ON QUASICONVEX SUBSETS OF HYPERBOLIC GROUPS

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Dedicated to my mother, Margarita, and my brother, Vardan.

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CHAPTER I

INTRODUCTION

The subject of this thesis belongs to the area of Geometric Group Theory. This field of mathematics has been developing rapidly since 1987, when M. Gromov introduced the concept of a (*word*) *hyperbolic group* [11]. Its main idea is to use geometrical and topological methods in the study of abstract algebraic objects – groups.

The idea was not new the time: similar approaches were exploited for investigating properties of fundamental groups of surfaces, lattices in Lie groups and small cancellation groups. Gromov's notion of hyperbolicity generalized many them and provided more convenient research tools. The potential of this theory was once again confirmed by A. Ol'shanskii, who showed that in a certain statistical sense almost every finitely presented group is hyperbolic [24].

Certain subgroups of hyperbolic groups, distinguished for having a lot of "nice" properties, are called *quasiconvex*. A subgroup H of a hyperbolic group G is quasiconvex if and only if its embedding into G is a *quasiisometry* (in other words, H is *undistorted* in G). For example, any finitely generated subgroup in a finite rank free group is quasiconvex. Quasiconvex subgroups have been studied thoroughly and a lot of information is known about them. Naturally, the research in this area could continue in two different directions. One would be to look at arbitrary subgroups, the other – to learn something about quasiconvex subsets. The first direction proved to be very complex and still remains practically unexplored. The goal of this work is to cast some light in the second direction.

In Chapters 2 and 3 we remind the reader basic definitions and facts concerning hyperbolic spaces, groups and quasiconvex subsets. The concept of *Gromov boundary* for a hyperbolic group plays an important role in our arguments, and Chapter 5 provides some information about it.

Let us recall that if H is a subgroup of a group G then its *virtual normalizer* (*commensurator*) is defined by the formula

$$VN_G(H) = \{g \in G \mid |H : (H \cap gHg^{-1})| < \infty, |gHg^{-1} : (H \cap gHg^{-1})| < \infty\}, \tag{1.1}$$

where $|H : (H \cap gHg^{-1})|$ denotes the index of the subgroup $K = (H \cap gHg^{-1})$ in H.

In Chapter 4 we establish a definition for a *subset commensurator* in an arbitrary group which generalizes (1.1). A result obtained by G. Arzhantseva [2] and, independently, I. Kapovich and H. Short [14], states that a quasiconvex subgroup of a hyperbolic group has a finite index in its commensurator. Natural analogues of this and other properties are proved for commensurators of quasiconvex subsets in Chapter 6.

The second half of this thesis studies homomorphisms of hyperbolic groups that preserve given quasiconvex subsets. A subgroup H is called a G-subgroup of a hyperbolic group G if for any finite subset $M \subset G$ there exists a homomorphism from G onto a non-elementary hyperbolic group G_1 that is surjective on H and injective on M. In his initial paper about hyperbolic groups Gromov claimed that every non-elementary subgroup possesses this property ([11, Thm. 5.5.A]). However, Ol'shanskii in [26] noted that this is not quite correct (if a hyperbolic group has non-trivial torsion, then it may have non-elementary subgroups that are not G-subgroups). In the same article he gave a description of all G-subgroups in any given non-elementary hyperbolic group. The paper [26] develops an important theory of small cancellations over hyperbolic groups and has a number of strong consequences. One of them is the fact that every torsion-free word hyperbolic group has a non-abelian quotient, all of whose proper subgroups are cyclic.

In Chapter 11 we combine small cancellation methods from [26] with hyperbolic boundary techniques to show that for the same class of G-subgroups the finiteness assumption on M (under certain natural conditions) can be replaced by an assumption of quasiconvexity. This allows to achieve several new embedding theorems for word hyperbolic groups. For example, we show that if G and H are hyperbolic groups and G is non-elementary, then there exists a simple quotient M of G that contains an isomorphic copy H' of H. If, in addition, the groups G and H are torsion-free, one is able obtain such a torsion-free quotient M with a much stronger property: H' is a proper malnormal subgroup of M and every proper subgroup of M is conjugated to a subgroup of H'. Another interesting result, we obtain, is the existence of a simple group which is a quotient of every non-elementary hyperbolic group and contains an isomorphic copy of each hyperbolic group as its subgroup.

Summarizing this work, we list the main results below.

We define a notion of a *tame* subset in a hyperbolic group (Definition 6.11), study its properties (Lemmas 6.2, 6.3), and use it to show that the commensurator subgroup of any sufficiently large quasiconvex subset is contained in a bounded neighborhood of this subset (Proposition 6.6). Further, we characterize all tame quasiconvex subsets of a hyperbolic group (Corollary 6.8).

Next, for any subgroup K of a fixed hyperbolic group G we establish a concept of a quasiconvex subset Q that is *small relatively to* K. In the case when $Q = H$ is a quasiconvex subgroup of G, Corollary 7.4 shows that H is small relatively to K if and only if the index $|K : (K \cap fHf^{-1})|$ is infinite for each $f \in G$. Then we prove that the property of being relatively small is stable under taking unions and products (Lemma 7.9). Consequently, we achieve Corollary 7.2 claiming that a hyperbolic group can not coincide with a product of finitely many quasiconvex subgroups having infinite index in it. This generalizes the previously known fact that any bounded-generated hyperbolic group is virtually cyclic.

The rest of the thesis is devoted to construction of residualizing homomorphisms preserving quasiconvex subsets (Theorem 11.1). We conclude with three new embedding theorems for hyperbolic groups (Theorems 12.1, 12.4 and 12.5) obtained as corollaries of the latter result.

CHAPTER II

PRELIMINARIES

In what follows we assume familiarity with basic concepts from combinatorial group theory (see [16]), topology (see [6]) and metric spaces (see [4]).

Hyperbolic Metric Spaces

In this section we discuss various definitions of a hyperbolic space and its basic properties.

Suppose $\mathcal{X}$ is a geodesic metric space with metric $d(\cdot,\cdot)$. If $Q \subset \mathcal{X}$ and $N \geq 0$, the *closed N-neighborhood* of Q in X will be denoted by

$$\mathcal{O}_N(Q) \stackrel{def}{=} \{x \in \mathcal{X} \mid \exists\ y \in Q \text{ s.t. } d(x,y) \leq N\}.$$

In [11] for any three elements $x, y, w \in \mathcal{X}$, M. Gromov defined the *Gromov product* of x and y with respect to w as follows:

$$(x|y)_w \stackrel{def}{=} \frac{1}{2}\Big(d(x,w) + d(y,w) - d(x,y)\Big)$$

Originally the concept of a hyperbolic metric space was introduced by Gromov in [11]:

Definition 2.1. the space $\mathcal{X}$ is called *hyperbolic* if there exists $\delta \geq 0$ such that for any four points $x, y, z, w \in \mathcal{X}$ their Gromov products satisfy

$$(x|y)_w \geq min\{(x|z)_w, (y|z)_w\} - \delta.$$

Later several other equivalent definitions were established. In addition to the definition above, we will be using two more below.

Let abc be a geodesic triangle in the space $\mathcal{X}$ and $[a,b]$, $[b,c]$, $[a,c]$ be its sides between the corresponding vertices. Then there exist "special" points $O_a \in [b,c]$, $O_b \in [a,c]$, $O_c \in [a,b]$ with the properties: $d(a,O_b) = d(a,O_c) = \alpha$, $d(b,O_a) = = d(b,O_c) = \beta$, $d(c,O_a) = d(c,O_b) = \gamma$. From a corresponding system of linear equations one can find that $\alpha = (b|c)_a$, $\beta = (a|c)_b$, $\gamma = (a|b)_c$. Two points $O \in [a,b]$ and $O' \in [a,c]$ are called *a-equidistant* if $d(a,O) = d(a,O') \leq \alpha$. The triangle abc is said to be *δ-thin* if for any two points O, O' lying on its sides and equidistant from one of its vertices, $d(O,O') \leq \delta$ holds (Figure 1).

Definition 2.2. ([8],[1]) $\mathcal{X}$ is said to be to be *hyperbolic* if all geodesic triangles in $\mathcal{X}$ are *δ-thin* for some fixed $\delta \geq 0$.

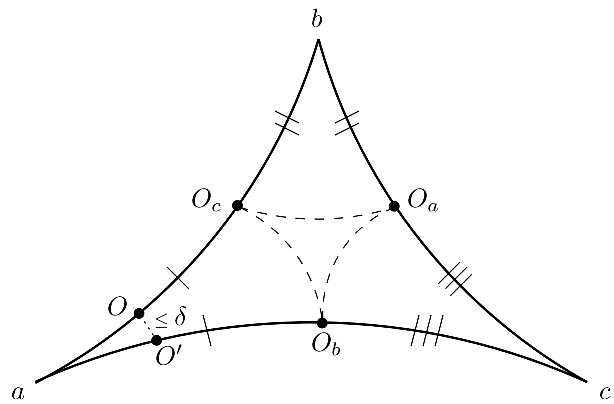

Figure 1: A δ-thin triangle.

A geodesic n-gon in the space $\mathcal{X}$ is said to be δ-*slim* if each of its sides belongs to a closed δ-neighborhood of the union of the others. The definition below is due to E. Rips:

Definition 2.3. ([8],[1]) $\mathcal{X}$ is said to be to be *hyperbolic* if all geodesic triangles in $\mathcal{X}$ are δ-*slim* for some fixed $\delta \geq 0$.

Remark 2.1. It is easy to see that the Definition 2.3 implies that any geodesic n-gon in the space $\mathcal{X}$ is $(n-2)\delta$-slim if $n \geq 3$.

It was shown in [8],[1] that the Definitions 2.1,2.2, 2.3 are equivalent, although the corresponding values of δ may vary.

Example 2.4. Consider a tree Γ with the usual path-length metric (every edge has length 1 and is isometric to the segment $[0,1] \subset \mathbb{R}$. Then Γ is a δ-hyperbolic metric space with $\delta = 0$.

Example 2.5. Evidently the Euclidean space $\mathbb{R}^n$ satisfies Definition 2.3 if and only if $n = 1$. Thus $\mathbb{R}^1$ is hyperbolic and $\mathbb{R}^n$ is not if $n > 1$.

For any two points $x, y \in \mathcal{X}$, $[x, y]$ will denote a geodesic path between them (if there are several such path, we choose one of them). Let p be a path in $\mathcal{X}$. Then p_-, p_+ will denote the startpoint and the endpoint of p, $\|p\|$ – its length. p^{-1} will be the inverse path to p, i.e., the path with the same set of points but traced in the opposite direction.

Definition 2.6. A path q is called (λ, c)- *quasigeodesic* if there exist $0 < \lambda \leq 1$, $c \geq 0$, such that for any subpath p of q the inequality $\lambda\|p\| - c \leq d(p_-, p_+)$ holds.

The latter concept is very important in the contemporary Geometric Group Theory and will be used a lot throughout this work.

The key property states that in a hyperbolic space quasigeodesics and geodesics with same ends are mutually close:

Lemma 2.2. ([8, 5.6,5.11],[1, 3.3]) *There is a constant* $\nu = \nu(\delta, \lambda, c)$ *such that for any* (λ, c)-*quasigeodesic path* p *in* $\Gamma(G, \mathcal{A})$ *and a geodesic* q *with* $p_- = q_-$, $p_+ = q_+$, *one has* $p \subset \mathcal{O}_\nu(q)$ *and* $q \subset \mathcal{O}_\nu(p)$.

Assume $(\mathcal{X}_1, d_1)$, $(\mathcal{X}_2, d_2)$ are two metric spaces and there is a map $f : \mathcal{X}_1 \to \mathcal{X}_2$. Recall that f is called a *quasiisometry* if there exist constants $b_1, b_2, c_1, c_2 > 0$ such that

$$b_1 d_1(x, y) + b_2 \leq d_2\left(f(x), f(y)\right) \leq c_1 d_1(x, y) + c_2 \text{ for any } x, y \in \mathcal{X}_1.$$

Theorem 2.3. ([8, Thm. 5.12]) *Consider two geodesic metric spaces* $\mathcal{X}_1$, $\mathcal{X}_2$ *and a quasi-isometry* $f : \mathcal{X}_1 \to \mathcal{X}_2$. *If the space* $\mathcal{X}_2$ *is hyperbolic then so is the space* $\mathcal{X}_1$.

Hyperbolic Groups

Let G be a group generated by a finite set $\mathcal{A}$ such that if $a \in \mathcal{A}$ then $a^{-1} \in \mathcal{A}$ (i.e., $\mathcal{A}$ is *symmetrized*), and $1_G \notin \mathcal{A}$ where 1_G denotes the identity element of G.

If $g \in G$, then the smallest number k such that $g = a_1 a_2 \cdots a_k$ where $a_i \in \mathcal{A}$, $i = 1, 2, \ldots, k$, will be denoted $|g|_G$ and called the *length* of g. $o(g)$ will denote the order of the element g in the group G.

Now one can define the *word metric* on G corresponding to $\mathcal{A}$: for any $x, y \in G$

$$d(x, y) \stackrel{def}{=} |x^{-1}y|_G. \tag{2.1}$$

The *Cayley graph* $\Gamma(G, \mathcal{A})$ for the group G with the generating set $\mathcal{A}$ is constructed as follows: $\Gamma(G, \mathcal{A})$ is a simplicial 1-complex without loops and multiple edges, whose vertices are the elements of G; two vertices x, y are connected by an edge if and only if $d(x, y) = 1$.

The edges of $\Gamma(G, \mathcal{A})$ can be endowed with metric of the interval $[0, 1] \subset \mathbb{R}$ which allows us to define the length metric on it. Thus, $\Gamma(G, \mathcal{A})$ becomes a proper geodesic metric space with a natural isometric embedding

$$G \hookrightarrow \Gamma(G, \mathcal{A}).$$

Obviously the group G isometrically (cocompactly and discreetly) acts on its Cayley graph by left translations.

Definition 2.7. Let $\delta \geq 0$ be given. The group G is said to be *δ-hyperbolic* if its Cayley graph $\Gamma(G, \mathcal{A})$ is a δ-hyperbolic metric space. G is hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

Hyperbolic groups were first introduced by M. Gromov in 1987 in his remarkable paper [11], and were studied quite thoroughly since then. It is well known that every such group is finitely presented and has solvable word problem (see [1],[8]).

Example 2.8. (see [11], [8]) Basic examples of hyperbolic groups are

- all finite groups;
- finitely generated free groups;
- fundamental groups of compact negatively curved Riemannian manifolds;
- groups with sufficiently small cancellation conditions (e.g., $C'(1/6)$-groups).

Two more examples (see below) are easy consequences of definitions.

Example 2.9. A direct product of a hyperbolic group with a finite group is a hyperbolic group. More generally, if a subgroup H has a finite index in G, then H is hyperbolic if and only if G is hyperbolic (because the Cayley graphs of H and G are quasiisometric).

Example 2.10. ([8, 1.34]) A free product of two hyperbolic groups is a hyperbolic group.

Assume $\mathcal{A}'$ is another finite symmetrized generating set of G. Then the spaces $\Gamma(G, \mathcal{A})$ and $\Gamma(G, \mathcal{A}')$ are quasiisometric. Consequently, an application of Theorem 2.3 leads to

Remark 2.4. ([11],[8, Cor. 5.14]) If the space $\Gamma(G, \mathcal{A})$ is hyperbolic then so is $\Gamma(G, \mathcal{A}')$.

Thus, hyperbolicity of a group is independent of the choice of a finite generating set.

An important property of cyclic subgroups of a hyperbolic group G states

Lemma 2.5. ([8, 8.21],[1, 3.2]) *For any word w representing an element $g \in G$ of infinite order there exist constants $\lambda > 0$, $c \geq 0$, such that any path with a label w^m in the Cayley graph of G is (λ, c)-quasigeodesic for arbitrary integer m.*

Recall that a group is called *elementary* if it has a cyclic subgroup of finite index. It is known that every element $g \in G$ of infinite order belongs to a unique *maximal elementary subgroup $E(g)$*. Then by [26, Lemmas 1.16,1.17]

$$E(g) = \left\{ \ x \in G \mid xg^n x^{-1} = g^{\pm n} \text{ for some } n \in \mathbb{N} \ \right\} \ \text{ and}$$

$$E(g) = \{\ x \in G \mid xg^k x^{-1} = g^l \text{ for some } k, l \in \mathbb{Z}\backslash\{0\}\ \}. \tag{2.2}$$

It is easy to see that the subgroup

$$E^+(g) = \{\ x \in G \mid xg^n x^{-1} = g^n \text{ for some } n \in \mathbb{N}\ \}$$

is of index at most 2 in $E(g)$.

Let $W_1, W_2, \ldots, W_l$ be words in $\mathcal{A}$ representing elements $g_1, g_2, \ldots, g_l$ of infinite order, where $E(g_i) \neq E(g_j)$ for $i \neq j$. The following statement will be useful:

Lemma 2.6. ([26, Lemma 2.3]) *There exist constants $\lambda = \lambda(W_1, W_2, \ldots, W_l) > 0$, $c = c(W_1, W_2, \ldots, W_l) \geq 0$ and $N = N(W_1, W_2, \ldots, W_l) > 0$ such that any path p in the Cayley graph $\Gamma(G, \mathcal{A})$ with label $W_{i_1}^{m_1} W_{i_2}^{m_2} \ldots W_{i_s}^{m_s}$ is (λ, c)-quasigeodesic if $i_k \neq i_{k+1}$ for $k = 1, 2, \ldots, s-1$, and $|m_k| > N$ for $k = 2, 3, \ldots, s-1$ (each i_k belongs to $\{1, \ldots, l\}$).*

The lemma below implies that orders of periodic elements in a hyperbolic group are uniformly bounded.

Lemma 2.7. ([1, Cor. 2.17]) *Any hyperbolic group has only finitely many conjugacy classes of elements of finite order.*

Lemma 2.8. ([8, 8.36]) *Any infinite subgroup of a hyperbolic group contains an element of infinite order.*

Thus, a word hyperbolic group can not have infinite periodic subgroups.

CHAPTER III

QUASICONVEX SUBSETS AND SUBGROUPS

This chapter speaks about definitions, examples and basic properties of quasiconvex subsets in hyperbolic groups.

Definitions

As usual, suppose G is a group generated by a finite symmetrized set $\mathcal{A}$.

Definition 3.1. A subset $Q \subseteq G$ is said to be *η-quasiconvex*, if any geodesic connecting two elements from Q belongs to a closed η-neighborhood $\mathcal{O}_\eta(Q)$ of Q in $\Gamma(G, \mathcal{A})$ for some $\eta \geq 0$. Q will be called *quasiconvex* if there exists $\eta > 0$ for which it is η-quasiconvex.

It turned out that the notion of quasiconvexity in a hyperbolic group does not depend on the choice of a finite generating set ([11]) and quasiconvex subgroups possess a lot of "nice" properties. For instance, such subgroups are finitely generated and undistorted (see below).

Lemma 3.1. ([1, 3.8], [5, 10.4.2]) *A quasiconvex subgroup H of a hyperbolic group G is finitely generated.*

Suppose $H = \langle \mathcal{B} \rangle$ is a subgroup of G with a finite generating set $\mathcal{B}$. If $h \in H$, then by $|h|_G$ and $|h|_H$ we will denote the lengths of the element h in the alphabets $\mathcal{A}$ and $\mathcal{B}$ respectively. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$. The *distortion function* (see [10]) $D_H : \mathbb{N}_0 \to \mathbb{N}_0$ of H in G is defined by

$$D_H(n) = max\{|h|_H \mid h \in H,\ |h|_G \leq n\}\ .$$

If $\alpha, \beta\ : \mathbb{N}_0 \to \mathbb{N}_0$ are two functions then we write $\alpha \preceq \beta$ if there are constants $K_1, K_2 > 0$ with $\alpha(n) \leq K_1 \beta(K_2 n)$ for every $n \in \mathbb{N}$. α and β are said to be equivalent if $\alpha \preceq \beta$ and $\beta \preceq \alpha$.

Evidently, the function D_H does not depend (up to this equivalence) on the choice of finite generating sets $\mathcal{A}$ of G and $\mathcal{B}$ of H. One can also notice that if H is infinite then $D_H(n)$ is at least linear.

Fix a linear function $L : \mathbb{N}_0 \to \mathbb{N}_0$ (for example, $L(n) = n$). If $D_H \preceq L$, H is called *undistorted*.

Lemma 3.2. ([19, Lemma 1.6], [7, Lemma 2.4]) *Let H be a finitely generated subgroup of a hyperbolic group G. Then H is quasiconvex if and only if H is undistorted in G.*

Thus a finitely generated subgroup H of a hyperbolic group G is quasiconvex if and only if its embedding into G is a quasiisometry. Combined with the claims of Lemma 3.1 and Theorem 2.3 this leads to

Remark 3.3. ([29]) A quasiconvex subgroup of a hyperbolic group is itself hyperbolic.

An easy but useful fact is formulated in

Remark 3.4. Suppose G_1 is a hyperbolic group and $H \leq G \leq G_1$. If H is quasiconvex in G and G is quasiconvex in G_1 then H is quasiconvex in G_1.

This follows from Lemma 3.2 and the observation that an undistorted subgroup of an undistorted subgroup is undistorted in the entire group.

A different (but equivalent) definition of a quasiconvex set can be obtained from the statement below.

Lemma 3.5. ([12, Lemma 3.9]) *Assume Q is a subset of a δ-hyperbolic group G. The following are equivalent:*

- *Q is η-quasiconvex for some $\eta \geq 0$;*
- *There exists $\eta' \geq 0$ such that for any $x \in Q$ and any geodesic path p between 1_G and x in $\Gamma(G, \mathcal{A})$, one has $p \subset \mathcal{O}_{\eta'}(Q)$.*

Proof. The sufficiency is trivial because all geodesic triangles in $\Gamma(G, \mathcal{A})$ are δ-slim (which implies that Q is $(\eta' + \delta)$-quasiconvex).

To demonstrate the necessity, assume Q is η-quasiconvex and fix an arbitrary element $y \in Q$. Take any $x \in Q$ and a geodesic path p connecting 1_G with x in $\Gamma(G, \mathcal{A})$. Denote $\eta' = \delta + \eta + |y|_G$ and consider a geodesic triangle $1_G xy$ with the side$[1_G, x] = p$. According to Definition 2.3, $p \subset \mathcal{O}_\delta\big([1_G, y] \cup [x, y]\big)$. Evidently, $[1_G, y] \subset \mathcal{O}_{|y|_G}(Q)$ and since Q is $\eta - quasiconvex$, $[x, y] \subset \mathcal{O}_\eta(Q)$. Thus, $p \subset \mathcal{O}_{\eta'}(Q)$. $\square$

Examples

Example 3.2. Any finite subset of G is d-quasiconvex where d is the diameter of this set.

Example 3.3. Let $A \subset G$ satisfy $G = \mathcal{O}_c(A)$ for some $c \geq 0$. Then A is quasiconvex because any point of $\Gamma(G, \mathcal{A})$ belongs to a closed $(c + 1/2)$-neighborhood of the subset A.

Example 3.4. In a hyperbolic group any cyclic subgroup is quasiconvex. This immediately follows from Lemmas 2.2 and 2.5.

Example 3.5. Let F be a finitely generated free group and H be its finitely generated subgroup. Since the Cayley graph of F is a tree, it takes an easy application of Lemma 3.5 to show that H is quasiconvex in F (see [29]).

Let us show that a non-elementary hyperbolic group has a lot of quasiconvex subgroups:

Lemma 3.6. *Assume H is a non-elementary subgroup of a δ-hyperbolic group G. Then H has a subgroup M of infinite index that is free of rank 2 and quasiconvex in G.*

Proof. Choose elements of infinite order $g_1, g_2 \in H$ with $E(g_1) \neq E(g_2)$ and let the words W_1, W_2 over the alphabet $\mathcal{A}$ represent them. Let λ, c and N be the corresponding constants from the claim of Lemma 2.6. Set $\nu = \nu(\delta, \lambda, c)$ as in Lemma 2.2.

Take an arbitrary $m \in \mathbb{N}$ with $m > c/\lambda$, and define the subgroup $M = \langle g_1^{mN}, g_2^{mN} \rangle \leq H$.

Consider a word

$$W \equiv W_{i_1}^{m_1} W_{i_2}^{m_2} \dots W_{i_s}^{m_s}, \quad \text{where } s \in \mathbb{N}, i_1, \dots, i_s \in \{1, 2\},\ i_k \neq i_{k+1} \tag{3.1}$$

for $k = 1 \dots, s-1$, and $m_1, \dots, m_s$ are non-zero integers divisible by mN. By Lemma 2.6 the path p in $\Gamma(G, \mathcal{A})$ starting at 1_G and labelled the word W is (λ, c)-quasigeodesic. Thus, $d(1_G, p_+) \geq \lambda \|p\| - c \geq \lambda m N - c > 0$.

Consider any non-trivial element $x \in M$ and a path $[1_G, x]$. Then x equals in G to some word W having the form (3.1). Let p be the corresponding (λ, c)-quasigeodesic path in the Cayley graph. Since $p_- = 1_G$ and $p_+ = x$, we have $[1_G, x] \subset \mathcal{O}_\nu(p)$. Evidently, $p \subset \mathcal{O}_\varkappa(M)$ where $\varkappa = mN \cdot \max\{|g_1|_G, |g_2|_G\}$. Thus, $[1_G, x] \subset \mathcal{O}_{\nu+\varkappa}(M)$. By Lemma 3.5 M is quasiconvex.

Now, suppose the word $v(a, b)$ is freely reduced in the two-generated free group $F(a, b)$. Then the word $W = v(W_1, W_2)$ is of the form (3.1). Hence if p is the corresponding path in $\Gamma(G, \mathcal{A})$, $|elem(p)|_G = d(1_G, p_+) > 0$. Consequently, $v(W_1^{mN}, W_2^{mN})$ represents a non-trivial element of the group G. This implies that M is free with free generators g_1^{mN} and g_2^{mN}.

Obviously, by choosing m large enough we can obtain $|H : M| = \infty$. □

Example 3.6. A basic example of a non-quasiconvex subgroup in any hyperbolic group is an infinite normal subgroup of infinite index. This follows from the property of an infinite quasiconvex subgroup H (first observed by M. Mihalik and W. Towle in [18]) stating that H has a finite index in its normalizer.

Operations with quasiconvex sets

The family of quasiconvex subsets of a hyperbolic group is closed under many subset operations.

Remark 3.7. Let $Q \subseteq G$ be η-quasiconvex, $g \in G$. Then

(a) the left shift $gQ = \{gx \mid x \in Q\}$ is quasiconvex with the same constant;

(b) the right shift $Qg = \{xg \mid x \in Q\}$ is quasiconvex (possibly, with a different quasiconvexity constant).

(a) holds because the metric $d(\cdot,\cdot)$ is left-invariant. $x, y \in Q$ if and only if $xg, yg \in Qg$. Applying Remark 2.1 we get

$$[xg, yg] \subset \mathcal{O}_{2\delta}\Big([x, xg] \cup [x, y] \cup [y, yg]\Big) \subset \mathcal{O}_{2\delta+|g|_G}\Big([x, y]\Big) \subset \mathcal{O}_{2\delta+|g|_G+\eta}(Q) \subset \mathcal{O}_{2\delta+2|g|_G+\eta}(Qg),$$

therefore (b) is true.

Hence a left coset of a quasiconvex subgroup and a conjugate subgroup to it are quasiconvex (in a hyperbolic group).

For two subsets A, B of G define their product: $A \cdot B = AB \overset{def}{=} \{ab \mid a \in A, b \in B\}$.

Now let us show that quasiconvexity is preserved under taking finite unions and products.

Lemma 3.8. ([12, Prop. 3.14],[19, Lemma 2.1,Prop. 0.1]) *Let G be a hyperbolic group and let A, B be its quasiconvex subsets. Then the subsets $A \cup B$ and AB are also quasiconvex.*

Proof. Assume that A is η_1-quasiconvex and B is η_2-quasiconvex. Find the corresponding η_1' and η_2' from Lemma 3.5, and set $\eta' = \max\{\eta_1', \eta_2'\}$. Then for any $x \in A \cup B$, $[1_G, x] \subset \mathcal{O}_{\eta'}(A \cup B)$. By Lemma 3.5, $A \cup B$ is quasiconvex.

Now, choose an arbitrary $xy \in AB$ with $x \in A$, $y \in B$. In $\Gamma(G, \mathcal{A})$ consider a geodesic triangle with vertices 1_G, x and xy. Since left translations by elements of G are isometries of the Cayley graph, the image $[1_G, y] = x^{-1} \circ ([x, xy])$ is a geodesic between 1_G and y. Hence

$$[x, xy] = x \circ ([1_G, y]) \subset x \circ (\mathcal{O}_{\eta'}(B)) = \mathcal{O}_{\eta'}(xB).$$

Finally, since our triangle is δ-slim, we obtain

$$[1_G, xy] \subset \mathcal{O}_\delta([1_G, x]) \cup \mathcal{O}_\delta([x, xy]) \subset \mathcal{O}_{\delta+\eta'}(A) \cup \mathcal{O}_{\delta+\eta'}(xB) \subset \mathcal{O}_{\delta+\eta'}(AB).$$

Thus, AB is quasiconvex by Lemma 3.5. □

Though intersection of two convex subsets of $\mathbb{R}^n$ is convex, an intersection of two quasiconvex subsets is not necessarily quasiconvex:

Example 3.7. Set $G = \langle a \rangle_\infty \cong \mathbb{Z}$ – infinite cyclic group, $A = \{a^{2k} \mid k \in \mathbb{Z}\}$, $B_1 = \{a^{2k+1} \mid k \in \mathbb{Z}\}$, $B_2 = \{a^{2^k} \mid k \in \mathbb{N}\}$. As we saw in Example 3.3 the subsets A and $B \overset{def}{=} B_1 \cup B_2$ are quasiconvex in G; but their intersection $A \cap B = B_2$ is not.

H. Short showed that the situation with quasiconvex subgroups is a lot better:

Lemma 3.9. ([29, Prop. 3]) *Let G be a group generated by a finite set $\mathcal{A}$. Let A, B be subgroups of G quasiconvex with respect to $\mathcal{A}$. Then $A \cap B$ is quasiconvex with respect to $\mathcal{A}$.*

In the next section we will define an important family of quasiconvex subsets that is closed under finite unions, products and intersections.

