TOPICS ON A LOGARITHMIC DIFFUSION EQUATION

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To my parents,

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CHAPTER I

INTRODUCTION AND MOTIVATIONS

I.1 Introduction

In this thesis we are interested in studying the following nonlinear partial differential equation

$$u_t - \Delta \ln u = 0. \tag{I.1.1}$$

The Cauchy problem for (I.1.1) is formulated as

$$\begin{aligned} &u_t - \Delta \ln u = 0 \quad \text{ in } \mathbb{R}^N \times \mathbb{R}^+; \\ &u(\cdot, 0) = u_o. \end{aligned}$$

When $u_o \in L^1(\mathbb{R}^2)$, necessary and sufficient conditions on the L^1-norm of u_o for a solution to exist is provided in [6]. When $N \geq 3$ it is shown in [20] that $u_o \in L^1(\mathbb{R}^N)$ does not generate a solution. Thus a more reasonable space to impose initial data seems to be $L^1_{loc}(\mathbb{R}^N)$. For $u_o \in L^1_{loc}(\mathbb{R}^N)$ that is radial, a necessary and sufficient condition on the growth of u_o is given in [7] in order to obtain a solution. The theory for non-radial initial data is still unclear.

The theory of solutions to (I.1.1) in bounded domains is even less known. Let E be a smooth bounded domain in $\mathbb{R}^N$. The Dirichlet Problem is formulated as

$$\begin{aligned} &u_t - \Delta \ln u = 0 \quad \text{ in } E_T; \\ &\ln u = \ln g \quad \text{ on } \partial E \times (0, T); \\ &u(\cdot, 0) = u_o. \end{aligned}$$

We will present our results on the solvability of Dirichlet problems for (I.1.1) in Chapter II. When assigning strictly positive boundary datum g, the existence of solution to the Dirichlet problem is considered in [13]. The new feature of our results is that g is allowed to vanish somewhere on $\partial E \times (0, T)$ in order to generate a solution. In particular, g can vanish on a set of positive H^{N-1}-measure on $\partial E \times (0,T)$.

Moreover we will show a non-existence theorem in the same chapter that if g vanishes on a subset of $\partial E \times (0,T)$ with positive H^N-measure then a solution does not exist in general. An essentially global version of nonexistence results concerning continuous solutions taking zero boundary datum is claimed in [20]. However, it is not clear whether our result covers that one since, the heart of the matter, the intended notion of solutions is not stated in [20] . It should be pointed out that our approach is entirely local and also independent of the interior continuity of solutions.

In Chapter III, we will list a number of local regularity properties we proved for local weak solutions to (I.1.1).

First of all, we use an example to explain that an estimate of the modulus of continuity of u over a compact subset K of E_T depending only on the bound of u and the distance from K to the boundary, in general, does not hold. Additional assumptions seem to be necessary. We provide a Harnack-type inequality in Section III.2. If the Harnack-type inequality holds, then we obtain the local smoothness of solutions to (I.1.1) by the classical theory ([17]).

A Harnack-type estimate for solutions to the Cauchy problem for (I.1.1) is established in [2]. The proof uses in an essential way the "globality" of the solutions in the whole $\mathbb{R}^N \times \mathbb{R}^+$. However our results seem to be the first local Harnack-type inequality. Our approach is entirely local and can be adapted to similar equations with full quasi-linear structures. In order to derive such a local Harnack-type inequality we need to assume $\ln u \in L^\infty_{loc}\big(0,T;L^p_{loc}(E)\big)$ for $p > N+2$. We will give an example in the same section that a continuous solution does not have to satisfy such a condition. Thus it is interesting to ask what is the minimal assumption for a local solution to be continuous. This deserves a future investigation.

With the aid of our Harnack-type inequality, we study the porous medium type approximation of (I.1.1) in Section III.3 and the local analyticity of solutions to (I.1.1) in Section III.4.

We give a Harnack inequality in the topology of $L^1_{loc}(E)$ in Section III.5. The proof is done for equations with full quasi-linear structures. It is assumed that $\ln u \in L^\infty_{loc}\big(0,T; L^2_{loc}(E)\big)$. We will use the example in Section III.1 to show that if such an assumption is removed then the L^1_{loc} Harnack inequality needs not to hold.

I.2 Motivations

Equation (I.1.1) arises from different physical models and a particularly interesting one is thin film dynamics. Suppose a viscous liquid film lies on a rigid plate and the thickness of the film is between 100Å and 1000Å. Double-layer forces are neglected. Assume the Navier-Stokes equations are applicable and the van der Waals force is considered as an external body force. It is modeled in [22] as a potential ϕ which is approximately the cube of the thickness u. Given an initial disturbance to the film and the van der Waals force eventually leads to the rupture of the film within a finite time. It is derived in [22] that

$$u_t - \Delta \ln u + \operatorname{div}(u^3 \nabla(\Delta u)) = 0 \quad \text{ in } \mathbb{R}^2 \times \mathbb{R}^+.$$

The fourth order term reflects the stabilizing effects of surface tension on the liquid-gas interface. Numerical studies in [22] suggest it is negligible.

Equation (I.1.1) also arises from geometry. Let $\{\Sigma; ds\}$ be a 2-dimensional, orientable, simply connected, noncompact Riemaniann surface with the metric $ds^2 = g_{ij}\, dx_i dx_j$, i, $j = 1, 2$, where g_{ij} is the first fundamental form of Σ. The Gauss curvature K is calculated from g_{ij}, and in a rectangular coordinate system takes the form

$$K = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left[\frac{\partial}{\partial x_1}\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial}{\partial x_1}g_{11} + \frac{\partial}{\partial x_2}\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial}{\partial x_2}g_{22}\right]. \tag{I.2.1}$$

When g_{ij} are sufficiently smooth, there is a positive, smooth function u defined on a subset of $\mathbb{R}^2$ such that

$$ds^2 = u\,dx_1dx_1 + u\,dx_2dx_2.$$

Thus

$$2K = -u^{-1}\Delta \ln u.$$

The Ricci flow for $\{\Sigma; ds\}$ describes the evolution of the metric ds^2 by its scalar curvature $R = 2K$, i.e.,

$$\frac{\partial}{\partial t} ds^2 = -R\,ds^2.$$

Equivalently, in rectangular coordinates, u satisfies

$$u_t = \Delta \ln u.$$

CHAPTER II

SOLVABILITY IN BOUNDED DOMAINS

II.1 Solvability of the Related Elliptic Equation

In this section, we will study the following Dirichlet problem.

$$\begin{aligned} &\lambda u - \Delta \ln u = f \quad \text{in } E; \\ &\ln u = \ln \Psi \quad \text{on } \partial E. \end{aligned} \tag{II.1.1}$$

Here $\lambda > 0$. Assume momentarily $f \geq 0$ and $\Psi \geq 0$ are measurable so that $\ln \Psi$ is well-defined as a measurable function on ∂E. This is related to the Dirichlet problem for the parabolic equation by the method of time discretization and we will discuss it later.

Different notions of solutions to (II.1.1) lead to different requirements on $\ln \Psi$ to generate a solution. Here we introduce two different notions of solutions.

First, a function $u \in L^1(E)$ is called a sub(super)-solution to (II.1.1) if $\ln u \in W^{1,2}(E)$ and

$$\lambda \int_E u\zeta \, dx + \int_E D \ln u D\zeta \, dx \leq (\geq) \int_E f\zeta \, dx \tag{II.1.2}$$

for all nonnegative

$$\zeta \in W_o^{1,2}(E).$$

In addition, $\ln u \leq (\geq) \ln \Psi$ in the sense of traces on ∂E. A solution is both a super-solution and sub-solution.

Second, a function $u \in L^1(E)$ is called a very weak sub(super)-solution to (II.1.1) if $\ln u \in L^1(E)$ and

$$\lambda \int_E u\zeta \, dx - \int_E \ln u \Delta\zeta \, dx \leq (\geq) \int_E f\zeta \, dx - \int_{\partial E} \ln \Psi \frac{\partial \zeta}{\partial \nu} \, d\sigma \tag{II.1.3}$$

for all $\zeta \in C^\infty(\bar{E})$ with $\zeta = 0$ on ∂E. A solution is both a super-solution and sub-solution.

When u is a smooth solution to (II.1.1), then

$$\int_{\partial E} \ln \Psi \frac{\partial \zeta}{\partial \nu}\, d\sigma = \int_{\partial E} \ln u \frac{\partial \zeta}{\partial \nu}\, d\sigma.$$

On the other hand, for any $h \in C^\infty(\partial E)$ there is $\zeta \in C^\infty(\bar{E})$ such that

$$\frac{\partial \zeta}{\partial \nu} = h \quad \text{on } \partial E;$$
$$\zeta = 0 \quad \text{on } \partial E.$$

From this we conclude that $\Psi = u$ on ∂E.

The extension claimed above is a basic fact of calculus. Indeed, suppose the origin $O \in \partial E$ and there is a local coordinate system such that ∂E is locally represented by

$$x_N = \phi(\bar{x}),\ |\bar{x}| < R, \quad \text{where } \bar{x} = (x_1, \ldots, x_{N-1}).$$

Since ϕ is smooth, the map

$$\Phi :\ (\bar{x}, s) \mapsto \big((\bar{x}, \phi(\bar{x}))\big) - s\vec{n}(\bar{x}), \quad \vec{n} = \frac{(D_{\bar{x}}\phi(\bar{x}), -1)}{\sqrt{1 + |D_{\bar{x}}\phi(\bar{x})|^2}}$$

is a diffeomorphism between $X = [|\bar{x}| < R] \times (0, \delta)$ and $\Phi(X)$ when $\delta > 0$ is small enough. Then we can extend h to the interior by defining

$$z(\bar{x}, s) = \int_0^s h\big((\bar{x}, \phi(\bar{x})) - t\vec{n}\big)\, dt.$$

Then the desired extension is obtained by a change of variables, i.e. $\zeta(y) = z(\Phi^{-1}(y))$ and an application of partition of unity.

II.1.1 Existence of Solutions to (II.1.1)

We state the general existence theorems regarding these two notions. We will discuss the uniqueness of these notions of solutions in a separate section.

Theorem II.1.1. *Let* $0 \leq f \in L^r(E)$ *and* $0 \leq \Psi \in L^r(\partial E)$ *such that* $\ln \Psi \in W^{\frac{1}{2},2}(\partial E)$*. Then there exists a solution* u *to* (II.1.1)*. Moreover,* $u \in C^{\beta}_{loc}(E)$ *for some* $0 < \beta < 1$ *depending on*

$$\{N,\, \lambda,\, \mathrm{diam}(E),\, \|f\|_r,\, \|\Psi\|_r\}.$$

Theorem II.1.2. *Let* $0 \leq f \in L^r(E)$ *and* $0 \leq \Psi \in L^r(\partial E)$ *such that* $\ln \Psi \in L^1(\partial E)$*. Then there exists a very weak solution* $u \in L^r(E)$ *to* (II.1.1)*. Moreover,* $u \in C^{\beta}_{loc}(E)$ *for some* $0 < \beta < 1$ *depending on*

$$\{N,\, \lambda,\, \mathrm{diam}(E),\, \|f\|_r,\, \|\Psi\|_r\}.$$

In order to prove general existence theorems we first need to use the Fixed Point Theorem below to solve a special case.

Theorem II.1.3. (Fixed Point Theorem [12]) *Let* $(X, \|\cdot\|)$ *be a Banach space and* H *be a closed and convex subset of* X*,* $x_o \in H$*, and* $T : H \times [0,1] \to H$ *be continuous and compact with* $T(\cdot, 0) = x_o$*. If there is a constant* M *such that* $\|x\| < M$ *apriori for all* $x \in H$ *and* $0 \leq \sigma \leq 1$ *satisfying* $T(x, \sigma) = x$*. Then there is a fixed point for* $T(x, 1)$*.*

For our convenience in applying the Fixed Point Theorem II.1.3, we use a transformation $v = \ln u$ to recast it into the following problem

$$\begin{cases} \lambda e^v - \Delta v = f & \text{in } E; \\ v = \ln \Psi & \text{in } \partial E. \end{cases}$$

Then we have

Lemma II.1.1. *Let* $f \in L^r(E)$ *be nonnegative, and* $\Psi \in C(\partial E)$ *such that* $\ln \Psi \in W^{\frac{1}{2},2}(\partial E)$ *and* $\mu_1 \leq \Psi \leq \mu_2$ *on* ∂E *for some* $\mu_1, \mu_2 > 0$*. Then there is a unique solution* u *to* (II.1.1) *and* $u \in C(\bar{E})$*.*

Proof. Let $H = \{v \in L^\infty(E) : |v| \leq M\}$ with M to be chosen. Then H is a closed and convex subset of $L^\infty(E)$. For a measurable function v on E, define v_n

as

$$v_n = \begin{cases} n & \text{if } v > n; \\ v & \text{if } \ -n \le v \le n; \\ -n & \text{if } \ v < -n. \end{cases}$$

Fix n and define an operator $w = T(v, \sigma)$ by the unique solution to

$$\begin{cases} \sigma\lambda e^{v_n} - \Delta w = \sigma f; \\ \\ w = \sigma \ln \Psi \qquad \text{on } \ \partial E. \end{cases}$$

Here Ψ is defined in Lemma II.1.1 and $w \in W^{1,2}(E) \cap C(\bar{E})$ is a solution in the weak sense.

We apply the stated Fixed Point Theorem to show Lemma II.1.1. First of all, an application of DeGiorgi's method (see [3], [17]) gives for some $C > 0$

$$|w| \le \sup_{\partial E} |\ln \Psi| + C(\|f\|_\infty + \lambda e^n) \stackrel{\text{def}}{=} M(n).$$

With such a choice of $M(n)$, $T(\cdot, \cdot)$ maps $H \times [0, 1]$ to H.

Secondly, we show $T(\cdot, \sigma) : H \to H$ is continuous. Let $w' = T(v'; \sigma)$ and $w'' = T(v''; \sigma)$ for $v', v'' \in H$, then

$$-\Delta(w' - w'') + \sigma\lambda(e^{v'_n} - e^{v''_n}) = 0, \qquad \text{and } w' - w'' = 0 \quad \text{ on } \partial E.$$

An application of DeGiorgi's method yields that

$$\|w' - w''\|_\infty \le Ce^n \|v'_n - v''_n\|_\infty.$$

Then the left hand side is small if $\|v' - v''\|_\infty$ is made small. The continuity with respect to σ is proved similarly.

Next, we show $T(\cdot, \sigma) : H \to H$ is compact uniformly in σ. In fact, since elements $v \in H$ are uniformly bounded by $M(n)$, functions from $T(H, \sigma)$ are equi-countinuous. Thus Arzela-Ascoli's theorem yields compactness.

As a result of the fixed point theorem, there exists a solution $v^{(n)} \in H \cap W^{1,2}(E) \cap C(\bar{E})$ satisfying in the weak sense

$$\begin{cases} \lambda e^{v_n^{(n)}} - \Delta v^{(n)} = f; \\ \\ v^{(n)} = \ln \Psi \qquad \text{on } \partial E. \end{cases} \tag{II.1.4}$$

In order to send $n \to \infty$ we show $|v^{(n)}|$ is uniformly bounded. We discard the first term in equation (II.1.4) by its non-negativity and an application of DeGiorgi's method gives the upper bound $A = \sup_{\partial E} \ln \Psi + C\|f\|_r$ for some C. Then another application of DeGiorgi's method shows we actually have

$$|v| \leq \sup_{\partial E} |\ln \Psi| + C(e^A + \|f\|_r) \stackrel{\text{def}}{=} A_1.$$

As a result, $v^{(n)}$ converges to some $v \in C(\bar{E})$ uniformly.

Let $\ln \Psi \in W^{1,2}(E)$ denote an extension of $\ln \Psi \in W^{\frac{1}{2},2}(\partial E)$. Using $(v^{(n)} - \ln \Psi)$ as a test function in the weak formulation of $v^{(n)}$ we have

$$\begin{aligned} &\int_E |\nabla v^{(n)}|^2 dx \\ &= \int_E \nabla v^{(n)} \nabla \ln \Psi dx + \int_E (v^{(n)} - \ln \Psi) f dx - \int_E e^{v_n^{(n)}} (v^{(n)} - \ln \Psi) dx \\ &\leq \frac{1}{2} \int_E |\nabla v^{(n)}|^2 dx + C(N,\, A_1,\, \|f\|_r) \end{aligned}$$

We conclude $\nabla v^{(n)}$ converges weakly in L^2 to some d. By the uniform convergence of $v^{(n)}$ we then have $d = \nabla v$.

