VARIETIES OF RESIDUATED LATTICES

By

Nikolaos Galatos

Dissertation
Submitted to the Faculty of the
Graduate School of Vanderbilt University
in partial fulfillment of the requirements
of the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

May, 2003

Nashville, Tennessee

Approved by:

Professor Constantine Tsinakis

Professor Ralph McKenzie

Professor Steven Tschantz

Professor Jonathan Farley

Professor Alan Peters

Dedicated to my beloved wife, Smaroula.

ACKNOWLEDGMENTS

This thesis wouldn't have been possible without the help and support of certain people. I could not possibly itemize my gratitude in detail, but I would like to mention some of the most important names.

First and foremost, I would like to thank my advisor, Constantine Tsinakis, for introducing me to the subject of residuated lattices, and for his guidance and advice. Despite his busy schedule and administrative duties, he would always set aside plenty of time to discuss mathematics with me and patiently monitor my attempts to do research. He has been most helpful in balancing between suggesting new problems and research directions on the one hand, and encouraging independence on the other. A major part of this thesis is joint research with him and all of my work has been influenced by his comments and suggestions. I would also like to express my gratitude to my advisor for enlightening me regarding some of the other necessary traits of being a mathematician, including engaging presentation techniques and proper proposal writing.

I would like to thank my co-advisor Peter Jipsen for his immeasurable contributions to my research and to this thesis. During his two-year visit at Vanderbilt University, I had the opportunity to have many long discussions and to engage in joint research with him. Even though not a member of my committee, he has truly been a second advisor for me, and a lot of the work in my dissertation is due to his suggestions and remarks. Peter has also been a good friend and it is always a pleasure and an intellectual stimulus to meet with him - even since his departure from Vanderbilt.

I am very grateful to my committee members. In particular, I was privileged to attend many of Ralph McKenzie's seminars on interesting topics of mathematics, to take my first graduate course in Universal Algebra from Steven Tschantz and to take a special topics course offered by Jonathan Farley. I would also like to thank Alan Peters for agreeing to be the extra-departmental member of my Ph.D. committee, and Professors Matthew Gould and Mark Sapir of Vanderbilt University for the interesting algebra courses that they offered. I was fortunate to collaborate with Professors Steve Seif and James Raftery, to whom I am grateful for the helpful discussions we have had.

During my study in Vanderbilt University, I was lucky to be surrounded by excellent fellow graduate students in algebra, many of whom became close friends. They have created a challenging and collaborative environment that has motivated me. In particular, I would like to thank Doctors Jac Cole, Kevin Blount, Miklós Maróti, Petar Marković, Patrick Bahls and David Jennings, as well as the doctoral students Marcin Kozik, Ashot Minasyan, Dmitriy Sonkin and Alexey Muranov. Special thanks go to Miklós and Kevin for their suggestions on the final version of my thesis, and to Jac, Petar and Marcin for the very helpful discussions we had throughout my studies.

I want to express my gratitude to Vanderbilt University for supporting my graduate studies with the University Graduate Fellowship, for the opportunity it gave to me to travel to Japan in November 2002, via the Dissertation Enhancement Grant and for the financial support it offered during the Summer of 2002 with the Summer Support Award.

Finally, I am indebted to my wife Smaroula for her understanding and constant support. Even during the long-distance parts of our relationship she has been the most valuable motivation for my attempts.

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CHAPTER I

INTRODUCTION

Residuated lattices are algebraic structures with strong connections to mathematical logic. This thesis studies properties of a number of collections of residuated lattices. The algebras under investigation combine the fundamental notions of multiplication, order and residuation, and include many well-studied ordered algebraic structures.

Residuated lattices were first considered, albeit in a more restrictive setting than the one we adopt here, by M. Ward and R. P. Dilworth in the 1930's. Their investigation stemmed from attempts to generalize properties of the lattice of ideals of a ring. On the other hand, work on residuation, a concept closely related to the notions of categorical adjunction and of Galois connection, was undertaken in algebra, with emphasis on multiplication, and in logic, with emphasis on implication, but without substantial communication between the fields. During relatively recent years, studies in relevant logic, linear logic and substructural logic as well as on the algebraic side draw attention to and establish strong connections between the fields. See, for example, [OK], [BvA], [RvA] and [JT].

The generality in the definition of residuated lattices is due to K. Blount and C. Tsinakis (see [BT]) who first developed a structure theory for these algebras. This thesis relies on their results and concentrates on subvarieties of residuated lattices.

After discussing, in Chapter II, the background needed for reading this thesis, in Chapter III we give the definition of residuated lattices and an extensive list of examples and constructions on residuated lattices. Also, we give a short overview of the description of congruence relations, presented in [BT], comment on the case of a finite residuated lattice and give two easy corollaries of the general theory. Furthermore, we define a number of interesting subvarieties of residuated lattices and discuss properties of the subvariety lattice. In particular, we establish a correspondence between positive universal formulas in the language of residuated lattices and residuated-lattice varieties and apply it to show, among other things, that the join of two finitely based commutative varieties of residuated lattices is also finitely based. We give a brief exposition of the fact that residuated lattices provide algebraic semantics for the full Lambek calculus, and review how this implies the decidability of the equational theory of residuated lattices, a fact proved in [JT]. Finally, we investigate the limitations of lexicographic orders on semidirect products, a useful tool for lattice-ordered groups, in the case of residuated lattices.

Chapter IV contains results to appear in [BCGJT]. In particular, we note that the class of residuated lattices with a cancellative monoid reduct is a variety, and we give a number of equational bases for the varieties of lattice-ordered groups and their negative cones and illuminate the connections between the two varieties.

In Chapter V we undertake an investigation of the atomic layer in the subvariety lattice of residuated lattices. We show that there exist only two cancellative atoms and provide a countably infinite list of commutative atoms. Moreover, we construct a continuum of atomic varieties that have an idempotent monoid reduct and are generated by totally ordered residuated lattices. We note that there are only two idempotent commutative atoms.

Chapter VI focuses on residuated lattices with a distributive lattice reduct. We mention that the variety of distributive residuated lattices has an undecidable quasi-equational theory, see [Ga], and remark that the same variety has a decidable equational theory, see [GR]. Moreover, we establish a Priestley-type duality for the category of distributive residuated bounded-lattices.

The collections of MV-algebras, lattice-ordered groups and their negative cones are generalized to the variety of GMV-algebras, in Chapter VII. We prove that a GMV-algebra decomposes into the Cartesian product of a lattice-ordered group and a nucleus-retraction on the negative cone of a lattice-ordered group. Moreover, we show that a GMV-algebra is the image of a monotone, idempotent map on a lattice ordered group. These characterizations and known results regarding lattice-ordered groups imply the decidability of the equational theory of GMV-algebras. Finally, we establish an equivalence between the category of GMV-algebras and a category of pairs of lattice-ordered groups and certain maps on them. We conclude our study with a list of open problems, in Chapter VIII.

An effort has been made so that the exposition can be understood by the non-specialist. Toward this goal we have tried to present proofs in full detail.

CHAPTER II

PRELIMINARIES

We assume familiarity with basic concepts from set theory, mathematical logic, topology and category theory. If h is a map from A to B and $C \subseteq A$, $D \subseteq B$, we set $h[C] = \{h(c) \,|\, c \in C\}$ and $h^{-1}[D] = \{a \in A \,|\, h(a) \in D\}$. In what follows, we give the basic notions and results that will be needed for the presentation of this thesis, organized according to three subject areas.

Universal algebra

We start with some basic definitions from universal algebra. For a detailed exposition of notions and results of the fields, the reader is referred to [MMT] and [BS].

An *(algebraic) language*, *signature*, or *(similarity) type* $\mathcal{F}$ is an indexed set of symbols F together with a map $\sigma : F \to \mathbb{N}$, called the *arity map*. An *operation* on a set A of *arity* n is a map from A^n to A. An *algebra* $\mathbf{A}$ of type $\mathcal{F}$ consists of a set A and an indexed set $\langle f^{\mathbf{A}} \rangle_{f \in F}$ of operations $f^{\mathbf{A}} : A^{\sigma(f)} \to A$ on A of *arity* $\sigma(f)$. The set A is called the *underlying set* or the *universe* of $\mathbf{A}$ and the maps $f^{\mathbf{A}}$ are called the *fundamental operations* of $\mathbf{A}$. We will be dealing with algebras over a finite similarity type. Such algebras will be denoted by $\mathbf{A} = \langle A, f_1^{\mathbf{A}}, f_2^{\mathbf{A}}, \ldots, f_n^{\mathbf{A}} \rangle$, and most of the times we will omit the superscript $\mathbf{A}$. Two algebras that have the same similarity type are called *similar*.

A *subuniverse* of an algebra $\mathbf{A}$ is a subset B of A that is closed under the fundamental operations, i.e., $f^{\mathbf{A}}(b_1, b_2, \ldots, b_{\sigma(f)}) \in B$, for all $b_1, \ldots b_{\sigma(f)} \in B$. If B is a subuniverse of of an algebra $\mathbf{A} = \langle A, f_1, f_2, \ldots, f_n \rangle$, then the algebra $\mathbf{B} = \langle B, f_1|_B, f_2|_B, \ldots, f_n|_B \rangle$, where $f_i|B$ is the restriction of $f_i^{\mathbf{A}}$ to $B^{\sigma(f)}$, is called a *subalgebra* of $\mathbf{A}$.

If $\mathcal{F}$ is a similarity type and G is a subset of F, the *G-reduct* of an algebra $\mathbf{A}$ with underlying set A, similarity type $\mathcal{F}$ and fundamental operations $f^{\mathbf{A}}$, $f \in F$ is the algebra $\mathbf{A}^G$ with underlying set A and fundamental operations $f^{\mathbf{A}}$, $f \in G$. A *G-subreduct* is a subalgebra of a G-reduct.

A *homomorphism* between two algebras $\mathbf{A}$ and $\mathbf{B}$ of the same similarity type $\mathcal{F}$ is a map $h : A \to B$, that commutes with all the fundamental operations, i.e., $h(f^{\mathbf{A}}(a_1, a_2, \ldots, a_{\sigma(f)})) = f^{\mathbf{B}}(h(a_1), h(a_2), \ldots, h(a_{\sigma(f)}))$, for all $a_1, a_2, \ldots, a_{\sigma(f)} \in A$ and for all $f \in F$. If h is an onto homomorphism form $\mathbf{A}$ to $\mathbf{B}$, then we say that $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$. The *kernel* of a homomorphism $h : \mathbf{A} \to \mathbf{B}$ is defined to be the set $Ker(h) = \{(x, y) \in A^2 \,|\, h(x) = h(y)\}$.

If $\mathcal{A} = \{\mathbf{A}_i \mid i \in I\}$ is an indexed set of algebras of a given similarity type $\mathcal{F}$, then the *product* of the algebras of $\mathcal{A}$ is the algebra $\mathbf{P} = \prod_{i\in I} \mathbf{A}_i$ with underlying set the Cartesian product of the underlying sets of the algebras in $\mathcal{A}$, similarity type $\mathcal{F}$ and fundamental operations $f^{\mathbf{P}}$, $f \in F$, defined by $f^{\mathbf{P}}(\langle a_{i1}\rangle_{i\in I}, \ldots, \langle a_{i\sigma(f)}\rangle_{i\in I}) = \langle f^{\mathbf{A}_i}(a_{i1}, \cdots, a_{i\sigma(f)})\rangle_{i\in I}$, for all $\mathbf{A}_i \in \mathcal{A}$, $a_{ij} \in A_i$, $i \in I$ and $j \in \{1, \ldots, \sigma(f)\}$.

A *congruence relation* on an algebra $\mathbf{A}$ of type $\mathcal{F}$ is an equivalence relation θ that is compatible with the fundamental operations of $\mathbf{A}$, i.e., for every fundamental operation $f^{\mathbf{A}}$, $f \in F$, and $a_1, a_2, \ldots, a_{\sigma(f)}$, $b_1, b_2, \ldots, b_{\sigma(f)} \in A$, if $a_1 \,\theta\, b_1$, $a_2 \,\theta\, b_2, \ldots,$ $a_{\sigma(f)} \,\theta\, b_{\sigma(f)}$ then $f(a_1, a_2, \ldots, a_{\sigma(f)}) \,\theta\, f(b_1, b_2, \ldots, b_{\sigma(f)})$. It is easy to see that the congruence relations on an algebra coincide with the kernels of homomorphisms on the algebra. The *congruence generated* by a set X of pairs of elements from an algebra $\mathbf{A}$ is the least congruence relation $Cg(X)$ containing X. The congruence generated by a singleton is called *principal*. The collection of all congruences on an algebra $\mathbf{A}$ forms a *lattice*, see definition below, denoted by $\mathbf{L}(\mathbf{A})$. Every non-trivial algebra has at least two congruences; the *universal congruence* A^2 and the *diagonal congruence* $\{(a, a) \mid a \in A\}$. If an algebra has exactly two congruences it is called *simple*. The class of all simple algebras of a class $\mathcal{K}$ is denoted by $\mathcal{K}_{Si}$.

If $\mathbf{A} = \langle A, f_1, f_2, \ldots, f_n\rangle$ is an algebra and θ a congruence on $\mathbf{A}$, we define the algebra $\mathbf{A}/\theta$ of the same similarity type as $\mathbf{A}$, with underlying set the set of all θ-congruence blocks $[a]_\theta$, $a \in A$, and fundamental operations $f_1^{\mathbf{A}/\theta}, \ldots, f_n^{\mathbf{A}/\theta}$, defined by $f_i^{\mathbf{A}/\theta}([a_1]_\theta, \ldots, [a_{\sigma(f_i)}]_\theta) = [f_i^{\mathbf{A}}(a_1, \ldots, a_{\sigma(f_i)})]_\theta$, for all $i \in \{1, 2, \ldots, n\}$ - the fact that θ is a congruence guarantees that the operations are well-defined. The algebra $\mathbf{A}/\theta$ is called the *quotient algebra* of $\mathbf{A}$ by θ.

A *subdirect product* of an indexed set $\mathcal{A} = \{\mathbf{A}_i \mid i \in I\}$ of algebras of a given similarity type $\mathcal{F}$, is a subalgebra $\mathbf{B}$ of the product of the algebras of $\mathcal{A}$, such that for every $i \in I$ and for every $a_i \in A_i$, there exists an element of B, whose i-th coordinate is a_i. In other words, the projection to the i-th coordinate map from B to A_i is onto. An non-trivial algebra is called *subdirectly irreducible*, if it is not a subdirect product of more than one non-trivial algebras. Looking at the kernels of the i-th projection maps, it can be seen that an algebra is subdirectly irreducible iff it has a minimum non-trivial congruence, called the *monolith*. The collection of all subdirectly irreducible members of a class of algebras $\mathcal{K}$ is denoted by $\mathcal{K}_{SI}$.

An *ultrafilter* over a set X is a *filter*, see definition below, in the power set $\mathcal{P}(X)$ of X that is maximal with respect to inclusion. If $\mathcal{A} = \{\mathbf{A}_i \,|\, i \in I\}$ is an indexed set of algebras of a given similarity type and $\mathcal{U}$ is an ultrafilter over the index set I, then the binary relation $\theta_{\mathcal{U}}$ on the product $\mathbf{P}$ of the algebras of $\mathcal{A}$, defined by $\langle a_i\rangle_{i\in I} \,\theta_{\mathcal{U}} \langle b_i\rangle_{i\in I}$ iff $\{i \in I \,|\, a_i = b_i\} \in \mathcal{U}$, is a congruence on $\mathbf{P}$. The quotient algebra $\mathbf{P}/\theta_{\mathcal{U}}$ is called the *ultraproduct* of $\mathcal{A}$ over the ultrafilter $\mathcal{U}$. The class of all ultraproducts of collections of algebras from a class $\mathcal{K}$ is denoted by $\mathbf{P}_u(\mathcal{K})$.

The ultraproduct construction preserves the *validity* of first-order formulas over the similarity type $\mathcal{F}$. A celebrated result due to B. Jónsson, known as *Jónsson's Lemma*, states that if a variety $\mathcal{V}$ is *congruence distributive*, i.e., the congruence lattice of every algebra is *distributive*, see definition below, then the subdirectly irreducible algebras of $\mathcal{V}$ are homomorphic images of subalgebras of ultraproducts of algebras of $\mathcal{V}$; in symbols $\mathcal{V}_{SI} \subseteq \mathbf{HSP}_u(\mathcal{V})$.

If $\mathcal{K}$ is a class of algebras we denote by $\mathbf{S}(\mathcal{K})$, $\mathbf{H}(\mathcal{K})$ and $\mathbf{P}(\mathcal{K})$ the classes of all algebras that are isomorphic to a subalgebra, a homomorphic image and a product of algebras of $\mathcal{K}$, respectively. A class of algebras is called a *variety* , if it is closed under the three operators $\mathbf{S}, \mathbf{H}$ and $\mathbf{P}$. We denote the composition $\mathbf{HSP}$ by $\mathbf{V}$. It is not hard to prove that a class $\mathcal{V}$ of algebras is a variety iff $\mathcal{V} = \mathbf{V}(\mathcal{V})$. Moreover, given a class $\mathcal{K}$ of similar algebras, the smallest variety containing $\mathcal{K}$ is $\mathbf{V}(\mathcal{K})$, the *variety generated* by $\mathcal{K}$. If $\mathcal{K} = \{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n\}$, we write $\mathbf{V}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n)$ for $\mathbf{V}(\mathcal{K})$.

Let X be a set of *variables*, $\mathcal{F}$ a similarity type and $(X \cup F)^*$ the set of all finite sequences of elements of $X \cup F$. The set $T_{\mathcal{F}}(X)$ of *terms* in $\mathcal{F}$ over X is the least subset of $(X \cup F)^*$ that contains X and if $f \in F$ and $t_1, t_2, \ldots, t_{\sigma(f)} \in T_{\mathcal{F}}(X)$, then the sequence $ft_1t_2 \ldots t_{\sigma(f)}$ is in $T_{\mathcal{F}}(X)$. Usually, we omit the set of variables and write $T_{\mathcal{F}}$, if it is understood or of no particular importance. Frequently, we will take the set of variables to be (bijective to) the set $\mathbb{N}$ of natural numbers. The *set of variables* $Var(t)$ of a term t in $\mathcal{F}$ over X - we avoid the clear inductive definition - is the indexed subset of variables of X that occur in t. The *term algebra* $\mathbf{T}_{\mathcal{F}}$ in $\mathcal{F}$ over X is the algebra with underlying set $T_{\mathcal{F}}$, similarity type $\mathcal{F}$ and fundamental operations $f^{\mathbf{T}_{\mathcal{F}}}$, for $f \in F$, defined by $f^{\mathbf{T}_{\mathcal{F}}}(t_1, t_2, \ldots, t_{\sigma(f)}) = ft_1t_2 \cdots t_{\sigma(f)}$, for all $t_i \in T_{\mathcal{F}}$.

If $\mathbf{A}$ is an algebra of type $\mathcal{F}$, t a term in $\mathcal{F}$ over a set of variables X and $Var(t) = \{x_1, x_2, \ldots, x_n\}$, we define the *evaluation*, or *term operation* $t^{\mathbf{A}}$ of t inductively on the subterms of t to be the operation from on A of arity n defined as follows: $x_i^{\mathbf{A}}$ is the i-th projection operation on A^n, and if $s = ft_1t_2 \ldots t_{\sigma(f)}$, where $f \in F$ and $t_1, t_2, \ldots, t_{\sigma(f)} \in T_{\mathcal{F}}$, then $s^{\mathbf{A}}$ is defined by $s^{\mathbf{A}}(a_1, a_2, \ldots, a_n) = f^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, a_2, \ldots, a_n), t_2^{\mathbf{A}}(a_1, a_2, \ldots, a_n), \ldots, t_{\sigma(f)}^{\mathbf{A}}(a_1, a_2, \ldots, a_n))$. If $t, t_1, t_2, \ldots, t_n$ are terms of $\mathbf{T}_{\mathcal{F}}$ and $n = |Var(t)|$, then the *substitution* of $t_1, t_2, \ldots, t_n$ into t is defined to be the element $t^{\mathbf{T}_{\mathcal{F}}}(t_1, t_2, \ldots, t_n)$. We also allow for substitutions of fewer terms than the variables. If $\mathbf{A}$ is an algebra of type $\mathcal{F}$ and t a term in $\mathcal{F}$, then the operation $t^{\mathbf{A}}$ on A is called a *term operation*. Two algebras of possibly different similarity types are called *term equivalent*, if every fundamental operation of one is a term operation of the other.

An *equation* in the similarity type $\mathcal{F}$ over a variable set X is a pair of terms of $T_{\mathcal{F}}$. If t, s are terms we write $t \approx s$ for the equation they define, instead of (t, s). We say that an equation $t \approx s$ in $\mathcal{F}$ over X is *valid* in an algebra $\mathbf{A}$ of type $\mathcal{F}$, or an *identity* of $\mathbf{A}$, or that it is *satisfied* by $\mathbf{A}$, in symbols $\mathbf{A} \models t \approx s$, if $t^{\mathbf{A}} = s^{\mathbf{A}}$. The notion of validity is extended to classes of algebras and sets of equations. A set $\mathcal{E}$ of equations in a type $\mathcal{F}$ is said to be *valid* in, a *set of identities* of, or *satisfied* by a class $\mathcal{K}$ of algebras of type $\mathcal{F}$, in symbols $\mathcal{K} \models \mathcal{E}$, if every equation of $\mathcal{E}$ is valid in every algebra of $\mathcal{K}$.

It is easy to see that if an equation is valid in an algebra then it also valid in any subalgebra and in any homomorphic image of the algebra. Moreover, if an equation is valid in a set of algebras then it is valid in their product. In other words, equations are preserved by the operators $\mathbf{S}, \mathbf{H}$ and $\mathbf{P}$.

A *theory* of equations, or *equational theory* $\mathcal{T}$ in a similarity type $\mathcal{F}$ is a congruence on $\mathbf{T}_{\mathcal{F}}$ closed under substitutions, i.e., if $(t \approx s) \in \mathcal{T}$, $Var(t) \cup Var(s) = \{x_1, x_2, \ldots, x_n\}$, and $t_1, t_2, \ldots, t_n \in T_{\mathcal{F}}$, then $(t^{\mathbf{T}_{\mathcal{F}}}(t_1, t_2, \ldots, t_n) \approx s^{\mathbf{T}_{\mathcal{F}}}(t_1, t_2, \ldots, t_n)) \in \mathcal{T}$. It is easy to see that if $\mathcal{K}$ is a class of algebras of similarity type $\mathcal{F}$, then $Th_{Eq}(\mathcal{K}) = \{(t \approx s) \in T_{\mathcal{F}} \mid \mathcal{K} \models t \approx s\}$ is an equational theory, called the *equational theory of* $\mathcal{K}$.

Given a set $\mathcal{E}$ of equations of a similarity type $\mathcal{F}$ the *equational class* axiomatized by $\mathcal{E}$ is defined to be the class $\mathrm{Mod}(\mathcal{E}) = \{\mathbf{A} \mid \mathbf{A} \models \mathcal{E}\}$ of algebras of type $\mathcal{F}$, that satisfy all equations of $\mathcal{E}$; the set $\mathcal{E}$ is called an *equational basis* for $\mathrm{Mod}(\mathcal{E})$. By previous observations, every variety is an equational class. G. Birkhoff's celebrated HSP-theorem of establishes that every equational class is a variety. Moreover, for every variety $\mathcal{V}$ of similar algebras, we have that $\mathrm{Mod}(\mathrm{Th}_{\mathrm{Eq}}(\mathcal{V})) = \mathcal{V}$, and for every theory $\mathcal{T}$ of equations in a given type, $Th_{Eq}(\mathrm{Mod}(\mathcal{T})) = \mathcal{T}$.

If $\mathcal{K}$ is a class of algebras of similarity type $\mathcal{F}$, then the quotient algebra $\mathbf{F}_{\mathcal{K}}(X) = \mathbf{T}_{\mathcal{F}}(X)/Th_{Eq}(\mathcal{K})$ is called the free algebra for $\mathcal{K}$ over X and has the following universal property: every map from X to an algebra $\mathbf{A}$ of $\mathcal{K}$ can be extended, in a unique way, to a homomorphism from $\mathbf{F}_{\mathcal{K}}(X)$ to $\mathbf{A}$. It can be shown that if $\mathcal{V}$ is a variety then $\mathbf{F}_{\mathcal{V}}(X)$ is in $\mathcal{V}$.

A *subvariety* is a subclass of a variety that is a variety. The class of all subvarieties of a variety $\mathcal{V}$ of algebras of type $\mathcal{F}$ is a set bijective to the set of all subtheories of $T_{\mathcal{F}}(\mathbb{N})$, that contain the theory $Th_{Eq}(\mathcal{V})$. Both of these sets form *lattices*, see definition below, under inclusion that are dually isomorphic. We denote the *lattice of subvarieties*, or *subvariety lattice* of a variety $\mathcal{V}$, by $\mathbf{L}(\mathcal{V})$. Note that $\mathbf{L}(\mathcal{V}) = \mathbf{L}(\mathbf{F}_{\mathcal{V}}(\mathbb{N}))$.

Order and lattice theory

Basic definitions and results in order and lattice theory can be found in [Gr].

A *(partial) order* relation $\leq$ on a set P is a subset of P^2 such that for all $x, y, z \in P$, (we write $x \leq y$ for $(x, y) \in \leq$)

1. $x \leq x$;

2. if $x \leq y$, then $y \leq x$; and

3. if $x \leq y$ and $y \leq z$, then $x \leq z$.

A *partially ordered set* or *poset* $\mathbf{P}$ is a set P with a partial order $\leq$ on it; $\mathbf{P} = \langle P, \leq \rangle$. It is easy to see that given a partial order $\leq$, the *converse* relation $\geq$ is also an order. The poset $\mathbf{P}^\partial = \langle P, \geq \rangle$ is called the *dual* of $\mathbf{P} = \langle P, \leq \rangle$. A subset X of P is called *increasing*, an *upset*, or an *order filter* if $p \in X$, whenever $x \leq p$, for some $x \in X$. A *decreasing* set, an *downset*, or an *order ideal* is the dual concept. The *interval* $[x, y]$ in $\mathbf{P}$ is defined to be the set $\{z \in P \mid x \leq z \leq y\}$.

An *upper bound* of a set X of elements in a poset $\mathbf{P}$ is an element p of P, such that $x \leq p$, for all $x \in X$. A *lower bound* is an upper bound of X in the dual poset. If there exists a least upper bound for a set X of elements in a poset $\mathbf{P}$, then it is called the *join* $\bigvee X$ of X. The greatest lower bound of X, if it exists, is called the *meet* $\bigwedge X$ of X. If X is a doubleton $\{x, y\}$, we denote its join by $x \vee y$ and its meet by $x \wedge y$. A *lattice* $\mathbf{L}$ is a poset, such that every pair of elements $x, y \in L$, has a join and a meet. In this case, the meet and the join can be considered as binary operations on L. The algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is also called a lattice . Every lattice satisfies the following equations:

1. $x \wedge x \approx x \approx x \vee x$;

2. $x \wedge y \approx y \wedge x$ and $x \vee y \approx y \vee x$; and

3. $x \wedge (x \vee y) \approx x \approx x \vee (x \wedge y)$.

It can be shown that if an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ satisfies these identities, then $\langle L, \leq \rangle$, where $x \leq y$ iff $x = x \wedge y$, is a lattice. We will be considering lattices as algebraic objects and think of the order as an auxiliary expressive tool, as defined above.

A *(lattice) ideal* in a lattice is an order ideal that is closed under joins. Obviously, a lattice ideal is a sublattice. The notion of a *(lattice) filter* is defined dually. A proper ideal I is called *prime*, if for every pair of elements x, y, $x \in I$ or $y \in I$, whenever $x \wedge y \in I$. The dual concept is that of a *prime filter*. The *Prime Ideal Theorem* states that if $I \cap F = \emptyset$, for an ideal I and filter F, then there exists a prime ideal J that contains I and $J \cap F = \emptyset$.

If $\mathbf{P} = \langle P, \leq_{\mathbf{P}} \rangle$ and $\mathbf{Q} = \langle Q, \leq_{\mathbf{Q}} \rangle$ are posets and f is a map from P to Q, then f is said to *preserve the order*, or that to be *order preserving*, if for all $x, y \in P$, $f(x) \leq_{\mathbf{Q}} f(y)$, whenever $x \leq_{\mathbf{P}} y$.

A *closure operator* on a lattice $\mathbf{L}$ is a map $\gamma : L \to L$, that satisfies the following conditions:

1. γ is extensive: $x \leq \gamma(x)$, for all $x \in L$.

2. γ is monotone: if $x \leq y$, then $\gamma(x) \leq \gamma(y)$, for all $x, y \in L$.

3. γ is idempotent: $\gamma(\gamma(x)) = \gamma(x)$, for all $x \in L$.

An *interior operator* on a lattice $\mathbf{L}$ is a map $\delta : L \to L$, that satisfies the following conditions:

1. δ is contracting: $\delta(x) \leq x$, for all $x \in L$.

2. δ is monotone: if $x \leq y$, then $\delta(x) \leq \delta(y)$, for all $x, y \in L$.

3. δ is idempotent: $\delta(\delta(x)) = \delta(x)$, for all $x \in L$.

We denote the image of an idempotent operator α on a lattice $\mathbf{L}$, by L_α. Note that $x \in L_\alpha$ iff $x = \alpha(x)$.

Residuation

For background in residuation theory we refer the reader to [Ro].

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset. A map $f : P \to P$ is called *residuated* if there exists a map $f^* : P \to P$, such that for all $x, y \in P$,

$$f(x) \leq y \;\Leftrightarrow\; x \leq f^*(y).$$

In this case, f^* is called the *residual* of f. It is not hard to see that if f is residuated then it preserves the order and existing joins. Note that if f^* is the residual of f, then $f^* \circ f$ is a closure operator and $f \circ f^*$ is an interior operator.

Let U be a set and $S \subseteq U^2$, a binary relation on U. For every subset X of U, we set $S[X] = S[X, \_\,] = \{y \in U \mid x \, S \, y, \text{ for some } x \in X\}$ and $S[\_\,, X] = \{y \in U \mid y \, S \, x, \text{ for some } x \in X\}$. We define the maps f_S, g_S on the power set of U, by $f_S(X) = S[X]$ and $g_S(X) = \{y \in U \mid S[\_\,, \{y\}] \subseteq X\}$. It is not hard to see that both f_S, g_S are residuated and that $f_S^*(X) = S[\_\,, X]$ and $g_S^*(X) = \{y \in U \mid S[\{y\}, \_\,] \subseteq X\}$.

A binary operation $*$ on a poset $\mathbf{P} = \langle P, \leq \rangle$ is called *residuated* if the maps l_x and r_x on P, defined by $l_x(y) = x * y$ and $r_x(y) = y * x$, are residuated, for all $x \in P$, i.e., if there exist binary operations $\backslash$ and $/$ on P, such that for all $x, y, z \in P$

$$x * y \leq z \;\Leftrightarrow\; y \leq x \backslash z \;\Leftrightarrow\; x \leq z / y.$$

Let U be a set and $R \subseteq U^3$, a ternary relation on U. We write $R(x, y, z)$ for $(x, y, z) \in R$ and $R[X, Y, \_\,]$ for $\{z \in P \mid R(x, y, z), \text{ for some } x \in X,\ y \in Y\}$. For X, Y subsets of U, we define the binary relations on U, $R_X = \{(y,z) \in P^2 \mid R(x,y,z)$, for some $x \in X\}$ and $R^Y = \{(x,z) \in P^2 \mid R(x,y,z)$, for some $y \in Y\}$, and the binary operation on the power set of U, $X * Y = R[X, Y, \_]$. It is easy to see that $*$ is residuated and the associated *residuals*, or division operations are $X \backslash Z = f^*_{R_X}(Z)$ and $Z/Y = f^*_{R^Y}(Z)$.

CHAPTER III

RESIDUATED LATTICES

We begin with the definition of residuated lattices and a list of their basic properties.

Definition

A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice; $\langle L, \cdot, e\rangle$ is a monoid; and for all $a, b, c \in L$,

$$a \cdot b \leq c \;\Leftrightarrow\; a \leq c/b \;\Leftrightarrow\; b \leq a\backslash c.$$

It is not hard to see that $\mathcal{RL}$, the class of all residuated lattices, is a variety and the identities

$$\begin{array}{lll} x \approx x \wedge (xy \vee z)/y, & x(y \vee z) \approx xy \vee xz, & (x/y)y \vee x \approx x \\ y \approx y \wedge x\backslash(xy \vee z), & (y \vee z)x \approx yx \vee zx, & y(y\backslash x) \vee x \approx x \end{array}$$

together with the monoid and the lattice identities form an equational basis for it.

In a residuated lattice term, multiplication has priority over the division operations, which, in turn, have priority over the lattice operations. So, for example, $x/yz \wedge u\backslash v$ means $[x/(yz)] \wedge (u\backslash v)$. We will be using the inequalitiy $t \leq s$ instead of the equalities $t = t \wedge s$ and $t \vee s = s$ to simplify the presentation, whenever appropriate.

The following lemma contains a number of identities useful in algebraic manipulations of residuated lattices. The proof can be found in [BT] and is left to the reader.

Lemma 3.1. *[BT] Residuated lattices satisfy the following identities:*

1. $x(y \vee z) \approx xy \vee xz$ *and* $(y \vee z)x \approx yx \vee zx$

2. $x\backslash(y \wedge z) \approx (x\backslash y) \wedge (x\backslash z)$ *and* $(y \wedge z)/x \approx (y/x) \wedge (z/x)$

3. $x/(y \vee z) \approx (x/y) \wedge (x/z)$ *and* $(y \vee z)\backslash x \approx (y\backslash x) \wedge (z\backslash x)$

4. $(x/y)y \leq x$ *and* $y(y\backslash x) \leq x$

5. $x(y/z) \leq xy/z$ *and* $(z\backslash y)x \leq z\backslash yx$

6. $(x/y)/z \approx x/(zy)$ *and* $z\backslash(y\backslash x) \approx (yz)\backslash x$

7. $(x/y)(y/z) \leq x/z$ *and* $(z\backslash y)(y\backslash x) \leq z\backslash x$

8. $x/y \leq (x/z)/(y/z)$ *and* $y\backslash x \leq (z\backslash y)\backslash(z\backslash x)$

9. $x/y \leq (z/x)\backslash(z/y)$ *and* $y\backslash x \leq (y\backslash z)/(x\backslash z)$

10. $x/y \leq xz/yz$ *and* $y\backslash x \leq zy\backslash zx$

11. $x \leq y/(x\backslash y)$ *and* $x \leq (y/x)\backslash y$

12. $x\backslash(y/z) \approx (x\backslash y)/z$

13. $x/e \approx x \approx e\backslash x$

14. $e \leq x/x$ *and* $e \leq x\backslash x$

15. $x(x\backslash x) \approx x \approx (x/x)x$

16. $(x\backslash x)^2 \approx (x\backslash x)$ *and* $(x/x)^2 \approx (x/x)$

Moreover, if a residuated lattice has a bottom element 0*, then it also has a top element* 1 *and for every element* a*, we have:*

(i) $a0 = 0a = 0$,

(ii) $a/0 = 0\backslash a = 1$ *and*

(iii) $1/a = a\backslash 1 = 1$.

It follows from (1), (2) and (3) of the lemma above that multiplication is order-preserving and that the two divisions are order-preserving in the numerator and order-reversing in the denominator. Moreover, it is shown in [BT] that the distribution in (1) and (2) holds for all existing meets and joins.

It is not difficult to see that the last condition in the definition of a residuated lattice is equivalent to the stipulation that multiplication is order-preserving and for any two elements y, z, the join of each of the sets $\{x \mid xy \leq z\}$ and $\{x \mid yx \leq z\}$ exists and is equal to z/y and $y\backslash z$, respectively.

The *dual* of a residuated lattice equation is the equation obtained by interchanging the two lattice operations. By the *opposite* of a residuated lattice equation we understand the "mirror image" of it, namely the equation written in reverse order, where the two division operations are interchanged. Examples of the opposite of an equation can be seen in properties (4)-(11) of Lemma 3.1; property (12) is self-opposite up to a permutation of the variables.

A residuated lattice is called *commutative* (respectively, *cancellative*, *idempotent*, *n-potent*), when its monoid reduct is commutative (respectively, cancellative, idempotent, n-potent). A residuated lattice is called *distributive* if it has a distributive lattice reduct; it is called *integral* if its lattice reduct is upper bounded by the multiplicative identity. Note that if a residuated lattice is commutative the two divisions operations coincide (each one is the opposite of the other). In this case we denote the element $x\backslash y = y/x$ by $x \to y$.

A *residuated bounded-lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, 0\rangle$ such that $\mathbf{L} = \langle L, \wedge, \vee, \cdot, e, \backslash, /\rangle$ is a residuated lattice and the nullary operation, 0, satisfies $x \vee 0 \approx x$. Note that $1 = 0\backslash 0 = 0/0$ is the top element of such an algebra, so the constant 1 can be conservatively added to the type.

An element a in a residuated lattice $\mathbf{L}$ is called *invertible*, if there exists an element a^{-1} such that $aa^{-1} = e = a^{-1}a$; a is called *integral*, if $e/a = a\backslash e = e$. We denote the set of invertible elements of $\mathbf{L}$ by $G(\mathbf{L})$ and the set of integral elements by $I(\mathbf{L})$. It is easy to see that a is invertible iff $a(a\backslash e) = e = (e/a)a$; in this case, $a^{-1} = e/a = a\backslash e$.

To establish an equality between two elements a, b of a residuated lattice, we will frequently prove that $x \leq a \ \Leftrightarrow \ x \leq b$, for every element x. By setting $x = a$, we have $a \leq b$. On the other hand, by setting $x = b$, we obtain $b \leq a$.

Lemma 3.2. *If a is invertible, then for all x, y we have*

1. $x/a = xa^{-1}$ *and* $a\backslash x = a^{-1}x$;
2. $a(x \wedge y) = ax \wedge ay$ *and* $(x \wedge y)a = xa \wedge ya$;
3. $a\backslash a = e$ *and* $a/a = e$;
4. $(x/a)y = x(a\backslash y)$; *and*
5. $a(a^{-1}x/a^{-1}y) = (x/y)a$ *and* $a(a^{-1}y\backslash a^{-1}x) = a(y\backslash x)$.

Moreover, (4) implies that a is invertible.

Proof. 1) For every element z we have

$$z \leq x/a \ \Leftrightarrow \ za \leq x \ \Leftrightarrow \ z \leq xa^{-1},$$

so $x/a = xa^{-1}$. Similarly, we get the opposite equality $a\backslash x = a^{-1}x$.

2) We have $a(x \wedge y) \leq ax, ay$, so $a(x \wedge y) \leq ax \wedge ay$. For the reverse inequality, note that

$$a^{-1}(ax \wedge ay) \leq a^{-1}ax \wedge a^{-1}ay = x \wedge y,$$

hence $ax \wedge ay \leq a(x \wedge y)$. Similarly we get the opposite equality.

3) This is a direct consequence of (1).

4) If a is invertible, then $(x/a)y = xa^{-1}y = x(a\backslash y)$. Conversely, if we set $x = e$ and $y = a$ in $(x/a)y = x(a\backslash y)$, we get $(e/a)e = a\backslash a$. Since, by Lemma 3.1(4) and (14), $(e/a)a \leq e$ and $a\backslash a \geq e$, we obtain $(e/a)a = a\backslash a = e$. Similarly, $(a\backslash e)a = e$.

5) For every z, we have

$$\begin{aligned} z \leq a(a^{-1}x/a^{-1}y) \quad &\Leftrightarrow\ a^{-1}z \leq a^{-1}x/a^{-1}y \\ &\Leftrightarrow\ a^{-1}za^{-1}y \leq a^{-1}x \\ &\Leftrightarrow\ za^{-1}y \leq x \\ &\Leftrightarrow\ za^{-1} \leq x/y \\ &\Leftrightarrow\ z \leq (x/y)a \end{aligned}$$

The opposite equation follows, since the definition of an invertible element is self-opposite.

□

Examples

In what follows we give a list of examples of residuated lattices, with the goal of enhancing the intuition of the reader.

<u>Known algebraic structures</u>

As mentioned before, residuated lattices generalize a class of well studied algebraic structures. In what follows we mention a few.

Example 3.1. *Lattice-ordered groups*

A *lattice-ordered group* or *ℓ-group* is an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, {}^{-1}, e\rangle$, such that $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, \cdot, {}^{-1}, e\rangle$ is a group and multiplication is order preserving. It can be easily shown, see [AF], that the last requirement is equivalent to the stipulation that multiplication distributes over binary meets and/or joins, hence the class of all ℓ-groups is a variety. It is easy to see that this variety is term equivalent to the subvariety of residuated lattices axiomatized by the identity $(e/x)x \approx e$, via $x^{-1} = e/x$ and $x/y = xy^{-1}$, $y\backslash x = y^{-1}x$, see Lemma 4.3.

Example 3.2. ***Generalized Boolean algebras***

A *generalized Boolean algebra* $\mathbf{B}$ is a lattice such that every principal filter is a Boolean algebra. We include in the type symbols for the lattice operations, the top element e and the binary operation that, given x, y in B, produces the complement of x in the Boolean algebra $[y, e]$. It is shown in Proposition 5.3 that the class of generalized Boolean algebras is term equivalent to the subvariety $\mathcal{GBA}$ of $\mathcal{RL}$ generated by $\mathbf{2}$, the two-element residuated lattice. This variety is shown to be an atom in the lattice of subvarieties of residuated lattices and an equational basis is provided for it.

Example 3.3. ***Brouwerian algebras***

A generalized Boolean algebra is a special case of a *Brouwerian algebra*. The latter is a lattice such that for any pair of elements x, y there exits an element z, which is maximum with respect to the property $x \wedge z \leq y$. This element is denoted by $x \rightarrow y$ and it is called the *relative pseudo-complement* of x with respect to y. As in the case of generalized Boolean algebras, the lattice operations, the top element and the relative pseudo-complement are considered as fundamental operations of the Brouwerian algebra. It is easy to see that the class of Brouwerian algebras is term equivalent to the subvariety $\mathcal{B}r$ of residuated lattices axiomatized by the equation $x \cdot y \approx x \wedge y$ and that the only atom below $\mathcal{B}r$ is $\mathcal{GBA}$. For a study of Brouwerian algebras we refer the reader to [BD].

