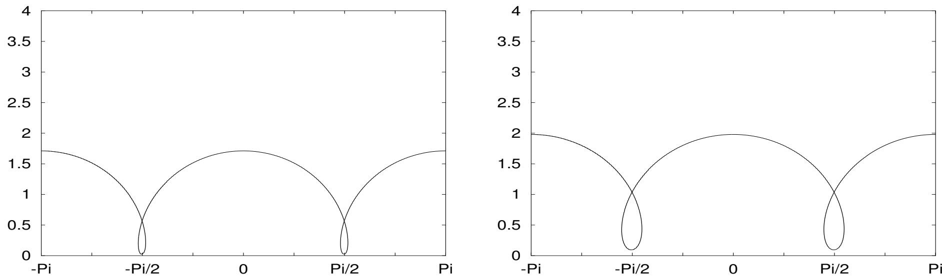

Figure 3.2: Shifted π periodic nodary curves with $B = 1.03$ and $B = 1.1$, respectively.

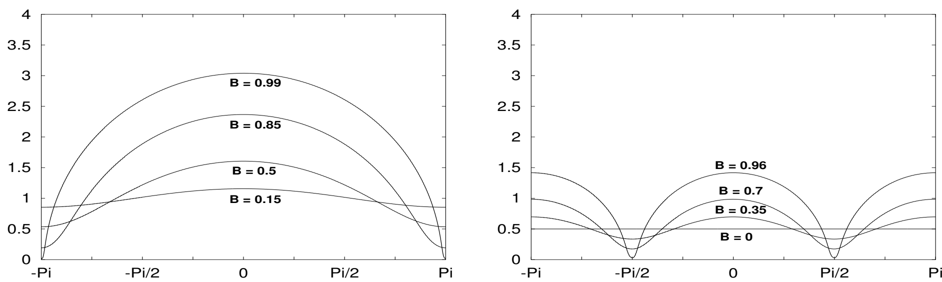

Figure 3.3: Shifted families of 2π and π periodic undulary curves with selected parameter values from $B = 0$ to $B = 0.99$, as indicated.

3.3 Stability Of Cylinders With Large Radius

As seen above, the constant function $r(x) \equiv r_\star$, for $r_\star > 0$, is an equilibrium of (3.3). Moreover, the constant function $r(x) \equiv r_\star$ is associated to the cylinder $\Gamma(r_\star)$ with radius $r_\star$, which is a stationary solution of the original surface diffusion problem (1.1). In this section, we establish tools for and carry out the investigation of nonlinear stability for these equilibria.

3.3.1 Preliminary Analysis and Definitions

Throughout this analysis, we consider an arbitrary $r_\star > 0$ and $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$, unless otherwise stated. Focusing on the properties of solutions near $r_\star$, we shift our equations, including the *shifted* operator

$$G_\star(\rho) := G(\rho + r_\star) = \frac{1}{\rho + r_\star}\, \partial_x \left[\frac{\rho + r_\star}{\sqrt{1+\rho_x^2}}\, \partial_x \mathcal{H}(\rho + r_\star) \right],$$

which maps $\rho \in E_1 \cap U_\star$ to E_0, where we consider $\rho = r - r_\star$, and is in the regularity class C^ω by Lemma 3.1.1; here we take $U_\star := V - r_\star := \{\rho - r_\star : \rho \in V\}$. Now we consider the surface diffusion problem shifted by $r_\star$,

$$\begin{cases} \rho_t(t,x) = G_\star(\rho(t,x)), & x \in \mathbb{T},\ t > 0, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{T}, \end{cases} \tag{3.14}$$

where $\rho_0 := r_0 - r_\star$. We say that

$$\rho = \rho(\cdot, \rho_0) \in C^1(\dot{J}, E_0) \cap C(\dot{J}, E_1) \cap C(J, E_\mu \cap U_\star)$$

is a solution to (3.14), with initial data $\rho_0 \in E_\mu \cap U_\star$, on the interval $J \subset \mathbb{R}_+$ if ρ satisfies (3.14) pointwise, for $t > 0$, and $\rho(0) = \rho_0$. We will investigate the mapping properties of $G_\star$ around 0 in order to gain information about the stability of $r_\star$ in (3.1).

Define the functional

$$F_\star(\rho) = F_\star(\rho; r_\star) := \int_{\mathbb{T}} \big(\rho(x) + r_\star\big)^2 dx,$$

which corresponds to the volume enclosed by the surface $\Gamma(\rho + r_\star)$. Considering the regularity of $F_\star$, it follows from the analyticity of multiplication and integration on little-Hölder spaces that $F_\star$ is of class C^ω from $h^\sigma(\mathbb{T})$ to $\mathbb{R}$, $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. The Fréchet derivative of $F_\star$ is

$$DF_\star(\rho) : \left[h \longmapsto 2 \int_{\mathbb{T}} \big(\rho(x) + r_\star\big) h(x) dx\right] \in \mathcal{L}\left(h^\sigma(\mathbb{T}), \mathbb{R}\right), \qquad \rho \in h^\sigma(\mathbb{T}, \mathbb{R}). \tag{3.15}$$

Moreover, it holds that $F_\star(\rho)$ is conserved along solutions to (3.14). Indeed, if $\rho = \rho(\cdot, \rho_0)$ is a solution to (3.14), then

$$\partial_t F_\star(\rho(t)) = 2 \int_{\mathbb{T}} \big(\rho(t,x) + r_\star\big) \rho_t(t,x) dx = 2 \int_{\mathbb{T}} \partial_x \left[\frac{\big(\rho(t,x) + r_\star\big)}{\sqrt{1 + \rho_x^2(t,x)}}\, \partial_x \mathcal{H}(\rho(t,x) + r_\star) \right] dx = 0,$$

for $t \in J(\rho_0) \setminus \{0\}$, where the last equality holds by periodicity. Thus, conservation of $F_\star$ along the solution ρ follows by continuity of $F_\star$ and convergence of ρ to the initial data ρ_0 in E_μ. From these properties, it follows that

$$\mathcal{M}_\eta^\sigma := \Big\{ \rho \in h^\sigma(\mathbb{T}) : F_\star(\rho) = F_\star(\eta) \Big\}, \qquad \eta \in \mathbb{R},\ \sigma \in \mathbb{R}_+ \setminus \mathbb{Z} \tag{3.16}$$

is a family of invariant level sets for (3.14).

Now we introduce the mapping

$$P_0\rho := \rho - \frac{1}{2\pi}\int_{\mathbb{T}} \rho(x)dx,$$

which defines a projection on $h^\sigma(\mathbb{T})$. We denote by $h_0^\sigma(\mathbb{T})$ the image $P_0\big(h^\sigma(\mathbb{T})\big)$, which exactly coincides with the zero-mean functions on $\mathbb{T}$ in the regularity class $h^\sigma(\mathbb{T})$, and we have the topological decomposition

$$h^\sigma(\mathbb{T}) = h_0^\sigma(\mathbb{T}) \oplus (1-P_0)\big(h^\sigma(\mathbb{T})\big) \cong h_0^\sigma(\mathbb{T}) \oplus \mathbb{R}\,.$$

In what follows, we equate the constant function $[\eta(x) \equiv \eta] \in (1-P_0)\big(h^\sigma(\mathbb{T})\big)$ with the value $\eta \in \mathbb{R}$, and we denote each simply as η.

Consider the operator

$$\Phi(\rho,\tilde{\rho},\eta) := \Big(P_0\rho - \tilde{\rho},\ F_\star(\rho) - F_\star(\eta)\Big),$$

which maps $h^\sigma(\mathbb{T}) \times h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ to $h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ and is of class C^ω, by regularity of the mappings $F_\star$ and P_0. Notice that $\Phi(0,0,0) = (0,0)$ and, using (3.15),

$$D_1\Phi(0,0,0) = \Big(P_0,\ 4\pi r_\star(1-P_0)\Big) \in \mathcal{L}_{isom}(h^\sigma(\mathbb{T}), h_0^\sigma(\mathbb{T}) \times \mathbb{R}), \tag{3.17}$$

i.e. the Fréchet derivative of Φ with respect to the first variable, at the origin, is a linear isomorphism. Hence, it follows from the implicit function theorem that there exist neighborhoods $(0,0) \in U = U_0 \times U_1 \subset h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ and $0 \in U_2 \subset h^\sigma(\mathbb{T})$ and a C^ω function $\psi : U \to U_2$ such that, for all $(\rho,\tilde{\rho},\eta) \in U_2 \times U$,

$$\Phi(\rho,\tilde{\rho},\eta) = (0,0) \qquad \text{if and only if} \qquad \rho = \psi(\tilde{\rho},\eta).$$

Remarks 3.3.1. We can immediately state the following properties of ψ, which follow directly from its definition and elucidate the relationship between P_0 and ψ.

a) $P_0\psi(\tilde{\rho},\eta) = \tilde{\rho}$ for all $(\tilde{\rho},\eta) \in U$.

b) Given $\rho \in \psi(U) \cap \mathcal{M}_\eta^\sigma$, it follows that $\psi(P_0\rho,\eta) = \rho\,$.

