ON THE ASYMPTOTIC BEHAVIOR OF THE OPTIMAL ERROR OF SPLINE INTERPOLATION OF MULTIVARIATE FUNCTIONS

By

Yuliya Babenko

Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August, 2006

Nashville, Tennessee

Approved:

Professor Larry L. Schumaker

Professor Edward B. Saff

Professor Gieri Simonett

Professor Akram Aldroubi

Professor Robert E. Bodenheimer

ACKNOWLEDGEMENTS

I would first like to thank my advisor, Professor Larry Schumaker, for his careful supervision, constant support and encouragement, patience and understanding during my work on this project.

I would also like to thank Akram Aldroubi, Dietmar Bisch, Mike Mihalik, and Edward Saff for the exceptional courses and for everything they have taught me.

I am indebted to the members of my PhD committee for their careful reading of my thesis, valuable suggestions, and references.

I am grateful to all of the faculty and students at the Department of Mathematics at Vanderbilt University with whom I have had the pleasure and honor to work during these five years.

Most importantly, I would like to thank my parents and my sister for everything they gave me. Dedication and persistence in research – those are the qualities my mother taught me, and qualities that proved to be invaluable in my career. The love, trust, and respect of my younger sister was always my biggest inspiration. The very special and important gift - love of mathematics and the profession of teaching - I received from my father, a great mathematician, exceptional teacher, wonderful person, and my role model.

Without infinite support of my family, their caring, trust, guidance, and unconditional love this work would not have been possible.

TABLE OF CONTENTS

Page

ACKNOWLEDGEMENTS ii

LIST OF TABLES iv

LIST OF FIGURES v

Chapter

I INTRODUCTION **1**

I.1 Main questions. 1

I.2 Organization of material and main results obtained. 3

I.3 History of results. 7

I.3.1 Univariate Case. 7

I.3.2 Multivariate Case. 8

II LINEAR SPLINES IN $\mathbb{R}^2$ **12**

II.1 Basics from Differential Geometry of a surface. 16

II.2 An estimate for the deviation of the second degree Taylor polynomial for C^2 functions defined on $[0,1]^2$. 18

II.3 Preliminaries. 21

II.4 Functions with positive Hessian. 23

II.4.1 General form for the error of interpolation of C^2 functions defined on $[0,1]^2$ by linear splines. Estimate from above. 25

II.4.2 General form for the error of interpolation of C^2 functions defined on $[0,1]^2$ by linear splines. Estimate from below. 32

II.4.3 Calculation of C_2^+. 34

II.4.4 Calculation of C_1^+. 38

II.4.5 Computing constants C_p^+ for other values of p. 41

II.5 Functions with negative Hessian and the case of the $L_{\infty,\Omega}$ norm. 42

II.5.1 Calculation of C_∞^-. 43

II.5.2 The error of linear interpolation of C^2 functions with negative Hessian. Estimate from above. 49

II.5.3 The error of linear interpolation of C^2 functions with negative Hessian. Estimate from below. 53

II.6 The L_∞-error of interpolation of C^2 functions with positive Hessian. 54

II.6.1 Calculation of C_∞^+. 55

II.6.2 Estimates of the error. 56

II.7 Algorithms. 61

II.8 Applications to quadrature formulae. 64

III LINEAR SPLINES IN $\mathbb{R}^d$ 66

III.1 An estimate for the deviation of the second degree Taylor polynomial for C^2 functions defined on $[0,1]^d$. 67

III.2 Necessary concepts and facts from Discrete and Computational Geometry.... 69

III.3 Delaunay triangulations in $\mathbb{R}^d$. 71

III.4 History. 73

III.5 Linear interpolation of quadratic functions. 74

III.6 Error of linear interpolation of C^2 functions defined on $[0,1]^d$. Estimate from above. 80

III.7 Error of linear interpolation of C^2 functions defined on $[0,1]^d$. Estimate from below. 84

III.8 Locally lattice-like systems of points. 85

IV BILINEAR SPLINES IN $\mathbb{R}^2$ 88

IV.1 Notation and Definitions. 88

IV.2 Two approaches: near interpolation and near continuity. 91

IV.3 Solution of the interpolation problem. 92

IV.4 Interpolation of quadratic functions Ax^2+By^2 with $AB>0$ by bilinear splines. 92

IV.4.1 The uniform norm. 94

IV.4.2 The L_1 norm. 96

IV.4.3 The L_2 norm. 98

IV.5 Interpolation of quadratic functions $Ax^2 - By^2$ with $AB > 0$ by bilinear splines. 100

IV.5.1 The L_∞-norm. 100

IV.5.2 The L_1-norm. 102

IV.5.3 The L_2-norm. 105

IV.6 General form of the error in the case of near interpolation of C^2 functions defined on $[0, 1]^2$ by bilinear splines. Estimate from above. 107

IV.7 General form of the error in the case of near interpolation of C^2 functions defined on $[0, 1]^2$ by bilinear splines. Estimate from below. 113

IV.8 Remarks: choice of an element. 115

V MULTILINEAR SPLINES IN $\mathbb{R}^d$ 116

V.1 Interpolation of quadratic functions with arbitrary signature. 117

V.2 Error of near interpolation of C^2 functions defined on $[0, 1]^d$. Estimate from above. 123

V.3 Error of near interpolation of C^2 functions defined on $[0, 1]^d$. Estimate from below. 127

VI BIQUADRATIC SPLINES 130

VI.1 Interpolation of cubic functions $Ax^3 + By^3$ with $AB > 0$ by biquadratic splines. 133

VI.1.1 The uniform norm. 133

VI.1.2 The L_1 norm. 136

VI.1.3 The L_2 norm. 138

VI.2 Interpolation of cubic functions $Ax^3 - By^3$ with $AB > 0$ by biquadratic splines. 140

VI.2.1 The L_∞-norm. 140

VI.2.2 The L_2 norm. 143

VI.3 Estimate of the deviation of the third degree Taylor polynomial for C^3 functions. 145

VI.4 Error of biquadratic interpolation of C^3 functions defined on $[0, 1]^2$. Estimate from above. 148

VI.5 Error of biquadratic interpolation of C^3 functions defined on $[0,1]^2$. Estimate from below. 152

VII CONCLUDING REMARKS **155**

BIBLIOGRAPHY 159

LIST OF TABLES

Table	Page

1 Previously known results in $\mathbb{R}^2$ 10

2 Previously known results in $\mathbb{R}^d$ 75

3 Known values of lattice covering density 87

Figure — Page

1 If the intersection of the triangle and a region is a quadrilateral, subdivide it without adding new vertices ... 28

2 Subdivision of triangles on the boundary without adding new vertices ... 29

3 Triangle T_u ... 35

4 Triangle T with one vertex at the origin and corresponding areas S_1, S_2, and S_3 ... 44

5 Chebyshev circle of the triangle T ... 55

6 Delaunay triangulation in $\mathbb{R}^2$ ... 72

7 Type of meshes we use for bilinear near interpolation (before subdivision) ... 89

8 Stitching of partitions on neighboring regions. ... 110

9 Interpolation scheme which uses biquadratic splines ... 130

INTRODUCTION

The question of adaptive approximation by splines has been studied for a number of years by various authors. The results obtained have numerous applications in computational and discrete geometry, computer aided geometric design, finite element methods for numerical solutions of partial differential equations, image processing, and mesh generation for computer graphics, among others.

In this dissertation we will investigate some questions of adaptive approximation by various classes of splines (linear, multilinear, biquadratic). In particular, we will study the asymptotic behavior of the optimal error of weighted approximation in different norms by interpolating splines from these classes.

I.1 Main questions.

To introduce the main questions addressed in this dissertation we need the following notation and definitions.

For simplicity, let the domain D be the unit cube $[0,1]^d \subset \mathbb{R}^d$. However, any bounded connected region that can be represented as a finite union of simplices (d-dimensional polytopes that have exactly $d+1$ distinct vertices) can be treated analogously.

Let $L_p(D)$, $1 \le p \le \infty$, be the standard Lebesgue space of functions defined on D with the usual norm $\|\cdot\|_p$ (sup-norm if $p = \infty$). Given a positive continuous function $\Omega(\mathbf{x})$ on D define a weighted norm $\|\cdot\|_{p,\Omega}$ as

$$\|f\|_{p,\Omega} := \begin{cases} \left(\int_D |f(\mathbf{x})|^p \Omega(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}, & p \neq \infty; \\ \max\limits_{\mathbf{x} \in D} \{|f(\mathbf{x})| \Omega(\mathbf{x})\}, & p = \infty. \end{cases}$$

The purpose of introducing a weight function is to allow more control of the quality of approximation over various regions of the domain.

Let us consider a function $f \in C^j(D)$, $j \in \mathbb{N}$. Given a partition $\diamondsuit_N = \{E_i\}_{i=1}^N$ of the domain D and a space of polynomials P of fixed degree define the space $S(\diamondsuit_N)$ of splines to be

$$S(\diamondsuit_N) := \{f \in C(D) : \forall i = 1, ..., N \ \exists \ p_i \in P \ \text{ s.t. } \ f|_{E_i} = p_i\}.$$

Let $s(f, \diamondsuit_N)$ denote the spline from $S(\diamondsuit_N)$ which interpolates the function $f \in C(D)$ at certain points (knots) associated with the partition $\diamondsuit_N$. Depending on the particular class of splines, various interpolation schemes are used.

Now let a function $f \in C^j(D)$, $j \in \mathbb{N}$, and a number $N \in \mathbb{N}$ of elements of partition $\diamondsuit_N$ be fixed. Define the *optimal* $L_{p,\Omega}$*-error* of the interpolation of the function f by the spline $s(f, \diamondsuit_N) \in S(\diamondsuit_N)$ to be

$$R_N(f, L_{p,\Omega}, S(\diamondsuit_N)) := \inf_{\diamondsuit_N} \|f - s(f, \diamondsuit_N)\|_{p,\Omega}.^1 \tag{1}$$

In this definition $\diamondsuit_N = \{E_i\}_{i=1}^N$ denotes either a simplicial partition $\triangle_N$ or a partition into d-dimensional boxes $\square_N$ (for precise definitions see the corresponding chapters).

A partition $\diamondsuit_N^0$ is called *optimal* for a given f if

$$\|f - s(f, \diamondsuit_N^0)\|_{p,\Omega} = R_N(f, L_{p,\Omega}, S(\diamondsuit_N)). \tag{2}$$

An explicit form and the exact value of $R_N(f, L_{p,\Omega}, S(\diamondsuit_N))$, as well as the explicit construction of the optimal partition, for every particular function f can be found only in exceptional situations. It was shown by Below, De Loera, and Richter-Gebert in 2000 [6] that it is not possible to construct an adaptive algorithm for optimal mesh (partition) generation that runs in polynomial time.

That is why the following two problems are interesting and important:

1. For each given function $f \in C^j(D)$, $j \in \mathbb{N}$, find the exact asymptotics of the optimal error $R_N(f, L_{p,\Omega}, S(\diamondsuit_N))$ as $N \to \infty$.

[1] In spite of the fact that the right-hand side does not depend on the partition $\diamondsuit_N$ we keep it in the notation on the left-hand side to illustrate what class of splines and on what partitions we use at the moment.

2. For each given function $f \in C^j(D)$, $j \in \mathbb{N}$, find an *asymptotically optimal* sequence of partitions, i.e. a sequence of partitions $\{\diamondsuit_N^*\}_{N=1}^{\infty}$ of D such that

$$\lim_{N\to\infty} \frac{\|f - s(f, \diamondsuit_N^*)\|_{p,\Omega}}{R_N(f, L_{p,\Omega}, S(\diamondsuit_N))} = 1. \tag{3}$$

Note that the problems formulated above are interesting for functions of arbitrary smoothness as well as for various classes of splines (for instance, for splines of higher order, interpolating splines, best approximating splines, best one-sided approximating splines, etc.). These problems have been thoroughly investigated in the univariate setting. In the multivariate situation, fewer results are known.

In this work we answer the above stated questions and describe the asymptotic behaviour of the optimal error in the case of interpolation by linear, multilinear, and biquadratic splines. The results obtained lead to algorithms for construction of the asymptotically optimal sequences of partitions (simplicial or box partitions, respectively).

As we already mentioned, these questions are interesting, in particular because of numerous applications in CAGD, adaptive mesh generation for numerical solutions of PDE, computational geometry, image processing, etc.. For more details see, for example, [8, 19, 41].

I.2 Organization of material and main results obtained.

The dissertation is organized as follows. Chapter I contains an introduction with definitions of necessary concepts, the main questions addressed, and the history of known results, together with an overview of the results obtained.

Chapter II contains results on interpolation by the class $S_1^0(\triangle_N)$ of linear splines on triangulations $\triangle_N$ in $\mathbb{R}^2$. First we consider functions with positive Hessian $H(f; x, y) := (f_{xx}f_{yy} - f_{xy}^2)(x, y)$ bounded away from zero and obtain the form of the error estimate (with a description of a constant) in the L_p case. We prove the following theorem.

Theorem. *Let $f \in C^2(D)$, $D = [0,1]^2$, and $H(f; x, y) \geq C^+ > 0$ for all $(x, y) \in D$.*

Let also the positive continuous weight function $\Omega(x,y)$ *be given. Then for all* $1 \le p < \infty$

$$\limsup_{N\to\infty} N\|f - s(f,\triangle_N)\|_{p,\Omega} \le \frac{C_p^+}{2}\left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}}\Omega(x,y)^{\frac{1}{p+1}}dxdy\right)^{\frac{p+1}{p}}, \tag{4}$$

where

$$C_p^+ = \min_T \frac{L_p - \text{error of linear interpolation of } x^2+y^2 \text{ on } T}{|T|^{1+\frac{1}{p}}}, \tag{5}$$

and min *in 5 is taken over all possible triangles* T.

Moreover, for any sequence $\{\Delta_N\}_{N=1}^{\infty}$ *of triangulations satisfying*

$$\sup_N \sqrt{N} \max_{T\in\triangle_N} \operatorname{diam}(T) < \infty \tag{6}$$

we have

$$\liminf_{N\to\infty} N\|f - s(f,\triangle_N)\|_{p,\Omega} \ge \frac{C_p^+}{2}\left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}}\Omega(x,y)^{\frac{1}{p+1}}dxdy\right)^{\frac{p+1}{p}}. \tag{7}$$

We then calculate the exact values of the constants C_p^+ in certain cases, namely in the cases of L_1, L_2, and L_∞ norms. In the case $p=\infty$ we do not need any additional assumptions of type (6).

Moreover, in the case $p=\infty$ we also prove the exact asymptotics of the error of linear interpolation for functions with negative Hessian bounded away from zero. Chapter 2 also contains an example of an algorithm to construct an asymptotically optimal sequence of triangulations which can be derived using our estimates. In addition, we provide one of the most natural applications of our results on adaptive mesh generation: obtaining quadrature formulas which are exact on piecewise linear functions corresponding to the partitions from the asymptotically optimal sequence of partitions.

Chapter III deals with similar questions in $\mathbb{R}^d$. Namely, we use the class $S_1^{d,0}(\triangle_N)$ of d-dimensional linear splines on simplicial partitions $\triangle_N$ of D to interpolate the function $f \in C^2(D)$. After introducing all necessary concepts from discrete geometry and proving auxiliary results, we arrive at the following result.

Theorem. *Let* $f \in C^2(D)$, $D=[0,1]^d \subset \mathbb{R}^d$, *and* $H(f;\mathbf{x}) := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})\right) \ge C^+ >$

0 *for all* $\mathbf{x} \in D$. *Let also the positive weight function* $\Omega(\mathbf{x}) \in C(D)$ *be given. Then*

$$\liminf_{N\to\infty} N^{\frac{2}{d}} \|f - s(f, \triangle_N)\|_{\infty,\Omega} \le \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d}\right)^{2/d} \left(\int_D H(f;\mathbf{x})^{\frac{1}{2}} \Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x}\right)^{\frac{2}{d}}, \tag{8}$$

where Θ_d *is the density of the thinnest covering of* $\mathbb{R}^d$ *by balls of fixed radius, and* κ_d *is the volume of the unit ball in* $\mathbb{R}^d$. *Moreover, for any sequence of sets of points and the corresponding sequence of Delaunay triangulations* $\{\triangle_N\}_{N=1}^{\infty}$ *such that*

$$\sup_N N^{\frac{1}{d}} \max_{T\in\triangle_N} \operatorname{diam}(T) < \infty \tag{9}$$

we have

$$\liminf_{N\to\infty} N^{\frac{2}{d}} \|f - s(f, \triangle_N)\|_{\infty,\Omega} \ge \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d}\right)^{2/d} \left(\int_D H(f;\mathbf{x})^{\frac{1}{2}} \Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x}\right)^{\frac{2}{d}}. \tag{10}$$

To prove these results we use the concepts of the thinnest covering of the space and Delaunay triangulations in $\mathbb{R}^d$.

In Chapter IV we consider the space $BS_1^0(\square_N)$ of bilinear (linear in each variable) splines in $\mathbb{R}^2$ and questions of so-called near interpolation (when we interpolate the given function at all but a few points) by these splines. Although the error of interpolation by linear splines on the triangulations and the error of near interpolation by bilinear splines on rectangular partitions have the same order, in all cases considered the constant was better for bilinear splines in the case of interpolating functions with negative Hessian (due to the hyperbolic nature of bilinear splines). In Chapter IV we obtain the exact asymptotics of the $L_{p,\Omega}$ error in the case of approximation of C^2 functions (with both positive and negative Hessian) by near interpolating continuous splines and we compute the explicit value of the constants in the cases $p = 1, 2, \infty$.

Chapter V is concerned with the problem of near interpolation of multivariate functions by splines from the class $BS_1^{d,0}(\square_N)$ which are linear in each variable. One of the most interesting results obtained here is the result about a sharp constant for the interpolation of a quadratic function in $\mathbb{R}^d$ which has an arbitrary signature. This is used for obtaining the exact asympotics of the error for any C^2 function whose quadratic part of the second degree Taylor polynomial has an arbitrary (but the same at every point of the domain) signature. (Recall that the signature of a form is the pair of numbers which represent the number of positive and negative coefficients in the form).

Theorem. *Let* $f \in C^2(D)$, *and* $D = [0,1]^d \subset \mathbb{R}^d$. *In addition, assume that at every point* $\mathbf{x}$ *the quadratic form* $\displaystyle\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x}) h_i^2$ *has the signature* $(k, d-k)$, $0 < k < d$, *and* $\displaystyle |H(f;\mathbf{x})| := \left| \prod_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x}) \right| \geq C^+ > 0$ *for all* $\mathbf{x} \in D$. *Let also a positive continuous weight function* $\Omega(\mathbf{x})$ *be given. Then in the case of interpolation by multilinear splines on box partitions in* $\mathbb{R}^d$ *we have*

$$\limsup_{N\to\infty} N^{\frac{2}{d}} \| f - s(f, \Box_N) \|_{\infty,\Omega} \leq \frac{k^{\frac{k}{d}} (d-k)^{1-\frac{k}{d}}}{8} \left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}} \Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x} \right)^{\frac{2}{d}}. \tag{11}$$

Moreover, for any sequences of box partitions $\{\Box_N\}$ *which satisfies*

$$\sup_N N^{\frac{1}{d}} \max_{R \in \Box_N} \operatorname{diam}(R) < \infty \tag{12}$$

we also have the estimate from below

$$\liminf_{N\to\infty} N^{\frac{2}{d}} \| f - s(f, \Box_N) \|_{\infty,\Omega} \geq \frac{k^{\frac{k}{d}} (d-k)^{1-\frac{k}{d}}}{8} \left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}} \Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x} \right)^{\frac{2}{d}}. \tag{13}$$

A similar theorem is proved for the case $k = d$.

Chapter VI deals with the class $QS_1^0(\Box_N)$ of biquadratic splines on rectangular partitions $\Box_N$ in $\mathbb{R}^2$ used in the construction of the popular serendipity finite elements. We prove the exact asymptotics of the error $R_N(f, L_{p,\Omega}, QS_1^{d,0}(\Box_N))$. We then compute the explicit values of the constant and provide the sketch of the explicit construction of the asymptotically optimal sequence of rectangular partitions for cases $p = 1, 2, \infty$.

Chapter VII is a general discussion of some potential generalizations and extensions of the results obtained, as well as related open questions.

Each chapter contains a short introduction with necessary definitions, the history of the particular problem considered in the chapter, and the description of the results obtained.

I.3 History of results.

I.3.1 Univariate Case.

The problem of best approximation by variable knot univariate splines has been studied for many years. By allowing the knots to vary, a function can be approximated much more closely for a given number of knots. The following is a brief survey of the work on related problems. More detailed information on this question can be found, for example, in the book of Schumaker [47].

In 1966 Powell [42] analyzed variable knot best L_2 spline approximation, and obtained conditions for optimal knot location. Phillips in 1970 obtained error estimates for best L_p polynomial and piecewise polynomial approximation.

Subbotin and Chernykh [48] in 1970 showed that by varying knots it is possible to improve the order of approximation by one:

$$\sup_{x \in W^{r+1}_{L_1[0,1]}} \inf_{\triangle_n([0,1])} \|x - s_r(x, \triangle_n[0,1])\|_{L_\infty[0,1]} = O\left(\frac{1}{n^{r+1}}\right)$$

where $r \in \mathbb{N}$ and $s_r(x, \triangle_n[0,1])$ denotes the spline of Subbotin and Chernykh on the optimal partition. On the other hand, we have only

$$\inf_{\triangle_n([0,1])} \sup_{x \in W^{r+1}_{L_1[0,1]}} \|x - sc_r(x, \triangle_n[0,1])\|_{L_\infty[0,1]} = O\left(\frac{1}{n^r}\right)$$

as $n \to \infty$ for splines over the uniform partition of the interval.

McClure thoroughly analyzed best L_2 piecewise polynomial approximation, with extensions to L_p and to spline approximation. Burchard and Hale obtained the asymptotic error estimate for best L_2 spline approximation, and analyzed the order of convergence. Barrow and Smith obtained an asymptotic error estimate for best L_2 approximation by splines, and Pence and Smith refined these results and extended them to the L_p approximation.

Observe that all previous results are optimal for the whole class of functions, and do not provide methods to find optimal knots for each particular function. This leads to the question about asymptotically optimal knots.

The first results about asymptotically optimal choice of knots for approximation of a particular function by splines appeared for the first time in 1976. These are results of Azarin and Barmin [1, 2], Grebennikov [29], Ligun and Storchai [36, 37]. The following theorem combines their results and gives the exact asymptotics for the error of interpolation by Hermite splines.

Theorem 1. *Let r be odd. Let also the function $x \in C^{r+2}[0,1]$ be such that $|x^{(r+1)}(t)| > 0$ for all $t \in [0,1]$. Then for $p \in [1,\infty]$ for $n \to \infty$*

$$\begin{aligned} \|x - sh_r(x, \triangle_n^*[0,1]\|_p &= \inf_{\triangle_n[0,1]} \|x - sh_r(x, \triangle_n[0,1]\|_p (1+o(1)) \\ &= \frac{C_{p,r,n}}{n^{r+1}} \|x^{(r+1)}\|_\beta (1+o(1)), \end{aligned} \tag{14}$$

where $sh_r(x, \triangle_n[0,1]$ denotes the Hermite spline over the partition $\triangle_n[0,1]$, $\beta = (r+1+\frac{1}{p})^{-1}$, and the constants are

$$C_{p,r,n} = \{(r+1)!^{-1}(\Gamma^2(p(r+1)/2+1)/\Gamma(rp+p+1)\}^{1/p}, \quad when \;\; p < \infty,$$

and $C_{p,r,n} = \left((r+1)!2^{r+1}\right)^{-1}$, when $p = \infty$.

A sequence of asymptotically optimal partitions $\{\triangle_n^*[0,1]\}_{n=1}^{\infty}$ can be found from the equation

$$\int_0^{t_{i,n}^*} |x^{(r+1)}(t)|^\beta dt = \frac{i}{n} \int_0^1 |x^{(r+1)}(t)|^\beta dt, \quad i = 0, 1, ..., n.$$

The above theorem was proved by Azarin and Barmin [1, 2] in the case $r = 1$, $p = 2$; by Grebennikov [27] in the case of odd r and $p = 1, \infty$; by Ligun and Storchai [36, 37] in general form.

To summarize let us say that in the univariate case general questions of this type have been investigated by many authors. The results obtained in this case are more or less complete (see, for example, [35, 47]) and have numerous applications (see, for example, [35]).

I.3.2 Multivariate Case.

Fewer results are known in the multivariate case.

The following classical results of L. Fejes Toth about approximation of convex bodies by inscribed polytopes can be considered as the first result in this direction. He indicated ([24], Ch. 5, §12) that for a body $C \subset \mathbb{R}^3$ with boundary of differentiability class C^2 and positive Gaussian curvature $K_C(x,y)$, the distance from C to its best inscribed polytope P_{ins} with at most n vertices in the Hausdorff metric is

$$d_H(C, P^n_{ins}) = \frac{1+o(1)}{3\sqrt{3}} \left(\int_{\partial C} K_C(x,y)^{1/2} d\sigma(x,y) \right) \frac{1}{n}$$

as $n \to \infty$, where σ is the surface area measure in $\mathbb{R}^3$. He also indicated that the distance of C to its best inscribed polytope with at most n vertices measured as the volume of the difference between C and the polytope is

$$d_1(C, P^n_{ins}) = \frac{1+o(1)}{4\sqrt{3}} \left(\int_{\partial C} K_C(x,y)^{1/4} d\sigma(x,y) \right)^2 \frac{1}{n}$$

as $n \to \infty$, where as above σ is the surface area measure in $\mathbb{R}^3$. These formulae were proved by Gruber in [28]. He also obtained a formula for the error in the Hausdorff metric of approximation of convex bodies in $\mathbb{R}^d$. He showed that if the body C in $\mathbb{R}^d$ has twice differentiable boundary with curvature $K_C > 0$ then

$$d_H(C, P^n_{inn}) = \frac{1+o(1)}{2} \left(\frac{\Theta_{d-1}}{\kappa_{d-1}} \int_{\partial C} K_C(\mathbf{x})^{1/2} d\sigma(\mathbf{x}) \right)^{2/d} \frac{1}{n},$$

where Θ_{d-1} is the minimum density of covering $\mathbb{R}^{d-1}$ with balls of fixed radius, and κ_{d-1} is the $(d-1)$-dimensional volume of a unit ball. In 1981 Schneider [46] proved this formula for the case when the boundary of C is three times differentiable. Schneider and Gruber discovered that the problem of approximation of C by inscribed polytopes with respect to d_H was intimately connected with the thinnest covering of ∂C with geodesic discs determined by a suitable Riemannian metric on ∂C. Thus, it is related to the thinnest covering of $\mathbb{R}^d$ with balls of fixed radius.

Table 1: Previously known results in $\mathbb{R}^2$

Author(s)	Class	Result	Norm	Const	Wgt	Alg.
Fejes Toth '72	$f \in C^2, K > 0$	asympt.	Hausd.	sharp	no	no
Nadler '86	$f \in C^3, K > 0$	asympt.	L_2, disc.	sharp	no	no
D'Azevedo-Simpson '89	quadr.,$\lvert K\rvert > 0$	local	L_∞	local	no	local
Gruber '92	$f \in C^2, K > 0$	asympt.	Hausd.	sharp	no	no
Pottmann-Hamann et al '00	quadr.,$\lvert K\rvert > 0$	local	L_∞	local	no	yes
Huang-Sun '03	$f \in C^2, K > 0$	mesh	L_2	no	no	no
Chen '04	$f \in C^2, K > 0$	order	L_p	no	no	no

Further investigations on asymptotically optimal errors for approximation of convex bodies by various classes of polytopes have been done by Böröczky, Ludwig, Gruber. A survey of further results in this direction can be found in [9, 28].

In his 1986 PhD. thesis Nadler [39] solved the problem of asymptotically optimal choice of a sequence of triangulations for approximation of C^3 functions by piecewise linear splines (which are not globally continuous) of best L_2-approximation.

D'Azevedo and Simpson in 1989 [17] studied the question of triangulating a given set of vertices for interpolation of a convex quadratic surface by piecewise linear functions. They showed that the Delaunay triangulation will be optimal for the error in the L_∞ norm. For the error in L_p norm this fact was proved by Rippa [44]. Chen and Xu [13] generalized this result to arbitrary dimensions. A Delaunay triangulation is therefore characterized as the optimal triangulation for piecewise linear interpolation to an isotropic function for a given set of points in the sense of minimizing the interpolation error in the L_p, $1 \leq p \leq \infty$, norm.

Later D'Azevedo [16] obtained local error estimates for functions with both positive and negative curvature. The same estimates were later obtained by Pottmann, Hamann *et al* [41] who studied the problem of optimally triangulating the plane for approximating quadratic functions by piecewise linear functions. They obtained local estimates (which repeated the result of D'Azevedo) and also suggested some algorithms for constructing function dependent triangulations of the whole domain.

Huang [32], and Huang and Sun [33] considered the problem of variational mesh adaptation in the numerical solutions of partial differential equations. This method utilizes a functional (the so-called monitor function) to determine the coordinate transformation needed for mesh generation. Using this method, they obtained asymptotic bounds on the interpolation error estimates in L_2 for adaptive meshes that satisfy regularity and equidistribution conditions.

Chen proved that if $f \in C^2(D)$ is a strictly convex (or concave) function defined on a bounded convex region D, and $\{T^N\}$ is a family of triangulations of D satisfying some additional assumptions (most of which are either not necessary or redundant) then there exists a constant C such that

$$\lim_{N\to\infty} N^{2/d}\|f - f_I^N\|_p = C\|\sqrt[n]{\det H}\|_{L_{\frac{pd}{2p+d}}}, \quad 1 \le p \le \infty,$$

where f_I^N is a linear interpolant to f constructed on T^N. Therefore, Chen in fact proved only the order of the error, not the exact asymptotic behavior.

Table 1 summarizes major results in the direction of investigating the asymptotics of the error of adaptive interpolation by linear functions in $\mathbb{R}^2$ (a similar table for $\mathbb{R}^d$ will be given in Chapter III).

LINEAR SPLINES IN $\mathbb{R}^2$

Let $D = [0,1]^2$ and $f \in C^2(D)$. Denote the *Hessian* of $f(x, y)$ by

$$H(f; x, y) := (f_{xx} f_{yy} - f_{xy}^2)(x, y).$$

Observe the connection between the Hessian $H(f; x, y)$ and the Gaussian curvature $K(x, y)$ of the surface which is the graph of the function $f(x, y)$:

$$K(x, y) = \frac{H(f; x, y)}{(1 + (f_x(x, y))^2 + (f_y(x, y))^2)^2}.$$

We shall use the following commonly accepted definition. A collection $\triangle_N = \triangle_N(D) = \{T_i\}_{i=1}^N$ of N triangles in the plane is called a *triangulation* of a set D provided that

1. any pair of triangles from $\triangle_N$ intersect at most at a common vertex or along a common edge,
2. $D = \cup_{i=1}^N T_i$.

Clearly, in the case of interpolation by linear splines, triangulations are the most natural partitions of the domain.

Let P_1 be the set of linear polynomials

$$p(x, y) = ax + by + c, \quad a, b, c \in \mathbb{R}.$$

Given a triangulation $\triangle_N$ define the space $S_1^0(\triangle_N)$ of linear splines to be

$$S_1^0(\triangle_N) := \{f \in C(D) : \forall i = 1, ..., N \ \ \exists \ \ p_i \in P_1 \ \text{ s.t. } \ f|_{T_i} = p_i\}.$$

Let $s(f, \triangle_N)$ denote the spline from $S_1^0(\triangle_N)$ which interpolates the function $f \in C(D)$

at the vertices of the triangulation $\triangle_N$. Note that the linear spline $s(f, \triangle_N)$ is uniquely defined by its values at the vertices of the triangulation $\triangle_N$.

Now let the function $f \in C^2(D)$ and the number of triangles $N \in \mathbb{N}$ be fixed. Define the *optimal* $L_{p,\Omega}$*-error* of the interpolation of the function f by the continuous piecewise linear function $s(f, \triangle_N) \in S_1^0(\triangle_N)$ to be

$$R_N(f, L_{p,\Omega}, S_1^0(\triangle_N)) := \inf_{\triangle_N} \|f - s(f, \triangle_N)\|_{p,\Omega}. \tag{15}$$

A triangulation $\triangle_N^0$ is called *optimal* for the given function f if

$$\|f - s(f, \triangle_N^0)\|_{p,\Omega} = R_N(f, L_{p,\Omega}, S_1^0(\triangle_N)). \tag{16}$$

In this chapter we will investigate the asymptotic behavior of the error of optimal interpolation of functions from the class $C^2(D)$ by linear splines in two cases: when the Hessian of the given function is positive at every point of the domain and when the Hessian is negative.

The chapter is organized as follows. In Sections II.4.1-II.4.2 we give the proof of the following two general theorems for the weighted L_p $(1 \le p < \infty)$ error for functions with positive Hessian.

Theorem 2. *Let $f \in C^2(D)$ and $H(f; x, y) \ge C^+ > 0$ for all $(x, y) \in D$. Let also the positive continuous weight function $\Omega(x, y)$ be given. Then for all $1 \le p < \infty$*

$$\limsup_{N\to\infty} N\|f - s(f, \triangle_N^*)\|_{p,\Omega} \le \frac{C_p^+}{2} \left(\int_D H(f; x, y)^{\frac{p}{2(p+1)}} \Omega(x, y)^{\frac{1}{p+1}} dxdy \right)^{\frac{p+1}{p}}, \tag{17}$$

where

$$C_p^+ = \min_T \frac{L_p - \textit{error of linear interpolation of } x^2 + y^2 \textit{ on } T}{|T|^{1+\frac{1}{p}}}. \tag{18}$$

To prove the lower bound we impose some (mild) additional restrictions which most likely can be avoided.

Theorem 3. *Let $f \in C^2(D)$ and $H(f; x, y) \ge C^+ > 0$ for all $(x, y) \in D$. Let also the positive continuous weight function $\Omega(x, y)$ be given. Then for all $1 \le p < \infty$ and for any* *sequence* $\{\Delta_N\}_{N=1}^{\infty}$ *of triangulations satisfying*

$$\sup_N \sqrt{N} \max_{T\in\triangle_N} \operatorname{diam}(T) < \infty \tag{19}$$

we have

$$\liminf_{N\to\infty} N\|f - s(f,\triangle_N)\|_{p,\Omega} \geq \frac{C_p^+}{2}\left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}}\Omega(x,y)^{\frac{1}{p+1}}dxdy\right)^{\frac{p+1}{p}}, \tag{20}$$

where C_p^+ *is defined in (18).*

The estimate from above is contained in Section II.4.1, and the estimate from below is in Section II.4.2. Sections II.4.3 – II.4.4 are devoted to calculating the constant C_p^+ for the cases $p = 1$, and $p = 2$.

For the case $p = \infty$ the result is contained in the following theorem.

Theorem 4. *Let* $f \in C^2(D)$ *and* $H(f;x,y) \geq C^+ > 0$ *for all* $(x,y) \in D$. *Let also the positive continuous weight function* $\Omega(x,y)$ *be given. Then*

$$R_N(f, L_{\infty,\Omega}, S_1^0(\triangle_N)) = \frac{2(1+o(1))}{3\sqrt{3}N}\int_D \sqrt{H(f;x,y)}\Omega(x,y)dxdy, \quad N\to\infty. \tag{21}$$

The proof of this theorem can be found in Section II.6.

For functions with negative everywhere Hessian we prove the following theorems in Section II.5.

Theorem 5. *Let* $f \in C^2(D)$ *and* $H(f;x,y) \leq C^- < 0$ *for all* $(x,y) \in D$. *Let also the positive continuous weight function* $\Omega(x,y)$ *be given. Then there exists a sequence of triangulations* $\{\triangle_N^*\}_{N=1}^{\infty}$ *satisfying*

$$\sup_N \sqrt{N} \max_{T\in\triangle_N^*} \operatorname{diam}(T) < \infty \tag{22}$$

such that

$$\limsup_{N\to\infty} N\|f - s_N(f,\triangle_N^*)\|_{\infty,\Omega} \leq \frac{1}{2\sqrt{5}}\int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy. \tag{23}$$

Theorem 6. *Let $f \in C^2(D)$ and $H(f;x,y) \le C^- < 0$ for all $(x,y) \in D$. Let also positive continuous weight function $\Omega(x,y)$ be given. Then for any sequence of triangulations $\{\triangle_N\}_{N=1}^{\infty}$ satisfying (22) we have*

$$\liminf_{N\to\infty} N\|f - s_N(f,\triangle_N)\|_{\infty,\Omega} \ge \frac{1}{2\sqrt{5}} \int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy. \tag{24}$$

Remark 1. Assumption (22) implies avoiding anisotropic meshes (triangulations with long and "skinny" triangles). However, such triangulations can be useful for interpolation of certain functions. For some details on using anisotropic meshes see, for example, [44].

Remark 2. In all results here we impose certain restrictions on the Hessian. We take it to be bounded away from zero. However, this is not a necessary assumption and can be removed using techniques similar to those used by Böröczky in [9].

In fact, in the case when the Hessian (or curvature) equals (or is close) to zero, the plane can be used as a local approximation for a function. In this case large elements of partition can be used and the order of approximation will be improved (to $o(N)$).

Remark 3. Although all theorems above are stated for only either convex or saddle-shaped surfaces, clearly the results can be combined (with the help of introducing a piecewise constant weight function) to obtain exact asymptotics of the error for interpolation of more complex surfaces (which have positive curvature on some regions, and negative curvature on others).

Remark 4. Theorems 4-6 for the special case $\Omega \equiv 1$ were proved in [4].

Let us describe the most essential part of obtaining estimates from above in these theorems. It consists of finding an appropriate sequence of "good" triangulations of D. This is done in the following way:

1. Divide D into a number m_N^2 (which is small in comparison with N) of equal subregions D_i^N. On each D_i^N, instead of f, consider its Taylor polynomial $P_{N,i}$ of second degree taken at the center of D_i^N.

2. To find an appropriate triangulation of D_i^N, first take any triangle T which solves the

following extremal problem:

$$\frac{L_p - \text{error of linear interpolation of } x^2 \pm y^2 \text{ on } T}{|T|^{1+\frac{1}{p}}} \to \min .$$

The value of the minimum will provide the constant $C_p^{\pm}$.

Then we reshape T depending on values of second derivatives of the function f at a point of D_i^N.

Moreover, we choose the size of T in such a way that the overall number n_i^N of triangles used for D_i^N is such that the sum $\sum_{i=1}^{m_N^2} n_i^N$ is approximately N, and the errors of interpolation on each D_i^N are approximately equal.

3. We obtain the final triangulation of D by "gluing" together triangulations of each region D_i^N possibly subdividing (without adding new vertices) triangles which have nonempty intersection with $\cup_i \partial D_i^N$ where ∂D_i^N denotes the boundary of D_i^N.

II.1 Basics from Differential Geometry of a surface.

In this section let us recall some notions from the differential geometry of a surface, see, for example, [14, 20].

Let us consider the quadratic form

$$Q(x,y) = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 = d^2 f. \qquad (25)$$

Let a surface $z = f(x,y)$ and a point (x_0, y_0, z_0) at which *grad* $f = 0$ be given (for simplicity we suppose that the $z-$axis is perpendicular to the tangent plane to the surface at (x_0, y_0, z_0). This condition, in fact, can be removed requiring more technical details). We define the *principal curvatures* of the surface at the point to be the eigenvalues of the matrix of the quadratic form (25). (These eigenvalues are real since the matrix is symmetric).

It was shown by Gauss (see for example [20]) that the Gaussian curvature K of the surface (which was defined earlier) is an "intrinsic" invariant of the surface, i.e. depends only on the internal metrical properties of the surface.

If we assume that the eigenvalues k_1, k_2 are distinct, then by Lemma 8.2.1 from [20] we have that corresponding eigenvectors are perpendicular. Hence, we can replace the coordinate system x, y by a new system x', y', with axes in the principal directions, obtained from the old system by means of a rotation of the (x, y)-plane through an angle φ. Then in terms of the new coordinates x', y', z we have

$$z = f(x(x', y'), y(x', y')),$$

where

$$x = x' \cos\varphi + y' \sin\varphi,$$

$$y = -x' \sin\varphi + y' \cos\varphi.$$

Relative to these new coordinates, the second fundamental form (25) becomes (at the point (x_0, y_0, z_0))

$$k_1(dx')^2 + k_2(dy')^2. \tag{26}$$

Let us also recall the classification of surfaces in $\mathbb{R}^3$ depending on the sign of the Gaussian curvature K, see [14]. When K is positive, the normal curvature (defined as the curvature of the curve in a normal cross-section) in this case never leaves the range from k_1 to k_2 and has the same sign in all directions. Thus, the surface is bending away from its tangent plane in all tangent directions. Such a surface is said to be *synclastic* (or "oval"). Ellipsoids, elliptic paraboloids and hyperboloids of two sheets are everywhere synclastic. In the case $K > 0$, the quadratic approximation to the surface $z = f(x, y)$ near point (x_0, y_0, z_0) is the paraboloid

$$2z = k_1 x^2 + k_2 y^2.$$

When K is negative, the normal curvature changes sign twice (during the rotation of the normal plane through a half-turn about the normal at a point); therefore, it is zero in the directions of two special tangents, called the *inflectional* tangents. Clearly, in this case the principal curvatures k_1 and k_2 have opposite signs. Such a surface is said to be *anticlastic* (or "saddle-shaped"). Non degenerate ruled quadrics (namely, hyperbolic paraboloids and hyperboloids of one sheet) are everywhere anticlastic. The quadratic approximation of such a surface near point (x_0, y_0, z_0) is a hyperboloid.

Surfaces more complicated than quadrics may be synclastic in some regions and anticlastic in others. Regions of the two kinds are then separated by a locus of *parabolic* points, at which $K = 0$.

Surfaces on which $K = 0$ everywhere are said to be *developable.* Such surfaces include cones and cylinders, and also the surfaces traced out by the tangents of any twisted curve. In the case when only one principal curvature is zero, the quadratic approximation is a ruled surface. When both principal curvatures vanish the quadratic approximation reduces simply to the plane.

II.2 An estimate for the deviation of the second degree Taylor polynomial for C^2 functions defined on $[0, 1]^2$.