Hence $v \in C(\bar{E})$ will be the desired solution. ■

If, in addition, we assume $f \in C^\infty(\bar{E})$ and $\Psi \in C^\infty(\partial E)$, then by Schauder's theory and a boot-strapping argument we conclude the obtained solution $u \in C^\infty(\bar{E})$.

Let $F(x, y)$ be the fundamental solution of the Laplacian in $\mathbb{R}^N$ with pole at

y. Green's function in E is defined as

$$G(x,y) = F(x,y) - \Phi(x,y), \quad N \geq 2$$

where $\Phi(x,\cdot) \in C^\infty(\bar{E})$ is the unique solution to

$$\begin{cases} \Delta_y \Phi(x,y) = 0 & \text{in } E; \\ \Phi(x,y) = F(x,y) & \text{for } y \in \partial E. \end{cases}$$

Poisson's kernel on ∂E is defined as

$$P(x,y) = -\frac{\partial}{\partial n(y)} G(x,y) \quad \text{for } x \in E,\, y \in \partial E.$$

We will use the well-known Poisson representation formula

$$w(x) = \int_{\partial E} w(y)P(x,y)\, d\sigma - \int_E \Delta w(y) G(x,y)\, dy \quad \text{when } x \in E$$

for all $w \in C^2(\bar{E})$.

To proceed, we need to discuss some integrability properties of $G(x,y)$ and $P(x,y)$. First of all, we note

$$0 \leq G(x,y) \leq F(x,y) \quad \text{for all } x, y \in E.$$

Moreover, there is a constant $C(r,\, N,\, \mathrm{diam}(E))$ such that

$$\|G(x,\cdot)\|_{\frac{r}{r-1}} \leq C \quad \text{for all } r > \frac{N}{2}.$$

Next, Poisson's kernel satisfies the following asymptotic behavior

$$P(x,y) \approx \frac{\mathrm{diam}(x, \partial E)}{|x-y|^N} \quad \text{for all } x \in E,\, y \in \partial E.$$

See [16] for an elementary proof of this fact. Therefore, there is a constant

$C(N, \operatorname{diam}(E))$ such that for all $y \in \partial E$

$$\int_E P(x,y)\,dx \le C.$$

Now we are ready to present

Proof of Theorem II.1.1. Let $k > 0$ and

$$\Psi_k = \begin{cases} k^{-1}, & \Psi < k^{-1}; \\ \Psi, & k^{-1} \le \Psi \le k; \\ k, & \Psi > k. \end{cases}$$

Since $\ln \Psi \in W^{\frac{1}{2},2}(\partial E)$, we also have $\ln \Psi_k \in W^{\frac{1}{2},2}(\partial E)$. See [4]. As a result, it admits an extension to a function in $W^{1,2}(E)$ which we still denote as $\ln \Psi_k$. Moreover we can find a sequence of functions $(\ln \Psi_k)_\epsilon \in C^\infty(\bar{E})$ such that as $\epsilon \to 0$

$$\begin{aligned}
&D(\ln \Psi_k)_\epsilon \to D \ln \Psi_k \quad \text{in } L^2(E); \\
&(\ln \Psi_k)_\epsilon \to \ln \Psi_k \quad \text{in } L^q(E) \quad \forall 1 \le q < \infty; \\
&(\ln \Psi_k)_\epsilon \to \ln \Psi_k \quad \text{in } L^r(\partial E); \\
&-\ln(2k) \le (\ln \Psi_k)_\epsilon \le \ln(2k) \quad \text{on } \partial E; \\
&e^{(\ln \Psi_k)_\epsilon} \to \Psi_k \quad \text{in } L^r(\partial E).
\end{aligned}$$

The first two convergences follow from the proof of the well-known approximation theorem for Sobolev functions when the boundary satisfies the *segment property*. The third one follows from the previous two and the trace inequality

$$\|w\|_{r,\partial E} \le \gamma(N)(\|Dw\|_2 + \|w\|_2)^{\frac{1}{r}} \|w\|_q^{1-\frac{1}{r}} \tag{II.1.5}$$

for all $w \in W^{1,2}(E) \cap L^q(E)$ where $q = 2(r-1)$. The fourth one comes from the previous one and the upper bound of Ψ_k. The last one follows from the third one and the fourth one in view of

$$\int_{\partial E} |e^{(\ln \Psi_k)_\epsilon} - \Psi_k|^r \, d\sigma \leq (2k)^r \int_{\partial E} |(\ln \Psi_k)_\epsilon - \ln \Psi_k|^r \, d\sigma.$$

See all basic theories of Sobolev functions in [4].

Assume first that $f \in C^\infty(\bar{E})$. According to the previous lemma, there is a unique solution $v^{k,\epsilon} \in C^\infty(\bar{E})$ to

$$\begin{cases} \lambda e^v - \Delta v = f & \text{in } E; \\ v = (\ln \Psi_k)_\epsilon & \text{on } \partial E. \end{cases} \tag{II.1.6}$$

We are going to use repeatedly Poisson's representation formula for any $x \in E$

$$v^{k,\epsilon}(x) = \int_{\partial E} (\ln \Psi_k)_\epsilon(y) P(x,y) \, d\sigma + \int_E G(x,y)(f(y) - \lambda e^{v^{k,\epsilon}(y)}) \, dy. \tag{II.1.7}$$

First of all, note that $P(\cdot, y) \, d\sigma(y)$ is a probability measure on ∂E; then, by Jensen's inequality

$$\begin{aligned} v^{k,\epsilon}(x) &\leq \int_{\partial E} (\ln \Psi_k)_\epsilon(y) P(x,y) \, d\sigma + \int_E G(x,y) f(y) \, dy \\ &\leq \ln \left(\int_{\partial E} e^{(\ln \Psi_k)_\epsilon(y)} P(x,y) \, d\sigma \right) + \|G(x,\cdot)\|_{\frac{r}{r-1}} \|f\|_r \\ &\leq \ln \left(\int_{\partial E} e^{(\ln \Psi_k)_\epsilon(y)} P(x,y) \, d\sigma \right) + \gamma(N, \, r, \, \mathrm{diam}(E)) \|f\|_r. \end{aligned}$$

This gives

$$\int_E e^{r v^{k,\epsilon}(x)} \, dx \leq \gamma(N, \, r, \, \mathrm{diam}(E)) e^{r\|f\|_r} \int_E \left(\int_{\partial E} e^{(\ln \Psi_k)_\epsilon(y)} P(x,y) \, d\sigma \right)^r dx.$$

The integral on the right hand side is estimated by Hölder's inequality

$$
\begin{aligned}
&\int_E \left(\int_{\partial E} e^{(\ln \Psi_k)_\epsilon(y)} P(x,y)\, d\sigma \right)^r dx \\
&\leq \int_E \left(\int_{\partial E} P(x,y)\, d\sigma \right)^{r-1} \left(\int_{\partial E} P(x,y) e^{r(\ln \Psi_k)_\epsilon(y)}\, d\sigma \right) dx \\
&= \int_E P(x,y)\, dx \int_{\partial E} e^{r(\ln \Psi_k)_\epsilon(y)}\, d\sigma \\
&\leq \gamma(N,\, \mathrm{diam}(E)) \int_{\partial E} \Psi^r\, d\sigma.
\end{aligned}
$$

Combining all these estimates we arrive at

$$
\int_E e^{r v^{k,\epsilon}(x)}\, dx \leq \gamma(N,\, r,\, \mathrm{diam}(E)) e^{r\|f\|_r} \int_{\partial E} \Psi^r(y)\, d\sigma \tag{II.1.8}
$$

Secondly, by taking power $p = 2$ at both sides of (II.1.7) and integrating in dx over E we obtain

$$
\begin{aligned}
\int_E |v^{k,\epsilon}(x)|^2\, dx &\leq \gamma \int_E \left(\int_{\partial E} (\ln \Psi_k)_\epsilon(y) P(x,y)\, d\sigma \right)^2 dx \\
&\quad + \gamma \int_E \left(\int_E G(x,y)(f(y) - \lambda e^{v^{k,\epsilon}(y)})\, dy \right)^2 dx \\
&= I_1 + I_2.
\end{aligned}
$$

By using the estimate of $\|e^{v^{k,\epsilon}}\|_r$, I_2 is easily seen to be bounded by

$$
\begin{aligned}
I_2 &\leq \gamma |E| \|G(x,\cdot)\|^2_{\frac{r}{r-1}} \left(\|f\|_r + \lambda \|e^{v^{k,\epsilon}}\|_r \right)^2 \\
&\leq \gamma(N,\, r,\, \mathrm{diam}(E)) \big(\lambda^2 e^{2\|f\|_r} \|\Psi\|^2_{r,\partial E} + \|f\|^2_r \big).
\end{aligned}
$$

On the other hand, I_1 is estimated by Hölder's inequality as

$$
\begin{aligned}
I_1 &\leq \gamma \int_E \left(\int_{\partial E} P(x,y)\, d\sigma \int_{\partial E} |(\ln \Psi_k)_\epsilon(y)|^2 P(x,y)\, d\sigma \right) dx \\
&= \gamma \int_{\partial E} |(\ln \Psi_k)_\epsilon(y)|^2\, d\sigma \int_E P(x,y)\, dx \\
&\leq \gamma(N,\, \mathrm{diam}(E)) \int_{\partial E} |(\ln \Psi_k)_\epsilon(y)|^2\, d\sigma.
\end{aligned}
$$

Thus there is a constant γ depending on $\{N,\, r,\, \mathrm{diam}(E)\}$ such that

$$\int_E |v^{k,\epsilon}(x)|^2\,dx \le \gamma\big(\lambda^2 e^{2\|f\|_r}\|\Psi\|^2_{r,\partial E} + \|f\|_r^2 + \|\ln\Psi\|^2_{2,\partial E}\big).$$

Now we show the L^2 norm of $Dv^{k,\epsilon}$ is also bounded. Indeed, if we take $(v^{k,\epsilon} - (\ln\Psi_k)_\epsilon)$ as a test function in (II.1.6), a standard calculation yields

$$\begin{aligned}\frac{1}{2}\int_E |Dv^{k,\epsilon}|^2\,dx &\le \frac{1}{2}\int_E |D(\ln\Psi_k)_\epsilon|^2\,dx + \int_E f v^{k,\epsilon}\,dx - \lambda\int_E e^{v^{k,\epsilon}} v^{k,\epsilon}\,dx\\ &\quad - \int_E f(\ln\Psi_k)_\epsilon\,dx + \lambda\int_E e^{v^{k,\epsilon}}(\ln\Psi_k)_\epsilon\,dx\\ &\le C(\|\ln\Psi\|_{1,2},\, \|\Psi\|_{r,\partial E},\, \lambda,\, \|f\|_r,\, \mathrm{diam}(E),\, N,\, r).\end{aligned}$$

Here, we have estimated the second integral on the right by

$$\begin{aligned}\int_E f v^{k,\epsilon}\,dx &\le \|f\|_r\|v^{k,\epsilon}\|_{\frac{r}{r-1}}\\ &\le C(N,\, r,\, |E|)\|f\|_r\|v^{k,\epsilon}\|_{\frac{2N}{N-2}}\\ &\le C(N,\, r,\, |E|)\|f\|_r\big[\|v^{k,\epsilon}\|_2 + \|Dv^{k,\epsilon}\|_2\big],\end{aligned}$$

and the term with the gradient is easily absorbed to the left-hand side by the Cauchy-Schwarz inequality.

Thus, by the Compact Imbedding Theorem we can conclude that there is some $v \in W^{1,2}(E)$ and a sub-sequence of $e^{v^{k,\epsilon}}$, which we still use the same symbol to denote, such that as $\epsilon \to 0$ and $k \to \infty$

$$\begin{aligned}&v^{k,\epsilon} \to v \quad \text{a.e. in } E;\\ &v^{k,\epsilon} \to v \quad \text{in } L^2(E);\\ &Dv^{k,\epsilon} \to Dv \quad \text{weakly in } L^2(E).\end{aligned}$$

Note the first convergence also implies that for any $e^{v^{k,\epsilon}} \to e^v$ a.e in E. The uniform boundedness of $\|e^{v^{k,\epsilon}}\|_r$ implies that there is some $u \in L^r(E)$ and a sub-sequence of $e^{v^{k,\epsilon}}$, which we still use the same symbol to denote, such that

$$e^{v^{k,\epsilon}} \to u \quad \text{weakly in } L^r(E).$$

Then $u = e^v$ a.e. in E. As a result, we are able to conclude that

$$v \in W^{1,2}(E) \quad \text{and} \quad e^v \in L^r(E),$$

and for all $\zeta \in W_o^{1,2}(E)$

$$\lambda \int_E e^v \zeta \, dx + \int_E DvD\zeta \, dx = \int_E f\zeta \, dx.$$

Finally the boundary datum $\ln \Psi$ is taken by $v \in W^{1,2}(E)$. In fact, we see from the trace inequality (II.1.5) that, when $k \to \infty$ and $\epsilon \to 0$,

$$\|(\ln \Psi_k)_\epsilon - v\|_{2,\partial E} \le C\|v^{k,\epsilon} - v\|_2 \to 0$$

where C depends on the uniform bound of the $\|v^{k,\epsilon}\|_{1,2}$. On the other hand,

$$(\ln \Psi_k)_\epsilon \to \ln \Psi_k \quad \text{in } L^r(\partial E) \quad \text{as } \epsilon \to 0;$$
$$\ln \Psi_k \to \ln \Psi \quad \text{in } L^r(\partial E) \quad \text{as } k \to \infty.$$

Thus $v = \ln u = \ln \Psi$ on ∂E.

For the local Hölder continuity of v in E we only need to observe that the first equation in (II.1.6) can be written as

$$-\Delta v = F \stackrel{\text{def}}{=} f - \lambda e^v \quad \text{with } F \in L^r(E).$$

Thus the classical theory of elliptic equations gives $v \in C^\beta_{loc}(E)$ for some β depending on

$$\{N,\, \lambda,\, \text{diam}(E),\, \|e^v\|_r,\, \|f\|_r\}.$$

In view of the bound on $\|e^v\|_r$, β depends on

$$\{N,\, \lambda,\, \mathrm{diam}(E),\, \|\Psi\|_{r,\partial E},\, \|f\|_r\}.$$

In view of all estimates depending only on $\|f\|_r$, the assumption $f \in C^\infty(\bar{E})$ can be removed by a proper approximation. ■

Next, we continue to present

Proof of Theorem II.1.2. Let $f_\epsilon \in C^\infty(\bar{E})$ be a sequence of approximation of f. Assume momentarily that $\delta \le \Psi \le M$ for some positive numbers δ, Λ. Let Ψ_ϵ be a smooth approximation of Ψ on ∂E such that

$$\begin{aligned} &\Psi_\epsilon \to \Psi \quad \text{a.e. in } \partial E; \\ &\delta/2 \le \Psi_\epsilon \le 2\Lambda \quad \text{uniformly in } \epsilon. \end{aligned}$$

By Lemma II.1.1 there is a classical solution $v_\epsilon \in C^\infty(\bar{E})$ to (II.1.1) such that

$$\delta/2 \le v_\epsilon \le 2\Lambda$$

and it satisfies (II.1.3) with u replaced by e^{v_ϵ} for all $\zeta \in C^\infty(\bar{E})$ and $\zeta = 0$ on ∂E. Since $\{v_\epsilon\}$ is uniformly bounded and, by the classical theory, it is also equicontinuous in the interior, there exists a bounded function v and a sub-sequence such that

$$v_{\epsilon'} \to v \quad \text{a.e. in } E.$$

Hence we are able to pass to the limit to obtain

$$\delta/2 \le v \le 2\Lambda$$

and v satisfies (II.1.3) with u replaced by e^v.

Moreover, we have Poisson's representation

$$v_\epsilon(x) = \int_{\partial E} P(x,y) \ln \Psi_\epsilon(y)\, d\sigma + \int_E G(x,y)(f(y) - \lambda e^{v_\epsilon(y)})\, dy,$$

from which we conclude, as in the proof of Theorem II.1.1,

$$\|e^v\|_r \leq \gamma(N,\, r,\, \mathrm{diam}(E))e^{\|f\|_r}\|\Psi\|_{r,\partial E}.$$

Taking absolute value on both sides and integrating over E, we have the uniform bound

$$\|v\|_1 \leq \gamma\big(\lambda^2 e^{2\|f\|_r}\|\Psi\|^2_{r,\partial E} + \|f\|^2_r + \|\ln\Psi\|_{1,\partial E}\big).$$

Now suppose $\Psi \geq \delta$ and let Ψ_k be the truncation of Ψ from above by k, namely, $\Psi_k = \min\{\Psi,\, k\}$.