Example 3.4. ***Reducts of MV-algebras***

MV-algebras are algebraic models of multi-valued logic. Among many term equivalent definitions, we chose the one given in the setting of residuated lattices. An MV-algebra is a commutative residuated bounded-lattice that satisfies the identity $(x \rightarrow y) \rightarrow y \approx x \vee y$, the relativized law of double negation. MV-algebras are generalizations of Boolean algebras and have been studied extensively; see [COM], [Ha] and [Mu]. In Chapter 7, we investigate a common generalization of MV-algebras and ℓ-groups.

Example 3.5. ***Reducts of relation algebras***

A *relation algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, {}^-, 0, 1, \cdot, e, {}^\cup \rangle$, such that $\langle A, \wedge, \vee, {}^-, 0, 1 \rangle$ is a Boolean algebra, $\langle A, \cdot, e \rangle$ is a monoid and for all $a, b, c \in A$

(i) $(a^\cup)^\cup = a$, $(ab)^\cup = b^\cup a^\cup$;

(ii) $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$, $(a \vee b)^\cup = a^\cup \vee b^\cup$; and

(iii) $a^\cup (ab)^- \leq b^-$.

The structure $\mathcal{R}(\mathbf{A}) = \langle A, \wedge, \vee, \cdot, \backslash, /, e\rangle$, where $a\backslash b = (a^{\cup}b^{-})^{-}$ and $b/a = (b^{-}a^{\cup})^{-}$ is a residuated lattice. The only thing to be checked is that the division operations are the residuals of multiplication, i.e., the last condition in the definition of a residuated lattice. If $ab \leq c$ then $c^{-} \leq (ab)^{-}$. So, $a^{\cup}c^{-} \leq a^{\cup}(ab)^{-} \leq b^{-}$, by (iii); hence $b \leq (a^{\cup}c^{-})^{-}$. On the other hand, if $b \leq (a^{\cup}c^{-})^{-}$, then $ab \leq a(a^{\cup}c^{-})^{-} \leq c$, by (iii), and the idempotency of $^{\cup}$ and $^{-}$.

General constructions on residuated lattices

Before proceeding to concrete examples, we mention some general constructions on existing residuated lattices that produce new ones.

Example 3.6. ***Subalgebras, products and homomorphic images***

As mentioned before, the class of all residuated lattices is equationally definable. Thus, by Birkhoff's Theorem, it is a variety, namely it is closed under the operations of subalgebras, products and homomorphic images.

Example 3.7. ***The negative cone***

The *negative cone* of a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is defined to be the algebra $\mathbf{L}^{-} = \langle L^{-}, \wedge, \vee, \cdot, \backslash_{\mathbf{L}^{-}}, /_{\mathbf{L}^{-}}, e\rangle$, where $L^{-} = \{x \in L \mid x \leq e\}$, $x\backslash_{\mathbf{L}^{-}}y = x\backslash y \wedge e$ and $x/_{\mathbf{L}^{-}}y = x/y \wedge e$. It is easy to check that $\mathbf{L}^{-}$ is also a residuated lattice, which is obviously integral. If $\mathcal{K}$ is a class of residuated lattices, we denote the class of negative cones of elements of $\mathcal{K}$ by $\mathcal{K}^{-}$.

Example 3.8. ***The opposite residuated lattice***

The *opposite* of a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is the algebra $\mathbf{L}^{op} = \langle L, \wedge, \vee, \cdot^{op}, \backslash^{op}, /^{op}, e\rangle$, where $x \cdot^{op} y = y \cdot x$, $x/^{op}y = y\backslash x$ and $y\backslash^{op}x = x/y$. The opposite of a residuated lattice is also a residuated lattice, because the defining identities of residuated lattices are self-opposite. We will use this symmetry frequently to obtain proofs of the opposites of already proved identities.

Example 3.9. [Ro] ***Nuclei retractions***

We first define an important notion in the context of residuated lattices.

A *nucleus* on a residuated lattice $\mathbf{L}$ is a closure operator γ on $\mathbf{L}$ such that $\gamma(a)\gamma(b) \leq \gamma(ab)$, for all $a, b \in L$.

The concept of a nucleus is not new to ordered algebraic structures. It was defined in the context of Brouwerian algebras, see [ST]. Recall that L_γ is the image of L under γ.

Lemma 3.3. *[Ro] If γ is a closure operator on a residuated lattice* $\mathbf{L}$*, then the following statements are equivalent:*

1. *γ is a nucleus.*

2. *$\gamma(\gamma(x)\gamma(y)) = \gamma(xy)$, for all $x, y \in L$.*

3. *x/y, $y\backslash x \in L_\gamma$, for all $x \in L_\gamma$, $y \in L$.*

Proof. (1) $\Rightarrow$ (2): Let $x \in L_\gamma$ and $y \in L$. Since γ is extensive and monotone, we have $\gamma(xy) \leq \gamma(\gamma(x)\gamma(y))$. On the other hand, by the defining property of a nucleus and monotonicity, we have $\gamma(\gamma(x)\gamma(y)) \leq \gamma(\gamma(xy))$. So, $\gamma(\gamma(x)\gamma(y)) \leq \gamma(xy)$, since γ is idempotent.

(2) $\Rightarrow$ (3): Since $x \in L_\gamma$, we get $\gamma(x) = x$. So,

$$\begin{aligned}
\gamma(x/y) \cdot y \quad &\leq \gamma(\gamma(x/y) \cdot \gamma(y)) && (\gamma \text{ is extensive}) \\
&= \gamma((x/y) \cdot y) && (2) \\
&\leq \gamma(x) && (\text{Lemma 3.1(4) and monotonicity}) \\
&= x.
\end{aligned}$$

So, $\gamma(x/y) \leq x/y$, by the defining property of residuated lattices. Since the reverse inequality follows by the extensivity of γ, we have $x/y = \gamma(x/y) \in L_\gamma$. Similarly, we get the result for the other division operation.

(3) $\Rightarrow$ (1): Since γ is extensive, $xy \leq \gamma(xy)$, so $x \leq \gamma(xy)/y$. By the monotonicity of γ and the hypothesis, we have $\gamma(x) \leq \gamma(xy)/y$. Using the defining property of residuated lattices, we get $y \leq \gamma(x)\backslash\gamma(xy)$. Invoking the monotonicity of γ and the hypothesis, once more, we obtain $\gamma(y) \leq \gamma(x)\backslash\gamma(xy)$, namely $\gamma(x)\gamma(y) \leq \gamma(xy)$. □

Actually, it can be shown that an arbitrary map γ on a residuated lattice $\mathbf{L}$ is a nucleus if and only if $\gamma(a)/b = \gamma(a)/\gamma(b)$ and $b\backslash\gamma(a) = \gamma(b)\backslash\gamma(a)$, for all $a, b \in L$.

If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice and γ a nucleus on $\mathbf{L}$, then the algebra $\mathbf{L}_\gamma = \langle L_\gamma, \wedge, \vee_\gamma, \circ_\gamma, \backslash, /, \gamma(e)\rangle$, where $x \circ_\gamma y = \gamma(x \cdot y)$ and $x \vee_\gamma y = \gamma(x \vee y)$, is called the *$\gamma$-retraction* of $\mathbf{L}$.

Proposition 3.4. *[Ro] If* $\mathbf{L}$ *is a residuated lattice and γ a nucleus on it, then the γ-retraction* $\mathbf{L}_\gamma$ *of* $\mathbf{L}$ *is a residuated lattice.*

Proof. Obviously, $\gamma(e)$ is the multiplicative identity of $\mathbf{L}_\gamma$ and $\mathbf{L}_\gamma$ is closed under $\circ_\gamma$ and $\wedge_\gamma$. By Lemma 3.3, it is also closed under the division operations. To prove that $\mathbf{L}_\gamma$ is closed under meets, note that for $x, y \in L_\gamma$, $\gamma(x \wedge y) \leq \gamma(x) \wedge \gamma(y) = x \wedge y$. The reverse inequality follows by the fact that γ is extensive, so $x \wedge y \in L_\gamma$. Thus, $\mathbf{L}_\gamma$ is closed under all operations, and it is a meet-subsemilattice of $\mathbf{L}$.

To show that $\mathbf{L}_\gamma$ is a lattice note that for elements $x, y, z \in L_\gamma$, $x, y \leq z$ is equivalent to $x \vee y \leq z$. Since $\gamma(z) = z$ and γ is extensive, this is, in turn, equivalent to $\gamma(x \vee y) \leq z$, namely to $x \vee_\gamma y \leq z$. Thus, $\vee_\gamma$ is the join in $\mathbf{L}_\gamma$.

We next show that multiplication is associative. Let $x, y, z \in L_\gamma$. Using Lemma 3.3 and the definition of multiplication, we get

$$\begin{aligned}
(x \circ_\gamma y) \circ_\gamma z &= \gamma(x \cdot y) \circ_\gamma z \\
&= \gamma(\gamma(x \cdot y) \cdot z) \\
&= \gamma(\gamma(x \cdot y) \cdot \gamma(z)) \\
&= \gamma((x \cdot y) \cdot z) \\
&= \gamma(x \cdot y \cdot z).
\end{aligned}$$

Similarly,

$$x \circ_\gamma (y \circ_\gamma z) = \gamma(x \cdot y \cdot z).$$

Hence, multiplication in $\mathbf{L}_\gamma$ is associative and $\langle L, \circ_\gamma, \gamma(e) \rangle$ is a monoid.

Finally, to check that $\circ_\gamma$ is residuated, consider $x, y, z \in L_\gamma$. We have

$$\begin{aligned}
x \circ_\gamma y \leq z &\Leftrightarrow \gamma(x \cdot y) \leq z \\
&\Leftrightarrow x \cdot y \leq z \qquad (x \cdot y \leq \gamma(x \cdot y) \text{ and } z = \gamma(z)) \\
&\Leftrightarrow y \leq x \backslash z.
\end{aligned}$$

Likewise, $x \circ_\gamma y \leq z \;\Leftrightarrow\; x \leq z/y$. ☐

If $\mathbf{L}$ is an algebra on the signature of residuated lattices without the constant e, then the concept of nucleus can be defined as above. In that case $\mathbf{L}_\gamma$ defines a residuated lattice, provided that it has a multiplicative identity.

The preceding construction is quite general as it can be seen in the following known result.

Proposition 3.5. *[Ro] Every complete residuated lattice is a nucleus retraction of the power set of a monoid. (See Example 3.15.)*

Example 3.10. ***Retraction to an interval***

Let $\mathbf{L}$ be a residuated lattice and $a \in L$ such that $a \leq e$. The structure $\mathbf{L}_a = \langle [a, e], \wedge, \vee, \circ_a, \backslash_a, /_a, e \rangle$, where $x \circ_a y = xy \vee a$, $x \backslash_a y = (x \backslash y) \wedge e$ and $y /_a x = (y/x) \wedge e$, is a residuated lattice.

The map γ on $\mathbf{L}^-$, defined by $\gamma(x) = x \vee a$ is obviously a closure operator. To see that it is a nucleus, note that if $x, y \leq a$, then $xa, ya, a^2 \leq a$, so

$$\gamma(x) \cdot \gamma(y) = (x \vee a)(y \vee a) = xy \vee xa \vee ay \vee a^2 \leq xy \vee a = \gamma(xy).$$

It is easy to observe that $\mathbf{L}_a = (\mathbf{L}^-)_\gamma$, which is a residuated lattice, by Proposition 3.4. Note that if $a \in L^-$, then $\mathbf{L}_a = (\mathbf{L}^-)_a$.

It is known, see [Mu], that if $\mathbf{L}$ is a commutative ℓ-group and a is a negative element of $\mathbf{L}$, then $\mathbf{L}_a$ is an MV-algebra.

Example 3.11. ***Kernel contractions***

A *kernel* δ on a residuated lattice $\mathbf{L}$ is an interior operator such that for all x, y in L

1. $\delta(\delta(x)\delta(y)) = \delta(x)\delta(y)$,
2. $\delta(e) = e$ and
3. $\delta(x) \wedge y = \delta(\delta(x) \wedge y)$.

Let $\mathbf{L}$ be a residuated lattice and δ a kernel on it. The *δ-contraction* of $\mathbf{L}$ is the algebra $\mathbf{L}_\delta = \langle L_\delta, \wedge, \vee, \cdot, \backslash_\delta, /_\delta, e \rangle$, where $x /_\delta y = \delta(x/y)$ and $x \backslash_\delta y = \delta(x \backslash y)$.

Proposition 3.6. *The δ-contraction $\mathbf{L}_\delta$ of a residuated lattice $\mathbf{L}$ under a kernel δ on $\mathbf{L}$ is a residuated lattice. Moreover, $\mathbf{L}_\delta$ is a lattice-ideal of $\mathbf{L}$.*

Proof. Note that L_δ is closed under join, since δ is an interior operator, and under multiplication, by the first property of a kernel. Moreover, it contains e and it is obviously closed under $\backslash_\delta$ and $/_\delta$.

By the third property of a kernel and the fact that it is closed under joins, L_δ is an ideal of $\mathbf{L}$. So, $\mathbf{L}_\delta$ is closed under all the operations.

Finally, $\mathbf{L}_\delta$ is residuated. Indeed, for all $x, y, z \in L_\delta$, $x \leq z /_\delta y$ is equivalent to $x \leq \delta(x/y)$, which in turn is equivalent to $x \leq z/y$, since δ is contracting and $x = \delta(x)$. $\square$

Note that under the weaker assumptions $\delta(\delta(x)\delta(y)) = \delta(x)\delta(y)$ and $\delta(e)\delta(x) = \delta(x) = \delta(x)\delta(e)$ on δ, the algebra $\langle L_\delta, \wedge, \vee_\delta, \cdot, \backslash_\delta, /_\delta, \delta(e) \rangle$, where $x \vee_\delta y = \delta(x \vee y)$, is a residuated lattice.

The δ-contraction construction, where δ is a kernel, is a generalization of the negative cone construction, defined in Example 3.7. The negative cone of a residuated lattice is its δ-contraction, where $\delta(x) = x \wedge e$.

Example 3.12. ***The dual of a residuated lattice with respect to an element***

Let $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e \rangle$ be a residuated lattice and $a \in L$ a *dualizing element*, i.e., an element of L such that

$$x = a/(x\backslash a) = (a/x)\backslash a,$$

for all $x \in L$. Then, the *dual* of $\mathbf{L}$ with respect to the element a is the algebra $\mathbf{L}^{\partial_a} = \langle L, \vee, \wedge, +, \div, \dot{-}, a \rangle$, where $x + y = x/(y\backslash a)$, $x \dot{-} y = x(y\backslash a)$ and $x \div y = (a/x)y$. (Notice that the underlying lattice of $\mathbf{L}^{\partial_a}$ is the dual of the lattice reduct of $\mathbf{L}$.)

Proposition 3.7. *The dual $\mathbf{L}^{\partial_a}$ of a residuated lattice $\mathbf{L}$ with respect to a dualizing element a of $\mathbf{L}$ is also a residuated lattice.*

Proof. First observe that

$$\begin{aligned} x + y &= x/(y\backslash a) \\ &= ((a/x)\backslash a)/((y\backslash a) \\ &= (a/x)\backslash(a/(y\backslash a)) \\ &= (a/x)\backslash y \end{aligned}$$

and that $e = a/(e\backslash a) = (a/e)\backslash a$, i.e., $e = a/a = a\backslash a$.

It is obvious that $\langle L, \vee, \wedge \rangle$ is a lattice. Multiplication is associative because

$$\begin{aligned} (x + y) + z &= [(a/x)\backslash y]/(z\backslash a) \\ &= (a/x)\backslash[y/(z\backslash a)] \\ &= x + (y + z); \end{aligned}$$

and a is the additive identity since

$$x + a = x/(a\backslash a) = x/e = x$$

and

$$a + x = (a/a)\backslash x = e\backslash x = x.$$

Finally multiplication is residuated, since

$$\begin{aligned} x + y \leq_{\mathbf{L}^{\partial_a}} z & \Leftrightarrow x + y \geq_{\mathbf{L}} z \\ & \Leftrightarrow x/(y\backslash a) \geq_{\mathbf{L}} z \\ & \Leftrightarrow x \geq_{\mathbf{L}} z(y\backslash a) \\ & \Leftrightarrow x \geq_{\mathbf{L}} z \dot{-} y \\ & \Leftrightarrow x \leq_{\mathbf{L}^{\partial_a}} z \dot{-} y \end{aligned}$$

and similarly for $-\!\!\cdot$. $\square$

The dual is a generalization of a construction for MV-algebras. The dual of an MV-algebra with respect to its least element is known to be an MV-algebra.

Example 3.13. ***Translations with respect to an invertible element***

Let $\mathbf{L}$ be a residuated lattice, a an invertible element of L and f_a the map on L defined by $f_a(x) = ax$. Note that the map f_a is invertible and $f_a^{-1}(x) = a^{-1}x$. Consider the structure $\mathbf{L}^a = \langle L, \wedge^a, \vee^a, \cdot^a, \backslash^a, /^a, e^a \rangle$, where $e^a = a$ and for every binary operation $\star \in \{\wedge, \vee, \cdot, \backslash, /\}$,

$$x \star^a y = f(f^{-1}(x) \star f^{-1}(y)).$$

By Lemma 3.2, we have

$$x \wedge^a y = a(a^{-1}x \wedge a^{-1}y) = aa^{-1}(x \wedge y) = x \wedge y.$$

Similarly, $\vee^a = \vee$. Moreover, by Lemma 3.2,

$$x \cdot^a y = a(a^{-1}xa^{-1}y) = xa^{-1}y,$$

$$x/^a y = a(a^{-1}x/a^{-1}y) = (x/y)a$$

and

$$y\backslash x = a(a^{-1}y\backslash a^{-1}x) = a(y\backslash x).$$

Note that if we take $g_a(x) = xa$, then we obtain the same structure, so $\mathbf{L}^a$ does not depend on the choice of left or right multiplication by a. The algebra $\mathbf{L}^a = \langle L, \wedge, \vee, \cdot^a, \backslash^a, /^a, a \rangle$ is called the *translation of* $\mathbf{L}$ *with respect to* a. We remark that we could have defined the operations as follows: $x \cdot^a y = (x/a)y$, $y\backslash^a x = (y/a)\backslash x$ and $x/^a y = x/(a\backslash y)$.

Proposition 3.8. *The translation* $\mathbf{L}^a$ *of a residuated lattice* $\mathbf{L}$ *with respect to an invertible element* a *is a residuated lattice.*

Proof. It is trivial to check that multiplication is associative and a is the multiplicative identity. To show that multiplication is residuated, let $x, y, z \in L$. We have

$$x \cdot^a y \leq z \Leftrightarrow xa^{-1}y \leq z \Leftrightarrow a^{-1}y \leq x\backslash z \Leftrightarrow y \leq a(x\backslash z) \Leftrightarrow y \leq x\backslash^a z$$

and similarly for the other division. □

Note that the translation by an invertible element and the negative cone constructions on a residuated lattice $\mathbf{L}$ commute, i.e., $(\mathbf{L}^a)^- = (\mathbf{L}^-)^a$.

Example 3.14. [Bl] ***The Dedekind-McNeille completion***

Let $\mathbf{L}$ be a residuated lattice and γ the map defined on $\mathcal{P}(\mathbf{L})$ by $\gamma(X) = X^{ul}$, where $A^u = \{x \in L \mid x \leq a \text{ for all } a \in A\}$ and $A^l = \{x \in L \mid x \geq a \text{ for all } a \in A\}$, for all $A \subseteq L$. It is shown in [Bl] that γ is a nucleus, so the Dedekind-McNeille completion $\mathcal{P}(\mathbf{L})_\gamma$, see Examples 3.15 and 3.9, of $\mathbf{L}$ is a residuated lattice. The Dedekind-McNeille completion is a complete residuated lattice and arbitrary existing joins and meets of $\mathbf{L}$ are preserved. In view of Proposition 3.5, this shows that every residuated lattice is a subalgebra of the nucleus image of the power set of a monoid.

For two more completions of residuated lattices see Examples 3.17 and 3.24, below.

<u>Subsets of monoids</u>

We now proceed to concrete examples of residuated lattices.

Example 3.15. ***The power set of a monoid***

Let $\mathbf{M} = \langle M, \cdot, e\rangle$ be a monoid. For any two elements X, Y of the power set $\mathcal{P}(M)$ of M, we denote their intersection, union and complex product respectively, by $X \cap Y$, $X \cup Y$ and $X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}$. Also, we define the sets $X/Y = \{z \mid \{z\} \cdot Y \subseteq X\}$ and $Y\backslash X = \{z \mid Y \cdot \{z\} \subseteq X\}$. It is easy to see that the algebra $\mathcal{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{e\}\rangle$ is a residuated lattice.

It follows that every monoid is a monoid subreduct of a residuated lattice. On the other hand no finite non-trivial group is the monoid reduct of a residuated lattice. It is an open problem to determine all monoid reducts of residuated lattices.

A *partial groupoid* $\langle S, *\rangle$ is a set S with a partially defined binary operation $*$ on it, namely a subset of S^3 such that if $(x, y, z) \in *$ and $(x, y, z') \in *$ then $z = z'$. If there is a z such that $(x, y, z) \in *$, we denote this unique z, the *product* of x, y, by $x * y$. We abbreviate the fact that such a z exists by $x * y \in S$.

A *partial semigroup* $\langle S, *\rangle$ is partial groupoid such that if any of the two sides of the associativity condition is defined, then the other side is also defined and they are equal. It is not hard to see that if the product of some elements of S exists with respect to a certain association of the parenthesis, then the product of the elements in the same order exists with respect to any other association of the parenthesis and the two products are equal.

A *partial monoid* $\langle M, *, e\rangle$ is a structure, such that $\langle M, *\rangle$ is a partial semigroup and $x * e = e * x = x$, for all $x \in M$.

Example 3.16. ***The power set of a partial monoid***

Let $\mathbf{M} = \langle M, R, E\rangle$ be a structure, where M is a set, $R \subseteq M^3$ a ternary relation on M and E is a subset of M. Define the following operations on the power set $\mathcal{P}(M)$ of M: $X \circ Y = R[X, Y, \_]$, $X/Y = \{z \in M \mid \{z\} \circ Y \subseteq X\}$ and $Y \backslash X = \{z \in M \mid Y \circ \{z\} \subseteq X\}$. It is not hard to see that the algebra $\mathcal{P}(\mathbf{M}) = \langle M, \cap, \cup, \circ, \backslash, /, E\rangle$ is a residuated lattice iff for all $x, y, z, w \in M$,

1. $R(x, e, y)$, for some $e \in E$, iff $x = y$, iff $R(e, x, y)$, for some $e \in E$; and
2. $R(x, y, u)$ and $R(u, z, w)$, for some $u \in M$ iff $R(x, v, w)$ and $R(y, z, v)$, for some $v \in M$.

In this case, the residuated lattice $\mathcal{P}(\mathbf{M})$ is called the power set of $\mathbf{M}$. If R is a partial operation, then E is a singleton and $\mathbf{M}$ is a partial monoid.

Example 3.17. ***Order ideals of a partially ordered monoid***

As a different generalization of Example 3.15, let $\mathbf{M} = \langle M, \cdot, e, \leq\rangle$ be a partially ordered monoid, namely a relational structure such that $\langle M, \cdot, e\rangle$ is a monoid, $\langle M, \leq\rangle$ is a partially ordered set and multiplication is order preserving. Moreover, let $\mathcal{O}$ be the set of all order ideals of the underlying partially ordered set. For every $X, Y \in \mathcal{O}$, set $X \bullet Y = \downarrow (X \cdot Y)$, the downset of their complex product. Then, the algebra $\mathcal{O}(\mathbf{M}) = \langle \mathcal{O}, \cap, \cup, \bullet, \backslash, /, \downarrow \{e\}\rangle$ is a residuated lattice.

To prove this we show that the map γ on $\mathcal{P}(\mathbf{M})$ defined by $\gamma(X) = \downarrow X$ is a nucleus. Indeed, if $z \in \gamma(X)\gamma(Y) = (\downarrow X)(\downarrow Y)$, then $z = ab$, $a \leq x$ and $b \leq y$, for some $x \in X$ and $y \in Y$. So, $z \leq xy$, namely $z \in \downarrow XY$. Finally notice that for any two order ideals X, Y, $\gamma(X \cup Y) = \downarrow (X \cup Y) = X \cup Y$. Thus, by Proposition 3.4, $\mathcal{O}(\mathbf{M}) = (\mathcal{P}(\mathbf{M}))_\gamma$.

In the case of a discrete order we obtain Example 3.15. Note that we could have taken order filters instead of order ideals.

If $\langle S, *\rangle$ is a partial semigroup, then for every two subsets X, Y of S we set $X * Y = \{x * y \,|\, x * y \in S,\ x \in X,\ y \in Y\}$, the *complex product* of X and Y, $\langle X\rangle_* = \{x_1 * x_2 * \cdots * x_n \in$

$S \mid n \in \mathbb{N},\, x_1, \ldots x_n \in X\}$, the *subsemigroup* generated by X, and $[X]_* = X \cup (S * X) \cup (X * S) \cup (S * X * S)$, the *semigroup ideal* generated by X.

Example 3.18. ***Ideals of a commutative partial semigroup***

Let $\mathbf{S} = \langle S, * \rangle$ be a commutative partial semigroup. Define γ on the power set of S, by $\gamma(X) = [X]_*$. Since $\mathbf{S}$ is commutative this simplifies to $\gamma(X) = X \cup (S * X)$.

It is easy to see that γ is a closure operator. Moreover, note that if $X, Y \subseteq S$, $a \in X \cup (S * X)$ and $b \in Y \cup (S * Y)$, then $a * b$ is in one of the two forms $x * y,\ s * x * y$, for some $s \in S,\ x \in X$ and $y \in Y$. In both cases $a * b$ is an element of $(X * Y) \cup (S * X * Y) = \gamma(X * Y)$. Finally, $S * X = X * S = X$, for every closed set X, i.e., S acts as an identity element. So, by the remark following Proposition 3.4, γ gives rise to the residuated lattice $\mathcal{I}_{\mathcal{L}}(\mathbf{S}) = (\mathcal{P}(\mathbf{S}))_\gamma$ of semigroup-ideals of $\mathbf{S}$.

In case that a partial semigroup is commutative and idempotent, namely a *partial semilattice*, we can look at its subsemigroups.

Example 3.19. ***Subsemilattices of a partial lower-bounded semilattice.***

Let $\mathbf{L} = \langle L, \vee \rangle$ be a partial lower-bounded join-semilattice and let γ be the map defined by $\gamma(X) = \langle X \rangle_\vee$, for every subset X of L. It is clear that γ is a closure operator. Moreover, if $a \in \gamma(X)$ and $b \in \gamma(Y)$, then

$$a = x_1 \vee x_2 \vee \ldots \vee x_n \text{ and } b = y_1 \vee y_2 \vee \ldots \vee y_m,$$

for some $n, m \in \mathbb{N},\ x_1, x_2, \ldots, x_n \in X$ and $y_1, y_2, \ldots, y_m \in Y$. So,

$$a \vee b = (x_1 \vee x_2 \vee \ldots \vee x_n) \vee (y_1 \vee y_2 \vee \ldots \vee y_m).$$

If $n \leq m$, using the commutativity and idempotency of join, we get

$$a \vee b = (x_1 \vee y_1) \vee (x_2 \vee y_2) \vee \ldots (x_n \vee y_n) \vee (x_1 \vee y_{n+1}) \vee \ldots \vee (x_1 \vee y_m),$$

which is an element of $\gamma(X \vee Y)$. If 0 is the lower bound of $\mathbf{L}$, then $\{0\} \vee X = X \vee \{0\} = X$, for all $X \subseteq L$; so invoking the remark after Proposition 3.4, we can see that the subsemilattices of $\mathbf{L}$ form a residuated lattice $\mathcal{S}(\mathbf{L}) = (\mathcal{P}(\mathbf{L}))_\gamma$.

Example 3.20. ***Subsemigroups of a partial semiring***

A *partial semiring* is a structure $\mathbf{S} = \langle S, *, e, + \rangle$ such that $\langle S, *, e \rangle$ is a monoid, $\langle S, + \rangle$ is a partial semigroup and $*$ distributes over existing binary sums, namely if $x + y \in S$, then

$x*z+x*y=(x+y)*z$ and $z*x+z*y=z*(x+y)$. Note, that it follows that $*$ distributes over finite existing sums.

Assume that $\mathbf{S}=\langle S,*,e,+\rangle$ is a partial semiring. By Example 3.15, $\mathcal{P}(\langle S,*,e\rangle)$ is a residuated lattice, which we denote by $\mathcal{P}(\mathbf{S})$. Define γ on $\mathcal{P}(S)$ by $\gamma(X)=\langle X\rangle_+$. We will show that γ is a nucleus on $\mathcal{P}(\mathbf{S})$.

It is clear that γ is a closure operator. To check that $\gamma(X)*\gamma(Y)\leq\gamma(X*Y)$, namely $\langle X\rangle_+*\langle Y\rangle_+\subseteq\langle X*Y\rangle_+$, let $a\in\langle X\rangle_+$ and $b\in\langle Y\rangle_+$. Then

$$a=x_1+x_2+\cdots+x_n \text{ and } b=y_1+y_2+\cdots+y_m,$$

for some $m,n\in\mathbb{N}$, $x_1,x_2\ldots x_n\in X$ and $y_1,y_2,\ldots y_m\in Y$. So,

$$a*b=(x_1+x_2+\cdots+x_n)(y_1+y_2+\cdots+y_m),$$

which, by the definition of a partial semiring, is equal to a sum of products of elements of X and Y. Thus, $a*b\in\langle X*Y\rangle_+$.

According to Proposition 3.4, γ gives rise to the residuated lattice $\mathcal{S}(\mathbf{S})=(\mathcal{P}(\mathbf{S}))_\gamma$ of the $+$-subsemigroups of $\mathbf{S}$.

Note that in a partial semiring $\mathbf{S}=\langle S,*,e,+\rangle$, multiplication coincides with addition iff $\mathbf{S}'=\langle S,*,e\rangle$ is a lower-bounded join-semilattice. Moreover, in this case the residuated lattice $\mathcal{S}(\mathbf{S})$ of subsemigroups of $\mathbf{S}$, given in Example 3.19, coincides with the residuated lattice of subsemilattices $\mathcal{S}(\mathbf{S}')$ of $\mathbf{S}'$ given in Example 3.20.

Example 3.21. ***Semigroup-ideals of a partial $+$-commutative semiring***

Assume that $\mathbf{S}=\langle S,*,e,+\rangle$ is a partial $+$-commutative semiring. Define the map $\gamma(X)=[X]_+$ on $\mathcal{P}(\mathbf{S})$. In view of commutativity of addition, γ simplifies to $\gamma(X)=X\cup(S+X)$.

Clearly γ is a closure operator. To see that it is a nucleus, let X,Y be subsets of S, $a\in\gamma(X)=X\cup(S+X)$ and $b\in\gamma(Y)=Y\cup(S+Y)$. Then $a=x$ or $a=s_1+x$, and $b=y$ or $b=s_2+y$, for some $s_1,s_2\in S$, $x\in X$ and $y\in Y$. If $a=s_1+x$ and $b=s_2+y$, then

$$a*b=(s_1+x)*(s_2+y)=s_1*s_2+s_1*y+x*s_2+x*y,$$

which is an element of $(X*Y)\cup[S+(X*Y)]=\gamma(X*Y)$. It is easy to see that $a*b\in\gamma(X*Y)$ in the other three cases, as well. Thus, by Proposition 3.4, the nucleus γ gives rise to the residuated lattice $\mathcal{I}_\mathcal{S}(\mathbf{S})=(\mathcal{P}(\mathbf{S}))_\gamma$ of semigroup-ideals of $\mathbf{S}$.

Note that a partially ordered monoid $\mathbf{M} = \langle M, \cdot, e, \leq \rangle$ can be identified with a partial semiring $\mathbf{M}' = \langle M, \cdot, e, \wedge \rangle$ such that $x \wedge x = x$ for all $x \in M$ and if $x \wedge y \in M$ then $x \wedge y = y \wedge x$. The definitional equivalence is given by $x \wedge y = x$ iff $x \leq y$. Moreover, $\downarrow X = [X]_\wedge$, namely the notions of order-ideal of $\mathbf{M}$ and semigroup-ideal of $\mathbf{M}'$ coincide. So, the residuated lattice $\mathcal{O}(\mathbf{M})$ of order-ideals of $\mathbf{M}$ given in Example 3.17 is a special case of the residuated lattice $\mathcal{I}_S(\mathbf{M})$ of semigroup-ideals of $\mathbf{M}'$ given in the previous example.

Example 3.22. ***Ideals of a partial semiring***

Assume that $\mathbf{S} = \langle S, *, e, + \rangle$ is a partial semiring. Set $\mathcal{I} = \{X \subseteq S \mid X = [\langle X \rangle_+]_*\}$, the collection of all *ideals* of $\mathbf{S}$. It is easy to see that $\mathcal{I}(\mathbf{S}) = \langle \mathcal{I}, \wedge, \vee, \cdot, /, \backslash, e \rangle$ is a subalgebra of $\mathcal{S}(\mathbf{S})$, given in Example 3.20. In the case where $\mathbf{S}$ is $*$-commutative, $\mathcal{I}(\mathbf{S})$ can be realized as the image of the power set of $\mathbf{S}$ under the nucleus defined by $\gamma(X) = [\langle X \rangle_+]_*$, the composition of the nuclei given in Examples 3.20 and 3.18.

In case that $\mathbf{S}$ is a *ring with unit* we get the residuated lattice of ideals of a ring. It was in this setting that (commutative, integral) residuated lattices were first considered by Ward and Dilworth, see [WD38] and [WD39].

Example 3.23. ***Ideals of a join-semilattice-ordered monoid***

A *join-semilattice-ordered monoid* $\mathbf{M} = \langle M, *, e, \vee \rangle$ is an algebra, such that $\langle M, *, e \rangle$ is a monoid, $\langle M, \vee \rangle$ is a join-semilattice and multiplication distributes over binary joins. Such an algebra is a special case of a partial semiring, so by Example 3.22 the join-closed subsets of $\mathbf{M}$ form a residuated lattice.

A *meet-semilattice-ordered monoid* is defined in a similar way.

Let $\mathbf{M} = \langle M, *, e, \vee \rangle$ be a join-semilattice-ordered monoid. Under the order induced by the join operation, $\mathbf{M}$ can be considered a partially ordered monoid. By Example 3.17 the map defined by $\gamma_1(X) = \downarrow X$ is a nucleus. Moreover, by the previous observation and Example 3.19, the map defined by $\gamma_2(X) = \langle X \rangle_\vee$ is also a nucleus. It is easy to see that the composition of two nuclei is also a nucleus. The composition γ of the two maps, which in our case commute, gives rise to the residuated lattice $\mathcal{I}_{\mathcal{L}}(\mathbf{M}) = (\mathcal{P}(\mathbf{M}))_\gamma$ of join-ideals of $\mathbf{M}$.

Note that the same holds for the filters of a meet-semilattice-ordered monoid.

In view of the remark following Example 3.21, a join-semilattice-ordered monoid can be viewed as a structure $\mathbf{M} = \langle M, *, e, \vee, \wedge \rangle$, where $\langle M, *, e, \vee \rangle$ and $\langle M, *, e, \wedge \rangle$ are partial semirings. It is mentioned that the map γ_2 is a special case of the nucleus of Example 3.19, that gives the $\vee$-subsemigroups of $\langle M, *, e, \vee \rangle$, while the map γ_1 can be considered a special case of the nucleus $\gamma(X) = [X]_\wedge$ in Example 3.17, that gives the $\wedge$-semigroup-ideals of $\langle M, *, e, \wedge\rangle$. Obviously, the join-ideals of a join-semilattice-ordered monoid are the $\vee$-subsemigroups that happen to be $\wedge$-ideals.

Example 3.24. ***Ideals of a lattice-ordered monoid***

A *lattice-ordered monoid* $\mathbf{M} = \langle M, *, e, \wedge, \vee\rangle$ is an algebra such that $\langle M\wedge, \vee\rangle$ is a lattice, and both $\langle M, *, e, \vee\rangle$ and $\langle M, *, e, \wedge\rangle$ are semilattice-ordered monoids.

As a special case of the previous example and of the remark following it, we obtain that the ideals of a lattice-ordered monoid $\mathbf{M}$ form a residuated lattice $\mathcal{I}_{\mathcal{L}}(\mathbf{M})$. The same holds for the filters of a lattice-ordered monoid.

Example 3.25. ***Ideals of a distributive lattice***

A bounded distributive lattice $\mathbf{L}$ can be viewed as a lattice or join-semilattice ordered monoid, where multiplication is meet. So the ideals of it form a residuated lattice. Even without the assumption of bounds the map defined on the power set of the semigroup $\langle L, \wedge\rangle$ by $\gamma(X) = \downarrow \langle X\rangle_\vee = [\langle X\rangle_\vee]_\wedge$ gives rise to an integral residuated lattice, actually to a Brouwerian algebra, in view of the remark following Proposition 3.4.

Example 3.26. ***Cancellative Monoids***

Let $\mathbf{K} = \langle K, \cdot, e\rangle$ be a cancellative monoid and set $M_K = K \cup \{0, 1\}$. We define an order on M_K, by $0 < k < 1$, for all $k \in K$, and extend the multiplication of K to M_K, by stipulating that 1 is an absorbing element for $K \cup \{1\}$ and 0 an absorbing element for the set M_K. Consider the algebra $\mathbf{M_K} = \langle M_K, \wedge, \vee, \cdot, \backslash, /, e\rangle$, where $x/y = \bigvee\{z \mid zy \leq x\}$ and $y\backslash x = \bigvee\{z \mid yz \leq x\}$. To see that $\mathbf{M_K}$ is a residuated lattice, note that it is isomorphic to $(\mathcal{P}(\mathbf{K}))_\gamma$, where $\gamma(X) = X$, if X has at most one element, and $\gamma(X) = K$ otherwise, for every $X \subseteq K$. The map γ is a nucleus since, if at least one of X, Y is the empty set, then both $\gamma(X)\gamma(Y)$ and $\gamma(XY)$ are empty. If X, Y are both singletons then $\gamma(X)\gamma(Y) = XY$, which is also a singleton, thus equal to $\gamma(XY)$. Finally if none is empty and at least one has more than one elements then XY has at least two elements, by cancellativity, so $\gamma(XY) = K \supseteq \gamma(X)\gamma(Y)$.

Note that the stipulation that K is cancellative is necessary, since otherwise if $ab = ac$ for some $a, b, c \in K$, then, by Lemma 3.1(1), $1 = a1 = a(b \vee c) = ab \vee ac = ab \in K$, a contradiction.

Other examples

We present a few more examples that we consider of special interest.

Example 3.27. ***Every bounded lattice with at least one completely join-irreducible element***

Let $\langle L, \wedge, \vee\rangle$ be a bounded lattice with at least one completely join-irreducible element e. Denote by 0 and 1 the least and greatest elements of the lattice and define multiplication on L by $xy = 0$, if both x, y are less than e; $xy = yx = y$ if y is less than e, but x is not; and $xy = 1$, if none of x, y is less than or equal to e. The element e is the multiplicative identity. One can easily check that multiplication is associative, order preserving and that the joins $x/y = \bigvee\{z \mid zy \leq x\}$ and $y\backslash x = \bigvee\{z \mid yz \leq x\}$ exist in L. Thus, $\langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice.

This example generalizes the example due to Peter Jipsen mentioned in [Bl], where e is stipulated to be an atom of L. As a special case we get that every dually algebraic lattice, in particular every finite lattice, can be *residuated*, i.e., it is the lattice reduct of a residuated lattice. Moreover, it follows that every lattice is a lattice subreduct of a residuated lattice. Actually, it is shown in [BCGJT] that every lattice is a lattice subreduct of a simple, cancellative residuated lattice.

Nevertheless, it is not the case that every lattice is the lattice reduct of a residuated lattice. By Lemma 3.1, if a residuated lattice has a bottom element then it must have a top element, as well. So, lattices that are lower, but not upper bounded cannot be residuated. An example of an algebraic lattice that cannot be residuated is given below.

Example 3.28. The lattice of a binary tree: a non-example

Consider an infinite binary tree and add a least element to it. The underlying set L of the lattice $\mathbf{L}$ obtained can be realized as the set of all finite words on two letters, that is the set of all functions from initial segments of the natural numbers to the two element set $\{1, 2\}$, together with a distinguished element 0. The order is defined as follows: a function f is greater than or equal to a function g iff the domain of f is a subset of the domain of g, and f, g agree on the domain of f. Moreover, the element 0 is less than any function.

Assume that $\mathbf{L}$ can be residuated and let e be the multiplicative identity. Every non-zero element of L has exactly two lower covers. Let a, b be the lower covers of e, and c one of the two lower covers of b. Since $a, b, c \leq e$, we have $ab \leq ae = a$ and $ab \leq b$, so $ab \leq a \wedge b = 0$. Moreover, $cb \leq ce = c$. By Lemma 3.1(1), $b = eb = (a \vee c)b = ab \vee cb \leq 0 \vee c = c$, a contradiction.

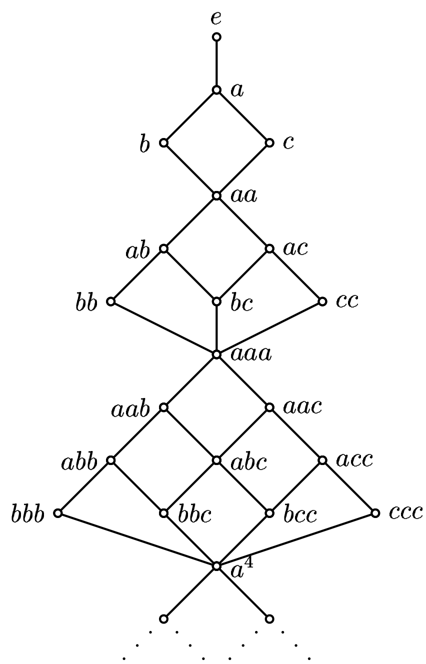

Figure 1: A non-distributive cancellative commutative example

It is an open problem to determine the lattice reducts of all residuated lattices.