Let us define the modulus of continuity of $f \in C^2(D)$ as follows

$$\omega(f, \delta) := \sup\{|f(x, y) - f(x', y')| : \ |x - x'| \le \delta, |y - y'| \le \delta, (x, y), (x', y') \in D\}. \quad (27)$$

Set

$$\omega_1(\delta) := \omega(f_{xx}, \delta), \ \ \omega_2(\delta) := \omega(f_{xy}, \delta), \ \ \omega_3(\delta) := \omega(f_{yy}, \delta),$$

and

$$\omega(\delta) := \max\{\omega_1(\delta), \omega_2(\delta), \omega_3(\delta)\}. \quad (28)$$

Lemma 1. *Let $f \in C^2(D)$. If $P_2(x, y)$ denotes the quadratic part of the Taylor polynomial for f at the center of a square $D_h \subset D$ with side length equal to h, then we have the following estimate*

$$|f(x, y) - P_2(x, y)| \le \frac{h^2}{2}\omega\left(\frac{h}{2}\right), \quad (x, y) \in D_h, \quad (29)$$

where $\omega(t)$ is defined in (28).

Proof: The Taylor formula for $f \in C^2(D)$ about the point (x_0, y_0) with remainder in the Lagrange form is given by

$$f(x, y) = P_1(x, y) + R_1(x, y), \tag{30}$$

where

$$P_1(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and for some $0 < \theta < 1$

$$\begin{aligned} R_1(x, y) = {} & \frac{(x - x_0)^2}{2} f_{xx}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)) \\ & + (x - x_0)(y - y_0) f_{xy}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)) \\ & + \frac{(y - y_0)^2}{2} f_{yy}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)). \end{aligned} \tag{31}$$

We add and subtract the term

$$\frac{(x - x_0)^2}{2} f_{xx}(x_0, y_0) + (x - x_0)(y - y_0) f_{xy}(x_0, y_0) + \frac{(y - y_0)^2}{2} f_{yy}(x_0, y_0) \tag{32}$$

to the right-hand side of (31). Then (30) can be rewritten as

$$\begin{aligned} f(x, y) = {} & P_2(x, y) + \frac{(x - x_0)^2}{2} (f_{xx}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)) - f_{xx}(x_0, y_0)) \\ & + (x - x_0)(y - y_0)(f_{xy}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)) - f_{xy}(x_0, y_0)) \\ & + \frac{(y - y_0)^2}{2} (f_{yy}(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)) - f_{yy}(x_0, y_0)) \\ = {} & P_2(x, y) + R_2(x, y), \end{aligned} \tag{33}$$

where

$$\begin{aligned} P_2(x, y) := {} & P_1(x, y) + \frac{(x - x_0)^2}{2} f_{xx}(x_0, y_0) + (x - x_0)(y - y_0) f_{xy}(x_0, y_0) \\ & + \frac{(y - y_0)^2}{2} f_{yy}(x_0, y_0), \end{aligned}$$

and

$$R_2(x,y) := \frac{(x-x_0)^2}{2}(f_{xx}(x_0+\theta(x-x_0), y_0+\theta(y-y_0)) - f_{xx}(x_0,y_0))$$
$$+ (x-x_0)(y-y_0)(f_{xy}(x_0+\theta(x-x_0), y_0+\theta(y-y_0)) - f_{xy}(x_0,y_0))$$
$$+ \frac{(y-y_0)^2}{2}(f_{yy}(x_0+\theta(x-x_0), y_0+\theta(y-y_0)) - f_{yy}(x_0,y_0)).$$

By the triangle inequality we have

$$|R_2(x,y)| \le \frac{(x-x_0)^2}{2}|f_{xx}(x_0+\theta(x-x_0), y_0+\theta(y-y_0)) - f_{xx}(x_0,y_0)|$$
$$+ |x-x_0||y-y_0||f_{xy}(x_0+\theta(x-x_0), y_0+\theta(y-y_0)) - f_{xy}(x_0,y_0)|$$
$$+ \frac{(y-y_0)^2}{2}|f_{yy}(x_0+\theta(x-x_0), y_0+\theta(y-y_0)) - f_{yy}(x_0,y_0)|.$$
$$|R_2(x,y)| \le \frac{(x-x_0)^2}{2}\omega(\max\{|\theta(x-x_0)|, |\theta(y-y_0)|\})$$
$$+ |x-x_0||y-y_0|\omega(\max\{|\theta(x-x_0)|, |\theta(y-y_0)|\})$$
$$+ \frac{(y-y_0)^2}{2}\omega(\max\{|\theta(x-x_0)|, |\theta(y-y_0)|\}). \tag{34}$$

Hence, on the square D_h with side length equal to h we have the following estimate

$$|f(x,y) - P_2(x,y)| \le \frac{h^2}{2}\omega\left(\frac{h}{2}\right), \tag{35}$$

and, therefore,

$$\|f - P_2\|_\infty \le \frac{h^2}{2}\omega\left(\frac{h}{2}\right).$$

□

Corollary. *Under the conditions of Lemma 1 we have*

$$\|f - P_2\|^p_{L_p(D_h)} = \int_0^h \int_0^h (f(x,y) - P_2(x,y))^p \, dxdy \le h^2 \|f - P_2\|^p_{L_\infty(D_h)}$$

and, hence,

$$\|f - P_2\|_p \le \frac{h^{2(1+\frac{1}{p})}}{2}\omega\left(\frac{h}{2}\right). \tag{36}$$

II.3 Preliminaries.

To investigate the asymptotic behavior of the error of the optimal piecewise linear interpolation of an arbitrary function from the class $C^2(D)$, $D = [0,1]^2$, we shall use linear interpolation of the piecewise quadratic functions which appear as intermediate approximations of f (see the idea of the proof in the previous section).

Some of the facts we shall present in this section are quite easy to see. However, we shall prove them, first of all for completeness, and secondly, because we shall use them in the construction of the asymptotically optimal sequence of triangulations.

First, observe that the error of linear interpolation of a quadratic function on a triangle is not affected by a shift of this triangle or a reflection about the midpoint of any side of this triangle. More precisely, we have the following almost obvious lemma.

Lemma 2. *For the given quadratic function*

$$Q(x,y) = Ax^2 + By^2 + 2Cxy, \tag{37}$$

an arbitrary triangle T, and any $(a,b) \in \mathbb{R}^2$, the L_p-errors ($1 \le p \le \infty$) of linear interpolation of $Q(x,y)$ on T, $(a,b)+T$, and a triangle $\tilde{T}$ which is symmetric to T with respect to the midpoint of any side of T, are equal.

Let $L_{Q,T}(x,y)$ denotes the linear function which interpolates $Q(x,y)$ at the vertices of the triangle T. Define

$$d_{Q,T,p} := \|Q - L_{Q,T}\|_p.$$

As we already mentioned, we need to solve the problem

$$\frac{d_{Q,T,p}}{|T|^{1+\frac{1}{p}}} \to \min, \tag{38}$$

and describe those triangles T which provide the minimum in (38). Observe that such an optimal triangle always exists.

Lemma 3. *There always exists an optimal triangle T which solves problem (38).*

Proof: Observe that for any triangle T $d_{Q,T,p}$ is a continuous function of six variables (vertices of triangle T). Therefore, $d_{Q,T,p}$ achieves its minimum and maximum values (call them $\min(d)$ and $\max(d)$, respectively) on the compact set which is the subset of $[0,1]^6$ of vertices of all triangles T with area equal to 1. From the definition of $d_{Q,T,p}$ it is easy to see that the function $\frac{d_{Q,T,p}}{|T|^{1+\frac{1}{p}}}$ is a homogeneous function in the sense that if we take an arbitrary triangle T and its scaled version αT then

$$\frac{d_{Q,T,p}}{|T|^{1+\frac{1}{p}}} = \frac{d_{Q,\alpha T,p}}{|\alpha T|^{1+\frac{1}{p}}}.$$

Therefore, for any triangle T, the minimum and maximum values of $\frac{d_{Q,T,p}}{|T|^{1+\frac{1}{p}}}$ will coincide with $\min(d)$ and $\max(d)$, respectively. $\square$

In Sections II.4.3-II.4.4 we shall solve problem (38) for sign definite quadratic forms, i.e. for forms $Q(x,y)$ such that $AB - C^2 > 0$, and for cases $p = 1, 2$. The case of the uniform norm is considered separately in Section II.6. For forms $Q(x,y)$ with $AB - C^2 < 0$ in case $p = \infty$ the solution will be given in Section II.5.1.

Now let

$$Q(x,y) = Ax^2 + By^2 + 2Cxy \tag{39}$$

be an arbitrary form such that either $AB - C^2 > 0$ or $AB - C^2 < 0$. Without loss of generality we may assume $A \geq 0$ ($A > 0$ in the case of the form with $AB - C^2 > 0$).

First let us find the eigenvalues and eigenvectors (take the lengths of the eigenvectors to be equal to 1) of the matrix of this quadratic form.

For eigenvalues we have

$$\lambda_{\max} = \frac{A+B}{2} + \sqrt{\left(\frac{A+B}{2}\right)^2 - (AB - C^2)}, \tag{40}$$

$$\lambda_{\min} = \frac{A+B}{2} - \sqrt{\left(\frac{A+B}{2}\right)^2 - (AB - C^2)}. \tag{41}$$

Observe that we have $0 < \lambda_{\min} < \lambda_{\max}$ for the form $Q(x,y)$ with $AB - C^2 > 0$, and

$\lambda_{\min} < 0 < \lambda_{\max}$ for the form $Q(x, y)$ with $AB - C^2 < 0$. Also note that

$$\lambda_{\min}\lambda_{\max} = AB - C^2.$$

Let (ξ_1, ξ_2) from the unit circle $\mathbb{S}^1$ be an eigenvector of $Q(x, y)$ corresponding to the eigenvalue $\lambda_{\max}$. Then $(\xi_2, -\xi_1) \in \mathbb{S}^1$ is an eigenvector corresponding to the eigenvalue $\lambda_{\min}$. Observe that

$$\lambda_{\max} = A\xi_1^2 + B\xi_2^2 + 2C\xi_1\xi_2,$$

$$\lambda_{\min} = A\xi_2^2 + B\xi_1^2 - 2C\xi_1\xi_2.$$

We shall define the new coordinate system (x', y') with the help of vectors (ξ_1, ξ_2) and $(\xi_2, -\xi_1)$ in the following way

$$F_1 : \quad x' = x\xi_1 + y\xi_2, \quad y' = x\xi_2 - y\xi_1. \tag{42}$$

The quadratic function (39) in this new system has the form

$$(Q \circ F_1^{-1})(x', y') = \lambda_{\max}(x')^2 + \lambda_{\min}(y')^2. \tag{43}$$

Note that the ratio $\frac{d_{Q,T,p}}{|T|}$ does not depend on the choice of the orthogonal coordinate system, i.e.,

$$\frac{d_{Q,T,p}}{|T|} = \frac{d_{Q,F_1T,p}}{|F_1T|}.$$

Therefore, we shall solve the problem of minimizing $\frac{d_{Q,T,p}}{|T|}$ in the coordinate system (x', y').

II.4 Functions with positive Hessian.

Let

$$Q(x, y) = Ax^2 + By^2 + 2Cxy \tag{44}$$

be a sign definite form. Without loss of generality we may assume (44) to be a positive definite form, i.e. such that $A > 0$ and $AB - C^2 > 0$.

To characterize triangles T which give a solution to problem (38) let us consider the following linear transformation

$$F_2: \quad u = \sqrt{\lambda_{\max}} x', \ \ v = \sqrt{\lambda_{\min}} y', \tag{45}$$

which converts (43) to

$$(Q \circ F_1^{-1} \circ F_2^{-1})(u, v) = u^2 + v^2.$$

Both the L_p-error of interpolation and the area of the triangle must be multiplied by the Jacobian of the transformation, i.e. $|F_2 T| = |T| \sqrt{\lambda_{\max} \lambda_{\min}}$ and

$$d_{Q \circ F_1^{-1} \circ F_2^{-1}, (F_2 \circ F_1) T, p} = d_{Q,T,p} (\lambda_{\max} \lambda_{\min})^{\frac{1}{2p}}.$$

Observe also that

$$\frac{d_{Q,T,p}}{|T|^{1+\frac{1}{p}}} = \frac{d_{Q \circ F_1^{-1} \circ F_2^{-1}, (F_2 \circ F_1) T, p} (\lambda_{\max} \lambda_{\min})^{\frac{1}{2p}}}{\left(\frac{1}{\sqrt{\lambda_{\max} \lambda_{\min}}} |(F_2 \circ F_1) T| \right)^{1+\frac{1}{p}}}.$$

Therefore, by definition of C_p^+ we obtain

$$\frac{d_{Q,T,p}}{|T|^{1+\frac{1}{p}}} \geq C_p^+ \sqrt{\lambda_{\max} \lambda_{\min}}. \tag{46}$$

This can be rewritten as

$$d_{Q,T,p} \geq C_p^+ |T|^{1+\frac{1}{p}} \sqrt{\lambda_{\max} \lambda_{\min}} \tag{47}$$

which will be used later.

In addition, we shall need the following lemma.

Lemma 4. *Let us consider the collection of quadratic forms of type (44) which satisfy the following conditions:*

$$0 < A \leq A^+, \ \ 0 < B \leq B^+, \ \ \text{and} \ \ H = AB - C^2 > C^+, \tag{48}$$

where A^+, B^+, C^+ are some positive numbers. Then for any such form

$$\lambda_{\min} \geq \frac{1}{2}(A^+ + B^+) - \sqrt{\left(\frac{1}{2}(A^+ + B^+) \right)^2 - C^+} > 0. \tag{49}$$

Proof: Due to (41) and the assumption that $H = AB - C^2 \geq C^+ > 0$, we have

$$\lambda_{\min} = \frac{1}{2}(A+B) - \sqrt{\left(\frac{1}{2}(A+B)\right)^2 - (AB - C^2)}$$

$$\geq \frac{1}{2}(A+B) - \sqrt{\left(\frac{1}{2}(A+B)\right)^2 - C^+}.$$

Let us consider the function

$$g(u) = u - \sqrt{u^2 - C^+}, \quad u > C^+.$$

Differentiating we obtain

$$g'(u) = 1 - \frac{u}{\sqrt{u^2 - C^+}} < 0.$$

Hence, $g(u)$ is a decreasing function, and achieves its minimum when u is maximal. Therefore,

$$\lambda_{\min} \geq \frac{1}{2}(A^+ + B^+) - \sqrt{\left(\frac{1}{2}(A^+ + B^+)\right)^2 - C^+} > 0.$$

□

Later we shall also need the following statement which follows from Lemmas 3 and 4.

Lemma 5. *For the collection of quadratic forms satisfying the assumptions of Lemma 4, the ratio of the diameter of the optimal triangle to the square root of the area of this triangle is uniformly bounded.*

II.4.1 General form for the error of interpolation of C^2 functions defined on $[0,1]^2$ by linear splines. Estimate from above.

Proof of Theorem 2: For a fixed $\varepsilon \in (0,1)$ and for every $N \in \mathbb{N}$ we define

$$m_N := \min\left\{m > 0 : \frac{1}{2m^{2(1+\frac{1}{p})}}\omega\left(\frac{1}{2m}\right) \leq \frac{\varepsilon}{N}\right\}, \tag{50}$$

where $\omega(\delta)$ is the function defined in (28).

Observe that clearly for m_N defined in (50) it is true that $m_N \to \infty$ as $N \to \infty$. In addition, note that

$$\frac{N}{m_N^{2(1+\frac{1}{p})}} \to \infty, \quad N \to \infty, \tag{51}$$

i.e. $m_N = o\left(N^{\frac{1}{2(1+\frac{1}{p})}}\right)$ as $N \to \infty$ and $\frac{1}{N} = o\left(\frac{1}{m_N^{2(1+\frac{1}{p})}}\right)$. Indeed, by the definition of m_N for all large enough N we have

$$\frac{N}{m_N^{2(1+\frac{1}{p})}} = 2\frac{(m_N-1)^{2(1+\frac{1}{p})}}{m_N^{2(1+\frac{1}{p})}} \frac{1}{\omega\left(\frac{1}{2(m_N-1)}\right)} \frac{1}{2} \frac{N}{(m_N-1)^{2(1+\frac{1}{p})}} \omega\left(\frac{1}{2(m_N-1)}\right)$$

$$\geq \varepsilon 2\frac{(m_N-1)^{2(1+\frac{1}{p})}}{m_N^{2(1+\frac{1}{p})}} \frac{1}{\omega\left(\frac{1}{2(m_N-1)}\right)} \to \infty, \text{ as } N \to \infty,$$

since $\left(\frac{m_N-1}{m_N}\right)^{2(1+\frac{1}{p})} \to 1$ and $\omega\left(\frac{1}{2(m_N-1)}\right) \to 0$ as $N \to \infty$. Hence, (51) is proved.

Divide the unit square $[0,1]\times[0,1]$ into squares with side length equal to $\frac{1}{m_N}$ and denote the resulting squares by D_i^N, $i = 1, \ldots, m_N^2$. Next take the center point (x_i^N, y_i^N) in each square D_i^N and set

$$A_i^N := \frac{1}{2} f_{xx}(x_i^N, y_i^N), \;\; B_i^N := \frac{1}{2} f_{yy}(x_i^N, y_i^N), \;\; C_i^N := f_{xy}(x_i^N, y_i^N).$$

Note that

$$H(x_i^N, y_i^N) := H(f; x_i^N, y_i^N) = 4(A_i^N B_i^N - (C_i^N)^2) \geq C^+, \;\; \forall i = 1, \ldots, m_N^2. \tag{52}$$

Set

$$n_i^N := \left\lceil \frac{N(1-\varepsilon) H(x_i^N, y_i^N)^{\frac{p}{2(p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{p+1}}}{\sum_{j=1}^{m_N^2} H(x_j^N, y_j^N)^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}} \right\rceil, \quad i = 1, ..., m_N^2. \tag{53}$$

The n_i^N, $i = 1, \ldots, m_N^2$, are determined by minimizing the sum of the errors of the interpolation of piecewise quadratic functions on each region, subject to the condition that the total number of triangles is N. For that purpose we shall use the method of Lagrange multipliers. Let us consider

$$\frac{\partial}{\partial \tilde{n}_i}\left(\sum_{i=1}^{m_N^2} \frac{H(x_i^N, y_i^N)^{\frac{p}{2}}}{m_N^{2(p+1)}(\tilde{n}_i^N)^{p+1}} \tilde{n}_i^N \Omega(x_i^N, y_i^N) + \lambda \sum_{i=1}^{m_N^2} \tilde{n}_i^N\right) = 0.$$

This can be rewritten as

$$-\frac{pH(x_i^N, y_i^N)^{\frac{p}{2}}}{m_N^{2(p+1)}(\tilde{n}_i^N)^{p+1}} \Omega(x_i^N, y_i^N) + \lambda = 0.$$

Hence,

$$\tilde{n}_i = \left(\frac{pH(x_i^N, y_i^N)^{\frac{p}{2}}\Omega(x_i^N, y_i^N)}{\lambda m_N^{2(p+1)}}\right)^{\frac{1}{p+1}}.$$

We shall find λ from the condition that the total number of triangles is N:

$$N = \sum_{i=1}^{m_N^2} \tilde{n}_i^N = \frac{p^{\frac{1}{p+1}}}{\lambda^{\frac{1}{p+1}} m_N^2} \sum_{i=1}^{m_N^2} H(x_i^N, y_i^N)^{\frac{p}{2(p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{p+1}}.$$

Solving for λ and plugging this value back into the expression for $\tilde{n}_i^N$, we obtain

$$\tilde{n}_i^N = \frac{N(1-\varepsilon)H(x_i^N, y_i^N)^{\frac{p}{2(p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{p+1}}}{\sum_{j=1}^{m_N^2} H(x_j^N, y_j^N)^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}}.$$

The method of Lagrange multipliers provides only the necessary conditions for the minimum. However, in this case the obtained $\tilde{n}_i^N$ indeed provides the minimum, because it clearly does not provide the maximum (the maximum can be explicitly constructed in a simple way) and the minimum exists.

Observe that all $n_i^N \to \infty$ when $N \to \infty$. This follows from the obvious estimate

$$n_i^N \geq \left[\frac{N(1-\varepsilon)(C^+)^{\frac{p}{2(p+1)}} \min_{(x,y)\in D}\{\Omega(x,y)\}^{\frac{1}{p+1}}}{m_N^2 \|H\|_\infty^{\frac{p}{2(p+1)}} \|\Omega\|_\infty^{\frac{1}{p+1}}}\right], \tag{54}$$

together with (51) (since if $\frac{N}{m_N^{2(1+\frac{1}{p})}} \to \infty$ then clearly $\frac{N}{m_N^2} \to \infty$ as $N \to \infty$), and $\min_{(x,y)\in D}\{\Omega(x,y)\} > 0$.

Given n_i^N for each square D_i^N we construct the triangulation $\triangle_N^*(D_i^N)$ of D_i^N in the following way:

1. Given the positive definite quadratic function

$$Q_i^N(x,y) = A_i^N x^2 + 2C_i^N xy + B_i^N y^2$$

   on D_i^N, consider transformations $F_{1,i}^N$ and $F_{2,i}^N$ of form (42) and (45) respectively, corresponding to the quadratic form $Q_i^N(x,y)$.

2. Take an arbitrary triangle T which solves problem (38), and consider $((F_{1,i}^N)^{-1} \circ (F_{2,i}^N)^{-1})T$.

3. Define T_i^N to be a rescaling of $((F_{1,i}^N)^{-1} \circ (F_{2,i}^N)^{-1})T$ so that

$$|T_i^N| = \frac{1}{m_N^2 n_i^N}.$$

4. Let $\tilde{T}_i^N$ be a triangle symmetric to T_i^N with respect to the midpoint of any side. Their union is a parallelogram.

5. Cover the square D_i^N with shifts of this parallelogram. We obtain the cover of D_i^N by the shifts of the triangle T_i^N and its reflection $\tilde{T}_i^N$.

6. If the intersection of D_i^N and a triangle T from this cover is a triangle, call it $\tilde{T}$, and include it in $\triangle_N^*(D_i^N)$.

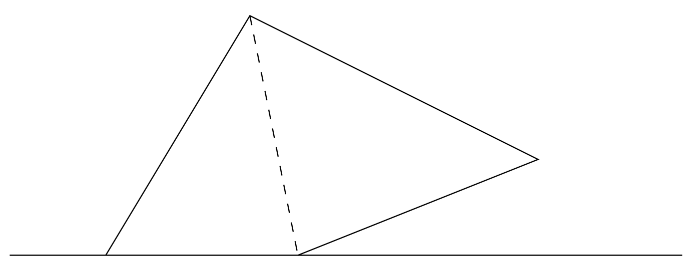

Figure 1: If the intersection of the triangle and a region is a quadrilateral, subdivide it without adding new vertices

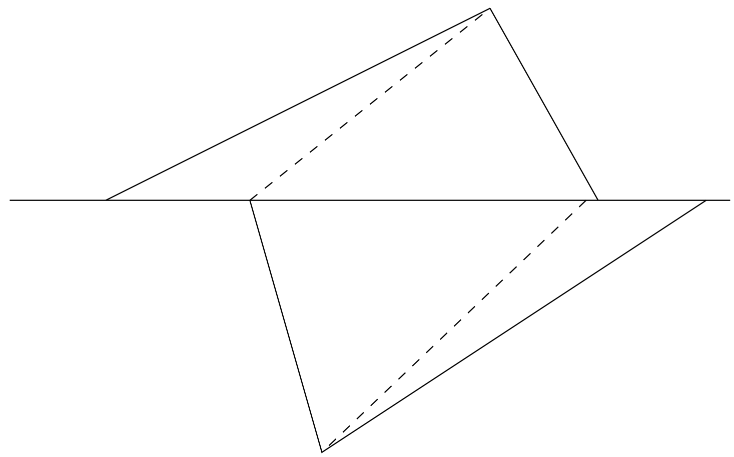

Figure 2: Subdivision of triangles on the boundary without adding new vertices

If the intersection of D_i^N and a triangle T is a quadrilateral, subdivide it into triangles without adding new vertices and include those triangles in $\triangle_N^*(D_i^N)$ (see Figure 1).

We obtain the triangulation $\triangle_N^*(D)$ of the whole domain D "gluing" together triangulations of each region without adding new vertices (see, Figure 2). Let us show that the triangulation $\triangle_N^*(D)$ is asymptotically optimal.

Everywhere below $c_1, c_2, \ldots,$ stand for constants independent of N.

Let us note that since for all $(x, y) \in D$ we have

$$0 < f_{xx}(x,y) < \|f_{xx}\|_\infty, \; 0 < f_{yy}(x,y) < \|f_{yy}\|_\infty, \text{ and } H(f;x,y) \geq C^+ > 0, \tag{55}$$

by Lemma 4 and Lemma 5 there exists a constant c_1 such that for any triangle T_i^N we have

$$\operatorname{diam}(T_i^N) \leq c_1 \sqrt{\frac{1}{m_N^2 n_i^N}}.$$

Let us consider a $c_1\sqrt{\frac{1}{m_N^2 n_i^N}}$-neighborhood of the boundary of D_i^N. Only those triangles that lie completely in this neighborhood may have nonempty intersection with the boundary of D_i^N. Therefore, if we denote their number by K_i^N then

$$K_i^N|T_i^N| = K_i^N \frac{1}{m_N^2 n_i^N} \leq 8\sqrt{\frac{1}{m_N^2 n_i^N}} \left(\frac{1}{m_N} + 2c_1\sqrt{\frac{1}{m_N^2 n_i^N}}\right).$$

This implies that

$$K_i^N \leq c_2\sqrt{n_i^N}.$$

After possible subdivision the number of triangles that have nonempty intersection with the boundary in the triangulation of D_i^N will be not greater than $c_3\sqrt{n_i^N}$ with some constant c_3. Hence, the total number of triangles that have nonempty intersection with the boundary is not greater than

$$c_3\sum_{i=1}^{m_N^2}\sqrt{n_i^N} \leq c_3\sum_{i=1}^{m_N^2}\sqrt{\frac{N(1-\varepsilon)H(x_i^N,y_i^N)^{\frac{p}{2(p+1)}}\Omega(x_i^N,y_i^N)^{\frac{1}{p+1}}}{\sum_{j=1}^{m_N^2} H(x_j^N,y_j^N)^{\frac{p}{2(p+1)}}\Omega(x_j^N,y_j^N)^{\frac{1}{p+1}}}}$$

$$\leq c_4\sqrt{N}m_N^2 \frac{\|H\|_\infty^{\frac{p}{4(p+1)}}\|\Omega\|_\infty^{\frac{1}{2(p+1)}}}{m_N (C^+)^{\frac{p}{4(p+1)}}\left(\min\limits_{(x,y)\in D}\{\Omega(x,y)\}\right)^{\frac{1}{2(p+1)}}} \leq c_5\sqrt{N}m_N = o\left(N^{\frac{3+\frac{1}{p}}{2(2+\frac{1}{p})}}\right),$$

as $N\to\infty$ (since $m_N = o\left(N^{\frac{1}{2(1+\frac{1}{p})}}\right)$ as $N\to\infty$ because of (51)).

Therefore, since $3+\frac{1}{p} \leq 2(2+\frac{1}{p})$, the number of triangles in the constructed triangulation will not exceed N, for all N large enough.

Let f_N denote the piecewise quadratic function constructed in the following way. On D_1^N we set f_N to be $A_1^N x^2 + 2C_1^N xy + B_1^N y^2$. Then for $i > 1$ on $D_i^N \setminus \cup_{j=1}^{i-1} D_j^N$ we set

$$f_N(x,y) := A_i^N x^2 + 2C_i^N xy + B_i^N y^2.$$

Observe that

$$R_N(f, L_{p,\Omega}, S_1^0(\triangle_N)) \leq \|f - s(f,\triangle_N^*)\|_{p,\Omega}$$

$$\leq \|f - f_N\|_{p,\Omega} + \|f_N - s(f_N,\triangle_N^*)\|_{p,\Omega} + \|s(f_N,\triangle_N^*) - s(f,\triangle_N^*)\|_{p,\Omega}.$$

Obviously,

$$\|s(f_N,\triangle_N^*)-s(f,\triangle_N^*)\|_{p,\Omega}\le\|f-f_N\|_{p,\Omega}.$$

Hence,

$$R_N(f,L_{p,\Omega},S_1^0(\triangle_N))\le 2\|f-f_N\|_{p,\Omega}+\|f_N-s(f_N,\triangle_N^*)\|_{p,\Omega}.$$

Let us estimate each term. First of all, by the corollary of Lemma 1 and the definition of m_N we have

$$\|f-f_N\|_{p,\Omega}\le\frac{\|\Omega\|_\infty^{\frac{1}{p}}}{2m_N^{2(1+\frac{1}{p})}}\omega\left(\frac{1}{2m_N}\right)\le\frac{\varepsilon}{N}\|\Omega\|_\infty^{\frac{1}{p}}.$$

Let us estimate the second term now. It is clear that for two embedded triangles the error of linear interpolation of a quadratic function with $H(f;x,y)\ge C^+>0$ is greater on the larger triangle. Therefore, we shall estimate this error on triangles that do not have intersection with the boundary.

Let us take a triangle $T_i^N\in\triangle_N^*(D_i^N)$ that does not have common points with the boundary of D_i^N. By (47), for every point $(x,y)\in T_i^N$ we have

$$|f_N(x,y)-s(f_N,\triangle_N^*;x,y)|^p\le\left(\frac{C_p^+}{2}\right)^p H(x_i^N,y_i^N)^{p/2}\frac{\Omega(x_i^N,y_i^N)}{(m_N^2n_i^N)^{p+1}}.$$

Hence, the p-power of the error on the whole D is bounded by

$$|f_N(x,y)-s(f_N,\triangle_N^*;x,y)|^p\le\left(\frac{C_p^+}{2}\right)^p\sum_{j=1}^{m_N^2}n_i^N\Omega(x_i^N,y_i^N)H(x_i^N,y_i^N)^{p/2}\frac{1}{(m_N^2n_i^N)^{p+1}}.$$

By the definition of n_i^N and by (54), for all large enough N, for all i, and for all $(x,y)\in D$, we have

$$|f_N(x,y)-s(f_N,\triangle_N^*;x,y)|^p\le$$

$$\le\left(\frac{C_p^+}{2}\right)^p\frac{(1+\varepsilon)}{m_N^{2(p+1)}}\sum_{j=1}^{m_N^2}H(x_i^N,y_i^N)^{p/2}\Omega(x_i^N,y_i^N)\left(\frac{\sum_{i=1}^{m_N^2}H(x_j^N,y_j^N)^{\frac{p}{2(p+1)}}\Omega(x_j^N,y_j^N)^{\frac{1}{p+1}}}{N(1-\varepsilon)H(x_i^N,y_i^N)^{\frac{p}{2(p+1)}}\Omega(x_i^N,y_i^N)^{\frac{1}{p+1}}}\right)^p=$$

$$= \left(\frac{C_p^+}{2}\right)^p \frac{(1+\varepsilon)}{N^p m_N^{2(p+1)}} \left(\sum_{j=1}^{m_N^2} H(x_j^N, y_j^N)^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}\right)^{p+1}.$$

Since this estimate does not depend on x and y, we obtain that

$$\|f_N - s(f_N, \triangle_N^*)\|_{p,\Omega} \le \frac{(1+\varepsilon)C_p^+}{2N} \left(\frac{1}{m_N^2}\sum_{j=1}^{m_N^2} H(x_j^N, y_j^N)^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}}.$$

Note that since $H(f;x,y)$ and $\Omega(x,y)$ both are Riemann integrable

$$\frac{1}{m_N^2}\sum_{j=1}^{m_N^2} H(x_j^N, y_j^N)^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}$$

$$= \sum_{j=1}^{m_N^2} |D_j^N| H(x_j^N, y_j^N)^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}} \to \int_D H(f;x,y)^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy \qquad (56)$$

as $N \to \infty$. Hence, for all N large enough we obtain

$$\|f_N - s(f_N, \triangle_N^*)\|_{p,\Omega} < \frac{C_p^+}{2N}\frac{1+2\varepsilon}{1-\varepsilon} \left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy\right)^{\frac{p+1}{p}}.$$

Therefore,

$$\|f - s(f, \triangle_N^*)\|_{p,\Omega} < \frac{2\varepsilon}{N}\|\Omega\|_\infty^{\frac{1}{p}} + \frac{C_p^+}{2N}\frac{1+2\varepsilon}{1-\varepsilon} \left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy\right)^{\frac{p+1}{p}}.$$

Because $\varepsilon > 0$ is arbitrary, we obtain the desired estimate. □

II.4.2 General form for the error of interpolation of C^2 functions defined on $[0,1]^2$ by linear splines. Estimate from below.

In what follows, the quantities m_N, n_i^N, D_i^N, A_i^N, B_i^N etc., are the same as defined in Section II.4.1.

Proof of Theorem 3: To obtain the estimate from below we shall consider an arbitrary sequence of triangulations $\{\triangle_N\}_{N=1}^\infty$ which satisfies (22). For any $\varepsilon > 0$ denote by $D_i^N(\varepsilon)$ the square congruent to D_i^N with side length equal to $\frac{1-\varepsilon}{m_N^2}$. Assumption (22) implies that

$m_N = o\left(\sqrt{N}\right)$ and, hence, for all large enough N there exists a triangle $T_{i_N}^N$ which lies completely in $D_i^N(\varepsilon)$.

For each N and i_N, set

$$f_{N,i_N}(x,y) := A_{i_N}^N x^2 + 2C_{i_N}^N xy + B_{i_N}^N y^2.$$

Observe that

$$\|f - s(f,\triangle_N)\|_{L_{p,\Omega}(T_{i_N}^N)} \geq \|f_{N,i_N} - s(f_{N,i_N},\triangle_N)\|_{L_{p,\Omega}(T_{i_N}^N)} - 2\|f - f_{N,i_N}\|_{L_{p,\Omega}(T_{i_N}^N)}.$$

By (47) we have for all N large enough

$$\|f_{N,i_N} - s(f_{N,i_N},\triangle_N)\|_{L_{p,\Omega}(T_{i_N}^N)}^p \geq \frac{(1-\varepsilon)(C_p^+)^p}{2^p(m_N^2 n_{i_N}^N)^{p+1}} H(x_{i_N}^N, y_{i_N}^N)^{p/2}\Omega(x_{i_N}^N, y_{i_N}^N).$$

By the definition of $n_{i_N}^N$ we have that the error on the whole D for all N large enough satisfies

$$\begin{aligned}
\|f_N - s(f_N,\triangle_N)\|_{p,\Omega}^p &\geq \sum_{i=1}^{m_N^2} \frac{(1-\varepsilon)(C_p^+)^p}{(m_N^2 n_{i_N}^N)^{p+1}} H(x_i^N, y_i^N)^{p/2}\Omega(x_i^N, y_i^N) \\
&\geq \frac{(C_p^+)^p(1-\varepsilon)}{m_N^{2(p+1)}} \sum_{i=1}^{m_N^2} H(x_i^N, y_i^N)^{p/2}\Omega(x_i^N, y_i^N) \left(\frac{\sum_{j=1}^{m_N^2} H(x_j^N, y_j^N)^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}}{N(1-\varepsilon) H(x_i^N, y_i^N)^{\frac{p}{2(p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{p+1}}} \right)^p \\
&= \frac{(C_p^+)^p}{(1-\varepsilon)^{p-1} N^p m_N^{2(p+1)}} \left(\sum_{j=1}^{m_N^2} H(x_j^N, y_j^N)^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}} \right)^{p+1} \\
&> \frac{(C_p^+)^p(1-\epsilon)}{N^p(1-\varepsilon)^{p-1}} \left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy \right)^{p+1}.
\end{aligned}$$

Hence, for all N large enough we obtain

$$\|f_N - s(f_N,\triangle_N)\|_{p,\Omega} > \frac{C_p^+}{2N}(1-c_6\varepsilon) \left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy \right)^{\frac{p+1}{p}}.$$

On the other hand

$$\|f - f_{N,i_N}\|_{L_{p,\Omega}(T^N_{i_N})} \le \|f - f_{N,i_N}\|_{L_{p,\Omega}(D^N_{i_N})} \le \frac{\|\Omega\|_\infty^{\frac{1}{p}}}{2m_N^{2+\frac{1}{p}}}\omega\left(\frac{1}{2m_N}\right) \le \frac{\varepsilon}{N}\|\Omega\|_\infty^{\frac{1}{p}}$$

due to the choice of m_N. Hence, we obtain that for all large enough N

$$\|f - s(f, \triangle_N)\|_{p,\Omega} \ge (1 - c_7\varepsilon)\frac{C_p^+}{2N}\left(\int_D H(f; x, y)^{\frac{p}{2(p+1)}}\Omega(x, y)^{\frac{1}{p+1}}dxdy\right)^{\frac{p+1}{p}}$$

with some positive constant c_7. Therefore,

$$\liminf_{N\to\infty} \frac{\|f - s(f, \triangle_N)\|_{p,\Omega}}{\frac{C_p^+}{2N}\left(\int_D H(f; x, y)^{\frac{p}{2(p+1)}}\Omega(x, y)^{\frac{1}{p+1}}dxdy\right)^{\frac{p+1}{p}}} \ge 1.$$

This completes the proof of the theorem. □

II.4.3 Calculation of C_2^+.

In this section we shall find the solution to the following extremal problem. Let

$$Q(x, y) = Ax^2 + By^2 + 2Cxy \tag{57}$$

be a sign definite quadratic form, i.e. such that $AB - C^2 > 0$. Without loss of generality we may take $A > 0$.

Let $L_{Q,T}(x, y)$ denote the linear function which interpolates $Q(x, y)$ at the vertices of the triangle T. Recall that we denoted by

$$d_{Q,T,2} := \|Q - L_{Q,T}\|_2 = \left(\int_T (Q(x, y) - L_{Q,T}(x, y))^2 dxdy\right)^{1/2}, \tag{58}$$

and as before let $|T|$ denote the area of triangle T. The problem is to describe those triangles T which solve the problem

$$\frac{d_{Q,T,2}}{|T|^{3/2}} \to \min. \tag{59}$$

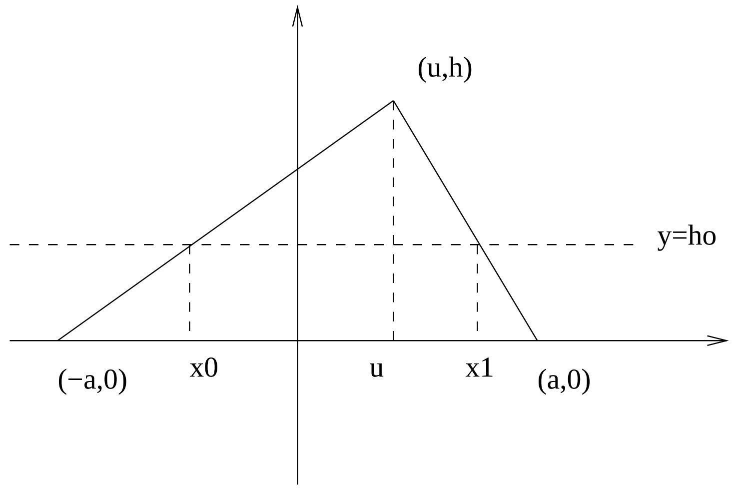

Figure 3: Triangle T_u

Lemma 6. *For the function $Q(x, y) = x^2 + y^2$ and an arbitrary triangle T we have*

$$\frac{d_{Q,T,2}}{|T|^{3/2}} \geq \frac{4}{\sqrt{45}}.$$

The equality occurs if and only if T is an equilateral triangle.

Remark. By definition of C_2^+ this lemma implies that

$$C_2^+ = \frac{4}{\sqrt{45}}. \tag{60}$$

Proof: First, let us consider the triangle T_u in Figure 1 with vertices $(-a, 0)$, $(a, 0)$ and (u, h), and let us investigate how the difference between function $Q(x, y) = x^2 + y^2$ and its interpolant $L_{Q,T_u}(x, y)$, denoted by

$$\delta_{T_u}(x, y) := Q(x, y) - L_{Q,T_u}(x, y),$$

behaves, depending on u.

First of all, observe that the equation of the interpolant $L_{Q,T_u}(x, y)$ to $x^2 + y^2$ at the vertices of the triangle T_u is

$$L_{Q,T_u}(x,y) = \frac{1}{h}(u^2 + h^2 - a^2)y + a^2. \tag{61}$$

Note that it is enough to consider the behavior of the difference $\delta_{T_u}(x,y)$ on the lines of the form $y = h_0$. The restriction of the elliptic paraboloid $Q(x,y)$ to this line is

$$z(x,h_0) = x^2 + h_0^2,$$

and the restriction of the interpolant $L_{Q,T_u}(x,y)$ to this line is

$$L_{Q,T_u}(x,h_0) = \frac{1}{h}(u^2 + h^2 - a^2)h_0 + a^2.$$

The points of intersection of the line $y = h_0$ with the sides of the triangle T_u are

$$\left(\frac{h_0}{h}(u+a) - a, h_0\right), \quad \left(\frac{h_0}{h}(u-a) + a, h_0\right).$$

For brevity, denote the x-coordinates of these points by $x_0(h)$ and $x_1(h)$, respectively. Observe that in this notation

$$d^2_{Q,T_u,2} = \int_0^h \int_{x_0(h)}^{x_1(h)} (\delta_{T_u}(x,y))^2\, dxdy.$$

Computing the square of the L_2 error on $y = h_0$, we obtain

$$q^2(u) := \int_{x_0(h)}^{x_1(h)} (\delta_{T_u}(x,h_0))^2\, dx = \int_{x_0}^{x_1} \left(\frac{1}{h}(u^2 + h^2 - a^2)h_0 + a^2 - x^2 - h_0^2\right)^2 dx \tag{62}$$

$$= 1/5 \left(\frac{h_0\ (u-a)}{h} + a\right)^5 - 1/5 \left(\frac{h_0\ (u+a)}{h} - a\right)^5$$

$$+1/3 \left(-2\,\frac{\left(u^2+h^2-a^2\right) h_0}{h} - 2\,a^2 + 2\,{h_0}^2\right) \left(\left(\frac{h_0\ (u-a)}{h} + a\right)^3 - \left(\frac{h_0\ (u+a)}{h} - a\right)^3\right)$$

$$+ \left(\frac{\left(u^2+h^2-a^2\right) h_0}{h} + a^2 - {h_0}^2\right)^2 \left(\frac{h_0\ (u-a)}{h} + 2\,a - \frac{h_0\ (u+a)}{h}\right).$$

The roots of the equation

$$\frac{d}{du}q^2(u) = 0$$

are

$$\begin{aligned}
u_1 &= 0, \\
u_2 &= 1/3\,\frac{\sqrt{-3\,h_0\,\left(3\,h^2h_0 + 2\,a^2h + 3\,a^2h_0\right)}}{h_0}, \\
u_3 &= -1/3\,\frac{\sqrt{-3\,h_0\,\left(3\,h^2h_0 + 2\,a^2h + 3\,a^2h_0\right)}}{h_0}.
\end{aligned}$$

Since the only real root is $u_1 = 0$, the minimum of the L_2 error of the difference between function and interpolant is obtained when $u = 0$ or, in other words, for isosceles triangle T.