c) $\psi(0,\eta) = \eta$, for $\eta \in U_1$. This and the preceding remark follow from the fact that $F_\star(\eta)$ is injective when restricted to $\eta \in (-r_\star, \infty) \subset \mathbb{R}$.

d) It follows from the identity $\Phi(\psi(\tilde{\rho},\eta),\tilde{\rho},\eta) = (0,0)$ and differentiating with respect to $\tilde{\rho}$ that $D_1\Phi(\psi(0,\eta),0,\eta)D_1\psi(0,\eta)h - (h,0) = (0,0)$. From this observation, and the fact that $D_1\Phi(\eta,0,\eta) = (P_0, 4\pi(r_\star+\eta)(1-P_0))$, it follows that

$$D_1\psi(0,\eta)h = h, \qquad h \in h_0^\sigma(\mathbb{T}),\ \eta \in U_1.$$

e) $\psi(U_0,\eta) \subset \mathcal{M}_\eta^\sigma$ for $\eta \in U_1$. Hence, $\psi(\cdot,\eta)$ can be taken as a (local) parametrization of $\mathcal{M}_\eta^\sigma$. Moreover, from the preceding remark and the bijectivity of $\psi(\cdot,\eta)$ from U_0 to $\mathcal{M}_\eta^\sigma \cap U_2$, we can see that $\mathcal{M}_\eta^\sigma \cap U_2$ is a Banach manifold over $h_0^\sigma(\mathbb{T})$ anchored at the point $\eta \in \mathbb{R}$.

f) For $(\tilde{\rho},\eta) \in U$, we have the representation

$$\psi(\tilde{\rho},\eta) = \Big(P_0 + (1-P_0)\Big)\psi(\tilde{\rho},\eta) = \tilde{\rho} + \frac{1}{2\pi}\int_{\mathbb{T}} \psi(\tilde{\rho},\eta)(x)dx,$$

and so we can see that $\mathcal{M}_\eta^\sigma \cap U_2$ can be realized (locally) as the graph of a $\mathbb{R}$-valued analytic function over the zero-mean functions $\tilde{\rho} \in h_0^\sigma(\mathbb{T})$.

g) Although $\psi(\cdot,\eta)$ depends upon the parameter σ, a priori, it follows easily from the preceding representation that

$$\psi(\cdot,\eta) : U_0 \cap h_0^{\tilde{\sigma}}(\mathbb{T}) \to h^{\tilde{\sigma}}(\mathbb{T}), \quad \tilde{\sigma} \in \mathbb{R}_+ \setminus \mathbb{Z},$$

so that ψ preserves the spacial regularity of functions regardless of the regularity parameter σ with which ψ was constructed. However, notice that the neighborhood U_0 will remain intrinsically linked with the parameter which was used to construct ψ.

With the established invariance and local structure of the sets $\mathcal{M}_\eta^\sigma$, it follows that the dynamics governing solutions to (3.1) manifest in the tangent space to the manifold $\mathcal{M}_\eta^\sigma \cap U_2$. Hence, if we reduce (3.1) to a local system **on** these manifolds, then we will have captured all of the dynamics of the problem. Remarks 3.3.1(d) is the first observation toward this reduced formulation. In fact, one can make use of the properties established in Remarks 3.3.1 to prove the following, even more general, result regarding the properties of the the tangent vectors to $\mathcal{M}_\eta^\sigma$. Although we use other tools to connect the reduced problem (3.18) below with the full problem (3.1), this remark provides good intuition into the nature of these manifolds.

Remark 3.3.2. Given $(\tilde{\rho}, \eta) \in U$ it follows that $D_1\psi(\tilde{\rho}, \eta) \circ P_0 = id_{T_{\psi(\tilde{\rho},\eta)}\mathcal{M}_\eta^\sigma}$, where $T_\rho\mathcal{M}_\eta^\sigma$ denotes the tangent space to the manifold $\mathcal{M}_\eta^\sigma$ at the point ρ.

3.3.2 The Reduced Problem

Fix $\alpha \in (0,1)$ and we denote the spaces

$$F_0 := h_0^\alpha(\mathbb{T}), \quad F_1 := h_0^{4+\alpha}(\mathbb{T}), \quad \text{and} \quad F_\mu := (F_0, F_1)_{\mu,\infty}^0, \quad \mu \in (0,1),$$

so that $F_\mu = P_0 E_\mu$ for $\mu \in [0,1]$. Define the operator

$$\mathcal{G}_\star(\tilde{\rho}, \eta) = \mathcal{G}_\star(\tilde{\rho}, \eta; r_\star) := P_0\, G\big(\psi(\tilde{\rho}, \eta) + r_\star\big),$$

which is defined for all $(\tilde{\rho}, \eta) \in U \subset F_0 \times \mathbb{R}$ with $\tilde{\rho} \in U_0 \cap F_1$.

Now we consider the *reduced* problem for the zero-mean functions

$$\begin{cases} \tilde{\rho}_t(t,x) = \mathcal{G}_\star(\tilde{\rho}(t,x), \eta), & t > 0,\ x \in \mathbb{T}, \\ \tilde{\rho}(0,x) = \tilde{\rho}_0(x), & x \in \mathbb{T}, \end{cases} \tag{3.18}$$

where $\tilde{\rho}_0 := P_0 r_0 = P_0(r_0 - r_\star)$. One will note that we should insist on $\psi(\tilde{\rho}, \eta)(x) > -r_\star$ for all $x \in \mathbb{T}$ in order to guarantee that $G(\psi(\tilde{\rho}, \eta) + r_\star)$ is well-defined. However, we can assume, without loss of generality, that the neighborhood U is chosen small enough to ensure this property holds for all $(\tilde{\rho}, \eta) \in U$.

Remark 3.3.3. Throughout most of the analysis that follows, we will treat the parameter η as a free parameter, although it has a very specific interpretation in relation to (3.3). If one is given initial data r_0 close to $r_\star$, then the parameter η is chosen so that

$$F_\star(\eta) = F_\star(r_0)\,.$$

a) Essentially, this parameter allows for the possibility that the volume enclosed by the surface $\Gamma(r_0)$ differs from that of the cylinder $\Gamma(r_\star)$, thereby allowing us to handle non-volume-preserving perturbations r_0 of the cylinder $r_\star$.

b) From a more general viewpoint, one can see that the family $\{\mathcal{M}^\sigma_\eta \cap \psi(U) : \eta \in U_1\}$ forms a dimension 1 foliation of a neighborhood of the positive real axis $\mathbb{R}_+ \subset h^\sigma(\mathbb{T})$ and the parameter η separates the leaves of the foliation.

For $\mu \in (0,1]$ and closed intervals $J \subseteq \mathbb{R}_+$ with $0 \in J$, define the spaces

$$\begin{aligned}\mathbb{E}_0(J) &:= BUC_{1-\mu}(J, E_0),\\ \mathbb{E}_1(J) &:= BUC^1_{1-\mu}(J, E_0) \cap BUC_{1-\mu}(J, E_1),\end{aligned}$$

and

$$\begin{aligned}\mathbb{F}_0(J) &:= BUC_{1-\mu}(J, F_0),\\ \mathbb{F}_1(J) &:= BUC^1_{1-\mu}(J, F_0) \cap BUC_{1-\mu}(J, F_1),\end{aligned}$$

within which we will discuss solutions to the shifted problem (3.14) and the reduced problem (3.18), respectively.

In order to connect these two problems, we will make use of the *lifting* map ψ, defined in the previous section. To ensure that ψ is well-defined on $\mathbb{F}_1$, we must restrict our attention to functions which map into an appropriate neighborhood $U_0 \subset F_0$ of 0. In particular, we assume that U_0 is given so that

$$\psi(\cdot, \eta) : U_0 \subset F_0 \to E_0, \quad \eta \in U_1,$$

is in the regularity class C^ω and, without loss of generality, we assume that U_0 is given sufficiently small so that ψ and the derivative $D_1\psi$ are bounded on $U = U_0 \times U_1$. More precisely, U_0 is chosen sufficiently small so that there exists a constant $N > 0$ for which

$$\|\psi(\tilde{\rho}, \eta)\|_{E_0} \leq N \quad \text{and} \quad \|D_1\psi(\tilde{\rho}, \eta)\|_{\mathcal{L}(F_0, E_0)} \leq N, \tag{3.19}$$

for all $(\tilde{\rho}, \eta) \in U = U_0 \times U_1$.

Lemma 3.3.4. *Fix $\eta \in U_1$ and $J := [0, T]$ for $T > 0$. Then*

$$\psi(\cdot, \eta) : \mathbb{F}_1(J) \cap C(J, U_0) \longrightarrow \mathbb{E}_1(J), \qquad \textit{with} \quad \psi(\tilde{\rho}, \eta)(t) := \psi(\tilde{\rho}(t), \eta).$$

Moreover, if $\tilde{\rho}_0 \in F_\mu$ and $\tilde{\rho} = \tilde{\rho}(\cdot, \tilde{\rho}_0) \in \mathbb{F}_1(J) \cap C(J, U_0)$ is a solution to (3.18)*, for some $\mu \in [1/2, 1]$, then $\rho := \psi(\tilde{\rho}, \eta)$ is the unique solution on the interval J to* (3.14)*, with initial data $\rho_0 := \psi(\tilde{\rho}_0, \eta) \in E_\mu$.*

Proof. First notice that the embeddings

$$\mathbb{F}_1(J) \hookrightarrow BUC(J, F_\mu) \hookrightarrow BUC(J, F_0), \qquad \mu \in [1/2, 1], \tag{3.20}$$

follow from [2, Theorem III.2.3.3] and the continuous embedding of little-Hölder spaces, respectively.

