Subgroups and Quotients of Fundamental Groups

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Introduction

Abstract topological spaces have been studied for over a century. A standard tool for studying path connected spaces is the fundamental group. There are questions about fundamental groups of Polish (separable, completely metrizable) spaces that can be answered using descriptive set theory. If one further restricts attention to the fundamental group of a Peano continuum (a connected, locally connected compact metrizable space) then there are interesting dichotomies. In [Sh] Shelah demonstrated the following dichotomy: The fundamental group of a Peano continuum is either finitely generated or of cardinality continuum. Using a theorem by Cannon and Conner [CC] one can replace the word "generated" in the conclusion of Shelah's theorem with the word "presented". Pawlikowski later gave a simplified proof of Shelah's result [P]. Using similar methods Conner and the author obtain the same result for first homology [CoCo].

In this thesis we are going to present improvements on these techniques and numerous applications, which give a great deal of information about fundamental groups of path connected Polish spaces.

In Chapter 2 we provide some preliminary definitions and key ideas, as well as some motivation for the results in the succeeding sections. Chapter 3 provides some useful general theorems which provide dichotomies on quotients of the fundamental group. In Chapter 4 we give some examples of topologically defined subgroups of the fundamental group and exhibit upper bounds on their topological complexity. An alternative characterization of the shape kernel is provided for locally path connected spaces. In Chapter 5 we define comonster groups and show an interesting dichotomy on the fundamental group of a Peano continuum. In Chapter 6 we show that subgroups of the fundamental group of a covering space are not more complex than their images under the covering map. Chapter 7 provides an array of results related to the first homology of Peano continua. One such result is the interesting fact that if the torsion-free quotient of $H_1(X)$ is not of cardinality continuum then $H_1(X)$ is a direct sum of cyclic groups. A definition for the strong abelianization of the fundamental group is also provided and computations. In Chapter 8 we demonstrate that subgroups of arbitrarily high Borel complexity exist in the fundamental group of the Hawaiian earring. Finally in Chapter 9 we provide some new theorems regarding n-slender groups. We present the concept of n-slenderness and prove that all torsion-free word hyperbolic groups are n-slender. We also exhibit stronger closure properties for the class of n-slender groups than were previously known.

Chapter 2

Preliminaries

Given a topological space X and distinguished point $x \in X$ we obtain the fundamental group $\pi_1(X,x)$ as follows. Recalls that a loop based at x in X is a continuous function $l : ([0,1],\{0,1\}) \to (X,x)$. Two loops l_0 and l_1 at x are homotopic if there exists a continuous function $H : [0,1] \times [0,1] \to X$ called a homotopy such that $H(s,0) = l_0(s)$, $H(s,1) = l_1(s)$ and $H(0,t) = H(1,t) = x$ for all $s,t \in [0,1]$. The relation defined by homotopy is an equivalence relation. Letting L_x denote the space of all loops at x in X we have the binary operation concatenation, denoted $*$, on L_x defined by $l_0 * l_1(s) = \begin{cases} l_0(2s) \text{ if } s \in [0,\frac{1}{2}] \\ l_1(2s-1) \text{ if } s \in [\frac{1}{2},1] \end{cases}$. This definition also works as a partial binary operation on paths, defined whenever the first path ends where the second path starts. For specificity, we mean $l_0 * (l_1 * (\cdots * (l_{n-1} * l_n) \cdots)$ when we write $l_0 * l_1 * \cdots * l_n$. There is also a unary operation $^{-1}$ given by $l^{-1}(s) = l(1-s)$. The fundamental group is the set L_x modulo homotopy, the binary operation is given by $[l_0] * [l_1] = [l_0 * l_1]$, the equivalence class of the constant loop is the identity and inverses are given by $[l]^{-1} = [l^{-1}]$. Clearly the fundamental group $\pi_1(X,x)$ is the same as the fundamental group of $\pi_1(C,x)$ where C is the path component of x. Thus we generally consider our spaces to be path connected to avoid trivialities.

We assume some familiarity with notions in topology such as metrizability and separability. Let Z be a topological space. A **pointclass** is a collection $\mathscr{P}$ of subsets of Z that are of a particular topological description, usually in terms of countable unions, countable intersections, complements, or projections. For example, the collection of open subsets (topology) of Z, the collection of closed sets of Z, and the collection of countable unions of closed sets of Z are all pointclasses of Z. Another example is the class of Borel subsets of Z. When we restrict our attention to specific types of topological spaces, we get more information about sets in pointclasses.

We now define the central concept for this paper. Take (X,d) to be a path connected metric space with distinguished point $x \in X$. Again, we let L_x denote the set of all loops at x, and topologize L_x by the sup metric. That is, the distance in L_x between loops l_0 and l_1 is $\sup_{s\in[0,1]} d(l_0(s), l_1(s))$. Since uniform convergence is equivalent to convergence in the compact-open topology, we may suppress the particular metric d on the space X (since any other metric gives the same topology on L_x). **Through the remainder of Section 2 we assume that X is metrizable and path connected.**

Definition 2.0.0.1. A subgroup $G \leq \pi_1(X,x)$ is of pointclass $\mathscr{P}$ if the collection of loops belonging to elements of G is in the pointset $\mathscr{P}$ in L_x. In other words, $G \leq \pi_1(X,x)$ is of pointclass $\mathscr{P}$ if $\bigcup G$ is in pointclass $\mathscr{P}$ in L_x.

Lemma 2.0.0.2. If $G \leq \pi_1(X,x)$ is open and $G \leq H \leq \pi_1(X,x)$ then H is open.

Proof. Let G be open and let $l \in \bigcup H$, with $\{l_n\}_{n\in\omega}$ a sequence in L_x converging to l. Since $l * l^{-1} \in \cup G$ there exists $\varepsilon > 0$ such that $B(l * l^{-1}, \varepsilon) \subseteq \bigcup G$. The sequence $\{l * l_n^{-1}\}_{n\in\omega}$ is eventually in $B(l * l^{-1}, \varepsilon)$, so that $\{l * l_n^{-1}\}_{n\in\omega}$ is eventually in $\bigcup G \subset \bigcup H$, so $\{l_n^{-1}\}_{n\in\omega}$ is eventually in $\bigcup H$, so $\{l_n\}_{n\in\omega}$ is eventually in $\bigcup H$. □

Lemma 2.0.0.3. If $\mathscr{P}$ is closed under continuous preimages and $H \leq \pi_1(X,x)$ is $\mathscr{P}$ then:

1. The equivalence relations $E, R \subseteq L_x \times L_x$ defined by $l_0 E l_1$ iff $[l_0]H = [l_1]H$ and $l_0 R l_1$ iff $H[l_0] = H[l_1]$ are $\mathscr{P}$.
2. Each equivalence class in E and R is $\mathscr{P}$.

Here we mean $[l]H$ to be the set of all loops based at x which are homotopic to a loop of the form $l * l'$ where $l' \in \bigcup H$ and the definition for $H[l]$ is analogous.

Proof. The function $L_x \times L_x \to L_x$ given by $(l_0, l_1) \mapsto (l_0)^{-1} * l_1$ is continuous and E is the preimage of $\bigcup H$ under this function, so by assumption we have E is $\mathscr{P}$. The proof that R is $\mathscr{P}$ is similar. This proves (1). For (2) we notice that for a fixed $l_0 \in L_x$ the function

$L_x \to L_x$ given by $l \mapsto (l_0)^{-1} * l$ is continuous and the set $[l_0]H$ is the continuous preimage of $\bigcup H$. $\square$

Lemma 2.0.0.4. If $H \leq \pi_1(X,x)$ is open then H is also closed.

Proof. Supposing H is open we have by Lemma 2.0.0.3 that the set $\bigcup_{l \notin \bigcup H} [l]H$ is a union of open sets in L_x, and this is precisely $L_x - (\bigcup H)$. $\square$

We notice that open and closed subgroups of a path connected metric space X enjoy the property of being basepoint-free in the following sense:

Lemma 2.0.0.5. Let $x, y \in X$ and ρ a path from y to x. Let $\phi : L_x \to L_y$ be the map such that $\phi(l) = \rho * l * \rho^{-1}$ and $\psi : L_y \to L_x$ be given by $\rho^{-1} * l * \rho$. Then the following hold:

1. ϕ and ψ are isometric embeddings and induce isomorphisms $\overline{\phi} : \pi_1(X,x) \to \pi_1(X,y)$, and $\overline{\psi} : \pi_1(X,y) \to \pi_1(X,x)$.

2. $G \leq \pi_1(X,x)$ is open (resp. closed) iff $\overline{\phi}(G)$ is.

3. $G \leq \pi_1(X,x)$ is open (resp. closed) iff every conjugate of G is.

Proof. The first part of (1) is clear, and the second is a standard exercise. We prove (2). Suppose G is not open. Let $l \in L_x$ be such that $[l] \in G$ and there exists a sequence of loops $\{l_n\}_{n \in \omega}$ such that $l_n \to l$ and $[l_n] \notin G$. Then $\rho * l_n * \rho^{-1} \to \rho * l * \rho^{-1}$ and $[\rho * l_n * \rho^{-1}] \notin \overline{\phi}(G)$, so $\overline{\phi}(G)$ is not open. If $\overline{\phi}(G)$ is not open then by the proof for the other direction we have that $\overline{\psi}\overline{\phi}(G) = G$ is not open.

Suppose that G is not closed and let $l \in L_x$ be such that $[l] \notin G$ and there exists a sequence $\{l_n\}_{n \in \omega}$ such that $[l_n] \in G$ and $l_n \to l$. Then $\rho * l_n * \rho^{-1} \to \rho * l * \rho^{-1}$ and $[\rho * l_n * \rho^{-1}] \in \phi(G)$ and $[\rho * l * \rho^{-1}] \notin \phi(G)$. Again, for the other direction we consider the application of the map $\overline{\psi}$.

The last claim is true by letting ρ be a loop from x to itself and applying the previous claim.

$\square$

By the above Lemma 2.0.0.5 we may consider open or closed normal subgroups as base point free. We shall see in Proposition 8 that if X is semilocally simply connect then all subgroups are open. For example, any subgroup of the fundamental group of the wedge of two circles is open.

Lemma 2.0.0.6. Let $G \trianglelefteq \pi_1(X)$ be open. Then there exists an open cover $\mathscr{U}$ of X such that any loop contained entirely in an element of $\mathscr{U}$ is in $\bigcup G$.

Proof. For each point $x \in X$ we have $G \trianglelefteq \pi_1(X,x)$ is open, and the constant loop c at x is in $\cup G$, so we may pick $\varepsilon_x > 0$ such that $B(c,\varepsilon_x) \subseteq \bigcup G$. Selecting the ε_x neighborhood $B(x,\varepsilon_x)$ around x gives the desired open cover $\mathscr{U} = \{B(x,\varepsilon_x)\}_{x\in X}$. □

The converse to the above lemma is not true in general, but there is a partial converse.

Definition 2.0.0.7. A topological space Z is **locally path connected** if for every $z \in Z$ and neighborhood U of z there exists a neighborhood $V \subseteq U$ of z such that V is path connected.

Lemma 2.0.0.8. Let X be locally path connected $G \trianglelefteq \pi_1(X)$. If there exists an open cover $\mathscr{U}$ of X such that any loop contained entirely in an element of $\mathscr{U}$ is in G then G is open.

Proof. Assume the hypotheses and fix $x \in X$. Let $l \in \bigcup G \subseteq L_x$. Cover the image of l with a finite subcollection $\{U_0,\ldots,U_{m-1}\} \subseteq \mathscr{U}$, so that the images of each inclusion $\iota_* : \pi_1(U_i) \to \pi_1(X)$ are in G. Let $\delta > 0$ be a Lebesgue number for the covering of the image of l by $\{U_0,\ldots,U_{m-1}\}$. Cover l with finitely many open balls $\{B_0,\ldots,B_{p-1}\}$ of radius $\frac{\delta}{2}$. Cover the image of l with finitely many path connected open sets $\{V_0,\ldots,V_q\}$, each of which is contained in one of the $\{B_0,\ldots,B_{p-1}\}$. Let ε be a Lebesgue number for the covering $\{V_0,\ldots,V_q\}$ of the image of l. Pick $N \in \omega$ sufficiently large so that for $0 \le n \le N-1$ we have that $l([\frac{n}{N},\frac{n+1}{N}])$ is contained inside V_{j_n}. Now assuming $l' \in L_x$ is less than distance ε from l we have that $d(l'(s),l(s)) < \varepsilon$. For each $1 \le n \le N-1$ let p_n be a path in V_{j_n} from $l(\frac{n}{N})$ to $l'(\frac{n}{N})$ and let p_0 and p_N be the constant path at x. Notice that the loop $l|[\frac{n}{N},\frac{n+1}{N}] * p_{n+1} * (l'|[\frac{n}{N},\frac{n+1}{N}])^{-1} * p_n^{-1}$ is contained in one of the U_i, and so is a representative of an element of G based potentially at a different point. Then $l^{-1} * l'$ is an element of $\bigcup G$, so $l' \in \bigcup G$. Thus G is open. $\square$

For the next proposition we recall the following definition.

Definition 2.0.0.9. A topological space Z is **semilocally simply connected** if for every $z \in Z$ there exists a neighborhood U of z such that the map induced by inclusion $\iota_* : \pi_1(U,z) \to \pi_1(Z,z)$ is the trivial map. For a locally path connected space we may obviously select U to be path connected.

Proposition 2.0.0.10. *Let X be locally path connected in addition to being metrizable and path connected. The following are equivalent:*

1. *The trivial subgroup of $\pi_1(X)$ is open.*

2. *All subgroups of $\pi_1(X,x)$ are open.*

3. *X is semilocally simply connected.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 2.0.0.2. For (2) $\Rightarrow$ (3) we let $x \in X$ be given along with a neighborhood U of x. Since in particular the trivial subgroup of $\pi_1(X,x)$ is open and the constant map $c : [0,1] \to \{x\}$ is trivial, we may select $\varepsilon > 0$ such that $B(c,\varepsilon) \subseteq \bigcup[c] \subseteq L_x$, where without loss of generality $B(x,\varepsilon) \subseteq U$. Now any loop with image in $B(x,\varepsilon)$ must be in $B(c,\varepsilon)$ and therefore nulhomotopic in X.

For (3) $\Rightarrow$ (1) we let $\mathscr{U}$ be an open cover of X by path connected open sets U whose inclusion maps induce a trivial map $\pi_1(U) \to \pi_1(X)$. Then we are in the situation of Lemma 2.0.0.8 and we see that the trivial subgroup is open, so we are done. $\square$

We now enter into some technical lemmas.

Definition 2.0.0.11. A topological space Z is **Polish** if it is completely metrizable and separable.

Many commonly used spaces such as the real line $\mathbb{R}$, compact metric spaces, and countable discrete spaces are Polish. Polish spaces are closed under countable disjoint union and countable products. When X is path connected and Polish the space L_x is also Polish. There are certain pointclasses defined for Polish spaces which are extremely well behaved, a discussion of which will be given in Section 8. We let H_x denote the space of homotopies of loops in L_x, topologized by the sup metric. This is also a Polish space. The following lemma provides a sense of base point independence as in Lemma 2.0.0.5.

Lemma 2.0.0.12. Suppose the pointclass $\mathscr{P}$ contains the closed sets and is closed under continuous images between Polish spaces, products, and finite intersections. Let X be Polish. Let ρ be a path from x to y in X. Letting ϕ be the map defined in Fact 2.0.0.5 a subgroup $G \leq \pi_1(X,x)$ is of type $\mathscr{P}$ if and only if $\phi(G)$ is.

Proof. Assume the hypotheses. We prove the forward direction of the biconditional and the other direction follows similarly. Let $G \leq \pi_1(X,x)$ be of type $\mathscr{P}$. Let $D \subseteq L_x \times H_x \times L_x$ be defined by $D = \{(l_0, H, l_1) : H$ is a homotopy from l_0 to $l_1\}$. It is easy to see that D is closed. Since the map $l \mapsto \rho^{-1} * l * \rho$ is an isometric embedding from L_x to L_y we have that $\rho^{-1} * G * \rho$ is in pointclass $\mathscr{P}$ in L_y by assumption. Then $\rho^{-1} * G * \rho \times H_y \times L_y$ is in pointclass $\mathscr{P}$ in $L_y \times H_y \times L_y$ by hypothesis. Then $D \cap \rho^{-1} * G * \rho \times H_y \times L_y$ is in pointclass $\mathscr{P}$. Letting $p_3 : L_x \times H_x \times L_x \to L_x$ be the projection to the third coordinate (obviously a continuous map), we have that $\bigcup \phi(G) = p_3(D \cap \rho^{-1} * G * \rho \times H_y \times L_y)$ is in the pointclass $\mathscr{P}$. □

For $K \subseteq L_x$ let $[K] \subseteq \pi_1(X,x)$ denote the subset of equivalence classes of loops which have representatives in K.

Lemma 2.0.0.13. Let $\mathscr{P}$ and X satisfy the hypotheses of Lemma 2.0.0.12. If $K \subseteq L_x$ is $\mathscr{P}$ then the set $\bigcup [K] \subseteq L_x$ is $\mathscr{P}$.

Proof. Letting $D = \{(l_0, H, l_1) : H$ homotopes l_0 to $l_1\} \subseteq L_x \times H_x \times L_x$ we have that D is closed, and therefore $\mathscr{P}$. The set K is $\mathscr{P}$ and therefore so is $K \times H_x \times L_x$. Then $(K \times H_x \times$

$L_x) \cap D$ is $\mathscr{P}$, and letting p_3 be projection in the third coordinate we have $p_3((K \times H_x \times L_x) \cap D) = \bigcup[K]$ is $\mathscr{P}$. □

Lemma 2.0.0.14. Let $\mathscr{P}$ and X satisfy the hypotheses of Lemma 2.0.0.12. Assume further that $\mathscr{P}$ is closed under countable unions. If $K \subseteq L_x$ is $\mathscr{P}$ then $\langle [K] \rangle$ is a $\mathscr{P}$ subgroup of $\pi_1(X,x)$.

Proof. Notice that the inversion map $l \mapsto l^{-1}$ is an isometry and therefore continuous. Thus K^{-1} is $\mathscr{P}$, and $K \cup K^{-1}$ is also $\mathscr{P}$. For each $n \in \omega$ let $m_n : \prod_{i=0}^{n-1} L_x \to L_x$ be given by $(l_0, \ldots, l_{n-1}) \mapsto l_0 * l_1 * \cdots * l_{n-1}$. This is clearly a continuous map. Each $m_n(\prod_{i=0}^{n-1}(K \cup K^{-1}))$ is of type $\mathscr{P}$. Thus $\bigcup_{n=0}^{\infty} m_n(\prod_{i=0}^{n-1}(K \cup K^{-1}))$ is $\mathscr{P}$. By Lemma 2.0.0.13 we have that $\bigcup[\bigcup_{n=0}^{\infty} m_n(\prod_{i=0}^{n-1}(K \cup K^{-1}))]$is $\mathscr{P}$. We are done since $\bigcup\langle [K] \rangle = \bigcup[\bigcup_{n=0}^{\infty} m_n(\prod_{i=0}^{n-1}(K \cup K^{-1}))]$. □

Lemma 2.0.0.15. Let $\mathscr{P}$ and X satisfy the hypotheses of Lemma 2.0.0.14. If $K \subseteq L_x$ is $\mathscr{P}$ then the normal closure $\langle\langle [K] \rangle\rangle$ is $\mathscr{P}$.

Proof. Let $c : L_x \times L_x \to L_x$ be given by $(l_0, l_1) \mapsto l_0 * l_1 * l_0^{-1}$. This is easily continuous. We have $L_x \times K$ is $\mathscr{P}$, and so is $c(L_x \times K)$. Then $\langle\langle [K] \rangle\rangle = \langle [c(L_x \times K)] \rangle$ is $\mathscr{P}$ by Lemma 2.0.0.14 . □

The preceeding lemmas motivate the following:

Definition 2.0.0.16. A pointclass $\mathscr{P}$ defined on Polish spaces is **nice** if it contains the closed sets, is closed under continuous images and preimages, and countable intersections.

Remark 2.0.0.17. A nice pointclass is also closed under countable products, for if $A_n \subseteq Z_n$ is of nice pointclass $\mathscr{P}$ for each $n \in \omega$ then $\prod_{n\in\omega} A_n = \bigcap_{n\in\omega} p_n^{-1}(A_n)$ is $\mathscr{P}$ in the polish space $\prod_{n\in\omega} Z_n$. A nice pointclass is also closed under countable unions, for suppose $A_n \subseteq Z$ are $\mathscr{P}$ for each $n \in \omega$. If $\bigcup_{n\in\omega} A_n = \emptyset$ then as $\emptyset$ is closed we have $\bigcup_{n\in\omega} A_n$ is $\mathscr{P}$. On the other hand if $\bigcup_{n\in\omega} A_n \neq \emptyset$ then say $A_m \neq \emptyset$ and let $\sqcup_{n\in\omega} Z$ be the disjoint union of countably many copies of Z. Letting $z \in A_m$ be some point in A_m we let $f : \sqcup_{n\in\omega} Y \to \prod_{n\in\omega} Z$ take

y_n to $(z,z,\ldots,z,y,z,z\ldots)$ (here y is in the nth coordinate) where y_n is a copy of y in the nth copy of Z in the disjoint union. This is easily seen to be continuous by the universal and couniversal properties of product and disjoint unions, respectively. Then $\prod_{n\in\omega} A_n$ is $\mathscr{P}$ as we have seen. Letting $g : \sqcup_{n\in\omega} Z \to Z$ map each copy of Z via identity we get that $g(f^{-1}(\prod_{n\in\omega} A_n)) = \bigcup_{n\in\omega} A_n$ is $\mathscr{P}$.

Under set inclusion, the smallest nice Polish pointclass is that of the analytic sets. If Z is Polish we say a $Y \subseteq Z$ is **analytic** if there exists a Polish space W and a continuous map $f : W \to Z$ such that $f(W) = Y$. All Borel sets of a Polish space are analytic (see [Ku]).

Lemma 2.0.0.18. If $X = \prod_{n\in\omega} X_n$ where each X_n is metrizable, then the loop space of X is homeomorphic to the product of the loop spaces of the spaces X_n and can be metrized thereby.

Proof. By applying a cutoff metric d_n to each space X_n we may assume $\mathrm{diam}(X_n) \leq 2^{-n}$. The metric $d(\{s_n\}_{n\in\omega}, \{t_n\}_{n\in\omega}) = \sum_{n=0}^{\infty} d_n(s_n,t_n)$ is compatible with the product topology on $\prod_n X_n$. Fix a point x_n in each X_n and let $x = \{x_n\}_{n\in\omega} \in \prod_n X_n$. The metric d induces the sup metric on the loop space L_x so that L_x is homeomorphic with the space $\prod_n L_{x_n}$ where the distance between loops $\{l_n\}_{n\in\omega}$ and $\{l_n'\}_\omega$ is $\sum_n \sup_{s\in[0,1]} d_n(l_n(s), l_n'(s))$. This follows from the fact that uniform convergence of a sequence of loops in L_x occurs precisely when the loops in each coordinate converge uniformly. Thus we metrize L_x with the metric defined by the metric on the product $\prod_n L_{x_n}$.

□

We cover some basic functoriality properties. Recall that if (X,x) and (Y,y) are two pointed spaces and $f : (X,x) \to (Y,y)$ is a continuous function then there is an induced homomorphism $f_* : \pi_1(X,x) \to (Y,y)$ defined by $f_*([l]) = [f \circ l]$. The map f also induces a continuous map $\overline{f} : L_x \to L_y$ given by $l \mapsto f \circ l$. We also recall that the wedge $(X,x) \vee (Y,y)$ is the topological space obtained by identifying the distinguished points, which has distinguished point corresponding to the identified points which we denote $x \vee y$. There are obvious inclusion maps from the spaces (X,x) and (Y,y) to the wedge as well as retraction maps from the wedge to the two spaces.