Now observe that, clearly, no equilateral triangle can be optimal. For every isosceles triangle which is not equilateral, using arguments similar to the arguments above, we can find a triangle with the same area but on which the L_2 norm of the difference will be smaller.

Next we calculate the square of the L_2 error of interpolation in the case when the triangle T is equilateral with given side length $2a$ (or fixed area $|T|$). In this case we have $u = 0$ and, therefore, the interpolant becomes $\frac{2\sqrt{3}a}{3}y + a^2$ and we obtain for the error

$$d_{Q,T,2}^2 = \int_0^{\sqrt{3}a} \int_{\sqrt{3}(a+x)}^{\sqrt{3}(a-x)} \left(\frac{2}{\sqrt{3}}ay + a^2 - x^2 - y^2\right)^2 dxdy = \frac{16\sqrt{3}}{15}a^6 \tag{63}$$

or, in terms of the area $|T| = \sqrt{3}a^2$,

$$(d_{Q,T,2})^2 = \frac{16}{45}|T|^3.$$

In other words, if T is an equilateral triangle then

$$\frac{d_{Q,T,2}}{|T|^{3/2}} = \frac{4}{3\sqrt{5}}.$$

□

Therefore, taking into consideration (47) note that we have proved the following lemma.

Lemma 7. *For the quadratic form*

$$Q(x, y) = Ax^2 + By^2 + 2Cxy$$

such that $AB - C^2 > 0$, *and for an arbitrary triangle* T *we have*

$$\frac{d_{Q,T,2}}{|T|^{3/2}} \geq \frac{4}{\sqrt{45}}(AB - C^2)^{1/2}. \tag{64}$$

Moreover, equality is obtained for triangles $T = (F_2 \circ F_1)^{-1}\tilde{T}$, *where* $\tilde{T}$ *is an arbitrary equilateral triangle, and only for them.*

II.4.4 Calculation of C_1^+.

The result of this section will repeat the result of Fejes Toth (see Section I.3.2). However, we shall need it later to prove the generalization of Fejes Toth's result to the case of integration with any positive continuous weight.

In this section we shall find the solution to the following extremal problem. Let

$$Q(x, y) = Ax^2 + By^2 + 2Cxy \tag{65}$$

be a sign definite quadratic form, i.e. such that $AB - C^2 > 0$. Without loss of generality we may take $A > 0$.

As before, let $L_{Q,T}(x, y)$ denote the linear function which interpolates $Q(x, y)$ at the vertices of the triangle T. Recall the notation

$$d_{Q,T,1} := \|Q - L_{Q,T}\|_1 = \int_D |Q(x, y) - L_{Q,T}(x, y)| dxdy, \tag{66}$$

and below let $|T|$ denote the area of the triangle T. The main goal is to find

$$\frac{d_{Q,T,1}}{|T|^2} \to \min \tag{67}$$

and to describe those triangles T which solve this minimization problem.

Lemma 8. *For the function* $Q(x,y) = x^2 + y^2$ *and an arbitrary triangle* T *we have*

$$\frac{d_{Q,T,1}}{|T|^2} \geq \frac{\sqrt{3}}{3}.$$

The equality occurs if and only if T *is an equilateral triangle.*

Remark. From this lemma it follows that

$$C_1^+ = \frac{\sqrt{3}}{3}. \tag{68}$$

Proof: First, let us consider the triangle T_u in Figure 1 with vertices $(-a, 0)$, $(a, 0)$ and (u, h), and let us investigate how the difference between function $Q(x,y) = x^2 + y^2$ and its interpolant $L_{Q,T_u}(x,y)$, denoted by

$$\delta_{T_u}(x,y) := Q(x,y) - L_{Q,T_u}(x,y),$$

behaves, depending on u.

First of all, observe that the equation of the interpolant $L_{Q,T_u}(x,y)$ to $x^2 + y^2$ at the vertices of the triangle T_u is

$$L_{Q,T_u}(x,y) = \frac{1}{h}(u^2 + h^2 - a^2)y + a^2. \tag{69}$$

Note that it is enough to consider the behavior of the difference $\delta_{T_u}(x,y)$ on the lines $y = h_0$. The restriction of the elliptic paraboloid $Q(x,y)$ to this line is

$$z(x, h_0) = x^2 + h_0^2,$$

and the restriction of the interpolant $L_{Q,T_u}(x,y)$ is

$$L_{Q,T_u}(x, h_0) = \frac{1}{h}(u^2 + h^2 - a^2)h_0 + a^2.$$

The points of intersection of the line $y = h_0$ with the sides of the triangle T_u are

$$\left(\frac{h_0}{h}(u+a) - a, h_0\right), \quad \left(\frac{h_0}{h}(u-a) + a, h_0\right).$$

For brevity, denote the x-coordinates of these points by $x_0(h)$ and $x_1(h)$, respectively.

Therefore, the value of the L_1 error on $y = h_0$ is

$$q(u) := \int_{x_0}^{x_1} |\delta(x, h_0)| dx = \int_{x_0}^{x_1} \left| \frac{1}{h}(u^2 + h^2 - a^2)h_0 + a^2 - x^2 - h_0^2 \right| dx \tag{70}$$

$$= \int_{x_0}^{x_1} \left(\frac{1}{h}(u^2 + h^2 - a^2)h_0 + a^2 - x^2 - h_0^2 \right) dx.$$

Differentiating this function of u, we obtain

$$\frac{d}{du} q(u) = \frac{4h_0 ua(h_0 - h)^2}{h^3}.$$

Obviously, the only solution to the equation

$$\frac{d}{du} q(u) = 0 \quad \text{or} \quad \frac{4h_0 ua(h_0 - h)^2}{h^3} = 0$$

is $u = 0$, and, hence, the minimum of the L_1 error of the difference between function and interpolant is obtained when $u = 0$ or, in other words, for isosceles triangle T_u.

Now observe that, clearly, no equilateral triangle can be optimal. For every isosceles triangle which is not equilateral, using arguments similar to the arguments above, we can find a triangle with the same area but on which the L_1 norm of the difference will be smaller.

Next we calculate the L_1 error of interpolation in the case when triangle T is equilateral with given side length $2a$ (or fixed area $|T|$). In this case we have $u = 0$ and, therefore, the interpolant becomes $\frac{2\sqrt{3}a}{3} y + a^2$ and we obtain for the error

$$d_{Q,T,1} = \int_0^{\sqrt{3}a} \int_{\sqrt{3}(a+x)}^{\sqrt{3}(a-x)} \left(\frac{2}{\sqrt{3}} ay + a^2 - x^2 - y^2 \right) dx dy = \sqrt{3}a^4 \tag{71}$$

or, in terms of the area $|T| = \sqrt{3}a^2$,

$$\frac{1}{2}d_{Q,T,1} = \frac{3}{\sqrt{3}}|T|^2.$$

In other words, if T is an equilateral triangle then

$$\frac{d_{Q,T,1}}{|T|^2} = \frac{3}{\sqrt{3}}.$$

□

Therefore, we have proved the following lemma.

Lemma 9. *For the quadratic form*

$$Q(x, y) = Ax^2 + By^2 + 2Cxy$$

such that $AB - C^2 > 0$*, and for an arbitrary triangle* T *we have*

$$\frac{d_{Q,T,1}}{|T|^2} \geq \frac{\sqrt{3}}{3}\sqrt{AB - C^2}. \tag{72}$$

Moreover, equality is obtained for triangles $T = (F_2 \circ F_1)^{-1}\tilde{T}$*, where* $\tilde{T}$ *is an arbitrary equilateral triangle, and only for them.*

II.4.5 Computing constants C_p^+ for other values of p.

Using a method similar those described in the two previous section we can also calculate constants for some other values of p. For example

$$C_3^+ = \left(\frac{124}{35 \cdot 3^{\frac{5}{2}}}\right)^{\frac{1}{3}}, \quad C_4^+ = \left(\frac{704}{4725}\right)^{\frac{1}{4}}, \quad C_5^+ = \left(\frac{256}{55 \cdot 3^{\frac{7}{2}}}\right)^{\frac{1}{5}}, \ldots.$$

However, at this point we do not have the explicit value of C_p^+ for an arbitrary p.

II.5 Functions with negative Hessian and the case of the $L_{\infty,\Omega}$ norm.

Now we shall consider the quadratic form

$$Q(x,y) = Ax^2 + By^2 + 2Cxy \tag{73}$$

with $A \geq 0$ and $AB - C^2 < 0$. Recall that in this case $\lambda_{\min} < 0 < \lambda_{\max}$.

Let us consider the following transformation

$$G_2: \quad u = \sqrt{\lambda_{\max}}x' - \sqrt{|\lambda_{\min}|}y', \quad v = \sqrt{\lambda_{\max}}x' + \sqrt{|\lambda_{\min}|}y'. \tag{74}$$

Under this transformation (73) becomes

$$(Q \circ F_1^{-1} \circ G_2^{-1})(u,v) = uv.$$

Note that the interpolation error does not change under this transformation, but to obtain the area of the new triangle G_2T we have to multiply the area $|T|$ by the Jacobian of the transformation G_2, i.e., $|G_2T| = 2\sqrt{|\lambda_{\max}\lambda_{\min}|}|T|$.

As before we are interested in the problem of minimizing the ratio

$$\frac{d_{Q,T,\infty}}{|T|}, \tag{75}$$

and characterizing those triangles which provide the minimum of this ratio.

Observe that for an arbitrary triangle T we have

$$\frac{d_{Q,T,\infty}}{|T|} = \frac{d_{Q\circ F_1^{-1},F_1T,\infty}}{|F_1T|} = \frac{d_{Q\circ F_1^{-1}\circ G_2^{-1},(G_2\circ F_1)T,\infty}}{\frac{1}{2\sqrt{|\lambda_{\max}\lambda_{\min}|}}|(G_2\circ F_1)T|}. \tag{76}$$

Therefore, if we denote

$$C_\infty^- := \min_T \frac{L_\infty - \text{error of linear interpolation of } uv \text{ on } T}{|T|} \tag{77}$$

then for an arbitrary quadratic form of type (73) we have

$$\frac{d_{Q,T,\infty}}{|T|} \geq 2C_\infty^- \sqrt{|\lambda_{\max}\lambda_{\min}|}. \tag{78}$$

II.5.1 Calculation of C_∞^-.

We shall need the following two simple auxiliary statements.

Lemma 10. *The L_∞-error of interpolation of the function $z = xy$ by a linear function on any triangle is attained on its boundary.*

Proof: For any fixed value $x = x_0$ the function $z = xy$ is a linear function of y:

$$z = x_0 y.$$

The restriction of the interpolant to this line $x = x_0$ is also linear. Hence, their difference is a linear function as well. Therefore, it achieves its maximal and minimal values at the end points of the interval (which is the intersection of the triangle with the line $x = x_0$). This is true for any value of x_0. Therefore, the maximal value of the error will be indeed attained on the boundary of the triangle. □

Remark. Clearly, the statement of the last lemma holds for any bilinear function on a convex set.

Lemma 11. *For any triangle T there exists a shift that maps one of the vertices of the triangle to the origin, and the remaining two to the same coordinate quadrant.*

Proof: For the given triangle T let us consider the rectangle of minimal area containing T whose sides are parallel to the coordinate axes. Because the area of the rectangle is minimal, every side of the rectangle should contain a vertex of the triangle T, and one of the vertices of T has to coincide with a vertex of the rectangle. This is the one that has to be placed at the origin. Clearly, setting the sides of the rectangle that contain this vertex on the coordinate axes, we obtain that the rectangle lies in one quadrant. □

Further, due to Lemma 2 and the fact that the error of interpolation will not change if we take the triangle which is symmetric to the given one with respect to any coordinate

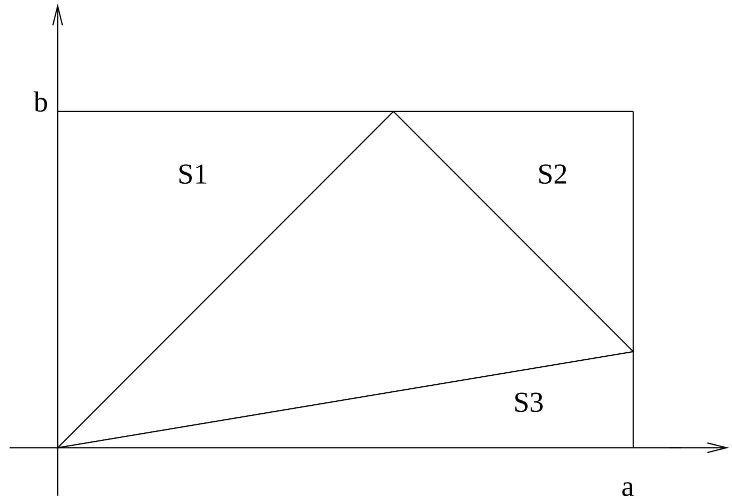

Figure 4: Triangle T with one vertex at the origin and corresponding areas S_1, S_2, and S_3

axis, without loss of generality, we consider a triangle in the first coordinate quadrant.

Let us now take an arbitrary triangle T. By Lemma 11, we may take T to have one vertex at the origin and to be inscribed in the rectangle with side length equal to a and b. Observe that this creates three right triangles whose union forms the complement of T with respect to the rectangle. Denote their areas by S_1, S_2, and S_3 (see Figure 4).

Note that the error of linear interpolation on each side of triangle T is equal to the error on the longest side of the right triangle which has this side common with T and complements T to the rectangle. The error on the longest side of this right triangle will not change if we place its right angle at the origin, and will be attained on its longest side (hypotenuse). The interpolant on this triangle is zero, and it is easy to calculate the maximum of the restriction of the function to the longest side. This shows, in particular, that the error of linear interpolation on each side of triangle T is equal to half of the area of a right triangle which has this side common with T and complements T to the rectangle.

Therefore, the problem of minimizing ratio (75) is equivalent to the extremal problem:

$$\frac{\max\{S_1, S_2, S_3\}}{2(ab - (S_1 + S_2 + S_3))} \to \min\,. \tag{79}$$

The following lemma provides the solution to this problem.

Lemma 12. *For an arbitrary $a > 0$ and $b > 0$, the solution to problem (79) is given by the triangle with vertices*

$$(0,0)\,, \quad \left(\frac{3-\sqrt{5}}{2}a, b\right), \quad \left(a, \frac{3-\sqrt{5}}{2}b\right). \tag{80}$$

Moreover, ratio (75) (and (79)) for this triangle is equal to $\frac{1}{2\sqrt{5}}$.

Proof: If we denote the coordinates of an arbitrary triangle T with one vertex at the origin and the other two vertices on the sides of the rectangle with sides a and b by (x, b) and (a, y) then (79) can be rewritten as

$$\frac{1}{2}\max\left\{\frac{bx}{ab-xy}, \frac{ay}{ab-xy}, \frac{(a-x)(b-y)}{ab-xy}\right\} \to \min\,. \tag{81}$$

We need to show that for an arbitrary triangle T, with vertices $(0,0)$, (x,b), and (a,y), where $x \in (0,a)$ and $y \in (0,b)$, we have

$$\frac{1}{2}\max\left\{\frac{bx}{ab-xy}, \frac{ay}{ab-xy}, \frac{(a-x)(b-y)}{ab-xy}\right\} \geq \frac{1}{2\sqrt{5}}. \tag{82}$$

As for the location of vertices of T the following four cases are possible:

1. $x \geq \frac{3-\sqrt{5}}{2}a$, $y \geq \frac{3-\sqrt{5}}{2}b$.
2. $x \leq \frac{3-\sqrt{5}}{2}a$, $y \leq \frac{3-\sqrt{5}}{2}b$.
3. $x \geq \frac{3-\sqrt{5}}{2}a$, $y \leq \frac{3-\sqrt{5}}{2}b$.
4. $x \leq \frac{3-\sqrt{5}}{2}a$, $y \geq \frac{3-\sqrt{5}}{2}b$.

Before we consider each case, observe that for $x \in (0,a)$ and $y \in (0,b)$ the function $\frac{bx}{ab-xy}$ is increasing with respect to x when y is fixed; the function $\frac{ay}{ab-xy}$ is increasing with respect to y when x is fixed; the function $\frac{(a-x)(b-y)}{ab-xy}$ is decreasing with respect to x when y is fixed and with respect to y when x is fixed.

Case 1. Clearly, in this case we have

$$\frac{1}{2}\max\left\{\frac{bx}{ab-xy},\frac{ay}{ab-xy},\frac{(a-x)(b-y)}{ab-xy}\right\}\geq\frac{bx}{2(ab-xy)}$$

$$\geq\frac{\frac{3-\sqrt{5}}{2}ab}{2(ab-\frac{3-\sqrt{5}}{2}a\frac{3-\sqrt{5}}{2}b)}=\frac{1}{2\sqrt{5}}.$$

Case 2. In this case we have

$$\frac{1}{2}\max\left\{\frac{bx}{ab-xy},\frac{ay}{ab-xy},\frac{(a-x)(b-y)}{ab-xy}\right\}\geq\frac{1}{2}\frac{(a-x)(b-y)}{ab-xy}$$

$$\geq\frac{1}{2}\frac{(a-\frac{3-\sqrt{5}}{2}a)(b-\frac{3-\sqrt{5}}{2}b)}{ab-\frac{3-\sqrt{5}}{2}a\frac{3-\sqrt{5}}{2}b}=\frac{1}{2\sqrt{5}}.$$

Cases 3 – 4. Note that it is enough to consider only one of the cases 3 and 4 because of the symmetry. Let us choose case 3.

First of all, observe that conditions of this case imply that $\frac{x}{a}\geq\frac{y}{b}$. Hence,

$$\frac{1}{2}\max\left\{\frac{bx}{ab-xy},\frac{ay}{ab-xy},\frac{(a-x)(b-y)}{ab-xy}\right\}$$

$$=\frac{1}{2}\max\left\{\frac{bx}{ab-xy},\frac{(a-x)(b-y)}{ab-xy}\right\}.$$

Clearly, since the first term is increasing and the second is decreasing, the maximum value will be greater than the value at the point where both terms are equal, i.e. when $x=\frac{a(b-y)}{2b-y}$. This gives

$$\frac{1}{2}\max\left\{\frac{bx}{ab-xy},\frac{(a-x)(b-y)}{ab-xy}\right\}\geq\frac{\frac{a(b-y)}{2b-y}}{ab-\frac{a(b-y)}{2b-y}y}=\frac{b(b-y)}{b^2+(b-y)^2}.$$

Note that the function $\frac{u}{1+u^2}$ is increasing for any u (in our case $u=b-y$). Therefore, it attains its minimum for the minimal possible value of u which corresponds to the maximal possible value of y, that is $\frac{3-\sqrt{5}}{2}b$. Plugging this value in, we obtain the desired inequality. This completes the proof of the lemma. □

With the help of Lemma 12, observations (76) and (78) we can prove the following lemma.

Lemma 13. *For the quadratic form*

$$Q(x,y) = Ax^2 + By^2 + 2Cxy \tag{83}$$

with $AB - C^2 < 0$ *and an arbitrary triangle* T *we have*

$$\frac{d_{Q,T,\infty}}{|T|} \geq \frac{\sqrt{|AB - C^2|}}{\sqrt{5}}.$$

The equality occurs for triangles $T = (F_1^{-1} \circ G_2^{-1})\bar{T}$ *where* $\bar{T}$ *is an arbitrary triangle with vertices (80) or symmetric to it with respect to any coordinate axis, and only for such triangles.*

Later in Section II.5 in the process of construction of the optimal triangulation we would like to avoid long and skinny triangles. This can be done due to the following lemma.

Lemma 14. *The parameters* a *and* b *in the triangle with vertices (80) can always be chosen so that ratios of lengths of each side of the triangle* $T = (F_1^{-1} \circ G_2^{-1})\bar{T}$ *where* $\bar{T}$ *is an arbitrary triangle of the form (80) to the square root of the area of the triangle are all bounded.*

Proof: Note that the orthogonal transformation F_1 does not affect the ratios in the statement of the lemma. Preimages of vertices $(0,0)$, $\left(\frac{3-\sqrt{5}}{2}a, b\right)$, $\left(a, \frac{3-\sqrt{5}}{2}b\right)$ with respect to the transformation G_2 are

$$\begin{aligned}
A_1 : &= (0,0), \\
A_2 : &= \left(\frac{1}{2\sqrt{\lambda_{\max}}}\left(\frac{3-\sqrt{5}}{2}a + b\right), \frac{1}{2\sqrt{|\lambda_{\min}|}}\left(b - \frac{3-\sqrt{5}}{2}a\right)\right), \\
A_3 : &= \left(\frac{1}{2\sqrt{\lambda_{\max}}}\left(a + \frac{3-\sqrt{5}}{2}b\right), \frac{1}{2\sqrt{|\lambda_{\min}|}}\left(\frac{3-\sqrt{5}}{2}b - a\right)\right),
\end{aligned} \tag{84}$$

respectively.

Calculating the lengths of the vectors defined by these vertices, we obtain

$$
\begin{aligned}
d_1^2 &= \|\overrightarrow{A_1A_2}\|_2^2 = \frac{1}{4\lambda_{\max}}\left(\frac{3-\sqrt{5}}{2}a+b\right)^2 + \frac{1}{4|\lambda_{\min}|}\left(b-\frac{3-\sqrt{5}}{2}a\right)^2, \\
d_2^2 &= \|\overrightarrow{A_1A_3}\|_2^2 = \frac{1}{4\lambda_{\max}}\left(a+\frac{3-\sqrt{5}}{2}b\right)^2 + \frac{1}{4|\lambda_{\min}|}\left(\frac{3-\sqrt{5}}{2}b-a\right)^2, \\
d_3^2 &= \|\overrightarrow{A_2A_3}\|_2^2 = \left(\frac{\sqrt{5}-1}{2}\right)^2\left(\frac{1}{4\lambda_{\max}}(a-b)^2+\frac{1}{4|\lambda_{\min}|}(a+b)^2\right).
\end{aligned}
\tag{85}
$$

For the area of the triangle $G_2^{-1}\bar{T}$ we have

$$
2S = (\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}) = \frac{1}{4\lambda_{\max}}\left(\frac{3-\sqrt{5}}{2}a+b\right)\left(a+\frac{3-\sqrt{5}}{2}b\right)
$$

$$
+\frac{1}{4|\lambda_{\min}|}\left(b-\frac{3-\sqrt{5}}{2}a\right)\left(\frac{3-\sqrt{5}}{2}b-a\right).
$$

The following two cases are possible:

Case 1. $\lambda_{\max} < |\lambda_{\min}|$

Case 2. $|\lambda_{\min}| < \lambda_{\max}$

Observe that because the Hessian $AB - C^2$ is bounded away from zero, both $\lambda_{\max}$ and $|\lambda_{\min}|$ cannot be small.

Case 1. Let us take

$$
a = b = \sqrt{\lambda_{\max}}.
$$

For these values of parameters we have

$$
2S \geq \frac{21}{16}\left(\frac{\sqrt{5}-1}{2}\right)^2, \quad d_1^2 \leq \frac{13}{2}\left(\frac{\sqrt{5}-1}{2}\right)^2, \quad d_2^2 \leq \frac{13}{2}\left(\frac{\sqrt{5}-1}{2}\right)^2, \quad d_3^2 \leq \left(\frac{\sqrt{5}-1}{2}\right)^2.
$$

Hence, the corresponding ratios can be bounded from above as follows

$$
\frac{d_1^2}{2S} \leq \frac{13}{168}, \qquad \frac{d_2^2}{2S} \leq \frac{13}{168}, \qquad \frac{d_3^2}{2S} \leq \frac{16}{21}.
$$

Therefore, under the conditions of the Case 1 we found values of parameters a and b such that all ratios of the lengths of each side to the square root of the area of the triangle are bounded. Case 2 can be considered analogously. □

II.5.2 The error of linear interpolation of C^2 functions with negative Hessian. Estimate from above.

Proof of Theorem 5: Let $\varepsilon > 0$ be fixed and m_N be defined as in (50). Recall that we showed in Section II.4.1 that $m_N = o(\sqrt{N})$ as $N \to \infty$. As before, without loss of generality we may assume $f_{xx}(x, y) \geq 0$ for all $(x, y) \in D$.

As in Section II.4.1 we shall divide the unit square $[0, 1] \times [0, 1]$ into squares with side length equal to $\frac{1}{m_N}$ and denote the resulting squares by D_i^N, $i = 1, \ldots, m_N^2$. Next we shall choose the center point (x_i^N, y_i^N) in each square D_i^N and set

$$A_i^N := \frac{1}{2} f_{xx}(x_i^N, y_i^N), \quad B_i^N := \frac{1}{2} f_{yy}(x_i^N, y_i^N), \quad C_i^N := f_{xy}(x_i^N, y_i^N).$$

Note that in this case

$$H(x_i^N, y_i^N) := H(f; x_i^N, y_i^N) = 4(A_i^N B_i^N - (C_i^N)^2) \leq C^-, \quad \forall i = 1, \ldots, m_N^2. \tag{86}$$

Set

$$n_i^N := \left\lfloor \frac{N(1-\varepsilon)\sqrt{|H(x_i^N, y_i^N)|}\Omega(x_i^N, y_i^N)}{\sum_{j=1}^{m_N^2} \sqrt{|H(x_j^N, y_j^N)|}\Omega(x_j^N, y_j^N)} \right\rfloor, \quad i = 1, ..., m_N^2. \tag{87}$$

We find n_i, $i = 1, ..., m_N^2$, by minimizing the overall $L_{\infty,\Omega}$-error on D, i.e. by equating errors on all regions D_i^N under additional condition that the total number of knots is N:

$$\frac{1}{\sqrt{5}} \frac{\sqrt{|H(x_i^N, y_i^N)|}}{m_N^2 \tilde{n}_i^N} \Omega(x_i^N, y_i^N) = \frac{1}{\sqrt{5}} \frac{\sqrt{|H(x_1^N, y_1^N)|}}{m_N^2 \tilde{n}_1^N} \Omega(x_1^N, y_1^N).$$

From this condition we can find $\tilde{n}_i^N$:

$$\tilde{n}_i^N = n_1^N \frac{\sqrt{|H(x_i^N, y_i^N)|}\Omega(x_i^N, y_i^N)}{\sqrt{|H(x_1^N, y_1^N)|}\Omega(x_1^N, y_1^N)}.$$

From the condition

$$\sum_{i=1}^{m_N^2} \tilde{n}_i^N = N$$

we have

$$N = \frac{\tilde{n}_1^N}{\sqrt{|H(x_1^N, y_1^N)|\Omega(x_1^N, y_1^N)}} \sum_{i=1}^{m_N^2} \sqrt{|H(x_i^N, y_i^N)|\Omega(x_i^N, y_i^N)}.$$

After solving this equation for $\tilde{n}_1^N$, then substituting the result in the expression for $\tilde{n}_i^N$, and finally taking the integer part we obtain the formula for n_i^N.

We can estimate (87) similary to (54) in the proof of Lemma 2 which will imply that all $n_i^N \to \infty$ as $N \to \infty$ because of (86).

Given n_i^N for each region D_i^N we construct a triangulation $\triangle_N^*(D_i^N)$ of D_i^N in the following way:

1. Given the quadratic function

$$A_i^N x^2 + 2C_i^N xy + B_i^N y^2 \tag{88}$$

   with $A_i^N B_i^N - (C_i^N)^2 \leq 0$ on D_i^N, let us consider transformations $F_{1,i}^N$ and $G_{2,i}^N$ of types (42) and (74) respectively, corresponding to the quadratic form $Q_i^N(x, y)$.

2. Take a triangle T with vertices as in (80) where parameters a and b are chosen depending on the quadratic form $Q_i^N(x, y)$ (see Lemma 14):

$$a = b = \min \left\{ \sqrt{\lambda_{i,\max}^N}, \sqrt{|\lambda_{i,\min}^N|} \right\},$$

   $\lambda_{i,\max}^N$ and $\lambda_{i,\min}^N$ are eigenvalues of $Q_i^N(x, y)$.

3. Define T_i^N to be a rescaling of $((F_{1,i}^N)^{-1} \circ (G_{2,i}^N)^{-1})T$ so that

$$|T_i^N| = \frac{1}{m_N^2 n_i^N}.$$

4. Let $\tilde{T}_i^N$ be a triangle symmetric to T_i^N with respect to the midpoint of any side. The union of T_i^N and $\tilde{T}_i^N$ is a parallelogram.

5. Cover the square D_i^N with shifts of this parallelogram. We obtain the cover of D_i^N by the shifts of the triangle T_i^N and its reflection $\tilde{T}_i^N$.

6. If the intersection of D_i^N and a triangle T from this cover is a triangle, denote it $\tilde{T}$ and include in $\triangle_N^*(D_i^N)$.

   If the intersection of D_i^N and a triangle T is a quadrilateral, subdivide it into triangles without adding new vertices and include those triangles in $\triangle_N^*(D_i^N)$.

Finally, the triangulation $\triangle_N^*(D)$ of the whole domain D is obtained by "gluing" together the partitions $\triangle_N^*(D_i^N)$ of the subdomains D_i^N without adding new vertices as was described before.

Because of Lemma 14 we have

$$\operatorname{diam}(T_i^N) \leq c_8\sqrt{\frac{1}{m_N^2 n_i^N}}. \tag{89}$$

Considering a $c_8\sqrt{\frac{1}{m_N^2 n_i^N}}$-neighborhood of the boundary of D_i^N and counting the number of triangles in it, we can show, as in II.4.1, that the number of triangles that have nonempty intersection with the boundary does not exceed $c_9\sqrt{n_i^N}$.

Hence, in this case as well as before, the total number of triangles that have nonempty intersection with the boundary will be $o(N)$ as $N \to \infty$.

Therefore, the total number of triangles in the constructed triangulation will not exceed N for N large enough. This fact together with (89) shows that the constructed triangulation satisfies condition (22) of Theorem 5.

Let f_N denote the piecewise quadratic function constructed in the following way. On D_1^N we set f_N to be $A_1^N x^2 + 2C_1^N xy + B_1^N y^2$. Then for $i = 2, \ldots, m_N^2$ on $D_i^N \setminus \cup_{j=1}^{i-1} D_j^N$ we set

$$f_N(x,y) := A_i^N x^2 + 2C_i^N xy + B_i^N y^2.$$

We observe that as in the proof of Lemma 2

$$\|f - s(f, \triangle_N^*)\|_{\infty,\Omega} \leq 2\|f - f_N\|_{\infty,\Omega} + \|f_N - s(f_N, \triangle_N^*)\|_{\infty,\Omega}.$$

Let us estimate each term. First of all, by Lemma 1 and the definition of m_N we have

$$\|f - f_N\|_{\infty,\Omega} \leq \frac{1}{2m_N^2}\omega\left(\frac{1}{2m_N}\right)\|\Omega\|_\infty \leq \frac{\varepsilon}{N}\|\Omega\|_\infty.$$

As for the second term, it is clear that for two embedded triangles the error of linear interpolation by quadratic function with $H(f;x,y) \leq C^- < 0$ is greater on the larger triangle. Therefore, it is enough to estimate this error on triangles that do not have common points with the boundary.

By Lemma 13 and the definition of n_i^N, for every triangle $T_i^N \in \triangle_N^*(D_i^N)$ that has an empty intersection with the boundary and for all large enough N we have

$$|f_N(x,y) - s(f_N, \triangle_N^*; x, y)|\Omega(x,y) \leq \|f_N(x,y) - s(f_N, \triangle_N^*)\|_{L_{\infty,\Omega}(T_i^N)}$$

$$= \frac{1}{2\sqrt{5}} \frac{\sqrt{|H(x_i^N, y_i^N)|}\Omega(x_i^N, y_i^N)}{m_N^2 n_i^N}$$

$$\leq \frac{1}{2\sqrt{5}} \frac{1}{m_N^2} \sqrt{|H(x_i^N, y_i^N)|}\Omega(x_i^N, y_i^N)(1+\varepsilon) \frac{\sum_{j=1}^{m_N^2} \sqrt{|H(x_j^N, y_j^N)|}\Omega(x_j^N, y_j^N)}{N(1-\varepsilon)\sqrt{|H(x_i^N, y_i^N)|}\Omega(x_i^N, y_i^N)}$$

$$= \frac{1+\varepsilon}{2\sqrt{5}N(1-\varepsilon)m_N^2} \sum_{j=1}^{m_N^2} \sqrt{|H(x_j^N, y_j^N)|}\Omega(x_j^N, y_j^N).$$

Note that

$$\frac{1}{m_N^2} \sum_{j=1}^{m_N^2} \sqrt{|H(x_j^N, y_j^N)|}\Omega(x_j^N, y_j^N) = \sum_{j=1}^{m_N^2} |D_j^N| \sqrt{|H(x_j^N, y_j^N)|}\Omega(x_j^N, y_j^N)$$

$$\to \int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy$$

as $N \to \infty$.

Hence, for all N large enough we obtain

$$\|f_N - s(f_N, \triangle_N^*)\|_{\infty,\Omega} < \frac{1}{2\sqrt{5}N} \frac{1+2\varepsilon}{1-\varepsilon} \int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy.$$

Therefore,

$$\|f - s(f, \triangle_N^*)\|_{\infty,\Omega} < \frac{2\varepsilon}{N}\|\Omega\|_\infty + \frac{1}{2\sqrt{5}N}\frac{1+2\varepsilon}{1-\varepsilon}\int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy.$$

Because $\varepsilon > 0$ is arbitrary we obtain the desired estimate

$$\limsup_{N\to\infty} \frac{\|f - s(f, \triangle_N^*)\|_{\infty,\Omega}}{\frac{1}{N}\frac{1}{2\sqrt{5}}\int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy} \le 1$$

which completes the proof. □

II.5.3 The error of linear interpolation of C^2 functions with negative Hessian. Estimate from below.

In this section all quantities m_N, D_i^N, n_i^N, A_i^N etc. are as defined in the previous sections.

Proof of Theorem 6: To prove Theorem 6 we shall consider an arbitrary sequence of triangulations $\{\triangle_N\}_{N=1}^\infty$ which satisfies (22).

Assumption (22) insures that for an arbitrary fixed $\varepsilon \in (0,1)$ and for large enough N there exists a triangle $T_{i_N}^N$ from the triangulation $\triangle_N$ which lies completely in $D_{i_N}^N$ with area greater than $(1-\varepsilon)\frac{1}{m_N^2 n_{i_N}^N}$ which lies completely in $D_{i_N}^N$.

Now for each such N and i_N let

$$f_{N,i_N}(x,y) := A_{i_N}^N x^2 + 2C_{i_N}^N xy + B_{i_N}^N y^2.$$

Observe that

$$\|f - s(f, \triangle_N)\|_{L_{\infty,\Omega}(T_{i_N}^N)} \tag{90}$$

$$\ge \|f_{N,i_N} - s(f_{N,i_N}, \triangle_N)\|_{L_{\infty,\Omega}(T_{i_N}^N)} - 2\|f - f_{N,i_N}\|_{L_{\infty,\Omega}(T_{i_N}^N)}.$$

By Lemma 13, we have

$$\|f_{N,i_N} - s(f_{N,i_N}, \triangle_N)\|_{L_{\infty,\Omega}(T_{i_N}^N)} \ge \frac{1}{2\sqrt{5}}\frac{\sqrt{|H(x_{i_N}^N, y_{i_N}^N)|}\Omega(x_{i_N}^N, y_{i_N}^N)}{m_N^2 n_{i_N}^N}(1-\varepsilon).$$

Clearly, since $|H(f;x,y)|$ is an integrable function, we have as before

$$\frac{\sqrt{|H(x_{i_N}^N,y_{i_N}^N)|}\Omega(x_{i_N}^N,y_{i_N}^N)}{m_N^2 n_{i_N}^N} \geq \frac{\sqrt{|H(x_{i_N}^N,y_{i_N}^N)|}\Omega(x_{i_N}^N,y_{i_N}^N)}{m_N^2} \frac{\sum_{j=1}^{m_N^2}\sqrt{|H(x_j^N,y_j^N)|}\Omega(x_{j_N}^N,y_{j_N}^N)}{N(1-\varepsilon)\sqrt{|H(x_{i_N}^N,y_{i_N}^N)|}\Omega(x_{i_N}^N,y_{i_N}^N)}$$

$$= \frac{1}{m_N^2 N(1-\varepsilon)}\sum_{j=1}^{m_N^2}\sqrt{|H(x_j^N,y_j^N)|}\Omega(x_{j_N}^N,y_{j_N}^N) \geq \frac{1}{N}\int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy$$

as $N\to\infty$. Therefore, for all large enough N

$$\left(\frac{\sqrt{|H(x_{i_N}^N,y_{i_N}^N)|}\Omega(x_{i_N}^N,y_{i_N}^N)}{m_N^2 n_{i_N}^N}\right) / \left(\frac{1}{N}\int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy\right) > 1.$$

Hence, for all large enough N we obtain

$$\|f_{N,i_N} - s(f_{N,i_N},\triangle_N)\|_{L_{\infty,\Omega}(T_{i_N}^N)} \geq (1-\varepsilon)\frac{1}{2\sqrt{5}N}\int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy.$$

On the other hand

$$\|f - f_{N,i_N}\|_{L_{\infty,\Omega}(T_{i_N}^N)} \leq \|f - f_{N,i_N}\|_{L_{\infty,\Omega}(D_{i_N}^N)} \leq \frac{1}{2m_N^2}\omega\left(\frac{1}{m_N}\right)\|\Omega\|_\infty \leq \frac{\varepsilon}{N}\|\Omega\|_\infty$$

due to the choice of m_N. Hence, combining these two estimates with (90) we obtain that for all large enough N

$$\|f - s(f,\triangle_N)\|_{\infty,\Omega} \geq (1-c_{10}\varepsilon)\frac{1}{2\sqrt{5}N}\int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy.$$

Therefore,

$$\liminf_{N\to\infty}\frac{\|f - s(f,\triangle_N)\|_{\infty,\Omega}}{\frac{1}{N}\frac{1}{2\sqrt{5}}\int_D \sqrt{|H(f;x,y)|}\Omega(x,y)dxdy} \geq 1,$$

which completes the proof of Theorem 6. □

II.6 The L_∞-error of interpolation of C^2 functions with positive Hessian.

In this section all quantities m_N, D_i^N, n_i^N, A_i^N etc. are as defined in the Section II.4.1.

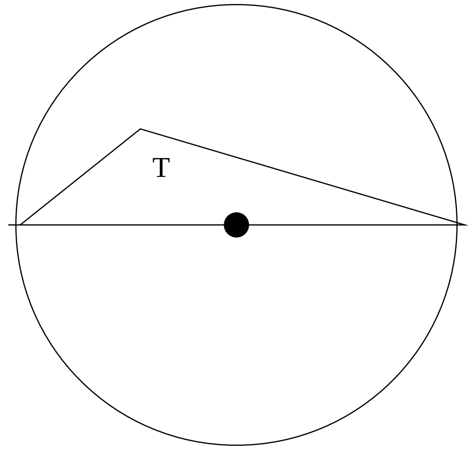

Figure 5: Chebyshev circle of the triangle T

II.6.1 Calculation of C_∞^+.

Lemma 15. *For the function*

$$Q(x,y) = x^2 + y^2,$$

and an arbitrary triangle T

$$\frac{d_{Q,T,\infty}}{|T|} \geq \frac{4}{3\sqrt{3}}. \tag{91}$$

Equality occurs only in the case when T *is an arbitrary equilateral triangle.*

Remark. This lemma implies that

$$C_\infty^+ = \frac{4}{3\sqrt{3}}. \tag{92}$$

Proof: Let T be an arbitrary triangle, let O be its Chebyshev center (which coincides with the center of the circumscribed circle if the center belongs to a triangle, and is the midpoint of the longest side if the center does not belong to a triangle), and let R be its Chebyshev radius. Observe that T is contained in the circle (call it S_T) centered at O with radius R. The error of linear interpolation of $Q(x,y)$ on T is equal to R^2. Clearly, the ratio in (38) will decrease if instead of T we consider the triangle which contains T and has all vertices on the circle S_T (the error does not change, but the area increases). As is well known, the equilateral triangle has the largest area among all triangles inscribed in a circle. Hence, in this case, equilateral triangles and only they solve problem (38). It is easy to calculate that the ratio of the left-hand side of (91) for an equilateral triangle is equal to $\frac{4}{3\sqrt{3}}$. □

Taking into consideration (46), note that we have proved the following lemma.

Lemma 16. *For the quadratic form*

$$Q(x, y) = Ax^2 + By^2 + 2Cxy \tag{93}$$

such that $AB - C^2 > 0$, *and for an arbitrary triangle* T *we have*

$$\frac{d_{Q,T,\infty}}{|T|} \geq \frac{4\sqrt{AB - C^2}}{3\sqrt{3}}.$$

Equality occurs for triangles $T = (F_1^{-1} \circ F_2^{-1})\bar{T}$, *where* $\bar{T}$ *is an arbitrary equilateral triangle, and only for them.*

II.6.2 Estimates of the error.

In this section we shall provide the proof of Theorem 4. The upper bound for the error is in the following lemma.

