MULTIPLICATION OPERATORS AND M-BEREZIN TRANSFORMS

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Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

May, 2006

Nashville, Tennessee

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ACKNOWLEDGEMENTS

CHAPTER I

INTRODUCTION

In classical mechanics, according to Newton's law, the position and velocity specify the state of a particle and the value of every observable (function) is completely determined by the state. In quantum mechanics, for a given state, observables (operators) have only probability distributions of values. Quantization problem is the problem of setting up a correspondence between classical observables (functions) and quantum observables (operators) such that the properties of the classical observables are reflected as much as possible in their quantum counterparts in a way consistent with the probabilistic interpretation of quantum observables. Operator theory was developed to provide a mathematics foundation for quantum mechanics.

Operator theory is about linear transformations between spaces. The realization of operators depends on the properties of the underlying spaces. If the underlying spaces are finite dimensional, then operators, under certain basis, are just matrices with finitely many rows and finitely many columns. Solving a system of linear equations has to involve the spectral theory of matrices. To solve certain differential or integral equations arising from physics, one has to deal with the corresponding theory of operators between spaces with infinitely many dimensions. Operators between infinite dimensional spaces can be viewed as matrices with infinitely many rows and infinitely many columns. Then the theory of matrices (with finitely many rows and columns) becomes a special case of operator theory.

Operator theory on function spaces studies operators on various function spaces such as Hardy spaces, Bergman spaces. The theory of Toeplitz operators originated in the 1910's, which had a development parallel to the theory of Wiener-Hopf operators. Toeplitz operators are of importance in applied mathematics such as system theory, and stationary stochastic processes [29]. They are also of importance in Quantum Mechanics, such as Weyl Quantization, Phase Operators [25]. They are, however, attractive to mathematicians as fascinating examples of the fruitful interplay between operator theory, operator algebras,

function theory, harmonic analysis, and complex analysis [22] .

One basic problem (the so called invariant subspace problem) in operator theory concerns the existence of a nontrivial invariant subspace for a given operator (bounded linear transformation). That is, if T is an operator on a Hilbert space H, does there exist a subspace (closed linear manifold) M of H different from both 0 and H such that M is invariant under T (That is, $TM \subseteq M$) [31]. For a linear transformation on a finite dimensional Hilbert space with dimension at least two, the existence of an eigenvalue and corresponding eigenvectors assures that there always exists a nontrivial invariant subspace. As pointed out by Halmos [31], the existence of eigenvalues is a deep property, derived by techniques far from the spirit of linear algebra. An eigenvalue, a geometric concept, is the same as a zero of the characteristic polynomial, an algebraic concept, and the existence of such zeros is guaranteed by the fundamental theorem of algebra, an analytic tool. For an operator on a separable infinite dimensional Hilbert space, the invariant subspace problem is still open. What can be achieved for Toeplitz operators?

The Bergman space L_a^2 is the Hilbert space consisting of analytic functions on the unit disk which are also square integrable with respect to the area measure. The Hardy space is the Hilbert space consisting of analytic functions on the unit disk with square integrable boundary values on the unit circle . If a Toeplitz operator is induced by a bounded analytic function, called the symbol of the Toeplitz operator, then the Toeplitz operator is just a multiplication operator (multiplication by the bounded analytic function). If the symbol is the position function, z, then the corresponding Toeplitz operator is a (unilateral) shift when the underlying space is the Hardy space. Whereas it is a (unilateral) weighted shift, called the Bergman shift, when the underlying space is the Bergman space.

In the case of the Hardy space , the structure of the invariant subspaces of the shift operator has been completely described by Beurling's famous theorem [13][22] in terms of inner functions. In the case of the Bergman space, although a Beurling-type theorem has been obtained [2], the structure of invariant subspaces of the Bergman shift is still too complicated to be understood completely. In fact, the existence of a nontrivial invariant subspace of any given operator acting on any given separable infinite dimensional Hilbert space is equivalent to the following: for any given two invariant subspaces of the Bergman

shift, one properly containing the other, there exists a third invariant subspace properly between these given two [9] [34]. It is natural to ask what can be said about the structure of reducing subspaces.

A reducing subspace of an operator is a subspace which is invariant under both the operator and its adjoint. The set of all reducing subspaces is called the reducing lattice of the operator. One way to characterize reducing subspaces of an operator is to determine the projections in the set of commutants of the operator which is the set of all operators commuting with the original operator. For the Hardy space, a lot of work has been done to determine the lattice of reducing subspaces of an analytic multiplication operator. For instance, Cowen proved that under some conditions the set of commutants of an analytic multiplication operator is exactly the same as the set of commutants of a multiplication operator induced by a finite Blaschke product [19]. A finite Blaschke product is a product of finitely many Blaschke factors. A Blaschke factor is a linear fractional transform (a conformal automorphism) of the unit disk to itself. The number of Blaschke factors in a finite Blaschke product is called the order of the Blaschke product. So both the shift and the Bergman shift are multiplication operators induced by Blaschke products of order one and it is not hard to prove that both of them have no nontrivial (other than 0 and the whole space) reducing subspaces.

A reducing subspace M is called minimal if the only reducing subspaces contained in M are M and 0. In the Hardy space, the multiplication operator induced by a finite Blaschke product of order greater than one has infinitely many minimal reducing subspaces. However, in the Bergman space, it was shown in [51], [58] that a multiplication operator induced by a Blaschke product of order two has only two nontrivial reducing subspaces. Zhu conjectured that, in the Bergman space, a multiplication operator induced by a finite Blaschke product of order n has exactly n nontrivial minimal reducing subspaces [58].

In Chapter II we study the structure of the reducing lattices of multiplication operators induced by finite Blaschke products. We will give complete descriptions of the reducing lattices of multiplication operators induced by Blaschke products of order three or order four. Our results give a negative answer to Zhu's conjecture.

The main idea here as in [32] and [53] is to realize (unitarily transform) the operator

M_φ as a Toeplitz type operator (a multiplication followed by a projection) acting on a nice subspace of $H^2(\mathbb{T}^2)$, the Hardy space of the torus. On $H^2(\mathbb{T}^2)$, the realization of M_φ has two extensions which are multiplication operators with finite Blaschke products as symbols and are doubly commuting pure isometries. The properties of these two isometries tell us the properties for M_φ. It should be pointed out that this basic idea originated from Douglas and Paulsen's work [23] and was further developed in [53].

Another basic problem about operators on function spaces is how to characterize the corresponding compact operators. Axler and Zheng gave a characterization about the compactness of a Toeplitz operator on the Bergman space of the unit disk in terms of the Berezin transform of the operator. In fact, they [5] proved that if an operator is a finite sum of finite product of Toeplitz operators, then the operator is compact if and only if its Berezin transform vanishes on the boundary of the unit disk. So their theorem raised an open question: does the characterization hold for operators in the Toeplitz algebra?

By introducing m-Berezin transform of a function, Ahern, Flore, Rudin [1] was able to prove that on the unit disk if the Berezin transform of a function is itself, then the function is harmonic. Suarez [49] [50] studied the m-Berezin transform of operators on the Bergman space of the unit disk, and proved that Axler-Zheng's theorem holds for a special kind of operators in the Toeplitz algebra, namely radial operators which commute with rotation operators.

In Chapter III we study the m-Berezin transform of operators acting on the Bergman spaces of the unit balls in higher complex spaces. We show that in the case of the unit ball Axler-Zheng's theorem still holds for radial operators in the Toeplitz algebra.

I.1   Main Results of Chapter II

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ be the boundary of $\mathbb{D}$, the unit circle. Let dA denote the Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in the space $L^2(\mathbb{D}, dA)$ of square integrable functions on $\mathbb{D}$. Since the nonnegative powers $\{z^n\}_{n=0}^\infty$ span the Bergman space, $\{\sqrt{n+1}z^n\}_{n=0}^\infty$ forms an

orthonormal basis for the Bergman space L_a^2.

If φ is a bounded analytic function on $\mathbb{D}$, the multiplication operator induced by φ and denoted by M_φ, is defined by

$$M_\varphi h = \varphi h$$

for any $h \in L_a^2$. φ is also called the symbol of M_φ.

The multiplication operator M_z with symbol z, the coordinate function, is called the Bergman shift. Indeed with respect to the standard orthonormal basis $\{e_n = \sqrt{n+1}z^n\}_{n=0}^\infty$,

$$M_z e_n = \sqrt{\frac{n+1}{n+2}} e_{n+1}.$$

That is, M_z is indeed a weighted shift with weights $\{\sqrt{\frac{n+1}{n+2}}\}_{n=0}^\infty$.

A subspace (a subspace always means a closed subspace) M is called an invariant subspace for an operator T if $TM \subseteq M$. The set of all invariant subspaces of T is called the lattice of T and denoted by $LatT$. About the structure of $LatM_z$, Aleman, Richter and Sundberg proved the following Beurling-type theorem.

Aleman-Richter-Sundberg Theorem [2]. *Let M be an invariant subspace of M_z acting on L_a^2. Then M is generated by $M \ominus M_z M$.*

On the other hand, Bercovici,Foias and Pearcy proved the following universal property of M_z.

Bercovici-Foias-Pearcy Theorem [9]. *For any strict contraction S on a separable Hilbert space H, there always exist a pair of invariant subspaces of M_z, M and N, such that S is unitarily equivalent to $P_{M \ominus N} M_z|_{M \ominus N}$, where $P_{M \ominus N}$ stands for the orthogonal projection of L_a^2 onto $M \ominus N$.*

Bercovici-Foias-Pearcy Theorem indicates that the structure of $LatM_z$ is very complicated, and furthermore implies that the positive answer to the invariant subspace problem for an operator acting on a separable Hilbert space is equivalent to whether $LatM_z$ is saturated, i.e., for any $M, N \in \text{Lat}M_z$, with $M \supset N$ and $\dim(M \ominus N) = \infty$, whether there exists some $\Omega \in LatM_z$ such that M contains properly Ω and Ω contains properly N. A natural question is what can be said about reducing subspaces.

A reducing subspace M for an operator T acting on a Hilbert space H is a subspace M of H such that $TM \subseteq M$ and $T^*M \subseteq M$ where T^* stands for the adjoint of T. The set of all reducing subspaces of T is called the reducing lattice of T. A reducing subspace M of T is called minimal if M and 0 are the only reducing subspaces contained in M.

A Blaschke factor is a Möbius function, or a Möbius transform of the unit disk to itself:

$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$$

for some $\alpha \in \mathbb{D}$. A finite Blaschke product is a product of finitely many Blaschke factors. The number of Blaschke factors in a finite Blaschke product is called the order of the Blaschke product. It was shown in [51] and [58] that for a Blaschke product φ of order two, the multiplication operator M_φ has only two nontrivial minimal reducing subspaces. Then it natural to ask about the reducing lattice of M_φ for a general finite Blaschke product and Zhu formulated the following conjecture.

Zhu's Conjeture [58]. *For a finite Blaschke product φ of order N, the reducing lattice of the operator M_φ acting on the Begman space is generated by N elements.*

In other words, Zhu conjectured that M_φ has exactly N nontrivial minimal reducing subspaces. However we will show that Zhu's conjecture is not true in general (see Section II.8).

For a finite Blaschke product φ, after composed with a Möbius transform from the right and a Möbius transform from the left, it can always has the following form (see the proof of Theorem 2 in Section II.8):

$$\varphi(z) = z^{n_0+1} \prod_{k=1}^{K} \left(\frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \right)^{n_k+1}$$

with $n_0 \geq 1$. The above form of φ might be up to multiplication of a constant with modulus one and in this chapter we always omit that constant since multiplying by a constant does not change the involved reducing lattice. Moreover the structure of the reducing lattice does not change after composition with a Möbius transform from the right or from the left (see Section II.8). So we can assume that φ have the above form without loss of generality.

Now we state our first main result as the following theorem.

Theorem 1. *Let* $\varphi = z^{n_0+1}\varphi_{\alpha_1}^{n_1+1}\cdots\varphi_{\alpha_K}^{n_K+1}$ *be a Blaschke product of order N with $n_0 \geq 1$, $K \geq 1$ and $\alpha_k \neq 0$ for $k = 1, ..., K$. Then M_φ cannot have N nontrivial reducing subspaces $\{M_i\}_{i=0}^{N-1}$ satisfying $L_a^2 = \bigoplus_{i=0}^{N-1} M_i$ and $M_i \perp M_j$ whenever $i \neq j$.*

For a holomorphic function h, we say that c is a critical point of h if its derivative vanishes at c. A finite Blaschke product, φ, of order N is an N to 1 conformal map of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$. Bochner's theorem [56], [57] says that φ has exactly $N-1$ critical points in the unit disk $\mathbb{D}$ and none on the unit circle. Let $\mathcal{C}$ denote the set of the critical points of φ in $\mathbb{D}$ and

$$\mathcal{F} = \varphi^{-1} \circ \varphi(\mathcal{C}).$$

Then $\mathcal{F}$ is a finite set, and $\varphi^{-1} \circ \varphi$ is an N-branched analytic function defined in $\mathbb{D}/\mathcal{F}$ and can be analytically continued to every point in $\mathbb{D}/\mathcal{F}$. The Riemann surface for $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$ is an N-sheeted cover of $\mathbb{D}$ with at most $N(N-1)$ branch points, and it is not connected if $N \geq 2$. In terms of the Riemann surface of $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$, we can state our another two main results as follows.

Theorem 2. *Let φ be a Blaschke product of order three. Then the number of nontrivial minimal reducing subspaces of M_φ equals the number of connected components of the Riemann surface of $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$.*

Theorem 3. *Let φ be a Blaschke product of order four. Then the number of nontrivial minimal reducing subspaces of M_φ equals the number of connected components of the Riemann surface of $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$.*

That we state the results in the above two theorems is because they need different detailed treatments. In fact the proof of Theorem 3 is much longer than that of Theorem 2. We strongly believe that there should be some more general results along this line.

I.2 Main Results of Chapter III

Let B denote the unit ball in n-dimensional complex space $\mathbb{C}^n$ and dz be normalized Lebesgue volume measure on B. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let $\langle z, w \rangle = \sum_{i=1}^n z_i \overline{w}_i$ and

$|z|^2 = \langle z, z \rangle$. The Bergman space of the ball, $L_a^2(B)$, is the space of analytic functions h on B which are square-integrable with respect to the normalized Lebesgue volume measure, dz. For $z \in B$, let P_z be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[z]$ generated by z and let $Q_z = I - P_z$. Then the map

$$\varphi_z(w) = \frac{z - P_z(w) - (1 - |z|^2)^{1/2}Q_z(w)}{1 - \langle w, z \rangle}$$

is the automorphism of B that interchanges 0 and z. The pseudo-hyperbolic metric on B is defined as $\rho(z, w) = |\varphi_z(w)|$.

The reproducing kernel in $L_a^2(B)$ is given by

$$K_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1}},$$

for $z, w \in B$ and the normalized reproducing kernel k_z is $K_z(w)/\|K_z(\cdot)\|_2$. That is, $\langle h, K_z \rangle = h(z)$, for every $h \in L_a^2(B)$ and $z \in B$. One fundamental property of the reproducing kernel $K_z(w)$ is

$$K_z(w) = \overline{k_\lambda(z)}K_{\varphi_\lambda(z)}(\varphi_\lambda(w))k_\lambda(w). \tag{I.1}$$

Given $f \in L^\infty(B, dz)$, the Toeplitz operator T_f is defined by $T_f h = P(fh)$ where P denotes the orthogonal projection of $L^2(B, dz)$ onto $L_a^2(B)$.

Let $\mathfrak{L}(L_a^2(B))$ be the algebra of bounded operators on $L_a^2(B)$. The Toeplitz algebra $\mathfrak{T}(L^\infty)$ is the closed subalgebra generated by $\{T_f : f \in L^\infty(B)\}$.

For $z \in B$, let U_z be the unitary operator given by

$$U_z f = (f \circ \varphi_z) \cdot J\varphi_z$$

where $J\varphi_z = (-1)^n k_z$. For $S \in \mathfrak{L}(L_a^2)$, set

$$S_z = U_z S U_z.$$

Observe that U_z is a selfadjoint unitary operator on $L^2(B, dz)$ and $L_a^2(B, dz)$, $U_z T_f U_z = T_{f \circ \varphi_z}$ for every $f \in L^\infty(B, dz)$.

Let T denote the class of trace operators on $L_a^2(B)$. For $T \in T$, we will denote the trace of T by $tr[T]$ and let $\|T\|_{C_1}$ denote the C_1 norm of T given by ([28])

$$\|T\|_{C_1} = tr[\sqrt{T^* T}].$$

Suppose f and g are in $L_a^2(B)$. Consider the operator $f \otimes g$ on $L_a^2(B)$ defined by

$$(f \otimes g)h = \langle h, g \rangle f,$$

for $h \in L_a^2(B)$. It is easy to prove that $f \otimes g$ is in T and with norm equal to $\|f \otimes g\|_{C_1} = \|f\|_2 \|g\|_2$ and

$$tr[f \otimes g] = \langle f, g \rangle.$$

For a nonnegative integer m, the m-Berezin transform of an operator $S \in \mathfrak{L}(L_a^2(B))$ is defined by

$$
\begin{aligned}
B_m S(z) &= C_n^{m+n} tr \left[S_z \left(\sum_{|k|=0}^{m} C_{m,k} \frac{n! k!}{(n+|k|)!} \frac{u^k}{\|u^k\|} \otimes \frac{u^k}{\|u^k\|} \right) \right] \qquad \text{(I.2)} \\
&= C_n^{m+n} tr \left[S_z \left(\sum_{|k|=0}^{m} C_{m,k} u^k \otimes u^k \right) \right]
\end{aligned}
$$

where $k = (k_1, \cdots, k_n) \in N^n$, N is the set of nonnegative integers, $|k| = \sum_{i=0}^{n} k_i$, $u^k = u_1^{k_1} \cdots u_n^{k_n}$, $k! = k_1! \cdots k_n!$,

$$C_n^{m+n} = \binom{m+n}{n} \quad \text{and} \quad C_{m,k} = C_{|k|}^m (-1)^{|k|} \frac{|k|!}{k_1! \cdots k_n!}.$$

Clearly, $B_m : \mathfrak{L}(L_a^2(B)) \to L^\infty(B, dz)$ is a bounded linear operator. We will obtain its norm in Section III.1 (see Theorem 59).

Given $f \in L^\infty(B, dz)$, define

$$B_m(f)(z) = B_m(T_f)(z).$$

$B_m(f)(z)$ equals the nice formula in [1]:

$$B_m(f)(z) = \int_B f \circ \varphi_z(u) d\nu_m(u),$$

for $z \in B$ where $d\nu_m(u) = C_n^{m+n}(1 - |u|^2)^m du$.

The Berezin transform of an operator S which is $B_0(S)$ by our notation was first introduced by Berezin in [10]. Because the Berezin transform encodes operator-theoretic information in function-theory in a striking but somewhat impenetrable way, the Berezin transform $B_0(S)$ has found useful applications in studying operators of "function-theoretic significance" on function spaces ([4], [5], [8], [11], [12], [24], and [46]). Suarez [49] introduced m-Berezin transforms of bounded operators on the Bergman space of the unit disk. We will show that our m-Berezin transform coincides with the one defined in [49] on the unit disk D by means of an integral representation of m-Berezin transform. The integral representation shows that many useful properties of the m-Berezin transforms inherit from the identity (I.1) of the reproducing kernel. On the unit ball, some useful properties of the m-Berezin transforms of functions were obtained by Ahern, Flores and Rudin [1]. Recently, Coburn [18] proved that $B_0(S)$ is Lipschitz with respect to the pseudo-hyperbolic distance $\rho(z, w)$. We will show that $B_m S(z)$ is Lipschitz with respect to pseudo-hyperbolic distance $\rho(z, w)$. We will show that the m-Berezin transforms B_m are invariant under the Möbius transform,

$$B_m(S_z) = (B_m S) \circ \varphi_z, \tag{I.3}$$

and commuting with each other,

$$B_j(B_m S)(z) = B_m(B_j S)(z) \tag{I.4}$$

for any nonnegative integers j and m. Properties (I.3) and (I.4) were obtained for $S = T_f$

in [1] and for operators S on the Bergman space of the unit disk [49].

On the unit disk, Axler and Zheng [4] showed that if the operator S equals the finite sum of finite products of Toeplitz operators with bounded symbols then S is compact if and only if $B_0(S)(z) \to 0$ as $z \to \partial D$. Englis extended this result to the unit ball even the bounded symmetric domains [24]. But the problem remains open whether the result is true if S is in the Toeplitz algebra. Recently, Suarez [50] solved the problem for radial operator S on the unit disk via the m-Berezin transform.

Using the m-Berezin transform, we will show that for a radial operator S in the Toeplitz algebra on the unit ball, S is compact if and only if $B_0 S(z) \to 0$ as $|z| \to 1$.

Let $S \in \mathfrak{T}(L^\infty)$ be a radial operator. Then S is compact if and only if $B_0 S \equiv 0$ on ∂B.

MULTIPLICATION OPERATORS

II.1 Outlines

In this section we introduce the spaces where we do most of the calculations and give the outlines of this chapter. We start with more notations.

Recall that $\mathbb{T}$ is the unit circle in the complex plane. The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$. Let $d\sigma$ be the rotation invariant Lebesgue measure on $\mathbb{T}^2$, normalized so that the measure of $\mathbb{T}^2$ equals 1. The Hardy space $H^2(\mathbb{T}^2)$ is the subspace of $L^2(\mathbb{T}^2, \, d\sigma)$, each function in $H^2(\mathbb{T}^2)$ can be identified with the boundary value of the function holomorphic in the bidisk $\mathbb{D}^2$ with the square summable Fourier coefficients. $H^2(\mathbb{T}^2)$ can also be viewed as the tensor product of $H^2(\mathbb{T})$ with itself. We often use $\{z^i w^j\}_{i=0,j=0}^{\infty}$ as a orthonormal basis of $H^2(\mathbb{T}^2)$. Let P be the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto $H^2(\mathbb{T}^2)$. The Toeplitz operator on $H^2(\mathbb{T}^2)$ with symbol f in $L^\infty(\mathbb{T}^2, d\sigma)$ is defined by

$$T_f(h) = P(fh),$$

for $h \in H^2(\mathbb{T}^2)$. Clearly, T_z and T_w are a pair of doubly commuting pure isometries on $H^2(\mathbb{T}^2)$. For each integer $n \geq 0$, let

$$p_n = p_n(z, w) = \sum_{i=0}^{n} z^i w^{n-i}.$$

Let $\mathcal{H}$ be the closed subspace of $H^2(\mathbb{T}^2)$ spanned by $\{p_n\}_{n=0}^{\infty}$. Then

$$H^2(\mathbb{T}^2) = \mathcal{H} \oplus cl\{(z - w)H^2(\mathbb{T}^2)\}.$$

Let $P_{\mathcal{H}}$ be the orthogonal projection from $L^2(\mathbb{T}^2, \, d\sigma)$ onto $\mathcal{H}$. It is easy to check that

$$P_{\mathcal{H}} T_z|_{\mathcal{H}} = P_{\mathcal{H}} T_w|_{\mathcal{H}}.$$

We always use $\boldsymbol{\mathcal{B}}$ to denote the operator above. It was shown explicitly in [53] and implicitly in [23] that $\boldsymbol{\mathcal{B}}$ is unitarily equivalent to the Bergman shift M_z on the Bergman space L_a^2 via the following unitary operator $U : L_a^2(\mathbb{D}) \to \boldsymbol{\mathcal{H}}$,

$$U e_n = \frac{p_n}{\sqrt{n+1}}.$$

So the Bergman shift is lifted up as operators induced by the coordinate functions on a nice subspace of $H^2(\mathbb{T}^2)$. Moreover for each Blaschke product $\varphi(z)$ of finite order, the multiplication operator M_φ on the Bergman space is unitarily equivalent to $\varphi(\boldsymbol{\mathcal{B}})$ on $\boldsymbol{\mathcal{H}}$. In fact, in [53] $\boldsymbol{\mathcal{B}}$ is said to be super-isometrically dilatable, and $\{T_z, T_w, H^2(\mathbb{T}^2)\}$ is called its super-isometric dilation. That is,

$$\boldsymbol{\mathcal{B}}^{n+m} = P_{\boldsymbol{\mathcal{H}}} T_z^n T_w^m |_{\boldsymbol{\mathcal{H}}}$$

for any non negative integers n and m, and

$$\boldsymbol{\mathcal{B}}^* = T_z^* |_{\boldsymbol{\mathcal{H}}} = T_w^* |_{\boldsymbol{\mathcal{H}}},$$

for the pair of doubly commuting pure isometries T_z and T_w on the Hardy space $H^2(\mathbb{T}^2)$. $H^2(\mathbb{T}^2)$ is where we do most of the calculations.

Our main idea as in [32] and [53] is to study the operator $\varphi(\boldsymbol{\mathcal{B}})$ on the Hardy space of the torus to get properties of the multiplication operator M_φ. This method seems to be effective since functions, especially inner functions, in the Hardy space of the torus behave better than the functions in the Bergman space.

In Section II.3, for a finite Blaschke product φ of order N, using the Wold decomposition of the pair of doubly commuting isometries $T_{\varphi(z)}$ and $T_{\varphi(w)}$ on the space

$$\boldsymbol{\mathcal{K}}_\varphi = span_{l,k \geq 0}\{\varphi^l(z)\varphi^k(w)\boldsymbol{\mathcal{H}}\},$$

we obtain

$$\boldsymbol{\mathcal{K}}_\varphi = \oplus_{l,k \geq 0}\varphi^l(z)\varphi^k(w)\boldsymbol{\mathcal{L}}_\varphi,$$

where $\boldsymbol{\mathcal{L}}_\varphi = ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)} \cap \boldsymbol{\mathcal{K}}_\varphi$ is the so called wandering space of $T_{\varphi(z)}$ and $T_{\varphi(w)}$ on $\boldsymbol{\mathcal{K}}_\varphi$. By means of the Fredholm theory introduced in [20], we are able to show that the dimension of $\boldsymbol{\mathcal{L}}_\varphi$ equals $2N - 1$ which is a key fact we need in the proof of our first main result.

For each e in the space $ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)} \cap \boldsymbol{\mathcal{H}}$ which is finite dimensional and denoted by L_0, we construct functions $\{d_e^k\}_{k=1}^\infty$ in Section II.4.1, d_e in Section II.4.2 and d_e^0 in Section II.4.3 such that for each $l \geq 1$,

$$p_l(\varphi(z), \varphi(w))e + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_e^{l-k} \in \boldsymbol{\mathcal{H}},$$

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e \in \boldsymbol{\mathcal{H}},$$

and

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e^0 \in \boldsymbol{\mathcal{H}}.$$

We have a precise formula of d_e^0 but we only know that d_e^k is orthogonal to $ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)} \cap \boldsymbol{\mathcal{H}}$.

In particular, for a given reducing subspace $\boldsymbol{\mathcal{M}}$ of $\varphi(\boldsymbol{\mathcal{B}})$, and $e \in \boldsymbol{\mathcal{M}}$, we have that

$$p_l(\varphi(z), \varphi(w))e + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_e^{l-k} \in \boldsymbol{\mathcal{M}}.$$

In Section II.5 we show that there is a unique reducing subspace denoted by $\boldsymbol{\mathcal{M}}_0$, called the distinguished reducing subspace of $\varphi(\boldsymbol{\mathcal{B}})$, such that $\varphi(\boldsymbol{\mathcal{B}})|_{\boldsymbol{\mathcal{M}}_0}$ is unitarily equivalent to the Bergman shift $\boldsymbol{\mathcal{B}}$.

The relation between $d_{\cdot}^1$ and $d_{\cdot}^0$ is given in Section II.4.3 and is used extensively in proving our main results.

We discuss the relation between weighted shifts and multiplication operators in Section II.7.

The proofs of our main results are in Section II.6 II.8 II.9 respectively.

II.2 Function theory

Let $H^2(\mathbb{T})$ be the Hardy space of the unit circle which consists of functions in $L^2(\mathbb{T})$ whose Fourier coefficients vanish for all the negative powers. $H^2(\mathbb{T})$ can also be viewed as the space of all analytic functions in the unit disk $\mathbb{D}$ whose Taylor coefficients are square summable.

For $\alpha \in \mathbb{D}$, let $k_\alpha = \frac{1}{1-\bar{\alpha}z}$ be the *reproducing kernel* of the Hardy space $H^2(\mathbb{T})$ at α. That is, for each function f in $H^2(\mathbb{T})$,

$$f(\alpha) = \langle f, k_\alpha \rangle.$$

For φ in $H^\infty(\mathbb{T})$, let $\hat{T}_\varphi$ denote the analytic Toeplitz operator on $H^2(\mathbb{T})$, with symbol φ, given by

$$\hat{T}_\varphi h = \varphi h,$$

for any $h \in H^2(\mathbb{T})$. Thus for each $h \in H^2(\mathbb{T})$,

$$
\begin{aligned}
\langle h, \hat{T}_\varphi^* k_\alpha \rangle &= \langle \hat{T}_\varphi h, k_\alpha \rangle \\
&= \langle \varphi h, k_\alpha \rangle \\
&= \varphi(\alpha) h(\alpha) \\
&= \langle h, \overline{\varphi(\alpha)} k_\alpha \rangle.
\end{aligned}
$$

So

$$\hat{T}_\varphi^* k_\alpha = \overline{\varphi(\alpha)} k_\alpha. \tag{II.1}$$

For an integer $s \geq 0$, define

$$k_\alpha^s(z) = \frac{s! z^s}{(1 - \bar{\alpha}z)^{s+1}}.$$

Note that $k_\alpha^0 = k_\alpha$.