The previous discussion gives a solution v_k which, by the comparison principle (Proposition II.1.1), is increasing along k to some function v. Taking into consideration the boundedness of $\|e^{v_k}\|_r$ and $\|v_k\|_1$, we are able to pass to the limit in the corresponding integral identity (II.1.3) and obtain v as a solution.

Now for a general Ψ that satisfies the conditions of Theorem II.1.1, we use Ψ_δ to denote the truncation from below by δ, namely, $\Psi_\delta = \max\{\Psi,\, \delta\}$.

By the previous argument, there is a solution v_δ which, by the comparison principle Proposition II.1.1, is decreasing to some v. Similar to the previous argument, we have uniform bounds for $\|v_\delta\|_1$ and $\|e^{v_\delta}\|_r$. Hence we can identify the limit function v as a very weak solution to II.1.1. ■

II.1.2 Uniqueness of Solutions

We have uniqueness for the notion of weak solutions. However, the uniqueness for the notion of very weak solutions is unclear. We omit the proof of the following proposition, since we prove a similar result for the parabolic equation in Proposition II.4.2.

Proposition II.1.1. *Let u_1 be a super-solution and u_2 be sub-solution to* (II.1.1) *in the sense of* (II.1.2)*. If $u_1 \geq u_2$ on ∂E, then $u_1 \geq u_2$ a.e. in E.*

II.1.3 Some Remarks

II.1.3.1 Global Boundedness of u.

It should be remarked that the mere requirement $\Psi \in L^r(\partial E)$ is not enough to insure global boundedness of u in E. However, if $f \in L^\infty(E)$ and $0 \le \Psi \le M$ for some M then $v^{k,\epsilon}$ is uniformly bounded above. Indeed, let us take

$$l \ge \max\{\ln M,\, \ln \frac{\|f\|_\infty}{\lambda}\}.$$

Multiply the equation by $(v^{k,\epsilon} - l)_+ \in W_o^{1,2}(E)$ and integrate in dx over E; we obtain

$$\int_E (\lambda e^{v^{k,\epsilon}} - f)(v^{k,\epsilon} - l)_+ \, dx + \int_E |D(v^{k,\epsilon} - l)_+|^2 \, dx = 0.$$

This implies by our assumptions on l that

$$0 \ge \int_E (\lambda e^{v^{k,\epsilon}} - f)(v^{k,\epsilon} - l)_+ \, dx \ge \lambda \int_E (e^{v^{k,\epsilon}} - e^l)(v^{k,\epsilon} - l)_+ \, dx \ge 0.$$

Hence

$$v^{k,\epsilon} \le \Lambda \stackrel{\text{def}}{=} \max\{\ln M,\, \ln \frac{\|f\|_\infty}{\lambda}\}. \tag{II.1.9}$$

Similarly, if f is strictly positive and $\Psi \ge \delta$ for some $\delta > 0$ then

$$v^{k,\epsilon} \ge \min\{\ln \delta, \ln \frac{\inf f}{\lambda}\}. \tag{II.1.10}$$

II.1.3.2 Interior Positivity of u in Terms of Ψ and f.

The representation (II.1.7) implies that when K is a compact subset of E and $x \in K$ there exists a constant $C_1(N,\, \text{dist}(K,\, \partial E))$ such that

$$v^{k,\epsilon}(x) \ge -C_1 \int_{\partial E} |\ln \Psi| \, d\sigma - \int_E G(x,y)(\lambda e^{v^{k,\epsilon}(x)} - f(x)) \, dx. \tag{II.1.11}$$

Then, since $f \geq 0$, (II.1.8) implies that there is $C_2(N, \operatorname{diam}(E), r)$ such that

$$\begin{aligned} v^{k,\epsilon}(x) &\geq -C_1 \int_{\partial E} |\ln \Psi|\, d\sigma - \lambda \|G(x,\cdot)\|_{\frac{r}{r-1}} \|e^{v^{k,\epsilon}(x)}\|_r \\ &\geq -C_1 \int_{\partial E} |\ln \Psi|\, d\sigma - C_2 \lambda e^{\|f\|_r} \|\Psi\|_{r,\partial E}. \end{aligned}$$

We see that the positivity of solutions constructed in this way hinges upon $\|\ln \Psi\|_{1,\partial E}$. In fact, we can formulate the following

Proposition II.1.2. *Let $u^\epsilon \in C^2(\bar{E})$ be a solution to* (II.1.1) *corresponding to $f_\epsilon \in L^r(E)$ uniformly and $M \geq \Psi_\epsilon \geq 0$. If*

$$\lim_{\epsilon \to 0} \int_{\partial E} |\ln \Psi_\epsilon(y)|\, d\sigma = \infty,$$

then $u^\epsilon \to 0$ uniformly in any compact subset K of E.

Proof. By Poisson's representation formula, for any $x \in K$

$$\begin{aligned} \ln u^\epsilon(x) &= \int_{\partial E} \ln \Psi_\epsilon(y) P(x,y)\, d\sigma + \int_E G(x,y)(f_\epsilon - \lambda u^\epsilon) \\ &\leq \int_{\partial E} \ln \Psi_\epsilon(y) P(x,y) \chi_{[\Psi_\epsilon < 1]}\, d\sigma + \int_{\partial E} \ln \Psi_\epsilon(y) P(x,y) \chi_{[\Psi_\epsilon \geq 1]}\, d\sigma \\ &+ \|G(x,\cdot)\|_{\frac{r}{r-1}} \|f_\epsilon\|_r \\ &\leq C(N, \operatorname{dist}(K, \partial E)) \int_{\partial E} \ln \Psi_\epsilon(y)\, d\sigma + \ln M + \|G(x,\cdot)\|_{\frac{r}{r-1}} \|f_\epsilon\|_r \end{aligned}$$

The right-hand side tends to $-\infty$ as $\epsilon \to 0$. ■

II.1.3.3 Zeros are Allowed on the Boundary.

The requirement $\ln \Psi \in W^{\frac{1}{2},2}(\partial E)$ allows Ψ to be not essentially bounded away from zeros on the boundary. For example, let ∂E have a local representation

$$x_N = \phi(\bar{x}) \in C^1(|\bar{x}| < 4R); \quad \bar{x} = (x_1, \ldots, x_{N-1}).$$

Then the functions

$$\Psi(x) = |\ln|x||^{-1}, \quad N \geq 2;$$
$$\Psi(x) = |x|^p, \quad p > 0,\, N \geq 3;$$
$$\Psi(x) = e^{-\frac{1}{|x|^{2\alpha}}}, \quad 0 < \alpha < \frac{N-3}{4},\, N \geq 4$$

will satisfy $\ln \Psi \in W^{\frac{1}{2},2}(\partial E)$ and $\Psi(x) \to 0$ as $x \to 0$.

Remark II.1.1. *Let $E = (a, b)$. If we assign $\Psi(a) = 0$ and $\Psi(b) = 1$ then there is no bounded solution u such that $\ln u \in W^{1,2}(a, b)$. Indeed, this implies also $u \in W^{1,2}(a, b)$. As a result u and $\ln u$ are absolutely continuous in (a, b) and*

$$\lim_{x \to a^+} u(x) = 0;$$
$$\lim_{x \to a^+} \ln u(x) \text{ is finite.}$$

Which is a contradiction. Note this is completely independent of the equation. An analogous argument for the case of multiple dimensions follows similarly. That is, if u, $\ln u \in W^{1,2}(E)$ then u cannot vanish on a set of positive H^{N-1}-measure of ∂E.

However, Lemma II.1.1 indicates $[\Psi = 0]$ is allowed to be of dimension $N - 2$. In fact, when $N \geq 2$ and

$$\Psi(x) = |\ln|\hat{x}||^{-1}, \quad \hat{x} = (x_1, x_2),$$

$\ln \Psi$ will be in $W^{1,2}(B_R(0))$. Note the set of zeros of this Ψ is an $N-2$ dimensional smooth manifold in $\mathbb{R}^N$. Let S be an $N - 1$ dimensional smooth manifold that contains $[\Psi = 0]$. Then $\ln \Psi \in W^{\frac{1}{2},2}(S)$ and Ψ contains a set of $N-2$ dimensional zeros. This mainly exhibits, up to introducing a local coordinate $x_N = \phi(\bar{x})$ as above, that if Ψ is defined on the sphere S^2 in $\mathbb{R}^3$, then it is allowed to have a set of zeros that occupies a one dimensional curve on S^2.

II.1.3.4 Notion of Distributional Solutions and Positivity.

Now suppose we are given a distributional solution u to

$$
\begin{aligned}
&u,\, \ln u \in L^1_{\mathrm{loc}}(E); \quad \lambda > 0 \\
&\lambda u - \Delta \ln u = f \in L^1_{\mathrm{loc}}(E) \qquad \text{distributionally in } E.
\end{aligned}
\tag{II.1.12}
$$

Denote by $(F)_\epsilon$ the usual mollification of a function F. Let $f \in L^r_{loc}(E)$; then, discarding the term containing λ, we have

$$
-\Delta(\ln u)_\epsilon \le f_\epsilon.
$$

The classical theory of elliptic equations yields that

$$
\|(\ln u)_{\epsilon,+}\|_{\infty,B_{\sigma\rho}} \le \frac{\gamma}{(1-\sigma)^{\frac{2Nr}{2r-N}}} \left(\fint_{B_\rho} |(\ln u)_{\epsilon,+}|^2 \, dx \right)^{\frac{1}{2}} + \gamma \rho^{1-\frac{N}{2r}} \|f_\epsilon\|_{r,B_\rho}
$$

An interpolation yields

$$
\|(\ln u)_{\epsilon,+}\|_{\infty,B_\rho} \le \gamma(N,r) \fint_{B_\rho} |(\ln u)_{\epsilon,+}| \, dx + \gamma \rho^{1-\frac{N}{2r}} \|f_\epsilon\|_{r,B_{2\rho}}
$$

Letting $\epsilon \to 0$ yields

$$
\|(\ln u)_{+}\|_{\infty,B_\rho} \le \gamma(N,r) \fint_{B_{2\rho}} |(\ln u)_{+}| \, dx + \gamma \rho^{1-\frac{N}{2r}} \|f\|_{r,B_{2\rho}}
$$

On the other hand, this implies

$$
-\Delta(\ln u)_\epsilon = f_\epsilon - \lambda u_\epsilon \in L^r_{loc}(E)
$$

Another application of the classical elliptic theory implies that

$$
\|(\ln u)_{\epsilon}\|_{\infty,B_{\sigma\rho}} \le \frac{\gamma}{(1-\sigma)^{\frac{2Nr}{2r-N}}} \left(\fint_{B_\rho} |(\ln u)_{\epsilon}|^2 \, dx \right)^{\frac{1}{2}} + \gamma \rho^{1-\frac{N}{2r}} \left(\|f_\epsilon\|_{r,B_\rho} + \lambda \|u_\epsilon\|_{r,B_\rho} \right)
$$

Again by interpolation we have

$$\|(\ln u)_\epsilon\|_{\infty,B_\rho} \le \gamma(N,\,r)\fint_{B_{2\rho}} |(\ln u)_\epsilon|\,dx + \gamma(N,\,r)\rho^{1-\frac{N}{2r}}\left(\|f_\epsilon\|_{r,B_{2\rho}} + \lambda\|u_\epsilon\|_{r,B_{2\rho}}\right)$$

Thus when letting $\epsilon \to 0$

$$\|\ln u\|_{\infty,B_\rho} \le \gamma(N,\,r)\fint_{B_{2\rho}} |(\ln u)_\epsilon|\,dx + \gamma(N,\,r)\rho^{1-\frac{N}{2r}}\left(\|f\|_{r,B_{2\rho}} + \lambda\|u\|_{r,B_{2\rho}}\right).$$

Since $\|u\|_r$ can be estimated using $\|f\|_r$ as above, we conclude that u is bounded above and below in B_ρ by constants depending on

$$\{N,\,\lambda,\,\|f\|_{r,B_{2\rho}},\,\|\ln u\|_{1,B_{2\rho}}\}$$

II.2 Nonexistence of Solutions for the Elliptic Equation (II.1.1)

Lemma II.1.1 implies that solutions could be generated even if the datum Ψ vanishes on a set of positive H^{N-2}-measure of ∂E provided $\ln\Psi \in W^{\frac{1}{2},2}(\partial E)$. The trace is taken by $\ln u \in W^{1,2}(E)$. Nevertheless, if Ψ vanishes on a set of positive H^{N-1}-measure of ∂E then it is impossible to generate a solution u such that both u and $\ln u$ take the boundary traces Ψ and $\ln\Psi$ in the sense that $u,\,\ln u \in W^{1,2}(E)$. This was independent of the equation. Now it is natural to ask if there exists a function u that solves the Dirichlet problem in a weaker sense. This is what we will explore next.

Define the notion of local weak solutions to the first of (II.1.1) irrespective of boundary data as

$$\begin{aligned} &u \in L^2_{loc}(E),\, \ln u \in W^{1,2}_{\text{loc}}(E), \qquad \lambda > 0; \\ &\lambda u - \Delta \ln u = f \in L^r(E) \qquad \text{weakly in } \; E. \end{aligned} \tag{II.2.1}$$

Let O be an open subset of ∂E. We have the following

Theorem II.2.1. *If Ψ vanishes on an open subset of ∂E, there is no nonnegative, bounded, local, weak solution u to* (II.2.1) *such that it takes zero boundary datum* *at O in the sense of trace of $u \in W^{1,2}_{loc}(E \cup O)$.*

This statement is entirely local and independent of the interior continuity or positivity of the solution. It indicates that zeros on the boundary propagate into the interior even if we only use u to take the trace in the Sobolev sense.

The proof of Theorem II.2.1 hinges on the uniform continuity of the solution at O and this is the content of the following proposition.

Proposition II.2.1. *Let u be a nonnegative, locally bounded, local, weak solution to* (II.1.1) *in E. Assume there is an open subset O of ∂E such that it satisfies the property of positive geometric density. If u vanishes in the sense of trace on O, then there exist constants γ and α so that u satisfies*

$$|u(x)| \leq \gamma |x-y|^{\alpha}$$

for any $x \in E$ and $y \in O$.

This proposition has a parabolic counterpart in Section II.5. We omit the proof for the elliptic equation while giving the proof of the parabolic case in Section II.6.2.

Proof of Theorem II.2.1. In the weak formulation of the local weak solutions to (II.2.1) we take φ_ϵ as a test function which is a usual smooth mollification of $\varphi \in C^\infty_o(E)$. Then we have

$$\lambda u_\epsilon - \Delta(\ln u)_\epsilon = f_\epsilon \quad \text{ in } E.$$

For $\delta > 0$, we define an interior region

$$E_\delta = \{x \in E : \operatorname{dist}(x, \partial E) > \delta\}.$$

Moreover, define a subset of ∂E_δ that corresponds to O as

$$\partial E_{\delta,O} = \{a - \delta\nu : a \in O, \, \nu \text{ is the outer normal at } a\}.$$

Note that, by Proposition II.2.1, for any $a \in O$ and the outer normal ν at a

$$u(a - \delta\nu) \leq \gamma\delta^{\alpha}.$$

Now Poisson's representation formula yields that for any $x \in E_\delta$

$$\begin{aligned}
(\ln u)_\epsilon(x) &= \int_{\partial E_\delta} P(x,y)(\ln u)_\epsilon(y)\,dy + \int_{E_\delta} G(x,y)(f_\epsilon(y) - \lambda u_\epsilon(y))\,dx \\
&\leq \int_{\partial E_{\delta,O}} P(x,y)(\ln u)_\epsilon(y)\,dy + \int_{\partial E_\delta \setminus \partial E_{\delta,O}} P(x,y)(\ln u)_\epsilon(y)\,dy \\
&+ \|G(x,\cdot)\|_{\frac{r}{r-1}} \|f\|_r \\
&\leq \ln(\gamma\delta^{\alpha}) + C(N,\, r,\, \mathrm{diam}(E),\, \|u\|_\infty,\, \|f\|_r).
\end{aligned}$$

Since the right-hand side is independent of ϵ, we have for a.e. $x \in E_\delta$

$$\begin{aligned}
\ln u(x) &\leq \ln(\gamma\delta^{\alpha}) + C(N,\, r,\, \mathrm{diam}(E),\, \|u\|_\infty,\, \|f\|_r) \\
&\to -\infty \qquad \text{as } \delta \to 0.
\end{aligned}$$

Hence $u = 0$ a.e. in E. ■

The main ingredients of the proof are the continuity at the portion of the boundary where $u = 0$ and a mollification of the PDE, which allows us to apply Poisson's representation in E_δ. A similar nonexistence result actually holds for distributional solutions if we know apriori that u takes zero boundary value on O uniformly continuously.