Lemma 17. *Let* $f \in C^2(D)$ *and* $H(f; x, y) \geq C^+ < 0$ *for all* $(x, y) \in D$. *Let also the positive continuous weight function* $\Omega(x, y)$ *be given. Then*

$$\limsup_{N\to\infty} N\|f - s_N(f, \triangle_N)\|_{\infty,\Omega} \leq \frac{2}{3\sqrt{3}} \int_D \sqrt{H(f; x, y)}\Omega(x, y)dxdy. \tag{94}$$

The proof of the upper bound is very similar to the proof of Theorem 5, so we shall not provide it here. However, the lower bound can be proven without any additional assumptions on the triangulations. Namely we can prove the following lemma.

Lemma 18. *Let* $f \in C^2(D)$ *and* $H(f; x, y) \geq C^+ < 0$ *for all* $(x, y) \in D$. *Let also the* *positive continuous weight function* $\Omega(x,y)$ *be given. Then*

$$\liminf_{N\to\infty} N\|f - s_N(f,\triangle_N)\|_{\infty,\Omega} \geq \frac{2}{3\sqrt{3}} \int_D \sqrt{H(f;x,y)}\Omega(x,y)dxdy. \tag{95}$$

For the proof of this lemma we shall need some auxiliary results.

Lemma 19. *If* $H(f;x,y) \geq C^+ > 0$ *for all* $(x,y) \in [0,1]\times[0,1]$, *then the second derivative of* $f \in C^2(D)$ *in any direction* $\xi = (\xi_1, \xi_2)$, $\xi_1^2 + \xi_2^2 = 1$, *is also bounded away from 0.*

Proof: For an arbitrary point (x,y) we have

$$\frac{\partial^2 f}{\partial \xi^2}(x,y) = f_{xx}(x,y)\xi_1^2 + 2f_{xy}(x,y)\xi_1\xi_2 + f_{yy}(x,y)\xi_2^2. \tag{96}$$

Let us consider the matrix of this form

$$M = M(x,y) := \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{pmatrix}$$

so that

$$\frac{\partial^2 f}{\partial \xi^2}(x,y) = (\xi, M(x,y)\xi) = (\xi, M\xi)\,.$$

As is known (see, for instance, [5]), for eigenvalues of this matrix $\lambda_{\min} < \lambda_{\max}$, we have

$$\lambda_{\max} = \max_{\xi} \frac{(\xi, M\xi)}{(\xi,\xi)},$$

and

$$\lambda_{\min} = \min_{\xi} \frac{(\xi, M\xi)}{(\xi,\xi)}.$$

Observe that coefficients of form (96) are satisfying the assumptions of Lemma 4 because of (55).

Therefore, if ξ belongs to the unit sphere $\mathbb{S}^1$ then by Lemma 4 we obtain

$$(\xi, M(x,y)\xi) \geq \lambda_{\min} \geq$$

$$\geq \frac{1}{2}(\|f_{xx}\|_\infty + \|f_{yy}\|_\infty) - \sqrt{\left(\frac{1}{2}(\|f_{xx}\|_\infty + \|f_{yy}\|_\infty)\right)^2 - C^+} > 0.$$

This implies that in any direction ξ the derivative $\frac{\partial^2 f}{\partial \xi^2}$ is bounded away from zero. $\square$

Lemma 20. *Let $f \in C^2[0,h]$ be such that $f''(x) \geq c > 0$ for all $x \in [0,h]$. Let $l(f,x)$ be a linear function which interpolates f at the end points 0 and h. Then*

$$\|f - l(f)\|_\infty \geq c\frac{h^2}{4}.$$

Proof: Let $p(x) = c\frac{x^2}{2}$. First of all, let us show that for all $x \in [0,h]$

$$l(f,x) - f(x) \geq l(p,x) - p(x).$$

Assume, to the contrary, that there exists x_0 such that

$$l(f,x_0) - f(x_0) \leq l(p,x_0) - p(x_0).$$

Consider the difference

$$\delta(x) := l(f,x) - f(x) - (l(p,x) - p(x)).$$

Clearly, we have $\delta(0) = \delta(h) = 0$ and $\delta(x_0) < 0$. By Rolle's theorem this implies that there exist points $x_1 \in [0, x_0)$ and $x_2 \in (x_0, h]$ such that $\delta'(x_1) < 0$ and $\delta'(x_2) > 0$. Therefore, there is a point $x_3 \in (x_1, x_2)$ such that $\delta''(x_3) = -f''(x_3) + c > 0$, i.e. $f''(x_3) < c$ and we obtain a contradiction with the assumption of the lemma.

Now calculating the maximum of the difference between $p(x)$ and $l(p,x)$ which interpolates $p(x)$ at 0 and h, we obtain the desired estimate. $\square$

Proof of Lemma 18: To obtain the estimate from below we shall consider an arbitrary sequence of triangulations $\{\triangle_N\}_{N=1}^\infty$.

The following two cases are possible:

Case 1. $L := \limsup_{N\to\infty} \frac{\max_i \operatorname{diam}(T_i^N)}{1/m_N} > 0.$

Case 2. $\max_i \operatorname{diam}(T_i^N) = o\left(\frac{1}{m_N}\right)$ when $N \to \infty$.

In the first case for any $0 < \varepsilon < L$ there exists a sequence $\{N_k\}$, $N_k \to \infty$ as $k \to \infty$, and a sequence of triangles $\{T_{i_k}^{N_k}\}$, $T_{i_k}^{N_k} \in \triangle(D_{N_k}^{N_k})$, such that

$$\frac{\operatorname{diam}\,(T_{i_k}^{N_k})}{1/m_{N_k}} > L - \varepsilon.$$

On every such triangle $T_{i_k}^{N_k}$ the deviation of the interpolant from the function will be not less than the deviation of the interpolant from the function on the longest side of the triangle. Taking into consideration the boundedness away from zero of the second derivative of f in any direction and using Lemma 20 we obtain that there exists a constant c_6 such that

$$\|f - s(f, \triangle_{N_k})\|_\infty \geq c_6 \frac{1}{m_{N_k}^2}.$$

Observe that

$$\frac{\|f - s(f, \triangle_{N_k})\|_\infty}{\frac{1}{N_k}} \geq c_6 \frac{N_k}{m_{N_k}^2}.$$

Recall that $\frac{N_k}{m_{N_k}^2} \to \infty$ as $k \to \infty$, hence,

$$\frac{\|f - s(f, \triangle_{N_k})\|_\infty}{\frac{1}{N_k}} \to \infty$$

as $k \to \infty$, so that triangulations satisfying the conditions of Case 1 provide a large error of interpolation and cannot be asymptotically optimal.

Next, we shall consider Case 2. Let $\varepsilon > 0$ be fixed. We shall estimate the sum of the areas of triangles in the triangulation $\triangle_N$ that have common points with the union of the boundaries of D_i^N, $i = 1, \ldots, m_N^2$.

The sum of these areas will not exceed

$$4(m_N + 1) \max_i \operatorname{diam}(T_i^N) = 4(m_N + 1) o\left(\frac{1}{m_N}\right) = o(1),$$

as $N \to \infty$.

We shall show that under the conditions of Case 2, for all large enough N there is a triangle $T_{i_N}^N$ in the region $D_{i_N}^N$ from the triangulation $\triangle_N$ with area greater than $(1-\varepsilon)\frac{1}{m_N^2 n_{i_N}^N}$. Indeed, if for some subsequence $N_k \to \infty$ all triangles in each D_i^N, $i = 1, \ldots, m_{N_k}^2$, have area that does not exceed $(1-\varepsilon)\frac{1}{m_{N_k}^2 n_i^{N_k}}$, then their sum will not exceed

$$\sum_{i=1}^{m_{N_k}^2} n_i^{N_k}(1-\varepsilon)\frac{1}{m_{N_k}^2 n_i^{N_k}} = 1-\varepsilon.$$

But this contradicts the fact that the sum of the areas of all triangles that have nonempty intersection with the union of boundaries $D_i^{N_k}$ is $o(1)$ as $N \to \infty$.

For each such N and i_N, set

$$f_{N,i_N}(x,y) := A_{i_N}^N x^2 + 2C_{i_N}^N xy + B_{i_N}^N y^2.$$

Observe that

$$\|f - s(f,\triangle_N)\|_{L_{\infty,\Omega}(T_{i_N}^N)} \geq \|f_{N,i_N} - s(f_{N,i_N},\triangle_N)\|_{L_{\infty,\Omega}(T_{i_N}^N)} - 2\|f - f_{N,i_N}\|_{L_{\infty,\Omega}(T_{i_N}^N)}.$$

By Lemma 15, we have

$$\|f_{N,i_N} - s(f_{N,i_N},\triangle_N)\|_{L_{\infty,\Omega}(T_{i_N}^N)} \geq \frac{2}{3\sqrt{3}}\frac{\sqrt{H(x_{i_N}^N, y_{i_N}^N)}\Omega(x_{i_N}^N, y_{i_N}^N)}{m_N^2 n_{i_N}^N}(1-\varepsilon).$$

By definition of $n_{i_N}^N$ we have that for all N large enough

$$\frac{\sqrt{H(x_{i_N}^N, y_{i_N}^N)}\Omega(x_{i_N}^N, y_{i_N}^N)}{m_N^2 n_{i_N}^N} \geq \frac{\sqrt{H(x_{i_N}^N, y_{i_N}^N)}}{m_N^2}\frac{\sum_{j=1}^{m_N^2}\sqrt{H(x_j^N, y_j^N)}\Omega(x_j^N, y_j^N)}{N(1-\varepsilon)\sqrt{H(x_{i_N}^N, y_{i_N}^N)}\Omega(x_{i_N}^N, y_{i_N}^N)}$$

$$= \frac{1}{m_N^2 N(1-\varepsilon)}\sum_{j=1}^{m_N^2}\sqrt{H(x_j^N, y_j^N)}\Omega(x_j^N, y_j^N) > \frac{1}{N}\int_D \sqrt{H(x,y)}\Omega(x,y)dxdy.$$

Hence, for all large enough N we obtain

$$\|f_{N,i_N} - s(f_{N,i_N},\triangle_N)\|_{L_{\infty,\Omega}(T_{i_N}^N)} \geq (1-\varepsilon)\frac{2}{3\sqrt{3}N}\int_D \sqrt{H(x,y)}\Omega(x,y)dxdy.$$

On the other hand

$$\|f - f_{N,i_N}\|_{L_{\infty,\Omega}(T^N_{i_N})} \le \|f - f_{N,i_N}\|_{L_{\infty,\Omega}(D^N_{i_N})} \le \frac{\|\Omega\|_\infty}{2m_N^2}\omega\left(\frac{1}{2m_N}\right) \le \frac{\varepsilon}{N}\|\Omega\|_\infty$$

due to the choice of m_N. Hence, we obtain that for all large enough N

$$\|f - s(f, \triangle_N)\|_{\infty,\Omega} \ge (1 - \tilde{c}\varepsilon)\frac{2}{3\sqrt{3}N}\int_D \sqrt{H(x,y)}\Omega(x,y)dxdy$$

with some positive constant $\tilde{c}$. Therefore,

$$\liminf_{N\to\infty} \frac{\|f - s(f, \triangle_N)\|_{\infty,\Omega}}{\frac{2}{3\sqrt{3}}\frac{1}{N}\int_D \sqrt{H(x,y)}\Omega(x,y)dxdy} \ge 1.$$

This completes the proof of the lemma. □

II.7 Algorithms.

In this section we shall provide an example of an algorithm for the construction of the asymptotically optimal sequence of triangulations for a spline interpolation. We shall also discuss the question of optimality of the presented algorithm.

We restrict ourselves to the case of functions with positive Hessian. Algorithms for functions with negative Hessian can be developed analogously.

There are two main questions in the problem of constructing an algorithm:

1. Given a precision ϵ, design an algorithm which provides this precision.

   In this section, in particular, we shall show why and in what sense our algorithm is optimal.

2. Given number N of elements of the parition (or number of measurements), construct an algorithm, which places these elements in an optimal way, and compute the error.

The answer to this question was already given in Sections II.4.1–II.4.2 and will be recalled in this section as well.

To develop an algorithm we have to impose some additional (not too restrictive) smoothness assumptions on the function. Namely, we need to assume that $f \in C^{2+\alpha}(D)$, $\alpha \in (0,1]$, where

$$C^{2+\alpha}(D) := \{f \in C^2(D) \ \ : \ \ \omega(t) \le K_f t^\alpha\}, \tag{97}$$

where K_f is some positive constant and the modulus of continuity $\omega(t)$ is defined in (28). Observe that for functions from this class Lemma 1 can be restated as follows.

Lemma 21. *Let $f \in C^{2+\alpha}(D)$. If $P_2(x,y)$ denotes the quadratic part of Taylor polynomial for f at the center of a square with side length equal to h, then in this square the following estimate holds:*

$$|f(x,y) - P_2(x,y)| \le 2K_f \left(\frac{h}{2}\right)^{2+\alpha}. \tag{98}$$

For simplicity we take the weight function $\Omega(x,y) \equiv 1$.

Let $\delta > 0$ and precision $\epsilon > 0$ be given. Taking δ in the place of ε in the proof of Theorem 2, we obtain after carrying out all the necessary constructions and computations (see, Section II.4.1) that for all N large enough

$$R_N(f, L_p, S_1^0(\triangle_N)) = \|f - s(f, \triangle_N^*)\|_p < \frac{2\delta}{N} K_f + \frac{K_p^+}{N}\frac{1+2\delta}{1-\delta}, \tag{99}$$

where

$$K_p^+ := \frac{C_p^+}{2}\left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}}\, dxdy\right)^{\frac{p+1}{p}}.$$

Given the precision $\epsilon > 0$ we set

$$\frac{2\delta}{N} + \frac{K_p^+}{N}\frac{1+2\delta}{1-\delta} = \epsilon. \tag{100}$$

Solving this equation for N we obtain

$$N = \frac{2K_p^+\delta + K_p^+ + 4\delta - 4\delta^2}{2\epsilon(1-\delta)} = \frac{K_p^+}{\epsilon}\left(1 + \left(\frac{2}{K_p^+} + 2\right)\delta\right) = \frac{K_p^+}{\epsilon}(1 + k\delta). \tag{101}$$

Therefore, we define the number of triangles to be

$$N_0 := \left[\frac{2K_p^+\delta + K_p^+ + 4\delta - 4\delta^2}{2\epsilon(1-\delta)}\right] + 1. \tag{102}$$

Next we choose m_{N_0} which is the linear size (the length of a side) of the initial grid for D. It is sufficient to take

$$m_{N_0} := \left[\left(\frac{N_0}{4\epsilon} \right)^{\frac{1}{3(1+\frac{1}{p})}} \right]. \tag{103}$$

Divide the unit square $[0,1]\times[0,1]$ into squares with side length equal to $\frac{1}{m_{N_0}}$ and denote the resulting squares by $D_i^{N_0}$, $i = 1, \ldots, m_{N_0}^2$. Next we take the center point $(x_i^{N_0}, y_i^{N_0})$ in each square $D_i^{N_0}$, and set

$$A_i^{N_0} := \frac{1}{2} f_{xx}(x_i^{N_0}, y_i^{N_0}), \;\; B_i^{N_0} := \frac{1}{2} f_{yy}(x_i^{N_0}, y_i^{N_0}), \;\; C_i^{N_0} := f_{xy}(x_i^{N_0}, y_i^{N_0}) \tag{104}$$

and

$$H(x_i^{N_0}, y_i^{N_0}) := H(f; x_i^{N_0}, y_i^{N_0}) = 4\left(A_i^{N_0} B_i^{N_0} - (C_i^{N_0})^2\right), \;\; \Omega_i^{N_0} := \Omega(x_i^{N_0}, y_i^{N_0}).$$

Set the number of triangles to be

$$n_i^{N_0} := \left[\frac{N_0(1-\varepsilon) H(x_i^{N_0}, y_i^{N_0})^{\frac{p}{2(p+1)}}}{\sum_{j=1}^{m_{N_0}^2} H(x_j^{N_0}, y_j^{N_0})^{\frac{p}{2(p+1)}}} \right], \quad i = 1, ..., m_{N_0}^2. \tag{105}$$

Given a number of triangles $n_i^{N_0}$ for each region $D_i^{N_0}$, $i = 1, ..., m_{N_0}^2$, we construct a triangulation of $D_i^{N_0}$ as described in the proof of Lemma 2.

By f_{N_0} denote the piecewise quadratic function constructed in the following way. On $D_1^{N_0}$ we set f_{N_0} to be $A_1^{N_0} x^2 + 2C_1^{N_0} xy + B_1^{N_0} y^2$. Then for $i > 1$ on $D_i^{N_0} \setminus \cup_{j=1}^{i-1} D_j^{N_0}$ we set

$$f_{N_0}(x, y) := A_i^{N_0} x^2 + 2C_i^{N_0} xy + B_i^{N_0} y^2.$$

By the choice of N_0 we have

$$R_N(f, L_p, S_1^0(\triangle_N)) < \epsilon,$$

and, hence, for the given precision $\epsilon > 0$ we gave an algorithms to construct a sequence of triangulations which provides this precision. This algorithm is optimal in the following sense. In (101) we observed that

$$N \approx \frac{\tilde{K}}{\epsilon}(1 + k\delta), \quad k > 0.$$

If instead we take

$$N \approx \frac{\tilde{K}}{\epsilon}(1 - l\delta), \quad l > 0,$$

then from the estimate from below (see the proof of Lemma 2) it will follow that for all $N \geq N_0$

$$R_N(f, L_p, S_1^0(\triangle_N)) \geq (1 - c_7)\frac{\tilde{K}}{\frac{\tilde{K}}{\epsilon}(1 - l\delta)} = \frac{1 - c_7\delta}{1 - l\delta}\epsilon > \epsilon$$

for $l \geq c_7$, and, therefore, we will not obtain the desired precision ϵ. This answers the question in what sense the described algorithm is optimal.

II.8 Applications to quadrature formulae.

One of the most natural applications of adaptive choice of knots is to design quadrature formulae which are optimal on the class of functions we consider and exact on a certain subset of it.

In this section we shall give a rule for computing an integral of an arbitrary function $f \in C^2(D)$, such that its Hessian $H(f; x, y)$ is positive and bounded away from zero, with positive weight $\Omega \in C(D)$, and obtain the estimate for the error. For convenience, in this section we take N to be the number of vertices.

Any triangulation generates a quadrature formula which is exact on the piecewise linear functions corresponding to this particular partition. To introduce the formula we shall first take an asymptotically optimal triangulation $\triangle_N^*$ constructed in Section II.4.1 and define the following functions. Let $l_{\triangle_N^*,i}$ be the linear spline defined on $\triangle_N^*$ which interpolates 1 at the vertex v_i^N, $i = 1, \dots, N$, and interpolates 0 at the rest of vertices. Set

$$c_{\triangle_N^*,i} = \int_D l_{\triangle_N^*,i}(x, y)\Omega(x, y)dxdy, \tag{106}$$

where $T_i^N \in \triangle_N^*$. Define the quadrature formula for a function $f \in C^2(D)$ to be

$$Q(f, \triangle_N^*) := \sum_{i=1}^{N} c_{\triangle_N^*,i} f(v_i^N). \tag{107}$$

Denote by

$$E(f, \triangle_N^*) := \left| Q(f, \triangle_N^*) - \int_D f(x,y)\Omega(x,y)dxdy \right|.$$

We have the following estimate:

$$E(f, \triangle_N^*) := \left| Q(f, \triangle_N^*) - \int_D f(x,y)\Omega(x,y)dxdy \right|$$

$$= \left| \sum_{i=1}^{N} c_{\triangle_N^*,i} f(v_i^N) - \int_D f(x,y)\Omega(x,y)dxdy \right|$$

$$= \left| \sum_{i=1}^{N} \left(\int_D l_{\triangle_N^*,i}(x,y)\Omega(x,y)dxdy \right) f(v_i^N) - \int_D f(x,y)\Omega(x,y)dxdy \right|$$

$$= \left| \int_D \left(\sum_{i=1}^{N} l_{\triangle_N^*,i}(x,y) f(v_i^N) - f(x,y) \right) \Omega(x,y)dxdy \right|$$

Observe that $\sum_{i=1}^{N} l_{\triangle_N^*,i}(x,y) f(v_i^N)$ is a spline $s_N(f)$ which interpolates f at the vertices of triangulation $\triangle_N^*$. Hence,

$$E(f, \triangle_N^*) \le \|\Omega\|_p^{1/p} R_N(f, L_{p,\Omega}, S_1^0(\triangle_N)).$$

Using Theorems 2-4, we can rewrite it in the form

$$E(f, \triangle_N^*) \le \|\Omega\|_p^{1/p} \frac{C_p^+}{2N} \left(\int_D H(f;x,y)^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy \right)^{\frac{p+1}{p}}.$$

In cases $p = \infty$, $p = 1$, and $p = 2$ we can provide explicit constants due to Lemmas 15, 6, and 8.

CHAPTER III

LINEAR SPLINES IN $\mathbb{R}^d$

Let $D = [0,1]^d$ and $f \in C^2(D)$. In this chapter we shall give the exact asymptotics of the uniform error of optimal linear spline interpolation in $\mathbb{R}^d$. More precisely, if by $S_1^{d,0}(\triangle_N)$ we denote the space of d-dimensional linear splines, then the problem is to investigate the asymptotic behavior of the optimal error $R_N(f, L_{\infty,\Omega}, S_1^{d,0}(\triangle_N))$, where $\triangle_N$ is a simplicial partition of D.

For the given function $f \in C^2(D)$ and $\mathbf{x} \in \mathbb{R}^d$ define the Hessian

$$H(f;\mathbf{x}) := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})\right), \quad i,j = 1,\ldots,d.$$

The main results of this chapter are in the following two theorems.

Theorem 7. *Let $f \in C^2(D)$, $D = [0,1]^d \subset \mathbb{R}^d$, and $H(f;\mathbf{x}) \geq C^+ > 0$ for all $\mathbf{x} \in D$. Let also positive weight function $\Omega(\mathbf{x}) \in C(D)$ be given. Then*

$$\liminf_{N\to\infty} N^{\frac{2}{d}} \|f - s(f,\triangle_N)\|_{\infty,\Omega} \leq \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d}\right)^{2/d}\left(\int_D H(f;\mathbf{x})^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}, \tag{108}$$

where Θ_d is the density of the thinnest covering of $\mathbb{R}^d$ by balls of fixed radius, and

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \tag{109}$$

is the volume of the unit ball in $\mathbb{R}^d$.

Theorem 8. *Let $f \in C^2(D)$, where $D = [0,1]^d \subset \mathbb{R}^d$, and $H(f;\mathbf{x}) \geq C^+ > 0$ for all $\mathbf{x} \in D$. Let also positive weight function $\Omega(\mathbf{x}) \in C(D)$ be given. Then for any sequence of sets of points and a corresponding sequence of Delaunay triangulations of D $\{\triangle_N\}_{N=1}^{\infty}$ satisfying*

$$\sup_N N^{\frac{1}{d}} \max_{T\in\triangle_N} \operatorname{diam}(T) < \infty \tag{110}$$

we have

$$\liminf_{N\to\infty} N^{\frac{2}{d}}\|f-s(f,\triangle_N)\|_{\infty,\Omega} \geq \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d}\right)^{2/d}\left(\int_D H(f;\mathbf{x})^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}, \tag{111}$$

where Θ_d is the density of the thinnest covering of $\mathbb{R}^d$ by balls of fixed radius, and and κ_d is the volume of the unit ball in $\mathbb{R}^d$.

After providing the necessary background in Sections III.1 – III.3, in Section III.4 we give the upper bound for the asymptotics of the error. To describe the sequence of asymptotically optimal simplicial partitions we use the thinnest coverings of $\mathbb{R}^d$ by balls of certain fixed radius and the corresponding (to centers of balls providing the thinnest covering) Delaunay triangulations. In Section III.6 we discuss the lower bound. Since for the cases $d = 2, 3, 4, 5$ the lattice covering density is known, in III.8 we restrict ourselves to considering the thinnest covering only by the lattice arrangements of balls, which enables us to provide explicit constants in certain cases.

III.1 An estimate for the deviation of the second degree Taylor polynomial for C^2 functions defined on $[0,1]^d$.

Let us define the modulus of continuity of $f \in C^2(D)$, $D \in \mathbb{R}^d$, as

$$\omega(f,\delta) := \sup\{|f(\mathbf{x}) - f(\mathbf{x}')| : \ |\mathbf{x}-\mathbf{x}'| \leq \delta, \ \mathbf{x},\mathbf{x}' \in D \subset \mathbb{R}^d\}, \tag{112}$$

where $|\mathbf{x}| := \max_{1\leq i\leq d} |x_i|$ for $\mathbf{x} \in \mathbb{R}^d$.

Set

$$\omega_{ij}(\delta) := \omega(f_{x_i x_j}, \delta), \quad i,j = 1,\dots,d,$$

and

$$\omega(\delta) := \max_{1\leq i,j\leq d}\{\omega_{ij}(\delta)\}. \tag{113}$$

Lemma 22. *Let $f \in C^2(D)$. If $P_2(\mathbf{x})$ denotes the quadratic Taylor polynomial for f at the center $\mathbf{x}_0$ of a cube $D_h \subset D$ in $\mathbb{R}^d$ with side length equal to h, then we have the following* *estimate:*

$$|f(\mathbf{x}) - P_2(\mathbf{x})| \le \frac{d^2}{2}\left(\frac{h}{2}\right)^2 \omega\left(\frac{h}{2}\right), \qquad \mathbf{x} \in D_h, \tag{114}$$

where $\omega(t)$ is defined at (113).

Proof: The Taylor formula for $f \in C^2(D)$ about point $\mathbf{x}_0 \in \mathbb{R}^d$ with remainder in the Lagrange form is

$$f(\mathbf{x}) = P_1(\mathbf{x}) + R_1(\mathbf{x}), \tag{115}$$

where $P_1(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{d} f_{x_i}(\mathbf{x})(x_i - x_{i_0})$ and

$$R_1(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} f_{x_i x_j}(\mathbf{x} + \theta(\mathbf{x} - \mathbf{x_0}))(x_i - x_{i_0})(x_j - x_{j_0}), \quad \theta \in (0,1). \tag{116}$$

We add and subtract the term

$$\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} f_{x_i x_j}(\mathbf{x})(x_i - x_{i_0})(x_j - x_{j_0})$$

to the right-hand side of (116). Then (115) can be rewritten as

$$f(\mathbf{x}) = P_2(\mathbf{x}) + R_2(\mathbf{x}), \tag{117}$$

where

$$P_2(\mathbf{x}) := P_1(\mathbf{x}) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} f_{x_i x_j}(\mathbf{x})(x_i - x_{i_0})(x_j - x_{j_0})$$

and

$$R_2(\mathbf{x}) := \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \left(f_{x_i x_j}(\mathbf{x} + \theta(\mathbf{x} - \mathbf{x_0})) - f_{x_i x_j}(\mathbf{x})\right)(x_i - x_{i_0})(x_j - x_{j_0})$$

By the triangle inequality we have

$$|R_2(\mathbf{x})| \le \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \left|f_{x_i x_j}(\mathbf{x} + \theta(\mathbf{x} - \mathbf{x_0})) - f_{x_i x_j}(\mathbf{x})\right| |x_i - x_{i_0}||x_j - x_{j_0}|.$$

In other words, on the cube centered at $\mathbf{x_0}$ with side length equal to h

$$|R_2(\mathbf{x})| \leq \frac{1}{2}\left(\frac{h}{2}\right)^2 \sum_{i=1}^{d}\sum_{j=1}^{d} \omega(|\mathbf{x}-\mathbf{x_0}|).$$

Hence, on the d-dimensional cube with side length equal to h we have the following estimate

$$|f(\mathbf{x}) - P_2(\mathbf{x})| \leq \frac{d^2}{2}\left(\frac{h}{2}\right)^2 \omega\left(\frac{h}{2}\right). \tag{118}$$

□

III.2 Necessary concepts and facts from Discrete and Computational Geometry.

Let us introduce the following notation and definitions.

Let $D = [0,1]^d$ be the unit cube in $\mathbb{R}^d$. In this section we shall take $N \in \mathbb{N}$ to be the number of vertices in the simplicial partition.

Let $\boldsymbol{F} = \{F\}$ be a family of sets, and U be a region in the space $\mathbb{R}^d$. We say that $\boldsymbol{F}$ is a *covering of* U if U is contained in the union $\cup_{F\in\boldsymbol{F}}F$ of all members of $\boldsymbol{F}$. If U is the whole Euclidean space $\mathbb{R}^d$ then a covering of U is simply called a *covering*.

If U is a bounded region, then the *density* $\Theta(\boldsymbol{F}|U)$ *of* $\boldsymbol{F}$ *relative to* U is defined as

$$\Theta(\boldsymbol{F}|U) := \frac{1}{V(U)} \sum_{F\in\boldsymbol{F}} V(F\cap U), \tag{119}$$

where $V(U)$ denotes the volume of set U. We shall also need concepts of *inner* and *outer density of* F *relative to* U:

$$\Theta(\boldsymbol{F}|U)_{inn} := \frac{1}{V(U)} \sum_{F\in\boldsymbol{F}, F\subset U} V(F\cap U),$$

$$\Theta(\boldsymbol{F}|U)_{out} := \frac{1}{V(U)} \sum_{F\in\boldsymbol{F}, F\cap U\neq\emptyset} V(F\cap U).$$

Define the *lower density* to be

$$\Theta_-(\boldsymbol{F}) := \liminf_{r\to\infty} \Theta_{inn}(\boldsymbol{F}|B(r)), \tag{120}$$

where $B(r)$ is a ball of radius r, and the *upper density* to be

$$\Theta_+(\boldsymbol{F}) := \limsup_{r\to\infty} \Theta_{out}(\boldsymbol{F}|B(r)). \tag{121}$$

If they coincide we call the common value the *density* of $\boldsymbol{F}$ and denote it by $\Theta(\boldsymbol{F})$.

The *covering density* $v(K)$ of a convex body K is defined to be

$$v(K) := \inf\{\Theta_-(C) \mid C \text{ is a covering of } \mathbb{R}^d \text{ with congruent copies of } K\}.$$

Recall that the *lattice* is the set of all integer linear combinations of a particular basis of $\mathbb{R}^d$. The *lattice arrangement* is the set of translates of a given set in $\mathbb{R}^d$ by all vectors of a lattice. The *lattice covering density* $v_L(K)$

$$v_L(K) := \inf\{\Theta_-(C) \mid C \text{ is a covering of } \mathbb{R}^d \text{ with lattice arrangements of } K\}.$$

By Θ_d denote the density of the thinnest covering of $\mathbb{R}^d$ by balls of equal radius.

By *polytope* P we understand the convex hull of a finite set of points. For a d-dimensional polytope its boundary consists of *faces* of dimension -1 (the empty set), 0 (vertices), 1 (edges), 2, ..., and $d-1$ (facets). A *d-dimensional simplex* is a d-polytope with exactly $d+1$ distinct vertices.

A *simplicial complex* Γ in $\mathbb{R}^d$ is a finite nonempty family of simplices in $\mathbb{R}^d$ such that

1. $\sigma \in \Gamma$ implies that $\tau \in \Gamma$ for every face τ of σ;

2. if $\sigma, \tau \in \Gamma$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ is a face of both σ and τ.

If the body of Γ is the polytope P, then we call Γ a *triangulation* of P.

Let S be a finite set of points. Let $R(S)$ be the maximal of all numbers R such that there exists a ball of radius R centered at a point of D, the interior of which does not contain points of the set S. Let $r(S)$ be the smallest of numbers r such that the balls of radius r centered at points of S cover D.

Lemma 23. *(see, for example, [3])*

$$R(S) = r(S).$$

Proof: Let balls of radius r centered at points of S cover D. Then for any point $M \in D$

$$\min_{M_k \in S} \rho(M, M_k) \leq r,$$

where $\rho(M, M_k)$ denotes the Euclidean distance between two points M and M_k in $\mathbb{R}^d$, i.e. $R(S) > r(S)$ is not possible. Therefore,

$$R(S) \leq r(S).$$

On the other hand, by the definition of $r(S)$, for every $\varepsilon > 0$ there exists a point $M_0 \in D$ such that

$$\min_{M_k \in S} \rho(M_0, M_k) \geq r(S) - \varepsilon.$$

This implies that the ball centered at M_0 with radius $r(S) - \varepsilon$ does not contain any points from S. Hence,

$$R(S) \geq r(S) - \varepsilon,$$

and since ε is arbitrary we obtain the desired

$$R(S) \geq r(S). \quad \square$$

III.3 Delaunay triangulations in $\mathbb{R}^d$.

We shall also need the definition of the *Delaunay triangulation* (in $\mathbb{R}^d$), since it is one of the tools used for the main construction in this chapter.

Sometimes we will refer to a d-dimensional simplex as a *triangle* in $\mathbb{R}^d$. A *triangulation*

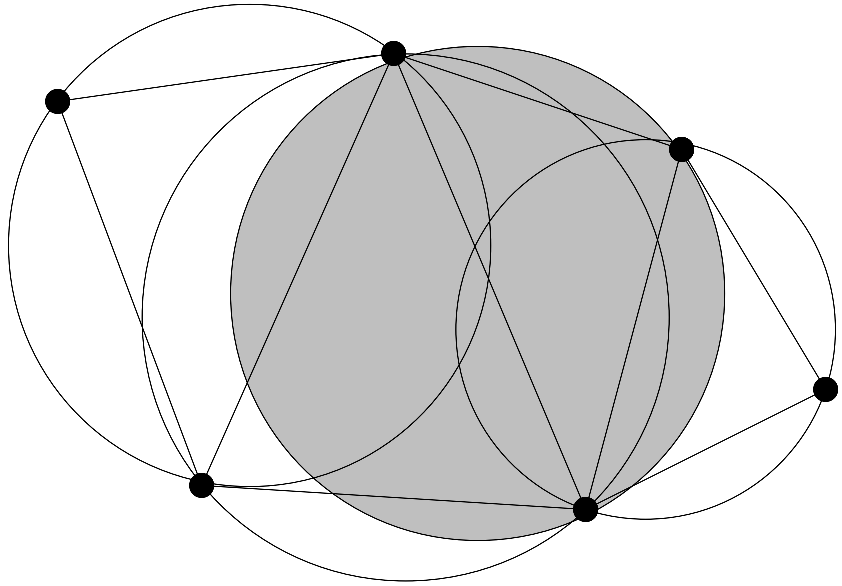

Figure 6: Delaunay triangulation in $\mathbb{R}^2$

of a set of points in $\mathbb{R}^d$ is a simplicial decomposition of the convex hull of the point set where the vertices of the tringles are contained in the point set. The *Delaunay triangulation* of a set of points in $\mathbb{R}^d$ is defined (see, for example, [26]) to be the triangulation such that the circumsphere of every triangle in the triangulation contains no point from the set in its interior (see Figure 6 for an example of a Delaunay triangulation of a set of points in $\mathbb{R}^2$). Such a triangulation exists for every point set in $\mathbb{R}^d$. The triangulation is unique if the points are in general position. (The set of sites (points) $S \subset \mathbb{R}^d$ is said to be in *general position* (or is *nondegenerate*) if no $d+2$ points lie on a common d-sphere and no $k+2$ points lie on a common k-flat, for $k < d$.)

In $\mathbb{R}^2$ Delaunay triangulations have been studied extensively (see, for example, [25, 7]). These triangulations possess many nice optimality properties. In particular, among all triangulations of a set of points in $\mathbb{R}^2$, the Delaunay triangulation lexicographically maximizes the minimum angle (Lawson), and also lexicographically minimizes the maximum circumradii.

Recall that in the Introduction we also mentioned the results on the optimality of Delaunay triangulations for the interpolation in L_p norm. In three and higher dimension very few results are known [26]. Most of the optimality properties of Delaunay triangulations do not generalize to higher dimensions. However, there is one exception: the Delaunay triangulation minimizes the maximum radius of a simplex enclosing sphere. Recall that the *enclosing sphere* (or *Chebyshev sphere*) is the smallest sphere containing a simplex. It is either the circumsphere, or the circumsphere of some face (see Figure 5).

Lemma 24. *(Rajan, [43]) The Delaunay triangulation minimizes the maximum radius of a simplex enclosing sphere (or Chebyshev radius).*

To compute the Delaunay triangulation for the given set of points several algorithms exist. In particular, using the connection with convexity, any $(d+1)$-dimensional convex hull algorithm can be used to compute a d-dimensional Delaunay triangulation. In fact, randomized incremental and gift-wrapping algorithms which are specialized convex hull algorithms, can be used.

The randomized incremental algorithm adds sites one by one, updating the Delaunay triangulation after each addition. The update consists of discovering all Delaunay faces whose circumspheres contain the new site. These faces are deleted and the empty region is partitioned into new faces, each of which has the new site as a vertex. An efficient algorithm requires a good data structure for finding the faces to be deleted. Then the running time is determined by the total number of faces updates, which depends upon site insertion order. Running time for the worst-case inputs is $O(N^{\lceil d/2 \rceil})$, and for inputs chosen uniformly at random is $O(N \log N)$.

The gift-wrapping algorithm is a specialization of the convex-hull gift-wrapping algorithm (or the graph traversal method) to Delaunay triangulations.

For an overview of these basic algorithms and their development see, for example, [26, 23].

III.4 History.

Let C_1 and C_2 be compact convex bodies in $\mathbb{R}^d$, h_{C_1} and h_{C_2} be the corresponding support functions (see, for example, [29]) on the unit sphere S^{d-1}. Define

$$\delta_H(C_1, C_2) := \|h_{C_1} - h_{C_2}\|_\infty = \sup\{|h_{C_1}(u) - h_{C_2}(u)| : u \in S^{d-1}\}. \tag{122}$$

This is called the *Hausdorff distance.* If C is a convex body with boundary (denoted by ∂C) of class C^2 for which the Gaussian curvature K_C is positive it was shown by Gruber [29] for $d > 3$ (for $d = 3$ this is the result of Fejes Toth, see Section I.3.2) that

$$\delta_H(C, P_{ins}^N) = \frac{1}{2}\left(\frac{\Theta_d}{N\kappa_d}\int_{\partial C} K_C(\mathbf{x})^{1/2} d\sigma(\mathbf{x})\right)^{\frac{2}{d}} + o\left(\frac{1}{N^{2/d}}\right), \qquad N \to \infty, \tag{123}$$

where P_{ins}^N is an iscribed polytope with N vertices, Θ_d is the density of the thinnest covering of $\mathbb{R}^d$ by balls of fixed radius, and κ_d is the volume of the unit ball in $\mathbb{R}^d$ as in (109). However, there is no algorithm to construct a sequence of inscribed polytopes which provides the asymptotically optimal error.

In general, questions of approximating the convex bodies by polytopes of various types are of special interest in convex and discrete geometry. In particular, Böröczky [9] obtained exact asymptotics of the optimal error for approximation of a convex body by *general* polytopes, i.e. polytopes that are not necessarily inscribed or circumsrbibed to the body C.

An interesting construction, somewhat similar to ours (presented later) was used by Gruber to prove a formula similar to (123) in the case of *symmetric difference metric* (also called Nikodym metric [29]). However, the obtained constants are maximal only for ellipsoids.

He considers first the approximation problem for paraboloids. His idea proceeds as follows. Given a thin covering in $\mathbb{R}^d$ with balls of different sizes, one may delete balls in such a way that the remaining balls form a packing where the total volume of the balls of the packing is not too small. For paraboloids the case of inscribed polytopes corresponds to Delaunay triangulations in $\mathbb{R}^d$. The transition from paraboloids to the convex body C is achieved by a version of Blaschke's "Schüttelung" (shaking) operator; best approximating

Table 2: Previously known results in $\mathbb{R}^d$

Author(s)	Class	Result	Norm	Const	Weight	Algor.
Dudley (1974)	convex	up. bound	Hausd.	no	no	no
Bronstein-Ivanov ('75)	convex	up. bound	Hausd.	no	no	no
Betke-Wills('79)	convex	up. bound	Hausd.	no	no	no
Schneider-Wieacker ('81)	$f \in C^3, K > 0$	asympt.	Hausd.	sharp	no	no
Gruber ('92)	$f \in C^2, K > 0$	asympt.	Hausd.	sharp	no	no
Chen ('04)	$f \in C^2, K > 0$	order	p	no	no	no

polytopes of paraboloids are transformed into polytopes which well approximate C locally and vice versa.

III.5 Linear interpolation of quadratic functions.

Let the domain be the d-dimensional unit cube $D = [0,1]^d \subset \mathbb{R}^d$.

Let P_1^d be a set of linear polynomials in d variables. In this chapter let N be the number of vertices of a simplicial partition, or simply partitions, rather than the number of simplicies in a partition.

Given a triangulation $\triangle_N$ of D define the space $S_1^{d,0}(\triangle_N)$ of d-dimensional linear splines to be

$$S_1^{d,0}(\triangle_N) := \left\{ f \in C(D) : \forall i = 1, ..., N \;\; \exists \;\; p \in P_1^d \;\text{ s.t. }\; f|_{T_i} = p|_{T_i} \right\}.$$

Let $s(f, \triangle_N)$ denote the spline from $S_1^{d,0}(\triangle_N)$ which interpolates the function $f \in C(D)$ at the vertices of the triangulation $\triangle_N$.

Let $f \in C^2(D)$ and number of vertices $N \in \mathbb{N}$ be fixed. Let M_N be a system of N points in the cube D that contains the vertices of D. Define

$$R(f, M_N) := \inf_{\triangle_N} \|f - s(f, \triangle_N)\|_\infty \tag{124}$$

and

$$R_N(f) := \inf_{M_N} R(f, M_N). \tag{125}$$

A sequence $\{\triangle_N^*\}_{N=1}^{\infty}$ of simplicial partitions of D is called asymptotically optimal if

$$\lim_{N\to\infty} \frac{\|f - s(f, \triangle_N^*)\|_\infty}{R_N(f)} = 1. \tag{126}$$

The goal of this chapter is to investigate the asymptotic behavior of $R_N(f)$ as $N \to \infty$ and to describe those sequences of simplicial partitions that provide exact asymptotics of the error.