Let φ be a finite Blaschke product of order N with zeros $\{\alpha_k\}_0^K$ and each α_k is a zero of multiplicity $n_k + 1$. That is,

$$\varphi(z) = \prod_{k=0}^K \left(\frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \right)^{n_k+1}.$$

The order of φ is given by

$$N = \sum_{i=0}^{K} (n_i + 1).$$

We assume that $\alpha_0 = 0$, and so $\varphi(z) = z\varphi_0(z)$ where φ_0 is the following Blaschke product:

$$\varphi_0(z) = z^{n_0} \prod_{k=1}^{K} \left(\frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \right)^{n_k+1}.$$

Letting ∂_z denote the partial derivative with respect to z, we have

$$k_\alpha^s(z) = \partial_{\bar{\alpha}}^s k_\alpha(z),$$

and (II.1) gives that for $h \in H^2(\mathbb{T})$,

$$\langle h, k_\alpha^s \rangle = h^{(s)}(\alpha).$$

Then for a positive integer s, (II.1) gives that

$$\hat{T}_z^* k_\alpha^s(z) = s k_\alpha^{s-1}(z) + \bar{\alpha} k_\alpha^s(z),$$

and

$$\hat{T}_z^* k_\alpha^0(z) = \bar{\alpha} k_\alpha^0(z).$$

More general we have the following lemma.

Lemma 4. *Let $s \geq 0$ be an integer. Then for each $f \in H^\infty(\mathbb{T})$,*

$$\hat{T}_f^* k_\alpha^s = \sum_{l=0}^{s} \frac{s!}{l!(s-l)!} \overline{f^{(l)}(\alpha)} k_\alpha^{s-l}.$$

Proof. For any $h \in H^2(\mathbb{T})$, we have

$$
\begin{aligned}
\langle h, \hat{T}_f^* k_\alpha^s \rangle &= \langle fh, k_\alpha^s \rangle \\
&= (fh)^{(s)}(\alpha) \\
&= \sum_{l=0}^s \frac{s!}{l!(s-l)!} f^{(l)}(\alpha) h^{(s-l)}(\alpha) \\
&= \langle h, \sum_{l=0}^s \frac{s!}{l!(s-l)!} \overline{f^{(l)}(\alpha)} k_\alpha^{(s-l)} \rangle.
\end{aligned}
$$

So the desired result holds.

For

$$
\varphi(z) = \prod_{k=0}^K \left(\frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \right)^{n_k+1},
$$

by the theory of the Toeplitz operators on the Hardy space of the unit circle, $\hat{T}_\varphi^*$ is Fredholm with Kernel of dimension N and Cokernel 0. Lemma 4 tells us that the kernel of the Toeplitz operator $\hat{T}_\varphi^*$ on the Hardy space of the unit circle is spanned by

$$
\{\{k_{\alpha_k}^{s_k}\}_{s_k=0,\cdots,n_k}\}_{k=0,\cdots,K}.
$$

Recall that $\mathcal{H}$ is the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^\infty$. The following lemma will be used from time to time to simplify the calculations involved elements in $\mathcal{H}$ or $\mathcal{H}^\perp$.

Lemma 5. *Let* $f \in H^2(\mathbb{T}^2)$. *If* $f(z,z) \in H^2(\mathbb{T})$, *then for each* $e \in \mathcal{H}$

$$
\begin{aligned}
\langle f(z,w), e(z,w) \rangle &= \langle f(z,z), e(z,0) \rangle \\
&= \langle f(w,w), e(0,w) \rangle.
\end{aligned}
$$

Proof. Writing $f(z,w) = \sum_{i=0,j=0}^\infty a_{ij} z^i w^j$ and $e = \sum_{j=0}^\infty b_j p_j$, we have

$$
\langle f(z,w), e(z,w) \rangle = \sum_{j=0}^\infty (a_{0j} + a_{1j-1} + \cdots + a_{j0}) \bar{b}_j.
$$

On the other hand,

$$f(z,z) = \sum_{j=0}^{\infty}(a_{0j} + a_{1j-1} + \cdots + a_{j0})z^j$$

and

$$e(z,0) = \sum_{j=0}^{\infty} b_j z^j,$$

so

$$\langle f(z,z), e(z,0)\rangle = \sum_{j=0}^{\infty}(a_{0j} + a_{1j-1} + \cdots + a_{j0})\bar{b}_j$$

and

$$\langle f(z,w), e(z,w)\rangle = \langle f(z,z), e(z,0)\rangle.$$

Similarly

$$\langle f(z,w), e(z,w)\rangle = \langle f(w,w), e(0,w)\rangle.$$

The proof is complete.

Lemma 6. *For $h(z,w) \in H^2(\mathbb{T}^2)$, $h \in \mathcal{H}^{\perp} \iff h(z,z) = 0$, for any $z \in \mathbb{D}$.*

Proof. As pointed out before,

$$\mathcal{H}^{\perp} = cl\{(z-w)H^2(\mathbb{T}^2)\}.$$

Let z be in $\mathbb{D}$. For each function $f(z,w) \in (z-w)H^2(\mathbb{T}^2)$, $f(z,z) = 0$. Thus $h(z,z) = 0$ for each $h \in \mathcal{H}^{\perp}$.

Conversely, assume that for a function $h \in H^2(\mathbb{T}^2)$, $h(z,z) = 0$, for any $z \in \mathbb{D}$. Then by writing

$$h(z,w) = \sum_{i=0,j=0}^{\infty} a_{ij} z^i w^j,$$

we have that

$$h(z,z) = \sum_{j=0}^{\infty}(a_{0j} + a_{1j-1} + \cdots + a_{j0})z^j.$$

Now $h(z,z) = 0$ implies $a_{0j} + a_{1j-1} + \cdots + a_{j0} = 0$ for all j which is the same as $\langle h, p_j\rangle = 0$

for all j. That is $h \perp \mathcal{H}$. We are done.

Lemma 7. *Suppose that $e(z, w)$ is in $\mathcal{H}$. If $e(z, z) = 0$ for each z in the unit disk, then $e(z, w) = 0$ for (z, w) on the torus.*

Proof. Writing $e(z, w) = \sum_{n=0}^{\infty} a_n p_n$, from $e(z, z) = a_0 + \sum_{n=1}^{\infty} n a_n z^n = 0$ we have that $a_n = 0$ for $n = 0, 1, 2, \dots$. That is, $e(z, w) = 0$. This completes the proof.

The above lemma tells us that a function in $\mathcal{H}$ is completely determined by its value on the diagonal. The following result implies that $e(z, w)$ is symmetric with respect to z and w.

Lemma 8. *If $e(z, w)$ is in $\mathcal{H}$, then*

$$e(z, w) = e(w, z).$$

Proof. The conclusion follows from that $p_n(z, w) = p_n(w, z)$ for all nonnegative integers n and that Each function $e(z, w)$ in $\mathcal{H}$ can be written as

$$e(z, w) = \sum_{n=0}^{\infty} a_n p_n(z, w)$$

for some sequence a_n.

Lemma 9. *Suppose $f(z, w)$ is in $\mathcal{H}$. Let $F(z) = f(z, 0)$. Then*

$$f(\lambda, \lambda) = \lambda F'(\lambda) + F(\lambda),$$

for each $\lambda \in \mathbb{D}$.

Proof. Let $f(z, w) = \sum_{n=0}^{\infty} a_n p_n(z, w)$. Then direct comparison of the Taylor expansion of $f(\lambda, \lambda)$ and $\lambda F'(\lambda) + F(\lambda)$ gives the proof.

II.3   Wold decomposition

For an operator T on a Hilbert space H, let $kerT$ denote the kernel of T. That is,

$$kerT = \{f : Tf = 0, f \in H\}.$$

Then $KerT^*$ is the same as the orthogonal complement of the range of T, TH. That is,

$$kerT^* = (TH)^{\perp}.$$

Given an isometry U on a Hilbert space H, the classical Wold decomposition theorem [34] states that H is the direct sum of two reducing subspaces of U,

$$H = H_u \oplus H_p,$$

so that U is unitary on H_u and U is pure on H_p, i.e., unitarily equivalent to a unilateral shift. In fact,

$$H_u = \cap_{n \geq 1} U^n H;$$

and

$$H_p = \oplus_{n \geq 0} U^n E,$$

where $E = H \ominus UH$ is called the wandering subspace for U. For a function φ in $H^{\infty}(\mathbb{D})$, we can view $\varphi(z)$ and $\varphi(w)$ as functions on the torus $\mathbb{T}^2$. While M_{φ} is not an isometry on the Bergman space of the unit disk, the analytic Toeplitz operators $T_{\varphi(z)}$ and $T_{\varphi(w)}$ are a pair of doubly commuting pure isometries on the Hardy space $H^2(\mathbb{T}^2)$ of torus. Since

$$
\begin{aligned}
T_z^* p_n &= T_w^* p_n \\
&= p_{n-1}
\end{aligned}
$$

for $n \geq 1$ and

$$T_z^* p_0 = T_w^* p_0$$
$$= 0,$$

$\mathcal{H}$ is an invariant subspace for both T_z^* and T_w^*. So $\mathcal{H}$ is also an invariant subspace for both $T_{\varphi(z)}^*$ and $T_{\varphi(w)}^*$. Let

$$\mathcal{K}_\varphi = span\{\varphi^l(z)\varphi^k(w)\mathcal{H}; l, k \geq 0\}.$$

Then $\mathcal{K}_\varphi$ is a reducing subspace for both $T_{\varphi(z)}$ and $T_{\varphi(w)}$, and so $T_{\varphi(z)}$ and $T_{\varphi(w)}$ are also a pair of doubly commuting isometries on $\mathcal{K}_\varphi$.

We consider the Wold decompositions for the pair $T_{\varphi(z)}$ and $T_{\varphi(w)}$ on both $\mathcal{K}_\varphi$ and

$$\mathcal{K}_\varphi^\perp = H^2(\mathbb{T}^2) \ominus \mathcal{K}_\varphi.$$

Let us first simplify the notation by denoting the wandering spaces

$$ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^* \cap \mathcal{K}_\varphi$$

and

$$ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^* \cap \mathcal{K}_\varphi^\perp$$

by $\mathcal{L}_\varphi$ and $\widehat{\mathcal{L}_\varphi}$ repectively.

The information about the dimension of the wandering space $\mathcal{L}_\varphi$ is crucial in the proof of our first main result in this chapter. To get the dimension of $\mathcal{L}_\varphi$ we first deal with the case when the zeros of φ are distinct and then use the Fredholm index theory for n-tuples developed in [20] to handle the general case. We start with the dimension of $ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^*$.

Lemma 10. *If $\varphi(z)$ is a Blaschke product with distinct zeros $\{\alpha_i\}_{i=1}^N$, then the intersection of the kernel of $T_{\varphi(z)}^*$ and $T_{\varphi(w)}^*$ is spanned by $\{k_{\alpha_i}(z)k_{\alpha_j}(w)\}_{i,j=1}^N$.*

Proof. Since $\varphi(z)$ is a Blaschke product with distinct zeros $\{\alpha_i\}_{i=1}^N$, as pointed out in the

previous section, the kernel of the Toeplitz operator $\hat{T}^*_{\varphi(z)}$ on the Hardy space of the unit circle is spanned by N linearly independent functions $\{k_{\alpha_i}(z)\}^N_{i=1}$. This gives that

$$ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)} \supset span\{k_{\alpha_i}(z)k_{\alpha_j}(w) : 1 \leq i, j \leq N\}.$$

To finish the proof, we need only to show

$$ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)} \subset span\{k_{\alpha_i}(z)k_{\alpha_j}(w) : 1 \leq i, j \leq N\}.$$

To do so, let $h(z, w)$ be a function in $ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)}$. Write

$$h(z, w) = \sum_{l=0}^{\infty} h_l(z)w^l,$$

where $h_l(z) \in H^2(T)$. Since $T^*_{\varphi(z)}h = 0$, we have

$$\begin{aligned} T^*_{\varphi(z)}h &= \sum_{l=0}^{\infty}[\hat{T}^*_{\varphi(z)}h_l](z)w^l \\ &= 0. \end{aligned}$$

Thus $[\hat{T}^*_{\varphi(z)}h_l](z) = 0$ for $l \geq 0$, and so $h_l(z)$ is in the kernel $\hat{T}^*_{\varphi(z)}$. Hence there are constants d_{li} such that

$$h_l(z) = \sum_{i=1}^{N} d_{li}k_{\alpha_i}(z),$$

to get

$$\begin{aligned} h(z, w) &= \sum_{l=0}^{\infty}\sum_{i=1}^{N} d_{li}k_{\alpha_i}(z)w^l \\ &= \sum_{i=0}^{N}(\sum_{l=0}^{\infty} d_{li}w^l)k_{\alpha_i}(z). \end{aligned}$$

Letting $g_i(w) = \sum_{l=0}^{\infty} d_{li}w^l$, we have

$$h(z, w) = \sum_{i=0}^{N} g_i(w)k_{\alpha_i}(z).$$

On the other hand, $h(z, w)$ is in the kernel of $T^*_{\varphi(w)}$. Thus

$$
\begin{aligned}
0 &= T^*_{\varphi(w)} h(z, w) \\
&= \sum_{i=1}^{N} [\hat{T}^*_{\varphi(w)} g_i](w) k_{\alpha_i}(z).
\end{aligned}
$$

So

$$
[\hat{T}^*_{\varphi(w)} g_i](w) = 0
$$

as $\{k_{\alpha_i}\}_{i=1}^{N}$ are linearly independent. Hence there are constants c_{ij} such that

$$
g_i = \sum_{j=1}^{N} c_{ij} k_{\alpha_j}(w),
$$

to get

$$
h(z, w) = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} k_{\alpha_i}(z) k_{\alpha_j}(w).
$$

We conclude that h is in the space spanned by $\{k_{\alpha_i}(z) k_{\alpha_j}(w)\}_{i,j=1}^{N}$, to finish the proof.

The following lemma is implicit in the proof of Theorem 3 in [53].

Lemma 11. *Let $\varphi(z)$ be a finite Blaschke product with distinct zeros $\{\alpha_i\}_{i=1}^{N}$. Then the dimension of $\widehat{\mathcal{L}_\varphi} = \ker T^*_{\varphi(z)} \cap \ker T^*_{\varphi(w)} \cap [H^2(\mathbb{T}^2) \ominus \mathcal{K}_\varphi]$ equals $(N-1)^2$.*

Proof. First we show

$$
\widehat{\mathcal{L}_\varphi} = \ker T^*_{\varphi(z)} \cap \ker T^*_{\varphi(w)} \cap \mathcal{H}^{\perp}.
$$

Since $\mathcal{H} \subset \mathcal{K}_\varphi$,

$$
\widehat{\mathcal{L}_\varphi} \subset \ker T^*_{\varphi(z)} \cap \ker T^*_{\varphi(w)} \cap \mathcal{H}^{\perp}.
$$

Conversely, if f is in $\ker T^*_{\varphi(z)} \cap \ker T^*_{\varphi(w)} \cap \mathcal{H}^{\perp}$, then f is in $\ker T^*_{\varphi(z)} \cap \ker T^*_{\varphi(w)}$ and orthogonal to $\mathcal{H}$. Thus for each $g(z, w) = \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k h_{kl} \in \mathcal{K}_\varphi$ where $h_{kl} \in \mathcal{H}$, we have

$$
\begin{aligned}
\langle f, g \rangle &= \sum_{k,l \geq 0} \langle f, \varphi(z)^l \varphi(w)^k h_{kl} \rangle \\
&= \sum_{k,l \geq 0} \langle [T^*_{\varphi(z)}]^l [T^*_{\varphi(w)}]^k f, h_{lk} \rangle \\
&= 0.
\end{aligned}
$$

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So f is also in $\widehat{\mathcal{L}_\varphi}$. Hence we have

$$\widehat{\mathcal{L}_\varphi} = ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)} \cap \mathcal{H}^\perp.$$

We are to prove that the dimension of $\widehat{\mathcal{L}_\varphi}$ is $(N-1)^2$. Without loss of generality, we assume that $\alpha_1 = 0$. By Lemma 10, the N^2 dimensional space $ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)}$ is spanned by $\{k_{\alpha_i}(z)k_{\alpha_j}(w)\}^N_{i,j=1}$. So it follows from Lemma 6 that $\widehat{\mathcal{L}_\varphi}$ consists of the elements h in $ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)}$ which satisfy $h(z,z) = 0$. That is,

$$\widehat{\mathcal{L}_\varphi} = \{h = \sum_{i=1}^N \sum_{j=1}^N c_{ij} k_{\alpha_i}(z)k_{\alpha_j}(w) : h(z,z) = \sum_{i=1}^N \sum_{j=1}^N c_{ij} k_{\alpha_i}(z)k_{\alpha_j}(z) = 0\}.$$

For any $h \in \widehat{\mathcal{L}_\varphi}$, taking the limit at infinity and testing the multiplicity at its poles $1/\bar{\alpha}_j$ of the function $h(z,z)$, we immediately have that $h(z,z) = 0$ implies $c_{jj} = 0$, $j = 1, 2..., N$. That is,

$$\widehat{\mathcal{L}_\varphi} = \{h = \sum_{i \neq j, i=1}^N \sum_{j=1}^N c_{ij} k_{\alpha_i}(z)k_{\alpha_j}(w) : h(z,z) = \sum_{i \neq j, i=1}^N \sum_{j=1}^N c_{ij} k_{\alpha_i}(z)k_{\alpha_j}(z) = 0\}.$$

Observe that $k_{\alpha_i}(z)k_{\alpha_j}(z) = a_{ij} k_{\alpha_i}(z) + b_{ij} k_{\alpha_j}(z)$ where $a_{ij} = \frac{\bar{a}_i}{\bar{a}_i - \bar{a}_j}$ and $b_{ij} = \frac{-\bar{a}_j}{\bar{a}_i - \bar{a}_j}$, and $k_{\alpha_2}(z), ..., k_{\alpha_N}(z)$ are linear independent. Write $h(z,z)$ as linear combination of $k_{\alpha_j}(z), j = 2, ..., N$, then all the coefficients of $k_{\alpha_j}(z)$ must be zero. So we have a system of another $N-1$ linear equations governing $c_{ij}, i \neq j, i, j = 1, ..., N$. Writing $\{c_{ij}\}_{i \neq j}$ as

$$(c_{12}, c_{21}, c_{13}, c_{31}, \cdots, c_{1N}, c_{N1}, c_{23}, c_{32}, \cdots, c_{2N}, c_{N2}, \cdots, c_{(N-1)N}, c_{N(N-1)})$$

gives the coefficient matrix of the system as:

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & * & * & \cdots & * & * & \cdots & * & * \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 & * & * & \cdots & * & * & \cdots & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 & * & * & \cdots & * & * & \cdots & * & *
\end{pmatrix}
$$

where $*$'s are some numbers. Clearly, the rank of the above matrix is $N - 1$. Hence the dimension of $\widehat{\mathcal{L}_\varphi}$ (as the solution space of $N^2 - N$ unknown variables governed by $N - 1$ linear independent equations) equals $N^2 - N - (N - 1)$. The proof is finished.

We are ready to prove our main result in the section.

Theorem 12. *Let φ be a finite Blaschke product of order N. Then*

$$\boldsymbol{\mathcal{K}}_\varphi = \oplus_{l,k \geq 0} \varphi^l(z) \varphi^k(w) \boldsymbol{\mathcal{L}}_\varphi,$$

and

$$H^2(\mathbb{T}^2) \ominus \boldsymbol{\mathcal{K}}_\varphi = \oplus_{l,k \geq 0} \varphi^l(z) \varphi^k(w) \widehat{\boldsymbol{\mathcal{L}}_\varphi}.$$

The dimension of $\widehat{\boldsymbol{\mathcal{L}}_\varphi}$ equals $(N - 1)^2$ and the dimension of $\boldsymbol{\mathcal{L}}_\varphi$ equals $2N - 1$.

Proof. Recall that $T_{\varphi(z)}$ and $T_{\varphi(w)}$ are a pair of doubly commuting isometries on both $\boldsymbol{\mathcal{K}}_\varphi$ and $H^2(\mathbb{T}^2) \ominus \boldsymbol{\mathcal{K}}_\varphi$. The Wold decomposition of $T_{\varphi(z)}$ on $\boldsymbol{\mathcal{K}}_\varphi$ gives

$$\boldsymbol{\mathcal{K}}_\varphi = \oplus_{l \geq} \varphi(z)^l E$$

where E is the wandering space for $T_{\varphi(z)}$ given by

$$
\begin{aligned}
E &= \boldsymbol{\mathcal{K}}_\varphi \ominus [T_{\varphi(z)} \boldsymbol{\mathcal{K}}_\varphi] \\
&= ker[T^*_{\varphi(z)} | \boldsymbol{\mathcal{K}}_\varphi] \\
&= ker T^*_{\varphi(z)} \cap \boldsymbol{\mathcal{K}}_\varphi.
\end{aligned}
$$

Since $T_{\varphi(z)}$ and $T_{\varphi(w)}$ are doubly commuting, E is a reducing subspace of $T_{\varphi(w)}$. Thus $T_{\varphi(w)}|_E$ is still an isometry. The Wold decomposition theorem again gives

$$E = \oplus_{k \geq 0} \varphi(w)^k E_1$$

where E_1 is the wandering space for $T_{\varphi(w)}|_E$ given by

$$
\begin{aligned}
E_1 &= E \ominus T_{\varphi(w)} E \\
&= ker T^*_{\varphi(w)} \cap E \\
&= ker T^*_{\varphi(z)} \cap T^*_{\varphi(w)} \cap \boldsymbol{\mathcal{K}}_\varphi.
\end{aligned}
$$

This gives

$$
\boldsymbol{\mathcal{K}}_\varphi = \oplus_{l,k \geq 0} \varphi^l(z) \varphi^k(w) \boldsymbol{\mathcal{L}}_\varphi.
$$

Considering the Wold decompositions of $T_{\varphi(z)}$ and $T_{\varphi(w)}$ on $H^2(\mathbb{T}^2) \ominus \boldsymbol{\mathcal{K}}_\varphi$, similarly we obtain

$$
H^2(\mathbb{T}^2) \ominus \boldsymbol{\mathcal{K}}_\varphi = \oplus_{l,k \geq 0} \varphi^l(z) \varphi^k(w) \widehat{\boldsymbol{\mathcal{L}}_\varphi}.
$$

Noting

$$
ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)} = \boldsymbol{\mathcal{L}}_\varphi \oplus \widehat{\boldsymbol{\mathcal{L}}_\varphi}
$$

we have

$$
dim[ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)}] = dim[\boldsymbol{\mathcal{L}}_\varphi] + dim[\widehat{\boldsymbol{\mathcal{L}}_\varphi}].
$$

By Lemma 10, the dimension of $ker T^*_{\varphi(z)} \cap ker T^*_{\varphi(w)}$ equals N^2. Hence

$$
dim[\boldsymbol{\mathcal{L}}_\varphi] = N^2 - dim[\widehat{\boldsymbol{\mathcal{L}}_\varphi}].
$$

We are to show that the dimension of $\widehat{\boldsymbol{\mathcal{L}}_\varphi}$ is $(N-1)^2$. To do so, we first interpret the dimension as some kind of Fredholm index.

For any given finite Blaschke product φ, let $index(T^*_{\varphi(z)}, T^*_{\varphi(w)})$ be the Fredholm index of the commuting pair $(T^*_{\varphi(z)}, T^*_{\varphi(w)})$ acting on the Hilbert space $H = H^2(\mathbb{T}^2) \ominus \boldsymbol{\mathcal{K}}_\varphi$. The Fredholm index of a commuting n-tuple was first introduced in [20].

Claim.

$$
dim \widehat{\boldsymbol{\mathcal{L}}_\varphi} = -index(T^*_{\varphi(z)}, T^*_{\varphi(w)}).
$$

Proof of the claim. Let $H = H^2(\mathbb{T}^2) \ominus \boldsymbol{\mathcal{K}}_\varphi$. Define $d_1 : H \longrightarrow H \oplus H$ and $d_2 :$

$H \oplus H \longrightarrow H$ by

$$d_1 f = (-T^*_{\varphi(w)} f, T^*_{\varphi(z)} f)$$

and

$$d_2(f, g) = T^*_{\varphi(z)} f + T^*_{\varphi(w)} g$$

respectively. Since $T^*_{\varphi(w)}$ commutes with $T^*_{\varphi(z)}$, we have

$$d_2 d_1 = 0,$$

to get the following complex (it is called Koszul complex)

$$H \longrightarrow H \oplus H \longrightarrow H \longrightarrow 0.$$

According to [20], the tuple $(T^*_{\varphi(z)}, T^*_{\varphi(w)})$ is Fredholm since

$$ker d_1 = \widehat{\mathcal{L}_\varphi}$$

is finite dimensional,

$$(ker d_2) \ominus d_1 H = \{0\},$$

and

$$H \ominus d_2(H \oplus H) = \{0\}.$$

The first equality is obvious. The last equality follows from that $T^*_{\varphi(w)}$ is onto. To show the second equality, let $(f, g) \in (ker d_2) \ominus d_1 H$. Then

$$T^*_{\varphi(z)} f + T^*_{\varphi(w)} g = 0$$

and (f, g) is orthogonal to $d_1 H$. So we have that for each $x \in H$,

$$\langle (f, g), (-T^*_{\varphi(w)} x, T^*_{\varphi(z)} x) \rangle = 0.$$

Thus

$$\langle -T_{\varphi(w)}f + T_{\varphi(z)}g, x \rangle = 0,$$

for each $x \in H$, and so

$$-T_{\varphi(w)}f + T_{\varphi(z)}g = 0.$$

This gives

$$
\begin{aligned}
0 &= T_{\varphi(z)}[T^*_{\varphi(z)}f + T^*_{\varphi(w)}g] \\
&= T_{\varphi(z)}T^*_{\varphi(z)}f + T^*_{\varphi(w)}T_{\varphi(z)}g \\
&= T_{\varphi(z)}T^*_{\varphi(z)}f + T^*_{\varphi(w)}T_{\varphi(w)}f \\
&= T_{\varphi(z)}T^*_{\varphi(z)}f + f
\end{aligned}
$$

Taking inner product of the above equation with f, we have

$$0 = ||T^*_{\varphi(z)}f||^2 + ||f||^2.$$

Hence $f = 0$ and so $g = 0$. It follows from Corollaries 6.2 and 7.2 in [20] that

$$
\begin{aligned}
&\quad index(T^*_{\varphi(z)}, T^*_{\varphi(w)}) \\
&= -dim[ker d_1] + dim[(ker d_2) \ominus d_1 H] - dim[H \ominus d_2(H \oplus H)] \\
&= -dim\widehat{\mathcal{L}_\varphi}.
\end{aligned}
$$

So the claim is proved. Now by Lemma 11, for a finite Blaschke product $\varphi(z)$ with distinct zeros, the dimension of $\widehat{\mathcal{L}_\varphi}$ equals $(N-1)^2$.

To finish the proof we need to show that this is also true for any finite Blaschke product φ of order N. To do so, recall that for a given $\lambda \in \mathbb{D}$, $\varphi_\lambda(z)$ is the Möbius transform:

$$\frac{z - \lambda}{1 - \bar{\lambda}z},$$

and $\varphi_\lambda \circ \varphi(z)$ is still a finite Blaschke product with N zeros in the unit disk and

$$T_{\varphi_\lambda \circ \varphi(z)} = (T_{\varphi(z)} - \lambda I)(I - \lambda T_{\varphi(z)})^{-1}.$$

Thus $\mathcal{K}_{\varphi_\lambda \circ \varphi} = \mathcal{K}_\varphi$.

It was also shown in [20] that the index is a continuous map from the set of the Fredholm tuples to the set of integers. Observe that

$$||\varphi_\lambda \circ \varphi(z) - \varphi(z)||_\infty \leq \frac{2|\lambda|}{1 - |\lambda|}.$$

Thus for a sufficiently small λ,

$$index(T^*_{\varphi_\lambda \circ \varphi(z)}, T^*_{\varphi_\lambda \circ \varphi(w)}) = index(T^*_{\varphi(z)}, T^*_{\varphi(w)}).$$

If λ is not in the critical value set $\{\mu \in \mathbb{D} : \mu = \varphi(z) \quad \text{and} \quad \varphi'(z) = 0 \quad \text{for some} \quad z \in \mathbb{D}\}$ of φ, then $\varphi_\lambda \circ \varphi(z)$ is a Blaschke product with N distinct zeros in $\mathbb{D}$. In this case, by Lemma 11,

$$-index(T^*_{\varphi_\lambda \circ \varphi(z)}, T^*_{\varphi_\lambda \circ \varphi(w)}) = dim\widehat{\mathcal{L}_{\varphi_\lambda \circ \varphi}} = (N - 1)^2.$$

Since by Bochner's theorem [57] there are only finitely many points in the critical value set, we conclude that

$$dim\widehat{\mathcal{L}_\varphi} = -index(T^*_{\varphi(z)}, T^*_{\varphi(w)}) = (N - 1)^2.$$

II.4 Basic constructions

In this section we will construct three functions d^1_e, d_e and d^0_e for each $e \in kerT^*_{\varphi(z)} \cap kerT^*_{\varphi(w)} \cap \mathcal{H}$, which have properties mentioned in Section II.1. We will obtain relations among d^1_e, d_e and d^0_e. Those relations are very useful for us to derive information about the reducing lattice of M_φ.

II.4.1 First Construction

First we will show that for a given reducing subspace $\mathcal{M}$ for $\varphi(\mathcal{B})$, for each $e \in \mathcal{M} \cap L_0$ and each integer $l \geq 1$, there are a family of functions $\{d_e^k\}_{k=1}^l$ such that

$$p_l(\varphi(z), \varphi(w))e + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_e^{l-k} \in \mathcal{M},$$

where

$$L_0 = ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^* \cap \mathcal{H}.$$

These functions are very useful in studying the structure of the multiplication operator M_φ on the Bergman space.