Proposition 2.0.0.19. *The following closure properties hold:*

1. *If $f:(X,x)\to(Y,y)$ is continuous, $\mathscr{P}$ is a pointclass closed under continuous preimages and $G\leq\pi_1(X,x)$ is $\mathscr{P}$, then $(f_*)^{-1}(G)$ is also $\mathscr{P}$.*

2. *If $G_0\leq\pi_1(X,x)$ and $G_1\subseteq\pi_1(Y,y)$ are both of pointclass $\mathscr{P}$ and $\mathscr{P}$ is closed under products, then $G_0\times G_1\leq\pi_1(X\times Y,(x,y))\simeq\pi_1(X,x)\times\pi_1(Y,y)$ is $\mathscr{P}$.*

3. *If $f:(X,x)\to(Y,y)$ is continuous between Polish spaces and $\mathscr{P}$ is nice and $G\leq\pi_1(X,x)$ is $\mathscr{P}$ then $f_*(G)$ is $\mathscr{P}$.*

4. *If $G_0\leq\pi_1(X,x)$ and $G_0\leq\pi_1(Y,y)$ are $\mathscr{P}$, with $\mathscr{P}$ nice, then the subgroup generated by the images of G_0 and G_1 under the inclusion maps is $\mathscr{P}$ in $(X,x)\vee(Y,y)$.*

Proof. (1) We notice that $\bigcup(f_*)^{-1}(G)=\overline{f}^{-1}(\bigcup G)$.

(2) This follows from lemma 2.0.0.18, and applies to countable products if $\mathscr{P}$ is closed under countable products.

(3) The map f induces the continuous map $\overline{f}$ from L_x to L_y by composition. The image of $\bigcup G$ under this map is $\mathscr{P}$ because $\mathscr{P}$ is nice, and $\bigcup f_*(G)=[\overline{f}(\bigcup G)]$.

(4) Follows immediately, since $\bigcup\langle\iota_{X*}(G_0)\cup\iota_{Y*}(G_1)\rangle=\bigcup\langle[\iota_X(\bigcup G_0)\cup\iota_Y(\bigcup G_1)]\rangle$ is evidently $\mathscr{P}$. □

The following theorem gives a catalogue of closure properties for nice subgroups.

Theorem 2.0.0.20. *Let $f:(X,x)\to(Y,y)$ be a continuous function between Polish spaces and let $\mathscr{P}$ be a nice pointclass. The following hold.*

1. *If $H\leq\pi_1(Y,y)$ is $\mathscr{P}$ then $f_*^{-1}(H)\leq\pi_1(X,x)$ is $\mathscr{P}$.*

2. *If $G\leq\pi_1(X,x)$ is $\mathscr{P}$ then $f_*(G)\leq\pi_1(Y,y)$ is $\mathscr{P}$.*

3. *The subgroups* 1 *and* $\pi_1(X,x)$ *are analytic.*

4. *If* $G_n \leq \pi_1(X,x)$ *are* $\mathscr{P}$ *then so are* $\bigcap_{n\in\omega} G_n$ *and* $\langle\bigcup_{n\in\omega} G_n\rangle$.

5. *Countable subgroups of* $\pi_1(X,x)$ *are analytic.*

6. *If* $G \leq \pi_1(X,x)$ *is* $\mathscr{P}$ *then so is* $\langle\langle G\rangle\rangle$.

7. *If* G *is* $\mathscr{P}$ *then so is any conjugate of* G.

8. *If* $w(x_0,\ldots,x_k)$ *is a reduced word in the free group* $F(x_0,\ldots,x_k)$ *and the groups* $G_0,\ldots,G_k \leq \pi_1(X,x)$ *are* $\mathscr{P}$ *then so is the subgroup* $\langle\{w(g_0,g_1,\ldots,g_k)\}_{g_i\in G_i}\rangle$.

9. *If* $G,H \leq \pi_1(X,x)$ *are* $\mathscr{P}$ *then so is the subgroup* $[G,H]$.

10. *If* G *is* $\mathscr{P}$ *we have that each countable index term of the derived series* $G^{(\alpha)}$ *and each term of the lower central series* G_n *are* $\mathscr{P}$. *Recall that the derived series is defined by letting* $G^{(0)} = G$, $G^{(\alpha+1)} = [G^{(\alpha)}, G^{(\alpha)}]$ *and* $G^{(\beta)} = \bigcap_{\alpha<\beta} G^{(\alpha)}$ *if* β *is a limit ordinal. The lower central series is defined by letting* $G_0 = G$ *and* $G_{n+1} = [G, G_n]$.

Proof. Claim (1) follows from (1) in Proposition 2.0.0.19. Claim (2) is claim (3) in Proposition 2.0.0.19. For (3) we have that $\pi_1(X,x)$ is a closed subgroup and 1 is the subgroup generated by the constant map to x, and so is analytic by Lemma 2.0.0.14 (since a singleton is closed in L_x). Claim (4) follows from the definition of nice pointclasses and Lemma 2.0.0.14. Claim (5) follows from the fact that singletons are closed in L_x and claim (4). Claim (6) is an instance of Lemma 2.0.0.15. Claim (7) is an instance of Lemma 2.0.0.12. For claim (8) we notice that the map $w : \prod_{i=0}^{k} L_x \to L_x$ given by $(l_0,\ldots,l_k) \mapsto w(l_0,\ldots,l_k)$ is continuous, and so $\{w(l_0,\ldots,l_k)\}_{l_i\in\bigcup G_i}$ is a $\mathscr{P}$ subset in L_x and the claim follows from Lemma 2.0.0.14. Claim (9) is an instance of claim (8). For claim (10) we iterate claim (9), applying claim (4) at limit ordinals. $\square$

Chapter 3

Some General Theorems

We begin with some definitions. If Z is a topological space, we say that $Y \subseteq Z$ is **nowhere dense** if $\overline{Y} \subseteq Z$ has empty interior, Y is **meager** if it is a union of countably many nowhere dense sets in Z, Y **has the property of Baire** (abbreviated BP) if there exists an open set $O \subseteq Z$ such that $Y \Delta O = (Y - O) \cup (O - Y)$ is meager, and Y is **comeager** if $Z - Y$ is meager. We say a pointclass $\mathscr{P}$ on Polish spaces has the property of Baire if each set in $\mathscr{P}$ has the property of Baire. For example, the pointclass of open sets obviously has BP. In fact, the class of analytic sets also has BP. Recall that a loop in a space is essential if it is not nulhomotopic.

The following was proven in [P], using a result from [M].

Lemma 3.0.0.21. Suppose $\approx$ is an equivalence relation on the Cantor set $\{0,1\}^\omega$ such that if α and β differ at exactly one coordinate then $\alpha \approx \beta$ fails. If $\approx$ has the property of Baire as a subset of $\{0,1\}^\omega \times \{0,1\}^\omega$, then $\approx$ has $2^{\aleph_0}$ many equivalence classes.

Throughout the rest of Section 3 we assume X is path connected and Polish.

Theorem 3.0.0.22. *Suppose that $G \trianglelefteq K \leq \pi_1(X,x)$ with G of pointclass $\mathscr{P}$ and K closed, that $\mathscr{P}$ has BP and is closed under continuous preimages in Polish spaces, and that there exist arbitrarily small loops at x which are in $\bigcup K$ and not in $\bigcup G$. Then K/G has cardinality $2^{\aleph_0}$.*

Proof. Assume the hypotheses and let $\{l_n\}_{n\in\omega}$ be a sequence of loops at x in $\bigcup(K-G)$ such that the diameter of l_n is $\leq 2^{-n}$. Let l_n^0 be the constant loop at x and let l_n^1 be the loop l_n. Given an element $\alpha \in \{0,1\}^\omega$ we define l^α to be the loop $l_0^{\alpha(0)} * (l_1^{\alpha(1)} * (l_2^{\alpha(2)} * (\cdots)))$ (which must also be in $\bigcup K$ as K is closed). In other words, l^α restricted to the interval $[0, \frac{1}{2}]$ is either the constant loop or l_0 in case $\alpha(0)$ is 0 or 1 respectively, l^α restricted to the interval $[\frac{1}{2},\frac{3}{4}]$ is either the constant loop or l_1 in case $\alpha(1)$ is 0 or 1 respectively, etc. The function from the Cantor set $\{0,1\}^\omega$ to L_x given by $\alpha \mapsto l^\alpha$ is clearly continuous. For $l,l' \in L_x$ letting $l \sim l'$ if and only if $[l]G = [l']G$, we have by Lemma 2.0.0.3 that $\sim \subseteq L_x \times L_x$ is of pointclass $\mathscr{P}$. Defining an equivalence relation $\approx$ on $\{0,1\}^\omega$ so that $\alpha \approx \beta$ if and only if $l^\alpha \sim l^\beta$, we see that $\approx \subseteq \{0,1\}^\omega \times \{0,1\}^\omega$ is of pointclass $\mathscr{P}$ as a continuous preimage. As $\mathscr{P}$ has BP we know that $\approx$ has BP. By Lemma 3.0.0.21 we shall be done if we show that if α and β differ at exactly one point then $\alpha \approx \beta$ fails. Suppose that $\alpha(n) \neq \beta(n)$ and that $\alpha(m) = \beta(m)$ whenever $m \neq n$ and that $l^\alpha \approx l^\beta$. Letting without loss of generality $\alpha(n) = 1$ and $\beta(n) = 0$ we see that $[(l^\beta)^{-1}l^\alpha] \in G$. Let $h = l_{n+1}^{\alpha(n+1)} * (l_{n+2}^{\alpha(n+2)} * (\cdots))$ and $g = l_0^{\alpha(0)} * (l_1^{\alpha(1)} * (\cdots l_{n-1}^{\alpha n-1}) \cdots)$. Then $[(l^\beta)^{-1}l^\alpha] = [h^{-1} * g^{-1} * g * l_n * h] = [h^{-1} * l_n * h] \in G$, so by normality of G in K we have $[l_n] \in G$, a contradiction. Thus there are at least $2^{\aleph_0}$ many elements in K/G by the above lemma, and there are at most $2^{\aleph_0}$ elements because there are at most $2^{\aleph_0}$ loops at x. □

Theorem 3.0.0.23. *Suppose X is locally path connected. If $G \trianglelefteq \pi_1(X,x)$ is $\mathscr{P}$ and $\mathscr{P}$ is nice with BP, then $\pi_1(X,x)/G$ is either of cardinality $\leq \aleph_0$ (in case G is open) or of cardinality $2^{\aleph_0}$ (in case G is not open).*

Proof. If G is open then the collection of left cosets $\{[l]G\}_{l\in L_x}$ is a covering of L_x by pairwise disjoint open sets, and since L_x is separable we know that the collection $\{[l]G\}_{l\in L_x}$ is countable. Supposing on the other hand that G is not open, we have by the contrapositive of Lemma 2.0.0.8 that there must exist some point $y \in X$ such that for any open neighborhood U of y there is a loop in U which is not in G. Since X is locally path connected we get a sequence of loops $\{l_n\}_{n\in\omega}$ based at y which are not in G. Considering G as a subgroup of $\pi_1(X,y)$ we see that G is $\mathscr{P}$ since $\mathscr{P}$ is nice, and thus we have satisfied the hypotheses of Theorem 3.0.0.22 and we are done. □

The above may be strengthened if X is also compact. Recall that a **Peano continuum** is a path connected, locally path connected compact metric space.

Theorem 3.0.0.24. *If X is a Peano continuum and $G \trianglelefteq \pi_1(X,x)$ is $\mathscr{P}$ with $\mathscr{P}$ nice with BP, then $\pi_1(X,x)/G$ is either finitely generated (in case G is open) or of cardinality $2^{\aleph_0}$ (in case G is not open).*

Proof. By the previous theorem we need only show that $\pi_1(X)/G$ is finitely generated if G is open. For this we will use a theorem from [CC] which will require a definition. Let $\phi : \pi_1(X) \to H$ be a group homomorphism. We say an open cover $\mathscr{U}$ is **2-set simple rel** ϕ if each element of $\mathscr{U}$ is path connected and any loop in the union of two elements of $\mathscr{U}$ is in the kernel of ϕ. This property of a cover implies that for any nerve associated with with $\mathscr{U}$ there is a homomorphism from the fundamental group of the nerve with the same image as ϕ. The following is a part of Theorem 7.3 in [CC]:

Theorem. Let X be path connected, $\phi : \pi_1(X) \to H$ a homomorphism and $\mathscr{U}$ a 2-set simple cover rel ϕ. If $\mathscr{U}$ is finite then $\phi(\pi_1(X))$ is finitely generated.

Now, assuming G is open we get by Lemma 2.0.0.6 an open cover $\mathscr{U}_1$ for X such that any loop contained in an element of $\mathscr{U}_1$ is in G. Let $\varepsilon > 0$ be a Lebesgue number for the cover $\mathscr{U}_1$ and let $\mathscr{U}_2$ be a cover of X by open balls of redius $\frac{\varepsilon}{4}$. By local path connectedness let $\mathscr{U}$ be an open cover of X by path connected sets, each of which is contained in an element of $\mathscr{U}_2$. By compactness we may pick $\mathscr{U}$ to be finite, and it is clear that $\mathscr{U}$ is 2-set simple rel the quotient projection $\pi_1(X) \to \pi_1(X)/G$. We are done by the theorem of Cannon and Conner that is quoted above. □

The conclusion of Theorem 3.0.0.24 cannot be strengthened by replacing "finitely generated" by "finitely presented" by the following basic example.

Example 3.0.0.25. Let X be the bouquet of two circles and G be a 2-generated group which is not finitely presented (for example, the lamplighter group). The fundamental group $\pi_1(X)$ is the free group of rank 2. Let $\phi : \pi_1(X) \to G$ be the homomorphism given by taking each of the free generators of $\pi_1(X)$ to a distinct generator of G. The space X is a semilocally simply connected Peano continuum and $\ker(\phi)$ is open by Proposition 8, but

$\pi_1(X)/\ker(\phi) \simeq G$ is not finitely presented. Similar examples can be given by replacing the number 2 by any finite number ≥ 2 and letting G be replaced by any other n-generated group which is not finitely presented.

The above theorems can be refined. Towards this we give the following technical lemma.

Lemma 3.0.0.26. Let $\mathscr{P}$ be a pointclass with BP which is closed under continuous preimages between Polish spaces. Let X be a path connected Polish space (as we have been assuming throughout this section). Suppose $N \trianglelefteq K \leq \pi_1(X,x)$ is such that $\bigcup N = \bigcup_{n=0}^{\infty} N_n$ with each N_n closed under inverses and homotopy and containing the trivial loop, and that K is closed. Assume also that $N_n N_m \subseteq N_{n+m}$. If each N_n is $\mathscr{P}$ and there exist loops at x of arbitrarily small diameter in $\bigcup K$ not contained in N_n, then K/N is of cardinality $2^{\aleph_0}$.

Proof. Let $\varepsilon > 0$ be given. By the proof of Theorem 3.0.0.22 we need only show that there is a loop at x in $\bigcup K$ of diameter less than ε that is not in $\bigcup N$, since N is $\mathscr{P}$. For contradiction we assume that no such loop exists. For each loop in $\bigcup K$ of diameter less than ε let ϕ map that loop to the minimal k such that $l \in N_k$. For two loops l_1, l_2 of radii less than ε we have that $\phi(l_1 * l_2) \leq \phi(l_1) + \phi(l_2)$ and $\phi(l_1) = \phi(l_1^{-1})$. Let $\{l_n\}$ be a sequence of loops such that $\text{diam}(l_n) < \varepsilon 2^{-n}$ and that $\phi(l_1) > 1$ and $\phi(l_n) > n + \sum_{m=0}^{n-1} \phi(l_m)$. In particular none of the l_n is nullhomotopic. Define l^{α} as before for each α in the Cantor set. Abuse notation by letting $\phi : \{0,1\}^{\omega} \to \omega$ be defined by $\phi(\alpha) = \phi(l^{\alpha})$.

Let $E_n = \{\alpha \in \{0,1\}^{\omega} : l^{\alpha} \in N_n\}$. As we are assuming that there is no loop in $\bigcup K$ of diameter less than ε that is not in $\bigcup N$, we have in particular that $\bigcup_{n=0}^{\infty} E_n = \{0,1\}^{\omega}$. We will derive our contradiction if we show that each E_n is meager, which would imply that $\{0,1\}^{\omega}$ is meager in itself. Each E_n is clearly $\mathscr{P}$, and so has the property of Baire. Supposing E_n is not meager there exists a nonempty open set in which E_n is comeager. In particular there exists a basic open set $U(\delta_0, \ldots, \delta_k) = \{\alpha \in \{0,1\}^{\omega} : \alpha(0) = \delta_0, \ldots, \alpha(k) = \delta_k\}$ such that $E_n \cap U(\delta_0, \ldots, \delta_k)$ is comeager in $U(\delta_0, \ldots, \delta_k)$. For each $p \geq k+1$ let $\beth_p : U(\delta_0, \ldots, \delta_k) \to$

$U(\delta_0,\ldots,\delta_k)$ be the homeomorphism that changes the p coordinate. Then $U(\delta_0,\ldots,\delta_k)\setminus \beth_p(E_n)$ is meager for each $p \geq k+1$. Then in fact there exists $\alpha \in U(\delta_1,\ldots,\delta_k)$ such that switching finitely many of the coordinates beyond the kth coordinate gives an element of E_n. It cannot be that the support of α is finite, for if $N \in \omega$ is a bound on the support of α (we can assume $N > 2n$), then $n \geq \phi(l^\alpha * f_{N+1}) \geq \phi(l_{N+1}) - \phi(l^\alpha) > N+1-n > n$, a contradiction. Thus taking a subsequence of the l_n, we may assume that $\alpha = (1,1,\ldots)$ and that $U(\delta_1,\ldots,\delta_k) = \{0,1\}^\omega$. We assume that this subsequence was the original sequence.

Let $\beta_k, \gamma_k \in \{0,1\}^\omega$ be given by $\beta_k(m) = \begin{cases} 0 \text{ if } m < k \\ 1 \text{ if } m \geq k \end{cases}$ and $\gamma_k(m) = \begin{cases} 1 \text{ if } m < k \\ 0 \text{ if } m \geq k \end{cases}$.
We have that $\phi(\gamma_k) \geq k$ and $\phi(\beta_k) \geq \phi(\gamma_k) - \phi(\alpha) \geq k - \phi(\alpha)$, so that if $k = 2n+1$ we have on the one hand that $\beta_k \in E_n$ and on the other hand $\phi(\beta_k) \geq k - \phi(\alpha) \geq (2n+1) - n$, a contradiction. □

This gives the following:

Theorem 3.0.0.27. *Let $\mathscr{P}$ be nice with BP and X be a Peano continuum. There does not exist a strictly increasing infinite sequence of $\mathscr{P}$ normal subgroups $\{G_n\}_{n\in\omega}$ of $\pi_1(X)$ such that $\bigcup_{n\in\omega} G_n = \pi_1(X)$.*

Proof. For each $n \in \omega$ let the set of loops $\bigcup G_n$ play the role of the N_n in the previous lemma. If $\pi_1(X,x)/G_n$ is finitely generated for some n, then the sequence $\{N_n\}_{n\in\omega}$ cannot be strictly increasing. Then $\pi_1(X,x)/G_n$ must be uncountable for each n, and so there exist arbitrarily small loops not in each G_n by the proof of Theorem 3.0.0.23. By picking an appropriate basepoint by local path connectedness, we are done by the previous lemma. □

Example 3.0.0.28. The dual analog of the previous theorem does not hold: there exists a Peano continuum with an infinite strictly descending chain of analytic (in fact closed) normal subgroups whose intersection is the trivial subgroup. The example can be constructed as follows. Let P be a projective plane and $X = \prod_\omega P$. Then X is a Peano continuum whose fundamental group is isomorphic to the countably infinite product of $\mathbb{Z}/2$ which we may denote $\{0,1\}^{\omega}$. We change the superscript ω for $\mathbb{Q}$ and the group $\{0,1\}^{\mathbb{Q}}$ remains unchanged since the cardinality of ω and $\mathbb{Q}$ are the same. Given any subset $S \subseteq \mathbb{Q}$ the subgroup of $\pi_1(X)$ corresponding to the subgroup $\{\alpha \in \{0,1\}^{\mathbb{Q}} : \alpha(q) = 1 \Rightarrow q \in S\} \leq \{0,1\}^{\mathbb{Q}}$ is closed. For each $r \in \mathbb{R}$ let $G_r \leq \pi_1(X)$ be the subgroup corresponding to the subgroup $\{\alpha \in \{0,1\}^{\mathbb{Q}} : \alpha(q) = 1 \Rightarrow q < r\}$. Then each G_r is a closed subgroup and the following hold:

1. $r_n \nearrow r$ implies $\bigcup_n G_{r_n} \lneq G_r$
2. $r_n \searrow r$ implies $\bigcap_n G_{r_n} = G_r$
3. $\bigcap_{r \in \mathbb{R}} G_r$ is the trivial subgroup

Picking a sequence $r_n \searrow -\infty$ gives a strictly descending sequence of normal analytic subgroups G_{r_n} as claimed. The subgroup $\bigcup_{r \in \mathbb{R}} G_r$ cannot be equal to $\pi_1(X)$ (else we could pick any sequence $r_n \nearrow \infty$ and the ascending chain G_{r_n} would contradict the previous theorem). For example the sequence over $\mathbb{Q}$ which is constantly 1 is not in $\bigcup_{r \in \mathbb{R}} G_r$.

We name some of the numerous applications of the above theorems.

Theorem 3.0.0.29. *Suppose X is a locally path connected, connected Polish space. The following groups are of cardinality $2^{\aleph_0}$ or $\leq \aleph_0$, and in case X is compact they are of cardinality $2^{\aleph_0}$ or are finitely generated.*

1. $\pi_1(X)$
2. $\pi_1(X)/(\pi_1(X))^{(\alpha)}$ *for any* $\alpha < \omega_1$ *(derived series)*
3. $\pi_1(X)/(\pi_1(X))_n$ *for any* $n \in \omega$ *(lower central series)*
4. $\pi_1(X)/N$ *where N is the normal subgroup generated by squares of elements, cubes of elements, or n-th powers of elements*

In case X is compact then countability of the fundamental group is equivalent to being finitely presented.

Proof. The noncompact case in parts (1)-(4) immediately follow from Theorem 3.0.0.23. For parts (2)-(4) in the compact case we apply Theorem 3.0.0.24. That $\pi_1(X)$ would be finitely presented follows from Theorem 7.3 in [CC] in part (1) assuming X is compact. □

Part (1) in the compact case is the main result of the papers [Sh] and [P], and part (2) with $\alpha = 1$ (both compact and noncompact cases) is proven in [CoCo].

We state another result of Continuum Hypothesis type:

Theorem 3.0.0.30. *Suppose X is path connected Polish and $K \leq \pi_1(X)$ is G_δ. Then $\pi_1(X)$ has either $\leq \aleph_0$ or $2^{\aleph_0}$ many left or right cosets over K.*

Proof. It is obviously sufficient to prove the claim for left cosets. We have by Lemma 2.0.0.3 that the relation $E \subseteq L_x \times L_x$ defined by lEl' iff $[l]K = [l']K$ is G_δ. Then by a theorem in [Sr] there is a Borel set $B \subseteq L_x$ such that $\text{card}(B \cap [l]K) = 1$ for all $l \in L_x$. Thus the number of left cosets is precisely the number $\text{card}(B)$. Since a Borel subset of a Polish space does not violate the Continuum Hypothesis we are done.