Lemma 25. *The uniform error of linear interpolation of the quadratic form* $\sum_{i=1}^{d} A_i x_i^2$ *on a simplex* T *is equal to the square of the Chebyshev radius of this simplex in the norm*

$$\|\mathbf{u}\| = \sqrt{\sum_{i=1}^{d} \frac{u_i^2}{A_i}}, \quad \mathbf{u} \in \mathbb{R}^d. \tag{127}$$

Proof: Let $\bar{\mathbf{x}}$ be the Chebyshev center, and R_A be the Chebyshev radius of the simplex T in the norm (127). The uniform error of linear interpolation on T of the given quadratic form $\sum_{i=1}^{d} A_i x_i^2$ and of the form

$$R_A^2 - \sum_{i=1}^{d} A_i (x_i - \bar{x}_i)^2 \tag{128}$$

coincide. The value of the form (128) at the point $\bar{\mathbf{x}}$ is equal to R_A^2. The values of it at the vertices of the simplex T (since they belong to the ball $B(\bar{\mathbf{x}}, R_A)$ in the norm (127)) are nonnegative. Therefore, the values of the interpolant on the whole simplex are nonnegative. Hence, at every point of T the error does not exceed R_A^2.

On the other hand, for the Chebyshev center $\bar{\mathbf{x}}$ there exists a face of a simplex (can be the whole simplex T) that has all its vertices on the sphere $S(\bar{\mathbf{x}}, R_A)$. Therefore, the values of the interpolant at $\bar{\mathbf{x}}$ is zero. □

Lemma 26. *For every* $\epsilon > 0$ *there exist numbers* N_0 *and* l *such that for all* $k \geq N_0$ *there exists a simplicial partition* $\{T_i\}_{i=1}^{k}$ *of the cube* P *such that every simplex is contained in a* *ball of radius*

$$r_k \leq (1+\epsilon)\left(\frac{\Theta_d V[P]}{k\kappa_d l}\right)^{\frac{1}{d}}, \tag{129}$$

where Θ_d *is the density of the thinnest covering of* $\mathbb{R}^d$ *by balls of equal radius, and* κ_d *is the volume of the unit ball in* $\mathbb{R}^d$ *as in (109).*

Moreover, the number of vertices in the partition satisfies

$$N(k, N_0) \leq (1+\varepsilon)^d k l^d.$$

Proof: Let $\epsilon > 0$ and l be fixed. We shall divide P into l^d of equal cubes. It follows from the Theorem 1.10 of [45] that there exists $N_0 = N_0(\epsilon)$ such that for all $k \geq N_0$ some system of balls $B = \{b_i + B\}_{i=1}^k$, $b_i \in P$, of shifts of the ball B centered at the origin and of radius r_k will cover one of the cubes of the partition with the volume l^d. In this case, we have

$$\frac{k r_k^d}{l^d \kappa_d} \leq (1+\epsilon)^d \Theta_d.$$

To construct the needed partition $\{T_i\}_{i=1}^k$ let us consider the system of balls $\{\frac{s}{l} + b_i + B\}$, where $s \in \mathbb{Z}^d$ and $i = 1, \ldots, k$. We shall take the projections of the centers of those balls that cover P on $(d-1)$-dimensional faces of P. Clearly, there will be $k2dl^{d-1}$ projections. Let V_k be the set of those centers of the balls of the system that belong to P, obtained projections, and vertices of the cube. It is easy to see that the number of elements in V_k will not exceed

$$kl^d + 2dkl^d + 2^d.$$

Using the empty ball method (see, for example, [18]) it is easy to see that there exists a simplicial partition of P (in fact, it is going to be exactly the Delaunay triangulation of V_k) with vertices at the points from V_k (and only them) such that every simplex is contained in the ball of radius r_k.

If we now chose l from the condition

$$1 + \frac{2d}{l} + \frac{2^d}{l^d} < (1+\varepsilon)^d$$

we obtain the desired partition $\{T_i\}_{i=1}^k$. The lemma is proved. □

Lemma 27. *For the quadratic function* $Q(\mathbf{x}) = \sum_{i=1}^{d} A_i x_i^2$ *on a parallelepiped* P *with sides parallel to coordinate axes and volume* $V[P]$ *the uniform error of linear interpolation* $R_N(Q)$ *on* P *satisfies*

$$\liminf_{N\to\infty} N^{\frac{2}{d}} R_N(Q) = \left(\frac{\Theta_d V[P]\sqrt{H}}{\kappa_d}\right)^{\frac{2}{d}}, \tag{130}$$

where $H = \prod_{i=1}^{d} A_i$.

Proof: For the given quadratic form $Q(\mathbf{x}) = \sum_{i=1}^{d} A_i x_i^2$ let us consider a linear transformation F such that

$$(Q \circ F)(\mathbf{u}) = \sum_{i=1}^{d} u_i^2. \tag{131}$$

In other words,

$$F(\mathbf{u}) = \left(\frac{u_1}{\sqrt{A_1}}, \ldots, \frac{u_d}{\sqrt{A_d}}\right). \tag{132}$$

Observe that the determinant of the inverse of this transformation is

$$\det(F^{-1}) = \sqrt{\prod_{i=1}^{d} A_i}. \tag{133}$$

Let us consider the parallelepiped $F^{-1}(P)$ with the volume

$$V[F^{-1}(P)] = V[P]\det(F^{-1}). \tag{134}$$

Let us also define the radius R to be such that

$$N\kappa_d R^d \approx \Theta_d V[F^{-1}(P)]. \tag{135}$$

More rigorously, from Lemma 26 it follows that for every $\epsilon > 0$ there exists a subsequence $\{N_k\}_{k=1}^{\infty}$ and the corresponding sequence $\{R_{N_k}\}_{k=1}^{\infty}$ of radii such that

$$R_{N_k} \leq (1+\epsilon)\left(\frac{\Theta_d V[F^{-1}(P)]}{N_k \kappa_d}\right)^{1/d}. \tag{136}$$

Let us take the thinnest covering of $\mathbb{R}^d$ by balls of radius R_{N_k}. The set of centers of these balls that fall into the parallelepiped $F^{-1}(P)$, their projections on the $(d-1)$ faces, and vertices of $F^{-1}(P)$, we denote by $F^{-1}(M_{N_k})$.

By Lemma 26 there exists a simplicial partition $T_{N_k} = \{T_i\}_{i=1}^{N_k}$ of $F^{-1}(P)$ such that the uniform error of linear interpolation of $Q \circ F$ on any simplex of (and, therefore, on the whole $F^{-1}(P)$) satisfies

$$R_{N_k}(Q \circ F) \leq (1+\epsilon)^2 \left(\frac{\Theta_d V[F^{-1}(P)]}{N_k \kappa_d} \right)^{2/d}. \tag{137}$$

Note that the uniform error will not change under the linear transformation F. Hence, the uniform error of linear interpolation of Q on $F(F^{-1}(P)) = P$ is also equal to (137). In other words, we found a sequence of simplicial partitions $\{F(T_i)\}_{i=1}^{\infty}$ on which

$$\liminf_{N\to\infty} N^{\frac{2}{d}} R_N(Q) \leq (1+\epsilon) \left(\frac{\Theta_d V[P]\sqrt{H}}{\kappa_d} \right)^{\frac{2}{d}}, \tag{138}$$

and because $\epsilon > 0$ is arbitrary, the estimate from above in the lemma is proved.

Observe also that the image of the thinnest covering by balls of radius R under linear transformation F becomes the thinnest covering of $\mathbb{R}^d$ by "ellipsoids" (the density Θ_d will be the same, see Theorem 1.9 in [45]).

Let us show now that in fact for all large enough N we have

$$\liminf_{N\to\infty} N^{\frac{2}{d}} R_N(Q) \geq (1-\epsilon) \left(\frac{\Theta_d V[P]\sqrt{H}}{\kappa_d} \right)^{\frac{2}{d}}.$$

Observe that a ball in the norm defined in (127) is in fact an ellipsoid in the usual Euclidean (l_2) norm.

If in Lemma 23 we use the distance function defined by

$$\rho(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\|, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \tag{139}$$

where $\|\cdot\|$ is as in (127), then the statement of the lemma will hold for radii of the balls in the new norm, i.e. if V_{N_k} is the set of centers obtained from the thinnest covering by balls in the new metric than

$$r(V_{N_k}) = R(V_{N_k}).$$

It is not hard to see that for all N_k large enough

$$r(V_{N_k}) \geq \left(\frac{\Theta_d V[P]\sqrt{H}}{N_k \kappa_d}\right)^{\frac{1}{d}},$$

since otherwise we would have for some $\delta > 0$, N_{k_0} and $T_{N_{k_0}}$ that

$$\frac{N_{k_0} r(V_{N_{k_0}})^d}{\Theta_d V[P]\sqrt{H}} \leq (1-\delta)\Theta_d.$$

This implies that there exists a covering of $\mathbb{R}^d$ with equal balls with the density less than Θ_d which contradicts the definition of Θ_d. □

III.6 Error of linear interpolation of C^2 functions defined on $[0,1]^d$. Estimate from above.

Proof of Theorem 7: For the fixed $\varepsilon \in (0,1)$ and for every N we define

$$m_N := \min\left\{m > 0 : \ \frac{d^2}{2}\left(\frac{1}{2m}\right)^2 \omega\left(\frac{1}{2m}\right) \leq \frac{\varepsilon}{N^{\frac{2}{d}}}\right\}, \tag{140}$$

where $\omega(\delta)$ is the function defined in (113).

Observe that, clearly, for m_N defined in (140) it is true that $m_N \to \infty$ as $N \to \infty$. In addition,

$$\frac{N^{\frac{2}{d}}}{m_N^2} \to \infty, \quad N \to \infty, \tag{141}$$

i.e. $m_N = o(N^{\frac{1}{d}})$ as $N \to \infty$. Indeed, by the definition of m_N for all large enough N we have

$$\frac{N^{\frac{2}{d}}}{m_N^2} = \frac{8}{d^2}\frac{(m_N-1)^2}{m_N^2}\frac{1}{\omega\left(\frac{1}{2(m_N-1)}\right)}\frac{d^2}{8}\frac{N^{\frac{2}{d}}}{(m_N-1)^2}\omega\left(\frac{1}{2(m_N-1)}\right)$$

$$\geq \varepsilon\frac{8}{d^2}\frac{(m_N-1)^2}{m_N^2}\frac{1}{\omega\left(\frac{1}{2(m_N-1)}\right)} \to \infty, \text{ as } N \to \infty,$$

since $\left(\frac{m_N-1}{m_N}\right)^2 \to 1$ and $\omega\left(\frac{1}{2(m_N-1)}\right) \to 0$ as $N \to \infty$. Hence, (141) is proved.

Divide the unit cube D into cubes with side length equal to $\frac{1}{m_N}$ and denote the resulting cubes by D_l^N, $l = 1, \ldots, m_N^d$. Next we take the center point $\mathbf{x}_l^N$ in each cube D_l^N and set

$$A_{i,j}^{N,l} := \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_l^N), \quad i, j = 1, \ldots, d, \quad l = 1, \ldots, m_N^d.$$

Set also

$$H(\mathbf{x}_l^N) := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_l^N)\right), \quad i, j = 1, \ldots, d, \quad l = 1, \ldots, m_N^d.$$

Set

$$n_l^N := \left\lfloor \frac{N(1-\varepsilon)H(\mathbf{x}_l^N)^{\frac{1}{2}}\Omega(\mathbf{x}_l^N)^{\frac{d}{2}}}{\sum_{j=1}^{m_N^d} H(\mathbf{x}_j^N)^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}} \right\rfloor, \quad l = 1, ..., m_N^d. \tag{142}$$

We find n_l^N by minimizing the overall error on D.

Set the radius

$$R_l^N := \left(\frac{\Theta_d}{\kappa_d n_l^N m_N^d}\right)^{\frac{1}{d}}, \quad l = 1, ..., m_N^d. \tag{143}$$

Let us consider the thinnest covering of the space $\mathbb{R}^d$ by balls of radius R_l^N and system of points which are centers of this balls. Next we shall take the Delaunay triangulation $\tilde{\triangle}$ of this system of points.

As we mentioned before, to compute the d-dimensional Delaunay triangulation one can use any $(d+1)$-dimensional convex hull algorithm. For example, either randomized incremental or "gift-wrapping" algorithms can be used (for detailed description see, for example, [26], pp. 516–518).

Recall that by Lemma 24 this triangulation has the following property: it has the smallest maximum of Chebyshev radii of the obtained simplices among all possible triangulations of V_l^N.

Given a positive definite quadratic form

$$Q_l^N(\mathbf{x}) = \sum_{k=1}^{d}\sum_{j=1}^{d} A_{k,j}^{N,l} x_k x_j,$$

consider the corresponding to this form orthogonal transformation $F_{1,l}^N$ which

$$F_{1,l}^N \ : \ Q_l^N(\mathbf{x}) \ \ \rightarrow \ \ \sum_{i=1}^d \lambda_i^{N,l} x_i^2, \quad l = 1, \ldots, m_N^d,$$

where $\lambda_i^{N,l}$, $i = 1, \ldots, d$ are the eigenvalues of the form $Q_l^N(\mathbf{x})$; and the linear transformation $F_{2,l}^N$ of the form (132) which

$$F_{2,l}^N \ : \ \sum_{i=1}^d \lambda_i x_i^2 \ \ \rightarrow \ \ \sum_{i=1}^d x_i^2, \quad l = 1, \ldots, m_N^d.$$

Taking the partition $((F_{1,l}^N)^{-1} \circ (F_{2,l}^N)^{-1})\tilde{\triangle}$ we construct the desired partition $\triangle_N^*(D_l^N)$ of D_l^N in the following way. If the intersection of $((F_{1,l}^N)^{-1} \circ (F_{2,l}^N)^{-1})\tilde{\triangle}$ and D_l^N is a simplex we include it in $\triangle_N(D_l^N)$. If the intersection of $((F_{1,l}^N)^{-1} \circ (F_{2,l}^N)^{-1})\tilde{\triangle}$ and D_l^N is a nonsimplex we subdivide it without adding new vertices and resulting simplices include in $\triangle_N(D_l^N)$.

We finally obtain the needed partition of D by gluing together partitions of each D_l^N without adding new vertices. Denote by $\triangle_N^*$ the partition of D obtained in this way.

Because of (143) and because, clearly, there will be only $o(N)$ vertices on the boundary, the total number of vertices in $\triangle_N^*$ is asymptotically N.

Recall that by Lemma 27 the error of linear interpolation of the quadratic form Q_l^N on D_l^N satisfies

$$\liminf_{N\to\infty} N^{\frac{2}{d}} R_N(Q_l^N) = \left(\frac{\Theta_d \sqrt{\det |A_{k,j}^{N,l}|}}{\kappa_d n_l^N m_N^d} \right)^{\frac{2}{d}}.$$

Let f_N denote the piecewise quadratic function constructed in the following way. On D_1^N we set f_N to be $Q_1^N(\mathbf{x})$. Then for $l > 1$ on $D_l^N \setminus \cup_{j=1}^{l-1} D_j^N$ we set

$$f_N(\mathbf{x}) := Q_l^N(\mathbf{x}).$$

As before, we have

$$\|f - s(f, \triangle_N^*)\|_{\infty,\Omega} \leq 2\|f - f_N\|_{\infty,\Omega} + \|f_N - s(f_N, \triangle_N^*)\|_{\infty,\Omega}.$$

By Lemma 22 and definition (140) of m_N

$$\|f - f_N\|_{\infty,\Omega} \le \|\Omega\|_\infty \frac{d^2}{2}\left(\frac{h}{2}\right)^2 \omega\left(\frac{h}{2}\right) \le \frac{\varepsilon\|\Omega\|_\infty}{N^{\frac{2}{d}}}.$$

Let us take a simplex $T_l^N \in \triangle_N(D_l^N)$ that does not have common points with the boundary of D_l^N. By Lemma 27, for every $\mathbf{x} \in D_l^N$ we have

$$|f_N(\mathbf{x}) - s(f_N, \triangle_N^*; \mathbf{x})|\Omega(\mathbf{x}) \le \frac{1}{2}\left(\frac{\Theta_d\sqrt{H(\mathbf{x}_l^N)}}{\kappa_d m_N^d n_l^N}\right)^{2/d} \Omega(\mathbf{x}_l^N).$$

By the definition of n_l^N, for all large enough N, for all $l = 1, \dots, m_N^d$, and for all $\mathbf{x} \in T_l^N$, we have

$$|f_N(\mathbf{x}) - s(f_N, \triangle_N^*; \mathbf{x})| \le \frac{1}{2}\left(\frac{\Theta_d\sqrt{H(\mathbf{x}_l^N)}}{\kappa_d}\right)^{\frac{2}{d}} \frac{1+\varepsilon}{m_N^2}\left(\frac{\sum_{j=1}^{m_N^d} H(\mathbf{x}_j^N)^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}}{N(1-\varepsilon)H(\mathbf{x}_l^N)^{\frac{1}{2}}\Omega(\mathbf{x}_l^N)^{\frac{d}{2}}}\right)^{\frac{2}{d}} \Omega(\mathbf{x}_l^N).$$

Simplifying the last expression and taking into consideration that this estimate does not depend on $\mathbf{x}$, we obtain that

$$\|f_N - s(f_N, \triangle_N^*)\|_{\infty,\Omega} \le \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d N(1-\varepsilon)}\right)^{\frac{2}{d}} \frac{1+\varepsilon}{m_N^2}\left(\sum_{j=1}^{m_N^d} H(\mathbf{x}_j^N)^{1/2}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}\right)^{\frac{2}{d}}.$$

Note that since $H(\mathbf{x})$ is Riemann integrable

$$\frac{1}{m_N^2}\left(\sum_{j=1}^{m_N^d} H(\mathbf{x}_j^N)^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}\right)^{\frac{2}{d}} = \left(\frac{1}{m_N^d}\sum_{j=1}^{m_N^d} H(\mathbf{x}_j^N)^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}\right)^{\frac{2}{d}} \to \left(\int_D H(f;\mathbf{x})^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x}\right)^{\frac{2}{d}} \tag{144}$$

as $N \to \infty$. Hence, for all N large enough we obtain

$$\|f_N - s(f_N, \triangle_N^*)\|_{\infty,\Omega} < \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d N(1-\varepsilon)}\right)^{\frac{2}{d}} (1+2\varepsilon)\left(\int_D H(f;\mathbf{x})^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x}\right)^{\frac{2}{d}}.$$

Therefore,

$$\|f - s(f, \triangle_N^*)\|_{\infty,\Omega} < \frac{2\varepsilon\|\Omega\|_\infty}{N} + \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d N(1-\varepsilon)}\right)^{\frac{2}{d}} (1+2\varepsilon)\left(\int_D H(f;\mathbf{x})^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x}\right)^{\frac{2}{d}}.$$

Because $\varepsilon > 0$ is arbitrary, we obtain the desired estimate. □

III.7 Error of linear interpolation of C^2 functions defined on $[0,1]^d$. Estimate from below.

Let m_N. n_i^N, D_i^N etc. be as defined in the previous section.

The proof of Theorem 8: Let us consider an arbitrary sequence $\{\triangle_N\}$ of simplicial partitions of D which satisfies (110). Recall that the notation $\triangle_N$ means that the partition has N vertices. As before, by $v(\triangle_N)$ denote the set of all vertices of the simplicial partition $\triangle_N$.

Since the total number of vertices is N there exists an index l_0 such that the number n_{l_0} of vertices in the cube whose center coincides with the center of $D_{n_{l_0}}^N$ but the side length is $\frac{1-2\epsilon}{m_N}$ satisfies

$$n_{l_0} \le \frac{N(1+\varepsilon)H(\mathbf{x}_{l_0}^N)^{\frac{1}{2}}\Omega(\mathbf{x}_{l_0}^N)^{\frac{d}{2}}}{\sum_{j=1}^{m_N^d} H(\mathbf{x}_j^N)^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}}. \tag{145}$$

For the radius of the covering by balls with the centers at these n_{l_0} vertices we have for all N large enough

$$r_N \ge \left(\frac{\Theta_d H(\mathbf{x}_{l_0}^N)^{\frac{1}{2}}}{\kappa_d n_{l_0}}\right)^{\frac{1}{d}} \frac{1-\epsilon}{m_N}. \tag{146}$$

Therefore, the error on a simplex from this region, and, hence, on the whole region will be not less than

$$\|f_N - s(f_N, \triangle_N^*)\|_{L_{\infty,\Omega}(D_{n_{l_0}}^N)} \ge \frac{1}{2} r_N^2 \Omega(\mathbf{x}_{l_0}^N) \ge \frac{1}{2}\left(\frac{\Theta_d H(\mathbf{x}_{l_0}^N)^{\frac{1}{2}}\Omega(\mathbf{x}_{l_0}^N)^{\frac{d}{2}}}{\kappa_d n_{l_0}}\right)^{\frac{2}{d}} \frac{(1-\epsilon)^2}{m_N^2}$$

$$\geq \frac{1}{2}\left(\frac{\Theta_d H(\mathbf{x}_{l_0}^N)^{\frac{1}{2}}\Omega(\mathbf{x}_{l_0}^N)^{\frac{d}{2}}\sum_{j=1}^{m_N^d} H(\mathbf{x}_j^N)^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}}{\kappa_d N(1+\varepsilon)H(\mathbf{x}_{l_0}^N)^{\frac{1}{2}}\Omega(\mathbf{x}_{l_0}^N)^{\frac{d}{2}}}\right)^{\frac{2}{d}} \frac{(1-\epsilon)^2}{m_N^2}$$

$$= \frac{1}{2}\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\left(\frac{\Theta_d}{\kappa_d}\right)^{\frac{2}{d}}\frac{1}{N^{2/d}}\left(\frac{1}{m_N^d}\sum_{j=1}^{m_N^d} H(\mathbf{x}_j^N)^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}\right)^{\frac{2}{d}}$$

$$\geq \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d}\right)^{\frac{2}{d}}\left(\frac{1-2\epsilon}{1+\epsilon}\right)^2\frac{1}{N^{2/d}}\left(\frac{1}{m_N^2}\int_D H(\mathbf{x})^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}.$$

Hence for all N large enough we have

$$\|f_N - s(f_N, \triangle_N^*)\|_{\infty,\Omega} \geq \frac{1}{2}\left(\frac{\Theta_d}{\kappa_d}\right)^{\frac{2}{d}}\frac{1}{N^{2/d}}\left(\frac{1-2\epsilon}{1+\epsilon}\right)^2\left(\frac{1}{m_N^2}\int_D H(\mathbf{x})^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}.$$

In addition, recall that by Lemma 22 and definition (140) of m_N we have

$$\|f - f_N\|_{\infty,\Omega} \leq \|\Omega\|_\infty \frac{d^2}{2}\left(\frac{h}{2}\right)^2\omega\left(\frac{h}{2}\right) \leq \frac{\varepsilon\|\Omega\|_\infty}{N^{\frac{2}{d}}}.$$

Hence,

$$\|f - s(f,\triangle_N)\|_{L_\infty,\Omega} \geq \|f_{N,i_N} - s(f_{N,i_N},\triangle_N)\|_{L_\infty,\Omega} - 2\|f - f_{N,i_N}\|_{L_\infty,\Omega}$$

$$\geq \frac{1}{2}(1-\tilde{c}\epsilon)\left(\frac{\Theta_d}{\kappa_d}\right)^{\frac{2}{d}}\frac{1}{N^{2/d}}\left(\frac{1}{m_N^2}\int_D H(\mathbf{x})^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}},$$

where $\tilde{c}$ is some positive constant.

Since $\epsilon > 0$ is arbitrary, we obtain the statement of the theorem. □

III.8 Locally lattice-like systems of points.

Particularly nice, explicit construction of an asymptotically optimal sequence of partitions can be given in the case when centers of the balls from the thinnest covering are arranged in a lattice.

Let the sequence $\{M_N\}$ of sets of N points be given. We say that this system is *asymptotically locally lattice-like* if there exists a sequence $\{\alpha_N\}$ ($\alpha_N = o(N)$) such that for each N there are α_N of lattices λ_k^N and

$$\sum_{k=1}^{\alpha_N} \#(M_N \cap \lambda_k^N) = (1 + o(1))N.$$

By $M^{l.l.}$ denote the collection of all such sets, and by $\triangle_N^{l.l.}$ the set of all possible simplicial partitions of set of point from $M^{l.l.}$ with N elements.

Define

$$R_N^{l.l.}(f) := \inf_{\triangle_N^{l.l.}} \|f - s(f, \triangle_N^{l.l.})\|_{\infty,\Omega}.$$

If we consider the question of asymptotic behavior of $R_N^{l.l.}(f)$ as $N \to \infty$, we shall arrive to results similar to Theorems 7 and 8. However, instead of density Θ_d, the lattice covering density (denote it by $v_L(B^d)$) will appear in the constant. Therefore, we obtain the following result.

Theorem 9. *Let $f \in C^2(D)$, $D = [0,1]^d \subset \mathbb{R}^d$, and $H(f;\mathbf{x}) \geq C^+ > 0$ for all $\mathbf{x} \in D$. Let also positive weight function $\Omega(\mathbf{x}) \in C(D)$ be given. Then there exist a sequence of sets of points and a sequence of triangulations $\{\triangle_N^{l.l.}\}_{N=1}^{\infty}$ such that*

$$R_N^{l.l.}(f) = \frac{1}{2N^{\frac{2}{d}}} \left(\frac{v_L(B^d)}{\kappa_d}\right)^{2/d} \left(\int_D H(f;\mathbf{x})^{\frac{1}{2}} \Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x}\right)^{\frac{2}{d}} + o\left(\frac{1}{N^{2/d}}\right), \tag{147}$$

where $v_L(B^d)$ is the lattice covering density of $\mathbb{R}^d$ by balls of fixed radius, and

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}$$

is the volume of the unit ball in $\mathbb{R}^d$.

The exact value of $v_L(B^d)$ is known for $d = 2, 3, 4, 5$ (see Table 3, [26]).

For $d \geq 6$ only the upper estimate for the $v_L(B^d)$ is known:

$$v_L(B^d) \leq cd(\ln d)^{\log_2 \sqrt{2\pi e}},$$

with some suitable constant c. This estimate was proved by Rogers [45].

Table 3: Known values of lattice covering density

Dim	$v_L(B^d)$	Author
2	$\frac{2\pi}{3\sqrt{3}}$	Kershner
3	$\frac{5\sqrt{5}\pi}{24}$	Bambah
4	$\frac{2\pi^2}{5\sqrt{5}}$	Delaunay and Ryshkov
5	$\frac{245\sqrt{35}\pi^2}{3888\sqrt{3}}$	Baranovskii and Ryshkov

Coxeter, Few, and Rogers ([45], Theorem 8.1) proved a dual counterpart to Roger's bound:

$$v_L(B^d) \geq v(B^d) \geq \tau_d,$$

where τ_d is the ratio between the total volume of the intersection of $d+1$ unit balls with the regular simplex of edge $\sqrt{2(d+1)/d}$ if their centers lie at the vertices of the simplex, and the volume of the simplex. Asymptotically,

$$\tau_d \asymp \frac{d}{e^{3/2}}.$$

BILINEAR SPLINES IN $\mathbb{R}^2$

In this chapter we shall consider the question of near interpolation of a bivariate C^2 function by bilinear spline surface. As for the linear splines in the previous chapters we shall investigate the asymptotics of the error (in L_1, L_2, and L_∞ norms) of near interpolation and consider sequences of rectangular partitions which provide this error.

The only other work known to us in the direction of analysis of the asymptotic behavior of the optimal error in the case of bilinear interpolation at the points associated with a rectangle is due to D'Azevedo [15]. In the case of the uniform error he locally compared the errors of interpolation by linear splines over triangles and by bilinear splines over quadrilaterals.

IV.1 Notation and Definitions.

Let the domain D be the unit square $D = [0,1]^2$. Let $f \in C^2(D)$. Define

$$H(f;x,y) := (f_{xx}f_{yy})(x,y). \tag{148}$$

By $\square_N = \{R_i\}_{i=1}^N$ we denote any rectangular partition of the set D consisting of N elements with sides parallel to the coordinate axis (see Figure 7 for a typical partition).

Let P_2 be the set of bilinear polynomials

$$p(x,y) = axy + bx + cy + d, \quad a,b,c,d \in \mathbb{R}.$$

Given a partition $\square_N(D)$ define the space $BS_1^0(\square_N)$ of *bilinear splines* to be

$$BS_1^0(\square_N) := \{f \in C(D) : \forall i = 1,...,N \;\; \exists \;\; p \in P_2 \;\; \text{s.t.} \;\; f|_{R_i} = p|_{R_i}\}.$$

Note that the bilinear spline $s(f,\square_N)$ is uniquely defined by its values at the vertices of

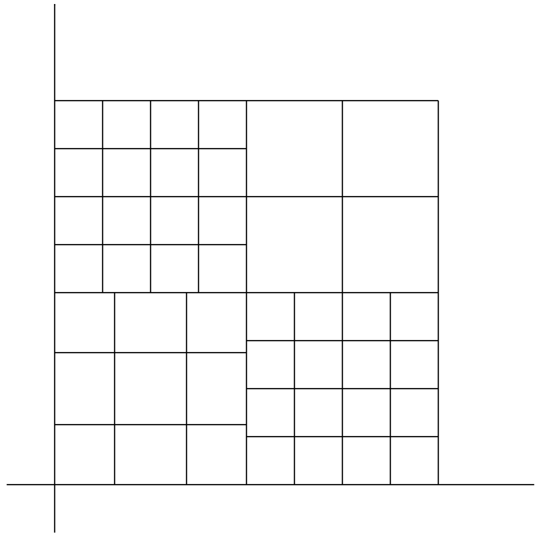

Figure 7: Type of meshes we use for bilinear near interpolation (before subdivision)

the rectangular partition.

Let $s(f, \Box_N) \in BS_1^0(\Box_N)$ denote a spline which interpolates the function f at all vertices of the partition $\Box_N$ except for $o(N)$ of them.

Now let $f \in C^2(D)$ and the number of rectangles $N \in \mathbb{N}$ be fixed. Define the error of optimal interpolation of the function f by the continuous piecewise bilinear function $s(f, \Box_N) \in BS_1^0(\Box_N)$ to be

$$R_N(f, L_{p,\Omega}, BS_1^0(\Box_N)) := \inf_{\Box_N} \|f - s(f, \Box_N)\|_{p,\Omega}, \tag{149}$$

where inf is taken over all rectangular partitions of D containing N rectangles. The main goal of this chapter is to investigate the asymptotic behaviour of the optimal error $R_N(f, L_{\infty,\Omega}, BS_1^0(\Box_N))$. The main theorems of this chapter are the following.

Theorem 10. *Let $f \in C^2(D)$ and $|H(f; x, y)| \geq C^+ > 0$ for all $(x, y) \in D$. Let also positive continuous weight function $\Omega(x, y)$ be given. Then for any $1 \leq p < \infty$*

$$\limsup_{N\to\infty} N\|f - s(f, \Box_N)\|_{p,\Omega} \leq \frac{M_p^{\pm}}{2} \left(\int_D |H(f; x, y)|^{\frac{p}{2(p+1)}} \Omega(x, y)^{\frac{1}{p+1}} dxdy \right)^{\frac{p+1}{p}}, \tag{150}$$

where

$$M_p^{\pm} = \min_R \frac{L_p - \textit{error of bilinear interpolation of } x^2 \pm y^2 \textit{ on } R}{|R|^{1+\frac{1}{p}}}. \tag{151}$$

Theorem 11. *Let $f \in C^2(D)$ and $|H(f;x,y)| \geq C^+ > 0$ for all $(x,y) \in D$. Let also positive continuous weight function $\Omega(x,y)$ be given. Then for any $1 \leq p < \infty$ and for any sequence $\{\square_N\}_{N=1}^{\infty}$ of rectangular partitions with sides parallel to the coordinate axes which satisfies*

$$\sup_N \sqrt{N} \max_{R \in \square_N} \operatorname{diam} R < \infty \tag{152}$$

we have

$$\liminf_{N\to\infty} N \|f - s(f, \square_N)\|_{p,\Omega} \geq \frac{M_p^{\pm}}{2} \left(\int_D |H(f;x,y)|^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy \right)^{\frac{p+1}{p}}, \tag{153}$$

where

$$M_p^{\pm} = \min_R \frac{L_p - \textit{error of bilinear interpolation of } x^2 \pm y^2 \textit{ on rectangle } R}{|R|^{1+\frac{1}{p}}}. \tag{154}$$

In the case of the uniform norm a similar theorem can be proved.

This theorem is a particular case of Theorem **??** in Chapter V, so we will not present a separate proof of it.

The main idea of the proofs (estimate from above) of these theorems is similar to the idea of proofs of corresponding statements for linear splines on $\mathbb{R}^2$ and proceeds as follows.

1. Divide D into a number m_N^2 (which is small in comparison with N) of equal subregions D_i^N. On each D_i^N instead of f we consider the quadratic part of its Taylor polynomial $P_{N,i}$ taken at the center of D_i^N.

2. To find an appropriate partition of D_i^N we shall use rectangles R which solve the following extremal problem:

$$\frac{L_p - \text{error of bilinear interpolation of } P_{N,i} \text{ on rectangle} R}{|R|^{1+\frac{1}{p}}} \to \min . \tag{155}$$

   Moreover, we choose the size of these rectangles in such a way that their number n_i^N

is such that the sum $\sum_{i=1}^{m_N^2} n_i^N$ is approximately N, and the errors of interpolation on each D_i^N are approximately equal.

3. We obtain the final partition of D by "gluing" together partitions of each region D_i^N possibly subdividing (without adding new vertices) rectangles which have nonempty intersection with the boundary $\cup_i \partial D_i^N$.

The chapter is organized as follows. Sections IV.4.1–IV.4.3 solve the optimization problem (155) for functions $Ax^2 + By^2$ in the cases $p = \infty, 1, 2$, respectively. Sections IV.5.1–IV.5.3 solve the optimization problem (155) for functions $Ax^2 - By^2$ in the cases $p = \infty, 1, 2$, respectively. Section IV.6 is devoted to the obtaining the estimate from above for the error $R_N(f, L_{p,\Omega})$. Section IV.7 is devoted to the obtaining the estimate from below for the error $R_N(f, L_{p,\Omega})$.

IV.2 Two approaches: near interpolation and near continuity.

For bilinear splines there are two approaches, depending on the problem.

1. If we take the "near continuity" approach, the result will be an interpolating spline which is discontinuous on $o(N)$ elements of a partition.

   In this case on every rectangle we take an interpolating spline and define it to be 0 outside of the rectangle. Taking thier average gives the interpolating but discontinuous bilinear spline.

2. In contrast to this, the "near interpolation" approach will provide a continuous spline which interpolates the original function at all but $o(N)$ vertices.

   In this case, the idea is to take a coarse partition of a square (domain), put a regular lattice on each region, refine "boundary elements", and interpolate not the original function but the spline already defined there. This way we gain continuity of the approximant but interpolate in all but $o(N)$ vertices. In this case we do have to introduce some restrictions on the rectangular partitions we work with (not to have to refine all of them).

Both methods provide asymptotically optimal error. So the difference is which we want: continuity or interpolation.

We shall demonstrate the first approach in the case of bilinear splines, and the second one in the case of biquadratic splines.

IV.3 Solution of the interpolation problem.

The fact that the problem of interpolation at four knots P_i, $i = 1, 2, 3, 4$, of the rectangular lattice is equivalent to the fact that there is no bilinear polynomial that takes zero values at these knots.

Let us assume that such a polynomial $\tilde{p}(x, y)$ exists, i.e.

$$\tilde{p}(x_i, y_i) = 0, \quad i = 1, 2, 3, 4. \tag{156}$$

Solving the equation

$$ax_iy_i + bx_i + cy_i + d = 0$$

for y_i, we obtain

$$y_i = -\frac{d + bx_i}{c + ax_i}. \tag{157}$$

This implies that the polynomial $\tilde{P}(x, y)$ annihilates at the points (x_i, y_i) if and only if (157) holds, i.e. these points lie on the hyperbola. However, we consider only rectangular lattices with sides parallel to the coordinate axis. Therefore, (157) is not possible.

IV.4 Interpolation of quadratic functions $Ax^2 + By^2$ with $AB > 0$ by bilinear splines.

Let the quadratic form

$$Q(x, y) = Ax^2 + By^2 \tag{158}$$

with $AB > 0$ be given. Observe that for interpolation by bilinear splines it is enough to consider quadratic functions of type (158), without the term "xy".

First of all, observe that the error of interpolation of quadratic functions on a rectangle does not depend on a shift of this rectangle. More precisely, we prove the following simple statement.

Lemma 28. *For the given quadratic function*

$$Q(x, y) = Ax^2 + By^2 \tag{159}$$

and an arbitrary rectangle R, the error (in any L_p norm) of bilinear interpolation on R and HR, where HR is obtained from R by the linear transformation

$$H: \quad x' = x + a, \quad y' = y + b, \tag{160}$$

is the same.

Proof: In the new coordinates function (159) has the form

$$Q(x', y') = A(x' - a)^2 + B(y' - b)^2 = A(x')^2 + B(y')^2 + L(x', y'), \tag{161}$$

where $L(x', y') = -2Aax' + Aa^2 - 2Bby' + Bb^2$ is a linear function and, hence, can be regarded as a part of an interpolant.

Since coefficients next to terms containing x and x', y and y' are the same, the errors of interpolation by bilinear splines on R and HR are the same in any metric. $\square$

We shall need a solution to the following extremal problem : minimize the ratio

$$\frac{L_p - \text{error of interpolation of } Q(x, y) \text{ on } R}{|R|^{1+\frac{1}{p}}}. \tag{162}$$

The solution to this problem in the case of $Q(x, y)$ with $AB > 0$ will be given in Section IV.4.1 (case of uniform norm), Section IV.4.2 (case of L_1 norm), and Section IV.4.3 (case of L_2 norm).

IV.4.1 The uniform norm.

Let $T_{Q,R}(x,y)$ denote the bilinear spline which interpolates the quadratic function $Q(x,y)$ at the vertices of the rectangle R. Set also

$$d_{Q,R,\infty} := \max_{(x,y)\in R} |Q(x,y) - T_{Q,R}(x,y)|, \tag{163}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R for which

$$\frac{d_{Q,R,\infty}}{|R|} \to \min, \tag{164}$$

and find the minimum of this ratio.

By Lemma 28, it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin, since any other rectangle can be brought to this by a linear transformation (shift).

The following lemma gives the answer to problem (164).

Lemma 29. *Let the quadratic form*

$$Q(x,y) = Ax^2 + By^2$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ we have

$$\frac{d_{Q,R,\infty}}{|R|} \geq \frac{1}{2}\sqrt{AB}. \tag{165}$$

Moreover, equality is obtained for rectangles with $\sqrt{A}h_1 = \sqrt{B}h_2$ and only for them.

Proof: Obviously, the interpolant to the function $Q(x,y)$ on the rectangle R is a constant equal to

$$T_{Q,R}(x,y) = Ah_1^2 + Bh_2^2.$$

In addition, note that the error in the uniform norm on the rectangle R is the same as the error on $[0, h_2] \times [0, h_1]$.

Denote the difference between function $Q(x, y)$ and interpolant $T_{Q,R}(x, y)$ by

$$\delta(x, y) := Ax^2 + By^2 - Ah_1^2 - Bh_2^2. \tag{166}$$

Clearly, the point $(0, 0)$ is a critical point of this function. The value of difference (166) at this point is

$$\delta(0, 0) = Ah_1^2 + Bh_2^2.$$

In addition, observe that on the boundary of $[0, h_2] \times [0, h_1]$ we have

$$\delta(x, h_2) = Ax^2 - Ah_1^2, \quad \text{and} \quad \delta(h_1, y) = By^2 - Bh_2^2$$

and, hence, the maximal values are

$$|\delta(0, h_2)| = Ah_1^2 \quad \text{and} \quad |\delta(h_1, 0)| = Bh_2^2.$$

Therefore, in the case of the uniform norm extremal problem (164) can be rewritten as follows. Find

$$d := \min_{h_1,h_2} \max\{Ah_1^2 + Bh_2^2, Ah_1^2, Bh_2^2\} \tag{167}$$

under the condition that the area of rectangle $[0, h_1] \times [0, h_2]$ is fixed, i.e. $h_1h_2 = S$. It is easy to see that this minimum will be attained when $\sqrt{A}h_1 = \sqrt{B}h_2$. From this fact and from the condition $h_1h_2 = S$ we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/4} \sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/4} \sqrt{S}. \tag{168}$$

Plugging this back into the expression for the error (167) gives the minimal value for the error over the rectangle of area S:

$$d = 2\sqrt{AB}S.$$

Therefore, for the error on the whole R (recall that $|R| = 4S$) we have

$$\frac{d_{Q,R,\infty}}{|R|} = \frac{1}{2}\sqrt{AB} \tag{169}$$

which gives the solution to problem (164) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_\infty^+ = \frac{1}{2}. \tag{170}$$

IV.4.2 The L_1 norm.

Let again $T_{Q,R}(x,y)$ denote the bilinear spline which interpolates the quadratic function $Q(x,y)$ at the vertices of the rectangle R. Let also

$$d_{Q,R,1} := \int_R\int |Q(x,y) - T_{Q,R}(x,y)|dxdy, \tag{171}$$

and let $|R|$ denote the area of the rectangle R. According to (162) for $p=1$, the problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,1}}{|R|^2} \to \min, \tag{172}$$

and find the minimum of it.

The following lemma gives the answer to this problem. Clearly, it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1,h_1] \times [-h_2,h_2]$ centered at the origin, since any other rectangle can be brought to this by a linear transformation (shift).

Lemma 30. *Let the quadratic form*

$$Q(x,y) = Ax^2 + By^2$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1,h_1] \times [-h_2,h_2]$ we have

$$\frac{d_{Q,R,1}}{|R|^2} \geq \frac{1}{3}\sqrt{AB}. \tag{173}$$

Moreover, equality is obtained for rectangles with $\sqrt{A}h_1 = \sqrt{B}h_2$ and only for them.