Proposition II.2.2. *If Ψ vanishes on an open subset of ∂E, then there is no nonnegative, bounded, distributional solution u to* (II.1.12) *such that it takes zero boundary datum at O uniformly continuously. Then $u = 0$ in E.*

II.3 Solvability of the Parabolic Problem

Consider the following Dirichlet problem for the logarithmic diffusion equation

$$\begin{aligned} & u_t - \Delta \ln u = 0 \quad \text{in } E_T; \\ & \ln u = \ln g \quad \text{on } \partial E \times (0,T); \\ & u(\cdot,0) = u_o. \end{aligned} \tag{II.3.1}$$

Here E is a smooth domain in $\mathbb{R}^N$ with $N \geq 2$, and $E_T = E \times (0,T)$ with $T > 0$. Let $\partial_p E_T = [E \times \{0\}] \cup [\partial E \times [0,T)]$ be the parabolic boundary of E_T and let $S_T = \partial E \times (0,T)$ be its lateral boundary. We will use $B_\rho(x)$ (or $K_\rho(x)$) to denote a ball (or a cube) centered at x with radius (or edge) ρ. We are interested in solving (II.3.1) when g is permitted to vanish on a subset of S_T. Assume momentarily $g \geq 0$ and $u_o \geq 0$ are bounded and measurable so that $\ln g$ is well-defined as a measurable function on S_T.

The existence or nonexistence of solutions to (II.3.1) hinges on the notion of solutions. A bounded measurable function u is called a weak sub(super)-solution to (II.3.1) if $\ln u \in L^2(0,T;W^{1,2}(E))$, and for almost all $0 < t < T$

$$\begin{aligned} & \int_E u\eta(x,t)\,dx + \int_0^t \int_E \big(- u\eta_\tau + D\ln u D\eta \big)\,dxd\tau \\ & \quad \leq (\geq) \int_E u_o \eta(x,0)\,dx \end{aligned} \tag{II.3.2}$$

for all nonnegative testing functions

$$\eta \in W^{1,1}(0,T;L^1(E)) \cap L^2(0,T;W_o^{1,2}(E)).$$

In addition, $\ln u(\cdot,t) \leq (\geq) \ln g(\cdot,t)$ in the sense of traces on ∂E for a.e. $0 < t < T$. Since we are only interested in the existence and nonexistence of solutions when g vanishes somewhere, we always assume the boundedness of g.

Theorem II.3.1. *Let $0 \leq u_o \in L^\infty(E)$ and $0 \leq g \in L^\infty(\partial E \times (0,T))$ and $\ln g$*

admits an extension to E_T *which we still denote as* $\ln g$ *such that*

$$\ln g \in W^{1,1}(0,T;L^1(E)) \cap L^2(0,T;W^{1,2}(E)).$$

Then there is a unique bounded solution to (II.3.1)*. If, in addition,*

$$\frac{g(\cdot,t)}{t} \text{ is decreasing on } \partial E \times (0,T), \tag{II.3.3}$$

then the solution obtained is smooth and positive in E_T *and*

$$u_t \leq \frac{u}{t} \quad \text{in } E_T. \tag{II.3.4}$$

The inequality (II.3.4) is called a semi-convexity inequality in time. An analogous inequality was first found by Aronson and Bénilan in [1] for global solutions to porous medium equations. A classical solution to (II.3.1) satisfies (II.3.4) provided $(\ln g)_t \leq 1/t$ on S_T. To see this, we let $w = (\ln u)_t - \frac{1}{t}$ and we can show by the maximum principle that $w \leq 0$ in E_T. See Remark 3.1 in [2] for details.

The condition $\ln \Psi \in W^{\frac{1}{2},2}(\partial E)$ yields an extension of $\ln \Psi$ to E and it is a natural condition to impose when seeking for a solution with $D \ln u \in L^2(E)$. We can consider a weaker notion of solutions.

A bounded function u is called a very weak solution to (II.3.1) if $\ln u \in L^1(E_T)$ and

$$\iint_{E_T} u\varphi_t + \ln u \Delta\varphi \, dxdt = \int_{\partial E} \ln g \frac{\partial \varphi}{\partial \nu} \, d\sigma dt - \int_E u_o \varphi(\cdot,0) \, dx \tag{II.3.5}$$

for all $\varphi \in C^\infty(\bar{E} \times [0,T])$ and $\zeta = 0$ on $[\partial E \times (0,T)] \cup [E \times \{T\}]$. Then we have the following

Theorem II.3.2. *Let* $0 \leq u_o \in L^\infty(E)$ *and* $0 \leq g \in L^\infty(\partial E \times (0,T))$ *and* $\ln g \in L^1(\partial E \times (0,T))$*. Then there exists a very weak solution to* (II.3.1)*.*

II.3.1 Proof of Theorem II.3.1

II.3.1.1 The δ-Problem

By assumption, $\ln g$ admits an extension to $E\times(0,T)$, which we still denote as $\ln g$, and it satisfies $\ln g \in L^2(0,T;W^{1,2}(E))\cap W^{1,1}(0,T;L^1(E))$. Assume momentarily that $\ln g \geq \ln\delta$ $u_o \geq \delta$ for a small $\delta > 0$.

For a positive integer n, slice the time interval $(0,T)$ into n equal sub-intervals with length $h = T/n$. Since $\ln g(\cdot,t) \in W^{\frac{1}{2},2}(\partial E)$ for a.e. $0 < t < T$ we may assume this is the case for any t of the form

$$\{\frac{iT}{n} : 1 \leq i \leq n,\ n = 1,2,3,\dots\}.$$

Construct a sequence of approximating solutions by setting $u(\cdot,0) = u_o$ and for $k = 0,1,2,\cdots,(n-1)$, the function $u(x,(k+1)h)$ is the solution to

$$\begin{cases} \dfrac{u(x,(k+1)h)}{h} - \Delta \ln u(x,(k+1)h) = \dfrac{u(x,kh)}{h}; \\ u(x,(k+1)h) = g(x,(k+1)h) \qquad \text{on } \partial E. \end{cases} \tag{II.3.6}$$

By the results from the elliptic problem, especially (II.1.9) and (II.1.10), there exist $u(x,kh)$ such that

$$\delta \leq u(x,kh) \leq \Lambda \overset{\text{def}}{=} \max\{\|u_o\|_\infty,\ \|g\|_\infty\}$$

and $\ln u(x,kh) \in W^{1,2}(E)$ for all $1 \leq k \leq n$; moreover for any $\varphi \in W_o^{1,2}(E)$

$$\int_E u_{\bar{t}}(x,kh)\varphi\,dx + \int_E D\ln u(x,kh)D\varphi\,dx = 0. \tag{II.3.7}$$

Here

$$u_{\bar{t}}(x,kh) = \frac{u(x,kh) - u(x,(k-1)h)}{h}.$$

The following identity is a discrete version of integration by parts in $(0, T)$.

$$h \sum_{k=1}^{n} e_{\bar{t}}(k) f(k) = e(n)f(n) - e(0)f(0) - h \sum_{k=0}^{n-1} f_{\bar{t}}(k+1) e(k). \tag{II.3.8}$$

Here f and e are mappings from $\{0, 1, \ldots, n\}$ to $\mathbb{R}$. Now choose φ to be $h\eta(x,t) \in C_o^\infty(E \times [0,T))$. Sum over k from 1 to n and use the identity (II.3.8) to obtain

$$\begin{aligned}
&- h \sum_{k=0}^{n-1} \int_E u(x, kh) \eta_{\bar{t}}(x, (k+1)h)\, dx - \int_E u_o \eta(x, 0)\, dx \\
&+ h \sum_{k=1}^{n} \int_E D \ln u(x, kh) D\eta(x, kh) dx = 0
\end{aligned}$$

If we denote by $(F)_n(x,t)$ a function that equals $F(x, kh)$ in the interval $[kh, (k+1)h)$. Note $(\ln u)_n = \ln(u)_n$ and $(D \ln u)_n = D(\ln u)_n = D \ln(u)_n$. Then the above equality can be rewritten as

$$- \int_0^T \int_E (u)_n (\eta_{\bar{t}})_n (\cdot, t+h)\, dxdt - \int_E u_o \eta(x, h)\, dx + \int_h^T \int_E D \ln(u)_n D(\eta)_n\, dxdt = 0. \tag{II.3.9}$$

As $n \to \infty$, $D(\eta)_n$ and $(\eta_{\bar{t}})_n$ will converge to $D\eta$ and η_t uniformly in E_T.

In order to pass to the limit, we need to identify the weak convergence of $(u)_n$ and $D \ln(u)_n$.

First of all, we find a uniform bound for $\|D \ln(u)_n\|_2$. Use $\varphi = h(\ln u(\cdot, kh) - g(\cdot, kh))$ in (II.3.7) and sum over k from 1 to n to obtain

$$\begin{aligned}
h \sum_{k=1}^{n} \int_E |D \ln u(x, kh)|^2\, dx &= h \sum_{k=1}^{n} \int_E D \ln u(x, kh) Dg(x, kh)\, dx \\
&+ h \sum_{k=1}^{n} \int_E u_{\bar{t}}(x, kh) \ln g(x, kh)\, dx \\
&- h \sum_{k=1}^{n} \int_E u_{\bar{t}}(x, kh) \ln u(x, kh)\, dx \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

The first integral is estimated by Young's inequality as

$$I_1 \leq \frac{h}{2}\sum_{k=1}^{n}\int_E |D\ln u(x,kh)|^2\,dx + \frac{h}{2}\sum_{k=1}^{n}\int_E |D\ln g(x,kh)|^2\,dx.$$

The second integral is estimated by using the identity (II.3.8). Then

$$\begin{aligned} I_2 = &\int_E u(x,nh)\ln g(x,nh)\,dx - \int_E u_o(x)\ln g(x,0)\,dx \\ &- h\sum_{k=0}^{n-1}\int_E (\ln g)_{\bar{t}}(x,(k+1)h)u(x,kh)\,dx. \end{aligned}$$

Finally

$$\begin{aligned} I_3 = &\int_E u_o(x)\ln u_o(x)\,dx - \int_E u(x,nh)\ln u(x,nh)\,dx \\ &+ h\sum_{k=0}^{n-1}\int_E (\ln u)_{\bar{t}}(x,(k+1)h)u(x,kh)\,dx. \end{aligned}$$

We estimate the last term using the elementary inequality

$$\ln(x+1) \leq x, \quad \forall x > -1.$$

In fact,

$$\begin{aligned} &h\sum_{k=0}^{n-1}\int_E (\ln u)_{\bar{t}}(x,(k+1)h)u(x,kh)\,dx \\ &= \sum_{k=0}^{n-1}\int_E u(x,kh)(\ln u(x,(k+1)h) - \ln u(x,kh))\,dx \\ &= \sum_{k=0}^{n-1}\int_E u(x,kh)\ln\left(\frac{u(x,(k+1)h)}{u(x,kh)} - 1 + 1\right)dx \\ &\leq \sum_{k=0}^{n-1}\int_E u(x,kh)\left(\frac{u(x,(k+1)h)}{u(x,kh)} - 1\right)dx \\ &= \sum_{k=0}^{n-1}\int_E [u(x,(k+1)h) - u(x,kh)]\,dx \\ &= \int_E u(x,nh)\,dx - \int_E u_o(x)\,dx. \end{aligned}$$

Thus, collecting all of these we have

$$\begin{aligned}
&\frac{h}{2}\sum_{k=1}^{n}\int_E |D\ln u(x,kh)|^2\,dx \\
&\quad\le \frac{h}{2}\sum_{k=1}^{n}\int_E |D\ln g(x,kh)|^2\,dx \\
&\quad+\int_E u(x,nh)\ln g(x,nh)\,dx-\int_E u_o(x)\ln g(x,0)\,dx \\
&\quad-h\sum_{k=1}^{n}\int_E (\ln g)_{\bar t}(x,kh)u(x,kh)\,dx \\
&\quad+\int_E u_o(x)\ln u_o(x)\,dx-\int_E u(x,nh)\ln u(x,nh)\,dx \\
&\quad+\int_E u(x,nh)\,dx-\int_E u_o(x)\,dx.
\end{aligned}$$

By our assumption that $\ln g \in W^{1,1}(0,T;L^1(E))$, $\ln g(\cdot,t)\in L^1(E)$ for all $0\le t\le T$. Hence the second and the third integrals on the right-hand side are bounded by $\Lambda(\|\ln g(\cdot,0^+)\|_1+\|\ln g(\cdot,T^-)\|_1)$. The first and fourth integrals on the right-hand side are bounded in view of the assumption on the extension of $\ln g$. The last four integrals on the right-hand side are easily seen to be bounded by a constant depending on $\{\Lambda,\ |E|\}$.

Hence

$$\|D\ln(u)_n\|_2 \le C \tag{II.3.10}$$

for some constant independent of δ and dependent on

$$\{\|D\ln g\|_2,\ \|(\ln g)_t\|_1,\ \Lambda,\ |E|,\ \|\ln g(\cdot,0^+)\|_1,\ \|\ln g(\cdot,T^-)\|_1\}$$

The estimate (II.3.10) implies

$$\|D\ln u_\delta\|_2 \le C.$$

Here C is independent of δ and depending on

$$\{\|D\ln g\|_2,\ \|(\ln g)_t\|_1,\ \Lambda,\ |E|,\ \|\ln g(\cdot,0^+)\|_1,\ \|\ln g(\cdot,T^-)\|_1\}.$$

Since we know $\|D\ln u_\delta - D(\ln g)_\delta\|_2$ is uniformly bounded in $L^2(0,T;W_o^{1,2}(E))$, the imbedding theorem implies $\ln u_\delta - (\ln g)_\delta$ is uniformly bounded in $L^2(E_T)$. Thus $\ln u_\delta$ is uniformly bounded in $L^2(E_T)$ by a constant depending only on

$$\{\|D\ln g\|_2,\ \|(\ln g)_t\|_1,\ \|\ln g\|_2,\ \Lambda,\ |E|,\ \|\ln g(\cdot,0^+)\|_1,\ \|\ln g(\cdot,T^-)\|_1\}.$$

Thus we have shown that the following quantities are bounded by quantities independent of δ.

$$\|(u)_n\|_\infty; \quad \|\ln(u)_n\|_2; \quad \|D\ln(u)_n\|_2.$$

By using the boundedness of the last quantity we show

Lemma II.3.1. *The discrete function $(u)_n$ constructed in the proof of Theorem II.3.1 converges to some u in $L^2(E_{T-h_1})$ for any $0 < h_1 < T$.*

Proof. Let $W^{-1,2}(E)$ be the dual space of $W_o^{1,2}(E)$. Then any $f \in L^2(E)$ can be seen as an element in $W^{-1,2}(E)$ in the sense that

$$[f,\varphi] = (f,\varphi).$$

Here $[\cdot,\cdot]$ means the pairing of $W^{-1,2}(E)$ and $W_o^{1,2}(E)$ and $(\cdot,\cdot)$ means the inner product in $L^2(E)$. The norm in $W^{-1,2}(E)$ is defined as

$$\|f\|_{W^{-1,2}(E)} = \sup_{\|\varphi\|_{W^{1,2}(E)}\le 1} [f,\varphi].$$

Hence $\|f\|_{W^{-1,2}(E)} \le \|f\|_2$. Let $(u)_{n,h}$ be the Steklov average of $(u)_n$. For h fixed, $\{(u)_{n,h}\}$ is precompact in $C(0,T-h;L^2(E))$. Indeed, by the general Ascoli-Arzela's theorem (p291, [4]), we only need to verify $(u)_{n,h}(\cdot,t)$ is equibounded in $L^2(E)$ for any fixed $t \in (0,T-h)$ and $(u)_{n,h}(\cdot,t)$ is equicontinuous at t in the topology of $L^2(E)$. All of them are clear since $(u)_n$ is uniformly bounded by Λ.