We start with the following lemma to show that for each reducing subspace $\mathcal{M}$ of $\varphi(\mathcal{B})$, the intersection of $\mathcal{M}$ and L_0 is nontrivial.

Lemma 13. *If $\mathcal{M}$ is a nontrivial reducing subspace for $\varphi(\mathcal{B})$, then the intersection $\mathcal{M} \cap L_0$ contains a nonzero function.*

Proof. Let $\mathcal{M}$ be a nontrivial reducing subspace for $\varphi(\mathcal{B})$. Suppose

$$\mathcal{M} \cap L_0 = \{0\}.$$

Since $\varphi(\mathcal{B})$ is unitarily equivalent to the multiplication operator M_φ on the Bergman space L_a^2, there is a unitary operator $U : L_a^2 \to \mathcal{H}$ such that $U^* M_\varphi U = \varphi(\mathcal{B})$. Let

$$\tilde{M} = U^* \mathcal{M},$$

and

$$\tilde{L}_0 = U^* L_0.$$

Thus $\tilde{M}$ is a reducing subspace of M_φ and the kernel of M_φ^* equals $\tilde{L}_0$. Moreover,

$$M_\varphi = [M_\varphi]|_{\tilde{M}} \oplus [M_\varphi]|_{\tilde{M}^\perp},$$

30

$$ker[M_\varphi]|_{\tilde{M}} = \{0\}$$

and

$$
\begin{aligned}
ker[M_\varphi^*]|_{\tilde{M}} &= \tilde{M} \cap \tilde{L}_0 \\
&= U^*[\mathcal{M} \cap L_0] \\
&= \{0\}.
\end{aligned}
$$

Noting that M_φ is Fredholm on L_a^2, we see that the restriction $[M_\varphi]|_{\tilde{M}}$ of M_φ on its reducing subspace $\tilde{M}$ is also Fredholm. Thus $\varphi\tilde{M} = \tilde{M}$. So every function in $\tilde{M}$ has φ^n as a factor for each $n \geq 1$ and then it vanishes at each zero of φ with infinite order. Consequently, it must be zero. This contradicts that $\mathcal{M}$ is a nontrivial reducing subspace for $\varphi(\mathcal{B})$.

Lemma 14. *If $\mathcal{M}$ is a reducing subspace for $\varphi(\mathcal{B})$, then $\varphi(\mathcal{B})^*\mathcal{M} = \mathcal{M}$.*

Proof. First note that for a Blaschke product $\varphi(z)$ with finite order, $\varphi(\mathcal{B})$ is Fredholm and the kernel of $\varphi(\mathcal{B})$ contains only zero. Thus

$$\varphi(\mathcal{B})^*\mathcal{H} = \mathcal{H}.$$

Suppose that $\mathcal{M}$ is a reducing subspace for $\varphi(\mathcal{B})$. Let $\mathcal{N} = \mathcal{M}^\perp$. Then

$$\varphi(\mathcal{B})^* = \varphi(\mathcal{B})^*|_{\mathcal{M}} \oplus \varphi(\mathcal{B})^*|_{\mathcal{N}}$$

under the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$. Since $\varphi(\mathcal{B})^*$ is surjective,

$$\varphi(\mathcal{B})^*|_{\mathcal{M}}\mathcal{M} = \mathcal{M}.$$

This completes the proof.

In the proof of the following theorem we will use the following fact: for each $f \in \mathcal{H}$,

$$
\begin{aligned}
\varphi(\mathcal{B})^* f &= T^*_{\varphi(z)} f \\
&= T^*_{\varphi(w)} f.
\end{aligned}
$$

Theorem 15. *Suppose that $\mathcal{M}$ is a reducing subspace for $\varphi(\mathcal{B})$. For a given $e \in \mathcal{M} \cap L_0$ there are a unique family of functions $\{d_e^k\} \subset \mathcal{L}_\varphi \ominus L_0$ such that*

$$
p_l(\varphi(z), \varphi(w))e + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_e^{l-k} \in \mathcal{M},
$$

for each $l \geq 1$.

Proof. By Lemma 14, for a given $e \in \mathcal{M} \cap L_0$, there is a unique function $e' \in \mathcal{M} \ominus L_0$ such that

$$
\begin{aligned}
T^*_{\varphi(z)} e' &= T^*_{\varphi(w)} e' \\
&= e.
\end{aligned}
$$

For a given $e \in \mathcal{M} \cap L_0$, we will use mathematical induction to construct a family of functions $\{d_e^k\}$. To do this, for each e in L_0, noting that

$$
T^*_{\varphi(z)}[(\varphi(z) + \varphi(w))e] = e,
$$

and

$$
T^*_{\varphi(w)}[(\varphi(z) + \varphi(w))e] = e,
$$

we have

$$
\begin{aligned}
T^*_{\varphi(z)}[e' - (\varphi(z) + \varphi(w))e] &= e - e \\
&= 0,
\end{aligned}
$$

and

$$T^*_{\varphi(w)}[e' - (\varphi(z) + \varphi(w))e] = e - e$$
$$= 0.$$

Letting $d_e^1 = e' - (\varphi(z) + \varphi(w))e$, the above two equalities give

$$d_e^1 \in \ker T^*_{\varphi(z)} \cap \ker T^*_{\varphi(w)}.$$

Because both e' and e are in $\mathcal{M}$, we have that d_e^1 is in $\mathcal{K}_\varphi$, and

$$(\varphi(z) + \varphi(w))e + d_e^1 = e' \in \mathcal{M}.$$

Thus d_e^1 is in $\mathcal{K}_\varphi$ and so it is in $\mathcal{L}_\varphi$. For each $f \in L_0$,

$$\langle d_e^1, f \rangle = \langle e' - (\varphi(z) + \varphi(w))e, f \rangle$$
$$= \langle e', f \rangle - \langle (\varphi(z) + \varphi(w))e, f \rangle$$
$$= 0 - \langle e, T^*_{\varphi(z)}f + T^*_{\varphi(w)}f \rangle$$
$$= 0.$$

The third equality follows from that $e' \in \mathcal{M} \ominus L_0$. Hence d_e^1 is in $\mathcal{L}_\varphi \ominus L_0$.

Assume that for $n < l$ there are a family of functions $\{d_e^k\}_{k=1}^n \subset \mathcal{L}_\varphi \ominus L_0$ such that

$$p_n(\varphi(z), \varphi(w))e + \sum_{k=0}^{n-1} p_k(\varphi(z), \varphi(w))d_e^{n-k} \in \mathcal{M}.$$

Let $E = p_n(\varphi(z), \varphi(w))e + \sum_{k=0}^{n-1} p_k(\varphi(z), \varphi(w))d_e^{n-k}$. By Lemma 14 again, there is a unique function $E' \in \mathcal{M} \ominus L_0$ such that

$$T^*_{\varphi(z)}E' = T^*_{\varphi(w)}E'$$
$$= E.$$

Let $F = p_{n+1}(\varphi(z), \varphi(w))e + \sum_{k=1}^{n} p_k(\varphi(z), \varphi(w))d_e^{n+1-k}$. Simple calculations give

$$
\begin{aligned}
T_{\varphi(z)}^* F &= T_{\varphi(w)}^* F \\
&= E.
\end{aligned}
$$

Thus

$$
\begin{aligned}
T_{\varphi(z)}^*(E' - F) &= T_{\varphi(w)}^*(E' - F) \\
&= E - E \\
&= 0.
\end{aligned}
$$

Letting $d_e^{n+1} = E' - F$, d_e^{n+1} is in $ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^*$. Noting E' is orthogonal to L_0, we have that for each $f \in L_0$,

$$
\begin{aligned}
\langle d_e^{n+1}, f \rangle &= \langle E', f \rangle - \langle F, f \rangle \\
&= -[\langle p_{n+1}(\varphi(z), \varphi(w))e, f \rangle + \sum_{k=1}^{n} \langle p_k(\varphi(z), \varphi(w))d_e^{n+1-k}, f \rangle] \\
&= 0,
\end{aligned}
$$

to get that d_e^{n+1} is in $\mathcal{L}_\varphi \ominus L_0$. Hence

$$
p_{n+1}(\varphi(z), \varphi(w))e + \sum_{k=1}^{n} p_k(\varphi(z), \varphi(w))d_e^{n+1-k} + d_e^{n+1} = E' \in \mathcal{M}.
$$

By induction, we get a family of functions $\{d_e^k\} \subset \mathcal{L}_\varphi \ominus L_0$ as desired to complete the proof.

In the special case for $\mathcal{H}$, as $\mathcal{H}$ is a reducing subspace for $\varphi(\mathcal{B})$, Theorem 15 immediately gives the following theorem.

Theorem 16. *For a given $e \in L_0$ there are a unique family of functions $\{d_e^k\} \subset \mathcal{L}_\varphi \ominus L_0$ such that*

$$
p_l(\varphi(z), \varphi(w))e + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_e^{l-k} \in \mathcal{H},
$$

for each $l \geq 1$.

Theorem 17. *If $\mathcal{M} \subset \mathcal{H}$ is a reducing subspace $\varphi(\mathcal{B})$ and $e \in \mathcal{M} \cap L_0$, let d_e^k be the function in Theorem 15, then*

$$P_{\mathcal{H}}[p_l(\varphi(z), \varphi(w))d_e^k] \in \mathcal{M}$$

for each $k \geq 1$, and $l \geq 0$.

Proof. Suppose that $\mathcal{M}$ is a reducing subspace of $\varphi(\mathcal{B})$ and $e \in \mathcal{M}$. A simple calculation gives

$$
\begin{aligned}
2\varphi(\mathcal{B})e \;&=\; P_{\mathcal{H}}(p_1(\varphi(z), \varphi(w))e) \\
&= P_{\mathcal{H}}(p_1(\varphi(z), \varphi(w))e + d_e^1) - P_{\mathcal{H}}(d_e^1) \\
&= p_1(\varphi(z), \varphi(w))e + d_e^1 - P_{\mathcal{H}}(d_e^1).
\end{aligned}
$$

This implies

$$P_{\mathcal{H}}(d_e^1) = [p_1(\varphi(z), \varphi(w))e + d_e^1] - 2\varphi(\mathcal{B})e \in \mathcal{M}.$$

Noting that $(d_e^1 - P_{\mathcal{H}}d_e^1)$ is in $\mathcal{H}^\perp$, Lemma 6 gives

$$(d_e^1 - P_{\mathcal{H}}d_e^1)(z, z) = 0.$$

Thus

$$
\begin{aligned}
[p_{l-1}(\varphi(z), \varphi(w))(d_e^1 - P_{\mathcal{H}}d_e^1)]|_{z=w} \;&=\; [p_{l-1}(\varphi(z), \varphi(z))[(d_e^1 - P_{\mathcal{H}}d_e^1)(z, z)] \\
&=\; 0.
\end{aligned}
$$

By Lemma 6 again, we have that

$$[p_{l-1}(\varphi(z), \varphi(w))(d_e^1 - P_{\mathcal{H}}d_e^1)] \in \mathcal{H}^\perp,$$

and so

$$P_{\mathcal{H}}[p_{l-1}(\varphi(z), \varphi(w))(d_e^1 - P_{\mathcal{H}}d_e^1)] = 0.$$

Hence

$$P_{\mathcal{H}}[p_{l-1}(\varphi(z), \varphi(w))(d_e^1)] = P_{\mathcal{H}}\{p_{l-1}(\varphi(z), \varphi(w))[P_{\mathcal{H}} d_e^1]\} \in \mathcal{M}.$$

Assume that $P_{\mathcal{H}}[p_l(\varphi(z), \varphi(w))d_e^k] \in \mathcal{M}$ for $k \leq n$ and any $l \geq 0$. To finish the proof by induction we need only to show that

$$P_{\mathcal{H}}[p_l(\varphi(z), \varphi(w))d_e^{n+1})] \in \mathcal{M}$$

for any $l \geq 0$.

A simple calculation gives

$$
\begin{aligned}
(n+2)\varphi(\mathcal{B})^{n+1}e \;\; &= P_{\mathcal{H}}[p_{n+1}(\varphi(z), \varphi(w))e + \sum_{k=0}^{n} p_k(\varphi(z), \varphi(w))d_e^{n+1-k}] \\
&\quad - \{P_{\mathcal{H}}[d_e^{n+1}] + P_{\mathcal{H}}[\sum_{k=1}^{n} p_k(\varphi(z), \varphi(w))d_e^{n+1-k}]\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
P_{\mathcal{H}}[d_e^{n+1}] \;\; &= P_{\mathcal{H}}[p_{n+1}(\varphi(z), \varphi(w))e + \sum_{k=0}^{n} p_k(\varphi(z), \varphi(w))d_e^{n+1-k}] - \\
&\quad \{(n+2)\varphi(\mathcal{B})^{n+1}e + P_{\mathcal{H}}[\sum_{k=1}^{n} p_k(\varphi(z), \varphi(w))d_e^{n+1-k}]\}.
\end{aligned}
$$

Theorem 15 gives that the first term in the last equality is $\mathcal{M}$, the induction hypothesis gives that the last term is in $\mathcal{M}$ and the second term belongs to $\mathcal{M}$ since $e \in \mathcal{M}$ and $\mathcal{M}$ is a reducing subspace for $\varphi(\mathcal{B})$. So $P_{\mathcal{H}}[d_e^{n+1}]$ is in $\mathcal{M}$. Therefore we conclude

$$P_{\mathcal{H}}[p_l(\varphi(z), \varphi(w))d_e^{n+1}] = P_{\mathcal{H}}[(p_l(\varphi(z), \varphi(w))(P_{\mathcal{H}} d_e^{n+1})] \in \mathcal{M},$$

to complete the proof.

Theorem 16 only gives the existence of the family of functions $\{d_e^{(k)}\} \subset \mathcal{L}_\varphi \ominus L_0$. It will be useful to know how those functions are constructed from e. Theorem 20 will give a recursive formula of $\{d_e^{(k)}\}$. First we need the following simple but useful lemma which is implicit in [32].

For two functions x, y in $H^2(\mathbb{T}^2)$, the symbol $x \otimes y$ is the operator on $H^2(\mathbb{T}^2)$ defined by

$$(x \otimes y)g = [\langle g, y \rangle_{H^2(\mathbb{T}^2)}]x$$

for $g \in H^2(\mathbb{T}^2)$.

Lemma 18. *On the Hardy space $H^2(\mathbb{T}^2)$, the identity operator equals*

$$
\begin{aligned}
I &= T_z T_z^* + \sum_{l \geq 0} w^l \otimes w^l \\
&= T_w T_w^* + \sum_{l \geq 0} z^l \otimes z^l.
\end{aligned}
$$

Proof. We will just verify the first equality in the lemma. The same argument will give the second equality. To do so, let $h \in H^2(\mathbb{T}^2)$. Write $h(z,w) = \sum_{j=0}^{\infty} h_j(w) z^j$ for some functions $h_j(w)$ in $H^2(\mathbb{T})$. Thus

$$
\begin{aligned}
T_z T_z^* h &= \sum_{j=0}^{\infty} h_j(w) T_z T_z^* z^j \\
&= \sum_{j=1}^{\infty} h_j(w) z^j,
\end{aligned}
$$

and

$$
\begin{aligned}
(w^l \otimes w^l)h &= \langle h, w^l \rangle w^l \\
&= \sum_{j=0}^{\infty} \langle h_j(w) z^j, w^l \rangle w^l \\
&= \langle h_0(w), w^l \rangle w^l.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(\sum_{l \geq 0} w^l \otimes w^l \right)h &= \sum_{l \geq 0} \langle h_0(w), w^l \rangle w^l \\
&= h_0(w).
\end{aligned}
$$

Consider

$$[T_z T_z^* + \sum_{l \geq 0} w^l \otimes w^l] h = \sum_{j=1}^{\infty} h_j(w) z^j + h_0(w)$$

$$= \sum_{j=0}^{\infty} h_j(w) z^j$$

$$= h.$$

This completes the proof.

Lemma 19. *Suppose that $\varphi(z) = z\varphi_0(z)$ for some finite Blaschke product $\varphi_0(z)$. If f is a function in $H^2(\mathbb{T}^2)$, then for each $l \geq 1$,*

$$T_{z-w}^*(p_l(\varphi(z), \varphi(w))f) = p_l(\varphi(z), \varphi(w))T_{z-w}^* f$$

$$+ \varphi_0(z) p_{l-1}(\varphi(z), \varphi(w)) f(0, w)$$

$$- \varphi_0(w) p_{l-1}(\varphi(z), \varphi(w)) f(z, 0).$$

Proof. Let $f \in H^2(T^2)$. By Lemma 18, we have

$$T_z^*(p_l(\varphi(z), \varphi(w))f)$$

$$= T_z^*[p_l(\varphi(z), \varphi(w))(T_z T_z^* + \sum_{i \geq 0} w^i \otimes w^i)f]$$

$$= T_z^*[p_l(\varphi(z), \varphi(w))(T_z T_z^* f)] + T_z^*[p_l(\varphi(z), \varphi(w))(\sum_{i \geq 0} w^i \otimes w^i)f)]$$

$$= p_l(\varphi(z), \varphi(w))(T_z^* f) + T_z^*[p_l(\varphi(z), \varphi(w))(\sum_{i \geq 0} w^i \otimes w^i)f)].$$

Noting

$$p_l(\varphi(z), \varphi(w)) = \sum_{k=0}^{l} \varphi(z)^k \varphi(w)^{l-k}$$

$$= \varphi(w)^l + \varphi(z) \sum_{k=1}^{l} \varphi(z)^{k-1} \varphi(w)^{l-k}$$

$$= \varphi(w)^l + z\varphi_0(z) \sum_{k=1}^{l} \varphi(z)^{k-1} \varphi(w)^{l-k},$$

and

$$(\sum_{i\geq 0} w^i \otimes w^i)f = f(0,w),$$

we obtain

$$T_z^*[p_l(\varphi(z),\varphi(w))(\sum_{i\geq 0} w^i \otimes w^i)f)]$$
$$= T_z^*[p_l(\varphi(z),\varphi(w))f(0,w)]$$
$$= T_z^*[\varphi(w)^l f(0,w)] + T_z^*[z\varphi_0(z)\sum_{k=1}^{l} \varphi(z)^{k-1}\varphi(w)^{l-k}f(0,w)]$$
$$= \varphi_0(z)[\sum_{k=1}^{l} \varphi(z)^{k-1}\varphi(w)^{l-k}]f(0,w)$$
$$= \varphi_0(z)p_{l-1}(\varphi(z),\varphi(w))f(0,w).$$

This gives

$$T_z^*(p_l(\varphi(z),\varphi(w))f) = p_l(\varphi(z),\varphi(w))(T_z^*f) + \varphi_0(z)p_{l-1}(\varphi(z),\varphi(w))f(0,w). \qquad \text{(II.2)}$$

Similarly, we also have

$$T_w^*(p_l(\varphi(z),\varphi(w))f) = p_l(\varphi(z),\varphi(w))(T_w^*f) + \varphi_0(w)p_{l-1}(\varphi(z),\varphi(w))f(z,0). \qquad \text{(II.3)}$$

Combining (II.2) and (II.3) yields

$$T_{z-w}^*(p_l(\varphi(z),\varphi(w))f) = p_l(\varphi(z),\varphi(w))T_{z-w}^*f$$
$$+\varphi_0(z)p_{l-1}(\varphi(z),\varphi(w))f(0,w) - \varphi_0(w)p_{l-1}(\varphi(z),\varphi(w))f(z,0)$$

as desired.

The following theorem gives a recursive formula for those functions $\{d_e^k\}$.

Theorem 20. *Suppose that e is in L_0 and $\{d_e^k\}$ are a family of functions in $H^2(\mathbb{T}^2)$. Then*

for a given integer $n \geq 1$,

$$p_l(\varphi(z), \varphi(w))e + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_e^{l-k} \in \mathcal{H},$$

for each $1 \leq l \leq n$, if and only if the following recursive formula holds

$$\varphi_0(z)e(0, w) - \varphi_0(w)e(z, 0) + T_{z-w}^* d_e^1(z, w) = 0;$$

and

$$\varphi_0(z)d_e^k(0, w) - \varphi_0(w)d_e^k(z, 0) + T_{z-w}^*(d_e^{k+1})(z, w) = 0,$$

for $1 \leq k \leq n - 1$.

Proof. For a given $e \in L_0$ and a family of functions $\{d_e^k\} \subset H^2(\mathbb{T}^2)$, for each integer $l \geq 1$, let

$$E_l = p_l(\varphi(z), \varphi(w))e + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_e^{l-k}.$$

E_l is in $\mathcal{H}$ for each $1 \leq l \leq n$, iff

$$T_{z-w}^* E_l = 0$$

for each $1 \leq l \leq n$. We need only show that for each $1 \leq l \leq n$,

$$T_{z-w}^* E_l = 0$$

is equivalent to the recursive formula in the theorem.

By Lemma 19, we have

$$T_{z-w}^* E_l$$

$$= T_{z-w}^*[p_l(\varphi(z), \varphi(w))e] + \sum_{k=0}^{l-1} T_{z-w}^*[p_k(\varphi(z), \varphi(w))d_e^{l-k}]$$

$$= p_l(\varphi(z), \varphi(w))T_{z-w}^* e + \varphi_0(z)p_{l-1}(\varphi(z), \varphi(w))e(0, w)$$

$$-\varphi_0(w)p_{l-1}(\varphi(z), \varphi(w))e(z, 0) + \sum_{k=1}^{l-1}[p_k(\varphi(z), \varphi(w))T_{z-w}^* d_e^{l-k}$$

$$+\varphi_0(z)p_{k-1}(\varphi(z), \varphi(w))d_e^{l-k}(0, w) - \varphi_0(w)p_{k-1}(\varphi(z), \varphi(w))d_e^{l-k}(z, 0)]$$

$$= p_{l-1}(\varphi(z), \varphi(w))[\varphi_0(z)e(0, w) - \varphi_0(w)e(z, 0) + T_{z-w}^* d_e^1]$$

$$+\sum_{k=0}^{l-2}[p_k(\varphi(z), \varphi(w))(T_{z-w}^* d_e^{l-k} + \varphi_0(z)d_e^{l-k-1}(0, w) - \varphi_0(w)d_e^{l-1-k}(z, 0))]$$

since e is in L_0. Thus $T_{z-w}^* E_l = 0$ for each $1 \le l \le n$ iff

$$\varphi_0(z)e(0, w) - \varphi_0(w)e(z, 0) + T_{z-w}^* d_e^1 = 0,$$

and

$$T_{z-w}^* d_e^{l-k} + \varphi_0(z)d_e^{l-k-1}(0, w) - \varphi_0(w)d_e^{l-1-k}(z, 0)) = 0,$$

for $1 \le k < l \le n$. This completes the proof.

II.4.2   Second Construction

Next for a given $e \in L_0$, we will show that there is a function $d_e \in \mathcal{L}_\varphi$ such that

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e \in \mathcal{H}$$

for each $l \ge 1$.

Recall that φ is a Blaschke product with zeros $\{\alpha_k\}_0^K$ and α_k repeats $n_k + 1$ times, and $\varphi(z) = z\varphi_0(z)$ where φ_0 is a Blaschke product with $N - 1$ zeros.

Given a point $\alpha \in \mathbb{D}$ and integer $s \ge 0$, recall

$$k_\alpha^s(z) = \frac{s! z^s}{(1 - \bar{\alpha}z)^{s+1}}.$$

For each $\alpha \in \mathbb{D}$ and integer $t \geq 0$, let

$$e_\alpha^t(z, w) = \sum_{s=0}^t \frac{t!}{s!(t-s)!} k_\alpha^s(z) k_\alpha^{t-s}(w). \tag{II.4}$$

The Mittag-Leffler expansion of the finite Blaschke product φ_0 is

$$\varphi_0(z) = \sum_{i=0}^K \sum_{t=0}^{n_i} c_i^t k_{\alpha_i}^t(z),$$

for some constants $\{c_i^t\}$. Define

$$e_0(z, w) = \sum_{i=0}^K \sum_{t=0}^{n_i} c_i^t e_{\alpha_i}^t(z, w).$$

Clearly,

$$e_0(z, 0) = \varphi_0(z).$$

Lemma 21. *For each $\alpha \in \mathbb{D}$ and $t \geq 0$, then*

$$e_\alpha^t(z, z) = \frac{(t+1)! z^t}{(1 - \bar{\alpha} z)^{t+2}}.$$

Proof. A simple calculation gives

$$
\begin{aligned}
e_\alpha^t(z, z) &= \sum_{s=0}^t \frac{t!}{s!(t-s)!} k_\alpha^s(z) k_\alpha^{t-s}(z) \\
&= \sum_{s=0}^t \frac{t!}{s!(t-s)!} \frac{s! z^s}{(1 - \bar{\alpha} z)^{s+1}} \frac{(t-s)! z^{t-s}}{(1 - \bar{\alpha} z)^{t-s+1}} \\
&= \sum_{s=0}^t \frac{t! z^t}{(1 - \bar{\alpha} z)^{t+2}} \\
&= \frac{(t+1)! z^t}{(1 - \bar{\alpha} z)^{t+2}}.
\end{aligned}
$$

This completes the proof.

Lemma 22. *For each $F(z, w) \in H^2(\mathbb{T}^2)$,*

$$\langle F, e_\alpha^t \rangle = [(\partial_z + \partial_w)^t F(z, w)]|_{z=w=\alpha}.$$

Proof. Let $F(z, w) \in H^2(\mathbb{T}^2)$. We have

$$\langle F, e_\alpha^t \rangle = \sum_{s=0}^{t} \frac{s!}{l!(s-l)!} \langle F, k_\alpha^s(z) k_\alpha^{t-s}(w) \rangle$$

$$= \sum_{s=0}^{t} \frac{s!}{l!(s-l)!} [\partial_z^s \partial_w^{t-s} F(z, w)]|_{z=w=\alpha}$$

$$= [\sum_{s=0}^{t} \frac{s!}{l!(s-l)!} \partial_z^s \partial_w^{t-s}][F(z, w)]|_{z=w=\alpha}$$

$$= [(\partial_z + \partial_w)^t F(z, w)]|_{z=w=\alpha}.$$

This completes the proof.

Noting that the dimension of L_0 is N and $\{e_{\alpha_i}^{t_i}(z, w) : 0 \leq i \leq K, \quad 0 \leq t_i \leq n_i\}$ are linearly independent, we immediately have the following lemma.

Lemma 23.

$$L_0 = span\{e_{\alpha_i}^{t_i}(z, w) : 0 \leq i \leq K, \quad 0 \leq t_i \leq n_i\}$$

Proof. By Lemma 4, those functions $\{e_{\alpha_i}^{t_i}(z, w) : 0 \leq i \leq K, \quad 0 \leq t_i \leq n_i\}$ are in the intersection $ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^*$. They are linearly independent since they are rational functions with different poles with multiplicity. So it suffices to show that they are in $\mathcal{H}$. To do so we need to show that

$$T_z^* e_\alpha^t = T_w^* e_\alpha^t.$$

A simple calculation gives

$$T_z^* e_\alpha^t = \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} (T_z^* k_\alpha^s(z)) k_\alpha^{t-s}(w)$$

$$= \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} [s k_\alpha^{s-1}(z) + \bar{\alpha} k_\alpha^s(z)] k_\alpha^{t-s}(w)$$

$$= \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} \bar{\alpha} k_\alpha^s(z) k_\alpha^{t-s}(w) + \sum_{s=1}^{t} \frac{t!}{(s-1)!(t-s)!} k_\alpha^{s-1}(z) k_\alpha^{t-s}(w)$$

$$= \bar{\alpha} e_\alpha^t + \sum_{l=0}^{t-1} \frac{t!}{l!(t-1-l)!} k_\alpha^l(z) k_\alpha^{t-1-l}(w)$$

$$= \bar{\alpha} e_\alpha^t + t e_\alpha^{t-1}.$$

The fourth equality follows from the substitution $l = s - 1$. Similarly, we also have

$$T_w^* e_\alpha^t = \bar{\alpha} e_\alpha^t + t e_\alpha^{t-1}.$$

Hence we conclude that $T_z^* e_\alpha^t = T_w^* e_\alpha^t$, to complete the proof.

Consequently, the above lemma gives the following lemma.

Lemma 24. *For each function* $F(z, w) \in ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^*$, *there is a function* $E(z, w) \in L_0$ *such that*

$$F(z, 0) = E(z, 0).$$

Proof. Suppose that $F(z, w)$ is in $ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^*$. Lemma 10 implies that there are constants c_{ij}^{sl} such that

$$F(z, w) = \sum_{i,j=0}^{K} \sum_{s=0, l=0}^{n_i, n_j} c_{ij}^{sl} k_{\alpha_i}^s(z) k_{\alpha_j}^l(w).$$

Since $k_{\alpha_j}^l(0) = 0$ for $l > 0$ and $k_{\alpha_j}^0(0) = 1$, we have

$$
\begin{aligned}
F(z, 0) &= \sum_{i,j=0}^{K} \sum_{s=0}^{n_i} c_{ij}^{s0} k_{\alpha_i}^s(z) \\
&= \sum_{i=0}^{K} \sum_{s=0}^{n_i} [\sum_{j=0}^{K} c_{ij}^{s0}] k_{\alpha_i}^s(z).
\end{aligned}
$$

Let

$$E(z, w) = \sum_{i=0}^{K} \sum_{s=0}^{n_i} [\sum_{j=0}^{K} c_{ij}^{s0}] e_{\alpha_i}^s(z, w).$$

Noting

$$e_\alpha^t(z, 0) = k_\alpha^t(z),$$

we conclude

$$F(z, 0) = E(z, 0),$$

to complete the proof.