Products of Quasiconvex Subgroups

Suppose $F_1, \ldots, F_n$ are quasiconvex subgroups of a hyperbolic group G and $g_0, \ldots, g_n$ are elements of G. The following notion was introduced in [19]:

Definition 3.8. The set

$$P = g_0 F_1 \cdot \cdots \cdot g_{n-1} F_n g_n = \{g_0 f_1 \cdot \cdots \cdot g_{n-1} f_n g_n \mid f_i \in F_i, i = 1, \ldots, n\}$$

is called a *quasiconvex product*. The quasiconvex subgroups F_i, $i = 1, 2, \ldots, n$, are *members* of the product P.

Applying statements of Remark 3.7 and Lemma 3.8 one immediately obtains

Remark 3.10. In a hyperbolic group a finite union of quasiconvex products is always a quasiconvex subset.

Let $\mathcal{K}$ denote the family of all subsets of G such that each of the subsets is equal to finite unions of quasiconvex products ($\emptyset \in \mathcal{K}$ is represented by the empty union). As it follows from the construction, $\mathcal{K}$ is closed under taking finite unions and products. It was proved in [19, Cor. 0.1] that an intersection of two subsets from $\mathcal{K}$ again belongs to $\mathcal{K}$. Thus, using terminology of Universal Algebra, $\mathcal{K}$ is a *lattice-ordered monoid*.

Let $U = \bigcup_{k=1}^{N} P_k$ be a finite union of quasiconvex products P_k, $k = 1, \ldots, N$.

Definition 3.9. A subgroup $F \leq G$ will be called a *member* of U, if F is a member of P_k for some $1 \leq k \leq N$.

In future, for any such set U we will fix its representation as a finite union of quasiconvex products and fix its members.

Study of quasiconvex products and family $\mathcal{K}$ plays a significant role in this work. Their properties were investigated by the author in [19] and [22]. A connection with profinite topology on hyperbolic groups has been recently found in [21].

CHAPTER IV

SOME CONCEPTS AND CONSTRUCTIONS

In this chapter we establish an equivalence relation on the set of subsets of any group, then use it to introduce and investigate the notion of a subset commensurator (a generalization of a virtual normalizer for a subgroup).

Equivalence of Subsets

Suppose G is an arbitrary group and 2^G is the set of all its subsets. Below we establish some auxiliary relations on 2^G. Assume $A, B \subseteq G$.

Definition 4.1. We will write $B \preceq A$ if there exist elements $x_1, \ldots, x_n \in G$ such that

$$B \subset Ax_1 \cup Ax_2 \cup \cdots \cup Ax_n.$$

Obviously, the relation "$\preceq$" is transitive and reflexive.

Definition 4.2. If $G \preceq A$, the subset A will be called *quasidense*.

Example 4.3. Any subgroup H of finite index in G is quasidense; the complement $H^{(c)} = G \backslash H$ in this case is also quasidense (if $H \neq G$) since it contains a left coset modulo H, and a shift (left or right) of a quasidense subset is quasidense.

On the other hand, if $H \leq G$ and $|G : H| = \infty$, the set of elements of H is not quasidense in G. There is $y \in G \backslash H$, hence for any $x \in G$ either $x \in G \backslash H$ or $xy \in G \backslash H$, thus $G = H^{(c)} \cup H^{(c)} y^{-1}$, i.e., $H^{(c)}$ is still quasidense.

Definition 4.4. A and B will be called *equivalent* if $A \preceq B$ and $B \preceq A$. In this case we will use the notation $A \approx B$.

It is easy to check that "$\approx$" is an equivalence relation on 2^G. Let $[A]$ denote the equivalence class of a subset $A \subseteq G$ and let $\mathcal{M}$ be the set of all such equivalence classes. Evidently, the relation "$\preceq$" induces a partial order on $\mathcal{M}$: $[A], [B] \in \mathcal{M}$, $[A] \leq [B]$ if and only if $A \preceq B$.

The group G acts on $\mathcal{M}$ as follows: $g \in G$, $A \subset G$, then $g \circ [A] = [gA]$. Indeed, the verification of the group action axioms is straightforward:

1. If $g, h \in G$, $[A] \in \mathcal{M}$ then $(gh) \circ [A] = g \circ (h \circ [A])$;

2. If $1_G \in G$ is the identity element and $[A] \in \mathcal{M}$ then $1_G \circ [A] = [A]$.

This action is well defined because if $A \approx B$ and $g \in G$, then $gA \approx gB$.

If the group G is finitely generated, one can fix a finite symmetrized generating set $\mathcal{A}$ and define the word metric $d(\cdot,\cdot)$ corresponding to $\mathcal{A}$ in the standard way (2.1). Now, for arbitrary two subsets $A, B \subseteq G$ one can establish

$$h(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{O}_\varepsilon(B), B \subset \mathcal{O}_\varepsilon(A)\} -$$

the Hausdorff distance between A and B. Where an infinum over the empty set is defined to be positive infinity.

In this case one can observe

Remark 4.1. For any $A, B \subseteq G$, $B \preceq A$ if and only if there exists $c > 0$ such that $B \subset \mathcal{O}_c(A)$. Therefore $A \approx B$ if and only if $h(A,B) < \infty$.

Indeed, suppose $B \subseteq Ag_1 \cup Ag_2 \cup \cdots \cup Ag_n$, where $g_i \in G$, $i = 1, 2, \ldots, n$. Denote $c = max\{|g_i|_G \ : \ 1 \leq i \leq n\}$. Then for any $x \in B$, there are $i \in \{1, \ldots, n\}$ and $y \in A$ with $x = yg_i$, hence $d(y,x) = |g_i|_G \leq c$.

For demonstrating the sufficiency, let $\{g_1, g_2, \ldots, g_n\}$ be the set of all elements in G of length at most c. Then for every $x \in B$ there exists $y \in A$ with $d(y,x) = |y^{-1}x|_G \leq c$; hence $y^{-1}x = g_i$ for some $i \in \{1, 2, \ldots, n\}$. Thus, $x = yg_i \in Ag_i$. Consequently, $B \subseteq Ag_1 \cup \cdots \cup Ag_n$.

Subset Commensurators

If $A \subseteq G$, the stabilizer of $[A] \in \mathcal{M}$ under the action defined in the previous section is the subgroup

$$St_G([A]) = \{g \in G \mid g \circ [A] = [A]\}.$$

Definition 4.5. For a given subset A of the group G the subgroup $St_G([A])$ will be called a *commensurator* of A in G and denoted $Comm_G(A)$. In other words,

$$Comm_G(A) = \{g \in G \mid gA \approx A\}.$$

Thus, for an arbitrary subset A of the group we find a corresponding subgroup in G. Now, let's list some properties of $Comm_G(A)$:

Remark 4.2. Let G be a group and $A \subseteq G$.

1) If $card(A) < \infty$ or A is quasidense then $Comm_G(A) = G$ (because any two finite non-empty subsets are equivalent and a left shift of a quasidense subset is quasidense);

2) If $A, B \subseteq G$ and $A \approx B$ then $Comm_G(A) = Comm_G(B)$;

3) The commensurator of $A \subset G$ contains (as its subgroups) the normalizer of A, $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$, and the stabilizer under the action of the group G on itself by left multiplication, $St_G(A) = \{g \in G \mid gA = A\}$.

4) For any $h \in G$, $Comm_G(hA) = hComm_G(A)h^{-1}$.

Lemma 4.3. *Let A, B be subgroups of G. Then $A \preceq B$ if and only if the index $|A : (A \cap B)|$ is finite.*

Proof. The sufficiency is trivial. To prove the necessity, suppose there exist $y_j \in G$, $j = 1, 2, \dots, m$, such that $A \subset By_1 \cup \dots \cup By_m$. Without loss of generality we can assume that $A \cap By_j \neq \emptyset$ for every $j = 1, 2, \dots, m$. Then for each $j = 1, 2, \dots, m$, there are $a_j \in A, b_j \in B$ such that $y_j = b_j a_j$. Hence $By_j = Ba_j$ for all j, and therefore

$$A = \bigcup_{j=1}^{m} By_j \cap A = \bigcup_{j=1}^{m} (Ba_j \cap A) = \bigcup_{j=1}^{m} (B \cap A)a_j,$$

i.e., $|A : (B \cap A)| < \infty$. □

For a subgroup $H \leq G$ the standard notion of the commensurator (virtual normalizer) subgroup of H is given by

$$VN_G(H) = \{g \in G \mid |H : (H \cap gHg^{-1})| < \infty, |gHg^{-1} : (H \cap gHg^{-1})| < \infty\} .$$

Now we are going to show that our new definition is just a generalization of it:

Lemma 4.4. *If H is a subgroup of the group G then $Comm_G(H) = VN_G(H)$.*

Proof. Indeed, let $g \in VN_G(H)$. Then, by definition,

$$H \preceq (H \cap gHg^{-1}) \preceq gHg^{-1} \preceq gH \quad \text{and} \quad gH \preceq gHg^{-1} \preceq (H \cap gHg^{-1}) \preceq H,$$

thus $H \approx gH$ and $g \in Comm_G(H)$. So, $VN_G(H) \subseteq Comm_G(H)$.

Now, suppose $g \in Comm_G(H)$, implying $H \approx gH$ but $gH \approx gHg^{-1}$, hence $H \preceq gHg^{-1}$ and $gHg^{-1} \preceq H$. By Lemma 4.3, $g \in VN_G(H)$. Therefore $VN_G(H) = Comm_G(H)$. □

Hyperbolic Case

In the special case, when G is δ-hyperbolic (hence finitely generated), we are able to make the additional observations below.

Remark 4.5. Suppose a subset $A \subset G$ is quasiconvex and $A \approx B$ for some $B \subset G$. Then B is also quasiconvex.

Indeed, as we saw in Remark 4.1, there exist $c_1, c_2 \geq 0$ such that $B \subset \mathcal{O}_{c_1}(A)$ and $A \subset \mathcal{O}_{c_2}(B)$. Consider arbitrary $x, y \in B$ and a geodesic segment $[x, y]$ connecting them. Then

$$x, y \in \mathcal{O}_{c_1}(A) = \bigcup_{g \in G, |g|_G \leq c_1} Ag \ ,$$

which is η-quasiconvex by Remark 3.7 and Lemma 3.8 for some $\eta \geq 0$. Therefore

$$[x, y] \subset \mathcal{O}_{c_1+\eta}(A) \subset \mathcal{O}_{c_1+\eta+c_2}(B)$$

implying that B is $(c_1 + c_2 + \eta)$-quasiconvex.

Remark 4.6. A subset Q of the group G is quasidense if and only if there exists $c \geq 0$ such that for every $x \in G$ the distance $d(x, Q) = \inf\{d(x, y) \mid y \in Q\}$ is at most c, i.e., $G \subseteq \mathcal{O}_c(Q)$.

Remark 4.7. A quasidense subset $Q \subseteq G$ is quasiconvex.

This follows from the previous Remark and was explained in Example 3.3.

CHAPTER V

GROMOV BOUNDARY AND LIMIT SETS

This chapter discusses the concept of a boundary for hyperbolic groups and speaks about its connections with the relation from Definition 4.1.

Boundaries of Hyperbolic Spaces

Let $\mathcal{X}$ be a proper geodesic metric space with metric $d(\cdot,\cdot)$. Assume also that $\mathcal{X}$ is δ-hyperbolic for some $\delta \geq 0$. Further in this work we will need Gromov's construction of a *boundary* $\partial\mathcal{X}$ for the space $\mathcal{X}$ introduced in [11] (for more detailed theory the reader is referred to the corresponding chapters in [8],[4]).

Establish an equivalence relation on the set of infinite geodesic geodesic rays

$$\{r : [0,\infty) \to X \mid r \text{ is geodesic}\}$$

as follows: two rays r_1, r_2 are *equivalent* if $\sup\{d(r_1(t), r_2(t))\} < \infty$.

Definition 5.1. The *Gromov boundary* $\partial\mathcal{X}$ is defined as the set of equivalence classes of all infinite geodesic rays.

For another definition of the boundary, fix a basepoint $p \in \mathcal{X}$. A sequence of points $(x_i)_{i\in\mathbb{N}}$ in $\mathcal{X}$ is called *converging to infinity* if

$$\lim_{i,j\to\infty} (x_i|x_j)_p = \infty.$$

Two sequences $(x_i)_{i\in\mathbb{N}}, (y_j)_{i\in\mathbb{N}}$ converging to infinity are said to be *equivalent* if

$$\lim_{i\to\infty} (x_i|y_i)_p = \infty.$$

Definition 5.2. The *boundary* $\partial\mathcal{X}$ is identified with the set of equivalence classes of sequences converging to infinity.

It is easy to see that this definition does not depend on the choice of a basepoint. If α is the equivalence class of $(x_i)_{i\in\mathbb{N}}$ we will write $\lim\limits_{i\to\infty} x_i = \alpha$.

It is known that the two objects given by Definitions 5.1 and 5.2 are homeomorphic through the map sending a geodesic ray $r : [0,\infty) \to \mathcal{X}$ into the sequence $\big(r(i)\big)_{i\in\mathbb{N}}$ ([4]).

Remark 5.1. Suppose $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ are two sequences in $\mathcal{X}$ and $(a_i)_{i\in\mathbb{N}}$ converges to infinity, $\lim\limits_{i\to\infty} a_i = \alpha \in \partial G$. If $\lim\limits_{i\to\infty}(a_i|b_i)_p = \infty$ then $(b_i)_{i\in\mathbb{N}}$ is also converging to infinity. Consequently, $\lim\limits_{i\to\infty} b_i = \alpha$.

Indeed, since $(a_i)_{i\in\mathbb{N}}$ converges to infinity, then, according to Definition 2.1, for any $i \in \mathbb{N}$ we have

$$(b_i|b_j)_p \geq \min\{(b_i|a_i)_p, (a_i|b_j)_p\} - \delta \geq \min\{(b_i|a_i)_p, (a_i|a_j)_p, (a_j|b_j)_p\} - 2\delta \to \infty \text{ as } i, j \to \infty.$$

As it is shown in [8, Prop. 7.6], the Gromov boundary of a hyperbolic proper geodesic metric space $\mathcal{X}$ possesses, so called, *visibility* property: for any two distinct points $\alpha, \beta \in \partial\mathcal{X}$ there exists at least one bi-infinite geodesic $r : (-\infty, +\infty) \to \mathcal{X}$ such that $\lim\limits_{i\to\infty} r(-i) = \alpha$ and $\lim\limits_{i\to\infty} r(i) = \beta$. We will say that this geodesic joins α and β; it will be denoted (α, β).

The space $\partial\mathcal{X}$ can be topologized so that it becomes compact, Hausdorff and metrizable (see [1],[8]).

Every isometry ψ of the space $\mathcal{X}$ induces a homeomorphism of $\partial\mathcal{X}$ in a natural way: for every equivalence class of geodesic rays $[r] \in \partial\mathcal{X}$ choose a representative $r : [0, \infty) \to \mathcal{X}$ and set $\psi([r]) = [\psi \circ r]$.

Definition 5.3. For a subset $A \subseteq \mathcal{X}$ the *limit set* $\Lambda(A)$ of A is the collection of the points $\alpha \in \partial\mathcal{X}$ that are limits of sequences (converging to infinity) from A.

Definition 5.4. Let Ω be a subset of $\partial\mathcal{X}$ containing at least two distinct points. We define the *convex hull* $CH(\Omega)$ of Ω to be the set of all points in $\mathcal{X}$ lying on bi-infinite geodesics that join elements from Ω.

Two important properties of convex hulls are formulated as follows:

Lemma 5.2. ([14, Lemmas 3.2,3.6]) *Let Ω be an arbitrary subset of $\partial\mathcal{X}$ having at least two elements. Then*

(a) *$CH(\Omega)$ is η-quasiconvex where $\eta \geq 0$ depends only on δ;*

(b) *If the subset Ω is closed then $\Lambda(CH(\Omega)) = \Omega$.*

Actions of Hyperbolic Groups on their Boundaries

Now let's assume that the space $\mathcal{X}$ is the Cayley graph $\Gamma(G, \mathcal{A})$ of some δ-hyperbolic group G with a fixed symmetrized finite generating set $\mathcal{A}$. Because of the natural embedding of G (as a metric subspace) into $\Gamma(G, \mathcal{A})$, we will identify subsets of G with subsets of its Cayley graph.

Definition 5.5. The *Gromov boundary* of the group G, ∂G, by definition, coincides with the boundary of $\Gamma(G, \mathcal{A})$.

Left multiplication by elements of the group induces an isometric action of G on $\Gamma(G, \mathcal{A})$. Hence G acts homeomorphically on the boundary ∂G as described above.

If $g \in G$ is an element of infinite order in G then the sequences $(g^i)_{i\in\mathbb{N}}$ and $(g^{-i})_{i\in\mathbb{N}}$ converge to infinity and we will use the notation

$$\lim_{i\to\infty} g^i = g^\infty \in \partial G, \ \lim_{i\to\infty} g^{-i} = g^{-\infty} \in \partial G.$$

Let us list some well-known properties of limit sets in ∂G.

Lemma 5.3. ([14],[30]) *Suppose A, B are arbitrary subsets of G, $g \in G$. Then*

(a) $\Lambda(A) = \emptyset$ *if and only if A is finite;*

(b) $\Lambda(A)$ *is a closed subset of the boundary ∂G;*

(c) $\Lambda(A \cup B) = \Lambda(A) \cup \Lambda(B)$*;*

(d) $\Lambda(Ag) = \Lambda(A)$, $g \circ \Lambda(A) = \Lambda(gA)$*;*

(e) *If $A \preceq B$ then $\Lambda(A) \subseteq \Lambda(B)$. Hence $A \approx B$ implies $\Lambda(A) = \Lambda(B)$.*

Proof. (b),(c) and (d) are easy consequences of the definition and (a) is obtained after a standard application of the Pigeon-Hole principle; (e) follows from (c) and (d). □

Lemma 5.4. *Suppose A and B are subsets of the hyperbolic group G and $\Lambda(A) \cap \Lambda(B) = \emptyset$. Then* $\sup_{a\in A,\ b\in B} \{(a|b)_{1_G}\} < \infty$.

Proof. This statement is an easy consequence of the definition of a limit set. Indeed, assume, by the contrary, that there are sequences of elements $(a_i)_{i\in\mathbb{N}} \subset A$ and $(b_i)_{i\in\mathbb{N}} \subset B$ such that $\lim_{i\to\infty}(a_i|b_i)_{1_G} = \infty$. Then the subsets $\{a_i \mid i \in \mathbb{N}\} \subset G$ and $\{b_i \mid i \in \mathbb{N}\} \subset G$ are infinite, hence each of them has at least one limit point (by Lemma 5.3.(a)). Thus, there are subsequences $(a_{i_j})_{j\in\mathbb{N}}$ of (a_i) and $(b_{i_j})_{j\in\mathbb{N}}$ of (b_i) satisfying

$$\lim_{j\to\infty} a_{i_j} = \alpha \in \Lambda(A), \ \lim_{j\to\infty} b_{i_j} = \beta \in \Lambda(B) \ .$$

But $\lim_{j\to\infty}(a_{i_j}|b_{i_j})_{1_G} = \infty$ by our assumption, hence $\alpha = \beta$. A contradiction with the assumption $\Lambda(A) \cap \Lambda(B) = \emptyset$. □

If H is a subgroup of G, it is known that $\Lambda(H)$ is either empty (if H is finite), or consists of two distinct points (if H is infinite elementary), or is uncountable (if H is non-elementary) – see [14],[8]. In the second case, when there exists $g \in H$ such that $o(g) = \infty$ (i.e., g has infinite order) and $|H : \langle g\rangle| < \infty$, one has $\Lambda H = \{g^{\infty}, g^{-\infty}\}$.

Lemma 5.5. ([14, Lemma 3.3]) *If H is an infinite subgroup of G then $\Lambda(H)$ contains at least two distinct points and the sets $\Lambda(H)$, $CH\big(\Lambda(H)\big)$ are H-invariant, i.e., for every $h \in H$, $h \circ \Lambda(H) = \Lambda(H)$, $h \cdot CH\big(\Lambda(H)\big) = CH\big(\Lambda(H)\big)$.*

The following statement indicates that infinite normal subgroups in hyperbolic groups are very large:

Lemma 5.6. ([14, Lemma 3.8]) *Let A be an infinite normal subgroup of a subgroup H in G. Then $\Lambda(A) = \Lambda(H)$.*

As the hyperbolic group G acts on its boundary, for every subset $\Omega \subset \partial G$ one can define the stabilizer subgroup by $St_G(\Omega) = \{g \in G \mid g \circ \Omega = \Omega\}$. For our convenience, we set $St_G(\emptyset) = G$.

It is proved in [8, 8.30] that for any point $\alpha \in \partial G$, $St_G(\{\alpha\})$ is an elementary subgroup of the group G (in fact, if $\alpha = g^{\infty}$ for some element of infinite order $g \in G$ then

$$St_G(\{\alpha\}) = E^+(g) = \{x \in G \mid \exists\, n \in \mathbb{N} \text{ such that } xg^nx^{-1} = g^n\} \leq E(g);$$

otherwise the subgroup $St_G(\{\alpha\})$ is finite). In addition, if $g \in G$, $o(g) = \infty$, then

$$St_G(\{g^{\infty}, g^{-\infty}\}) = E(g).$$

Now we can make

Remark 5.7. For an arbitrary subset A of G, $Comm_G(A) \subseteq St_G\big(\Lambda(A)\big)$.

Indeed, if $g \in Comm_G(A)$, then $gA \approx A$, hence after applying claims (d),(e) of Lemma 5.3, we obtain $g \circ \Lambda(A) = \Lambda(gA) = \Lambda(A)$, i.e., $g \in St_G\big(\Lambda(A)\big)$.

Remark 5.8. Suppose $\Omega \subseteq \partial G$ has at least two distinct points. Denote by $cl(\Omega) \subseteq \partial G$ the closure of Ω in the topology of the group boundary. Then $\Lambda\left(CH(\Omega)\right) = cl(\Omega)$.

Indeed, since $CH(\Omega) \subseteq CH\left(cl(\Omega)\right)$ we obtain

$$\Lambda\left(CH(\Omega)\right) \subseteq \Lambda\left(CH\big(cl(\Omega)\big)\right) = cl(\Omega),$$

where the last equality is achieved using Lemma 5.2. Finally, $\Lambda\left(CH(\Omega)\right)$ is a closed subset of ∂G containing Ω (by part (b) of Lemma 5.3), which implies the statement of Remark 5.8.

The following lemma will be used quite often (in a somewhat different form it can be found in [30, Cor. to Lemma 13]):

Lemma 5.9. *Suppose $\Omega \subset \partial G$ is a subset having at least two distinct points. Then $\Lambda\big(St_G(\Omega)\big) \subseteq cl(\Omega)$.*

Proof. Since Ω has at least two points, it makes sense to consider the convex hull $CH(\Omega)$. Observe that for any $g \in St_G(\Omega)$, $gCH(\Omega) \subseteq CH(\Omega)$: the left translation by the element $g \in G$ is an isometry of $\Gamma(G, \mathcal{A})$, therefore a bi-infinite geodesic (α, β), $\alpha, \beta \in \Omega$ goes to a bi-infinite geodesic $(g \circ \alpha, g \circ \beta) \subset CH(\Omega)$ since Ω is $St_G(\Omega)$-invariant.

Fix any point $x \in CH(\Omega)$. By our observation above, $St_G(\Omega)x \subset CH(\Omega)$, hence $\Lambda\big(St_G(\Omega)x\big) \subset \Lambda\big(CH(\Omega)\big)$. The claim of the lemma now follows by applying Lemma 5.3.(d) and Remark 5.8. $\square$

It is a well-known fact that the set of all rational points $\{g^\infty \mid g \in G^0\}$ is dense in the group boundary ∂G (see, for example, [3, Theorem], [11, 4,8.2D]). Later we will be using a stronger statement:

Lemma 5.10. *Assume H is a non-elementary subgroup of a hyperbolic group G and $\alpha \in \partial G$. Then $\Lambda(H) \subseteq cl(H \circ \alpha)$ where $H \circ \alpha$ is the orbit of α under the action of H and $cl(H \circ \alpha)$ is its closure inside ∂G.*

Proof. Since H is non-elementary, the set $H \circ \alpha$ consists of more than one point. By definition, $H \subset St_G(H \circ \alpha)$, hence after applying Lemma 5.9 we achieve

$$\Lambda(H) \subseteq cl(H \circ \alpha).$$

Q.e.d. $\square$

CHAPTER VI

COMMENSURATORS AND QUASICONVEXITY

In this chapter we list some known results concerning limit sets and subgroup commensurators; then we extend these results to larger classes of subsets.

Known Results and Examples

Result 6.1. *Let A and B be quasiconvex subgroups of a hyperbolic group G. Then $A \approx B$ if and only if $\Lambda(A) = \Lambda(B)$.*

Indeed, the necessity follows by Lemma 5.3.(e). For proving the sufficiency we note that by [30, Thm. 8] (see Lemma 9.1 of this thesis), $\Lambda(A \cap B) = \Lambda(A) \cap \Lambda(B) = \Lambda(A) = \Lambda(B)$. But $A \cap B \leq A$ and $A \cap B \leq B$, so, by Lemma 3.9 and [30, Thm. 4],

$$|A : (A \cap B)| < \infty,\ |B : (A \cap B)| < \infty,$$

i.e., the subgroups A and B are commensurable. Hence $A \approx B$ (by Lemma 4.3).

However, if one removes at least one of the conditions on A and B, the claim of the Result 6.1 fails:

Example 6.2. Let $G = F(x, y)$ – the free group with two free generators x, y. Define $A = \{x^n \mid n \geq 0\}$, $B = \{x^n y^m \mid 0 \leq m \leq n\}$. According to Lemma 3.5these are quasiconvex subsets (not subgroups) of G because any prefix of an element from one of these sets is still contained in the same set. Evidently, $\Lambda(A) = \{x^\infty\}$. Suppose $(x^{n_i} y^{m_i})_{i\in\mathbb{N}}$, $0 \leq m_i \leq n_i$, $i \in \mathbb{N}$, is a sequence converging to infinity in B. If the sequence of integers $(n_i)_{i\in\mathbb{N}}$ is bounded then the sequence $(m_i)_{i\in\mathbb{N}}$ is also bounded, hence the set the group of elements in $(x^{n_i} y^{m_i})_{i\in\mathbb{N}}$ is finite which contradicts to the definition of a sequence that converges to infinity. Thus, $\sup\limits_{i\in\mathbb{N}}\{n_i\} = \infty$ and, passing to a subsequence, we can assume $\lim\limits_{i\to\infty} n_i = \infty$. Then

$$(x^{n_i} y^{m_i} | x^{n_i})_{1_G} = n_i \to \infty \text{ as } i \to \infty\ ,$$

thus, $\lim\limits_{i\to\infty}(x^{n_i} y^{m_i}) = \lim\limits_{i\to\infty} x^{n_i} = x^\infty$. Therefore $\Lambda(B) = \{x^\infty\} = \Lambda(A)$ but $A \not\approx B$.

Example 6.3. If G is an arbitrary hyperbolic group and H is its infinite normal subgroup of infinite index, then H is not quasiconvex (see Example 3.6) and $\Lambda(H) = \Lambda(G) = \partial G$ by Lemma 5.6. Thus the quasiconvexity of A, B in Result 6.1 is also important.

Result 6.4. *If A is a quasiconvex subgroup of a hyperbolic group G then we have an equality in claim of Remark 5.7: $Comm_G(A) = St_G\big(\Lambda(A)\big)$.*

By [30, Thm. 17] or [14, Cor. 3.10], $VN_G(A) = St_G\big(\Lambda(A)\big)$ and $VN_G(A) = Comm_G(A)$ by Lemma 4.4.

Again, both of the requirements for A to be a subgroup and to be quasiconvex are not redundant:

Example 6.5. Choose $G = F(x, y)$ and let

$$B = \{x^n y^m \mid 0 \leq m \leq n^2, n \geq 0\},\ C = \{x^{-n} \mid n \in \mathbb{N}\},\ A = B \cup C\ .$$

A is quasiconvex since B and C are, $\Lambda(A) = \Lambda(B) \cup \Lambda(C) = \{x^\infty, x^{-\infty}\}$ ($\Lambda(B) = \{x^\infty\}$ by a similar argument to the one presented in Example 6.2). Then $St_G\big(\Lambda(A)\big) = \langle x \rangle$ – the infinite cyclic subgroup generated by x.

Let's show that $Comm_G(A) = \{1_G\}$. By Remark 5.7 and since $Comm_G(A)$ is a subgroup, it is enough to prove that $x^{-k} \notin Comm_G(A)$ for any integer $k > 0$. Indeed, for any $n > k$ $x^{n-k}y^{n^2} \in x^{-k}A$ and

$$d\left(x^{n-k}y^{n^2}, A\right) = d\left(x^{n-k}y^{n^2}, x^{n-k}y^{(n-k)^2}\right) = n^2 - (n-k)^2 = 2nk - k^2 \to \infty$$

when $n \to \infty$. Implying that $x^{-k}A \not\approx A$.

Example 6.6. Consider a finitely generated group M containing a normal subgroup $N \lhd M$ and an infinite subnormal subgroup $K \lhd N$ such that $|M : N| = \infty$, $|N : K| = \infty$ and for any $x \in M \backslash N$, $xKx^{-1} \cap K = \{1_M\}$ (for example, one can take $M = \mathbb{Z}\ wr\ \mathbb{Z}$). Then M is isomorphic to a quotient of some free group G of finite rank by its normal subgroup H: $M \cong G/H$. Let $\phi : G \to G/H$ be the natural homomorphism and $A, B \leq G$ be the preimages of K and N under ϕ correspondingly. Then $H \lhd A \lhd B \lhd G$, $|G : A| = \infty$. $\Lambda(A) = \Lambda(B) = \Lambda(G) = \partial G$ by Lemma 5.6, hence $St_G\big(\Lambda(A)\big) = G$. We claim that $Comm_G(A) = B$. As we know $Comm_G(A) = VN_G(A)$, therefore $B \subset Comm_G(A)$. Now, for an arbitrary $g \in G \backslash B$, by construction, one has $\phi(A \cap gAg^{-1}) = \{1_M\}$, hence $(A \cap gAg^{-1}) \subset H$. Since K is infinite, we get $|A : H| = \infty$, and thus, $|A : (A \cap gAg^{-1})| = \infty$, so, $g \notin Comm_G(A)$.