Proof: As we already mentioned, the interpolant on the rectangle R is a constant equal to

$$T_{Q,R}(x,y) = Ah_1^2 + Bh_2^2.$$

In the L_1 case, extremal problem (172) can be rewritten as follows: Find

$$d := \min_{h_1,h_2} \int_0^{h_1} \int_0^{h_2} \left| Ax^2 + By^2 - Ah_1^2 - Bh_2^2 \right| dxdy \tag{174}$$

under the condition that the area of the rectangle $[0,h_1] \times [0,h_2]$ is fixed, i.e. $h_1 h_2 = S$. We have

$$\int_0^{h_1} \int_0^{h_2} |Ax^2 + By^2 - Ah_1^2 - Bh_2^2| dxdy = \int_0^{h_1} \int_0^{h_2} (Ah_1^2 + Bh_2^2 - Ax^2 - By^2) dxdy$$

$$= \frac{2}{3}(Ah_1^2 + Bh_2^2) h_1 h_2.$$

Hence, we have to minimize the function $\frac{2}{3}(Ah_1^2 + Bh_2^2)h_1 h_2$ under the condition $h_1 h_2 = S$. Using the method of Lagrange multipliers we obtain that the minimum is attained for $\sqrt{A}h_1 = \sqrt{B}h_2$. From here and condition $h_1 h_2 = S$ we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/4} \sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/4} \sqrt{S}.$$

Plugging this back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d = \frac{4}{3}\sqrt{AB} S^2.$$

Note that $|R| = 4S$. Therefore, for the error on the whole R we have

$$\frac{d^1_{Q,R}}{|R|^2} = \frac{1}{3}\sqrt{AB} \tag{175}$$

which gives solution to (172) and completes the proof of the lemma. □

Remark. The previous lemma provides

$$M_1^+ = \frac{1}{3}. \tag{176}$$

IV.4.3 The L_2 norm.

Let again $T_{Q,R}(x,y)$ denote the bilinear spline which interpolates quadratic function $Q(x,y)$ at the vertices of the rectangle R. Let also

$$d_{Q,R,2} := \int_R \int (Q(x,y) - T_{Q,R}(x,y))^2 dxdy, \tag{177}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,2}}{|R|^{3/2}} \to \min, \tag{178}$$

and find the minimum of it.

The following lemma gives the answer to this problem (again it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin).

Lemma 31. *Let the quadratic form*

$$Q(x,y) = Ax^2 + By^2$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ we have

$$\frac{d_{Q,R,2}}{|R|^{3/2}} \geq \sqrt{\frac{11}{90}}\sqrt{AB}. \tag{179}$$

Moreover, equality is obtained for rectangles with $\sqrt{A}h_1 = \sqrt{B}h_2$ and only for them.

Proof: As we mentioned before, the interpolant on the rectangle R is a constant equal to

$$T_{Q,R}(x,y) = Ah_1^2 + Bh_2^2.$$

In the case of L_2 norm, extremal problem (178) can be rewritten as follows. Find

$$d^2 := \min_{h_1,h_2} \int_0^{h_1} \int_0^{h_2} \left(Ax^2 + By^2 - Ah_1^2 - Bh_2^2\right)^2 dxdy \tag{180}$$

under the condition that the area of rectangle $[0, h_1] \times [0, h_2]$ is fixed, i.e. $h_1 h_2 = S$. We have

$$\int_0^{h_1} \int_0^{h_2} \left(Ax^2 + By^2 - Ah_1^2 - Bh_2^2\right)^2 dxdy = h_1 h_2 \left(\frac{8}{15} B^2 h_2^4 + \frac{8}{9} AB h_1^2 h_2^2 + \frac{8}{15} A^2 h_1^4\right).$$

Hence, we have to minimize the function

$$h_1 h_2 \left(\frac{8}{15} B^2 h_2^4 + \frac{8}{9} AB h_1^2 h_2^2 + \frac{8}{15} A^2 h_1^4\right)$$

under the condition $h_1 h_2 = S$. Considering the corresponding Lagrangian and partial derivatives with respect to h_1 and h_2, we obtain that the minimum is attained for $\sqrt{A} h_1 = \sqrt{B} h_2$. From here and condition $h_1 h_2 = S$ we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/4} \sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/4} \sqrt{S}.$$

Plugging this back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d^2 = \frac{88}{45} ABS^3.$$

Therefore, for the error on the whole R (recall that $|R| = 4S$) we have

$$\frac{d_{Q,R,2}}{|R|^{3/2}} = \sqrt{\frac{11}{90}} \sqrt{AB} \tag{181}$$

which gives solution to (178) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_2^+ = \sqrt{\frac{11}{90}}. \tag{182}$$

IV.5 Interpolation of quadratic functions $Ax^2 - By^2$ with $AB > 0$ by bilinear splines.

IV.5.1 The L_∞-norm.

Let $T_{Q,R}(x, y)$ denote the bilinear spline which interpolates the quadratic function $Q(x, y)$ at the vertices of the rectangle R. Set also

$$d_{Q,R,\infty} := \max_{(x,y)\in R} |Q(x, y) - T_{Q,R}(x, y)|, \tag{183}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,\infty}}{|R|} \to \min, \tag{184}$$

and find the minimum of it.

By Lemma 28, it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin, since any other rectangle can be brought to this one by a linear transformation (shift).

The following lemma gives the answer to problem (184).

Lemma 32. *Let the quadratic form*

$$Q(x, y) = Ax^2 - By^2$$

with $AB > 0$ *be given. For an arbitrary rectangle* $R = [-h_1, h_1] \times [-h_2, h_2]$ *we have*

$$\frac{d_{Q,R,\infty}}{|R|} \geq \frac{1}{4}\sqrt{AB}. \tag{185}$$

Moreover, equality is obtained for rectangles with $\sqrt{A}h_1 = \sqrt{B}h_2$ *and only for them.*

Proof: Obviously, the interpolant to the function $Q(x, y)$ on a rectangle R is a constant equal to

$$T_{Q,R}(x, y) = Ah_1^2 - Bh_2^2.$$

Observe that the error in the uniform norm on R is the same as the error on $[0, h_2] \times [0, h_1]$.

Denote the difference between the function $Q(x, y)$ and the interpolant $T_{Q,R}(x, y)$ by

$$\delta(x, y) := Ax^2 - By^2 - Ah_1^2 + Bh_2^2. \tag{186}$$

Clearly, the point $(0, 0)$ is a critical point of this function. The value of the difference (186) at this point is

$$|\delta(0, 0)| = |Ah_1^2 - Bh_2^2|.$$

Also observe that on the boundary of $[0, h_2] \times [0, h_1]$ we have

$$\delta(x, h_2) = Ax^2 - Ah_1^2, \quad \text{and} \quad \delta(h_1, y) = By^2 - Bh_2^2$$

and, hence, maximal values are

$$|\delta(0, h_2)| = Ah_1^2 \quad \text{and} \quad |\delta(h_1, 0)| = Bh_2^2.$$

Therefore, in the case of uniform norm the extremal problem (164) can be rewritten as follows. Find

$$d := \min_{h_1, h_2} \max\{|Ah_1^2 - Bh_2^2|, Ah_1^2, Bh_2^2\} \tag{187}$$

under the condition that the area of rectangle $[0, h_1] \times [0, h_2]$ is fixed, i.e. $h_1 h_2 = S$. It is easy to see that this minimum will be attained when $\sqrt{A}h_1 = \sqrt{B}h_2$. From here and condition $h_1 h_2 = S$ we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/4} \sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/4} \sqrt{S}.$$

Plugging this back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d = \sqrt{AB}S.$$

Therefore, for the error on the whole R we have

$$\frac{d_{Q,R,\infty}}{|R|} = \frac{1}{4}\sqrt{AB} \tag{188}$$

which gives the solution to (184) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_\infty^- = \frac{1}{4}. \tag{189}$$

IV.5.2 The L_1-norm.

Let $T_{Q,R}(x,y)$ denote the bilinear spline which interpolates quadratic function $Q(x,y)$ at the vertices of the rectangle R. Let also

$$d_{Q,R,1} := \int_R\int |Q(x,y) - T_{Q,R}(x,y)|dxdy, \tag{190}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,1}}{|R|^2} \to \min \tag{191}$$

and find the minimum of it.

The following lemma gives the answer to this problem. Clearly, it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin, since any other rectangle can be brought to this one by a linear transformation (shift).

Lemma 33. *Let the quadratic form*

$$Q(x,y) = Ax^2 - By^2$$

with $AB>0$ be given. For an arbitrary rectangle $R=[-h_1,h_1]\times[-h_2,h_2]$ we have

$$\frac{d_{Q,R,1}}{|R|^2}\geq\frac{1}{12}\sqrt{AB}. \tag{192}$$

Moreover, equality is obtained for rectangles with $\sqrt{A}h_1=\sqrt{B}h_2$ and only for them.

Proof: As we already mentioned, the interpolant on a rectangle R is a constant equal to

$$T_{Q,R}(x,y)=Ah_1^2-Bh_2^2.$$

The extremal problem (191) can be rewritten as follows: find

$$d:=\min_{h_1,h_2}\int_0^{h_1}\int_0^{h_2}|Ax^2-By^2-Ah_1^2+Bh_2^2|dxdy \tag{193}$$

under the condition that the area of rectangle $[0,h_1]\times[0,h_2]$ is fixed, i.e. $h_1h_2=S$ (assume for definiteness $h_2>h_1$, another case is analogous). Set

$$r(x):=\sqrt{\frac{Ax^2-Ah_1^2+Bh_2^2}{B}}.$$

With this notation we have

$$\begin{aligned}
&\int_0^{h_1}\int_0^{h_2}\left|Ax^2-By^2-Ah_1^2+Bh_2^2\right|dydx\\
=&\int_0^{h_1}\int_0^{r(x)}\left(Ah_1^2-Bh_2^2-Ax^2+By^2\right)dydx-\int_0^{h_1}\int_{r(x)}^{h_2}\left(Ah_1^2-Bh_2^2-Ax^2+By^2\right)dydx\\
=&\int_0^{h_1}\left(h_2\left(Bh_2^2-Ah_1^2+Ax^2\right)-\frac{B}{3}h_2^3-\frac{4}{3\sqrt{B}}\left(Bh_2^2-Ah_1^2+Ax^2\right)^{3/2}\right)dx. \qquad (194)
\end{aligned}$$

Hence, we have to minimize the function

$$\int_0^{h_1}\left(h_2\left(Bh_2^2-Ah_1^2+Ax^2\right)-\frac{B}{3}h_2^3-\frac{4}{3\sqrt{B}}\left(Bh_2^2-Ah_1^2+Ax^2\right)^{3/2}\right)dx$$

under the condition $h_1h_2=S$. Clearly, the minimum of this function exists since the function is continuous on a compact set. Let us show that the minimum is obtained for $\sqrt{A}h_1=\sqrt{B}h_2$. Assume to the contrary, that the minimum is attained for the point (h_1,h_2)

such that $\sqrt{A}h_1 \neq \sqrt{B}h_2$. For this point necessary conditions of the minimum have to be satisfied. Considering the corresponding Lagrangian and setting partial derivatives with respect to h_1 and h_2 equal to 0, we have:

$$-\frac{2}{3}Bh_2^3 - 2Ah_1^2h_2 + 4\sqrt{\frac{A}{B}}h_1\left(\frac{\sqrt{AB}}{2}h_1h_2 + \frac{Bh_2^2 - Ah_1^2}{2}\ln\frac{\sqrt{A}h_1 + \sqrt{B}h_2}{\sqrt{Bh_2^2 - Ah_1^2}}\right) + \lambda h_2 = 0.$$

$$-\frac{2}{3}Ah_1^3 + 2Bh_2^2h_1 - 4\sqrt{\frac{B}{A}}h_2\left(\frac{\sqrt{AB}}{2}h_1h_2 + \frac{Bh_2^2 - Ah_1^2}{2}\ln\frac{\sqrt{A}h_1 + \sqrt{B}h_2}{\sqrt{Bh_2^2 - Ah_1^2}}\right) + \lambda h_1 = 0.$$

Multiplying the first equation by h_1, the second one by $-h_2$ and adding, we obtain after simplifying

$$(Ah_1^2 - Bh_2^2)\left(\frac{2}{3}h_1h_2 - 2\frac{(Bh_2^2 + Ah_1^2)}{\sqrt{AB}}\ln\frac{\sqrt{A}h_1 + \sqrt{B}h_2}{\sqrt{Bh_2^2 - Ah_1^2}}\right) = 0.$$

Due to the assumption $\sqrt{A}h_1 \neq \sqrt{B}h_2$ we obtain

$$\left(\frac{2}{3}h_1h_2 - 2\frac{(Bh_2^2 + Ah_1^2)}{\sqrt{AB}}\ln\frac{\sqrt{A}h_1 + \sqrt{B}h_2}{\sqrt{Bh_2^2 - Ah_1^2}}\right) = 0.$$

First of all observe that it is enough to consider the case $A = B = 1$ (due to the change of variables) and $h_1h_2 = 1$. Then the system of equations can be rewritten

$$(h_2^2 + h_1^2)\ln\frac{h_1 + h_2}{h_2 - h_1} = \frac{2}{3}$$

and $h_1h_2 = 1$. Set $h_1 = h < 1$ and, hence, $h_2 = \frac{1}{h} > 1$. Therefore,

$$\ln\frac{1 + h^2}{1 - h^2} = \frac{2h^2}{3(h^4 + 1)}.$$

Denote the left-hand side of the last equation by $f(h)$ and right-hand side by $g(h)$. Note that $f(0) = g(0) = 0$. Let us show that for all $h \in (0, 1)$

$$f'(h) > g'(h). \tag{195}$$

Indeed, calculating the derivatives we obtain

$$f'(h) = \frac{4h}{1-h^4}, \quad g'(h) = \frac{4}{3}h\frac{1-h^4}{(1+h^4)^2}.$$

Therefore, we obtain an obvious inequality and (195) holds. This means that the only solution of the system is $\sqrt{A}h_1 = \sqrt{B}h_2$ or

$$h_1 = \left(\frac{B}{A}\right)^{1/4}\sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/4}\sqrt{S}.$$

Plugging this back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d = \frac{1}{3}\sqrt{AB}S^2.$$

Note that $|R| = 4S$. Therefore, for the error on the whole R we have

$$\frac{d_{Q,R,1}}{|R|^2} = \frac{1}{12}\sqrt{AB} \tag{196}$$

which gives solution to (191) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_1^- = \frac{1}{12}. \tag{197}$$

IV.5.3 The L_2-norm.

Let again $T_{Q,R}(x,y)$ denote the bilinear spline which interpolates quadratic function $Q(x,y)$ at the vertices of the rectangle R. Let also

$$d_{Q,R,2} := \left(\int_R\int (Q(x,y) - T_{Q,R}(x,y))^2 dxdy\right)^{1/2}, \tag{198}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,2}}{|R|^{3/2}} \to \min, \tag{199}$$

and find the minimum of it.

The following lemma gives the answer to this problem (again it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin).

Lemma 34. *Let the quadratic form*

$$Q(x, y) = Ax^2 - By^2$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ we have

$$\frac{d_{Q,R,2}}{|R|^{3/2}} \geq \sqrt{\frac{1}{90}}\sqrt{AB} \tag{200}$$

Moreover, the equality is obtained for rectangles with $\sqrt{A}h_1 = \sqrt{B}h_2$ and only for them.

Proof: As we mentioned before, the interpolant on the rectangle R is a constant equal to

$$T_{Q,R}(x, y) = Ah_1^2 - Bh_2^2.$$

In the case of L_2 norm the extremal problem (199) can be rewritten as follows. Find

$$d^2 := \min_{h_1, h_2} \int_0^{h_1} \int_0^{h_2} (Ax^2 - By^2 - Ah_1^2 + Bh_2^2)^2 dxdy \tag{201}$$

under the condition that the area of rectangle $[0, h_1] \times [0, h_2]$ is fixed, i.e. $h_1 h_2 = S$. We have

$$\int_0^{h_1} \int_0^{h_2} (Ax^2 - By^2 - Ah_1^2 + Bh_2^2)^2 dxdy = h_1 h_2 \left(\frac{8}{15} B^2 h_2^4 - \frac{8}{9} ABh_1^2 h_2^2 + \frac{8}{15} A^2 h_1^4 \right).$$

Hence, we have to minimize the function

$$h_1 h_2 \left(\frac{8}{15} B^2 h_2^4 - \frac{8}{9} ABh_1^2 h_2^2 + \frac{8}{15} A^2 h_1^4 \right)$$

under the condition $h_1 h_2 = S$. Using the method of Lagrange multipliers we obtain that the minimum is attained for $\sqrt{A}h_1 = \sqrt{B}h_2$. From this and from the condition $h_1 h_2 = S$

we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/4} \sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/4} \sqrt{S}.$$

Plugging these back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d^2 = \frac{8}{45} ABS^3.$$

Therefore, for the error on the whole R (recall that $|R| = 4S$) we have

$$\frac{d_{Q,R,2}}{|R|^{3/2}} = \sqrt{\frac{1}{90}} \sqrt{AB} \tag{202}$$

which gives the solution to (199) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_2^- = \sqrt{\frac{1}{90}}. \tag{203}$$

IV.6 General form of the error in the case of near interpolation of C^2 functions defined on $[0,1]^2$ by bilinear splines. Estimate from above.

We restart numbering of the constants $c_1, c_2, \ldots$.

Proof of Theorem 10: For an arbitrary fixed $\varepsilon \in (0,1)$ and for every $N \in \mathbb{N}$ we define

$$m_N := \min\left\{ m > 0 : \ \frac{1}{2m^{2(1+\frac{1}{p})}} \omega\left(\frac{1}{2m}\right) \leq \frac{\varepsilon}{N} \right\}, \tag{204}$$

where $\omega(\delta)$ is a function defined in (28).

Observe that clearly for m_N defined in (204) it is true that $m_N \to \infty$ as $N \to \infty$. In addition, as before we can show

$$\frac{N}{m_N^{2(1+\frac{1}{p})}} \to \infty, \quad N \to \infty, \tag{205}$$

i.e. $m_N = o\left(N^{\frac{1}{2(1+\frac{1}{p})}}\right)$ as $N \to \infty$ and $\frac{1}{N} = o\left(\frac{1}{m_N^{2(1+\frac{1}{p})}}\right)$.

Divide the unit square $[0,1]\times[0,1]$ into squares with side length equal to $\frac{1}{m_N}$ and denote the resulting squares by D_i^N, $i = 1, \ldots, m_N^2$. Next we take the center point (x_i^N, y_i^N) in each square D_i^N and set

$$A_i^N := \frac{1}{2} f_{xx}(x_i^N, y_i^N), \quad B_i^N := \frac{1}{2} f_{yy}(x_i^N, y_i^N).$$

Note that

$$|H(x_i^N, y_i^N)| := 4|A_i^N B_i^N| \geq C^+, \quad \forall i = 1, \ldots, m_N^2. \tag{206}$$

Set

$$n_i^N := \left[\frac{N(1-\varepsilon)|H(x_i^N, y_i^N)|^{\frac{p}{2(p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{p+1}}}{\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}} \right], \quad i = 1, \ldots, m_N^2. \tag{207}$$

The n_i^N, $i = 1, \ldots, m_N^2$, are determined by minimizing the sum of the errors of the interpolation of piecewise quadratic functions on each region subject to the condition the total number of rectangles is N.

To find the minimum we shall use the method of Lagrange multipliers. Let us consider

$$\frac{\partial}{\partial \tilde{n}_i} \left(\sum_{i=1}^{m_N^2} \frac{|H(x_i^N, y_i^N)|^{\frac{p}{2}}}{m_N^{2(p+1)} (\tilde{n}_i^N)^{p+1}} \tilde{n}_i^N \Omega(x_i^N, y_i^N) + \lambda \sum_{i=1}^{m_N^2} \tilde{n}_i^N \right) = 0.$$

This can be rewritten as

$$-\frac{p|H(x_i^N, y_i^N)|^{\frac{p}{2}}}{m_N^{2(p+1)} (\tilde{n}_i^N)^{p+1}} \Omega(x_i^N, y_i^N) + \lambda = 0.$$

Hence,

$$\tilde{n}_i = \left(\frac{p|H(x_i^N, y_i^N)|^{p/2} \Omega(x_i^N, y_i^N)}{\lambda m_N^{2(p+1)}} \right)^{\frac{1}{p+1}}.$$

We shall find λ from the condition that the sum of all rectangles is N:

$$N = \sum_{i=1}^{m_N^2} \tilde{n}_i^N = \frac{p^{\frac{1}{p+1}}}{\lambda^{\frac{1}{p+1}} m_N^2} \sum_{i=1}^{m_N^2} |H(x_i^N, y_i^N)|^{\frac{p}{2(p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{p+1}}.$$

Solving for λ and plugging this value back into the expression for $\tilde{n}_i^N$, we obtain

$$\tilde{n}_i^N = \frac{N(1-\varepsilon)|H(x_i^N, y_i^N)|^{\frac{p}{2(p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{p+1}}}{\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}}.$$

The method of Lagrange multipliers provides only necessary conditions for the minimum. However, in this case the obtained $\tilde{n}_i^N$ indeed provides the minimum, because it clearly does not provide the maximum (the maximum can be explicitly constructed in a simple way) and the minimum exists.

Observe that all $n_i^N \to \infty$ when $N \to \infty$.

Having the number of rectangles n_i^N on each region D_i^N, $i = 1, ..., m_N^2$, we construct a partition $\Box_N^*(D_i^N)$ of D_i^N in the following way.

Let $h_{i,1}^N$ and $h_{i,2}^N$ be the linear sizes of the optimal rectangle on D_i^N (call it R_i^N), i.e. the one which solves the problem

$$\frac{L_p - \text{error of biquadratic interpolation of } \; A_i^N x^3 \pm B_i^N y^3 \; \text{ on } \; R_i^N}{|R_i^N|^{\frac{3}{2}+\frac{1}{p}}}. \tag{208}$$

Observe that due to Lemmas 29-34 we can provide the explicit expressions for $h_{i,1}^N$ and $h_{i,2}^N$ in cases $p = 1, 2, \infty$:

$$h_{i,1}^N = \left(\frac{B_i^N}{A_i^N}\right)^{1/4} \frac{1}{m_N \sqrt{n_i^N}}, \quad h_{i,2}^N = \left(\frac{A_i^N}{B_i^N}\right)^{1/4} \frac{1}{m_N \sqrt{n_i^N}}, \quad i = 1, ..., m_N^2. \tag{209}$$

Most likely these expressions will also provide optimal parameters for other values of p as well.

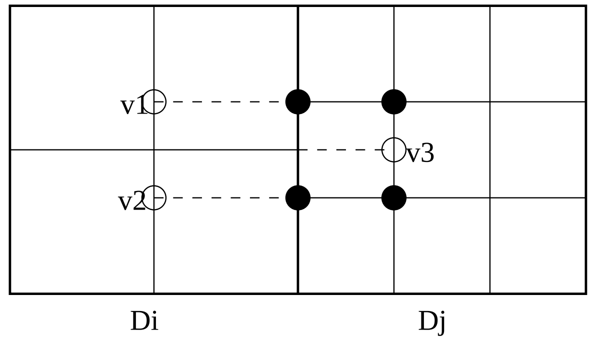

Figure 8: Stitching of partitions on neighboring regions.

The intersection of the lattice

$$[kh_{i,1}^N, (k+1)h_{i,1}^N] \times [lh_{i,2}^N, (l+1)h_{i,2}^N], \quad k, l \in \mathbb{Z} \tag{210}$$

with D_i^N gives us the partition $\square_N^*(D_i^N)$ of D_i^N, $i = 1, ..., m_N^2$.

Stitching of partitions of neighboring regions proceeds as follows.

Let us consider two neighboring regions D_i^N and D_j^N with corresponding interpolants S_i^N and S_j^N. If the parameters of the grid on them are different, we have to subdivide rectangles that have nonempty intersection with the boundary to ensure the global continuity of the approximant (see Figure 8). For that purpose we need to continue each line (horizontal in this case) of the grid from D_i^N which intersects the boundary between D_i^N and D_j^N to the intersection with the first line (vertical in this case) of the grid of D_j^N and vice versa. Newly created in this way vertices (v_1, v_2 and v_3 on the picture) we shall call "irregular" vertices, and all vertices of both partitions before subdivision we call "regular". We shall also call "irregular" those rectangles from $\square_N^*(D_i^N)$ that have sides on the boundary of D_i^N and "regular" - all the rest.

To obtain the final approximant and to ensure the global continuity of it we shall interpolate the original function f at the regular vertices, and we shall interpolate original splines S_i^N and S_j^N at irregular vertices.

Let us calculate the number of irregular rectangles (denote it by K_i^N). Recalling (313) and the fact that the area of a rectangle from $\square_N^*(D_i^N)$ is $\frac{1}{m_N^2 n_i^N}$ we have that

$$K_i^N = 2\sqrt{\frac{A_i^N}{B_i^N}}\sqrt{n_i^N} + 2\sqrt{\frac{B_i^N}{A_i^N}}\sqrt{n_i^N} = c_1\sqrt{n_i^N}$$

with $c_1 = 2\left(\sqrt{\frac{A_i^N}{B_i^N}} + \sqrt{\frac{B_i^N}{A_i^N}}\right)$. The number of irregular rectangles in the partition of D_i^N is not greater than $c_1\sqrt{n_i^N}$. Hence, the total number of irregular rectangles is not greater than

$$c_2\sum_{i=1}^{m_N^2}\sqrt{n_i^N} \le c_2\sum_{i=1}^{m_N^2}\sqrt{\frac{N(1-\varepsilon)|H(x_i^N,y_i^N)|^{\frac{p}{2(p+1)}}\Omega(x_i^N,y_i^N)^{\frac{1}{p+1}}}{\sum_{j=1}^{m_N^2} H(x_j^N,y_j^N)^{\frac{p}{2(p+1)}}\Omega(x_j^N,y_j^N)^{\frac{1}{p+1}}}}$$

$$\le c_3\sqrt{N}m_N^2\frac{\|H\|_\infty^{\frac{p}{4(p+1)}}\|\Omega\|_\infty^{\frac{1}{2(p+1)}}}{m_N(C^+)^{\frac{p}{4(p+1)}}\left(\min_{(x,y)\in D}\{\Omega(x,y)\}\right)^{\frac{1}{2(p+1)}}} \le c_4\sqrt{N}m_N = o\left(N^{\frac{3+\frac{1}{p}}{2(2+\frac{1}{p})}}\right),$$

as $N\to\infty$ (since $m_N = o\left(N^{\frac{1}{2(1+\frac{1}{p})}}\right)$ as $N\to\infty$ because of (205)).

Therefore, since $3+\frac{1}{p} \le 2\left(2+\frac{1}{p}\right)$, the number of rectangles in the constructed partition will not exceed N for all N large enough.

By f_N denote the piecewise quadratic function constructed in the following way. On D_1^N we set f_N to be $A_1^N x^2 + B_1^N y^2$. Then for $i>1$ on $D_i^N \setminus \cup_{j=1}^{i-1} D_j^N$ we set

$$f_N(x,y) := A_i^N x^2 + B_i^N y^2.$$

If a function f is such that $f_{xx}f_{yy} > 0$ for all $(x,y)\in D$ then all $B_i^N > 0$, $i=1,\ldots,m_N^2$. If a function f is such that $f_{xx}f_{yy} < 0$ for all $(x,y)\in D$ then all $B_i^N < 0$, $i=1,\ldots,m_N^2$.

To estimate $R_N(f, L_{p,\Omega}, BS_1^0(\square_N))$ we observe that

$$R_N(f, L_{p,\Omega}, BS_1^0(\square_N)) \le \|f - s(f,\square_N^*)\|_{p,\Omega} \le \|f - f_N\|_{p,\Omega} + \|f_N - s(f_N,\square_N^*)\|_{p,\Omega}$$

$$+\|s(f_N,\square_N^*) - s(f,\square_N^*)\|_{p,\Omega}.$$

Obviously,

$$\|s(f_N, \square_N^*) - s(f, \square_N^*)\|_{p,\Omega} \le \|f - f_N\|_{p,\Omega}.$$

Hence,

$$R_N(f, L_{p,\Omega}, BS_1^0(\square_N)) \le 2\|f - f_N\|_{p,\Omega} + \|f_N - s(f_N, \square_N^*)\|_{p,\Omega}.$$

Let us estimate each term. First of all, by Lemma 1 and the definition of m_N we have

$$\|f - f_N\|_{p,\Omega} \le \frac{\|\Omega\|_\infty^{\frac{1}{p}}}{2m_N^{2(1+\frac{1}{p})}} \omega\left(\frac{1}{2m_N}\right) \le \frac{\varepsilon}{N}\|\Omega\|_\infty^{\frac{1}{p}}.$$

Let us estimate the second term now. It is clear that for two embedded rectangles the error of linear interpolation of quadratic function with $|H(f;x,y)| \ge C^+ > 0$ will be greater on the larger rectangle. Therefore, we shall estimate this error on rectangles that do not have an intersection with the boundary.

Let us take a rectangle $R_i^N \in \square_N^*(D_i^N)$ that does not have common points with the boundary of D_i^N. By (47), for every $(x,y) \in R_i^N$ we have

$$|f_N(x,y) - s(f_N, \square_N^*; x, y)|^p \le \left(\frac{M_p^\pm}{2}\right)^p |H(x_i^N, y_i^N)|^{p/2} \frac{1}{(m_N^2 n_i^N)^{p+1}}.$$

Hence, the p-power of the error on the whole D is bounded by

$$|f_N(x,y) - s(f_N, \square_N^*; x, y)|^p \le \left(\frac{M_p^\pm}{2}\right)^p \sum_{i=1}^{m_N^2} n_i^N \Omega(x_i^N, y_i^N) |H(x_i^N, y_i^N)|^{p/2} \frac{1}{(m_N^2 n_i^N)^{p+1}}.$$

By the definition of n_i^N and by (54), for all large enough N, for all i, and for all $(x,y) \in D$, we have

$$|f_N(x,y) - s(f_N, \square_N^*; x, y)|^p$$

$$\le \left(\frac{M_p^\pm}{2}\right)^p \frac{(1+\varepsilon)}{m_N^{2(p+1)}} \sum_{i=1}^{m_N^2} |H(x_i^N, y_i^N)|^{p/2} \Omega(x_i^N, y_i^N) \left(\frac{\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}}{N(1-\varepsilon)|H(x_i^N, y_i^N)|^{\frac{p}{2(p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{p+1}}} \right)^p$$

$$= \left(\frac{M_p^\pm}{2}\right)^p \frac{(1+\varepsilon)}{N^p m_N^{2(p+1)}} \left(\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}\right)^{p+1}.$$

Since this estimate does not depend on x and y, we obtain that

$$\|f_N - s(f_N, \Box_N^*)\|_{p,\Omega} \le \frac{(1+\varepsilon)M_p^\pm}{2N} \left(\frac{1}{m_N^2}\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}}\right)^{\frac{p+1}{p}}.$$

Note that since $H(f; x, y)$ and $\Omega(x, y)$ are Riemann integrable

$$\frac{1}{m_N^2}\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}} \tag{211}$$

$$= \sum_{j=1}^{m_N^2} |D_j^N||H(x_j^N, y_j^N)|^{\frac{p}{2(p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{p+1}} \to \int_D |H(f; x, y)|^{\frac{p}{2(p+1)}} \Omega(x, y)^{\frac{1}{p+1}} dxdy$$

as $N \to \infty$. Hence, for all N large enough we obtain

$$\|f_N - s(f_N, \Box_N^*)\|_{p,\Omega} < \frac{M_p^\pm}{2N}\frac{1+2\varepsilon}{1-\varepsilon} \left(\int_D |H(f; x, y)|^{\frac{p}{2(p+1)}} \Omega(x, y)^{\frac{1}{p+1}} dxdy\right)^{\frac{p+1}{p}}.$$

Therefore,

$$\|f - s(f, \Box_N^*)\|_{p,\Omega} < \frac{2\varepsilon}{N}\|\Omega\|_\infty^{\frac{1}{p}} + \frac{M_p^\pm}{2N}\frac{1+2\varepsilon}{1-\varepsilon} \left(\int_D |H(f; x, y)|^{\frac{p}{2(p+1)}} \Omega(x, y)^{\frac{1}{p+1}} dxdy\right)^{\frac{p+1}{p}}.$$

Because $\varepsilon > 0$ is arbitrary, we obtain the desired estimate. □

IV.7 General form of the error in the case of near interpolation of C^2 functions defined on $[0, 1]^2$ by bilinear splines. Estimate from below.

In this section, let m_N, n_i^N, D_i^N etc. be as defined in Section IV.6.

Proof of Theorem 11: To obtain the estimate from below we shall consider an arbitrary sequence of partitions $\{\Box_N\}_{N=1}^\infty$ which satisfies (152).

For the fixed $\varepsilon > 0$ denote by $D_i^N(\varepsilon)$ the square congruent to D_i^N with side length equal to $\frac{1-\varepsilon}{m_N^2}$. Assumption (22) implies that for all large enough N and for any $\varepsilon > 0$ there exists a rectangle $R_{i_N}^N$ which lies completely in $D_i^N(\varepsilon)$.

For each such N and i_N, set

$$f_{N,i_N}(x,y) := A^N_{i_N} x^2 + B^N_{i_N} y^2.$$

If a function f is such that $f_{xx}f_{yy} > 0$ for all $(x,y) \in D$ then all $B^N_i > 0$, $i = 1, \dots, m^2_N$. If a function f is such that $f_{xx}f_{yy} < 0$ for all $(x,y) \in D$ then all $B^N_i < 0$, $i = 1, \dots, m^2_N$.

Observe that

$$\|f - s(f, \square_N)\|_{L_{p,\Omega}(R^N_{i_N})} \geq \|f_{N,i_N} - s(f_{N,i_N}, \square_N)\|_{L_{p,\Omega}(R^N_{i_N})} - 2\|f - f_{N,i_N}\|_{L_{p,\Omega}(R^N_{i_N})}.$$

By Lemmas 29–34 we have for all N large enough

$$\|f_{N,i_N} - s(f_{N,i_N}, \square_N)\|^p_{L_{p,\Omega}(R^N_{i_N})} \geq \frac{(1-\varepsilon)(M^{\pm}_p)^p}{2^p(m^2_N n^N_{i_N})^{p+1}} |H(x^N_i, y^N_i)|^{p/2} \Omega(x^N_i, y^N_i)$$

By the definition of $M^{\pm}_p$, by definition of $n^N_{i_N}$, and (315) we have that for all N large enough

$$\begin{aligned}
&\frac{(1-\varepsilon)(M^{\pm}_p)^p}{(m^2_N n^N_{i_N})^{p+1}} |H(x^N_i, y^N_i)|^{p/2} \Omega(x^N_i, y^N_i) \\
\geq \quad & \frac{(M^{\pm}_p)^p}{m_N^{2(p+1)}} \sum_{j=1}^{m^2_N} |H(x^N_i, y^N_i)|^{p/2} \Omega(x^N_i, y^N_i) \left(\frac{\sum_{j=1}^{m^2_N} |H(x^N_j, y^N_j)|^{\frac{p}{2(p+1)}} \Omega(x^N_j, y^N_j)^{\frac{1}{p+1}}}{N(1-\varepsilon)|H(x^N_i, y^N_i)|^{\frac{p}{2(p+1)}} \Omega(x^N_i, y^N_i)^{\frac{1}{p+1}}} \right)^p \\
= \quad & \frac{(M^{\pm}_p)^p}{(1-\varepsilon)^p N^p m_N^{2(p+1)}} \left(\sum_{j=1}^{m^2_N} |H(x^N_j, y^N_j)|^{\frac{p}{2(p+1)}} \Omega(x^N_j, y^N_j)^{\frac{1}{p+1}} \right)^{p+1} \\
> \quad & \frac{(M^{\pm}_p)^p}{N^p} \left(\int_D |H(f; x,y)|^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy \right)^{p+1}.
\end{aligned}$$

Hence, for all N large enough we obtain

$$\|f_N - s(f_N, \square^*_N)\|_{p,\Omega} > \frac{M^{\pm}_p}{2N}(1-\varepsilon) \left(\int_D |H(f; x,y)|^{\frac{p}{2(p+1)}} \Omega(x,y)^{\frac{1}{p+1}} dxdy \right)^{\frac{p+1}{p}}.$$

On the other hand

$$\|f - f_{N,i_N}\|_{L_{p,\Omega}(R^N_{i_N})} \le \|f - f_{N,i_N}\|_{L_{p,\Omega}(D^N_{i_N})} \le \frac{\|\Omega\|_\infty^{\frac{1}{p}}}{2m_N^{2(1+\frac{1}{p})}} \omega\left(\frac{1}{2m_N}\right) \le \frac{\varepsilon}{N}\|\Omega\|_\infty^{\frac{1}{p}}$$

due to the choice of m_N. Hence, we obtain that for all large enough N

$$\|f - s(f, \Box_N)\|_{p,\Omega} \ge (1 - c_6\varepsilon)\frac{M_p^\pm}{2N}\left(\int_D |H(f; x, y)|^{\frac{p}{2(p+1)}}\Omega(x, y)^{\frac{1}{p+1}} dxdy\right)^{\frac{p+1}{p}}$$

with some positive constant c_6. Therefore,

$$\liminf_{N\to\infty} \frac{\|f - s(f, \Box_N)\|_{p,\Omega}}{\frac{M_p^\pm}{2N}\left(\int_D |H(f; x, y)|^{\frac{p}{2(p+1)}}\Omega(x, y)^{\frac{1}{p+1}} dxdy\right)^{\frac{p+1}{p}}} \ge 1.$$

This completes the proof of the theorem. □

IV.8 Remarks: choice of an element.

The choice of whether to use a triangular or rectangular partition depends primarily on the setting of the problem and on the shape of the domain. Triangles are more flexible, however, in solid mechanics (the study of load carrying members in terms of forces, deformations, and stability, where there are often prefered directions because of certain geometric relations) rectangular elements are generally preferred.

MULTILINEAR SPLINES IN $\mathbb{R}^d$

In $\mathbb{R}^d$ we understand the space of multilinear splines to be the collection of functions of d variables which are linear in each variable, i.e. have the form

$$a\prod_{i=1}^{d} x_i + \sum_{j=1}^{d} b_j \prod_{i\neq j} x_i + \sum_{j=1}^{d}\sum_{k\neq j} c_{kj} \prod_{i\neq j,k} x_i + \dots .$$

Denote this space by $BS_1^{d,0}$. Let the d-dimensional box partition $\square_N$ of $D = [0,1]^d$ with sides parallel to the coordinate axes be fixed. By $BS_1^{d,0}(\square_N)$ denote the space of multilinear splines on this partition. Let $s(f, \square_N) \in BS_1^{d,0}(\square_N)$ denote a spline which interpolates the function f at all vertices of the partition $\square_N$ except for $o(N)$ of them.

The main question of this chapter is to investigate the asymptotic behaviour of the optimal error $R_N(f, L_{p,\Omega}, BS_1^{d,0}(\square_N))$.

In this chapter the main theorems are the following.

Theorem 12. *Let $f \in C^2(D)$, and $D = [0,1]^d \subset \mathbb{R}^d$. In addition, assume that at every point $\mathbf{x}$ the quadratic form $\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x}) h_i^2$ has the signature $(k, d-k), 0 < k < d$. Let also $|H(f;\mathbf{x})| \geq C^+ > 0$ for all $\mathbf{x} \in D$ and a positive continuous weight function $\Omega(\mathbf{x})$ be given. Then*

$$\limsup_{N\to\infty} N^{\frac{2}{d}} \|f - s(f, \square_N^*)\|_{\infty,\Omega} \leq \frac{1}{8} k^{\frac{k}{d}} (d-k)^{1-\frac{k}{d}} \left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}} \Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x} \right)^{\frac{2}{d}}. \tag{212}$$

Theorem 13. *Let $f \in C^2(D)$, and $D = [0,1]^d \subset \mathbb{R}^d$. In addition, assume that at every point $\mathbf{x}$ the quadratic form $\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x}) h_i^2$ has the signature $(k, d-k), k > 0$. Let also $|H(f;\mathbf{x})| \geq C^+ > 0$ for all $\mathbf{x} \in D$ and a positive continuous weight function $\Omega(\mathbf{x})$ be given. Then for any sequences of box partitions $\{\square_N\}$ which satisfies*

$$\sup_N N^{\frac{1}{d}} \max_{R\in\square_N} \operatorname{diam}(R) < \infty \tag{213}$$

we have

$$\liminf_{N\to\infty} N^{\frac{2}{d}}\|f - s(f,\Box_N)\|_{\infty,\Omega} \geq \frac{1}{8}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}\left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}. \tag{214}$$

Let us remark here that the interpolation problem on the partitions with sides parallel to the coordinate axes has a solution since clearly the interpolation matrix is nonsingular.

V.1 Interpolation of quadratic functions with arbitrary signature.

Let the following quadratic form

$$Q(\mathbf{x}) = \sum_{i=1}^{d} \sigma_i x_i^2 \tag{215}$$

be given. Assume that $\sigma_i = 1$ for all $1 \leq i \leq k$ and $\sigma_i = -1$ for all $k+1 \leq i \leq d$. In this case we say that the quadratic form (215) has the *signature* $(k, d-k)$.

Lemma 35. *The error of interpolation of the quadratic form (215) with signature* $(k, d-k)$, $0 \leq k \leq d$, *by multilinear splines at the vertices of the d-dimensional box* $\prod_{i=1}^{d}[-h_i, h_i]$ *is*

$$\max\left\{h_1^2 + \cdots + h_k^2, h_{k+1}^2 + \cdots + h_d^2\right\}. \tag{216}$$

Proof: We shall proceed by induction. The basis of induction, i.e. the statement of the theorem in the case $d = 2$ and $k = 1$ has been proved in Lemma 32.

Next let us consider the form (215) with signature $(k, d-k)$. The value of the error at the center is

$$\delta(\mathbf{0}) = \left|\sum_{i=1}^{k} h_i^2 - \sum_{i=k+1}^{d} h_i^2\right|. \tag{217}$$

Let us consider the error on the boundary.