Example 3.29. [BCGJT] **A commutative, non-distributive residuated lattice on a free monoid**

Let $\mathbf{F} = \langle \{a^i b^j c^k : i, j, k \in \mathbb{N}\}, \cdot, e\rangle$ be the 3-generated free commutative monoid. For a word $w \in \mathbf{F}$, we denote the length of w by $|w|$, and for $x \in \{a, b, c\}$, we define $|w|_x$ to be the number of occurrences of x in w. The order on $\mathbf{F}$ is defined by $w \leq v$ if $|w| > |v|$, or $|w| = |v|$, $|w|_b \leq |v|_b$ and $|w|_c \leq |v|_c$ (see Figure 1). In [BCGJT] it is verified that $\mathbf{F}$ defines a residuated lattice.

We refer the reader to [Co2], for general constructions of residuated lattices, whose monoid reduct is a free monoid.

Example 3.30. ([Bl], [Le]) **Residuated maps**

Let $\mathbf{L}$ be a complete residuated lattice and let $\mathcal{R}es(\mathbf{L}, \mathbf{L})$ be the set of all residuated maps on $\mathbf{L}$. If we order all such maps point-wise and define multiplication to be composition then it can be shown, see [Bl], that the algebra $\mathbf{L}^{\mathbf{L}} = \langle \mathcal{R}es(\mathbf{L}, \mathbf{L}), \wedge, \vee, \circ, \backslash, /\rangle$ is a residuated lattice, where $f/g = \bigvee\{h \mid h \circ g \leq f\}$ and $g\backslash f = \bigvee\{h \mid g \circ h \leq f\}$.

Structure theory

The structure theory of residuated lattices was developed by K. Blount and C. Tsinakis in [BT]. We review their basic result, specialize it to the finite case and give two easy corollaries of their descriptions of congruence relations on residuated lattices.

Congruences as convex normal subalgebras

Congruence relations are in one-to-one correspondence with normal subgroups, in the setting of group theory and with ideals in ring theory, but generally it is not the case that congruences correspond to special subalgebras. It is shown in [BT] that residuated-lattice congruences correspond to *convex normal* subalgebras.

Lemma 3.9. *[BT] Let* $\mathbf{L}$ *be a residuated lattice and* $\theta \in Con(\mathbf{L})$. *Then the following are equivalent:*

1. $a \,\theta\, b$
2. $[a/b \wedge e] \,\theta\, e$ *and* $[b/a \wedge e] \,\theta\, e$
3. $[a\backslash b \wedge e] \,\theta\, e$ *and* $[b\backslash a \wedge e] \,\theta\, e$

Let $\mathbf{L}$ be a residuated lattice, Y a set of variables. For $x \in L \cup Y \cup \{e\}$, where e is the constant in the language of residuated lattices and $y \in Y$, we define the polynomials

$$\rho_x(y) = xy/x \wedge e \text{ and } \lambda_x(y) = x\backslash yx \wedge e,$$

the *right* and *left conjugate* of y with respect to x. An *iterated conjugate* is a composition of a number of left and right conjugates - we consider composition of conjugates with respect to their arguments. For X, A subsets of $L \cup Y \cup \{e\}$, we define the sets $\Gamma_X^0 = \{\lambda_e\}$,

$$\Gamma_X^n = \{\gamma_{x_1} \circ \gamma_{x_2} \circ ... \gamma_{x_n} \mid \gamma_{x_i} \in \{\lambda_{x_i}, \rho_{x_i}\},\ x_i \in X \cup \{e\},\ i \in \mathbb{N}\},$$

$$\Gamma_X^n(A) = \{\gamma(a) \mid \gamma \in \Gamma_X^n,\ a \in A\},$$

$$\Gamma_X = \bigcup\{\Gamma_X^n \mid n \in \mathbb{N}\},$$

$$\Gamma_X(A) = \bigcup\{\Gamma_X^n(A) \mid n \in \mathbb{N}\}.$$

Note that if $\mathbf{L}$ is a residuated lattice, $\lambda_e(x) = \rho_e(x) = x \wedge e$, $\gamma(x) \leq e$ and $\gamma(e) = e$, for all $x \in L$ and for every iterated conjugate $\gamma \in \Gamma_L$. In particular, if x is negative, $\lambda_e(x) = \rho_e(x) = x$. If $\mathbf{L}$ is commutative, then $x \wedge e \leq \gamma(x)$, for all $x \in L$ and $\gamma \in \Gamma_L$.

A subset N of L is called *normal*, if it is closed under conjugation, i.e., $\gamma(N) \subseteq N$, for all $\gamma \in \Gamma_L$. A subset X of L is called *convex*, if for every x, y in X and z in L, $x \leq z \leq y$ implies that z is in X.

Theorem 3.10. *[BT]*

1. *The convex normal subalgebras of a residuated lattice* $\mathbf{L}$ *form a lattice* $\mathbf{CNS}(\mathbf{L})$*, which is isomorphic to the congruence lattice* $\mathbf{ConL}$ *of* $\mathbf{L}$ *via*

$$S \mapsto \theta_S = \{(a, b) \in L^2 | \, (a/b \wedge e)(b/a \wedge e) \in H\}$$

   and $\theta \mapsto [e]_\theta$*, the* θ*-class of* e*.*

   Moreover, for each $a \in L$*, the principal congruence generated by* (a, e) *corresponds to the convex normal subalgebra generated by* a*.*

2. *The convex normal (in* $\mathbf{L}$*) submonoids of the negative cone of a residuated lattice* $\mathbf{L}$ *form a lattice* $\mathbf{CN_L SM}(\mathbf{L}^-)$*, which is isomorphic to* $\mathbf{CNS}(\mathbf{L})$*, via* $S \mapsto S^-$ *and* $M \mapsto S_M = \{x \in L \,|\, m \leq x \leq e/m, \; m \in M\}$*.*

   The convex normal submonoid generated by a negative element corresponds to the convex normal subalgebra generated by that element.

3. *If* $A \subseteq L^-$ *then the convex normal (in* $\mathbf{L}$*) submonoid of the negative cone of* $\mathbf{L}$ *is* $M(A) = \{x \in L \,|\, g_1 g_2 ... g_n \leq x \leq e,$ *for some* $n \in \mathbb{N}$ *and* $g_1, \ldots, g_n \in \Gamma_L(A)\}$*.*

The description of congruences in a residuated lattice by convex normal subalgebras is pivotal. For finite residuated lattices, we can get a more concrete representation than the one in the general case.

Let $\mathbf{L}$ be a residuated lattice and $S \subseteq L$. We denote the set of *idempotent* elements of S by $E(S) = \{a \in S \,|\, a^2 = a\}$ and the set of *central idempotents* of S by $CE(S) = \{a \in S \,|\, a^2 = a$ and $ax = xa$, for all $x \in L\}$.

Lemma 3.11. *Let* $\mathbf{L}$ *be a residuated lattice. If* $a \in CE(L^-)$*, then* $[a, e/a]$ *is the universe of a convex normal subalgebra of* $\mathbf{L}$*. Conversely, if* N *is the universe of a convex normal subalgebra of* $\mathbf{L}$ *with a least element* a*, then* $N = [a, e/a]$ *and* $a \in CE(L^-)$*.*

Proof. Let $a \in CE(L^-)$. Note that $a(e/a) = (e/a)a \leq e$, so $e/a \leq a\backslash e$. Similarly we get $a\backslash e \leq e/a$, hence, $e/a = a\backslash e$. Moreover, since a is negative, $e \leq e/a$, so by Lemma 3.1,

$$e/a \leq (e/a)(e/a) \leq (e/a)e/a \leq (e/a)/a = e/a^2 = e/a,$$

hence $e/a \in E(L)$. For every $x, y \in [a, e/a]$, we have

$$a = a^2 \leq xy \leq (e/a)(e/a) = e/a,$$

thus, $xy \in [a, e/a]$. Moreover,

$$a = a^2 \leq a/(e/a) \leq x/y \leq (e/a)/a = e/a^2 = e/a,$$

that is $x/y \in [a, e/a]$. Since, $x \vee y, x \wedge y, e \in [a, e/a]$, the interval $[a, e/a]$ is a subuniverse, which is obviously convex. To prove that $[a, e/a]$ is normal, let $x \in [a, e/a]$ and $z \in L$. We have,

$$a = a \wedge e \leq az/z \wedge e = za/z \wedge e \leq zx/z \wedge e \leq e,$$

that is $\rho_z(x) \in [a, e/a]$. Similarly, we show that $\lambda_z(x) \in [a, e/a]$.

Conversely, assume that $\mathbf{N}$ is a convex normal subalgebra with a least element a. The element a is in the negative cone, so $a^2 \leq a$. Since $a^2 \in N$, we get $a = a^2$, i.e., $a \in E(L)$. By the normality of N, for all $z \in L$, $za/z \wedge e$ is an element of N, hence $a \leq za/z \wedge e$. Since a is already negative, this is equivalent to $a \leq za/z$, thus $az \leq za$ for all $z \in L$. Symmetrically, we get $za \leq az$ for all $z \in L$, so $a \in CE(L^-)$. Moreover, since N is a convex subalgebra $[a, e/a] \subseteq N$. On the other hand, for every $b \in N$, we have $e/b \in N$, so $a \leq e/b$, i.e., $ab \leq e$. By the centrality of a we get $ba \leq e$, i.e., $b \leq e/a$, hence $b \in [a, e/a]$. Consequently, $[a, e/a] = N$. □

The next theorem shows that the congruence lattice of a finite residuated lattice is dually isomorphic to a join-subsemilattice of $\mathbf{L}$.

Theorem 3.12. *Let* $\mathbf{L}$ *be a finite residuated lattice. Then the structure* $\mathbf{CE}(L^-) = \langle CE(L^-), \cdot, \vee \rangle$ *is a lattice and* $\mathbf{ConL} \cong (\mathbf{CE}(L^-))^\partial$.

Proof. It is easy to see that $\mathbf{CE}(L^-)$ is a lattice and that for all $a, b \in CE(L^-)$,

$$a = ab \Leftrightarrow a \leq b \Leftrightarrow a \vee b = b.$$

We define the map $\phi : CE(L^-) \to CNS(\mathbf{L})$, by $\phi(a) = [a, e/a]$. If follows from the previous lemma that ϕ is well defined. If $\phi(a) = \phi(b)$ for some $a, b \in CE(L^-)$, then $[a, e/a] = [b, e/b]$, so $a = b$; hence ϕ is one-to-one. If $N \in CNS(L)$, then, by the previous lemma, $N = [a, e/a]$, for some $a \in CE(L^-)$, so ϕ is onto. The map ϕ reverses the order, since if $a \leq b$, then $[b, e/b] \subseteq [a, e/a]$. Moreover, if $[a, e/a] \subseteq [b, e/b]$ then $b \leq a$, so ϕ is a lattice anti-isomorphism. Using the isomorphism between **ConL** and **CNS**(**L**) provided in Theorem 3.10, we get an anti-isomorphism between **ConL** and **CE**(L^-). □

In the commutative case we do not need the centrality assumption.

Corollary 3.13. *Let* **L** *be a finite commutative residuated lattice. Then* **E**(L^-) *is a lattice with multiplication as meet and* **ConL** $\cong$ (**E**(L^-))$^\partial$.

Note that the statement is false without the assumption of finiteness. For example, $|\mathbf{Con}\mathbb{Z}^-| = 2$, but $|CE(\mathbb{Z}^-)| = 1$.

Varieties with equationally definable principal congruences

We use the description of congruence relations to characterize the commutative varieties of residuated lattices that have EDPC.

For two elements a, b in a residuated lattice, set $a\Delta b = (a/b \wedge e)(b/a \wedge e)$.

Lemma 3.14. *If a variety $\mathcal{V}$ satisfies the identity $(x \wedge e)^k y \approx y(x \wedge e)^k$, for some $k \in \mathbb{N}^*$, then for every $\mathbf{L} \in \mathcal{V}$ and for all $a, b, c, d \in L$, $(a, b) \in Cg(c, d)$ is equivalent to $(c\Delta d)^l \leq a\Delta b$, for some $l \in \mathbb{N}$*

Proof. Let **L** be a residuated lattice and $a, b \in L$. It follows from Lemma 3.9 that $a\theta b$ iff $(a\Delta b)\theta e$. Consequently, $Cg(a, b) = Cg(a\Delta b, e)$; moreover, $(a, b) \in Cg(c, d)$ iff $a\Delta b \in [e]_{Cg(c\Delta d, e)}$. Since $a\Delta b$ is negative, $(a, b) \in Cg(c, d)$ is equivalent to $a\Delta b \in M(c\Delta d)$, by Theorem 3.10. Using the description of the convex, normal submonoid $M(s)$ generated by a negative element s given Theorem 3.10(3), we see that this is in turn equivalent to $\prod_{i=1}^{m} \gamma_i(c\Delta d) \leq a\Delta b$, for some $m \in \mathbb{N}$ and some iterated conjugates $\gamma_1, ..., \gamma_n \in \Gamma_L$. Recall that $f \leq \gamma(f)$, for every negative element $f \in L$ and for every iterated conjugate $\gamma \in \Gamma_L$, so,

$$(c\Delta d)^{km} = ((c\Delta d)^k)^m \leq \prod_{i=1}^{m} \gamma_i((c\Delta d)^k) \leq \prod_{i=1}^{m} \gamma_i(c\Delta d),$$

thus $(a, b) \in Cg(c, d)$ is equivalent to $(c\Delta d)^l \leq a\Delta b$, for some $l \in \mathbb{N}$. □

We say that a variety has *equationally definable principal congruences* or *EDPC* if there is a conjunction $\phi(x, y, z, w)$ of equations such that for every algebra in the variety and for all elements a, b, c, d in the algebra, (a, b) is in the congruence generated by (c, d) iff $\phi(a, b, c, d)$ holds.

Proposition 3.15. *Let $\mathcal{V}$ be a variety that satisfies $(x \wedge e)^k y \approx y(x \wedge e)^k$, for some $k \in \mathbb{N}^*$ and let . Then, $\mathcal{V}$ has EDPC iff $\mathcal{V}$ satisfies $(x \wedge e)^n \approx (x \wedge e)^{n+1}$, for some $n \in \mathbb{N}$.*

Proof. Assume that $\mathcal{V}$ satisfies $(x \wedge e)^n \approx (x \wedge e)^{n+1}$, for some $n \in \mathbb{N}$ and let $\mathbf{L} \in \mathcal{V}$ and $a, b, c, d \in L$. Since, $(c\Delta d)^n \leq (c\Delta d)^l$, for every l, by Lemma 3.14 we get

$$(a, b) \in Cg(c, d) \;\Leftrightarrow\; (c\Delta d)^n \leq a\Delta b.$$

Conversely, if $\mathcal{V}$ has EDPC given by a conjunction ϕ of equations and $(x \wedge e)^n \approx (x \wedge e)^{n+1}$ fails for every natural number n, then for every n there exist $A_n \in \mathcal{V}$ and $a_n \in A_n$, $a_n < e$, such that $a_n^{n+1} < a_n^n$. Let $A = \prod_{i=1}^n A_n$, $a = (a_n)_{n\in\mathbb{N}}$ and $b = (a_n^{n+1})_{n\in\mathbb{N}}$. Since A_n satisfies $\phi(a_n^{n+1}, e, a_n, e)$, for all n, it follows that A satisfies $\phi(b, e, a, e)$, that is $(b, e) \in Cg(a, e)$. By Lemma 3.14, this is equivalent to $a^l \leq b$, for some number l. Thus, $a_l^l \leq a_l^{l+1}$, for some l, a contradiction. □

Corollary 3.16. *A variety of commutative residuated lattices has EDPC iff the negative cones of the algebras in the variety are n-potent, for some natural number n.*

The congruence extension property

A variety has the *congruence extension property* or *CEP* if for every algebra $\mathbf{A}$ in the variety, for any subalgebra $\mathbf{B}$ of $\mathbf{A}$ and for any congruence θ on $\mathbf{B}$, there exists a congruence $\bar{\theta}$ on $\mathbf{A}$, such that $\bar{\theta} \cap B^2 = \theta$.

Note that in view of Theorem 3.10 congruences of subalgebras can be extended to the whole algebra iff convex normal subalgebras can be extended.

Lemma 3.17. *If a variety satisfies $(x \wedge e)^k y \approx y(x \wedge e)^k$ then it enjoys the congruence extension property. In particular $\mathcal{CRL}$ has the CEP.*

Proof. Recall that by Theorem 3.10, congruences on a residuated lattice are in one-to-one correspondence with convex normal (in the whole residuated lattice) submonoids of the negative cone. Let $\mathbf{A}$ be a residuated lattice, $\mathbf{B}$ a subalgebra of it and N a convex normal submonoid of $\mathbf{B}$. If N' is the convex normal submonoid of $\mathbf{A}$ generated by N, it suffices to show that $N = N' \cap B$. For the non-obvious inclusion, let $b \in N' \cap B$. Then $\prod_{i=1}^n \gamma_i(a_i) \leq b$, for some $a_1, ..., a_n \in N$ and some iterated conjugates $\gamma_1, ..., \gamma_n$. Since, k-powers of the negative cone are in the center, $a_i^k \leq \gamma_i(a_i^k)$. Moreover, $\gamma_i(a_i^k) \leq \gamma_i(a_i)$, because a_i are in the negative cone. Thus, $\prod_{i=1}^n a_i^k \leq b$. Since, $a_i \in N$ and $b \in B$, we get $b \in N$. □

Not every residuated lattice satisfies the CEP. Let $A = \{0, c, b, a, e\}$ and $0 < c < b < a < e$. Define $a^2 = a, b^2 = ba = ab = b, ac = bc = c$, and let all other non-trivial products be 0. It is easy to see that $\mathbf{A}$ is a residuated lattice and $B = \{e, a, b\}$ defines a subalgebra of it. $\mathbf{B}$ has the non-trivial congruence $\{\{e, a\}, \{b\}\}$, while $\mathbf{A}$ is simple. To see that, let θ be a non-trivial congruence and $a\theta e$; then $(ca/c)\theta ce/c$, namely $c\theta e$. So, $0\theta e$, hence $\theta = A \times A$.

The subvariety lattice

In this section we define a number of interesting subvarieties of $\mathcal{RL}$ and investigate their relative position in $\mathbf{L}(\mathcal{RL})$. Also, we describe a correspondence between positive universal formulas of residuated lattices and subvarieties, and we apply it to get equational basis for joins of varieties in $\mathbf{L}(\mathcal{RL})$. Finally, we provide sufficient conditions for the join of two finitely based varieties to be finitely based and give examples where the join of two varieties is their Cartesian product.

We denote the class of commutative, cancellative, distributive, and integral residuated lattices, by $\mathcal{C}RL$, $\mathcal{C}an\mathcal{RL}$, $\mathcal{DRL}$ and $\mathcal{IRL}$, respectively. It is clear that all these classes, except possibly for $\mathcal{C}an\mathcal{RL}$, are varieties (in particular, $\mathcal{IRL} = \mathrm{Mod}(x \wedge e \approx x) = \mathrm{Mod}(e/x \approx e)$). We will show in Lemma 4.1 that $\mathcal{C}an\mathcal{RL}$ is a variety as well. Also, let $\mathcal{RL}^C$ be the variety generated by the class of all totally ordered residuated lattices.

Theorem 3.18. *([BT], [JT]) The equation $\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) \approx e$ constitutes an equational basis for $\mathcal{RL}^C$.*

Definition 3.19. *A generalized BL-algebra (GBL-algebra) is a residuated lattice that satisfies the identities*

$$((x \wedge y)/y)y \approx x \wedge y \approx y(y\backslash(x \wedge y)).$$

A generalized MV-algebra (GMV-algebra) is a residuated lattice that satisfies the identities

$$x/((x \vee y)\backslash x) \approx x \vee y \approx (x/(x \vee y))\backslash x.$$

We denote the varieties of all GBL-algebras and all GMV-algebras, by $\mathcal{GBL}$ and $\mathcal{GMV}$, respectively. GBL-algebras generalize BL-algebras, the algebraic counterpart of basic logic (see [Ha]).

It is noted in [Bl] that the variety $\mathcal{RL}$ is arithmetical; in particular the subvariety lattice $\mathbf{L}(\mathcal{RL})$ is distributive. We give a partial picture of the subvariety lattice. Inclusions that have not been discussed will be proved in subsequent chapters.

Varieties generated by positive universal classes

A variety $\mathcal{V}$ is called a *discriminator variety* if there exists a term $t(x, y, z)$ in the language of $\mathcal{V}$, such that if an algebra $\mathbf{A}$ of $\mathcal{V}$ is subdirectly irreducible then $t(a, a, c) = c$ and $t(a, b, c) = a$, for all $a, b, c \in \mathbf{A}$, with $a \neq b$.

If $\mathcal{V}$ is a discriminator variety, to every first order formula corresponds a variety with the property that a subdirectly irreducible algebra is in the variety iff it satisfies the first order

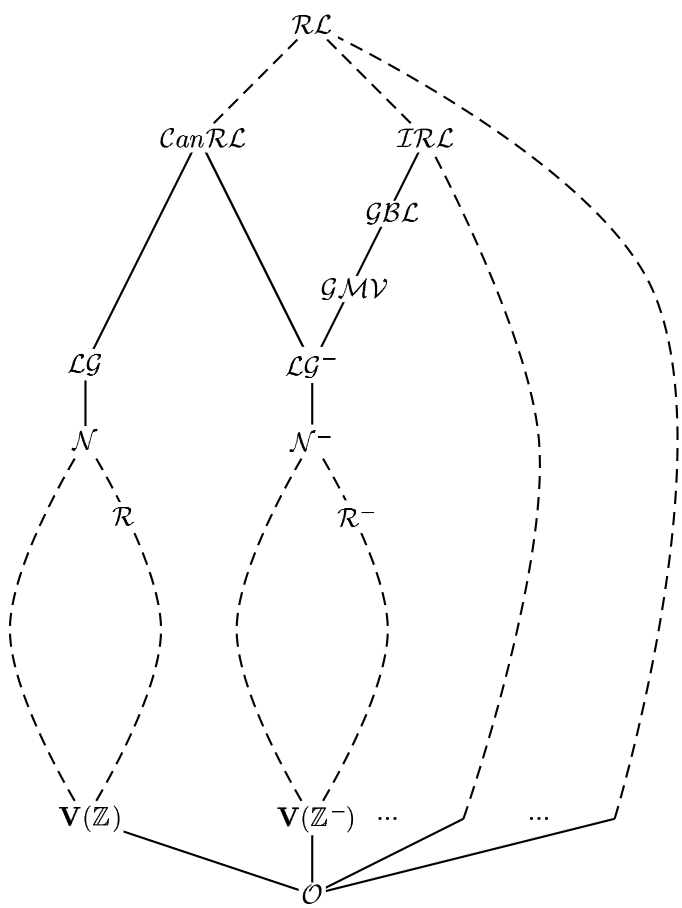

Figure 2: Inclusions between some subvarieties of $\mathcal{RL}$

formula. In this case it is easy to construct an equational basis for the variety generated by the class of all models of a first order formula. Moreover, all subdirectly irreducible algebras are simple.

Residuated lattices do not form a discriminator variety, since e.g. not all subdirectly irreducible residuated lattices are simple. Nevertheless, a similar correspondence can be developed for positive universal formulas. We construct an equational basis for the variety generated by an arbitrary positive universal class in a recursive way. The main tool in the proof is the lattice isomorphism between congruence relations and certain subalgebras of a residuated lattice developed in [BT], see Theorem 3.10. Even though the produced basis of equations is infinite it reduces to a finite one for certain classes.

Lemma 3.20. *Let* $\mathbf{L}$ *be a residuated lattice and* $A_1, ..., A_n$ *finite subsets of* L. *If* $a_1 \vee ... \vee a_n = e$, *for all* $a_i \in A_i$, $i \in \{1, ..., n\}$, *then for all* $i \in \{1, \ldots, n\}$, $n_i \in \mathbb{N}$, *and for all* $a_{i1}, a_{i2}, \ldots, a_{in_i} \in A_i$, *we have* $p_1 \vee ... \vee p_n = e$, *where* $p_i = a_{i1} a_{i2} \cdots a_{in_i}$.

Proof. The proof is a simple induction argument. □

An *open positive universal formula* in a given language is an open first order formula that can be written as a disjunction of conjunctions of equations in the language. A *(closed) positive universal formula* is the universal closure of an open one. A positive universal class is the collection of all models of a set of positive universal formulas.

Lemma 3.21. *Every open (closed) positive universal formula,* ϕ, *in the language of residuated lattices is equivalent to (the universal closure of) a disjunction,* ϕ', *of equations of the form* $e \approx r$, *where the evaluation of the term* r *is negative in all residuated lattices.*

Proof. Every equation $t \approx s$ in in the language of residuated lattices, where t, s are terms, is equivalent to the conjunction of the two inequalities $t \leq s$ and $s \leq t$, which in turn is equivalent to the conjunction of the inequalities $e \leq s/t$ and $e \leq t/s$. Moreover, a conjunction of a finite number of inequalities of the form $e \leq t_i$, for $1 \leq i \leq n$ is equivalent to the inequality $e \leq t_1 \wedge ... \wedge t_n$. So, a conjunction of a a finite number of equations is equivalent to a single inequality of the form $e \leq p$, which in turn is equivalent to the equation $e \approx r$, where $r = p \wedge e$. □

Recall the definition of the set Γ_Y^m of conjugate terms on the variable set Y.

For a positive universal formula $\phi(\bar{x})$ and a countable set of variables Y, we define

$$B_Y^m(\phi'(\bar{x})) = \{e \approx \gamma_1(r_1(\bar{x})) \vee ... \vee \gamma_n(r_n(\bar{x})) \mid \gamma_i \in \Gamma_Y^m\}$$

and $B_Y(\phi'(\bar{x})) = \bigcup\{B_Y^m(\phi'(\bar{x})) \mid \ m \in \mathbb{N}\}$, where $\phi'(\bar{x}) = (r_1(\bar{x}) = e$ or ... or $r_n(\bar{x}) = e)$ is the equivalent to $\phi(\bar{x})$ formula, given in Lemma 3.21.

Theorem 3.22. *Let ϕ be a positive universal formula in the language of residuated lattices and $\mathbf{L}$ a residuated lattice.*

1. *If $\mathbf{L}$ satisfies $(\forall \bar{x})(\phi(\bar{x}))$, then $\mathbf{L}$ satisfies $(\forall \bar{x}, \bar{y})(\varepsilon(\bar{x}, \bar{y}))$, for all $\varepsilon(\bar{x}, \bar{y})$ in $B_Y(\phi'(\bar{x}))$ and $\bar{y} \in Y^l$, for some appropriate $l \in \mathbb{N}$.*

2. *If $\mathbf{L}$ is subdirectly irreducible, then $\mathbf{L}$ satisfies $(\forall \bar{x})(\phi(\bar{x}))$ iff $\mathbf{L}$ satisfies the equation $(\forall \bar{x}, \bar{y})(\varepsilon(\bar{x}, \bar{y}))$, for all $\varepsilon(\bar{x}, \bar{y})$ in $B_Y(\phi'(\bar{x}))$ and $\bar{y} \in Y^l$.*

Proof. 1) Let $\mathbf{L}$ be a residuated lattice that satisfies $(\forall \bar{x})(\phi(\bar{x}))$. Moreover, let $\varepsilon(\bar{x}, \bar{y})$ be an equation in $B_Y(\phi'(\bar{x}))$, $\bar{c} \in L^k$ and $\bar{d} \in L^l$. We will show that $\varepsilon(\bar{c}, \bar{d})$ holds in $\mathbf{L}$. Since $\mathbf{L}$ satisfies $(\forall \bar{x})(\phi(\bar{x}))$, $\phi'(\bar{c})$ holds in $\mathbf{L}$. So, $r_i(\bar{c}) = e$, for some $i \in \{1, 2, \ldots, n\}$; hence $\gamma(r_i(\bar{c})) = e$, for all $\gamma \in \Gamma_Y$. Thus, $\varepsilon(\bar{c}, \bar{d})$ holds.

2) Let $\mathbf{L}$ be a subdirectly irreducible that satisfies $B_Y(\phi'(\bar{x}))$ and $\bar{c} \in L^k$, and let $a_i = r_i(\bar{c})$. We will show that $a_i = e$ for some i.

Let $b \in M(a_1) \cap ... \cap M(a_n)$, where $M(x)$ symbolizes the convex normal submonoid of the negative cone generated by x. Using Theorem 3.10(3), we have that for all $i \in \{1, 2, \ldots, n\}$, $\prod_{j=1}^{s_i} g_{ij} \leq b \leq e$, for some $s_1, s_2, \ldots, s_n \in \mathbb{N}$ and $g_{i1}, g_{i2}, \ldots, g_{is_i} \in \Gamma_L(a_i)$. So,

$$\prod_{j=1}^{s_1} g_{1j} \vee \prod_{j=1}^{s_2} g_{2j} \vee ... \vee \prod_{j=1}^{s_n} g_{2j} \leq b \leq e.$$

On the other hand,

$$\gamma_1(a_1) \vee \gamma_2(a_2) \vee ... \vee \gamma_n(a_n) = e,$$

for all $\gamma_i \in \Gamma_L$, since every equation of $B_Y(\phi'(\bar{x}))$ holds in $\mathbf{L}$. Thus, for all $i \in \{1, 2, \ldots, n\}$ and $g_i \in \Gamma_L(a_i)$, we have $g_1 \vee g_2 \vee ... \vee g_n = e$ and, by Lemma 3.20,

$$\prod_{j=1}^{s_1} g_{1j} \vee \prod_{j=1}^{s_2} g_{2j} \vee ... \vee \prod_{j=1}^{s_n} g_{2j} = e.$$

Thus, $b = e$ and $M(a_1) \cap ... \cap M(a_n) = \{e\}$.

Using the lattice isomorphisms of Theorem 3.10, we obtain

$$\Theta(a_1, e) \cap \Theta(a_2, e) \cap ... \cap \Theta(a_n, e) = \Delta,$$

where $\Theta(a,e)$ denotes the principal congruence generated by (a,e) and Δ denotes the diagonal congruence. Since $\mathbf{L}$ is subdirectly irreducible, this implies that $\Theta(a_i,e)=\Delta$, i.e., $a_i=e$, for some i. Thus, $(\forall\bar{x})(\phi'(\bar{x}))$ holds in $\mathbf{L}$. $\square$

Corollary 3.23. *Let $\{\phi_i \mid i\in I\}$ be a collection of positive universal formulas. Then, $\bigcup\{B(\phi_i')|i\in I\}$ is an equational basis for the variety generated by the (subdirectly irreducible) residuated lattices that satisfy ϕ_i, for every $i\in I$.*

Proof. By the previous theorem a subdirectly irreducible residuated lattice satisfies ϕ_i iff it satisfies all the equations in $B(\phi_i')$, so

$$\begin{aligned}(\mathrm{Mod}(\textstyle\bigcup\{\phi_i \mid i\in I\}))_{SI} &= \textstyle\bigcap\{(\mathrm{Mod}(\phi_i))_{SI} \mid i\in I\}\\ &= \textstyle\bigcap\{(\mathrm{Mod}(B(\phi_i')))_{SI} \mid i\in I\}\\ &= (\mathrm{Mod}(\textstyle\bigcup\{B(\phi_i') \mid i\in I\}))_{SI},\end{aligned}$$

where for every variety $\mathcal{V}$ and every set of equations $\mathcal{E}$, $\mathcal{V}_{SI}$ denotes the class of all subdirectly irreducible algebras of $\mathcal{V}$ and $\mathrm{Mod}(\mathcal{E})$ denotes the variety of all models of $\mathcal{E}$. Consequently,

$$\begin{aligned}\mathbf{V}((\mathrm{Mod}(\textstyle\bigcup\{\phi_i \mid i\in I\}))_{SI}) &= \mathbf{V}((\mathrm{Mod}(\textstyle\bigcup\{B(\phi_i') \mid i\in I\}))_{SI})\\ &= \mathrm{Mod}(\textstyle\bigcup\{B(\phi_i') \mid i\in I\}),\end{aligned}$$

where $\mathbf{V}(\mathcal{K})$ denotes the variety generated by a class $\mathcal{K}$ of similar algebras. $\square$

Note that the equational basis for the variety generated by the models of a positive universal formula is recursive.

The basis given in Theorem 3.22 is by no means of minimal cardinality. It is always infinite, while, as it can be easily seen, for commutative subvarieties it simplifies to the conjunction of commutativity and the equation of $B^0(\phi')$. So, for example, the variety generated by the commutative residuated lattices, whose underlying set is the union of its positive and negative cone, is axiomatized by $xy\approx yx$ and $e\approx(x\wedge e)\vee(e/x\wedge e)$.

Equational bases for joins of subvarieties

We can apply the correspondence to the join of two residuated lattice varieties to obtain an equational basis for it, given equational bases for the two varieties. In particular, we provide sufficient conditions for a variety so that the join of any two of its finitely based subvarieties is also finitely based.

Corollary 3.24. *If $B_1,B_2,\dots B_n$ are equational bases for the varieties $\mathcal{V}_1,\mathcal{V}_2$, ...,$\mathcal{V}_n$, such that the sets of variables in each basis are pairwise disjoint, then $\bigcup\{B(\phi_i') \mid i\in I\}$ is an* *equational basis for the join $\mathcal{V}_1 \vee \mathcal{V}_2 \vee \ldots \vee \mathcal{V}_n$, where ϕ_i ranges over all possible disjunctions of n equations, one from each of $B_1, B_2, \ldots, B_n$.*

Proof. The variety $\mathcal{RL}$ is congruence distributive, so, by Jónsson's Lemma, a subdirectly irreducible residuated lattice in the join of finitely many varieties is in one of the varieties. Moreover, by the definition of ϕ_i, it is clear that a subdirectly irreducible residuated lattice satisfies every ϕ_i, for $i \in I$, if and only if it is in one of the varieties $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$. So,

$$\begin{aligned}\mathcal{V}_1 \vee \mathcal{V}_2 \vee \ldots \vee \mathcal{V}_n &= \mathbf{V}((\mathcal{V}_1 \vee \mathcal{V}_2 \vee \ldots \vee \mathcal{V}_n)_{SI}) \\ &= \mathbf{V}((\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_n)_{SI}) \\ &= \mathbf{V}(\mathrm{Mod}(\bigcup\{\phi_i \mid i \in I\})_{SI}) \\ &= \mathrm{Mod}(\bigcup\{B(\phi_i') \mid i \in I\}).\end{aligned}$$

□

In the case of the join of finitely based varieties the situation is simpler.

Corollary 3.25. *If $B_1, B_2, \ldots B_n$ are finite equational bases for the varieties $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, then $B(\phi')$ is an equational basis for the join $\mathcal{V}_1 \vee \mathcal{V}_2 \vee \ldots \vee \mathcal{V}_n$ of the varieties, where $\phi = (\bigwedge B_1 \vee \bigwedge B_2 \vee \cdots \vee \bigwedge B_n)$ and for every $i \in \{1, 2, \cdots, n\}$, $\bigwedge B_i$ denotes the conjunction of the equations in B_i.*

Proof. Retaining the notation of Corollary 3.24, we see that $\bigcup\{\phi_i \mid i \in I\}$ is equivalent to ϕ and $\bigcup\{B(\phi_i') \mid i \in I\}$ is equivalent to $B(\phi')$. □

Corollary 3.26. *The join of finitely many finitely based varieties of residuated lattices is recursively based.*

We define the varieties $\mathcal{C}_k^-\mathcal{RL} = \mathrm{Mod}((x \wedge e)^k (y \wedge e) \approx (y \wedge e)(x \wedge e)^k)$ and $\mathcal{C}an\mathcal{C}_1^-\mathcal{RL} = \mathcal{C}an\mathcal{RL} \cap \mathcal{C}_1^-\mathcal{RL}$.

Theorem 3.27.

1. *The join of two finitely based subvarieties of $\mathcal{LG} \vee \mathcal{C}an\mathcal{C}_1^-\mathcal{RL}$ is also finitely based.*

2. *The join of two finitely based subvarieties of $\mathcal{RL}^C \vee \mathcal{C}_k^-\mathcal{RL}$ is also finitely based, for every $k \geq 1$.*

Proof. 1) Note that $\mathcal{LG}$ satisfies $\lambda_z(\lambda_w(x)) \approx \lambda_{wz}(x)$ and $\rho_z(x) \approx \lambda_{z^{-1}}(x)$, since

$$\begin{aligned}
\lambda_z(\lambda_w(x)) &= z\backslash(w\backslash xw \wedge e)z \wedge e \\
&= z\backslash(w\backslash xw)z \wedge z\backslash z \wedge e \\
&= z^{-1}w^{-1}xwz \wedge e \\
&= (wz)^{-1}xwz \wedge e \\
&= wz\backslash xwz \wedge e \\
&= \lambda_{wz}(x)
\end{aligned}$$

and

$$\rho_z(x) = zx/z \wedge e = zxz^{-1} \wedge e = z^{-1}\backslash xz^{-1} \wedge e = \lambda_{z^{-1}}(x).$$

So, $\lambda_z(\lambda_w(x \wedge e)) \approx \lambda_{wz}(x \wedge e)$ and $\rho_z(x \wedge e) \approx \lambda_{z^{-1}}(x \wedge e)$ hold in $\mathcal{LG}$. The same two equations hold in $\mathcal{C}an\mathcal{C}_1^-\mathcal{RL}$, since for any negative element a and any element b, $\lambda_b(a) = b\backslash ab \wedge e = b\backslash ba \wedge e = a \wedge e = a$ and $\rho_b(a) = a \wedge e = a$. Thus, these equations hold in the join $\mathcal{LG} \vee \mathcal{C}an\mathcal{C}_1^-\mathcal{RL}$.

If $\mathcal{V}_1$, $\mathcal{V}_2$ are subvarieties of $\mathcal{LG} \vee \mathcal{C}an\mathcal{C}_1^-\mathcal{RL}$ with finite equational bases B_1, B_2, their join satisfies the two equations, which together with the equations in $B^2(\phi')$ imply every equation of $B(\phi')$, where $\phi = \bigwedge B_1 \vee \bigwedge B_2$.

2) The variety $\mathcal{RL}^C$ satisfies the implication

$$x \vee y = e \;\Rightarrow\; \lambda_z(x) \vee \rho_w(y) = e,$$

by Theorem 3.18. We will show that the same implication holds in $\mathcal{C}_k^-\mathcal{RL}$. If $x \vee y = e$, then, by Lemma 3.20, $x^k \vee y^k = e$. Since, $x \leq e$, we have $x^k \leq x \leq e$; so, for all z, $x^k z = zx^k$, hence $x^k \leq z\backslash x^k z$ and $x^k \leq zx^k/z$. Since $x^k \leq e$, this implies $x^k \leq z\backslash x^k z \wedge e$ and $x^k \leq zx^k/z \wedge e$, i.e., $x^k \leq \lambda_z(x^k)$ and $x^k \leq \rho_z(x^k)$, for all z. Thus, $\lambda_z(x^k) \vee \rho_w(y^k) = e$. Moreover, left and right conjugates are increasing in their arguments, so $\lambda_z(x) \vee \rho_w(y) = e$.

All subdirectly irreducible residuated lattices in the join $\mathcal{RL}^C \vee \mathcal{C}_k^-\mathcal{RL}$ coincide with the subdirectly irreducibles in the union $\mathcal{RL}^C \cup \mathcal{C}_k^-\mathcal{RL}$, so all of them satisfy the implication. Since every residuated lattice in the join $\mathcal{RL}^C \vee \mathcal{C}_k^-\mathcal{RL}$ is a subdirect product of subdirectly irreducible algebras, and quasi-equations are preserved under products and subalgebras, the join satisfies the above implication.

Now, if $\mathcal{V}_1$, $\mathcal{V}_2$ are subvarieties of $\mathcal{RL}^C \vee \mathcal{C}_k^-\mathcal{RL}$ with finite equational basis B_1, B_2, then their join also satisfies the implication. Since, $B(\phi')$ is an equational basis for $\mathcal{V}_1 \vee \mathcal{V}_2$, where $\phi = \bigwedge B_1 \vee \bigwedge B_2$, the implication is a consequence of a finite subset B of $B(\phi')$, by compactness. It is clear that $B \cup B^0(\phi')$ is a finite equational basis for $\mathcal{V}_1 \vee \mathcal{V}_2$. □

Corollary 3.28. *The join of two finitely based commutative varieties of residuated lattices is finitely based.*

It is an open problem whether the join of two finitely based varieties of residuated lattices is finitely based.

Direct product decompositions

Certain pairs of subvarieties of $\mathcal{RL}$ are so different that their join decomposes into their the Cartesian product of the two varieties, i.e., the class of all Cartesian products of algebras of the two varieties. Such a pair is the variety of ℓ-groups and the variety of their negative cones. First we give a general lemma that allows us to obtain such decompositions of the join of two varieties from two projection-terms.

The following proposition is in the folklore of the subject and easy to prove.

Proposition 3.29. *Let $\mathcal{V}_1, \mathcal{V}_2$ be subvarieties of $\mathcal{RL}$ with equational bases B_1 and B_2, respectively, and let $\pi_1(x), \pi_2(x)$ be unary terms, such that $\mathcal{V}_1$ satisfies $\pi_1(x) \approx x$ and $\pi_2(x) \approx e$ and $\mathcal{V}_2$ satisfies $\pi_1(x) \approx e$ and $\pi_2(x) \approx x$. Then $\mathcal{V}_1 \vee \mathcal{V}_2 = \mathcal{V}_1 \times \mathcal{V}_2$ and the following list, $B_1 * B_2$, of equations is an equational basis for it.*
i) $\pi_1(x) \cdot \pi_2(x) \approx x$
ii) $\pi_i(\pi_j(x)) \approx e$, for $i, j \in \{1, 2\}, i \neq j$ and $\pi_i(\pi_i(x)) \approx \pi_i(x)$ for $i \in \{1, 2\}$.
iii) $\pi_i(x \star y) \approx \pi_i(x) \star \pi_i(y)$, where $\star \in \{\wedge, \vee, \cdot, /, \backslash\}$ and $i \in \{1, 2\}$
iv) $\varepsilon(\pi_1(x_1), ..., \pi_1(x_n))$, for all equations $\varepsilon(x_1, ..., x_n)$ of B_1
v) $\varepsilon(\pi_2(x_1), ..., \pi_2(x_n))$, for all equations $\varepsilon(x_1, ..., x_n)$ of B_2

For any pair of subvarieties of $\mathcal{V}_1, \mathcal{V}_2$, the same decomposition holds for their join, and if B_1, B_2 are finite, then so is $B_1 * B_2$.

Proof. It is easy to see that the equations in $B_1 * B_2$ hold both in $\mathcal{V}_1$ and $\mathcal{V}_2$, hence they hold in $\mathcal{V}_1 \vee \mathcal{V}_2$, also. Now, suppose that the residuated lattice A satisfies the equations $B_1 * B_2$; we will show that A is in $\mathcal{V}_1 \times \mathcal{V}_2$.