In this example the subgroup A of G is not quasiconvex and

$$|St_G\big(\Lambda(A)\big) : Comm_G(A)| = \infty.$$

Result 6.7. ([2, Thm. 2],[14, Lemma 3.9]) *If A is an infinite quasiconvex subgroup of a hyperbolic group G then A has a finite index in its commensurator $Comm_G(A)$.*

By Lemma 4.3, the condition $|Comm_G(A) : A| < \infty$ is equivalent to $Comm_G(A) \preceq A$. It is easy to construct an example of a quasiconvex subset (not subgroup) A with exactly one limit point demonstrating that the latter fails, more precisely, $Comm_G(A)$ can have two limit points:

Example 6.8. Let $G = F(x, y)$ and $A = \{x^n \mid n \in \mathbb{N}\}$ – quasiconvex in G. Then $\Lambda(A) = \{x^\infty\}$ and $Comm_G(A) = \langle x \rangle$. Obviously, $Comm_G(A) \not\preceq A$.

However, in the next section Result 6.7 will be extended to the class of all quasiconvex subsets A with $card\big(\Lambda(A)\big) \geq 2$.

Result 6.9. *Let A be a quasiconvex subgroup of a hyperbolic group G. Then $Comm_G(A)$ is quasiconvex.*

If the subgroup A is infinite, this is a consequence of the Result 6.7 by Remark 4.5. On the other hand, if A is finite, then $Comm_G(A) = G$.

Below we give an example of an infinite quasiconvex set $A \subset G$ such that $Comm_G(A)$ is not quasiconvex.

Example 6.10. We use E. Swenson's Counterexample 12 from [30]. Again, let $G = F(x, y)$ be the free group of rank 2. Let $K = \langle x^n y x^{-n} \mid n \geq 0 \rangle$. It is shown in [30] that $\Lambda(K)$ is not a limit set of a quasiconvex subgroup in G (because $\Lambda(K)$ is not "symmetric": $x^\infty \in \Lambda(K)$ but $x^{-\infty} \notin \Lambda(K)$). As the subgroup K is infinite, we can consider the convex hull $A = CH\big(\Lambda(K)\big)$. By Lemma 5.2, A is quasiconvex and $\Lambda(A) = \Lambda(K)$ ($\Lambda(K) \subset \partial G$ is closed by the claim (b) of Lemma 5.3). A is K-invariant (Lemma 5.5), consequently $K \subset Comm_G(A)$. Remark 5.7 and Lemma 5.9 imply

$$\Lambda(K) \subset \Lambda\big(Comm_G(A)\big) \subset \Lambda\left(St_G\big(\Lambda(A)\big)\right) \subset \Lambda(A) = \Lambda(K)\ .$$

Thus $\Lambda\big(Comm_G(A)\big) = \Lambda\left(St_G\big(\Lambda(A)\big)\right) = \Lambda(K)$, therefore the subgroups $Comm_G(A)$ and $St_G\big(\Lambda(A)\big)$ are not quasiconvex.

In the next section we are going to extend Results 6.1, 6.4 and 6.7 to a broader class of quasiconvex subsets of a hyperbolic group. In particular, we will substitute the requirement for A and B to be subgroups with a weaker condition.

Tame Subsets

Again, let G be a δ-hyperbolic group with fixed finite symmetrized generating set $\mathcal{A}$.

Definition 6.11. A subset A of the group G will be called *tame* if A has at least two limit points on ∂G and $A \preceq CH\big(\Lambda(A)\big)$. I.e., there exists $\nu \geq 0$ such that $A \subset \mathcal{O}_\nu(C)$ where $C = CH\big(\Lambda(A)\big)$.

In particular, this definition implies that any tame subset is infinite.

Remark 6.1. If A and D are subsets of G such that $A \approx D$ and A is tame then D is also tame.

Indeed, by Lemma 5.3.(e), $\Lambda(A) = \Lambda(D)$. Hence

$$D \preceq A \preceq CH\big(\Lambda(A)\big) = CH\big(\Lambda(D)\big).$$

Thus "tameness" of a subset is preserved under the equivalence relation "$\approx$".

Lemma 6.2. *Let A, B, C, D be non-empty subsets of the group G where A and B are tame, C is finite and D is arbitrary. Let $H \leq G$ be an infinite subgroup. Then the following sets are tame:* 1) $A \cup B$; 2) $A \cup C$; 3)$A \cdot C$; 4) $D \cdot A$; 5) H.

Proof. 1) Since $\Lambda(A), \Lambda(B) \subset \Lambda(A \cup B)$, we have

$$CH\big(\Lambda(A)\big) \cup CH\big(\Lambda(B)\big) \subseteq CH\big(\Lambda(A \cup B)\big).$$

$A \preceq CH\big(\Lambda(A)\big)$ and $B \preceq CH\big(\Lambda(B)\big)$ by assumptions of the lemma, hence

$$A \cup B \preceq CH\big(\Lambda(A)\big) \cup CH\big(\Lambda(B)\big) \preceq CH\big(\Lambda(A \cup B)\big)\ ,$$

which shows that $A \cup B$ is tame.

2) and 3) are immediate consequences of the fact that $A \cup C \approx A$, $A \cdot C \approx A$, and Remark 6.1.

4) Denote $K = CH\big(\Lambda(A)\big)$. By definition, $A \preceq K$, therefore $DA \preceq DK$. Now, since for every $y \in D$, $yK = CH\big(\Lambda(yA)\big) \subset CH\big(\Lambda(DA)\big)$, we obtain $DK \subset CH\big(\Lambda(DA)\big)$. Hence $DA \preceq CH\big(\Lambda(DA)\big)$.

5) The set $CH\big(\Lambda(H)\big)$ is H-invariant by Lemma 5.5, therefore for any $x \in CH\big(\Lambda(H)\big)$ we have $Hx \subset CH\big(\Lambda(H)\big)$. But $H \preceq Hx$, hence H is a tame subset. □

Example 6.12. Lemma 6.2 shows that any infinite set U that is a finite union of quasiconvex products in G is tame.

Example 6.13. In Example 6.5 we constructed a quasiconvex subset A in the group $G = F(x,y)$ with exactly two limit points $x^\infty, x^{-\infty}$. Therefore $CH\big(\Lambda(A)\big)$ consists of one bi-infinite geodesic and $CH\big(\Lambda(A)\big) \cap G = \{x^n \mid n \in \mathbb{Z}\}$. Now, for each $n \in \mathbb{N}$, $x^n y^{n^2} \in A$ and

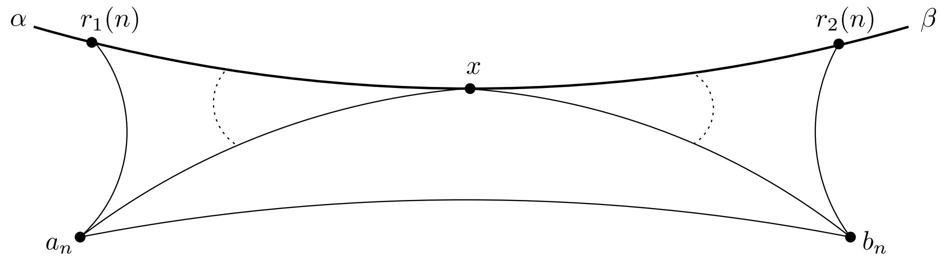

Figure 2: Illustration of Lemma 6.3

$d\left(x^n y^{n^2}, CH\big(\Lambda(A)\big)\right) = n^2 \to \infty$, as $n \to \infty$. Thus, the subset A from Example 6.5 is not tame.

Lemma 6.3. *Suppose A is a tame subset of a hyperbolic group G and $B \subseteq G$ is a quasiconvex subset such that $\Lambda(A) \subseteq \Lambda(B)$. Then $A \preceq B$.*

Proof. By the conditions of the lemma, $A \preceq CH\big(\Lambda(A)\big) \preceq CH\big(\Lambda(B)\big)$. Therefore it remains to show that $CH\big(\Lambda(B)\big) \preceq B$, i.e., there exists $\varkappa \geq 0$ such that $CH\big(\Lambda(B)\big) \subset \mathcal{O}_{\varkappa}(B)$.

Let η be the quasiconvexity constant for B. Consider any $x \in CH\big(\Lambda(B)\big)$. By definition, there exist $\alpha, \beta \in \Lambda(B)$ such that $x \in (\alpha, \beta)$. Let $r_1, r_2 : [0, \infty) \to \Gamma(G, \mathcal{A})$ be the geodesic half-lines obtained by bisecting (α, β) at the point x. Thus, $r_1(0) = r_2(0) = x$, $\lim\limits_{i\to\infty} r_1(i) = \alpha$, $\lim\limits_{i\to\infty} r_2(i) = \beta$ (see Figure 2).

There are sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ in B converging to infinity such that $\lim\limits_{i\to\infty} a_i = \alpha$, $\lim\limits_{i\to\infty} b_i = \beta$. Hence $(r_1(i)|a_i)_x \to \infty$, $(r_2(i)|b_i)_x \to \infty$ as $i \to \infty$. Consequently, for some $n \in \mathbb{N}$ we have

$$(r_1(n)|a_n)_x > 2\delta, \quad (r_2(n)|b_n)_x > 2\delta\ . \tag{6.1}$$

Remark 6.4. Let PQR be a geodesic triangle in the Cayley graph $\Gamma(G, \mathcal{A})$ and $(P|Q)_R > 2\delta$. Then $d(R, [P, Q]) > 2\delta$.

Indeed, assume, by contradiction, that there exists $S \in [P, Q]$ satisfying $d(R, S) \leq 2\delta$. By definition of the Gromov product,

$$\begin{aligned} (P|Q)_R = \frac{1}{2}\big(d(P,R) + d(Q,R) - d(P,Q)\big) \\ \leq \frac{1}{2}\big(d(P,S) + d(S,R) + d(Q,S) + d(S,R) - d(P,Q)\big). \end{aligned}$$

But $d(P,S) + d(Q,S) = d(P,Q)$ since $[P,Q]$ is a geodesic segment, therefore $(P|Q)_R \leq d(S,R) \leq 2\delta$. A contradiction.

Consider, now, the geodesic quadrangle in $\Gamma(G, \mathcal{A})$ with vertices a_n, $r_1(n)$, $r_2(n)$, b_n.

$x \in [r_1(n), r_2(n)]$. Applying (6.1) and Remark 6.4 we obtain

$$d\big(x, [a_n, r_1(n)]\big) > 2\delta, \; d\big(x, [b_n, r_2(n)]\big) > 2\delta.$$

Since the Cayley graph $\Gamma(G, \mathcal{A})$ is δ-hyperbolic, all quadrangles are 2δ-slim, thus

$$[r_1(n), r_2(n)] \subset \mathcal{O}_{2\delta}\big([a_n, r_1(n)] \cup [b_n, r_2(n)] \cup [a_n, b_n]\big).$$

Consequently, $d(x, [a_n, b_n]) \leq 2\delta$. $a_n, b_n \in B$ and B is η-quasiconvex, therefore $[a_n, b_n] \subset \mathcal{O}_\eta(B)$.

So, $d(x, B) \leq 2\delta + \eta$ for every $x \in CH\big(\Lambda(B)\big)$. After denoting $\varkappa = 2\delta + \eta$ we achieve $CH\big(\Lambda(B)\big) \subset \mathcal{O}_\varkappa(B)$. Thus, Lemma 6.3 is proved. □

Corollary 6.5. *Let $A \preceq B$ be subsets of G where A has at least two limit points on ∂G and B is quasiconvex. Then $Comm_G(A) \preceq B$, $St_G\big(\Lambda(A)\big) \preceq B$.*

Proof. By Remark 5.7 it is enough to prove the second inequality.

If $St_G\big(\Lambda(A)\big)$ is finite, there is nothing to prove. So, assume it is infinite. Using Lemmas 5.9, 5.3 we get

$$\Lambda\Big(St_G\big(\Lambda(A)\big)\Big) \subseteq \Lambda(A) \subseteq \Lambda(B).$$

According to Lemma 6.2, any subgroup is a tame subset, therefore by Lemma 6.3 we have $St_G\big(\Lambda(A)\big) \preceq B$. □

The latter corollary immediately implies the following generalization of Result 6.7:

Proposition 6.6. *Let G be a hyperbolic group and let $A \subset G$ be a quasiconvex subset that has at least two distinct limit points on the boundary ∂G. Then $St_G\big(\Lambda(A)\big) \preceq A$. Consequently, $Comm_G(A) \preceq A$.*

Now we are going to extend Result 6.1 to all tame quasiconvex subsets.

Proposition 6.7. *Suppose A and B are tame quasiconvex subsets of a hyperbolic group G. Then $A \approx B$ if and only if $\Lambda(A) = \Lambda(B)$.*

Proof. The necessity is given by Lemma 5.3.(e); the sufficiency immediately follows from Lemma 6.3. □

Now we are able to characterize all tame quasiconvex subsets of a hyperbolic group.

Corollary 6.8. *Let A be a subset of a hyperbolic group G having at least two distinct limit points on ∂G. Then the following two conditions are equivalent:*

1) *A is tame and quasiconvex;*

2) *$A \approx CH\big(\Lambda(A)\big)$;*

3) *$A \approx CH\big(\Omega\big)$ for some closed subset Ω of ∂G with $card(\Omega) \geq 2$.*

Proof. Suppose $\Omega \subset \partial G$ is closed, $card(\Omega) \geq 2$ and $C = CH(\Omega)$. Then from Lemma 5.2.(b) we get $\Lambda(C) = \Omega$. Hence $CH\big(\Lambda(C)\big) = C$ implying that $C = CH(\Omega)$ is tame. $CH(\Omega)$ is quasiconvex by Lemma 5.2.(a).

Since the properties of quasiconvexity and tameness are preserved by equivalence relation "$\approx$", 3) implies 1).

$\Lambda\left(CH\big(\Lambda(A)\big)\right) = \Lambda(A)$ by Lemma 5.2.(b). $CH\big(\Lambda(A)\big)$ is tame and quasiconvex as noted in the beginning of the proof, hence Proposition 6.7 shows that 1) implies 2).

The implication 2) $\Rightarrow$ 3) holds true because of to Lemma 5.3.(b). $\square$

Finally, Result 6.4 is generalized as follows:

Proposition 6.9. *For any tame quasiconvex subset A of a hyperbolic group G, $Comm_G(A) = St_G\big(\Lambda(A)\big)$.*

Proof. By Remark 5.7 it is enough to show that $St_G\big(\Lambda(A)\big) \subseteq Comm_G(A)$. Take an arbitrary $g \in St_G\big(\Lambda(A)\big)$. Then $\Lambda(gA) = g \circ \Lambda(A) = \Lambda(A)$. The subset gA is tame and quasiconvex since A is so, hence by Proposition 6.7, $gA \approx A$. Thus, $g \in Comm_G(A)$. Q.e.d. $\square$

Now, it is easy to see that the set A from Example 6.10 is tame and quasiconvex, thus, $Comm_G(A) = St_G\big(\Lambda(A)\big)$. However $Comm_G(A)$ is not quasiconvex. Thus we can not extend Result 6.9 in the same way we did the other ones.

CHAPTER VII

COMPARING SUBSETS WITH SUBGROUPS

The goal of this chapter is to provide tools allowing to compare subgroups with quasiconvex subsets in a given word hyperbolic group G. In particular, we are going to prove

Theorem 7.1. *Assume that U is a finite union of quasiconvex products in a hyperbolic group G and the subgroups $H_1, H_2, \dots, H_s$ are all the members of U. If K is a subgroup of G and $K \subseteq U$ then for some $g \in G$ and $j \in \{1, 2, \dots, s\}$ one has $|K : (K \cap H_j^g)| < \infty$.*

Definition 7.1. We will say that a finite union of quasiconvex products has *infinite index* in G if each of its members has infinite index in G.

Recall that a group G is called *bounded generated* if it coincides with a product of finitely many cyclic subgroups, i.e., there exist elements $g_1, \dots, g_n \in G$ such that

$$G = \langle g_1 \rangle \langle g_2 \rangle \cdot \dots \cdot \langle g_n \rangle.$$

It is known that a word hyperbolic group is bounded generated if and only if it is elementary (see [19, Cor. 4.3]). Since any cyclic subgroup of a hyperbolic group is quasiconvex, the following statement is more general:

Corollary 7.2. *Let G be a hyperbolic group and U be a finite union of quasiconvex products of infinite index in G. Then U is a proper subset of G, i.e., $G \neq U$.*

Proof. This can be obtained as a particular case of Theorem 7.1 when $K = G$. □

Relatively Small Subsets

Definition 7.2. Let H be a subgroup of the group G and $Q \subseteq G$ be a quasiconvex subset. The subset Q will be called *small relatively to H* if for any two finite subsets P_1, P_2 of the group G one has

$$H \nsubseteq \left(P_1 \cdot Q^{-1} \cdot Q \cdot P_2\right). \qquad (*)$$

At the first glance this definition may seem unnatural and sophisticated but it turns out to be quite useful and will be applied later in this work.

Example 7.3. Obviously if the subgroup H is infinite then any finite subset Q of G is small relatively to H.

Example 7.4. A theorem of B. Neumann [23] states that a group G can not be covered by finitely many cosets to subgroups of infinite index. Thus, if $Q \leq G$ is a subgroup and $|G : Q| = \infty$ then Q is small relatively to G.

Proposition 7.3. *Let $K, H_1, H_2, \dots, H_s$ be subgroups of a group G such that $K \preceq \bigcup_{i=1}^{s} H_i$. Then $|K : (K \cap H_i)| < \infty$ for some $i \in \{1, \dots, s\}$.*

Proof. Fist, let us observe that it is enough to consider the case when $K = G$ (i.e., when $H_i \leq K$ for each i). Indeed, suppose

$$K \subset \bigcup_{i=1}^{s} H_i P,$$

for some finite subset $P = \{g_1, \dots, g_m\} \subset G$. Then

$$K = \bigcup_{i=1}^{s} (K \cap H_i P) = \bigcup_{i=1}^{s} \bigcup_{j=1}^{m} (K \cap H_i g_j).$$

During the proof of Lemma 4.3 we saw that for any i and $g \in G$, $K \cap H_i g$ is either empty or equal to $(K \cap H_i)h$ for some $h \in H$. Therefore for each i we can find a finite subset P_i of K satisfying

$$K = \bigcup_{i=1}^{s} \big((K \cap H_i) P_i\big) \subset \left(\bigcup_{i=1}^{s} (K \cap H_i) \right) P',$$

where $P' = P_1 \cup \dots \cup P_s \subset K$, $card(P') < \infty$.

Now the claim of the proposition follows from Neumann's theorem mentioned in Example 7.4. We will present a proof for completeness.

So, let us have $G = H_1 P \cup \dots \cup H_s P$ for a finite $P \subset G$. Induction on s. The case $s = 1$ is trivial. So, suppose $s > 1$. If $|G : H_s| < \infty$ there is nothing to prove. Otherwise, there must exist $x \in G \backslash (H_s P)$. Hence $H_s x \cap H_s P = \emptyset$, implying $H_s x \subset \bigcup_{i=1}^{s-1} H_i P$, $H_s \subset \bigcup_{i=1}^{s-1} H_i P x^{-1}$. Therefore

$$G = \bigcup_{i=1}^{s-1} H_i P \cup \left(\bigcup_{i=1}^{s-1} H_i P x^{-1} \right) P = \bigcup_{i=1}^{s-1} H_i (P \cup P x^{-1} P).$$

Since the subset $P \cup P x^{-1} P$ is finite, we can apply the inductions hypothesis to find $i \in \{1, \dots, s-1\}$ such that $|G : H_i| < \infty$. Q.e.d. $\square$

Corollary 7.4. *Suppose $K, H \leq G$ where H is quasiconvex. Then H is small relatively to K if and only if $|K : (K \cap f H f^{-1})| = \infty$ for all $f \in G$.*

Proof. The necessity trivially follows from the definition.

To show the sufficiency, suppose that $P_1 = \{f_1, \ldots, f_l\}$ and P_2 are arbitrary finite subsets of G. Arguing by contradiction, assume $K \subset P_1 H^{-1} H P_2$. Since $H^{-1}H = H$ one has

$$K \subset \bigcup_{i=1}^{l} \left(f_i H f_i^{-1}\right)\left(f_i P_2\right) \subset \left(\bigcup_{i=1}^{l} f_i H f_i^{-1}\right)\left(f_1 P_2 \cup \cdots \cup f_l P_2\right) \preceq \bigcup_{i=1}^{l} f_i H f_i^{-1}.$$

Applying Proposition 7.3 to the latter formula, we achieve a contradiction with the assumption $|K : (K \cap f_i H f_i^{-1})| = \infty$, $i = 1, \ldots, l$. □

Auxiliary Lemmas

Let G be a δ-hyperbolic group, $Q \subseteq G$ – η-quasiconvex subset.

Lemma 7.5. *The subset $Q^{-1}Q \subseteq G$ is $(\eta + \delta)$-quasiconvex.*

Proof. Consider arbitrary $x \in Q^{-1}Q$, $x = u^{-1}v$ where $u, v \in Q$. Then $[u, v] \subset \mathcal{O}_\eta(Q)$. Since the metric on $\Gamma(G, \mathcal{A})$ is invariant under the action of G by left translations, we have

$$[1_G, x] = u^{-1} \circ [u, v] \subset \mathcal{O}_\eta(u^{-1}Q) \subset \mathcal{O}_\eta(Q^{-1}Q). \tag{7.1}$$

Since the geodesic triangles in $\Gamma(G, \mathcal{A})$ are δ-slim, for any two $x_1, x_2 \in Q^{-1}Q$ using (7.1) one obtains

$$[x_1, x_2] \subset \mathcal{O}_\delta([1_G, x_1] \cup [1_G, x_2]) \subseteq \mathcal{O}_{\delta+\eta}(Q^{-1}Q)\ .$$

The lemma is proved. □

Lemma 7.6. *Suppose $S, Q \subseteq G$ and the subset Q is η-quasiconvex. Then on the boundary ∂G of the group G one has*

$$\Lambda(S \cdot Q) \subseteq \Lambda(S) \,\cup\, (S \cdot Q) \circ \Lambda(Q^{-1} \cdot Q)\ ,$$

$$\Lambda(S \cdot Q^{-1}) \subseteq \Lambda(S) \,\cup\, (S \cdot Q^{-1} \cdot Q) \circ \Lambda(Q^{-1})\ .$$

Proof. Let $P \subseteq G$, consider an arbitrary limit point $\alpha \in \Lambda(SP)$. There is a sequence $(z_i)_{i \in \mathbb{N}}$ converging to infinity in G with $z_i = x_i y_i$, $x_i \in S$, $y_i \in P$ for all $i \in \mathbb{N}$, and $\lim\limits_{i \to \infty} z_i = \alpha$.

I. Suppose, first, that $\sup_{i \in \mathbb{N}} (z_i | x_i)_{1_G} = \infty$. Then one can find a sequence $(i_j)_{j \in \mathbb{N}}$ of natural numbers such that

$$\lim_{j \to \infty} (z_{i_j} | x_{i_j})_{1_G} = \infty\ .$$

But $\lim_{j\to\infty} z_{i_j} = \lim_{i\to\infty} z_i = \alpha$, which implies that $(x_{i_j})_{j\in\mathbb{N}}$ also converges to infinity and

$$\lim_{j\to\infty} x_{i_j} = \lim_{j\to\infty} z_{i_j} = \alpha \ .$$

Thus, $\alpha \in \Lambda(S)$.

II. Therefore we can now assume that there is a number $M \geq 0$ such that $(z_i|x_i)_{1_G} \leq M$ for every $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ consider a geodesic triangle in $\Gamma(G, \mathcal{A})$ with vertices 1_G, x_i and z_i. It is δ-thin, hence $d(1_G, [x_i, z_i]) \leq M + \delta$.

a) Suppose $P = Q$. Fix an arbitrary element $q \in Q$ and let $\varkappa = |q|_G$. Then

$$[1_G, y_i] \in \mathcal{O}_\delta([1_G, q] \cup [q, y_i]) \subset \mathcal{O}_{\delta+\varkappa}([q, y_i]) \subset \mathcal{O}_{\delta+\varkappa+\eta}(Q) \ .$$

Using the left translation-invariance of the word metric, we get

$$[x_i, z_i] = [x_i, x_i y_i] \subset \mathcal{O}_{\delta+\varkappa+\eta}(x_i Q) \ .$$

Consequently, there exists $q_i \in Q$ satisfying

$$d(1_G, x_i q_i) = |x_i q_i|_G \leq M + 2\delta + \varkappa + \eta \ \text{ for every } i \in \mathbb{N}.$$

The group G has only finitely many elements in a ball of finite radius, hence by passing to a subsequence, we can assume that $x_i q_i = p \in SQ$ for all $i \in \mathbb{N}$. Thus, $z_i = x_i q_i q_i^{-1} y_i = p q_i^{-1} y_i \in p Q^{-1} Q$ for every i, which implies

$$\alpha \in \Lambda(pQ^{-1}Q) = p \circ \Lambda(Q^{-1}Q) \subset (SQ) \circ \Lambda(Q^{-1}Q) \ .$$

b) Assume, $P = Q^{-1}$. Then $y_i^{-1} \in Q$, hence

$$[x_i, z_i] = [z_i y_i^{-1}, z_i] \subset \mathcal{O}_{\delta+\varkappa+\eta}(z_i Q) \ .$$

So, there are elements $q_i \in Q$ such that

$$d(1_G, z_i q_i) = |z_i q_i|_G \leq M + 2\delta + \varkappa + \eta \ \text{ for every } i \in \mathbb{N}.$$

As before, we can suppose that $z_i q_i = p \in SQ^{-1}Q$ for all $i \in \mathbb{N}$. Thus $z_i = p q_i^{-1} \in pQ^{-1}$ for every i, implying

$$\alpha \in \Lambda(pQ^{-1}) = p \circ \Lambda(Q^{-1}) \subset (SQ^{-1}Q) \circ \Lambda(Q^{-1}) \ .$$

Q.e.d. $\square$

On Condition $(*)$

We are now going to explore the properties of condition $(*)$ introduced on page 30. In particular, we will show that a quasiconvex subset Q is small relatively to a subgroup H if and only if $\Lambda(H)$ is not covered by the orbit of $\Lambda(Q^{-1}Q)$ under the action of G on ∂G.

Lemma 7.7. *Assume that H is a subgroup and A is a non-empty quasiconvex subset of a hyperbolic group G. The following conditions are equivalent:*

1. *There are finite subsets $P_1, P_2 \subset G$ such that $H \subseteq P_1 \cdot A \cdot P_2$;*

2. $\Lambda(H) \subseteq G \circ \Lambda(A)$.

Proof. The implication $1 \Rightarrow 2$ is an immediate consequence of Lemma 5.3 since

$$\Lambda(P_1 \cdot A \cdot P_2) = \Lambda(P_1 A) = P_1 \circ \Lambda(A) \subset G \circ \Lambda(A) \ .$$

Now let's show that 2 implies 1. Denote $\Omega = \Lambda(A)$.

If the subgroup H is finite then the claim is trivial.

If H is infinite elementary then $card\big(\Lambda(H)\big) = 2$, hence according to the condition 2, there are elements $g_1, g_2 \in G$ such that

$$\Lambda(H) \subset g_1 \circ \Lambda(A) \cup g_2 \circ \Lambda(A) = \Lambda(g_1 A \cup g_2 A) \ .$$

The subset $g_1 A \cup g_2 A \subset G$ is quasiconvex by Lemma 3.8, H is tame (Lemma 6.2), hence we can apply Lemma 6.3 to find a finite subset P_2 of G satisfying $H \subseteq (g_1 A \cup g_2 A) P_2 = P_1 A P_2$ where $P_1 = \{g_1, g_2\}$.

Thus we can assume that the subgroup H is non-elementary.

Case 1. Suppose that for some $g \in G$, $g \circ \Omega$ contains a non-empty open set U of the subspace $\Lambda(H)$, i.e., $U = U' \cap \Lambda(H)$ for some open set $U' \subseteq \partial G$. Then, by Lemma 5.10, for any $\beta \in \Lambda(H)$ there exists $h \in H$ such that $h \circ \beta \in U'$. On the other hand, $h \circ \beta \in \Lambda(H)$ by Lemma 5.5, thus, $h \circ \beta \in U$, i.e., $\beta \in h^{-1} \circ U$. Consequently,

$$\Lambda(H) \subseteq \bigcup_{h \in H} h \circ U. \tag{7.2}$$

The space $\Lambda(H)$ is a closed subspace of the compact metric space ∂G, hence it is compact itself and one can choose a finite subcover of the open cover from (7.2). Thus

$$\Lambda(H) \subseteq \bigcup_{i=1}^{N} h_i \circ U \subseteq \bigcup_{i=1}^{N} h_i \circ (g \circ \Omega) = \bigcup_{i=1}^{N} h_i \circ \Lambda\big(gA\big) = \Lambda\left(\bigcup_{i=1}^{N} h_i g A\right) = \Lambda(P_1 A)$$

according to Lemma 5.3, where $P_1 = \bigcup_{i=1}^{N} h_i g \subset G$, $card(P_1) < \infty$.

The set $P_1 A = \bigcup_{y \in P_1} yA$ is quasiconvex as a finite union of quasiconvex sets, therefore we are able to apply Lemma 6.3. Thus, there is a finite subset P_2 of the group G such that $H \subseteq P_1 \cdot A \cdot P_2$, as we needed.

Hence we can proceed to

Case 2. For every $g \in G$, $g \circ \Omega$ contains no non-empty open subsets of $\Lambda(H)$. Ω is a closed subset of the boundary ∂G by Lemma 5.3.(b), thus $g \circ \Omega$ is also closed and, hence $\big(g \circ \Omega\big) \cap \Lambda(H)$ is a closed nowhere dense subset of the compact metric space $\Lambda(H)$. Evidently, $\Lambda(H)$ is a Baire space (it is locally compact and Hausdorff). Since the group G is countable, the set

$$(G \circ \Omega) \cap \Lambda(H) = \bigcup_{g \in G} (g \circ \Omega) \cap \Lambda(H)$$

is of the first category in the space $\Lambda(H)$, hence by a well-know theorem from topology (see, for instance, [6, Ch. XI, Thm. 10.5]),

$$\Lambda(H) \neq (G \circ \Omega) \cap \Lambda(H) \ ,$$

therefore $\Lambda(H) \nsubseteq G \circ \Lambda(A)$ which is a contradiction to our assumptions. Thus, Case 2 is impossible. □

Remark 7.8. In the notations of Lemma 7.7, assume that H is non-elementary. Then the following are equivalent:

1. There exist no finite subsets P_1, P_2 of G such that $H \subset P_1 A P_2$;

2. On the hyperbolic boundary ∂G for any $g \in G$ the set $\big(g \circ \Lambda(A)\big) \cap \Lambda(H)$ is nowhere dense in $\Lambda(H)$.