On the face $x_i = h_i$ in the case when $i \leq k$ the form (215) becomes

$$\sum_{j=1, j\neq i}^{k} x_j^2 - \sum_{j=k+1}^{d} x_j^2 \tag{218}$$

and the error by hypothesis of induction is

$$\max\left\{\sum_{j=1,j\neq i}^{k} h_j^2, \sum_{j=k+1}^{d} h_j^2\right\}. \tag{219}$$

On the face $x_i = h_i$ in the case when $i > k$ the form (215) becomes

$$\sum_{j=1}^{k} x_j^2 - \sum_{j=k+1,j\neq i}^{d} x_j^2 \tag{220}$$

and the error by hypothesis of induction is

$$\max\left\{\sum_{j=1}^{k} h_j^2, \sum_{j=k+1,j\neq i}^{d} h_j^2\right\}. \tag{221}$$

Therefore, the global error is

$$\begin{aligned}
\Delta &= \max\left\{\left|\sum_{i=1}^{k} h_i^2 - \sum_{i=k+1}^{d} h_i^2\right|, \max_i \max\left\{\sum_{j=1,j\neq i}^{k} h_j^2, \sum_{j=k+1}^{d} h_j^2\right\}, \max_i \max\left\{\sum_{j=1}^{k} h_j^2, \sum_{j=k+1,j\neq i}^{d} h_j^2\right\}\right\} \\
&= \max\left\{\max_{i\leq k} \max\left\{\sum_{j=1,j\neq i}^{k} h_j^2, \sum_{j=k+1}^{d} h_j^2\right\}, \max_{i>k} \max\left\{\sum_{j=1}^{k} h_j^2, \sum_{j=k+1,j\neq i}^{d} h_j^2\right\}\right\} \\
&= \max\left\{\max\left\{\max_{i\leq k} \sum_{j=1,j\neq i}^{k} h_j^2, \sum_{j=k+1}^{d} h_j^2\right\}, \max\left\{\sum_{j=1}^{k} h_j^2, \max_{i>k} \sum_{j=k+1,j\neq i}^{d} h_j^2\right\}\right\} \\
&= \max\left\{\max_{i\leq k} \sum_{j=1,j\neq i}^{k} h_j^2, \sum_{j=k+1}^{d} h_j^2, \sum_{j=1}^{k} h_j^2, \max_{i>k} \sum_{j=k+1,j\neq i}^{d} h_j^2\right\} \\
&= \max\left\{\sum_{j=k+1}^{d} h_j^2, \sum_{j=1}^{k} h_j^2\right\}.
\end{aligned}$$

The lemma is proved. □

Next we shall compute the value of the minimal l_∞ error. Denote by

$$\Delta := \min_{h_i}\left\{\sum_{j=1}^{k} h_j^2, \sum_{j=k+1}^{d} h_j^2\right\}, \tag{222}$$

where min is taken over all h_i such that

$$2^d \prod_{i=1}^{d} h_i = V. \tag{223}$$

In the next two lemmas we shall compute the value of Δ in the case of positive definite quadratic form and in the case of the quadratic form with arbitrary signature.

Lemma 36. *The minimal error of interpolation of the quadratic form (215) with signature $(d, 0)$, by multilinear splines at the vertices of the d-dimensional box $\prod_{i=1}^{d}[-h_i, h_i]$ of volume V is*

$$\Delta = \frac{d}{4} V^{\frac{2}{d}}. \tag{224}$$

Proof: Clearly, the minimum of the function $\sum_{i=1}^{d} h_i^2$ with the additional assumption (223) is achieved when all h_i are equal, i.e.

$$h_1 = h_2 = \cdots = h_d := h.$$

In this case, we have

$$\Delta = \min_{h_i} \left\{ \sum_{i=1}^{d} h_i^2 \right\} = dh^2.$$

From condition (223) we also have

$$h = \frac{V^{\frac{1}{d}}}{2},$$

and, hence,

$$\Delta = d\frac{V^{\frac{2}{d}}}{4}.$$

□

Lemma 37. *The minimal error of interpolation of the quadratic form (215) with signature $(k, d-k)$, $0 < k < d$, by multilinear splines on the d-dimensional box $\prod_{i=1}^{d}[-h_i, h_i]$ of volume V is*

$$\Delta = \frac{1}{4} k^{\frac{k}{d}} (d-k)^{1-\frac{k}{d}} V^{\frac{2}{d}}. \tag{225}$$

Proof: To prove the lemma, i.e. to minimize the $l_\infty-$norm of (216), we minimize the l_p

norm (with an arbitrary p) of (216) under assumption that the volume of the parralelepiped is fixed ($= V$) and take the value of the minimum when $p = \infty$:

$$\left(\sum_{j=k+1}^{d} h_j^2\right)^p + \left(\sum_{j=1}^{k} h_j^2\right)^p, \tag{226}$$

since

$$\lim_{p\to\infty} \|x\|_{L_p(\mathbb{R}^d)} = \|x\|_{L_\infty(\mathbb{R}^d)},$$

and, therefore, if the actual maximum value was less than the value obtained in this way then it would also be true for all p large enough.

The assumption of volume being fixed is equivalent to

$$2^{2d} \prod_{i=1}^{d} h_i^2 = V^2. \tag{227}$$

To minimize the function in (226), we shall use the method of Lagrange multipliers. Let us consider the function (we substitute $x_i = h_i^2$)

$$f(\mathbf{x}) = \left(\sum_{j=k+1}^{d} x_j\right)^p + \left(\sum_{j=1}^{k} x_j\right)^p + \lambda \prod_{j=1}^{d} x_j, \tag{228}$$

and its partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= p\left(\sum_{j=1}^{k} x_j\right)^{p-1} + \lambda \prod_{j=1, j\neq i}^{d} x_j = 0, \quad i = 1, \ldots, k \\ \frac{\partial f}{\partial x_i} &= p\left(\sum_{j=k+1}^{d} x_j\right)^{p-1} + \lambda \prod_{j=1, j\neq i}^{d} x_j = 0, \quad i = k+1, \ldots, d. \end{aligned} \tag{229}$$

Multiplying the first equation by x_i we obtain

$$px_i \left(\sum_{j=1}^{k} x_j\right)^{p-1} + \lambda V^2 2^{-2d} = 0, \quad i = 1, \ldots, k. \tag{230}$$

Analogously,

$$px_i\left(\sum_{j=k+1}^{d} x_j\right)^{p-1} + \lambda V^2 2^{-2d} = 0, \quad i = k+1, \ldots, d. \tag{231}$$

Therefore, the only solution is

$$x_1 = \cdots = x_k =: x,$$

$$x_{k+1} = \cdots = x_d =: y.$$

Taking into consideration assumption (227)

$$x^k y^{d-k} = V^2 2^{-2d},$$

and using equations (230) and (231), we can find x and y:

$$x(kx)^{p-1} = y((d-k)y)^{p-1}.$$

It is easy to check that we obtain

$$x = \left(\frac{d-k}{k}\right)^{(1-\frac{k}{d})(1-\frac{1}{p})} \frac{V^{\frac{2}{d}}}{4}, \tag{232}$$

and

$$y = \left(\frac{d-k}{k}\right)^{-\frac{k}{d}(1-\frac{1}{p})} \frac{V^{\frac{2}{d}}}{4}. \tag{233}$$

In the case $p = \infty$ we have

$$x = \frac{1}{4}\left(\frac{d-k}{k}\right)^{(1-\frac{k}{d})} V^{\frac{2}{d}},$$

$$y = \frac{1}{4}\left(\frac{d-k}{k}\right)^{-\frac{k}{d}} V^{\frac{2}{d}}.$$

Therefore,

$$h_i = \frac{1}{2}\left(\frac{d-k}{k}\right)^{\frac{d-k}{2d}} V^{\frac{1}{d}}, \quad i \le k, \tag{234}$$

$$h_j = \frac{1}{2}\left(\frac{d-k}{k}\right)^{-\frac{k}{2d}} V^{\frac{1}{d}}, \quad j > k. \tag{235}$$

Hence, the global error is

$$\Delta = \frac{1}{4}k\left(\frac{d-k}{k}\right)^{(1-\frac{k}{d})} V^{\frac{2}{d}} = \frac{1}{4}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}V^{\frac{2}{d}}. \tag{236}$$

□

Now let the quadratic form

$$Q(\mathbf{x}) = \sum_{i=1}^{d} A_i x_i^2 \tag{237}$$

be given. Assume that $A_i > 0$ for all $1 \leq i \leq k$ and $A_i < 0$ for all $k+1 \leq i \leq d$, i.e. the form $Q(\mathbf{x})$ has the sign $(k, d-k)$.

Lemma 38. *The error of interpolation of quadratic form (237) with signature $(k, d-k)$, $0 < k < d$, by multilinear splines on the d-dimensional box P of volume $V[P]$ is*

$$\frac{1}{4}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}\left(V\sqrt{\prod_{i=1}^{d} A_i}\right)^{\frac{2}{d}}. \tag{238}$$

Proof: For the given quadratic form $Q(\mathbf{x}) = \sum_{i=1}^{d} A_i x_i^2$ let us consider a linear transformation F such that

$$(Q \circ F)(\mathbf{u}) = \sum_{i=1}^{d} u_i^2. \tag{239}$$

In other words,

$$F(\mathbf{u}) = \left(\frac{u_1}{\sqrt{A_1}}, \ldots, \frac{u_d}{\sqrt{A_d}}\right). \tag{240}$$

Observe that the determinant of the inverse of this transformation is

$$\det(F^{-1}) = \sqrt{\prod_{i=1}^{d} A_i}. \tag{241}$$

Let us consider the box $F^{-1}(P)$ which clearly has the volume

$$V[F^{-1}(P)] = V[P]\det(F^{-1}). \tag{242}$$

Combining the result of the previous lemma about the error of interpolation on the box

$F^{-1}(P)$ with (242) and (241), we obtain (243). □

Similarly, in the case of positive definite form we obtain the following statement.

Lemma 39. *The error of interpolation of the positive definite quadratic form by multilinear splines on the d-dimensional box P of volume $V[P]$ is*

$$\frac{d}{4}\left(V\sqrt{\prod_{i=1}^{d} A_i}\right)^{\frac{2}{d}}. \tag{243}$$

V.2 Error of near interpolation of C^2 functions defined on $[0,1]^d$. Estimate from above.

In this section we restart the numbering of constants.

Proof of Theorem 12: For a fixed $\varepsilon \in (0,1)$ and for every N we define as before

$$m_N := \min\left\{m > 0 : \frac{d^2}{2}\left(\frac{1}{2m}\right)^2 \omega\left(\frac{1}{2m}\right) \le \frac{\varepsilon}{N^{\frac{2}{d}}}\right\}, \tag{244}$$

where $\omega(\delta)$ is a function defined in (113).

Observe that clearly for m_N defined in (244) it is true that $m_N \to \infty$ as $N \to \infty$. In addition, we can show as in Section II.4.1 that

$$\frac{N^{\frac{2}{d}}}{m_N^2} \to \infty, \quad N \to \infty, \tag{245}$$

i.e. $m_N = o(N^{\frac{1}{d}})$ as $N \to \infty$.

Divide the unit cube D into cubes with side length equal to $\frac{1}{m_N}$ and denote the resulting cubes by D_l^N, $l = 1, \dots, m_N^d$. Next take the center point $\mathbf{x}_l^N$ in each cube D_l^N and set

$$A_{i,j}^{N,l} := \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_l^N), \quad i,j = 1, \dots, d, \quad l = 1, \dots, m_N^d.$$

Observe that, clearly,

$$A_{i,j}^{N,l} = A_{j,i}^{N,l}, \quad i,j = 1, \dots, d, \quad l = 1, \dots, m_N^d.$$

In addition, denote by

$$H(\mathbf{x}_l^N) := \prod_{i=1}^{d} A_{i,i}^{N,l}, \qquad l = 1, \ldots, m_N^d.$$

Set

$$n_l^N := \left\lfloor \frac{N(1-\varepsilon)|H(\mathbf{x}_l^N)|^{\frac{1}{2}}\Omega(\mathbf{x}_l^N)^{\frac{d}{2}}}{\sum_{j=1}^{m_N^d} |H(\mathbf{x}_j^N)|^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}} \right\rfloor, \quad l = 1, \ldots, m_N^d. \tag{246}$$

We find n_l^N by minimizing the overall error on D.

Clearly, the minimal overall error in the uniform norm will be achieved when errors on each region are equal. Using this condition and Lemma 39 we arrive at the following equations for all $l = 1, \ldots, m_N^d$

$$\frac{1}{4}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}} \left(\frac{\sqrt{|H(\mathbf{x}_l^N)|}}{m_N^d \tilde{n}_l^N} \right)^{2/d} \Omega(\mathbf{x}_l^N) = \frac{1}{4}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}} \left(\frac{\sqrt{|H(\mathbf{x}_1^N)|}}{m_N^d \tilde{n}_1^N} \right)^{2/d} \Omega(\mathbf{x}_1^N).$$

From this we find the number $\tilde{n}_l^N$:

$$\tilde{n}_l^N = \tilde{n}_1^N \frac{|H(\mathbf{x}_l^N)|^{1/2}\Omega(\mathbf{x}_l^N)^{d/2}}{|H(\mathbf{x}_1^N)|^{1/2}\Omega(\mathbf{x}_1^N)^{d/2}}.$$

From the condition $\sum_{l=1}^{m_N^d} \tilde{n}_l^N = N$ we have

$$N = \frac{\tilde{n}_1}{|H(\mathbf{x}_1^N)|^{1/2}\Omega(\mathbf{x}_1^N)^{d/2}} \sum_{l=1}^{m_N^d} |H(\mathbf{x}_l^N)|^{1/2}\Omega(\mathbf{x}_l^N)^{d/2}$$

Solving for $\tilde{n}_1^N$, substituting it in the expression for $\tilde{n}_l^N$, and taking the integer part of it gives us the formula for n_l^N.

We have shown in Section II.4.1 that $n_l^N \to \infty$ when $N \to \infty$.

Now set

$$h^N_{l,i} := \frac{1}{2}\left(\frac{d-k}{k}\right)^{\frac{d-k}{2d}}\left(\frac{1}{m^d_N n^N_l}\right)^{\frac{1}{d}}, \quad i \le k, \;\; l = 1, \dots, m^d_N, \tag{247}$$

$$h^N_{l,j} := \frac{1}{2}\left(\frac{d-k}{k}\right)^{-\frac{k}{2d}}\left(\frac{1}{m^d_N n^N_l}\right)^{\frac{1}{d}}, \quad j > k, \;\; l = 1, \dots, m^d_N. \tag{248}$$

The intersection of the lattice

$$\prod_{i \le k}[L_i h^N_{l,i}, (L_i+1)h^N_{l,i}] \times \prod_{i > k}[L_i h^N_{l,i}, (L_i+1)h^N_{l,i}], \quad L_i \in \mathbb{Z}, \;\; l = 1, \dots, m^d_N, \tag{249}$$

with D^N_l gives us the desired partition $\square^*_N(D^N_l)$ of D^N_l, $l = 1, ..., m^2_N$. The stitching of the partitions on the neighboring regions proceeds as was described in Section IV.3 for $\mathbb{R}^2$ with the obvious corresponding changes. Denote by $\square^*_N(D)$ the obtained partition of D. Let us show that the sequence of obtained in such a way partitions $\{\square^*_N(D)\}$ will be asymptotically optimal.

By f_N denote the piecewise quadratic function constructed in the following way. On D^N_1 we set f_N to be $\sum_{i=1}^{d} A^{N,1}_{i,i} x_i^2$. Then for $l > 1$ on $D^N_l \setminus \cup_{j=1}^{l-1} D^N_j$ we set

$$f_N(\mathbf{x}) := \sum_{i=1}^{d} A^{N,l}_{i,i} x_i^2.$$

To estimate $R_N(f, L_{\infty,\Omega}, BS^{d,0}_1(\square_N))$ we observe that

$$R_N(f, L_{\infty,\Omega}, BS^{d,0}_1(\square_N)) \le \|f - s(f, \square^*_N)\|_{\infty,\Omega} \le \|f - f_N\|_{\infty,\Omega} + \|f_N - s(f_N, \square^*_N)\|_{\infty,\Omega}$$

$$+\|s(f_N, \square^*_N) - s(f, \square^*_N)\|_{\infty,\Omega} \le 2\|f - f_N\|_{\infty,\Omega} + \|f_N - s(f_N, \square^*_N)\|_{\infty,\Omega}.$$

Let us estimate each term. First of all, by Lemma 22 and the definition of m_N we have

$$\|f - f_N\|_{\infty,\Omega} \le \frac{d^2}{2}\left(\frac{1}{2m}\right)^2 \omega\left(\frac{1}{2m_N}\right)\|\Omega\|_\infty \le \frac{\varepsilon}{N^{\frac{2}{d}}}\|\Omega\|_\infty.$$

Let us estimate the second term now. It is clear that for two embedded elements of partition the error of multilinear interpolation of a quadratic function is greater on the larger element. Therefore, we shall estimate this error on elements that do not have intersection with the boundary.

Let us take an element $R_l^N \in \Box_N^*(D_l^N)$ that does not have common points with the boundary of D_l^N. By Lemma 37, for every $\mathbf{x} \in R_l^N$ we have

$$|f_N(\mathbf{x}) - s(f_N, \Box_N^*; \mathbf{x})|\Omega(\mathbf{x}_l^N) \le \|f_N - s(f_N, \Box_N^*; \cdot)\|_{\infty,\Omega}$$

$$= \frac{1}{8}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}\left(\frac{1}{m_N^d n_l^N}\sqrt{|H(\mathbf{x}_l^N)|}\right)^{\frac{2}{d}}\Omega(\mathbf{x}_l^N).$$

By the definition of n_l^N, for all large enough N, for all l, and for all $\mathbf{x} \in R_l^N$, we have

$$|f_N(\mathbf{x}) - s(f_N, \Box_N^*; \mathbf{x})|\Omega(\mathbf{x}_i^N) \le$$

$$\le \frac{1}{8}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}\left(\frac{\sum_{j=1}^{m_N^d}|H(\mathbf{x}_j^N)|^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}}{m_N^d N(1-\varepsilon)|H(\mathbf{x}_l^N)|^{\frac{1}{2}}\Omega(\mathbf{x}_l^N)^{\frac{d}{2}}}\sqrt{|H(\mathbf{x}_l^N)|}\right)^{\frac{2}{d}}\Omega(x_l^N).$$

Since this estimate does not depend on $\mathbf{x}$, we obtain

$$\|f_N - s(f_N, \Box_N^*; \cdot)\|_{\infty,\Omega} \le \frac{k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}}{8(N(1-\varepsilon))^{2/d}}\left(\frac{1}{m_N^2}\sum_{j=1}^{m_N^d}|H(\mathbf{x}_j^N)|^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}\right)^{\frac{2}{d}}.$$

Note that

$$\frac{1}{m_N^2}\sum_{j=1}^{m_N^d}|H(\mathbf{x}_j^N)|^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}} \to \int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}, \quad \text{as} \quad N \to \infty.$$

Hence, for all N large enough we have

$$\|f_N - s(f_N, \Box_N^*; \cdot)\|_{\infty,\Omega} \le \frac{k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}}{8(N(1-\varepsilon))^{2/d}}\left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}.$$

Therefore,

$$\|f - s(f, \Box_N^*; \cdot)\|_{\infty,\Omega} \le \frac{2\varepsilon}{N^{\frac{2}{d}}}\|\Omega\|_\infty + \frac{k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}}{8(N(1-\varepsilon))^{\frac{2}{d}}}\left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}.$$

Because $\varepsilon > 0$ is arbitrary, we obtain the desired estimate (250). □

Remark. Observe that in the case $k = d$, i.e. in the case when the quadratic part of Taylor polynomial is a positive definite form, we obtain the idefinite form 0^0. However, all arguments will hold true if we set $0^0 = 1$. Therefore, in the case $k = d$ we have the following result.

Lemma 40. *Let $f \in C^2(D)$, and $D = [0,1]^d \subset \mathbb{R}^d$. In addition, assume that at every point $\mathbf{x}$ the quadratic form $\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x})h_i^2$ has the signature $(d, 0)$. Let also $|H(f;\mathbf{x})| \ge C^+ > 0$ for all $\mathbf{x} \in D$ and a positive continuous weight function $\Omega(\mathbf{x})$ be given. Then*

$$\limsup_{N\to\infty} N^{\frac{2}{d}}\|f - s(f, \Box_N^*)\|_{\infty,\Omega} \le \frac{d}{8}\left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}. \qquad (250)$$

V.3 Error of near interpolation of C^2 functions defined on $[0,1]^d$. Estimate from below.

Let m_N, n_i^N, D_i^N etc. be as defined in the previous section.

Proof of Theorem 13: To obtain the estimate from below we shall consider an arbitrary sequence of box partitions $\{\Box_N\}_{N=1}^\infty$ which satisfies (213). This assumption implies that for all N large enough there exists l_N and $D_{l_N}^N$ that completely contains an element $R_{l_N}^N \in \Box_N$.

For each such N and l_N, set

$$f_{N,l_N}(\mathbf{x}) := \sum_{i=1}^{d} A_{i,i}^{N,l} x_i^2.$$

Observe that

$$\|f - s(f, \Box_N)\|_{L_{\infty,\Omega}(R_{l_N}^N)} \ge \|f_{N,l_N} - s(f_{N,l_N}, \Box_N)\|_{L_{\infty,\Omega}(R_{l_N}^N)} - 2\|f - f_{N,l_N}\|_{L_{\infty,\Omega}(R_{l_N}^N)}.$$

By Lemma 37 we have

$$\|f_{N,l_N} - s(f_{N,i_N}, \square_N)\|_{L_{\infty,\Omega}(R^N_{l_N})} \geq (1-\varepsilon)\frac{1}{8}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}\left(\frac{1}{m_N^d n_l^N}\sqrt{|H(\mathbf{x}_l^N)|}\right)^{\frac{2}{d}}\Omega(\mathbf{x}_l^N).$$

By the definition of $n^N_{l_N}$ we have that for all N large enough

$$\frac{1}{8}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}\left(\frac{1}{m_N^d n_l^N}\sqrt{|H(\mathbf{x}_l^N)|}\right)^{\frac{2}{d}}\Omega(\mathbf{x}_l^N)$$

$$= \frac{1}{8}k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}\left(\frac{\sum_{j=1}^{m_N^d}|H(\mathbf{x}_j^N)|^{\frac{1}{2}}\Omega(\mathbf{x}_j^N)^{\frac{d}{2}}}{m_N^d N(1-\varepsilon)|H(\mathbf{x}_l^N)|^{\frac{1}{2}}\Omega(\mathbf{x}_l^N)^{\frac{d}{2}}}\sqrt{|H(\mathbf{x}_l^N)|}\right)^{\frac{2}{d}}\Omega(x_l^N)$$

$$> \frac{k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}}{8N^{\frac{2}{d}}}\left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}.$$

Hence, for all N large enough we obtain

$$\|f_N - s(f_N, \square_N)\|_{\infty,\Omega} > (1-\varepsilon)\frac{k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}}{8N^{\frac{2}{d}}}\left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}.$$

On the other hand

$$\|f - f_{N,i_N}\|_{L_{\infty,\Omega}(R^N_{i_N})} \leq \|\Omega\|_\infty \frac{\varepsilon}{N^{\frac{2}{d}}}$$

due to the choice of m_N. Hence, we obtain that for all large enough N

$$\|f - s(f, \square_N)\|_{\infty,\Omega} \geq (1-c_2\epsilon)\frac{k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}}{8N^{\frac{2}{d}}}\left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}$$

with some positive constant c_2. Therefore,

$$\liminf_{N\to\infty} \frac{\|f - s(f, \square_N)\|_{\infty,\Omega}}{\frac{k^{\frac{k}{d}}(d-k)^{1-\frac{k}{d}}}{8N^{\frac{2}{d}}}\left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}}\Omega(\mathbf{x})^{\frac{d}{2}}d\mathbf{x}\right)^{\frac{2}{d}}} \geq 1.$$

This completes the proof of the lemma. □

Analogously, the following statement can be proved.

Lemma 41. *Let $f \in C^2(D)$, and $D = [0,1]^d \subset \mathbb{R}^d$. In addition, assume that at every point* $\mathbf{x}$ *the quadratic form* $\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x}) h_i^2$ *has the signature $(d, 0)$. Let also $|H(f;\mathbf{x})| \geq C^+ > 0$ for all $\mathbf{x} \in D$ and let a positive continuous weight function $\Omega(\mathbf{x})$ be given. Then for any sequences of box partitions $\{\square_N\}$ which satisfies (213) we have*

$$\liminf_{N\to\infty} N^{\frac{2}{d}} \|f - s(f, \square_N)\|_{\infty,\Omega} \geq \frac{d}{8} \left(\int_D |H(f;\mathbf{x})|^{\frac{1}{2}} \Omega(\mathbf{x})^{\frac{d}{2}} d\mathbf{x} \right)^{\frac{2}{d}}. \tag{251}$$

CHAPTER VI

BIQUADRATIC SPLINES

A popular interpolation scheme on rectangles consists of piecewise polynomials of degree 3 whose restriction to the edges are quadratic polynomials, i.e. functions of the type

$$ax^2y + bxy^2 + cx^2 + dxy + ey^2 + fx + gy + h.$$

There are eight parameters or eight degrees of freedom. They can be used to interpolate a given function at the vertices of the rectangle and at the midpoints of the sides (see Figure 9). Using the language of finite element methods, this element is called the *eight node element* or the *serendipity element.*

Using this space of biquadratic polynomials to interpolate a given C^3 function we shall ask the same question as before: what are the exact asymptotics of the optimal error of interpolation the given function on the rectangular grid with sides parallel to the coordinate axes? We shall also provide a construction of the sequence of partitions which will provide the asymptotically optimal error.

Let P_3 be the set of biquadratic polynomials, i.e. functions of the type

$$p(x, y) = ax^2y + bxy^2 + cx^2 + dxy + ey^2 + fx + gy + h, \quad a, b, c, d \in \mathbb{R}.$$

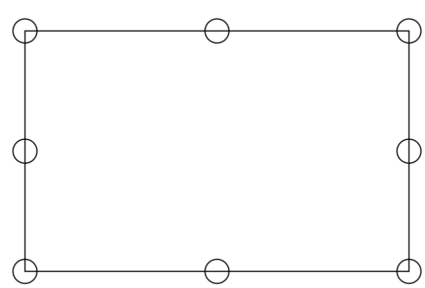

Figure 9: Interpolation scheme which uses biquadratic splines

Given a partition $\Box_N(D)$ define the space $QS_1^0(\Box_N)$ of *bilinear splines* to be

$$QS_1^0(\Box_N) := \{f \in C(D) : \forall i = 1, ..., N \ \exists \ p \in P_3 \ \text{s.t.} \ f|_{R_i} = p|_{R_i}\}.$$

Note that the bilinear spline $s(f, \Box_N)$ is uniquely defined by its values at the vertices of the rectangular partition.

Let $s(f, \Box_N) \in QS_1^0(\Box_N)$ denote a spline which interpolates the function f at all vertices of the partition $\Box_N$ except for $o(N)$ of them.

Now let $f \in C^3(D)$, $D = [0,1]^2$, and the number of rectangles $N \in \mathbb{N}$ be fixed. Define the error of optimal interpolation of the function f by the continuous piecewise bilinear function $s(f, \Box_N) \in QS_1^0(\Box_N)$ to be

$$R_N(f, L_{p,\Omega}, QS_1^0(\Box_N)) := \inf_{\Box_N} \|f - s(f, \Box_N)\|_{p,\Omega}, \tag{252}$$

where inf is taken over all rectangular partitions of D containing N rectangles. The main goal of this chapter is to investigate the asymptotics of the optimal error $R_N(f, L_{p,\Omega}, QS_1^0(\Box_N))$. The following two theorems give the answer to this question.

Theorem 14. *Let $f \in C^3(D)$, $D = [0,1]^2$, and $|H(f; x, y)| := |f_{xxx} f_{yyy}(x, y)| \geq C^+ > 0$ for all $(x, y) \in D$. Let also positive continuous weight function $\Omega(x, y)$ be given. Then for any $1 \leq p \leq \infty$ in the case of interpolation by biquadratic splines on rectangular partition with sides parallel to the coordinate axes we have*

$$\limsup_{N\to\infty} N^{\frac{3}{2}} \|f - s(f, \Box_N)\|_{p,\Omega} \leq \frac{M_p^{\pm}(1 + o(1))}{6} \left(\int_D |H(f; x, y)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x, y)^{\frac{1}{\frac{3}{2}p+1}} dxdy \right)^{\frac{3}{2}+\frac{1}{p}}, \tag{253}$$

where

$$M_p^{\pm} = \min_R \frac{L_p - \textit{error of biquadratic interpolation of } x^3 \pm y^3 \textit{ on rectangle } R}{|R|^{\frac{3}{2}+\frac{1}{p}}}. \tag{254}$$

Theorem 15. *Let $f \in C^3(D)$, $D = [0,1]^2$, and $|H(f; x, y)| := |f_{xxx} f_{yyy}(x, y)| \geq C^+ > 0$ for all $(x, y) \in D$. Let also positive continuous weight function $\Omega(x, y)$ be given. Then for* *any* $1 \le p \le \infty$ *and for any sequence of rectangular partitions* $\{\Box_N\}_{N=1}^{\infty}$ *satisfying*

$$\sup_N \sqrt{N} \max_{R \in \triangle_N} \operatorname{diam}(R) < \infty, \tag{255}$$

we have

$$\liminf_{N\to\infty} N^{\frac{3}{2}} \|f - s(f, \Box_N)\|_{p,\Omega} \ge \frac{M_p^{\pm}(1+o(1))}{6} \left(\int_D |H(f;x,y)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x,y)^{\frac{1}{\frac{3}{2}p+1}} dxdy \right)^{\frac{3}{2}+\frac{1}{p}}. \tag{256}$$

To prove these theorems we need to investigate the question of biquadratic interpolation of cubic functions of the type

$$Q(x,y) = Ax^3 + By^3. \tag{257}$$

Remark. Observe that it is enough to solve the interpolation problem for the cubic functions of the type (257) since the terms x^2y and xy^2 belong to the biquadratic interpolant.

It is not hard to check that the interpolant to the function $Q(x,y)$ on the rectangle $R = [-h_1, h_2] \times [-h_2, h_2]$ is

$$I_{Q,R}(x,y) = Ah_1^2 x + Bh_2^2 y. \tag{258}$$

Moreover, observe that the error of interpolation of cubic functions on a rectangle does not depend on a shift of this rectangle. More precisely, we have the following simple lemma.

Lemma 42. *For the given cubic function*

$$Q(x,y) = Ax^3 + By^3 \tag{259}$$

and an arbitrary rectangle R*, the error (in any* L_p *norm) of biquadratic interpolation on* R *and* HR*, where* HR *is obtained from* R *by linear transformation*

$$H: \quad x' = x + a, \quad y' = y + b, \tag{260}$$

is the same.

Proof: In new coordinates, function (259) looks like

$$Q(x', y') = A(x' - a)^3 + B(y' - b)^3 = A(x')^3 + B(y')^3 + L(x', y'), \tag{261}$$

where

$$L(x', y') = -3a(x')^2 + 3a^2x' - 3b(y')^2 + 3b^2y' - b^3 - a^3$$

is a quadratic function and, hence, can be regarded as a part of the interpolant.

Since coefficients next to the terms containing x and x', y and y' are the same, the errors of interpolation by biquadratic splines on R and HR are the same in any metric. □

We shall need a solution to the following extremal problem : minimize the ratio

$$\frac{L_p - \text{error of interpolation of } Q(x, y) \text{ on } R}{|R|^{\frac{3}{2}+\frac{1}{p}}}. \tag{262}$$

The solution to this problem in the case of $Ax^3 + By^3$ with $AB > 0$ will be given in Section VI.1.1 (the case of the uniform norm), Section VI.1.2 (the case of the L_1 norm) and Section VI.1.3 (the case of the L_2 norm).

The solution in the case of $Ax^3 - By^3$ with $AB > 0$ will be given in Section VI.2.1 (the case of uniform norm) and Section VI.2.2 (the case of L_2 norm).

VI.1 Interpolation of cubic functions $Ax^3 + By^3$ with $AB > 0$ by biquadratic splines.

VI.1.1 The uniform norm.

Let $I_{Q,R}(x, y)$ denote the biquadratic spline which interpolates the cubic function $Q(x, y)$ at the vertices and at the midpoints of the sides of the rectangle R. Set also

$$d_{Q,R,\infty} := \max_{(x,y)\in R} |Q(x, y) - I_{Q,R}(x, y)|, \tag{263}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R for which

$$\frac{d_{Q,R,\infty}}{|R|^{3/2}} \to \min, \tag{264}$$

and find the minimum of this ratio.

By Lemma 42, it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin, since any other rectangle can be brought to this form by a linear transformation (shift).

The following lemma gives the answer to problem (264).

Lemma 43. *Let the function*

$$Q(x, y) = Ax^3 + By^3$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ we have

$$\frac{d_{Q,R,\infty}}{|R|^{3/2}} \geq \frac{1}{2}\sqrt{AB}. \tag{265}$$

Moreover, equality is obtained for rectangles with $\sqrt[3]{A}h_1 = \sqrt[3]{B}h_2$ and only for them.

Proof: As we already mentioned, the interpolant to the function $Q(x, y)$ on the rectangle R has the form

$$I_{Q,R}(x, y) = Ah_1^2 x + Bh_2^2 y.$$

In addition, note that because of the symmetry the L_∞-error in the uniform norm on the rectangle R is the same as the error on $[0, h_1] \times [0, h_2]$.

Denote the difference between the function $Q(x, y)$ and the interpolant $T_{Q,R}(x, y)$ by

$$\delta(x, y) := Ax^3 + By^3 - Ah_1^2 x - Bh_2^2 y. \tag{266}$$

Taking partial derivatives of $\delta(x, y)$ with respect to x and y we obtain that this function has a critical point at $\left(\frac{h_1}{\sqrt[3]{3}}, \frac{h_2}{\sqrt[3]{3}}\right)$. The value of the difference (266) at this point is

$$\left|\delta\left(\frac{h_1}{\sqrt[3]{3}}, \frac{h_2}{\sqrt[3]{3}}\right)\right| = \frac{2}{3^{3/2}}\left(Ah_1^3 + Bh_2^3\right).$$

In addition, observe that on the boundary of $[0, h_1] \times [0, h_2]$ we have

$$\delta(x, h_2) = Ax^3 - Ah_1^2 x, \quad \text{and} \quad \delta(h_1, y) = By^3 - Bh_2^2 y$$

and, hence, maximal values are obtained at the points

$$\left(\frac{h_1}{\sqrt[3]{3}}, h_2\right), \quad \left(h_1, \frac{h_2}{\sqrt[3]{3}}\right)$$

respectively, and are equal to

$$\left|\delta\left(\frac{h_1}{\sqrt[3]{3}}, h_2\right)\right| = \frac{2}{3\sqrt{3}} A h_1^3 \quad \text{and} \quad \left|\delta\left(h_1, \frac{h_2}{\sqrt[3]{3}}\right)\right| = \frac{2}{3\sqrt{3}} B h_2^3.$$

Obviously,

$$\left|\delta\left(\frac{h_1}{\sqrt[3]{3}}, \frac{h_2}{\sqrt[3]{3}}\right)\right| \geq \left|\delta\left(\frac{h_1}{\sqrt[3]{3}}, h_2\right)\right| \quad \text{and} \quad \left|\delta\left(\frac{h_1}{\sqrt[3]{3}}, \frac{h_2}{\sqrt[3]{3}}\right)\right| \geq \left|\delta\left(h_1, \frac{h_2}{\sqrt[3]{3}}\right)\right|.$$

Therefore, in the case of the uniform norm, extremal problem (264) can be rewritten as follows. Find

$$d := \min_{h_1, h_2} \left\{ \frac{2}{3^{3/2}} \left(Ah_1^3 + Bh_2^3\right) \right\} \tag{267}$$

under the condition that the area of the rectangle $[0, h_1] \times [0, h_2]$ is fixed, i.e. $h_1 h_2 = S$. It is easy to see that this minimum will be attained when $\sqrt[3]{A} h_1 = \sqrt[3]{B} h_2$. From this fact and from the condition $h_1 h_2 = S$ we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/6} \sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/6} \sqrt{S}. \tag{268}$$

Plugging this back into the expression for error (267) gives us the minimal value for the error over the rectangle of area S:

$$d = \frac{4}{3\sqrt{3}} \sqrt{AB} S^{3/2}.$$

Therefore, for the error on the whole R (recall that $|R| = 4S$) we have

$$\frac{d_{Q,R,\infty}}{|R|^{3/2}} = \frac{1}{6\sqrt{3}}\sqrt{AB} \tag{269}$$

which gives the solution to problem (264) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_{\infty}^{+} = \frac{1}{6\sqrt{3}}. \tag{270}$$

VI.1.2 The L_1 norm.

Let again $I_{Q,R}(x, y)$ denote the biquadratic spline which interpolates the cubic function $Q(x, y)$ at the vertices of the rectangle R. Let also

$$d_{Q,R,1} := \int_R\int |Q(x, y) - T_{Q,R}(x, y)|dxdy, \tag{271}$$

and let $|R|$ denote the area of the rectangle R. According to (262) for $p = 1$, the problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,1}}{|R|^{5/2}} \to \min, \tag{272}$$

and find the minimum of it.

The following lemma gives the answer to this problem. Clearly, again it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin, since any other rectangle can be brought to this form by a linear transformation (shift).

Lemma 44. *Let the function*

$$Q(x, y) = Ax^3 + By^3$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ we have

$$\frac{d_{Q,R,1}}{|R|^{5/2}} \geq \frac{1}{16}\sqrt{AB}. \tag{273}$$

Moreover, equality is obtained for rectangles with $\sqrt[3]{A}h_1 = \sqrt[3]{B}h_2$ *and only for them.*

Proof: As we already mentioned, the interpolant on the rectangle R is

$$I_{Q,R}(x,y) = Ah_1^2 x + Bh_2^2 y.$$

In the case of L_1 norm, extremal problem (272) can be rewritten as follows: find

$$d := \min_{h_1,h_2} \int_0^{h_1} \int_0^{h_2} \left| Ax^3 + By^3 - Ah_1^2 x - Bh_2^2 y \right| dxdy \tag{274}$$

under the condition that the area of rectangle $[0, h_1] \times [0, h_2]$ is fixed, i.e. $h_1 h_2 = S$. We have

$$\int_0^{h_1} \int_0^{h_2} |Ax^3 + By^3 - Ah_1^2 x - Bh_2^2 y| dxdy = \int_0^{h_1} \int_0^{h_2} (Ah_1^2 x + Bh_2^2 y - Ax^3 - By^3) dxdy$$

$$= \frac{1}{4}(Ah_1^3 + Bh_2^3) h_1 h_2.$$

Hence, we have to minimize function $\frac{1}{4}(Ah_1^3 + Bh_2^3)h_1 h_2$ under the condition $h_1 h_2 = S$. To that end, we shall use the method of Lagrange multipliers. Let us consider the function

$$L(h_1, h_2) = \frac{1}{4}(Ah_1^3 + Bh_2^3) h_1 h_2 - \lambda h_1 h_2$$

and its partial derivatives

$$\frac{\partial}{\partial h_1} L = \frac{3}{4} A h_1^3 h_2 + \frac{1}{4} h_2 (Ah_1^3 + Bh_2^3) - \lambda h_2,$$

$$\frac{\partial}{\partial h_2} L = \frac{3}{4} B h_1 h_2^3 + \frac{1}{4} h_1 (Ah_1^3 + Bh_2^3) - \lambda h_1.$$

Setting the derivatives equal to zero we see that the minimum is attained when

$$Ah_1^3 = Bh_2^3.$$

Taking into consideration this fact, together with the condition $h_1 h_2 = S$, we can find h_1

and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/6} \sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/6} \sqrt{S}.$$

Plugging this back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d = \frac{1}{2}\sqrt{AB} S^{5/2}.$$

Note that $|R| = 4S$. Therefore, for the error on the whole R we have

$$\frac{d_{Q,R,1}}{|R|^{5/2}} = \frac{1}{16}\sqrt{AB} \tag{275}$$

which gives solution to (272) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_1^+ = \frac{1}{16}. \tag{276}$$

VI.1.3 The L_2 norm.

Let again $I_{Q,R}(x, y)$ denote the biquadratic spline which interpolates the cubic function $Q(x, y)$ at the vertices of the rectangle R. Let also

$$d_{Q,R,2} := \int_R \int (Q(x, y) - I_{Q,R}(x, y))^2 dx dy, \tag{277}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,2}}{|R|^2} \to \min, \tag{278}$$

and find the minimum of it.

The following lemma gives the answer to this problem (again it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin).

Lemma 45. *Let the function*

$$Q(x,y) = Ax^3 + By^3 \tag{279}$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ we have

$$\frac{d_{Q,R,2}}{|R|^2} \geq \sqrt{\frac{11}{90}}\sqrt{AB} \tag{280}$$

Moreover, equality is obtained for rectangles with $\sqrt[3]{A}h_1 = \sqrt[3]{B}h_2$ and only for them.

Proof: As we mentioned before, the interpolant to $Q(x,y)$ in (279) on the rectangle R is

$$I_{Q,R}(x,y) = Ah_1^2 x + Bh_2^2 y.$$

In the case of L_2 norm, extremal problem (278) can be rewritten as follows. Find

$$d^2 := \min_{h_1,h_2} \int_0^{h_1} \int_0^{h_2} \left(Ax^3 + By^3 - Ah_1^2 x - Bh_2^2 y\right)^2 dxdy \tag{281}$$

under the condition that the area of rectangle $[0, h_1] \times [0, h_2]$ is fixed, i.e. $h_1 h_2 = S$. We have

$$\int_0^{h_1} \int_0^{h_2} \left(Ax^3 + By^3 - Ah_1^2 x - Bh_2^2 y\right)^2 dxdy = \frac{1}{840} h_1 h_2 \left(64A^2 h_1^6 + 105AB h_1^3 h_2^3 + 64B^2 h_2^6\right).$$

Hence, we have to minimize the function

$$\frac{1}{840} h_1 h_2 \left(64A^2 h_1^6 + 105AB h_1^3 h_2^3 + 64B^2 h_2^6\right)$$

under the condition $h_1 h_2 = S$. Considering the corresponding Lagrangian and partial derivatives with respect to h_1 and h_2, we obtain

$$\frac{1}{840} h_1 h_2 \left(384A^2 h_1^5 + 315AB h_1^2 h_2^3\right) + \frac{1}{840} h_2 \left(64A^2 h_1^6 + 105AB h_1^3 h_2^3 + 64B^2 h_2^6\right) - \lambda h_2 = 0,$$

$$\frac{1}{840} h_1 h_2 \left(315AB h_1^3 h_2^3 + 384B^2 h_2^6\right) + \frac{1}{840} h_1 \left(64A^2 h_1^6 + 105AB h_1^3 h_2^3 + 64B^2 h_2^6\right) - \lambda h_1 = 0,$$

and, it is easy to check that the minimum is attained for $\sqrt[3]{A}h_1 = \sqrt[3]{B}h_2$. From here and condition $h_1 h_2 = S$ we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/6} \sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/6} \sqrt{S}.$$

Plugging this back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d^2 = \frac{233}{840} ABS^4.$$

Therefore, for the error on the whole R (recall that $|R| = 4S$) we have

$$\frac{d_{Q,R,2}}{|R|^2} = \sqrt{\frac{233}{53760}} \sqrt{AB} \tag{282}$$

which gives solution to (278) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_2^+ = \sqrt{\frac{233}{53760}}. \tag{283}$$

VI.2 Interpolation of cubic functions $Ax^3 - By^3$ with $AB > 0$ by biquadratic splines.