Then $\{(u)_{n,h}\}$ is precompact in $L^2(0,T-h;W^{-1,2}(E))$ automatically. Next we show $(u)_{n,h} \to (u)_n$ in $L^2(0,T-h;W^{-1,2}(E))$ uniformly in n. Indeed, from the equation (II.3.7), we have

$$\|u(\cdot,kh)-u(\cdot,(k-1)h)\|_{W^{-1,2}(E)} \le h\|D\ln u(\cdot,k)\|_2.$$

For an integer l, let $s=(l+1)h$. Summing over k from j_o+1 to j_o+l we obtain

$$\begin{aligned}\|u(\cdot,(j_o+l)h)-u(\cdot,j_oh)\|_{W^{-1,2}(E)} &\le \sum_{k=j_o+1}^{j_o+l} h\|D\ln u(\cdot,kh)\|_2\\ &\le s^{\frac{1}{2}}\bigg(\sum_{k=j_o+1}^{j_o+l} h\|D\ln u(\cdot,kh)\|_2^2\bigg)^{\frac{1}{2}}.\end{aligned}$$

Then taking power 2 and multiplying by h at both sides and summing over j_o from 0 to $n-l$ we obtain

$$\begin{aligned}&\sum_{j_o=0}^{n-l} h\|u(\cdot,j_oh+s)-u(\cdot,j_oh+h)\|^2_{W^{-1,2}(E)}\\ &\le s\sum_{j_o=0}^{n-l} h\int_{j_oh}^{j_oh+s}\int_E |D\ln(u)_n|^2\,dxdt\\ &\le sT\iint_{E_T}|D\ln(u)_n|^2\,dxdt.\end{aligned}$$

This gives

$$\|(u)_n(x,t+s)-(u)_n(x,t)\|^2_{L^2(0,T-s;W^{-1,2}(E))}=O(s)\to 0 \quad \text{as } s\to 0$$

uniformly in n. Thus an application of the triangle inequality yields that $\{(u)_n\}$ is precompact in $L^2(0,T-h_1;W^{-1,2}(E))$ for any $0<h_1<T$.

In order to show $\{(u)_n\}$ is precompact in $L^2(E_{T-h_1})$, we show that for any $\epsilon>0$ there is a constant C_ϵ depending only on ϵ such that

$$\|v\|_{L^2(E)}\le\epsilon\|v\|_{W^{1,2}(E)}+C_\epsilon\|v\|_{W^{-1,2}(E)} \qquad \text{for all } v\in W^{1,2}(E).$$

Suppose this is false, then there exist $\epsilon_o>0$, $C_i\to\infty$ and $v_i\in W^{1,2}(E)$ such that

$$\|v_i\|_{L^2(E)}\ge\epsilon_o\|v_i\|_{W^{1,2}(E)}+C_i\|v_i\|_{W^{-1,2}(E)}.$$

Let $w_i = v_i/\|v_i\|_{W^{1,2}(E)}$ then we have

$$\|w_i\|_{L^2(E)} \geq \epsilon_o + C_i\|w_i\|_{W^{-1,2}(E)}. \tag{II.3.11}$$

Since the left-hand side is bounded independent of i, we get $\|w_i\|_{W^{-1,2}(E)} \to 0$. However the boundedness of w_i in $W^{1,2}(E)$ and the Compact Imbedding Theorem imply that w_i converges to some w in $L^2(E)$, and hence in $W^{-1,2}(E)$. This forces $w = 0$. Thus we reach a contradiction in (II.3.11).

Since $\{(u)_n\}$ is precompact in $L^2(0, T-h_1; W^{-1,2}(E))$, it is totally bounded. See Proposition 17.6 on p48 of [4]. That means for any $\delta > 0$ we have a finite set $\{(u)_{n_i}\} \subset \{(u)_n\}$ such that for any $(u)_n$ there is $(u)_{n_i}$ satisfying

$$\|(u)_n - (u)_{n_i}\|_{L^2(0,T-h_1;W^{-1,2}(E))} < \delta.$$

This joint with the previous interpolation inequality yield that

$$\begin{aligned}
\|(u)_n - (u)_{n_i}\|_{L^2(E_{T-h_1})} &< \epsilon\|(u)_n - (u)_{n_i}\|_{L^2(0,T-h_1;W^{1,2}(E))} \\
&\quad + C_\epsilon\|(u)_n - (u)_{n_i}\|_{L^2(0,T-h_1;W^{-1,2}(E))} \\
&\leq \epsilon M + C_\epsilon \delta.
\end{aligned}$$

Here M is the uniform bound of $\{(u)_n\}$ in $L^2(0,T;W^{1,2}(E))$. Choosing ϵ and δ appropriately we find $\{(u)_n\}$ is totally bounded in $L^2(0, T-h_1; W^{1,2}(E))$. Hence the proof is concluded by Proposition 17.6 on p48 of [4]. ■

By the boundedness of $\|D\ln(u)_n\|_2$, there is a vector $\vec{a}$ with each component in $L^2(E_T)$ such that

$$D\ln(u)_n \to \vec{a} \quad \text{weakly in } L^2(E_T).$$

Noting $\ln(u)_n$ is uniformly bounded by a constant depending on δ and Λ and

$\ln(u)_n \to \ln u$ a.e. in E_T, we have by the Dominated Convergence Theorem

$$\ln(u)_n \to \ln u \quad \text{in } L^2(E_T).$$

Thus $\vec{a} = D \ln u$.

Using these, we can pass to the limit in (II.3.9) to obtain

$$-\iint_{E_T} u\eta_t - \int_E u_o \eta(x,0)\,dx + \iint_{E_T} D\ln u D\eta\,dxdt = 0.$$

Next, we show $\ln u$ takes $\ln g$ as its trace on ∂E. To this end, we consider the trace inequality (II.1.5). A time integration over $(0,T)$ yields that

$$\|\ln(g)_n - \ln u\|_{2,S_T} \le C\|\ln(u)_n - \ln u\|_{2,E_T} \to 0 \quad \text{as } n \to \infty.$$

Here C depends on the uniform bound of $\ln(u)_n$ in $L^2(0,T;W^{1,2}(E))$. On the other hand,

$$\ln(g)_n \to \ln g \quad \text{a.e. in } \partial E$$

and in view of the assumption $\delta \le g \le \Lambda$ and the dominated convergence theorem, we have for any $1 \le p < \infty$

$$\ln(g)_n \to \ln g \quad \text{in } L^p(S_T).$$

Then an application of the triangle inequality yields $\ln u = \ln g$ on S_T.

II.3.1.2 When $\delta \to 0$.

Now let us consider the case when g vanishes somewhere on the boundary. By our assumptions, $\ln g$ admits an extension to E_T, which we still denote as $\ln g$, such that

$$\ln g \in L^2(0,T;W^{1,2}(E)) \cap W^{1,1}(0,T;L^1(E)).$$

We take the truncations $(u_o)_\delta = \max\{u_o,\, \delta\}$ and $(\ln g)_\delta = \ln \max\{g,\, \delta\}$, then $(\ln g)_\delta$ is in the same functional spaces as above. By our previous argument, $(u_o)_\delta$

and $(\ln g)_\delta$ generate a solution u^δ and it satisfies

$$-\iint_{E_T} u_\delta \eta_t - \int_E (u_o)_\delta \eta(x,0)\,dx + \iint_{E_T} D\ln u_\delta D\eta\,dxdt = 0$$

for all $\eta \in C_o^\infty(E\times[0,T))$.

Note also $u_{\delta_1} \le u_{\delta_2}$ if $\delta_1 \le \delta_2$. Indeed, let $(u)_{\delta_1,n}$ and $(u)_{\delta_2,n}$ be the approximating solutions corresponding to the initial-boundary data truncated by δ_1 and δ_2 respectively. Since a comparison principle holds for the elliptic equation (II.1.1), we have for any fixed n, $(u)_{\delta_1,n} \le (u)_{\delta_2,n}$ in E_T. Then letting $n\to\infty$ yields $u_{\delta_1} \le u_{\delta_2}$.

Suppose the limit of u_δ is u and then $\ln u_\delta \to \ln u$ a.e. in E_T. As we have pointed out earlier, since $D\ln u_\delta$ is uniformly bounded in $L^2(E_T)$, we know $\ln u_\delta - (\ln g)_\delta$ is uniformly bounded in $L^2\big(0,T;W_o^{1,2}(E)\big)$. Thus $\ln u_\delta$ is uniformly bounded in $L^2(E_T)$. From the uniform boundedness of $\ln u_\delta$ in $L^2(0,T;W^{1,2}(E))$, we can extract a sub-sequence and a vector $\vec{a}$ with all components in $L^2(E_T)$ such that

$$D\ln u_{\delta'} \to \vec{a} \quad \text{weakly in } L^2(E_T).$$

On the other hand, by the Dominated Convergence Theorem $\ln u_{\delta'} \to \ln u$ in $L^2(E_T)$. Hence $\vec{a} = D\ln u$.

Finally, $\ln u(\cdot,t)$ takes trace $\ln g(\cdot,t)$ and this is proved in a similar fashion as in the case of u_δ.

Now we consider the trace inequality (II.1.5) applied to $\ln u(\cdot,t) - \ln u_\delta(\cdot,t)$. A time integration over $(0,T)$ yields that

$$\|(\ln g)_\delta - \ln u\|_{2,S_T} \le C\|\ln u_\delta - \ln u\|_{2,E_T} \to 0 \quad \text{as } n\to\infty.$$

Here C depends on the uniform bound of $\ln u_\delta$ in $L^2(0,T;W^{1,2}(E))$. On the other hand, the dominated convergence theorem yields

$$(\ln g)_\delta \to \ln g \quad \text{in } L^2(S_T).$$

Then an application of the triangle inequality yields $\ln u = \ln g$ on S_T.

II.3.1.3 Locally Smooth Solutions with Semi-convexity in t

Now we consider the case when $g(\cdot, t)/t$ is decreasing. It implies a discrete version of the semi-convexity inequality (II.3.3); for any $0 \le k \le n-1$

$$\frac{u(x,(k+1)h)}{k+1} \le \frac{u(x,kh)}{k} \quad \forall x \in E.$$

This can be proved using induction. See the proof of Lemma 6.1 in [6]. Now, we use (II.1.11) with

$$\lambda = \frac{1}{h} \quad \text{and} \quad f = \frac{1}{h} u(y,(k-1)h)$$

to conclude that for any compact subset K of E there is a constant $C_1(N, \operatorname{dist}(K, \partial E))$ such that

$$\ln u(x,kh) \ge -C_1 \int_{\partial E} |\ln g(x,kh)|\, d\sigma - \frac{1}{h} \int_E G(x,y)[u(y,kh) - u(y,(k-1)h)]\, dy$$

for all $x \in K$. Multiplying both sides by h and summing over k from a positive integer j_o to $j_1 \le n$ we obtain

$$\begin{aligned} h \sum_{k=j_o}^{j_1} \ln u(x,kh) &\ge -C_1 h \sum_{k=j_o}^{j_1} \int_{\partial E} |\ln g(x,kh)|\, d\sigma \\ &\quad - \int_E G(x,y)[u(y,j_1 h) - u(y,(j_o-1)h)]\, dy \\ &\ge -C_1 \int_0^T \int_{\partial E} |\ln g|\, d\sigma dt - C_2 \end{aligned}$$

where C_2 depends on $\{N, \operatorname{diam}(E), \Lambda\}$ and Λ is the uniform bound of $(u)_n$.

On the other hand, the discrete semi-convexity inequality implies that for any

$0 < j_o < j_1 \le n$

$$
\begin{aligned}
h \sum_{k=j_o}^{j_1} \ln u(x,kh) &= h \sum_{k=j_o}^{j_1} \left(\ln \frac{u(x,kh)}{k} + \ln k \right) \\
&\le h(j_1 - j_o + 1) \ln \frac{u(x,j_o h)}{j_o} + h(j_1 - j_o + 1) \ln j_1 \\
&\le T \ln \frac{u(x,j_o h)}{j_o} + T \ln j_1
\end{aligned}
$$

Combining these estimates yields for any $0 < j_o < j_1 \le n$

$$
\ln u(x,j_o h) \ge -\frac{C_1}{T} \int_0^T \int_{\partial E} |\ln g| \, d\sigma dt - \frac{C_2}{T} + \ln \frac{j_o}{j_1}.
$$

Since this estimate is independent of n, we conclude, for any compact subset $K \times [t_1, t_2]$, there is a constant C depending on

$$
\{\| \ln g\|_{1,\partial E \times (0,T)},\, T,\, \mathrm{diam}(E),\, \mathrm{dist}(K, \partial E),\, t_1,\, t_2,\, \Lambda\}
$$

such that the discrete step function

$$
(u)_n(x,t) \ge e^{-C} \quad \forall (x,t) \in K \times [t_1, t_2].
$$

The rest of the proof follows easily. ■

II.3.2 Proof Theorem II.3.2

II.3.2.1 When $\ln g \in L^1(\partial E \times (0,T))$.

Assume momentarily $u_o, g \ge \delta$ and denote a pointwise approximation of g by $g_\epsilon \in C^\infty(\partial E \times (0,T))$ and

$$
\delta/2 \le g_\epsilon \le 2\Lambda = 2 \max\{\|u_o\|_\infty, \|g\|_\infty\}.
$$

Let u_ϵ be the corresponding classical solution such that

$$
\delta/2 \le u_\epsilon \le 2\Lambda
$$

and it satisfies (II.3.5) with u replaced by u_ϵ. Since $\{u_\epsilon\}$ is uniformly bounded above and below, and hence equicontinuous in the interior, there is a function u and a sub-sequence such that

$$\delta/2 \le u \le 2\Lambda;$$

$$u_{\epsilon'} \to u \quad \text{a.e. in } E_T.$$

Thus we can pass to the limit in the integral identity to obtain u as a solution.

For the general case, we use g_δ to denote the truncation from below of g by δ. Then there is a corresponding solution u_δ and it is decreasing to some function u as $\delta \to 0$. Since we have a uniform bound

$$\| \ln u_\delta \|_1 \le \gamma_1 \| \ln g \|_{1,S_T} + \gamma(N, \operatorname{diam}(E), \Lambda),$$

we can pass to the limit in the corresponding integral identity and obtain u as a solution. ■

II.3.3 Failure of Constructive Approximations

The bound from below in the above proof hinges upon the $L^1(\partial E \times (0,T))$ norm of $\ln g$. It is natural to ask what happens if $\ln g$ is not in $L^1(\partial E \times (0, t_o))$ for some $t_o < T$. Let g_ϵ be a smooth function on S_T and $u_{o,\epsilon}$ is a nonnegative bounded function in E. Let us consider a family of solutions to

$$\begin{aligned} &u_{\epsilon,t} - \Delta \ln u_\epsilon = 0 \quad \text{in } E_T; \\ &\ln u_\epsilon = \ln g_\epsilon \quad \text{on } \partial E \times (0,T); \\ &u_\epsilon(\cdot, 0) = u_{o,\epsilon}. \end{aligned} \tag{II.3.12}$$

We will see in the following proposition that this sequence of approximating problems does not give any meaningful solution when $t \ge t_o$ as the limit function will always be zero in the interior for $t \ge t_o$.

Proposition II.3.1. *Suppose u_ϵ is a decreasing sequence of $C^2(\bar{E} \times (0,T))$ solu-*

tions to (II.3.12). *Assume* u_ϵ *is uniformly bounded by* M *and satisfies the semi-convexity inequality* (II.3.3). *If there is a* $0 < t_o < T$ *such that*

$$\lim_{\epsilon\to 0}\int_0^{t_o}\int_{\partial E} |\ln g_\epsilon(y,t)|\, d\sigma dt = \infty,$$

then the limit function $u(x,t) = 0$ *for all* $(x,t) \in E \times [t_o, T)$.