Define $A_1 = \{x \in A | \pi_2(x) = e\}$ and $A_2 = \{x \in A | \pi_1(x) = e\}$. Using (iii) and (i), it is easy to see that A_1 and A_2 are subalgebras of A. Define the map $f : A \to A_1 \times A_2$, by $f(x) = (\pi_1(x), \pi_2(x))$. It is easy to check that f is well defined, using (ii); that it is a homomorphism, using (iii); one-to-one, using (i); and onto, using (iii) and (i). Thus, A is isomorphic to $A_1 \times A_2 \in \mathcal{V}_1 \times \mathcal{V}_2 \subseteq \mathcal{V}_1 \vee \mathcal{V}_2$. □

Corollary 3.30. *If $B_1 = \{(e/x)x \approx e\}$ and $B_2 = \{e \wedge x \approx x\}$, then $B_1 * B_2$ is an equational basis for $\mathcal{LG} \vee \mathcal{IRL} = \mathcal{LG} \times \mathcal{IRL}$.*

Proof. Let $\pi_1(x) = e/(e/x)$ and $\pi_2(x) = (e/x)x$. It is easy to see that $\mathcal{LG}$ satisfies $e/(e/x) \approx e(ex^{-1})^{-1} \approx x$ and $(e/x)x \approx x^{-1}x \approx e$, and that $\mathcal{IRL}$ satisfies $(e/x)x \approx ex \approx x$ and $e/(e/x) \approx e$. $\square$

Substructural logics and the decidability of the equational theory

In this section we discuss the connections of residuated lattices to logical sequent calculi and mention the derivation of the decidability of the equational theory of $\mathcal{RL}$ from this analysis, given in [JT].

Let $\mathcal{L}$ be the similarity type of residuated lattices and $T_{\mathcal{L}}$ the set of all residuated lattice terms. A *sequent* is a sequence of the form

$$w(\gamma_1, \dots \gamma_n, t_1(\bar{x}), \dots t_m(\bar{x})) \vdash t(\bar{x}),$$

where $\bar{x} = (x_1, \dots x_l)$, $n, m, l \in \mathbb{N}$, w is a monoid word on its arguments; $t, t_1, \dots, t_m$ are residuated lattice terms; and $\gamma_1, \dots, \gamma_n$ are distinct symbols. An *instance* of a sequent is obtained by substituting $(t_{1k}, \dots, t_{i_k k})$ for γ_k and s_i for x_i, where $i \in \mathbb{N}_l$, $k \in \mathbb{N}_n$, $i_k \in \mathbb{N}$, $s_i, t_{ij} \in T_{\mathcal{L}}$. A *sequent rule* or *Gentzen rule* is a sequence of the form

$$\Gamma_1, \dots \Gamma_m, \Gamma_{m+1},$$

where $m \in \mathbb{N}$ and Γ_i are sequents for all $i \in \mathbb{N}_{m+1}$. An *instance* of the rule is obtained by substituting instances of the sequents in it. We denote the empty monoid word by ε and the empty sequence of sequents by space. A Gentzen rule R is usually written in fraction notation:

$$\frac{\Gamma_1\ \Gamma_2\ \dots\ \Gamma_m}{\Gamma_{m+1}}\ R.$$

A *sequent calculus* or *Gentzen system* is a set of Gentzen rules.

Let Σ be a set of instances of sequents, Γ an instance of a sequent and S a Gentzen system. We call Γ an *immediate consequence* of Σ via S, if there are $\Gamma_1, \dots, \Gamma_m \in \Sigma$ and $R \in S$, such that $\frac{\Gamma_1\ \Gamma_2\ \dots\ \Gamma_m}{\Gamma}$ is an instance of R. We say that Γ *is provable from* Σ *via* S, if there is a sequence $\Gamma_1, \dots, \Gamma_n = \Gamma$, such that for all $i \in \mathbb{N}_n$, Γ_i is an immediate consequence of $\Sigma \cup \{\Gamma_1, \dots \Gamma_{i-1}\}$, via S. If Γ is provable from $\emptyset$ via S, we say that Γ is *provable in* S.

We define the interpretation, $[\,]$, of a sequence of terms, an instance of a sequent and an instance of a Gentzen rule in the following way:

$$[\gamma] = [(t_1, t_2, \dots, t_l)] = t_1 \cdot t_2 \cdots t_l;\ [\varepsilon] = e;$$

$$\begin{aligned}[\Gamma] &= [w(\gamma_1, \ldots, \gamma_n, t_1(\bar{s}), \ldots, t_m(\bar{s})) \vdash t(\bar{s})] \\ &= (\bar{w}([\gamma_1], [\gamma_2], \ldots, [\gamma_n], t_1(\bar{s}), \ldots, t_m(\bar{s})) \leq [t(\bar{s})]),\end{aligned}$$

where $\bar{w}$ is the evaluation of w in residuated lattices, $\gamma_k = (t_{1k}, \ldots, t_{i_k k})$ and $\bar{s} = (s_1, \ldots, s_k)$;

$$\left[\frac{\Gamma_1\, \Gamma_2\, \ldots\, \Gamma_m}{\Gamma}\right] = (([\Gamma_1] \text{ and } [\Gamma_2] \text{ and } \ldots \text{ and } [\Gamma_m]) \text{ imply } [\Gamma]).$$

Consider the following Gentzen system, G:

$$\frac{}{t \vdash t}\,(\mathrm{Id}) \qquad \frac{\gamma\delta \vdash u}{\gamma e \delta \vdash u}\,(e\text{-left}) \qquad \frac{}{\varepsilon \vdash e}\,(e\text{-right})$$

$$\frac{\gamma s t \delta \vdash u}{\gamma(s \cdot t)\delta \vdash u}\,(\cdot\mathrm{left}) \qquad \frac{\gamma \vdash s \quad \delta \vdash t}{\gamma\delta \vdash s \cdot t}\,(\cdot\mathrm{right})$$

$$\frac{\sigma \vdash s \quad \gamma t \delta \vdash u}{\gamma\sigma(s\backslash t)\delta \vdash u}\,(\backslash\mathrm{left}) \qquad \frac{s\gamma \vdash t}{\gamma \vdash s\backslash t}\,(\backslash\mathrm{right})$$

$$\frac{\sigma \vdash s \quad \gamma t \delta \vdash u}{\gamma(t/s)\sigma\delta \vdash u}\,(/\mathrm{left}) \qquad \frac{\gamma s \vdash t}{\gamma \vdash t/s}\,(/\mathrm{right})$$

$$\frac{\gamma s \delta \vdash u \quad \gamma t \delta \vdash u}{\gamma(s \vee t)\delta \vdash u}(\vee\mathrm{left}) \qquad \frac{\gamma \vdash s}{\gamma \vdash s \vee t}(\vee\mathrm{right}_1) \qquad \frac{\gamma \vdash t}{\gamma \vdash s \vee t}(\vee\mathrm{right}_2)$$

$$\frac{\gamma s \delta \vdash u}{\gamma(s \wedge t)\delta \vdash u}(\wedge\mathrm{left}_1) \qquad \frac{\gamma t \delta \vdash u}{\gamma(s \wedge t)\delta \vdash u}(\wedge\mathrm{left}_2) \qquad \frac{\gamma \vdash s \quad \gamma \vdash t}{\gamma \vdash s \wedge t}(\wedge\mathrm{right})$$

This system lacks the three structural rules of weakening, contraction and exchange, so it describes a substructural logic. Actually, the corresponding logic is the unbounded version of the Full Lambek calculus. It is easy to check that if Γ is provable in G, then $[\Gamma]$ is true in $\mathcal{RL}$, by verifying that the interpretation of every immediate consequence of Σ is satisfied, if the interpretation of every element of Σ is satisfied. But more than soundness of the rules is true; the following completeness theorem is a generalization to the non-commutative case of a theorem for a fragment of intuitionistic linear logic, proved in [OT]. The details can be found in [JT].

Theorem 3.31. *[JT] For a residuated lattice term p, the inequality $e \leq p$ is satisfied in $\mathcal{RL}$ iff the sequent $\varepsilon \vdash p$ is provable in G.*

Corollary 3.32. *[JT] The equational theory of residuated lattices is decidable.*

Proof. Let t, s be residuated lattice terms. Note that an equation $t \approx s$ is equivalent to the conjunction of the inequalities $e \leq s/t$ and $e \leq t/s$. Thus, to decide whether $t \approx s$ holds in $\mathcal{RL}$ it suffices to decide whether $\mathcal{RL} \models e \leq p$, where p is a term. By Theorem 3.31, this is equivalent to deciding whether $\varepsilon \vdash p$ is provable in G. Note that provability in G is decidable, because if Γ is provable then there is a sequence of immediate consequences, of which Γ is the last member. There are only finitely many choices for the rule used in the last step. Actually, only the rules for which there is an instance with Γ as the denominator are candidates. Moreover, there are finitely many ways in which Γ can be a denominator of a given instance of a rule. Thus, we have finitely many collections of finitely many sequent instances as the only choices for numerators of rule instances that can produce Γ. Additionally, all these sequent instances have strictly lower complexity than Γ, where the complexity of a sequent instance could be taken as the sum of the heights of the terms that are members of it (a sequent instance is a sequence of terms and $\vdash$). Of course the same argumentation applies to the candidate sequent instances. This process of checking possible elements for being denominators has to stop because of the decreasing-complexity nature of it. If a possible route, which can be visualized as a branch of a search tree, leads to the numerator of the Id-rule or of the e-right rule, then Γ is provable in G. Otherwise, if all routes stop at some sequent that is not a denominator of any instance of a rule in G, then Γ cannot be provable. Thus, it is decidable whether $\varepsilon \vdash p$ is provable in G, for every term p. $\square$

Lexicographic orders on semidirect products of residuated lattices

We conclude this section with an observation on semidirect products of residuated lattices under the lexicographic order. Lexicographic orders on ℓ-groups turn out to be a useful tool, see [AF], [Me]. We provide conditions for the semidirect product of two residuated lattices under the lexicographic order to be a residuated lattice.

Let $\mathbf{K}$ and $\mathbf{Q}$ be residuated lattices and θ a monoid homomorphism from the monoid reduct of $\mathbf{Q}$ to the endomorphism monoid of the monoid reduct of $\mathbf{K}$. It is easy to check that the set $K \times Q$ together with the multiplication given by $(a, b) \cdot (c, d) = (a \cdot \theta_b(c), bd)$ and the order defined by $(a, b) < (c, d) \Leftrightarrow b < d$ or ($b = d$ and $a < c$), the reverse lexicographic order, is actually a monoid with identity element $(1_{\mathbf{K}}, 1_{\mathbf{Q}})$ and a partially ordered set. We call this structure the semidirect product of $\mathbf{K}$ by $\mathbf{Q}$ over θ and we symbolize it by $\mathbf{K} \overleftarrow{\times}_\theta \mathbf{Q}$. (Note that multiplication is not necessarily compatible with the order.)

Proposition 3.33. *Let $\mathbf{K}$ and $\mathbf{Q}$ be residuated lattices and θ a monoid homomorphism from the monoid reduct of $\mathbf{Q}$ to the endomorphism monoid of the monoid reduct of $\mathbf{K}$. Then, $\mathbf{K} \overleftarrow{\times}_\theta \mathbf{Q}$ defines a residuated lattice iff all the following conditions hold*

1. $\mathbf{Q}$ *is a chain or* $\mathbf{K}$ *is bounded;*

2. θ_x *is a residuated map, for all* $x \in Q$*;*

3. $\mathbf{Q}$ *is cancellative or* K *is a singleton; and*

4. $\mathbf{Q}$ *is an* ℓ*-group or* $\mathbf{K}$ *has a maximum element.*

Proof. Assume all the conditions hold. The first condition guarantees that the structure is a lattice. In both cases the join and the meet are easy to compute.
If $\mathbf{K}$ is bounded then

$$(a,b) \vee (c,d) = \begin{cases} (a \vee c, b) & \text{if } b = d \\ (0_{\mathbf{K}}, b \vee d) & \text{if } b \neq d \end{cases}$$

and

$$(a,b) \wedge (c,d) = \begin{cases} (a \wedge c, b) & \text{if } b = d \\ (1_{\mathbf{K}}, b \wedge d) & \text{if } b \neq d \end{cases}$$

If $\mathbf{Q}$ is a chain then

$$(a,b) \vee (c,d) = \begin{cases} (a \vee c, b) & \text{if } b = d \\ (a,b) & \text{if } b > d \\ (c,d) & \text{if } b < d \end{cases}$$

and

$$(a,b) \wedge (c,d) = \begin{cases} (a \wedge c, b) & \text{if } b = d \\ (c,d) & \text{if } b > d \\ (a,b) & \text{if } b < d \end{cases}$$

If $\mathbf{K}$ is a singleton, multiplication is vacuously order preserving. In view of the third condition assume that $\mathbf{Q}$ is cancellative. Let $a, c, f \in K,\ b, d, g \in Q$ and $(a,b) < (c,d)$, i.e., $b < d$ or ($b = d$ and $a < c$). We will show that $(a,b)(f,g) \leq (c,d)(f,g)$ and $(f,g)(a,b) \leq (f,g)(c,d)$, namely that $(a\theta_b(f), bg) \leq (c\theta_d(f), dg)$ and that $(f\theta_g(a), gb) \leq (f\theta_g(c), gd)$.

If $b < d$ then $bg < dg$ and $gb < gd$, because of cancellativity, so both of the inequalities hold. If $b = d$ and $a < c$, then $bg = dg$ and $gb = gd$. Additionally, $a\theta_b(f) = a\theta_d(f) < c\theta_d(f)$ and $\theta_g(a) \leq \theta_g(c)$, so $f\theta_g(a) \leq f\theta_g(c)$. Thus, the inequalities hold in this case, as well.

By the second condition, $\theta_x = \theta(x)$ is a residuated map for all $x \in Q$; let $\theta^*(x)$ denote the residual of θ_x. To prove that multiplication is residuated, in view of the last condition, suppose first that Q is an l-group. Let $k, l, m \in K, x, y, z \in Q$. We will show that the pair $(\theta^*_x(k\backslash l), x^{-1}y)$ is the maximum (m,z) with respect to the property $(k,x)(m,z) \leq (l,y)$. Note that

$$(k,x)(\theta^*_x(k\backslash l), x^{-1}y) = (k\theta_x(\theta^*_x(k\backslash l)), xx^{-1}y) \leq (k(k\backslash l), y) \leq (l,y).$$

Conversely, if $(k,x)(m,z) \leq (l,y)$, then $(k\theta_x(m), xz) \leq (l,y)$. So either $xz < y$, or $xz = y$ and $k\theta_x(m) \leq l$. In the first case, $z < x^{-1}y$, so $(m,z) \leq (\theta_x^*(k\backslash l), x^{-1}y)$. In the second case, $z = x^{-1}y$ and $\theta_x(m) \leq k\backslash l$, so $m \leq \theta^*(k\backslash l)$, hence $(m,z) \leq (\theta_x^*(k\backslash l), x^{-1}y)$. Thus, $(k,x)\backslash(l,y) = (\theta_x^*(k\backslash l), x^{-1}y)$. Similarly, we can show that $(l,y)/(k,x) = (yx^{-1}, \theta_{yx^{-1}}(k)\backslash l)$.

In the case that K has a top element, working as above we can see that

$$(k,x)\backslash(l,y) = \begin{cases} (1_{\mathbf{K}}, x\backslash y) & \text{if } x(x\backslash y) < y \\ (\theta_x^*(k\backslash l), x\backslash y) & \text{if } x(x\backslash y) = y \end{cases}$$

and

$$(l,y)/(k,x) = \begin{cases} (1_{\mathbf{K}}, y/x) & \text{if } (y/x)x < y \\ (l/\theta_{y/x}(k), y/x) & \text{if } (y/x)x = y \end{cases}$$

Conversely, suppose that $K \overleftarrow{\times}_\theta Q$ is a residuated lattice. If $\mathbf{Q}$ is not a chain, then there is a pair of incomparable elements x, y. Let $a \in K$ and $(a,x) \vee (a,y) = (b,z)$, for some $b \in K$ and $z \in Q$. Since, $(a,x), (a,y) \leq (b,z)$ we get $x, y \leq z$, thus $x \vee y \leq z$. If $x \vee y < z$ then $(a, x \vee y)$ would be a common upper bound of (a,x) and (a,y), but strictly less than their join, a contradiction. So $z = x \vee y$. Since x, y are incomparable $(a,x), (a,y) < (c, x \vee y)$ for all $c \in K$, thus $(b, x \vee y) = (a,x) \vee (a,y) \leq (c, x \vee y)$ for all $c \in K$; hence $b \leq c$, for all $c \in K$, namely b is the least element of $\mathbf{K}$. Similarly, one can prove that $\mathbf{K}$ is upper bounded; thus the first condition holds.

If Q is not an l-group, then there exists an element x in Q such that $x(x\backslash e) < e$. Let $k, l, m \in K$, $z \in Q$ and $(m,z) = (k,x)\backslash(l,e)$. Then $(k\theta_x(m), xz) \leq (l,e)$, so $xz \leq e$, i.e., $z \leq x\backslash e$. Hence, $(m,z) \leq (m, x\backslash e)$. Moreover,

$$(k,x)(m, x\backslash e) = (k\theta_x(m), x(x\backslash e)) < (l,e),$$

hence $(m,z) = (m, x\backslash e)$. Furthermore, for every $n \in K$,

$$(k,x)(n, x\backslash e) = (k\theta_x(n), x(x\backslash e)) < (l,e),$$

hence $(n, x\backslash e) \leq (m, x\backslash e)$. Thus, $n \leq m$, for all $n \in K$, i.e., K has an upper bound, a fact that establishes the last condition.

To prove the second condition, we need to show that for all $x \in Q$, the map θ_x is residuated, i.e., there exists a map θ'_x,such that for all $k, n \in K$, $\theta_x(n) \leq l$ iff $n \leq \theta'_x(l)$. We define $\theta'_x(l)$ to be the first coordinate of the element $(e,x)\backslash(l,x)$. It is easy to see that the second coordinate of this element is $x\backslash x$, so $(\theta'_x(l), x\backslash x) = (e,x)\backslash(l,x)$. Given that

$x \cdot (x\backslash x) = x$ holds in every residuated lattice, by Lemma 3.1(15), we have

$$
\begin{aligned}
n \leq \theta'_x(l) \quad &\Leftrightarrow \; (n, x\backslash x) \leq (\theta'_x(l), x\backslash x) \\
&\Leftrightarrow \; (n, x\backslash x) \leq (e, x)\backslash(l, x) \\
&\Leftrightarrow \; (e, x)(n, x\backslash x) \leq (l, x) \\
&\Leftrightarrow \; (e \cdot \theta_x(n), x \cdot (x\backslash x)) \leq (l, x) \\
&\Leftrightarrow \; \theta_x(n) \leq l
\end{aligned}
$$

Assume, now, that K is not a singleton. We will show that $\mathbf{Q}$ is cancellative. Note first that for all $q \in Q$, there are $m, n \in K$ such that $m < n$ and $\theta_q(m) \neq \theta_q(n)$, because otherwise $\theta_q(x) = \theta_q(y)$, for all $x, y \in K$, a contradiction since no constant map is residuated, unless it is defined over a singleton. Suppose, by way of contradiction, that $qs = qr$, for some $q, s, r \in Q$. If $t = s \vee r$, then $qt = q(s \vee r) = qs \vee qr = qs$. Since multiplication is order preserving and $(n, s) \leq (m, t)$, we get $(e, q)(n, s) \leq (e, q)(m, t)$, namely $(\theta_q(n), qs) \leq (\theta_q(m), qt)$. Since $qs = qt$, we get $\theta_q(n) \leq \theta_q(m)$, while from $m < n$ and the fact that θ_q is order preserving we have $\theta_q(m) \leq \theta_q(n)$, a contradiction. Now, suppose that $sq = tq$ and $s \leq t$, for some $q, s, t \in Q$. From $(n, s) \leq (m, t)$, we get $(n, s)(e, q) \leq (m, t)(e, q)$, namely $(n, sq) \leq (m, tq)$. Since $sq = tq$, we get $n \leq m$, a contradiction. Thus, $\mathbf{Q}$ is both left and right cancellative. $\square$

Since the direct product is a special case of a semidirect product under a residuated map, the same conditions apply.

For a study of semidirect products under different order relations, we refer the reader to the work in progress [JoT] of B. Jónsson and C. Tsinakis.

CHAPTER IV

CANCELLATIVE RESIDUATED LATTICES

This section contains a brief exposition of cancellative residuated lattices and of the connections between ℓ-groups and their negative cones. Most of the results will appear in the author's joint paper [BCGJT] and will be used in Chapter VII.

Note that every non-trivial cancellative residuated lattice is infinite. Indeed, since a non-trivial residuated lattice has a non-trivial negative cone (it can have a trivial positive cone), and multiplication is order preserving and cancellative, it follows that all the powers of a strictly negative element have to be distinct.

It turns out that the language of residuated lattices has enough descriptive power to express equationally the property of cancellativity ($ac = bc \Rightarrow a = c$).

Lemma 4.1. *[BCGJT] A residuated lattice is right cancellative as a monoid if and only if it satisfies the identity $xy/y \approx x$.*

Proof. The identity $(xy/y)y \approx xy$ holds in every residuated lattice since $xy/y \leq xy/y$ implies $(xy/y)y \leq xy$, and $xy \leq xy$ implies $x \leq xy/y$, hence $xy \leq (xy/y)y$. By right cancellativity, we have $xy/y = x$. Conversely, suppose $xy/y \approx x$ holds, and consider elements a, b, c such that $ac = bc$. Then $a = ac/c = bc/c = b$, so right cancellativity is satisfied. □

Thus, a residuated lattice is cancellative if it satisfies both $x\backslash xy \approx y$ and $yx/x \approx y$. Consequently, the class $\mathcal{C}an\mathcal{RL}$ of cancellative residuated lattices is a variety.

In [AF], it is shown that the lattice reduct of an ℓ-group is distributive. Example 3.29 shows that this is not the case for, even commutative, cancellative residuated lattices. Actually, it is shown in [BCGJT] that $\mathcal{C}an\mathcal{RL}$ satisfies no non-trivial lattice identity.

The following proposition shows that $\mathbf{V}(\mathbb{Z}^-)$, the variety of residuated lattices generated by the non-positive integers under addition and the natural order, and the variety of ℓ-groups form a splitting pair in the subvariety lattice of cancellative residuated lattices.

Proposition 4.2. *For every cancellative residuated lattice, either it has $\mathbb{Z}^-$ as a subalgebra or it is an ℓ-group.*

Proof. Let $\mathbf{A}$ be a cancellative residuated lattice. In view of Lemma 3.1(4) either there exists a strictly negative element a of A, such that $e/a = e$ or for every strictly negative element x of A, $e < e/x$. It is easy to see that in the first case the subalgebra generated by a is isomorphic to $\mathbb{Z}^-$. In the second case for every element a of A, consider the element $x = (e/a)a$. It cannot be strictly negative because $e/x = e/(e/a)a = (e/a)/(e/a) = e$, by cancellativity; so $x = e$. Thus, $\mathbf{A}$ is an ℓ-group. $\square$

Lattice-ordered groups

The most well studied examples of cancellative residuated lattices are ℓ-groups. As mentioned in Example 3.1, the class $\mathcal{LG}$ of ℓ-groups is axiomatized, in the context of residuated lattices, by the identity $x(x\backslash e) \approx e$. Below we provide alternative axiomatizations of $\mathcal{LG}$.

Lemma 4.3. *Each of the following sets of equations forms an equational basis for $\mathcal{LG}$.*

1. $(e/x)x \approx e$
2. $x \approx e/(x\backslash e)$ *and* $x/x = e$
3. $x \approx y/(x\backslash y)$
4. $x/(y\backslash e) \approx yx$
5. $x/(y\backslash e) \approx xy$ *and* $e/x \approx x\backslash e$
6. $(y/x)x \approx y$
7. $x/(y\backslash z) \approx (z/x)\backslash y)$

Proof. Recall that an ℓ-group has a group and a lattice reduct and multiplication distributes over joins. Obviously all the equations are valid in ℓ-groups, if we define $x/y = xy^{-1}$ and $y\backslash x = y^{-1}x$.

A residuated lattice that satisfies the first identity is a monoid such that every element has a right inverse, so it is a group. Multiplication distributes over joins, by Lemma 3.1, so we obtain an ℓ-group. Using the two identities of (2) and Lemma 3.1 we get

$$\begin{aligned} (e/x)/(e/x) \approx e \quad &\Rightarrow\ e/(e/x)x \approx e \\ &\Rightarrow\ [e/(e/x)x]\backslash e \approx e\backslash e \\ &\Rightarrow\ (e/x)x \approx e. \end{aligned}$$

So, (2) implies (1). Setting $y = e$ in (3), we obtain $x \approx e/(x\backslash e)$; setting $x = e$, we get $e \approx y/y$. So (3) implies (2). Setting $x = e$ in (4), we have $e/(y\backslash e) \approx y$; setting $y = e/x$, we get $x/[(e/x)\backslash e] \approx (e/x)x$, so $x/x \approx (e/x)x$. It follows from Lemma 3.1(4) and (14) that x/x is an element of the positive cone and that $(e/x)x$ is an element of the negative cone, so

$x/x \approx (e/x) \approx e$. For (5) we work in a similar way. Identity (6) yields (1) for $y = e$. Finally, for $x = z = e$ in (7), we get $y = e/(y\backslash e)$ and for $z = x, y = e$ we have $e \leq x/x = (x/x)\backslash e$, so $x/x \leq e$, so $x/x = e$. Thus, (7) implies (1). $\square$

The subvariety lattice of $\mathcal{LG}$ has a unique atom, the variety $\mathcal{CLG}$ of commutative ℓ-groups. This varitey is known (see [AF]) to be equal to $\mathbf{V}(\mathbb{Z})$, the variety of ℓ-groups generated by the integers under addition and the natural order. Moreover, $\mathcal{LG}$ has a unique lower cover $\mathcal{N}$. It is well known that $\mathcal{LG}$ has a decidable equational theory.

Negative cones of ℓ-groups

Recall the definition of the negative cone of a residuated lattice and of a class of residuated lattices given in Example 3.7. We present a characterization of the negative cones of ℓ-groups, that allows to conclude that $\mathcal{LG}^-$ is a variety. Moreover, we investigate the similarities between the subvariety lattices of $\mathcal{LG}$ and $\mathcal{LG}^-$.

Recall the definition of a generalized BL-algebra and of a generalized MV-algebra, from page 34.

Theorem 4.4. *[BCGJT] For a residuated lattice* $\mathbf{L}$*, the following statements are equivalent.*

1. $\mathbf{L}$ *is the negative cone of an* ℓ*-group.*
2. $\mathbf{L}$ *is a cancellative integral GMV-algebra.*
3. $\mathbf{L}$ *is a cancellative integral GBL-algebra.*

Corollary 4.5. *[BCGJT] The class* $\mathcal{LG}^-$ *is a variety, axiomatized relative to* $\mathcal{RL}$ *by the identities* $xy/y \approx x \approx y\backslash yx$ *and* $(x/y)y \approx x \wedge y \approx y(y\backslash x)$*. Alternatively, the last two identities can be replaced by* $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$*.*

Recall that for a class $\mathcal{K}$ of residuated lattices, $\mathbf{H}(\mathcal{K})$, $\mathbf{S}(\mathcal{K})$, $\mathbf{P}(\mathcal{K})$ and $\mathcal{K}^-$ denote, respectively, the class of homomorphic images, subalgebras, products and negative cones of members of $\mathcal{K}$.

Theorem 4.6. *[BCGJT] The map* $\mathcal{K} \mapsto \mathcal{K}^-$*, defined on classes of* ℓ*-groups, commutes with the operators* $\mathbf{H}$*,* $\mathbf{S}$ *and* $\mathbf{P}$*, and restricts to a lattice isomorphism between the subvariety lattices of* $\mathcal{LG}$ *and* $\mathcal{LG}^-$*.*

Corollary 4.7. *[BCGJT] The variety* $\mathcal{V}(\mathbb{Z}^-)$ *consists of all negative cones of Abelian* ℓ*-groups.*

We now show how equational bases of varieties are translated by the isomorphism of the subvariety lattices of $\mathcal{LG}$ and $\mathcal{LG}^-$.

For a residuated lattice term t, we define a translated term t^- by

$$\begin{array}{lll} x^- = x \wedge e & e^- = e & \\ (s/t)^- = s^-/t^- \wedge e & (s\backslash t)^- = s^-\backslash t^- \wedge e & \\ (st)^- = s^-t^- & (s \vee t)^- = s^- \vee t^- & (s \wedge t)^- = s^- \wedge t^- \end{array}$$

Lemma 4.8. *[BCGJT] For any $\mathbf{L} \in \mathcal{RL}$, $\mathbf{L}^- \models s \approx t$ iff $\mathbf{L} \models s^- \approx t^-$.*

Theorem 4.9. *[BCGJT] Let $\mathcal{V}$ be a subvariety of $\mathcal{LG}^-$, defined by a set $\mathcal{E}$ of identities and let $\mathcal{W} = \mathrm{Mod}(\mathcal{E}^-) \cap \mathcal{LG}$, where $\mathcal{E}^- = \{s^- \approx t^- \mid (s \approx t) \in \mathcal{E}\}$. Then $\mathcal{W}^- = \mathcal{V}$.*

Note that since $\cdot$ and $^{-1}$ distribute over $\vee$ and $\wedge$, any $\mathcal{LG}$ identity is equivalent to a conjunction of two identities of the form $e \leq p(g_1, \ldots, g_n)$, where p is a lattice term and $g_1, \ldots, g_n$ are group terms. Since ℓ-groups are distributive, this can be further reduced to a finite conjunction of inequalities of the form $e \leq g_1 \vee \cdots \vee g_n$.

For a term $t(x_1, \ldots, x_m)$ and a variable z distinct from $x_1, \ldots, x_m$, let

$$\bar{t}(z, x_1, \ldots, x_m) = t(z^{-1}x_1, \ldots, z^{-1}x_m).$$

Every group term g can be written in the form $p_1q_1^{-1}p_2q_2^{-1} \cdots p_nq_n^{-1}$ where the p_i and q_i are products of variables (without inverses). Define

$$\hat{g} = q_n \cdots q_2q_1\backslash[q_n(\cdots(q_2(q_1p_1/q_1)p_2/q_2)\cdots)p_n/q_n].$$

Theorem 4.10. *[BCGJT] Let $\mathcal{V}$ be a subvariety of $\mathcal{LG}$, defined by a set $\mathcal{E}$ of identities, which we may assume to be of the form $e \leq g_1 \vee \ldots \vee g_n$. Let*

$$\bar{\mathcal{E}} = \{e \approx \widehat{\bar{g}_1} \vee \ldots \vee \widehat{\bar{g}_n} \mid e \leq g_1 \vee \ldots \vee g_n \text{ is in } \mathcal{E}\}.$$

Then $\bar{\mathcal{E}}$ is an equational basis for $\mathcal{V}^-$ relative to $\mathcal{LG}^-$.

For example consider the variety $\mathcal{R} = \mathcal{RL}^C \cap \mathcal{LG}$ of *representable ℓ-groups* which (by definition) is generated by the class of totally ordered groups (see [AF] for more details). An equational basis for this variety is given by $e \leq x^{-1}yx \vee y^{-1}$ (relative to $\mathcal{LG}$). Applying the translation above, we obtain $e \approx zx\backslash(zy/z)x \vee y\backslash z$ as as equational basis for $\mathcal{R}^-$.

Corollary 4.11. *[BCGJT] The map $\mathcal{V} \mapsto \mathcal{V}^-$ from $\mathbf{L}(\mathcal{LG})$ to $\mathbf{L}(\mathcal{LG}^-)$ sends finitely based subvarieties of $\mathcal{LG}$ to finitely based subvarieties of $\mathcal{LG}^-$.*

CHAPTER V

ATOMIC SUBVARIETIES

In this chapter we investigate the atomic varieties in the subvariety lattice of residuated lattices. In particular, we give infinitely many commutative atoms and note that there are only two cancellative ones. Moreover, we present a continuum of atoms that satisfy the idempotency law for multiplication and are in $\mathcal{RL}^C$. Finally, we observe that there are only two commutative idempotent atomic varieties.

A non-trivial algebra $\mathbf{A}$ is called *strictly simple* if it lacks non-trivial proper subalgebras and congruences. Recall that, by Theorem 3.10, congruences on residuated lattices correspond to convex normal subalgebras. So, the absence of non-trivial proper subalgebras is enough to establish the strict simplicity of a residuated lattice.

Proposition 5.1. *Let a be a non-identity element of a strictly simple, lower bounded residuated lattice, $\mathbf{A}$, and let $t(x)$ be a term such that $\mathbf{A}$ satisfies $t(x) = a$, if $x \neq e$. Then, the variety generated by $\mathbf{A}$ is an atom in the subvariety lattice.*

Proof. Let $\mathcal{V}$ be the variety generated by $\mathbf{A}$. By Jónsson's Lemma the subdirectly irreducible algebras of $\mathcal{V}$ are contained in $\mathbf{HSP}_u(\mathbf{A})$. So, if $\mathbf{D} \in \mathcal{V}_{SI}$, there exists, an ultrapower $\mathbf{B}$ of $\mathbf{A}$ and a non-trivial subalgebra $\mathbf{C}$ of $\mathbf{B}$ such that $\mathbf{D} = f(\mathbf{C})$ for some homomorphism f. Since $\mathbf{A}$ is strictly simple, thus generated by any of its non-identity elements, we can assume, without loss of generality, that $a = 0$, the least element of $\mathbf{A}$. Note that $\mathbf{A}$ satisfies the first order formula:

$$(\forall\, x, y, z)(x \neq e \neq y \rightarrow t(x) = t(y) \leq z),$$

thus, so does $\mathbf{B}$, by the remark on page 5. So, $\mathbf{B}$ has a least element $0'$, which is actually contained in all non-trivial subalgebras of $\mathbf{B}$.

Since the least element is term definable and $\mathbf{A}$ is generated by 0, the subalgebra $\mathbf{F}$ of $\mathbf{C}$ generated by $0'$ is isomorphic to $\mathbf{A}$, hence $\mathbf{F}$ is strictly simple. If any two elements of F have the same image under f, then $f(F) = \{e\}$; thus $f(0') = f(e)$. Since the identity element of a residuated lattice is its least element only if the residuated lattice is trivial, we get $f(C) = \{e\}$, a contradiction. Consequently, $f(\mathbf{F}) \subseteq \mathbf{D}$ is isomorphic to $\mathbf{F}$. Thus, $\mathbf{A}$ is isomorphic to a subalgebra of every subdirectly irreducible member of $\mathcal{V}$, hence $\mathcal{V}$ is an atom. □

The following lemma describes the finitely generated atoms of $\mathbf{L}(\mathcal{RL})$.

Corollary 5.2. *Let $\mathcal{V}$ be a finitely generated variety. Then $\mathcal{V}$ is an atom in $\mathbf{L}(\mathcal{RL})$ iff $\mathcal{V} = \mathcal{V}(\mathbf{L})$, for some finite strictly simple $\mathbf{L}$.*

Proof. Let $\mathcal{V}$ be an atomic variety generated by a finite algebra K. If $\mathbf{K}$ is not strictly simple, then there is a minimal non-trivial subalgebra $\mathbf{L}$ of $\mathbf{K}$. Since $\mathcal{V}$ is an atom, it is generated by $\mathbf{L}$. The converse is a direct consequence of the previous lemma; the necessary term exists because $\mathbf{L}$ is strictly simple and finite. □

Commutative atoms

The simplest non-trivial residuated lattice is $\mathbf{2}$. The underlying set is $2 = \{0, e\}$, 0 is the least element and e the multiplicative identity. Recall the definition of a generalized Boolean algebra from Example 3.2. We prove that the class $\mathcal{GBA}$ of generalized Boolean algebras is a variety and it is generated by $\mathbf{2}$. Additionally, we provide equational bases for this variety.

Proposition 5.3. *Let $\mathbf{L}$ be a residuated lattice. The following statements are equivalent.*

1. *$\mathbf{L}$ is a generalized Boolean algebra.*
2. *$\mathbf{L}$ is in the variety $\mathbf{V}(\mathbf{2})$.*
3. *$\mathbf{L}$ satisfies the identities*
   (a) $x \cdot y \approx x \wedge y$, *and*
   (b) $x/(x \vee y) \vee (x \vee y) \approx e$.
4. *$\mathbf{L}$ satisfies the identities*
   (a) $x \cdot y \approx x \wedge y$, *and*
   (b) $(x \wedge y)/y \vee y \approx e$.
5. *$\mathbf{L}$ satisfies the identities*
   (a) $xy \approx x \wedge y$, *and*
   (b) $y/(y/x) \approx x \vee y$.
6. *$\mathbf{L}$ satisfies $x/(x\backslash y) \approx x \approx (y/x)\backslash x$.*

Proof. We will show that (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (5) $\Rightarrow$ (1) and that (6) $\Leftrightarrow$ (1).

(1) $\Rightarrow$ (3): We assume that every principal ideal is a Boolean algebra. In particular, $\mathbf{L}$ has a top element e. Consider arbitrary $x, y \in L$. Since the element $x \vee y$ is in the interval

$[x, e]$, it has a complement z in $[x, e]$. Note that $x/(x \vee y) \wedge (x \vee y) \leq x$, by Lemma 3.1, and $x/(x \vee y)$ is the maximum element with this property. Since z also satisfies this property, we have $z \leq x/(x \vee y)$. So, $e = z \vee (x \vee y) \leq x/(x \vee y) \vee (x \vee y) \leq e$, hence $x/(x \vee y) \vee (x \vee y) = e$.

(3) $\Rightarrow$ (4): The identity (4)(b) follows from the identity (3)(b). by substituting $x \wedge y$ for x.

(4) $\Rightarrow$ (2): Let P be a prime filter of $\mathbf{L}$ and $f_P : \mathbf{L} \rightarrow\ 2$ be defined by $f(x) = 1$ iff $x \in P$. We will show that f is a residuated lattice homomorphism. It is clear that f_P is a lattice homomorphism, thus a monoid homomorphism as well. To prove that it preserves the division operations, given their behavior on $\mathbf{2}$, we only need to show that

$$x/y \notin P \text{ iff } x \notin P \text{ and } y \in P.$$

Assume that $x/y \notin P$ and $y \notin P$. Since, $x/y = (x \wedge y)/y$, and P is prime we have $e = (x \wedge y)/y \vee y \notin P$, a contradiction. Assume that $x/y \notin P$ and $x \in P$, then $x \leq x/y$ and, since P is a filter, $x/y \in P$, a contradiction. Conversely, if $x \notin P$, $y \in P$ and $x/y \in P$, then $x \geq (x/y) \wedge y \in P$, hence $x \in P$, a contradiction.

Since f_P is a homomorphism, $Ker(f_P)$ is a congruence on $\mathbf{L}$. In order to prove that $\mathbf{L}$ is a subdirect product of copies of $\mathbf{2}$, we need only show that the intersection of the congruences above is the diagonal. This follows from the fact that any pair of elements (a, b) in a distributive lattice can be separated by a prime filter, i.e., there exists a prime filter P such that $a \in P$ and $b \notin P$, or such that $b \in P$ and $a \notin P$. Thus $\mathbf{L}$ is in $\mathbf{V}(\mathbf{2})$.

(2) $\Rightarrow$ (5): It is trivial to check that $\mathbf{2}$ satisfies the identities in (5).

(5) $\Rightarrow$ (1): Assume that x, y are elements of $\mathbf{L}$, such that $x \leq y$. We will show that x/y is the complement of y in $[x, e]$. We have $x \leq x/y$, since $x \wedge y \leq x$; so $x \leq y \wedge (x/y)$. Moreover, $y \wedge (x/y) \leq x$, hence $y \wedge (x/y) = x$. Additionally,

$$y \vee (x/y) = (x/y)/((x/y)/y) = (x/y)/(x/(y \wedge y)) = (x/y)/(x/y) = e.$$

(1) $\Leftrightarrow$ (6): Having established the equivalence of (1) and (2), note that the algebra $\mathbf{2}$ satisfies the identity (6). Conversely, suppose the equation (6) holds in $\mathbf{L}$. For every element y of it we have $e = e/(e \backslash y)$, so $e \leq e/y$, i.e., $y \leq e$. So, $\mathbf{L}$ is an integral residuated lattice. Moreover, we have

$$\begin{aligned} x \approx x/(x\backslash y) \quad &\Rightarrow\ x\backslash(x\backslash y) \approx (x/(x\backslash y))\backslash(x\backslash y) \\ &\Rightarrow\ x^2\backslash y \approx x\backslash y \\ &\Rightarrow\ x^2 \approx x. \end{aligned}$$

Together with integrality this gives $xy = x \wedge y$, for all $x, y \in L$. Assume now that $y \leq x$.

We will show that the complement of x in $[y, e]$ is $x\backslash y$. Note that $y \leq x\backslash y$, by integrality, so $y \leq x \wedge x\backslash y$. On the other hand we have $x \wedge (x\backslash y) \leq y$, by Lemma 3.1, thus $x \wedge x\backslash y = y$. Moreover,

$$\begin{aligned} x \vee x\backslash y &= (x/(x \vee x\backslash y))\backslash(x \vee x\backslash y) \\ &= (x/x \wedge x/(x\backslash y))\backslash(x \vee x\backslash y) \\ &= (x/x \wedge x)\backslash(x \vee x\backslash y) \\ &\geq x\backslash x \geq e. \end{aligned}$$

So, $x \vee x\backslash y = e$. □

Recall that $\mathcal{B}r$ denotes the variety of Brouwerian algebras, see Example 3.3, and $\mathcal{GMV}$ the variety of generalized MV-algebras, see page 34. By (5)(b) of the previous lemma, we have $\mathcal{GBA} = \mathcal{B}r \cap \mathcal{GMV}$. Moreover, $\mathcal{GBA}$ is an atom in the subvariety lattice, since $\mathbf{2}$ is strictly simple. It is easy to see that it is the only atom below $\mathcal{B}r$.

We denote by $\mathbf{n}$ the integral residuated lattice defined by the monoid on the set $\{e, a, a^2, \ldots, a^{n-1}\}$, under the obvious linear order. It is easy to see that $\mathbf{n}$ is an n-potent GMV-algebra.