Lemma 25. *If for a function* $f \in \mathcal{H}$, $p_l(\varphi(z), \varphi(w)) f \in \mathcal{H}$, *for each* $l \geq 0$, *then* $f(z, 0) =$

$\lambda \varphi_0(z)$, *for constant λ.*

Proof. Suppose that $p_l(\varphi(z), \varphi(w))f \in \mathcal{H}$, for each $l \geq 0$. Let $d_f^k = 0$. Then

$$p_l(\varphi(z), \varphi(w))f + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_f^{l-k} \in \mathcal{H},$$

for each $l \geq 1$. By Theorem 20, we have

$$\varphi_0(z)f(0, w) - \varphi_0(w)f(z, 0) = 0.$$

This gives

$$\frac{f(z, 0)}{\varphi_0(z)} = \frac{f(0, w)}{\varphi_0(w)}$$

holds for all $(z, w) \in \mathbb{D} \times \mathbb{D}$ except for a finite vertical or horizontal lines. Thus the equality holds for an open subset of D^2, and so there is a constant λ such that $f(z, 0) = \lambda \varphi_0(z)$ on the unit disk. This completes the proof.

Theorem 26. *For a given $e \in L_0$, there is a unique function $d_e \in \mathcal{L}_\varphi \ominus e_0$ such that*

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e \in \mathcal{H}$$

for each $l \geq 1$. If e is linearly independent of e_0, then $d_e \neq 0$. Moreover, the mapping

$$e \to d_e$$

is a linear operator from L_0 into $\mathcal{L}_\varphi \ominus e_0$.

Proof. First we show the existence of d_e. For the given e, by Theorem 16, there is a function $d_e^1 \in \mathcal{L}_\varphi$ such that

$$p_1(\varphi(z), \varphi(w))e + d_e^1 \in \mathcal{H}.$$

By Theorem 20 we have

$$\varphi_0(z)e(0, w) - \varphi_0(w)e(z, 0) + T_{z-w}^* d_e^1(z, w) = 0. \tag{II.5}$$

Since $e(z, w)$ is in $\mathcal{H}$, by Lemma 8, $d_e^1(z, w)$ is symmetric with respect to z and w. In addition, $p_1(\varphi(z), \varphi(w))$ is also symmetric with respect to z and w. This gives

$$d_e^1(z, w) = d_e^1(w, z).$$

Thus

$$d_e^1(z, 0) = d_e^1(0, z).$$

By Lemma 24, choose a function $\tilde{e}(z, w) \in L_0$ such that

$$d_e^1(z, 0) = \tilde{e}(z, 0).$$

Hence

$$d_e^1(0, z) = \tilde{e}(0, z),$$

because $\tilde{e}(z, w)$ is also symmetric with respect to z and w. Let $d_e = d_e^1 - \tilde{e}$. Clearly,

$$p_1(\varphi(z), \varphi(w))e + d_e \in \mathcal{H},$$

and

$$
\begin{aligned}
d_e(z, 0) &= d_e(0, z) \\
&= d_e^1(z, 0) - \tilde{e}(z, 0) \\
&= 0.
\end{aligned}
$$

Letting $\tilde{d}_e^1 = d_e$ and $\tilde{d}_e^k = 0$, for $k > 1$, by (II.5), we have following equations:

$$\varphi_0(z)e(0, w) - \varphi_0(w)e(z, 0) + T_{z-w}^* \tilde{d}_e^1(z, w)$$

$$= \varphi_0(z)e(0, w) - \varphi_0(w)e(z, 0) + T_{z-w}^*[d_e^1(z, w) - \tilde{e}(z, w)]$$

$$= 0,$$

$$\varphi_0(z)\tilde{d}_e^k(0, w) - \varphi_0(w)\tilde{d}_e^k(z, 0) + T_{z-w}^*(\tilde{d}_e^{k+1})(z, w)$$

$$= 0 - 0 - 0$$

$$= 0$$

for $1 \leq k \leq l - 1$. The last equality in the first equation follows from that $T_{z-w}^* \tilde{e}(z, w) = 0$. By Theorem 20, we conclude that

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e \in \mathcal{H},$$

as desired.

Next we show that if there is another function $b_e \in \mathcal{L}_\varphi$ such that

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))b_e \in \mathcal{H},$$

for each $l \geq 1$, then $d_e - b_e = \mu e_0$ for some constant μ.

Since

$$p_{l-1}(\varphi(z), \varphi(w))[d_e - b_e] = p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e$$
$$- (p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))b_e) \in \mathcal{H},$$

letting $f = d_e - b_e$, we have that $f \in \mathcal{H}$ and

$$p_l(\varphi(z), \varphi(w))f \in \mathcal{H}.$$

By Theorem 31, we obtain that $f = \lambda e_0$ to conclude

$$d_e = b_e + \lambda e_0.$$

If $d_e = 0$, i.e.,

$$p_l(\varphi(z), \varphi(w))e \in \mathcal{H},$$

then Theorem 31 again implies that $e = \lambda e_0$. This gives that if e is linearly independent of e_0, then $d_e \neq 0$.

As showed above, we know that the mapping $e \to d_e$ is well-defined from L_0 into $\mathcal{L}_\varphi \ominus e_0$. To finish the proof we need to show that the mapping is linear. To do so, let e_1 and e_2 be in L_0. For given constants c_1 and c_2, we have

$$p_l(\varphi(z), \varphi(w))e_1 + p_{l-1}(\varphi(z), \varphi(w))d_{e_1} \in \mathcal{H}$$

$$p_l(\varphi(z), \varphi(w))e_2 + p_{l-1}(\varphi(z), \varphi(w))d_{e_2} \in \mathcal{H}$$

$$p_l(\varphi(z), \varphi(w))[c_1 e_1 + c_2 e_2] + p_{l-1}(\varphi(z), \varphi(w))d_{c_1 e_1 + c_2 e_2} \in \mathcal{H}.$$

Thus

$$p_{l-1}(\varphi(z), \varphi(w))[c_1 d_{e_1} + c_2 d_{e_2} - d_{c_1 e_1 + c_2 e_2}] \in \mathcal{H},$$

for each $l \geq 1$. By Theorem 31,

$$c_1 d_{e_1} + c_2 d_{e_2} - d_{c_1 e_1 + c_2 e_2} = c_3 e_0,$$

for some constant c_3. But d_{e_1}, d_{e_2}, and $d_{c_1 e_1 + c_2 e_2}$ are orthogonal to e_0. We conclude

$$c_1 d_{e_1} + c_2 d_{e_2} - d_{c_1 e_1 + c_2 e_2} = 0,$$

to complete the proof.

By Theorem 20, the function d_e can be constructed from e by a formula.

Corollary 27. *Let e be in L_0. For a function $d_e \in H^2(\mathbb{T}^2)$,*

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e \in \mathcal{H}$$

for each $l \geq 1$, iff

$$\varphi_0(z)e(0, w) - \varphi_0(w)e(z, 0) + T^*_{z-w}d_e(z, w) = 0,$$

and

$$\varphi_0(z)d_e(0, w) - \varphi_0(w)d_e(z, 0) = 0.$$

II.4.3   Third Construction

In this section, for a given element $e(z, w)$ in L_0, we will obtain a simple formula of another function, denoted by d_e^0, such that

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e^0 \in \mathcal{H},$$

for $l \geq 1$. Again, we first consider the example where the zeros of φ are distinct.

   Example. Let $\{\alpha_i\}_{i=1}^{n-1}$ be nonzero distinct points in $\mathbb{D}$. Let φ be the Blaschke product $z \prod_{i=1}^{n-1} \frac{z-\alpha_i}{1-\overline{\alpha}_i z}$, and $\varphi_0 = \prod_{i=1}^{n-1} \frac{z-\alpha_i}{1-\overline{\alpha}_i z}$. The Mittag-Leffler expansion of φ_0 is

$$\varphi_0(z) = c_0 + \sum_{i=1}^{n-1} c_i k_{\alpha_i}(z)$$

for some constants c_i and hence

$$e_0(z, w) = c_0 + \sum_{i=1}^{n-1} c_i k_{\alpha_i}(z) k_{\alpha_i}(w).$$

   For each $e \in L_0$, we will find a function d_e^0 such that

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e^0 \in \mathcal{H},$$

for $l \geq 1$.

To do so, write

$$e(z, w) = \sum_{i=0}^{n-1} u_i k_{\alpha_i}(z) k_{\alpha_i}(w),$$

for some constants u_i. We shall solve the following equation

$$\varphi_0(z)e(0, w) - \varphi_0(w)e(z, 0) + T_{z-w}^* \tilde{d}_e(z, w) = 0 \qquad (\text{II.6})$$

for $\tilde{d}_e(z, w)$ in the form

$$\tilde{d}_e(z, w) = \sum_{i<j} \gamma_{ij}[k_{\alpha_i}(z)k_{\alpha_j}(w) + k_{\alpha_j}(z)k_{\alpha_i}(w)]$$

where γ_{ij} are constants. Since

$$e(z, 0) = \sum_{i=0}^{n-1} u_i k_{\alpha_i}(z)$$

and

$$\varphi_0(z) = c_0 + \sum_{i=1}^{n-1} c_i k_{\alpha_i}(z),$$

(II.6) gives

$$\sum_{i,j}(c_i u_j - c_j u_i)k_{\alpha_i}(z)k_{\alpha_j}(w) + T_{z-w}^* \tilde{d}_e(z, w) = 0.$$

Grouping the first sum and noting that

$$T_{z-w}^* k_{\alpha_i}(z)k_{\alpha_j}(w) = (\bar{\alpha}_i - \bar{\alpha}_j)k_{\alpha_i}(z)k_{\alpha_j}(w),$$

we obtain

$$\sum_{i<j}(c_i u_j - c_j u_i)[k_{\alpha_i}(z)k_{\alpha_j}(w) - k_{\alpha_j}(z)k_{\alpha_i}(w)] +$$
$$\sum_{i<j}\gamma_{ij}(\bar{\alpha}_i - \bar{\alpha}_j)[k_{\alpha_i}(z)k_{\alpha_j}(w) - k_{\alpha_j}(z)k_{\alpha_i}(w)] = 0.$$

Because $\{k_{\alpha_i}(z)k_{\alpha_j}(w) - k_{\alpha_j}(z)k_{\alpha_i}(w)\}_{i<j}$ are linearly independent, the above equation implies

$$\gamma_{ij} = -\frac{c_i u_j - c_j u_i}{\bar{\alpha}_i - \bar{\alpha}_j}.$$

This gives

$$\tilde{d}_e = \sum_{i<j} -[\frac{c_i u_j - c_j u_i}{\bar{\alpha}_i - \bar{\alpha}_j}][k_{\alpha_i}(z)k_{\alpha_j}(w) + k_{\alpha_j}(z)k_{\alpha_i}(w)].$$

Let

$$d_e^0 = \sum_{i<j} -[\frac{c_i u_j - c_j u_i}{\bar{\alpha}_i - \bar{\alpha}_j}]\beta_{ij}(z,w),$$

where

$$\beta_{ij}(z,w) = k_{\alpha_i}(z)k_{\alpha_j}(w) + k_{\alpha_j}(z)k_{\alpha_i}(w) - k_{\alpha_i}(z)k_{\alpha_i}(w) - k_{\alpha_j}(z)k_{\alpha_j}(w).$$

Since $k_{\alpha_i}(z)k_{\alpha_i}(w)$ is in $\mathcal{H}$, $\tilde{d}_e - d_e^0$ is in $\mathcal{H}$. Thus d_e^0 is also a solution of (II.6). If let $d_e^1 = d_e^0$ and $d_e^k = 0$ for $k > 1$, then those functions satisfy the equations in Theorem 20 and

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e^0 \in \mathcal{H},$$

for $l \geq 1$.

The above example suggests the following result.

Theorem 28. *Let $e(z,w)$ be in L_0. Then*

$$d_e^0(z,w) = we(0,w)e_0(z,w) - w\varphi_0(w)e(z,w)$$

is a function such that

$$p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e^0 \in \mathcal{H},$$

for $l \geq 1$.

Proof. Note that $d_e^0(z,0) = d_e^0(0,w) = 0$. In order to show that $p_l(\varphi(z), \varphi(w))e + p_{l-1}(\varphi(z), \varphi(w))d_e^0 \in \mathcal{H}$, for $l \geq 1$, by Corollary 27 we need only to show that

$$\varphi_0(z)e(0,w) - \varphi_0(w)e(z,0) + [T_z^* - T_w^*]d_e^0 = 0.$$

By Lemma 18, simple calculations give

$$
\begin{aligned}
T_w^* d_e^0 &= T_w^* [w \varphi_0(w)(T_w T_w^* + \sum_{l \geq 0} z^l \otimes z^l) e(z, w)] \\
&\quad - e(0, w)(T_w T_w^* + \sum_{l \geq 0} z^l \otimes z^l) e_0(z, w) \\
&= w \varphi_0(w)(T_w^* e)(z, w) \\
&\quad + \varphi_0(w) e(z, 0) - w e(0, w)(T_w^* e_0)(z, w) - e(0, w) e_0(z, 0),
\end{aligned}
$$

and

$$
T_z^* d_e^0 = w \varphi_0(w)(T_z^* e)(z, w) - w e(0, w)(T_z^* e_0)(z, w).
$$

Noting that $T_z^* e = T_w^* e$, and $T_z^* e_0 = T_w^* e_0$, by the above two equations we have

$$
\begin{aligned}
[T_z^* - T_w^*] d_e^0 &= e(0, w) e_0(z, 0) - \varphi_0(w) e(z, 0) \\
&= \varphi_0(z) e(0, w) - \varphi_0(w) e(z, 0).
\end{aligned}
$$

The last equality follows from that $\varphi_0(z) = e_0(z, 0)$. This gives the desired result.

In [32], it was shown that distinguished reducing subspace equals

$$
\mathcal{M}_0 = span_{l \geq 0} \{ p_l(\varphi(z), \varphi(w)) e_0 \}.
$$

We will give more details about $\mathcal{M}_0$ in the next section.

Theorem 29. *If $\mathcal{M} \subset \mathcal{H}$ is a reducing subspace of $\varphi(\mathcal{B})$ orthogonal to $\mathcal{M}_0$, for $e \in \mathcal{M} \cap L_0$, let d_e be the function in Theorem 26, then*

$$
p_l(\varphi(z), \varphi(w)) e + p_{l-1}(\varphi(z), \varphi(w)) d_e \in \mathcal{M}
$$

for each $l \geq 1$, and there is $\tilde{e} \in \mathcal{M} \cap L_0$ such that

$$
d_e^1 = d_e + \tilde{e}.
$$

Proof. Since $\mathcal{M}$ is orthogonal to $\mathcal{M}_0$, we have

$$
\begin{aligned}
\mathcal{H} \quad &= \mathcal{M}_0 \oplus \mathcal{M}_0^\perp \\
&= \mathcal{M}_0 \oplus \mathcal{M} \oplus [\mathcal{M}_0^\perp \cap \mathcal{M}^\perp].
\end{aligned}
$$

Thus

$$
L_0 = Ce_0 \oplus [\mathcal{M} \cap L_0] \oplus [\mathcal{M}_0^\perp \cap \mathcal{M}^\perp \cap L_0].
$$

So e is orthogonal to e_0, and

$$
L_0 \ominus e_0 = [\mathcal{M} \cap (L_0 \ominus e_0)] \oplus [\mathcal{M}_0^\perp \cap \mathcal{M}^\perp \cap (L_0 \ominus e_0)].
$$

By Theorem 16, there is a function $d_e^k \in \mathcal{L}_\varphi \ominus L_0$ such that

$$
p_l(\varphi(z), \varphi(w))e + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_e^{l-k} \in \mathcal{M},
$$

for each $l \geq 1$. Thus

$$
d_e - d_e^1 = p_1(\varphi(z), \varphi(w))e + d_e - (p_1(\varphi(z), \varphi(w))e + d_e^1) \in \mathcal{H}.
$$

So $d_e - d_e^1$ is in $L_0 \ominus e_0$. Write

$$
d_e - d_e^1 = e' + e''
$$

for $e' \in \mathcal{M} \cap (L_0 \ominus e_0)$ and $e'' \in \mathcal{M}^\perp \cap (L_0 \ominus e_0)$. Thus

$$
\begin{aligned}
&p_2(\varphi(z), \varphi(w))e + p_1(\varphi(z), \varphi(w))d_e \\
= \quad &[p_2(\varphi(z), \varphi(w))e + p_1(\varphi(z), \varphi(w))d_e^1 + d_e^2] + [p_1(\varphi(z), \varphi(w))e' \\
&+ d_{e'}^1] + [p_1(\varphi(z), \varphi(w))e'' + d_{e''}^1] - (d_e^2 + d_{e'}^1 + d_{e''}^1).
\end{aligned}
$$

Since

$$
p_2(\varphi(z), \varphi(w))e + p_1(\varphi(z), \varphi(w))d_e^1 + d_e^2 \in \mathcal{M},
$$

$$p_1(\varphi(z), \varphi(w))e' + d^1_{e'} \in \mathcal{M},$$

and

$$p_1(\varphi(z), \varphi(w))e'' + d^1_{e''} \in \mathcal{M}^\perp,$$

we have

$$d^2_e + d^1_{e'} + d^1_{e''} \in \mathcal{H} \cap ker T^*_{\varphi(z)} \cap T^*_{\varphi(w)} = L_0.$$

Noting

$$d^2_e + d^1_{e'} + d^1_{e''} \in \mathcal{L}_\varphi \ominus L_0,$$

we have

$$d^2_e + d^1_{e'} + d^1_{e''} = 0,$$

to get

$$P_{\mathcal{H}} d^1_{e''} = -(P_{\mathcal{H}} d^1_{e'} + P_{\mathcal{H}} d^2_e).$$

But Theorem 17 gives that

$$P_{\mathcal{H}} d^1_{e'} + P_{\mathcal{H}} d^2_e \in \mathcal{M},$$

and

$$P_{\mathcal{H}} d^1_{e''} \in \mathcal{M}^\perp.$$

Thus

$$P_{\mathcal{H}} d^1_{e''} = 0,$$

and so

$$
\begin{aligned}
\|d^1_{e''}\|^2 &= \langle d^1_{e''}, d^1_{e''} \rangle \\
&= \langle d^1_{e''}, p_1(\varphi(z), \varphi(w))e'' + d^1_{e''} \rangle \\
&= \langle d^1_{e''}, P_{\mathcal{H}}[p_1(\varphi(z), \varphi(w))e'' + d^1_{e''}] \rangle \\
&= \langle P_{\mathcal{H}}(d^1_{e''}), p_1(\varphi(z), \varphi(w))e'' + d^1_{e''} \rangle = 0.
\end{aligned}
$$

This gives that $d^1_{e''} = 0$. We have that $p_1(\varphi(z), \varphi(w))e'' \in \mathcal{H}$. Theorem 31 gives that

$e'' = \lambda e_0$, for some constant λ. Since $e'' \in \mathcal{M}^{\perp} \cap (L_0 \ominus e_0)$ we conclude that $e'' = 0$. Hence $d_e = d_e^1 + e'$. Letting $\tilde{e} = -e'$ we obtain $d_e^1 = d_e + \tilde{e}$, as desired. Now we write

$$p_n(\varphi(z), \varphi(w))e + p_{n-1}(\varphi(z), \varphi(w))d_e$$
$$= \ p_n(\varphi(z), \varphi(w))e + p_{n-1}(\varphi(z), \varphi(w))d_e^1 + p_{n-1}(\varphi(z), \varphi(w))e'$$
$$= \ [p_n(\varphi(z), \varphi(w))e + \sum_{k=0}^{n-1} p_k(\varphi(z), \varphi(w))d_e^{n-k}]$$
$$+ [p_{n-1}(\varphi(z), \varphi(w))e' - \sum_{k=0}^{n-2} p_k(\varphi(z), \varphi(w))d_e^{n-k}].$$

Because $p_n(\varphi(z), \varphi(w))e + p_{n-1}(\varphi(z), \varphi(w))d_e$ is in $\mathcal{H}$, the above equality becomes

$$p_n(\varphi(z), \varphi(w))e + p_{n-1}(\varphi(z), \varphi(w))d_e$$
$$= \ [p_n(\varphi(z), \varphi(w))e + \sum_{k=0}^{n-1} p_k(\varphi(z), \varphi(w))d_e^{n-k}]$$
$$+ P_{\mathcal{H}}[p_{n-1}(\varphi(z), \varphi(w))e' - \sum_{k=0}^{n-2} p_k(\varphi(z), \varphi(w))d_e^{n-k}]$$
$$= \ [p_n(\varphi(z), \varphi(w))e + \sum_{k=0}^{n-1} p_k(\varphi(z), \varphi(w))d_e^{n-k}] + P_{\mathcal{H}}[p_{n-1}(\varphi(z), \varphi(w))e']$$
$$- \sum_{k=0}^{n-2} P_{\mathcal{H}}(p_k(\varphi(z), \varphi(w))d_e^{n-k}).$$

Theorem 15 gives

$$p_n(\varphi(z), \varphi(w))e + \sum_{k=0}^{n-1} p_k(\varphi(z), \varphi(w))d_e^{n-k} \in \mathcal{M}$$

and Theorem 17 gives
$$\sum_{k=0}^{n-2} P_{\mathcal{H}}(p_k(\varphi(z), \varphi(w))d_e^{n-k}) \in \mathcal{M}.$$

Since $\mathcal{M}$ is a reducing subspace of $\varphi(\mathcal{B})$ and e' is in $\mathcal{M}$, we have

$$P_{\mathcal{H}}[p_{n-1}(\varphi(z), \varphi(w))e'] \in \mathcal{M},$$

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to conclude

$$p_n(\varphi(z), \varphi(w))e + p_{n-1}(\varphi(z), \varphi(w))d_e \in \mathcal{M}$$

for each $n \geq 0$. This completes the proof.

Combining Theorem 26 with Theorems 28 and 29 gives the following important relation between d_e^1 and d_e^0.

Theorem 30. *If $\mathcal{M}$ is a reducing subspace of $\varphi(\mathcal{B})$ orthogonal to the distinguished reducing subspace $\mathcal{M}_0$, then for each $e \in \mathcal{M} \cap L_0$, there is an element $\tilde{e} \in \mathcal{M} \cap L_0$ and a number λ such that*

$$d_e^1 = d_e^0 + \tilde{e} + \lambda e_0.$$

II.5   The distinguished reducing subspace

Theorems 16 and 20 are useful in studying reducing subspaces of $\varphi(\mathcal{B})$. In this section we will use them to show that there always exists a unique reducing subspace $\mathcal{M}_0$ for $\varphi(\mathcal{B})$ such that the restriction of $\varphi(\mathcal{B})$ on $\mathcal{M}_0$ is unitarily equivalent to the Bergman shift. The existence of such a $\mathcal{M}_0$ is the main result in [32]. Furthermore, we will show that such kind of reducing subspace is unique. We call $\mathcal{M}_0$ the distinguished reducing subspace for $\varphi(\mathcal{B})$.

Theorem 31. *If for a function $f \in \mathcal{H}$, $p_l(\varphi(z), \varphi(w))f \in \mathcal{H}$, for each $l \geq 0$, then there exists a constant λ such that $f = \lambda e_0$.*

Proof. Suppose that $p_l(\varphi(z), \varphi(w))f \in \mathcal{H}$, for each $l \geq 0$. By Lemma 25, then $f(z,0) = \lambda \varphi_0(z)$, for constant λ. Thus $f(z,0) = \lambda e_0(z,0)$. Since $f - \lambda e$ is in $\mathcal{H}$, Lemma 9 implies that $f(z,z) - \lambda e_0(z,z) = 0$. By Lemma 7, we conclude that $f = \lambda e_0$.

Lemma 32. *Let f be a function in $H^2(\mathbb{T}^2)$. Then*

$$P_{\mathcal{H}}[\varphi(z)p_n(\varphi(z), \varphi(w))f] = \frac{n+1}{n+2}P_{\mathcal{H}}[p_{n+1}(\varphi(z), \varphi(w))f].$$

Proof. Write

$$\varphi(z)p_n(\varphi(z),\varphi(w)) = \varphi(z)^{n+1} + \varphi(z)^n\varphi(w) + \cdots + \varphi(z)\varphi(w)^n$$

$$= \frac{n+1}{n+2}[\varphi(z)^{n+1} + \varphi(z)^n\varphi(w) + \cdots + \varphi(z)\varphi(w)^n + \varphi(w)^{n+1}]$$

$$+ \frac{1}{n+2}[(\varphi(z)^{n+1} - \varphi(w)^{n+1}) + (\varphi(z)^n\varphi(w) - \varphi(w)^{n+1})$$

$$+ \cdots + (\varphi(z)\varphi(w)^n - \varphi(w)^{n+1})]$$

$$= \frac{n+1}{n+2}p_{n+1}(\varphi(z),\varphi(w)) + J(z,w),$$

where

$$J(z,w) = \frac{1}{n+2}[(\varphi(z)^{n+1} - \varphi(w)^{n+1}) + (\varphi(z)^n\varphi(w) - \varphi(w)^{n+1}) + \cdots$$

$$+ (\varphi(z)\varphi(w)^n - \varphi(w)^{n+1})].$$

Observe $J(z,z) = 0$. Thus $[J(z,w)f(z,w)]|_{w=z} = 0$. By Lemma 6 we have that $J(z,w)f(z,w)$ is in $\mathcal{H}^\perp$, to get $P_{\mathcal{H}}(J(z,w)f(z,w)) = 0$. Hence we conclude

$$P_{\mathcal{H}}[\varphi(z)p_n(\varphi(z),\varphi(w))f] = P_{\mathcal{H}}[\frac{n+1}{n+2}p_{n+1}(\varphi(z),\varphi(w))f + J(z,w)f]$$

$$= \frac{n+1}{n+2}P_{\mathcal{H}}[p_{n+1}(\varphi(z),\varphi(w))f]$$

to complete the proof.

Now we are ready to prove the main result in this section.

Theorem 33. *There is a unique reducing subspace $\mathcal{M}_0$ for $\varphi(\mathcal{B})$ such that $\varphi(\mathcal{B})|_{M_0}$ is unitarily equivalent to the Bergman shift. In fact,*

$$\mathcal{M}_0 = span_{l \geq 0}\{p_l(\varphi(z),\varphi(w))e_0\},$$

and $\{\frac{p_l(\varphi(z),\varphi(w))e_0}{\sqrt{l+1}\|e_0\|}\}_0^\infty$ form an orthonormal basis of $\mathcal{M}_0$.

Proof. First we show that there exists a reducing subspace $\mathcal{M}_0$ of $\varphi(\mathcal{B})$ such that $\varphi(\mathcal{B})|_{M_0}$

is unitarily equivalent to the Bergman shift. Let

$$e_0(z,w) = \sum_{i=0}^{K} \sum_{t=0}^{n_i} c_i^t e_{\alpha_i}^t(z,w).$$

Then $e_0(z,w)$ is a nonzero function in L_0 and $e_0(z,0) = \varphi_0(z)$. Letting $d_{e_0}^k = 0$ for each $k \geq 1$, we obtain that $\{d_{e_0}^k\}$ satisfy the following recursive formula:

$$\varphi_0(z)e_0(0,w) - \varphi_0(w)e_0(z,0) + T_{z-w}^* d_{e_0}^1(z,w) = 0$$

and

$$\varphi_0(z)d_{e_0}^k(0,w) - \varphi_0(w)d_{e_0}^k(z,0) + T_{z-w}^*(d_{e_0}^{k+1})(z,w) = 0,$$

for $1 \leq k \leq n-1$. Theorem 20 gives that $p_l(\varphi(z),\varphi(w))e_0 \in \mathcal{H}$. Note that $T_{\varphi(z)}^* e_0 = T_{\varphi(w)}^* e_0 = 0$. A simple calculation gives

$$\|p_l(\varphi(z),\varphi(w))e_0\|_2^2 = (l+1)\|e_0\|_2^2,$$

and

$$\langle p_l(\varphi(z),\varphi(w))e_0, p_n(\varphi(z),\varphi(w))e_0 \rangle = 0,$$

for $n \neq l$. Let $E_n = \frac{p_n(\varphi(z),\varphi(w))e_0}{\sqrt{(n+1)}\|e_0\|_2}$, and $\mathcal{M}_0 = span_{n \geq 0}\{p_n(\varphi(z),\varphi(w))e_0\}$. Thus $\{E_n\}$ are an orthonormal basis of $\mathcal{M}_0$. Noting

$$\begin{aligned} T_{\varphi(z)}^*(p_n(\varphi(z),\varphi(w))e_0) &= T_{\varphi(w)}^*(p_n(\varphi(z),\varphi(w))e_0) \\ &= p_{n-1}(\varphi(z),\varphi(w))e_0, \end{aligned}$$

we see that $\mathcal{M}_0$ is a reducing subspace of $\varphi(M)$. By Lemma 32 we have

$$\begin{aligned} \varphi(\mathcal{B})[p_n(\varphi(z),\varphi(w))e_0] &= P_{\mathcal{H}}[\varphi(z)p_n(\varphi(z),\varphi(w))e_0] \\ &= P_{\mathcal{H}}[\frac{n+1}{n+2}p_{n+1}(\varphi(z),\varphi(w))e_0] \\ &= \frac{n+1}{n+2}p_{n+1}(\varphi(z),\varphi(w))e_0, \end{aligned}$$

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to obtain

$$\begin{aligned}
\varphi(\mathcal{B})E_n &= \frac{\varphi(\mathcal{B})[p_n(\varphi(z),\varphi(w))e_0]}{\sqrt{(n+1)}\|e_0\|_2} \\
&= \frac{n+1}{n+2}\frac{p_{n+1}(\varphi(z),\varphi(w))e_0}{\sqrt{(n+1)}\|e_0\|_2} \\
&= \sqrt{\frac{n+1}{n+2}}E_{n+1}.
\end{aligned}$$

Clearly, $\varphi(\mathcal{B})^*E_0 = 0$. This implies that $\varphi(\mathcal{B})|_{\mathcal{M}_0}$ is unitarily equivalent to the Bergman shift.