□

Chapter 4

Some Examples of Topologically Defined Subgroups

We give some standard examples, and introduce some new examples, of subgroups of the fundamental group which are topologically defined. These are intended to illustrate the richness of the theory and give a grab-bag of examples to which to apply the theorems.

4.1 The Shape Kernel

One well known subgroup of the fundamental group is the shape kernel. We discuss this subgroup by first giving preliminary definitions towards defining the shape group and the shape kernel and then prove that the shape group is a closed subgroup.

We assume some familiarity with geometric simplicial complexes. Given a topological space X and a open cover $\mathscr{U}$ of X let $N(\mathscr{U})$ denote the **nerve** of the cover-that is, the geometric simplicial complex which has a distinct vertex v_U for every $U \in \mathscr{U}$ and which contains the n-simplex $[v_{U_0}, v_{U_1}, \ldots, v_{U_n}]$ if and only if $U_0 \cap U_1 \cap \cdots \cap U_n \neq \emptyset$. If $\mathscr{V}$ is an open cover of X that refines $\mathscr{U}$ (i.e. for each $V \in \mathscr{V}$ there is a $U \in \mathscr{U}$ such that $V \subseteq U$) then any map from the vertices of $N(\mathscr{V})$ to the vertices of $N(\mathscr{U})$ such that $v_V \mapsto v_U$ implies $V \subseteq U$ extends to a simplicial map from $N(\mathscr{V})$ to $N(\mathscr{U})$.

If the topological space has a distinguished basepoint x, then one can distinguish an element U in an open cover $\mathscr{U}$ such that $x \in U$, which in turn gives a distinguished vertex in the nerve $N(\mathscr{U})$. With this added structure, if $\mathscr{V}$ refines $\mathscr{U}$ with distinguished elements V and U such that $V \subseteq U$ then a simplicial map as described above extending a vertex assignment satisfying $v_V \to v_U$, say $p_{(\mathscr{V},V),(\mathscr{U},U)} : (N(\mathscr{V}), v_V) \to (N(\mathscr{U}), v_U)$ preserves basepoint and is unique up to basepoint preserving homotopy. Assuming X is path connected all nerves are connected, and the refinement relation on open covers gives an inverse directed system

$(\pi_1(N(\mathscr{U}), v_U), p_{(\mathscr{V},V),(\mathscr{U},U)}*)$. The shape group of X is defined as the inverse limit

$$\check{\pi}_1(X,x) = \lim_{\leftarrow}(\pi_1(N(\mathscr{U}), v_U), p_{(\mathscr{V},V),(\mathscr{U},U)}*)$$

The index of the inverse limit will generally be of uncountable cardinality. Assuming (X,x) is also paracompact and Hausdorff we have for each open cover with distinguished neighborhood of x, $(\mathscr{U},U)$, a refinement $(\mathscr{V},V)$ such that V is the unique element of $\mathscr{V}$ containing x. A partition of unity subordinate to $\mathscr{V}$ (which necessarily exists by our assumption of paracompactness and Hausdorffness) induces a barycentric map $f_{\mathscr{U}} : (X,x) \to (N(\mathscr{V}), v_V)$ which is unique up to based homotopy. The induced map $f_{\mathscr{U}*}$ on the fundamental group $\pi_1(X,x)$ can be checked to commute with the maps of the inverse system in the apprporiate way, and since the set of all such open covers $\mathscr{V}$ described are cofinal in the inverse system we get a map $\Psi : \pi_1(X,x) \to \check{\pi}_1(X,x)$.

The natural object used to assess the loss of information when passing from the fundamental to the shape group is the shape kernel ker(Ψ). The following demonstrates an alternate characterizaion of the shape kernel.

Theorem 4.1.0.31. *Suppose (X,x) is a path connected metrizable space. Then the shape kernel is a closed subgroup of $\pi_1(X,x)$. If in addition X is locally path connected then the shape kernel is equal to the folowing two subgroups:*

1. *$\bigcap_f \ker(f_*)$ where f is taken over all continuous maps to semilocally simply connected spaces*
2. $\displaystyle\bigcap_{G \text{ open, normal}} G$

Proof. It is clear from the definition that the shape kernel is equal to $\bigcap_f \ker(f_*)$ where f is taken over all baricentric maps of open covers. Fix a barycentric map f to the nerve $N(\mathscr{V})$. As $N(\mathscr{V})$ is a geometric simplicial complex it is not difficult for each loop $l \in L_x$ to select $\varepsilon > 0$ such that for each $l' \in L_x$ that is ε close to l we have $f \circ l$ is homotopic to $f \circ l'$ in

$N(\mathscr{V})$. This shows that $\ker(f_*)$ is open, and therefore also closed by Lemma 2.0.0.4. Then the shape kernel is a closed subgroup as an intersection of closed subgroups.

Now suppose that X is also locally path connected. Since each nerve is a geometric simplicial complex, each nerve is also semilocally simply connected. Thus the shape kernel contains the subgroup (1). Furthermore, if $f : X \to Y$ is continuous with Y semilocally simply connected, then we can find an open covering $\mathscr{U}$ of X such that the image of any loop in an element of $\mathscr{U}$ has nulhomotpic image under the map f. This gives an open cover satisfying the criteria of Lemma opencover and since X is locally path connected we have that $\ker(f_*)$ is open. Thus subgroup (1) contains subgroup (2).

We conclude by proving that subgroup (2) contains the shape kernel. Let G be an open normal subgroup in $\pi_1(X)$ (since G is open, normal we may consider $\pi_1(X,x)$ as basepoint free by Lemma 2.0.0.5). Let $q : \pi_1(X) \to \pi_1(X)/G$ be the canonical quotient homomorphism. We introduce some terms and a theorem given in [CC].

Assuming Y is a path connected topological space and $\phi : \pi_1(Y) \to H$ is a group homomorphism we say an open cover $\mathscr{V}$ of Y by path connected sets is **2-set simple rel ϕ** provided any loop whose image lies in the union of two elements of $\mathscr{V}$ is in $\ker(\phi)$ (as defined in the proof of Theorem 3.0.0.24). Two paths p_0 and p_1 are **$\mathscr{V}$-related** if there is some parametrization for p_0 and p_1 such that for all $s \in [0,1]$ are in a common element of $\mathscr{V}$. To be $\mathscr{V}$-related is not necessarily an equivalence relation; we say that paths p_0 and p_1 are **$\mathscr{V}$-equivalent** if they are in the same class under the equivalence class generated by $\mathscr{V}$-relatedness. The following is part (1) of Theorem 7.3 in [CC]:

Theorem. Let Y be a path connected topological space, $\phi : \pi_1(Y) \to H$ a homomorphism and $\mathscr{V}$ a 2-set simple cover of Y rel ϕ. If two loops $l, l' \in L_y$ are $\mathscr{V}$-equivalent then $h([l]) = h([l'])$.

By Lemma 2.0.0.6 we have an open cover $\mathscr{U}_0$ of X such that each loop in an element of $\mathscr{U}_0$ is in G. For each $z \in X$ we may select $U_z \in \mathscr{U}_0$ satisfying $z \in U_z$. Define $r_0(z) = d(z, X - U_z)$. Letting $\mathscr{U}_1 = \{B(z, \frac{r_0(z)}{3})\}_{z \in X}$ it is straightforward to check that if for $U, U' \in$

$\mathscr{U}_1$ we have $U \cap U' \neq \emptyset$ then $U \cup U'$ is contained in an element of $\mathscr{U}_0$. By local path connectedness we let $\mathscr{U}_2$ be a refinement of $\mathscr{U}_1$ by path connected open sets. It is easy to see that $\mathscr{U}_2$ is a 2-set simple cover rel q. For each $z \in X$ pick a $U_z \in \mathscr{U}_2$ such that $z \in U_z$ and let $r_2(z) = d(z, X - U_z)$. Letting $\mathscr{U}_3 = \{B(z, \frac{r_2(z)}{5})$ it is straightforward to check that if $U, U', U'' \in \mathscr{U}_3$ satisfy $U \cap U' \neq \emptyset$ and $U' \cap U'' \neq \emptyset$ then $U \cup U' \cup U''$ is contained entirely in an element of $\mathscr{U}_2$. Let $\mathscr{U}_4$ be a refinement of $\mathscr{U}_3$ by path connected open sets. Finally for each $z \in X$ select a $U_z \in \mathscr{U}_4$ such that $z \in U_z$, let $r_4(z) = d(z, X - U_z)$ and $\mathscr{U} = \{B(z, \frac{r_4(z)}{3})\}_{z \in X}$. Again, it is straightforward to see that if $U, U' \in \mathscr{U}$ satisfy $U \cap U' \neq \emptyset$ then $U \cup U'$ is entirely contained in an element of $\mathscr{U}_4$. Without loss of generality we can assume $\mathscr{U}$ is refined so that x is contained in exactly one element of the cover $\mathscr{U}$.

Let $b : X \to N(\mathscr{U})$ be a barycentric map associated to some partition of unity subordinated to $\mathscr{U}$. Then $b(x) = v_U$ where $U \in \mathscr{U}$ is unique such that $x \in U$. We define a map f from the 1-skeleton $N(\mathscr{U})^1$ to X. Let $f(v_U) = x$ and for all other vertices $v_{U'} \in N(\mathscr{U})^0$ simply let $f(v_{U'}) \in U'$. By our choice of $\mathscr{U}$ if $[v_{U'}, v_{U''}]$ is a 1-simplex in $N(\mathscr{U})$ then $U' \cap U'' \neq \emptyset$ and so there exists a path contained entirely in an element of $\mathscr{U}_4$ from $f(v_{U'})$ to $f(v_{U''})$. Let $f|[v_{U'}, v_{U''}]$ map via this path.

We will be done if we show that $\ker(b_*) \leq G$. Suppose now that $l \in L_x$ is such that $[l] \in \ker(b_*)$. Then $b \circ l$ is a loop in $N(\mathscr{U})$ based at v_U which is nulhomotopic. Recall that b has the property that $b^{-1}(\operatorname{St} v_{U'}) \subseteq U'$ where $\operatorname{St} v_{U'}$ is the open star of the vertex $v_{U'}$. There exists a combinatorial loop $p(v_U, v_{U_1}, v_{U_2}, \ldots, v_{U_{n-1}}, v_{U_n} = v_U)$ which is homotopic to $b \circ l$ such that $b \circ l(s) \in \operatorname{St} v_{U_k}$ where $s \in [\frac{k}{n}, \frac{k+1}{n}]$. Letting $l_0 : [0,1] \to N(\mathscr{U})$ be a topological realization of this loop we see that l is $\mathscr{U}_2$-related to $f \circ l_0$.

By assumption there exists a nulhomotopy of l_0, and so in particular there exists a combinatorial nulhomotopy of $p(v_U, v_{U_1}, v_{U_2}, \ldots, v_{U_{n-1}}, v_{U_n} = v_U)$. In other words, there exists a finite sequence of combinatorial paths:

$$p_0 = p(v_U, v_{U_1}, v_{U_2}, \ldots, v_{U_{n-1}}, v_{U_n} = v_U)$$
$$p_1 = p(v_U, v_{U_{1,1}}, v_{U_{1,2}}, \ldots, v_{U_{1,n_1}} = v_U)$$

$$p_2 = p(v_U, v_{U_{2,1}}), v_{U_{2,2}}, \ldots, v_{U_{2,n_2}} = v_U)$$
$$\vdots$$
$$p_m = p(v_U)$$

such that one obtains p_k from p_{k-1} by performing one of the following elementary path homotopies:

1. Exchanging the subpath $v_{U_p}, v_{U_{p+1}}$ for the subpath v_{U_p} assuming $U_p = U_{p+1}$, or vice versa.

2. Exchanging the subpath $v_{U_p}, v_{U_{p+1}}, v_{U_{p+2}}$ for the subpath v_{U_p} assuming $U_{p+2} = U_p$, or vice versa.

3. Exchanging the subpath $v_{U_p}, v_{U_{p+1}}, v_{U_{p+2}}$ for the subpath $v_{U_p}, v_{U_{p+2}}$ assuming $[v_{U_p}, v_{U_{p+1}}, v_{U_{p+2}}]$ is a 2-simplex in $N(\mathscr{U})$, or vice versa.

Letting $l_k : [0,1] \to N(\mathscr{U})$ be a topological realization of the combinatorial path p_k, it is easy to see that $f \circ l_k$ is $\mathscr{U}_2$ related to $f \circ l_{k+1}$. By the theorem of Cannon and Conner quoted above, we have that $q([l]) = q([f \circ l_0]) = q([f \circ l_1]) = \cdots = q([f \circ l_m]) = q(1)$, and so $[l] \in G$. □

As a direct consequence of Theorem 3.0.0.30 we get the following

Corollary 4.1.0.32. *If X is a path connected Polish space then the quotient of $\pi_1(X)$ by the shape kernel is of cardinality $\leq \aleph_0$ or $2^{\aleph_0}$.*

4.2 The Spanier Group

Another useful subgroup of the fundamental group is the Spanier group, which we denote $\pi_1^s(X,x)$ (first defined in [Spa]). We give the necessary definitions for this group, then give some results about the topological properties.

Let X be a path connected topological space and $x \in X$. If $\mathscr{U}$ is an open cover of X we define $\pi_1(\mathscr{U},x)$ to be the subgroup of $\pi_1(X,x)$ generated by loops of the form $\rho * l * \rho^{-1}$ where $\rho(0) = x$ and l is a loop based at $\rho(1)$ and contained in some element of $\mathscr{U}$. This subgroup is easily seen to be normal. The Spanier group is defined to be $\pi_1^s(X,x) = \bigcap_{\mathscr{U}} \pi_1(\mathscr{U},x)$ where the parameter $\mathscr{U}$ is taken over all open covers. The first of the following two lemmas does not assume metrizability of X.

Lemma 4.2.0.33. If X is path connected then $\pi_1^s(X,x)$ is contained in the shape kernel.

Proof. Let b be a barycentric map from X to some nerve. Since a nerve is semilocally simply connected we have a cover $\mathscr{U}$ of X such that any loop contained in an element of $\mathscr{U}$ is in $\ker(b_*)$. Obviously $\pi_1(\mathscr{U},x) \leq \ker(b_*)$ and taking the appropriate intersections gives the claim.

□

Lemma 4.2.0.34. Let X be a path connected metric space, $\mathscr{U}$ an open cover of X and $x \in X$.

1. If X is locally path connected then $\pi_1(\mathscr{U},x)$ is open.

2. If X is Polish and $\mathscr{U}$ is countable then $\pi_1(\mathscr{U},x)$ is analytic.

Proof. Assume the hypotheses for part (1). The open cover $\mathscr{U}$ is such that any loop contained in an element thereof (considering loops to be base point free) is an element of $\pi_1(\mathscr{U})$ (we switch here to a basepoint free notation for emphasis). Then by Lemma 2.0.0.8 we have that $\pi_1(\mathscr{U})$ is an open subgroup.

Assume the hypotheses for part (2). Let $L_{x,U,n} = \{l \in L_x : (\forall s \in [0,\frac{1}{2}])[l(s) = l(1-s)] \wedge (\forall s \in [\frac{1}{3},\frac{2}{3}])[d(l(s), X-U) \geq \frac{1}{n}]\}$ where $U \in \mathscr{U}$ and $n \in \omega$. It is clear that $L_{x,U,n}$ is closed as a subset of L_x. The set $\bigcup_{U\in\mathscr{U},n\in\omega} L_{x,U,n}$ is a countable union of closed sets (and therefore analytic). Then $\pi_1(\mathscr{U},x) = \langle\langle \bigcup_{U\in\mathscr{U},n\in\omega} L_{x,U,n}\rangle\rangle$ is analytic by Lemma 2.0.0.15. □

That the shape kernel is equal to the Spanier group holds for all locally path connected, path connected paracompact Hausdorff spaces was recently shown in [BF]. Part (1) of the following theorem gives a rather short proof of a slightly less general fact.

Theorem 4.2.0.35. *The following hold:*

1. *If X is a path connected, locally path connected metric space then $\pi_1^s(X,x)$ is equal to the shape kernel, and in particular closed.*
2. *If X is a path connected compact metric space then $\pi_1^s(X,x)$ is analytic.*

Proof. (1) Assume the hypotheses. That the Spanier group is contained in the shape kernel was proved in Lemma 4.2.0.33. That the shape kernel is contained in the Spanier group follows from characterization (2) of Theorem 4.1.0.31 and from Lemma 4.2.0.34 part (1).

For (2) we assume the hypotheses. As X is a compact metric space there exists a sequence $\{\mathscr{U}_n\}_{n\in\omega}$ of finite open covers such that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$ and which is cofinal in the inverse directed system of open covers. Thus $\pi_1^s(X,x) = \bigcap_{n\in\omega}\pi_1(\mathscr{U}_n, x)$ is analytic as a countable intersection of analytic subgroups (Lemma 4.2.0.34 part (2) and Theorem 2.0.0.20 part (4)). □

4.3 Subgroups reflecting local behavior

We give a couple of subgroups that can be thought of as indicating local behavior. Firstly, recall that a space X is **homotopically Hausdorff at** x if each loop based at x which can be homotoped into any neighborhood of x is in fact nulhomotopic. This notion has found many uses (as, for example, in [BS] and [FZ]). If X is a Polish space, let $L_{x,n}$ be the set of all loops given by $l \in L_{x,n}$ if and only if $(\forall s \in [0,1])[d(l(s),x) \leq \frac{1}{n}]$. Then $L_{x,n}$ is clearly a closed subset of L_x, so the subgroup $\langle [L_{x,n}]\rangle$ is analytic by Lemma 2.0.0.14. The subgroup $\bigcap_{n\in\omega}\langle [L_{x,n}]\rangle$ is trivial if and only if X is homotopically Hausdorff at x. This subgroup is analytic and can be thought of as the indicator subgroup for the property.

If X is compact, metrizable and path connected, then it is easy to see that the cone over X, $\mathscr{C}X = X \times [0,1]/X \times \{1\}$, is also compact, metrizable and path connected. We shall consider X as a subset of $\mathscr{C}X$ by identifying X with $X \times \{0\}$.

Let $S \subseteq X$ be nonempty. Fixing a metric on $\mathscr{C}X$ we let $Y_{n,S} \subseteq \mathscr{C}X$ be given by $X \cup (\mathscr{C}X - B(S, \frac{1}{n}))$. Let $f_{n,S}$ be the inclusion map from X to $Y_{n,S}$. Then $f_{n,S}$ is a continuous map to a compact metric space, and $\ker(f_{n,S*})$ is analytic. Since $Y_{n,S} = Y_{n,\overline{S}}$ there is no generality lost in assuming that S is compact. Also, the choice of metric on $\mathscr{C}X$ does not change $\bigcup_n \ker(f_{n,S*})$ (by compactness). Let $N(S)$ denote the normal subgroup $\bigcup_n \ker(f_{n,S*})$. This subgroup is intended to convey a sense of the importance of the subspace S in the fundamental group of X. If the subgroup $N(S)$ is all of $\pi_1(X)$ then the points of S carry little significance in the fundamental group. If $N(S)$ is trivial, then the points of S can be thought of as holding importance. If $S \subseteq S'$ then $Y_{n,S} \supseteq Y_{n,S'}$ and so $N(S) \leq N(S')$.

Example 4.3.0.36. Let X be compact, metrizable and path connected. Letting $S = X$ we get that for every $n \in \omega - \{0\}$, the path component in $Y_{n,S}$ including all elements of X is simply the subset X. Thus any nulhomotopy of a loop in X taking place in $Y_{n,S}$ must in fact already take place in X, so $N(S)$ is trivial.

For the next example we first prove the following.

Example 4.3.0.37. Let $S \subseteq X$ be a compactum such that any map $f : S^1 \to X$ can be homotoped to have image disjoint from S. Then given $x \in X$ and a loop $l \in L_x$ there is a homotopy of l to a loop $\rho * l' * \rho^{-1}$ such that l' is a loop with image disjoint from S. By compactness there is some positive distance between S and the image of l', and so l' can be nulhomotoped in $Y_{n,S}$ for some n, so that l is also nulhomotopic in $Y_{n,S}$. Then $N(S) = \pi_1(X)$.

Example 4.3.0.38. Let $X = S^1$ and $S = \{x\}$ be any singleton. For each $n \in \omega - \{0\}$ there is a superset $Z \supseteq Y_{n,S}$ such that Z strongly deformation retracts to the set X, so that $N(S)$ is trivial. This holds true as well if X is a wedge of finitely many circles and x is the wedge point by the same proof.

Lemma 4.3.0.39. If $r : X \to Y$ is a retraction with $Y \supset S$ then the monomorphism induced by inclusion $\pi_1(Y) \to \pi_1(X)$ induces a monomorphism $\pi_1(Y)/N_Y(S) \to \pi_1(X)/N_X(S)$ (here we use the subscript to denote the ambient space).

Proof. This follows from the fact that the retraction r extends to a retraction R of the cones $R : C(X) \to C(Y)$ given by $R(x,t) = (r(x),t)$ where $t \in [0,1]$. □

Example 4.3.0.40. Let X be the Hawaiian earring and $S = \{x\}$ where x is the wedge point. The wedge Y_m of the outer m circles is a retract of X and each $N_{Y_m}(S)$ is trivial by the previous example. Then $N_X(S)$ has no elements of the cannonical free group retracts. Then $N_X(S)$ is trivial by the standard fact that the Hawaiian earring fundamental group injects naturally into the inverse limit of the retract free groups.

Example 4.3.0.41. Let X be the Hawaiian earring again and $S = \{x\}$ be any other point in X besides the wedge point. Then for some $n \in \omega - \{0\}$ the ball $B(x, \frac{1}{n})$ does not intersect any other circle on the Hawaiian earring besides that on which x lies. Then $N(S)$ contains the kernel of the retract induced homomorphism r_* where r fixes the circle on which x lies and takes all other points to the wedge point. On the other hand, $N(S)$ must be precisely the kernel of the induced homomorphism by the lemma.

Chapter 5

Comonster Groups

As an application of the above theory we give the following definition.

Definition 5.0.0.42. We say a group G is **comonster** if for every finite subset $S \subseteq G$ we have $\langle\langle S\rangle\rangle \neq G$. More generally G is κ-**comonster** if for every $S \subseteq G$ with S of cardinality $< \kappa$ we have $\langle\langle S\rangle\rangle \neq G$.

Thus comonster groups are $\aleph_0$-comonster groups. One easily sees that any abelian group of cardinality $\kappa > \aleph_0$ is κ-comonster. Also, if $h : G \to H$ is an epimorphism with H comonster (respectively κ-comonster), then G is also comonster (resp. κ-comonster).