In the proof of Lemma 7.7 the condition 1 automatically puts us into the Case 2, thus $1 \Rightarrow 2$. Now, if the property 2 holds and the property 1 doesn't, we can find some finite subsets $P_1, P_2 \subset G$ satisfying $H \subset P_1 A P_2$. Therefore by Lemma 5.3,

$$\Lambda(H) \subset \Lambda(P_1 A P_2) = \Lambda(P_1 A) \ .$$

Hence $\Lambda(H) = \bigcup_{g\in P_1} \big(g \circ \Lambda(A) \cap \Lambda(H)\big)$ contradicting the assumption 2 because of the fact that a finite union of nowhere dense subsets is nowhere dense in $\Lambda(H)$. Hence $2 \Rightarrow 1$.

The next lemma states that a union and a product of two relatively small subsets is again relatively small:

Lemma 7.9. *Suppose that H is a non-elementary subgroup and Q, S are quasiconvex subsets of a hyperbolic group G. Assume that for any two finite subsets P_1, P_2 of the group G*

$$H \nsubseteq P_1 Q^{-1} Q P_2 \text{ and } H \nsubseteq P_1 S^{-1} S P_2. \tag{7.3}$$

Then the (quasiconvex) subsets $T_1 = Q \cup S$ and $T_2 = QS$ satisfy the same property: for any $i \in \{1,2\}$ and arbitrary finite $P_1, P_2 \subset G$ one has

$$H \nsubseteq P_1 T_i^{-1} T_i P_2.$$

Proof. a) Since $T_1^{-1} = Q^{-1} \cup S^{-1}$, we can apply Lemmas 5.3.(c) and 7.6 to obtain

$$\Lambda\left(T_1^{-1}T_1\right) = \Lambda\left(Q^{-1}Q \cup Q^{-1}S \cup S^{-1}Q \cup S^{-1}S\right) =$$

$$= \Lambda\left(Q^{-1}Q\right) \cup \Lambda\left(Q^{-1}S\right) \cup \Lambda\left(S^{-1}Q\right) \cup \Lambda\left(S^{-1}S\right) \subseteq$$

$$\subseteq \Lambda\left(Q^{-1}Q\right) \cup \Lambda(Q^{-1}) \cup G \circ \Lambda\left(S^{-1}S\right) \cup \Lambda(S^{-1}) \cup G \circ \Lambda\left(Q^{-1}Q\right) \cup \Lambda\left(S^{-1}S\right) =$$

$$= G \circ \Lambda\left(S^{-1}S\right) \cup G \circ \Lambda\left(Q^{-1}Q\right) = G \circ \Lambda\left(S^{-1}S \cup Q^{-1}Q\right)$$

(here we used the fact that if $s \in S$ then $S^{-1}s \subset S^{-1}S$, and by Lemma 5.3.(d), $\Lambda(S^{-1}) = \Lambda(S^{-1}s) \subset \Lambda(S^{-1}S)$; similarly for Q). Thus, $G \circ \Lambda\left(T_1^{-1}T_1\right) \subseteq G \circ \Lambda\left(S^{-1}S \cup Q^{-1}Q\right)$.

The conditions (7.3) imply (by Remark 7.8) that for any $g \in G$ the subsets $g \circ \Lambda(Q^{-1}Q) \cap \Lambda(H)$ and $g \circ \Lambda(S^{-1}S) \cap \Lambda(H)$ are nowhere dense in $\Lambda(H)$, therefore the set $G \circ \Lambda(Q^{-1}Q \cup S^{-1}S) \cap \Lambda(H)$ is of the first category in the compact metric space $\Lambda(H)$. Consequently,

$$\Lambda(H) \nsubseteq G \circ (Q^{-1}Q \cup S^{-1}S)\ . \tag{7.4}$$

Hence $\Lambda(H) \nsubseteq G \circ (T_1^{-1}T_1)$. The subset $T_1 \subset G$ is quasiconvex according to Lemma 3.8. To finish the proof it remains to Lemma 7.5 to $T_1^{-1}T_1$ and then Lemma 7.7 to $T_1^{-1}T_1$ and H.

b) The proof for T_2 is similar. Note that $T_2^{-1} = S^{-1}Q^{-1}$, hence by Lemma 7.6,

$$\Lambda\left(T_2^{-1}T_2\right) = \Lambda\left(S^{-1}Q^{-1}QS\right) \subset \Lambda\left(S^{-1}Q^{-1}Q\right) \cup G \circ \Lambda\left(S^{-1}S\right)\ .$$

$Q^{-1}Q \subset G$ is quasiconvex by Lemma 7.5 and $(Q^{-1}Q)^{-1} = Q^{-1}Q$. Applying Lemma 7.6 one more time we obtain

$$\Lambda\left(T_2^{-1}T_2\right) \subset \Lambda\left(S^{-1}\right) \cup G \circ \Lambda\left(Q^{-1}Q\right) \cup G \circ \Lambda\left(S^{-1}S\right) = G \circ \left(Q^{-1}Q \cup S^{-1}S\right) \ .$$

Recalling (7.4), one obtains $\Lambda(H) \nsubseteq G \circ (T_2^{-1}T_2)$. Since T_2 is a quasiconvex subset of G (Lemma 3.8), T_2 and H satisfy the needed property by Lemmas 7.5 and 7.7. $\square$

Finally, we note that in the special case when both Q and Q^{-1} are quasiconvex, condition $(*)$ can be simplified: the product $Q^{-1}Q$ in it can be replaced by just Q.

Corollary 7.10. *Let Q be a quasiconvex subset of a hyperbolic group G and H be a non-elementary subgroup of G. Assume, in addition, that $Q^{-1} \subset G$ is also quasiconvex. Then the following properties are equivalent:*

1. *For arbitrary finite subsets P_1, P_2 of G, $H \nsubseteq P_1QP_2$;*

2. *For arbitrary finite subsets P_1, P_2 of G, $H \nsubseteq P_1Q^{-1}QP_2$.*

Proof. Evidently, 2 implies 1. So, let's assume that 1 holds and prove 2. Since the subset Q^{-1} is quasiconvex, we are able to apply Lemma 7.6 to achieve

$$\Lambda\left(Q^{-1}Q\right) \subset \Lambda\left(Q^{-1}\right) \cup G \circ \Lambda\left(Q\right).$$

Thus, $G \circ \Lambda\left(Q^{-1}Q\right) \subset G \circ (\Lambda(Q^{-1}) \cup \Lambda(Q))$. Observe that the property 1 is equivalent to $H \nsubseteq P_1Q^{-1}P_2$ for any finite $P_1, P_2 \subset G$ (because $H^{-1} = H$). Consequently, by Remark 7.8,

$$\Lambda(H) \nsubseteq G \circ \left(\Lambda(Q^{-1}) \cup \Lambda(Q)\right), \quad \text{hence} \quad \Lambda(H) \nsubseteq G \circ \Lambda\left(Q^{-1}Q\right).$$

After applying Lemma 7.6 one can conclude that the property 2 holds. $\square$

Example 7.5. We observe that the implication $1 \Rightarrow 2$ in the latter corollary may fail if Q^{-1} is not quasiconvex: let $G = F(x, y)$ be the free group with free generators x, y. Set Q to be the set of all reduced words w over the alphabet $\{x^{\pm 1}, y^{\pm 1}\}$ satisfying the property: if $k \in \mathbb{N}$ and $2^k \leq \|w\|$ then the letter on 2^k-th place in w is x. Thus,

$$Q = \{x, x^{-1}, y, y^{-1}, x^2, yx, y^{-1}x, x^3, x^2y, x^2y^{-1}, yxy, yxy^{-1}, y^{-1}xy, y^{-1}xy^{-1}, \dots\} \subset G.$$

The subset Q is quasiconvex in G since any prefix of a word from Q belongs to Q. It is not difficult to show that $G \nsubseteq P_1QP_2$ for any finite subsets $P_1, P_2 \subset G$; nevertheless, $G = Q^{-1}Q$ (because any reduced word is a suffix of some word from Q).

Proof of Theorem 7.1

Proof. If K is finite, there is nothing to prove.

Now, if K is infinite elementary, then there is and element of infinite order $g \in K$ such that $\Lambda(K) = \{g^{\infty}, g^{-\infty}\} \subset \partial G$. Observe that Lemma 7.6 implies

$$\Lambda(U) \subset G \circ \Lambda(H_1) \cup \cdots \cup G \circ \Lambda(H_s).$$

Since $K \subseteq U$, there are $x, y \in G$ and $i, j \in \{1, \ldots, s\}$ with the property $g^{\infty} \in xH_i$ and $g^{-\infty} \in yH_j$. Then $\Lambda(K) \subset \Lambda(xH_i \cup yH_j)$. By Remark 3.7 and Lemma 3.8 the subset $xH_i \cup yH_j \subset G$ is quasiconvex. The subgroup K is tame (Lemma 6.2), hence one is able to apply Lemma 6.3 to achieve $K \preceq xH_i \cup yH_j$. And since $xH_i \cup yH_j \preceq xH_ix^{-1} \cup yH_jy^{-1}$, we have $K \preceq xH_ix^{-1} \cup yH_jy^{-1}$. Now the claim of the theorem follows by Proposition 7.3.

So, we can suppose that the subgroup K is non-elementary.

Assume $|K : (K \cap gH_ig^{-1})| = \infty$ for all $g \in G$ and $i \in \{1, \ldots, s\}$. Then H_1, H_2, $\ldots$, H_s are small relatively to K (Corollary 7.4). The set U can be obtained from the collection of subsets $\{H_1, \ldots, H_s, \{g\} \mid g \in G\}$ after performing finitely many subset operations: products and unions. Since all the members of the collection are small relatively to K, then, by Lemma 7.9, so is U. This leads to a contradiction to the assumption $K \subseteq U$ of the theorem. □

Using argumentation similar to the one above, it is not difficult to obtain

Corollary 7.11. *Assume U is a finite union of quasiconvex products in a hyperbolic group G with members $H_1, \ldots, H_s$ and $K \leq G$. If $H_1, \ldots, H_s$ are all small relatively to the subgroup K then so is U.*

CHAPTER VIII

G-SUBGROUPS OF HYPERBOLIC GROUPS

Let us start with a notion defined by A. Ol'shanskii in [26]:

Definition 8.1. A subgroup H is called a G-subgroup of a hyperbolic group G if for any finite subset $M \subset G$ there exists a homomorphism from G onto a non-elementary hyperbolic group G_1 that is surjective on H and injective on M.

In the original work [11] Gromov stated that if G is a word hyperbolic groups then every non-elementary subgroup satisfies the above property. Afterwards he noted that this would allow to construct new examples of groups, including a non-abelian group all of whose proper subgroups are cyclic, finitely generated infinite torsion groups, etc.

However, in [26] Ol'shanskii gave the following

Example 8.2. Define $G = \Big(F(x,y) \times \langle a \rangle_2\Big) * \langle b \rangle$, $H = F(x,y)$, $M = \{1, [a,b]\}$, where $F(x,y)$ is the free group of rank 2, $\langle a \rangle_2$ is the cyclic group of order 2 and $\langle b \rangle$ is the infinite cyclic group. G is hyperbolic as a free product of two hyperbolic groups and for any homomorphism $\phi : G \to G_1$, with $\phi(H) = G_1$, one has $M \subset ker(\phi)$. Thus, the non-elementary subgroup H is not a G-subgroup of G.

The above example outlines a possible obstacle. In general, the problem can only be caused by a finite subgroup of G which is normalized by H.

Description of G-subgroups

Suppose G is a δ-hyperbolic group with a fixed finite symmetrized generating set $\mathcal{A}$.

If G is a groups and $S \subseteq G$, $C_H(S)$ will denote the centralizer subgroup of S in H, i.e.,

$$C_H(S) = \{h \in H \mid hg = gh \ \ \forall\, g \in S\}.$$

For any subgroup H of G denote by H^0 the set of elements of infinite order in H. If $g \in G^0$, $E(g)$ is the maximal elementary subgroup of G containing g (see page 7).

Now, set $E(H) = \bigcap_{x \in H^0} E(x)$. If H is a non-elementary subgroup of G, then $E(H)$ is the unique maximal finite subgroup of G normalized by H ([26, Prop. 1]). Hence H acts on $E(H)$ by conjugation and we have a homomorphism of H into the permutation group on the set of elements of $E(H)$. The kernel of that homomorphism is $C_H\big(E(H)\big)$, which sometimes will be denoted by $K(H)$. The index $|H : K(H)|$ is finite because of the finiteness of $E(H)$.

The following characterization of all G-subgroups was given by A. Ol'shanskii:

Theorem 8.1. ([26, Thm. 1]) *A non-elementary subgroup H of a hyperbolic group G is a G-subgroup if and only if $E(H) = E(G)$ and $|H : K(H)| = |G : K(G)|$ (i.e., the actions by conjugation of H and G on $E(H) = E(G)$ are similar: for every $g \in G$ there exists an element $h \in H$ with $gag^{-1} = hah^{-1}$ for all $a \in E(G)$).*

The sufficiency in the previous statement was provided by

Theorem 8.2. ([26, Thm. 2]) *Let $H_1, H_2, \ldots, H_k$ be G-subgroups of a non-elementary hyperbolic group G and $H'_1, \ldots, H'_{k'}$ be some non-elementary subgroups of G. Then for any finite subset $M \subset G$, there is a quotient G_1 of the group G, such that,*

1) *G_1 is a non-elementary hyperbolic group;*

2) *The natural homomorphism $\phi : G \to G_1$ injective on M;*

3) *ϕ is surjective on each of the subgroups $H_1, \ldots, H_k$, i.e., $\phi(H_i) = G_1$ for each $i = 1, 2, \ldots, k$;*

4) *ϕ-images of two elements from M are conjugate in G_1 if and only if these elements are conjugate in G;*

5) *The centralizer $C_{G_1}\big(\phi(a)\big)$ for every $a \in M$ is the ϕ-image of the centralizer $C_G(a)$;*

6) *$ker(\phi)$ is a torsion-free subgroup;*

7) *ϕ induces a bijective map on sets of conjugacy classes of elements having finite orders in G and G_1 respectively;*

8) *$\phi(H'_1), \ldots, \phi(H'_{k'})$ are non-elementary subgroups of G_1.*

For the next few chapters our goal will be to find and prove an analog of Theorem 8.2 where a finite subset M is replaced by a quasiconvex subset Q.

The main tool used for proving Theorem 8.2 is generalized Small Cancellation Theory for hyperbolic groups developed in [26]. Some of this theory will be discussed Chapter 10.

Suitable Elements

An important role in the theory of G-subgroups plays the notion of suitable elements.

If $g \in G^0$, the subset of $E(g)$ consisting of all elements having finite order is a (finite) subgroup $T(g) \leq E(g)$.

Definition 8.3. Let G be a hyperbolic group and H be a non-elementary subgroup. An element $g \in H^0$ will be called *H-suitable* if $E(H) = T(g)$ and

$$E(g) = E^+(g) = C_G(g) = T(g) \times \langle g \rangle.$$

In particular, if the element g is H-suitable then $g \in C_H\big(E(H)\big)$.

Definition 8.4. Two elements $g, h \in G$ of infinite order are called *commensurable* if

$$g^k = ah^l a^{-1}$$

for some non-zero integers k, l and some $a \in G$.

Now let's recall the statement of [26, Lemma 3.8]:

Lemma 8.3. *Every non-elementary subgroup H of a hyperbolic group G contains an infinite set of pairwise non-commensurable H-suitable elements.*

We will need the following modification of [26, Lemma 3.7]:

Lemma 8.4. *Let g be an H-suitable element in a non-elementary subgroup H of a hyperbolic group G. Suppose $l \in \mathbb{N}$ and K is a non-elementary subgroup of H. Then for any number $C_1 \geq 0$ there exist elements $x_i \in K$, $i = 1, \dots, l$, satisfying the following properties:*

0) $|x_i|_G > C_1$ *for every* $i = 1, 2, \dots, l$;

1) $x_i \notin E(g)$ *for every* $i = 1, 2, \dots, l$;

2) $x_i \in C_G(E(H))$ *for every* $i = 1, 2, \dots, l$;

3) $ax_i = x_i b$ *for* $a, b \in E(g)$ *implies that* $a = b \in E(H)$, $i = 1, 2, \dots, l$;

4) *if* $a, b \in E(g)$ *and* $ax_i = x_j b$ *for some* $i, j \in \{1, \dots, l\}$ *then* $i = j$.

Proof. Indeed, it is shown in the proof of [26, Lemma 3.7] that if the elements g, h_1, h_2, $\dots$, $h_l \in H$ are pairwise non-commensurable in G then for any sufficiently large $t \in \mathbb{N}$, the elements $x_i = h_i^t$ satisfy the conditions $1) - 4)$. By Lemma 8.3 we can choose such $h_1, \dots, h_l$ inside of K, thus $x_i = h_i^t \in K$. Obviously, if $t \in \mathbb{N}$ is sufficiently large, the property 0) will be satisfied as well. □

CHAPTER IX

AUXILIARY FACTS

In this chapter we prove a number of facts that will be applied in Chapter 11.

Intersections of Quasiconvex Subgroups and their Limit Sets

In [30] E. Swenson showed that intersection of the limit sets of two quasiconvex subgroups is equal to the limit set of their intersection. We will utilize this property later, so we present its proof it for completeness.

Lemma 9.1. ([30, Thm. 8]) *Let A, B be η-quasiconvex subgroups of a δ-hyperbolic group G. Then on the boundary ∂G the equality $\Lambda(A) \cap \Lambda(B) = \Lambda(A \cap B)$ holds.*

Proof. Obviously, $\Lambda(A \cap B) \subseteq \Lambda(A) \cap \Lambda(B)$, so we have to prove the reverse inclusion.

Define a finite subset Φ of G by $\Phi = \{g \in AB \mid |g|_G \leq \delta + 2\eta\}$. For each $g \in \Phi$ pick a pair $(x, y) \in A \times B$ such that $x^{-1}y = g$, and let $\Omega \subset A \times B$ denote the (finite) set of the chosen pairs. Define Ω_1 to be the projection of Ω on the first coordinate, i.e., $\Omega_1 = \{x \in A \mid \exists\ y \in B \text{ with } (x, y) \in \Omega\}$.

By construction, $card(\Omega_1) < \infty$, and thus, $D \stackrel{def}{=} max\{|x|_G \mid x \in \Omega_1\} < \infty$.

Choose an arbitrary $\alpha \in \Lambda(A) \cap \Lambda(B)$. By definition, there exist two sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ converging to infinity in A and B respectively, such that

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = \alpha. \tag{9.1}$$

For every $i \in \mathbb{N}$ consider a geodesic triangle Δ_i in $\Gamma(G, \mathcal{A})$ with vertices 1_G, a_i and b_i. Let $\hat{c}_i$, $\hat{d}_i$ be the special points of Δ_i on the sides $[1_G, a_i]$, $[1_G, b_i]$ correspondingly. Since Δ_i is δ-thin, one has $d(\hat{c}_i, \hat{d}_i) \leq \delta$ for all $i \in \mathbb{N}$. Also, according to the assumptions of the lemma, there are elements $c_i \in A$ and $d_i \in B$ satisfying $d(\hat{c}_i, c_i) \leq \eta$ and $d(\hat{d}_i, d_i) \leq \eta$.

Note that $d(c_i, d_i) = |c_i^{-1}d_i|_G \leq \delta + 2\eta$ and $c_i^{-1}d_i \in \Phi$, therefore there is a pair of elements $(x_i, y_i) \in \Omega$ such that $c_i^{-1}d_i = x_i^{-1}y_i$ (Figure 3). Hence

$$e_i \stackrel{def}{=} c_i x_i^{-1} = d_i y_i^{-1} \in A \cap B.$$

By construction, $d(e_i, \hat{c}_i) \leq d(e_i, c_i) + d(c_i, \hat{c}_i) \leq |x_i|_G + \eta \leq D + \eta$ for every $i \in \mathbb{N}$. Using

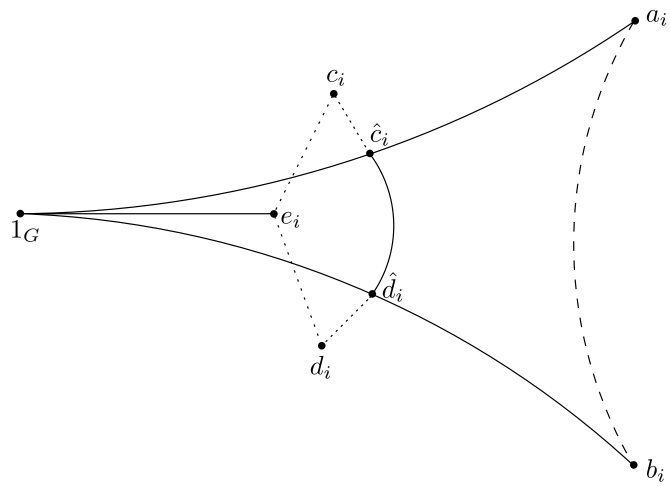

Figure 3: The triangle Δ_i

(9.1), the definition of Gromov product and the triangle inequality we obtain

$$
\begin{aligned}
(e_i|a_i)_{1_G} = \frac{1}{2}\big(d(1_G,e_i)+d(1_G,a_i)-d(e_i,a_i)\big) \geq \\
\frac{1}{2}\big(d(1_G,\hat{c}_i)-d(e_i,\hat{c}_i)+d(1_G,a_i)-d(\hat{c}_i,a_i)-d(e_i,\hat{c}_i)\big) \geq \\
\frac{1}{2}\big(d(1_G,\hat{c}_i)+d(1_G,a_i)-d(\hat{c}_i,a_i)-2D-2\eta\big) = d(1_G,\hat{c}_i)-D-\eta = \\
(a_i|b_i)_{1_G}-D-\eta \to \infty \quad \text{as } i \to \infty.
\end{aligned}
$$

By Remark 5.1, the sequence $(e_i)_{i\in\mathbb{N}}$ converges to infinity and $\lim\limits_{i\to\infty} e_i = \lim\limits_{i\to\infty} a_i = \alpha$. Thus, $\alpha \in \Lambda(A \cap B)$. □

As a corollary of the previous statement one can obtain

Lemma 9.2. *Let G be a δ-hyperbolic and let A, B be quasiconvex subgroups of G. If one has card$(H_1 \cap H_2) < \infty$ then* $\sup\{(a|b)_{1_G} \mid a \in A, b \in B\} < \infty$.

Proof. Indeed, applying Lemma 9.1, we get $\Lambda(A) \cap \Lambda(B) = \emptyset$. The claim now follows by Lemma 5.4. □

Broken Lines and Quasigeodesics

Assume $\mathcal{X}$ is a δ-hyperbolic metric space with metric $d(\cdot,\cdot)$.

If $X_1, X_2, \ldots, X_n$ are points in $\mathcal{X}$, the notation $X_1X_2\ldots X_n$ will be used for a geodesic n-gon with vertices X_i, $i = 1,\ldots,n$, and sides $[X_i, X_{i+1}]$, $i = 1,2,\ldots,n-1$, $[X_n, X_1]$. $[X_1, X_2, \ldots, X_n]$ will denote the broken line with these vertices in the corresponding order (i.e., the path $[X_1, X_2, \dots, X_n]$ will consist of consecutively concatenated geodesic segments $[X_i, X_{i+1}]$, $i = 1, 2, \dots, n-1$).

The following statement is an analog of the fact that in a hyperbolic space k-local geodesics are quasigeodesics if k is sufficiently large (see [1], [4]):

Lemma 9.3. ([27, Lemma 21]) *Let $p = [X_0, X_1, \dots, X_n]$ be a broken line in $\mathcal{X}$ such that $\|[X_{i-1}, X_i]\| > C_1 \; \forall\, i = 1, \dots, n$, and $(X_{i-1}|X_{i+1})_{X_i} \le C_0 \; \forall\, i = 1, \dots, n-1$, where $C_0 \ge 14\delta$, $C_1 > 12(C_0 + \delta)$. Then p is contained in the closed $2C_0$-neighborhood $\mathcal{O}_{2C_0}([X_0, X_n])$ of the geodesic segment $[X_0, X_n]$.*

Lemma 9.4. *In the conditions of Lemma 9.3, $\|[X_0, X_n]\| \ge \|p\|/2$.*

Proof. Induction on n. If $n = 1$ the statement is trivial. So, assume $n > 1$. By the induction hypothesis $\|[X_0, X_{n-1}]\| \ge \|q\|/2$ where q is the broken line $[X_0, X_1, \dots, X_{n-1}]$. It is shown in the proof of [27, Lemma 21] that our conditions imply $(X_0|X_n)_{X_{n-1}} \le C_0 + \delta$, hence

$$\|[X_0, X_n]\| = \|[X_0, X_{n-1}]\| + \|[X_{n-1}, X_n]\| - 2(X_0|X_n)_{X_{n-1}} \ge$$
$$\|q\|/2 + \|[X_{n-1}, X_n]\|/2 + C_1/2 - 2(C_0 + \delta) \ge \|p\|/2 \;.$$

□

Lemma 9.5. *Suppose $\varkappa \ge 0$, $X, Y, Z, X', Y' \in \mathcal{X}$ and $X' \in \mathcal{O}_\varkappa([X, Z])$, $Y' \in \mathcal{O}_\varkappa([Y, Z])$. Then $(X'|Y')_Z \le (X|Y)_Z + 2\varkappa$.*

Proof. Let $X'' \in [X, Z]$, $Y'' \in [Y, Z]$ satisfy $d(X', X'') \le \varkappa$, $d(Y', Y'') \le \varkappa$. According to the triangle inequality,

$$(X'|Y')_Z = \frac{1}{2}\big(d(X', Z) + d(Y', Z) - d(X', Y')\big) \le$$
$$\frac{1}{2}\big(d(X'', Z) + d(Y'', Z) - d(X'', Y'') + 2d(X', X'') + 2d(Y', Y'')\big).$$

Now, since $d(X, Z) = d(X'', Z) + d(X'', X)$ and $d(Y, Z) = d(Y'', Z) + d(Y'', Y)$, we achieve

$$(X'|Y')_Z \le \frac{1}{2}\big(d(X, Z) + d(Y, Z) - [d(X'', Y'') + d(X'', X) + d(Y'', Y)]\big) + 2\varkappa \le$$
$$\frac{1}{2}\big(d(X, Z) + d(Y, Z) - d(X, Y)\big) + 2\varkappa = (X|Y)_Z + 2\varkappa.$$

Q.e.d. □

Lemma 9.6. *Let $\bar\lambda > 0$, $\bar c \geq 0$, $C_0 \geq 14\delta$, $C_1 = 12(C_0+\delta)+\bar c+1$ be given. Then for $\lambda = \bar\lambda/4 > 0$ there exist $c = c(\bar\lambda, \bar c, C_0) \geq 0$ satisfying the statement below.*
Assume $N \in \mathbb{N}$, $X_i \in \mathcal{X}$, $i = 0, \dots, N$, and q_i are $(\bar\lambda, \bar c)$-quasigeodesic paths between X_{i-1} and X_i in $\mathcal{X}$, $i = 1, \dots, N$. If $\|q_i\| \geq (C_1+\bar c)/\bar\lambda$, $i = 1, \dots, N$, and $(X_{i-1}|X_{i+1})_{X_i} \leq C_0$ for all $i = 1, \dots, N-1$, then the path q obtained as a consequent concatenation of $q_1, q_2, \dots, q_N$ is (λ, c)-quasigeodesic.

Proof. Let the number $\nu = \nu(\delta, \bar\lambda, \bar c) \geq 0$ be chosen according to the claim of Lemma 2.2. Set $c = \frac{5}{2}(\nu + C_1) \geq 0$.

Suppose p is an arbitrary subpath of q. Then $p_- \in q_j$, $p_+ \in q_k$ for some $1 \leq j \leq k \leq N$. If $j = k$, p is a subpath of q_j and therefore it is $(\bar\lambda, \bar c)$-quasigeodesic, hence it is (λ, c)-quasigeodesic.

Now let's assume that $j < k$. By our conditions and the choice of ν, there are points $U \in [X_{j-1}, X_j]$, $V \in [X_{k-1}, X_k]$ such that $d(p_-, U) \leq \nu$ and $d(p_+, V) \leq \nu$. $\|[X_{j-1}, X_j]\| \geq \bar\lambda\|q_j\| - \bar c \geq C_1$, similarly, $[X_{k-1}, X_k] \geq C_1$, hence after shifting the points U and V along the segments $[X_{j-1}, X_j]$ and $[X_{k-1}, X_k]$ (correspondingly) by distances at most C_1 we will obtain $\|[U, X_j]\| \geq C_1$, $\|[X_{k-1}, V]\| \geq C_1$, $d(U, p_-) \leq \nu + C_1$, $d(V, p_+) \leq \nu + C_1$ and, therefore $d(p_-, p_+) \geq d(U, V) - 2\nu - 2C_1$.

According to Lemma 9.5, all the conditions of Lemma 9.4 applied to the broken line $[U, X_j, \dots, X_{k-1}, V]$ are satisfied, hence

$$d(U, V) \geq \frac{1}{2}\|[U, X_j, \dots, X_{k-1}, V]\|.$$

Consequently,

$$d(p_-, p_+) \geq \frac{1}{2}\left(d(U, X_j) + \sum_{i=j}^{k-2} d(X_i, X_{i+1}) + d(X_{k-1}, V)\right) - 2\nu - 2C_1.$$

Finally, we observe that $d(U, X_j) \geq \frac{1}{2}d(U, X_j) + \frac{C_1}{2} > \frac{1}{2}\big(d(U, X_j) + \bar c\big)$ and analogously for the other summands. Denote by q'_j, q'_k the segments of q_j and q_k from p_- to X_j and from X_{k-1} to p_+ correspondingly. We obtain

$$d(p_-,p_+) \geq \frac{1}{4}\left(d(U,X_j) + \bar{c} + \sum_{i=j}^{k-2}\Big(d(X_i,X_{i+1}) + \bar{c}\Big) + d(X_{k-1},V) + \bar{c}\right) - 2\nu - 2C_1 \geq$$

$$\frac{1}{4}\left(d(p_-,X_j) + \bar{c} + \sum_{i=j}^{k-2}\Big(d(X_i,X_{i+1}) + \bar{c}\Big) + d(X_{k-1},p_+) + \bar{c}\right) - \frac{5}{2}\nu - \frac{5}{2}C_1 \geq$$

$$\frac{1}{4}\left(\bar{\lambda}\|q_j'\| + \sum_{i=j}^{k-2}\bar{\lambda}\|q_{i+1}\| + \bar{\lambda}\|q_k'\|\right) - \frac{5}{2}\nu - \frac{5}{2}C_1 \geq \frac{\bar{\lambda}}{4}\|p\| - \frac{5}{2}(\nu + C_1).$$

The statement is proved. □

We will also need the fact below concerning quadrangles in a hyperbolic space.