Suppose that M_1 is a reducing subspace of $\varphi(\mathcal{B})$ and $\varphi(M)|_{M_1}$ is unitarily equivalent to the Bergman shift, i.e., there is an orthonormal basis $\{E_n\}$ of M_1 such that

$$\varphi(\mathcal{B})E_n = \sqrt{\frac{n+1}{n+2}}E_{n+1}.$$

Next we will show that $M_1 = \mathcal{M}_0$. Observe

$$\begin{aligned}
P_{\mathcal{H}}[(\varphi(z)+\varphi(w))E_0] &= 2\varphi(\mathcal{B})E_0 \\
&= \frac{2}{\sqrt{2}}E_1.
\end{aligned}$$

Thus

$$\|P_{\mathcal{H}}[(\varphi(z)+\varphi(w))E_0]\|^2 = 2.$$

Since

$$\begin{aligned}
T^*_{\varphi(z)}E_0 &= \varphi(\mathcal{B})^*E_0 \\
&= 0,
\end{aligned}$$

a simple calculation gives

$$\begin{aligned}
\|(\varphi(z) + \varphi(w))E_0\|^2 \ &= \ \langle(\varphi(z) + \varphi(w))E_0, (\varphi(z) + \varphi(w))E_0\rangle \\
&= \ \langle\varphi(z)E_0, \varphi(z)E_0\rangle + \langle\varphi(w)E_0, \varphi(w)E_0\rangle \\
&\qquad + \langle\varphi(z)E_0, \varphi(w)E_0\rangle + \langle\varphi(w)E_0, \varphi(z)E_0\rangle \\
&= \ 2\langle E_0, E_0\rangle \\
&= \ 2.
\end{aligned}$$

Thus we obtain

$$P_{\boldsymbol{\mathcal{H}}^\perp}[(\varphi(z) + \varphi(w))E_0] = 0$$

because

$$\|(\varphi(z) + \varphi(w))E_0\|^2 = \|P_{\boldsymbol{\mathcal{H}}}[(\varphi(z) + \varphi(w))E_0]\|^2 + \|P_{\boldsymbol{\mathcal{H}}^\perp}[(\varphi(z) + \varphi(w))E_0]\|^2.$$

So $(\varphi(z) + \varphi(w))E_0$ is in $\boldsymbol{\mathcal{H}}$. By Theorem 20, we have

$$\varphi_0(z)E_0(0, w) - \varphi_0(w)E_0(z, 0) = 0.$$

Let $d_{E_0}^k = 0$ for each $k \geq 1$. The family $\{d_{E_0}^k\}$ satisfy the recursive formula in Theorem 20. Hence $p_l(\varphi(z), \varphi(w))E_0$ is in $\boldsymbol{\mathcal{H}}$. By Theorem 31, we obtain that

$$E_0(z, w) - \lambda e_0(z, w) = 0,$$

for some constant λ. Thus $\boldsymbol{\mathcal{M}}_0 \subset M_1$ and so $\boldsymbol{\mathcal{M}}_0$ is a reducing subspace of $\varphi(\boldsymbol{\mathcal{B}})|_{M_1}$, which is unitarily equivalent to the Bergman shift. Since the Bergman shift is irreducible, we conclude that $M_1 = \boldsymbol{\mathcal{M}}_0$.

II.6   Structure of reducing subspaces

In this section, we first derive some information about the minimal reducing subspaces, then we prove our first main result about the structure of reducing lattice for any multiplication

operator induced by a finite Blaschke product.

II.6.1 Minimal reducing subspaces

We start with a theorem which will be used in the proof of our second and third main result. The theorem says that every nontrivial minimal reducing subspace of $\varphi(\boldsymbol{\mathcal{B}})$ is either $\boldsymbol{\mathcal{M}}_0$ or orthogonal to $\boldsymbol{\mathcal{M}}_0$. Then we prove our main theorem of this subsection, Theorem 40, which gives a picture of how three minimal reducing subspaces are related. Theorem 40 will be used in a key step to completely determine the structure of reducing lattice involved Blaschke product of order four.

Theorem 34. *Suppose that Ω is a nontrivial minimal reducing subspace for $\varphi(\boldsymbol{\mathcal{B}})$. If Ω does not equal $\boldsymbol{\mathcal{M}}_0$ then Ω is a subspace of $\boldsymbol{\mathcal{M}}_0^\perp$.*

Proof. By Lemma 13, there is a function e in $\Omega \cap L_0$ such that $e = \lambda e_0 + e_1$ for some constant λ and a function e_1 in $\boldsymbol{\mathcal{M}}_0^\perp \cap L_0$. By Theorem 15

$$p_1(\varphi(z), \varphi(w))e + d_e^1 \in \Omega.$$

Here d_e^1 is the function constructed in Theorem 15. Let

$$E = \varphi(\boldsymbol{\mathcal{B}})^*[\varphi(\boldsymbol{\mathcal{B}})e] - \frac{1}{2}e.$$

Since $p_1(\varphi(z), \varphi(w))e_0$ is in $\boldsymbol{\mathcal{H}}$, we obtain

$$\varphi(\boldsymbol{\mathcal{B}})e_0 = \frac{p_1(\varphi(z), \varphi(w))e_0}{2}.$$

Hence

$$
\begin{aligned}
E &= \varphi(\mathcal{B})^*[\varphi(\mathcal{B})[\lambda e_0 + e_1] - \frac{1}{2}[\lambda e_0 + e_1] \\
&= \lambda\{\varphi(\mathcal{B})^*[\varphi(\mathcal{B})e_0] - \frac{1}{2}e_0\} + \varphi(\mathcal{B})^*[\varphi(\mathcal{B})e_1] - \frac{1}{2}e_1 \\
&= \varphi(\mathcal{B})^*[P_{\mathcal{H}}(\varphi(z)e_1)] - \frac{1}{2}e_1 \\
&= \frac{1}{2}[\varphi(\mathcal{B})^*[P_{\mathcal{H}}(p_1(\varphi(z),\varphi(w))e_1)] - e_1] \\
&= \frac{1}{2}[\varphi(\mathcal{B})^*[P_{\mathcal{H}}(p_1(\varphi(z),\varphi(w))e_1 + d_{e_1}^1 - d_{e_1}^1)] - e_1] \\
&= \frac{1}{2}\{\varphi(\mathcal{B})^*[p_1(\varphi(z),\varphi(w))e_1 + d_{e_1}^1] - \varphi(\mathcal{B})^*P_{\mathcal{H}}d_{e_1}^1 - e_1\} \\
&= \frac{1}{2}\{P[\overline{\varphi(z)}p_1(\varphi(z),\varphi(w))e_1 + \overline{\varphi(z)}d_{e_1}^1] - \varphi(\mathcal{B})^*P_{\mathcal{H}}d_{e_1}^1 - e_1\} \\
&= \frac{1}{2}\{e_1 - \varphi(\mathcal{B})^*P_{\mathcal{H}}d_{e_1}^1 - e_1\} \\
&= -\frac{1}{2}\varphi(\mathcal{B})^*P_{\mathcal{H}}d_{e_1}^1.
\end{aligned}
$$

The sixth equality holds because that $p_1(\varphi(z),\varphi(w))e_1 + d_{e_1}^1 \in \mathcal{H}$. The eighth equality follows from that $d_{e_1}^1$ is in $\mathcal{L}_\varphi$. We claim that $E \neq 0$. If this is not true, we would have

$$
\frac{1}{2}\varphi(\mathcal{B})^*P_{\mathcal{H}}d_{e_1}^1 = 0.
$$

This gives that $P_{\mathcal{H}}d_{e_1}^1$ is in L_0. And hence

$$
\begin{aligned}
0 &= \langle P_{\mathcal{H}}d_{e_1}^1, d_{e_1}^1 \rangle \\
&= \langle P_{\mathcal{H}}d_{e_1}^1, p_1(\varphi(z),\varphi(w))e_1 + d_{e_1}^1 \rangle \\
&= \langle d_{e_1}^1, p_1(\varphi(z),\varphi(w))e_1 + d_{e_1}^1 \rangle \\
&= \langle d_{e_1}^1, p_1(\varphi(z),\varphi(w))e_1 + d_{e_1}^1 \rangle \\
&= \langle d_{e_1}^1, d_{e_1}^1 \rangle \\
&= \|d_{e_1}^1\|^2.
\end{aligned}
$$

This gives that $d_{e_1}^1 = 0$. Thus we obtain that $p_1(\varphi(z),\varphi(w))e_1 \in \mathcal{H}$. By Theorem 20,

$$
p_l(\varphi(z),\varphi(w))e_1 \in \mathcal{H},
$$

for $l \geq 0$. Hence by Theorem 31, we get that e_1 is linearly dependent on e_0. This contradicts that $e_1 \in \mathcal{M}_0^\perp$. By Theorem 17, $P_{\mathcal{H}} d_{e_1}^1$ is in $\mathcal{M}$ and so is $E = -\frac{1}{2}\varphi(\mathcal{B})^* P_{\mathcal{H}} d_{e_1}^1$. This implies that E is in $\Omega \cap \mathcal{M}_0^\perp$. We conclude that $\Omega \cap \mathcal{M}_0^\perp = \Omega$ since Ω is minimal to complete the proof.

The structure of reducing subspaces of a Blaschke product of order 2 was completely described in [51] and in [58] by different methods. We state their main result as the following Corollary and give another proof based on the methods we have developed so far.

Corollary 35. *Let φ be a Blaschke product of order 2. Then M_φ has exactly two minimal reducing subspaces $\mathcal{M}_0$ and $\mathcal{M}_0^\perp$.*

Proof. By Theorem 34, we only need to show that $\mathcal{M}_0^\perp$ is minimal. That $\mathcal{M}_0^\perp$ is minimal is obvious since the dimension of L_0 is two and by Lemma 13 $\mathcal{M}_0^\perp$ cannot split furthermore.

For a given reducing subspace $\mathcal{M}$ of $\varphi(\mathcal{B})$, define

$$\widetilde{\mathcal{M}} = span\{\varphi(z)^l \varphi(w)^k \mathcal{M}, l, k \geq 0\}.$$

Since $\mathcal{M}$ is a reducing subspace of $\varphi(\mathcal{B})$ and $\widetilde{\mathcal{M}}$ is a reducing subspace of both the pair of doubly commuting isometries $T_{\varphi(z)}$ and $T_{\varphi(w)}$, by the Wold decomposition of the pair of isometries on $\widetilde{\mathcal{M}}$, we have

$$\widetilde{\mathcal{M}} = \oplus_{l,k \geq 0}\varphi(z)^l \varphi(w)^k L_{\widetilde{\mathcal{M}}},$$

where $L_{\widetilde{\mathcal{M}}}$ is the wandering space

$$L_{\widetilde{\mathcal{M}}} = ker T_{\varphi(z)}^* \cap ker T_{\varphi(w)}^* \cap \widetilde{\mathcal{M}}.$$

Lemma 36. *If $\mathcal{M}$ and $\mathcal{N}$ are two mutually orthogonal reducing subspaces of $\varphi(\mathcal{B})$, then $\widetilde{\mathcal{M}}$ is also orthogonal to $\widetilde{\mathcal{N}}$.*

Proof. Let $f = \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k m_{lk}$ and $g = \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k n_{lk}$ for finite numbers of elements $m_{lk} \in \mathcal{M}$ and $n_{lk} \in \mathcal{N}$. Then

$$\langle f, g \rangle \quad = \langle \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k m_{lk}, \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k n_{lk} \rangle$$

$$= \sum_{l,k \geq 0} \sum_{l_1, k_1 \geq 0} \langle \varphi(z)^{l-l_1} \varphi(w)^{k-k_1} m_{lk}, n_{l_1 k_1} \rangle.$$

Since $\mathcal{M}$ is orthogonal to $\mathcal{N}$ and both $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces of $T^*_{\varphi(z)}$ and $T^*_{\varphi(w)}$, the above inner product $\langle f, g \rangle$ must be zero. Thus we conclude that $\widetilde{\mathcal{M}}$ is orthogonal to $\widetilde{\mathcal{N}}$ to complete the proof.

Theorem 37. *Suppose that $\mathcal{M}$ is a reducing subspace of $\varphi(\mathcal{B})$ which is orthogonal to $\mathcal{M}_0$. If $\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}\}$ is a basis of $\mathcal{M} \cap L_0$, then*

$$L_{\widetilde{\mathcal{M}}} = span\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}; d^1_{e_1^{(M)}}, \cdots, d^1_{e_{q_M}^{(M)}}\},$$

and

$$dim L_{\widetilde{\mathcal{M}}} = 2q_M.$$

Proof. Suppose that $\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}\}$ form a basis of $\mathcal{M} \cap L_0$. First we show

$$span\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}; d^1_{e_1^{(M)}}, \cdots, d^1_{e_{q_M}^{(M)}}\} \subset L_{\widetilde{\mathcal{M}}}.$$

Note that $\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}; d^1_{e_1^{(M)}}, \cdots, d^1_{e_{q_M}^{(M)}}\}$ are contained in $\mathcal{L}_\varphi$. It suffices to show

$$\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}; d^1_{e_1^{(M)}}, \cdots, d^1_{e_{q_M}^{(M)}}\} \subset \widetilde{\mathcal{M}}.$$

Since $\mathcal{M} \cap L_0$ contains $\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}\}$, for each $l, k \geq 0$, $\varphi(z)^l \varphi^k(w) e_i^{(M)}$ is in $\widetilde{\mathcal{M}}$ for $1 \leq i \leq q_M$. Thus $p_1(\varphi(z), \varphi(w)) e_i^{(M)}$ is in $\widetilde{\mathcal{M}}$. By Theorem 15, we have

$$p_1(\varphi(z), \varphi(w)) e_i^{(M)} + d^1_{e_i^{(M)}} \in \mathcal{M}.$$

So we have that $d^1_{e_i^{(M)}} \in \widetilde{\mathcal{M}}$, to obtain

$$span\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}; d^1_{e_1^{(M)}}, \cdots, d^1_{e_{q_M}^{(M)}}\} \subset L_{\widetilde{\mathcal{M}}}.$$

Next we will show that $\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}; d^1_{e_1^{(M)}}, \cdots, d^1_{e_{q_M}^{(M)}}\}$ are linearly independent. Suppose that for some constants λ_i and μ_i,

$$\sum_{i=1}^q \lambda_i e_i^{(M)} + \sum_{i=1}^q \mu_i d^1_{e_i^{(M)}} = 0.$$

Thus

$$\sum_{i=1}^q \lambda_i e_i^{(M)} = -\sum_{i=1}^q \mu_i d^1_{e_i^{(M)}}.$$

The right hand side of the above equality is in L_0 but the left hand side of the equality is orthogonal to L_0. So we have

$$\sum_{i=1}^q \lambda_i e_i^{(M)} = 0,$$

and

$$\sum_{i=1}^q \mu_i d^1_{e_i^{(M)}} = 0.$$

The first equality gives that $\lambda_i = 0$ and the second equality gives

$$d^1_{\sum_{i=1}^q \mu_i e_i^{(M)}} = 0.$$

Because $\mathcal{M}$ is orthogonal to $\mathcal{M}_0$, by Theorem 31, we have

$$\sum_{i=1}^q \mu_i e_i^{(M)} = 0.$$

This gives that $\mu_i = 0$. Hence $\{e_1^{(M)}, \cdots, e_{q_M}^{(M)}; d^1_{e_1^{(M)}}, \cdots, d^1_{e_{q_M}^{(M)}}\}$ are linearly independent. So far, we have obtained

$$dim L_{\widetilde{\mathcal{M}}} \geq 2q_M.$$

To finish the proof, we need only to show that

$$dim L_{\widetilde{\mathcal{M}}} \le 2q_M.$$

To do so, we consider the decomposition of $\mathcal{H}$,

$$\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M} \oplus [\mathcal{M}_0^\perp \cap \mathcal{M}^\perp],$$

and

$$L_0 = [\mathcal{M}_0 \cap L_0] \oplus [\mathcal{M} \cap L_0] \oplus \{[\mathcal{M}_0^\perp \cap \mathcal{M}^\perp] \cap L_0\}.$$

Then

$$dim\{[\mathcal{M}_0^\perp \cap \mathcal{M}^\perp] \cap L_0\} \quad = dim L_0 - dim[\mathcal{M}_0 \cap L_0] - dim[\mathcal{M} \cap L_0]$$

$$= N - 1 - q_M.$$

Letting $\mathcal{N} = [\mathcal{M}_0^\perp \cap \mathcal{M}^\perp]$, Lemma 36 gives

$$\mathcal{K}_\varphi = \widetilde{\mathcal{M}_0} \oplus \widetilde{\mathcal{M}} \oplus \widetilde{\mathcal{N}},$$

and

$$\mathcal{L}_\varphi = L_{\widetilde{\mathcal{M}_0}} \oplus L_{\widetilde{\mathcal{M}}} \oplus L_{\widetilde{\mathcal{N}}}.$$

Replacing $\mathcal{M}$ by $\mathcal{N}$ in the above argument gives

$$dim L_{\widetilde{\mathcal{N}}} \ge 2(N - 1 - q_M).$$

By Theorem 12, so we have

$$2N - 1 = 1 + dim[L_{\widetilde{\mathcal{M}}}] + dim[L_{\widetilde{\mathcal{N}}}].$$

Hence

$$\begin{aligned}
dim[L_{\widetilde{\mathcal{M}}}] \quad &= 2N - 2 - dim[L_{\widetilde{\mathcal{N}}}] \\
&\leq 2N - 2 - 2(N - 1 - q_M) \\
&= 2q_M.
\end{aligned}$$

This completes the proof.

Lemma 38. *Suppose that* $\mathcal{M}$, $\mathcal{N}$, *and* Ω *are three distinct nontrivial minimal reducing subspaces of* $\varphi(\mathcal{B})$ *such that*

$$\Omega \subset \mathcal{M} \oplus \mathcal{N}.$$

If $\mathcal{M}$, $\mathcal{N}$, *and* Ω *are orthogonal to* $\mathcal{M}_0$, *then*

$$\widetilde{\mathcal{M}} \cap \widetilde{\Omega} = \widetilde{\mathcal{N}} \cap \widetilde{\Omega} = \{0\}.$$

Proof. Since the intersection $\widetilde{\mathcal{M}} \cap \widetilde{\Omega}$ is also a reducing subspace of the pair of isometries $T_{\varphi(z)}$ and $T^*_{\varphi(w)}$, the Wold decomposition of the pair of isometries on $\widetilde{\mathcal{M}} \cap \widetilde{\Omega}$ gives

$$\widetilde{\mathcal{M}} \cap \widetilde{\Omega} = \oplus_{l,k \geq 0} \varphi(z)^l \varphi(w)^k L_{\widetilde{\mathcal{M}} \cap \widetilde{\Omega}},$$

where $L_{\widetilde{\mathcal{M}} \cap \widetilde{\Omega}}$ is the wandering space given by

$$\begin{aligned}
L_{\widetilde{\mathcal{M}} \cap \widetilde{\Omega}} \quad &= ker T^*_{\varphi(z)} \cap T^*_{\varphi(w)} \cap \widetilde{\mathcal{M}} \cap \widetilde{\Omega} \\
&= [ker T^*_{\varphi(z)} \cap T^*_{\varphi(w)} \cap \widetilde{\mathcal{M}}] \cap [ker T^*_{\varphi(z)} \cap T^*_{\varphi(w)} \cap \widetilde{\Omega}] \\
&= L_{\widetilde{\mathcal{M}}} \cap L_{\widetilde{\Omega}}.
\end{aligned}$$

To prove that $\widetilde{\mathcal{M}} \cap \widetilde{\Omega} = \{0\}$, it suffices to show

$$L_{\widetilde{\mathcal{M}}} \cap L_{\widetilde{\Omega}} = \{0\}.$$

To do this, let $q \in L_{\widetilde{\mathcal{M}}} \cap L_{\widetilde{\Omega}}$. By Theorem 37, there are functions $e_M, \tilde{e}_M \in \mathcal{M} \cap L_0$ and

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$e_\Omega, \tilde{e}_\Omega \in \Omega \cap L_0$ such that

$$
\begin{aligned}
q \quad &= e_M + d^1_{\tilde{e}_M} \\
&= e_\Omega + d^1_{\tilde{e}_\Omega}.
\end{aligned}
$$

The above two equalities give

$$
e_M - e_\Omega = d^1_{\tilde{e}_M - \tilde{e}_\Omega}.
$$

On the other hand, $d^1_{\tilde{e}_M - \tilde{e}_\Omega}$ is orthogonal to L_0. Thus

$$
\begin{aligned}
d^1_{\tilde{e}_M - \tilde{e}_\Omega} \quad &= \quad e_M - e_\Omega \\
&= \quad 0.
\end{aligned}
$$

This gives

$$
e_M = e_\Omega
$$

But e_M is in $\mathcal{M}$ and e_Ω is in Ω and hence both e_M and e_Ω are zero. Since $d^1_{\tilde{e}_M - \tilde{e}_\Omega} = 0$, Theorem 31 implies that $\tilde{e}_M - \tilde{e}_\Omega$ linearly depends on e_0. Since both $\mathcal{M}$ and Ω are orthogonal to $\mathcal{M}_0$, we have that $\tilde{e}_M = \tilde{e}_\Omega$. Thus we obtain $\tilde{e}_M = 0$ to conclude that $q = 0$, as desired. So

$$
\widetilde{\mathcal{M}} \cap \widetilde{\Omega} = \{0\}.
$$

Similarly we obtain

$$
\widetilde{\mathcal{N}} \cap \widetilde{\Omega} = \{0\}.
$$

Lemma 39. *Suppose that $\mathcal{M}$, $\mathcal{N}$, and Ω are three distinct nontrivial minimal reducing subspaces of $\varphi(\mathcal{B})$ such that*

$$
\Omega \subset \mathcal{M} \oplus \mathcal{N}.
$$

If $\mathcal{M}$, $\mathcal{N}$, and Ω are orthogonal to $\mathcal{M}_0$, then

$$
P_{\widetilde{\mathcal{M}}} L_{\widetilde{\Omega}} = L_{\widetilde{M}},
$$

and

$$P_{\widetilde{\mathcal{N}}}L_{\widetilde{\Omega}} = L_{\widetilde{N}},$$

where $P_{\widetilde{\mathcal{M}}}$ *denotes the orthogonal projection from* $H^2(\mathbb{T}^2)$ *onto* $\widetilde{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ is orthogonal to $\mathcal{N}$, Lemma 36 gives that $\widetilde{\mathcal{M}}$ is orthogonal to $\widetilde{\mathcal{N}}$ and

$$\widetilde{\Omega} \subset \widetilde{\mathcal{M}} \oplus \widetilde{\mathcal{N}}.$$

We will show that $P_{\widetilde{\mathcal{M}}}L_{\widetilde{\Omega}} = L_{\widetilde{M}}$.

Since $\Omega \subset \mathcal{M} \oplus \mathcal{N}$, we have

$$\Omega \cap L_0 \subset [\mathcal{M} \cap L_0] \oplus [\mathcal{N} \cap L_0].$$

For each $e^{(\Omega)} \in \Omega \cap L_0$, there are two functions $e^{(M)} \in \mathcal{M} \cap L_0$ and $e^{(N)} \in \mathcal{N} \cap L_0$ such that

$$e^{(\Omega)} = e^{(M)} + e^{(N)}$$

$$d^1_{e^{(\Omega)}} = d^1_{e^{(M)}} + d^1_{e^{(N)}}.$$

By Theorem 37, $d^1_{e^{(M)}}$ is in $\widetilde{\mathcal{M}}$ and $d^1_{e^{(N)}}$ is in $\widetilde{\mathcal{N}}$. Since $\mathcal{M}$, $\mathcal{N}$, and Ω are orthogonal to $\mathcal{M}_0$, the above decompositions are unique. Thus

$$P_{\widetilde{\mathcal{M}}}e^{(\Omega)} = e^{(M)},$$

and

$$P_{\widetilde{\mathcal{M}}}d^1_{e^{(\Omega)}} = d^1_{e^{(M)}}.$$

So for each $f = e^{(\Omega)} + d^1_{\tilde{e}^{(\Omega)}} \in L_{\widetilde{\Omega}}$, where $e^{(\Omega)}$ and $\tilde{e}^{(\Omega)}$, we have

$$P_{\widetilde{\mathcal{M}}}f = e^{(M)} + d^1_{\tilde{e}^{(M)}}$$

is in $L_{\widetilde{\mathcal{M}}}$ to obtain

$$P_{\widetilde{\mathcal{M}}}L_{\tilde{\Omega}} \subset L_{\widetilde{\mathcal{M}}}.$$

To prove that $P_{\widetilde{\mathcal{M}}} L_{\tilde{\Omega}} = L_{\widetilde{\mathcal{M}}}$, it suffices to show that

$$P_{\widetilde{\mathcal{M}}} : L_{\tilde{\Omega}} \to L_{\widetilde{\mathcal{M}}}$$

is surjective. If this is not so, by Theorem 37, there are two functions $e, \tilde{e} \in \mathcal{M} \cap L_0$ such that $0 \neq e + d_{\tilde{e}}^1$ is orthogonal to $P_{\widetilde{\mathcal{M}}} L_{\tilde{\Omega}}$.

Assume that $\{e_1, \cdots, e_{q_\Omega}\}$ are a basis of $\Omega \cap L_0$. Then

$$P_{\widetilde{\mathcal{M}}} L_{\tilde{\Omega}} = span\{e_1^{(M)}, \cdots, e_{q_\Omega}^{(M)}; d_{e_1^{(M)}}^1, \cdots, d_{e_{q_\Omega}^{(M)}}^1\}.$$

If $e \neq 0$, then $\langle e, e_i^{(M)} \rangle = 0$, for $1 \leq i \leq q_\Omega$. Thus

$$
\begin{aligned}
0 \quad &= \langle e, e_i^{(M)} \rangle \\
&= \langle e, e_i^{(M)} + e_i^{(N)} \rangle \\
&= \langle e, e_i \rangle,
\end{aligned}
$$

and

$$\langle e, d_{e_i}^1 \rangle = 0,$$

for each $1 \leq i \leq q_\Omega$. So e is orthogonal to $L_{\tilde{\Omega}} = span\{e_1, \cdots, e_{q_\Omega}; d_{e_1}^1, \cdots, d_{e_{q_\Omega}}^1\}$. Noting e is in L_0, we see that e is orthogonal to $\varphi(z)^l \varphi(w)^k L_{\tilde{\Omega}}$, for each $l > 0$ or $k > 0$. This gives that e is orthogonal to $\tilde{\Omega}$ and hence orthogonal to Ω. Since e is in $\mathcal{M}$, e must be orthogonal to the closure of $P_{\mathcal{M}} \Omega \subset \mathcal{M}$, which is also a reducing subspace of $\varphi(\mathcal{B})$. Therefore e is orthogonal to $\mathcal{M}$, which is a contradiction.

If $e = 0$, then $d_{\tilde{e}}^1 \neq 0$ and

$$
\begin{aligned}
0 \quad &= \langle d_{\tilde{e}}^1, d_{e_i^{(M)}}^1 \rangle \\
&= \langle d_{\tilde{e}}^1, P_{\widetilde{\mathcal{M}}} d_{e_i}^1 \rangle \\
&= \langle d_{\tilde{e}}^1, d_{e_i}^1 \rangle,
\end{aligned}
$$

and

$$\langle d_{\tilde{e}}^1, e_i \rangle = 0,$$

for each $1 \leq i \leq q_\Omega$. This gives that $d_{\tilde{e}}^1$ is orthogonal to $L_{\tilde{\Omega}}$. But $d_{\tilde{e}}^1$ is also in $\mathcal{L}_\varphi$. We have that for any $f \in L_{\tilde{\Omega}}$,

$$\langle d_{\tilde{e}}^1, \varphi(z)^l \varphi(w)^k f \rangle = 0,$$

for $l > 0$ or $k > 0$. We have that $d_{\tilde{e}}^1$ is orthogonal to $\tilde{\Omega}$ and hence orthogonal to Ω to obtain that $P_{\mathcal{H}} d_{\tilde{e}}^1$ is orthogonal to Ω. On the other hand, by Theorem 17, $P_{\mathcal{H}} d_{\tilde{e}}^1$ is in $\mathcal{M}$. Thus $P_{\mathcal{H}} d_{\tilde{e}}^1$ is orthogonal to the closure of $P_{\mathcal{M}} \Omega$ and so $P_{\mathcal{H}} d_{\tilde{e}}^1$ must be zero because the closure of $P_{\mathcal{M}} \Omega$ equals $\mathcal{M}$. Therefore,

$$
\begin{aligned}
0 &= \langle P_{\mathcal{H}} d_{\tilde{e}}^1, p_1(\varphi(z), \varphi(w)) \tilde{e} + d_{\tilde{e}}^1 \rangle \\
&= \langle d_{\tilde{e}}^1, p_1(\varphi(z), \varphi(w)) \tilde{e} + d_{\tilde{e}}^1 \rangle \\
&= \langle d_{\tilde{e}}^1, d_{\tilde{e}}^1 \rangle = \| d_{\tilde{e}}^1 \|^2.
\end{aligned}
$$

The second equality follows from that $p_1(\varphi(z), \varphi(w)) \tilde{e} + d_{\tilde{e}}^1$ is in $\mathcal{H}$ and the third equality follows that $d_{\tilde{e}}^1$ is orthogonal to $p_1(\varphi(z), \varphi(w)) \tilde{e}$. This gives that $d_{\tilde{e}}^1 = 0$, which is a contradiction. We have obtained that $P_{\widetilde{\mathcal{M}}} : L_{\tilde{\Omega}} \to L_{\widetilde{\mathcal{M}}}$ is surjective and hence

$$P_{\widetilde{\mathcal{M}}} L_{\tilde{\Omega}} = L_{\widetilde{\mathcal{M}}}.$$

Similarly we obtain

$$P_{\widetilde{\mathcal{N}}} L_{\tilde{\Omega}} = L_{\widetilde{\mathcal{N}}}.$$

This completes the proof.