To see that the mapping property for $\psi(\cdot, \eta)$ holds, let $\tilde{\rho} \in \mathbb{F}_1(J) \cap C(J, U_0)$. Uniform continuity and differentiability of the function $\psi(\tilde{\rho}(\cdot), \eta)$ follows from the regularity of ψ and $\tilde{\rho}$, and compactness of the interval J. Hence we focus on demonstrating that $\psi(\tilde{\rho}(\cdot), \eta)$ satisfies the boundedness conditions for $\mathbb{E}_1(J)$. In the case $\mu \in [1/2, 1)$, it follows from Remarks 3.3.1(f) and (3.19) that, for $t \in \dot{J}$,

$$\begin{aligned} t^{1-\mu} \|\psi(\tilde{\rho}(t), \eta)\|_{E_1} &\leq t^{1-\mu} \|\tilde{\rho}(t)\|_{F_1} + \frac{t^{1-\mu}}{2\pi} \int_{\mathbb{T}} |\psi(\tilde{\rho}(t), \eta)(x)| dx \\ &\leq \|\tilde{\rho}\|_{\mathbb{F}_1(J)} + t^{1-\mu} \|\psi(\tilde{\rho}(t), \eta)\|_{C(\mathbb{T})} \\ &\leq \|\tilde{\rho}\|_{\mathbb{F}_1(J)} + T^{1-\mu} N, \\ \text{and} \qquad \lim_{t \to 0} t^{1-\mu} &\|\psi(\tilde{\rho}(t), \eta)\|_{E_1} = 0. \end{aligned} \tag{3.21}$$

From (3.21) we conclude that $\psi(\tilde{\rho}, \eta) \in BUC_{1-\mu}(J, E_1)$. Meanwhile, looking at the time derivative of $\psi(\tilde{\rho}, \eta)$, we note that $\partial_t \psi(\tilde{\rho}(t), \eta) = D_1 \psi(\tilde{\rho}(t), \eta) \partial_t \tilde{\rho}(t)$ and so we again make use of (3.19) to see that

$$
\begin{aligned}
t^{1-\mu}\|\partial_t\psi(\tilde{\rho}(t),\eta)\|_{E_0} &\leq \|D_1\psi(\tilde{\rho}(t),\eta)\|_{\mathcal{L}(F_0,E_0)}t^{1-\mu}\|\partial_t\tilde{\rho}(t)\|_{F_0} \\
&\leq N\|\tilde{\rho}\|_{\mathbb{F}_1(J)} < \infty, \\
\text{and} \qquad \lim_{t\to 0} t^{1-\mu}\|\partial_t\psi(\tilde{\rho}(t),\eta)\|_{E_0} &= 0.
\end{aligned}
$$

Hence, making use of the embedding $F_1 \hookrightarrow F_0$, we see that $\psi(\tilde{\rho},\eta) \in \mathbb{E}_1(J)$, as desired. Meanwhile, when $\mu = 1$ we again get continuity and differentiability from the regularity of the mappings $\tilde{\rho}$ and ψ.

To see that the second part of the lemma holds, observe by (3.20) that $\rho_0 := \psi(\tilde{\rho}_0,\eta) \in E_\mu \cap U_\star$. Hence, by Theorem 3.1.1, there exists a unique maximal solution

$$
r(\cdot,\rho_0) \in C^1_{1-\mu}(J(\rho_0),E_0) \cap C_{1-\mu}(J(\rho_0),E_1)
$$

to (3.14) on some maximal interval of existence $J(\rho_0) = [0,t^+(\rho_0))$. It suffices to show that $\rho_t(t) = G_\star(\rho(t))$ for $t \in \dot{J} := (0,T]$, since this will imply that $\rho(t) = r(t,\rho_0)$ by uniqueness and maximality of the solution $r(\cdot,\rho_0)$. In order to conclude this, let $t \in \dot{J}$ and consider the auxiliary problem

$$
\begin{cases}
\dot{\gamma}(\tau) = G_\star(\gamma(\tau)), \quad \text{for } \tau \in [0,\varepsilon], \\
\gamma(0) = \rho(t),
\end{cases}
$$

which has a unique solution $\gamma \in C^1([0,\varepsilon],E_0) \cap C([0,\varepsilon],E_1)$ by Theorem 3.1.1, provided we choose $\varepsilon > 0$ sufficiently small for the particular value $\rho(t) \in E_1$. Note that, by the regularity of γ we have

$$
\dot{\gamma}(0) = G_\star(\gamma(0)) = G_\star(\rho(t)).
$$

Further, note that $\rho(t) \in \mathcal{M}^{4+\alpha}_\eta$, from which we conclude that $\gamma(\tau) \in \mathcal{M}^{4+\alpha}_\eta$ and by Remarks 3.3.1

we have the representation $\gamma(\tau) = \psi(P_0\gamma(\tau), \eta)$, $\tau \in [0, \varepsilon]$. Finally, we see that

$$\begin{aligned} G_\star(\rho(t)) = \dot\gamma(0) = \partial_\tau \left(\psi(P_0\gamma(\tau), \eta)\right)\Big|_{\tau=0} &= D_1\psi(P_0\gamma(0), \eta)P_0\dot\gamma(0) \\ &= D_1\psi(P_0\rho(t), \eta)P_0G_\star(\rho(t)) = D_1\psi(\tilde\rho(t), \eta)\mathcal{G}_\star(\tilde\rho(t), \eta) \\ &= \partial_t \left(\psi(P_0\rho(t), \eta)\right) = \rho_t(t), \end{aligned}$$

which concludes the proof. □

3.3.3 Mapping Properties of $D_1\mathcal{G}_\star(0, \eta)$

Notice that the points $(0, \eta) \in U$ are equilibria of (3.18), and they correspond to the cylinders $\Gamma(r_\star + \eta)$. We are interested in the spectral properties of the linearization of $\mathcal{G}_\star$ about these equilibria. In particular, we compute the Fréchet derivative

$$D_1\mathcal{G}_\star(0, \eta)h = P_0\, DG_\star(\psi(0, \eta))\, D_1\psi(0, \eta)h = P_0\, DG_\star(\eta)\, D_1\psi(0, \eta)h\,,$$

for $h \in F_1$. Hence, by Remarks 3.3.1(d) we derive the formula

$$D_1\mathcal{G}_\star(0, \eta) = P_0DG(\eta)\big|_{F_1} = DG_\star(\eta)\big|_{F_1}, \tag{3.22}$$

where the last equality is verified by applying the divergence theorem to the linearization

$$DG_\star(\eta) = -\partial_x^2\left(\frac{1}{(r_\star + \eta)^2} + \partial_x^2\right). \tag{3.23}$$

Utilizing the Fourier series representation of functions in $h^\sigma(\mathbb{T})$, c.f. Section 2.1.2, we find the eigenvalues of this linearized operator. In particular, for $h \in E_1$,

$$\begin{aligned} (\lambda - DG_\star(\eta))\, h &= \left(\lambda + \partial_x^2\left(\frac{1}{(r_\star + \eta)^2} + \partial_x^2\right)\right)\sum_{k\in\mathbb{Z}} \hat h(k)e_k \\ &= \sum_{k\in\mathbb{Z}}\left(\lambda - k^2\left(\frac{1}{(r_\star + \eta)^2} - k^2\right)\right)\hat h(k)e_k \\ \implies \qquad \sigma_p(DG_\star(\eta)) &= \left\{k^2\left(\frac{1}{(r_\star + \eta)^2} - k^2\right) : k \in \mathbb{Z}\right\}. \end{aligned}$$

Noting that the embedding $E_1 \hookrightarrow E_0$ is compact, it follows that the resolvent $R(\lambda) := (\lambda - DG_\star(\eta))^{-1}$ is a compact operator, λ in the resolvent set $\rho(DG_\star(\eta))$, c.f. [32, Proposition II.5.8]. It follows from classic theory of linear operators that the spectrum $\sigma(DG_\star(\eta))$ consists entirely of isolated eigenvalues of finite multiplicity, see Kato [46, Theorem III.6.29] for instance. Hence, $\sigma_p(DG_\star(\eta)) = \sigma(DG_\star(\eta))$

Remark 3.3.5. If $r_\star + \eta > 1$, then $\sigma(DG_\star(\eta)) \subset (-\infty, 0]$, however the spectrum will always contain 0. The presence of this 0 eigenvalue can be seen as a consequence of the fact that the equilibria $r_\star + \eta$ are not isolated in the space E_1. Hence, by passing to the operator $\mathcal{G}_\star$, which acts on an open subset of the zero-mean functions F_1, we eliminate the nontrivial equilibria (since the only constant function in F_1 is the zero function) and thereby eliminate the zero eigenvalue. In particular, one easily computes that

$$\sigma(D_1\mathcal{G}_\star(0,\eta)) = \left\{ k^2 \left(\frac{1}{(r_\star + \eta)^2} - k^2 \right) : k \in \mathbb{Z} \setminus \{0\} \right\}, \qquad \eta \in U_1. \tag{3.24}$$

Before we return to the problem (3.1), we state the following maximal regularity result for the linearization $D_1\mathcal{G}_\star(0,\eta)$. For this result, we define the exponentially weighted maximal regularity spaces

$$\mathbb{F}_j(\mathbb{R}_+, \omega) := \Big\{ f : (0,\infty) \to F_0 \text{ s.t. } [t \mapsto e^{\omega t} f(t)] \in \mathbb{F}_j(\mathbb{R}_+) \}, \qquad \omega \in \mathbb{R},\ j = 0, 1,$$

which are Banach spaces when equipped with the norms $\|u\|_{\mathbb{F}_j(\mathbb{R}_+,\omega)} := \|e^{\omega t} u\|_{\mathbb{F}_j(\mathbb{R}_+)}$, $j = 0, 1$.