We have the following:

Theorem 5.0.0.43. *Let X be a Peano continuum and $N \trianglelefteq \pi_1(X)$ be of type $\mathscr{P}$ with $\mathscr{P}$ nice with BP. If $\pi_1(X)/N$ is comonster then $\pi_1(X)/N$ is $\aleph_1$-comonster. In particular, if $\pi_1(X)$ is comonster, then $\pi_1(X)$ is $\aleph_1$-comonster.*

Proof. Suppose for contradiction that $\pi_1(X)/N$ is comonster but not $\aleph_1$-comonster. Let $S = \{g_0, \ldots\} \subseteq \pi_1(X,x)$ be a countably infinite set such that $\langle\langle S\rangle\rangle = \pi_1(X,x)$. The normal groups $G_n = \langle\langle N \cup \{g_0, \ldots, g_n\}\rangle\rangle$ are easily seen to be $\mathscr{P}$ and $\bigcup_n G_n = \pi_1(X)$. On the other hand the sequence G_n cannot stabilize since $\pi_1(X,x)$ is comonster. Thus one can pick a strictly increasing subsequence of normal $\mathscr{P}$ subgroups whose union is $\pi_1(X)$, contradicting Theorem 3.0.0.27. $\square$

Example 5.0.0.44. Let **HE** denote the Hawaiian earring. We have an epimorphism $h : \pi_1(\mathbf{HE}) \to \prod_\omega \mathbb{Z}$ given by letting the n-th coordinate of $h([l])$ be given by counting the number of times a loop traverses the n-th circle of the infinite wedge that defines **HE** in an oriented direction. Then $\pi_1(\mathbf{HE})$ is $2^{\aleph_0}$-comonster, since $\prod_\omega \mathbb{Z}$ is abelian of cardinality $2^{\aleph_0}$.

Example 5.0.0.45. If X is a one-dimensional Peano continuum with $\pi_1(X)$ uncountable, then X retracts to a subspace that is homeomorphic to **HE**, so that again $\pi_1(X)$ is $2^{\aleph_0}$-comonster.

Even if $\pi_1(X)$ is uncountable, it may still be the case that $\pi_1(X)$ is not comonster, as illustrated in the following example.

Example 5.0.0.46. Let Y be a Peano continuum with $\pi_1(Y) \simeq A_5$. This can be done by taking a finite presentation for A_5 and constructing the finite 2-dimensional CW complex by letting loops correspond to generators in the presentation and gluing on the boundary of a disc along a path that gives the relators. Such a space is compact, metrizable, path connected and locally path connected. Then Y is a Peano continuum, and so is $X = \prod_\omega Y$. We have $\pi_1(X) \cong \prod_\omega A_5$. Letting $g \in \prod_\omega A_5$ have every entry be the 3-cycle (123), we claim that $\langle\langle g \rangle\rangle = \prod_\omega A_5$. This demonstrates that $\pi_1(X)$ is not comonster.

To see that $\langle\langle g \rangle\rangle = \prod_\omega A_5$, notice that all 3-cycles are conjugate (in A_5) to each other. Thus for each $h \in \prod_\omega A_5$ whose each entry is a 3-cycle we have $h \in \langle\langle g \rangle\rangle$. Each 3-cycle is a product of two 3 cycles (if (abc) is a 3-cycle then $(abc) = (abc)^{-1}(abc)^{-1} = (cba)(cba)$). Since the trivial element in A_5 is a product of two three cycles and each 5-cycle and each product of two disjoint transpositions $(ab)(cd)$ is a product of two 3-cycles then in fact every element in $\prod_\omega A_5$ is a product of exactly two conjugates of g and we are done.

In all the above examples of Peano continua with comonster fundamental group, we used the fact that if the abelianization is uncountable, then the fundamental group is comonster.

Question 5.0.0.47. Does there exist a Peano continuum whose first homology is trivial and whose fundamental group is comonster?

A negative answer would be very interesting as it would imply a theorem for finitely presented perfect groups (groups with trivial abelianization).

Theorem 5.0.0.48. *Suppose the answer to the above question is no. Let $\mathscr{P}_n$ be the class of groups whose elements are products of n or fewer commutators. For each $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that if $G \in \mathscr{P}_n$ is finitely presented there exists a set $F \subseteq G$ with $|F| \leq k(n)$ and $\prod_{k(n)} F^G = G$ (each element of G is a product of $k(n)$ or fewer conjugates of elements of F).*

Proof. Suppose for contradiction that for some $n \in \mathbb{N}$ there is no such $k(n)$. Select finitely presented groups $G_m \in \mathscr{P}_n$ such that for any $F \subseteq G_m$ with $|F| \leq m$ we have that $\prod_m F^{G_m} \neq G_m$. For each m there is a finite CW complex Y_m of dimension at most two whose fundamental group is isomorphic to G_m. Each Y_m is a Peano continuum. Then $\prod_m Y_m$ is a Peano continuum, with fundamental group isomorphic to $\prod_m G_m$. It is easy to see that $\prod_m G_m \in \mathscr{P}_n$ and is also comonster. □

This is adjacent to a question of Wiegold: Does every finitely generated perfect group contain an element which normally generates the group?

Chapter 6

Covering Spaces

We begin by reminding the reader of some relevant concepts.

Definition 6.0.0.49. Let X be a topological space. A **covering space** for X is a topological space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ such that for every $x \in X$ there exists an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets $\{U_j\}_{j \in J}$ such that $p|U_j$ is a homeomorphism onto U for each $j \in J$. Such a U is said to be evenly covered.

It is easy to see that the map p is a continuous, open mapping. The covering space map satisfies the following conditions proven in the classical theory:

Lemma 6.0.0.50. If $p : \tilde{X} \to X$ is a covering map, $x \in X$ and $p(\tilde{x}) = x$ then $p_* : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is injective. If $[l] \in p_*(\pi_1(\tilde{X}, \tilde{x}))$ then l lifts to a loop $\tilde{l} \in L_{\tilde{x}}$.

Recall the classic metrization theorem of Nagata and Smirnov:

Theorem. A space is metrizable if and only if it is regular and has a basis which is countably locally finite.

A collection of subsets $\mathscr{B}$ of X is locally finite if for each $x \in X$ there is an open neighborhood U of x such that the set of elements of $\mathscr{B}$ with nonempty intersection with U is finite. A collection $\mathscr{B}$ of subsets of X is countably locally finite if it can be written as a countable union $\mathscr{B} = \bigcup_{n \in \omega} \mathscr{B}_n$ where each $\mathscr{B}_n$ is locally finite.

Assuming X is a metrizable space, the above theorem implies that a covering space $p : \tilde{X} \to X$ is also metrizable. The regularity of $\tilde{X}$ follows easily from the normality of X. Let $\mathscr{B}$ be a countably locally finite basis, say $\mathscr{B} = \bigcup_n \mathscr{B}_n$. The set of those elements of $\mathscr{B}$ which are evenly covered by p is also a basis and obviously also countably locally finite, so we may assume that elements of $\mathscr{B}$ are evenly covered. Letting $\tilde{\mathscr{B}}_n = \{U \subseteq \tilde{X} : p|U$ is a homeomorphism onto an element of $\mathscr{B}_n\}$ and $\tilde{\mathscr{B}} = \bigcup_n \tilde{\mathscr{B}}_n$ it is easy to see that each $\tilde{\mathscr{B}}_n$ is locally finite and $\tilde{\mathscr{B}}$ is a basis.

In keeping with the rest of the paper, we shall focus on path connected metrizable spaces and those covers which are also path connected (and metrizable).

Theorem 6.0.0.51. *If $p : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering map with X path connected metrizable, then $p_*(\pi_1(\tilde{X}, \tilde{x}))$ is an open subgroup in $\pi_1(X, x)$.*

Proof. Fix a metric d on X. Let $l \in L_x$ be such that $[l] \in p_*(\pi_1(\tilde{X}, \tilde{x}))$ and $\tilde{l} \in L_{\tilde{x}}$ be the lift of l. By compactness of $l([0,1])$ let $\delta > 0$ be a number small enough that for all $s \in [0,1]$ we have $B(l(s), \delta)$ is evenly covered by p. Let $U_s \subseteq \tilde{X}$ be the unique open set such that $\tilde{l}(s) \in U_s$ and $p|U_s$ is a homeomorphism onto $B(l(s), \delta)$.

Suppose l' is distance less than δ away from l in L_x. Then $l'(s) \in B(l(s), \delta)$ for all $s \in [0,1]$. Let $\rho : [0,1] \to \tilde{X}$ be the unique path such that $p \circ \rho = l'$ and $\rho(0) = \tilde{x}$. Notice that $\{s \in [0,1] : \rho(s) \in U_s\}$ is open in $[0,1]$. Also, $\{s \in [0,1] : \rho(s) \notin U_s\}$ is also open in $[0,1]$ since if $\rho(s) \notin U_s$ then $\rho(s)$ is in another, disjoint neighborhood U_s' such that $p|U_s'$ is a homeomorphism with $B(l(s), \delta)$ (this is because $l'(s) \in B(l(s), \delta)$.) By connectedness we have $\rho(s) \in U_s$ for all $s \in [0,1]$. Then $\rho(1) = \tilde{x}$ and $[l']$ is in $p_*(\pi_1(\tilde{X}, \tilde{x}))$. □

Assuming $\tilde{X}$ is path connected we know each fiber $p^{-1}(z)$ is of the same cardinality and is equal to the index $\pi_1(X) : p_*(\pi_1(\tilde{X}, \tilde{x}))$. By lemma 2.0.0.3 this index is countable since the set of left cosets is a partition of the separable space L_x into disjoint open sets. Thus if $Q \subseteq X$ is countable, dense we easily see that $p^{-1}(Q)$ is as well. Thus any path connected cover of a path connected separable metrizable space is also separable, metrizable. To see that the property of being Polish lifts as well, we use the following (see [Ke] 8.19).

Theorem. (Sierpiński) Let Z be Polish and W separable metrizable. If there is a continuous open surjection of Z onto W then W is Polish.

Proposition 6.0.0.52. *If X is path connected Polish and $\tilde{X}$ is a path connected then $\tilde{X}$ is Polish.*

Proof. We have already seen that $\tilde{X}$ is separable, metrizable. Let $\mathscr{U}$ be a countable open cover of X by open evenly covered sets. Let Z be the topological disjoint union of countably many copies of each of the $U \in \mathscr{U}$. As Z is a disjoint union of Polish spaces we know Z is itself Polish. Let $f : Z \to \tilde{X}$ be any map such that $f|U'$ is a homeomorphism to one of the stacked disjoint open sets which compose $p^{-1}(U)$ for each of the countably many copies of each $U \in \mathscr{U}$ and such that f is surjective. It is clear that f is continuous and open. We are done by the quoted theorem. □

Theorem 6.0.0.53. *Suppose $\mathscr{P}$ is a Polish pointclass containing the open or the closed sets which is closed under continuous preimages and let $p : (\tilde{X}, \tilde{x}) \to (X, x)$ be a covering map with X and $\tilde{X}$ Polish path connected. Then $G \leq \pi_1(\tilde{X}, \tilde{x})$ is $\mathscr{P}$ if and only if $p_*(G)$ is $\mathscr{P}$.*

Proof. As p_* is injective, assuming $p_*(G)$ is $\mathscr{P}$ we have that $G = p_*^{-1}(p_*(G))$ is also $\mathscr{P}$. Let $\overline{p} : L_{\tilde{x}} \to \bigcup L_x$ be the induced map on the loop spaces given by composition with p. We show that $\overline{p}$ is a homeomorphism onto $\bigcup p_*(\pi_1(\tilde{X}, \tilde{x})) \subseteq L_x$. We know that p_* is continuous, and also onto $\bigcup p_*(\pi_1(\tilde{X}, \tilde{x}))$ (by considering the lift of a loop one detects the preimage). As lifts are unique we get that $\overline{p}$ is injective as well. Finally, suppose that loops $\{l_n\}_{n\in\omega}$ in L_x converge to l. We must show that the lifts $\{\tilde{l}_n\}_{n\in\omega}$ converge to the lift $\tilde{l}$. Letting $\{U_0, \dots, U_m\}$ be an open cover of the image of l by evenly covered open sets, we pick $N \in \omega$ large enough that for each $0 \leq k \leq N-1$ we have $l([\frac{k}{N}, \frac{k+1}{N}]) \subseteq U_{i_k}$. By passing to a subsequence if necessary, we may assume that $l_n([\frac{k}{N}, \frac{k+1}{N}]) \subseteq U_{i_k}$ for all $n \in \omega$. For each $0 \leq i \leq m$ let $U_i' \subseteq \tilde{X}$ be such that $p|U_i'$ is a homeomorphism to U_i and $\tilde{l}([\frac{k}{N}, \frac{k+1}{N}]) \subseteq U_{i_k}'$.

Since $p|U_{i_0}'$ is a homeomorphism we know that $\tilde{l}_n|[\frac{0}{N}, \frac{1}{N}]$ converges pointwise to $\tilde{l}$. In particular we have $\tilde{l}_n(\frac{1}{N})$ is eventually inside U_{i_1}'. Since $p|U_{i_1}'$ is a homeomorphism we have that $\tilde{l}_n|[\frac{1}{N}, \frac{2}{N}]$ must eventually converge pointwise to $\tilde{l}|[\frac{1}{N}, \frac{2}{N}]$. Continuing in this manner, we see that the restrictions of the $\tilde{l}_n$ to the closed intervals $[\frac{k}{N}, \frac{k+1}{N}]$ converge pointwise, which is sufficient for $\tilde{l}_n$ converging pointwise to $\tilde{l}$. Thus $\overline{p}$ is a homeomorphism. Then assuming G is $\mathscr{P}$ we have that $p_*(G)$ is a subset of the clopen subset $\bigcup p_*(\pi_1(\tilde{X}, \tilde{x})) \subseteq L_x$.

If G is not all of $\pi_1(\tilde{X},\tilde{x})$ then we select $l \in L_{\tilde{x}} - \bigcup G$ and since $p_*(\pi_1(\tilde{X},\tilde{x}))$ is clopen we may retract L_x to $\bigcup p_*(\pi_1(\tilde{X},\tilde{x}))$ by letting $L_x - \bigcup p_*(\pi_1(\tilde{X},\tilde{x}))$ map to the point $p \circ l$. If $G = \pi_1(\tilde{X},\tilde{x})$ then $p_*(G)$ is clopen and again we are done. □

Theorem 6.0.0.54. *If X is a path connected, locally path connected metric space and $G \trianglelefteq \pi_1(X,x)$ is open, then there exists a covering space $p : (\tilde{X},\tilde{x}) \to (X,x)$ associated with G (i.e. $p_*(\pi_1(\tilde{X},\tilde{x})) = G$.)*

Proof. Let P_x be the set of all paths $\gamma : [0,1] \to X$ such that $\gamma(0) = x$. Notice that L_x is a subset of P_x. Define an equivalence relation $\sim$ on P_x by letting $\gamma_0 \sim \gamma_1$ if and only if $\gamma_0(1) = \gamma_1(1)$ and $\gamma_0 * \gamma_1 \in \bigcup G$. This relation is clearly reflexive. If $\gamma_0 \sim \gamma_1$ then $\gamma_0 * \gamma_1^{-1} \in \bigcup G$, so that $(\gamma_0 * \gamma_1^{-1})^{-1} = \gamma_1 * \gamma_0^{-1} \in \bigcup G$, so symmetry holds. If $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$ then $\gamma_0 * \gamma_2^{-1}$ is homotopic to $\gamma_0 * \gamma_1^{-1} * \gamma_1 * \gamma_2^{-1} \in \bigcup G$ and so transitivity holds as well. Obviously if two paths are homotopic rel endpoints then they are equivalent under $\sim$.

Letting $\tilde{X}$ be the set $P_x/\sim$ and $\tilde{x}$ be the equivalence class of the constant path at x, we have a function $p : (\tilde{X},\tilde{x}) \to (X,x)$ given by $p : \gamma \mapsto \gamma(1)$. We describe the topology of $\tilde{X}$.

By Lemma 2.0.0.6 we have an open cover $\mathscr{U}$ of X such than any loop with image in an element of $\mathscr{U}$ is an element of G (considering G as basepoint free by normality.) By taking a finer cover if necessary we may assume the elements of $\mathscr{U}$ are path connected. Given $\gamma \in p^{-1}(U)$ we let $U_\gamma \subseteq P_x/\sim$ be defined to be set of all those paths $\gamma_0 \in P_x$ such that for some $\gamma_0' : [0,1] \to U$ we have $\gamma_0'(0) = \gamma(1)$ and $\gamma * \gamma_0' \sim \gamma_0$. Notice that the restriction $p|U_\gamma$ is injective, for if $\gamma_0, \gamma_1 \in U_\gamma$ with γ_0' and γ_1' witnessing this, we have $\gamma_1 * (\gamma_1')^{-1} * \gamma^{-1} \in G$ and $\gamma_0 * (\gamma_0')^{-1} * \gamma^{-1} \in G$. Supposing $p(\gamma_0) = p(\gamma_1)$ we have that $\gamma_0 * (\gamma_0')^{-1} * \gamma^{-1} * \gamma * \gamma_1' * \gamma_1^{-1} \in \bigcup G$, which is homotopic to $\gamma_0 * (\gamma_0')^{-1} * \gamma_1' * \gamma_1^{-1}$ and since $(\gamma_0')^{-1} * \gamma_1'$ is a loop in U we know it represents an element of G with basepoint $p(\gamma_0) = p(\gamma_1)$, and so $\gamma_0 * \gamma_1^{-1}$ represents an element of G, so $\gamma_0 \sim \gamma_1$. By path connectedness of U we see that $p|U_\gamma$ is also onto. Notice that if γ_0, γ_1 represent elements in U_γ then by definition there exist paths γ_0', γ_1' such that $\gamma * \gamma_0' \sim \gamma_0$ and similarly for γ_1'. We have that $\gamma_1 \sim \gamma_0 * (\gamma_0')^{-1} * \gamma_1'$, so that defining U_{γ_0} in the comparable way as U_γ we get that γ_1 represents an element of U_{γ_0} as well. Thus in fact $U_\gamma \subseteq U_{\gamma_0}$ and the proof of the other inclusion is symmetric.

Define each U_γ to be open and declare each restriction $p|U_\gamma$ to be a homeomorphism. This defines a basis for a topology on $\tilde{X}$ under which p is a continuous local homeomorphism. Moreover, for each point $y \in X$ we may pick $U \in \mathscr{U}$ such that $y \in U$ and notice that the U_γ associated with U evenly cover U (they are disjoint since we have shown that any nonempty intersection gives equality.)

Letting $l \in L_{\tilde{x}}$ be a loop, the loop $p \circ l$ as an element of the path space P_x is such that $p \circ l$ is $\sim$ the constant loop, so $p \circ l$ is a loop in $\bigcup G$. On the other hand, given a loop $l \in \bigcup G$ we consider $l \in P_x$ and notice that by definition $l(1) \in \tilde{x}$. Then the lift of l in $L_{\tilde{x}}$ is a closed loop. We have shown $p_*(\pi_1(\tilde{X}, \tilde{x}) = G$.

$\square$

Chapter 7

Applications to First Homology

The theory developed around the pointclasses of subgroups of the fundamental group has applications to the structure of the abelianization of the fundamental group of a path connected Polish space: the first homology. To start, we have the following which first appeared in [CoCo], which is an instance of Theorem 3.0.0.29.

Theorem 7.0.0.55. *If X is a path connected, locally path connected Polish space then $H_1(X)$ is either of cardinality $\leq \aleph_0$ or $2^{\aleph_0}$. In case such an X is also compact, then $\pi_1(X)$ is either a finite direct sum of cyclic groups or of cardinality $2^{\aleph_0}$.*

We also give a theorem which can be interpreted as a type of small loop compactness.

Definition 7.0.0.56. Recall that in a group G the **commutator length** of $g \in G$, which we will denote $\mathrm{cl}(g)$, is the smallest number n such that g can be written as a product of n commutators. We write $\mathrm{cl}(g) = \infty$ in case g is not in the commutator subgroup of G.

Theorem 7.0.0.57. *If X is a Peano continuum with $H_1(X)$ of cardinality $< 2^{\aleph_0}$ then there exists $\varepsilon > 0$ and $N \in \omega$ such that any loop of of diameter less than ε is of commutator length $\leq N$.*

Proof. Suppose the conclusion fails. We may then pick a sequence of loops $\{l_n\}_{n\in\omega}$ such that $\mathrm{diam}(l_n) \leq 2^{-n}$ and the commutator length of $[l_n] \in \pi_1(X, x_n)$ is greater than n. As X is locally path connected, we may pass to a subsequence if necessary and assume that all loops are based at the same point, say x. Letting N_n be the set of all loops at x which are of commutator length at most n, we have that $\bigcup[\pi_1(X,x), \pi_1(X,x)] = \bigcup_{n\in\omega} N_n$. It is clear that $N_n N_m = N_{m+n}$ and the set of products of n commutators is easily analytic, so by Lemma 3.0.0.26 we get that $H_1(X)$ is of cardinality $2^{\aleph_0}$. □

If A is an abelian group let $\mathrm{Tor}(A)$ denote the subgroup of A consisting of the torsion elements. Let $\mathrm{Torfree}(A)$ denote the quotient $A/\mathrm{Tor}(A)$. It is not always the case that $\mathrm{Tor}(H_1(X))$ is a direct summand, by the following example adapted from [Fu].

Example 7.0.0.58. Let P be the set of primes and for each $p \in P$ let X_p be a Peano continuum with fundamental group isomorphic to $\mathbb{Z}/p$. Then $X = \prod_{p\in P} X_p$ has fundamental group isomrophic to $\prod_{p\in P}\mathbb{Z}/p$, so $H_1(X) \simeq \prod_{p\in P}\mathbb{Z}/p$ as well.

Suppose $a \in \prod_{p\in P}\mathbb{Z}/p$ is torsion. If the p coordinate of a is nonzero then p divides the order of a, so in particular a must have finite support. Conversely any finite supported element is torsion, so $\mathrm{Tor}(\prod_{p\in P}\mathbb{Z}/p) = \bigoplus_{p\in P}\mathbb{Z}/p$. Now for $a = (1,1,1,,\ldots) \in \prod_{p\in P}\mathbb{Z}/p$ we have that for any $n \in \omega$ there exists $b \in \prod_{p\in P}\mathbb{Z}/p$ such that $b^n = a$ in $\prod_{p\in P}\mathbb{Z}/p/\bigoplus_{p\in P}\mathbb{Z}/p$. This is seen by letting $b(p)$ be such that $b(p)^n = 1 \mod p$ for all p that do not divide n and $b(p) = 0$ otherwise. In $\prod_{p\in P}\mathbb{Z}/p$ there is no nontrivial element which has all roots (for example a p-th root would not exist if the p-th coordinate is not zero). Thus $\bigoplus_{p\in P}\mathbb{Z}/p$ cannot be a direct summand.

Theorem 7.0.0.59. *If X is a path connected locally, path connected Polish space then* $\mathrm{Torfree}(H_1(X))$ *is either of cardinality* $\leq \aleph_0$ *or* $2^{\aleph_0}$. *If in addition X is compact then* $H_1(X)/\mathrm{Tor}(H_1(X))$ *is a finite rank free abelian group or of cardinality* $2^{\aleph_0}$.

Proof. It suffices to show that the kernel of the map $\phi : \pi_1(X) \to H_1(X)/\mathrm{Tor}(H_1(X))$ is analytic. Notice that $l \in \bigcup \ker(\phi)$ if and only if $(\exists n \in \omega - \{0\})[l^n \in \bigcup[\pi_1(X),\pi_1(X)]]$. Since $\bigcup[\pi_1(X),\pi_1(X)]$ is analytic and each map $l \mapsto l^n$ is continuous we have that $\ker(\phi)$ is a countable union of continuous preimages of analytic sets, and is therefore analytic. $\square$

In case $\mathrm{Torfree}(H_1(X))$ is countable, it is a finite rank free group and we get the splitting of the short exact sequence of abelian groups

$$0 \to \mathrm{Tor}(H_1(X)) \to H_1(X) \to \mathrm{Torfree}(H_1(X)) \to 0$$

so that $H_1(X) \simeq \mathrm{Tor}(H_1(X)) \oplus \bigoplus_{m=0}^{n}\mathbb{Z}$.

Remark 7.0.0.60. One can obtain the same dichotomies to the above theorem by considering $\mathrm{Tor}_m(H_1(X))$ (the subgroup of all elements whose order divides $m \in \omega$) by arguing similarly.