Theorem 40. *Suppose that Ω, $\mathcal{M}$ and $\mathcal{N}$ are three distinct nontrivial minimal reducing subspaces for $\varphi(\mathcal{B})$ and*

$$\Omega \subset \mathcal{M} \oplus \mathcal{N}.$$

If they are all contained in $\mathcal{M}_0^\perp$, then there is a unitary operator $U : \mathcal{M} \to \mathcal{N}$ such that U commutes with $\varphi(\mathcal{B})$ and $\varphi(\mathcal{B})^*$.

Proof. First we will show

$$P_{\mathcal{M}} = P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}}.$$

Let $\mathcal{N}_1$ denote the orthogonal complementary of $\mathcal{M} \oplus \mathcal{N}$ in $\mathcal{H}$. Write

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{N}_1.$$

Lemma 36 gives

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{M}} \oplus \widetilde{\mathcal{N}} \oplus \widetilde{\mathcal{N}_1}.$$

For each function in $H^2(\mathbb{T}^2)$, write

$$
\begin{aligned}
f \quad &= f_{\widetilde{\mathcal{H}}} \oplus f_2 \\
&= f_{\widetilde{\mathcal{M}}} \oplus f_{\widetilde{\mathcal{N}}} \oplus f_{\widetilde{\mathcal{N}_1}} \oplus f_2,
\end{aligned}
$$

where f_2 is orthogonal to $\widetilde{\mathcal{H}}$, $f_{\widetilde{\mathcal{H}}} \in \widetilde{\mathcal{H}}$, $f_{\widetilde{\mathcal{M}}} \in \widetilde{\mathcal{M}}$, $f_{\widetilde{\mathcal{N}}} \in \widetilde{\mathcal{N}}$, and $f_{\widetilde{\mathcal{N}_1}} \in \widetilde{\mathcal{N}_1}$. Since $\widetilde{\mathcal{M}}$ contains $\mathcal{M}$, we write

$$f_{\widetilde{\mathcal{M}}} = f_{\mathcal{M}} \oplus f_3,$$

for two functions $f_{\mathcal{M}} \in \mathcal{M}$ and $f_3 \in \widetilde{\mathcal{M}} \ominus \mathcal{M}$. Thus f_3 is orthogonal to both $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}_1}$ and hence orthogonal to both $\mathcal{N}$ and $\mathcal{N}_1$. So f_3 is orthogonal to

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{N}_1.$$

This gives that $P_{\mathcal{H}} f_3 = 0$. We have

$$
\begin{aligned}
P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}} f \quad &= P_{\mathcal{H}} f_{\widetilde{\mathcal{M}}} \\
&= P_{\mathcal{H}} f_{\mathcal{M}} + P_{\mathcal{H}} f_3 \\
&= P_{\mathcal{H}} f_{\mathcal{M}} \\
&= f_{\mathcal{M}},
\end{aligned}
$$

and

$$P_{\mathcal{M}} f = f_{\mathcal{M}},$$

to get

$$P_{\mathcal{M}} = P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}}.$$

Next we will show that $P_{\mathcal{M}}$ is surjective from Ω onto $\mathcal{M}$. For each $q \in \mathcal{M}$, by Lemma 39, there are functions $q_{lk} \in L_{\tilde{\Omega}}$ such that

$$q = \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k m_{lk},$$

and

$$\|q\|^2 = \sum_{l,k \geq 0} \|m_{lk}\|^2 < \infty,$$

where $m_{lk} = P_{\widetilde{\mathcal{M}}} q_{lk}$. Since $L_{\tilde{\Omega}}$ and $L_{\widetilde{M}}$ are finite dimension spaces, there are two positive constants c_1 and c_2 such that

$$c_1 \|q_{lk}\| \leq \|m_{lk}\| \leq c_2 \|q_{lk}\|.$$

Define

$$\tilde{q} = \sum_{l,k \geq 0} \varphi(z)^k \varphi(w)^l q_{lk}.$$

Thus

$$
\begin{aligned}
\|\tilde{q}\|^2 &= \sum_{l,k \geq 0} \|q_{lk}\|^2 \\
&\leq c_2 \sum_{l,k \geq 0} \|m_{lk}\|^2 \\
&< \infty.
\end{aligned}
$$

So we obtain that $\tilde{q}$ is in $\tilde{\Omega}$, and

$$
\begin{aligned}
\tilde{q} &= \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k q_{lk} \\
&= \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k [P_{\widetilde{\mathcal{M}}} q_{lk} + P_{\widetilde{\mathcal{N}}} q_{lk}] \\
&= \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k m_{lk} + \sum_{l,k \geq 0} \varphi(z)^l \varphi(w)^k [P_{\widetilde{\mathcal{N}}} q_{lk}] \\
&= q + q_N,
\end{aligned}
$$

where $q_N = \sum_{l,k \geq 0} \varphi(z)^k \varphi(w)^l [P_{\widetilde{\mathcal{N}}} q_{lk}]$ is in $\widetilde{N}$. Hence $P_{\widetilde{\mathcal{M}}} \tilde{q} = q$. We have

$$
\begin{aligned}
P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}} \tilde{q} &= P_{\mathcal{H}} q \\
&= q,
\end{aligned}
$$

to obtain

$$
\begin{aligned}
P_{\mathcal{M}} \tilde{q} &= P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}} \tilde{q} \\
&= q.
\end{aligned}
$$

Since $\mathcal{M}$ is a subspace of $\mathcal{H}$, $P_{\mathcal{M}} = P_{\mathcal{M}} P_{\mathcal{H}}$. Thus

$$
\begin{aligned}
P_{\mathcal{M}} P_{\mathcal{H}} \tilde{q} &= P_{\mathcal{M}} \tilde{q} \\
&= q.
\end{aligned}
$$

Writing $q_{lk} = e_{kl}^{(\Omega)} + d_{\tilde{e}_{kl}^{(\Omega)}}^1$ for functions $e_{kl}^{(\Omega)}, \tilde{e}_{kl}^{(\Omega)} \in \Omega \cap L_0$, we have

$$
\begin{aligned}
P_{\mathcal{H}} \tilde{q} &= \sum_{l,k \geq 0} P_{\mathcal{H}}(\varphi(z)^l \varphi(w)^k q_{lk}) \\
&= \sum_{l,k \geq 0} P_{\mathcal{H}} \varphi(z)^l \varphi(w)^k (e_{kl}^{(\Omega)} + d_{\tilde{e}_{kl}^{(\Omega)}}^1) \\
&= \sum_{l,k \geq 0} (P_{\mathcal{H}} \varphi(z)^l \varphi(w)^k e_{kl}^{(\Omega)}) + \sum_{l,k \geq 0} (P_{\mathcal{H}} \varphi(z)^l \varphi(w)^k d_{\tilde{e}_{kl}^{(\Omega)}}^1) \\
&= \sum_{l,k \geq 0} (P_{\mathcal{H}} \varphi(z)^l \varphi(w)^k e_{kl}^{(\Omega)}) + \sum_{l,k \geq 0} [P_{\mathcal{H}} \varphi(z)^l \varphi(w)^k (P_{\mathcal{H}} d_{\tilde{e}_{kl}^{(\Omega)}}^1)]
\end{aligned}
$$

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The last equality follows from that $\varphi(z)^l \varphi(w)^k (1 - P_{\mathcal{H}}) d^1_{\tilde{e}^{(\Omega)}_{kl}}$ is orthogonal to $\mathcal{H}$. The the first sum in the last equality is in Ω and Theorem 17 gives that the second sum in the equality is in Ω also. Letting $\omega = P_{\mathcal{H}}\tilde{q}$, we have proved that $P_{\mathcal{M}}\omega = q$ to get that

$$P_{\mathcal{M}}\Omega = \mathcal{M}.$$

On the other hand, $ker[P_{\mathcal{M}}|_{\Omega}] \subset \Omega$ is a reducing subspace of $\varphi(\mathcal{B})$. Since Ω is a nontrivial minimal reducing spaces of $\varphi(\mathcal{B})$, we see that $ker[P_{\mathcal{M}}|_{\Omega}] = \{0\}$. This implies that $P_{\mathcal{M}} : \Omega \to \mathcal{M}$ is bijective and bounded. By the closed graph theorem we conclude that $P_{\mathcal{M}}|_{\Omega}$ is invertible.

Similarly we can show that that $P_{\mathcal{N}}|_{\Omega}$ is invertible. Define

$$S = [P_{\mathcal{N}}|_{\Omega}][P_{\mathcal{M}}|_{\Omega}]^{-1}.$$

Then S is an invertible operator from $\mathcal{M}$ onto $\mathcal{N}$. Both S and S^* commute with $\varphi(\mathcal{B})$ because Ω, $\mathcal{M}$ and $\mathcal{N}$ are three distinct nontrivial minimal reducing subspaces for $\varphi(\mathcal{B})$. Thus S^*S commutes with $\varphi(\mathcal{B})$. Making the polar decomposition of S, we write

$$S = U|S|,$$

for some unitary operator U from $\mathcal{M}$ onto $\mathcal{N}$, where $|S| = [S^*S]^{1/2}$. So U commutes with both $\varphi(\mathcal{B})$ and $\varphi(\mathcal{B})^*$. This completes the proof.

II.6.2 Structure of reducing subspaces

For a finite Blaschke product φ, by Bochner's theorem [56], $\varphi(z)$ always has a critical point, denoted by $-c$, in the unit disk. Let $\lambda = \varphi(-c)$. Then

$$\varphi_\lambda \circ \varphi \circ \varphi_c(z) = z^{n_0+1} \prod_{k=1}^{K} \left(\frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \right)^{n_k+1}$$

with $n_0 \geq 1$. Moreover the structure of the reducing lattice of M_φ is the same as that of $M_{\varphi_\lambda \circ \varphi \circ \varphi_c(z)}$. So we can always assume that φ has the form as in the following Theorem 41

as long as the structures of reducing lattices are concerned.

Theorem 41. *Let $\varphi = z^{n_0+1}\varphi_{\alpha_1}^{n_1+1}\cdots\varphi_{\alpha_K}^{n_K+1}$ be a Blaschke product of order N with $n_0 \geq 1$, $K \geq 1$ and $\alpha_k \neq 0$. Then $\varphi(\mathcal{B})$ cannot have N nontrivial reducing subspaces $\{M_i\}_{i=0}^{N-1}$ satisfying $\mathcal{H} = \bigoplus_{i=0}^{N-1} M_i$ and $M_i \perp M_j$ whenever $i \neq j$.*

Proof. Write

$$\varphi = z\varphi_0 = z^{n_0+1}\varphi_1,$$

where

$$\varphi_0 = z^{n_0}\varphi_{\alpha_1}^{n_1+1}\cdots\varphi_{\alpha_K}^{n_K+1}$$

and

$$\varphi_1 = \varphi_{\alpha_1}^{n_1+1}\cdots\varphi_{\alpha_K}^{n_K+1}.$$

Then

$$L_0 = span\{1, p_1, ..., p_{n_0}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K}\}.$$

Assume that $\varphi(\mathcal{B})$ has N nontrivial reducing subspaces $\{M_i\}_{i=0}^{N-1}$ such that

$$\mathcal{H} = \bigoplus_{i=0}^{N-1} M_i$$

and

$$M_i \perp M_j$$

whenever $i \neq j$.

By Lemma 13, for each i, there is an $e_i \neq 0$ such that $e_i \in M_i \cap L_0$. Thus

$$L_0 = span\{e_0, e_1, ..., e_{N-1}\}.$$

By Theorem 15 in the first construction, there are functions $\{d_{e_i}^1\} \subset \mathcal{L}_\varphi \ominus L_0$ such that

$$p_1(\varphi(z), \varphi(w))e_i + d_{e_i}^1 \in M_i.$$

For $i \neq j$, by $M_i \perp M_j$, we have

$$\langle p_1(\varphi(z), \varphi(w))e_i + d_{e_i}^1, p_1(\varphi(z), \varphi(w))e_j + d_{e_j}^1 \rangle = 0$$

On the other hand, a simple calculation gives

$$\langle p_1(\varphi(z), \varphi(w))e_i + d_{e_i}^1, p_1(\varphi(z), \varphi(w))e_j + d_{e_j}^1 \rangle = \langle d_{e_i}^1, d_{e_j}^1 \rangle.$$

So

$$\langle d_{e_i}^1, d_{e_j}^1 \rangle = 0$$

for any $i \neq j$.

If each $d_{e_i}^1 \neq 0, i = 0, ..., N-1$, then the linear independence of $\{d_{e_i}^1\}_{i=0}^{N-1}$ will imply that the dimension of $\mathcal{L}_\varphi$ is at least $2N$. But we know from the Theorem 12 in Section II.3 that the dimension of $\mathcal{L}_\varphi$ is $2N - 1$. Hence at least one $d_{e_i}^1$ is zero. On the other hand if $d_{e_i}^1 = 0$, then the corresponding subspace M_i is the distinguished reducing subspace. But we have only one such distinguished reducing subspace. Hence we have one and only one $d_{e_i}^1 = 0$. By Theorems 26 and 29, without loss of generality, we may assume that M_0 is the distinguished reducing subspace $\mathcal{M}_0$ of $\varphi(\mathcal{B})$ and e_0 is exactly the element e_0 in the distinguished reducing subspace $\mathcal{M}_0$.

So each $d_{e_i}^1 \neq 0$ for $i = 1, ..., N-1$ and

$$\{d_{e_i}^1\}_{i=1}^{N-1} \subset \mathcal{L}_\varphi \ominus L_0$$

are linearly independent.

By Theorem 30, there are numbers β_i, λ_i such that

$$d_{e_i}^1 = d_{e_i}^0 + \beta_i e_i + \lambda_i e_0, i = 1, \cdots, N-1. \tag{II.7}$$

Observe that for $0 \leq k \leq n_0$,

$$-\langle d_{e_i}^0, p_k \rangle = \langle \varphi(w)e_i - we_i(0, w)e_0, p_k \rangle$$

$$= \langle \varphi(w)e_i(w, w), p_k(0, w) \rangle - \langle we_i(0, w)e_0(w, w), p_k(0, w) \rangle$$

$$= \langle \varphi(w)e_i(w, w), w^k \rangle - \langle we_i(0, w)(w\varphi_0'(w) + \varphi_0(w)), w^k \rangle$$

$$= \langle w^{n_0+1-k}\varphi_1(w)e_i(w, w), 1 \rangle$$

$$- \langle w^{n_0+1-k}[w\varphi_1'(w) + (n_0 + 1)\varphi_1(w)]e_i(0, w), 1 \rangle$$

$$= 0.$$

The second equality follows from Lemma 5 and the third equality follows from Lemma 9.

Since $e_{\alpha_j}^t$ is in the kernel of $T_{\varphi(w)}^*$, $\varphi^{(s)}(\alpha_j) = 0$ for $0 \leq s \leq n_j$ gives that for $0 \leq t \leq n_j - 1$, $j = 1, ..., K$,

$$\begin{aligned}
\langle d_{e_i}^0, e_{\alpha_j}^t \rangle &= \langle we_i(0, w)e_0(w, w) - \varphi(w)e_i, e_{\alpha_j}^t \rangle \\
&= \langle we_i(0, w)e_0(w, w), e_{\alpha_j}^t(0, w) \rangle \quad \text{(by Lemma 5)} \\
&= \langle we_i(0, w)[w\varphi_0'(w) + \varphi_0(w)], e_{\alpha_j}^t(0, w) \rangle \quad \text{(by Lemma 9)} \\
&= \langle we_i(0, w)\varphi', k_{\alpha_j}^t \rangle \quad \text{(by (II.4))} \\
&= (we_i(0, w)\varphi')^{(t)}|_{w=\alpha_j} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle d_{e_i}^0, e_{\alpha_j}^{n_j} \rangle &= [we_i(0, w)\varphi'(w)]^{(n_j)}|_{\alpha_j} \\
&= \alpha_j e_i(0, \alpha_j)\varphi^{(n_j+1)}(\alpha_j).
\end{aligned}$$

These give that

$$d_{e_i}^0 \perp \{1, p_1, ..., p_{n_0-1}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1-1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K-1}\}. \tag{II.8}$$

We also have that for $0 \leq k \leq n_0 - 1$

$$
\begin{aligned}
\langle e_0, p_k \rangle &= \langle e_0(w, w), p_k(0, w) \rangle \\
&= \langle \varphi'(w), w^k \rangle \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\langle e_0, p_{n_0} \rangle &= \frac{1}{n_0!} \varphi^{(n_0+1)}(0) \\
&\neq 0.
\end{aligned}
$$

A simple calculation gives that for $j = 1, ..., K$, $0 \leq t \leq n_j - 1$

$$
\begin{aligned}
\langle e_0, e_{\alpha_j}^t \rangle &= [e_0(w, w)]^{(t)}|_{\alpha_j} \\
&= \varphi^{(t+1)}(\alpha_j) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\langle e_0, e_{\alpha_j}^{n_j} \rangle &= \varphi^{(n_j+1)}(\alpha_j) \\
&\neq 0.
\end{aligned}
$$

These give

$$
e_0 \perp \{1, p_1, ..., p_{n_0-1}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1-1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K-1}\}. \tag{II.9}
$$

If β_{i_0} does not equal 0 for some i_0, (II.7) yields

$$
e_{i_0} = \frac{1}{\beta_{i_0}} [d_{e_{i_0}}^1 - d_{e_{i_0}}^0 - \lambda_{i_0} e_0].
$$

Noting that $d_{e_i}^1 \perp L_0$, by (II.8) and (II.9) we have

$$e_{i_0} \perp \{1, p_1, ..., p_{n_0-1}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1-1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K-1}\}.$$

Thus

$$e_{i_0} \perp \{1, p_1, ..., p_{n_0-1}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1-1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K-1}, e_0\}. \qquad \text{(II.10)}$$

Hence there are at most K nonzero β_i's.

On the other hand if $\beta_i = 0$, then (II.7) gives

$$d_{e_i}^1 = d_{e_i}^0 + \lambda_i e_0.$$

Since p_{n_0} is in L_0 and $d_{e_i}^1 \perp L_0$, we have that $d_{e_i}^0 \perp p_{n_0}$, and

$$\langle e_0, p_{n_0} \rangle \neq 0,$$

to obtain that $\lambda_i = 0$ and $d_{e_i}^0 = d_{e_i}^1$ is orthogonal to L_0. By Theorem 43, there is at least one nonzero β_i.

Without loss of generality, assume that for some m, $\beta_{N-j} \neq 0$ for $1 \leq j \leq m$ and $\beta_j = 0$ for $1 \leq j \leq N - m - 1$. (II.10) gives

$$e_{N-j} \perp \{1, p_1, ..., p_{n_0-1}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1-1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K-1}, e_0\}$$

for $1 \leq j \leq m$. Now we extend

$$\{1, p_1, ..., p_{n_0-1}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1-1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K-1}, e_0, e_{N-1}, ..., e_{N-m}\}$$

to a basis of L_0:

$$\{1, p_1, ..., p_{n_0-1}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1-1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K-1}, e_0, e_{N-1}, ..., e_{N-m}, f_1, ..., f_{K-m}\}$$

by adding some elements $f_1, ..., f_{K-m}$ in L_0. Let $\{g_j\}_{j=1}^{N-m-1}$ denote

$$\{1, p_1, ..., p_{n_0-1}, e_{\alpha_1}^0, ..., e_{\alpha_1}^{n_1-1}, ..., e_{\alpha_K}^0, ..., e_{\alpha_K}^{n_K-1}, f_1, ..., f_{K-m}\}.$$

Since for $1 \leq j \leq N - m - 1$, e_j is in L_0 and

$$e_j \perp \{e_0, e_{N-1}, ..., e_{N-m}\}$$

we have that e_j is in the subspace $span\{1, g_2, ..., g_{N-m-1}\}$ of L_0. This implies that there are numbers $\{c_{jl}\}_{j,l=1}^{N-m-1}$ such that for $1 \leq j \leq N - m - 1$

$$e_j = c_{j1} + c_{j2}g_2 + \cdots + c_{jN-m-1}g_{N-m-1}. \tag{II.11}$$

On the other hand, because $\beta_j = 0$ for $1 \leq j \leq N - m - 1$, we have that $d_{e_j}^0 = d_{e_j}^1$ is orthogonal to L_0, and

$$\begin{aligned} \langle d_{e_j}^0, e_{\alpha_1}^{n_1} \rangle &= \alpha_1 e_j(0, \alpha_1)\varphi^{(n_1+1)}(\alpha_1) \\ &= 0. \end{aligned}$$

This implies that $e_j(0, \alpha_1) = 0$. Hence (II.11) gives

$$\begin{aligned} e_j(0, \alpha_1) &= c_{j1}1 + c_{j2}g_2(0, \alpha_1) + \cdots + c_{jN-m-1}g_{N-m-1}(0, \alpha_1) \\ &= 0 \end{aligned}$$

for $1 \leq j \leq N - m - 1$. Thus the determinant $det[c_{jk}]$ of the coefficient matrix of the above system must be zero. So There is a nonzero vector $(x_1, \cdots, x_{N-m-1})$ such that

$$c_{1l}x_1 + c_{2l}x_2 + \cdots + c_{N-m-1l}x_{N-m-1} = 0$$

for $1 \leq l \leq N - m - 1$. This implies

$$x_1e_1 + x_2e_2 + \cdots + x_{N-m-1}e_{N-m-1} = 0.$$

We obtain a contradiction that $e_1, ..., e_{N-m-1}$ are linearly independent to complete the proof.

II.7   Weighted shifts

In this section we will characterize multiplication operators on the Bergman space which is unitarily equivalent to a weighted shift of finite multiplicity. In fact, a weighted shift of finite multiplicity is unitarily equivalent to a direct sum of finitely many weighted shifts.

A weighted shift T of finite multiplicity n on Hilbert space H is an operator that maps each vector in an orthonormal basis $\{e_k\}_{k=0}^{\infty}$ of H into a scaler multiple of the next nth vector,

$$Te_k = w_k e_{k+n},$$

for all k. The sequence $\{w_k\}$ is called the weight of the weighted shift T. In fact, T is unitarily equivalent to the multiplication operator by z^n on some Hilbert space of analytic functions on the unit disk (see [39] and [40]).

Theorem 42. *Suppose that φ is a Blaschke product of order N. If there are N mutually orthogonal reducing subspaces $\{M_i\}_{i=1}^{N}$ of $\varphi(\mathcal{B})$ such that $\varphi(\mathcal{B})|_{M_i}$ is unitarily equivalent to a weighted shift, then for each $e_i \in M_i \cap L_0$, those $d_{e_i}^l$ obtained in Theorem 16 satisfy $d_{e_i}^l = 0$ for $l > 1$.*

Proof. By Theorem 33 we may assume that $\varphi(\mathcal{B})|_{M_1}$ is unitarily equivalent to the Bergman shift. Let e_i be a nonzero vector in $M_i \cap L_0$. By Theorem 16, there are functions $d_{e_i}^l \in \mathcal{L}_\varphi \ominus L_0$ such that

$$p_l(\varphi(z), \varphi(w))e_i + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_{e_i}^{l-k} \in M_i.$$

Theorem 33 implies that $d_{e_1}^l = 0$ for $l \geq 1$ and $d_{e_i}^1 \neq 0$, for $i > 1$. Let

$$E_{il} = p_l(\varphi(z), \varphi(w))e_i + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_{e_i}^{l-k}.$$

Then E_{il} is in M_i, and

$$
\begin{aligned}
\varphi(\mathcal{B})^* E_{il} &= T^*_{\varphi(z)} E_{il} \\
&= P[\overline{\varphi(z)}(p_l(\varphi(z), \varphi(w))e_i + \sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))d_{e_i}^{l-k})] \\
&= p_{l-1}(\varphi(z), \varphi(w))e_i + \sum_{k=0}^{l-2} p_k(\varphi(z), \varphi(w))d_{e_i}^{l-k} \\
&= E_{i(l-1)}.
\end{aligned}
$$

The last equality follows from that $P(\overline{\varphi(z)}e_i) = 0$, and $P(\overline{\varphi(z)}d_{e_i}^l) = 0$. Thus $\{E_{il}\}_l$ are orthogonal to $\{E_{jl}\}_l$ for $i \neq j$ and so $\{d_{e_i}^l\}_l$ are orthogonal to $\{d_{e_j}^l\}_l$. Since $dim[\mathcal{L}_\varphi \ominus L_0]$ equals $N-1$ and $d_{e_i}^1$ does not equal zero for $i > 1$, $\{d_{e_i}^1\}$ form an orthogonal basis of $\mathcal{L}_\varphi \ominus L_0$. This gives that there are constants β_{il} such that

$$
d_{e_i}^l = \beta_{il} d_{e_i}^1.
$$

Because $\varphi(\mathcal{B})|_{M_i}$ is a weighted shift, there are an orthonormal basis $\{F_l\}$ of M_i such that

$$
\varphi(\mathcal{B})F_l = a_l F_{l+1}
$$

where $\{a_l\}$ are weights of $\varphi(\mathcal{B})$ on M_i. Thus F_0 is in the kernel of $[\varphi(\mathcal{B})|_{M_i}]^*$, and so $F_0 = \lambda_0 e_i$ for some constant λ_0. Since $\varphi(\mathcal{B})^* F_1 = a_0 F_0$, we have

$$
\varphi(\mathcal{B})^*[F_1 - a_0\lambda_0 E_{i1}] = 0.
$$

Thus

$$
F_1 = a_0\lambda_0 E_{i1} + \mu_1 e_i.
$$

But both F_1 and E_{i1} are orthogonal to e_i. So $\mu_1 = 0$. Hence there is a constant λ_1 such that

$$
F_1 = \lambda_1 E_{i1}.
$$

By induction, we obtain that there are constants λ_l such that

$$F_l = \lambda_l E_{il}.$$

This implies that $\{E_{il}\}$ are an orthogonal set. Note

$$E_{il} = p_1(\varphi(z), \varphi(w))e_i + [\sum_{k=0}^{l-1} p_k(\varphi(z), \varphi(w))\beta_{i(l-k)}]d^1_{e_i}.$$

We conclude that $\beta_{il} = 0$ for $l > 1$. This gives

$$E_{il} = p_1(\varphi(z), \varphi(w))e_i + p_{l-1}(\varphi(z), \varphi(w))d^1_{e_i} \in M_i$$

and $d^l_{e_i} = 0$ for $l > 1$.

Theorem 43. *Suppose that φ is a finite Blaschke product and $\varphi(0) = 0$. If φ has a nonzero root α, then there is a function $e \in L_0$ such that d^0_e is not orthogonal to L_0.*

Proof. Assume that for each $e \in L_0$, d^0_e is orthogonal to L_0. We will derive a contradiction.

Since $\{\{e^{s_k}_{\alpha_k}\}_{s_k=0,\cdots,n_k}\}_{k=0,\cdots,K}$ form a basis for L_0, for each $e \in L_0$ there is a vector $(u^0_0, \cdots, u^{n_0}_0, \cdots, u^0_{\alpha_K}, \cdots, u^{n_K}_{\alpha_K}) \in C^N$ such that

$$e(z, w) = \sum_{i=0}^{K} \sum_{t=0}^{n_i} u^t_{\alpha_i} e^t_{\alpha_i}(z, w).$$

Noting that $dim L_0 = N$, we see that

$$e \to (u^0_0, \cdots, u^{n_0}_0, \cdots, u^0_{\alpha_K}, \cdots, u^{n_K}_{\alpha_K})$$

is a linear invertible mapping from L_0 onto C^N.