Proof. Fix $x_o \in E$; it suffices to show $u(x_o, \tau) = 0$ for all $\tau \geq t_o$. By the Poisson's representation formula

$$\ln u_\epsilon(x_o, t) = \int_{\partial E} \ln g_\epsilon(y,t) P(x_o, y)\, d\sigma - \int_E G(x_o, y) u_{\epsilon,t}(y,t)\, dy.$$

Integrate this formula in dt over $(0, \tau)$ for $\tau \geq t_o$ to obtain

$$\begin{aligned}\int_0^\tau \ln u_\epsilon(x_o,t)\, dt &= \int_0^\tau \int_{\partial E} \ln g_\epsilon(y,t) P(x_o,y)\, d\sigma dt - \int_0^\tau \int_E G(x_o,y) u_{\epsilon,t}(y,t)\, dy \\ &\leq \int_0^\tau \int_{\partial E} \ln g_\epsilon(y,t) P(x_o,y)\, d\sigma dt + \int_E G(x_o,y) u_{o,\epsilon}(y)\, dy.\end{aligned}$$

The left-hand side is estimated by using the semi-convexity inequality (II.3.3), namely,

$$\begin{aligned}\int_0^\tau \ln u_\epsilon(x_o,t)\, dt &= \int_0^\tau \ln \frac{u_\epsilon(x_o,t)}{t}\, dt + \int_0^\tau \ln t\, dt \\ &\geq \tau \ln u_\epsilon(x_o,\tau) - \tau \ln \tau + \int_0^\tau \ln t\, dt.\end{aligned}$$

Thus

$$\ln u_\epsilon(x_o, \tau) \leq \fint_0^\tau \int_{\partial E} \ln g_\epsilon(y,t) P(x_o,y)\, d\sigma dt + C(N,\, \tau,\, M,\, \mathrm{diam}(E)).$$

From this we conclude that $u(x_o, \tau) = 0$ for all $\tau \geq t_o$. ■

Remark II.3.1. *When* $g_\epsilon = \epsilon$ *and* $N = 1$ *Proposition II.3.1 has been reported in [18]. However, we have given a more general criterion for all dimensions and a rigorous proof.*

II.3.4 Locally Continuous Solutions with Quenching Times

In this section, we present a construction of continuous solutions that is not covered by the previous section. Continue to assume ∂E is smooth. Consider the function

$$S(x,t) = \frac{\eta(t)}{C_1 + C_2\phi(x)}$$

where ϕ is the positive solution to the first eigenvalue problem of $-\Delta$ in E with eigenvalue λ_1 and η is a nonnegative absolutely continuous function satisfying

$$|\eta'(t)| \leq M \quad \text{and} \quad \int_0^T |\ln \eta(t)|^2 \, dt < \infty.$$

For example, $\eta(t) = (1-t)^2$ will satisfy the above conditions.

Note that $-\Delta\phi = \lambda_1\phi \geq 0$ and by Hopf's lemma $|D\phi(x)| \geq -D\phi(x)\cdot\nu(x) > 0$ in ∂E for any $x \in \partial E$. See Proposition 5.1 on p.53 of [3]. Thus there is $\delta > 0$ such that $\lambda_1\phi^2 + |D\phi|^2 \geq \delta$ in $\bar{E}$. A direct calculation yields

$$\begin{aligned}
S_t - \Delta \ln S &= \frac{1}{(C_1+C_2\phi)^2}\left[(C_1+C_2\phi)(\eta'(t) - \lambda_1 C_2\phi) - C_2^2|D\phi|^2\right] \\
&\leq \frac{1}{(C_1+C_2\phi)^2}\left[(C_1+C_2\phi)(M - \lambda_1 C_2\phi) - C_2^2|D\phi|^2\right] \\
&\leq \frac{1}{(C_1+C_2\phi)^2}\left[M(C_1+C_2\phi) - \lambda_1 C_1 C_2\phi - C_2^2(\lambda_1\phi^2 + |D\phi|^2)\right] \\
&\leq \frac{1}{(C_1+C_2\phi)^2}\left[M(C_1+C_2\phi) - \lambda_1 C_1 C_2\phi - C_2^2\delta\right].
\end{aligned}$$

Now the right-hand side is easily seen to be non-positive if C_2 is large enough. More precisely,

$$C_2 \geq \frac{M\|\phi\|_\infty + \sqrt{M^2\|\phi\|_\infty^2 + 4\delta M C_1}}{2\delta}.$$

Let $C_o > 0$ and functions η and ϕ be as above. Consider another function

$$U(x,t) = C_o\eta(t)(1+\phi(x)) \quad \text{in } E_T.$$

A similar calculation yields

$$\begin{aligned} U_t - \Delta \ln U &= C_o \eta'(t)(1+\phi) + (1+\phi)^{-2}[\lambda_1(1+\phi)\phi + |D\phi|^2] \\ &\geq (1+\phi)^{-2}[-C_o M(1+\phi)^3 + \lambda_1 \phi(1+\phi) + |D\phi|^2]. \end{aligned}$$

Thus, in order to guarantee U to be a super-solution we only need to choose C_o such that

$$C_o \leq \inf_E \frac{\lambda_1 \phi(1+\phi) + |D\phi|^2}{M(1+\phi)^3} \tag{II.3.13}$$

or

$$C_o \leq \frac{\delta}{M(1+\|\phi\|_\infty)^3}$$

Typically η is a nonnegative function, which might be zero at discrete points such that $\ln \eta \in L^2(0,T)$. In such a case, it is worth noting that U, S, $\ln U$, $\ln S \in L^2(0,T;W^{1,2}(E))$ and the boundary trace of $U(\cdot,t)$ and $S(\cdot,t)$ can be taken for every t, while the boundary trace of $\ln S(\cdot,t)$ and $\ln U(\cdot,t)$ can be taken almost everywhere except on the set $[\eta(t)=0]$.

Recall that C_1 is still left to be chosen. We choose C_o first according to (II.3.13) and then choose C_1 so that $C_1 C_o \geq 1$. With the aid of the sub-solution S and the super-solution U we are able to establish the following theorem.

Theorem II.3.3. *Let $u_o(x) = 0$ in E and $\ln g(x,t) \in L^2(0,T;W^{\frac{1}{2},2}(\partial E))$ satisfy*

$$C_o \eta(t) \geq g(x,t) \geq \frac{\eta(t)}{C_1} \qquad \text{in } \partial E \times (0,T).$$

Then there is a unique nonnegative, bounded, locally continuous solution u to (II.3.1) *such that*

$$S \leq u \leq U \qquad \text{in } E_T.$$

Proof. The construction of the unique solution is in Theorem II.3.1. In addition, since $S(x,t) \leq u(x,t) \leq U(x,t)$ for $(x,t) \in \partial E_T$, this is true in E_T by the comparison principle. If $\eta(t_o) = 0$ for some $0 \leq t_o \leq T$, then u will be forced to tend to zero continuously with the same rate. ■

II.4 Uniqueness of Solutions to (II.3.1)

In this section we list some basic facts about uniqueness and comparison principles which are satisfied by solutions to (II.3.1).

Proposition II.4.1. *Let u and v be two bounded weak solutions to* (II.3.1) *in the sense of* (II.3.2) *with the same boundary and initial datum. Then $u = v$ a.e. in $E \times (0,T)$.*

Proof. Take the difference of the weak formulations for u and v; we obtain

$$\int_0^t \int_E \big[-(u-v)\eta_\tau(x,\tau) + D(\ln u - \ln v)D\eta \big]\, dxd\tau = 0$$

for all

$$\eta \in W^{1,1}(0,T;L^1(E)) \cap L^2(0,T;W_o^{1,2}(E)).$$

Now take

$$\eta(x,\tau) = \begin{cases} \displaystyle\int_\tau^t (\ln u(x,s) - \ln v(x,s))\, ds, & 0 \le \tau < t; \\ 0, & \tau \ge t. \end{cases}$$

It is straightforward to verify this function is an admissible test function. Then

$$\int_0^t \int_E (u-v)(\ln u - \ln v)\, dxd\tau + \frac{1}{2}\int_E \left| \int_0^t D(\ln u - \ln v)\, ds \right|^2 dx = 0$$

Thus

$$\int_0^t \int_E (u-v)(\ln u - \ln v)\, dxd\tau = 0$$

and $u = v$ a.e. in E_T. ■

Proposition II.4.2. *Let u be a super-solution and v be sub-solution to* (II.3.1). *If $u \ge v$ on $\partial_p E_T$ and $v_t, u_t \in L^1(E_T)$, then $u \ge v$ a.e. in E_T.*

Proof. Let $S_n(\cdot)$ be an approximation to the Heaviside function. Namely, $S_n(c)$ equals 0 when $c \le 0$ and 1 when $c \ge \frac{1}{n}$ and it is linear when $0 \le c \le \frac{1}{n}$.

Take $S_n((\ln v - \ln u)_+)$ as a test function then

$$\begin{aligned}&\iint_{E_t} (v-u)_t S_n((\ln v - \ln u)_+)\, dxdt \\ &= -\iint_{E_t} S_n'((\ln v - \ln u)_+)|D(\ln v - \ln u)_+|^2\, dxdt \le 0.\end{aligned}$$

Then letting $n \to \infty$ we have for any $0 < t < T$

$$\int_E (v-u)_+(\cdot, t)\, dx \le 0.$$

■

II.5 Nonexistence of Solutions for the Parabolic Equation

We call u a local weak solution of the logarithmic diffusion equation (II.3.1) if

$$\begin{aligned}&u \in C_{\text{loc}}\big(0, T; L^2_{\text{loc}}(E)\big), \quad \ln u \in L^2_{\text{loc}}\big(0, T; W^{1,2}_{\text{loc}}(E)\big); \\ &u_t - \Delta \ln u = 0 \qquad \text{weakly in } E_T.\end{aligned} \tag{II.5.1}$$

Then we have

Theorem II.5.1. *If g vanishes on an open subset of S_T, then there is no solution to* (II.3.1) *in the sense that* (II.5.1) *is satisfied in the interior and the zero boundary datum is taken in the sense of traces.*

The proof hinges on the continuity of the solution at O and this is the content of the following proposition.

Proposition II.5.1. *Let u be a nonnegative, bounded, local, weak solution to* (II.5.1) *in E_T. Assume there is an open subset O of the smooth boundary S_T. If u vanishes in the sense of trace on O, then there exist constants γ and α depending only on N so that u satisfies*

$$|u(x,t)| \le \gamma(|x-y| + \|u\|_{\infty, E_T}^{-\frac{1}{2}} |t-s|^{\frac{1}{2}})^\alpha$$

for any $(x,t) \in E_T$ and $(y,s) \in O$.

Local boundedness is enough to reach the same conclusion. The proof is a boundary version adaption of the interior arguments in [11], where the interior Hölder continuity is proved for the porous medium type equations. The interior Hölder continuity of local weak solutions to the logarithmic diffusion equation cannot be shown as in [11], and the main difficulty is generated by working with the truncated function $(u-k)_-$. In order to derive a DeGiorgi-type lemma for $(u-k)_-$ we have to assume a proper extra integrability of $\ln u$; see [8] for such a lemma and see how it fails in the remark following Lemma II.6.1 in Section II.6. However, the situation for our current case is much simpler. First of all, the zero boundary trace allows us to circumvent working with $(u-k)_-$. Also, the local logarithmic estimates are avoided since we assume the boundary satisfies the property of positive geometric density. The complete proof will be reported in the next section.

Proof of Theorem II.5.1 Assuming Proposition II.5.1. Assume without loss of generality that $O = \Gamma \times (t_1, t_2)$ where Γ is a open subset of ∂E. Let $\varphi \in C_o^\infty(E_T)$ and $K_\epsilon(x,t)$ be a mollifying kernel. Let φ_ϵ be the space-time convolution of φ with K_ϵ. In the weak formulation of the local weak solutions to (II.5.1) we take φ_ϵ as a test function. Then we have

$$u_{\epsilon,t} - \Delta(\ln u)_\epsilon = 0 \quad \text{ in } E_T.$$

Define an interior region in E as

$$E_\delta = \{x \in E : \operatorname{dist}(x, \partial E) > \delta\} \quad \text{for some } \delta > 0.$$

Moreover, define

$$\partial E_{\delta,O} = \{a - \delta\nu : a \in \Gamma, \, \nu \text{ is the outer normal at } a\}.$$

Note that, by Proposition II.5.1, for any $(y,t) \in \partial E_{\delta,O}^{t_1,t_2}$

$$u(y,t) \leq \gamma \delta^\alpha.$$

Continue to denote by P the Poisson kernel and G the Green function for E_δ. By Poisson's representation formula and a time integration over $(\tau_1, \tau_2) \subset (t_1, t_2)$ yield that for any $x \in E_{\delta_1}$ with $\delta_1 > \delta$

$$\begin{aligned}
\int_{\tau_1}^{\tau_2} (\ln u)_\epsilon(x,t)\,dt &= \int_{\tau_1}^{\tau_2} \int_{\partial E_\delta} P(x,y)(\ln u)_\epsilon(y,t)\,d\sigma dt - \int_{\tau_1}^{\tau_2} \int_{E_\delta} G(x,y) u_{\epsilon,t}(y,t)\,dydt \\
&\le \int_{\tau_1}^{\tau_2} \int_{\partial E_{\delta,O}} P(x,y)(\ln u)_\epsilon(y,t)\,d\sigma dt + C(N,\,\mathrm{diam}(E),\,\|u\|_\infty) \\
&\le \ln(\gamma\delta^\alpha)|\tau_2 - \tau_1| + C(N,\,\mathrm{diam}(E),\,\|u\|_\infty).
\end{aligned}$$

Here we have used the fact that

$$\int_{\partial E_\delta} P(x,y)\,d\sigma \quad \text{and} \quad \int_{E_\delta} G(x,y)\,dy$$

are bounded by a constant independent of δ.

Since the right-hand side is independent of ϵ, we have for a.e. $x \in E_{\delta_1}$

$$\begin{aligned}
\int_{\tau_1}^{\tau_2} \ln u(x,t)\,dt &\le \ln(\gamma\delta^\alpha)|\tau_2 - \tau_1| + C(N,\,\mathrm{diam}(E), \|u(\cdot,\tau)\|_\infty) \\
&\to -\infty \quad \text{as } \delta \to 0.
\end{aligned}$$

As a result, for any $x \in E_{\delta_1}$ and arbitrary $(\tau_1, \tau_2) \subset (t_1, t_2)$

$$\int_{\tau_1}^{\tau_2} \ln u(x,t)\,dt = -\infty.$$

Hence $u = 0$ a.e. in $E \times (t_1, t_2)$. ∎

As in the elliptic case, the main ingredients of the proof are the continuity at the portion of the boundary where $u = 0$ and a mollification of the PDE, which allows us to apply Poisson's representation in E_δ. A similar nonexistence result actually holds for distributional solutions if we know apriori that u takes zero boundary value uniformly continuously. A distributional solution is defined as

$$\begin{aligned}
&u,\, \ln u \in L^1_{\mathrm{loc}}(E_T); \\
&u_t - \Delta \ln u = 0 \qquad \text{distributionally in } E_T.
\end{aligned} \tag{II.5.2}$$

Proposition II.5.2. *If u is a nonnegative, locally bounded, distributional solution to* (II.5.1) *such that it takes zero boundary datum at O uniformly continuously. Then $u = 0$ a.e. in $E \times (t_1, t_2)$.*

II.6 Proof of Proposition II.5.1

II.6.1 An Energy Estimate

Let u be a local solution to (II.5.1). The logarithmic diffusion equation satisfies the notion of *parabolicity* defined in [5] and [11]. Thus, $(u-k)_+$ is a local sub-solution to the logarithmic diffusion equation in the sense that

$$\int_K \frac{\partial}{\partial t}(u-k)_{+,h}\varphi + \left[\frac{D(u-k)_+}{u}\right]_h D\varphi \, dx \le 0$$

for any $\varphi \in W_o^{1,2}(K)$ and any compact set $K \subset E$. Here, we have used *Steklov averages*. See [5] for such a notion and its basic properties.

If we know $u(\cdot, t) \in W^{1,2}_{\mathrm{loc}}(E \cup O_t)$, then since $[\dots]_h$ is always in $L^2_{\mathrm{loc}}(E \cup O_t)$, the above integral inequality holds for any $\varphi \in W_o^{1,2}(\Omega)$ and any compact set $\Omega \times \{t\} \subset E \cup O_t$.

Let us assume $(y, s) \in O$ and consider the cylinder $(y, s) + Q_\rho(\theta) \overset{\text{def}}{=} K_\rho(y) \times (s - \theta\rho^2, s]$ with $\theta\,, \rho > 0$ so small that $\{[y + K_{2\rho}] \cap \partial E\} \times (s - \theta(2\rho)^2, s] \subset O$. By a translation we may assume (y, s) coincides with $(0, 0)$. We may obtain an energy estimate by taking the test function

$$\varphi_h = (u-k)_{+,h}\zeta^2,$$

in the weak formulation of (II.3.1) and integrating over $Q_\rho(\theta)$ and then letting $h \to 0$. Such a choice of test function is admissible since for a.e. $t \in (-\theta\rho^2, 0]$ we know $x \to \zeta(x, t)$ vanishes on the boundary of K_ρ but not on the boundary of $K_\rho \cap E$, and for any $k \ge 0$

$$(u-k)_+(\cdot, t) = 0 \quad \text{in the sense of trace on } \partial K_\rho \cap E.$$

Thus

$$(u(\cdot,t)-k)_+\zeta^2(\cdot,t) \in W_o^{1,2}(K_\rho \cap E). \tag{II.6.1}$$

With such a choice of k we can establish the following energy estimate near S_T.