Theorem 3.3.1. *Suppose $r_\star > 1$ and $\mu \in (0,1]$. There exist nonzero positive constants $\delta = \delta(r_\star)$ and $\omega = \omega(r_\star, \delta)$ such that*

$$\Big(\mathbb{F}_0(\mathbb{R}_+, \omega), \mathbb{F}_1(\mathbb{R}_+, \omega) \Big)$$

is a pair of maximal regularity for $-D_1\mathcal{G}_\star(0,\eta)$, given any $\eta \in (-\delta, \delta)$. I.e. the property

$$(\partial_t - D_1\mathcal{G}_\star(0,\eta), \gamma) \in \mathcal{L}_{isom}\left(\mathbb{F}_1(\mathbb{R}_+, \omega), \mathbb{F}_0(\mathbb{R}_+, \omega) \times h_0^{4\mu+\alpha}(\mathbb{T}) \right),$$

holds uniformly for $\eta \in (-\delta, \delta)$.

Proof. Fix $\delta > 0$ so that $(-\delta, \delta) \subset U_1 \cap (1 - r_\star, \infty)$. Following the notation and definitions of

Chapter 2, it is clear from the representation (3.23) that $-DG_\star(\eta)$ is a uniformly elliptic operator from which we see, by Theorem 2.4.4, that $DG_\star(\eta)$ generates an analytic semigroup on $h^\alpha(\mathbb{T},\mathbb{C})$ with domain $h^{4+\alpha}(\mathbb{T},\mathbb{C})$. Since $h_0^\alpha(\mathbb{T},\mathbb{C})$ inherits the topology of $h^\alpha(\mathbb{T},\mathbb{C})$ and the projection P_0 commutes with $DG_\star$, the analogous resolvent estimates hold for $D_1\mathcal{G}_\star(0,\eta)$ and so we conclude that $D_1\mathcal{G}_\star(0,\eta)$ generates an analytic semigroup on $h_0^\alpha(\mathbb{T},\mathbb{C})$ with domain $h_0^{4+\alpha}(\mathbb{T},\mathbb{C})$. Moreover, from (3.24) we see that $type(D_1\mathcal{G}_\star(0,\eta)) < 0$ for all $\eta \in (-\delta,\delta)$, where $type(B)$ denotes the spectral type of the semigroup generator B. In particular, it follows that

$$type(D_1\mathcal{G}_\star(0,\eta)) < \frac{1-(r_\star-\delta)^2}{(r_\star-\delta)^2} < 0, \qquad \eta \in (-\delta,\delta).$$

Now, choose $\omega \in \left(0, \frac{(r_\star-\delta)^2-1}{(r_\star-\delta)^2}\right)$ and the remainder of the result follows from [2, Theorem III.3.4.1 and Remarks 3.4.2(b)] and the restriction of maximal regularity to the subspaces $h_0^\sigma(\mathbb{T})$. Notice, the characterization $\gamma\mathbb{F}_1(\omega) = h_0^{4\mu+\alpha}(\mathbb{T})$ follows from stability of little-Hölder spaces under continuous interpolation. □

3.3.4 Exponential Stability of Cylinders With Radius $r_\star > 1$

Our main result regarding stability of cylinders in the axisymmetric surface diffusion flow (3.1) establishes asymptotic stability under perturbations which maintain the prescribed periodic boundary conditions and symmetry about the same axis of rotation (which we are taking to be the x-axis in our setting). One feature of our result that we point out is the fact that it establishes a form of stability which allows for perturbations which are not volume-preserving. In particular, we refer to *asymptotic stability* of the cylinder $\Gamma(r_\star)$ by which we mean that small perturbations of $\Gamma(r_\star)$ will have global solutions to (3.1) which converge to a cylinder $\Gamma(r_\star+\eta)$, where $r_\star \neq r_\star+\eta$ in general.

Theorem 3.3.2 (Exponential Stability)**.** *Fix $\alpha \in (0,1)$, $\mu \in [1/2,1]$, so that $4\mu+\alpha \notin \mathbb{Z}$, and $r_\star > 1$. There exist nonzero positive constants $\varepsilon = \varepsilon(r_\star)$, $\delta = \delta(r_\star)$ and $\omega = \omega(r_\star,\delta)$, such that problem* (3.3) *with initial data $r_0 \in \mathbb{B}_{E_\mu}(r_\star,\varepsilon)$ has a unique global solution*

$$r(\cdot,r_0) \in C^1_{1-\mu}(\mathbb{R}_+,E_0) \cap C_{1-\mu}(\mathbb{R}_+,E_1),$$

and there exists $\eta = \eta(r_0) \in (-\delta, \delta)$ *and* $M = M(\alpha) > 0$ *for which the bound*

$$t^{1-\mu}\|r(t,r_0) - (r_\star + \eta)\|_{E_1} + \|r(t,r_0) - (r_\star + \eta)\|_{E_\mu} \le e^{-\omega t} M \|r_0 - r_\star\|_{E_\mu}$$

holds uniformly for $t > 0$.

Proof. **(i)** Let δ, $\omega > 0$ be the constants given by Theorem 3.3.1 and consider the operator

$$\mathcal{K}(\tilde{\rho}, \tilde{\rho}_0, \eta) := \Big(\partial_t \tilde{\rho} - \mathcal{G}_\star(\tilde{\rho}, \eta),\ \gamma\tilde{\rho} - \tilde{\rho}_0\Big),$$

acting on $\mathbb{U} := \Big(\mathbb{F}_1(\mathbb{R}_+, \omega) \cap C(\mathbb{R}_+, U_0)\Big) \times \Big(U_0 \cap F_\mu\Big) \times U_1$ which is open in the Banach space $\mathbb{F}_1(\mathbb{R}_+, \omega) \times F_\mu \times \mathbb{R}$.

Considering the space into which $\mathcal{K}$ maps, first notice that

$$\gamma : \mathbb{F}_1(\mathbb{R}_+, \omega) \to (F_0, F_1)^0_{\mu,\infty}$$

follows from [17, Lemma 2.2(a)]. Hence, $\gamma\tilde{\rho} \in F_\mu$ and ∂_t maps $\mathbb{F}_1(\mathbb{R}_+, \omega)$ into $\mathbb{F}_0(\mathbb{R}_+, \omega)$ by definition of the spaces $BUC^1_{1-\mu}(J, E)$. To see that $\mathcal{G}_\star(\cdot, \eta)$ maps $\mathbb{U}$ into $\mathbb{F}_0(\mathbb{R}_+, \omega)$, choose $\tilde{\rho} \in \mathbb{U}$ and notice that $\tilde{\rho}(t) \in U_0 \cap h^{2+\alpha}_0(\mathbb{T})$, for $t > 0$, from the embeddings (3.20). Utilizing the explicit quasilinear representation of the operator G, as given by (3.4)–(3.5), whereby

$$\mathcal{G}_\star(\tilde{\rho}(t), \eta) = P_0\Big(- \mathcal{A}\Big(\psi(\tilde{\rho}(t), \eta) + r_\star\Big)(\psi(\tilde{\rho}(t), \eta) + r_\star) + f\big(\psi(\tilde{\rho}(t), \eta) + r_\star\big)\Big),$$

one will easily conclude the desired mapping property for the operator $\mathcal{G}_\star$. For instance, we have seen that $\mathcal{A}(\rho)\rho = b_1(\rho)\partial_x^4\rho + b_2(\rho)\partial_x^3\rho$, where the functions b_i only depend on ρ, ρ_x and ρ_{xx}, $i = 1, 2$. Hence, it follows that

$$\begin{aligned}
&e^{\omega t} t^{1-\mu} \left\| \mathcal{A}\Big(\psi(\tilde{\rho}(t), \eta) + r_\star\Big)(\psi(\tilde{\rho}(t), \eta) + r_\star) \right\|_{E_0} \\
&\qquad \le e^{\omega t} t^{1-\mu} \left\| \partial_x^4 \psi(\tilde{\rho}(t), \eta) \right\|_{E_0} \left\| b_1\big(\psi(\tilde{\rho}(t), \eta) + r_\star\big) \right\|_{E_0} \\
&\qquad\quad + e^{\omega t} t^{1-\mu} \left\| \partial_x^3 \psi(\tilde{\rho}(t), \eta) \right\|_{E_0} \left\| b_2\big(\psi(\tilde{\rho}(t), \eta) + r_\star\big) \right\|_{E_0},
\end{aligned}$$

for $t > 0$. From here, we take advantage of the boundedness of $\psi(\tilde{\rho}(t), \eta)$ in the topology of $F_{1/2}$, in conjunction with the explicit formulas for b_i, in order to bound the terms $\|b_i(\psi(\tilde{\rho}(t),\eta)+r_\star)\|_{E_0}$, uniformly in t. Meanwhile, the representation given by Remarks 3.3.1(d) and the fact that $\tilde{\rho} \in \mathbb{F}_1(\mathbb{R}_+,\omega)$ yield the bounds

$$e^{\omega t}t^{1-\mu}\|\partial_x^k \psi(\tilde{\rho}(t),\eta)\|_{E_0} = e^{\omega t}t^{1-\mu}\|\partial_x^k \tilde{\rho}(t)\|_{F_0} \leq \|e^{\omega t}\tilde{\rho}\|_{\mathbb{F}_1(\mathbb{R}_+)}, \qquad k = 1,\ldots,4.$$