Let α_j be a nonzero root of φ with multiplicity $n_j + 1$. Then

$$\varphi^{(t)}(\alpha_j) = \langle \varphi, k^t_{\alpha_j} \rangle = 0$$

for $0 \leq t \leq n_j$ and

$$\varphi^{(n_j+1)}(\alpha_j) = \langle \varphi, k_{\alpha_j}^{n_j+1} \rangle \neq 0.$$

Because d_e^0 is orthogonal to L_0 and $\{e_{\alpha_j}^t\}_{t=0}^t$ is in L_0, we have

$$
\begin{aligned}
0 &= \langle d_e^0, e_{\alpha_j}^t \rangle \\
&= \langle [w\varphi_0(w)e(z,w) - we(0,w)e_0(z,w)], e_{\alpha_j}^t \rangle \\
&= \langle w\varphi_0(w)e(z,w), e_{\alpha_j}^t \rangle - \langle we(0,w)e_0(z,w), e_{\alpha_j}^t \rangle.
\end{aligned}
$$

By Lemma 22,

$$
\begin{aligned}
\langle w\varphi_0(w)e(z,w), e_{\alpha_j}^t \rangle &= \{[\partial_z + \partial_w]^t \varphi(w)e(z,w)\}|_{z=w=\alpha_j} \\
&= \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} \varphi^{(s)}(\alpha_j)\{[\partial_z + \partial_w]^{t-s} e(z,w)\}|_{z=w=\alpha_j} \\
&= 0.
\end{aligned}
$$

Thus

$$\langle we(0,w)e_0(z,w), e_{\alpha_j}^t \rangle = 0$$

for $0 \leq t \leq n_j$. By Lemma 22 again, we have

$$
\begin{aligned}
0 &= \langle we(0,w)e_0(z,w), e_{\alpha_j}^t \rangle \\
&= \{[\partial_z + \partial_w]^t we(0,w)e_0(z,w)\}|_{z=w=\alpha_j} \\
&= \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} (we(0,w))^{(s)}(\alpha_i)\{[\partial_z + \partial_w]^{t-s} e_0(z,w)\}|_{z=w=\alpha_j} \quad \text{(II.12)}
\end{aligned}
$$

for $0 \leq t \leq n_j$. When $t = 0$, the above equation gives

$$\alpha_j e(0,\alpha_j)e_0(\alpha_j,\alpha_j) = 0.$$

Noting that $\alpha_j e(0,\alpha_j) = 0$ is equivalent to

$$\sum_{i=0}^{K} \sum_{t=0}^{n_i} u_i^t e_{\alpha_i}^t(0,\alpha_j) = 0,$$

we see that there is a function e in L_0 such that

$$\alpha_j e(0, \alpha_j) \neq 0.$$

Hence $e_0(\alpha_j, \alpha_j) = 0$. Letting $t = 1$, (II.12) gives

$$\alpha_j e(0, \alpha_j)\{[\partial_z + \partial_w]e_0(z, w)\}|_{z=w=\alpha_j} + (we(0, w))^{(1)}|_{w=\alpha_j} e_0(\alpha_j, \alpha_j) = 0,$$

Thus

$$\{[\partial_z + \partial_w]e_0(z, w)\}|_{z=w=\alpha_j} = 0.$$

By induction we obtain

$$\{[\partial_z + \partial_w]^t e_0(z, w)\}|_{z=w=\alpha_j} = 0,$$

for $0 \leq t \leq n_j$. In particular,

$$0 = \{[\partial_z + \partial_w]^{n_j} e_0(z, w)\}|_{z=w=\alpha_j}.$$

A simple calculation gives

$$
\begin{aligned}
\{[\partial_z + \partial_w]^{n_j} e_0(z, w)\}|_{z=w=\alpha_j} &= \langle e_0, e_{\alpha_j}^{n_j} \rangle \\
&= \langle \overline{e_{\alpha_j}^{n_j}} e_0(z, w), 1 \rangle \\
&= \langle P_{\mathcal{H}}[\overline{e_{\alpha_j}^{n_j}(z, w)} e_0(z, w)], 1 \rangle.
\end{aligned}
$$

Because $e_{\alpha_j}^{n_j}$ is in $H^\infty(\mathbb{T}^2)$ and $e_0(z, w)$ is in $\mathcal{H}$, we have

$$P_{\mathcal{H}}[\overline{e_{\alpha_j}^{n_j}(z, w)} e_0(z, w)] = P_{\mathcal{H}}[\overline{e_{\alpha_j}^{n_j}(z, z)} e_0(z, w)].$$

Thus

$$
\begin{aligned}
\{[\partial_z + \partial_w]^{n_j} e_0(z, w)\}|_{z=w=\alpha_j} &= \langle P_{\mathcal{H}}[\overline{e_{\alpha_j}^{n_j}(z, z)} e_0(z, w)], 1\rangle \\
&= \langle \overline{e_{\alpha_j}^{n_j}(z, z)} e_0(z, w), 1\rangle \\
&= \langle e_0(z, w), e_{\alpha_j}^{n_j}(z, z)\rangle \\
&= \langle e_0(z, 0), e_{\alpha_j}^{n_j}(z, z)\rangle \\
&= \langle \varphi_0(z), \frac{(n_j + 1)! z^{n_j}}{(1 - \bar{\alpha}_j z)^{n_j + 2}}\rangle.
\end{aligned}
$$

On the other hand, we also have

$$
\begin{aligned}
0 &= \varphi_0^{(n_j)}(\alpha_j) \\
&= \langle \varphi_0, k_{\alpha_j}^{n_j}\rangle \\
&= \langle \varphi_0, \frac{n_j! z^{n_j}}{(1 - \bar{\alpha}_j z)^{n_j + 1}}\rangle.
\end{aligned}
$$

Combining the above two equalities gives

$$
\begin{aligned}
0 &= \langle \varphi_0(z), [\frac{z^{n_j}}{(1 - \bar{\alpha}_j z)^{n_j + 2}} - \frac{z^{n_j}}{(1 - \bar{\alpha}_j z)^{n_j + 1}}]\rangle \\
&= \langle \varphi_0(z), \frac{\bar{\alpha}_j z^{n_j + 1}}{(1 - \bar{\alpha}_j z)^{n_j + 2}}\rangle.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\varphi_0^{(n_j + 1)}(\alpha_j) &= \langle \varphi_0(z), k_{\alpha_j}^{n_j + 1}(z)\rangle \\
&= \frac{(n_j + 1)!}{\bar{\alpha}_j} \langle \varphi_0(z), \frac{\bar{\alpha}_j z^{n_j + 1}}{(1 - \bar{\alpha}_j z)^{n_j + 2}}\rangle \\
&= 0.
\end{aligned}
$$

This contradicts that α_j is a nonzero root of φ_0 with multiplicity $n_j + 1$.

Now we can prove our main result in this section.

Theorem 44. *If φ is a bounded analytic function in $\mathbb{D}$ and the multiplication operator M_φ is unitarily equivalent to a direct sum of N weighted shifts, then $\varphi = c\varphi_\lambda^N$, for a constant c*

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and some Möbius transform $\varphi_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$.

Proof. After multiplying φ by a constant if necessary, we may assume that $\|M_\varphi\| = 1$. Suppose that M_φ is unitarily equivalent to the direct sum $\oplus_{i=1}^N W_i$ where W_i is a weighted shift. Then

$$dim ker M_\varphi^* = \sum_i dim ker W_i^*$$

and the essential spectrum of M_φ is

$$\sigma_e(M_\varphi) = \cup_{i=1}^N \sigma_e(W_i).$$

Noting that W_i is subnormal, we see that the essential spectrum of W_i is a circle with center 0. So $\cup_{i=1}^N \sigma_e(W_i)$ is a union of circles with the same center 0. On the other hand, by Corollary 20 [47], the essential spectrum of M_φ is connected. Thus $\cup_{i=1}^N \sigma_e(W_i)$ is the unit circle and $|\varphi(z)| = 1$ on $\mathbb{T}$. So φ is an inner function.

We claim that φ is continuous on $\overline{\mathbb{D}}$, therefore a Blaschke product. If φ is not so, there is a singularity $z_0 \in \mathbb{T}$ of $\varphi(z)$ where $\varphi(z)$ does not extend analytically, by Theorem 6.6 in [27], the cluster set of $\varphi(z)$ is the closed unit disk. Note that a point η in the cluster set of $\varphi(z)$ at z_0 iff there are points z_n in $\mathbb{D}$ tending to z_0 such that $\varphi(z_n)$ converges to η. This implies that the cluster set of $\varphi(z)$ at every point z_0 on the unit circle is contained in the essential spectrum of M_φ, which is a contradiction.

Now φ is a finite Blaschke product, after composing with φ a Möbius transform from the right if necessary, we may assume that $\varphi(0) = 0$ as in the Theorem 43.

By Theorem 42, there are N linear independent functions $\{e_i\}$ of L_0 such that $\{d_{e_i}\}$ are orthogonal to L_0 and

$$p_l(\varphi(z), \varphi(w))e_i + p_{l-1}(\varphi(z), \varphi(w))d_{e_i} \in \mathcal{H}.$$

Also we have

$$p_l(\varphi(z), \varphi(w))e_i + p_{l-1}(\varphi(z), \varphi(w))d_{e_i}^0 \in \mathcal{H}.$$

Thus

$$p_l(\varphi(z), \varphi(w))(d_{e_i} - d_{e_i}^0) \in \mathcal{H}.$$

So there are constants λ_i such that

$$d_{e_i} = d_{e_i}^0 + \lambda_i e_0.$$

Since $e_0^{n_0}$ is in L_0 and d_{e_i} is orthogonal to L_0, we have

$$
\begin{aligned}
0 &= \langle d_{e_i}, e_0^{n_0} \rangle \\
&= \langle d_{e_i}^0, e_0^{n_0} \rangle + \lambda_i \langle e_0, e_0^{n_0} \rangle.
\end{aligned}
$$

On the other hand, Lemma 22 gives

$$
\begin{aligned}
\langle e_0, e_0^{n_0} \rangle &= \langle e_0(z, w), e_0^{n_0}(z, z) \rangle \\
&= \langle e_0(z, 0), e_0^{n_0}(z, z) \rangle \\
&= (n_0 + 1)! \langle \varphi_0(z), z^{n_0} \rangle \\
&= (n_0 + 1)! \varphi_0^{(n_0)}(0) \neq 0, \\
\langle d_{e_i}^0, e_0^{n_0} \rangle &= \langle w \varphi_0(w) e_i(z, w) - w e_i(0, w) e_0(z, w), e_0^{n_0}(z, w) \rangle \\
&= \langle \varphi(w) e_i(z, w), e_0^{n_0}(z, w) \rangle - \langle w e_i(0, w) e_0(z, w), e_0^{n_0}(z, w) \rangle.
\end{aligned}
$$

The Leibniz rule and Lemma 22 give

$$
\begin{aligned}
\langle \varphi(w) e_i(z, w), e_0^{n_0}(z, w) \rangle &= [(\partial_z + \partial_w)^{n_0} (\varphi(w) e_i(z, w))]|_{z=w=0} \\
&= \sum_{s=0}^{n_0} \frac{n_0!}{s!(n_0 - s)!} \varphi^{(s)}(0) [(\partial_z + \partial_w)^{n_0 - s} e_i](0, 0) \\
&= 0.
\end{aligned}
$$

The last equality follows from that 0 is a root of φ with multiplicity $n_0 + 1$. Similarly, we

have

$$\langle we_i(0,w)e_0(z,w), e_0^{n_0}(z,w)\rangle$$

$$= [(\partial_z + \partial_w)^{n_0}(we_i(0,w)e_0(z,w))]|_{z=w=0}$$

$$= \sum_{s=0}^{n_0} \frac{n_0!}{s!(n_0-s)!}(we_i(0,w))^{(s)}(0)[(\partial_z + \partial_w)^{n_0-s}e_0](0,0).$$

Lemmas 22 and 21 give

$$[(\partial_z + \partial_w)^{n_0-s}e_0](0,0) = \langle e_0(z,w), e_0^{n_0-s}(z,w)\rangle$$

$$= \langle e_0(z,w), e_0^{n_0-s}(z,z)\rangle$$

$$= \langle e_0(z,0), e_0^{n_0-s}(z,z)\rangle$$

$$= \langle \varphi_0(z), (n_0-s+1)!z^{n_0-s}\rangle$$

$$= 0$$

for $0 < s \leq n_0$. The second equality follows from

$$P_{\mathcal{H}}[\overline{e_0^{n_0-s}(z,w)}e_0(z,w)] = P_{\mathcal{H}}[\overline{e_0^{n_0-s}(z,z)}e_0(z,w)].$$

Thus

$$\sum_{s=0}^{n_0} \frac{n_0!}{s!(n_0-s)!}(we_i(0,w))^{(s)}(0)[(\partial_z + \partial_w)^{n_0-s}e_0](0,0) = 0,$$

and so

$$\langle we_i(0,w)e_0(z,w), e_0^{n_0}(z,w)\rangle = 0.$$

Hence we have that the constant $\lambda_i = 0$. Therefore $d_{e_i}^0$ is orthogonal to L_0 for each i. Noting that $\{e_i\}_{i=1}^N$ forms a basis for L_0 we see that d_e^0 is orthogonal to L_0 for each $e \in L_0$. By Theorem 43, we must have that $\varphi = z^N$. That is, $\varphi = c\varphi_\lambda^N$, for a constant c and some Möbius transform φ_λ. The proof is complete.

II.8 Blaschke products of order three

Now we can prove the second main result of this chapter (Theorem 2) which is about the structure of reducing subspaces of a multiplication operator on the Bergman space induced by a Blaschke product of order three.

Suppose that φ is a Blaschke product of order three. As pointed out in section II.1, the multiplication operator, M_φ, on the Bergman space is unitarily equivalent to the operator, $\varphi(\mathcal{B})$, on $\mathcal{H}$. So we will only need to consider $\varphi(\mathcal{B})$.

First, observe that for $\lambda \in \mathbb{D}$ and a subspace M of $\mathcal{H}$, M is a reducing subspace of $\varphi(\mathcal{B})$ if and only if M is a reducing subspace of $\varphi_\lambda \circ \varphi(\mathcal{B})$.

Then, observe that for $\mu \in \mathbb{D}$, $\varphi(\mathcal{B})$ is unitarily equivalent to $\varphi \circ \varphi_\mu(\mathcal{B})$.

Therefore, for any two numbers $\lambda, \mu \in \mathbb{D}$, the structures of reducing lattices of $\varphi(\mathcal{B})$ and $\varphi_\lambda \circ \varphi \circ \varphi_\mu(\mathcal{B})$ are the same.

It follows from Bochner's theorem [56], [57] that φ has 2 critical numbers (counting multiplicity) in the unit disk $\mathbb{D}$ and has no critical numbers on the unit circle.

If $\varphi(z)$ has a multiple critical number in the unit disk, then

$$\varphi = \varphi_\lambda \circ z^3 \circ \varphi_\mu$$

for two numbers $\lambda, \ \mu \in \mathbb{D}$. Thus without loss of generality, we may assume that

$$\varphi = z^3.$$

In this case, it follows from the main result in [44] that $\varphi(\mathcal{B})$ has exactly three minimal reducing subspaces [44]. It is obvious that for $\varphi = z^3$, the Riemann surface of $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$ has exactly three connected components corresponding to the three branches, $z, ze^{\frac{2\pi}{3}\sqrt{-1}}, ze^{\frac{4\pi}{3}\sqrt{-1}}$ respectively.

If φ have two distinct critical numbers in the unit disk, let $-c$ be one of them and let $\lambda = \varphi(-c)$, then

$$\varphi_\lambda \circ \varphi \circ \varphi_c(z) = z^2 \frac{z-a}{z-\bar{a}z},$$

for some nonzero point $a \in \mathbb{D}$. So without loss of generality, we may assume that

$$\varphi = z^2 \frac{z-a}{z - \bar{a}z}, a \neq 0$$

In this case, by the example in [41], except for the trivial branch z, nontrivial branches of $\varphi^{-1} \circ \varphi$ are all continuations of one another. Thus the Riemann surface of $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$ has exactly two connected components.

To finish the proof, we only need to show that if $\varphi = z^2 \frac{z-a}{z-\bar{a}z}$, $a \in \mathbb{D}$ and $a \neq 0$, then $\varphi(\boldsymbol{\mathcal{B}})$ has exactly two minimal reducing subspaces. To show this, by our Theorem 34 we only need to show that $\boldsymbol{\mathcal{M}}_0^\perp$, the orthogonal complement of the distinguished reducing subspace $\boldsymbol{\mathcal{M}}_0$, is minimal. If $\boldsymbol{\mathcal{M}}_0^\perp$ is not minimal, then we assume that M_1 is a nontrivial reducing subspace properly contained in $\boldsymbol{\mathcal{M}}_0^\perp$ and have a third nontrivial reducing subspace, $M_2 = \boldsymbol{\mathcal{M}}_0^\perp \ominus M_1$, such that $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{M}}_0 \oplus M_1 \oplus M_2$. However, this contradicts Theorem 41, and the proof is finished.

II.9   Blaschke products of order four

In this section we will prove our third main result of this chapter, Theorem 3, which is about the structure of reducing subspaces of a multiplication operator, M_φ, on the Bergman space, where φ a Blaschke product of order four. The proof consists of two parts. One part is about the Riemann surface for $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$ which will be dealt with in the first subsection. Another part is about the minimal reducing subspaces of M_φ which will be addressed in the remaining three subsections.

II.9.1   Riemann surfaces

In this subsection, we study Riemann surfaces for $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$. The main result is Theorem 48.

We start with the Riemann surface of $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$ for any finite Blaschke product of order N. Write $\varphi = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are two polynomials. The degree of $P(z)$ is N and the degree of $Q(z)$ is less than or equal to N. Observe that the multi-valued function $w = \varphi^{-1} \circ \varphi(z)$ for $z \in \mathbb{D}$ is the same as the one that is determined by the equation

$\varphi(w) - \varphi(z) = 0$ for $z \in \mathbb{D}$. Since we are only concerned with $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$, the function is also determined by the following equation

$$Q(z)P(w) - P(z)Q(w) = 0$$

with $z \in \mathbb{D}, w \in \mathbb{D}$.

Denote $Q(z)P(w) - P(z)Q(w)$ by $f(z,w)$. Note that $f(z,w)$ is a polynomial of z, w. The degree of $f(z,w)$ with respect to w is N and its degree is less than or equal to N with respect to z. So we will only need to study the Riemann surface over $\mathbb{D}$, denoted by S_φ, for the function determined by the equation

$$f(z,w) = 0.$$

We will first describe the construction of S_φ by cut and paste (For general case, see [3], [14], [26]).

Let C denote the set of the critical points of φ in $\mathbb{D}$ and F denote the set

$$\varphi^{-1} \circ \varphi(C) = \{z_1, \cdots, z_m\}$$

with $m \leq (N-1)N$. Then F is the set of all possible branch points and $\varphi^{-1} \circ \varphi$ is an N-branched analytic function defined and can be analytically continued to $\mathbb{D}/F$. Not all of the branches of $\varphi^{-1} \circ \varphi$ can be continued to a different branch, for example z is a single valued branch of $\varphi^{-1} \circ \varphi$. The Riemann surface S_φ for $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$ is an N-sheeted cover of $\mathbb{D}$ with at most $N(N-1)$ branch points, and is not connected.

We begin with N copies of the unit disk $\mathbb{D}$, called sheets. The sheets are labeled $\mathbb{D}_1, \cdots, \mathbb{D}_N$ and stacked up over $\mathbb{D}$. Cut each $\mathbb{D}_j$ along a piecewise smooth curve passing through all $z_1, \cdots, z_m$ and a fixed point z_0 on the unit circle, in such a way that $\mathbb{D}_j/P$ is simply connected. We may assume that P consists of smooth arcs l_i connecting z_{i-1} to z_i for $i = 1, 2, ..., N-1$. Each l_i has two edges. By Theorem 12.3 [14], N distinct solutions

$\rho_k(z)$, $k = 1, \cdots, N$ of the algebraic equation

$$f(z, w) = 0$$

are holomorphic functions in $\mathbb{D}/P$. Each sheet is associated with a branch. One sheet is glued to another one along the edges of l_i, $i = 1, 2, ..., m$ according to the analytic continuation from one branch to the another one, maybe the same branch. With the point in the k-th sheet over a value z in $\mathbb{D}/P$ we associate the pair of values $(z, \rho_k(z))$. In this way a one-to-one correspondence is set up between the points in S_φ over $\mathbb{D}/P$ and the pair of points on the N sheets over $\mathbb{D}/P$. In order to make the correspondence continuous along the cuts exclusive of their ends, let two regions R_1 and R_2 be defined in a neighborhood of each cut, for example, l_i. In the region formed by R_1, R_2 and the cut l_i between them exclusive of its ends, the values of the algebraic function $w = g(z)$ form again N distinct holomorphic functions $\rho_k(z)$ $(k = 1, \cdots, N)$, and these can be so numbered that $g_l(z) = \rho_l(z)$ in R_1. In the region R_2 the functions $g_k(z)$ are the functions of the set $\{\rho_k(z)\}$ but possibly in a different order. We joint the edge of cut bounding R_1 in the k-th sheet to the edge bounding R_2 in the l-th sheet, where l is so determined that $g_k(z) = \rho_l(z)$ in R_2. The continuous Riemann surface so formed has the property that points in the Riemann surface S_φ over non-branch points $D/\{z_1, \cdots, z_m\}$ are in one-to-one continuous correspondence with the nonsingular points (z, w) which satisfies the equation $f(z, w) = 0$.

We are interested in the number of connected components of the Riemann surface S_φ. The following theorem implies that the number of connected components equals the number of irreducible factors of $f(z, w)$. This result holds for Riemann surfaces over the complex plane (see page 78 [14] and page 374 [26]).

Theorem 45. *Let $\varphi = \frac{P(z)}{Q(z)}$ be a finite Blaschke product. Suppose that $p(z, w)$ is a factor of $f(z, w) = Q(z)P(w) - P(z)Q(w)$. Then the Riemann surface S_p over $\mathbb{D}$ for the function defined by $p(z, w) = 0$ is connected if and only if $p(z, w)$ is irreducible.*

Proof. Let $\{z_j\}_{j=1}^m$ be the branch points of the function determined by the equation $p(z, w) = 0$ in $\mathbb{D}$. Bochner's Theorem [57] says that those points $\{z_j\}_{j=1}^m$ is contained in a compact

subset of $\mathbb{D}$. Cut $\mathbb{D}$ along a piecewise smooth curve joining all $z_j, j = 1, 2, ...m$ and a fixed point on the circle. If the Riemann surface S_p is not connected, let $\{\rho_k(z)\}_{k=1}^n$ be n distinct branches of $p(z, w) = 0$ over $\mathbb{D}/P$. Then $\{\rho_k(z)\}_{k=1}^n$ are also roots of the equation

$$\varphi(w) - \varphi(z) = 0.$$

So they are analytic in a neighborhood of the unit circle and map the unit circle into the unit circle.

Suppose a connected component of S_p is made up of the sheets corresponding to $\{\rho_1, \cdots, \rho_{n_1}\}$ ($n_1 < n$). Let $\sigma_s(x_1, \cdots, x_{n_1})$ be elementary symmetric functions of variables $x_1, \cdots, x_{n_1}$ with degree s. Then every $\sigma_s(z) = \sigma_s(\rho_1(z), \cdots, \rho_{n_1}(z))$ is a holomorphic function well-defined on $\mathbb{D}/\{z_j\}_{j=1}^m$ although $\rho_j(z)$ is defined only on $\mathbb{D}/P$.

Note that $\rho_j(z)$ is in $\mathbb{D}$. Thus $\sigma_s(\rho_1(z), \cdots, \rho_{n_1}(z))$ is bounded on $\mathbb{D}/\{z_j\}_{j=1}^m$. By the Riemann removable theorem, $\sigma_s(\rho_1(z), \cdots, \rho_{n_1}(z))$ extends analytic on $s\mathbb{D}$ for some $s > 1$. Now we extend $\sigma_s(\rho_1(z), \cdots, \rho_{n_1}(z))$ to the complex plane C. For each $z \in C/\mathbb{D}$, define

$$f_s(z) = \sigma_s(\frac{1}{\rho_1(\frac{1}{\bar{z}})}, \cdots, \frac{1}{\rho_{n_1}(\frac{1}{\bar{z}})}).$$

By Theorem 11.1 on page 25 [14], near an ordinary point $z = a$, each function $\rho_j(z)$ has a power series of $z - a$. By Lemma 13.1 on page 29 [14], each function $\rho_j(z)$ has a Laurent series of a fraction power of $(z - a)$ but the number of terms with negative exponents must be finite. Thus $f_s(z)$ is a meromorphic function in $C/\mathbb{D}$. Note that $\rho_i(z)$ is analytic and does not vanish in a neighborhood of the unit circle and so $f_s(z)$ is analytic in $t\mathbb{D}/r\mathbb{D}$ for $0 < r < 1 < t$. If z is on the unit circle,

$$\frac{1}{\rho_i(\frac{1}{\bar{z}})} = \rho_i(z)$$

for each i. Thus

$$\sigma_s(\rho_1(z), \cdots, \rho_{n_1}(z)) = f_s(z).$$

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for z on the unit circle. So

$$\sigma_s(\rho_1(z), \cdots, \rho_{n_1}(z)) = f_s(z).$$

in a neighborhood of the unit circle. Define

$$F_s(z) = \begin{cases} \sigma_s(\rho_1(z), \cdots, \rho_{n_1}(z)) & if \ z \in \bar{\mathbb{D}} \\ f_s(z) & if \ z \in C/\mathbb{D}. \end{cases}$$

Thus $F_s(z)$ is a rational function of z and so is $\sigma_s(\rho_1(z), \cdots, \rho_{n_1}(z))$ in $\mathbb{D}$.

Now consider the polynomial

$$f_1(z, w) = w^{n_1} - \sigma_1(z)w^{n_1-1} + \cdots + (-1)^{n_1}\sigma_{n_1}(z) = \prod_{j=1}^{n_1}(w - \rho_j(z))$$

whose coefficients are rational functions of z. Thus

$$p(z, w) = f_1(z, w)f_2(z, w)$$

for another polynomial $f_2(z, w)$. This implies that $p(z, w)$ is reducible.

If $P(z, w) = P_1(z, w) \cdots P_k(z, w)$ is reducible, the continuations of the roots, $\rho_1(z), \cdots,$ $\rho_{n_1}(z)$, of $P_1(z, w)$ are always roots of $P_1(z, w)$. Hence the set of roots permutes into itself across the cuts, and the Riemann surface S_p has k connected components, one for each of the factors $P_1, \cdots, P_k$. This completes the proof.

The above theorem and its proof give the following corollary.

Corollary 46. *Let $\varphi = \frac{P(z)}{Q(z)}$ be a finite Blaschke product and*

$$f(z, w) = Q(z)P(w) - P(z)Q(w).$$

If

$$f(z, w) = p_1(z, w)^{n_1}p_2(z, w)^{n_2} \cdots p_m(z, w)^{n_m}$$

for some irreducible polynomials $p_1, ..., p_m$, then the number of connected components of S_φ,

the Riemann surface over $\mathbb{D}$ for the function $(\varphi)^{-1} \circ \varphi$, is m.

We say that two Blaschke products φ_1 and φ_2 are equivalent if there are two points λ and c in $\mathbb{D}$ such that

$$\varphi_1 = \varphi_\lambda \circ \varphi_2 \circ \varphi_c.$$

By the same reason as in the proof of our first main result (see section II.8), M_{φ_1} and M_{φ_2} share the same structure of reducing lattice.

For each $\lambda \in \mathbb{D}$, it is easy to see that

$$(\varphi_\lambda \circ \varphi)^{-1} \circ \varphi_\lambda \circ \varphi = \varphi^{-1} \circ \varphi$$

So for any finite Blaschke product φ, the Riemann surface S_φ is the same as the Riemann surface $S_{\varphi_\lambda \circ \varphi}$. Hence the Riemann surface S_φ is isomorphic to the Riemann surface $S_{\varphi_\lambda \circ \varphi \circ \varphi_c}$ for any $c \in \mathbb{D}$.

Now suppose φ is a Blaschke product of order four. Let $\lambda = \varphi(-c)$ be a critical value of φ in the unit disk for some critical point $-c$ in the unit disk. Then there are two numbers α and β in the unit disk such that

$$\varphi_\lambda \circ \varphi \circ \varphi_c(z) = z^2 \varphi_\alpha(z) \varphi_\beta(z).$$

Therefore, without loss of generality, from now on in this chapter we always assume that

$$\varphi(z) = z^2 \varphi_\alpha(z) \varphi_\beta(z)$$

for some fixed $\alpha, \beta \in \mathbb{D}$. The corresponding $f(z, w)$ as in the above Corollary is denoted by

$$f_{\alpha,\beta}(z, w) = w^2(w - \alpha)(w - \beta)(1 - \bar{\alpha}z)(1 - \bar{\beta}z) - z^2(z - \alpha)(z - \beta)(1 - \bar{\alpha}w)(1 - \bar{\beta}w).$$

Theorem 47. *Let α and β be in $\mathbb{D}$.*

(1) If both α and β are zero, then

$$f_{\alpha,\beta}(z, w) = (w - z)(w + z)(w - iz)(w + iz).$$

(2) If both α and β are nonzero, and $\alpha = \beta$ or $\alpha = -\beta$, then

$$f_{\alpha,\beta}(z, w) = (w - z)p(z, w)q(z, w)$$

for two distinct irreducible polynomials $p(z, w)$ and $q(z, w)$.

(3) If only one of α and β is zero, say $\beta = 0$ and $\alpha \neq 0$, then

$$f_{\alpha,\beta}(z, w) = (w - z)p(z, w)$$

for some irreducible polynomial $p(z, w)$.

(4) If both α and β are nonzero, and α does not equal either β or $-\beta$, then

$$f_{\alpha,\beta}(z, w) = (w - z)p(z, w)$$

for some irreducible polynomial $p(z, w)$.

Proof. We observe first that $(w - z)$ is a factor of the polynomial $f_{\alpha,\beta}(z, w)$. A long division gives

$$f_{\alpha,\beta}(z, w) \;=\; (w - z)g_{\alpha,\beta}(z, w),$$

where

$$g_{\alpha,\beta}(z, w) \;=\; (1 - \bar{\alpha}z)(1 - \bar{\beta}z)w^3 + (z - (\alpha + \beta))(1 - \bar{\alpha}z)(1 - \bar{\beta}z)w^2$$
$$+ (z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z)w + z(z - \alpha)(z - \beta).$$

Clearly, (1) holds.

To prove (2), we note that if $\alpha = \beta$, then

$$g_{\alpha,\beta}(z, w) \;=\; [(1 - \bar{\alpha}z)w + (z - \alpha)][w(w - \alpha)(1 - \bar{\alpha}z) + z(z - \alpha)(1 - \bar{\alpha}w)].$$

It is routine to check that $w(w - \alpha)(1 - \bar{\alpha}z) + z(z - \alpha)(1 - \bar{\alpha}w)$ is irreducible if $\alpha \neq 0$.

If $\alpha = -\beta \neq 0$, we also have

$$g_{\alpha,\beta}(z,w) \;=\; (w+z)[(1-\bar{\alpha}^2 z^2)w^2 + (z^2 - \alpha^2)],$$

and that $(1 - \bar{\alpha}^2 z^2)w^2 + (z^2 - \alpha^2)$ is irreducible.