Lemma 5.4. *The following list is an equational basis for* $\mathbf{V}(\mathbf{n}+\mathbf{1})$.

1. $\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) = e$
2. $x^{n+1} \approx x^n$
3. $x \wedge y = x(x\backslash y) = (y/x)x$
4. $x^n = x^n/(x^n\backslash y^n)$
5. $(x^n/y^n)^2 = (x^n/y^n)$ *and* $(y^n\backslash x^n)^2 = y^n\backslash x^n$
6. $xy \approx yx$

Proof. Obviously, the algebra $\mathbf{n}+\mathbf{1}$ satisfies all the identities. Conversely, assume that $\mathbf{L}$ is a subdirectly irreducible residuated lattice that satisfies the identities. By (1) and (2), $\mathbf{L}$ is an n-potent chain. It is easy to see, and it will be proved in Lemma 7.5, that $\mathbf{L}$ is an integral GBL-algebra, by (3). Note that the idempotent elements are of the form x^n and that they form a subalgebra of $\mathbf{L}$. Indeed, they are closed under division by (5), obviously closed under the lattice operations, and the product of any two such elements is their meet - if $a \leq b$, then $a = a^2 \leq ab \leq a$. By the fourth identity and Proposition 5.3(6), this subalgebra is a generalized Boolean algebra. Since it is also totally ordered it is isomorphic to $\mathbf{2}$.

We will, now, show that $\mathbf{L}$ is generated by a single element as a monoid. Assume that there are non-identity elements $a \leq b$ that are not powers of a common element. Define $a_1 = a$, $b_1 = b$, $a_{k+1} = b_k \wedge b_k \backslash a_k$ and $b_{k+1} = b_k \vee b_k \backslash a_k$. Obviously, $a_k \leq b_k$, $a_k = a_{k+1} b_{k+1}$, because of (6) and (3), for all k, and b_k is an increasing sequence. So, $a = a_1 = a_2 b_2 = a_3 b_3 b_2 = \cdots = a_{n+1} b_{n+1} b_n b_{n-1} \cdots b_3 b_2 \leq e(b_{n+1})^n = (b_{n+1})^n$. Since there are only two idempotent elements in $\mathbf{L}$, either $(b_{n+1})^n = 0$, or $(b_{n+1})^n = e$. In the first case, $a = 0 = b^n$, so both a and b are powers of b. In the second case $b_{n+1} = e$. Since $b_{n+1} = b_n \vee b_n \backslash a_n = b_{n-1} \vee b_{n-1} \backslash a_{n-1} \vee \ldots \vee b_n \backslash a_n = \cdots = b \vee b_1 \backslash a_1 \vee \ldots \vee b_n \backslash a_n$, we have $b_k \backslash a_k = e$, for some k. We have $b_k \leq a_k$, so $a_k = b_k$. Using the fact that $b_k \in \{a_{k+1}, b_{k+1}\}$ and $a_k = a_{k+1} b_{k+1}$, for all k, and induction, it is not hard to see that both b and a are powers of b^k. □

Any strictly simple finite residuated lattice different from $\mathbf{2}$ has to have a top element different than e. This is because otherwise $\{0, e\}$ would be a subalgebra isomorphic to $\mathbf{2}$. We give below an infinite list of examples of finite commutative totally ordered residuated lattices that are strictly simple and generate distinct atoms in $\mathbf{L}(\mathcal{RL})$.

For every natural number n set $T_n = \{1, e\} \cup \{u_k \mid k \in \mathbb{N}_n\}$. Define an order relation on T_n by $u_k \leq u_l$ iff $k \geq l$, and $u_k < e < 1$, for all natural numbers $k \leq n$. Also, define multiplication by $x1 = 1x = x$, for all $x \neq e$; $u_k u_l = u_{min\{n,k+l\}}$, for all $k, l \in \mathbb{N}_n$; and the two division operations by $x/y = \bigvee\{z \in T_n \mid zy \leq x\}$ and $y \backslash x = \bigvee\{z \in T_n \mid yz \leq x\}$. Note that multiplication is order preserving and, since T_n is dually well ordered, $\mathbf{T}_n = \langle T_n, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice.

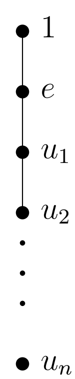

Figure 3: The residuated lattice $\mathbf{T}_n$.

Lemma 5.5. *The variety $\mathbf{V}(\mathbf{T}_n)$ is an atom in the subvariety lattice of $\mathcal{RL}$, for every natural number n.*

Proof. Note that $\mathbf{T}_n$ is generated by any of its non-identity elements. If $x < e$, then $e/x = 1$; moreover, $e/1 = u$ and $u_k = u^k$, for all $k \leq n$. So, $\mathbf{T}_n$ is strictly simple, hence it generates an atom by Corollary 5.2. $\square$

Set $\overline{1}(x) = x \vee (e/x)$. If $a \in T_n - \{e\}$, then $\overline{1}(a) = 1$.

Proposition 5.6. *For every n, the following list B_n of equations is a finite equational basis for $\mathcal{V}(\mathbf{T}_n)$.*

1. $\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) \approx e$
2. $x^{n+1} \approx x^n$
3. $(x \vee e)^2 \approx (x \vee e)$
4. $e/((x \vee e)\backslash e) \approx x \vee e$
5. $(e/\overline{1}(x))^n \cdot x \approx (e/\overline{1}(x))^n$
6. $x \wedge y \wedge e \approx (x \wedge e)((x \wedge e)\backslash(y \wedge e)) \approx ((y \wedge e)/(x \wedge e))(x \wedge e)$
7. $(x \wedge e)^n \approx (x \wedge e)^n/((x \wedge e)^n\backslash(y \wedge e)^n \wedge e) \wedge e$
8. $(x^n/y^n)^2 \approx (x^n/y^n)$ *and* $(y^n\backslash x^n)^2 \approx y^n\backslash x^n$
9. $xy \approx yx$

Proof. Obviously, $\mathcal{V}(\mathbf{T}_n)$ satisfies B_n. Let $\mathbf{L}$ be a subdirectly irreducible residuated lattice that satisfies B_n. $\mathbf{L}$ has to be a chain, because of the first equation and Theorem 3.18, and its negative cone is isomorphic to $\mathbf{n}$, because of equations (2), (6), (7), (8) and (9). Assume that the negative cone is $\{e, u, u^2, ..., u^n = 0\}$. Observe that $\mathbf{L}$ has a strictly positive element a. Otherwise, $\mathbf{L}$ would be integral, so $e/x = e$, for all $x \in L$, hence $\overline{1}(x) = e$. In that case, (5) would imply $e \cdot x = e$, for all $x \in A$, a contradiction. By (3), we get $a^2 = a$. Since $\mathbf{L}$ has a bottom element, it also has a top element 1. For every strictly positive element b of L, we have $u = eu \leq bu$. If $e \leq bu$, we have $e \leq beu \leq bbuu \leq ... \leq b^n u^n = b^n 0 = 0$, a contradiction. So $bu = u$, hence $b\backslash e = u$. Using equation (4), we have $b = e/(b\backslash e) = e/u = 1$, so there is a unique strictly positive element and $\mathbf{L}$ is isomorphic to $\mathbf{T}_n$ $\square$

Working toward a partial description of finite, commutative, strictly simple, residuated chains, we note that they have similar properties as the algebras $\mathbf{T}_n$.

Lemma 5.7. *Let $\mathbf{L}$ be a finite, commutative, strictly simple member of $\mathcal{RL}^C$ and let 1 be its top element. Then $x1 = x$, $x/1 = x$, $x/x = 1$ and $x(e/1) \leq x \wedge (e/1)$, for all $x \neq e$. Moreover, 1 covers e and e covers $e/1$.*

Proof. Obviously, $\mathbf{L}$ is a subdirectly irreducible element of $\mathcal{RL}^C$, so $\mathbf{L}$ is chain. If $\mathbf{L} \cong \mathbf{2}$, then the conclusion is obvious. Otherwise, $\mathbf{L}$ has a top element $1 \neq e$. If $e = e/1$, then $e \leq e/1$, i.e., $1 \leq e$, a contradiction. So, $e \neq e/1$. Note that $e/1 = e/1^2 = (e/1)/1$, so $e/1 \leq (e/1)/1$, hence $(e/1)1 \leq e/1$. On the other hand, $e/1 \leq (e/1)1$, since $e \leq 1$; so $e/1 = (e/1)1$.

It is easy to show that if $x1 = x$ and $y1 = y$, then $xy1 = xy$, $(x/y)1 = x/y$ and $(e/x)1 = e/x$. By the assumption of strict simplicity, for every element of $a \neq e$ of $\mathbf{L}$, there exists a term t_a, such that $a = t_a(1)$. It is easy to prove that $x1 = x$, for all $x \neq e$, by induction on the complexity of t_a. Consequently, $x \leq x/1$. Since, $x/1 \leq x/e = x$, we have $x/1 = x$. Moreover, $1x \leq x$ implies $1 \leq x/x$, so $x/x = 1$. Obviously, $e/1 \leq e$, so $x(e/1) \leq x$. Also, $e/1 \leq e/x$, since $x \leq 1$, i.e., $x(e/1) \leq e$. So, $x(e/1)1 \leq e$, hence $x(e/1) \leq e/1$. Thus, $x(e/1) \leq x \wedge (e/1)$. To show that e is covered by 1, note that if $x > e$, then $1 \leq 1x = x$. It is obvious that $e/1 \leq e$. If $x < e$, then $1x \leq e$, so $x \leq e/1$, hence e is a cover of $e/1$. $\square$

Corollary 5.8. *Let $\mathcal{V} \models (e/(e/x))^n \leq x$ and $(x \wedge e)^n \approx (x \wedge e)^{n+1}$. Then $\mathcal{CanIRL} \vee \mathcal{V} = \mathcal{CanIRL} \times \mathcal{V}$. Hence, $\mathcal{CanIRL} \vee \mathcal{V}(\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, ..., \mathbf{T}_{i_k}) = \mathcal{CanIRL} \times \mathcal{V}(\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, ..., \mathbf{T}_{i_k})$.*

Proof. Let $\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e$ and $\pi_2(x) = (e/(e/x))^n \vee x$. Note that $\mathcal{CanIRL}$ satisfies $\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e \approx (e \wedge x) \wedge e \approx x$ and $\pi_2(x) = (e/(e/x))^n \vee x \approx e^n \vee x \approx e$. On the other hand, $\mathcal{V}$ satisfies $\pi_2(x) = (e/(e/x))^n \vee x \approx x$ and $\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e \approx ((e \wedge x)^n/(e \wedge x)^n) \wedge e \approx e$.

If $n \geq m$ and $x \in \mathbf{T}_m$, then, $\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e = (0/0) \wedge e = 1 \wedge e = e$, for $x \neq e, 1$, $\pi_1(e) = e$ and $\pi_1(1) = (e/e) \wedge e = e$. Thus, $\mathcal{V}(\mathbf{T}_m)$ satisfies $\pi_1(x) \approx e$. Moreover, if $x \in \mathbf{T}_m$, then $\pi_2(x) = (e/(e/x))^n \vee x = (e/1)^n \vee x = 0 \vee x = x$, for $x \neq e, 1$, $\pi_2(e) = e$ and $\pi_2(1) = (e/(e/1))^n \vee 1 = 1$. So $\mathcal{V}(\mathbf{T}_m)$ satisfies $\pi_2(x) \approx x$. If we pick $n \geq \max\{i_1, ..., i_k\}$ then $\mathcal{V}(\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, ..., \mathbf{T}_{i_k})$ satisfies $\pi_1(x) = e$ and $\pi_2(x) = x$. $\square$

Note that $\mathcal{IRL} \vee \mathcal{V}(\mathbf{T}_1) \neq \mathcal{IRL} \times \mathcal{V}(\mathbf{T}_1)$ since $\mathbf{A} \in \mathbf{S}(\mathbf{T}_1 \times \mathbf{2}) - (\mathcal{IRL} \times \mathcal{V}(\mathbf{T}_1))$, where $A = \{(1,1), (1,e), (1,0), (0,0)\}$.

Idempotent atoms

It is well known and easy to observe that the variety $\mathbf{V}(\mathbb{Z})$ generated by the ℓ-group of the integers under addition is the only ℓ-group atom. It is shown in [BCGJT] that the variety $\mathbf{V}(\mathbb{Z}^-)$ generated by the negative cone of $\mathbb{Z}$ is the only atom below the variety of negative cones of ℓ-groups. Both of these atoms are cancellative. It follows from Proposition 4.2 that they are actually the only atoms below the variety of cancellative residuated lattices.

Corollary 5.9. *The varieties $\mathbf{V}(\mathbb{Z})$ and $\mathbf{V}(\mathbb{Z}^-)$ are the only cancellative atoms.*

In view of this observation, it makes sense to investigate the other end of the spectrum of atoms, i.e., varieties that are n-potent for some n. We will provide a continuum of idempotent atoms, that are actually distributive.

For every set of integers S, set $N_S = \{a_i \mid i \in \mathbb{Z}\} \cup \{b_i \mid i \in \mathbb{Z}\} \cup \{e\}$. We define an order on N_S, by $b_i < b_j < e < a_k < a_l$, for all $i, j, k, l \in \mathbb{Z}$, such that $i < j$ and $k > l$. Obviously, this is a total order on N_S. We also define a multiplication by

$$a_i a_j = a_{min\{i,j\}}, \qquad b_i b_j = b_{min\{i,j\}}$$

and

$$b_j a_i = \begin{cases} b_j & \text{if } j < i, \text{ or } i = j \in S \\ a_i & \text{if } i < j, \text{ or } i = j \notin S \end{cases}, \qquad a_i b_j = \begin{cases} a_i & \text{if } i < j, \text{ or } i = j \in S \\ b_j & \text{if } i > j, \text{ or } i = j \notin S \end{cases}$$

The division operations are defined in the usual way by $x/y = \bigvee\{z \mid xz \leq y\}$ and $y\backslash x = \bigvee\{z \mid zx \leq y\}$.

It is easy to see that multiplication is associative, order preserving and residuated. So, we can define a residuated lattice $\mathbf{N}_S$ with underlying set N_S and operations the ones described above.

We will investigate for which sets S the variety generated by $\mathbf{N}_S$ is an atom in the subvariety lattice of residuated lattices.

Define the following residuated lattice terms:

$$\ell(x) = x\backslash e, \; r(x) = e/x,$$

$$t(x) = e/x \vee x\backslash e,$$

$$m(x) = \ell\ell(x) \wedge \ell r(x) \wedge r\ell(x) \wedge rr(x),$$

$$p(x) = \ell\ell(x) \vee \ell r(x) \vee r\ell(x) \vee rr(x).$$

Moreover, define three binary relations by,

$$x \xrightarrow{r} y \;\Leftrightarrow\; r(x) = y,$$

$$x \xrightarrow[\ell]{} y \;\Leftrightarrow\; \ell(x) = y,$$

$$x \rightarrow y \;\Leftrightarrow\; r(x) = y \text{ or } \ell(x) = y.$$

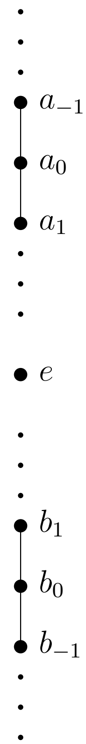

Figure 4: The residuated lattice $\mathbf{N}_S$.

A *word* over $\{0,1\}$ is a function $w : A \to \{0,1\}$, where A is a subinterval of $\mathbb{Z}$; A is called the *support*, *supp*(w), of w. We call w *finite* (*infinite*, *bi-infinite*) if $|A| < \omega$ ($A = \mathbb{N},\ A = \mathbb{Z}$, respectively). If w is a bi-infinite and v a finite word, we say v is a *subword* of w, if there exists an integer k, such that $v(i) = w(i+k)$ for all $i \in supp(v)$. Note that the characteristic function w_S of a subset S of $\mathbb{Z}$ is a bi-infinite word. For two bi-infinite words w_1, w_2, define $w_1 \leq w_2$ iff every finite subword of w_1 is a subword of w_2. Obviously, $\leq$ is a pre-order. Define $w_1 \cong w_2$ iff $w_1 \leq w_2 \leq w_1$. We call a bi-infinite word w *minimal* with respect to the pre-order $\leq$, if $w \cong w'$, whenever $w \leq w'$, for some bi-infinite word w'.

Proposition 5.10. *The following properties hold for $\mathbf{N}_S$, for every S.*

1. *For all $i \in \mathbb{Z}$, $m(b_i) = b_{i-1}$, $p(b_i) = b_{i+1}$, $m(a_i) = a_{i+1}$, $p(a_i) = a_{i-1}$. Moreover, $t(a_i) = b_i$ and $t(b_i) = a_i$.*

2. *It is totally ordered.*

3. *For every x, $\{xt(x), t(x)x\} = \{x, t(x)\}$.*

4. *If $x < e < y$, then $m(x) \prec x \prec p(x) < e < m(y) \prec y \prec p(y)$ and $t(y) < e < t(x)$.*

5. *For every x, $m(t(x)) = t(p(x))$, $p(t(x)) = t(m(x))$, $m(p(x)) = p(m(x)) = x$ and $t(t(x)) = x$.*

6. *If x is negative, then $xy = yx = \begin{cases} x & \text{for } x \leq y < t(x) \\ y & \text{for } y \leq x \text{ or } t(x) < y. \end{cases}$*

   If x is positive, then $xy = yx = \begin{cases} x & \text{for } t(x) < y \leq x \\ y & \text{for } y < t(x) \text{ or } x \leq y. \end{cases}$

7. *For all x, y; $x \wedge y, x \vee y, xy \in \{x, y\}$.*

8. *For all x, y; $x/y, y\backslash x \in \{x, m(x), p(x), t(x), m(t(x)), t(y), m(t(y)), p(t(y)\}$.*

9. *For every finite word v there exists a universal first order formula $\phi(v)$, such that v is not a subword of w_S iff $\phi(v)$ is satisfied in $\mathbf{N}_S$.*

Proof. It is easy to see that

$$b_{i-1} \underset{\ell}{\leftarrow} a_i \underset{\ell}{\overset{r}{\longleftrightarrow}} b_i \overset{r}{\rightarrow} a_{i+1} \qquad (i \in S)$$

$$b_{i-1} \overset{r}{\leftarrow} a_i \underset{\ell}{\overset{r}{\longleftrightarrow}} b_i \underset{\ell}{\rightarrow} a_{i+1} \qquad (i \notin S)$$

It follows directly that $t(b_i) = a_i \vee a_{i+1} = a_i$ and $t(a_i) = b_{i-1} \vee b_i = b_i$.
Moreover,

$$\{r(r(b_i)), r(\ell(b_i)), \ell(r(b_i)), \ell(\ell(b_i))\} = \{b_{i-1}, b_i, b_{i+1}\},$$

so $m(b_i) = b_{i-1}$ and $p(b_i) = b_{i+1}$. Similarly, $m(a_i) = a_{i+1}$ and $p(a_i) = a_{i-1}$. So (1) holds. Moreover, (2) is obvious from the definition; (3)-(7) follow from (1); and (8) is easy to check. Finally for (9), the first order formula associated to a finite word v is

$$\phi_v = (\ \forall x_1, ...x_n, y_1, ..., y_n)[(x_1 \prec x_2 \prec ... \prec x_n < e < y_n \prec ... \prec y_1)$$

$$\&\ (t(x_1) = y_1\ \& \cdots \&\ t(x_n) = y_n) \rightarrow \neg(x_1 y_1 = s_1\ \& \cdots \&\ x_n y_n = s_n)],$$

where n is the length of v and $s_i = x_i$, if $v(i) = 1$ and $s_i = y_i$ if $v(i) = 0$. Note that ϕ_v is equivalent to a universally quantified formula in the language of residuated lattices. □

Corollary 5.11. *The residuated lattice $\mathbf{N}_S$ is strictly simple, for every set of integers S.*

Proof. For all $a, b \in N_S - \{e\}$, (a, b) is in the transitive closure of the relation $\rightarrow$ defined above. Thus, $\mathbf{N}_S$ is strictly simple. □

Lemma 5.12. *Every non-trivial one-generated subalgebra of an ultrapower of* $\mathbf{N}_S$ *is isomorphic to* $\mathbf{N}_{S'}$*, for some set of integers* S'*.*

Proof. Every first order formula true in $\mathbf{N}_S$ is also true in an ultrapower of it. Since properties (2)-(8) of Proposition 5.10 can be expressed as first order formulas, they hold in any ultrapower of $\mathbf{N}_S$.

By property (2), any ultrapower $\mathbf{B}$ of $\mathbf{N}_S$ is totally ordered, so the same holds for every subalgebra of $\mathbf{B}$. Let $\mathbf{A}$ be a non-trivial one-generated subalgebra of $\mathbf{B}$ and let a be a generator for $\mathbf{A}$. The element a can be taken to be negative, since if a is positive, $t(a)$ is negative and, by property (4), it generates $\mathbf{A}$, because $t(t(a)) = a$, by property (5).

By properties (7) and (8), A is the set of evaluations of terms composed by the terms m, p, t and the constant term e. By property (5), these compositions reduce to one of the forms $m^n(x), p^n(x), p^n(t(x))$ and $m^n(t(x))$, for n a natural number.

Set $b_{-n} = m^n(a), b_n = p^n(a), a_{-n} = p^n(t(a))$ and $a_n = m^n(t(a))$, for all natural numbers n. By the remark above, A consists of exactly these elements together with e. Define a subset S' of $\mathbb{Z}$, by $m \in S'$ iff $b_m a_m = b_m$ and consider the following map $f : A \to \mathbf{N}_{S'}$, $f(b_i) = b'_i$, $f(a_i) = a'_i, f(e) = e'$, where $N_{S'} = \{b'_i \mid i \in \mathbb{Z}\} \cup \{a'_i \mid i \in \mathbb{Z}\} \cup \{e'\}$. By property (4), this map is an order isomorphism and, since $\mathbf{A}$ is totally ordered, a lattice isomorphism, as well. Moreover, it is easy to check that it is a monoid homomorphism, using properties (3) and (6). Any lattice isomorphism preserves existing joins, so f preserves the two division operations. Thus, $\mathbf{A}$ is isomorphic to $\mathbf{N}_{S'}$. $\square$

Theorem 5.13. *Let* $\mathbf{A}$ *be a one-generated residuated lattice and* S *a subset of* $\mathbb{Z}$*. Then,* $\mathbf{A} \in \mathbf{HSP}_u(\mathbf{N}_S)$ *iff* $\mathbf{A} \cong \mathbf{N}_{S'}$*, for some* S' *such that* $w_{S'} \leq w_S$*.*

Proof. Let S' be a set of integers, such that $w_{S'} \leq w_S$. Also, let $\mathbf{B} = (\mathbf{N}_S)^{\mathbb{N}}/\mathcal{F}$, where $\mathcal{F}$ is an ultrafilter over $\mathbb{N}$ that extends the filter of co-finite subsets, and $N_S = \{b_i | i \in \mathbb{Z}\} \cup \{a_i | i \in \mathbb{Z}\} \cup \{e\}$. We will show that $\mathbf{N}_{S'} \in \mathbf{ISP}_u(\mathbf{N}_S)$.

For every natural number n, define the finite approximations, v_n, of the bi-infinite word $w_{S'}$, by $v_n(i) = w_{S'}(i)$, for all $i \in [-n, n]_{\mathbb{Z}}$. Since, $w_{S'} \leq w_S$, the words v_n are subwords of w_S, so for every natural number n there exists an integer K_n, such that $v_n(i) = w_S(K_n + i)$, for all $i \in supp(v_n) = [-n, n]_{\mathbb{Z}}$.

Let $\bar{b} = (b_{K_n})_{n \in \mathbb{N}}$, where $b_{K_n} \in N_S$. By Lemma 5.12, the subalgebra of $\mathbf{B}$ generated by $\tilde{b} = [\bar{b}]$, the equivalence class of b under $\mathcal{F}$, is isomorphic to $\mathbf{N}_{\tilde{S}}$, $N_{\tilde{S}} = \{\tilde{b}_i \mid i \in \mathbb{Z}\} \cup \{\tilde{a}_i \mid i \in \mathbb{Z}\} \cup \{\tilde{e}\}$, for some subset $\tilde{S}$ of $\mathbb{Z}$. We identify the subalgebra generated by $\tilde{b}$ with $\mathbf{N}_{\tilde{S}}$ and we can actually choose $\tilde{S}$ such that $\tilde{b}_0 = \tilde{b}$. We will show that $\tilde{S} = S'$.

We pick representatives $\bar{b}_m$ and $\bar{a}_m$, for $\tilde{b}_m$ and $\tilde{a}_m$, and we adopt a double subscript notation for their coordinates. So, there exist $\bar{b}_{mn}$ and $\bar{a}_{mn}$ in $\mathbf{N}_S$, such that $\tilde{b}_m = [\bar{b}_m] =$

$[(\bar{b}_{mn})_{n\in\mathbb{N}}]$ and $\tilde{a}_m = [\bar{a}_m] = [(\bar{a}_{mn})_{n\in\mathbb{N}}]$.

It is easy to prove that $\tilde{b}_m = [(b_{K_n+m})_{n\in\mathbb{N}}]$ and $\tilde{a}_m = [(a_{K_n+m})_{n\in\mathbb{N}}]$, using the definition of $\tilde{b}$, Proposition 5.10(1), basic induction and the following facts:

$$\tilde{a}_m = t(\tilde{b}_m) = t([(\bar{b}_{mn})_{n\in\mathbb{N}}]) = [(t(\bar{b}_{mn}))_{n\in\mathbb{N}}]$$

$$\tilde{b}_{m+1} = p(\tilde{b}_m) = p([(\bar{b}_{mn})_{n\in\mathbb{N}}]) = [(p(\bar{b}_{mn}))_{n\in\mathbb{N}}]$$

$$\tilde{b}_{m-1} = m(\tilde{b}_m) = m([(\bar{b}_{mn})_{n\in\mathbb{N}}]) = [(m(\bar{b}_{mn}))_{n\in\mathbb{N}}]$$

Now, for $|m| < n$, i.e., $m \in supp(v_n)$, we have

$$\begin{aligned} K_n + m \in S \quad &\Leftrightarrow\ w_s(K_n + m) = 1 \\ &\Leftrightarrow\ v_n(m) = 1 \\ &\Leftrightarrow\ w_{S'}(m) = 1 \\ &\Leftrightarrow\ m \in S'. \end{aligned}$$

Since, $b_{K_n+m}a_{K_n+m} = b_{K_n+m}$ exactly when $K_n + m \in S$, we get that if $|m| < n$, then $b_{K_n+m}a_{K_n+m} = b_{K_n+m}$ is equivalent to $m \in S'$.

In other words,

$$\{n \mid |m| < n\} \subseteq \{n \mid b_{K_n+m}a_{K_n+m} = b_{K_n+m} \ \Leftrightarrow\ m \in S'\}.$$

Since the first set is in $\mathcal{F}$, so is the second one. It is not hard to check that this means that: $\{n \mid b_{K_n+m}a_{K_n+m} = b_{K_n+m}\} \in \mathcal{F}$ is equivalent to $m \in S'$. So, $\tilde{b}_m\tilde{a}_m = \tilde{b}_m$ is equivalent to $m \in S'$; hence $m \in \tilde{S}$ iff $m \in S'$. Thus, $\tilde{S} = S'$

For the converse, we will prove the implication for $\mathbf{A} \in \mathbf{SP}_u(\mathbf{N}_S)$. This is sufficient since under a homomorphism every one generated subalgebra will either map isomorphically or to the identity element, because of the strictly simple nature of the algebras $\mathbf{N}_{S'}$. Let $\mathbf{A}$ be a subalgebra of an ultrapower of $\mathbf{N}_S$. By Lemma 5.12, $\mathbf{A}$ is isomorphic to $\mathbf{N}_{S'}$, for some subset S' of $\mathbb{Z}$.

To show that $w_{S'} \leq w_S$ it suffices to show that, for every finite word v, if v is not a subword of w_S, then it is not a subword of $w_{S'}$ either. If v is not a subword of w_S, then $\mathbf{N}_S$ satisfies ϕ_v of Proposition 5.10(9); hence so does every ultrapower of $\mathbf{N}_S$. Since ϕ_v is universally quantified it is also satisfied by any subalgebra of an ultrapower of $\mathbf{N}_S$ and in particular by $\mathbf{N}_{S'}$. Thus, v is not a subword of $w_{S'}$. $\square$

Corollary 5.14. *Let S, S' be sets of integers, then*

1. $\mathbf{V}(\mathbf{N}_{S'}) \subseteq \mathbf{V}(\mathbf{N}_S)$ *if and only if* $w_{S'} \leq w_S$, *and*

2. *if* w_S *is minimal with respect to* $\leq$, *then* $\mathcal{V} = \mathbf{V}(\mathbf{N}_S)$ *is an atom in the subvariety lattice of* $\mathcal{RL}$.

Proof. 1) If $w_S \leq w_{S'}$ then, by Theorem 5.13, $\mathbf{N}_{S'} \in \mathbf{HSP}_u(\mathbf{N}_S) \subseteq \mathbf{V}(\mathbf{N}_S)$, so $\mathbf{V}(\mathbf{N}_{S'}) \subseteq \mathbf{V}(\mathbf{N}_S)$. Conversely, if $\mathbf{V}(\mathbf{N}_{S'}) \subseteq \mathbf{V}(\mathbf{N}_S)$, then $\mathbf{N}_{S'} \in \mathbf{V}(\mathbf{N}_S)$. $\mathbf{N}_{S'}$ is subdirectly irreducible, by Lemma 5.11, so, by Jónsson's Lemma, $\mathbf{N}_{S'} \in \mathbf{HSP}_u(\mathbf{N}_S)$. By Theorem 5.13, $w_{S'} \leq w_S$.

2) If $\mathbf{L}$ is a subdirectly irreducible of $\mathcal{V}$, then $\mathbf{L} \in \mathbf{HSP}_u(\mathbf{N}_S)$, by Jónsson's Lemma. Every one-generated subalgebra $\mathbf{A}$ of $\mathbf{L}$ is a member of $\mathbf{SHSP}_u(\mathbf{N}_S) \subseteq \mathbf{HSP}_u(\mathbf{N}_S)$, because $\mathbf{SH} \leq \mathbf{HS}$; so, by Theorem 5.13, $\mathbf{A}$ is isomorphic to some $\mathbf{N}_{S'}$, where $w_{S'} \leq w_S$. Since w_S is minimal with respect to the pre-order $\leq$, we have $w_{S'} \cong w_S$; hence $\mathbf{V}(\mathbf{N}_{S'}) = \mathbf{V}(\mathbf{N}_S)$, by (i). Thus, $\mathcal{V} = \mathbf{V}(\mathbf{N}_{S'}) = \mathbf{V}(\mathbf{A}) \subseteq \mathbf{V}(\mathbf{L}) \subseteq \mathcal{V}$. Since $\mathcal{V} = \mathbf{V}(\mathbf{L})$, for every subdirectly irreducible $\mathbf{L}$ in $\mathcal{V}$, $\mathcal{V}$ is an atom. □

The following corollary generalizes a result of [JT].

Corollary 5.15. *There are uncountably many atoms in the subvariety lattice of* $\mathcal{RL}^C \cap \mathtt{Mod}(x^2 \approx x)$.

Proof. There are uncountably many minimal bi-infinite words that are not related by $\cong$, by [Lo]. □

The proof of the previous result relies heavily on the non-commutativity of the generating algebras. If we add the restriction of commutativity or even the weaker condition $e/x \approx x\backslash e$, we get only finitely many atoms, actually only two, even without the hypothesis that they are in $\mathcal{RL}^C$.

Theorem 5.16. *The only atoms below the variety* $\mathtt{Mod}(x^2 \approx x, e/x \approx x\backslash e)$ *are the varieties generated by the residuated lattices on the chains* $\{0, e\}$ *and* $\{0, e, \top\}$.

Proof. Assume $\mathbf{A}$ is a non trivial member of $\mathrm{Mod}(x^2 \approx x, e/x \approx x\backslash e)$ and let a be a negative element of $\mathbf{A}$. If $e/a = e$, then $\{e, a\}$ is a subalgebra of $\mathbf{A}$. If $e < e/a$, set $T = e/a$ and $b = e/T$. We have $a \leq b \leq bT = (e/T)T \leq e$ and $bT = bbT \leq b$, so $bT = b$. Since $e/T = T\backslash e$, we also get $Tb = b$. Additionally, $T \leq e/b$. If $S = e/b$, then $Sa \leq Sb \leq e$, so $S \leq e/a = T$; thus, $T = e/b$. Moreover, $b \leq b/T \leq (b/T)T \leq b$, so $b/T = b$. Also, $a \leq aa \leq ba \leq a$, so $T/b = (e/a)/b = e/ba = e/a = T$. Thus, $\{b, e, T\}$ is a subalgebra of $\mathbf{A}$. □

CHAPTER VI

DISTRIBUTIVE RESIDUATED LATTICES

Many interesting examples of residuated lattices are distributive. In this section we discuss the undecidability of the quasi-equational theory of the variety $\mathcal{DRL}$ of distributive residuated lattices and provide a duality theory for distributive residuated bounded-lattices.

We start with some sufficient conditions for distributivity.

Corollary 6.1. *[BCGJT] For residuated lattices, any of the following sets of identities implies the distributive law:*

1. $x/x \approx e$ *and* $(x \vee y)/z \approx x/z \vee y/z$
2. $x(x\backslash(x \wedge y)) \approx x \wedge y$
3. $x\backslash xy \approx y$, $xy \approx yx$ *and* $x(y \wedge z) \approx xy \wedge xy$.

Undecidability of the quasi-equational theory

The results in this section are due to the author and can be found in [Ga]. We present only the main theorem and its consequences and refer the reader to the paper for details.

Let $\mathbf{V}$ be a vector space. The residuated lattice $\mathcal{P}(\mathbf{V})$ on the power set of the monoid reduct of $\mathbf{V}$, given in Example 3.15, is distributive.

Theorem 6.2. *[Ga] Let $\mathcal{V}$ be a variety of distributive residuated lattices, containing $\mathcal{P}(\mathbf{V})$, for some infinite-dimensional vector space $\mathbf{V}$. Then, there is a finitely presented residuated lattice in $\mathcal{V}$, with unsolvable word problem.*

The proof of the theorem uses the notion of an n-frame and results on distributive lattices to reduce the decidability of the word problem for semigroups to the decidability of the word problem for distributive residuated lattices.

Corollary 6.3. *[Ga] If $\mathcal{V}$ is a variety such that $\mathbf{HSP}(\mathcal{P}(\mathbf{V})) \subseteq \mathcal{V} \subseteq \mathcal{DRL}$, for some infinite-dimensional vector space $\mathbf{V}$, then $\mathcal{V}$ has an undecidable quasi-equational theory.*

Corollary 6.4. *[Ga] The word problem and the quasi-equational theory of distributive and for commutative distributive residuated lattices are undecidable.*

This result becomes more interesting given that, as recently proved, the equational theory of commutative distributive residuated lattices is decidable. The details can be found in [GR].

Duality Theory for distributive residuated bounded-lattices

In what follows we try to extend the Priestley duality for bounded distributive lattices to distributive residuated bounded-lattices. The ideas stem from [Ur], where a duality theory for *bounded distributive lattice-ordered semigroups* is developed.

Priestley duality

H. Priestley introduced, see e.g. [DP], a duality between the category of bounded distributive lattices and certain ordered topological spaces. The theory is useful because it presents an alternative understanding of distributive lattices and suggests a different approach to problems about them.

A structure $\mathbf{S} = \langle S, \tau, \leq \rangle$ is called a *Priestley space* if $\langle S, \tau \rangle$ is a compact topological space, $\langle S, \leq \rangle$ is a bounded partially ordered set and $\mathbf{S}$ is *totally order-disconnected*, i.e., for all $x, y \in S$, if $x \not\leq y$, then there exists a clopen increasing set containing y, but not x.

A map $h : \mathbf{S}_1 \to \mathbf{S}_2$ between two Priestley spaces is a *Priestley map* if it is order-preserving, continuous and preserves the bounds.

If $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice, then its *dual space* is the structure $\bar{\mathcal{S}}(\mathbf{L}) = \langle \mathcal{S}(L), \tau, \leq \rangle$, where $\mathcal{S}(L)$ is the set of all prime filters of $\mathbf{L}$; τ is the topology having the family of all sets of the form $f(l)$ and $(f(l))^c$, $l \in L$, as sub-basis, where $f(l) = \{X \in \mathcal{S}(L) |\ l \in X\}$, for $l \in L$; and $\leq$ is set inclusion.

If $\mathbf{S} = \langle S, \tau, \leq \rangle$ is a Priestley space then its *dual lattice* is the structure $\bar{\mathcal{L}}(\mathbf{S}) = \langle \mathcal{L}(S), \cap, \cup, \emptyset, S \rangle$, where $\mathcal{L}(S)$ is the set of all clopen increasing subsets of S.

Theorem 6.5. *[DP]*

1. *The dual space of a bounded distributive lattice* $\mathbf{L}$ *is a Priestley space and* $\mathcal{L}(\mathcal{S}(L)) = \{f(l) |\ l \in L\} \cup \{\emptyset, \mathcal{S}(L)\}$.

2. *The dual lattice of a Priestley space is a bounded distributive lattice.*

Theorem 6.6. *[DP]*

1. *If* $\mathbf{L}_1, \mathbf{L}_2$ *are bounded distributive lattices and* $h : \mathbf{L}_1 \to \mathbf{L}_2$ *is a bounded-lattice homomorphism, then the map* $\bar{\mathcal{S}}(h) : \bar{\mathcal{S}}(\mathbf{L}_2) \to \bar{\mathcal{S}}(\mathbf{L}_1)$, *defined by* $\bar{\mathcal{S}}(h)(X) = h^{-1}[X]$, *is a Priestley map.*

2. *If* $\mathbf{S}_1, \mathbf{S}_2$ *are Priestley spaces and* $h : \mathbf{S}_1 \to \mathbf{S}_2$ *is a Priestley map, then the map* $\bar{\mathcal{L}}(h) : \bar{\mathcal{L}}(\mathbf{S}_2) \to \bar{\mathcal{L}}(\mathbf{S}_1)$, *defined by* $\bar{\mathcal{L}}(h)(A) = h^{-1}[A]$, *is a bounded-lattice homomorphism.*

Theorem 6.7. *[DP] The categories of bounded distributive lattices with bounded-lattice homomorphisms and of Priestley spaces with Priestley maps are dual.*

Duality for distributive residuated bounded-lattices

For X, Y subsets of a residuated lattice, we denote by $X \cdot Y$ their complex product and define $X \bullet Y = \uparrow (X \cdot Y)$. Note that if Z is a filter then $X \cdot Y \subseteq Z$ iff $X \bullet Y \subseteq Z$.

Lemma 6.8. *If X, Y are filters in a residuated lattice, then $X \bullet Y$ is also a filter.*

Proof. The set $X \bullet Y$ is obviously increasing. Moreover, if $k_1, k_2 \in X \bullet Y$, then $a_1 b_1 \leq k_1$ and $a_2 b_2 \leq k_2$, for some $a_1, a_2 \in X$, $b_1, b_2 \in Y$. Thus,

$$(a_1 \wedge a_2)(b_1 \wedge b_2) \leq (a_1 \wedge a_2) b_1 \wedge (a_1 \wedge a_2) b_2 \leq a_1 b_1 \wedge a_2 b_2 \leq k_1 \wedge k_2$$

and $a_1 \wedge a_2 \in X$, $b_1 \wedge b_2 \in Y$; hence, $k_1 \wedge k_2 \in X \bullet Y$. □

Lemma 6.9. *If X, Y, Z are filters in a distributive residuated lattice $\mathbf{L}$, Z is a prime filter and $XY \subseteq Z$, then there are prime filters X', Y', such that $X \subseteq X'$, $Y \subseteq Y'$, $X'Y \subseteq Z$ and $XY' \subseteq Z$.*

Proof. We first show that $I = \{l \in L | \, lY \not\subseteq Z\}$ is a down-set. Indeed, if $l' \leq l \in I$ and $l' \notin I$, then $l'Y \subseteq Z$, i.e., $l'y \in Z$, for all $y \in Y$. Since $l'y \leq ly$ and Z is a filter, we have $ly \in Z$ for all $y \in Y$, i.e., $lY \subseteq Z$. So, $l \notin I$, a contradiction.

Furthermore, if $l_1, l_2 \in I$ then $l_1 Y \not\subseteq Z$ and $l_2 Y \not\subseteq Z$, i.e., $l_1 y_1 \notin Z$ and $l_1 y_1 \notin Z$, for some $y_1, y_2 \in Y$. Since Z is prime, we have $l_1 y_1 \vee l_2 y_2 \notin Z$. Moreover,

$$(l_1 \vee l_2)(y_1 \wedge y_2) = l_1(y_1 \wedge y_2) \vee l_2(y_1 \wedge y_2) \leq l_1 y_1 \vee l_2 y_2,$$

so $(l_1 \vee l_2)(y_1 \wedge y_2) \notin Z$, because Z is a filter. Consequently, $(l_1 \vee l_2)Y \not\subseteq Z$, i.e., $l_1 \vee l_2 \in I$. Thus, I is an ideal.

Note that $X \cap I = \emptyset$, since $xY \subseteq Z$ for all x in X, so by the Prime Ideal Theorem, there exists a prime filter X', such that $X \subseteq X'$ and $X' \cap I = \emptyset$, i.e., $x \notin I$, for all x in X'. So $xY \subseteq Z$, for all x in X', i.e., $X'Y \subseteq Z$. The existence of Y' is proved similarly. □

If R is a ternary relation on a set S, x, y, z are elements of S and A, B are subsets of S, we write $R(x, y, z)$ for $(x, y, z) \in R$. Moreover, we define

$$R[A, B, \_] = \{z \in S | \, (\exists x \in A)(\exists y \in B)(R(x, y, z))\},$$

$$R[x, B, \_] = R[\{x\}, B, \_] \text{ and } R[A, y, \_] = R[A, \{y\}, \_].$$

Multiplication in a residuated lattice corresponds to a suitable ternary relation in the dual space. Recall the definition of $\mathcal{L}(\mathbf{S})$.