Analogous methods work for the remaining terms of the function $G_\star\big(\psi(\tilde{\rho}(t),\eta)\big)$, since we can always isolate an element of the form $\partial_x^k\psi(\tilde{\rho}(t),\eta)$, and bound the remaining elements using boundedness in $F_{1/2}$. We conclude the result by noting that the linear projection P_0 adds no complexity to acquiring the necessary bounds.

With the establishment of the spaces into which the operator $\mathcal{K}$ maps, we move on with our analysis. Regarding the regularity of $\mathcal{K}$, it can be shown that $\mathcal{G}_\star$ is C^ω via substitution operators and the derivative ∂_t and the trace operator γ are linear. Hence, it follows that

$$\mathcal{K} \in C^\omega\Big(\mathbb{U}, \mathbb{F}_0(\mathbb{R}_+,\omega) \times F_\mu\Big).$$

Meanwhile, notice that $\mathcal{K}(0,0,0) = (0,0)$ and

$$D_1\mathcal{K}(0,0,0) = \Big(\partial_t - D_1\mathcal{G}_\star(0,0), \gamma\Big) \in \mathcal{L}_{isom}\Big(\mathbb{F}_1(\mathbb{R}_+,\omega), \mathbb{F}_0(\mathbb{R}_+,\omega) \times F_\mu\Big),$$

by Theorem 3.3.1. Hence, we conclude from the implicit function theorem that there exists an open neighborhood $0 \in \tilde{U} \subset F_\mu \times \mathbb{R}$ and a C^ω mapping $\kappa : \tilde{U} \to \mathbb{F}_1(\mathbb{R}_+,\omega)$ such that

$$\mathcal{K}(\kappa(\tilde{\rho}_0,\eta),\tilde{\rho}_0,\eta) = (0,0) \quad \text{for all} \quad (\tilde{\rho}_0,\eta) \in \tilde{U}.$$

In particular, $\kappa(\tilde{\rho}_0,\eta)$ is a global solution to (3.18) with parameter η and initial data $\tilde{\rho}_0 \in F_\mu$, where we assume, without loss of generality, that $\tilde{U} \subseteq U$.

(ii) Choose $\varepsilon > 0$ so that for every $r_0 \in \mathbb{B}_{F_\mu}(r_\star,\varepsilon)$, there exists $\eta \in (-r_\star,\infty)$ for which

$$(P_0 r_0,\eta) \in \tilde{U} \quad \text{and} \quad F_\star(r_0 - r_\star; r_\star) = F_\star(\eta; r_\star).$$

The existence of such a constant ε is guaranteed by the continuity of P_0 and $F_\star$, injectivity of $F_\star(\eta; r_\star)$ for $\eta \in (-r_\star, \infty)$ and the fact that $P_0 r_\star = 0$.

Let $r_0 \in \mathbb{B}_{F_\mu}(r_\star, \varepsilon)$ and fix $\eta = \eta(r_0)$ as mentioned so that $F_\star(r_0 - r_\star; r_\star) = F_\star(\eta; r_\star)$. Define the function

$$r := \psi(\kappa(P_0 r_0, \eta), \eta) + r_\star, \tag{3.25}$$

where $\psi(\kappa(P_0 r_0, \eta), \eta)(t) := \psi(\kappa(P_0 r_0, \eta)(t), \eta)$, and we will demonstrate that r satisfies the desired properties claimed in the theorem.

To see that r is the unique global solution to (3.3) with initial data r_0, first fix $T > 0$ and consider the interval $J := [0, T]$. By the choice of $\varepsilon > 0$ we know that $(P_0 r_0, \eta) \in \tilde{U}$ and so it follows from part **(i)** above that $\kappa(P_0 r_0, \eta) \in \mathbb{F}_1(\mathbb{R}_+, \omega)$. From this we see that $\kappa(P_0 r_0, \eta) \in \mathbb{F}_1(J)$ is a solution to (3.18) with initial data $P_0 r_0 \in F_\mu$. Thus it follows, by Lemma 3.3.4, that $r \in \mathbb{E}_1(J)$ is the solution on J to the problem (3.3) with initial data

$$\psi(P_0 r_0, \eta) + r_\star = \psi(P_0(r_0 - r_\star), \eta) + r_\star = r_0,$$

where we use Remarks 3.3.1(b) and the fact that $r_0 - r_\star \in \mathcal{M}_\eta^{4\mu+\alpha}$. The claim now follows by the fact that $T > 0$ was arbitrary and by definition of the Fréchet spaces $C_{1-\mu}(\mathbb{R}_+, E)$.

Now, to see that r satisfies the exponential bounds in the second part of the claim, first notice that $\kappa(0, \eta) \equiv 0$ for $\eta \in U_1$. Then, by definition of r, Remarks 3.3.1, and application of the mean value theorem, we see that the expression

$$\begin{aligned}
r(t) - (r_\star + \eta) &= \psi(\kappa(P_0 r_0, \eta)(t), \eta) - \eta = \psi(\kappa(P_0 r_0, \eta)(t), \eta) - \psi(\kappa(0, \eta)(t), \eta) \\
&= \Big(P_0 + (1 - P_0)\Big)\Big(\psi(\kappa(P_0 r_0, \eta)(t), \eta) - \psi(\kappa(0, \eta)(t), \eta)\Big) \\
&= \kappa(P_0 r_0, \eta)(t) + \frac{1}{2\pi}\int_{\mathbb{T}} \Big(\psi(\kappa(P_0 r_0, \eta)(t, x), \eta) - \psi(\kappa(0, \eta)(t, x), \eta)\Big) dx \\
&= \kappa(P_0 r_0, \eta)(t) + \frac{1}{2\pi}\int_{\mathbb{T}}\int_0^1 D_1\psi\big(\tau\kappa(P_0 r_0, \eta)(t), \eta\big)\kappa(P_0 r_0, \eta)(t, x)\, d\tau\, dx,
\end{aligned}$$

holds for all $t > 0$. Notice that

$$e^{\omega t} t^{1-\mu} \|\kappa(P_0 r_0, \eta)(t)\|_{F_1} \leq \|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)}$$

and

$$\sup_{t\in\mathbb{R}_+} \|e^{\omega t}\kappa(P_0 r_0,\eta)(t)\|_{F_\mu}$$

are finite quantities by the fact that $\kappa(P_0 r_0,\eta) \in \mathbb{F}_1(\mathbb{R}_+,\omega)$ and the embedding (3.20). We note that the reference for (3.20) does not explicitly include the unbounded interval $J = \mathbb{R}_+$, however the methods of the proof extend to this unbounded case with little trouble. Meanwhile, the remaining term in $r(t)-(r_\star+\eta)$ above is scalar-valued, so we bound $D_1\psi(\tau\kappa(P_0 r_0,\eta)(t),\eta)\kappa(P_0 r_0,\eta)(t)$ in the $C(\mathbb{T})$-topology, which are then bounded in the $h^\sigma(\mathbb{T})$-topology, for any $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. In particular, observe that, by (3.19),

$$\sup_{\tilde\rho\in U_0} \|D_1\psi(\tilde\rho,\eta)\kappa(P_0 r_0,\eta)(t)\|_{h^\alpha} \le N\|\kappa(P_0 r_0,\eta)(t)\|_{h_0^\alpha}, \qquad t>0,$$

and we conclude that the bounds

$$e^{\omega t} t^{1-\mu}\|r(t)-(r_\star+\eta)\|_{E_1} \le \Big(1+c_1 N\Big)\|\kappa(P_0 r_0,\eta)\|_{\mathbb{F}_1(\mathbb{R}_+,\omega)} \tag{3.26}$$

and

$$e^{\omega t}\|r(t)-(r_\star+\eta)\|_{E_\mu} \le \Big(c_2+c_3 N\Big)\|\kappa(P_0 r_0,\eta)\|_{\mathbb{F}_1(\mathbb{R}_+,\omega)}, \tag{3.27}$$

hold uniformly for $t>0$. Here the constant c_1 comes from the embedding $F_1 \hookrightarrow F_0$, and the constants c_2 and c_3 come from the embeddings (3.20). Finally, by the regularity of κ, we may assume that $\tilde U$ was chosen sufficiently small to ensure that $D_1\kappa$ is uniformly bounded from $\tilde U$ into $\mathbb{F}_1(\mathbb{R}_+,\omega)$. Recalling that $\kappa(0,\eta)=0$, it follows that

$$\begin{aligned}\|\kappa(P_0 r_0,\eta)\|_{\mathbb{F}_1(\mathbb{R}_+,\omega)} &\le \int_0^1 \|D_1\kappa(\tau P_0 r_0,\eta)P_0 r_0\|_{\mathbb{F}_1(\mathbb{R}_+,\omega)}\ d\tau \\ &\le \tilde M\|P_0 r_0\|_{F_\mu} \le M\|r_0-r_\star\|_{E_\mu},\end{aligned} \tag{3.28}$$

where $M := \|P_0\|\sup_{(\tilde\rho,\eta)\in\tilde U}\|D_1\kappa(\tilde\rho,\eta)\|_{\mathcal{L}(F_\mu,\mathbb{F}_1(\mathbb{R}_+,\omega))}$. The claim now follows from (3.28) and the inequalities (3.26)–(3.27). □