Proposition II.6.1. *Let u be a nonnegative, local, weak solution to* (II.3.1) *in $Q_\rho(\theta)$ and ζ is a cutoff function vanishing on the parabolic boundary of $Q_\rho(\theta)$. There exists a constant γ depending only on N such that for every $(y,s) \in O$, for every cylinder $(y,s)+Q(\theta,\rho)$ such that $s-\theta\rho^2 > 0$ and every level $k \geq 0$, the following inequality holds:*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta\rho^2<t<s} \int_{[y+K_\rho]\cap E} (u-k)_+^2\zeta^2(x,t)\,dx \\
&\qquad + \iint_{[(y,s)+Q_\rho(\theta)]\cap E_T} u^{-1}|D(u-k)_+|^2\zeta^2\,dxdt \\
&\qquad \leq \gamma \iint_{[(y,s)+Q_\rho(\theta)]\cap E_T} (u-k)_+^2\zeta|\zeta_t|\,dxdt \\
&\qquad + \iint_{[(y,s)+Q_\rho(\theta)]\cap E_T} u^{-1}(u-k)_+^2|D\zeta|^2\,dxdt.
\end{aligned} \tag{II.6.2}$$

II.6.2 Proof of the Proposition

For a cylinder $Q_{2\rho}(\theta) \overset{\text{def}}{=} K_{2\rho} \times (-\theta(2\rho)^2, 0]$ and a point $(y,s) \in O$ we define

$$\mu_+ = \operatorname*{ess\,sup}_{[(y,s)+Q_{2\rho}(\theta)]\cap E_T} u.$$

Since we always have

$$\operatorname*{ess\,inf}_{[(y,s)+Q_{2\rho}(\theta)]\cap E_T} u = 0$$

the essential oscillation ω over the cylinder $Q_{2\rho}(\theta)$ satisfies $\omega = \mu_+$. Let ξ and a be constants in $(0,1)$.

Lemma II.6.1. *Let u be a nonnegative, locally bounded, local, weak solution to* (II.3.1) *in E_T. There exists a positive number ν, depending on ω, θ, ξ, a and N such that if*

$$|[u \geq \mu_+ - \xi\omega] \cap [(y,s)+Q_{2\rho}(\theta)] \cap E_T| \leq \nu |Q_{2\rho}(\theta) \cap E_T|,$$

then

$$u \le \mu_+ - a\xi\omega \qquad a.e.\ in\ [(y,s)+Q_\rho(\theta)] \cap E_T.$$

Proof. Assume without loss of generality that $(y,s)=(0,0)$ and for $n = 0, 1, \ldots$. Set

$$\rho_n = \rho + \frac{\rho}{2^n}, \quad K_n = K_{\rho_n}, \quad Q_n = K_n \times (-\theta\rho_n^2, 0].$$

Apply (II.6.2) over K_n and Q_n to $(u-k_n)_+$, for the levels

$$k_n = \mu_+ - \xi_n\omega \qquad \text{where } \xi_n = a\xi + \frac{1-a}{2^n}\xi.$$

The cutoff function $\zeta(x,t)$ takes value 1 on Q_{n+1} and vanishes on the parabolic boundary of Q_n such that

$$|D\zeta| \le \frac{2^{n+1}}{\rho} \qquad \text{and} \qquad |\zeta_t| \le \frac{2^{2(n+1)}}{\theta\rho^2}.$$

Then (II.6.2) gives

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta_n\rho_n^2<t<0} \int_{K_n\cap E} (u-k_n)_+^2 \zeta^2(x,t)\,dx + \iint_{Q_n\cap E_T} u^{-1}|D(u-k_n)_+\zeta]|^2\,dxdt \\
&\le \gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^2 \iint_{Q_n\cap E_T}\left(\frac{1}{(1-\xi)\omega}+\frac{1}{\theta}\right)\chi_{[u>k_n]}\,dxdt \\
&\le \gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^2\frac{1}{\omega(1-\xi)}\left(1+\frac{\omega}{\theta}\right)|[u>k_n]\cap Q_n\cap E_T|.
\end{aligned}$$

The second integral on the left hand side is estimated by

$$\iint_{Q_n\cap E_T} u^{-1}|D(u-k_n)_+\zeta]|^2\,dxdt \ge \omega^{-1}\iint_{Q_n\cap E_T}|D[(u-k_n)_+\zeta]|^2\,dxdt.$$

Setting

$$A_n = [u>k_n]\cap Q_n\cap E_T \qquad \text{and} \qquad Y_n = \frac{|A_n|}{|Q_n\cap E_T|},$$

then

$$\operatorname*{ess\,sup}_{-\theta_n\rho_n^2<t<0}\int_{K_n\cap E}(u-k_n)_+^2\zeta^2(x,t)\,dx+\omega^{-1}\iint_{Q_n\cap E_T}|D(u-k_n)_+\zeta]|^2\,dxdt$$
$$\leq\gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^2\frac{1}{\omega(1-\xi)}\left(1+\frac{\omega}{\theta}\right)|A_n|.$$

An application of the parabolic embedding (Chap. 1, [5]) yields

$$\begin{aligned}\left(\frac{1-a}{2^{n+1}}\right)^2(\xi\omega)^2|A_{n+1}|&\leq\iint_{Q_{n+1}\cap E_T}(u-k_n)_+^2\,dxdt\\&\leq\left(\iint_{Q_n\cap E_T}[(u-k_n)_+\zeta]^{2\frac{N+2}{N}}\,dxdt\right)^{\frac{N}{N+2}}|A_n|^{\frac{2}{N+2}}\\&\leq\gamma\left(\iint_{Q_n\cap E_T}|D[(u-k_n)_+\zeta]|^2\,dxdt\right)^{\frac{N}{N+2}}\\&\times\left(\operatorname*{ess\,sup}_{-\theta\rho_n^2<t<0}\int_{K_n}[(u-k_n)_+\zeta]^2(x,t)\,dx\right)^{\frac{2}{N+2}}|A_n|^{\frac{2}{N+2}}\end{aligned}$$

for a constant γ depending only on N. This joint with the previous estimate gives

$$|A_{n+1}|\leq\frac{\gamma 2^{4n}}{(1-a)^2\rho^2}\frac{\omega^{\frac{-2}{N+2}}}{1-\xi}\left(1+\frac{\omega}{\theta}\right)|A_n|^{1+\frac{2}{N+2}}.$$

In terms of Y_n this can be rewritten as

$$Y_{n+1}\leq\frac{\gamma 2^{4n}}{(1-a)^2(1-\xi)}\frac{1+\theta\omega^{-1}}{(\theta\omega^{-1})^{\frac{N}{N+2}}}Y_n^{1+\frac{2}{N+2}}.$$

Thus using Lemma 4.1 on p12 of [5] we conclude that $Y_n\to 0$ provided

$$Y_o=\frac{|A_o|}{|Q_o\cap E_T|}\leq\left[\frac{(1-a)^2(1-\xi)}{\gamma 4^{N+2}}\right]^{\frac{N+2}{2}}\frac{(\theta\omega^{-1})^{\frac{N}{2}}}{(1+\theta\omega^{-1})^{\frac{N+2}{2}}}\overset{\text{def}}{=}\nu.$$

■

Since O is smooth, there is a constant β such that

$$|\{x\in[y+K_\rho]\cap E:u(x,t)>0\}|\leq|K_\rho\cap E|\leq(1-\beta)|K_\rho| \tag{II.6.3}$$

for all

$$s - \theta\rho^2 < t \le s.$$

Lemma II.6.2. *Let $\theta = \omega$. For every $\nu \in (0,1)$ there exists a positive integer q depending on ν and N such that*

$$|[u > \mu_+ - \frac{\omega}{2^q}] \cap [(y,s) + Q_{2\rho}(\theta)] \cap E_T| < \nu |Q_{2\rho}(\theta) \cap E_T|.$$

Proof. Assume $(y,s) = (0,0)$ and set $Q = Q_\rho(\theta)$ and $Q' = Q_{2\rho}(\theta)$, and use the energy estimate for the functions

$$(u - k_j)_+ \quad \text{where} \quad k_j = \mu_+ - \frac{\omega}{2^j} \quad \text{for} \quad j = 1, \ldots, q,$$

over the pair of cylinders Q and Q'. The cutoff function ζ is taken to be one on Q, vanishing on the parabolic boundary of Q' such that

$$|D\zeta| \le \frac{1}{\rho} \quad \text{and} \quad 0 \le \zeta_t \le \frac{2}{\theta\rho^2}, \quad \theta = \omega.$$

Then the energy estimate (II.6.2) gives

$$\omega^{-1} \iint_{Q \cap E_T} |D(u - k_j)_+|^2 \, dxdt \le \gamma \frac{\omega^{-1}}{\rho^2} \left(\frac{\omega}{2^j}\right)^2 |Q|. \tag{II.6.4}$$

Now apply the discrete isoperimetric inequality (see p5, [5]) to the function $x \to u(x,t)$, for $-\theta\rho^2 < t < 0$, over the cube K_ρ, for the levels k_j and k_{j+1}. Taking into account (II.6.3) this gives

$$\begin{aligned}
&\frac{\omega}{2^{j+1}} |[u(\cdot,t) > k_{j+1}] \cap K_\rho \cap E| \\
&\le \frac{\gamma\rho^{N+1}}{|[u(\cdot,t) < k_j] \cap K_\rho \cap E_T|} \int_{[k_j < u(\cdot,t) < k_{j+1}] \cap K_\rho \cap E} |Du| \, dx \\
&\le \frac{\gamma}{\beta}\rho \left(\int_{[k_j < u(\cdot,t) < k_{j+1}] \cap K_\rho} |Du(\cdot,t)|^2 \, dx \right)^{\frac{1}{2}} \\
&\times |([u(\cdot,t) > k_j] - [u(\cdot,t) > k_{j+1}]) \cap K_\rho \cap E|^{\frac{1}{2}}.
\end{aligned}$$

Set

$$|A_j| = |[u > k_j] \cap Q \cap E_T| = \int_{-\theta\rho^2}^{0} |[u(\cdot, t) > k_j] \cap K_\rho \cap E|\, dt$$

and integrate the above estimate in t over the interval $(-\theta\rho^2, 0)$. We have

$$\frac{\omega}{2^j}|A_{j+1}| \leq \frac{\gamma}{\beta}\rho\left(\iint_{Q\cap E_T} |D(u-k_j)_+|^2\, dxdt\right)^{\frac{1}{2}} (|A_j| - |A_{j+1}|)^{\frac{1}{2}}$$

This together with (II.6.4) gives

$$|A_{j+1}|^2 \leq \frac{\gamma}{\beta^2}|Q|(|A_j| - |A_{j+1}|).$$

Add these from $j = 1$ to $j = q-1$; we have

$$q|A_q|^2 \leq \sum_{j=0}^{q} |A_{j+1}|^2 \leq \frac{\gamma}{\beta^2}|Q|^2.$$

From this

$$|A_q| \leq \frac{\gamma}{\sqrt{q}\beta}|Q|.$$

Now, we can choose q from

$$\frac{\gamma}{\sqrt{q}\beta} = \nu$$

$\blacksquare$

Next we choose ν as in the Lemma II.6.1 and note ν depends only on N since we choose $\theta = \omega$. Then fix q as in the Lemma II.6.2 and

$$\xi = \frac{1}{2^q} \quad \text{and} \quad a = \frac{1}{2}$$

then we have

$$u \leq \mu_+ - \frac{\omega}{2^{q+1}} = \left(1 - \frac{1}{2^{q+1}}\right)\omega \quad \text{in } (y, s) + Q_\rho(\theta) \cap E_T.$$

This quantitative bound yields

$$\operatorname*{ess\,osc}_{(y,s)+Q_\rho(\theta)\cap E_T} u = \operatorname*{ess\,sup}_{(y,s)+Q_\rho(\theta)\cap E_T} u \leq \left(1 - \frac{1}{2^{q+1}}\right)\omega.$$

We conclude by using a standard induction argument based on the above recursive inequality. See p45, [5].

CHAPTER III

ON THE LOCAL BEHAVIOR OF SOLUTIONS TO THE PARABOLIC PROBLEM

Recall we have defined u as a local weak solution to the logarithmic diffusion equation (II.3.1) if

$$\begin{aligned} &u \in C_{\mathrm{loc}}\big(0,T;L^2_{\mathrm{loc}}(E)\big), \quad \ln u \in L^2_{\mathrm{loc}}\big(0,T;W^{1,2}_{\mathrm{loc}}(E)\big); \\ &u_t - \Delta \ln u = 0 \qquad \text{weakly in } \ E_T. \end{aligned} \tag{III.0.1}$$

III.1 Local Continuity of (III.0.1)

Set

$$w_m = \begin{cases} \dfrac{u^m - 1}{m} & \text{for } \ 0 < |m| < 1; \\ \\ \ln u & \text{for } \ m = 0, \end{cases}$$

then (III.0.1) can be viewed as the formal limit, as $|m| \to 0$, of non-negative solutions to the family of porous medium equations

$$\begin{aligned} &u \in C_{\mathrm{loc}}\big(0,T;L^2_{\mathrm{loc}}(E)\big), \quad w_m \in L^2_{\mathrm{loc}}\big(0,T;W^{1,2}_{\mathrm{loc}}(E)\big) \\ &u_t - \Delta w_m = 0 \quad \text{weakly in } \ E_T, \quad 0 < |m| < 1. \end{aligned} \tag{PDE$_m$}$$

When $m > 0$, it is well-known that a nonnegative, locally bounded, local weak solution to $(\mathrm{PDE})_m$ is locally Hölder continuous for all $0 < m < 1$ and the modulus of continuity $\omega(\cdot)$ over a compact subset K of E_T is estimated by the bound of the local solution on K and the distance of K to $\partial_p E_T$. See [11]. Nevertheless, this need not be true for the logarithmic diffusion equation, even though it is a formal limit of the porous medium equation as $m \to 0$. The main technical reason lies in that it is not known how to establish a DeGiorgi-type lemma for $(u - k)_-$, without assuming sufficient integrability of $\ln u$. Lemma II.6.1 is a boundary version DeGiorgi-type lemma for $(u-k)_+$.

To see the failure of such a local continuity estimate, let us consider (II.3.1) with the initial datum $u_o = 1$ and the boundary datum

$$g_\epsilon(x,t) = \begin{cases} 1, & 0 \le t < 1; \\ -\dfrac{1}{\epsilon}(t-1)+1, & 1 < t < 1+\epsilon-\epsilon^2; \\ \epsilon, & t \ge 1+\epsilon-\epsilon^2. \end{cases} \tag{III.1.1}$$

We can further use a mollification to generate a smooth decreasing g_ϵ that equals one when $t \le 1$ and ϵ when $t \ge 1+\epsilon-\epsilon^2$. By Theorem II.5.1, this generates a family of smooth solutions u_ϵ that satisfy the semi-convexity inequality (II.3.3). By the maximum principle, the sequence of solutions u_ϵ decreases to a function u. If the modulus of continuity of u_ϵ over a compact subset K of E_T could be estimated only in term of the uniform bound and the distance of K to $\partial_p E_T$, then u should be a continuous function in the interior by Arzela-Ascoli's theorem. However, we clearly have $u = 0$ when $t > 1$ by Proposition II.3.1 and, by the maximum principle, $u \ge 1$ when $0 \le t \le 1$. A contradiction. Thus, such a local continuity estimate can never be attained. It would be interesting to see an explicit solution that shows the local continuity fails irrespective of possible boundary data. A construction is claimed in [21]. However, the heart of the matter, the intended notion of solutions, is not clear and the topology is not identified by which the limit process takes place in the construction.