A *distributive residuated bounded-lattice space* (*DRbL-space*) is a structure $\mathbf{S} = \langle S, \tau, \leq, R, E\rangle$, where $\langle S, \tau, \leq\rangle$ is a Priestley space, R is a ternary relation on S and $E \subseteq S$, such that for all $x, y, z, w \in S$ the following properties hold:

1. $R(x, y, u)$ and $R(u, z, w)$ for some $u \in S$ iff $R(y, z, v)$ and $R(x, v, w)$ for some $v \in S$.
2. If $x \leq y$ and $R(y, z, w)$, then $R(x, z, w)$;
   if $x \leq y$ and $R(z, y, w)$, then $R(z, x, w)$; and
   if $x \leq y$ and $R(z, w, x)$, then $R(z, w, y)$.
3. The failure of $R(x, y, z)$ is witnessed by some $A, B \in \mathcal{L}(S)$, namely $x \in A$, $y \in B$ and $z \notin R[A, B, \_]$.
4. If $A, B \in \mathcal{L}(S)$ then the sets $R[A, B, \_]$, $\{z \in S | R[z, B, \_] \subseteq A\}$ and $\{z \in S | R[B, z, \_] \subseteq A\}$ are clopen.
5. $E \in \mathcal{L}(S)$ and for all $K \in \mathcal{L}(S)$, $R[K, E, \_] = R[E, K, \_] = K$.

The dual space of a distributive residuated bounded-lattice is an extension of the Priestley space of the distributive lattice reduct. Note that the third condition states that the space is totally disconnected with respect to the ternary relation. This condition for ternary relations is the analogue of the assumption that the space is totally disconnected with respect to the order. The first and last conditions are reminiscent of the conditions in Example 3.16.

Let $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, 0, 1\rangle$ be a distributive residuated bounded-lattice. We denote by $\mathcal{S}(L)$ the set of all prime filters of $\mathbf{L}$ and set

$$f(k) = \{X \in \mathcal{S}(L) | \, k \in X\},$$

for every $k \in L$. Let τ be the topology whose sub-basis is the family of all sets of the form $f(l)$ and $(f(l))^c$, $l \in L$. Also, let $E = f(e)$ and $R(X, Y, Z) = (X \bullet Y \subseteq Z)$, for all $X, Y, Z \in \mathcal{S}(L)$. The structure $\mathcal{S}(\mathbf{L}) = \langle \mathcal{S}(L), \tau, \subseteq, R, E\rangle$ is called the *dual space* of $\mathbf{L}$.

Let $\mathbf{S} = \langle S, \tau, \leq, R, E\rangle$ be a DRbL-space and $\mathcal{L}(S)$, as mentioned before, the set of all clopen increasing subsets of S. For $A, B \in \mathcal{L}(\mathbf{S})$, define the operations $A \circ B = R[A, B, \_]$, $A/B = \{z \in S \mid z \circ B \subseteq A\}$, and $B\backslash A = \{z \in S \mid B \circ z \subseteq A\}$. The structure $\mathcal{L}(\mathbf{S}) = \langle \mathcal{L}(S), \cap, \cup, \circ, \backslash, /, E, \emptyset, S\rangle$, is called the *dual algebra* of $\mathbf{S}$.

In the proofs of the results in this section we will use variables for different structures. To make the exposition as clear as possible, we use specific letters for each set of variables. In particular, we use lower-case letters toward the end of the alphabet for elements of DRbL-spaces and letters in the first part of the alphabet for bounded residuated lattices. For the dual of a structure we use uppercase letters and for the second dual lower-case Greek letters. For example we will use x, y, z for elements of a DRbL-space and A, B, C, for elements of its dual.

Lemma 6.10. *Let* $\mathbf{L}$ *be a bounded distributive residuated lattice. Then, for all* $k, l \in L$,

1. $f(k \cdot l) = f(k) \circ f(l)$,
2. $f(k/l) = f(k)/f(l)$ *and*
3. $f(k \backslash l) = f(k) \backslash f(l)$.

Proof. 1) If $W \in f(k) \circ f(l) = R[f(k), f(l), \_\,]$, then there exist $U \in f(k)$ and $V \in f(l)$ such that $R(U, V, W)$, i.e., there exist prime filters U, V of $\mathbf{L}$ such that $k \in U$, $l \in V$ and $U \cdot V \subseteq W$. Hence, $kl \in U \cdot V \subseteq W$, i.e., $W \in f(kl)$.

Conversely, if $W \in f(kl)$, i.e., W is a prime filter of $\mathbf{L}$ that contains the element kl, then $(\uparrow k)(\uparrow l) \subseteq W$. By Lemma 6.9, there exist prime filters U, V such that $k \in U$, $l \in V$ and $UV \subseteq W$, i.e., there exist $U \in f(k)$ and $V \in f(l)$ such that $R(U, V, W)$; hence $W \in f(k) \circ f(l)$.

2) We have $f(k/l) \circ f(l) = f((k/l)l) = \{W \in \mathcal{S}(L) |\ (k/l)l \in W\}$. Since $(k/l)l \leq k$, we have $f(k/l) \circ f(l) \subseteq \{W \in \mathcal{S}(L) \,|\, k \in W\} = f(k)$. So, if $X \in f(k/l)$, then $\{X\} \circ f(l) \subseteq f(k)$, i.e., $X \in f(k)/f(l)$.

Conversely, assume that $X \in f(k)/f(l)$ and let $\overline{U} = \{\bar{l} \in L \mid (\exists a \in X)(a\bar{l} \leq k)\}$. If $l' \leq \bar{l} \in \overline{U}$, then there is an $a \in X$ such that $a\bar{l} \leq k$. So, $al' \leq a\bar{l} \leq k$, hence $l' \in \overline{U}$. Moreover, if $l_1, l_2 \in \overline{U}$, then $a_1 l_1 \leq k$ and $a_2 l_2 \leq k$, for some $a_1, a_2 \in X$. Note that

$$(a_1 \wedge a_2)(l_1 \vee l_2) = (a_1 \wedge a_2)l_1 \vee (a_1 \wedge a_2)l_2 \leq a_1 l_1 \vee a_2 l_2 \leq k \vee k = k$$

and $(a_1 \wedge a_2) \in X$, since X is a filter; so, $l_1 \vee l_2 \in \overline{U}$. Consequently, $\overline{U}$ is an ideal.

We will show that $l \in \overline{U}$. Assume, by way of contradiction, that $l \notin \overline{U}$. Then $\overline{U} \cap \uparrow l = \emptyset$ and, by the Prime Ideal Theorem, there exists a prime filter U such that $\uparrow l \subseteq U$ and $U \cap \overline{U} = \emptyset$. In particular, U is contained in the complement of $\overline{U}$, hence for all $l' \in U$ and for all $a \in X$, we have $a\bar{l} \not\leq k$. Consequently, $k \notin X \bullet U$, hence $\downarrow k \cap X \bullet U = \emptyset$. Since $X \bullet U$ is a filter by Lemma 6.8, there is a prime filter V such that $X \bullet U \subseteq V$ and $\downarrow k \cap V = \emptyset$.

Note that $l \in U$, since $l' \in U'$, and that $k \notin V$, since $\downarrow k \cap V = \emptyset$. Summarizing, there exist prime filters U and V such that $X \bullet U \subseteq V$, $U \in f(l)$ and $V \notin f(k)$. This shows that $R[X, f(l), \_\,] \nsubseteq f(k)$, i.e., $X \circ f(l) \nsubseteq f(k)$, a contradiction, since $\{X\} \in f(k)/f(l)$.

Consequently, $l \in \overline{U}$, i.e., $al \leq k$, for some $a \in X$; hence $a \leq k/l$, for some $a \in X$. Since X is a filter, we get $(k/l) \in X$, i.e., $X \in f(k/l)$.

3) We obtain the proof of the last property in a similar way. □

Theorem 6.11.

1. *The dual algebra $\mathcal{L}(\mathbf{S})$ of a DRbL-space $\mathbf{S}$ is a distributive residuated bounded-lattice.*

2. *The dual space $\mathcal{S}(\mathbf{L})$ of a distributive residuated bounded-lattice $\mathbf{L}$ is a DRbL-space.*

Proof. 1) Assume that $\mathbf{S}$ is a DRbL-space. By Priestley duality for bounded distributive lattices, we have that $\mathcal{L}(S)$ has a bounded distributive lattice reduct.

We will show that $\mathcal{L}(S)$ is closed under multiplication and the two division operations. If A, B are clopen and and increasing then $A \circ B, A/B, B\backslash A$ are clopen, by the the fourth property of a DRbL-space. Moreover, if $x \in A \circ B$ and $x \leq y$, then $x \in R[A, B, \_\,]$, i.e., $R(a, b, x)$ for some $a \in A$ and $b \in B$. By the second property of a DRbL-space, we get $R(a, b, y)$, i.e., $y \in A \circ B$. Thus, $A \circ B$ is increasing. Additionally, if $x \in A/B$ and $x \leq y$, then $R[x, B, \_\,] \subseteq A$ and $R[y, B, \_\,] \subseteq R[x, B, \_\,]$. So, $R[y, B, \_\,] \subseteq A$, i.e., $y \in A/B$. Consequently, A/B is increasing. The opposite set $B\backslash A$ is also increasing; thus, $A \circ B, A/B, B\backslash A \in \mathcal{L}(S)$.

To see that multiplication is associative, note that $w \in (A \circ B) \circ C$, namely $w \in R[R[A, B, \_\,], C, \_\,]$, is equivalent to $R(x, y, u)$ and $R(u, z, w)$, for some $u \in L$, $x \in A$, $y \in B$ and $z \in C$. By the first property of a DRbL-space, this is equivalent to $R(x, v, w)$ and $R(y, z, v)$, for some $v \in L$, $x \in A$, $y \in B$ and $z \in C$. Finally, this is in turn equivalent to $w \in R[A, R[B, C, \_\,], \_\,]$, namely $w \in A \circ (B \circ C)$. Thus, $(A \circ B) \circ C = A \circ (B \circ C)$.

To show that $/$ is the right residual of multiplication, we need to show that $A \circ B \subseteq C$ iff $A \subseteq C/B$, i.e., that

$$R[A, B, \_\,] \subseteq C \text{ iff } A \subseteq \{x \in S |\, R[x, B, \_\,] \subseteq C\}.$$

For the forward direction, assume that $x \in A$. If $z \in R[x, B, \_\,]$, then there is a $y \in B$ such that $R(x, y, z)$; thus $z \in R[A, B, \_\,]$. By assumption $R[A, B, \_\,] \subseteq C$, so $z \in C$. Conversely, assume that $z \in R[A, B, \_\,]$. Then, there is an $x \in A$ such that $z \in R[x, B, \_\,]$. By our assumption $R[x, B, \_\,] \subseteq C$, for all $x \in A$, hence $z \in C$.

Likewise, we have that $\backslash$ is the left residual of multiplication. Finally, it follows from the last property of a DRbL-space that E is the multiplicative identity.

2) By Theorem 6.5, $\langle \mathcal{S}(L), \tau, \leq \rangle$ is a Priestley space and $\mathcal{L}(\mathcal{S}(L)) = \{f(l) |\ l \in L\} \cup \{\emptyset, \mathcal{S}(L)\}$. In what follows we verify all the properties of a DRbL-space.

For the first property, suppose $R(X, Y, U)$ and $R(U, Z, W)$ hold, i.e., $X \bullet Y \subseteq U$ and $U \bullet Z \subseteq W$. Let $V' = Y \bullet Z$. If $d \in X \bullet V'$, then there exist $a \in X$ and $g \in V'$ such that $ag \leq d$. Since $V' = Y \bullet Z$, there exist $b \in Y$ and $c \in Z$ such that $bc \leq g$. Moreover, $ab \in U$, since $X \bullet Y \subseteq U$, hence $abc \in U \bullet Z$. Consequently, $ag \in U \bullet Z$, so $d \in W$. Thus, $X \bullet V' \subseteq W$. By Lemma 6.9, there exists a prime filter V such that $X \bullet V \subseteq Z$, $V' \subseteq V$ and $Y \bullet Z \subseteq V$, i.e., $R(X, V, W)$ and $R(Y, Z, V)$.

To see that the second property holds, let $X, Y, U, V \in \mathcal{S}(L)$ and $X \subseteq Y$. If $R(Y, U, V)$ holds, i.e., $Y \bullet U \subseteq V$, then

$$X \bullet U = \uparrow (X \cdot U) \subseteq \uparrow (Y \cdot U) = Y \bullet U \subseteq V,$$

thus $X \bullet U \subseteq V$, i.e., $R(X, U, V)$ holds. The other two implications are proved similarly.

To prove the third property, note first that if there exist $A, B \in \mathcal{L}(S)$, such that $x \in A$, $y \in B$ and $z \notin R[A, B, \_\,]$, then $R(U, V, Z)$ fails, for all $U \in \alpha$ and $V \in \beta$. In particular $R(X, Y, Z)$ is false. Conversely, if $R(X, Y, Z)$ fails for some $X, Y, Z \in \mathcal{S}(L)$, then $\uparrow (X \cdot Y) \not\subseteq Z$, i.e., there exists a c in L such that $c \in \uparrow (X \cdot Y)$ and $c \notin Z$. So, $ab \leq c$ and $c \notin Z$, for some $c \in L$, $a \in X$ and $b \in Y$. If $\alpha = f(a)$, $\beta = f(b)$, then, $X \in \alpha$ and $Y \in \beta$. We will show that $Z \notin R[\alpha, \beta, \_\,]$. If $Z \in R[\alpha, \beta, \_\,]$, then $R(X', Y', Z)$, i.e., $\uparrow (X' \cdot Y') \subseteq Z$, for some $X' \in \alpha, Y' \in \beta$. Since $a \in X'$, $b \in Y'$ and $ab \leq c$, we get $c \in Z$, a contradiction.

The fourth property follows from Lemma 6.10, since if $\alpha, \beta \in \mathcal{L}(\mathcal{S}(L))$, then there exist $a \in L$ and $b \in L$, such that $\alpha = f(a)$ and $\beta = f(b)$). So,

$$R[\alpha, \beta, \_\,] = \alpha \circ \beta = f(a) \circ f(b) = f(ab) \in \mathcal{L}(\mathcal{S}(\mathcal{L}));$$

$$\{c \in X |\ \{c\} \circ \beta \subseteq \alpha\} = \alpha / \beta = f(a)/f(b) = f(a/b) \in \mathcal{L}(\mathcal{S}(\mathcal{L}));$$

and

$$\{c \in X |\ \beta \circ \{c\} \subseteq \alpha\} = \beta \backslash a = f(b) \backslash f(a) = f(b \backslash a) \in \mathcal{L}(\mathcal{S}(\mathcal{L})).$$

To verify the last property, let K be a clopen increasing subset of $\mathcal{S}(L)$; then, $K = f(k)$, for some $k \in L$. We have,

$$R[K, E, \_\,] = R[f(k), f(e), \_\,] = f(k) \circ f(e) = f(ke) = f(k) = K$$

and similarly $R[E, K, \_\,] = K$. $\square$

The following theorem shows that we can recover the original structure from the dual.

Theorem 6.12.

1. *The dual algebra $\mathcal{L}(\mathcal{S}(\mathbf{L}))$ of the dual space of a distributive residuated bounded-lattice $\mathbf{L}$ is isomorphic to $\mathbf{L}$.*

2. *The dual space $\mathcal{S}(\mathcal{L}(\mathbf{S}))$ of the dual algebra of a DRbL-space $\mathbf{S}$ is homeomorphic to $\mathbf{S}$ under a map that respects and preserves the order, the ternary and the unary relation.*

Proof. 1) Let $f : L \to \mathcal{L}(\mathcal{S}(\mathbf{L}))$ be the map $l \mapsto f(l)$. Note that f is a lattice isomorphism, by Theorem 6.7; $\mathcal{L}(\mathcal{S}(\mathbf{L}))$ is a distributive residuated bounded-lattice, by Theorem 6.11; and f preserves multiplication and both division operations, by Lemma 6.10. Since $f(e) = E$, the map f is a residuated lattice isomorphism.

2) Define $g : \mathbf{S} \to \mathcal{S}(\mathcal{L}(\mathbf{S}))$ by $g(x) = \{A \in \mathcal{L}(S) | \, x \in A\}$. Notice that, by Theorem 6.7, g is a topological homeomorphism that is also an order-isomorphism. We will show that g is a R-isomorphism, as well, i.e.,

$$R_{\mathbf{S}}(x, y, z) \text{ iff } R_{\mathcal{S}(\mathcal{L}(\mathbf{S}))}(g(x), g(y), g(z)),$$

or equivalently that

$$R_{\mathbf{S}}(x, y, z) \Leftrightarrow \uparrow (g(x) \circ g(y)) \subseteq g(z).$$

For the forward direction, assume that $C \in \uparrow (g(x) \circ g(y))$. Then, $A \circ B \subseteq C$, for some $A \in g(x)$ and $B \in g(y)$, i.e., $R_{\mathbf{S}}[A, B, \_\,] \subseteq C$, for some prime filters A, B, such that $x \in A$ and $y \in B$. By the hypothesis, $z \in R_{\mathbf{S}}[A, B, \_\,]$, thus $z \in C$, i.e., $C \in g(z)$.

Conversely, if $R_{\mathbf{S}}(x, y, z)$ is false, then there exist $A, B \in \mathcal{L}(S)$, such that $x \in A$, $y \in B$ and $z \notin R[A, B, \_\,] = A \circ B$, i.e., $A \circ B \notin g(z)$, for some $A \in g(x)$ and $B \in g(y)$. So, $g(x) \circ g(y) \nsubseteq g(z)$; a fortiori, $\uparrow (g(x) \circ g(y)) \nsubseteq g(z)$.

Finally, $x \in E_{\mathbf{S}}$ iff $E_{\mathbf{S}} \in g(x)$ iff $g(x) \in f(E_{\mathbf{S}})$ iff $g(x) \in E_{\mathcal{S}(\mathcal{L}(\mathbf{S}))}$, namely g is an E-isomorphism. $\square$

Let $\mathbf{S}_1 = \langle S_1, \tau_1, \leq_1, R_1, E_1 \rangle$ and $\mathbf{S}_2 = \langle S_2, \tau_2, \leq_2, R_2, E_2 \rangle$ be two DRbL-spaces. A *DRbL-map*, is a Priestley map $h : S_1 \to S_2$ that satisfies the following conditions.

1. If $R_1(x, y, z)$, then $R_2(h(x), h(y), h(z))$.

2. If $R_2(u, v, h(z))$, then $u \leq h(x)$, $v \leq h(y)$ and $R_1(x, y, z)$, for some $x, y \in S_1$.

3. For all $B, C \in \mathcal{L}(S_2)$ and for all $x \in S_1$, if $R_1[x, h^{-1}[B], \_\,] \subseteq h^{-1}[C]$ then $R_2[h(x), B, \_\,] \subseteq C$.

4. $h^{-1}[E_2] = E_1$.

We can now prove the main result in this section.

Theorem 6.13. *The categories of distributive residuated bounded-lattices with residuated bounded-lattice homomorphisms that preserve the lattice bounds and DRbL-spaces with DRbL maps are dual.*

Proof. The restrictions $\mathcal{S}$ and $\mathcal{L}$ of the functors $\bar{\mathcal{S}}$ and $\bar{\mathcal{L}}$ that arise from Priestley duality, given in Theorem 6.7, are bijective on objects of the subcategories of distributive residuated bounded-lattices and DRbL-spaces, by Theorems 6.11 and 6.12. We show that these restrictions map morphisms to morphisms on the subcategories.

Let $\mathbf{L}_1, \mathbf{L}_2$ be distributive residuated bounded-lattices and $h : \mathbf{L}_1 \to \mathbf{L}_2$ a residuated lattice homomorphism that preserves the lattice bounds. We define the map $\mathcal{S}(h) : \mathcal{S}(\mathbf{L}_2) \to \mathcal{S}(\mathbf{L}_1)$, by $\mathcal{S}(h)(A) = h^{-1}[A]$. By Theorem 6.6, it is a Priestley map. To show that $\mathcal{S}(h)$ is a DRbL-map we verify the four conditions of the definition.

If $R_2(X, Y, Z)$, i.e., $X \cdot Y \subseteq Z$, then $h^{-1}[X \cdot Y] \subseteq h^{-1}[Z]$. So $h^{-1}[X] \cdot h^{-1}[Y] \subseteq h^{-1}[Z]$, i.e., $R_1(h^{-1}[X], h^{-1}[Y], h^{-1}[Z])$.

For the second condition, assume that $R_1(U, V, h^{-1}[Z])$, i.e., $UV \subseteq h^{-1}[Z]$, holds and set $X' = \uparrow(h[U])$ and $Y' = \uparrow(h[V])$. If $a \in U$, then $h(a) \in h[U] \subseteq \uparrow(h[U]) = X'$, so $a \in h^{-1}[X']$. Thus $U \subseteq h^{-1}[X']$ and similarly $V \subseteq h^{-1}[Y']$. Moreover, if $a \in X'$ and $b \in Y'$, then there are $c \in U$, $d \in V$ such that $h(c) \leq a$ and $h(d) \leq b$; so, $h(c) \cdot h(d) \leq ab$ and $cd \in UV \subseteq h^{-1}[Z]$, i.e., $h(cd) \in Z$. Since h is a homomorphism and Z is an upset we have $ab \in Z$. Thus, $X' \cdot Y' \subseteq Z$. Finally, by Lemma 6.9, there are prime filters X, Y such that $X' \subseteq X$, $Y' \subseteq Y$ and $XY \subseteq Z$, i.e., $U \subseteq h^{-1}[X]$, $V \subseteq h^{-1}[Y']$ and $R_1(X, Y, Z)$.

To show the third condition, let β, γ be clopen increasing subsets of $S_1 = \mathcal{S}(L_1)$, let $X \in S_2 = \mathcal{S}(L_2)$ and assume that $R_2[X, (\mathcal{S}(h))^{-1}[\beta], \_\,] \subseteq (\mathcal{S}(h))^{-1}[\gamma]$. We will show that $R_1[\mathcal{S}(h)[X], \beta, \_] \subseteq \gamma$. For that purpose let $Z \in R_1[h^{-1}[X], \beta, \_]$, namely $R_1(h^{-1}[X], Y, Z)$, for some $Y \in \beta$. By definition, $\beta = f(b)$ and $\gamma = f(c)$, for some $b, c \in L_2$, so $\uparrow(h^{-1}[X]Y) \subseteq Z$, for some prime filter Y of $\mathbf{L}_1$, such that $b \in Y$. We will show that $Z \in \gamma$, i.e., that $c \in Z$.

We will first show that $c/b \in h^{-1}[X]$. If this is not the case, then $c/b \notin h^{-1}[X]$, i.e., $h(c)/h(b) \notin X$, since h is a homomorphism. Since X is increasing, we have that there is no element a of X, such that $a \leq h(c)/h(b)$, i.e., such that $ah(b) \leq h(c)$. So, $h(c) \notin \uparrow(X \cdot \uparrow(h(b)))$ and in particular

$$\uparrow(X \cdot \uparrow(h(b))) \cap \downarrow(h(c)) = \emptyset.$$

Note that $\uparrow(X \cdot \uparrow(h(b)))$ is a filter, by Lemma 6.8, so by the Prime Ideal Theorem there exists a prime filter W such that $\uparrow(X \cdot \uparrow(h(b))) \subseteq W$ and $W \cap \downarrow(h(c)) = \emptyset$, i.e., such that $\uparrow(X \cdot \uparrow(h(b))) \subseteq W$ and $h(c) \notin W$. By Lemma 6.9, there is a prime filter V, such that

$h(b) \in V$ and $XV \subseteq W$, i.e., such that $b \in h^{-1}[V]$ and $XV \subseteq W$. So, there is a prime filter V, such that $h^{-1}[V] \in \beta$, $XV \subseteq W$ and $c \notin h^{-1}[W]$, i.e., there exist a $V \in \mathcal{S}(h))^{-1}[\beta]$, such that $XV \subseteq W$ and $h^{-1}[W] \notin \gamma$. Consequently, $W \in R_2[X, (\mathcal{S}(h))^{-1}[\beta], \_]$, but $W \notin (\mathcal{S}(h))^{-1}[\gamma]$, a contradiction to our hypothesis. So, $c/b \in h^{-1}[X]$.

Now, note that since $b \in Y$ and $c/b \in h^{-1}[X]$, we have $(c/b)b \in h^{-1}[X]Y$. Moreover, $(c/b)b \leq c$, so $c \in \uparrow (h^{-1}[X]Y)$. Thus, $c \in Z$.

Finally, for the last condition, we will show that $(\mathcal{S}(h))^{-1}[E_1] = E_2$. Note that a clopen increasing set X is in the first set iff $X \in (\mathcal{S}(h))^{-1}(Y)$, for some $Y \in E_1$, i.e., iff $\mathcal{S}(h)(X) = Y$, for some clopen increasing set Y, such that $e_1 \in Y$. Recalling the definition of $\mathcal{S}(h)$, this is equivalent to $h^{-1}(X) = Y$ and $e_1 \in Y$, i.e., to $e_1 \in h^{-1}(X)$. This is in turn equivalent to $h(e_1) \in X$, namely to $e_2 \in X$. In view of the definition of E_2, this is a restatement of $X \in E_2$.

For the reverse direction, let $\mathbf{S}_1, \mathbf{S}_2$ be DRbL-spaces and $h : \mathbf{S}_1 \to \mathbf{S}_2$ a DRbL-map. We define the map $\mathcal{L}(h) : \mathcal{L}(\mathbf{S}_2) \to \mathcal{L}(\mathbf{S}_1)$, by $\mathcal{L}(h)(A) = h^{-1}[A]$. By Theorem 6.6, $\mathcal{L}(h)$ is a lattice homomorphism that preserves lattice bounds. To show that it is a residuated lattice homomorphism we need to demonstrate that it preserves multiplication, both division operations and the identity.

We first show that

$$h^{-1}[A] \circ h^{-1}[B] = h^{-1}(A \circ B).$$

If $z \in h^{-1}[A] \circ h^{-1}[B]$, i.e., $z \in R_1[h^{-1}[A], h^{-1}[B], \_]$, then $(R_1(x, y, z)$, for some $x \in h^{-1}[A]$ and $y \in h^{-1}[B]$. By the first property of a DRbL-map, we get $R_2(h(x), h(y), h(z))$, for some $x, y \in S_1$ such that $h(x) \in A$ and $h(y) \in B$, hence $h(z) \in R_2[A, B, \_]$. So, $h(z) \in A \circ B$, i.e., $z \in h^{-1}(A \circ B)$.

Conversely, if $z \in h^{-1}(A \circ B)$, then $h(z) \in A \circ B = R_2[A, B, \_]$, i.e., $R_2(x, y, h(z))$, for some $x \in A$ and $y \in B$. By the second property for h, we have $R_1(u, v, z)$, for some $x \in A$, $y \in B$ and for some $u, v \in S_1$ such that $x \leq h(u)$ and $y \leq h(v)$. In other words $R_1(u, v, z)$, for some $u, v \in S_1$ such that $h(u) \in A$ and $h(v) \in B$, i.e., such that $u \in h^{-1}[A]$ and $v \in h^{-1}[B]$. So, $z \in R_1[h^{-1}[A], h^{-1}[B], \_] = h^{-1}[A] \circ h^{-1}[B]$.

Next we show that

$$h^{-1}[C/B] \circ h^{-1}[B] = h^{-1}[C]/h^{-1}[B].$$

First, note that $h^{-1}[C/B] \circ h^{-1}[B] = h^{-1}[(C/B) \circ B] \subseteq h^{-1}[C]$. Consequently, we have $h^{-1}[C/B] \subseteq h^{-1}[C]/h^{-1}[B]$. Conversely, if $x \in h^{-1}[C]/h^{-1}[B]$, then $x \circ h^{-1}[B] \subseteq h^{-1}[C]$, i.e., $R_1[x, h^{-1}[B], \_] \subseteq h^{-1}[C]$. By the third property of h, we have $R_2[h(x), B, \_] \subseteq C$, i.e., $h(x) \circ B \subseteq C$. Thus, $h(x) \in C/B$, namely $x \in h^{-1}[C/B]$.

Finally the last condition for h gives $h^{-1}[E_2] = E_1$.

Thus, the correspondences $\mathcal{S}$ and $\mathcal{L}$ are the restrictions of the functors $\bar{\mathcal{S}}$ and $\bar{\mathcal{L}}$ on the objects and on the morphisms of the subcategories. Since they are actually restrictions of a duality, they induce a duality between the category of distributive residuated bounded-lattices and homomorphisms, and the category of DRbL-spaces and DRbL-maps. $\square$

CHAPTER VII

GENERALIZED MV-ALGEBRAS

As we have seen before, generalized BL-algebras encompass ℓ-groups and Brouwerian algebras. Also, generalized Boolean algebras and ℓ-groups are special cases of generalized MV-algebras. In this chapter we study GBL and GMV-algebras and show that they decompose into Cartesian products of ℓ-groups and integral residuated lattices. Moreover, we characterize the integral factor of a GMV-algebra as a nucleus retraction on the negative cone of an ℓ-group. From the analysis we get that every GMV-algebra is equivalent to an image of a *core* map on an ℓ-group. Both of the correspondences, in the integral and in the general case, extend to categorical equivalences. Finally, we observe that the close connection of the variety $\mathcal{GMV}$ with those of ℓ-groups and of their negative cones guarantees the decidability of its equational theory.

Definitions and basic properties

Recall the definition of a GBL and of a GMV-algebra from page 34. Note that the equational bases for the varieties $\mathcal{GBL}$ of generalized BL-algebras and $\mathcal{GMV}$ of generalized MV-algebras have the following more simple quasi-identity formulations, respectively:

$$x \leq y \;\Rightarrow\; (x/y)y = x = y(y\backslash x)$$

and

$$x \leq y \;\Rightarrow\; x/(y\backslash x) = y = (x/y)\backslash x.$$

Moreover, it is noted in [BCGJT] that the following are equivalent bases of equations for the two varieties, respectively:

$$x(x\backslash y \wedge e) \approx x \wedge y \approx (y/x \wedge e)x$$

and

$$x/(y\backslash x \wedge e) = x \vee y = (x/y \wedge e)\backslash x$$

Also note that the first set of identities is also equivalent to the property of *divisibility*:

$$x \leq y \;\Rightarrow\; (\exists z, w)(zy = x = yw),$$

in the setting of residuated lattices.

Lemma 7.1. *[BCGJT] Every GMV-algebra is also a GBL-algebra.*

Proof. Let x, y be elements of L such that $x \leq y$. Set $z = (x/y)y$ and note that, by Lemma 3.1, $z \leq x$ and $y/z \leq x/z$.

Using the equivalent quasi-equation for GMV-algebras and Lemma 3.1(12), (6), we have the following:

$$\begin{aligned} z \leq x \quad &\Rightarrow (z/x)\backslash z = x \\ &\Rightarrow ((z/x)\backslash z)/y = x/y \\ &\Rightarrow (z/x)\backslash(z/y) = x/y \\ &\Rightarrow (z/y)/((z/x)\backslash(z/y)) = (z/y)/(x/y) \\ &\Rightarrow z/x = z/(x/y)y \\ &\Rightarrow (z/x)\backslash z = (z/(x/y)y)\backslash z \\ &\Rightarrow x = (x/y)y \end{aligned}$$

Thus, $x \leq y$ implies $x = (x/y)y$. Likewise, $x \leq y$ implies $y(y\backslash x) = x$. □

Lattice-ordered groups and their negative cones are examples of cancellative GMV-algebras. Non-cancellative examples include generalized Boolean algebras.

Lemma 7.2. *Let* $\mathbf{L}$ *be a GBL-algebra. Then,*

1. *Every positive element of* $\mathbf{L}$ *is invertible.*
2. $\mathbf{L}$ *satisfies the identities* $x/x \approx x\backslash x \approx e$.
3. $\mathbf{L}$ *satisfies* $e/x \approx x\backslash e$.

Proof. For the first property, let a be a positive element; by the defining identity for GBL-algebras, we get $a(a\backslash e) = e = (e/a)a$; that is, a is invertible. By (1) and Lemma 3.1(14), x/x and $x\backslash x$ are invertible for every x. Hence, by Lemma 3.1(16), $x/x = e = x\backslash x$. Finally, by (2) and Lemma 3.1(5), $x(e/x) \leq x/x = e$, hence $e/x \leq x\backslash e$. Likewise, $x\backslash e \leq e/x$. □

Lemma 7.3. *If* x, y *are elements of a GBL-algebra and* $x \vee y = e$, *(*x, y *are orthogonal), then* $xy = x \wedge y$.

Proof. We have,

$$x = x/e = x/(x \vee y) = x/x \wedge x/y = e \wedge x/y = y/y \wedge x/y = (y \wedge x)/y.$$

So, $xy = ((x \wedge y)/y)y = x \wedge y$. □

Lemma 7.4. *Every GBL-algebra has a distributive lattice reduct.*

Proof. Let $\mathbf{L}$ be a GBL-algebra and $x, y, z \in L$. Using Lemma 3.1, we have

$$\begin{aligned} x \wedge (y \vee z) &= [(x \wedge (y \vee z))/(y \vee z)](y \vee z) \\ &= [x/(y \vee z) \wedge e](y \vee z) \\ &= [x/(y \vee z) \wedge e]y \vee [x/(y \vee z) \wedge e]z \\ &\leq (x/y \wedge e)y \vee (x/z \wedge e)z \\ &= (x \wedge y) \vee (x \wedge z), \end{aligned}$$

for all x, y, z. We have proved that the lattice reduct of $\mathbf{L}$ is distributive. □

We denote the variety of integral GBL-algebras by $\mathcal{IGBL}$ and the variety of integral GMV-algebras by $\mathcal{IGMV}$.

Lemma 7.5.

1. *The variety $\mathcal{IGBL}$ is axiomatized, relative to $\mathcal{RL}$, by the equations*

$$(x/y)y \approx x \wedge y \approx y(y\backslash x).$$

2. *The variety $\mathcal{IGMV}$ is axiomatized by the equations*

$$x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x.$$

Proof. In view of the alternative axiomatizations of $\mathcal{GBL}$ and $\mathcal{GMV}$, the proposed equations hold in the corresponding varieties. For the reverse direction we verify that the proposed identities imply integrality. This is obvious for the first set of identities for $y = e$. For the second set observe that for every x,

$$e \leq e \vee e/x = e/((e/x)\backslash e) = e/(e \vee x);$$

so $e \vee x \leq e$, i.e., $x \leq e$. □

Negative cones of ℓ-groups are examples of integral GMV-algebras, hence also of integral GBL-algebras. Moreover, they are cancellative residuated lattices. Note that, by Corollary 4.4, $\mathcal{LG}^- = \mathcal{IGMV} \cap \mathcal{C}an\mathcal{RL} = \mathcal{IGBL} \cap \mathcal{C}an\mathcal{RL}$.

It is easy to see that $\mathcal{IGBL}$ contains all Brouwerian algebras. Also, it was mentioned before that $\mathcal{GBA} = \mathbf{V}(\mathbf{2}) = \mathcal{IGMV} \cap \mathcal{B}r$.

Lemma 7.6.

1. *Every integral GBL-algebra satisfies the identity $(y/x)\backslash(x/y) \approx x/y$ and its opposite.*

2. *Every integral GMV-algebra satisfies the identity $x/y \vee y/x \approx e$ and its opposite.*

3. *Every commutative integral GMV-algebra is in $\mathcal{RL}^C$. Consequently, the subdirectly irreducible commutative integral GMV-algebras are totally ordered.*

Proof. 1) For every integral GBL-algebra, $y/x \leq e$, so $(y/x)\backslash(x/y) \geq x/y$.

To show the reverse inequality, we need to check that

$$((y/x)\backslash(x/y))y \leq x.$$

By Lemma 3.1(12), it suffices to show that

$$(((y/x)\backslash x)/y)y \leq x.$$

Using one of the the defining equations, $(u/v)v \approx (v/u)u$, of integral GBL-algebras, we see that the last equation is equivalent to

$$(y/((y/x)\backslash x))((y/x)\backslash x) \leq x,$$

which in turn is equivalent to

$$y/((y/x)\backslash x) \leq x/((y/x)\backslash x).$$

To show that this holds note that

$$y/((y/x)\backslash x) \leq y/x,$$

since $y/x \leq e$, and that

$$y/x \leq x/((y/x)\backslash x),$$

since $u \leq v/(u\backslash v)$ is valid in any residuated lattice, by Lemma 3.1(11).

2) Using one of the defining equations, $u \vee v \approx u/(v\backslash u)$, for integral GMV-algebras, $x/y \vee y/x$ equals $(x/y)/((y/x)\backslash(x/y))$, which simplifies to $(x/y)/(x/y)$, by invoking (1) and the fact that integral GMV-algebras are integral GBL-algebras. Finally, the last term equals to e in integral residuated lattices.

3) By Lemma 3.18, an equational basis for $\mathcal{RL}^C$, relative to $\mathcal{RL}$, is

$$z\backslash(x/(x \vee y))z \vee w(y/(x \vee y))/w \approx e,$$

which simplifies to

$$x/(x \vee y) \vee y/(x \vee y) \approx e,$$

under commutativity and integrality. In every residuated lattice

$$x/(x \vee y) \vee y/(x \vee y) = (x/x \wedge x/y) \vee (y/x \wedge y/y),$$

which in turn equals $x/y \vee y/x$, under integrality. By (2), every commutative integral GMV-algebra satisfies the last equation. $\square$

Bosbach's embedding theorem

The results of this section are due to B. Bosbach, see [BoRG] and [BoCA]. Our presentation is a variant of his exposition.

A *cone algebra* is an algebra $\mathbf{C} = \langle C, \backslash, /, e\rangle$, that satisfies:

$$\begin{array}{cc} (x\backslash y)\backslash(x\backslash z) \approx (y\backslash x)\backslash(y\backslash z) & (z/y)/(x/y) \approx (z/x)/(y/x) \\ e\backslash y \approx y & y \approx y/e \\ x\backslash(y/z) \approx (x\backslash y)/z & x/(y\backslash x) \approx (y/x)\backslash y \\ x\backslash x \approx e & x/x \approx e \end{array}$$

Lemma 7.7. *([BoRG], [BoCA]) If $\mathbf{C} = \langle C, \backslash, /, e\rangle$ is a cone algebra, then*

1. *for all $a, b \in C$, $a\backslash b = e$ iff $b/a = e$;*
2. *the relation $\leq$ on C defined by $a \leq b \Leftrightarrow a\backslash b = e$ is a semilattice order with $a \vee b = a/(b\backslash a)$; in particular $a \leq e$, for all a;*
3. *if $a \leq b$, then $c\backslash a \leq c\backslash b$ and $a/c \leq b/c$.*

If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, e, \backslash, /, \rangle$, is an integral GMV-algebra, then $\langle L, \backslash, /\rangle$ is a cone algebra, called *the cone algebra of* $\mathbf{L}$.

It will be shown that every cone algebra is a subalgebra of the cone algebra of the negative cone of an ℓ-group. In the following construction, the negative cone is defined as the union of an ascending chain $\langle \mathbf{C}_n\rangle_{n\in\mathbb{N}}$ of cone algebras, each of which is a subalgebra of its successor. In the process of constructing the algebras $\mathbf{C}_n$, we also define in $\mathbf{C}_{n+1}$ binary products of elements of $\mathbf{C}_n$. Each such product is identified with the congruence class of the corresponding ordered pair. The definition below of the division operations becomes transparent if we note that negative cones of ℓ-groups satisfy the law $ab\backslash cd = (b\backslash(a\backslash c)) \cdot (((a\backslash c)\backslash b)\backslash((c\backslash a)\backslash d))$ and its opposite.

Let $\mathbf{C}$ be a cone algebra. Define the operations $\backslash$ and $/$ and the relations θ and θ' on $C \times C$, by

$$(a,b)\backslash(c,d) = (b\backslash(a\backslash c), ((a\backslash c)\backslash b)\backslash((c\backslash a)\backslash d))$$

$$(d,c)/(b,a) = ((d/(a/c))/(b/(c/a)), (c/a)/b)$$

$$(a,b)\theta(c,d) \;\Leftrightarrow\; (a,b)\backslash(c,d) = (e,e) \text{ and } (c,d)\backslash(a,b) = (e,e)$$

$$(a,b)\theta'(c,d) \;\Leftrightarrow\; (a,b)/(c,d) = (e,e) \text{ and } (c,d)/(a,b) = (e,e)$$

Lemma 7.8. *([BoRG], [BoCA]) Let* $\mathbf{C}$ *be a cone algebra. Then:*
i) $\theta = \theta'$*.*
ii) θ *is a congruence relation.*
iii) $s(\mathbf{C}) = \langle C \times C, \backslash, /\rangle/\theta$ *is a cone algebra.*
iv) $\mathbf{C}$ *can be embedded in* $s(\mathbf{C})$*.*

Let $\mathbf{C}_0 = \mathbf{C}$, $\mathbf{C}_{n+1} = s(\mathbf{C}_n)$, for every natural number n, and $\overline{\mathbf{C}} = \bigcup \mathbf{C}_n$, the directed union of the $\mathbf{C}_n$'s.

We can now establish the main result of [BoCA].

Theorem 7.9. *[BoCA] Every cone algebra* $\mathbf{C}$ *is a subalgebra of the cone algebra of the negative cone* $\widehat{\mathbf{C}}$ *of an* ℓ*-group. Moreover, every element of* $\widehat{C}$ *is a product of elements of* C*.*

Proof. We will show that $\overline{\mathbf{C}}$ is the cone algebra, i.e., the $\{\backslash, /\}$-reduct of the negative cone $\widehat{\mathbf{C}}$ of an ℓ-group.

For two elements of $\overline{C}$, we define their product, $a \cdot b$, to be the element $[(a,b)]_\theta$. This is well defined, because of the embedding of $\mathbf{C}_n$ into $\mathbf{C}_{n+1}$, for every n. Let $\widehat{\mathbf{C}} = \langle \overline{C}, \wedge, \vee, \cdot, \backslash, /, e\rangle$, where $\backslash = \backslash_{\overline{\mathbf{C}}}$, $/ = /_{\overline{\mathbf{C}}}$, $x \vee y = x/(y\backslash x)$ and $x \wedge y = (x/y) \cdot y$. We will show that $\widehat{\mathbf{C}}$ is the negative cone of an ℓ-group.

By the definition of the operations in $\widehat{\mathbf{C}}$ and Lemma 7.7(2), $\widehat{\mathbf{C}}$ is a join semilattice. Note that $ab\backslash cd = (b\backslash(a\backslash c)) \cdot (((a\backslash c)\backslash b)\backslash((c\backslash a)\backslash d))$. In particular, $ab\backslash c = b\backslash(a\backslash c)$ and $a\backslash ab = b$. The dual equations hold, as well. Finally, note that $e/a = e = a\backslash e$.