3.4 Instability of Cylinders With Radius $0 < r_\star < 1$

In this section we turn our attention to the stability of cylinders with small radius. Again, taking advantage of the reduced problem (3.18) and the connection we have established between it and the original problem (3.1), we proceed with the following result regarding instability of cylinders with radius $0 < r_\star < 1$, in the setting of F_μ. Because of the nature of the instability result, differences in volume between the initial data r_0 and the cylinder $r_\star$ are not a factor in the following argument. In light of this, we will assume that the parameter η, associated with the reduced problem (3.18), is simply taken to be zero for this section.

Theorem 3.4.1 (Instability)**.** *Let $r_\star \in (0,1)$ and $\mu \in [1/2, 1]$ be fixed with $4\mu + \alpha \notin \mathbb{Z}$. Then the equilibrium 0 of (3.18) is unstable in the topology of F_μ for initial values in F_μ.*

Proof. **(i)** Let $r_\star \in (0,1)$ be fixed, and let $L := D_1\mathcal{G}_\star(0,0)$ be the linearization of $\mathcal{G}_\star$ at $\tilde{\rho} = 0$. We can restate the evolution equation (3.18) in the following equivalent form

$$\begin{cases} \tilde{\rho}_t - L\tilde{\rho} = g(\tilde{\rho}), \quad t > 0 \\ \tilde{\rho}(0) = \tilde{\rho}_0, \end{cases} \tag{3.29}$$

where $g(\tilde{\rho}) := \mathcal{G}_\star(\tilde{\rho}, 0) - L\tilde{\rho}$. Using the quasilinear structure of $[\tilde{\rho} \mapsto \mathcal{G}_\star(\tilde{\rho}, 0)]$ it is not difficult to see that for every $\beta > 0$ there exists a number $\varepsilon_0 = \varepsilon_0(\beta) > 0$ such that

$$\|g(\tilde{\rho})\|_{F_0} \le \beta \|\tilde{\rho}\|_{F_1}, \quad \tilde{\rho} \in \mathbb{B}_{F_\mu}(0, \varepsilon_0) \cap F_1, \tag{3.30}$$

where we will be assuming throughout that $\tilde{\rho} \in U_0$, to guarantee that $\mathcal{G}_\star(\tilde{\rho}, 0)$, and subsequently $g(\tilde{\rho})$, is defined. It follows from (3.24) that

$$\sigma(L) \cap [\operatorname{Re} z > 0] \neq \emptyset,$$

and we may choose numbers $\omega, \gamma > 0$ such that

$$[\omega - \gamma \le \operatorname{Re} z \le \omega + \gamma] \cap \sigma(L) = \emptyset \quad \text{and} \quad \sigma_+ := [\operatorname{Re} z > \omega + \gamma] \cap \sigma(L) \neq \emptyset,$$

i.e. the strip $[\omega-\gamma \leq \operatorname{Re} z \leq \omega+\gamma]$ does not intersect $\sigma(L)$ and there is at least one point of $\sigma(L)$ to the right of the line $[\operatorname{Re} z = \omega+\gamma]$.

We define P_+ to be the spectral projection, in F_0, with respect to the spectral set σ_+, and let $P_- := 1 - P_+$. Then $P_+(F_0)$ is finite dimensional and the topological decomposition

$$F_0 = P_+(F_0) \oplus P_-(F_0)$$

reduces L, so that $L = L_+ \oplus L_-$, where $L_\pm$ is the part of L in $P_\pm(F_0)$, respectively, with the domains $D(L_\pm) = P_\pm(F_1)$. Moreover, $P_\pm$ decomposes F_1 by the embedding $F_1 \hookrightarrow F_0$, and, without loss of generality, we can take the norm on F_1 so that

$$\|v\|_{F_1} = \|P_+ v\|_{F_1} + \|P_- v\|_{F_1}.$$

We note that

$$\sigma(L_-) \subset [\operatorname{Re} z < \omega - \gamma], \qquad \sigma(L_+) = \sigma^+ \subset [\operatorname{Re} z > \omega + \gamma].$$

This implies that there is a constant $M_0 \geq 1$ such that

$$\begin{aligned} \|e^{L_- t} P_-\|_{\mathcal{L}(F_0)} &\leq M_0 e^{(\omega-\gamma)t}, \\ \|e^{-L_+ t} P_+\|_{\mathcal{L}(F_0)} &\leq M_0 e^{-(\omega+\gamma)t}, \qquad t \geq 0, \end{aligned} \tag{3.31}$$

where $\{e^{L_- t} : t \geq 0\}$ is the analytic semigroup in $P_-(F_0)$ generated by L_- and $\{e^{L_+ t} : t \in \mathbb{R}\}$ is the group in $P_+(F_0)$ generated by the bounded operator L_+.

From (3.22)–(3.23) and Chapter 2 one sees that $\big(\mathbb{F}_0(J), \mathbb{F}_1(J)\big)$ is a pair of maximal regularity for $-L$ and it is easy to see that $-L_-$ inherits the property of maximal regularity. In particular, the pair $\big(P_-(\mathbb{F}_0(J)), P_-(\mathbb{F}_1(J))\big)$ is a pair of maximal regularity for $-L_-$. In fact, since $type(-\omega + L_-) < -\gamma < 0$ we see that $\big(P_-(\mathbb{F}_0(\mathbb{R}_+)), P_-(\mathbb{F}_1(\mathbb{R}_+))\big)$ is a pair of maximal regularity for $(\omega - L_-)$. This, in turn, implies the a priori estimate

$$\|e^{-\omega t} w\|_{\mathbb{F}_1(J_T)} \leq M_1 \Big(\|w_0\|_{F_\mu} + \|e^{-\omega t} f\|_{\mathbb{F}_0(J_T)} \Big) \tag{3.32}$$

for $J_T := [0, T]$, any $T \in (0, \infty)$ (or $J_T = \mathbb{R}_+$ for $T = \infty$), with a universal constant $M_1 > 0$, where

w is a solution of the linear Cauchy problem

$$\begin{cases} \dot{w} - L_{-}w = f, \\ w(0) = w_0, \end{cases}$$

with $(f, w_0) \in \Big(C\big((0,T), P_{-}F_0\big), P_{-}U_0\Big)$.

(ii) By way of contradiction, suppose that the equilibrium 0 is stable for (3.18). Then for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that (3.29) admits for each $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0,\delta)$ a global solution

$$\tilde{\rho} = \tilde{\rho}(\cdot, \tilde{\rho}_0) \in C^1_{1-\mu}(\mathbb{R}_+, F_0) \cap C_{1-\mu}(\mathbb{R}_+, F_1) \cap C(\mathbb{R}_+, U_0),$$

which satisfies

$$\|\tilde{\rho}(t)\|_{F_\mu} < \varepsilon, \qquad t \geq 0. \tag{3.33}$$

We can assume without loss of generality that β and ε are chosen such that

$$2C_0(M_0 + M_1\gamma)\beta \leq \gamma \qquad \text{and} \qquad \varepsilon \leq \varepsilon_0(\beta), \tag{3.34}$$

where $C_0 := \max\{\|P_{-}\|_{\mathcal{L}(F_0)}, \|P_{+}\|_{\mathcal{L}(F_0)}\}$. As $P_{+}(F_0)$ is finite dimensional, we may also assume that

$$\|P_{+}v\|_{F_\nu} = \|P_{+}v\|_{F_0}, \qquad v \in F_0, \quad \nu \in \{\mu, 1\},$$

where we also use the fact that $P_{+}F_0 \subset D(L^n)$ for every $n \in \mathbb{N}$, c.f. [51, Proposition A.1.2].