Remark III.1.1. *In general, we do not have a DeGiorgi-type lemma that states there is a constant ν_- depending only on given data, such that, if*

$$|[u < \xi\omega] \cap [(y,s) + Q_{2\rho}(\theta)]| \le \nu_- |Q_{2\rho}(\theta)|, \tag{III.1.2}$$

then

$$u \ge a\xi\omega \quad a.e.\ in\ [(y,s)+Q_\rho(\theta)].$$

Indeed, let us consider the sequence of solutions u^ϵ constructed after Proposition

II.3.1. Fix $a = \xi\omega = \theta = \frac{1}{2}$ *and choose* $(y, s) \in E \times (1, T)$ *such that*

$$(y, s) + Q_{2\rho}(\theta) \subset E_T$$

and

$$|[(y, s) + Q_{2\rho}(\theta)] \cap [E \times (1, T)]| \leq \nu_- |Q_{2\rho}(\theta)|.$$

By Proposition II.3.1 and the discussion that followed, u^ϵ *will satisfy* (III.1.2), *while at the same time the intended conclusion*

$$u^\epsilon \geq \frac{1}{4} \qquad \textit{in } (y, s) + Q_\rho(\theta)$$

does not hold as

$$u^\epsilon \to 0 \qquad \textit{uniformly in } [(y, s) + Q_\rho(\theta)] \cap E \times (1, T).$$

III.2 A Harnack-type Inequality

For the porous medium equation $(\text{PDE})_m$ it is known that the classical Harnack's inequality does not hold when

$$0 < m \leq \frac{(N-2)_+}{N}.$$

See [11] for details. However such a principle could present a different form. Our result ([8]) gives a partial answer in this direction for all $|m| < 1$. We present here the case $m = 0$.

For $\rho > 0$ let K_ρ be the cube of center the origin on $\mathbb{R}^N$ and edge ρ and for $y \in \mathbb{R}^N$ let $K_\rho(y)$ denote the homothetic cube centered at y. For positive ρ and θ set

$$Q_\rho^-(\theta) = K_\rho \times (-\theta\rho^2, 0], \qquad Q_\rho^+(\theta) = K_\rho \times (0, \theta\rho^2]$$

and for $(y,s)\in\mathbb{R}^N\times\mathbb{R}$

$$
\begin{aligned}
(y,s)+Q_\rho^-(\theta)&=K_\rho(y)\times(s-\theta\rho^2,s]\\
(y,s)+Q_\rho^+(\theta)&=K_\rho(y)\times(s,s+\theta\rho^2].
\end{aligned}
$$

Now let u be a non-negative, local, weak solution to (II.3.1). Having fixed $(x_o,t_o)\in E_T$, and $K_{4\rho}(x_o)\subset E$, introduce the quantity

$$
\theta_o\overset{\text{def}}{=}\varepsilon\Big(\fint_{K_\rho(x_o)}u^q(\cdot,t_o)dx\Big)^{\frac{1}{q}}
$$

where $\varepsilon\in(0,1)$ is to be chosen, and $q>1$ is arbitrary. If $\theta_o>0$ assume that

$$
(x_o,t_o)+Q_{8\rho}^-(\theta_o)=K_{8\rho}(x_o)\times(t_o-\theta_o(8\rho)^2,t_o]\subset E_T
$$

and set

$$
\eta\overset{\text{def}}{=}\left[\frac{\Big(\fint_{K_\rho(x_o)}u^q(\cdot,t_o)dx\Big)^{\frac{1}{q}}}{\Big(\fint_{K_{4\rho}(x_o)}u^r(\cdot,t_o-\theta_o\rho^2)dx\Big)^{\frac{1}{r}}}\right]^{\frac{2r}{2r-N}},
$$

where $r>1$ satisfies

$$
u\in L^r_{\text{loc}}(E)\qquad\text{for }\ r>\max\{1;\tfrac12 N\}
$$

Assume that u is locally bounded in E_T and satisfies in addition

$$
\ln u\in L^\infty_{loc}(0,T;L^p_{loc}(E))\qquad\text{for}\qquad p\geq1. \tag{III.2.1}
$$

Set

$$
\Lambda=\left[\sup_{t_o-\theta_o(8\rho)^2<t<t_o}\fint_{K_{8\rho}(x_o)}\left(\ln\frac{\sup\limits_{(x_o,t_o)+Q_{8\rho}^-(\theta_o)}u}{u}\right)^p dx\right]^{\frac1p}.
$$

Theorem III.2.1. **([8])** *Let u be a non-negative, locally bounded, local, weak solution to* (III.0.1)*, satisfying (III.2.1) with $p>N+2$ and assume that $\theta_o>0$. There exist a constant $\varepsilon\in(0,1)$, depending only on the parameters $\{N,p,r\}$, and* *a continuous, increasing function* $\eta \to f(\eta, \Lambda_p)$ *defined in* $\mathbb{R}^+$ *and vanishing at* $\eta = 0$*, that can be quantitatively determined a priori only in terms of* $\{N, p, r\}$ *such that*

$$\inf_{K_{4\rho}} u(\cdot, t) \geq f(\eta) \sup_{K_{2\rho}\times(-\rho^2\theta_o,0]} u$$

$$\textit{for all } t \in (-\tfrac{1}{16}\theta_o\rho^2\,,\,0].$$

This is not a Harnack inequality *per se* since the $f(\eta)$ depends on the solution. However, it implies a spreading of positivity, from some region at time $t = 0$, backwards to a larger region over a period of time. From this we get continuity as a byproduct by the classical theory [17]. However continuous solutions may violate the condition (III.2.1).

To see an example, we let $\lambda > 0$. We can solve the following equation using separation of variables.

$$\begin{cases} u_t - \Delta \ln u = 0; \\ u(x,t) = \lambda t \quad \text{ on } \partial E \times (0, \infty); \\ u(x, 0) = 0. \end{cases}$$

Indeed, $u = \lambda t \phi(x)$ is a solution where ϕ is the unique solution to

$$\begin{cases} \lambda\phi - \Delta \ln \phi = 0; \\ \phi = 1 \quad \text{on } \partial E. \end{cases}$$

The unique solution u satisfies $u \in C^\infty(\overline{E_\infty})$ and $\ln u \in L^2_{\text{loc}}(\overline{\mathbb{R}_+}; W^{1,2}(E))$. Consider next the explicit solution

$$v = \frac{2(N-2)(-t)_+^{\frac{N}{N-2}}}{\lambda_o + (-t)_+^{\frac{2}{N-2}}|x|^2}, \quad \lambda_o > 0,\, t \leq 0,\, N \geq 3.$$

Denote by w the patched function over $E\times\mathbb{R}$. One verifies that $\ln w \in L^2_{\text{loc}}(\mathbb{R}; W^{1,2}(E))$ and $w \in C(\mathbb{R}; L^2(E))$. Then w will be a continuous solution to (II.3.1) across $t = 0$, though w_t is not continuous. Thus we have an explicit continuous solution which violates our assumption for the Harnack-type inequality from [8], namely,

$\ln u \in L^\infty_{loc}(\mathbb{R}; L^p_{loc}(E))$ for $p > N+2$. Accordingly, the conclusion of the Harnack-type inequality fails since $u(\cdot, 0) = 0$. On the other hand, further regularity, C^1 in time, may not be obtained if such an assumption is missing. See Section III.4.

III.3 The Limiting Process as $m \to 0$

Using this Harnack-type inequality, we can make precise the topology in which the porous medium equation tends to the (III.0.1) as m goes to 0. See [9]. Indeed the Harnack-type inequality gives a uniform lower bound. Together with an upper bound estimate such a limiting process, mentioned above, would be guaranteed by the classical parabolic theory.

Theorem III.3.1. ([9]) *Let $\{u_m, w_m\}$ be a family of non-negative, local, weak solution to (PDE)$_m$ satisfying*

$$u_m \in L^\infty_{\text{loc}}\big(0, T; L^r_{\text{loc}}(E)\big) \qquad \textit{for some } r > \max\{1; \tfrac{1}{2}N\}$$
$$w_m \in L^\infty_{\text{loc}}\big(0, T; L^p_{\text{loc}}(E)\big) \qquad \textit{for some } p > N+2$$

uniformly in m. Assume moreover that there exists an open set $E_o \subset E$ and a positive number $\sigma_{E_o;T}$ such that

$$\int_{E_o} u_m(\cdot, T)dx \geq \sigma_{E_o;T} \quad \textit{uniformly in } m.$$

Then $\{u_m\}$ is locally bounded above and below in E_T, uniformly in m, and there exists a sub-sequence $\{u_{m'}\} \subset \{u_m\}$, converging, as $|m'| \to 0$, to a local solution u to (III.0.1) *in E_T, in the sense*

$$\begin{array}{ll} \{u_{m'}\} \to u & \textit{in } C^{\alpha, \frac{1}{2}\alpha}_{loc}(E_T) \\ \{w_{m'}\} \to \ln u & \textit{weakly in } L^2_{\text{loc}}\big(0, T; W^{1,2}_{\text{loc}}(E)\big) \\ (PDE)_{m'} \longrightarrow (\text{III.0.1}) & \textit{in } \mathcal{D}'(E_T). \end{array}$$

The limit is identified as a local classical solution to (III.0.1) *in E_T.*

Such a limiting process has been studied in [15] in the case of $N = 1, 2$ for Cauchy problems and in [14] for initial-boundary value problems. In all these cases the approach consists in prescribing suitable initial data or boundary data which permits uniform bounds above and below. In contrast, our approach is entirely local and independent of any initial or boundary data.

III.4 Spacial Analyticity

Despite the singular nature of (III.0.1) a local solution is actually spatially analytic provided the assumptions of the Harnack-type inequality is satisfied.

Theorem III.4.1. ([10]) *Let u be a non-negative, local, weak solution to* (III.0.1), *satisfying the integrability conditions (III.2.1), and assume $\theta_o > 0$. There exist two parameters C and H, that have a polynomial dependence on $f(\eta)$, $[f(\eta)]^{-1}$, N, such that for every N-dimensional multi-index α*

$$|D^\alpha u(x_o, t_o)| \leq \frac{CH^{|\alpha|}|\alpha|!}{\rho^{|\alpha|}} u(x_o, t_o). \tag{III.4.1}$$

Moreover, for every non-negative integer k

$$\left|\frac{\partial^k}{\partial t^k} u(x_o, t_o)\right| \leq \frac{CH^{2k}(2k)!}{\rho^{2k}} u(x_o, t_o)^{1-k}.$$

The above estimates can be recovered from the analogous ones for solutions to $(\mathrm{PDE})_m$ whenever the limit is identified as in Theorem III.3.1.

III.5 L^1_{loc}-type Harnack Inequality

The following weak form Harnack inequality is of independent interest.

Proposition III.5.1. ([10]) *Let u be a non-negative, local, weak solution to* (III.0.1) *satisfying in addition (III.2.1) for some $p \geq 2$. There exists a positive constant γ depending only on $\{N, r, p\}$ and Λ_1 and Λ_2, such that for all cylinders $K_{2\rho}(y) \times [s, t] \subset E_T$, there holds*

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x, \tau) dx \leq \gamma \left(\inf_{s<\tau<t} \int_{K_{2\rho}(y)} u(x, \tau) dx + \frac{t-s}{\rho^\lambda} \right), \tag{III.5.1}$$

where

$$\lambda = 2 - N. \tag{III.5.2}$$

As usual the estimates above can be recovered from the analogous ones for solutions to $(\text{PDE})_m$ whenever the limit is identified as in Theorem III.3.1.

It is assumed (III.2.1) holds for some $p \geq 2$. The constant γ depends on N and the $L^{\infty}_{loc}(0, T; L^p_{loc}(E))$ norm of $\ln u$.

The failure of the L^1_{loc} Harnack inequality, if such a requirement on $\ln u$ is not assumed, also follows from the example constructed in Section III.1. Indeed, suppose the conclusion holds with a constant γ depending only on N. Then (III.5.1) is satisfied by u_ϵ which is constructed using the boundary datum (III.1.1). Fix ρ and take $s = 1$ and $t = 1 + \sigma$ then

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u_\epsilon(x, \tau)dx \geq \int_{K_\rho(y)} u_\epsilon(x, 1)dx \geq |K_\rho|.$$

On the other hand, for any given $\delta > 0$, σ can be chosen so small that

$$\frac{t-s}{\rho^{2-N}} < \delta.$$

Now fix such a σ; we can choose ϵ so small that

$$\inf_{s<\tau<t} \int_{K_{2\rho}(y)} u_\epsilon(x, \tau)dx \leq \int_{K_{2\rho}(y)} u_\epsilon(x, 1+\sigma)dx < \delta.$$

Thus, when ρ is fixed we have

$$|K_\rho| < 2\gamma\delta \quad \forall \delta > 0.$$

A contradiction.

III.6 Geometry of the Set $[u = 0]$

We have seen in Section II.3.4 that locally continuous solutions may have a set of zeros in the interior of the form of a horizontal hyperplane $K_\rho(x_o) \times \{t_o\}$. This is permitted, roughly speaking, because we have less restriction in the time direction. As we will see next, the geometry of an admissible set where $u = 0$ is much more restricted along the t-axis.

Since we do not know a local weak solution to (II.5.1) is locally continuous, the meaning of the set $[u = 0]$ is ambiguous. Here we define

$$[u = 0] \stackrel{\text{def}}{=} \{(x, t) \in E_T : \lim_{r \to 0} \int_{t-\rho}^{t+\rho} \int_{B_\rho(x)} u(y, s)\, dyds = 0\}.$$

A point (x, t) is called a vanishing point, denoted as $u(x, t) = 0$, if the above limit holds.

Proposition III.6.1. *Let u be a nonnegative, locally bounded, distributional solution to* (II.5.2)*. Then, for any $x \in E$, the set $V(x) = \{t \in (0, T) : u(x, t) = 0\}$ cannot occupy a set of positive H^1-measure.*

Proof. We may assume that $u < 1$. Fix $x \in E$ and suppose $V(x)$ contains a set of positive H^1-measure. In the weak formulation of the distributional solutions to (II.5.2) we take φ_ϵ as a test function where $\varphi \in C_o^\infty(E_T)$ and φ_ϵ is a smooth mollification of φ in space and time. Then we have

$$u_{\epsilon,t} - \Delta(\ln u)_\epsilon = 0 \quad \text{ in } E_T.$$

Now, if $B_\rho(x)$ is a ball centered at x with radius ρ, Poisson's representation and a time integration over (τ_1, τ_2) yield

$$\begin{aligned}
\int_{\tau_1}^{\tau_2} (\ln u)_\epsilon(x, t)\, dt &= \int_{\tau_1}^{\tau_2} \int_{\partial B_\rho(x)} P(x, y)(\ln u)_\epsilon(y, t)\, d\sigma dt \\
&\quad - \int_{\tau_1}^{\tau_2} \int_{B_\rho(x)} G(x, y) u_{\epsilon,t}(y, t)\, dydt \\
&\geq \int_{\tau_1}^{\tau_2} \fint_{\partial B_\rho(x)} (\ln u)_\epsilon(y, t)\, d\sigma dt - \|G(x, \cdot)\|_1 \|u(\cdot, \tau_2)\|_\infty.
\end{aligned}$$

Integrate both sides in $d\rho$ over $(0,R)$; we obtain that

$$\int_{\tau_1}^{\tau_2} (\ln u)_\epsilon(x,t)\,dt \geq \int_{\tau_1}^{\tau_2} \fint_{B_R(x)} (\ln u)_\epsilon(y,t)\,dydt \\ - \|G(x,\cdot)\|_1 \|u(\cdot,\tau_2)\|_\infty.$$

This implies that

$$\limsup_{\epsilon\to 0} \int_{\tau_1}^{\tau_2} (-\ln u)_\epsilon(x,t)\,dt \leq \liminf_{\epsilon\to 0} \int_{\tau_1}^{\tau_2} \fint_{B_R(x)} (-\ln u)_\epsilon(y,t)\,dydt \\ + \|G(x,\cdot)\|_1 \|u(\cdot,\tau_2)\|_\infty.$$

Fatou's lemma yields

$$\int_{\tau_1}^{\tau_2} \liminf_{\epsilon\to 0} (-\ln u)_\epsilon(x,t)\,dt \leq \liminf_{\epsilon\to 0} \int_{\tau_1}^{\tau_2} (-\ln u)_\epsilon(x,t)\,dt$$

If (x,t) is a vanishing point, then $(-\ln u)_\epsilon(x,t) \to \infty$. If there is a set of positive H^1-measure of t in (τ_1,τ_2) such that $u(x,t)=0$, then we conclude, by combining the previous two estimates, that

$$\liminf_{\epsilon\to 0} \int_{\tau_1}^{\tau_2} \fint_{B_R(x)} (-\ln u)_\epsilon(y,t)\,dydt = \infty.$$

Since $\ln u \in L^1_{\text{loc}}(E_T)$, we reach a contradiction. ■

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