To see that multiplication is order preserving, let $a \le c$. We have $e = a\backslash c$, by the definition of $\le$. To show that $ab \le cb$, we note that $ab\backslash cb = b\backslash[(c\backslash a)\backslash b] = [(c\backslash a)b]\backslash b$. Moreover,

$$b/[(c\backslash a)b] = (b/b)/(c\backslash d) = e/(c\backslash d) = e.$$

This yields successively, $(c\backslash a)b \le b$, $[(c\backslash a)b]\backslash b = e$, $ab\backslash cb = e$ and $ab \le cb$. Likewise $a \le c$

implies $ba \leq bc$. Also, multiplication is associative, since

$$\begin{aligned}(ab)c \leq d \quad &\Leftrightarrow\; ab \leq d/c \\ &\Leftrightarrow\; b \leq a\backslash(d/c) \\ &\Leftrightarrow\; b \leq (a\backslash d)/c \\ &\Leftrightarrow\; bc \leq (a\backslash d) \\ &\Leftrightarrow\; a(bc) \leq d.\end{aligned}$$

To see that multiplication is residuated, note that $a(a\backslash c) \leq c$, since $[a(a\backslash c)]\backslash c = (a\backslash c)\backslash(a\backslash c) = e$. If $ab \leq c$, then $a\backslash ab \leq a\backslash c$, so $b \leq a\backslash c$. Conversely, if $b \leq a\backslash c$, then $ab \leq a(a\backslash c) \leq c$. The equivalence for right division is the opposite of the one established.

To show that the operation $\wedge$ that we have defined above is the meet operation, note that it was proved above that $a(a\backslash b) \leq b$. Moreover, $a(a\backslash b) \leq ae = a$. On the other hand, if $c \leq a$ and $c \leq b$, then $e = c\backslash a = c\backslash b$. We have,

$$c\backslash a(a\backslash b) = (c\backslash a) \cdot [(a\backslash c)\backslash(a\backslash c)] = (c\backslash a)\backslash(c\backslash b) = e,$$

so $c \leq a(a\backslash b)$. Interchanging the roles of a and b we get that $c \leq a, b \;\Leftrightarrow\; c \leq b(b\backslash a)$. The opposites of these properties are obtained in a similar way.

Thus, $\widehat{\mathbf{C}}$ is a residuated lattice. Since it satisfies $x\backslash xy \approx y \approx yx/x$ and $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$, it is the negative cone of an ℓ-group, by Corollary 4.5. Finally, by construction, every element of $\widehat{C}$ is the product of elements of C. □

The algebra $\widehat{\mathbf{C}}$ is called the *product extension* of $\mathbf{C}$.

Decomposition of GBL-algebras

We now show that every GBL-algebra decomposes into a direct product.

Lemma 7.10. *Every GBL-algebra satisfies the identity $x \approx (x \vee e)(x \wedge e)$.*

Proof. Setting $y = e$ into the equivalent axiomatization of GBL-algebras, we have that $(e/x \wedge e)x = x \wedge e$. Moreover, by Lemma 7.2(1), $x \vee e$ is invertible and $(x \vee e)^{-1} = e/(x \vee e) = e/x \wedge e$. Thus, $(x \vee e)^{-1}x = x \wedge e$, i.e., $x = (x \vee e)(x \wedge e)$. □

We say that an algebra $\mathbf{A}$ is the *direct sum* of two of its subuniverses B, C, in symbols $\mathbf{A} = B \oplus C$, if the map $f : \mathbf{B} \times \mathbf{C} \to \mathbf{A}$, defined by $f(x, y) = xy$ is an isomorphism.

Recall the definition of the set $G(\mathbf{L})$ of invertible and $I(\mathbf{L})$ of integral elements of $\mathbf{L}$.

Theorem 7.11. *Every GBL-algebra, $\mathbf{L}$, is equal to the direct sum $G(\mathbf{L}) \oplus I(\mathbf{L})$.*

Proof. We begin with a series of claims.

Claim 1: $G(\mathbf{L})$ is a subuniverse of $\mathbf{L}$.

Let x, y be invertible elements. It is clear that xy is invertible. Additionally, by Lemma 3.2, $x/y = xy^{-1}$ and $y\backslash x = y^{-1}x$ are invertible.

Lastly, $x \vee y = (xy^{-1} \vee e)y$. So, $x \vee y$ is invertible, since every positive element is invertible, by Lemma 7.2(1), and the fact that the product of two invertible elements is invertible. By Lemma 7.2(3), $x \wedge y = e/(x^{-1} \vee y^{-1})$, which is invertible, since we have already shown that $G(\mathbf{L})$ is closed under joins and the division operation. We have verified that $G(\mathbf{L})$ is a subuniverse of $\mathbf{L}$.

Claim 2: $I(\mathbf{L})$ is a subuniverse of $\mathbf{L}$.

Note that every integral element a is negative, since $e = e/a$ implies $e \leq e/a$ and $a \leq e$. For $x, y \in I(\mathbf{L})$, using Lemma 3.1 repeatedly, we get:

$$e/xy = (e/y)/x = e/x = e, \text{ so } xy \in I(\mathbf{L}).$$

$$e/(x \vee y) = e/x \wedge e/y = e, \text{ so } x \vee y \in I(\mathbf{L}).$$

$$e \leq e/x \leq e/(x \wedge y) \leq e/xy = e, \text{ so } x \wedge y \in I(\mathbf{L}).$$

$$e = e/(e/y) \leq e/(x/y) \leq e/(x/e) = e/x = e, \text{ so } x/y \in I(\mathbf{L}).$$

We have shown that $I(\mathbf{L})$ is a subuniverse of $\mathbf{L}$.

Claim 3: For every $g \in (G(\mathbf{L}))^-$ and every $h \in I(\mathbf{L})$, $g \vee h = e$.

Let $g \in (G(\mathbf{L}))^-$ and $h \in I(\mathbf{L})$. We have $e/(g \vee h) = e/g \wedge e/h = e/g \wedge e = e$, since $e \leq e/g$. Moreover, $g \leq g \vee h$, so $e \leq g^{-1}(g \vee h)$. Thus, by the GBL-algebra identities

$$\begin{aligned} e &= (e/[g^{-1}(g \vee h)])[g^{-1}(g \vee h)] \\ &= ([e/(g \vee h)]/g^{-1})g^{-1}(g \vee h) \\ &= (e/g^{-1})g^{-1}(g \vee h) \\ &= gg^{-1}(g \vee h) \\ &= g \vee h. \end{aligned}$$

Claim 4: For every $g \in (G(\mathbf{L}))^-$ and every $h \in I(\mathbf{L})$, $gh = g \wedge h$.

In light of Lemma 7.10, $g^{-1}h = (g^{-1}h \vee e)(g^{-1}h \wedge e)$. Multiplication by g yields $h = (h \vee g)(g^{-1}h \wedge e)$. Using Claim 3 and Lemma 3.2(2), we have $gh = g(g^{-1}h \wedge e) = h \wedge g$.

Claim 5: For every $g \in G(\mathbf{L})$ and every $h \in I(\mathbf{L})$, $gh = hg$.

The statement is true if $g \leq e$, by Claim 4. If $g \geq e$ then $g^{-1} \leq e$, thus $g^{-1}h = hg^{-1}$, hence $hg = gh$. For arbitrary g, note that both $g \vee e$ and $g \wedge e$ commute with h. Using Lemma 7.10, we get $gh = (g \vee e)(g \wedge e)h = (g \vee e)h(g \wedge e) = h(g \vee e)(g \wedge e) = hg$.

Claim 6: For every $x \in L$, there exist $g_x \in G(\mathbf{L})$ and $h_x \in I(\mathbf{L})$, such that $x = g_x h_x$.

By Lemma 7.10, $x = (x \vee e)(x \wedge e)$. Since $e \leq x \vee e$ and $e \leq e/(x \wedge e)$, by Lemma 7.2(1), these elements are invertible. Set $g_x = (x \vee e)(e/(x \wedge e))^{-1}$ and $h_x = (e/(x \wedge e))(x \wedge e)$. It is clear that $x = g_x h_x$, g_x is invertible and h_x is integral.

Claim 7: For every $g_1, g_2 \in G(\mathbf{L})$ and $h_1, h_2 \in I(\mathbf{L})$, $g_1h_1 \leq g_2h_2$ if and only if $g_1 \leq g_2$ and $h_1 \leq h_2$.

For the non-trivial direction we have

$$g_1h_1 \leq g_2h_2 \;\Rightarrow\; g_2^{-1}g_1h_1 \leq h_2 \;\Rightarrow\; g_2^{-1}g_1 \leq h_2/h_1 \leq e \;\Rightarrow\; g_1 \leq g_2.$$

Moreover,

$$\begin{aligned} g_2^{-1}g_1 \leq h_2/h_1 \quad &\Rightarrow\; e \leq g_1^{-1}g_2(h_2/h_1) \\ &\Rightarrow\; e = [e/g_1^{-1}g_2(h_2/h_1)]g_1^{-1}g_2(h_2/h_1) \\ &\Rightarrow\; e = [(e/(h_2/h_1))/g_1^{-1}g_2]g_1^{-1}g_2(h_2/h_1) \\ &\Rightarrow\; e = g_2^{-1}g_1g_1^{-1}g_2(h_2/h_1) \\ &\Rightarrow\; e = h_2/h_1 \\ &\Rightarrow\; h_1 \leq h_2. \end{aligned}$$

By Claims 1 and 2, $\mathbf{G}(\mathbf{L})$ and $\mathbf{I}(\mathbf{L})$ are subalgebras of $\mathbf{L}$. Define $f : \mathbf{G}(\mathbf{L}) \times \mathbf{I}(\mathbf{L}) \to \mathbf{L}$ by $f(g, h) = gh$. We will show that f is an isomorphism. It is onto by Claim 6 and an order isomorphism by Claim 7. So, it is a lattice isomorphism, as well. To verify that f preserves the other operations note that, by Claim 5 and 7, for all $g, g' \in G(\mathbf{L})$ and $h, h' \in I(\mathbf{L})$, $gg'hh' = ghg'h'$, $gh/g'h' = (g/g')(h/h')$ and $g'h' \backslash gh = (g' \backslash g)(h' \backslash h)$. □

Corollary 7.12. *The varieties $\mathcal{GBL}$ and $\mathcal{GMV}$ decompose as follows:*
$\mathcal{GBL} = \mathcal{LG} \times \mathcal{IGBL} = \mathcal{LG} \vee \mathcal{IGBL}$ *and* $\mathcal{GMV} = \mathcal{LG} \times \mathcal{IGMV} = \mathcal{LG} \vee \mathcal{IGMV}$

Taking intersections with $\mathcal{C}an\mathcal{RL}$ and recalling Theorem 4.4, we get:

Corollary 7.13. $\mathcal{C}an\mathcal{GMV} = \mathcal{C}an\mathcal{GBL} = \mathcal{LG} \times \mathcal{LG}^-$.

This simplifies the equational basis obtained by Corollary 3.30. Moreover, in conjunction with Lemma 7.6(3) and Theorem 3.18, we have:

Corollary 7.14. $\mathcal{CGMV} \subseteq \mathcal{RL}^C$. *Thus, every commutative GMV-algebra is a subdirect product of totally ordered GMV-algebras.*

Representation theorems

In this section we establish two related representation theorems for generalized MV-algebras, by first characterizing integral GMV-algebras.

Direct product representation

The first representation decomposes a generalized MV-algebra into the direct product of an ℓ-group and the nucleus image of the negative cone of an ℓ-group.

Recall the definition of a nucleus from Example 3.3.

Theorem 7.15. *If* $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ *is a GMV-algebra and* γ *a nucleus on it, then*

1. $\mathbf{L}_\gamma = \langle L_\gamma, \wedge, \vee, \circ_\gamma, \backslash, /, e\rangle$ *and* $\mathbf{L}_\gamma$ *is a GMV-algebra,*

2. $\mathbf{L}_\gamma$ *is a filter in* $\mathbf{L}$ *and*

3. γ *is join-preserving.*

Proof. 1) By Lemma 3.4, $\mathbf{L}_\gamma$ is a residuated lattice. Since γ is extensive, $e \leq \gamma(e)$. Hence, $\gamma(e)$ is invertible, by Lemma 7.2(i). By the fact that γ is a nucleus, we get $\gamma(e)\gamma(e) \leq \gamma(e)$, so $\gamma(e) \leq e$. Thus, $\gamma(e) = e$.

Since $\mathbf{L}$ is a GMV-algebra, if $x \in L_\gamma$, then $x \vee y = x/((x \vee y)\backslash x) \in L_\gamma$, by Lemma 3.3(2). Thus, $\vee_\gamma$ is the restriction of $\vee$ on $\mathbf{L}_\gamma$. Finally, $\mathbf{L}_\gamma$ is a GMV-algebra, because the join and division operations of $\mathbf{L}_\gamma$ are the restrictions of the ones in $\mathbf{L}$, and $\mathbf{L}$ is a GMV-algebra.

2) If $x \in L_\gamma$, $y \in L$ and $x \leq y$, then by Lemma 3.3, $y = x \vee y = x/((x \vee y)\backslash x)$ is an element of L_γ. Since $\mathbf{L}_\gamma$ is also a sublattice, it is a filter.

3) For all $x, y \in L$ we have

$$\gamma(x) \vee \gamma(y) \leq \gamma(x \vee y),$$

by the monotonicity of γ. So,

$$\gamma(\gamma(x) \vee \gamma(y)) \leq \gamma(x \vee y),$$

by the monotonicity and idempotency of γ. The reverse inequality is also true, since γ is extensive and monotone, so

$$\gamma(x \vee y) = \gamma(\gamma(x) \vee \gamma(y)).$$

Finally, since $\gamma(x) \vee \gamma(y)$ is an element of L_γ,

$$\gamma(\gamma(x) \vee \gamma(y)) = \gamma(x) \vee \gamma(y).$$

Thus,

$$\gamma(x \vee y) = \gamma(x) \vee \gamma(y).$$

□

Corollary 7.16. *If $\mathbf{L}$ is an integral GMV-algebra and γ is a nucleus on it, then $\mathbf{L}_\gamma$ is an integral GMV-algebra, as well.*

Lemma 7.17. *Let $\mathbf{L}$ be the negative cone of an ℓ-group and γ a nucleus on it. If $z \in L$ and $u = \gamma(z)$, then γ agrees with γ_u on the principal filter generated by z, where $\gamma_u(x) = u \vee x$.*

Proof. Let $x \geq z$. We will show that $\gamma(x) = u \vee x$. On the one hand, $u \vee x = \gamma(z) \vee x \leq \gamma(x)$, since γ is monotone and extensive. Moreover, $x \leq u \vee x$, so $\gamma(x) \leq \gamma(u \vee x) = u \vee x$, because L_γ is a filter, by Theorem 7.15(2). □

Corollary 7.18. *Every nucleus on a GMV-algebra is a lattice homomorphism.*

Proof. In view of Theorem 7.15(3), we need only show that γ preserves meets. Let x, y be elements of the GMV-algebra and set $z = x \wedge y$ and $u = \gamma(z)$. By Lemma 7.17, we have

$$\gamma(x \wedge y) = \gamma_u(x \wedge y) = u \vee (x \wedge y) = (u \vee x) \wedge (u \vee y) = \gamma_u(x) \wedge \gamma_u(y) = \gamma(x) \wedge \gamma(y).$$

We used the fact that GMV-algebras have a distributive lattice reduct; this follows from Lemmas 7.1 and 7.4. □

Theorem 7.19. *The residuated lattice $\mathbf{M}$ is an integral GMV-algebra if and only if $\mathbf{M} \cong \mathbf{L}_\gamma$, for some $\mathbf{L} \in \mathcal{LG}^-$ and some nucleus γ on it.*

Proof. One direction follows from the previous corollary. For the other direction, let $\mathbf{M} = \langle M, \wedge, \vee, \bullet, \backslash, /, e\rangle$ be an integral GMV-algebra. Using Lemma 7.1, Lemma 7.5(2), Lemma 3.1(6), Lemma 7.2(2), Lemma 3.1(8),(7) and Lemma 7.5(1), we see that $\langle M, \backslash, /\rangle$ is a cone algebra. So, by Theorem 7.9, it is a subreduct of the negative cone $\mathbf{L} = \widehat{\mathbf{M}}$ of an ℓ-group, such that the monoid generated by M is equal to L.

Since the division operations of $\mathbf{M}$ are the restrictions of the division operations of $\mathbf{L}$ we use the symbols $\backslash$ and $/$ for the latter, as well. Moreover, the same holds for the join and the constant e, because in integral GMV-algebras they are term definable by the division operations ($x \vee y \approx x/(y\backslash x)$ and $e \approx x/x$). We denote multiplication in $\mathbf{L}$ by $\cdot$.

Since M generates L as a monoid, for every $x \in L$, there exist elements $x_1, \ldots, x_n \in M$, such that $x = x_1 \cdot x_2 \cdots x_n$. We prove the following Claim.

Claim: If $z \in M, x \in L$ and $x = x_1 \cdots x_n$, then $z \vee x = z \vee x_1 \bullet \cdots \bullet x_n$.

$$\begin{aligned}
z \vee x &= z/(x\backslash z) && \text{(axiom of IGMV-algebras)} \\
&= z/((x_1 \cdots x_n)\backslash z) && \\
&= z/[x_n\backslash(\ldots(x_2\backslash(x_1\backslash z))\ldots)] && \text{(Lemma 3.1(6))} \\
&= z/((x_1 \bullet \cdots \bullet x_n)\backslash z) && \text{(Lemma 3.1(4))} \\
&= z \vee x_1 \bullet \cdots \bullet x_n && \text{(axiom of IGMV-algebras)}
\end{aligned}$$

Suppose now that $x = x_1 \cdots x_n = y_1 \cdots y_n$, with $x_i, y_i \in M$. Then,

$$x_1 \bullet \cdots \bullet x_n \vee y_1 \bullet \cdots \bullet y_n = x_1 \bullet \cdots \bullet x_n \vee x_1 \bullet \cdots \bullet x_n,$$

by the preceding claim. Hence, $y_1 \bullet \cdots \bullet y_n \leq x_1 \bullet \cdots \bullet x_n$. Likewise, $x_1 \bullet \cdots \bullet x_n \leq y_1 \bullet \cdots \bullet y_n$, hence $x_1 \bullet \cdots \bullet x_n = y_1 \bullet \cdots \bullet y_n$.

Retaining the notation established in the preceding paragraph, we define $\gamma(x) = x_1 \bullet \cdots \bullet x_n$. By the previous paragraph this map is well-defined. We will show that it is a nucleus on $\mathbf{L}$, $L_\gamma = M$ and $\mathbf{L}_\gamma \cong \mathbf{M}$.

Note that $\gamma(x) \in M$, for all $x \in L$, so by setting $z = \gamma(x)$ to the claim above, we get $\gamma(x) \vee x = \gamma(x)$. So, $x \leq \gamma(x)$, for all $x \in L$. If $x \leq y$, then

$$\begin{aligned}
\gamma(x) &\leq \gamma(y) \vee \gamma(x) && \\
&= \gamma(y) \vee x && \text{(Claim for } z = \gamma(y)) \\
&\leq \gamma(y) \vee y && (x \leq y) \\
&\leq \gamma(y) && \text{(extensivity of } \gamma)
\end{aligned}$$

So, γ is monotone. We also have

$$\gamma(\gamma(x)) = \gamma(x_1 \bullet \cdots \bullet x_n) = x_1 \bullet \cdots \bullet x_n = \gamma(x).$$

We have shown that γ is idempotent, hence γ is a closure operator. Finally, if $x = x_1 \cdots x_n$ and $y = y_1 \ \cdots y_n$, then

$$\begin{aligned}
\gamma(x) \cdot \gamma(y) &\leq \gamma(\gamma(x) \cdot \gamma(y)) && \text{(extensivity)} \\
&= \gamma((x_1 \bullet \cdots \bullet x_n) \cdot (y_1 \bullet \cdots \bullet y_m)) && \text{(definition of } \gamma) \\
&= (x_1 \bullet \cdots \bullet x_n) \bullet (y_1 \bullet \cdots \bullet y_n) && \text{(definition of } \gamma) \\
&= \gamma(x \cdot y) && \text{(definition of } \gamma)
\end{aligned}$$

Thus, γ is a nucleus. By definition, $\gamma(x) \in M$, for every $x \in L$. So, $L_\gamma \subseteq M$. Conversely, if $x \in M$, then $\gamma(x) = x$, that is $x \in L_\gamma$. We have established that $L_\gamma = M$.

By the remarks at the beginning of the proof and the definition of $\mathbf{L}_\gamma$, we see that the division operations, join and e agree on $\mathbf{L}_\gamma$ and $\mathbf{M}$. Moreover, for $x, y \in M$, $x \circ_\gamma y = \gamma(x \cdot y) = x \bullet y$. Finally, the meet operation on the two structures is the same, since integral GMV-algebras satisfy the identity $x \wedge y \approx (x/y) \cdot y$. Thus, the two structures are identical. $\square$

As an example, we note that the collection of all co-finite subsets of $\mathbb{N}$ is the universe of a generalized Boolean algebra $\mathbf{A}$, hence an integral GMV-algebra. It is easy to see that $\mathbf{A} \cong ((\mathbb{Z}^-)^{\mathbb{N}})_\gamma$, where $\gamma((x_n)_{n\in\mathbb{N}}) = (x_n \vee (-1))_{n\in\mathbb{N}}$.

Combining Theorem 7.11 and Theorem 7.19, we obtain the following.

Theorem 7.20. *A residuated lattice* $\mathbf{M}$ *is a GMV-algebra if and only if it has a direct product decomposition* $\mathbf{M} \cong \mathbf{G} \times \mathbf{H}^-_\gamma$*, where* $\mathbf{G}, \mathbf{H}$ *are* ℓ*-groups and* γ *is a nucleus on* $\mathbf{H}^-$.

Representation as a retraction

In what follows we obtain a second characterization of GMV-algebras. A generalized MV-algebra is shown to be the image of an ℓ-group under an idempotent monotone operator.

Recall the definition of a kernel from Example 3.11.

Lemma 7.21. *If* $\mathbf{L}$ *is a GMV-algebra or a GBL-algebra and* δ *a kernel on it, then so is the* δ*-contraction of* $\mathbf{L}$.

Proof. If $\mathbf{L}$ is a GMV-algebra, then

$$(x \vee y)\backslash x = x\backslash x \wedge y\backslash x = e \wedge y\backslash x \leq e.$$

Since L_δ is an ideal that contains e, we have

$$\delta((x \vee y)\backslash x) = (x \vee y)\backslash x.$$

So,

$$x/_\delta[(x \vee y)\backslash_\delta x] = \delta(x/\delta((x \vee y)\backslash x)) = \delta(x/((x \vee y)\backslash x)) = \delta(x \vee y) = x \vee y.$$

Similarly, if $\mathbf{L}$ is a GBL-algebra, $(x \wedge y)/y \leq e$, so

$$((x \wedge y)/_\delta y)y = \delta((x \wedge y)/y)y = ((x \wedge y)/y)y = x \wedge y.$$

The opposite properties are obtained similarly. $\square$

Theorem 7.22. *A residuated lattice* $\mathbf{L}$ *is a GMV-algebra iff* $\mathbf{L} \cong (\mathbf{G}_\delta)_\gamma$, *for some* ℓ*-group* $\mathbf{G}$, *some kernel* δ *on* $\mathbf{G}$ *and some nucleus* γ *on* $\mathbf{G}_\delta$.

Proof. By the previous lemma, if $\mathbf{G}$ is an ℓ-group and δ a kernel on it, then $\mathbf{G}_\delta$ is a GMV-algebra. Moreover, by Theorem 7.16, $(\mathbf{G}_\delta)_\gamma$ is a GMV-algebra, as well.

Conversely, let $\mathbf{L}$ be a GMV-algebra. By Corollary 7.22, $\mathbf{L} \cong \mathbf{K} \times \mathbf{H}^-_\gamma$, for some ℓ-groups $\mathbf{K}$ and $\mathbf{H}$, and a nucleus γ on $\mathbf{H}^-$. Define a map δ on $K \times H$, by $\delta(k,h) = (k, h \wedge e)$. We will show that δ is a kernel. It is obviously an interior operator and $\delta(e,e) = (e,e)$. Note that

$$\delta(k,h)\delta(k',h') = (k, h \wedge e)(k', h' \wedge e) = (kk', (h \wedge e)(h' \wedge e)) = (kk', hh' \wedge h \wedge h' \wedge e)$$

and $\delta(kk', hh' \wedge h \wedge h' \wedge e) = (kk', hh' \wedge h \wedge h' \wedge e)$. Similarly

$$\delta(k,h) \wedge (k',h') = (k, h \wedge e) \wedge (k', h) = (k \wedge k', h \wedge e \wedge h')$$

and $\delta(k \wedge k', h \wedge e \wedge h') = (k \wedge k', h \wedge e \wedge h')$.

Observe that the underlying set of $(\mathbf{K} \times \mathbf{H})_\delta$ is $K \times H^-$. Define $\bar{\gamma}$ on $K \times H^-$, by $\bar{\gamma}(k,h) = (k, \gamma(h))$. We will show that $\bar{\gamma}$ is a nucleus on $(\mathbf{K} \times \mathbf{H})_\delta$. It is obviously a closure operator. Moreover,

$$\begin{aligned}
\bar{\gamma}(k,h)\bar{\gamma}(k',h') &= (k, \gamma(h))(k', \gamma(h')) \\
&= (kk', \gamma(h)\gamma(h')) \\
&\leq (kk', \gamma(hh')) \\
&= \bar{\gamma}(kk', hh') \\
&= \bar{\gamma}((k,h)(k',h')).
\end{aligned}$$

Notice that $\bar{\gamma}((K \times H)_\delta) = \bar{\gamma}(K \times H^-) = K \times H^-_\gamma$. So, the underlying set of $\mathbf{K} \times \mathbf{H}^-_\gamma$ and $((\mathbf{K} \times \mathbf{H})_\delta)_{\bar{\gamma}}$ coincide. Recalling the constructions of the δ-contraction and γ-retraction, we see that the lattice operations on the two algebras coincide. To show that the other operations are the same, as well, note that for all $(k,h), (k'h') \in K \times H^-_\gamma$,

$$\begin{aligned}
(k,h) \bullet_{((\mathbf{K} \times \mathbf{H})_\delta)_{\bar{\gamma}}} (k',h') &= (k,h) \circ_{\bar{\gamma}} (k',h') \\
&= \bar{\gamma}((k,h) \cdot (k',h')) \\
&= \bar{\gamma}(kk', hh') \\
&= (kk', \gamma(hh')) \\
&= (kk', h \circ_\gamma h') \\
&= (k,h) \bullet_{\mathbf{K} \times \mathbf{H}^-_\gamma} (k',h')
\end{aligned}$$

$$
\begin{aligned}
(k,h)\backslash_{((\mathbf{K}\times\mathbf{H})_\delta)_{\bar\gamma}}(k',h') &= \delta((k,h)\backslash_{\mathbf{K}\times\mathbf{H}}(k',h'))\\
&= \delta((k\backslash_{\mathbf{K}}k', h\backslash_{\mathbf{H}}h'))\\
&= (k\backslash_{\mathbf{K}}k', h\backslash_{\mathbf{H}}h' \wedge e)\\
&= (k\backslash_{\mathbf{K}}k', h\backslash_{\mathbf{H}^-}h')\\
&= (k\backslash_{\mathbf{K}}k', h\backslash_{\mathbf{H}^-_\gamma}h')\\
&= (k,h)\backslash_{\mathbf{K}\times\mathbf{H}^-_\gamma}(k',h')
\end{aligned}
$$

and likewise for the other division operation. □

We investigate the action of nuclei and kernels on GMV-algebras, before we characterize their compositions.

Corollary 7.23. *If δ is a kernel on an ℓ-group $\mathbf{G}$, then there exist ℓ-groups $\mathbf{K}, \mathbf{H}$, such that $\mathbf{G} = \mathbf{K} \times \mathbf{H}$ and $\delta(k,h) = (k, h \wedge e)$, for all $(k,h) \in K \times H$. Thus, $\mathbf{G}_\delta = \mathbf{K} \times \mathbf{H}^-$.*

Proof. Since $\mathbf{G}_\delta$ is a GMV-algebra, by Theorem 7.15, there are ℓ-groups $\mathbf{K}, \mathbf{H}$ and a nucleus γ on $\mathbf{H}^-$, such that $\mathbf{G}_\delta = \mathbf{K} \times \mathbf{H}^-_\gamma$ and the submonoid generated by H^-_γ is $\mathbf{H}^-$. Since $K \times H^-_\gamma$ is contained in G, the ℓ-subgroup generated by $K \times H^-_\gamma$ is contained in G, as well. So, K and the ℓ-subgroup generated by H^-_γ are contained in G. Since the submonoid generated by H^-_γ is $\mathbf{H}^-$ and the ℓ-subgroup generated by H^- is H, H is contained in G. So, $\mathbf{K} \times \mathbf{H}$ is contained in $\mathbf{G}$. By Theorem 7.21, $\mathbf{G}_\delta$ is a lattice ideal of $\mathbf{G}$. Since $(k,h) \in G$, for $k \in K$ and $h \in H^-$, and $(k,e) \in K \times H^-_\gamma = G_\delta$, we get $(k,h) \in G_\delta$. So, $K \times H^-$ is contained in $G_\delta = K \times H^-_\gamma$, which in turn is contained in $K \times H^-$. Thus, $G_\delta = K \times H^-$. If $x \in G^-$, we get $x \in G_\delta$, since $e \in G_\delta$. So, G^- is contained in $G_\delta = K \times H^-$, hence G^- is contained in $K^- \times H^-$, which in turn is contained in G. So, $G^- = K^- \times H^-$, thus $G = K \times H$.

Consequently, $(K \times H)_\delta = K \times H^-$, so $\delta(K \times H) = \delta'(K \times H)$, where $\delta'(g,h) = (g, h \wedge e)$ is a interior operator. Since an interior operator is defined by its image, we get $\delta(g,h) = (g, h \wedge e)$. □

Lemma 7.24.

1. *The identity map is the only nucleus on an ℓ-group.*

2. *The identity is the only kernel on an integral GMV-algebra.*

Proof. 1) Assume γ is a nucleus on the ℓ-group $\mathbf{G}$. Since $\mathbf{G}$ is a GMV-algebra, by Theorem 7.15, $e = \gamma(e) \in G_\gamma$. Moreover, by Lemma 3.3, for every $x \in G$, $e/x \in G_\gamma$, that is $x^{-1} \in G_\gamma$. Thus, $G_\gamma = G$. Since a closure operator is uniquely defined by its image, γ is the identity on G.

2) Assume that δ is a kernel on an integral GMV-algebra $\mathbf{M}$. By Lemma 7.21, M_δ is an ideal of M. Moreover, $e = \delta(e) \in M_\delta$. So, $M_\delta = M$, hence δ is the identity. □

Corollary 7.25. *If δ is a kernel on a GMV-algebra $\mathbf{M}$, then there exist a GMV-algebra $\mathbf{N}$ and an ℓ-group $\mathbf{H}$, such that $\mathbf{M} = \mathbf{N} \times \mathbf{H}$ and $\delta(n, h) = (n, h \wedge e)$, for all $(n, h) \in N \times H$. Thus, $\mathbf{M}_\delta = \mathbf{N} \times \mathbf{H}^-$.*

Proof. By Theorem 7.20, there are ℓ-groups $\mathbf{G}, \mathbf{L}$, and a nucleus γ on $\mathbf{L}^-$, such that $\mathbf{M} = \mathbf{G} \times \mathbf{L}^-_\gamma$. The coordinate maps of δ, which we denote by δ, as well, on $\mathbf{G}$ and $\mathbf{L}^-_\gamma$ are kernels, because of the equational definition of a kernel. By Corollary 7.23, there exist ℓ-groups $\mathbf{K}, \mathbf{H}$, such that $\mathbf{G} = \mathbf{K} \times \mathbf{H}$ and $\delta(k, h) = (k, h \wedge e)$, for all $(k, h) \in K \times H$. So, $\mathbf{M} = \mathbf{K} \times \mathbf{H} \times \mathbf{L}^-_\gamma$. Moreover, by Lemma 7.24(2), δ on $\mathbf{L}^-_\gamma$ is the identity. If we identify isomorphic algebras and set $\mathbf{N} = \mathbf{K} \times \mathbf{L}^-_\gamma$, we get $\mathbf{M} = \mathbf{N} \times \mathbf{H}$ and $\delta(n, h) = (n, h \wedge e)$, for all $(n, h) \in N \times H$. □

Corollary 7.26.

1. *A residuated lattice $\mathbf{L}$ is a cancellative GMV-algebra iff $\mathbf{L} \cong \mathbf{G}_\delta$, for some ℓ-group $\mathbf{G}$ and some kernel δ on $\mathbf{G}$.*

2. *A residuated lattice $\mathbf{L}$ is a GMV-algebra iff $\mathbf{L} \cong \mathbf{K}_\gamma$, for some cancellative GMV-algebra $\mathbf{K}$ and some nucleus γ on $\mathbf{K}$.*

Proof. 1) One direction follows from Corollary 7.23 and Corollary 7.13. For the other direction, assume that $\mathbf{L}$ is a cancellative GMV-algebra. By Corollary 7.13, $\mathbf{L} = \mathbf{K} \times \mathbf{H}^-$, for some ℓ-groups $\mathbf{K}, \mathbf{H}$. It is easy to see that the map δ on $\mathbf{K} \times \mathbf{H}$, defined by $\delta(k, h) = (k, h \wedge e)$ is a kernel and that $(\mathbf{K} \times \mathbf{H})_\delta = \mathbf{K} \times \mathbf{H}^- = \mathbf{L}$.
2) One direction follows from Theorem 7.15. Conversely, if $\mathbf{L}$ is a GMV-algebra, by Theorem 7.20 , there exist ℓ-groups $\mathbf{G}, \mathbf{H}$ and a nucleus on $\mathbf{H}^-$, such that $\mathbf{L} = \mathbf{G} \times \mathbf{H}^-_\gamma$. It is easy to check that the map $\bar{\gamma}$ on $\mathbf{G} \times \mathbf{H}^-$ defined by $\bar{\gamma}(g, h) = (g, \gamma(h))$ is a nucleus and that $(\mathbf{G} \times \mathbf{H}^-)_{\bar{\gamma}} = \mathbf{G} \times \mathbf{H}^-_\gamma = \mathbf{L}$. Finally, $\mathbf{K} = \mathbf{G} \times \mathbf{H}^-$ is a cancellative GMV-algebra, by Corollary 7.13. □

A *core*, defined below, on a GMV-algebra is a typical composition of a nucleus and a kernel.

A map β on a residuated lattice is called a *core* if

1. $\beta(x)\beta(y) \leq \beta(xy)$,

2. $\beta(e) = e$,

3. $(\beta(x) \wedge x)(\beta(y) \wedge y) \leq \beta((\beta(x) \wedge x)(\beta(y) \wedge y))$,

4. $\beta(x) \wedge x \wedge y \leq \beta(\beta(x) \wedge x \wedge y)$ and

5. $\beta(\beta(x) \wedge x) = \beta(x)$.

If δ is a map on a residuated lattice $\mathbf{L}$ and γ a map on $\delta(L)$, define $\beta_{(\gamma,\delta)}$ on L, by $\beta_{(\gamma,\delta)}(x) = \gamma(\delta(x))$. Moreover, β is a map on a residuated lattice, define δ_β on L and γ_β on $\delta_\beta(L)$, by $\delta_\beta(x) = \beta(x) \wedge x$ and $\gamma_\beta(x) = \beta(x)$.

Lemma 7.27. *Let $\mathbf{L}$ be a GMV-algebra. If δ is a kernel on $\mathbf{L}$, γ a nucleus on $\mathbf{L}_\delta$, and β a core on $\mathbf{L}$, then*

1. *γ_β is a nucleus on $\delta_\beta(L)$ and δ_β is a kernel on L,*
2. *$\delta_{\beta_{(\gamma,\delta)}} = \delta$, $\gamma_{\beta_{(\gamma,\delta)}} = \gamma$ and $\beta_{(\gamma_\beta,\delta_\beta)} = \beta$,*
3. *$\beta_{(\gamma,\delta)}$ is a core on $\mathbf{L}$.*

Proof. 1) Since γ_β is the restriction of β, we have $\gamma_\beta(x)\gamma_\beta(y) \leq \gamma_\beta(xy)$, by the first property of a core. So, γ_β is a nucleus.

Obviously, $\delta_\beta(e) = \beta(e) \wedge e = e$, by the second property of a core. The remaining two properties of a kernel state that $\delta_\beta(x)\delta_\beta(y)$ and $\delta_\beta(x) \vee y$ are elements fixed by δ_β. It is easy to see that for every x, $\delta_\beta(x) = x$ iff $x \leq \beta(x)$. So, the remaining properties are equivalent to properties (3) and (4) of the definition of a core, which hold for β. Thus, δ_β is a kernel.

2) We have $\delta_{\beta_{(\gamma,\delta)}}(x) = \beta_{(\gamma,\delta)}(x) \wedge x = \gamma(\delta(x)) \wedge x$. In view of Corollary 7.25, to show that $\delta_{\beta_{(\gamma,\delta)}} = \delta$, it suffices to verify that $\gamma(\delta(x)) \wedge x = \delta(x)$, only for the cases $\delta(x) = x$ and $\delta(x) = x \wedge e$. In the first case, the equation holds, by the extensivity of γ. In the second case, the equation reduces to $\gamma(x \wedge e) \wedge x = x \wedge e$. By the extensivity of γ we have $x \wedge e = x \wedge e \wedge x \leq \gamma(x \wedge e) \wedge x$ and by the monotonicity of γ we get $\gamma(x \wedge e) \wedge x \leq \gamma(e) \wedge x = e \wedge x$, by Theorem 7.15(1).

For every x in the range of $\delta_{\beta_{(\gamma,\delta)}} = \delta$, namely $\delta(x) = x$, we have $\gamma_{\beta_{(\gamma,\delta)}}(x) = \beta_{(\gamma,\delta)}(x) = \gamma(\delta(x)) = \gamma(x)$. Finally, $\beta_{(\gamma_\beta,\delta_\beta)}(x) = \gamma_\beta(\delta_\beta(x)) = \beta(\beta(x) \wedge x) = \beta(x)$.

3) For the first property of a core we have

$$\beta(x)\beta(y) = \gamma(\delta(x))\gamma(\delta(y)) \leq \gamma(\delta(x)\delta(y)) = \gamma(\delta(\delta(x)\delta(y))) \leq \gamma(\delta(xy)) = \beta(xy).$$

Also, $\beta(e) = \gamma(\delta(e)) = \gamma(e) = e$, by Theorem 7.15(1).

Since for every x, $x \leq \beta_{(\gamma,\delta)}(x)$ iff $\delta_{\beta_{(\gamma,\delta)}}(x) = x$, properties (3) and (4) of the definition of a core hold for $\beta_{(\gamma,\delta)}$ iff and only if the last two properties of a kernel hold for $\delta_{\beta_{(\gamma,\delta)}}$. This is a true statement, since $\delta_{\beta_{(\gamma,\delta)}} = \delta$, by (2).

The last property of a core for $\beta_{(\gamma,\delta)}$ is equivalent to $\beta_{(\gamma,\delta)}(\delta_{\beta_{(\gamma,\delta)}}(x)) = \beta_{(\gamma,\delta)}(x)$, that is $\beta_{(\gamma,\delta)}(\delta(x)) = \beta_{(\gamma,\delta)}(x)$, namely $\gamma(\delta(\delta(x))) = \gamma(\delta(x))$, which follows from the idempotency of δ. □

For a residuated lattice $\mathbf{L}$ and a core β on it, define $\mathbf{L}_\beta = (\mathbf{L}_{\delta_\beta})_{\gamma_\beta}$.

Corollary 7.28. *A residuated lattice $\mathbf{L}$ is a GMV-algebra iff $\mathbf{L} \cong \mathbf{G}_\beta$, for some ℓ-group $\mathbf{G}$ and some core β on $\mathbf{G}$.*

Categorical equivalences

In this section we show that the representations of integral GMV-algebras and of GMV-algebras extend to categorical equivalences.

Let **IGMV** be the category with objects integral GMV-algebras and morphisms residuated lattice homomorphisms. Also, let $\mathbf{LG}^-_*$ be the category with objects algebras $\langle \mathbf{L}, \gamma \rangle$, such that $\mathbf{L}$ is the negative cone of an ℓ-group and γ is a nucleus on it, whose image generates L as a monoid. Let the morphisms of this category be homomorphisms between these algebras.

Moreover, let **GMV** be the category with objects GMV-algebras and morphisms residuated lattice homomorphisms. Also, let $\mathbf{LG}^*$ be the category with objects algebras $\langle \mathbf{G}, \beta \rangle$, such that $\mathbf{G}$ is an ℓ-group and β is core on $\mathbf{G}$, whose image generates $\mathbf{G}$. Let the morphisms of this category be homomorphisms between these algebras.

The two main results of this section, Theorem 7.43 and Theorem 7.44, assert that the two pairs of categories defined above are pairs of equivalent categories.

Lemma 7.29. *For a, b, c in the negative cone of an ℓ-group, $ab = c$ iff ($a = c/b$ and $c \leq b$) iff ($b = a\backslash c$ and $c \leq a$). Moreover, the negative cone of every ℓ-group satisfies the identity, $x/(y \wedge z) \approx x/y \vee x/z$ and its opposite.*

Proof. If $ab = c$, then $ab/b = c/b$, so, by Theorem 4.5, $a = c/b$. Moreover, $c = ab \leq eb \leq b$, by integrality. Conversely, if $a = c/b$, then $ab = (c/b)b$. So, since negative cones of ℓ-groups are integral GBL-algebras, $ab = c \wedge b$. Since $c \leq b$, we get $ab = c$.
Assume that $\mathbf{G}$ is an ℓ-group and recall the definition of a negative cone. For elements $x, y, z \in G^-$, we have

$$\begin{aligned} x/(y \wedge z) &= x(y \wedge z)^{-1} \wedge e \\ &= x(y^{-1} \vee z^{-1}) \wedge e \\ &= (xy^{-1} \vee xz^{-1}) \wedge e \\ &= (xy^{-1} \wedge e) \vee (xz^{-1} \wedge e) \\ &= x/y \vee x/z. \end{aligned}$$

For the opposite equation we work similarly. $\square$

Definition 7.30 and lemmas 7.31, 7.32, 7.33, 7.36 and 7.39 are non-commutative, unbounded generalizations of concepts and results in [Mu].