CLAIM 1: For any initial value $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0,\delta)$, $P_{+}\tilde{\rho}$ admits the representation

$$P_{+}\tilde{\rho}(t) = -\int_t^\infty e^{L_{+}(t-s)} P_{+}g(\tilde{\rho}(s))\, ds \qquad t \geq 0. \tag{3.35}$$

For this we first establish that, for $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0,\delta)$,

$$e^{-\omega t}\tilde{\rho} \in BC_{1-\mu}(\mathbb{R}_+, F_1) := \left\{ u \in C((0,\infty), F_1) : \sup_{t \in \mathbb{R}_+} t^{1-\mu}\|u(t)\|_{F_1} < \infty \right\}.$$

First notice that the mapping property

$$g : \mathbb{F}_1(J_T) \cap C(J_T, U_0) \to \mathbb{F}_0(J_T), \qquad 0 < T < \infty,$$

which follows in the same way as the mapping property derived for $\mathcal{G}_\star$ in the proof of Theorem 3.3.2 above, together with the inequalities (3.30) and (3.32) yield

$$\begin{aligned} &\|e^{-\omega t} P_- \tilde{\rho}\|_{C_{1-\mu}(J_T, F_1)} \\ &\leq M_1 \Big(\|P_- \tilde{\rho}_0\|_{F_\mu} + C_0 \beta \|e^{-\omega t} P_+ \tilde{\rho}\|_{C_{1-\mu}(J_T, F_1)} + C_0 \beta \|e^{-\omega t} P_- \tilde{\rho}\|_{C_{1-\mu}(J_T, F_1)} \Big) \end{aligned} \tag{3.36}$$

for any $0 < T < \infty$. Due to (3.34), we have $M_1 C_0 \beta \leq 1/2$ and can further conclude

$$\|e^{-\omega t} P_- \tilde{\rho}\|_{C_{1-\mu}(J_T, F_1)} \leq 2M_1 \Big(\|P_- \tilde{\rho}_0\|_{F_\mu} + C_0 \beta \|e^{-\omega t} P_+ \tilde{\rho}\|_{C_{1-\mu}(J_T, F_1)} \Big). \tag{3.37}$$

It follows from (3.33) that

$$t^{1-\mu} \|e^{-\omega t} P_+ \tilde{\rho}(t)\|_{F_1} \leq t^{1-\mu} e^{-\omega t} C_0 \|\tilde{\rho}(t)\|_{F_\mu} \leq C_0 C_1 \varepsilon$$

where $C_1 := \sup\{t^{1-\mu} e^{-\omega t} : t \geq 0\} < \infty$. Inserting this result into (3.37) yields

$$\|e^{-\omega t} \tilde{\rho}\|_{C_{1-\mu}(J_T, F_1)} \leq 2M_1 \|P_- \tilde{\rho}_0\|_{F_\mu} + (2M_1 C_0 \beta + 1) C_0 C_1 \varepsilon \leq C_2 \tag{3.38}$$

for any $0 < T < \infty$. However, since T is arbitrary and (3.38) is independent of T we conclude that $e^{-\omega t} \tilde{\rho} \in BC_{1-\mu}(\mathbb{R}_+, F_1)$, for any initial value $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$. Next we note that, for $s \geq t$, by (3.31)

$$\begin{aligned} \|e^{L_+(t-s)} P_+ g(\tilde{\rho}(s))\|_{F_0} &\leq M_0 C_0 \beta e^{(\omega+\gamma)(t-s)} \|\tilde{\rho}(s)\|_{F_1} \\ &\leq M_0 C_0 \beta e^{\omega t} e^{\gamma(t-s)} s^{\mu-1} \|e^{-\omega s} \tilde{\rho}\|_{C_{1-\mu}(\mathbb{R}_+, F_1)}, \end{aligned} \tag{3.39}$$

which shows that the integral in (3.35) exists for any $t \geq 0$, with convergence in F_1. Moreover,

$$\Big\| \int_t^\infty e^{L_+(t-s)} P_+ g(\tilde{\rho}(s))\, ds \Big\|_{F_0} \leq e^{\omega t} M_0 C_0 C_3 \beta \|e^{-\omega t} \tilde{\rho}\|_{C_{1-\mu}(\mathbb{R}_+, F_1)}, \tag{3.40}$$

where $C_3 := \sup\left\{ \int_t^\infty e^{\gamma(t-s)} s^{\mu-1}\, ds \,:\, t \geq 0\right\} < \infty$. Noting that $w = P_+\tilde{\rho}$ solves the Cauchy problem

$$\begin{cases} \dot{w} - L_+ w = P_+ g(\tilde{\rho}), \\ w(0) = P_+\tilde{\rho}_0, \end{cases}$$

it follows from the variation of parameters formula that, for $t \geq 0$ and $\tau > 0$,

$$P_+\tilde{\rho}(t) = e^{L_+(t-\tau)} P_+\tilde{\rho}(\tau) + \int_\tau^t e^{L_+(t-s)} P_+ g(\tilde{\rho}(s))\, ds.$$

Since this representation holds for any $\tau > 0$, the claim follows from (3.31) and (3.33) by sending τ to ∞.

CLAIM 2: If $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$ and $\|\tilde{\rho}(t, \tilde{\rho}_0)\|_{F_\mu} < \varepsilon$ for all $t \geq 0$, then it must hold that

$$\|P_+\tilde{\rho}_0\|_{F_\mu} \leq 2M_0 M_1 C_3 \|P_-\tilde{\rho}_0\|_{F_\mu}.$$

From (3.35) and (3.39) follows

$$\begin{aligned} &\|e^{-\omega t} P_+\tilde{\rho}\|_{C_{1-\mu}(\mathbb{R}_+, F_0)} \\ &\leq \frac{M_0 C_0 \beta}{\gamma} \Big(\|e^{-\omega t} P_+\tilde{\rho}\|_{C_{1-\mu}(\mathbb{R}_+, F_1)} + \|e^{-\omega t} P_-\tilde{\rho}\|_{C_{1-\mu}(\mathbb{R}_+, F_1)} \Big) \end{aligned} \tag{3.41}$$

where we have used the fact that $\sup_{t\geq 0}\left\{ t^{1-\mu} \int_t^\infty e^{\gamma(t-s)} s^{\mu-1}\, ds \right\} \leq 1/\gamma$. Adding the estimates in (3.36) and (3.41) and employing (3.34) yields

$$\|e^{-\omega t}\tilde{\rho}\|_{C_{1-\mu}(\mathbb{R}_+, F_1)} \leq 2M_1 \|P_-\tilde{\rho}_0\|_{F_\mu}. \tag{3.42}$$

The representation (3.35) in conjunction with (3.40) and (3.42) then implies

$$\|P_+\tilde{\rho}_0\|_{F_\mu} \leq M_0 C_0 C_3 \beta \|e^{-\omega t}\tilde{\rho}\|_{C_{1-\mu}(\mathbb{R}_+, F_1)} \leq M_0 C_3 \|P_-\tilde{\rho}_0\|_{F_\mu}, \tag{3.43}$$

where the last inequality follows from the fact that $2C_0 M_1 \beta \leq 1$. We have thus demonstrated the claim.

Notice that the preceding claim contradicts the stability assumption. In particular, if $\tilde{\rho}_0 \in$

$\mathbb{B}_{F_\mu}(0,\delta)$ is chosen such that $P_-\tilde{\rho}_0 = 0$, then it must hold that $P_+\tilde{\rho}_0 = 0$, and hence $\tilde{\rho}_0 = 0$, which contradicts the assumption of stability for arbitrary $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0,\delta)$. □

We can immediately state the following corollary, which establishes instability of small cylinders for the original problem (3.1). The corollary is easily proved by use of the projection P_0, which serves as a connection between the problems (3.1) and (3.18), and application of the result established in the theorem above. In fact, the same techniques used to prove Theorem 3.4.1 can also be employed to prove the corollary directly.

Corollary 3.4.1. *Let $r_\star \in (0,1)$ and $\mu \in [1/2,1]$ be fixed with $4\mu + \alpha \notin \mathbb{Z}$. Then the equilibrium $r_\star$ of* (3.1) *is unstable in the topology of E_μ for initial values in E_μ.*

3.5 Bifurcation Results

In this section we turn our attention back to the general equilibria of (3.1). In particular, we are interested in the interactions between the family of cylinders and the family of unduloids. We have already seen that the radius $r_\star = 1$ plays a critical role in the dynamics of the cylinders. The change of stability for cylinders above and below this critical radius suggests that there is a bifurcation at $r_\star = 1$. Indeed, we will confirm this bifurcation, using results of Crandall and Rabinowitz [21] and go on to investigate further bifurcation behavior for cylinders. Herein we will take the parameter $\lambda := 1/r_\star$ as our bifurcation parameter, $r_\star > 0$.

With the tools and reductions developed in Section 3.3, we see that it suffices to study bifurcation of the equation

$$\bar{\mathcal{G}}(\tilde{\rho},\lambda) := \mathcal{G}_\star(\tilde{\rho},0) = P_0 G(\psi(\tilde{\rho},0) + r_\star) = 0, \qquad \lambda = 1/r_\star, \tag{3.44}$$

in the setting of $(\tilde{\rho},\lambda) \in F_1 \times (0,\infty)$. However, recalling the explicit characterization (3.24), we note that the eigenvalues of $D_1\mathcal{G}_\star(0,0)$ all have multiplicity two in the setting of F_1, regardless of the value of the parameter $r_\star$. From this observation we see that the techniques of [21], where the authors derive results for operators with simple eigenvalues, are not directly applicable in this setting. We may choose at this point to employ more general bifurcation results for high dimensional kernels, such as the results contained in Kielhöfer [48, Section I.19], or we can simplify the setting in which we are working in order to make the results of Crandall and Rabinowitz accessible to us.