Recall that a torsion group is a p-group if the order of every element is divisible by p. A basic fact about torsion abelian groups is the following (see, for example, Theorem 2.1 in [Fu]):

Theorem. Every abelian torsion group A may be decomposed into a direct sum of the p-groups $A_p \leq A$ where A_p is the subgroup of elements whose order is divisible by p.

Theorem 7.0.0.61. *Let X be a Peano continuum. If* $\mathrm{card}(\mathrm{Torfree}(H_1(X))) < 2^{\aleph_0}$ *then* $H_1(X)$ *is a direct sum of cyclic groups and* $\mathrm{Tor}(H_1(X))$ *is of bounded exponent.*

Proof. Let $P = \{p_0, p_1, \ldots\}$ be the set of prime numbers. Notice that for each prime p the p-subgroup $H_1(X)_p \leq H_1(X)$ is such that $[l] \in \pi_1(X)$ maps to $H_1(X)_p$ if and only if $(\exists k \in \omega)[l^{p^k} \in \bigcup[\pi_1(X), \pi_1(X)]]$. Then the kernels of each of the maps $\phi_p : \pi_1(X) \to H_1(X)/H_1(X)_p$ are analytic subgroups. Writing $H_1(X) \simeq \mathrm{Tor}(H_1(X)) \oplus \bigoplus_{m=0}^{k} \mathbb{Z}$ we select loops $l_0, \ldots, l_k$ such that $[l_k]$ generates the k-th copy of $\mathbb{Z}$ in $\bigoplus_{m=0}^{k} \mathbb{Z}$. Then the map $\pi_1(X) \to H_1(X)/\bigoplus_{m=0}^{k} \mathbb{Z}$ has kernel which is precisely $\langle \{[l_0], \ldots, [l_n]\} \cup [\pi_1(X), \pi_1(X)] \rangle$, and so this kernel is also analytic.

Now the kernels of the maps

$$\pi_1(X) \to H_1(X)/(\bigoplus_{m=0}^{k} \mathbb{Z} \oplus H_1(X)_{p_0} \oplus \cdots \oplus H_1(X)_{p_j})$$

are all analytic, and their union is all of $\pi_1(X)$. Thus by Theorem 3.0.0.27 we have that the ascending sequence eventually stabilizes. Then we have that $H_1(X) \simeq \bigoplus_{m=0}^{k} \mathbb{Z} \oplus H_1(X)_{p_0} \oplus \cdots \oplus H_1(X)_{p_j}$ for some $j \in \omega$. For a torsion abelian group A write A_{p^q} for the subgroup of those elements whose order divides p^q, where p is a prime. We similarly have that the kernels of the maps

$$\pi_1(X) \to H_1(X)/(\bigoplus_{m=0}^{k} \mathbb{Z} \oplus H_1(X)_{p_0^q} \oplus H_1(X)_{p_1} \cdots \oplus H_1(X)_{p_j})$$

are analytic for all $q \in \omega - \{0\}$. The union of all these subgroups is the whole of $\pi_1(X)$, so the sequence must stabilize. Proceeding similarly for p_1, p_2, etc., we get that $\mathrm{Tor}(H_1(X))$ is a finite direct sum of p-subgroups, each of bounded power. Thus $\mathrm{Tor}(H_1(X))$ is a torsion group of bounded power, and so is a direct sum of cyclic groups by a theorem of Prüfer (see [Fu] Theorem 11.2). □

A theorem of Kulikov (see Theorem 12.2 of [Fu]) states that subgroups of direct sums of cyclic groups are direct sums of cyclic groups. Thus if $\mathrm{card}(\mathrm{Torfree}(H_1(X))) < 2^{\aleph_0}$ we see that $H_1(X)$ has no nontrivial divisible subgroup. Recall that an abelian group is divisible if each element has all roots.

Since the divisible abelian groups are injective, each torsion-free abelian group A decomposes as a direct sum $A = \mathrm{Div}(A) \oplus \mathrm{Red}(A)$ where $\mathrm{Div}(A)$ is the maximal divisible group in A and R is a reduced subgroup (i.e. contains no nontrivial divisible subgroups.) The reduced subgroup is not necessarily unique as a subgroup of A, but is unique up to isomorphism as it is isomorphic to $A/\mathrm{Div}(A)$.

Theorem 7.0.0.62. *If X is a Peano continuum then* $\mathrm{Red}(\mathrm{Torfree}(H_1(X)))$ *is either a free abelian group of finite rank or of cardinality* $2^{\aleph_0}$.

Proof. We need only show that the kernel K of the map $\pi_1(X) \to \mathrm{Red}(\mathrm{Torfree}(H_1(X)))$ is analytic. We have already seen that the kernel K_1 of the map $\pi_1(X) \to \mathrm{Torfree}(H_1(X))$ is analytic. Now l is in $\bigcup K$ if and only $(\forall n \in \omega)(\exists l_1 \in L_x)[(l_1)^n l^{-1} \in \bigcup K_1]$. Then K is an analytic subgroup as a countable intersection of countinuous preimages of analytic subsets of L_x. □

From the example earlier where $\pi_1(X) \simeq \prod_{p \in P} \mathbb{Z}/p$ we have that reach element of

Torfree$(H_1(X))$ is divisible by the same explanation as given for $(1,1,\ldots)$. Thus Torfree$(H_1(X))$ is a torsion-free divisible group, of cardinality $2^{\aleph_0}$ by cardinality considerations. Thus by considering Torfree$(H_1(X))$ as a vector space over $\mathbb{Q}$ we may select a basis, so that Torfree$(H_1(X)) \simeq \bigoplus_{2^{\aleph_0}} \mathbb{Q}$. In this case the reduced summand of Torfree$(H_1(X))$ is trivial, and by taking a product of X with a finite dimensional torus T^n we easily get that Torfree$(H_1(X \times T^n)) \simeq \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \prod_{i=0}^{n-1} \mathbb{Z}$. By taking the product of X with the infinite torus T^∞ we get Torfree$(H_1(X \times T^n)) \simeq \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \prod_{n\in\omega} \mathbb{Z}$, so all cases of the previous theorem obtain.

Given an abelian group A there is a largest quotient of A with no infinitely divisible elements. To see this, let S_0 be the set of infinitely divisible elements in A. Let S_1 be the set of those elements in A which map under the quotient map $A \to A/\langle S_0\rangle$ to an infinitely divisible element. In general let $S_{\alpha+1}$ be the set of those elements which map to an infinitely divisible element under the quotient map $A \mapsto A/\langle S_\alpha\rangle$ and $S_\beta = \bigcup_{\alpha<\beta} S_\alpha$ for a limit ordinal. It is clear that $S_0 \subseteq \langle S_0\rangle \subseteq S_1 \subseteq \langle S_1\rangle \subseteq \cdots$. The subgroup $B = \bigcup_{\alpha<\omega_1} S_\alpha = \bigcup_{\alpha<\omega_1} \langle S_\alpha\rangle$ is evidently such that A/B has no infinitely divisible elements and any homomorphism to an abelian group with no infinitely divisible elements must evidently contain B. Let InfFree(A) denote this maximal quotient with no infinitely divisible elements.

For the next result we state Theorem 4.4 in [CC]:

Theorem. Let X be a topological space, let $\phi : \pi_1(X,x_0) \to L$ a be a homomorphism to the group L, $U_0 \supseteq U_1 \supseteq \cdots$ be a countable local basis for X at x_0 and G_i be the image of the natural map of $\pi_1(U_i,x_0)$ into $\pi_1(X,x_0)$. If L is of cardinality $< 2^{\aleph_0}$ and abelian with no infinitely divisible elements then $\phi(G_n) = 0$ for some $n \in \omega$.

Theorem 7.0.0.63. *If X is a Peano continuum with* card(InfFree$(H_1(X))) < 2^{\aleph_0}$ *then* InfFree$(H_1(X))$ *is a free abelian group of finite rank.*

Proof. The homomorphism $\pi_1(X) \to$ InfFree$(H_1(X))$ is, by the previously stated theorem, such that given any point $x \in X$ there is a neighborhood O_x such that any loop in O_x maps trivially under the map. This implies by Lemma 2.0.0.8 that the kernel is open, and so $\mathrm{InfFree}(H_1(X))$ is a finitely generated abelian group with no torsion. $\square$

Theorem 7.0.0.64. *If X is a Peano continuum then* $\mathrm{Hom}(\pi_1(X), \mathbb{Z}/p)$ *is either countable or of cardinality* $2^{2^{\aleph_0}}$ *and* $\mathrm{Hom}(\pi_1(X), \mathbb{Q})$ *is either countable or of cardinality* $2^{2^{\aleph_0}}$.

Proof. Notice that the set $\{l^p : l \in L_x\}$ is an analytic set in L_x as the continuous image of L_x. The subgroup $\langle[\{l^p : l \in L_x\}] \cup [\pi_1(X), \pi_1(X)]\rangle$ is analytic, being generated by two analytic sets. Any homomorphism $\pi_1(X) \to \mathbb{Z}/p$ must factor through a homomorphism $\pi_1(X)/\langle[\{l^p : l \in L_x\}] \cup [\pi_1(X), \pi_1(X)]\rangle \to \mathbb{Z}/p$. Thus the homomorphisms $\pi_1(X) \to \mathbb{Z}/p$ are in correspondence with the homomorphisms $\pi_1(X)/\langle[\{l^p : l \in L_x\}] \cup [\pi_1(X), \pi_1(X)]\rangle \to \mathbb{Z}/p$. The group $\pi_1(X)/\langle[\{l^p : l \in L_x\}] \cup [\pi_1(X), \pi_1(X)]\rangle$ is a vector space over $\mathbb{Z}/p$, so we may pick a basis and write $\pi_1(X)/\langle[\{l^p : l \in L_x\}] \cup [\pi_1(X), \pi_1(X)]\rangle \simeq \bigoplus_T \mathbb{Z}/p$. Now by Theorem 3.0.0.24 we know T is either finite or of cardinality $2^{\aleph_0}$. In case T is finite, there are finitely many homomorphisms from $\bigoplus_T \mathbb{Z}/p$ to $\mathbb{Z}/p$, and in case T is of cardinality $2^{\aleph_0}$ there are $2^{2^{\aleph_0}}$ many.

For the claim regarding $\mathbb{Q}$ we notice that any homomorphism $\pi_1(X) \to \mathbb{Q}$ must factor through $\mathrm{Torfree}(H_1(X))$. We have seen that $\mathrm{Torfree}(H_1(X))$ is either a finite rank free group or of cardinality $2^{\aleph_0}$. In case $\mathrm{Torfree}(H_1(X))$ is a finite rank free group we have $\mathrm{Hom}(\pi_1(X), \mathbb{Q})$ countable. Suppose $\mathrm{Torfree}(H_1(X))$ has cardinality $2^{\aleph_0}$. Recall that the rank of an abelian group A is the rank of the largest free abelian group in A, and satisfies the inequality $\mathrm{card}(A) \leq \mathrm{rank}(A)\aleph_0$. Thus $\mathrm{Torfree}(H_1(X))$ has a free abelian subgroup F of rank $2^{\aleph_0}$. There are $2^{2^{\aleph_0}}$ many homomorphisms from F to $\mathbb{Q}$, and since $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, each of these homomorphisms extends to all of $\mathrm{Torfree}(H_1(X))$. This establishes $2^{2^{\aleph_0}}$-many distinct homomorphisms from $\mathrm{Torfree}(H_1(X))$ to $\mathbb{Q}$, and there cannot be more than $2^{2^{\aleph_0}}$, so we are done. $\square$

We introduce a notion of strong abelianization:

Definition 7.0.0.65. If X is a path connected metrizable space, define $\overline{H}_1(X)$ to be the quotient $\pi_1(X)/\overline{[\pi_1(X),\pi_1(X)]}$ where $\overline{G}$ is defined to be the smallest closed subgroup containing $G \leq \pi_1(X)$.

The strong abelianization $\overline{H}_1(X)$ corresponds more with our intuition of abelianization for the fundamental group of a space and we shall see that it is easier to understand. We establish some functorial properties.

Theorem 7.0.0.66. *Suppose that X_n is a path connected metrizable space for each $n \in \omega$. Then $\overline{H}_1(\prod_n X_n) \simeq \prod_n \overline{H}_1(X)$.*

Proof. For each $n \in \omega$ fix a point $x_n \in X_n$ and assume without loss of generality that $\mathrm{diam}(X_n) \leq 2^{-n}$. We recall that by lemma 2.0.0.18 the loop space $L_{\{x_n\}_{n\in\omega}}$ is homeomorphic to the space $\prod_{n\in\omega} L_{x_n}$ and so can be metrized with the metric inherited by the product metric. Let $p_n : L_x \to L_{x_n}$ denote projection to the n-th coordinate.

Supposing $l, l' \in L_x$ we have that the loop $l * l' * l^{-1} * (l')^{-1}$ projects under p_n to $p_n(l) * p_n(l') * p_n(l)^{-1} * p_n(l')^{-1}$. Thus by taking products we see that the commutator subgroup $[\pi_1(X),\pi_1(X)]$ is naturally a subgroup of $\prod_n [\pi_1(X_n),\pi_1(X_n)]$. For each $n \in \omega$ the map $\iota_n L_{x_n} \to L_x$ which takes a loop $l \in L_{x_n}$ to the loop $l' \in L_x$ such that $p_n(l'(s)) = l(s)$ and $p_m(l'(s)) = x_m$ for $m \neq n$ demonstrates the inclusion $\bigoplus_n [\pi_1(X_n),\pi_1(X_n)] \leq [\pi_1(X),\pi_1(X)]$.

For each n we have that the continuous preimage $p_{n*}^{-1}(\overline{[\pi_1(X_n),\pi_1(X_n)]}) = \prod_{m<n} \pi_1(X_m) \times \overline{[\pi_1(X_n),\pi_1(X_n)]} \times \prod_{m>n} \pi_1(X_m)$ is closed and contains $\prod_n [\pi_1(X_n),\pi_1(X_n)]$. Hence the intersection of such groups, $\prod_n \overline{[\pi_1(X_n),\pi_1(X_n)]}$, contains the subgroup $\overline{[\pi_1(X),\pi_1(X)]}$. On the other hand we have $\overline{\bigoplus_n [\pi_1(X_n),\pi_1(X_n)]} \leq \overline{[\pi_1(X),\pi_1(X)]}$. Since the set of loops $\bigcup \bigoplus_n [\pi_1(X_n),\pi_1(X_n)]$ is dense in the set of loops $\bigcup \prod_n [\pi_1(X_n),\pi_1(X_n)]$ we see that $\overline{\bigoplus_n [\pi_1(X_n),\pi_1(X_n)]} = \overline{\prod_n [\pi_1(X_n),\pi_1(X_n)]}$. The equality $\overline{\prod_n [\pi_1(X_n),\pi_1(X_n)]} = \prod_n \overline{[\pi_1(X_n),\pi_1(X_n)]}$ is similarly clear and so we see that $\prod_n \overline{[\pi_1(X_n),\pi_1(X_n)]} = \overline{[\pi_1(X),\pi_1(X)]}$ from which we get the isomorphism $\overline{H}_1(\prod_n X_n) \simeq \prod_n \overline{H}_1(X)$. □

Example 7.0.0.67. The circle S^1 is a semilocally simply connected metric space whose fundamental group is isomorphic to $\mathbb{Z}$, and so we can compute the strong abelianization of the infinite torus by the above theorem: $\overline{H}_1(T^\infty) \simeq \prod_{n\in\omega} \overline{H}_1(S^1) \simeq \prod_{n\in\omega} \mathbb{Z}$.

The standard abelianization of abstract groups does not behave in nearly so nice a manner. If $G = \prod_{n\in\omega} G_n$ is a product of groups, then the abelianization of G needn't be the product of abelianizations of the G_n, since an element of $[G,G]$ needs to have finite commutator length and elements of $\prod_n [G_n, G_n]$ can have infinite commutator length.

Definition 7.0.0.68. Let X_n be a sequence of metrizable spaces with distinguished points x_n. Using a cutoff metric if necessary we endow each space with metric d_n such that $\mathrm{diam}(X_n) \leq 2^{-n}$. Define the shrinking wedge of spaces $\bigvee^s_{n\in\omega}(X_n, x_n)$ to be the set which identifies the points x_n with topology given by the metric $d(y,z) = \begin{cases} d_n(y,z) \text{ if } y,z \in X_n \\ d_n(y,x_n) + d_m(z,x_m) \text{ if } y \in X_n - \{x_n\}, z \end{cases}$
It is not difficult to see that the topology does not depend on the metrics chosen and is homeomorphic under any reordering of the index set. If we let all but finitely many of the spaces X_n be a single point then we obtain the standard (finitary) wedge of spaces.

Theorem 7.0.0.69. *If $\{(X_n, x_n)\}_{n\in\omega}$ is a collection of path connected, metrizable pointed spaces then $\overline{H}_1(\bigvee^s_{n\in\omega}(X_n, x_n)) \simeq \prod_{n\in\omega} \overline{H}_1(X_n)$.*

Proof. Let $f : \bigvee^s_{n\in\omega}(X_n, x_n) \to \prod_{n\in\omega} X_n$ be the obvious map. That f_* is onto follows from the fact that $d_n \leq 2^{-n}$. As f is continuous we know $(f_*)^{-1}(\overline{[\pi_1(\prod_{n\in\omega} X_n), \pi_1(\prod_{n\in\omega} X_n)]})$ is a closed subgroup of $\pi_1(\bigvee^s_{n\in\omega}(X_n, x_n))$ and must contain the commutator subgroup. For each m let $r_m : \bigvee^s_{n\in\omega}(X_n, x_n) \to (X_m, x_m)$ be the retraction which takes all subspaces X_n to the point x_m whenever $n \neq m$.

Suppose $l \in L_x$. For each $m \in \omega$ let $\mathscr{I}_m$ be the set of those maximal closed intervals $[a,b] \subseteq [0,1]$ with nonempty interior such that $l|[a,b]$ is a loop in X_m with $a = \inf\{s \in [a,b] : l(s) \neq x_m\}$ and $b = \sup\{s \in [a,b] : l(s) \neq x_m\}$. Write $\mathscr{I}_m = \{I_{m,0}, I_{m,1}, I_{m,2}, \}$ so that the length of $I_{m,k}$ is at least as great as the length of $I_{m,k+1}$. Then each $\mathscr{I}_m$ consists of disjoint closed intervals and the collection $\mathscr{I} = \bigcup_m \mathscr{I}_m$ consists of nonoverlapping intervals, which has a natural ordering by comparing elements in the interior of I and I' under the natural ordering of $[0,1]$. The loop l is equivalent over $[\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n)),\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n))]$ to the loop l' which has $\inf(I_{0,0})=0$, and $\sup(I_{0,0})$ equal to the length of $I_{0,0}$ and all the other relative positions of the elements of $\mathscr{I}$ unchanged. This loop l' is in turn equivalent over $[\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n)),\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n))]$ to the loop l'' which has $I_{0,0}$ in the same position as l' has and $I_{0,1}$ immediately to the right of $I_{0,0}$ (now $I_{0,0}$ and $I_{0,1}$ are no longer disjoint). Continuing in this manner we get loops l''', etc which are equivalent to l over $[\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n)),\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n))]$ with more and more of the elements of $\mathscr{I}_0$ adjacent to each other. Taking the limit of these loops we see that l is equivalent to a loop l_0 over $\overline{[\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n)),\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n))]}$ such that $l_0|[0,s_0]$ is a loop lying in X_0 with $l_0(s)\notin X_0-\{x_0\}$ for all $s\in[s_0,1]$. Now we perform the same process to the loop $l_0|[s_0,1]$ to obtain a loop l_1 which is equivalent over $\overline{[\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n)),\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n))]}$ to l such that $l_0|[0,s_0]=l_1|[0,s_0]$ and $l_1|[s_0,s_1]$ is a loop in X_1 and $l_1(s)\notin X_1-\{x_1\}$ for all $s>s_1$. Continue in this process to get loops $l_2,l_3,\cdots$ such that the analogous relations hold. Since the lengths of the elements of $\mathscr{I}$ must add to a number at most 1, we may take a limit again and see that l is equivalent over $\overline{[\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n)),\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n))]}$ to a loop $\tilde{l}$ such that $\tilde{l}|[0,s_0]$ is a loop in X_0 and $\tilde{l}(s)\notin X_0-\{x_0\}$ for all $s>s_0$, $\tilde{l}|[s_0,s_1]$ is a loop in X_1 such that $\tilde{l}(s)\notin X_1-\{x_1\}$ for $s\notin[s_0,s_1]$, and in general $\tilde{l}|[s_{n-1},s_n]$ is a loop in X_n such that $\tilde{l}(s)\notin X_n-\{x_n\}$ for $s\notin[s_{n-1},s_n]$.

Now consider a loop $l\in\bigcup f_*^{-1}(\overline{[\pi_1(\prod_{n\in\omega}X_n),\pi_1(\prod_{n\in\omega}X_n)]})$ where without loss of generality l is of the form $\tilde{l}$ as in the preceeding paragraph. We have $p_0\circ f\circ l|[0,s_0]=r_0\circ l|[0,s_0]$ is a loop in $\overline{[\pi_1(X_0),\pi_1(X_0)]}$ by the proof of Theorem 7.0.0.66. Then $l|[0,s_0]$ is a loop in $\overline{[\pi_1(X_0),\pi_1(X_0)]}$ by considering the inclusion map $(X_0,x_0)\to\bigvee^s_{n\in\omega}(X_n,x_n)$. Then l is equivalent over $[\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n)),\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n))]$ to a loop l_0 such that $l|[s_0,1]=l_0|[s_0,1]$ and $l(s)=x$ for all $s\in[0,s_0]$. Continuing in this fashion and taking a limit we see that l is equivalent over $[\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n)),\pi_1(\bigvee^s_{n\in\omega}(X_n,x_n))]$ to the constant loop at x. This gives us the reverse containment. □

Example 7.0.0.70. Recall that the Hawaiian Earring X is a shrinking wedge of countably many circles. Thus $\overline{H}_1(X) \simeq \prod_{n\in\omega}\mathbb{Z}$. This computation concides with the standard short proof that $H_1(X)$ is uncountable which is given by mapping onto the fundamental group of the infinite torus. The first homology was computed by K. Eda in [E1] to be isomorphic to $\prod_{n\in\omega}\mathbb{Z}\oplus\bigoplus_{2^{\aleph_0}}\mathbb{Q}\oplus\bigoplus_p A_p$ where A_p is the p-dic completion of the free abelian group of rank $2^{\aleph_0}$.

We note that first homology does not behave this nicely even under a wedge of two spaces, by observing the following theorem of Eda (see [E2])

Theorem. Letting (X,x) and (Y,y) be arbitrary pointed spaces we have that $H_1((X,x)\vee(Y,y)) \simeq H_1(X)\oplus H_1(Y)\oplus H_1(C(X,x)\vee C(Y,y))$ where $C(W,w)$ denotes the cone of the space W with distinguished point $(w,0)$ in the cone.

The homology group $H_1(C(X,x)\vee C(Y,y))$ is often nontrivial.

Chapter 8

Complexity of Subgroups

We have seen that in case X is path connected, locally path connected and semi-locally simply connected, every subgroup of $\pi_1(X)$ is clopen (Proposition). If one considers the famous Griffiths Space [Gri] (a wedge of two Hawaiian earrings with an independent cone over each of the two Hawaiian earrings) then it is straightforward to check that each loop based at the wedge point is a limit of nulhomotopic loops. Thus the only closed subgroup of the fundamental group is the entire fundamental group. In particular, there is no subgroup which is closed and not open. It seems natural to ask whether restrictions exist on subgroups of arbirtrary path connected Polish spaces. In other words, does there exist a path connected Polish space on which there exist subgroups of, say, arbitrarily high Borel complexity? Is there a space in which one can find a a subgroup which is analytic and not Borel? In case both questions have a positive answer, is there a space in which both phenomena occur, and can we find such a space which is a commonly occuring example in the literature?