Lemma 9.7. *Consider a geodesic quadrangle $X_1X_2X_3X_4$ in $\mathcal{X}$, whose sides satisfy the inequality $d(X_2,X_3) > d(X_1,X_2) + d(X_3,X_4)$. Then there are points $U,V \in [X_2,X_3]$ such that $d(X_2,U) \leq d(X_1,X_2)$, $d(V,X_3) \leq d(X_3,X_4)$ and the geodesic subsegment $[U,V]$ of $[X_2,X_3]$ lies 2δ-close to the side $[X_1,X_4]$.*

Proof. Since $(X_1|X_3)_{X_2} \leq d(X_1,X_2)$ and $(X_1|X_4)_{X_3} \leq d(X_3,X_4)$, one can choose points $U,V \in [X_2,X_3]$ satisfying $d(X_2,U) = (X_1|X_3)_{X_2}$, $d(X_3,V) = (X_1|X_4)_{X_3}$. The triangle $X_1X_3X_2$ is δ-thin, therefore after taking $V' \in [X_1,X_3]$ at the distance $d(X_3,V)$ from X_3, one obtains $[U,V] \subset \mathcal{O}_\delta([X_1,V'])$. Finally, since V' is the special point of triangle $X_1X_3X_4$ by construction, $[X_1,V']$ is in the closed δ-neighborhood of the side $[X_1,X_4]$, and thus, $[U,V] \subset \mathcal{O}_{2\delta}([X_1,X_4])$. □

Double Cosets of Quasiconvex Subgroups

Let G be a δ-hyperbolic group and $\Gamma(G,\mathcal{A})$– its Cayley graph corresponding to some finite symmetrized generating set $\mathcal{A}$.

In the case when A is a quasiconvex subgroup of the group G, the statement below was proved in [9, Lemma 1.2]. In situations that we will be considering, however, A will be just a quasiconvex subset.

Lemma 9.8. *Let A be an infinite η-quasiconvex set in G and $g \in G$. Then, if the intersection $A \cap gAg^{-1}$ is infinite, there exists an element $r \in G$ with $|r|_G \leq 4\delta + 2\eta + 2\varkappa$ such that $g \in ArA^{-1}$, where $\varkappa$ is the length of a shortest element from A.*

Proof. Note, at first, that for every $a \in A$ the geodesic segment $[1_G,a]$ belongs to a closed $(\delta+\eta+\varkappa)$-neighborhood of A in $\Gamma(G,\mathcal{A})$. Indeed, pick $b \in A$ with $d(1_G,b) = |b|_G = \varkappa$ and consider the geodesic triangle $1_G ab$. Using δ-hyperbolicity of the Cayley graph one achieves

$$[1_G, a] \subset \mathcal{O}_\delta([a,b] \cup [1_G, b]) \subset \mathcal{O}_{\delta+\varkappa}([a,b]) \subset \mathcal{O}_{\delta+\varkappa+\eta}(A) \ .$$

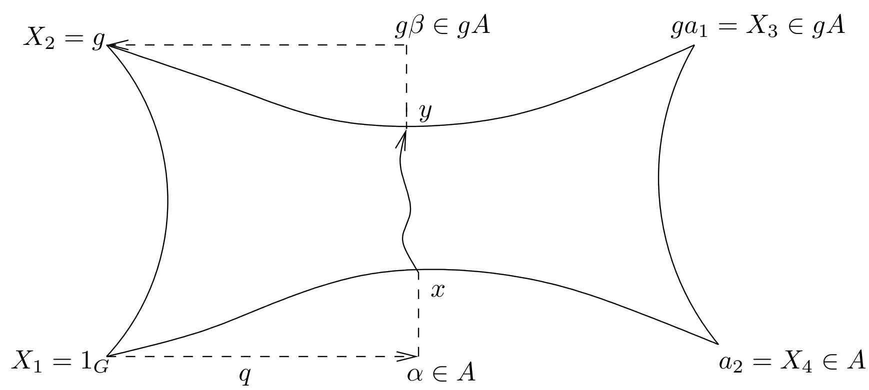

Figure 4: Finding a short double coset representative.

By the assumptions of the lemma there is an element $a_1 \in A$ such that $ga_1g^{-1} = a_2 \in A$ and $|a_1|_G > 2|g|_G$. Set $X_1 = 1_G$, $X_2 = g$, $X_3 = ga_1$,$X_4 = a_2$ (Figure 4). Then $d(X_2, X_3) = |a_1|_G$, $d(X_1, X_2) = |g|_G = |a_2^{-1}ga_1|_G = d(X_3, X_4)$ and in the geodesic quadrangle $X_1X_2X_3X_4$ one has $d(X_2, X_3) > d(X_1, X_2) + d(X_3, X_4)$ and, so, by Lemma 9.7 there exist $x \in [X_1, X_4]$, $y \in [X_2, X_3]$ with $d(x, y) \le 2\delta$. As we showed above, $[1_G, a_i] \subset \mathcal{O}_{\delta+\varkappa+\eta}(A)$ for $i = 1, 2$, hence there is $\alpha \in A$ such that $d(\alpha, x) \le \delta + \eta + \varkappa$. A left shift is an isometry of $\Gamma(G, \mathcal{A})$, thus, $[X_2, X_3] = [g, ga_1] \subset \mathcal{O}_{\delta+\varkappa+\eta}(gA)$ and we can obtain an element $\beta \in A$ such that $d(y, g\beta) \le \delta + \eta + \varkappa$.

Consider the broken line $q = [X_1, \alpha, g\beta, g]$ in $\Gamma(G, \mathcal{A})$; then $elem(q) = g$ in G. $d(\alpha, g\beta) \le 4\delta + 2\eta + 2\varkappa$ by construction, hence we have achieved $g = elem(q) = \alpha \cdot r \cdot \beta^{-1}$ where $r = elem([\alpha, g\beta])$, $|r|_G \le d(\alpha, g\beta) \le 4\delta + 2\eta + 2\varkappa$. □

Lemma 9.9. ([2, Lemma 10],[9, Lemma 1.3]) *For any integer $m \ge 1$ and non-negative numbers δ, η, C, there exists $A = A(m, \delta, \eta, C) \ge 0$ with the following property.*

Let G be a δ-hyperbolic group with a generating set containing at most m elements and H a η-quasiconvex subgroup of G. Let $g_1, \ldots, g_n, s$ be elements of G such that

(i) *cosets Hg_i and Hg_j are different for $i \ne j$;*

(ii) *g_n is a shortest representative of the coset Hg_n;*

(iii) *$|g_i|_G \le |g_n|_G$ for $1 \le i < n$;*

(iv) *for $i \ne n$, all the products $g_ig_n^{-1}$ belong to the same double coset HsH with $|s|_G \le C$.*

Then $n \le A = A(m, \delta, \eta, C)$.

An useful property of double cosets modulo quasiconvex subgroups states:

Lemma 9.10. *Let G be a δ-hyperbolic group, H and K – its subgroups where H is quasiconvex. If* $K \subset \bigcup_{j=1}^{N} H s_j H$ *for some* $s_1, \ldots, s_N \in G$ *then* $K \preceq H$*, i.e.,* $|K : (K \cap H)| < \infty$.

Proof. By contradiction, assume $K = \bigsqcup_{i=1}^{\infty} (K \cap H) x_i$ – disjoint union of right cosets with $x_i \in K$ for all $i \in \mathbb{N}$. For every i choose a shortest representative g_i of the coset Hx_i in G. Then for arbitrary $i \neq k$, $Hg_i = Hx_i \neq Hx_k = Hg_k$ and $x_i x_k^{-1} \in H s_j H$ for some $j \in \{1, 2, \ldots, N\}$, $j = j(i,k)$, hence $g_i g_k^{-1} \in H s_j H$.

Let A_j be the constants corresponding to $H s_j H$, $j = 1, \ldots, N$, from Lemma 9.9. Pick a natural number $n > \sum_{j=1}^{N} A_j$ and consider $g_1, g_2, \ldots, g_n$. Without loss of generality, assume $|g_n|_G \geq |g_i|_G$ for $1 \leq i < n$.

By the choice of n, there exits $l \in \{1, \ldots, N\}$ such that

$$card \left\{ i \in \{1, 2, \ldots, n-1\} \mid g_i g_n^{-1} \in H s_l H \right\} \geq A_l.$$

This leads to a contradiction with the claim of Lemma 9.9. Q.e.d. $\square$

An interesting consequence of the latter fact is the following

Corollary 9.11. *Suppose H is a quasiconvex subgroup of infinite index in a hyperbolic group G. Then H contains no infinite normal subgroups of G.*

Proof. Indeed, assume $N \trianglelefteq G$ and $N \subset H$. By Lemma 9.10 applied to the case $K = G$, there is a double coset HrH, $r \in G$, with the length of a shortest representative greater than $(4\delta + 2\eta)$ (η is the quasiconvexity constant of H). Thus, according to the Lemma 9.8, $N \subset H \cap rHr^{-1}$ is finite. $\square$

Another application of Lemma 9.10 allows to prove

Lemma 9.12. *Suppose G is a hyperbolic group, H is its non-elementary subgroup and $\alpha_1, \ldots, \alpha_n$ are points on the boundary ∂G. Then there exists a subgroup M of H possessing the following properties:*

- *M is free of rank 2;*
- *M is quasiconvex in G;*
- *$\alpha_i \notin \Lambda(M) \subset \partial G$ for every $i = 1, 2, \ldots, n$.*

Proof. Induction on n. Let $n = 1$. One can apply Lemma 3.6 to find an infinite index subgroup $M_1 \leq H$ satisfying the first two properties. If $\alpha_1 \notin \Lambda(M_1)$, there is nothing to prove. So, assume $\alpha_1 \in \Lambda(M_1)$. By Lemmas 9.8 and 9.10 there exists $h \in H$ such that $card(M_1 \cap hM_1h^{-1}) < \infty$. The subgroup $hM_1h^{-1} \leq H$ is non-elementary and quasiconvex in G, hence using Lemmas 9.1 and 5.3.(a) we obtain

$$\Lambda(M_1) \cap \Lambda(hM_1h^{-1}) = \emptyset \text{ in } \partial G.$$

Consequently, $\alpha_1 \notin \Lambda(hM_1h^{-1})$.

Assume, now, that $n > 1$. And the induction hypothesis is verified for $\alpha_1, \dots, \alpha_{n-1} \in \partial G$. I.e., there is a non-elementary subgroup $M' \leq H$ with $\alpha_i \notin \Lambda(M')$, $1 \leq i \leq n-1$. Using the base of our induction, we obtain a free of rank 2 subgroup $M \leq M' \leq H$ that is quasiconvex in G and $\alpha_n \notin \Lambda(M)$. Since $\Lambda(M) \subseteq \Lambda(M')$, M satisfies all the properties required. □

Modification of Suitable Elements

Assume G is a δ-hyperbolic group for some $\delta \geq 0$. As usual, $\mathcal{A}$ will denote some finite symmetrized generating set of G.

Lemma 9.13. *Suppose $g \in G$ is an element of infinite order and $y \in G \backslash E(g)$. Then there is a constant $C_0 \geq 14\delta$ such that $(g^{-n}y^{-1}|yg^n)_{1_G} \leq C_0$ for every $n \in \mathbb{N}$.*

Proof. Observe that $E(g) \neq E(ygy^{-1})$ because, otherwise, we would have $yg^ky^{-1} = g^l$ for some non-zero integers k, l, and (2.2) would imply $y \in E(g)$ which is not true by the assumptions of the lemma. Therefore $E(g) \cap E(ygy^{-1})$ is finite, hence by Lemma 9.2 there is $C_0' \geq 0$ such that $(g^l|yg^ky^{-1})_{1_G} \leq C_0'$ for any $k, l \in \mathbb{Z}$. Set $C_0 = C_0' + 2|y|_G + 14\delta$, then $C_0 \geq 14\delta$ and

$$\begin{aligned}
(g^{-n}y^{-1}|yg^n)_{1_G} &= \frac{1}{2}\big(|g^{-n}y^{-1}|_G + |yg^n|_G - |yg^nyg^n|_G\big) \\
&\leq \frac{1}{2}\big(|g^{-n}|_G + |y|_G + |yg^ny^{-1}|_G + |y|_G - |g^nyg^ny^{-1}|_G + 2|y|_G\big) \\
&\leq (g^{-n}|yg^ny^{-1})_{1_G} + 2|y|_G \leq C_0 \quad \forall\, n \in \mathbb{N}.
\end{aligned}$$

□

Lemma 9.14. *Let $g \in G$ be an element of infinite order and $y \in G \backslash E(g)$. Then there is $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$ the element $yg^n \in G$ has infinite order.*

Proof. In the notation of Lemma 9.13, set $C_1 = 12(C_0 + \delta) + 1$. Evidently, there exists $N_1 \in \mathbb{N}$ such that $|yg^n|_G \geq C_1$ for all $n \geq N_1$.

Suppose $(yg^n)^t = 1_G$ for some $t \in \mathbb{N}$ and $n \geq N_1$.
Consider the broken line $[X_0, X_1, \ldots, X_t]$ in $\Gamma(G, \mathcal{A})$ with $X_i = (yg^n)^i$, $i = 0, 1 \ldots, t$. By Lemma 9.13 there is an estimate

$$\big(X_{i-1}|X_{i+1}\big)_{X_i} = \big((yg^n)^{i-1}|(yg^n)^{i+1}\big)_{(yg^n)^i} = \big(g^{-n}y^{-1}|yg^n\big)_{1_G} \leq C_0.$$

Therefore $[X_0, X_1, \ldots, X_t]$ satisfies all the assumptions of Lemma 9.4, thus

$$\|[X_0, X_t]\| \geq \frac{1}{2}\|[X_0, X_1, \ldots, X_t]\| \geq C_1/2 > 0.$$

But we had assumed $X_0 = X_t$. A contradiction. Hence the element yg^n has infinite order for each $n \geq N_1$. $\square$

Lemma 9.15. *Let $g_i \in G$ have infinite order and $y_i \in G \backslash E(g_i)$, $i = 1, 2$. There exists $N_2 \in \mathbb{N}$ satisfying the following property. Assume $x(y_1 g_1^n)^k x^{-1} = (y_2 g_2^n)^m$ for some $x \in G$, $n \geq N_2$ and $k, m \in \mathbb{Z}\backslash\{0\}$. Then one can find $z \in G$ and integers $\sigma, \sigma' \neq 0$, ζ, ξ, ζ', ξ' such that $z g_1^{\sigma'} z^{-1} = g_2^{\sigma}$ and $x = (y_2 g_2^n)^{\zeta} g_2^{\xi} z g_1^{\xi'} (y_1 g_1^n)^{\zeta'}$.*

Proof. According to Lemma 9.13, there is $C_0 \geq 14\delta$ such that

$$(g_i^{-n} y_i^{-1} | y_i g_i^n)_{1_G} \leq C_0 \quad \forall\, n \in \mathbb{N},\ i = 1, 2. \tag{9.2}$$

Let w_1, w_2, w_3, w_4 be shortest words in the alphabet $\mathcal{A}$ representing y_1, g_1, y_2 and g_2 correspondingly. By Lemma 2.5 there exist $\bar\lambda > 0$ and $\bar c' \geq 0$ such that any path in $\Gamma(G, \mathcal{A})$ labelled by the word w_i^n is $(\bar\lambda, \bar c')$-quasigeodesic for any $n \in \mathbb{N}$, $i = 2, 4$. Consequently, any path labelled by $w_{i-1} w_i^n$, $i = 2, 4$, is $(\bar\lambda, \bar c)$-quasigeodesic where $\bar c = \bar c' + \max\{2\|w_1\|, 2\|w_3\|\}$. Set $C_1 = 12(C_0 + \delta) + \bar c + 1$. Suppose $n \geq (C_1 + \bar c)/\bar\lambda$. Then $\|w_1 w_2^n\| \geq C_1$ and, by (9.2), we can apply Lemma 9.6 to find $\lambda > 0$ and $c \geq 0$ (not depending on n) such that any path labelled by $(w_1 w_2^n)^t$ is (λ, c)-quasigeodesic for any $t \in \mathbb{Z}$. Similarly, for any $t' \in \mathbb{Z}$, any path in $\Gamma(G, \mathcal{A})$ labelled by $(w_3 w_4^n)^{t'}$ is (λ, c)-quasigeodesic.

According to our assumptions, $x(y_1 g_1^n)^{lk} x^{-1} = (y_2 g_2^n)^{lm}$ for every $l \in \mathbb{N}$. Thus, we can consider a geodesic quadrangle $Y_1 Y_2 Y_3 Y_4$ in $\Gamma(G, \mathcal{A})$ with $Y_1 = 1_G$, $Y_2 = x$, $Y_3 = x(y_1 g_1^n)^{lk}$, $Y_4 = x(y_1 g_1^n)^{lk} x^{-1}$, and (λ, c)-quasigeodesic paths p between Y_2 and Y_3 and q between Y_1 and Y_4 labelled by the words $(w_1 w_2^n)^{lk}$ and $(w_3 w_4^n)^{lm}$ respectively. Choose $\nu = \nu(\delta, \lambda, c)$ to be the constant given by Lemma 2.2. Thus,

$$p \subset \mathcal{O}_\nu([Y_2, Y_3]),\ [Y_2, Y_3] \subset \mathcal{O}_\nu(p),\ q \subset \mathcal{O}_\nu([Y_1, Y_4]),\ [Y_1, Y_4] \subset \mathcal{O}_\nu(q).$$

Obviously, by taking the number l sufficiently large, one can find a subpath r of p labelled by w_2^n with its endpoints r_- and r_+ having distances at least $(|x|_G + \nu)$ from both of the vertices Y_2 and Y_3. Then an application of Lemma 9.7 will give us

$$r_-, r_+ \in \mathcal{O}_{\nu+2\delta}([Y_1, Y_4]) \subset \mathcal{O}_{2\nu+2\delta}(q).$$

Let u, v denote the points on the path q with $d(r_-, u) \le 2\nu + 2\delta$ and $d(r_+, v) \le 2\nu + 2\delta$, and let r' be the subpath of q (or q^{-1}) starting at u, ending at v. The lengths of r and the of r' are related as follows:

$$\|r'\| \ge d(r'_-, r'_+) \ge d(r_-, r_+) - 2(2\nu + 2\delta) \ge \bar{\lambda}\|r\| - \bar{c}' - 4(\nu + \delta) \ge \bar{\lambda} n - \bar{c}' - 4(\nu + \delta).$$

If n is sufficiently large, then $\bar{\lambda} n/3 > \bar{c}' + 4(\nu + \delta) + \|w_3\|$. Therefore r' will have a subpath q' labelled by w_4^t, $t \in \mathbb{Z}$, and (in the worst case, when the segment labelled by w_3 is in the middle of r') $|t| \ge \bar{\lambda} n/(3\|w_4\|)$ (Figure 5).

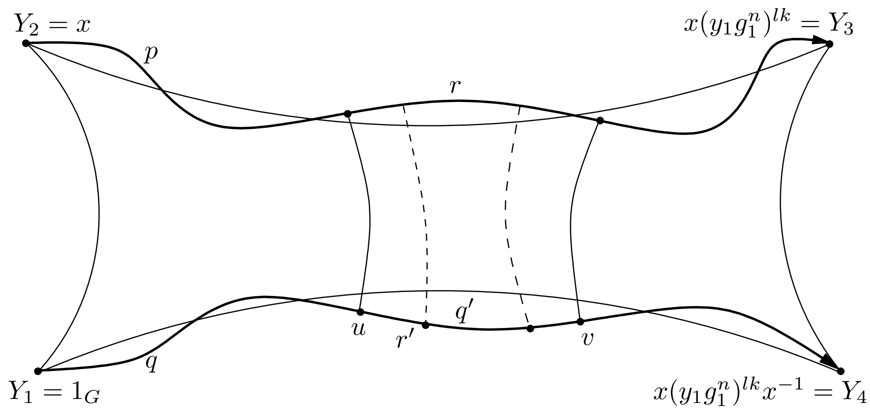

Figure 5: Illustration of Lemma 9.15

Since the quadrangles in $\Gamma(G, \mathcal{A})$ are 2δ-slim, we achieve

$$[u, v] \subset \mathcal{O}_{2\delta}([r_-, r_+] \cup [r_-, u] \cup [r_+, v]) \subset \mathcal{O}_{2\nu+4\delta}([r_-, r_+]), \quad \text{hence,}$$

$$q' \subset \mathcal{O}_{\nu}([u, v]) \subset \mathcal{O}_{3\nu+4\delta}([r_-, r_+]) \subset \mathcal{O}_{4\nu+4\delta}(r).$$

Consider the vertices $a_0 = q'_-, a_2, \ldots, a_{|t|} = q'_+$ of the path q' such that the subpaths between a_{i-1} and a_i are labelled by w_4 (respectively, w_4^{-1} if $t < 0$) for every $1 \le i \le |t|$ (we will call them *phase* vertices). Then each of them is at distance at most $(4\nu + 4\delta + \|w_2\|/2)$ from some phase vertex of r. There are only finitely many words over the alphabet $\mathcal{A}$ of length at most $(4\nu + 4\delta + \|w_2\|/2)$, therefore if n is sufficiently large (and, consequently, t is large too) there will be two paths α and β connecting two different phase vertices of q' with some vertices of r having the same word w written on them. Thus we achieve the equality in the group G:

$$w_4^\sigma = w w_2^{\sigma'} w^{-1} \text{ for some } \sigma, \sigma' \in \mathbb{Z}\backslash\{0\}.$$

So, if z denotes the element of G represented by the word w, we have $z g_1^{\sigma'} z^{-1} = g_2^\sigma$. According to the construction, $x = (y_2 g_2^n)^\zeta g_2^\xi z g_1^{\xi'} (y_1 g_1^n)^{\zeta'}$ for some $\zeta, \xi, \zeta', \xi' \in \mathbb{Z}$. Q.e.d. $\square$

Lemma 9.16. *Let H be a non-elementary subgroup of a hyperbolic group G, and g be an H-suitable element. If $y \in C_H\big(E(H)\big)\backslash E(g)$ then there exists $N \in \mathbb{N}$ such that the element yg^n has infinite order in H and is H-suitable for every $n \geq N$.*

Proof. By Lemma 9.14, $yg^n \in H^0$ for all $n \geq N_1$ for some $N_1 \in \mathbb{N}$. Suppose $x \in E(yg^n)$, $n \geq N_1$. Then there are integers is $k, m \in \mathbb{Z}\backslash\{0\}$ such that $x(yg^n)^k x^{-1} = (yg^n)^m$. Now one is able to apply Lemma 9.15 to the situation when $g_1 = g_2 = g$, $y_1 = y_2 = y$, to find $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $zg^{\sigma'}z^{-1} = g^\sigma$ and $x = (yg^n)^\zeta g^\xi z g^{\xi'} (yg^n)^{\zeta'}$ for some $\sigma, \sigma' \in \mathbb{Z}\backslash\{0\}$ and $\zeta, \xi, \zeta', \xi' \in \mathbb{Z}$.

According to (2.2), $z \in E(g)$. Note that $(yg^n)^{-\zeta} x (yg^n)^{-\zeta'} \in E(yg^n)$ and $g^\xi z g^{\xi'} \in E(g)$. Observe that $E(yg^n) \neq E(g)$ because, otherwise, we would obtain $yg^n \in E(g)$ which implies $y \in E(g)$, contradicting to the assumptions of the lemma. Hence

$$E(g) \cap E(yg^n) \subset T(g) = E(H) \subset E(yg^n), \quad \text{thus} \quad E(g) \cap E(yg^n) = E(H).$$

Consequently, $g^\xi z g^{\xi'} = (yg^n)^{-\zeta} x (yg^n)^{-\zeta'} \in E(yg^n) \cap E(g) = E(H)$.

Finally, since $yg^n \in C_H\big(E(H)\big)$, we obtain $x \in \langle yg^n \rangle \cdot E(H) = \langle yg^n \rangle \times E(H)$ for arbitrary x from $E(yg^n)$. This implies that yg^n is H-suitable. $\square$

Lemma 9.17. *Let G be a hyperbolic group, $s \in \mathbb{N}$ and let $g_1, g_2, \dots, g_s$ be pairwise non-commensurable elements of G. Consider $y_i \in G\backslash E(g_i)$ for each $i = 1, 2, \dots, s$. Then there exists $N \in \mathbb{N}$ such that the elements $y_1 g_1^n, \dots, y_s g_s^n$ have infinite order and are pairwise non-commensurable if $n \geq N$.*

Proof. The elements $y_1 g_1^n, \dots, y_s g_s^n$ have infinite order in the group G for any sufficiently large n by Lemma 9.14.

Suppose that $y_i g_i^n$ is commensurable with $y_j g_j^n$, for some i, j, $1 \leq i < j \leq s$. Then there is $x \in G$ satisfying $x(y_i g_i^n)^k x^{-1} = (y_j g_j^n)^m$ for some $k, m \in \mathbb{Z}\backslash\{0\}$. In this case Lemma 9.15 claims that if n is sufficiently large, the elements g_i and g_j are commensurable. The latter yields a contradiction to our assumptions. $\square$

CHAPTER X

SMALL CANCELLATIONS OVER HYPERBOLIC GROUPS

In this chapter we discuss the results and tools developed by A. Ol'shanskii in [26] that are essential for our approach.

Generalized Small Cancellation Conditions

Suppose $\mathcal{A}$ is a finite symmetrized alphabet. If U and V are two words over $\mathcal{A}$, we will write $U \equiv V$ if there is a *graphical* (i.e., letter-by-letter) equality between them.

Let us recall the definitions from [26, Ch. 4]:

Definition 10.1. A set of words $\mathcal{R}$ over $\mathcal{A}$ is called *symmetrized* if it satisfies the following conditions:

1) for every $R \in \mathcal{R}$, $R^{-1} \in \mathcal{R}$;

2) $R \in \mathcal{R}$, $R \equiv R_1R_2$ implies $R' \equiv R_2R_1 \in \mathcal{R}$.

Assume the group G is generated by $\mathcal{A}$, $\mathcal{R}$ is a finite symmetrized set of words over $\mathcal{A}$ and ε is a non-negative number.

Definition 10.2. A subword U of is said to be an ε*-piece* of a word $R \in \mathcal{R}$ with respect to G if there exists a word $R' \in \mathcal{R}$ such that

(a) $R \equiv UV$, $R' \equiv U'V'$ for some V, U', V';

(b) $U' = YUZ$ in G for some words Y, Z where $\|Y\|, \|Z\| \le \varepsilon$;

(c) $YRY^{-1} \neq R'$ in the group G.

Definition 10.3. A subword U of is said to be an ε'*-piece* of a word $R \in \mathcal{R}$ with respect to G if

(a′) $R \equiv UVU'V'$ for some words V, U', V' over $\mathcal{A}$;

(b′) $U' = YU^{\pm 1}Z$ in the group G for some words Y, Z where $\|Y\|, \|Z\| \le \varepsilon$.

A word W over $\mathcal{A}$ is called (λ, c)*-quasigeodesic* if some (or, equivalently, any) path p in $\Gamma(G, \mathcal{A})$ labelled by W is (λ, c)-quasigeodesic.

Definition 10.4. We will say that the system $\mathcal{R}$ satisfies $C(\varepsilon, \mu, \lambda, c, \rho)$-condition for some $\mu, \lambda, \rho > 0$, $c \geq 0$, if

(1) $\|R\| \geq \rho$, for each $R \in \mathcal{R}$;

(2) any word $R \in \mathcal{R}$ is (λ, c)-quasigeodesic;

(3) for any ε-piece of any word $R \in \mathcal{R}$ the inequalities $\|U\|, \|U'\| < \mu\|R\|$ hold (in the notations of Definition 10.2).

For planar non-simply connected diagrams a stronger condition will be used:

Definition 10.5. $\mathcal{R}$ meets the condition $C_1(\varepsilon, \mu, \lambda, c, \rho)$ for some $\mu, \lambda, \rho > 0$, $c \geq 0$, if $\mathcal{R}$ satisfies $C(\varepsilon, \mu, \lambda, c, \rho)$, i.e., conditions (1)-(3) of Definition 10.4, together with

(4) for any ε'-piece U of any word $R \in \mathcal{R}$ (see Definition 10.3), $\|U\|, \|U'\| < \mu\|R\|$.

Consider non-elementary subgroups $H_1, \dots, H_k$ (not necessarily distinct) of a hyperbolic group G, elements $g_i \in H_i$ chosen according to the claim of Lemma 8.3 and arbitrary elements $x_{i0} \in C_G(E(H_i)) \backslash E(g_i)$, $i = 1, \dots, k$. Let g_i, $x_{i0}, \dots, x_{il}$ be represented by words W_i, $X_{i0}, \dots, X_{il}$ over the alphabet $\mathcal{A}$ of minimal length, $i = 1, \dots, k$.

As the set of words $\mathcal{R} = \mathcal{R}_{k,l,m}(W_1, \dots, W_k, X_{10}, \dots, X_{kl}, m)$ consider all cyclic permutations of $R_i^{\pm 1}$ where

$$R_i \equiv X_{i0} W_i^m X_{i1} W_i^m \dots X_{il} W_i^m, \quad i = 1, 2, \dots, k.$$

Then we have

Lemma 10.1. ([26, Lemma 4.2]) *For the words $W_1, \dots, W_k$, $X_{10}, \dots, X_{k0}$ given above, there exist $\lambda > 0$ such that for any $\mu > 0$ there are $l \in \mathbb{N}$ and $c \geq 0$ such that for any $\varepsilon \geq 0$, $\rho > 0$ there are $m_0 \in \mathbb{N}$ and words $X_{11}, \dots, X_{kl}$ such that the system $\mathcal{R}_{k,l,m}$ satisfies $C(\varepsilon, \mu, \lambda, c, \rho)$ and $C_1(\varepsilon, \mu, \lambda, c, \rho)$-conditions if $m \geq m_0$.*

Remark 10.2. From the proof of this lemma it follows that the elements $x_{i1}, \dots, x_{il} \in H_i$ can be chosen right after the choice of $l \in \mathbb{N}$ to be any elements that satisfy properties 1)-4) of Lemma 8.4 for $g = g_i$ and $H = H_i$, $i = 1, 2, \dots, k$.

Diagrams over Group Presentations

Below we assume that the concepts of a *van Kampen* (*circular*) *diagram* and a *Schupp* (*annular*) *diagram* over a group presentation are known to the reader (see, for instance, [16]).

As usual, we assume that G is a group generated by a finite symmetrized set $\mathcal{A}$.