Whether we choose to simplify our current setting or access the more general techniques of Kielhöfer, we can make good use of the following observation. Due to the periodicity enforced in the problem, the set of equilibria of (3.1) is invariant under shifts along the axis of rotation. More precisely, recalling the translation operators T_a as discussed in the proof of Theorem 3.1.3, one can easily verify that $G(T_a\bar{r}) = 0$ if and only if $G(\bar{r}) = 0$, $a \in \mathbb{R}$. Obviously, this invariance carries over to the reduced problem (3.18) and subsequently to the bifurcation equation (3.44).

One can take advantage of this shift invariance of equilibria in the context of bifurcation with high dimensional kernels by constructing a two dimensional bifurcation parameter $\tilde{\lambda} = (1/r_\star, a)$ and eventually observe two dimensional bifurcating *surfaces* of equilibria, c.f. [48, Theorem I.19.2, Remarks I.19.3]. On the other hand, we will make use of this invariance to simplify the setting in which we are looking for equilibria and make accessible the methods of Crandall and Rabinowitz for operators with simple eigenvalues. The specific simplification that we apply to our setting is supported by the following observation which allows us to consider functions in the class

$$F_{1,e} := h^{4+\alpha}_{0,e}(\mathbb{T})$$

of functions which are even, i.e. symmetric about the *y–axis*, and $h^{4+\alpha}_0$ regular.

Proposition 3.5.1. *For every equilibrium $\bar{\rho}$ of* (3.18)*, there exists $x_0 = x_0(\bar{\rho}) \in \mathbb{T}$ for which the translation $T_{x_0}\bar{\rho}$ is in the space $F_{1,e} := h^{4+\alpha}_{0,e}(\mathbb{T})$ of even functions on $\mathbb{T}$ in the class F_1. I.e. up to translations on $\mathbb{T}$, all equilibria of* (3.18) *are even functions.*

Proof. By Theorem 3.2.1 and Remarks 3.2.1(e), we can see that any equilibrium $\bar{\rho}$ can be realized as the projection of an undulary curve. More precisely, recalling the formula (3.12), we have

$$\bar{\rho} = P_0 R(\cdot\,; \mathcal{H}, B), \qquad \text{for some} \quad \mathcal{H} > 0,\, B \geq 0,$$

with $\mathcal{H}$ and B related according to the relationship (3.13). Now, taking $s_0 := \pi/2\mathcal{H}$, one readily verifies that $y(s)$ and $x'(s)$ are symmetric about $s = s_0$. It follows that $x_0 := -x(s_0)$ satisfies the properties claimed. □

From this proposition, we see that there is no loss of generality if we focus our bifurcation analysis on the setting of $\tilde{\rho} \in F_{1,e}$. One benefit of working in this setting is that we have the

Fourier series representation

$$\tilde{\rho}(x) = \sum_{k\geq 1} a_k \cos(kx), \quad \{a_k\} \subset \mathbb{R} \qquad \text{for all} \quad \tilde{\rho} \in F_{1,e}.$$

Theorem 3.5.1. *For every $\ell \in \mathbb{N}$, $(0,\ell) \in F_{1,e} \times (0,\infty)$ is a bifurcation point for the equation* (3.5.1)*. In particular, there exists a positive constant $\delta = \delta(\ell) > 0$ and a nontrivial analytic curve*

$$\{(\tilde{\rho}(s), \lambda(s)) : s \in (-\delta,\delta), (\tilde{\rho}(0), \lambda(0)) = (0,\ell)\}, \tag{3.45}$$

such that

$$\bar{\mathcal{G}}(\tilde{\rho}(s), \lambda(s)) = 0 \qquad \text{for all} \quad s \in (-\delta,\delta),$$

and all solutions of (3.44) *in a neighborhood of $(0,\ell)$ are either a trivial solution $(0,\lambda)$ or an element of the nontrivial curve* (3.45)*. Moreover, if $\lambda \in (0,\infty) \setminus \mathbb{N}$, then $(0,\lambda)$ is not a bifurcation point for* (3.44)*.*

Proof. We first note that bifurcation can only occur at points $(0,\lambda)$ for which $D_1\bar{\mathcal{G}}(0,\lambda)$ is not bijective. We can see from (3.22)-(3.23) that

$$D_1\bar{\mathcal{G}}(0,\lambda) = -\partial_x^2 \left(\lambda^2 + \partial_x^2\right)\Big|_{F_{1,e}}, \tag{3.46}$$

and so it follows that bifurcation can only occur at points of the form $(0,\ell)$, $\ell \in \mathbb{N}$. Now we proceed to verify that $(0,\ell)$ is indeed a bifurcation point for every value $\ell \in \mathbb{N}$.

Utilizing the expression (3.2) we compute the Fréchet derivative of the shifted operator $G_\star := G(\cdot + r_\star)$ with respect to the shifted value $r_\star$. It follows that

$$\begin{aligned} D_{r_\star} G(\rho + r_\star) = \frac{-1}{\rho + r_\star} \Bigg[G(\rho + r_\star) - \partial_x \Bigg(& \frac{1}{\sqrt{1+\rho_x^2}}\, \partial_x \mathcal{H}(\rho + r_\star) \\ & - \frac{\rho + r_\star}{\sqrt{1+\rho_x^2}}\, \partial_x \left[\frac{1}{(\rho + r_\star)^2 \sqrt{1+\rho_x^2}} \right] \Bigg) \Bigg], \end{aligned}$$

for all $(\rho, r_\star) \in E_1 \times (0,\infty)$. Because the derivative is acting on $\mathbb{R}$, we equate the operator $D_{r_\star}G(\rho + r_\star)$, mapping $\mathbb{R}$ into E_0, with its evaluation at the identity in $\mathbb{R}$, namely $D_{r_\star}G(\rho + r_\star)1$ in E_0. Now, we see that

$$D_2\bar{\mathcal{G}}(\tilde{\rho},\lambda) = P_0 D_{r_\star} G(\psi(\tilde{\rho},0)+r_\star)\Big|_{F_{1,e}}, \qquad \lambda = 1/r_\star, \tag{3.47}$$

and, after some computation and simplifications, we have the mixed derivatives

$$D_{12}\bar{\mathcal{G}}(0,\ell) = 2P_0(\ell^3\partial_x^2 - \ell\partial_x^4)\Big|_{F_{1,e}}, \qquad \ell \in \mathbb{N}. \tag{3.48}$$

By compactness of the resolvent operators $R(\lambda) := (\lambda - DG_\star(0))^{-1}$, $\lambda \in \rho(DG_\star(0))$, it follows that $D_1\bar{\mathcal{G}}(0,\ell)$ is a Fredholm operator of index zero for every $\ell \in \mathbb{N}$. Further, we see that

$$\begin{aligned} N_\ell &:= N(D_1\bar{\mathcal{G}}(0,\ell) = \overline{\mathrm{span}}\{\cos(\ell x)\}, \qquad \ell \in \mathbb{N}, \\ R_\ell &:= R(D_1\bar{\mathcal{G}}(0,\ell)) = \overline{\mathrm{span}}\left\{\cos(kx) : k \geq 1, k \neq \ell\right\}, \end{aligned}$$

where $N(B)$ and $R(B)$ denote the kernel and the range, respectively, of the operator B. Since $h^\sigma(\mathbb{T}) \hookrightarrow L_2(\mathbb{T})$, we can borrow the L_2-inner product to realize N_ℓ as a topological complement to R_ℓ as subspaces of $F_{1,e}$, $\ell \in \mathbb{N}$. Finally, take $\hat{v}_0 := \cos(\ell\,\cdot) \in N_\ell$ and it is straight forward to see that

$$D_{12}\bar{\mathcal{G}}(0,\ell)\hat{v}_0 = -4\ell^5\ \cos(\ell\,\cdot) \notin R_\ell\,, \qquad \ell \in \mathbb{N}.$$

Hence, the claim follows by [21, Theorem 1.7], or [48, Theorem I.5.1]. □

Remarks 3.5.2. Beyond simply establishing the existence of bifurcation points $(0,\ell)$, we can immediately state several conclusions regarding the bifurcating equilibria. Namely:

a) It follows from the characterization of equilibria developed in Section 3.2 that the bifurcating branches of equilibria have to be the *non-trivial* unduloids, i.e. the even versions of the functions $R(\cdot\,;\mathcal{H},B)$, as defined by (3.12) and the Proposition 3.5.1, with $0 \leq B < 1$.

b) For every $\ell \in \mathbb{N}$ we note that the bifurcation point $(\tilde{\rho}(0),\lambda(0)) = (0,\ell)$ corresponds to the cylinder of radius $r_\star = 1/\ell$, which also coincides with $R(\cdot\,;\mathcal{H},0)$. Hence, it follows from Remarks 3.2.1 that the equilibria which bifurcate from $(0,\ell)$ are precisely the family of unduloids which satisfy the relation $\mathcal{H} = \ell$ when $B = 0$. From this relation, one can conclude that all of the bifurcating equilibria $\tilde{\rho}(s)$ have fundamental periodicity $2\pi/\ell$, $s \in (-\delta(\ell),\delta(\ell))$.

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