We show that the fundamental group of the Hawaiiam earring exhibits subgroups of an arbitrarily high Borel complexity, as well as a subgroup which is analytic and not Borel. We first explore subgroups of a particular space: a countably infinite product of projective planes. Let P denote the projective plane, which is a compact manifold with fundamental group of order 2. Let d_P be a metric on P, $x \in P$, $l \in L_x$ be an essential loop and let 0 denote the trivial loop at x. We have a continuous map $f : L_x \to \{0,1\}$ defined by $[0] \mapsto 0$ and $[l] \mapsto 1$, where the codomain is given the discrete topology. Considering $\{0,1\}$ as an additive group, we see that f essentially defines an isomorphism from the fundamental group to $\{0,1\}$.

Endowing the product $X = \prod_\omega P$ with the Tychonov topology we get a compact space.

We endow X with the distinguished point $\overline{x}$ consisting of x in each entry. For each $n \in \omega$ we have the continuous projection map $p_n : X \to P$ which projects the nth coordinate. The map f above gives a continuous map $\phi : L_{\overline{x}} \to \{0,1\}^\omega$ satisfying $f \circ p_n \circ l' = \phi(l')(n)$. We have another map $\psi : \{0,1\}^\omega \to L_{\overline{x}}$ given by $(p_n \circ \psi(\alpha))(s) = \begin{cases} l(s) \text{ if } \alpha(n) = 1 \\ x \text{ if } \alpha(n) = 0 \end{cases}$. The map $\psi \circ \phi : L_{\overline{x}} \to L_{\overline{x}}$ gives a topological retraction from $L_{\overline{x}}$ to a selection of representatives of elements of $\pi_1(X, \overline{x})$. The map $\phi \circ \psi : \{0,1\}^\omega \to \{0,1\}^\omega$ is identity. Thus we have an isomorphism $\overline{\phi} : \pi_1(X, \overline{x}) \to \{0,1\}^\omega$ induced by a continuous map on the loop space $L_{\overline{x}}$. It shall sometimes be convenient to consider $\{0,1\}^\omega$ as isomorphic to the group given by the powerset on the natural numbers $P(\omega)$ with the group operation given by symmetric difference: $A \Delta B = (A - B) \cup (B - A)$. The isomorphism is given by mapping a sequence to its support: $\alpha \mapsto \text{supp}(\alpha)$.

The group $\{0,1\}$ is a Polish group (a topological group with Polish topology) with compatible metric given by the Kronecker delta δ. The Polish group $\{0,1\}^\omega$ has as compatible metric $d(\alpha, \beta) = \sum_{n \in \omega} \frac{\delta(\alpha(n), \beta(n))}{(n+1)^{n+1}}$. One can define pointclass subgroups of a Polish group G in a completely analogous way: $H \leq G$ is of pointclass $\mathscr{P}$ if and only if H is of pointclass $\mathscr{P}$ as a subset of G.

Lemma 8.0.0.71. If $\mathscr{P}$ is a Polish pointclass and closed under continuous preimages between Polish spaces, then $G \leq \pi_1(X, \overline{x})$ is $\mathscr{P}$ if and only if $\overline{\phi}(G)$ is $\mathscr{P}$.

Proof. If $G \leq \pi_1(X, \overline{x})$ is $\mathscr{P}$ then $\bigcup G \subseteq L_{\overline{x}}$ is $\mathscr{P}$, so $\psi^{-1}(\bigcup G) = \overline{\phi}(G)$ is $\mathscr{P}$. If $\overline{\phi} \leq \{0,1\}^\omega$ is $\mathscr{P}$ then $\phi^{-1}(\overline{\phi}(G)) = \bigcup G$ is $\mathscr{P}$, so $G \leq \pi_1(X, \overline{x})$ is $\mathscr{P}$. □

One of the simplest ways of constructing a subgroup of $\{0,1\}^\omega$ from another subgroup is by selecting a group complement, as defined below.

Definition 8.0.0.72. If $G \leq \{0,1\}^\omega$ then $H \leq \{0,1\}^\omega$ is a **group complement** for G if $H \oplus G = \{0,1\}^\omega$. In other words, $H \cap G = 0$ and $\{0,1\}^\omega$ is generated by the set $H \cup G$.

We say H is a **sub-group complement** for G if $H \cap G = 0$. Clearly H is a (sub-)group complement for G if and only if G is a (sub)-group complement for H.

Group complements need not be unique, but the following lemma implies that they always exist.

Lemma 8.0.0.73. If H is a sub-group complement of G then there exists a complement H' of G containing H.

Proof. Let $\mathscr{H}$ be the collection of all sub-group complements of G containing H. It is clear that $\mathscr{H}$ is a poset under set inclusion, $H \in \mathscr{H} \neq \emptyset$ and each chain in $\mathscr{H}$ has an upper bound (the union over the chain). Let H' be a maximal element of $\mathscr{H}$ by Zorn's Lemma. If $\alpha \in \{0,1\}^{\omega}$ is not in the subgroup $H' + G$ then we consider the subgroup $H'' = \{\alpha, 0\} + H'$. If $\beta \in H'' \cap G$ with $\beta \neq 0$ then $\beta = \alpha + h$ for some $h \in H'$, and $\beta = \alpha = h \in G$ as well. Then $\alpha = \beta + h \in G + H'$, a contradiction.

Alternatively we know that H and G are $\mathbb{Z}/2$ submodules of $\{0,1\}^{\omega}$ so we may pick bases B_0 and B_1 for H and G respectively. Extend the linearly independent set $B_0 \cup B_1$ to a basis B of $\{0,1\}^{\omega}$. Letting H' be the linear span of $B - B_1$ one can check that H' is a group complement for G and clearly contains H. □

In particular since 0 is a sub-group complement for any subgroup of $\{0,1\}^{\omega}$ we see that each subgroup has a group complement. The fact that the proof of the above lemma used nonconstructive techniques ought to alert the reader that group complements might no satisfy a nice topological descripition (e.g. have BP.)

We give a few trivial observations:

Observation 8.0.0.74. $G \leq \{0,1\}^{\omega}$ is open iff there exists $N \in \omega$ such that $0^N \times \{0,1\}_{n=N+1}^{\infty} \subseteq G$ iff G is of finite index and of a nice pointclass $\mathscr{P}$ with BP iff G is of countable index and of a nice pointclass $\mathscr{P}$ with BP.

Thus the group complement of an open subgroup is finite, and therefore closed. One might ask whether a group complement of a closed subgroup need be closed.

Example 8.0.0.75. Let $\alpha = (1,1,1,1,\dots)$ and consider the finite closed subgroup $G = \{0,\alpha\}$. Let H be the subgroup of $\{0,1\}^{\omega}$ of sequences which are eventually zero. Then H is a sub-group complement of G and is therefore contained in a group complement H' for G. Then H' cannot have BP since it is of finite index and not open (since H' is clearly not closed). Thus even a group complement of a finite subgroup might be very complicated. Each subgroup $\prod_{n=0}^{m-1}\{0,1\} \times 0 \times \prod_{n=m+1}^{\infty}\{0,1\}$ is a closed group complement to G.

Example 8.0.0.76. Letting H be the subgroup as in the previous example, we have by the above observation that any group complement of H cannot have BP (since it would have countable index and not be open). Thus there exists an F_{σ} subgroup which does not have a group complement which has BP.

The metric d we gave to $\{0,1\}^{\omega}$ satisfies the property that for all $\alpha,\beta,\gamma \in \{0,1\}^{\omega}$, $d(\alpha,\beta) = d(\alpha,\gamma)$ implies $\beta = \gamma$. Assuming $G \leq \{0,1\}^{\omega}$ is closed, let $r : \{0,1\}^{\omega} \to G$ be the map which takes α to the closest point in G (which is uniquely defined by the previous sentence). We have the following

Proposition 8.0.0.77. *The map $r : \{0,1\}^{\omega} \to G$ is a continuous homomorphic retract from $\{0,1\}^{\omega}$ to the subgroup G. Moreover the kernel of r is a closed group complement to G of form $\prod_{n=0}^{\infty} H_n \leq \{0,1\}^{\omega}$ with H_n either 0 or $\{0,1\}$. In particular, every closed subgroup of $\{0,1\}^{\omega}$ has a closed group complement.*

Proof. Observe first that r is a continuous map. If $\{\alpha_n\}_{n\in\omega}$ is a sequence which converges to α, suppose for contradiction that $r(\alpha_n)$ does not converge to $r(\alpha)$. Picking a subsequence of $\{\alpha_n\}_{n\in\omega}$ we may assume without loss of generality that $d(r(\alpha_n), r(\alpha)) > \varepsilon$ for all n. Since G is a compact metric space we may again pass to a subsequence if necessary so that we can assume that $r(\alpha_n))$ converges to $\beta \in G$. Now $d(r(\alpha),\alpha) = d(G,\alpha) = \lim_{n\to\infty} d(G,\alpha_n) = \lim_{n\to\infty} d(r(\alpha_n),\alpha_n) = d(\beta,\alpha)$, so that in fact $r(\alpha) = \beta$, a contradiction.

We show that r is a homomorphism and we will be done, since $\ker(r)$ will be a closed group complement. We do this by showing that $r' : \alpha \mapsto \alpha + r(\alpha)$ is a homomorphism, which is clearly sufficient. Consider the cosets $\{\alpha + G\}_{\alpha \in \{0,1\}^\omega}$. If $k = \min d(\alpha + G, \beta + G)$ then there exist $\alpha_1 \in \alpha + G$ and $\beta_1 \in \beta + G$ such that the minimum distance between the compact sets $\alpha + G$ and $\beta + G$ is $d(\alpha_1, \beta_1) = k$. The regular action of $\{0,1\}^\omega$ on itself is by isometries. Thus for any $\gamma \in G$ we have that $d(\alpha_1 + \gamma, \beta_1 + \gamma) = d(\alpha_1, \beta_1)$ and $d(\alpha + \beta, \beta) = d(0, \alpha)$. Then the minimum distance between $\alpha + G$ and $\beta + G$ is attained between any point of $\alpha + G$ and a particular point of $\beta + G$, and vice versa. Then $r'(\alpha) = \alpha + r(\alpha)$ is the closest point on $\alpha + G$ to the point 0. Thus the image of r' is $\{\alpha \in \{0,1\}^\omega : d(\alpha, G) = d(\alpha, 0)\}$. In other words, $\alpha \in \mathrm{Im}(r')$ iff

$$(\forall \gamma \in G - 0)[\alpha(\min(\mathrm{supp}(\gamma))) = 0]$$

Thus the image of r' is a subgroup of form $\prod_{n=0}^{\infty} H_n \leq \{0,1\}^\omega$ with H_n either 0 or $\{0,1\}$. We noticed that $r'(\alpha)$ is the closest member of $\alpha + G$ to 0. For any $\alpha, \beta \in \{0,1\}^\omega$ we have that $r'(\alpha) + r'(\beta) = \alpha + r(\alpha) + \beta + r(\beta) \in \alpha + \beta + G$ and $r'(\alpha) + r'(\beta) \in \mathrm{Im}(r')$ as $\mathrm{Im}(r')$ was shown to be a subgroup. Then $r'(\alpha) + r'(\beta)$ is the closest element of $\alpha + \beta + G$ to 0, so $r'(\alpha + \beta) = r'(\alpha) + r'(\beta)$. Thus r' is a homomorphism and so is r. It is easy to see that $\mathrm{Im}(r') = \ker(r)$, so we have our description of $\ker(r)$. □

Let $f : \mathbf{HE} \to T^\infty$ be the standard mapping from the Hawaiian earring determined by mapping the n-th circle to the circle in T^∞ at the n-th coordinate. This is a continuous, onto mapping. Let $g : T^\infty \to \prod_{n \in \omega} P$ be the continuous map which takes the circle in the n-th coordinate to the n-th copy of P in such a way that the generator of the fundamental group of the circle maps to the generator of the fundamental group of P.

By viewing $\mathbb{Z}$ as a discrete topological group, $\prod_{n \in \omega} \mathbb{Z}$ inherits a Polish group topology.

The following lemma together with lemma 8.0.0.71 ties the pointclasses of subgroups of the infinite torus, product of projective planes, and the Hawaiian earring in a natural way.

Lemma 8.0.0.78. Suppose $\mathscr{P}$ is a Polish pointclass which is closed under continuous preimages. A subgroup $G \leq \prod_{n\in\omega}\mathbb{Z}$ is $\mathscr{P}$ if and only if the corresponding subgroup in $\pi_1(T^\infty)$ is $\mathscr{P}$. A subgroup $G \leq \pi_1(T^\infty)$ is $\mathscr{P}$ if and only if $f_*^{-1}(G)$ is $\mathscr{P}$.

Proof. The claim in the second sentence has the same proof as lemma 8.0.0.71. If $G \leq \pi_1(T^\infty)$ is $\mathscr{P}$ then $f_*^{-1}(G)$ is $\mathscr{P}$ by Proposition 2.0.0.19 part (1). Suppose on the other hand that $f_*^{-1}(G)$ is $\mathscr{P}$. Let $x \in HE$ be the wedge point and let $y \in T^\infty$ be some point. Let $h : L_y \to L_x$ be given by mapping the loop $l \in L_y$ to the loop $l' \in L_x$ given by letting $l'|[1-2^{-n}, 1-2^{-n-1}]$ trace the loop $p_n \circ l$ on the n-th circle of the Hawaiian earring. This map is easily checked to be continuous.

We prove that $h^{-1}(\bigcup f_*^{-1}(G)) = \bigcup G$, which will conclude the proof of the lemma. If $l \in \bigcup G$ then it is easily seen that $f \circ h(l)$ is homotopic to l, so $h(l) \in \bigcup f_*^{-1}(G)$, so $l \in h^{-1}(\bigcup f_*^{-1}(G))$. Given $l \in h^{-1}(\bigcup f_*^{-1}(G))$ we have $h(l) \in \bigcup f_*^{-1}(G)$, so $[h(l)] \in f_*^{-1}(G)$, so $[f \circ h(l)] \in G$ and since $f \circ h(l)$ is homomotopic to l we get $l \in G$. $\square$

We recall the Borel pointclass hierarchy. If Z is a Polish space let $\Sigma_1^0(Z)$ denote the collection of open subsets of X and $\Pi_1^0(Z)$ the collection of closed subsets. For each ordinal $\gamma < \omega_1$ define the following:

1. $\Sigma_\gamma^0(Z)$ is the collection of countable unions of sets in $\bigcup_{\varepsilon<\gamma}\Pi_\varepsilon^0(Z)$

2. $\Pi_\gamma^0(Z)$ is the collection of countable intersections of sets in $\bigcup_{\varepsilon<\gamma}\Sigma_\varepsilon^0(Z)$

3. $\Delta_\gamma^0(Z) = \Pi_\gamma^0(Z) \cap \Sigma_\gamma^0(Z)$

These pointclasses arrange neatly into an array scheme

$$\Sigma^0_1(Z) \qquad \Sigma^0_2(Z) \qquad \cdots$$

$$\Delta^0_1(Z) \qquad \Delta^0_2(Z) \qquad \cdots$$

$$\Pi^0_1(Z) \qquad \Pi^0_2(Z) \qquad \cdots$$

Each pointclass contains all pointclasses to the left, and if Z is an uncountable Polish space the containments are strict. The class of Borel subsets of Z is easily checked to be $\bigcup_{\gamma<\omega_1} \Sigma^0_\gamma$, which is equal to $\bigcup_{\gamma<\omega_1} \Pi^0_\gamma$. These pointclasses give a natural way of organizing the Borel sets according to complexity. We shall say a subset $W \subseteq Z$ is a **true** $\mathscr{P}$ set provided it is a $\mathscr{P}$ set and is not in any of the pointclasses to the left of $\mathscr{P}$. Each pointclass is closed under continuous preimages.

The class $\Sigma^1_1(Z)$ is defined to be the class of all analytic subsets of Z, $\Pi^1_1(Z)$ is the class of complements of analytic sets and $\Delta^1_1(Z) = \Sigma^1_1(Z) \cap \Pi^1_1(Z)$. As all Borel sets are analytic we know the class of Borel subsets of Z is contained in $\Delta^1_1(Z)$, and a theorem of Suslin states that in fact $\Delta^1_1(Z)$ is precisely the class of Borel sets. Analogously a subset $W \subseteq Z$ is **true analytic** provided W is analytic and not Borel.

We have the following theorem of Farah and Solecki (Theorem 2.1 in [FS]):

Theorem. If Γ is an uncountable Polish group then for each $\omega_1 > \gamma \geq 2$ there exists a subgroup which is true Σ^0_γ in Γ and for each $\omega_1 > \gamma \neq 2$ there is a subgroup which is true Π^0_γ as a subspace of Γ.

As a consequence of Theorem 1.2 in [DL] we also get the following:

Theorem. If Γ is an uncountable abelian Polish group then Γ has a subgroup which is true analytic.

These results seem to lend a robustness to the theory of Borel and analytic subgroups of Polish groups. As a consequence, a comparable robustness holds for subgroups of the Hawaiian Earring fundamental group, which is summarized in the following:

Theorem 8.0.0.79. *The fundamental group $\pi_1(\boldsymbol{HE})$ has subgroups of the following types:*

1. *true Σ^0_γ for $\omega_1 > \gamma \geq 2$*

2. *true Π^0_γ for each $\omega_1 > \gamma \neq 2$*

3. *true analytic*

Also, $\pi_1(\boldsymbol{HE})$ has a normal subgroup N such that $\pi_1(\boldsymbol{HE})/N$ is a countably infinite group of exponent 2 (which is not of a nice pointclass with BP since $\pi_1(\boldsymbol{HE})/N$ is not finitely generated). Finally, $\pi_1(\boldsymbol{HE})$ has $2^{2^{\aleph_0}}$-many subgroups of index 2 which are not of a nice pointclass with BP.

Proof. The items (1) and (2) follow directly from the stated theorem of Farah and Solecki, for if $G \leq \prod_{n\in\omega}\mathbb{Z}$ is true $\mathscr{P}$, then considering G as a subgroup of $\pi_1(T^\infty)$ we have that G is true $\mathscr{P}$, and so $f_*^{-1}(G)$ must also be true $\mathscr{P}$. Item (3) follows in the same way, from the paper of [DL]. For the subgroup N we let N be the kernel of the composition of the surjective maps $f_* : \pi_1(\mathbf{HE}) \to \pi_1(T^\infty)$, $g_* : \pi_1(T^\infty) \to \pi_1(\prod_{n\in\omega} P)$, $\phi : \pi_1(\prod_{n\in\omega} P) \to \{0,1\}^\omega$ and $\{0,1\}^\omega \to \{0,1\}^\omega/H'$ where H' is any group complement of the subgroup $H \leq \{0,1\}^\omega$ where H consists of those sequences which are eventually 0.

The last claim was essentially proved in [CS]. Let $\pi_1(\mathbf{HE}) \to \{0,1\}^\omega$ be the onto homomorphism described above. Each nonprinciple ultrafilter on $\{0,1\}^\omega$ gives a distinct homomorphism to the group of order 2 by mapping a sequence to 1 if and only if it is the characteristic function of an element in the ultrafilter. These maps take arbitrarily small loops to the nontrivial element $1 \in \{0,1\}$. The kernel of the composition of the two maps is therefore not open, and yet it is of index 2 in $\pi_1(\mathbf{HE})$. This kernel cannot be of a nice pointclass with BP by Theorem 3.0.0.24. As there are $2^{2^{\aleph_0}}$-many ultrafilters on ω and each produces a distinct homomorphism to $\{0,1\}$ we get the appropriate number of subgroups of index 2 which are not of a nice pointclass with BP. $\square$

Chapter 9

Noncommutatively Slender Groups

Certain groups that allow for infinite multiplication exhibit a curious behavior, namely that maps to particular well understood groups are always boring. This phenomenon was first noticed by Specker in [Spe], who proved that for each integer valued homomorphism from the countable product of integers $\phi : \prod_{\omega} \mathbb{Z} \to \mathbb{Z}$ there exists a natural number N such that the projection $p_N : \prod_{\omega} \mathbb{Z} \to \prod_{n=0}^{N} \mathbb{Z}$ satisfies $\phi = \phi \circ p_N$. The phenomenon was studied by Łos and lead him to define a slender group to be a torsion-free abelian group A for which any homomorphism $\phi : \prod_{\omega} \mathbb{Z} \to A$ has an $N \in \omega$ for which $\phi = \phi \circ p_N$. These groups have been extensively studied and also classified via subgroups (see for example [Fu] volume 2, Sections 94, 95).

The term noncommutatively slender (we will use the contraction n-slender to be short) was introduced by K. Eda. The first examples of such groups were free groups as demonstrated by Higman (in [H]) 40 years before Eda defined such groups. Eda's idea is essentially the same as with the abelian case, with the domain being replaced by the fundamental group of the Hawaiian earring. He showed that n-slender groups are necessarily torsion-free and all abelian n-slender groups are indeed slender in the abelian sense. Also, the class of n-slender groups is closed under direct sums and free products (see [E3] for the definition and such results).

In contrast to slender groups, no nice characterization for n-slender groups via subgroups is known. Eda has noted that n-slenderness is an open question even for finitely presented groups [E']. This note gives a broad class of finitely presented groups which are n-slender, and shows that most finitely generated groups are n-slender in the few-relator sense (as used in [AO]). The theorem is the following:

Theorem 9.0.0.80. *If G is a a torsion-free word hyperbolic group then G is n-slender.*

The hypothesis torsion-free cannot be dropped, as any group with torsion fails to be n-slender ([CS] provides some interesting examples of homomorphisms to torsion). The proof of the theorem uses an interesting theorem about torsion-free hyperbolic groups and a modification of a theorem in [H]. In section 1 some background definitions and results are provided. We define the universal monotone condition (u.m.), which is that there exists a length function on the group such that for any real number r there exists a power K_r such that g^{K_r} is of length at least r for any $g \neq 1$. We show that this condition is sufficient to imply that a group is n-slender. In section 2 we prove torsion-free hyperbolic groups are u.m. which concludes the proof of Theorem 9.0.0.80.

In section 3 we prove the following theorem, which generalizes the fact that n-slender groups are closed under direct sums and direct products:

Theorem 9.0.0.81. *The class of n-slender groups is closed under taking graph products.*

In section 4 we motivate the question of whether Thompson's group F is n-slender. We also show that the natural length function defined by the characterization of F as a diagram group is not u.m. Finally in section 5 we give a family of examples to show that even very uncompicated n-slender groups can fail to be u.m.

9.1 The Hawaiian earring group and uniformly monotone length length functions

In this section we give a characterization of the Hawaiian earring fundamental group, define n-slenderness, and prove a modification of Higman's original theorem of the n-slenderness of free groups.

The Hawaiian earring is a shrinking wedge of countably-infinitely-many circles. More formally, given a point $p \in \mathbb{R}^2$ and $r \in (0,\infty)$ we let $C(p,r)$ denote the circle centered at p of radius r and define the Hawaiian earring to be the subspace $\mathbf{HE} = \bigcup_{n\in\omega} C((0,\frac{1}{n+2}),\frac{1}{n+2})$ of $\mathbb{R}^2$. Naively the fundamental group of **HE** might seem to be a free group of countably infinite rank (one free generator for each circle in the union defining **HE**), but the fundamental group is in fact uncountable. We give a combinatorial characterization of this group using countable words.