VI.2.1 The L_∞-norm.

Let $I_{Q,R}(x, y)$ denote the biquadratic spline which interpolates the cubic function

$$Q(x, y) = Ax^3 - By^3$$

at the vertices of the rectangle R. Set also

$$d_{Q,R,\infty} := \max_{(x,y)\in R} |Q(x, y) - I_{Q,R}(x, y)|, \tag{284}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,\infty}}{|R|^{3/2}} \to \min, \tag{285}$$

and find the minimum of it.

By Lemma 42, it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin, since any other rectangle can be brought to this one by a linear transformation (shift).

The following lemma gives the answer to problem (285).

Lemma 46. *Let the quadratic form*

$$Q(x, y) = Ax^2 - By^2$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ we have

$$\frac{d_{Q,R,\infty}}{|R|^{3/2}} \geq \frac{1}{12\sqrt{3}}\sqrt{AB}. \tag{286}$$

Moreover, equality is obtained for rectangles with $\sqrt{A}h_1 = \sqrt{B}h_2$ and only for them.

Proof: It is easy to see that the interpolant to the function $Q(x, y)$ on a rectangle R is

$$I_{Q,R}(x, y) = Ah_1^2 x - Bh_2^2 y.$$

In addition, note that the error in the uniform norm on R is the same as the error on $[0, h_2] \times [0, h_1]$.

Denote the difference between function $Q(x, y)$ and interpolant $I_{Q,R}(x, y)$ by

$$\delta(x, y) := Ax^3 - By^3 - Ah_1^2 x + Bh_2^2 y. \tag{287}$$

Taking the derivatives of $\delta(x, y)$ with respect to x and y and setting them equal to zero, we obtain that the point $\left(\frac{h_1}{\sqrt{3}}, \frac{h_2}{\sqrt{3}}\right)$ is a critical point of this function inside the rectangle

$[0, h_2] \times [0, h_1]$. The value of difference (287) at this point is

$$\left|\delta\left(\frac{h_1}{\sqrt{3}}, \frac{h_2}{\sqrt{3}}\right)\right| = \frac{2}{3\sqrt{3}}\left|Bh_2^3 - Ah_1^3\right|.$$

Also observe that on the boundary of $[0, h_2] \times [0, h_1]$ we have

$$\delta(x, h_2) = \delta(x, 0) = Ax^3 - Ah_1^2 x,$$

$$\delta(h_1, y) = \delta(0, y) = -By^3 + Bh_2^2 y.$$

Hence, the maximal values are

$$\left|\delta\left(\frac{h_1}{\sqrt{3}}, h_2\right)\right| = \frac{2}{3\sqrt{3}}Ah_1^3$$

$$\left|\delta\left(h_1, \frac{h_2}{\sqrt{3}}\right)\right| = \frac{2}{3\sqrt{3}}Bh_2^3.$$

Therefore, in the case of the uniform norm the extremal problem (285) can be rewritten as follows. Find

$$d := \min_{h_1,h_2} \max\left\{\frac{2}{3\sqrt{3}}\left|Bh_2^3 - Ah_1^3\right|, \frac{2}{3\sqrt{3}}Ah_1^3, \frac{2}{3\sqrt{3}}Bh_2^3\right\} \tag{288}$$

under the condition that the area of rectangle $[0, h_1] \times [0, h_2]$ is fixed, i.e. $h_1 h_2 = S$. It is easy to see that this minimum will be attained when $\sqrt[3]{A}h_1 = \sqrt[3]{B}h_2$. From here and the condition $h_1 h_2 = S$ we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/6}\sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/6}\sqrt{S}.$$

Plugging this back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d = \frac{2}{3\sqrt{3}}\sqrt{AB}S^{\frac{3}{2}}.$$

Therefore, for the error on the whole R we have

$$\frac{d_{Q,R,\infty}}{|R|^{\frac{3}{2}}} = \frac{1}{12\sqrt{3}}\sqrt{AB} \tag{289}$$

which gives solution to (285) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_{\infty}^{-} = \frac{1}{12\sqrt{3}}. \tag{290}$$

VI.2.2 The L_2 norm.

Let again $I_{Q,R}(x, y)$ denote the biquadratic spline which interpolates the cubic function

$$Q(x, y) = Ax^3 - By^3$$

with $AB > 0$ at the vertices and at the midpoints of the rectangle R. Let also

$$d_{Q,R,2} := \int_R \int (Q(x, y) - I_{Q,R}(x, y))^2 dxdy, \tag{291}$$

and let $|R|$ denote the area of the rectangle R. The problem is to characterize those rectangles R which minimize the ratio

$$\frac{d_{Q,R,2}}{|R|^2} \to \min, \tag{292}$$

and find the minimum of it.

The following lemma gives the answer to this problem (again it is enough to consider the error of interpolation on the symmetric rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ centered at the origin).

Lemma 47. *Let the function*

$$Q(x, y) = Ax^3 - By^3$$

with $AB > 0$ be given. For an arbitrary rectangle $R = [-h_1, h_1] \times [-h_2, h_2]$ we have

$$\frac{d_{Q,R,2}}{|R|^2} \geq \sqrt{\frac{23}{53760}} \sqrt{AB}. \tag{293}$$

Moreover, the equality is obtained for rectangles with $\sqrt[3]{A}h_1 = \sqrt[3]{B}h_2$ and only for them.

Proof: As we mentioned before, the interpolant on the rectangle R is

$$I_{Q,R}(x,y) = Ah_1^2x - Bh_2^2y.$$

In the case of L_2 norm, extremal problem (292) can be rewritten as follows. Find

$$d^2 := \min_{h_1,h_2} \int_0^{h_1} \int_0^{h_2} \left(Ax^3 - By^3 - Ah_1^2x + Bh_2^2y\right)^2 dxdy \tag{294}$$

under the condition that the area of rectangle $[0,h_1] \times [0,h_2]$ is fixed, i.e. $h_1h_2 = S$. We have

$$\int_0^{h_1} \int_0^{h_2} \left(Ax^3 - By^3 - Ah_1^2x + Bh_2^2y\right)^2 dxdy = \frac{1}{840}h_1h_2\left(64A^2h_1^6 - 105ABh_1^3h_2^3 + 64B^2h_2^6\right).$$

Hence, we have to minimize the function

$$\frac{1}{840}h_1h_2\left(64A^2h_1^6 - 105ABh_1^3h_2^3 + 64B^2h_2^6\right)$$

under the condition $h_1h_2 = S$. Considering the corresponding Lagrangian and setting its partial derivatives with respect to h_1 and h_2 equal to zero, we obtain

$$\frac{1}{840}h_1h_2\left(384A^2h_1^5 - 315ABh_1^2h_2^3\right) + \frac{1}{840}h_2\left(64A^2h_1^6 - 105ABh_1^3h_2^3 + 64B^2h_2^6\right) - \lambda h_2 = 0$$

$$\frac{1}{840}h_1h_2\left(-315ABh_1^3h_2^3 + 384B^2h_2^6\right) + \frac{1}{840}h_1\left(64A^2h_1^6 - 105ABh_1^3h_2^3 + 64B^2h_2^6\right) - \lambda h_1 = 0$$

and, it is easy to check that the minimum is attained for $\sqrt[3]{A}h_1 = \sqrt[3]{B}h_2$. From here and the condition $h_1h_2 = S$ we can find h_1 and h_2:

$$h_1 = \left(\frac{B}{A}\right)^{1/6}\sqrt{S}, \quad h_2 = \left(\frac{A}{B}\right)^{1/6}\sqrt{S}.$$

Plugging this back into the expression for the error gives the minimal value for the error over the rectangle of area S:

$$d^2 = \frac{23}{840}ABS^4.$$

Therefore, for the error on the whole R (recall that $|R| = 4S$) we have

$$\frac{d_{Q,R,2}}{|R|^2} = \sqrt{\frac{23}{53760}}\sqrt{AB} \tag{295}$$

which gives the solution to (292) and completes the proof of the lemma. □

Remark. The previous lemma gives

$$M_2^+ = \sqrt{\frac{23}{53760}}. \tag{296}$$

VI.3 Estimate of the deviation of the third degree Taylor polynomial for C^3 functions.

Let us define the modulus of continuity of $f \in C^3(D)$ as follows

$$\omega(f,\delta) := \sup\{|f(x,y) - f(x',y')| : \ |x - x'| \leq \delta, |y - y'| \leq \delta, \ (x,y),(x',y') \in D\}. \tag{297}$$

Set

$$\omega_1(\delta) := \omega(f_{xxx},\delta), \ \ \omega_2(\delta) := \omega(f_{xxy},\delta),$$

$$\omega_3(\delta) := \omega(f_{xyy},\delta), \ \ \omega_4(\delta) := \omega(f_{yyy},\delta)$$

and

$$\omega(\delta) := \max\{\omega_1(\delta),\omega_2(\delta),\omega_3(\delta),\omega_4(\delta)\}. \tag{298}$$

Lemma 48. *Let $f \in C^3(D)$. If $P_3(x,y)$ denotes the Taylor polynomial of third degree for f at the center of a square D_h with side length equal to h, then we have the following estimate*

$$|f(x,y) - P_3(x,y)| \leq \frac{h^3}{6}\omega\left(\frac{h}{2}\right), \tag{299}$$

where $\omega(t)$ is defined at (298).

Proof: The Taylor formula for $f \in C^3(D)$ about point (x_0,y_0) with remainder in the Lagrange form is

$$f(x,y) = P_2(x,y) + R_2(x,y), \tag{300}$$

where

$$P_2(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f_{xx}(x_0,y_0)\frac{(x-x_0)^2}{2}$$

$$+f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)\frac{(y-y_0)^2}{2}$$

and

$$\begin{aligned} R_2(x,y) &= \frac{1}{3!}f_{xxx}(x_0+\theta(x-x_0), y_0+\theta(y-y_0))(x-x_0)^3 \\ &+ \frac{1}{2}f_{xxy}(x_0+\theta(x-x_0), y_0+\theta(y-y_0))(x-x_0)^2(y-y_0) \\ &+ \frac{1}{2}f_{xyy}(x_0+\theta(x-x_0), y_0+\theta(y-y_0))(x-x_0)(y-y_0)^2 \\ &+ \frac{1}{3!}f_{yyy}(x_0+\theta(x-x_0), y_0+\theta(y-y_0))(y-y_0)^3 \end{aligned} \tag{301}$$

with $\theta \in (0,1)$. We add and subtract the term

$$\begin{aligned} &\frac{1}{3!}f_{xxx}(x_0,y_0)(x-x_0)^3 + \frac{1}{2}f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0) \\ + &\frac{1}{2}f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 + \frac{1}{3!}f_{yyy}(x_0,y_0)(y-y_0)^3 \end{aligned} \tag{302}$$

to the right-hand side of (301). Then (300) can be rewritten as

$$f(x,y) = P_3(x,y) + R_3(x,y),$$

where

$$\begin{aligned} P_3(x,y) &= P_2(x,y) + \frac{1}{3!}f_{xxx}(x_0,y_0)(x-x_0)^3 + \frac{1}{2}f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0) \\ &+ \frac{1}{2}f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 + \frac{1}{3!}f_{yyy}(x_0,y_0)(y-y_0)^3, \end{aligned} \tag{303}$$

and

$$\begin{aligned}R_3(x,y) &= \frac{(x-x_0)^3}{3!}\left(f_{xxx}(x_0+\theta(x-x_0),y_0+\theta(y-y_0))-f_{xxx}(x_0,y_0)\right)\\ &+ \frac{(x-x_0)^2(y-y_0)}{2}\left(f_{xxy}(x_0+\theta(x-x_0),y_0+\theta(y-y_0))-f_{xxy}(x_0,y_0)\right)\\ &+ \frac{(x-x_0)(y-y_0)^2}{2}\left(f_{xyy}(x_0+\theta(x-x_0),y_0+\theta(y-y_0))-f_{xyy}(x_0,y_0)\right)\\ &+ \frac{(y-y_0)^3}{3!}\left(f_{yyy}(x_0+\theta(x-x_0),y_0+\theta(y-y_0))-f_{yyy}(x_0,y_0)\right). \qquad (304)\end{aligned}$$

By (298) and the triangle inequality we obtain the estimate

$$\begin{aligned}|R_3(x,y)| &\le \frac{(x-x_0)^3}{3!}\omega(\max\{|x_0+\theta(x-x_0)|,|y_0+\theta(y-y_0)|\})\\ &+ \frac{(x-x_0)^2(y-y_0)}{2}\omega(\max\{|x_0+\theta(x-x_0)|,|y_0+\theta(y-y_0)|\})\\ &+ \frac{(x-x_0)(y-y_0)^2}{2}\omega(\max\{|x_0+\theta(x-x_0)|,|y_0+\theta(y-y_0)|\})\\ &+ \frac{(y-y_0)^3}{3!}\omega(\max\{|x_0+\theta(x-x_0)|,|y_0+\theta(y-y_0)|\}). \qquad (305)\end{aligned}$$

In other words, for every (x, y) from the square D_h centered at (x_0, y_0) with the side length equal to h we obtain the estimate

$$|f(x,y)-P_3(x,y)| \le \frac{h^3}{6}\omega\left(\frac{h}{2}\right). \qquad (306)$$

□

Corollary. Under the assumptions of the theorem we have

$$\|f-P_3\|^p_{L_p(D)} = \int_0^h\int_0^h (f(x,y)-P_3(x,y))^p\,dxdy \le h^2\|f-P_3\|^p_{L_\infty(D)}$$

and, hence,

$$\|f-P_3\|_p \le \frac{h^{3+\frac{2}{p}}}{6}\omega\left(\frac{h}{2}\right). \qquad (307)$$

VI.4 Error of biquadratic interpolation of C^3 functions defined on $[0,1]^2$. Estimate from above.

Proof of Theorem 14: For the fixed $\varepsilon \in (0,1)$ and for every $N \in \mathbb{N}$ we define

$$m_N := \min\left\{ m > 0 : \quad \frac{1}{6m^{2\ \frac{3}{2}+\frac{1}{p}}} \omega\left(\frac{1}{2m}\right) \le \frac{\varepsilon}{N^{\frac{3}{2}}} \right\}, \tag{308}$$

where $\omega(\delta)$ is the function defined in (298).

Observe that clearly for m_N defined in (308) it is true that $m_N \to \infty$ as $N \to \infty$. In addition,

$$\frac{N^{\frac{3}{2}}}{m_N^{3+\frac{2}{p}}} \to \infty, \quad N \to \infty, \tag{309}$$

i.e. $m_N = o\left(N^{\frac{3}{2(3+\frac{2}{p})}}\right)$ as $N \to \infty$.

Indeed, by the definition of m_N for all large enough N we have

$$\frac{N^{\frac{3}{2}}}{m_N^{3+\frac{2}{p}}} = 6\frac{(m_N-1)^{3+\frac{2}{p}}}{m_N^{3+\frac{2}{p}}} \frac{1}{\omega\left(\frac{1}{2(m_N-1)}\right)} \frac{1}{6} \frac{N^{\frac{3}{2}}}{(m_N-1)^{3+\frac{2}{p}}} \omega\left(\frac{1}{2(m_N-1)}\right)$$

$$\ge \varepsilon 6 \frac{(m_N-1)^{3+\frac{2}{p}}}{m_N^{3+\frac{2}{p}}} \frac{1}{\omega\left(\frac{1}{2(m_N-1)}\right)} \to \infty, \quad \text{as } N \to \infty,$$

since $\left(\frac{m_N-1}{m_N}\right)^{3+\frac{2}{p}} \to 1$ and $\omega\left(\frac{1}{2(m_N-1)}\right) \to 0$ as $N \to \infty$. Hence, (309) is proved.

Divide the unit square $[0,1] \times [0,1]$ into squares with side length equal to $\frac{1}{m_N}$ and denote the resulting squares by D_i^N, $i = 1, \ldots, m_N^2$. Next we take the center point (x_i^N, y_i^N) in each square D_i^N and set

$$A_i^N := \frac{1}{6} f_{xxx}(x_i^N, y_i^N), \quad B_i^N := \frac{1}{6} f_{yyy}(x_i^N, y_i^N).$$

Note that

$$|H(x_i^N, y_i^N)| := |H(f; x_i^N, y_i^N)| = 6|A_i^N B_i^N| \ge C^+, \quad \forall i = 1, \ldots, m_N^2. \tag{310}$$

Set

$$n_i^N := \left\lceil \frac{N(1-\varepsilon)|H(x_i^N, y_i^N)|^{\frac{p}{3p+2}}\Omega(x_i^N, y_i^N)^{\frac{2}{3p+2}}}{\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{3p+2}}\Omega(x_j^N, y_j^N)^{\frac{2}{3p+2}}} \right\rceil, \quad i = 1, ..., m_N^2. \tag{311}$$

The n_i^N, $i = 1, \ldots, m_N^2$, are determined by minimizing the sum of the errors of the interpolation of piecewise quadratic functions on each region subject to the condition the total number of rectangles is N.

Observe that all $n_i^N \to \infty$ when $N \to \infty$.

Having the number of rectangles n_i^N on each region D_i^N, $i = 1, ..., m_N^2$, we construct a partition $\Box_N^*(D_i^N)$ of D_i^N in the following way. Let $h_{i,1}^N$ and $h_{i,2}^N$ be the linear sizes of the optimal rectangle on D_i^N, i.e. the one which solves the problem

$$\frac{L_p - \text{error of biquadratic interpolation of} \quad A_i^N x^3 \pm B_i^N y^3 \;\; \text{on} \;\; R}{|R|^{\frac{3}{2}+\frac{1}{p}}}. \tag{312}$$

Observe that due to Lemmas 44-47 we can provide explicit formulas for $h_{i,1}^N$ and $h_{i,2}^N$ in the cases $p = 1, 2, \infty$

$$h_{i,1}^N = \left(\frac{B_i^N}{A_i^N}\right)^{1/6} \frac{1}{m_N\sqrt{n_i^N}}, \qquad h_{i,2}^N = \left(\frac{A_i^N}{B_i^N}\right)^{1/6} \frac{1}{m_N\sqrt{n_i^N}}, \qquad i = 1, ..., m_N^2. \tag{313}$$

Most likely these formulae will provide the optimal parameters for all other values of p as well.

The intersection of the lattice

$$[kh_{i,1}^N, (k+1)h_{i,1}^N] \times [lh_{i,2}^N, (l+1)h_{i,2}^N], \quad k, l \in \mathbb{Z}, \tag{314}$$

with D_i^N gives us the partition $\Box_N^*(D_i^N)$ of D_i^N, $i = 1, ..., m_N^2$.

Define the biquadratic spline on each element of the partition obtained by interpolating the original function $f(x, y)$ at the eight points as shown on Figure 9.

By f_N denote the piecewise cubic function constructed in the following way. On D_1^N we set f_N to be $A_1^N x^3 + B_1^N y^3$. Then for $i > 1$ on $D_i^N \setminus \cup_{j=1}^{i-1} D_j^N$ we set

$$f_N(x, y) := A_i^N x^3 + B_i^N y^3.$$

If a function f is such that $f_{xxx} f_{yyy} > 0$ for all $(x, y) \in D$ then all $B_i^N > 0$, $i = 1, \ldots, m_N^2$. If a function f is such that $f_{xxx} f_{yyy} < 0$ for all $(x, y) \in D$ then all $B_i^N < 0$, $i = 1, \ldots, m_N^2$.

Observe that

$$\|f - s(f, \square_N^*)\|_{p,\Omega} \leq 2\|f - f_N\|_{p,\Omega} + \|f_N - s(f_N, \square_N^*)\|_{p,\Omega}.$$

Let us estimate each term. First of all, by Lemma 48 and the definition of m_N we have

$$\|f - f_N\|_{p,\Omega} \leq \frac{\|\Omega\|_\infty^{\frac{1}{p}}}{2m_N^{3+\frac{2}{p}}} \omega\left(\frac{1}{2m_N}\right) \leq \frac{\varepsilon}{N^{\frac{3}{2}}} \|\Omega\|_\infty^{\frac{1}{p}}.$$

Let us estimate the second term now. It is clear that for two embedded rectangles the error of linear interpolation of a quadratic function with $|H(f; x, y)| \geq C^+ > 0$ will be greater on the larger rectangle. Therefore, we shall estimate this error on rectangles that do not have an intersection with the boundary.

Let us take a rectangle $R_i^N \in \square_N^*(D_i^N)$ that does not have common points with the boundary of D_i^N. By Lemmas 44-47, for every $(x, y) \in R_i^N$ we have

$$|f_N(x, y) - s(f_N, \square_N^*; x, y)|^p \leq \left(\frac{M_p^\pm}{6}\right)^p |H(x_i^N, y_i^N)|^{p/2} \frac{1}{(m_N^2 n_i^N)^{\frac{3}{2}p+1}}.$$

Hence, the p-power of the error on the whole D is bounded by

$$|f_N(x, y) - s(f_N, \square_N^*; x, y)|^p \Omega(x, y) \leq \left(\frac{M_p^\pm}{6}\right)^p \sum_{i=1}^{m_N^2} n_i^N \Omega(x_i^N, y_i^N) |H(x_i^N, y_i^N)|^{p/2} \frac{1}{(m_N^2 n_i^N)^{\frac{3}{2}p+1}}.$$

By the definition of n_i^N and by (54), for all large enough N, for all i, and for all $(x, y) \in D$, we have

$$
\begin{aligned}
&|f_N(x,y)-s(f_N,\square_N^*;x,y)|^p\Omega(x,y)\\
\le\ & \left(\frac{M_p^{\pm}}{6}\right)^p \frac{(1+\varepsilon)}{m_N^{2(\frac{3}{2}p+1)}}\sum_{i=1}^{m_N^2}|H(x_i^N,y_i^N)|^{p/2}\Omega(x_i^N,y_i^N)\left(\frac{\sum_{j=1}^{m_N^2}|H(x_j^N,y_j^N)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x_j^N,y_j^N)^{\frac{1}{\frac{3}{2}p+1}}}{N(1-\varepsilon)|H(x_i^N,y_i^N)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x_i^N,y_i^N)^{\frac{1}{\frac{3}{2}p+1}}}\right)^{\frac{3}{2}p}\\
\le\ & \left(\frac{M_p^{\pm}}{6}\right)^p \frac{(1+2\varepsilon)}{N^{\frac{3}{2}p}m_N^{2(\frac{3}{2}p+1)}}\left(\sum_{j=1}^{m_N^2}|H(x_j^N,y_j^N)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x_j^N,y_j^N)^{\frac{1}{\frac{3}{2}p+1}}\right)^{\frac{3}{2}p+1}.
\end{aligned}
$$

Since this estimate does not depend on x and y, we obtain that

$$
\|f_N-s(f_N,\square_N^*)\|_{p,\Omega}\le\frac{(1+2\varepsilon)M_p^{\pm}}{6N^{\frac{3}{2}}}\left(\frac{1}{m_N^2}\sum_{j=1}^{m_N^2}|H(x_j^N,y_j^N)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x_j^N,y_j^N)^{\frac{1}{\frac{3}{2}p+1}}\right)^{\frac{3}{2}+\frac{1}{p}}.
$$

Note that since $H(f;x,y)$ and $\Omega(x,y)$ are Riemann integrable

$$
\begin{aligned}
&\frac{1}{m_N^2}\sum_{j=1}^{m_N^2}|H(x_j^N,y_j^N)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x_j^N,y_j^N)^{\frac{1}{\frac{3}{2}p+1}} \qquad (315)\\
=\ & \sum_{j=1}^{m_N^2}|D_j^N||H(x_j^N,y_j^N)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x_j^N,y_j^N)^{\frac{1}{\frac{3}{2}p+1}}\to\int_D|H(f;x,y)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x,y)^{\frac{1}{\frac{3}{2}p+1}}dxdy
\end{aligned}
$$

as $N\to\infty$. Hence, for all N large enough we obtain

$$
\|f_N-s(f_N,\square_N^*)\|_{p,\Omega}<\frac{M_p^{\pm}}{6N^{\frac{3}{2}}}\frac{1+2\varepsilon}{1-\varepsilon}\left(\int_D|H(f;x,y)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x,y)^{\frac{1}{\frac{3}{2}p+1}}dxdy\right)^{\frac{3}{2}+\frac{1}{p}}.
$$

Therefore,

$$
\|f-s(f,\square_N^*)\|_{p,\Omega}<\frac{2\varepsilon}{N^{\frac{3}{2}}}\|\Omega\|_\infty^{\frac{1}{p}}+\frac{M_p^{\pm}}{6N^{\frac{3}{2}}}\frac{1+2\varepsilon}{1-\varepsilon}\left(\int_D|H(f;x,y)|^{\frac{p}{2(\frac{3}{2}p+1)}}\Omega(x,y)^{\frac{1}{\frac{3}{2}p+1}}dxdy\right)^{\frac{3}{2}+\frac{1}{p}}.
$$

Because $\varepsilon>0$ is arbitrary, we obtain the desired estimate. $\square$

VI.5 Error of biquadratic interpolation of C^3 functions defined on $[0,1]^2$. Estimate from below.

Let m_N, n_i^N, D_i^N etc. be as defined in the previous section. We shall also restart the numbering of constants.

Proof of Theorem 15: To obtain the estimate from below we shall consider an arbitrary sequence of partitions $\{\square_N\}_{N=1}^{\infty}$ satisfying (255).

For the fixed $\varepsilon > 0$ denote by $D_i^N(\varepsilon)$ the square congruent to D_i^N with side length equal to $\frac{1-\varepsilon}{m_N^2}$. Assumption (22) implies that for all large enough N and for any $\varepsilon > 0$ there exists a rectangle $R_{i_N}^N$ which lies completely in $D_i^N(\varepsilon)$.

For each such N and i_N, set

$$f_{N,i_N}(x,y) := A_{i_N}^N x^3 + B_{i_N}^N y^3.$$

If the function f is such that $(f_{xxx}f_{yyy})(x,y) > 0$ for all $(x,y) \in D$ then all $B_i^N > 0$, $i = 1,\ldots,m_N^2$. If the function f is such that $(f_{xxx}f_{yyy})(x,y) < 0$ for all $(x,y) \in D$ then all $B_i^N < 0$, $i = 1,\ldots,m_N^2$.

Observe that

$$\|f - s(f,\square_N)\|_{L_{p,\Omega}(R_{i_N}^N)} \geq \|f_{N,i_N} - s(f_{N,i_N},\square_N)\|_{L_{p,\Omega}(R_{i_N}^N)} - 2\|f - f_{N,i_N}\|_{L_{p,\Omega}(R_{i_N}^N)}.$$

By Lemmas 44-47 we have

$$\|f_{N,i_N} - s(f_{N,i_N},\square_N)\|^p_{L_{p,\Omega}(R_{i_N}^N)} \geq \frac{(1-\varepsilon)(M_p^{\pm})^p}{6^p(m_N^2 n_{i_N}^N)^{\frac{3}{2}p+1}} |H(x_{i_N}^N, y_{i_N}^N)|^{p/2}\Omega(x_{i_N}^N, y_{i_N}^N).$$

By the definition of $M_p^{\pm}$, (315), and by the definition of $n_{i_N}^N$ we have that for all N large enough the error on the whole region D_i^N

$$\begin{aligned}
&\sum_{i=1}^{m_N^2} \frac{(1-\varepsilon)(M_p^{\pm})^p}{(m_N^2 n_{i_N}^N)^{\frac{3}{2}p+1}} |H(x_i^N, y_i^N)|^{p/2} \Omega(x_i^N, y_i^N) n_{i_N}^N \\
\geq\ & (1-\varepsilon)\frac{(M_p^{\pm})^p}{m_N^{2(\frac{3}{2}p+1)}} \sum_{i=1}^{m_N^2} |H(x_i^N, y_i^N)|^{p/2} \Omega(x_i^N, y_i^N) \left(\frac{\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{\frac{3}{2}p+1}}}{N(1-\varepsilon)|H(x_i^N, y_i^N)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x_i^N, y_i^N)^{\frac{1}{\frac{3}{2}p+1}}} \right)^{\frac{3}{2}p} \\
=\ & \frac{(M_p^{\pm})^p (1-\varepsilon)}{(1-\varepsilon)^{\frac{3}{2}p} N^{\frac{3}{2}p} m_N^{2(\frac{3}{2}p+1)}} \left(\sum_{j=1}^{m_N^2} |H(x_j^N, y_j^N)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x_j^N, y_j^N)^{\frac{1}{\frac{3}{2}p+1}} \right)^{\frac{3}{2}p+1} \\
>\ & \frac{(M_p^{\pm})^p}{N^{\frac{3}{2}p}} \left(\int_D |H(f;x,y)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x,y)^{\frac{1}{\frac{3}{2}p+1}} dxdy \right)^{\frac{3}{2}p+1}.
\end{aligned}$$

Hence, for all N large enough we obtain

$$\|f_N - s(f_N, \Box_N^*)\|_{p,\Omega} > \frac{M_p^{\pm}}{6N^{\frac{3}{2}}}(1-\varepsilon) \left(\int_D |H(f;x,y)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x,y)^{\frac{1}{\frac{3}{2}p+1}} dxdy \right)^{\frac{3}{2}+\frac{1}{p}}.$$

On the other hand

$$\|f - f_{N,i_N}\|_{L_{p,\Omega}(R_{i_N}^N)} \leq \|f - f_{N,i_N}\|_{L_{p,\Omega}(D_{i_N}^N)} \leq \frac{\|\Omega\|_\infty^{\frac{1}{p}}}{6m_N^{2(\frac{3}{2}+\frac{1}{p})}} \omega\left(\frac{1}{2m_N}\right) \leq \frac{\varepsilon}{N^{\frac{3}{2}}} \|\Omega\|_\infty^{\frac{1}{p}}$$

due to the choice of m_N. Hence, we obtain that for all large enough N

$$\|f - s(f, \Box_N)\|_{p,\Omega} \geq (1 - c_2\varepsilon)\frac{M_p^{\pm}}{6N^{\frac{3}{2}}} \left(\int_D |H(f;x,y)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x,y)^{\frac{1}{\frac{3}{2}p+1}} dxdy \right)^{\frac{3}{2}+\frac{1}{p}}$$

with some positive constant c_2. Therefore,

$$\liminf_{N\to\infty} \frac{\|f - s(f, \Box_N)\|_{p,\Omega}}{\frac{M_p^{\pm}}{6N^{\frac{3}{2}}} \left(\int_D |H(f;x,y)|^{\frac{p}{2(\frac{3}{2}p+1)}} \Omega(x,y)^{\frac{1}{\frac{3}{2}p+1}} dxdy \right)^{\frac{3}{2}+\frac{1}{p}}} \geq 1.$$

This completes the proof of the theorem. □

Similarly (with obvious corresponding changes) the following lemmas can be proved.

Lemma 49. *Let $f \in C^3(D)$ and $|H(f;x,y)| \geq C^+ > 0$ for all $(x,y) \in D$. Let also positive* *continuous weight function $\Omega(x,y)$ be given. Then*

$$\limsup_{N\to\infty} N^{\frac{3}{2}}\|f - s(f,\Box_N)\|_{\infty,\Omega} \le \frac{M_\infty^\pm}{6}\left(\int_D |H(f;x,y)|^{\frac{1}{3}}\Omega(x,y)^{\frac{2}{3}}dxdy\right)^{\frac{3}{2}}, \tag{316}$$

where $M_\infty^\pm$ is as in (270) and (290).

Lemma 50. *Let $f \in C^3(D)$ and $|H(f;x,y)| \ge C^+ > 0$ for all $(x,y) \in D$. Let also positive continuous weight function $\Omega(x,y)$ be given. Then for any sequence of rectangular partitions which satisfies (255) we have*

$$\liminf_{N\to\infty} N^{\frac{3}{2}}\|f - s(f,\Box_N)\|_{\infty,\Omega} \ge \frac{M_\infty^\pm}{6}\left(\int_D |H(f;x,y)|^{\frac{1}{3}}\Omega(x,y)^{\frac{2}{3}}dxdy\right)^{\frac{3}{2}}, \tag{317}$$

where $M_\infty^\pm$ is as in (270) and (290).

CHAPTER VII

CONCLUDING REMARKS

In this dissertation we investigate the asympotic behavior of the L_p-error of interpolation and near interpolation of functions (bivariate and multivariate) by various classes of splines (linear, multilinear, biquadratic). The proofs of results obtained lead to algorithms for the construction of asymptotically optimal sequences of partitions (triangulations or rectangular partitions, respectively).

The results obtained as well as the algorithms based on them might find interesting applications in computational geometry, the problems of approximation of convex bodies by polytopes, computer-aided geometric design, etc.

As a next step it would be interesting to try to use techniques similar to those developed to obtain the exact asymptotics of the error in different norms for higher order splines (quadratic, cubic, etc.) and other types of splines (best approximating splines, best one-sided approximating splines etc.) in arbitrary dimensions, and to develop corresponding algorithms. We are also interested in considering the splines of maximal defect (discontinuous splines) in arbitrary dimensions that might have interesting applications in edge detection problems.

Because of the numerous applications, it would be interesting to write such algorithms for scattered data and for parametric surfaces. As an idea, we could use a least squares fit instead of the quadratic part of the Taylor polynomial at the stage of intermediate approximation if we have scattered data (as opposed to the continuous function).

We also plan to investigate other (different from those considered) interpolation schemes due to numerous applications in finite element methods for numerical solutions of PDE's.

[1] Azarin, V. S.; Barmin, V. I. *Approximation by piecewise linear functions* Mathematics collection, Izdat. "Naukova Dumka", Kiev, 1976, pp. 25–26.

[2] Azarin, V. S.; Barmin, V. I. *The approximation by piecewise linear functions* Teor. Funkcii, Funkcional. Anal. i Prilozhen. Vyp. 25 (1976), pp. 5–14.

[3] Babenko V.F., *Interpolation of Continuous Functions by Piecewise Linear Ones*, Math. Notes, 24, no.1, (1978) 43–53.

[4] Babenko V., Babenko Yu., Ligun A., Shumeiko A., *On asymptotical behavior of the optimal linear spline interpolation error of C^2 functions*, East J. Approx., V. 12, N. 1 (2006), 71–101.

[5] Beckenbach E., Bellman R., Inequalities, Berlin, Springer, 1961.

[6] Below A., De Loera J., Richter-Gebert J. *The complexity of finding small triangulations of convex 3-polytopes.* SODA 2000 special issue. J. Algorithms 50 (2004), no. 2, 134–167.

[7] Bern M., Eppstein D., *Mesh generation and optimal triangulation*, manuscript.

[8] Bertram M., Barnes J., Hamann B., Joy K., Pottmann H., Wushour D., *Piecewise optimal triangulation for the approximation of scattered data in the plane*, Comput. Aided Geom. Design 17, no. 8, (2000) 767–787.

[9] Böröczky K.,*Approximation of general smooth convex bodies*, Adv. in Math., 153 (2000) 325–341.

[10] Böröczky K., Finite packing and covering, Cambridge University Press, 2004.

[11] Böröczky K., Ludwig M., *Approximation of Convex Bodies and a Momentum Lemma for Power Diagrams*, Monatshefte für Mathematik, V. 127, N. 2, (1999) 101–110.

[12] Chen L. *Optimal Delaunay triangulations*, J. Comp. Math., Vol. 22, No. 2, 2004, 299-308.

[13] Chen L., Sun P., Xu J. *Optimal anisotropic meshes for minimizing interpolation errors in L_p-norm*, Math. Comp.,

[14] Coxeter H.S.M., Introduction to Geometry, 1969.

[15] D'Azevedo, E. F. *Are bilinear quadrilaterals better than linear triangles?* SIAM J. Sci. Comput. 22 (2000), no. 1, 198–217.

[16] D'Azevedo, E. F. *Optimal triangular mesh generation by coordinate transformation.* SIAM J. Sci. Statist. Comput. 12 (1991), no. 4, 755–786.

[17] D'Azevedo, E. F.; Simpson, R. B. *On optimal interpolation triangle incidences*. SIAM J. Sci. Statist. Comput. 10 (1989), no. 6, 1063–1075.

[18] Delone, B. N.; Ryshkov, S. S. *Extremal problems of the theory of positive quadratic forms*. (Russian) Collection of articles dedicated to Academician Ivan Matveevich Vinogradov on his eightieth birthday, I. Trudy Mat. Inst. Steklov. 112 (1971), 203–223, 387.

[19] Demaret L., Dyn N., Iske A., *Image compression by linear splines over adaptive triangulations*, IEEE Transactions on Image Processing.

[20] Dubrovin B.A., Fomenko A.T., Novikov S.P., Modern Geometry - methods and applications, Springer, 1984.

[21] Dyn N., Levin D., Rippa S., *Data dependent triangulations for piecewise linear interpolation,* IMA J. Numer. Anal., 10, no. 1 (1990) 137–154.

[22] Dyn N., Levin D., Rippa S., *Boundary correction for piecewise linear interpolation defined over data-dependent triangulations*, Journal of Computational and Applied Mathematics, 39, 1992, pp. 179–192.

[23] Edelsbrunner H., Algorithms in Combinatorial Geometry, EATCS Monographs on Theoretical Computer Science, Volume 10, Springer-Verlag, 1987.

[24] Fejes Toth L., Lagerungen in der Ebene, auf der Kugel und im Raum, 2nd edn. Berlin: Springer, 1972.

[25] Fortune, S. *Voronoi diagrams and Delaunay triangulations.* Handbook of discrete and computational geometry, 377–388, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, 1997.

[26] Goodman J., O'Rourke J., (editors) Handbook of Discrete and Computational Geometry, CRC Press, 2004.

[27] Grebennikov, A. I. *The choice of nodes in the spline approximation of functions* Ž. Vyčisl. Mat. i Mat. Fiz. 16 (1976), no. 1, 219–223, 278.

[28] Gruber P., *Volume approximation of convex bodies by inscribed polytopes*, Math. Ann., V. 281, 1988, pp.229–245.

[29] Gruber P., *Aspects of approximation of convex bodies.* In: GRUBER P.M., WILLS J (eds) Handbook of Convex Geometry A, Amsterdam: North-Holland, (1999) 319–345.

[30] Gruber P., *Error of asymptotic formulae for volume approximation of convex bodies in E^d,* Monatsh. Math. 135 (2002) 279-304.

[31] Handscomb, D.C., *Errors of linear interpolation on a triangle*, Report N. 95, 1995.

[32] Huang, W. *Variational mesh adaptation: isotropy and equidistribution.* J. Comput. Phys. 174 (2001), no. 2, 903–924.

[33] Huang, W.; Sun, W. *Variational mesh adaptation. II. Error estimates and monitor functions.* J. Comput. Phys. 184 (2003), no. 2, 619–648

[34] Ligun A.A., Shumeiko A.A., *Optimal choice of knots for approximation by splines*, Dokl. AN USSR, A, N. 6, (1984) 18–22.

[35] Ligun A.A., Shumeiko A.A., Asymptotic methods of curve recovery, *Kiev. Inst. of Math. NAS of Ukraine*, 1997. (in Russian)

[36] Ligun, A. A.; Storčaĭ, V. F. *Best choice of nodes in the approximation of functions by local Hermite splines* Ukrain. Mat. Zh. 32 (1980), no. 6, pp. 824–830.

[37] Ligun, A. A.; Storchaĭ, V. F. *Interpolation of functions by cubic Hermitian splines* Izv. Vyssh. Uchebn. Zaved. Mat. 1982, no. 6, pp. 26–29.

[38] Manzi, C.; Rapetti, F.; Formaggia, L. *Function approximation on triangular grids: some numerical results using adaptive techniques.* (English. English summary) Numerical grid generation-technologies for advanced simulations (Berlin, 1997). Appl. Numer. Math. 32 (2000), no. 4, 389–399

[39] Nadler E., *Piecewise linear best L_2 approximation on triangles,* in: Chui, C.K., Schumaker, L.L. and Ward, J.D. (Eds.), Approximation Theory V, Academic Press, (1986) 499–502.

[40] Nadler E., *Piecewise linear approximation on triangulations of a planar region*, PhD thesis, 1986.

[41] Pottmann H., Krasauskas R., Hamann B. Joy K., Seibold W., *On piecewise linear approximation of quadratic functions*, J. Geom. Graph. 4, no. 1, (2000) 31–53.

[42] Powell, M. J. D. *On best L_2 spline approximations* 1968 Numerische Mathematik. Differentialgleichungen. Approximationstheorie (Tagungen, Oberwolfach, 1966) Birkhuser, Basel, pp. 317–339.

[43] Rajan V. T., *Optimality of the Delaunay triangulations in $\mathbb{R}^d$*, Discrete Comput. Geom., 12: 189–202, 1994.

[44] Rippa, S., *Long and thin triangles can be good for linear interpolation*, SIAM J. Num. An., Vol. 29, No. 1 (1992), 257–270.

[45] Rogers, C.A., Packing and Covering, Cambridge University Press, Cambridge, 1964.

[46] Schneider R. *Zur optimalen Approximation konvexer Hyperflchen durch Polyeder.* Math. Ann. 256 (1981), no. 3, 289–301.

[47] Schumaker L. L., Spline Functions : Basic Theory, 1981, p. 553.

[48] Subbotin, Ju. N.; Černykh, N. I. *The order of the best spline approximations of certain classes of functions* Mat. Zametki 7, 1970, pp. 31–42.