Let $\mathcal{O}$ denote the system of all relations (not only defining) in the group G. Let $\mathcal{R}$ be some symmetrized set of additional relations over the alphabet $\mathcal{A}$. The group G_1 will be defined by its presentation:

$$G_1 = \langle \mathcal{A} \parallel \mathcal{O} \cup \mathcal{R} \rangle. \tag{10.1}$$

Thus, G_1 is a quotient of the group G modulo the subgroup $\mathcal{N} = \langle \mathcal{R}^G \rangle$ which is a normal closure of the set of elements in G represented by words from $\mathcal{R}$.

Van Kampen proved that if a word W that is trivial in the group G_1 then there exists a circular diagram over the presentation (10.1) whose boundary label is letter-by-letter equal to W (see [16, Ch. 5, Thm. 1.1]). Later, Schupp noted that if two words U and V are conjugate in G_1 then there is an annular diagram over the presentation (10.1) whose boundary contours have labels (letter-by-letter) equal to U and V respectively ([16, Ch. 5, Lemma 5.2]).

Suppose Δ is a diagram over the presentation (10.1). Inheriting the terminology from [26], the faces of Δ with boundary labels from $\mathcal{O}$ (from $\mathcal{R}$) will be called 0-faces ($\mathcal{R}$-faces). An edge e from Δ can be labelled by a letter from $\mathcal{A}^{\pm 1}$ or by the letter 1 corresponding to the identity element of G. In the latter case the edge e is said to be a 0-edge.

Now we recall the elementary transformations of diagrams defined in [26, Ch. 5]:

I. Let $o = e_+ = f_-$ for some edges $e \neq f$ of a diagram Δ, where o is a vertex of degree 2. Suppose e (or f) is a 0-edge. then o can be deleted from the set of vertices by declaring ef to be a single edge u with $lab(u) \equiv lab(f)$ (or $lab(u) \equiv lab(e)$).

II. Assume a vertex o has degree 1 in a diagram Δ. Then one can delete o from Δ together with the edge adjacent to it.

III. If two different 0-faces Π_1 and Π_2 have a common edge e on their boundaries, then one may replace them with a single (appropriate) 0-face Π.

IV. Denote by $\mathrm{I}', \mathrm{II}', \mathrm{III}'$ the converse transformations to I, II and III, where the transformation III$'$ is only permitted when the faces Π_1 and Π_2 arising from Π correspond to some relators of G.

Suppose Π_1 and Π_2 are two different $\mathcal{R}$-faces of a diagram Δ with boundary labels R_1 and R_2 reading clockwise, starting from vertices $o_i \in \partial\Pi_i$, $i = 1, 2$. The face Π_1, Π_2 are said to be *opposite* if after a series of elementary transformations one can find a simple path s in Δ such that $s_- = o_1$, $s_+ = o_2$ and $lab(s)^{-1} R_1 lab(s) R_2 = 1_G$ in G. Clearly, after removing Π_1 and Π_2 from Δ and cutting it along s one obtains a hole whose boundary label is equal to 1_G in G. By van Kampen's lemma this hole can be tessellated with 0-faces. As a result, on gets a diagram Δ' with fewer number of $\mathcal{R}$-faces than Δ. Performing the above operation finitely many times, we achieve a diagram without opposite $\mathcal{R}$-faces.

A circular diagram is said to be *reduced* if it has no opposite $\mathcal{R}$-faces.

Definition 10.6. Consider a simple closed path $o = p_1q_1p_2q_2$ in a diagram Δ over G_1, such that q_1 is a subpath of the boundary cycle of an $\mathcal{R}$-faceΠ and q_2 is a subpath of a section q of $\partial\Delta$. Let Γ denote the subdiagram of Δ bounded by o. Assuming that Γ has no holes, no $\mathcal{R}$-faces and $\|p_1\|, \|p_2\| \leq \varepsilon$, it will be called an *$\varepsilon$-contiguity subdiagram* of Π to q. The ratio $\|q_1\|/\|\partial\Pi\|$ will be called the *contiguity degree* of Π to q and denoted (Π, Γ, q).

Now, suppose Δ is an annular diagram over G_1. Following [26], an $\mathcal{R}$-face Π of Δ will be called an *(ε, μ)-extra face* if there is a contiguity subdiagram Γ of Π to itself with contour $zuy^{-1}u'$ (where u, u' are the contiguity arcs of Γ and $uvu'v'$ is the contour of Π), such that $\|u\| \geq \mu\|\partial\Pi\|$ or $\|u'\| \geq \mu\|\partial\Pi\|$ and $elem(yv)$ is conjugate in G to $elem(yu^{-1}v'^{-1}u'^{-1})$. After removing such an extra face from Δ and making a cut along the path y, one can insert an annular diagram (corresponding to the pair $elem(yv)$, $elem(yu^{-1}v'^{-1}u'^{-1})$ of conjugated elements) over G in Δ, reducing the number of $\mathcal{R}$-faces in it. Thus, further we can consider only *reduced* annular diagrams which have no opposite faces and no (ε, μ)-extra faces.

Diagrams with Small Cancellations

Later in this work we will consider diagrams over G and G_1 (with the presentations $G = \langle \mathcal{A} \parallel \mathcal{O} \rangle$ and (10.1)), and in both cases we will use the same alphabet $\mathcal{A}$.

The boundary $\partial\Delta$ of a diagram Δ will be divided into at most 4 distinguished subpaths (called *sections*) each of which will be $(\lambda, c)-$quasigeodesic (for some given $\lambda > 0$, $c \geq 0$).

Suppose $\varepsilon \geq 0$ is a given number. The following analog of Grindlinger's lemma is proved in [26, Lemma 6.6] (here we include a correction mentioned in [28]):

Lemma 10.3. *For any hyperbolic group G and any $\lambda > 0$ there is $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ and $c \geq 0$ there are $\varepsilon \geq 0$ and $\rho > 0$ with the following property:*

Let the symmetrized presentation (10.1) *satisfy the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition. Furthermore, let Δ be a reduced circular diagram over G_1 whose boundary is decomposed into a product of (λ, c)-quasigeodesic sections $q^1, \ldots, q^r$ where $1 \leq r \leq 4$. Then, provided Δ has an $\mathcal{R}$-face, there exists an $\mathcal{R}$-face Π in Δ and disjoint ε-contiguity subdiagrams $\Gamma_1, \ldots, \Gamma_r$ (some of them may be absent) of Π to $q^1, \ldots, q^r$ respectively, such that*

$$(\Pi, \Gamma_1, q^1) + \cdots + (\Pi, \Gamma_r, q^r) > 1 - 23\mu \ . \tag{10.2}$$

The next lemma is an analog of the previous one for annular diagrams.

Lemma 10.4. [26, Lemma 8.1] *For any hyperbolic group G and any $\lambda > 0$ there is $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ and $c \geq 0$ there are $\varepsilon \geq 0$ and $\rho > 0$ with the following property:*

Let the symmetrized presentation (10.1) *satisfy the $C_1(\varepsilon, \mu, \lambda, c, \rho)$-condition. Further, let Δ be a reduced annular diagram over G_1 with boundary contours $p = p_1p_2$, $q = q_1q_2$ such that p_1, p_2, q_1, q_2 are (λ, c)-quasigeodesic. Then, provided Δ has an $\mathcal{R}$-face, there exists an $\mathcal{R}$-face Π in Δ and disjoint ε-contiguity subdiagrams $\Gamma_1, \ldots, \Gamma_4$ (some of them may be absent) of Π to p_1, p_2, q_1, q_2 respectively, such that*

$$(\Pi, \Gamma_1, p_1) + (\Pi, \Gamma_2, p_2) + (\Pi, \Gamma_3, q_1) + (\Pi, \Gamma_4, q_2) > 1 - 23\mu \ . \qquad (10.3)$$

Collecting together the claims of Lemmas 6.7,7.4 and 7.5 from [26] we obtain

Lemma 10.5. *Suppose G is a non-elementary hyperbolic group and $H'_1, \ldots, H'_{k'}$ – its non-elementary subgroups. Then for any $\lambda > 0$ there is $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ and $c \geq 0$ there is $\varepsilon \geq 0$ such that for any $N > 0$ there exists $\rho > 0$ with the following property:*

Let the symmetrized presentation (10.1) *satisfy the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition. Then the quotient G_1* (10.1) *is a non-elementary hyperbolic group and the images of the subgroups $H'_1, \ldots, H'_{k'}$ are non-elementary in G_1. Moreover, $W = 1$ in G_1 if and only if $W = 1$ in G for every word W with $\|W\| \leq N$.*

It is also possible to describe all periodic elements in the group G_1:

Lemma 10.6. [26, Lemma 7.2] *Let G be a hyperbolic group and $\lambda \geq 0$. Then there is $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ and $c \geq 0$ there are $\varepsilon \geq 0$ and $\rho > 0$ with the following property:*

Suppose the symmetrized presentation (10.1) *satisfies the $C_1(\varepsilon, \mu, \lambda, c, \rho)$-condition or the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition. Then each $R \in \mathcal{R}$ represents an element of infinite order in G. In addition, a word X has a finite order in the group G_1 if and only if X is conjugate in G_1 to a word having finite order in G, or to a word belonging to the centralizer $C_G(R) \leq G$ for some $R \in \mathcal{R}$.*

CHAPTER XI

RESIDUALIZING HOMOMORPHISMS PRESERVING QUASICONVEXITY

As we know, any generating set induces a left-invariant metric on the set of elements of a group. So, if G_1 is a quotient of G, the group G_1 will be generated by the image of $\mathcal{A}$ under the natural homomorphism $\phi : G \to G_1$. Therefore, later G_1 will be assigned the metric corresponding to the generating set $\phi(\mathcal{A})$.

The main goal of this chapter is to prove

Theorem 11.1. *Let $H_1, H_2, \dots, H_k$ be G-subgroups of a non-elementary hyperbolic group G and $H'_1, \dots, H'_{k'}$ be some non-elementary subgroups of G. Assume $Q \subseteq G$ is an η-quasiconvex subset (for some $\eta \geq 0$) that is small relatively to H_i for every $i = 1, 2 \dots, k$. Then there exist a group G_1 and an epimorphism $\phi : G \to G_1$ such that*

1) *G_1 is a non-elementary hyperbolic group;*

2) *The homomorphism ϕ is an isometry between Q and $\phi(Q)$ (if the metrics on G and G_1 are chosen as explained above) and for any quasiconvex subset $S \subseteq Q$, its image $\phi(S)$ is quasiconvex in G_1. In particular, ϕ is injective on Q;*

3) *ϕ is surjective on each of the subgroups $H_1, \dots, H_k$, i.e., $\phi(H_i) = G_1$ for each $i = 1, 2, \dots, k$;*

4) *ϕ-images of two elements from Q are conjugate in G_1 if and only if these elements are conjugate in G;*

5) *The centralizer $C_{G_1}\big(\phi(a)\big)$ for every $a \in Q$ is the ϕ-image of the centralizer $C_G(a)$;*

6) *$ker\phi$ is a torsion-free subgroup;*

7) *ϕ induces a bijective map on sets of conjugacy classes of elements having finite orders in G and G_1 respectively;*

8) *$\phi(H'_1), \dots, \phi(H'_{k'})$ are non-elementary subgroups of G_1;*

9) *$E(G_1) = \phi\big(E(G)\big)$.*

If $card(Q) < \infty$, Q will be small relatively to any infinite subgroup of G, thus, parts 1)-8) of Theorem 11.1 generalize Ol'shanskii's Theorem 8.2. We added part 9) to the above statement because it is useful for applications.

Before proving Theorem 11.1, we would like to demonstrate that the assumption of Q being small relatively to H_i is indeed required.

Necessity of Assumptions

Recall that a subgroup $H \le G$ is called a *retract* if there is a homomorphism $\phi : G \to H$ such that $\phi|_H = id_H$. In other words, $G = HN$, where $N = ker(\phi)$, and $H \cap N = \{1_G\}$.

In the case of word hyperbolic groups, it makes sense to use the more general notion below:

Definition 11.1. A subgroup H of a group G will be called a *quasiretract* of G if there exists a normal subgroup $N \lhd G$ such that $|G : HN| < \infty$ and the intersection $H \cap N$ is finite.

In particular, any retract is a quasiretract. Now we would like to observe

Lemma 11.2. *Assume that G is a hyperbolic group and H is a quasiretract of G. Then the subgroup H is quasiconvex in G.*

Proof. Let H be a quasiretract of the hyperbolic group G and suppose the normal subgroup $N \lhd G$ satisfies $|G : HN| < \infty$, $card(H \cap N) < \infty$. Denote $\hat{G} = HN \le G$ ($\hat{G}$ is hyperbolic – see Example 2.9). Then for the quotient group $K = \hat{G}/N$ there is a natural epimorphism $\phi : \hat{G} \to K$ such that $K = \phi(H)$ and $M \stackrel{def}{=} ker(\phi) \cap H$ is finite.

The group $\hat{G}$ is generated by a finite set $\hat{\mathcal{A}}$ (because it is of finite index in a finitely generated group G). Hence K is generated by the finite set $\mathcal{C} = \phi(\hat{\mathcal{A}})$. For every element $x \in \mathcal{C}$ choose one element $y \in H$ from its preimage under ϕ and denote by $\bar{\mathcal{C}}$ the subset of H consisting of them. Since

$$H/M \cong K,$$

H is generated by the finite set $\mathcal{B} = \bar{\mathcal{C}} \cup M$.

Now one can define the corresponding length functions $|\cdot|_H$ and $|\cdot|_{\hat{G}}$ which satisfy the following properties:

$$\forall\ g \in \hat{G}, \quad |\phi(g)|_K \le |g|_{\hat{G}},$$

$$\forall\ h \in H, \quad |h|_H \le |\phi(h)|_K + 1.$$

Combining these inequalities we get

$$\forall\ h \in H \quad |h|_H \le |h|_{\hat{G}} + 1.$$

Therefore $D_H(n) \le n + 1$, i.e., H is undistorted in $\hat{G}$. By Lemma 3.2, H is quasiconvex in $\hat{G}$. The group G is hyperbolic and any its subgroup of finite index is quasiconvex, thus, $\hat{G}$ is quasiconvex in G. Finally, according to Remark 3.4, H is quasiconvex in G. $\square$

One can observe that if the group G is torsion-free then every non-elementary subgroup is a G-subgroup. However, by far, not every subgroup in G will be quasiconvex (or a quasiretract). As the next proposition shows, demanding Q to be small relatively to H_i is necessary if one doesn't impose additional limitations on the subgroups H_i, $i = 1, 2, \ldots, k$.

Proposition 11.3. *Let H be an infinite subgroup of a hyperbolic group G and $Q \subset G$ be a subset (not necessarily quasiconvex). Suppose that*

$$H \subseteq P_1 Q^{-1} Q P_2$$

for some finite subsets P_1, P_2 of G and there is a group G_1 and an epimorphism $\phi : G \to G_1$ such that ϕ is surjective on H and $\phi|_Q$ is a quasiisometry between Q and $\phi(Q)$. Then the subgroup H is a quasiretract of G.

Proof. It is enough to show that $M = ker\phi \cap H$ is finite. By the conditions,

$$M \subset P_1 Q^{-1} Q P_2 = \bigcup_{x \in P_1, y \in P_2} x Q^{-1} Q y.$$

Proving by contradiction, assume that M is infinite. Then, since the subsets P_1, P_2 are finite, there are elements $g \in P_1$ and $h \in P_2$ such that the intersection

$$A \stackrel{def}{=} M \cap g Q^{-1} Q h$$

is infinite. Therefore $B = g^{-1} A h^{-1}$ is an infinite subset of $Q^{-1}Q$ satisfying

$$\phi(B) = \{\phi(g^{-1}h^{-1})\} \text{ -- a one-element subset.}$$

It is easy to see that the latter is impossible if ϕ is a quasiisometry between Q and $\phi(Q)$ (since for any $u, v \in Q$, $d(u, v) = |u^{-1}v|_G$, $d_1(\phi(u), \phi(v)) = |\phi(u^{-1}v)|_{G_1}$, $u^{-1}v \in Q^{-1}Q$). $\square$

Main Construction

Assume, now, that we are in the conditions of Theorem 11.1. The δ-hyperbolic group G is generated by a symmetrized set $\mathcal{A} = \{a'_1, \ldots, a'_r\}$. Set $s = kr$ and define $a_1, \ldots, a_s$, $\hat{H}_1, \ldots, \hat{H}_s$ as follows:

$$a_{ir+j} = a'_j, \ \hat{H}_{ir+j} = H_{i+1} \ \text{ if } \ 1 \le j \le r, \ 0 \le i \le k-1, \tag{11.1}$$

i.e., $a_1 = a'_1, \dots, a_r = a'_r, a_{r+1} = a'_1, a_{r+2} = a'_2, \dots,$ $\hat{H}_1 = H_1, \dots,$ $\hat{H}_r = H_1,$ $\hat{H}_{r+1} = H_2,$ $\hat{H}_{r+2} = H_2, \dots$.

Since every $\hat{H}_i$ is a G-subgroup, we can find $b_i \in \hat{H}_i$ such that $a_i b_i^{-1} \in C_G\big(E(\hat{H}_i)\big)$ (such a choice is possible because $E(\hat{H}_i) = E(G)$ and $|\hat{H}_i : K(\hat{H}_i)| = |G : K(G)|$).

For each $i = 1, 2, \dots, s$, the subgroup $F_i = C_{\hat{H}_i}\big(E(\hat{H}_i)\big)$ has finite index in $\hat{H}_i$, hence $\Lambda(\hat{H}_i) = \Lambda(F_i)$ by parts (c) and (d) of Lemma 5.3. The set $Q^{-1}Q$ is quasiconvex by Lemma 7.5, thus, according to the assumptions of Theorem 11.1, we can apply Lemma 7.7 to find the points on the boundary ∂G:

$$\alpha_i \in \Lambda(F_i) \backslash \big(G \circ \Lambda(Q^{-1}Q)\big)$$

and a sequence $\left(y_j^{(i)}\right)_{j \in \mathbb{N}} \subset F_i$ with $\lim\limits_{j \to \infty} y_j^{(i)} = \alpha_i,\ i = 1, 2, \dots, s.$

The set $\{y_j^{(i)} \mid j \in \mathbb{N}\}$ is infinite, therefore the set $\{(y_j^{(i)})^{-1} \mid j \in \mathbb{N}\}$ is also infinite, hence by Lemma 5.3.(a), it has at least one limit point $\beta_i \in \partial G$. So, after passing to a subsequence, we can assume that

$$\lim_{j \to \infty} (y_j^{(i)})^{-1} = \beta_i,\ i = 1, 2, \dots, s.$$

Using Lemma 8.3 one can find an $\hat{H}_i$-suitable element $g_i \in \hat{H}_i^0$ for every $i = 1, \dots, s$, so that the elements $g_1, \dots, g_s$ are pairwise non-commensurable and on the Gromov boundary of the group G we have

$$\{g_i^{\infty}, g_i^{-\infty}\} \cap \{\alpha_i, \beta_i\} = \emptyset,\ i = 1, 2, \dots, s \tag{11.2}$$

(recall that if for two elements of infinite order $g, h \in G$ one has $g^{\infty} = h^{\pm\infty}$ then $g^m = h^l$ for some $m, l \in \mathbb{Z}\backslash\{0\}$ – by Lemma 9.1).

$\hat{H}_i$ is non-elementary, therefore $F_i \le \hat{H}_i$ is also non-elementary, $i = 1, \dots, s$. Now we use Lemma 9.12 to obtain a non-elementary subgroup $K_i \le F_i$ such that

$$\{\alpha_i,\ g_i^{-\infty},\ (b_i a_i^{-1}) \circ g_i^{-\infty}\} \cap \Lambda(K_i) = \emptyset \text{ in } \partial G\ ,\ i = 1, 2, \dots, s. \tag{11.3}$$

According to (11.2) and (11.3) we can use the claim of Lemma 5.4 to show that

$$C_{01i} = \sup\left\{ \left((y_j^{(i)})^{-1} | g_i^n \right)_{1_G} \ : \ j, n \in \mathbb{N} \right\} < \infty,$$

$$C_{02i} = \sup\left\{ \left(y_j^{(i)} | g_i^{-n} \right)_{1_G} \ : \ j, n \in \mathbb{N} \right\} < \infty,$$

$$C_{03i} = \sup\left\{ \left(x | y_j^{(i)} \right)_{1_G} \ : \ j \in \mathbb{N}, x \in K_i \right\} < \infty,$$

$$C_{04i} = \sup\left\{ \left(g_i^{-n}|x\right)_{1_G} \ : \ n \in \mathbb{N}, x \in K_i \right\} < \infty \ \text{ and}$$

$$C_{05i} = \sup\left\{ \left(b_i a_i^{-1} g_i^{-n}|x\right)_{1_G} \ : \ n \in \mathbb{N}, x \in K_i \right\} < \infty$$

for each $i = 1, 2 \ldots, s$. Finally, define

$$C_0 = \max_{1\le i\le s} \left\{ C_{01i}, C_{02i}, C_{03i} + |b_i a_i^{-1}|_G, C_{04i}, C_{05i} + |b_i a_i^{-1}|_G \right\} + 14\delta.$$

Denote $\bar{\lambda} = 1$, $\bar{c} = 0$ and

$$C_1 = 12(C_0 + \delta) + \bar{c} + 1. \tag{11.4}$$

Let

$$\lambda = \bar{\lambda}/4 = 1/4, \ \ c = c(\bar{\lambda}, \bar{c}, C_0) \ge 0 \ \text{ be the constants from Lemma 9.6,} \tag{11.5}$$

$$\text{and let } \nu = \nu(\delta, \lambda, c) \text{ be the constant from the claim of Lemma 2.2.} \tag{11.6}$$

By the assumptions of Theorem 11.1, the subset Q is η-quasiconvex for some $\eta \ge 0$.

$$\text{Let } \varkappa \text{ be the length of a shortest element from } Q. \tag{11.7}$$

$$\text{Set } \ A = \{g \in G \ : \ |g|_G \le 3\delta + \nu + \eta + \varkappa\}, \tag{11.8}$$

then $card(A) < \infty$. By the construction of α_i and Lemma 5.3,

$$\alpha_i \notin \bigcup_{g\in A} g \circ \Lambda(Q^{-1}Q) = \Lambda\left(\bigcup_{g\in A} gQ^{-1}Q\right) = \Lambda(AQ^{-1}Q).$$

Hence according to Lemma 5.4, one can define

$$C_3 = \max_{1\le i\le s} \sup\{ \left(y_j^{(i)}|x\right)_{1_G} : \ j \in \mathbb{N}, x \in AQ^{-1}Q\} < \infty. \tag{11.9}$$

For every $i \in \{1, \ldots, s\}$, $|y_j^{(i)}|_G \to \infty$ as $j \to \infty$, and the intersection $\{y_j^{(i)} \mid j \in \mathbb{N}\} \cap E(g_i)$ is finite by (11.2), therefore for some j_0 (depending on i), after setting $y_i = y_{j_0}^{(i)} \in F_i$, we will have

$$|y_i|_G > 3\delta + \nu + \eta + \varkappa + C_1 + 2C_3 \ \text{ and } \ y_i \in C_{\hat{H}_i}\big(E(\hat{H}_i)\big) \backslash E(g_i). \tag{11.10}$$

Applying Lemmas 9.16 and 9.17 we can find $n \in \mathbb{N}$ such that the elements

$$w_i = y_i g_i^n$$

have infinite order, are $\hat{H}_i$-suitable and pairwise non-commensurable when $1 \le i \le s$. Obviously, in addition, we can demand that $|g_i^n|_G, |w_i|_G > C_1$ for every i.

The subgroups $K_i \le G$ are non-elementary, hence we can find elements $c_i \in K_i \le C_{\hat{H}_i}\big(E(\hat{H}_i)\big)$ for which

$$x_{i0} \overset{def}{=} a_i b_i^{-1} c_i \in C_G\big(E(\hat{H}_i)\big) \backslash E(w_i) \text{ and } |x_{i0}|_G > C_1 \text{ for } i = 1, \dots, s. \tag{11.11}$$

Note that $b_i^{-1} c_i \in \hat{H}_i$ for each i.

Let $w_i \in \hat{H}_i^0$, $x_{i0} \in G$, $x_{i1}, \dots, x_{il} \in \hat{H}_i$ be represented by words W_i, X_{i0}, $X_{i1}, \dots, X_{il}$ over the alphabet $\mathcal{A}$ of minimal length, $i = 1, \dots, s$.

Finally, define the system of relations $\mathcal{R} = \mathcal{R}_{s,l,m}(W_1, \dots, W_s, X_{10}, \dots, X_{sl}, m)$ as the set of all cyclic permutations of $R_i^{\pm 1}$ where

$$R_i \equiv X_{i0} W_i^m X_{i1} W_i^m \dots X_{il} W_i^m, \quad i = 1, 2, \dots, s. \tag{11.12}$$

Properties of Relations (11.12)

Lemma 11.4. *Fix $i \in \{1, \dots, s\}$. Let λ, c be the constants defined in* (11.5) *and the elements x_{ij}, $j = 1, \dots, l$, satisfy the properties:*

$$x_{ij} \in K_i \text{ and } |x_{ij}|_G > C_1 \text{ for every } j = 1, \dots, l$$

($K_i \le \hat{H}_i$ and $C_1 > 0$ are as above). Then for any $n \in \mathbb{N}$, any path q in $\Gamma(G, \mathcal{A})$ labelled by a word $R_i^{\pm n}$ is (λ, c)-quasigeodesic.

Proof. It is enough to prove this lemma for the case when q is labelled by R_i^n because if $lab(q) \equiv R_i^{-n}$ then $lab(q^{-1}) \equiv R_i^n$ and if q^{-1} is (λ, c)-quasigeodesic then so is q.

For convenience, assume that $i = 1$. Since left translations are isometries, we can suppose that $q_- = 1_G$. Set $t = (m+1)(l+1)$. The path q is a broken line $[z_0, z_1, \dots, z_{nt}]$ in the Cayley graph $\Gamma(G, \mathcal{A})$ where

$z_0 = q_- = 1_G$, $z_1 = x_{10}$, $z_2 = x_{10} w_1$, $z_3 = x_{10} w_1^2$,

$\dots\dots$

$z_{m+1} = x_{10} w_1^m$, $z_{m+2} = x_{10} w_1^m x_{11}$, $z_{m+3} = x_{10} w_1^m x_{11} w_1$,

$\dots\dots$

$z_t = x_{10} w_1^m x_{11} w_1^m \dots x_{1l} w_1^m$, $z_{t+1} = x_{10} w_1^m x_{11} w_1^m \dots x_{1l} w_1^m x_{10}$,

$\dots\dots$

$z_{nt} = x_{10} w_1^m \dots x_{1l} w_1^m x_{10} w_1^m \dots x_{1l} w_1^m \dots x_{10} w_1^m \dots x_{1l} w_1^m = q_+$.

By construction, $\|[z_{j-1}, z_j]\| > C_1$, $j = 1, 2, \ldots, nt$. In order to apply Lemma 9.6 to the path q it remains to verify that $(z_{j-1}|z_{j+1})_{z_j} \leq C_0$ for every $l = 1, \ldots, nt-1$. There are several types of Gromov products that appear when j changes from 1 to $nt-1$. Below we compute them in the order of their occurrence.

Type I. $(z_0|z_2)_{z_1} = (1_G|x_{10}w_1)_{x_{10}} = (x_{10}^{-1}|w_1)_{1_G}$. Recall that $w_1 = y_1 g_1^n$. By Gromov's definition of a hyperbolic space,

$$(x_{10}^{-1}|y_1)_{1_G} \geq \min\{(x_{10}^{-1}|w_1)_{1_G}, (y_1|w_1)_{1_G}\} - \delta.$$

Now, we observe that

$$(x_{10}^{-1}|y_1)_{1_G} = (c_1^{-1}b_1a_1^{-1}|y_1)_{1_G} \leq (c_1^{-1}|y_1)_{1_G} + |b_1a_1^{-1}|_G \leq C_{031} + |b_1a_1^{-1}|_G.$$

Hence

$$\min\{(x_{10}^{-1}|w_1)_{1_G}, (y_1|w_1)_{1_G}\} \leq C_{031} + |b_1a_1^{-1}|_G + \delta \leq C_0. \tag{11.13}$$

From the geodesic triangle $1_G y_1 w_1$ in $\Gamma(G, \mathcal{A})$ we obtain

$$(y_1|w_1)_{1_G} = |y_1|_G - (1_G|w_1)_{y_1} = |y_1|_G - (y_1^{-1}|g_1^n)_{1_G} \geq C_1 - C_0 > C_0.$$

Combining the latter inequality with (11.13) we achieve

$$(z_0|z_2)_{z_1} = (x_{10}^{-1}|w_1)_{1_G} \leq C_0.$$

Type II. $(z_1|z_3)_{z_2} = (x_{10}|x_{10}w_1^2)_{x_{10}w_1} = (w_1^{-1}|w_1)_{1_G}$.

Again, applying the definition of hyperbolicity twice, we obtain

$$(g_1^{-n}|y_1)_{1_G} \geq \min\{(g_1^{-n}|w_1)_{1_G}, (y_1|w_1)_{1_G}\} - \delta \geq \\ \min\{(w_1^{-1}|w_1)_{1_G}, (g_1^{-n}|w_1^{-1})_{1_G}, (y_1|w_1)_{1_G}\} - 2\delta.$$

By construction, $(g_1^{-n}|y_1)_{1_G} \leq C_{021} \leq C_0 - 2\delta$. As we showed above, $(y_1|w_1)_{1_G} > C_0$. Considering the geodesic triangle $1_G g_1^{-n} w_1^{-1}$ we get

$$(g_1^{-n}|w_1^{-1})_{1_G} = |g^{-n}|_G - (1_G|w_1^{-1})_{g^{-n}} = |g^n|_G - (g^n|y_1^{-1})_{1_G} \geq C_1 - C_0 > C_0.$$

So, combining these inequalities, we achieve

$$(z_1|z_3)_{z_2} = (w_1^{-1}|w_1)_{1_G} \leq C_0.$$

Type III. $(z_m|z_{m+2})_{z_{m+1}} = (w_1^{-1}|x_{11})_{1_G} \le C_0$.

Type IV. $(z_{m+1}|z_{m+3})_{z_{m+2}} = (x_{11}^{-1}|w_1)_{1_G} \le C_0$.

These two inequalities are proved in the same way as we proved the inequality for Type I (the proofs even easier since $x_{11} \in K_1$).

The last possibility is

Type V. $(z_{t-1}|z_{t+1})_{z_t} = (w_1^{-1}|x_{10})_{1_G}$.