Definition 7.30. *Let* $\mathbf{L}$ *be the negative cone of an* ℓ*-group and* u, x *elements of it. Define the elements* x_n^u *and* b_n^u*, for every natural number* n*, inductively, by* $b_0^u = x$ *and* $x_{k+1}^u = u \vee b_k^u$*,* $b_{k+1}^u = x_{k+1}^u \backslash b_k^u$*, for all* $k \geq 0$*.*

Lemma 7.31. *Let* $\mathbf{L}$ *be the negative cone of an* ℓ*-group and* u, x *elements of it. For all natural numbers* n*,*

1. $b_n^u = u^n \backslash x$*,*

2. $b_n^u = (x_1^u x_2^u \cdots x_n^u) \backslash x$*,*

3. $x \leq x_1^u x_2^u \cdots x_n^u$*.*

Proof. Statement (1) is obvious for $n = 0$; we proceed by induction. Assume the statement is true for $n = k$. To show that it is true for $n = k + 1$, note that, using properties (3) and (6) of Lemma 3.1, we get

$$\begin{aligned} b_{k+1}^u &= x_{k+1}^u \backslash b_k^u = (u \vee b_k^u) \backslash b_k^u \\ &= u \backslash b_k^u \wedge b_k^u \backslash b_k^u = u \backslash b_k^u \wedge e \\ &= u \backslash (u^k \backslash x) = u^{k+1} \backslash x. \end{aligned}$$

The second statement is clear from the definition of b_n^u and Lemma 3.1(6). We prove the third statement by induction. For $n = 1$ we have $x = b_0^u \leq u \vee b_0^u = x_1$. If $x \leq x_1^u x_2^u \cdots x_n^u$, then,

$$\begin{aligned} x &= x_1^u x_2^u \cdots x_n^u \wedge x \\ &= x_1^u x_2^u \cdots x_n^u [(x_1^u x_2^u \cdots x_n^u) \backslash x] \\ &= x_1^u x_2^u \cdots x_n^u b_n^u \\ &\leq x_1^u x_2^u \cdots x_n^u (u \vee b_n^u) \\ &\leq x_1^u x_2^u \cdots x_n^u \cdot x_{n+1}^u. \end{aligned}$$

Thus, $x \leq x_1^u x_2^u \cdots x_n^u$ holds for all natural numbers. □

Lemma 7.32. *Let* $\mathbf{L}$ *be the negative cone of an* ℓ*-group and* u, x *elements of it. If* $u^n \leq x$*, for some natural number* n*, then*

1. $b_{k-1}^u = e$*, for all* $k > n$*,*

2. $x_k^u = e$*, for all* $k > n$*,*

3. $x = x_1^u x_2^u \cdots x_n^u$*,*

4. If $x \leq y$, then $x_i^u \leq y_i^u$, for all i.

Proof. For the first property note that $u^{k-1} \leq u^n \leq x$, so $e \leq u^{k-1}\backslash x$. Hence, $b_{k-1}^u = u^{(k-1)}\backslash x = e$. As a consequence we have $x_k^u = u \vee b_{k-1}^u = u \vee e = e$. Moreover, by Lemma 7.29 and Lemma 7.31(3), $x = x_1^u x_2^u \cdots x_n^u$,so $e = b_{k-1}^u = (x_1^u x_2^u \cdots x_n^u)\backslash x$. Finally, by Lemma 7.31(1), we have $x_i^u = u \vee u^{i-1}\backslash x \leq u \vee u^{i-1}\backslash y = y_i^u$, for all i. $\square$

Lemma 7.33. *Let* $\mathbf{L}$ *be the negative cone of an ℓ-group and γ a nucleus on it, such that L_γ generates L as a monoid. If $x \in L$ and $u \leq \gamma(x)$, then $x = x_1^u x_2^u \cdots x_n^u$, for some n.*

Proof. By the monoid generation property, we have $x = x_1 x_2 \cdots x_n$, for some elements $x_1, \ldots x_n$ of L_γ and some natural number n. So,

$$u \leq \gamma(x) = \gamma(x_1 \cdots x_n) = x_1 \circ_\gamma \cdots \circ_\gamma x_n \leq x_i,$$

for all i. Thus, $u^n \leq x_1 x_2 \cdots x_n = x$. The lemma follows from Lemma 7.32(3). $\square$

Lemma 7.34. *Let* $\mathbf{L}$ *be the negative cone of an ℓ-group and γ a nucleus on it, such that L_γ generates L as a monoid. Also, let $z, x \in L$, $x \leq z$ and $u = \gamma(z)$. Then, the elements x_i^u are the unique elements x_i that satisfy $x = x_1 \cdots x_n$, for some n, and $x_i \circ_\gamma x_{i+1} = x_i$, for all $i \geq 1$.*

Proof. Note that $x = x_1^u \cdots x_n^u$, for some n, by Lemma 7.33, since $u = \gamma(z) \leq \gamma(x)$, by the monotonicity of γ. Additionally, $x_i^u \circ_\gamma x_{i+1}^u = \gamma(x_i^u x_{i+1}^u) = \gamma_u(x_i^u x_{i+1}^u)$, by Lemma 7.17, since $z \leq x \leq x_i^u x_{i+1}^u$, which in turn equals $u \vee x_i^u x_{i+1}^u = u \vee x_i^u(u \vee b_i^u) = u \vee x_i^u u \vee x_i^u b_i^u = u \vee x_i^u u \vee b_{i-1}^u = u \vee b_{i-1}^u = x_i^u$.

Conversely, if $x = x_1 \cdots x_n$, for some n, and $x_i \circ_\gamma x_{i+1} = x_i$, for all i, then $\gamma(x) = \gamma(x_1 \cdots x_n)$. So, since $z \leq x$, by Lemma 7.17, $\gamma_u(x) = x_1 \circ_\gamma \cdots \circ_\gamma x_n$, hence $u \vee x = x_1$, namely $x_1 = x_1^u$. We proceed by induction. If $x_i = x_i^u$, for all $i \leq k$, then

$$\begin{aligned}
(x_1 \cdots x_k)\backslash x = x_{k+1} \cdots x_n \quad & \Rightarrow\ (x_1^u \cdots x_k^u)\backslash x = x_{k+1} \cdots x_n \\
& \Rightarrow\ b_k^u = x_{k+1} \cdots x_n \geq x \geq z \\
& \Rightarrow\ \gamma(b_k^u) = \gamma(x_{k+1} \cdots x_n) \text{ and } z \leq b_k^u \\
& \Rightarrow\ \gamma(b_k^u) = x_{k+1} \circ_\gamma \cdots \circ_\gamma x_n \text{ and } \gamma(b_k^u) = \gamma_u(b_k^u) \\
& \Rightarrow\ \gamma(b_k^u) = x_{k+1} \text{ and } \gamma(b_k^u) = u \vee b_k^u \\
& \Rightarrow\ x_{k+1} = x_{k+1}^u.
\end{aligned}$$

Thus, the decomposition is unique. $\square$

Corollary 7.35. *Let* $\mathbf{L}$ *be the negative cone of an ℓ-group and γ a nucleus on it, such that L_γ generates L as a monoid. If $z, x \in L$ and $z \leq x$, then, for all $i \geq 1$, $x_i^{\gamma(z)} = x_i^{\gamma(x)}$.*

Lemma 7.36. *Let* $\mathbf{L}$ *be the negative cone of an* ℓ*-group and* γ *a nucleus on it, such that* L_γ *generates* L *as a monoid. Also, let* x, y, t *be elements of* L*, such that* $t \leq x \wedge y$ *and* $u = \gamma(t)$*. Then, for some natural number* s*,*

$$x \wedge y = \prod_{i=1}^{s} (x_i^u \wedge y_i^u).$$

Proof. Set $z = x \wedge y$. Then, by Lemma 7.33, for some s,

$$x = \prod_{i=1}^{s} x_i^u, \; y = \prod_{i=1}^{s} y_i^u \text{ and } z = \prod_{i=1}^{s} z_i^u.$$

Obviously,

$$\prod_{i=1}^{s} (x_i^u \wedge y_i^u) \leq \prod_{i=1}^{s} x_i^u \wedge \prod_{i=1}^{s} y_i^u = x \wedge y = z$$

Moreover, $z \leq x, y$, so $z_i^u \leq x_i^u \wedge y_i^u$, for all i, by Lemma 7.32(4). Consequently,

$$z = \prod_{i=1}^{s} z_i^u \leq \prod_{i=1}^{s} (x_i^u \wedge y_i^u).$$

Thus, $z = \prod_{i=1}^{s} (x_i^u \wedge y_i^u)$. □

Let $\mathbf{L}$ be the negative cone of an ℓ-group and $a_i, b_j, c_{ij} \in L$. We say that the matrix $C = [c_{ij}]$, $1 \leq i \leq n$, $1 \leq j \leq m$ is an *orthogonal decomposition* of the factors of the equation $a_1 \cdot a_2 \cdots a_n = b_1 \cdot b_2 \cdots b_m$, in symbols,

$$\begin{array}{cc} & \begin{array}{ccc} a_1 & \dots & a_n \end{array} \\ \begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} & \left[\left[\begin{array}{ccc} c_{11} & .. & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & .. & c_{mn} \end{array}\right]\right] \end{array}$$

if for all i, j,

$$a_i = \prod_{j=1}^{m} c_{ij}, \qquad b_j = \prod_{i=1}^{n} c_{ij},$$

and the (i, j)-*orthogonality condition*,

$$\prod_{k=i+1}^{n} c_{kj} \vee \prod_{l=j+1}^{m} c_{il} = e$$

holds, for all i, j; that is the product of the elements to the right of c_{ij} is orthogonal to the product of elements below it.

Lemma 7.37. *Let* $\mathbf{L}$ *be the negative cone of an* ℓ*-group and* $a_i, b_j, c_{ij} \in L$*. If the matrix* C *is an orthogonal decomposition of the factors of the equation* $a_1 \cdot a_2 \cdots a_n = b_1 \cdot b_2 \cdots b_m$*, then the equation holds.*

Proof. For $m = n = 2$, we have $a_1 a_2 = c_{11} c_{12} c_{21} c_{22} = c_{11} c_{21} c_{12} c_{22} = b_1 b_2$. We proceed by induction on the pair (m, n). Assume the lemma is true for all pairs (m, k), where $m \geq 2$ and $k < n$. We will show it is true for the pair (m, n).

Suppose that the matrix $C = [c_{ij}],\ 1 \leq i \leq n,\ 1 \leq j \leq m$ is an orthogonal decomposition of the factors of the equation $a_1 \cdot a_2 \cdots a_n = b_1 \cdot b_2 \cdots b_m$. It is easy to see that

$$\begin{array}{cc} & \begin{array}{ccc} a_2 & \dots & a_n \end{array} \\ \begin{array}{c} c_1 \\ \vdots \\ c_m \end{array} & \left[\left[\begin{array}{ccc} c_{12} & \cdot\cdot & c_{1n} \\ \vdots & & \vdots \\ c_{m2} & \cdot\cdot & c_{mn} \end{array}\right]\right] \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{cc} a_1 & c \end{array} \\ \begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} & \left[\left[\begin{array}{cc} c_{11} & c_1 \\ \vdots & \vdots \\ c_{m1} & c_m \end{array}\right]\right] \end{array}$$

where $c = c_1 \cdots c_m$. So, $a_1 \cdot a_2 \cdots a_n = a_1 \cdot (c_2 \cdots c_m) = a_1 c = b_1 \cdot b_2 \cdots b_m$ □

The following refinement lemma, can be found in [Fu]. For completeness, we give the proof in the language of negative cones of ℓ-groups.

Lemma 7.38. *([Fu], Theorem 1, p. 68) Let* $\mathbf{L}$ *be the negative cone of an* ℓ*-group and let* $a_1, ..., a_n, b_1, ..., b_m$ *be elements of* L*. Then,* $a_1 \cdot a_2 \cdots a_n = b_1 \cdot b_2 \cdots b_m$ *iff there exists an orthogonal decomposition of the factors of the equation.*

Proof. One direction is given by the previous lemma. For the other direction we use induction. We first prove it for $m = n = 2$. Assume that $a_1 a_2 = b_1 b_2 = c$. Set

$$c_{11} = a_1 \vee b_1, \quad c_{21} = a_2 / c_{22}$$
$$c_{12} = c_{11} \backslash a_1, \quad c_{22} = a_2 \vee b_2$$

Using Lemma 7.29 and Lemma 3.1 we get

$$\begin{aligned}
c_{21} &= a_2/c_{22} = a_2/(a_2 \vee b_2)\\
&= (a_1\backslash c)/(a_1\backslash c \vee b_1\backslash c)\\
&= (a_1\backslash c)/((a_1 \wedge b_1)\backslash c)\\
&= a_1\backslash[c/((a_1 \wedge b_1)\backslash c)]\\
&= a_1\backslash[(a_1 \wedge b_1) \vee c]\\
&= a_1\backslash(a_1 \wedge b_1) = a_1\backslash a_1 \wedge a_1\backslash b_1\\
&= e \wedge a_1\backslash b_1 = a_1\backslash b_1 \wedge b_1\backslash b_1\\
&= (a_1 \vee b_1)\backslash b_1 = c_{11}\backslash b_1
\end{aligned}$$

Similarly, we show that $c_{12} = b_2/c_{22}$. Consequently, we can compute the products

$$c_{11}c_{12} = c_{11}(c_{11}\backslash a_1) = c_{11} \wedge a_1 = (a_1 \vee b_1) \wedge a_1 = a_1$$

$$c_{21}c_{22} = (a_2/c_{22})c_{22} = a_2 \wedge c_{22} = a_2$$

$$c_{11}c_{21} = c_{11}(c_{11}\backslash b_1) = c_{11} \wedge b_1 = b_1$$

$$c_{12}c_{22} = (b_2/c_{22})c_{22} = b_2 \wedge c_{22} = b_2$$

Finally, $c_{12} \vee c_{21} = c_{11}\backslash a_1 \vee c_{11}\backslash b_1 = c_{11}\backslash(a_1 \vee b_1) = c_{11}\backslash c_{11} = e$.

For the general case, we proceed by induction on the pair (m, n). Assume that the statement is true for all pairs (m, k), where $m \geq 2$ and $k < n$. We will show it is true for the pair (m, n).

Assume that $a_1 \cdot a_2 \cdots a_n = b_1 \cdot b_2 \cdots b_m$ and set $a = a_2 \cdot a_3 ... a_n$. So, $a_1 a = b_1 \cdot b_2 \cdots b_m$. By the induction hypothesis, we get

$$\begin{array}{cc}
 & \begin{array}{cc} a_1 & a \end{array} \\
\begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} & \left[\left[\begin{array}{cc} c_{11} & c_{12} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{array}\right]\right]
\end{array}
\quad \text{and} \quad
\begin{array}{cc}
 & \begin{array}{ccc} a_2 & \dots & a_n \end{array} \\
\begin{array}{c} c_{12} \\ \vdots \\ c_{m2} \end{array} & \left[\left[\begin{array}{ccc} d_{12} & .. & d_{1n} \\ \vdots & & \vdots \\ d_{m2} & .. & d_{mn} \end{array}\right]\right]
\end{array}$$

So, we have,

$$
\begin{array}{cc}
 & \begin{array}{cccc} a_1 & a_2 & \dots & a_n \end{array} \\
\begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} & \left[\left[\begin{array}{cccc} c_{11} & d_{12} & \dots & d_{1n} \\ \vdots & & & \vdots \\ c_{m1} & d_{m2} & \dots & d_{mn} \end{array}\right]\right]
\end{array}
$$

□

Lemma 7.39. *Let* $\mathbf{L}$ *be the negative cone of an* ℓ*-group,* γ *a nucleus on it and* $a, a_1, \dots, a_n$ *in* $\mathbf{L}_\gamma$*. Then,* $a = a_1 \cdot a_2 \cdots a_n$ *iff* $a = a_1 \circ_\gamma a_2 \circ_\gamma \cdots \circ_\gamma a_n$ *and* $a_k = (a_k \circ_\gamma a_{k+1} \circ_\gamma \cdots \circ_\gamma a_n)/(a_{k+1} \circ_\gamma a_{k+2} \circ_\gamma \cdots \circ_\gamma a_n)$*, for all* $1 \leq k < n$*.*

Proof. We use induction on n. For $n = 2$, if $a = a_1a_2$, then $\gamma(a) = \gamma(a_1a_2)$, so $a = a_1 \circ_\gamma a_2$. Moreover, by Lemma 7.29, $a_1 = a/a_2$, so $a_1 = (a_1 \circ_\gamma a_2)/a_2$. Conversely, if $a = a_1 \circ_\gamma a_2$, then $a = \gamma(a_1a_2) \leq \gamma(a_2) = a_2$. Since $a_1 = a/a_2$, we get $a = a_1a_2$, by Lemma 7.29.

Assume, now, that the statement is true for all numbers less than n.

$$
\begin{aligned}
a = a_1(a_2 \cdots a_n) \quad &\Leftrightarrow \quad a = a_1 b \text{ and } b = a_2 \cdots a_n \\
&\Leftrightarrow \quad a = a_1 \circ_\gamma b,\ a_1 = a/b,\ b = a_2 \circ_\gamma \cdots \circ_\gamma a_n \text{ and} \\
&\qquad\ a_k = (a_k \circ_\gamma a_{k+1} \circ_\gamma \cdots \circ_\gamma a_n)/(a_{k+1} \circ_\gamma a_{k+2} \circ_\gamma \cdots \circ_\gamma a_n), \\
&\qquad\ \text{for all } 2 \leq k < n \\
&\Leftrightarrow \quad a = a_1 \circ_\gamma a_2 \circ_\gamma \cdots \circ_\gamma a_n \text{ and} \\
&\qquad\ a_k = (a_k \circ_\gamma a_{k+1} \circ_\gamma \cdots \circ_\gamma a_n)/(a_{k+1} \circ_\gamma a_{k+2} \circ_\gamma \cdots \circ_\gamma a_n), \\
&\qquad\ \text{for all } 1 \leq k < n.
\end{aligned}
$$

□

Lemma 7.40. *Assume* $\mathbf{K}, \mathbf{L}$ *are negative cones of* ℓ*-groups,* γ_1, γ_2 *are nuclei and* K_{γ_1}*,* L_{γ_2} *generate* $\mathbf{K}$ *and* $\mathbf{L}$ *respectively as monoids. Moreover, let* $f : \mathbf{K}_{\gamma_1} \to \mathbf{L}_{\gamma_2}$ *be a residuated lattice homomorphism and let* $a_1, \dots, a_n, b_1, \dots, b_m$ *be elements of* M*, such that* $a_1a_2 \cdots a_n = b_1b_2 \cdots b_m$*, where multiplication is in* $\mathbf{K}$*. Then,* $f(a_1)f(a_2) \cdots f(a_n) = f(b_1)f(b_2) \cdots f(b_m)$*, where multiplication is in* $\mathbf{L}$*.*

Proof. First note that, for all $c_1, c_2, \dots, c_n \in K_{\gamma_1}$, if $c_1c_2 \cdots c_n \in K_{\gamma_1}$, then

$$f(c_1c_2 \cdots c_n) = f(c_1)f(c_2) \cdots f(c_n).$$

To see that, notice that by Lemma 7.39, $c = c_1c_2 \cdots c_n$ is equivalent to a system of IGMV-algebra equations in $\mathbf{K}_{\gamma_1}$. Since f is a homomorphism, the same equations hold for the images of the elements under f. Applying Lemma 7.39 again, we get $f(c) = f(c_1)f(c_2) \cdots f(c_n)$.

By Lemma 7.38, there exist $c_{ij} \in K_{\gamma_1}$, such that if for all i, j,

$$a_i = \prod_{j=1}^{m} c_{ij},\ b_j = \prod_{i=1}^{n} c_{ij} \text{ and } \prod_{k=i+1}^{n} c_{kj} \vee \prod_{l=j+1}^{m} c_{il} = e.$$

Using the observation above and the fact that f preserves joins (join in $\mathbf{K}_{\gamma_1}$ is the restriction of join in $\mathbf{K}$), we get that, for all i, j,

$$f(a_i) = \prod_{j=1}^{m} f(c_{ij}),\ f(b_j) = \prod_{i=1}^{n} f(c_{ij}) \text{ and } \prod_{k=i+1}^{n} f(c_{kj}) \vee \prod_{l=j+1}^{m} f(c_{il}) = e.$$

Applying Lemma 7.38 again, we get

$$f(a_1)f(a_2)\cdots f(a_n) = f(b_1)f(b_2)\cdots f(b_m),$$

where multiplication is calculated in $\mathbf{L}$. □

Lemma 7.41. *Assume $\mathbf{K}, \mathbf{L}$ are negative cones of ℓ-groups, γ_1, γ_2 are nuclei, K_{γ_1}, L_{γ_2} generate $\mathbf{K}$ and $\mathbf{L}$ respectively as monoids, and $f : \mathbf{K}_{\gamma_1} \to \mathbf{L}_{\gamma_2}$ is a residuated lattice homomorphism. Then, the map $\bar{f} : \mathbf{K} \to \mathbf{L}$, defined by $\bar{f}(x_1 x_2 \cdots x_n) = f(x_1)f(x_2)\cdots f(x_n)$, is a homomorphism, such that $\bar{f} \circ \gamma_1 = \gamma_2 \circ \bar{f}$.*

Proof. By Lemma 7.40, $\bar{f}$ is well defined and it obviously preserves multiplication. If $x \in K$, then there exist $x_1, \ldots, x_n \in L_{\gamma_1}$ such that $x = x_1 \cdots x_n$. Hence,

$$\begin{aligned}
\bar{f}(\gamma_1(x)) \quad &= f(\gamma_1(x)) \\
&= f(\gamma_2(x_1 \cdots x_n)) \\
&= f(x_1 \circ_{\gamma_1} \cdots \circ_{\gamma_1} x_n) \\
&= f(x_1) \circ_{\gamma_2} \cdots \circ_{\gamma_2} f(x_n) \\
&= \gamma_2(f(x_1) \cdots f(x_n)) \\
&= \gamma_2(\bar{f}(x)).
\end{aligned}$$

Thus, $\bar{f} \circ \gamma_1 = \gamma_2 \circ \bar{f}$. Note that $\bar{f}$ is order preserving. If $x \leq y$ and $u = \gamma(x \wedge y)$, then

$$\bar{f}(x) = f(x_1^u) \cdots f(x_n^u) \leq f(y_1^u) \cdots f(y_n^u) = \bar{f}(y),$$

by Lemma 7.32(4). Note that if $u = \gamma_1(z),\ z \leq x$, then, by Lemma 7.34,

$$x = x_1^u \cdots x_n^u \text{ and } x_i^u \circ_{\gamma_1} x_{i+1}^u = x_i^u.$$

So,

$$\bar{f}(x) = f(x_1^u) \cdots f(x_n^u) \text{ and } f(x_i^u) \circ_{\gamma_2} f(x_{i+1}^u) = f(x_i^u).$$

Applying Lemma 7.34 again, we get that for all i,

$$f(x_i^u) = (\bar{f}(x))_i^{\gamma_2(\bar{f}(x))}.$$

Since $\bar{f}$ preserves order, $\bar{f}(z) \leq \bar{f}(x)$. So, by Corollary 7.35,

$$f(x_i^u) = (\bar{f}(x))_i^{\gamma_2(\bar{f}(z))}.$$

We can now show that $\bar{f}$ preserves meets.
Let $z = x \wedge y,\ u = \gamma_1(z)$

$$\begin{aligned}
\bar{f}(x \wedge y) &= \bar{f}(\prod_{i=1}^{s}(x_i^u \wedge y_i^u)) \text{ (Lemma 7.36)} \\
&= \prod_{i=1}^{s} f(x_i^u \wedge y_i^u) \\
&= \prod_{i=1}^{s}(f(x_i^u) \wedge f(y_i^u)) \\
&= \prod_{i=1}^{s}[(\bar{f}(x))_i^{\gamma_2(\bar{f}(z))} \wedge (\bar{f}(y))_i^{\gamma_2(\bar{f}(z))}] \\
&= \bar{f}(x) \wedge \bar{f}(y),
\end{aligned}$$

where the last equality is given by Lemma 7.36, since

$$\gamma_2(\bar{f}(z)) \leq \gamma_2(\bar{f}(x)) \wedge \gamma_2(\bar{f}(y)).$$

Thus, $\bar{f}$ is a map between the negative cones of two ℓ-groups that preserves multiplication and meet. By Theorem 1.4.5 of [BKW], $\bar{f}$ is a homomorphism. □

Corollary 7.42. *Under the hypothesis of the previous theorem, if f is an isomorphism, then so is $\bar{f}$.*

Proof. To show that $\bar{f}$ is onto, let $y \in L$. There exist $y_1, \ldots, y_n \in K_{\gamma_2}$, such that $y = y_1 \cdots y_n$. Moreover, there exist $x_1, \ldots, x_n \in K_{\gamma_1}$, such that $f(x_i) = y_i$ for all i. Then, $\bar{f}(x_1 \cdots x_n) = f(x_1) \cdots f(x_n) = y_1 \cdots y_n = y$.

If $\bar{f}(x) = \bar{f}(y)$, namely $f(x_1^u) \cdots f(x_n^u) = f(y_1^u) \cdots f(y_m^u)$ then, by the preservation of the uniqueness of the decomposition under $\bar{f}$, given in the proof of the previous theorem, we get $f(x_i^u) = f(y_i^u)$ for all i. By the injectivity of f we get $x_i^u = y_i^u$, for all i, so $x = y$. □

Theorem 7.43. *The categories* $\mathbf{LG}_*^-$ *and* $\mathbf{IGMV}$ *are equivalent.*

Proof. For an object $\langle \mathbf{K}, \gamma \rangle$ of $\mathbf{LG}_*^-$, let $\Gamma(\langle \mathbf{K}, \gamma \rangle) = \mathbf{K}_\gamma$ and for a homomorphism $f : \langle \mathbf{K}, \gamma_1 \rangle \to \langle \mathbf{L}, \gamma_2 \rangle$ let $\Gamma(f)$ be the restriction of f to K_{γ_1}.

By Corollary 7.16, $\Gamma(\langle \mathbf{K}, \gamma \rangle)$ is an object. Using the fact that f commutes with the nuclei it is easy to see that $\Gamma(f)$ is a morphism of $\mathbf{IGMV}$. To check, for example, that it preserves multiplication, note that

$$\Gamma(f)(x \circ_{\gamma_1} y) = f(\gamma_1(xy)) = \gamma_2(f(xy)) = \gamma_2(f(x)f(y)) = f(x) \circ_{\gamma_2} f(y).$$

Moreover, it is obvious that $\Gamma(f \circ g) = \Gamma(f) \circ \Gamma(g)$ and that $\Gamma(id_{\mathbf{K}_{\gamma_1}}) = id_{\mathbf{K}_{\gamma_1}}$. Thus, Γ is a functor between the two categories. By Theorem 7.19, Γ is onto the objects of $\mathbf{IGMV}$ and by Lemma 7.41, Γ is full. Finally, Γ is faithful, because if two morphisms agree on a generating set, they agree on the whole negative cone of the ℓ-group. Thus, by [Ml], Γ is a categorical equivalence between the two categories. $\square$

Since the category of ℓ-groups and the category of their negative cones are equivalent, by [BCGJT], one can consider for objects pairs $(\mathbf{G}, \gamma)$, where $\mathbf{G}$ is an ℓ-group, but all other conditions remain the same (i.e., γ is a nucleus on $\mathbf{G}^-$ and morphisms are homomorphisms between negative cones), and still obtain a categorical equivalence between the categories $\mathbf{LG}_*$ and $\mathbf{IGMV}$.

The categorical equivalence holds also for the full subcategories of $\mathbf{IGMV}$ and $\mathbf{LG}_*^-$ (or $\mathbf{LG}_*$), where we consider only bounded IGMV-algebras, also known as pseudo-MV-algebras (category $\mathbf{bIGMV}$), and nuclei γ such that $\gamma(x) = u \vee x$, for some u (category $\mathbf{bLG}_*^-$ or $\mathbf{bLG}_*$).

Moreover, the categorical equivalence holds also for the subcategories of $\mathbf{IGMV}$ and $\mathbf{LG}_*^-$ (or $\mathbf{LG}_*$), where we consider only homomorphisms such that the order filter generated by the image is the co-domain (categories $\mathbf{IGMVb}$, and $\mathbf{LG}_*^-\mathbf{b}$ or $\mathbf{LG}_*\mathbf{b}$).

Finally, the same holds if we make both of these restrictions to obtain the categories $\mathbf{bIGMVb}$, and $\mathbf{bLG}_*^-\mathbf{b}$ or $\mathbf{bLG}_*\mathbf{b}$. This final categorical equivalence is the one established by Dvurečenskij in [Dv]. If we restrict further to the commutative sub-case, we obtain Mundici's result, in [Mu].

Theorem 7.44. *The categories* $\mathbf{LG}^*$ *and* $\mathbf{GMV}$ *are equivalent.*

Proof. For an object $\langle \mathbf{G}, \beta \rangle$ of $\mathbf{LG}^*$, define $\Gamma(\langle \mathbf{G}, \beta \rangle) = \mathbf{G}_\beta$. For a morphism f of $\mathbf{LG}^*$ with domain $\langle \mathbf{G}, \beta \rangle$, define $\Gamma(f)$ to be the restriction of f to G_β.

Let $\delta = \delta_\beta$ and $\gamma = \gamma_\beta$. By Corollary 7.21 and Theorem 7.15, $\Gamma(\langle \mathbf{G}, \beta \rangle)$ is an object of **GMV**. Actually, $\mathbf{G}_\beta = \langle (G_\delta)_\gamma, \wedge, \vee, \circ_\gamma, \backslash_\delta, /_\delta, e \rangle$. To show that $\Gamma(f)$ is a morphism of **GMV** we use the fact that f commutes with β - we use the same symbol for the cores in the domain and in the co-domain.

First note that if $x = \beta(x)$, then $x = \gamma(x) = \delta(x)$. In this case $f(x) = \delta(f(x)) = \gamma(f(x))$. By Lemma 7.27,

$$\delta(f(x)) = \beta(f(x)) \wedge f(x) = f(\beta(x)) \wedge f(x) = f(\beta(x) \wedge x) = f(\delta(x)) = f(x).$$

Moreover, $\gamma(f(x)) = \gamma(\delta(f(x))) = f(\gamma(\delta(x))) = f(x)$.

We can now show that f preserves multiplication. For $x, y \in \beta(G)$, $x = \delta(x) = \gamma(x)$ and $y = \delta(y)\gamma(y)$, so $\delta(xy) = \delta(\delta(x)\delta(y)) = \delta(x)\delta(y) = xy$. Thus,

$$\begin{aligned} f(x \circ_\gamma y) &= f(\gamma(xy)) = f(\gamma(\delta(xy)) = f(\beta(xy)) \\ &= \beta(f(xy)) = \gamma(\delta(f(xy))) = \gamma(f(xy)) \\ &= \gamma(f(x)f(y)) = f(x) \circ_\gamma f(y) \end{aligned}$$

Additionally,

$$\begin{aligned} f(x/_\delta y) &= f(\gamma(x)/_\delta \gamma(y)) = f(\gamma(x/_\delta y)) \\ &= f(\gamma(\delta(x/y))) = \gamma(\delta(f(x/y))) \\ &= \gamma(\delta(f(x)/f(y))) = \gamma(f(x)/_\delta f(y)) \\ &= \gamma(f(x))/_\delta \gamma(f(y)) = f(x)/_\delta f(y). \end{aligned}$$

For the other division we work similarly. $\Gamma(f)$ preserves the lattice operations, because they are restrictions of the lattice operations of the ℓ-group, so $\Gamma(f)$ is a homomorphism.

By Theorem 7.28, Γ is onto the objects of **GMV**. Moreover, Γ is faithful, because if two morphisms agree on a generating set, they agree on the whole ℓ-group.

To see that Γ is full, let $g : \mathbf{M} \to \mathbf{N}$, be a morphism in **GMV**. By Corollary 7.20, there exist ℓ-groups $\mathbf{K}, \mathbf{H}, \overline{\mathbf{K}}, \overline{\mathbf{H}}$ and nuclei γ_1 on $\mathbf{H}^-$ and γ_2 on $\overline{\mathbf{H}}^-$, such that $\mathbf{M} = \mathbf{K} \times \mathbf{H}^-_{\gamma_1}$ and $\mathbf{N} = \overline{\mathbf{K}} \times \overline{\mathbf{H}}^-_{\gamma_2}$. Moreover, by the proof of Theorem 7.22, there exist kernels δ_1 on $\mathbf{K} \times \mathbf{H}$, δ_2 on $\overline{\mathbf{K}} \times \overline{\mathbf{H}}$, and nuclei $\overline{\gamma_1}$ on $(\mathbf{K} \times \mathbf{H})_{\delta_1}$ and $\overline{\gamma_2}$ on $(\overline{\mathbf{K}} \times \overline{\mathbf{H}})_{\delta_2}$, such that $\delta_i(k, h) = (k, h \wedge e)$ and $\overline{\gamma_i}(k, h) = (k, \gamma_i(h))$, $i \in \{1, 2\}$. So, there are homomorphisms $g_1 : \mathbf{G} \to \overline{\mathbf{G}}$ and $g_2 : \mathbf{H}^-_{\gamma_1} \to \overline{\mathbf{H}}^-_{\gamma_2}$, such that $g = (g_1, g_2)$. By Theorem 7.41, there exists a homomorphism $f_2^- : \mathbf{H}^- \to \overline{\mathbf{H}}^-$, that extends g_2 and commutes with the γ's. By the results in [BCGJT], there exists a homomorphism $f_2 : \mathbf{H} \to \overline{\mathbf{H}}$ that extends f_2^-. Let $f : \mathbf{K} \times \mathbf{H} \to \overline{\mathbf{K}} \times \overline{\mathbf{H}}$ be defined by $f = (g_1, f_2)$. It is clear that $\Gamma(f) = g$. We will show that $g(\beta_1(x)) = \beta_2(g(x))$, where $\beta_i(x) = \overline{\gamma_i}(\delta_i(x))$. Let $(k, h) \in K \times H^-_{\gamma_1}$.

$$\begin{aligned}
g(\beta_1(k,h)) &= g(\overline{\gamma_1}(\delta_1(k,h))) = g(k, \gamma_1(h \wedge e)) \\
&= (g_1(k), g_2(\gamma_1(h \wedge e))) = (g_1(k), \gamma_2(g_2(h \wedge e))) \\
&= (g_1(k), \gamma_2(g_2(h) \wedge e)) = \overline{\gamma_2}(g_1(k), g_2(h) \wedge e) \\
&= \overline{\gamma_2}(\delta_2((g_1(k), g_2(h)))) = \beta_2(g(k,h)).
\end{aligned}$$

Thus, by [Ml], Γ is a categorical equivalence between the two categories. $\square$

Decidability of the equational theory

We obtain the decidability of the equational theory of $\mathcal{GMV}$ as an easy application of the representation theorem, established above.

For a residuated lattice term t and a variable $z \notin Var(t)$, we define the term t_z inductively on the complexity of a term, by

$$\begin{array}{lll}
x_z = x \vee z & e_z = e & \\
(s \vee r)_z = s_z \vee r_z & (s \wedge r)_z = s_z \wedge r_z & \\
(s/r)_z = s_z/r_z & (s\backslash r)_z = s_z\backslash r_z & (sr)_z = s_z r_z \vee z,
\end{array}$$

for every variable x and every pair of terms s, r.

Recall the definition of the term operation $t^{\mathbf{A}}$ on an algebra $\mathbf{A}$ induced by a term t over the (ordered) set of variables $\{x_i \mid i \in \mathbb{N}\}$, given on page 5.

For a residuated lattice term t, a residuated lattice $\mathbf{L}$ and a map γ on $\mathbf{L}$, we define the operation t_γ on L, of arity equal to the number of variables in t, by

$$\begin{array}{lll}
x_\gamma = x^{\mathbf{L}} & e_\gamma = e^{\mathbf{L}} & \\
(s \vee r)_\gamma = s_\gamma \vee r_\gamma & (s \wedge r)_\gamma = s_\gamma \wedge r_\gamma & \\
(s/r)_\gamma = s_\gamma/r_z & (s\backslash r)_\gamma = s_\gamma\backslash r_\gamma & (sr)_\gamma = \gamma(s_\gamma r_\gamma)
\end{array}$$

for every variable x and every pair of terms s, r.

Essentially, t_γ is obtained from $t^{\mathbf{L}}$ by replacing every product sr by $\gamma(sr)$, and t_z is obtained from t by replacing every product sr by $sr \vee z$ and every variable x by $x \vee z$. We extend the above definitions to every residuated lattice identity $\varepsilon = (t \approx s)$ by $\varepsilon_z = (t_z \approx s_z)$, for a variable z that does not occur in ε. Moreover, we define $\varepsilon_\gamma(\bar{a}) = (t_\gamma(\bar{a}) = s_\gamma(\bar{a}))$, where $\bar{a}$ is an element of an appropriate power of L.

Lemma 7.45. *An identity ε holds in $\mathcal{IGMV}$ iff the identity ε_z holds in $\mathcal{LG}^-$, where $z \notin$*

$Var(\varepsilon)$.

Proof. We prove the contrapositive of the lemma. Let ε be an identity that fails in $\mathcal{IGMV}$. Then there exists an integral generalized MV-algebra $\mathbf{M}$, and an $\bar{a}$ in an appropriate power, n, of M, such that $\varepsilon(\bar{a})$ is false. By Theorem 7.19, there exists a negative cone $\mathbf{L}$ of an ℓ-group and a nucleus γ on $\mathbf{L}$, such that $\mathbf{M} = \mathbf{L}_\gamma$. By the definition of $\mathbf{L}_\gamma$, it follows that $\varepsilon_\gamma(\bar{a})$ is false in $\mathbf{L}$. Let p be the meet of all products $t_\gamma(\bar{a})s_\gamma(\bar{a})$, where t, s range over all subterms of ε and $u = \gamma(p)$. By Lemma 7.17, γ and γ_u agree on the upset of p. Since the arguments of all occurences of γ in $\varepsilon_\gamma(\bar{a})$ are of the form $t_\gamma(\bar{a})s_\gamma(\bar{a})$, where t, s are subterms of ε, and $t_\gamma(\bar{a})s_\gamma(\bar{a})$ are in the upset of p, we can replace, working inductively inwards, all occurences of γ in $\varepsilon_\gamma(\bar{a})$ by γ_u. So, $\varepsilon_{\gamma_u}(\bar{a}) = \varepsilon_\gamma(\bar{a})$, hence $\varepsilon_{\gamma_u}(\bar{a})$ is false in $\mathbf{L}$. Note that p is below $\bar{a}(i)$, for all $i \in \{1, \ldots, n\}$, so $u = \gamma(p) \leq \gamma(\bar{a}(i)) = \bar{a}(i)$, hence $\bar{a}(i) = \bar{a}(i) \vee u$, for all $i \in \{1, \ldots, n\}$. Consequently, $\varepsilon_{\gamma_u}(\bar{a}) = (\varepsilon_z)^{\mathbf{L}}(\bar{a}, u)$, thus ε_z fails in $\mathbf{L}$; i.e., ε_z fails in $\mathcal{LG}^-$.

Conversely, if ε_z, fails in $\mathcal{LG}^-$, then there exists a negative cone $\mathbf{L}$ of an ℓ-group, $\bar{a}$ in an appropriate power, n, of L and $u \in L$, such that $(\varepsilon_q)^{\mathbf{L}}(\bar{a}, u)$ is false. Obviously, γ_u is a nucleus on $\mathbf{L}$, so $\mathbf{L}_{\gamma_u}$ is an integral generalized MV-algebra. Let $\bar{b}$ be the element of L^n, defined by $\bar{b}(i) = \bar{a}(i) \vee u$, for all $i \in \{1, \ldots, n\}$. Note that $(\varepsilon_z)^{\mathbf{L}}(\bar{a}, u) = \varepsilon_{\gamma_u}(\bar{b}) = \varepsilon^{\mathbf{L}_{\gamma_u}}(\bar{b})$ and $u, \bar{b}(i) \in \mathbf{L}_{\gamma_u}$, for all $i \in \{1, \ldots, n\}$. So ε fails in $\mathbf{L}_{\gamma_u}$, hence it fails in $\mathcal{IGMV}$. $\square$

In view of Theorem 7.20 we have the following corollary.

Corollary 7.46. *An identity ε holds in $\mathcal{GMV}$ iff ε holds in $\mathcal{LG}$ and ε_z holds in $\mathcal{LG}^-$, where $z \notin Var(\varepsilon)$.*

The variety of ℓ-groups has a decidable equational theory by [HM]. Based on this fact, it is shown in [BCGJT] that the same holds for $\mathcal{LG}^-$. So, we get the following result.

Corollary 7.47. *The equational theories of the varieties $\mathcal{IGMV}$ and $\mathcal{GMV}$ are decidable.*

CHAPTER VIII

CONCLUDING REMARKS AND OPEN PROBLEMS

In this thesis we have tried to present a range of subvarieties of residuated lattices. Our goal was not to exhaust the topic, but rather to stimulate interest for this area of mathematics that is algebraic in nature and has connections to logic. The vastness of the topic is apparent considering that many well and not well-studied classes of algebras are examples of residuated lattices. We believe that the context of residuated lattices is ideal for formulating and proving general results about its subclasses.

The connections to logic (substructural, relevant, linear etc.) have not been explored fully. It is promising that lately researchers concentrate on the interactions mentioned above. Certain results seem to have easier, or only, logic proofs, i.e., see [JT], [GR].

We mention below a number of open problems that have come up from our study. We believe that a lot of them have relative easy answers, but we suspect that some are very hard.

1. Is there a continuum of commutative atomic subvarieties of residuated lattices?

2. Are there infinitely many integral atoms in the subvariety lattice of $\mathcal{RL}$?

3. Is the equational theory of distributive or cancellative residuated lattices decidable? Are there cut-free Gentzen systems for the corresponding logics?

4. Do commutative cancellative integral residuated lattices satisfy any non-trivial lattice identity?

5. Is the join of any two finitely based residuated lattice varieties also finitely based?

6. Which varieties of residuated lattices have EDPC. Which satisfy the CEP or the AP?

7. Is there a good description of all monoid or lattice reducts of residuated lattices?

INDEX

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