Let $\{a_n^{\pm 1}\}_{n=0}^{\infty}$ be a countably infinite set with formal inverses, the elements of which we call **letters**. A map $W : \overline{W} \to \{a_n^{\pm 1}\}_{n=0}^{\infty}$ from a countable totally ordered set $\overline{W}$ is a **word** if for each $n \in \omega$ the set $W^{-1}(\{a_n^{\pm 1}\})$ is finite. Two words U and V are isomorphic, denoted $U \simeq V$, if there exists an order isomorphism of the domains of each word $f : \overline{U} \to \overline{V}$ such that $U(t) = V(f(t))$. We identify isomorphic words. The class of isomorphic words is a set of cardinality continuum which we denote $\mathscr{W}$. For each $N \in \omega$ there is a projection map p_N to the set of finite words given by letting $p_N(W) = W|\{t \in \overline{W} : W(t) \in \{a_n^{\pm 1}\}_{n=0}^{N}\}$. For words $U, V \in \mathscr{W}$ we let $U \sim V$ if for each $N \in \omega$ we have that $p_N(U) = p_N(V)$ in the free group $F(\{a_0, \ldots, a_N\})$. This is an equivalence relation. For each word U there is an inverse word U^{-1} whose domain is the totally ordered set $\overline{U}$ under the reverse order and $U^{-1}(t) = U(t)^{-1}$. Given two words $U, V \in \mathscr{W}$ there is a natural way to form the concatenation UV. In particular, one takes the domain of UV to be the disjoint union of $\overline{U}$ with $\overline{V}$, with order extending that of $\overline{U}$ and $\overline{V}$ and placing all elements of $\overline{U}$ before those of $\overline{V}$, and $UV(t) = \begin{cases} U(t) \text{ if } t \in \overline{U} \\ V(t) \text{ if } t \in \overline{V} \end{cases}$. The set $\mathscr{W}/\sim$ now has a group structure with binary operation given by $[U][V] = [UV]$, inverses defined by $[U]^{-1} = [U^{-1}]$ and the trivial element given by the equivalence class of the empty word.

Let **HEG** denote the group $\mathscr{W}/\sim$. The free group $F(\{a_0, \ldots, a_N\})$, which we shall denote $\mathbf{HEG}_N$, may be though of as a subgroup in **HEG** in the obvious way. Moreover, the word map p_N defines a group retraction $\mathbf{HEG} \to \mathbf{HEG}_N$ which we denote p_N by abuse of notation. There is another word map p^N given by $p^N(W) = W|\{t \in \overline{W} : W(t) \in \{a_n^{\pm 1}\}_{n=N+1}^{\infty}\}$ which gives a group retraction from **HEG** to the subgroup $\mathbf{HEG}^N$ consisting of those equivalence classes which contain words involving no letters in $\{a_n^{\pm 1}\}_{n=0}^{N}$. We again abuse notation by calling this retraction p^N. There is a canonical isomorphism $\mathbf{HEG} \simeq \mathbf{HEG}_N * \mathbf{HEG}^N$ obtained by considering a word W as a concatenation of finitely many words in the letters $\{a_n^{\pm 1}\}_{n=0}^{N}$ and finitely many words in the letters $\{a_n^{\pm 1}\}_{n=N+1}^{\infty}$.

Definition 9.1.0.82. A group G is noncommutatively slender (or n-slender) if for each homomorphism $\phi : \mathbf{HEG} \to G$ there exists $N \in \omega$ such that $\phi = \phi \circ p_N$.

In other words, G is n-slender if for each homomorphism $\phi : \mathbf{HEG} \to G$ there exists N so that the restriction of ϕ to $\mathbf{HEG}^N$ is the trivial homomorphism.

For our purposes a **length function** on a group G is a map $l : G \to \mathbb{R}$ such that the following hold:

1. $l(g) \geq 0$ with $l(1) = 0$
2. $l(g) = l(g^{-1})$
3. $l(gh) \leq l(g) + l(h)$

As an example, if G has a generating set X one may define a length function l_X on G by letting $l_X(g)$ be the distance from 1 to g in the Cayley graph $\Gamma(G, X)$. In other words, $l_X(g)$ is the length of a minimal word in the generators X that is needed to represent the group element g in G. We say that a length function l is **universally monotone** if for each $r \in \mathbb{R}$ there exists $K_r \in \omega$ such that for each $g \in G - \{1\}$ we have that $l(g^{K_r}) \geq l(g) + r$. In particular, for a universally monotone length function we have $l(g) = 0$ if and only if $g = 1$. We say a group is **universally monotone** (or **u.m.**) if it has a length function which is universally monotone.

The following is the critical theorem, the ideas of which are in [H]:

Theorem 9.1.0.83. *If the group G is u.m. then G is n-slender.*

Proof. Let l be a universally monotone length function on G. Let $\phi : \mathbf{HEG} \to G$ be a homomorphism and suppose for contradiction that the restriction of ϕ to each $\mathbf{HEG}^N$ is nontrivial. Select a sequence of words $\{W_m\}_{m=0}^{\infty}$ such that W_m uses only letters in $\{a_n^{\pm 1}\}_{n=m}^{\infty}$ and $\phi([W_m]) \neq 1$. Let $r_m = l(\phi([W_m]))$ and $k_m \in \omega$ be such that $g \in G - \{1\}$ implies $l(g^{k_m}) \geq l(g) + r_m + 1$.

Consider the word $U = U_0$ defined by the equations $U_{p-1} = W_p U_p^{k_m}$. In other words, one can think of U as being of form $U = W_1(W_2(W_3(\cdots)^{k_3})^{k_2})^{k_1}$. Notice that if $\phi([U_p]) \neq 1$ we have

$$l(\phi([U_p]^{k_p})) = l((\phi([U_p]))^{k_p}) \geq l(\phi([U_p])) + r_p + 1 = l(\phi([U_p])) + l(\phi([W_p])) + 1$$

from which we have

$$l(\phi([U_{p-1}])) = l(\phi([W_p])\phi([U_p]^{k_p})) \geq l(\phi([U_p]^{k_p})) - l(\phi([W_p])) \geq l(\phi([U_p])) + 1$$

Thus $\phi([U_{p-1}]) \neq 1$ and the argument may be repeated. By induction we get that $l(\phi([U_{p-p'}])) \geq l(\phi([U_p])) + p'$ for $p' \leq p$. Thus if $p > l(\phi([U_0]))$ we have $\phi([U_p]) = 1$, which gives $1 = \phi([U_p]) = \phi([W_{p+1}])(\phi([U_{p+1}]))^{k_{p+1}} = \phi([W_{p+1}])$, a contradiction.

□

9.2 Hyperbolic Groups

We review some basic concepts related to hyperbolic groups and prove that every torsion-free word hyperbolic group is u.m.

Recall that a metric space (Z,d) is **hyperbolic** if there exists a δ such that for all $p,x,y,z \in Z$ we have

$$(x,z)_p \geq \min\{(x,y)_p, (y,z)_p\} - \delta$$

where $(x,y)_p = \frac{1}{2}(d(x,p) + d(y,p) - d(x,y))$ is the Gromov product. A geodesic metric space (Z,d) is hyperbolic if and only if there exists a δ such that for all points $x,y,z \in S$, and geodesics $[x,y]$ and $[x,z]$, the points $v \in [x,y]$ and $w \in [x,z]$ satisfying $d(x,v) = d(x,w) = (y,z)_x$ also satisfy $d(v,w) \leq 2\delta$. The δ used in the alternative criterion for geodesic spaces is not necessarily the same as in the original definition. Bounded spaces and the classical hyperbolic metric spaces $\mathbb{H}^n$ are examples of hyperbolic spaces.

A finitely generated group G is word hyperbolic if for some finite generating set X the Cayley graph $\Gamma(G,X)$ is a hyperbolic space under the combinatorial path metric (under which $\Gamma(G,X)$ is a geodesic space). It turns out that for a hyperbolic group G it is the case that for any finite generating set X the Cayley graph $\Gamma(G,X)$ is hyperbolic.

Now we fix some notation. Let G be a group with generating set X. Any word W in the letters $X^{\pm 1}$ gives an element of the group G by performing the necessary multiplication of the letters. Write $W =_G g$ if the word W represents the element $g \in G$ and $W =_G U$ if the words W and U represent the same element in G. Let $\|W\|$ denote the length of a word W in the letters $X^{\pm 1}$. Let l_X be the length function induced by X on G, that is $l_X(g) = \min\{\|W\| : W =_G g\}$, and by abuse of notation let $l_X(W) = l_X(g)$ where $W =_G g$. Obviously $l_X(W) \leq \|W\|$. Given words V, W in the letters $X^{\pm 1}$ we say that V is W-periodic if V is a subword of a power of $W^{\pm 1}$. We say a word W in $X^{\pm 1}$ is **cyclically minimal** if the equality $W = VUV^{-1}$ in G implies that $l_X(W) \leq \|U\|$. For $R \in \mathbb{R}$ let $D(R) = \{g \in G : l_X(g) \leq R\}$.

We use the following two results which appear as Lemmas 21 and 26 respectively in [O]:

Lemma 9.2.0.84. Let G be a word hyperbolic group and δ be a constant such that for all $p,x,y,z \in Z$ we have $(x,z)_p \geq \min\{(x,y)_p, (y,z)_p\} - \delta$. Let $K \geq 14\delta$ and $K_1 > 12(K+\delta)$ and suppose that a geodesic n-gon $[x_1,\ldots,x_n]$ satisfies the conditions $d(x_{i-1},x_i) > K_1$ for $i = 2,\ldots,n$ and $(x_{i-2},x_i)_{x_{i-1}} < K$ for $i = 3,\ldots,n$ (if $n \geq 3$). Then the polygonal line $p = [x_1,x_2] \cup [x_2,x_3] \cup \cdots \cup [x_{n-1},x_n]$ is contained in a $2K$-neighborhood of the side $[x_1,x_n]$ and the side $[x_1,x_n]$ is contained in a 14δ-neighborhood of p.

Lemma 9.2.0.85. For every hyperbolic group G with finite generating set X and every $\theta > 0$ there exists a number C such that for every W-periodic word V, where W is a cyclically minimal word with $\|W\| > C$ it is true that $l_X(V) \geq (1-\theta)\|V\|$.

Recall the following classical facts:

Lemma 9.2.0.86. If G is word hyperbolic, generated by the finite set X, and $g \in G$ is of infinite order then the following two conditions hold:

1. There exists $\lambda > 0$ such that $|n| \leq \lambda l_X(g^n)$ for all $n \in \mathbb{Z}$.

2. There exists $L \geq 0$ such that each geodesic γ in $\Gamma(G,X)$ between two elements in the subgroup $\langle g \rangle$ is within the L-neighborhood of $\langle g \rangle$.

The unifom monotonicity of word hyperbolic groups follows immediately from the following theorem, which together with Theorem 9.1.0.83 proves Theorem 9.0.0.80.

Theorem 9.2.0.87. *If G is a torsion-free word hyperbolic group and X a finite generating set there exists a constant $N \in \omega$ such that if $g \in G - \{1\}$ then $l_X(g^N) > l_X(g)$.*

Proof. Fix δ which satisfies both the original and the geodesic definitions of hyperbolic space. Let $l_X(\cdot) = l(\cdot)$ for simplicity of notation. In Lemma 9.2.0.85 we let $\theta = \frac{1}{1000}$ and pick C accordingly where without loss of generality $C > 1000\delta$. For each $h \in D(C) - \{1\}$ pick $\lambda_h, L_h \in \omega - \{0\}$ as in the statement of Lemma 9.2.0.86. Let $\lambda = \max\{\lambda_h\}_{h \in D(C)-\{1\}}$ and $L \geq \max\{L_h\}_{h \in D(C)-\{1\}}, C$. Let $N \geq 100, 241L\lambda$. Let $g \in G - \{1\}$. We treat cases. In Cases 1a and 1b we use the fact that $N \geq 100$ and in Cases 2a and 2b we use the fact that $N \geq 241L\lambda$. Pick h which is conjugate to g and of minimal length. Pick x of minimal length such that $g = xhx^{-1}$.

Case 1a. Suppose $l(h) > C$ and $l(x) \leq 7l(h)$. Then we have that

$$\begin{aligned} l(g^N) &\geq l(h^N) - 2l(x) \geq l(h^N) - 14l(h) \\ &\geq 100(\tfrac{1}{2}l(h)) - 14l(h) \text{ (here we are using } N \geq 100 \text{ and } \theta \leq \tfrac{1}{2}) \\ &\geq 36l(h) > l(h) + 14l(h) \geq l(h) + 2l(x) \geq l(g) \end{aligned}$$

Case 1b. Suppose $l(h) > C$ and $l(x) > 7l(h)$. Consider the geodesic n-gon $[x, xh, xh^2, \ldots, xh^N]$, which is an isometric translate of the geodesic n-gon $[1, h, h^2, \ldots, h^N]$. Notice that

$$\begin{aligned} (h^i, h^{i+2})_{h^{i+1}} &= (1, h^2)_h \\ &= \tfrac{1}{2}(l(h) + d(h, h^2) - l(h^2)) \\ &\leq \tfrac{1}{2}(2l(h) - (1 - \tfrac{1}{100})2l(h)) \\ &= \tfrac{1}{100}l(h) < \tfrac{1}{50}l(h) \end{aligned}$$

for $0 \leq i \leq N-2$. Also, $d(h^i, h^{i+1}) = l(h)$. Thus letting $K = \frac{1}{50}l(h)$ and $K_1 = \frac{1}{2}l(h)$ in Lemma 9.2.0.84, we have that $[x, xh^N]$ is in the 14δ-neighborhood of $[x, xh] \cup [xh, xh^2] \cup \cdots \cup [xh^{N-1}, xh^N]$.

We use Lemma 9.2.0.84 again. Notice that $l(x) \leq l(xh^i)$ for all $i \in \mathbb{Z}$ by the minimality of the length of x (else $(xh^i)h(xh^i)^{-1} = g$ and we have a contradiction). Letting $v \in [x, xh^N]$ be such that $d(x, v) = (1, xh^N)_x$ we may pick $v' \in [x, xh] \cup [xh, xh^2] \cup \cdots \cup [xh^{N-1}, xh^N]$ such that $d(v, v') \leq 14\delta$. For some $0 \leq i \leq N$ we have that $d(v', xh^i) \leq \frac{l(h)}{2}$. Thus $d(v, xh^i) \leq \frac{l(h)}{2} + 14\delta$. Then

$$\begin{aligned} d(1, v) &\geq d(1, xh^i) - d(v, xh^i) \\ &\geq l(x) - \tfrac{l(h)}{2} - 14\delta \end{aligned}$$

Letting $w \in [1, x]$ be such that $d(x, w) = (1, xh^N)_x$ we have that $d(w, v) \leq 2\delta$, and so

$$\begin{aligned} d(1, w) &\geq d(1, v) - d(v, w) \\ &\geq l(x) - \tfrac{l(h)}{2} - 14\delta - 2\delta \\ &= l(x) - \tfrac{l(h)}{2} - 16\delta \end{aligned}$$

Thus $(1, xh^N)_x \leq \frac{l(h)}{2} + 16\delta$. The similar argument shows that $(x, g^N)_{xh^N} \leq \frac{l(h)}{2} + 16\delta$. Now letting $K = \frac{l(h)}{2} + 17\delta$ and $K_1 = 7l(h)$, we see that $K_1 = 7l(h) \geq 12(\frac{l(h)}{2} + 18\delta)$ since $l(h) > C > 1000\delta$. Considering the geodesic quadrangle $[1, x, xh^N, g^N = xh^N x^{-1}]$ we see by Lemma 9.2.0.84 that $[1, x] \cup [x, xh^N] \cup [xh^N, g]$ is in the $2K$-neighborhood of $[1, g^N]$. Pick $s_0, s_1 \in [1, g^N]$ such that $d(x, s_0) \leq 2K = l(h) + 34\delta$ and $d(xh^N, s_1) \leq 2K = l(h) + 34\delta$. It is easy to see that $s_0 \in [1, s_1] \subseteq [1, g^N]$.

Now

$$\begin{aligned} l(g^N) &= d(1, s_0) + d(s_0, s_1) + d(s_1, g^N) \\ &\geq (l(x) - l(h) - 34\delta) + (l(h^N) - 2l(h) - 68\delta) + (l(x) - l(h) - 34\delta) \\ &= 2l(x) + l(h^N) - 4l(h) - 136\delta \\ &\geq 2l(x) + 100(\tfrac{999}{1000}l(h)) - 4l(h) - 136\delta \end{aligned}$$

$$
\begin{aligned}
&> 2l(x) + 99l(h)) - 4l(h) - 136\delta \\
&> 2l(x) + l(h) \\
&\geq l(g)
\end{aligned}
$$

so that we are done in this case.

Case 2a. Suppose that $l(h) \leq C$ and $l(x) \leq 60L$. Then we have

$$
\begin{aligned}
l(g^N) &\geq l(h^N) - 2l(x) \\
&\geq l(h^N) - 120L \\
&\geq \tfrac{N}{\lambda} - 120L \\
&> 241L - 120L = L + 120L \geq l(h) + 2l(x) \geq l(g)
\end{aligned}
$$

Case 2b. Suppose that $l(h) \leq C$ and $l(x) > 60L$. Let $v \in [x, xh^N]$ be such that $d(x,v) = (1, xh^N)_x$. As $[x, xh^N]$ is an isometric translation of $[1, h^N]$ we have that there is some $i \in \mathbb{Z}$ such that $d(v, xh^i) \leq L$. We know $l(xh^i) \geq l(x)$ by the minimality condition on $l(x)$. Pick $w \in [1, x]$ such that $d(x,w) = (1, xh^N)_x$, so that $d(v,w) \leq 2\delta$. Then

$$
\begin{aligned}
(1, xh^N)_x &= l(x) - d(1,w) \\
&\leq l(x) - (d(1,v) - 2\delta) \\
&\leq l(x) + 2\delta - (l(xh^i) - L) \\
&\leq l(x) + 2\delta + L - l(x) = 2\delta + L
\end{aligned}
$$

The condition $(x, g^N)_{xh^N} \leq 2\delta + L$ is proven similarly. Now we employ Lemma 9.2.0.84 again using $K = 2\delta + 2L$ and $K_1 = 24L + 36\delta$, so that $[1,x] \cup [x, xh^N] \cup [xh^N, g^N]$ is within the $2K = 4\delta + 4L$-neighborhood of $[1, g^N]$. Select $s_0, s_1 \in [1, g^N]$ so that $d(x, s_0), d(xh^N, s_1) \leq 4\delta + 4L$. Now

$$
\begin{aligned}
l(g^N) &= d(1, s_0) + d(s_0, s_1) + d(s_1, g^N) \\
&\geq (l(x) - 4\delta - 4L) + (l(h^N) - 8\delta - 8L) + (l(x) - 4\delta - 4L) \\
&\geq 2l(x) + l(h^N) - 32L > 2l(x) + l(h) \geq l(g)
\end{aligned}
$$

so we are done in this case as well.

□

9.3 Graph Products of n-slender Groups

We recall the definition of a graph product of groups and some machinery, then prove Theorem 9.0.0.81. Suppose $\Gamma = (V, E)$ is a graph (we allow the sets of vertices and edges to be of arbitrary cardinality but do not allow an edge to connect a vertex to itself) and to each vertex $v \in V$ we associate a group G_v. We call the G_v the generating groups. The graph product $G = \Gamma(\{G_v\}_{v\in V})$ is defined by taking the free product $*_{v\in V} G_v$ and modding out by the normal closure of the set $\{[g_{v_0}, g_{v_1}]\}_{g_{v_0}\in G_{v_0}, g_{v_1}\in G_{v_1}, \{v_0,v_1\}\in E}$. Thus free products of groups and direct sums of groups are examples of graph products of groups, with the graphs having either no edges or being complete in the respective cases.

Each G_v is a retract subgroup of G and G is generated by the elements of the generating subgroups G_v. Thus each element $g \in G$ has a representation as a word $g =_G g_0 g_1 g_2 \cdots g_{n-1}$ with each g_i in a generating group. In such a word we call each g_i a syllable. Given two generating groups G_{v_0} and G_{v_1} it is easy to see that the subgroup $\langle G_{v_0} \cup G_{v_1}\rangle \leq G$ is a retract of G and is either isomorphic to $G_{v_0} * G_{v_1}$ or $G_{v_0} \times G_{v_1}$, the first being the case if and only if $\{v_0, v_1\} \notin E$. Thus for nontrivial elements $g_0 \in G_{v_0}$ and $g_1 \in G_{v_1}$ we have that $[g_0, g_1] = 1$ if and only if $\{v_0, v_1\} \in E$.

We present some machinery found in [Gre], where graph products were first introduced. We say a word $g_0 g_1 \cdots g_{n-1}$ in elements of the generating groups is reduced if the following hold:

1. Each g_i is a nontrivial element in a generating group and g_i and g_{i+1} are in different generating groups for all $0 \leq i < n-1$

2. If $i \leq k < j$ and $[g_i, g_{i+1}] = [g_i, g_{i+2}] = \cdots = [g_i, g_k] = 1 = [g_{k+1}, g_j] = [g_{k+2}, g_j] = \cdots = [g_{j-1}, g_j]$ then g_i and g_j are in different generating groups.

We say that two reduced words w_0, w_1 are equivalent, if one can obtain w_1 from w_0 by a permutation of syllables as allowed in the group (i.e. one can permute the syllables g_i and g_{i+1} if and only if $[g_i, g_{i+1}] = 1$). Clearly the equivalence of w_0 to w_1 implies that w_0 and w_1 have the same word length and $w_0 =_G w_1$. Using $\bigcup_{v\in V} G_v$ as a generating set for G we get a length function l on G.

The following result combines the statements of Theorem 3.9 and Corollary 3.13 of [Gre]:

Lemma 9.3.0.88. Each $g \neq 1$ has a reduced word representation $g =_G g_0 g_1 \cdots g_{n-1}$ which is unique up to equivalence, with $l(g) = n$.

We give a lemma before proving Theorem 9.0.0.81.

Lemma 9.3.0.89. An n-slender by n-slender group is n-slender.

Proof. Suppose that $1 \to K \to^{\iota} G \to^{q} Q \to 1$ is a short exact sequence of groups with K and Q n-slender groups, where for simplicity we identify K with its image in G. Let $\phi : \mathbf{HEG} \to G$ be a homomorphism. By the n-slenderness of Q we see that for $q \circ \phi$ there exists an $N' \in \omega$ such that $\phi|\mathbf{HEG}^{N'}$ maps into the kernel of q. In other words, ϕ maps $\mathbf{HEG}^{N'}$ into K. As $\mathbf{HEG} \simeq \mathbf{HEG}_{N'} * \mathbf{HEG}^{N'}$ we may define a homomorphism $\phi' : \mathbf{HEG} \to G$ by letting $\phi'|\mathbf{HEG}_{N'}$ be the trivial map and $\phi'|\mathbf{HEG}^{N'} = \phi|\mathbf{HEG}^{N'}$. By n-slenderness of K there exists an $N \in \omega$, without loss of generality $N > N'$ such that $\phi'|\mathbf{HEG}^{N}$ is the trivial map. Then $\phi|\mathbf{HEG}^{N}$ is the trivial map. □

Proof. (of Theorem 9.0.0.81) Let G_v be n-slender for each $v \in V$. Let $\sigma : G \to \bigoplus_{v\in V} G_v$ be the obvious surjective map. Eda proved in [**?**] that $\bigoplus_{v\in V} G_v$ is n-slender. Thus by the previous lemma we will be done if we can show that $\ker(\sigma)$ is n-slender. We prove in fact that $\ker(\sigma)$ is u.m.