As before, we have

$$(g_1^{-n}|x_{10})_{1_G} \ge \min\{(w_1^{-1}|x_{10})_{1_G}, (w_1^{-1}|g_1^{-n})_{1_G}\} - \delta.$$

$$(g_1^{-n}|x_{10})_{1_G} = (g_1^{-n}|a_1 b_1^{-1} c_1)_{1_G} = (b_1 a_1^{-1} g_1^{-n}|c_1)_{b_1 a_1^{-1}} \le$$
$$(b_1 a_1^{-1} g_1^{-n}|c_1)_{1_G} + |b_1 a_1^{-1}|_G \le C_{051} + |b_1 a_1^{-1}|_G \le C_0 - \delta.$$

We showed while considering the Type II, that $(g_1^{-n}|w_1^{-1})_{1_G} > C_0$. Therefore

$$(z_{t-1}|z_{t+1})_{z_t} = (w_1^{-1}|x_{10})_{1_G} \le C_0.$$

It is easy to see that for arbitrary $j \in \{1, 2, \dots, nt-1\}$ the Gromov product $(z_{j-1}|z_{j+1})_{z_j}$ is equal to a Gromov product of one of the Types I-V, thus it is not larger than C_0.

Therefore recalling that the constant C_1 was defined by formula (11.4), we can use the Lemma 9.6 to show that the path q is (λ, c)-quasigeodesic, where $\lambda > 0$ and $c \ge 0$ are defined in (11.5). Q.e.d. □

Below we have an analog of the Lemma 10.1 needed for our proof:

Lemma 11.5. *Suppose $W_1, \dots, W_s$, $X_{10}, \dots, X_{s0}$ and $\lambda > 0$, $c \ge 0$ are the words and the constants defined above. Then for any $\mu > 0$ there are $l \in \mathbb{N}$ and words $X_{11}, \dots, X_{sl}$ (X_{ij} represents an element $x_{ij} \in \hat{H}_i$, $j = 1, \dots, l$, $i = 1, \dots, s$) such that for any $\varepsilon \ge 0$, $\rho > 0$ there is $m_0 \in \mathbb{N}$ such that the system $\mathcal{R}_{s,l,m}$ (11.12) satisfies $C(\varepsilon, \mu, \lambda, c, \rho)$ and $C_1(\varepsilon, \mu, \lambda, c, \rho)$-conditions if $m \ge m_0$.*

Proof. By Lemma 10.1 there exist $\lambda' > 0$ such that for any $\mu > 0$ there are $l \in \mathbb{N}$ and $c' \ge 0$ such that for any $\varepsilon \ge 0$, $\rho > 0$ there are $m_0 \in \mathbb{N}$ and words $X_{11}, \dots, X_{sl}$ such that the system $\mathcal{R}_{s,l,m}$ satisfies the generalized small cancellation conditions $C(\varepsilon, \mu, \lambda', c', \rho)$ and $C_1(\varepsilon, \mu, \lambda', c', \rho)$ if $m \ge m_0$.

According to the Remark 10.2 and Lemma 8.4, the elements $x_{i1}, \dots, x_{is}$ can be chosen right after l, inside of the subgroup K_i, with an additional property $|x_{ij}|_G > C_1$ (the constant C_1 was defined in (11.4)) for every $j = 1, \dots, l$, $i = 1, \dots, s$.

Consider any word $R \in \mathcal{R}_{s,l,m}$. By definition, R is a subword of a word $R_i^{\pm 2}$ for some $i \in \{1, \dots, s\}$. By Lemma 11.4 the word $R_i^{\pm 2}$ is (λ, c)-quasigeodesic (where λ and c are defined in (11.5)), hence so is R. Therefore according to Definitions 10.4 and 10.5, the system $\mathcal{R}_{s,l,m}$ satisfies the conditions $C(\varepsilon, \mu, \lambda, c, \rho)$ and $C_1(\varepsilon, \mu, \lambda, c, \rho)$, provided $m \geq m_0$. $\square$

Lemma 11.6. *Let $\mathcal{R} = \mathcal{R}_{s,l,m}(W_1, \dots, W_s, X_{10}, \dots, X_{sl}, m)$ be the system of additional relations defined in* (11.12). *Then for any $\varepsilon \geq 0$ and $\xi > 0$ there exists $m_1 \in \mathbb{N}$ such that for any $m \geq m_1$ the following property holds:*

Suppose Δ is a diagram over the presentation (10.1) *and q – a subpath of $\partial\Delta$ such that the corresponding path q' in the Cayley graph $\Gamma(G, \mathcal{A})$ of the group G with the same label as q is geodesic (in other words, $\|q\| = |elem(q)|_G$) and $elem(q) \in Q$ in G. Then for arbitrary $\mathcal{R}$-face Π of Δ and an ε-contiguity subdiagram Γ between Π and q, one has* $(\Pi, \Gamma, q) \leq \xi$.

Proof. Assume λ, c and ν are the constants from (11.5) and (11.6). Let $\partial\Gamma = p_1 q_1 p_2 q_2$ where q_1, q_2 are subpaths of $\partial\Pi$ and q correspondingly and $\|p_1\|, \|p_2\| \leq \varepsilon$. Fix an arbitrary $\xi > 0$. Obviously, assuming that $l \in \mathbb{N}$ in the definition (11.12) is fixed, there is $m_1 \in \mathbb{N}$ such that for any $m \geq m_1$ the inequality $\|q_1\|/\|\partial\Pi\| > \xi$ implies that the path q_1 has a subpath o labelled by the word $W_i^{\pm 1}$ for some $i \in \{1, \dots, s\}$ and, moreover, the subpaths o_1, o_2 of q_1 (with $(o_1)_- = (q_1)_-$, $(o_1)_+ = o_-$, $(o_2)_- = o_+$, $(o_2)_- = (q_1)_+$) satisfy

$$\|o_j\| > (\varepsilon + c + \nu)/\lambda, \; j = 1, 2. \tag{11.14}$$

We are going to obtain a contradiction with the definitions of elements w_i and y_i.

Since the diagram Γ contains only 0-faces (i.e., it is a diagram over the group G), we can consider the corresponding picture in $\Gamma(G, \mathcal{A})$ with a geodesic path q' starting at 1_G (its subpath q_2'), (λ, c)-quasigeodesic path q_1' (its subpaths o', o_1', o_2') and paths p_1', p_2' of lengths at most ε with $(p_1')_- = (q_2')_+$, $(p_1')_+ = (q_1')_-$, $(p_2')_- = (q_1')_+$, $(p_2')_+ = (q_2')_-$ (i.e., for every path r from Δ we construct a corresponding path r' in $\Gamma(G, \mathcal{A})$ with the same label; see Figure 6).

Pick any $z \in Q$ with $|z|_G = \varkappa$ (the constant $\varkappa$ was defined in (11.7)). Then $q'_+ = elem(q') = elem(q) \in Q$. Hence since the triangles are δ-slim, one obtains

$$q' \subset \mathcal{O}_\delta([1, z] \cup [z, elem(q')]) \subset \mathcal{O}_{\delta+\varkappa}([z, elem(q')]) \subset \mathcal{O}_{\delta+\varkappa+\eta}(Q) \; .$$

Denote $u = (q_1')_-$, $v = (q_1')_+$. Then $u, v \subset \mathcal{O}_\varepsilon(q_2')$.

Since $o' \subset q_1' \subset \mathcal{O}_\nu([u, v])$ and q' is geodesic, using (11.14) and Lemma 9.7, we obtain

$$o'_-, o'_+ \in \mathcal{O}_{\nu+2\delta}(q') \subset \mathcal{O}_{3\delta+\nu+\varkappa+\eta}(Q).$$

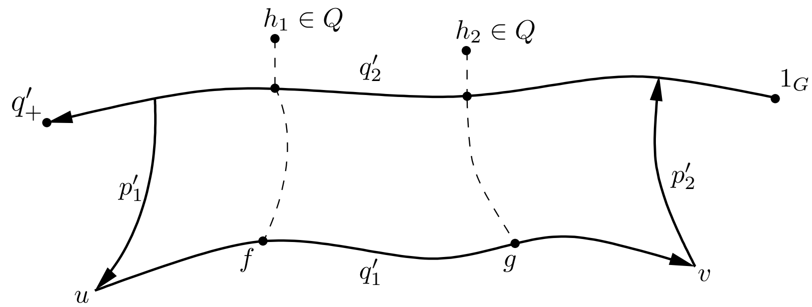

Figure 6: Contiguity in $\Gamma(G, \mathcal{A})$

Recall that $lab(o') = W_i^{\pm 1}$ by construction. So, if $lab(o') = W_i$, define the points $f = o'_-$, $g = o'_+$ and if $lab(o') = W_i^{-1}$, define $g = o'_-$ and $f = o'_+$. Thus there are elements $h_1, h_2 \in Q$ such that

$$d(f, h_1) = |f^{-1}h_1|_G \leq 3\delta + \nu + \varkappa + \eta, \; d(g, h_2) \leq 3\delta + \nu + \varkappa + \eta.$$

By the definition of w_i, we have $fy_ig_i^n = g$. Using Definition 2.1, we achieve

$$(h_2|fy_i)_f \geq \min\{(h_2|g)_f, (g|fy_i)_f\} - \delta.$$

Observe that $(h_2|fy_i)_f = (f^{-1}h_2|y_i)_{1_G}$ and $x = f^{-1}h_2 = (f^{-1}h_1)h_1^{-1}h_2 \in AQ^{-1}Q$ (the set A was defined in (11.8)). Inequality (11.10) implies

$$(g|fy_i)_f = \|[f, fy_i]\| - (f|g)_{fy_i} = |y_i|_G - (y_i^{-1}|g_i^n)_{1_G} \geq$$
$$|y_i|_G - C_0 > 5\delta + \nu + \varkappa + \eta + 2C_3$$

(here we used that $C_1 - C_0 > 2\delta$). Note that $d(f, g) \geq (g|fy_i)_f$, hence

$$(h_2|g)_f \geq \frac{1}{2}\big(d(f,g) - d(g,h_2)\big) \geq \frac{1}{2}\big((g|fy_i)_f - (3\delta + \nu + \varkappa + \eta)\big) > C_3 + \delta \; .$$

Combining the above formulas, we finally obtain $(h_2|fy_i)_f = (x|y_i)_{1_G} > C_3$, contradicting to the definition (11.9) of C_3. Therefore $\|q_1\|/\|\partial\Pi\| = (\Pi, \Gamma, q) \leq \xi$.

The lemma is proved. $\square$

For the last property of relations (11.12) we will need to recall some auxiliary notions from [26].

Suppose the words $D_1, \ldots, D_l$ and $\bar{D}_1, \ldots, \bar{D}_l$ represent elements $d_1, \ldots, d_l \in G^0$ and $\bar{d}_1, \ldots, \bar{d}_l \in G^0$ respectively. Consider a closed path $p_1q_1p_2q_2$ in $\Gamma(G, \mathcal{A})$ where q_1 has a decomposition $q_1 = s_0 t_1 s_1 \dots t_l s_l$, and q_2 – a decomposition $q_2 = \bar{s}_0 \bar{t}_1 \bar{s}_1 \dots \bar{t}_l \bar{s}_l$. Let $lab(t_i) \equiv D_i^{m_i}$, $lab(\bar{t}_i) \equiv \bar{D}_i^{\bar{m}_i}$, $m_i, \bar{m}_i \in \mathbb{Z}$, $i = 1, \dots, l$; $lab(s_j) = X_j$, $lab(\bar{s}_j) = \bar{X}_j$, $j = 0, \dots, l$; satisfying $X_i^{-1} D_i X_i \notin E(D_{i+1})$ and $\bar{X}_i^{-1} \bar{D}_i \bar{X}_i \notin E(\bar{D}_{i+1})$ in G, $i = 1, \dots, l-1$. As in the proof of Lemma 9.15, we will say that o is a phase vertex of a path t_i if the subpath of t_i from $(t_i)_-$ to o is labelled by some power of the word D_i (and similarly for $\bar{t}_i$).

Paths t_i and $\bar{t}_j$ will be called *compatible* if there is a path v_i in $\Gamma(G, \mathcal{A})$ joining some phase vertices of t_i and $\bar{t}_j$ such that $elem(v_i) \bar{d}_j^a elem(v_i)^{-1} = d_i^b$ in G, for some $a, b \in \mathbb{N}$. Such a path v_i is said to be *matching*.

Lemma 11.7. ([26, Lemma 2.5]) *Assume q_1, q_2, p_1, p_2 are as above and $\|p_1\|, \|p_2\| \le C$ for some C. Then there exist integers M and k, $|k| \le 1$, such that the paths t_i and $\bar{t}_{i+k}$ are compatible for all $i = 2, \dots, l-1$, if $m_2, \dots, m_{l-1} \ge M$.*

The fact below was implicitly used during the proof of Theorem 2 in [26] but has not been formulated there.

Lemma 11.8. *Let the elements $x_{i1}, \dots, x_{is}$ be chosen according to the claim of Lemma 8.4 applied to the situation when $g = w_i$, $H = \hat{H}_i$, $i = 1, \dots, s$. Suppose $r \in G$ is the element represented by a word $R \in \mathcal{R}_{s,l,m}$, s, l are as above and m is sufficiently large. Then the maximal elementary subgroup $E(r) \le G$ is generated by the infinite cyclic subgroup $\langle r \rangle$ and $E(G)$ (thus $E(r)$ is their semidirect product).*

Proof. Recall that r has infinite order by Lemmas 11.5 and 10.6.

Without loss of generality, we can assume $R \equiv R_1$ since the other cases are completely similar. Choose an arbitrary $z \in E(r)$. Then $zr^a z^{-1} = r^{\epsilon a}$ in G for some $a \in \mathbb{N}$ and $\epsilon \in \{1, -1\}$. Thus, $zr^{ab} z^{-1} = r^{\epsilon ab}$ for every $b \in \mathbb{N}$. Consider paths γ_1, γ_2, γ_3 and γ_4 in the Cayley graph of G where $(\gamma_1)_- = 1_G$, $lab(\gamma_1) \equiv R_1^{ab}$, $(\gamma_3)_- = z$, $lab(\gamma_3) \equiv R_1^{\epsilon ab}$ and γ_2, γ_4 are geodesics connecting $(\gamma_1)_-$ with $(\gamma_3)_-$ and $(\gamma_1)_+$ with $(\gamma_3)_+$ respectively. The paths γ_1 and γ_3 are (λ, c)-quasigeodesic by Lemma 11.4, hence γ_1, γ_2, γ_3 and γ_4 are sides of a (λ, c)-quasigeodesic quadrangle $\mathfrak{P}$. Obviously lengths of γ_1 and γ_3 grow with increasing b and $\|\gamma_2\| = \|\gamma_4\| = |z|_G$, therefore if b is large compared to $|z|_G$ we can use Lemma 9.7 (as in the proof of Lemma 9.15) to find a long segment of γ_1 that is C-close to γ_3 where C depends on δ, λ and c but doesn't depend on z and m. Thus, there will be subpaths q_1 of γ_1 and q_2 of γ_3, with $lab(q_1) \equiv R_1$ and $d\big((q_1)_-, (q_2)_-\big), d\big((q_1)_+, (q_2)_+\big) \le C$.

It is easy to see that q_1 and q_2 satisfy the assumptions of Lemma 11.7, hence, for any sufficiently large m, one can find subpaths t_i, t_{i+1} of q_1 compatible with $\bar{t}_j, \bar{t}_{j+1}$ of q_2 where $lab(t_i) \equiv lab(t_{i+1}) \equiv W_1^m$, $lab(\bar{t}_j) \equiv lab(\bar{t}_{j+1}) \equiv W_1^{\epsilon m}$, together with matching paths v_i, v_{i+1} between some phase vertices of t_i, t_{i+1} and $\bar{t}_j$, $\bar{t}_{j+1}$ correspondingly. Since $E(w_1) = E^+(w_1)$,

ϵ must be equal to 1. Consequently we have the following equality in the group G:

$$w_1^{\zeta_1} x_{1i} w_1^{\zeta_2} = elem(v_i) w_1^{\zeta_3} x_{1j} w_1^{\zeta_4} elem(v_{i+1})^{-1},$$

for some non-negative integers $\zeta_1, \zeta_2, \zeta_3, \zeta_4$. By definition, $elem(v_i), elem(v_{i+1}) \in E(w_1)$. Thus,

$$f_1 x_{1i} = x_{1j} f_2, \text{ where } f_1 = w_1^{-\zeta_3} elem(v_i)^{-1} w_1^{\zeta_1} \in E(w_1), \; f_2 = w_1^{\zeta_4} elem(v_{i+1})^{-1} w_1^{-\zeta_2} \in E(w_1).$$

Recalling the definition of elements x_{1i} and x_{1j}, we get $i = j$ and $f_1 = f_2 \in E(\hat{H}_1) = E(G)$. From the quadrangle $\mathfrak{P}$ we achieve

$$z = r^{\theta_1} p f_1 p^{-1} r^{\theta_2},$$

where $\theta_1, \theta_2 \in \mathbb{Z}$ and p is an element of G corresponding to the prefix $X_{10} W_1^m \dots X_{1,i-1} W_1^m$ of the word R_1. Since $E(G) \lhd G$, the observation $p f_1 p^{-1} \in E(G)$ completes our proof. $\square$

Proof of Theorem 11.1

Proof. The group G_1 is generated by $\phi(\mathcal{A})$, so let $|x|_{G_1}$ be the corresponding length function for elements $x \in G_1$, and $d_1(\cdot, \cdot)$ be the corresponding metric on the Cayley graph of the group G_1. It will be convenient for us to identify $\mathcal{A}$ and $\phi(\mathcal{A})$ for G_1, so $\Gamma(G_1, \mathcal{A})$ will be the Cayley graph of G_1.

Since ϕ is a homomorphism, from the definition of the word metric it follows that

$$\forall\, x, y \in G, \quad d_1\big(\phi(x), \phi(y)\big) \le d(x, y). \tag{11.15}$$

Define the elements $a_1, \dots, a_s \in G$ and the subgroups $\hat{H}_1, \dots, \hat{H}_s$ as in (11.1). Construct g_i, y_i, w_i and x_{i0}, $i = 1, 2, \dots, s$, as described before. Then we can find the constants $\lambda > 0$ and $c \ge 0$ according to (11.5).

Suppose that W_i, $X_{i0}, \dots, X_{il}$ are shortest words in the alphabet $\mathcal{A}$ representing w_i, $x_{i0}, \dots, x_{il}$, $i = 1, \dots, s$. As the system of additional relations, consider the set

$$\mathcal{R} = \mathcal{R}_{s,l,m}(W_1, \dots, W_s, X_{10}, \dots, X_{sl}, m)$$

of all cyclic permutations of $R_i^{\pm 1}$, $i = 1, \dots, s$, established in (11.12).

Define the group G_1 according to (10.1), thus, $G_1 \cong G/\langle \mathcal{R}^G \rangle$. Let ϕ be the natural epimorphism from G to G_1.

By Lemma 11.5 one can find $l, m_0 \in \mathbb{N}$ and elements $x_{ij} \in \hat{H}_i$, $j = 1, \dots, l$, $i = 1, \dots, s$, such that the group G_1 satisfies all of the conditions of Lemmas 10.3 and 10.5 if $m \geq m_0$. Therefore we obtain the parts 1) and 8) of Theorem 11.1.

It is easy to see that the relation R_i implies $\phi(a_i z_i) = 1$ in G_1 for some $z_i \in \hat{H}_i$, hence $\phi(a_i) \in \phi(\hat{H}_i)$ for $i = 1, \dots, s$.

Due to the choice of $a_1, \dots, a_s$ and $\hat{H}_1, \dots, \hat{H}_s$ we obtain $\phi(\mathcal{A}) \subset \phi(H_j)$ for every $j \in \{1, 2, \dots, k\}$. Consequently, $G_1 = \phi(H_j)$, $j = 1, \dots, k$, so part 3) of the theorem is proved.

Let us now prove the property 2). Let $\mu_0 > 0$, $\varepsilon \geq 0$ be chosen according to Lemma 10.3. Since we can take any μ inside of the interval $(0, \mu_0]$, we can demand it to satisfy the inequality $1/(\lambda + 1) < 1 - 23\mu$. Choose $\xi > 0$ in such a way that

$$\frac{1}{\lambda + 1} < 1 - 23\mu - 2\xi \ . \tag{11.16}$$

Denote $\theta = 1 - 23\mu - 2\xi > 0$. Then (11.16) implies that $(\lambda + 1)\theta - 1 > 0$. Set $L_0 = \min\{\|R\| \mid R \in \mathcal{R}\}$. Evidently, L_0 depends on m and there exists $m_2 \in \mathbb{N}$ such that for any $m \geq m_2$

$$\big((\lambda + 1)\theta - 1\big) L_0 > c + 4\varepsilon \ . \tag{11.17}$$

Now, let's apply the statement Lemma 11.6 to find $m_1 = m_1(\varepsilon, \xi) \in \mathbb{N}$.

By taking any $m \geq \max\{m_0, m_1, m_2\}$ we can further assume that the claims of Lemmas 10.3 and 11.6 hold together with the inequality (11.17).

Consider arbitrary elements $u, v \in Q$. We need to show that $d(u, v) = d_1\big(\phi(u), \phi(v)\big)$.

Observe that, by definition, $d(u, v) = |u^{-1}v|_G$, $d_1(\phi(u), \phi(v)) = |\phi(u^{-1}v)|_{G_1}$. Obviously, $|u^{-1}v|_G \geq |\phi(u^{-1}v)|_{G_1}$, so assume, by contradiction, that

$$|u^{-1}v|_G > |\phi(u^{-1}v)|_{G_1}. \tag{11.18}$$

Thus, if U, V are shortest words representing u, v in G, there is a word Z such that $U^{-1}V = Z$ in the G_1 but not in G (Z is a word of minimal length representing the element $\phi(u^{-1}v)$ in G_1).

Consider a reduced circular diagram Δ over G_1 whose boundary is labelled by the word $U^{-1}VZ^{-1}$. Let q^1, q^2, q^3 be the (geodesic) sections of the boundary $\partial\Delta$ labelled by the words U, V, Z respectively.

This diagram must contain at least one $\mathcal{R}$-face since $U^{-1}VZ^{-1} \neq 1$ in G. Therefore by Lemma 10.3 there exists an $\mathcal{R}$-face Π in Δ and ε-contiguity subdiagrams Γ_1,Γ_2,Γ_3 between Π and the sections q^1,q^2,q^3 (for our convenience, for each of the sections q^j we can choose a corresponding orientation of $\partial\Pi$, $j = 1, 2, 3$) satisfying

$$(\Pi, \Gamma_1, q^1) + (\Pi, \Gamma_2, q^2) + (\Pi, \Gamma_3, q^3) > 1 - 23\mu \ .$$

Since $elem(q^1) = u \in Q$, $elem(q^2) = v \in Q$ and $m \geq m_1$, we have $(\Pi, \Gamma_1, q^1) \leq \xi$ and $(\Pi, \Gamma_2, q^2) \leq \xi$. Hence

$$(\Pi, \Gamma_3, q^3) > 1 - 23\mu - 2\xi = \theta. \tag{11.19}$$

Now we are going to obtain a contradiction with the choice of Z. Let $\partial(\Gamma_3) = p_1 r_1 p_2 o_2$ where $\partial\Pi = r_1 r_2$, $q^3 = o_1 o_2 o_3$, $\|p_1\|, \|p_2\| \leq \varepsilon$ (Figure 7).

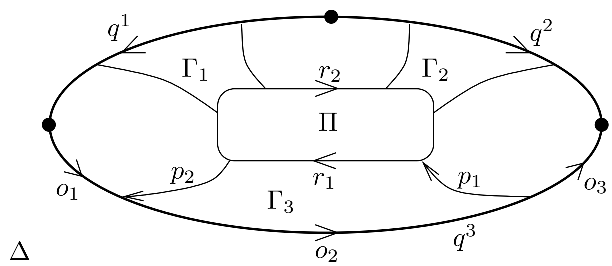

Figure 7: Diagram Δ from the proof of property 2)

Let L denote the length of $\partial\Pi$. (11.19) implies

$$\|r_1\| > \theta L \ , \quad \|r_2\| = L - \|r_1\| < (1 - \theta)L. \tag{11.20}$$

Now, since Γ_3 is a diagram over the group G, the equality

$$elem(o_2^{-1}) = elem(p_1)elem(r_1)elem(p_2)$$

holds in G. The path q^3 is geodesic, therefore its subpath o_2 is also geodesic, thus,

$$\|o_2\| = \|o_2^{-1}\| = |elem(o_2^{-1})|_G \geq |elem(r_1)|_G - |elem(p_1)|_G - |elem(p_2)|_G \ , \text{ hence}$$

$$\|o_2\| \geq |elem(r_1)|_G - 2\varepsilon.$$

The path r_1 is (λ, c)-quasigeodesic as a subpath of the face contour $\partial\Pi$ by Lemma 11.4, consequently $|elem(r_1)|_G \geq \lambda\|r_1\| - c$. Combining the last two inequalities with (11.20) we obtain

$$\|o_2\| \geq \lambda\theta L - c - 2\varepsilon. \tag{11.21}$$

Consider the subdiagram Ω of Δ bounded by the closed path $p_2^{-1} r_2 p_1^{-1} o_2^{-1}$. It corresponds to the following equality in the group G_1:

$$elem(o_2) = elem(p_2^{-1}) \cdot elem(r_2) \cdot elem(p_1^{-1}).$$

Thus, $\|o_2\| = |elem(o_2)|_{G_1} \leq |elem(p_2^{-1})|_{G_1} + |elem(r_2)|_{G_1} + |elem(p_1^{-1})|_{G_1} \leq$
$$\|r_2\| + 2\varepsilon \leq (1-\theta)L + 2\varepsilon.$$

Comparing the latter inequality with (11.21) we get

$$\lambda\theta L - c - 2\varepsilon \leq (1-\theta)L + 2\varepsilon.$$

Or, equivalently,

$$\big((\lambda+1)\theta - 1\big)L \leq c + 4\varepsilon.$$

Since $L \geq L_0$ this contradicts to the inequality (11.17).

Therefore the assumption (11.18) was incorrect and $d(u,v) = d_1\big(\phi(u), \phi(v)\big)$ for arbitrary $u, v \in Q$. Thus $\phi|_Q$ is an isometry.

By 1), G_1 is δ_1-hyperbolic for some $\delta_1 \geq 0$. Take any ω-quasiconvex (in G) subset $S \subseteq Q$. Let's show that $\phi(S) \subset G_1$ is $(\omega + \delta_1)$-quasiconvex.

Consider arbitrary two elements $u, v \in S$ and let p be a geodesic path in $\Gamma(G, \mathcal{A})$ connecting them. Then

$$p \subset \mathcal{O}_\omega(S) \ \text{ in } \Gamma(G, \mathcal{A}).$$

Let p_1 be the path in $\Gamma(G_1, \mathcal{A})$ starting at $\phi(u)$ with the same label as p. Then $(p_1)_+ = \phi(v)$ (this is equivalent to the equality $\phi(u) \cdot elem(p_1) = \phi(v)$ which follows from $u \cdot elem(p) = v$). Now, since ϕ is an isometry between S and $\phi(S)$,

$$\|p_1\| = \|p\| = d(u,v) = d_1\big(\phi(u), \phi(v)\big).$$

Therefore p_1 is a geodesic path between $\phi(u)$ and $\phi(v)$ in $\Gamma(G_1, \mathcal{A})$. (11.15) implies

$$p_1 \subset \mathcal{O}_\omega\big(\phi(S)\big) \ \text{ in } \Gamma(G_1, \mathcal{A}).$$

The space $\Gamma(G_1, \mathcal{A})$ is δ_1-hyperbolic, hence for any geodesic path q between $\phi(u)$ and $\phi(v)$ we have $q \subset \mathcal{O}_{\delta_1}(p_1)$. Consequently,

$$q \subset \mathcal{O}_{\omega+\delta_1}\big(\phi(S)\big) \ \text{ in } \Gamma(G_1, \mathcal{A}).$$

The proof of part 2) is complete.

Let us now prove property 4). Suppose U, V are shortest words representing some elements $u, v \in Q$ in G such that their images $\phi(u)$ and $\phi(v)$ are conjugate in G_1. Then there exists a reduced annular diagram Δ with boundary contours labelled by U and V correspondingly. Now, if Δ has no $\mathcal{R}$-cells then u and v are conjugate in G and there is nothing to prove. So, assume that Δ has at least one $\mathcal{R}$-cell. Let p_1 and q_2 denote the boundary contours of Δ with $lab(p_1) \equiv U$, $lab(p_2) \equiv V$. Fix a number $\mu \in (0, 1/23)$. Applying Lemmas 11.5 and 10.4, for any sufficiently large m we can find an $\mathcal{R}$-face Π in Δ, and two contiguity subdiagrams Γ_1, Γ_2 between Π and p_1, p_2 respectively, satisfying

$$(\Pi, \Gamma_1, p_1) + (\Pi, \Gamma_2, p_2) > 1 - 23\mu.$$

Therefore $(\Pi, \Gamma_i, p_i) > (1 - 23\mu)/2 > 0$ for all sufficiently large m and some $j \in \{1, 2\}$. The latter is a contradiction to the claim of Lemma 11.6. Hence, Δ has no $\mathcal{R}$-faces.

To show 5), assume the equality $AXA^{-1}X^{-1} = 1$ holds in G_1 for some words A and X, where A is a shortest word representing some element $a \in Q$. Then there is an annular diagram Δ with both contours labelled by A and a path x connecting the contours with $lab(x) \equiv X$.

The diagram Δ may not be reduced. If it has a pair of opposite faces Π_1 and Π_2 joined by a path t, then one can replace x with a homotopic path x' which avoids t (perhaps after a number of elementary transformations of Δ). Therefore $elem(x) = elem(x')$ in G_1 and reducing the pair (Π_1, Π_2) one obtains an annular diagram Δ' corresponding to the conjugacy of A with A by means of $X' \equiv lab(x')$.

There can be no (ε, μ)-extra face in Δ (or in Δ') because of the $C_1(\varepsilon, \mu, \lambda, c, \rho)$-condition. Hence it is possible to get a reduced annular diagram $\bar{\Delta}$ over G_1 corresponding to the equality $A\bar{X}A^{-1}\bar{X}^{-1} = 1$ where the word $\bar{X}$ represents the same element of G_1 as the word X.

Because of Lemmas 10.4 and 11.6, we can use a similar argument as before to show that $\bar{\Delta}$ doesn't have any $\mathcal{R}$-faces, provided m is sufficiently large. Thus $A\bar{X} = \bar{X}A$ in G confirming the property 5).

According to Lemma 2.7, there is a finite subset $T \subset G$ containing a representative from each conjugacy class of elements having finite order in G. Obviously, we can enlarge the subset Q by joining it with $T' = T \cup \{1_G\}$: set $Q' = Q \cup T'$. Q' is still quasiconvex and small relatively to $H_1, \ldots, H_k$. Constructing the corresponding homomorphism ϕ and using its property 2) we see that $ker(\phi) \cap T' = \{1_G\}$. Thus, property 6) of Theorem 11.1 is proved.

Let $C(G)$ and $C(G_1)$ denote the sets of conjugacy classes of elements having finite order in G and G_1 respectively. Include in Q a finite set of representatives of conjugacy classes from $C(G)$ together with the identity element. Then the homomorphism ϕ will induce a well-defined map $\bar{\phi} : C(G) \to C(G_1)$. Using property 4) we achieve injectivity of $\bar{\phi}$. On the other hand, when m is sufficiently large, surjectivity of $\bar{\phi}$ follows from Lemma 10.6, an observation that $C_G(R) \leq E(R)$ for every $R \in \mathcal{R}$ and Lemma 11.8. Thus we get property 7).

Finally, let's derive property 9). By Lemma 8.3, we can choose a G-suitable element $g \in G$. Then, by definition, $T(g) = E(G)$. Denote $S = \langle g \rangle$ – a quasiconvex subgroup of the group G. Then for any $h \in G$,

$$|H_i : (H_i \cap hSh^{-1})| = \infty$$

since H_i is non-elementary for every $i = 1, \ldots, k$. Hence, according to Corollary 7.4, S is small relatively to H_i, $i = 1, \ldots, s$. By Lemmas 3.8 and 7.9, the union

$$Q' = Q \cup S = Q \cup \langle g \rangle$$

is quasiconvex and small relatively to H_1,...,H_s.

Since the properties 1)-8) were already proved, we can further use them for the elements of Q'. Therefore, $ker(\phi) \cap Q' = \{1_G\}$, implying that $\phi(g)$ has infinite order in G_1.

Consider arbitrary $x \in E(G_1)$. Then, in particular, $x \in E\big(\phi(g)\big)$. By definition, there exists $n \in \mathbb{N}$ such that $x(\phi(g))^n x^{-1} = (\phi(g))^{\pm n}$.

If $x\phi(g)^n x^{-1} = \phi(g)^{-n}$ then by property 4) the elements $g^n, g^{-n} \in Q'$ must be conjugate in G which fails because $E(g) = E^+(g)$. Hence, $x(\phi(g))^n x^{-1} = (\phi(g))^n$, i.e., $x \in C_{G_1}\big(\phi(g^n)\big)$.

Since $g^n \in Q'$, one can apply property 5) to find $y \in C_G(g^n)$ with $\phi(y) = x$. $g \in G$ is G-suitable, therefore $C_G(g^n) \leq E(g) = T(g) \times \langle g \rangle$. G_1 is non-elementary, therefore the subgroup $E(G_1) \leq G_1$ is finite, thus x has a finite order in G_1. It follows that y has a finite order in G, because, otherwise, we would get $y^{l_1} = g^{l_2}$ for some $l_1, l_2 \in \mathbb{Z}\backslash\{0\}$ and $x^{l_1} = \phi(y^{l_1}) = \phi(g^{l_2})$ where $\phi(g^{l_2})$ has infinite order in G_1. Consequently, $y \in T(g) = E(G)$ and

$$x = \phi(y) \in \phi\big(E(G)\big).$$

The proof of Theorem 11.1 is finished. □

CHAPTER XII

EMBEDDING THEOREMS FOR HYPERBOLIC GROUPS

In this chapter we will apply Theorem 11.1 to obtain several new theorems concerning embeddability of a word hyperbolic group into a simple quotient of another non-elementary hyperbolic group.

First, lets us recall the notion of a direct (or inductive) limit of a sequence of groups. Let $\mathcal{A}$ be an alphabet and $\mathcal{R}_i$ – subsets of words in $\mathcal{A}^{\pm 1}$, $i \in \mathbb{N}$, satisfying $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ for all i. Let the groups G_i have presentations

$$G_i = \langle \mathcal{A} \parallel \mathcal{R}_i \rangle, \ i \in \mathbb{N}.$$

Then $G_{i+1} \cong G_i/N_i$ where $N_i \lhd G_i$ is the normal closure of $\mathcal{R}_{i+1} \backslash \mathcal{R}_i$ in G_i, i.e., there is an epimorphism $\phi_i : G_i \to G_{i+1}$ with $ker(\phi_i) = N_i$, $i \in \mathbb{N}$.

Set $\mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{R}_i$. The group M defined by the presentation

$$M = \langle \mathcal{A} \parallel \mathcal{R} \rangle$$

is said to be an *direct limit* of the groups G_i, $i \in \mathbb{N}$. Thus we obtain an infinite sequence of epimorphisms

$$G_1 \stackrel{\phi_1}{\to} G_2 \stackrel{\phi_2}{\to} G_3 \stackrel{\phi_3}{\to} \dots$$

and $M = \varinjlim (G_i, \phi_i)$.

The group M, as a direct limit of groups G_i, inherits a lot of their properties because any word over $\mathcal{A}^{\pm 1}$ that is trivial in M has to be trivial in one of G_i's. For example, if each G_i is torsion-free, then so is M.

The construction of the direct limit is very useful in Combinatorial Group Theory. Many difficult examples of groups can be created with its help (e.g., a finitely generated divisible group [13], a finitely generated verbally complete group [17], etc.). A. Ol'shanskii showed that the free Burnside group of a sufficiently large odd exponent can be obtained as a direct limit of word hyperbolic groups [27].

The groups we construct next are direct limits of hyperbolic groups. The main drawback of such limit groups is the property that they are never finitely presented (provided $M \neq G_i$ for every i).

Simple Quotients

Let us start with formulating the following consequence of Theorem 11.1:

Theorem 12.1. *Suppose G, H are hyperbolic groups and G is non-elementary. Then H can be isomorphically embedded into some simple quotient M of the group G. Moreover, the group M is a direct limit of hyperbolic groups.*

The proof requires the two auxiliary statements below.

Lemma 12.2. *Suppose N is an infinite normal subgroup of a hyperbolic group G and H is a quasiconvex subgroup of G such that $|G : H| = \infty$. Then H is small relatively to N.*

Proof. According to Corollary 7.4, it is enough to show that $|N : (N \cap gHg^{-1})| = \infty$ for arbitrary $g \in G$.

Since a conjugate of a quasiconvex subgroup of infinite index is again a quasiconvex subgroup of infinite index, it suffices to consider the case when $g = 1_G$. Assume, by the contrary, that $|N : (N \cap H)| < \infty$. Then there exist elements $g_1, \dots, g_n \in N$ such that $N \subseteq Hg_1 \cup \dots \cup Hg_n$. Applying Lemmas 5.6 and 5.3 we achieve

$$\Lambda(G) = \Lambda(N) \subseteq \Lambda(Hh_1 \cup \dots \cup Hh_n) = \Lambda(H).$$

Hence, by Lemmas 6.2.5) and 6.3, $G \subset K \cdot P = \bigcup_{p \in P} Hp$ for some finite subset P of G, which implies that $|G : H| < \infty$ – a contradiction to the assumptions. □

Lemma 12.3. *Assume that A is a non-elementary normal subgroup of a subgroup H in a hyperbolic group G. Then $E(A) = E(H)$.*

Proof. By definition, $E(H) \le E(A)$. Since $A \lhd H$, for every $h \in H$ the conjugate subgroup $hE(A)h^{-1}$ is normalized by A in G. But $E(A)$ is the unique maximal finite subgroup of G normalized by A (see [26, Prop. 1]), hence $hE(A)h^{-1} = E(A)$. Thus $E(A)$ is normalized by H, implying $E(A) \le E(H)$. □

Proof of Theorem 12.1. First, since $E(G)$ is the maximal finite normal subgroup of G, we can consider the quotient $\hat{G} = G/E(G)$. It is an easy exercise to show that the natural homomorphism $\psi : G \to \hat{G}$ is a quasiisometry between G and $\hat{G}$, therefore $\hat{G}$ is a non-elementary hyperbolic group (Theorem 2.3) without non-trivial finite normal subgroups. Consequently, $E(\hat{G}) = \{1_{\hat{G}}\}$.

Now, consider the free product $F = \hat{G} * H$. F is hyperbolic as a free product of hyperbolic groups ([8, 1.34]) and non-elementary. Identify $\hat{G}$ and H with their canonical copies inside of F. Evidently, we have $E(\hat{G}) = E(F) = \{1_F\}$ in F, hence $\hat{G}$ is a G-subgroup of F. By Lemma 2.8 one can find an element $g \in \hat{G} \leq F$ of infinite order. Then

$$\langle g \rangle \cap H = \{1_F\} \text{ in } F.$$

As it follows from the normal forms of elements of a free product, the subgroup H is undistorted in F, hence, by Lemma 3.2, H is a quasiconvex subgroup of F. Define the quasiconvex subset $Q \subset F$ by $Q = H \cup \langle g \rangle$. Obviously, no non-trivial element of $\hat{G}$ is conjugate to an element of H in F, therefore, according to Corollary 7.4, H and $\langle g \rangle$ are small relatively to $\hat{G}$ in F. By Lemmas 3.8 and 7.9, the subset Q is quasiconvex and small relatively to $\hat{G}$ in the group F.

Hence one can apply Theorem 11.1 to find a non-elementary hyperbolic quotient G_1 of F and an epimorphism $\phi_0 : F \to G_1$ that is surjective on $\hat{G}$, injective on Q; $\phi_0(H)$ is quasiconvex in G_1 and

$$E(G_1) = \phi_0\big(E(F)\big) = \{1_{G_1}\}, \tag{12.1}$$

$$\langle \phi_0(g) \rangle \cap \phi_0(H) = \{1_{G_1}\}. \tag{12.2}$$

In particular, $\phi_0(H) \cong H$.

Let $\{\chi_j \mid j \in \mathbb{N}\}$ denote the set of all non-trivial conjugacy classes of elements in the group G_1. Let N_1 be the normal subgroup of G_1 generated by χ_1. Observe that (12.1) implies that N_1 is infinite, consequently, it is non-elementary (because $\Lambda(N_1) = \Lambda(G) = \partial G$ according to Lemma 5.6 and this set is uncountable, but the limit set of an infinite elementary subgroup consists of only two points).

By Lemma 12.3 $E(N_1) = E(G_1)$ is trivial, hence, N_1 is a G-subgroup of the group G_1. Denote $g_1 = \phi_0(g) \in G_1$, $H_1 = \phi_0(H) \leq G_1$, $Q_1 = \langle g_1 \rangle \cup H_1$. The order of g_1 in the group G_1 is infinite, hence (12.2) implies that $|G_1 : H_1| = \infty$. Therefore, $|N_1 : (N_1 \cap h\langle g_1 \rangle h^{-1})| = \infty$ and $|N_1 : (N_1 \cap hH_1h^{-1})| = \infty$ for any $h \in G_1$ (by Lemma 12.2). Thus, by Corollary 7.4 and Lemma 7.9, we can apply Theorem 11.1 again and achieve a non-elementary hyperbolic quotient G_2 of G_1 together with an epimorphism $\phi_1 : G_1 \to G_2$ satisfying $\phi_1(N_1) = G_2$, ϕ_1 is injective on Q_1, $H_2 = \phi_1(H_1)$ is a quasiconvex subgroup of G_2, $E(G_2) = \{1_{G_2}\}$ and $\langle g_2 \rangle \cap H_2 = \{1_{G_2}\}$ where $g_2 = \phi_1(g_1)$.

Now, let $j_1 = 1$ and $j_2 > j_1$ be the smallest index such that $\phi_1(\chi_{j_2})$ is non-trivial in G_2. Set $N_2 = \langle\langle \phi_1(\chi_{j_2}) \rangle\rangle \lhd G_2$. We can apply the same argument as before to get a non-elementary hyperbolic quotient G_3 of G_2 with the natural epimorphism $\phi_2 : G_2 \to G_3$ satisfying the properties we need (as above). And so on.

Thus, we obtain an infinite sequence of epimorphisms

$$G \xrightarrow{\psi} \hat{G} \xrightarrow{\phi_0} G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \dots$$

where each epimorphism ϕ_i is injective on the image of $\phi_{i-1}(H)$, $i \in \mathbb{N}$.

Denote by M the corresponding direct limit of non-elementary hyperbolic groups. Then M is a quotient of G. As it is evident from the construction, M is a simple group and the group H is isomorphically embedded into M. Q.e.d. $\square$

Using similar techniques, one can prove even a more general statement:

Theorem 12.4. *There exists a simple group M that is a quotient of every non-elementary hyperbolic group and contains every hyperbolic group (isomorphically embedded).*

Proof. Let $A_1, A_2, A_3, \dots$ be an enumeration of all non-elementary hyperbolic groups and $B_1, B_2, B_3, \dots$ – an enumeration of all hyperbolic groups (there are countably many of them since every hyperbolic group is finitely presented [1]). Denote $\hat{A}_i = A_i/E(A_i)$, $i = 1, 2, \dots$.

Set $F_1 = \hat{A}_1 * B_1$. Then, applying Theorem 11.1, we can obtain a non-elementary hyperbolic group G_1 and an epimorphism $\phi_0 : F \to G_1$ that is surjective on $\hat{A}_1$ and injective on B_1 (as before, we can demand that $\phi_0(B_1)$ is quasiconvex in G_1, $|G_1 : \phi_0(B_1)| = \infty$ and $E(G_1) = \{1_{G_1}\}$).

Again, let the $\{\chi_j \mid j \in \mathbb{N}\}$ be the set of all non-trivial conjugacy classes of elements in the group G_1, $N_1 = \langle \chi_1 \rangle \lhd G_1$. By Theorem 11.1 we obtain a (non-elementary hyperbolic) quotient $\hat{G}_1$ with the natural epimorphism $\psi_1 : G_1 \to \hat{G}_1$ that is surjective on N_1 and injective on the image $\phi_0(B_1)$ of B_1 in G_1.

Next, define $F_2 = \hat{G}_1 * \hat{A}_2 * B_2$. Let G_2 be a non-elementary hyperbolic quotient of F_2 such that the natural epimorphism $\phi_1 : F_2 \to G_2$ is surjective on the subgroups $\hat{G}_1, \hat{A}_2 \leq F_2$ and injective on B_2 and the image of $B_1 \leq G_1 \leq F_2$.

Now, let $j_1 = 1$ and $j_2 > j_1$ be the smallest index such that the image of χ_{j_2} (under the composition $\phi_1 \circ \psi_1$) is non-trivial in G_2. Let $N_2 = \langle (\phi_1 \circ \psi_1)(\chi_{j_2}) \rangle \lhd G_2$. Then we can find an epimorphism $\psi_2 : G_2 \to \hat{G}_2$ onto a non-elementary hyperbolic group $\hat{G}_2$ that is surjective on N_2 and injective on the images of B_1, B_2.

And so on. Thus we achieve a sequence of epimorphisms

$$\hat{G}_1 \to \hat{G}_2 \to \hat{G}_3 \to \dots.$$

It is easy to see that the direct limit of this sequence enjoys all of the desired properties. $\square$

Thrifty Embeddings

In the paper [25] A. Ol'shanskii introduced, so called, *thrifty* embeddings of groups, and proved that any countable group H can be embedded into a 2-generated simple group M so that every proper subgroup of M is either infinite cyclic or infinite dihedral or is conjugate to a subgroup of H in M ([25, Thm. 2]).

Recall that a non-trivial proper subgroup H of a group G is called *malnormal* if for any $g \in G \backslash H$ the intersection $H \cap gHg^{-1}$ is trivial.

In the torsion-free word hyperbolic case Theorem 11.1 allows us to obtain even more efficient embeddings:

Theorem 12.5. *Suppose G, H are torsion-free hyperbolic groups, G is non-elementary and H is non-trivial. Then there exists a simple torsion-free quotient M of G and an injective homomorphism $\pi : H \to M$ such that $\pi(H)$ is malnormal in M and any proper subgroup of M is conjugate (in M) to a subgroup of $\pi(H)$.*

For the special case when the subgroup H is infinite cyclic, the finitely generated group M will satisfy the following property: every element of M is conjugate to some power of a fixed element $g \in M$. First examples of such groups were constructed by V. Guba in 1986 [13].

Observe that since any elementary torsion-free group is cyclic, maximal elementary subgroups are malnormal in any torsion-free non-elementary hyperbolic group.

Remark 12.6. Let H be a malnormal subgroup of a group G, $g \in G$. Then

(a) The conjugate subgroup $gHg^{-1} \leq G$ is also malnormal;

(b) If $K \leq G$ is an infinite subgroup and $|K : (K \cap gHg^{-1})| < \infty$ then $g^{-1}Kg \leq H$;

(c) For any $h \in H \backslash \{1_G\}$, $C_G(h) \leq H$;

(d) If $f \in G$ and $fHf^{-1} \cap gHg^{-1} \neq \{1_G\}$ then $fHf^{-1} = gHg^{-1}$.

For the proof of Theorem 12.5 we will need the statement below addressing malnormality in HNN-extensions.

Lemma 12.7. *Suppose G is a group and H, A, B are its subgroups. Assume that H and B are malnormal in G, $H \cap gBg^{-1} = \{1_G\}$ for any $g \in G$ and there is an isomorphism $\tau : A \to B$. Then the natural image of H in the HNN-extension*

$$G_1 = \langle G, t \mid tAt^{-1} = B \rangle \overset{def}{=} \langle G, t \mid tat^{-1} = \tau(a),\ a \in A \rangle$$

is malnormal.

Proof. Identify G and H with their canonical images in G_1. Assume that there exists $w \in G_1 \backslash H$ and non-trivial elements $x, y \in H$ such that $wxw^{-1} = y$. Then we can write

$$w = u_0 t^{\epsilon_1} u_1 t^{\epsilon_2} \cdot \dots \cdot t^{\epsilon_{n-1}} u_{n-1} t^{\epsilon_n} u_n \quad \text{in } G_1, \tag{12.3}$$

where $u_0, u_n \in G$, $u_1, \dots, u_{n-1} \in G \backslash \{1_G\}$, $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$, and this representation is reduced (i.e., it contains no occurrences of the form tut^{-1} or $t^{-1}vt$ where $u \in A$, $v \in B$).

Observe that $n \geq 1$ since $w \notin G$ (by malnormality of H in G) and

$$u_0 t^{\epsilon_1} \cdot \dots \cdot t^{\epsilon_n} u_n x u_n^{-1} t^{-\epsilon_n} \cdot \dots \cdot t^{-\epsilon_2} u_0^{-1} y^{-1} = 1_{G_1}. \tag{12.4}$$

By Britton's Lemma ([16]) the left-hand side in (12.4) is not reduced, hence $u_n x u_n^{-1}$ belongs to A or B. But this element is a conjugate of $x \in H$ therefore, according to the assumptions of the lemma, it has to be in A and $\epsilon_n = 1$. Consequently, $t^{\epsilon_n} u_n x u_n^{-1} t^{-\epsilon_n} = v \in B \backslash \{1_G\}$. Since no element of B is conjugate to the element $y \in H$ in the group G, the number n from the representation (12.3) must be at least 2 and

$$wxw^{-1}y^{-1} \stackrel{G_1}{=} u_0 t^{\epsilon_1} \cdot \dots \cdot t^{\epsilon_{n-1}} u_{n-1} v u_{n-1}^{-1} t^{-\epsilon_{n-1}} \cdot \dots \cdot t^{-\epsilon_2} u_0^{-1} y^{-1} = 1_{G_1}.$$

Applying Britton's Lemma again, we get that the element $u_{n-1} v u_{n-1}^{-1}$ either belongs to A (and $\epsilon_{n-1} = 1$) or to B (and $\epsilon_{n-1} = -1$). So, if it is in A, then $t^{\epsilon_{n-1}} u_{n-1} v u_{n-1}^{-1} t^{-\epsilon_{n-1}} \in B$ and n has to be at least 3; thus we can proceed as before. This process will end after finitely many steps because each time we eliminate a $t^{\pm 1}$-element from the representation (12.3) of w. Therefore, we can assume that $u_{n-1} v u_{n-1}^{-1} \in B$ and $\epsilon_{n-1} = -1$. But the subgroup B was malnormal in G and $v \in B \backslash \{1_G\}$, hence $u_{n-1} \in B$. Hence $t^{\epsilon_{n-1}} u_{n-1} t^{\epsilon_n} \equiv t^{-1} u_{n-1} t \in A$ which contradicts to our assumption that the right-hand side of (12.3) is reduced.

The lemma is proved. □

Lemma 12.8. ([17, Thm. 3],[15, Cor. 1]) *Let G be a hyperbolic group with isomorphic infinite elementary subgroups A and B, and let τ be an isomorphism from A to B. The HNN-extension $G_1 = \langle G, t \mid tat^{-1} = \tau(a), a \in A \rangle$ of G with associated subgroups A and B is hyperbolic if and only if the following two conditions hold:*
1) *either A or B is a maximal elementary subgroup of G;*
2) *for all $g \in G$ the subgroup $gAg^{-1} \cap B$ is finite.*

Lemma 12.9. ([15, Thm. 4]) *Let the HNN-extension $G_1 = \langle G, t \mid tAt^{-1} = B \rangle$ be hyperbolic with A quasiconvex in G_1. Then G is quasiconvex in G_1.*

Proof of Theorem 12.5. Consider the free product $F = G * H$. Then F is a non-elementary torsion-free hyperbolic group, G is a G-subgroup of F and H is quasiconvex in F (because it is undistorted). H is non-trivial by the assumptions of the theorem, hence there is an element $y \in H$ of infinite order. Pick any $f \in G \backslash \{1_F\}$ and set $x = fyf^{-1} \in F$. From normal forms of elements of the free product F it follows that H is malnormal in F, $gHg^{-1} \cap G = \{1_F\}$ for any $g \in F$ and the infinite cyclic subgroup of F generated by x has trivial intersection with H. Denote $Q = \langle x \rangle \cup H$ – a quasiconvex subset of F.

As before, all the assumptions of Theorem 11.1 are satisfied, hence there exists a non-elementary hyperbolic quotient G_0 of F and an epimorphism $\psi_0 : F \to G_0$ with the properties $1) - 9)$ from the claim of Theorem 11.1. Thus $\psi_0(G) = G_0$, ψ_0 is injective on Q, G_0 is torsion-free (by the property 7)), $\psi_0(H)$ is quasiconvex in G_0, $\psi_0(x) \in (G_0)^0$ and $\psi_0(H) \cap \langle \psi_0(x) \rangle = \{1_{G_0}\}$.

Suppose for some non-trivial $z \in G_0$ there are non-trivial $a, b \in H$ such that $z\psi_0(a)z^{-1} = \psi_0(b)$. By property 4) from the claim of Theorem 11.1, there exists an element $u \in F$ such that $uau^{-1} = b$. H was malnormal in F, therefore $u \in H$ and $z^{-1}\psi_0(u)\psi_0(a)\psi_0(u)^{-1}z = \psi_0(a)$, i.e., $z^{-1}\psi_0(u) \in C_{G_0}\big(\psi_0(a)\big)$. Then, according to property 5), there is $v \in C_G(a)$ satisfying $\psi_0(v) = z^{-1}\psi_0(u)$. Also, by Remark 12.6, $v \in H$. Thus, $z = \psi_0(u)\psi_0(v)^{-1} \in \psi_0(H)$, i.e., $\psi_0(H)$ is malnormal in G_0.

Enumerate all non-trivial elements of the group G_0: $g_1, g_2, \dots$, and all its two-generated non-elementary subgroups: $K_1, K_2, \dots$.

The group M will be constructed as an inductive limit of groups G_i, $i = 0, 1, \dots$. Assume, the non-elementary hyperbolic torsion-free quotient G_{i-1} of G_0 has already been constructed, $i \geq 1$, and it satisfies the following properties: the natural epimorphism $\pi_{i-1} : G_0 \to G_{i-1}$ ($\pi_0 = id_{G_0} : G_0 \to G_0$) is injective on $\psi_0(H) \cup \langle \psi_0(x) \rangle$, the image of $\psi_0(H)$ is quasiconvex and malnormal in G_{i-1}; images of the elements $g_1, \dots, g_{i-1}$ are conjugate in G_{i-1} to some elements from $\pi_{i-1}\big(\psi_0(H)\big)$, and images of the subgroups $K_1, \dots, K_{i-1}$ either coincide with G_{i-1} or are conjugate in G_{i-1} to a subgroup of $\pi_{i-1}\big(\psi_0(H)\big)$, or are elementary.

Let us now construct the group G_i. Consider the element $\pi_{i-1}(g_i) \in G_{i-1}$. To simplify the notation, identify H and $\pi_{i-1}\big(\psi_0(H)\big)$. If $\pi_{i-1}(g_i)$ is conjugate in G_{i-1} to an element from H then set $F_i = G_{i-1}$.

If not, then the element $\pi_{i-1}(g_i)$ has infinite order in G_{i-1} and the maximal elementary subgroup $B = E\big(\pi_{i-1}(g_i)\big)$ is infinite cyclic (because G_{i-1} is torsion-free) and malnormal in G_{i-1}. Part (b) of Remark 12.6 implies that $H \cap gBg^{-1} = \{1_{G_{i-1}}\}$ for any $g \in G_{i-1}$. Denote by $A \leq G_{i-1}$ the infinite cyclic subgroup of H generated by the element y chosen in the beginning of the proof. Then we can construct an HNN-extension

$$F_i = \langle G_{i-1}, t \mid tAt^{-1} = B \rangle \ .$$

According to Lemma 12.8 and basic properties of HNN-extensions (see [16, Ch. IV]), F_i is a torsion-free non-elementary hyperbolic group. Since any cyclic subgroup in F_i is quasiconvex, Lemma 12.9 implies that the natural image of G_{i-1} in F_i is also quasiconvex. By Lemma 12.7 and Remark 3.4, H is malnormal and quasiconvex in F_i. Note that the latter implies $|G_{i-1} : (G_{i-1} \cap gHg^{-1})| = \infty$ for any $g \in F_i$ because, otherwise, by part (b) of Remark 12.6, $G_{i-1} \leq gHg^{-1}$ and since $H \leq G_{i-1}$ is non-trivial, part (d) of the same Remark would claim that $G_{i-1} \leq gHg^{-1} = H$. This leads to a contradiction with the fact that $x \in G_{i-1} \backslash H$.

By construction, $\pi_{i-1}(g_i)$ is conjugate to some element of H in F_i.

Now consider the subgroup $\pi_{i-1}(K_i) \leq G_{i-1} \leq F_i$. If this subgroup is elementary or conjugate to a subgroup of H in F_i, then we apply Theorem 11.1 to obtain a torsion-free non-elementary hyperbolic group G_i and an epimorphism $\psi_i : F_i \to G_i$ such that $\psi_i(G_{i-1}) = G_i$, ψ_i is injective on $H \cup \langle x \rangle$, $\psi_i(H)$ is quasiconvex and malnormal (as before) in G_i. Then $\psi_i\big(\pi_{i-1}(K_i)\big) \leq G_i$ is either elementary or conjugate to a subgroup of $\psi_i(H)$ in G_i.

Thus, we can assume that $\pi_{i-1}(K_i)$ is non-elementary and not conjugate to a subgroup of H in F_i. Then, by Remark 12.6,

$$|\pi_{i-1}(K_i) : (\pi_{i-1}(K_i) \cap gHg^{-1})| = \infty \quad \text{for any } g \in F_i,$$

hence $H \cup \langle x \rangle$ is small relatively to K_i in F_i and we can use Theorem 11.1 to get an epimorphism ψ_i of F_i onto a non-elementary torsion-free hyperbolic group G_i satisfying the following conditions: $\psi_i\big(\pi_{i-1}(K_i)\big) = G_i$ (consequently, $\psi_i(G_{i-1}) = G_i$), ψ_i is injective on $H \cup \langle x \rangle$ and $\psi_i(H)$ is quasiconvex and malnormal in G_i.

Thus, we have constructed the group G_i for every $i = 0, 1, 2, \dots$.

Set $M = \varinjlim(G_i, \psi_{i+1})$. It remains to prove that M satisfies the properties required. There is a natural epimorphism $\pi : G_0 \to M$. Note that if a word w is trivial in M, then (by the definition of an inductive limit) w is trivial in G_i for some i, hence M is torsion-free, π is injective on H, $\pi(x) \neq 1_M$ and $\pi(H) \cap \langle \pi(x) \rangle = \{1_M\}$ (we identify H and x with their images in G_0). Therefore $\pi(H)$ is a proper subgroup of M and, since the image of H was malnormal in each G_i, $\pi(H)$ will be malnormal in M.

Denote $P = \pi(H) \leq M$ and assume that L is a proper non-trivial subgroup of M. Then there exists $a \in L \backslash \{1_M\}$. Suppose that for every $b \in L$ there exists $g_b \in M$ such that $g_b \langle a, b \rangle g_b^{-1} \leq P$. Set $g = g_{1_M}$ and pick an arbitrary $b \in L$. Then

$$a \in g_b^{-1} P g_b \cap g^{-1} P g \neq \{1_M\} \ .$$

Therefore, applying Remark 12.6, we get $g_b^{-1} P g_b = g^{-1} P g$, thus $b \in g^{-1} P g$ for any $b \in L$, hence $gLg^{-1} \leq P$. So, if L is not conjugate to a subgroup of P then there should exist $b \in L$

such that the subgroup $\langle a, b\rangle \leq L$ is not conjugate to any subgroup from P. Choose arbitrary elements $c, d \in G_0$ with $\pi(c) = a$, $\pi(d) = b$. Then $\pi(\langle c, d\rangle) = \langle a, b\rangle$ and the image of $\langle c, d\rangle$ in G_i is not conjugate to a subgroup of the (corresponding) image of H for all i. Thus, this image is non-elementary (i.e., non-cyclic) in G_i for all i (since every cyclic subgroup will eventually be conjugate to some cyclic subgroup from an image of H). Consequently, $\langle c, d\rangle = K_j$ for some $j \in \mathbb{N}$ and the homomorphism $\pi_j : G_0 \to G_j$ will be surjective on K_j. It follows that $\langle a, b\rangle = \pi(\langle c, d\rangle) = M$ – a contradiction with the condition $L \neq M$. So, we showed that any proper subgroup L of M is conjugate to some subgroup of P.

Finally, if $N \lhd M$ and $N \neq M$ then, applying the above, we obtain an element $g \in M$ such that $N = gNg^{-1} \leq P$. But this implies that $N = \{1_M\}$ because P is malnormal. Thus, M is simple. $\square$

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