The length function l on G described above restricts to a length function on $\ker(\sigma)$. We show that for $g \in \ker(\sigma) - \{1\}$ we have $l(g^2) > l(g)$, which is sufficient to show that l is a uniformly monotone length function. Let $g \in \ker(\sigma) - \{1\}$ be given and $g =_G g_0 g_1 \cdots g_{n-1}$

be a reduced word representation as described in Lemma 9.3.0.88. We permute the syllables of the word $g_0g_1\cdots g_{n-1}$ to get a possibly nicer reduced word representation. To start, suppose that for some $0 \le i < j \le n-1$ we have that $[g_i, g_{i-1}] = [g_i, g_{i-1}] = \cdots = [g_i, g_0] = 1 = [g_j, g_{j+1}] = \cdots = [g_j, g_{n-1}]$ and $g_i = g_j^{-1}$. Then permute the syllables and relabel them so that $i = 0$ and $j = n-1$ and $g_0 = g_{n-1}^{-1}$. Perform the same process on the word $g_1g_2\cdots g_{n-2}$, moving a pair of mutual inverses to the front and rear of the word if possible. Continue this process until it is impossible to go further, so that by relabeling we get a (possibly empty, in case the process can never be performed) initial segment $g_0g_1\cdots g_{k-1}$ and a (possibly empty) terminal segment $g_{n-k}\cdots g_{n-1}$ such that for $0 \le i < k$ we have $g_i = g_{n-i-1}^{-1}$ and the process cannot be performed on the word $g_kg_{k+1}\cdots g_{n-k-1}$.

We next manipulate the word $g_kg_{k+1}\cdots g_{n-k-1}$. If there exists $k \le i < j \le n-k$ such that $[g_i, g_{i-1}] = [g_i, g_{i-2}] = \cdots = [g_i, g_k] = 1 = [g_j, g_{j+1}] = \cdots = [g_j, g_{n-k-1}]$ and both g_i and g_j are in the same generating group, then move the syllable g_i to the front and the syllable g_j to the rear of the word $g_kg_{k+1}\cdots g_{n-k-1}$ so that by relabeling we may assume $i = k$ and $j = n-k-1$. For the word $g_{k+1}g_{k+2}\cdots g_{n-k-2}$ consider whether there exist $k+1 \le i < j \le n-k-2$ such that $[g_i, g_{i-1}] = [g_i, g_{i-2}] = \cdots = [g_i, g_k] = 1 = [g_j, g_{j+1}] = \cdots = [g_j, g_{n-k-1}]$ and both g_i and g_j are in the same generating group. If so, permute the syllables of the word $g_{k+1}g_{k+2}\cdots g_{n-k-2}$ so that the syllable g_i is now in the front and the syllable g_j is at the rear. By relabeling we our modified word we may assume that $i = k+1$ and $j = n-2$. Perform the same process on the word $g_{k+2}g_{k+3}\cdots g_{n-k-3}$, and continue the process until it becomes impossible. Thus we obtain a (possibly empty, in case the process cannot be performed) initial segment $g_kg_{k+1}\cdots g_{k+p-1}$ and (possibly empty) terminal segment $g_{n-k-p}\cdots g_{n-k-1}$ of the word $g_kg_{k+1}\cdots g_{n-k-1}$ such that all elements of the set $\{g_k, \ldots, g_{k+p-1}\}$ commute with each other and for $k \le i \le k+p$ we have that g_i is in the same generating group as g_{n-i-1}. For $k \le i \le k+p-1$ let h_i be the element $g_{n-i-1}g_i$ in the generating group containing g_i. By the first process that was performed, we have that $h_i \ne 1$.

Notice that the syllables g_{k+p-1} and g_{n-k-p} cannot be side by side, since otherwise we have that all syllables of $g_k g_{k+1} \cdots g_{n-k-1}$ commute with each other and thus the word $g_0 g_1 \cdots g_{n-1}$ was not reduced. Thus necessarily $n-k-p > k+p$ and there is a nonempty word w_0 in between g_{k+p-1} and g_{n-k-p} such that for each $k \leq i \leq k+p-1$there is a syllable in w_0 which does not commute with the syllable g_i (else the word $g_0 \cdots g_{n-1}$ was not reduced). Let w_1 be the word $g_k \cdots g_{k+p-1}$, w_1' be the word $g_{n-k-p} \cdots g_{n-k-1}$, w_1'' be the word $h_k \cdots h_{k+p-1}$, and w_2 be the word $g_0 \cdots g_{k-1}$. We already have that $w_2 w_1 w_0 w_1' (w_2)^{-1}$ is a reduced word representation for g. The equalities

$$\begin{aligned} g^2 &=_G w_2 w_1 w_0 w_1' (w_2)^{-1} w_2 w_1 w_0 w_1' (w_2)^{-1} \\ &=_G w_2 w_1 w_0 w_1' w_1 w_0 w_1' (w_2)^{-1} \\ &=_G w_2 w_1 w_0 w_1'' w_0 w_1' (w_2)^{-1} \end{aligned}$$

are clear.

We claim that the word $w_2 w_1 w_0 w_1'' w_0 w_1' (w_2)^{-1}$ is reduced. Each of the words $w_0, w_1, w_1', w_1'', w_2, (w_2)^{-1}$ is reduced. The words $w_0, w_1, w_1', w_2, (w_2)^{-1}$ are reduced since they are subwords of a reduced word. The word w_1'' is reduced since the h_i constituting w_1'' are nontrivial, commute with each other, are in the same generating groups as the syllables of w_1 (or w_1'), and w_1 is reduced. No syllable of the word w_2 can be permuted to be next to a syllable of the same generating group in the word w_1, the first occurence of w_0, or w_1'' since the original word $w_2 w_1 w_0 w_1' (w_2)^{-1}$ was reduced and the h_i syllables that constitute the word w_1'' are from precisely the same generating groups as those syllables that constitute w_1 and w_1'. Also, no syllable of the word w_2 can be permuted next to a syllable of the same generating group in the second occurence of w_0 since the same is true of the first occurence of w_0. No syllable of w_2 can be permuted next to a syllable of the same generating group in the words w_1' and $(w_2)^{-1}$ since the original word $w_2 w_1 w_0 w_1' (w_2)^{-1}$ was reduced. That no syllable in w_1 can be permuted next to a syllable of the same generating group in any of the words to the right of w_1 follows similar lines. No syllable in the first occurence of w_0 can be permuted next to a syllable of the same generating group in w_1' since the subword $w_0 w_1$

of the word $w_2 w_1 w_0 w_1'(w_2)^{-1}$ is reduced. If the syllable g_q of the first occurence of w_0 can be permuted next to a syllable $g_{q'}$ in the second occurence of w_0, where $g_q, g_{q'} \in G_v$, then g_q must commute with all the syllables of w_1 (and of w_1' and w_1''). By the second process, which was performed on the word $g_k \cdots g_{n-k-1}$, it must be that $g_q = g_{q'}$ and thus in fact g_q commutes with all syllables in the word $g_k \cdots g_{n-k-1}$. We have that all other syllables of the word $g_k \cdots g_{n-k-1}$ are not in G_v (since $g_k \cdots g_{n-k-1}$ is reduced) and so $\sigma(g_k \cdots g_{n-k-1}) = g_q =\neq 1$ is conjugate to $\sigma(g) = 1$, a contradiction. The remaining cases are straightforward to check and follow the same lines.

Thus we have that

$$l(g^2) = 2l(w_2) + 3l(w_1) + 2l(w_0) > 2l(w_2) + 2l(w_1) + l(w_0) = l(g)$$

since, although $l(w_2)$ and/or $l(w_1)$ might be zero, we demonstrated that $l(w_0)$ is not zero.

□

9.4 Thompson's Group

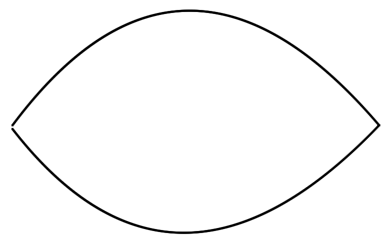
The group F of R. Thompson is a well-known finitely presented group which satisfies many curious properties, and about which many open problems remain. The group has no infinitely divisible elements and has the unique extraction of roots property (i.e. if $g^n = h^n$ and $n > 0$ then $g = h$). Thus one might ask the following:

Question. Is Thompson's group F n-slender?

We show that the natural length function defined on diagram groups is not universally monotone. This does not rule out the possibility of a universally monotone length function, nor the n-slenderness of F.

We begin with a discussion of diagram groups (essentially following [GS]), of which F is an example. Start with an alphabet X. Given two words u, v in the elements of X a **cell** $(u \to v)$ is a directed planar graph consisting of exactly two directed paths with the same

Figure 9.1: The cell $(u \to v)$

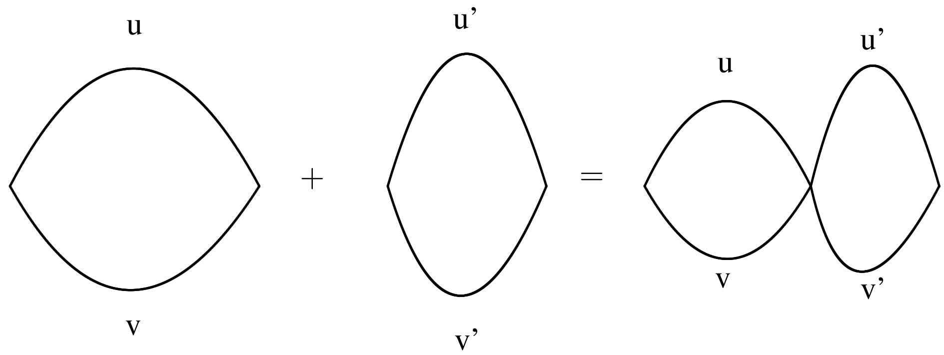


initial and terminal vertices and which share no other vertices, which share no edges, and with the two paths labeled by the words u and v. The path labeled by u is called the top path and that labeled by v is called the bottom path. A **trivial diagram** is a single directed path labeled by a word in the elements of X; if that word is u we denote the trivial diagram by $\varepsilon(u)$. For a trivial diagram we say that the path defining the trivial diagram is both the top and the bottom path.

We declare that diagrams are defined only up to planar isotopy and that cells and trivial diagrams are diagrams. Thus the diagrams that we have so far have an initial and a terminal vertex, that is, any two maximal paths begin and end at the same vertices. Also, each diagram has a top and a bottom path. Given a diagram Δ we let $\iota(\Delta)$ and $\tau(\Delta)$ denote the initial and terminal vertices respectively. All diagrams will similarly have an initial and terminal vertex as well as a top and a bottom path. In addition to the cells and the trivial diagrams, we close the collection of diagrams under the following three operations:

1. Addition. Given two diagrams Δ_0 and Δ_1 we let $\Delta_0 + \Delta_1$ be the planar graph created by identifying $\tau(\Delta_0)$ with $\iota(\Delta_1)$. Thus the top path of $\Delta_0 + \Delta_1$ is the concatenation of the top paths of Δ_0 and Δ_1, and similarly for the bottom paths. Also we have $\iota(\Delta_0 + \Delta_1) = \iota(\Delta_0)$ and $\tau(\Delta_0 + \Delta_1) = \tau(\Delta_1)$. The operation $+$ is clearly associative. If $u = x_0 x_1 \cdots x_k$ then we may write $\varepsilon(u) = \varepsilon(x_0) + \varepsilon(x_1) + \cdots + \varepsilon(x_k)$.

Figure 9.2: The sum of diagrams

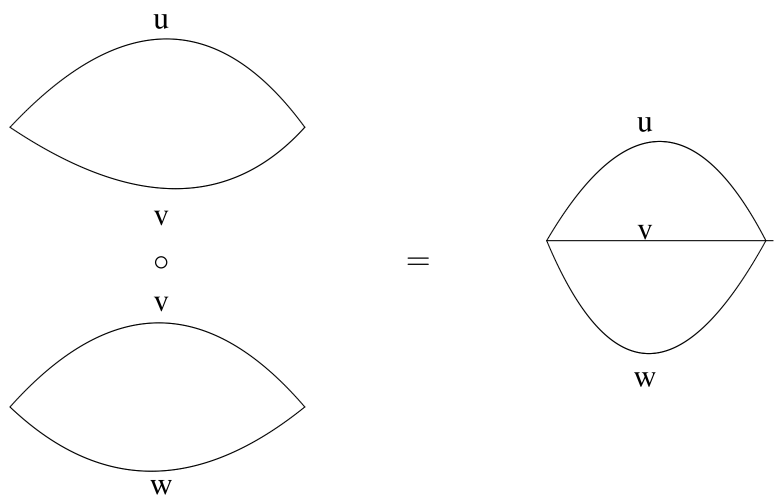


2. Multiplication. Given diagrams Δ_0 and Δ_1 such that the bottom path of Δ_0 has the same label as the top path of Δ_1 we let $\Delta_0 \circ \Delta_1$ be the planar graph obtained by identifying the bottom path of Δ_0 with the top of Δ_1. Thus under the identification we have $\iota(\Delta_0 \circ \Delta_1) = \iota(\Delta_0) = \iota(\Delta_1)$ (and similarly for τ), the top of $\Delta_0 \circ \Delta_1$ is the top of Δ_0 and the bottom of $\Delta_0 \circ \Delta_1$ is the bottom of Δ_1.

3. Inversion. Given a diagram Δ we define Δ^{-1} to be the diagram obtained by flipping the diagram Δ about a horizontal line, so that the top path becomes the bottom path and vice versa.

By definition, the class of diagrams (over X) is built out of cells and trivial diagrams using the above operations. If we wish, we can restrict our attention to those diagrams which are built only from trivial diagrams and cells in a set P and the three operations above and let $D(P)$ denote this class. If a diagram Δ has two cells such that the top of the second is identified with the bottom of the first, and the first and second cells are inverses of each other, then we call this pair of cells a **dipole**. Notice that if one eliminates the two cells from the diagram and identifies the top of the first cell with the bottom of the second, then we have a new diagram Δ' and say that Δ and Δ' are equivalent. This induces an equivalence relation on $D(P)$ by making the relation reflexive, symmetric and transitive.

Figure 9.3: The product of diagrams



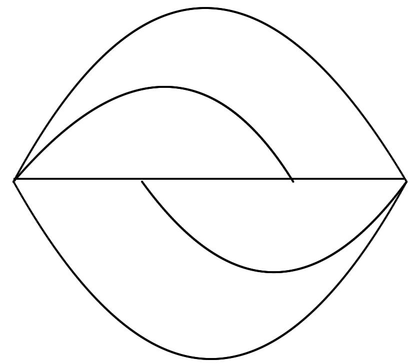
We say Δ is **reduced** in case Δ has no dipoles, and note that every diagram is equivalent to a unique reduced diagram (see [GS]).

Given a word u and a collection of cells P we let $D(P,u)$ denote the collection of reduced diagrams built by using cells in P, trivial diagrams, and the above three operations and whose top and bottom paths are labeled by the word u. This forms a group by letting the binary operation be given by $\Delta_0\Delta_1 = \Delta$, where Δ is the reduced diagram equivalent to $\Delta_0 \circ \Delta_1$ (see [GS] for a proof). The identity element and the inverse operation are clear.

Now we state a characterization of F as a diagram group. We shall use this as our working definition of F, and the isomorphism of F with the group we define is given in [GS]. Letting $X = \{x\}$ and $P = \{(x^2 \to x)\}$ it is shown that $D(P,x) \simeq F$. Given a reduced diagram $\Delta \in F$ we let $l(\Delta)$ be the number of cells in Δ. It is easy to check that l is a length function.

We now show that l is not universally monotone. Letting $n > 1$ be given we give an example of a diagram Δ such that $l(\Delta) > l(\Delta^n)$. Since the alphabet X includes only the letter x, we may assume that each arc in our diagrams is labeled by the letter x as read from left to right. Let ρ denote the $(1,2)$ diagram. Let θ be the reduced diagram with 4 cells as

Figure 9.4: The diagram θ



pictured.

For $n \in \mathbb{Z}$ let θ^n denote the reduced diagram associated with multiplying θ with itself n times. The diagrams for θ^2 and respectively for θ^m for $m \geq 1$ are straightforward to compute and are pictured in Figure 9.5, having 6 and $2+2m$ cells, resp.

Define Δ_n to be $\underbrace{\theta^{-1}+\theta^{-1}\cdots+\theta^{-1}}_{n-1\text{times}}+\theta^{n-1}$. Select $k \in \omega$ large enough that $2n^2 < 2k+2$. Let $k_1, k_2 > n+1$ be such that $1+k_1+nk+k_2 = 2^m$ for some $m \in \omega$. Let Ψ be the diagram pictured in Figure 9.6 with top path of length 1 and bottom path of length 2^m+2.

Figure 9.5: Powers of θ

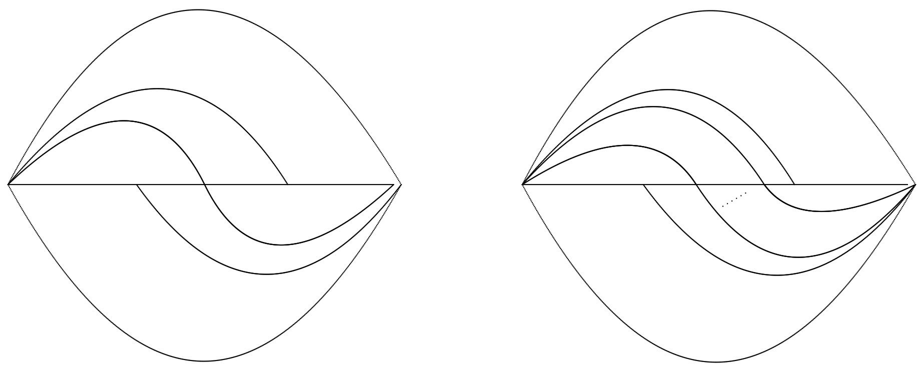


Figure 9.6: The diagram Ψ

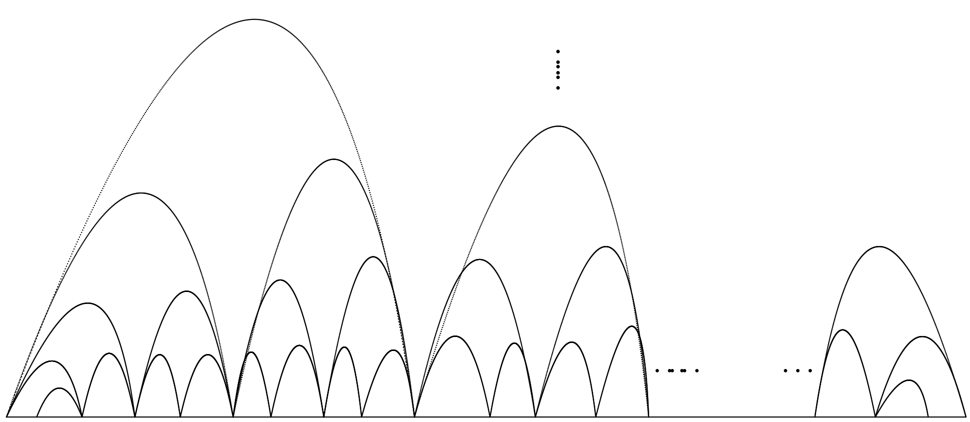
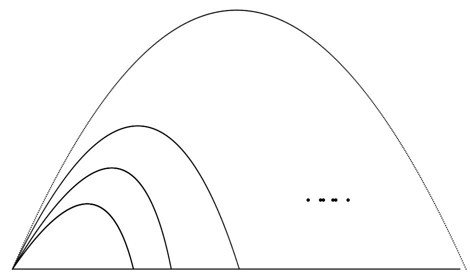


Figure 9.7: The diagram Γ_1



Let χ be the diagram $\rho + \varepsilon k_1 + \underbrace{\Delta_n + \Delta_n + \cdots + \Delta_n}_{k\text{times}} + \varepsilon(k_2) + \rho^{-1}$ and finally let $\Delta = \Psi \circ \chi \circ \Psi^{-1}$. It is straightforward to check that Δ is a reduced diagram with top and bottom path of length 1, so Δ is in the Thompson group F. To compute $l(\Delta)$ we note that each Δ_n has $2+6(n-1)$ cells in it, so that χ has $1+k(2+6(n-1))+1$ cells in it. The diagram Ψ contains 2^m+1 cells, so that $l(\Delta) = 1+k(2+6(n-1))+1+2(2^m+1) = 4-4k+6kn+2^{m+1}$. Clearly Δ^n is equivalent to $\Psi \circ \chi^n \circ \Psi$. We count the number of cells in $\Psi \circ \chi^n \circ \Psi$ to obtain an upper bound on $l(\Delta^n)$. First of all we have $2(2^m+1)$ cells in Ψ and Ψ^{-1} combined. The reduced diagram χ^n is given by the equality

$$\chi^n =$$
$$\Gamma_1 + \varepsilon(k_1-n) + \theta^{-1} + \theta^{-2} + \cdots + \theta^{1-n} + \varepsilon((k-1)n+2) + \theta + \theta^2 + \cdots + \theta^{n-1} + \varepsilon(k_2) + \Gamma_2$$

where Γ_1 is the diagram with n cells pictured below and Γ_2 is the rotation of Γ_1 by 180 degrees.

Thus χ^n has $n+(4+6+\cdots+(2+2(n-1)))+(4+6+\cdots+(2+2(n-1)))+n = 2n+2(n^2-n) = 2n^2$ cells. Hence $l(\Delta^n) \leq 2n^2+2(2^m+1) < 4-4k+6kn+2^{m+1} = l(\Delta)$ by our choice of k, as desired.

9.5 A Family of Non-Examples

Although the uniformly monotone condition gives a very nice sufficient condition for n-slenderness, it is not a necessary condition. We give an example of a very basic family of groups that are n-slender but which are not uniformly monotone.

The countable slender abelian groups have the following criterion (see [Fu]):

Lemma 9.5.0.90. A countable abelian group A is slender if and only if A is torsion-free and reduced (i.e. $\bigcap_{m=1}^{\infty} mA = \bigcap_{m=1}^{\infty} \{ma : a \in A\}$ is trivial.)

Theorem 9.5.0.91. *The groups $BS(1,n)$ are n-slender but not u.m. for $n > 1$. Moreover, these groups are HNN extensions of the u.m. group $\mathbb{Z}$.*

Proof. Recall that $BS(1,n) = \langle a, b | bab^{-1} = a^n \rangle$. The retraction map $q : BS(1,n) \to \langle b \rangle$ defined by $b \mapsto b$ and $a \mapsto 1$ has kernel which is easily seen to be isomorphic to the additive group of the n-adic rational numbers, $\mathbb{Z}[\frac{1}{n}]$.

The group $\mathbb{Z}[\frac{1}{n}]$ is clearly torsion-free and countable. To see that $\mathbb{Z}[\frac{1}{n}]$ is reduced we notice that an element $\frac{r}{n^k}$ with $r \in \mathbb{Z}$ is not a p-th power for any p that divides neither n nor r. Thus $\mathbb{Z}[\frac{1}{n}]$ is slender and therefore n-slender, and since $\mathbb{Z} \simeq \langle b \rangle$ is also n-slender we know that $BS(1,n)$ is n-slender as an n-slender by n-slender group (Lemma 9.3.0.89).

However we know that $BS(1,n)$ cannot be u.m. since it contains a subgroup isomorphic to $\mathbb{Z}[\frac{1}{n}]$. Each element of $\mathbb{Z}[\frac{1}{n}]$ is infinitely divisible, and a u.m. group cannot have any infinitely divisible elements besides the identity element.

□

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