To prove (3), We only need to show that $g_{\alpha,\beta}(z,w)$ is irreducible. Note that in this case

$$
\begin{aligned}
g_{\alpha,\beta}(z,w) \;=\;& (1-\bar{\alpha}z)w^3 + (z-\alpha)(1-\bar{\alpha}z)w^2 \\
& + z(z-\alpha)(1-\bar{\alpha}z)w + z^2(z-\alpha).
\end{aligned}
$$

If we can factor $g_{\alpha,\beta}(z,w)$ as the product of two polynomials $p(z,w)$ and $q(z,w)$ of degree one and two respectively. We may assume that

$$
\begin{aligned}
p(z,w) \;&=\; a_1(z)w + a_0(z) \\
q(z,w) \;&=\; b_2(z)w^2 + b_1(z)w + b_0(z)
\end{aligned}
$$

where $a_j(z)$ and $b_j(z)$ are polynomials of z. Comparing coefficients of w^k for $k = 2,1,0$ in both sides of the equation

$$g_{\alpha,\beta}(z,w) = p(z,w)q(z,w)$$

gives

$$
\begin{aligned}
a_1(z)b_2(z) \;&=\; (1-\bar{\alpha}z), & \text{(II.13)} \\
a_1(z)b_1(z) + a_0(z)b_2(z) \;&=\; (z-\alpha)(1-\bar{\alpha}z), & \text{(II.14)} \\
a_1(z)b_0(z) + a_0(z)b_1(z) \;&=\; z(z-\alpha)(1-\bar{\alpha}z), & \text{(II.15)} \\
a_0(z)b_0(z) \;&=\; z^2(z-\alpha). & \text{(II.16)}
\end{aligned}
$$

Equation (II.13) gives that (up to a non zero constant) either

$$
\begin{aligned}
a_1(z) \;&=\; (1-\bar{\alpha}z) \quad or \\
a_1(z) \;&=\; 1.
\end{aligned}
$$

If $a_1(z) = 1 - \bar{\alpha}z$, then $b_2(z) = 1$ and equation (II.14) gives that $1 - \bar{\alpha}z$ is a factor of $a_0(z)$. But equation (II.16) says it is impossible for $1 - \bar{\alpha}z$ to be a factor of $a_0(z)$.

If $a_1(z) = 1$, then $b_2(z) = 1 - \bar{\alpha}z$ and equation (II.14) gives that $1 - \bar{\alpha}z$ is a factor of $b_1(z)$. So it follows from equation (II.15) that $1 - \bar{\alpha}z$ is also a factor of $b_0(z)$. But equation (II.16) says this is impossible.

To prove (4), we will show that $g_{\alpha,\beta}(z,w)$ is irreducible.

If we can factor $g_{\alpha,\beta}(z,w)$ as the product of two polynomials $p(z,w)$ and $q(z,w)$ of degree one and two with respect to w. We may assume that

$$p(z,w) = a_1(z)w + a_0(z)$$
$$q(z,w) = b_2(z)w^2 + b_1(z)w + b_0(z)$$

where $a_j(z)$ and $b_j(z)$ are polynomials of z. Since

$$g_{\alpha,\beta}(z,w) = p(z,w)q(z,w),$$

comparing coefficients of w^k in both sides of the above equation gives

$$a_1(z)b_2(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z), \tag{II.17}$$
$$a_1(z)b_1(z) + a_0(z)b_2(z) = (z - (\alpha + \beta))(1 - \bar{\alpha}z)(1 - \bar{\beta}z), \tag{II.18}$$
$$a_1(z)b_0(z) + a_0(z)b_1(z) = (z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z), \tag{II.19}$$
$$a_0(z)b_0(z) = z(z - \alpha)(z - \beta). \tag{II.20}$$

Equation (II.17) gives that (up to some nonzero constant) either

$$a_1(z) = (1 - \bar{\alpha}z) \quad or$$
$$a_1(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z) \quad or$$
$$a_1(z) = 1.$$

If $a_1(z) = (1 - \bar{\alpha}z)$, then $b_2(z) = (1 - \bar{\beta}z)$. Thus by Equation (II.18), we have

$$a_0(z)(1 - \bar{\beta}z) = (1 - \bar{\alpha}z)[(z - (\alpha + \beta))(1 - \bar{\beta}z) - b_1(z)].$$

So $(1 - \bar{\alpha}z)$ is a factor of $a_0(z)$ and hence a factor of $z(z - \alpha)(z - \beta)$ by equation (II.20). This is impossible.

If $a_1(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z)$, then $b_2(z) = 1$. Thus either the degree of $b_1(z)$ or the degree of $b_0(z)$ must be one while the degrees of $b_1(z)$ and $b_0(z)$ are at most one. So the degree of $a_0(z)$ is at most two. Also $a_0(z)$ does not equal zero. Equation (II.18) gives

$$(1 - \bar{\alpha}z)(1 - \bar{\beta}z)b_1(z) + a_0(z) = (z - (\alpha + \beta))(1 - \bar{\alpha}z)(1 - \bar{\beta}z).$$

Thus

$$a_0(z) = c_1(1 - \bar{\alpha}z)(1 - \bar{\beta}z)$$

for constant c_1. But Equation (II.20) gives

$$c_1(1 - \bar{\alpha}z)(1 - \bar{\beta}z)b_0(z) = z(z - \alpha)(z - \beta).$$

Either $c_1 = 0$ or $(1 - \bar{\alpha}z)(1 - \bar{\beta}z)$ is a factor of $z(z - \alpha)(z - \beta)$. This is impossible.

If $a_1(z) = 1$, then $b_2(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z)$. Since the root w of $f_{\alpha,\beta}(w, z)$ is a nonconstant function of z, the degree of $a_0(z)$ must be 1. Thus the degrees of $b_1(z)$ and $b_0(z)$ are at most 2. Equation (II.18) gives

$$(1 - \bar{\alpha}z)(1 - \bar{\beta}z)a_0(z) + b_1(z) = (z - (\alpha + \beta))(1 - \bar{\alpha}z)(1 - \bar{\beta}z).$$

This implies

$$b_1(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z)[(z - (\alpha + \beta)) - a_0(z)].$$

Since the degree of $b_1(z)$ is at most 2, we have

$$\begin{aligned} a_0(z) &= (z - (\alpha + \beta)) - c_0; \\ b_1(z) &= c_0(1 - \bar{\alpha}z)(1 - \bar{\beta}z). \end{aligned}$$

Equation (II.20) gives

$$[(z - (\alpha + \beta)) - c_0]b_0(z) = z(z - \alpha)(z - \beta).$$

Equation (II.19) gives

$$\begin{aligned} &b_1(z)[(z - (\alpha + \beta)) - c_0] + b_0(z) \\ &= (z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z). \end{aligned}$$

Multiplying the both sides of the above equality by $[(z - (\alpha + \beta)) - c_0]$ gives

$$\begin{aligned} &b_1(z)[(z - (\alpha + \beta)) - c_0]^2 + z(z - \alpha)(z - \beta) \\ &= [(z - (\alpha + \beta)) - c_0](z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z). \end{aligned}$$

Hence

$$\begin{aligned} &c_0(1 - \bar{\alpha}z)(1 - \bar{\beta}z)[(z - (\alpha + \beta)) - c_0]^2 + z(z - \alpha)(z - \beta) \\ &= [(z - (\alpha + \beta)) - c_0](z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z). \end{aligned}$$

If $c_0 \neq 0$, then $(z - \alpha)(z - \beta)$ is a factor of $[(z - (\alpha + \beta)) - c_0]^2$. This is also impossible.

If $c_0 = 0$, then

$$z(z - \alpha)(z - \beta) = [(z - (\alpha + \beta))](z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z).$$

This forces that $\bar{\alpha} + \bar{\beta} = 0$ and hence $\alpha = -\beta$. This is impossible

This completes the proof.

Combining Corollary 46 with Theorem 47 leads to the main theorem in this section.

Theorem 48. *Let α and β be in $\mathbb{D}$, and $\varphi = z^2 \varphi_\alpha \varphi_\beta$. Then*

(1) If both α and β are zero, then the Riemann surface S_φ has four connected components.

(2) If both α and β are nonzero, and $\alpha = \beta$ or $\alpha = -\beta$, then the Riemann surface S_φ has three connected components.

(3) If only one of α and β is zero, say $\beta = 0$ and $\alpha \neq 0$, then the Riemann surface S_φ has two connected components.

(4) If both α and β are nonzero, and α does not equal either β or $-\beta$, then the Riemann surface S_φ has two connected components.

II.9.2   Reducing subspaces

Now we turn to the part about reducing subspaces. Let us state our main result in this part as the following theorem.

Theorem 49. *Let α and β be in $\mathbb{D}$, and $\varphi = z^2 \varphi_\alpha \varphi_\beta$. Then*

(1) If both α and β are zero, then M_φ has exactly four nontrivial minimal reducing subspaces.

(2) If both α and β are nonzero, and $\alpha = \beta$ or $\alpha = -\beta$, then M_φ has exactly three nontrivial minimal reducing subspaces.

(3) If only one of α and β is zero, say $\beta = 0$ and $\alpha \neq 0$, then M_φ has exactly two nontrivial minimal reducing subspaces.

(4) If both α and β are nonzero, and α does not equal either β or $-\beta$, then M_φ has exactly two nontrivial minimal reducing subspaces.

The proof of Theorem 49 is long and will be finished in Section II.9.3 and II.9.4 . Before starting the proof, we make two remarks.

The first remark is that combining Theorem 48 and Theorem 49 yields our third main result in this chapter which we restate as the following theorem.

Theorem 50. *Let φ be a Blaschke product of order four. Then the number of nontrivial minimal reducing subspaces of M_φ equals the number of connected components of the Riemann surface of $\varphi^{-1} \circ \varphi$ over $\mathbb{D}$.*

For a Blaschke product φ we say that φ is decomposable if there are two Blaschke products η and ψ with degrees greater than 1 such that

$$\varphi(z) = \eta \circ \psi(z).$$

Recall that two Blaschke products φ_1 and φ_2 are equivalent if there are two points λ and c in $\mathbb{D}$ such that

$$\varphi_1 = \varphi_\lambda \circ \varphi_2 \circ \varphi_c.$$

Let $M_0(\varphi)$ be the distinguished reducing subspace of M_φ on which the restriction of M_φ is unitarily equivalent to the Bergman shift.

Our second remark is that we can say a little bit more about the reducing subspaces of M_φ in terms of decomposability as stated in the following theorem.

Theorem 51. *Let φ be a Blaschke product of order four. One of the following holds.*

(1) If φ is equivalent to z^4, then M_φ has only four nontrivial minimal reducing subspaces.

(2) If φ is not equivalent to z^4, but is decomposable, i.e, $\varphi = \eta \circ \psi$ for two Blaschke products η and ψ of order 2, then M_φ has only three nontrivial minimal reducing subspaces $M_0(\varphi), M_0(\psi) \ominus M_0(\varphi)$ and $M_0(\psi)^\perp$.

(3) If φ is not decomposable, then M_φ has only two nontrivial minimal reducing subspaces $M_0(\varphi)$ and $M_0(\varphi)^\perp$.

Proof of Theorem 51 by Theorem 49. As commented before, we may assume that $\varphi(z) = z^2 \varphi_\alpha \varphi_\beta$ for two points α, β in $\mathbb{D}$.

Given Theorem 49, we only need to show that, if $\varphi = \eta \circ \psi$ for two Blaschke products η and ψ of order 2, then α equals either β or $-\beta$. To show this we may suppose that $\eta(0) = 0$, $\psi(0) = 0$. Then taking derivative at 0 gives that $0 = \varphi^{'}(0) = \eta^{'}(0)\psi^{'}(0)$. So either $\eta^{'}(0) = 0$ or $\psi^{'}(0) = 0$. $\eta^{'}(0) = 0$ implies that $\alpha = \beta$. $\psi^{'}(0) = 0$ implies that $\alpha = -\beta$. We are done.

Proof of Theorem 49.

(1) If $\varphi = z^4$, then it follows from Theorem B in [44] that M_φ has exactly four nontrivial minimal reducing subspaces M_0, M_1, M_2, M_3 such that

$$\mathcal{H} = M_0 \oplus M_1 \oplus M_2 \oplus M_3$$

and each reducing subspace is a direct sum of some M_j's. In fact,

$$M_j = span\{z^{4k+3-j} : k = 0, 1, 2, \cdots, \}, j = 0, 1, 2, 3.$$

(2) If $\alpha = \beta \neq 0$, then $\varphi = z^2 \varphi_\alpha^2 = (z\varphi_\alpha)^2 = \eta \circ \psi$ with $\eta = z^2$ and $\psi = z\varphi_\alpha$ which are Blaschke products of order 2, and $\varphi(\mathcal{B}) = \eta(\psi(\mathcal{B}))$. It follows from Corollary 35 (also see the main theorem in [51] or in [58]) that $\psi(\mathcal{B})$ has exactly two nontrivial minimal reducing subspaces M_0 and $M_1 = M_0^\perp$. Of course, M_0 and M_1 are also reducing subspaces for $\varphi(\mathcal{B})$. By Theorem 33 we may assume that the restriction of $\psi(\mathcal{B})$ to it, $\psi(\mathcal{B})|_{M_0}$, is unitarily equivalent to the Bergman shift $\mathcal{B}$. So the restriction of $\varphi(\mathcal{B}) = \eta(\psi(\mathcal{B}))$ to M_0, $\eta(\psi(\mathcal{B}))|_{M_0}$, is unitarily equivalent to $\eta(\mathcal{B})$. Hence by Lemma 35 again $\varphi(\mathcal{B})|_{M_0}$ has two nontrivial minimal reducing subspaces M_{00} and M_{01} with $M_0 = M_{00} \oplus M_{01}$ such that $\varphi(\mathcal{B})|_{M_00}$ is unitarily equivalent to the Bergman shift. Therefore $\varphi(\mathcal{B})$ has three nontrivial reducing subspaces M_{00}, M_{01}, and M_1 such that $\mathcal{H} = M_{00} \oplus M_{01} \oplus M_1$ and the restriction of $\varphi(\mathcal{B})$ to M_{00} is a Bergman shift. That is, M_{00} is the distinguished reducing subspace for $\varphi(\mathcal{B})$. Now it follows from Theorem 41 that the each of the three nontrivial reducing subspaces is minimal.

To prove that they are the only nontrivial minimal reducing subspaces for $\varphi(\mathcal{B})$, we assume that there is another one and derive a contradiction. Observe that $L_0 = M_{00} \cap L_0 \oplus M_{01} \cap L_0 \oplus M_1 \cap L_0$ and the dimension of L_0 is four and the dimension of $M_{00} \cap L_0$ is one . So we may assume that the dimension of $M_{01} \cap L_0$ is one and the dimension of $M_1 \cap L_0$ is two. If there were another minimal reducing subspace Ω other than the known three, then first by the Theorem 34

$$\Omega \subset M_{01} \oplus M_1$$

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and then by the Theorem 40 there is a unitary operator

$$U : M_{01} \longrightarrow M_1$$

such that U commutes with both $\varphi(\boldsymbol{\mathcal{B}})$ and $\varphi(\boldsymbol{\mathcal{B}})^*$. Therefore we would have the dimension of $M_{01} \cap L_0$ is the same as the dimension of $M_1 \cap L_0$, a contradiction.

If $\alpha = -\beta \neq 0$, then $\varphi = z^2 \varphi_\alpha \varphi_{-\alpha} = \eta \circ \psi$. Here $\eta = z\varphi_{\alpha^2}$ and $\psi = z^2$ are also two Blaschke product of order 2. By the same argument as above we know that in this case $\varphi(\boldsymbol{\mathcal{B}})$ also has exactly three nontrivial minimal reducing subspaces. The proof is finished.

The proofs of (3) and (4) are long and we put them in Section II.9.3 and II.9.4 respectively. By Theorems 52 in Section II.9.3 and Theorem 53 in Section II.9.4 , M_φ has only two nontrivial minimal reducing subspaces.

II.9.3 Reducing subspaces of $M_{z^3 \varphi_\alpha}$

In this section we restate item (3) of Theorem 49 as Theorem 52 and prove it. Recall that $\boldsymbol{\mathcal{M}}_0$ is the distinguished reducing subspace of $\varphi(\boldsymbol{\mathcal{B}})$ as in Theorem 33.

Theorem 52. *Let $\varphi = z^3 \varphi_\alpha$ for a nonzero point $\alpha \in \mathbb{D}$. Then $\varphi(\boldsymbol{\mathcal{B}})$ has only two nontrivial reducing subspaces $\boldsymbol{\mathcal{M}}_0$ and $\boldsymbol{\mathcal{M}}_0^\perp$.*

Proof. By Theorem 34, every minimal reducing subspace other than $\boldsymbol{\mathcal{M}}_0$ is contained in $\boldsymbol{\mathcal{M}}_0^\perp$. So we only need to show that $\boldsymbol{\mathcal{M}}_0^\perp$ is a minimal reducing subspace for $\varphi(\boldsymbol{\mathcal{B}})$.

Assume that $\boldsymbol{\mathcal{M}}_0^\perp$ is not a minimal reducing subspace for $\varphi(\boldsymbol{\mathcal{B}})$. Then by the same argument as in Section II.8 we may assume

$$\boldsymbol{\mathcal{H}} = \bigoplus_{i=0}^{2} M_i$$

such that each M_i is a nontrivial reducing subspace for $\varphi(\boldsymbol{\mathcal{B}})$, $M_0 = \boldsymbol{\mathcal{M}}_0$ is the distinguished reducing subspace for $\varphi(\boldsymbol{\mathcal{B}})$ and

$$M_0^\perp = M_1 \oplus M_2.$$

Recall that

$$\varphi(z) = z\varphi_0(z),$$

$$\varphi_0(z) = z^2\varphi_\alpha(z),$$

$$L_0 = span\{1, p_1, p_2, k_\alpha(z)k_\alpha(w)\},$$

and

$$L_0 = (L_0 \cap M_0) \oplus (L_0 \cap M_1) \oplus (L_0 \cap M_2).$$

We further assume that

$$dim(M_1 \cap L_0) = 1$$

and

$$dim(M_2 \cap L_0) = 2.$$

Take $0 \neq e_1 \in M_1 \cap L_0$, e_2, $e_3 \in M_2 \cap L_0$ such that $\{e_2, e_3\}$ are a basis for $M_2 \cap L_0$, then

$$L_0 = span\{e_0, e_1, e_2, e_3\}$$

By Theorem 28 in section II.4.3, we have

$$d_{e_j}^0 = we_j(0, w)e_0 - \varphi(w)e_j.$$

Direct computations show that

$$
\begin{aligned}
\langle d_{e_j}^0, p_k \rangle &= \langle we_j(0, w)e_0 - \varphi(w)e_j, p_k \rangle \\
&= \langle we_j(0, w)e_0, p_k \rangle \quad (by \ T_{\varphi(w)}^* p_k = 0) \\
&= \langle we_j(0, w)e_0(w, w), p_k(0, w) \rangle \\
&= \langle we_j(0, w)\varphi'(w), w^k \rangle \\
&= \langle w^3 e_j(0, w)(w\varphi_\alpha'(w) + 3\varphi_\alpha(w)), w^k \rangle \\
&= \langle w^{3-k} e_j(0, w)(w\varphi_\alpha'(w) + 3\varphi_\alpha(w)), 1 \rangle \\
&= 0
\end{aligned}
$$

for $0 \leq k \leq 2$, and

$$
\begin{aligned}
\langle d_{e_j}^0, k_\alpha(z) k_\alpha(w) \rangle &= \alpha e_j(0, \alpha) e_0(\alpha, \alpha) \\
&= \alpha e_j(0, \alpha) \frac{\alpha^3}{1 - |\alpha|^2}.
\end{aligned}
$$

This implies that those functions $d_{e_j}^0$ are orthogonal to $\{1, p_1, p_2\}$.

Simple calculations give

$$
\langle e_0, p_k \rangle = 0
$$

for $0 \leq k \leq 1$,

$$
\begin{aligned}
\langle e_0, p_2 \rangle &= \langle e_0(0, w), p_2(w, w) \rangle \\
&= \frac{3}{2} \varphi_0''(0) \\
&= -3\alpha \neq 0
\end{aligned}
$$

and

$$
\begin{aligned}
\langle e_0, k_\alpha(z) k_\alpha(w) \rangle &= e_0(\alpha, \alpha) \\
&= \varphi'(\alpha) \\
&= \frac{\alpha^3}{1 - |\alpha|^2} \neq 0
\end{aligned}
$$

By Theorem 30, there are numbers μ, λ_j such that

$$
d_{e_1}^1 = d_{e_1}^0 + \mu e_1 + \lambda_1 e_0
$$

$$
d_{e_2}^1 = d_{e_2}^0 + \tilde{e}_2 + \lambda_2 e_0
$$

$$
d_{e_3}^1 = d_{e_3}^0 + \tilde{e}_3 + \lambda_3 e_0
$$

where $\tilde{e}_2, \tilde{e}_3 \in M_2 \cap L_0$.

Now we consider two cases. In each case we will derive a contradiction.

Case 1. $\mu \neq 0$. In this case, we get that e_1 is orthogonal to $\{1, p_1\}$. So $\{1, p_1, e_0, e_1\}$

form an orthogonal basis for L_0.

First we show that $\tilde{e}_2 = 0$. If $\tilde{e}_2 \neq 0$, then we get that $\{1, p_1, e_0, \tilde{e}_2\}$ are also an orthogonal basis for L_0. Thus

$$\tilde{e}_2 = ce_1$$

for some nonzero number c. However, $\tilde{e}_2$ is orthogonal to e_1 since $\tilde{e}_2 \in M_2$ and $e_1 \in M_1$. This is a contradiction. Thus

$$d_{e_2}^1 = d_{e_2}^0 + \lambda_2 e_0.$$

Since both $d_{e_2}^1$ and $d_{e_2}^0$ are orthogonal to p_2 and

$$\langle e_0, p_2 \rangle = -3\alpha \neq 0,$$

we have that $\lambda_2 = 0$ to get that $d_{e_2}^0 = d_{e_2}^1$ is orthogonal to L_0. On the other hand,

$$\langle d_{e_2}^0, k_\alpha(z) k_\alpha(w) \rangle = \alpha e_2(0, \alpha) \frac{\alpha^3}{1 - |\alpha|^2}.$$

Thus

$$e_2(0, \alpha) = 0.$$

Similarly we get that

$$e_3(0, \alpha) = 0.$$

Moreover, since e_2 and e_3 are orthogonal to $\{e_0, e_1\}$, write

$$e_2 = c_{11} + c_{12} p_1,$$

$$e_3 = c_{21} + c_{22} p_1.$$

Thus

$$e_2(0, \alpha) = c_{11} + c_{12}\alpha = 0,$$

$$e_3(0, \alpha) = c_{21} + c_{22}\alpha = 0.$$

This gives that e_2 and e_3 are linearly dependent. So we get a contradiction in this case.

Case 2. $\mu = 0$. In this case we have

$$d_{e_1}^1 = d_{e_1}^0 + \lambda_1 e_0.$$

Similarly to the proof in **Case 1** we get that $\lambda_1 = 0$,

$$d_e^1 = d_{e_1}^0 \perp L_0 \tag{II.21}$$

and

$$e_1(0, \alpha) = 0.$$

Theorem 43 in Section II.7 gives that at least one $\tilde{e}_j$, say $\tilde{e}_2 \neq 0$. Assume that $\tilde{e}_2 \neq 0$, write

$$\tilde{e}_2 = d_{e_2}^1 - d_{e_2}^0 - \lambda_2 e_0.$$

Note that we have shown above that both $d_{e_2}^0$ and e_0 are orthogonal to both 1 and p_1. Thus

$$\tilde{e}_2 \perp \{1, p_1\}$$

and

$$L_0 = span\{1, p_1, e_0, \tilde{e}_2\}.$$

Since e_1 is orthogonal to $\{e_0, \tilde{e}_2\}$ we have

$$e_1 = c_1 + c_2 p_1.$$

Noting that $e_1(0, \alpha) = c_1 + c_2\alpha = 0$ we get

$$e_1 = c_2(-\alpha + p_1).$$

Without loss of generality we assume that

$$e_1 = -\alpha + p_1. \tag{II.22}$$

Letting e be in $M_2 \cap L_0$ such that e is a nonzero function orthogonal to $\tilde{e}_2$, we have that e is orthogonal to $\{e_0, \tilde{e}_2\}$. Thus e must be in the subspace $span\{1, p_1\}$. So there are two constants b_1 and b_2 such that

$$e = b_1 + b_2 p_1.$$

Noting

$$
\begin{aligned}
0 &= \langle e, e_1 \rangle \\
&= -b_1 \bar{\alpha} + 2b_2
\end{aligned}
$$

we have

$$e = \frac{b_1}{2}(2 + \bar{\alpha} p_1).$$

Hence we may assume that

$$e = 2 + \bar{\alpha} p_1. \tag{II.23}$$

By Theorem 30 we have

$$d_e^1 = d_e^0 + \tilde{e} + \lambda e_0$$

for some number λ and $\tilde{e} \in M_2 \cap L_0$. Thus

$$
\begin{aligned}
0 &= \langle d_{e_1}^1, d_e^1 \rangle \\
&= \langle d_{e_1}^1, d_e^0 + \tilde{e} + \lambda e_0 \rangle \\
&= \langle d_{e_1}^1, d_e^0 \rangle \\
&= \langle d_{e_1}^0, d_e^0 \rangle \quad (by \ (II.21)).
\end{aligned}
$$

However, a simple computation gives

$$
\begin{aligned}
\langle d^0_{e_1}, d^0_e \rangle &= \langle d^0_{e_1}, we(0,w)e_0 - \varphi(w)e \rangle \\
&= \langle d^0_{e_1}, we(0,w)e_0 \rangle \quad (by \ \ T^*_{\varphi(w)} d^0_{e_1} = 0) \\
&= \langle we_1(0,w)e_0 - \varphi(w)e_1, we(0,w)e_0 \rangle \\
&= \langle we_1(0,w)e_0, we(0,w)e_0 \rangle - \langle \varphi(w)e_1, we(0,w)e_0 \rangle.
\end{aligned}
$$

We need to calculate the two terms in the right hand of the above equality. By (II.22) and (II.23), the first term becomes

$$
\begin{aligned}
&\langle we_1(0,w)e_0, we(0,w)e_0 \rangle \\
&= \langle w(-\alpha+w)e_0, w(2+\bar{\alpha}w)e_0 \rangle \\
&= \langle (-\alpha+w)e_0, (2+\bar{\alpha}w)e_0 \rangle \\
&= \langle -\alpha e_0, 2e_0 \rangle + \langle we_0, 2e_0 \rangle + \langle -\alpha e_0, \bar{\alpha}we_0 \rangle + \langle we_0, \bar{\alpha}we_0 \rangle \\
&= -\alpha \langle e_0, e_0 \rangle + 2\langle we_0, e_0 \rangle - \alpha^2 \langle e_0, we_0 \rangle.
\end{aligned}
$$

The first term in right hand of the last equality is

$$
\begin{aligned}
\langle e_0, e_0 \rangle &= \langle e_0(w,w), e_0(0,w) \rangle \\
&= \langle w\varphi'_0 + \varphi_0, \varphi_0 \rangle \\
&= \langle w(2w\varphi_\alpha + w^2 \varphi'_\alpha), w^2\varphi_\alpha \rangle + \langle \varphi_0, \varphi_0 \rangle. \\
&= 2 + \langle w\varphi'_\alpha, \varphi_\alpha \rangle + 1 \\
&= 4.
\end{aligned}
$$

The last equality follows from

$$
\begin{aligned}
\varphi_\alpha &= -\frac{1}{\bar{\alpha}} + \frac{\frac{1}{\bar{\alpha}} - \alpha}{1 - \bar{\alpha}w} \\
&= -\frac{1}{\bar{\alpha}} + (\frac{1}{\bar{\alpha}} - \alpha)K_\alpha(w).
\end{aligned}
$$

Similarly, we have

$$\langle we_0, e_0 \rangle = \langle we_0(w, w), e_0(0, w) \rangle$$
$$= \langle w(w\varphi_0' + \varphi_0), \varphi_0 \rangle$$
$$= \alpha.$$

This gives

$$\langle we_1(0, w)e_0, we(0, w)e_0 \rangle = \langle e_1(0, w)e_0, e(0, w)e_0 \rangle$$
$$= \langle (-\alpha + w)e_0, (2 + \bar{\alpha}w)e_0 \rangle$$
$$= -2\alpha\langle e_0, e_0 \rangle - \alpha^2\langle e_0, we_0 \rangle$$
$$+ 2\langle we_0, e_0 \rangle + \alpha\langle we_0, we_0 \rangle$$
$$= -8\alpha - \alpha|\alpha|^2 + 2\alpha + 4\alpha$$
$$= -2\alpha - \alpha|\alpha|^2$$

A simple calculation gives that the second term becomes

$$\langle \varphi(w)e_1, we(0, w)e_0 \rangle$$
$$= \langle \varphi_0(w)e_1, (2 + \bar{\alpha}w)e_0 \rangle$$
$$= \langle \varphi_0(w)e_1, 2e_0 \rangle + \langle \varphi_0(w)e_1, \bar{\alpha}we_0 \rangle$$
$$= 2\langle \varphi_0(w)e_1(w, w), e_0(0, w) \rangle + \alpha\langle \varphi_0(w)e_1(w, w), we_0(0, w) \rangle$$
$$= 2\langle e_1(w, w), 1 \rangle + \alpha\langle e_1(w, w), w \rangle$$
$$= 2\langle -\alpha + 2w, 1 \rangle + \alpha\langle -\alpha + 2w, w \rangle = -2\alpha + 2\alpha = 0.$$

Thus we conclude

$$\langle d_{e_1}^0, d_e^0 \rangle = \langle we_1(0, w)e_0, we(0, w)e_0 \rangle - \langle \varphi(w)e_1, we(0, w)e_0 \rangle$$
$$= -2\alpha - \alpha|\alpha|^2$$
$$= -\alpha(2 + |\alpha|^2) \neq 0$$

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to get a contradiction in this case. The proof is finished.

II.9.4 Reducing subspaces of $M_{z^2 \varphi_\alpha \varphi_\beta}$

In this section we prove item (4) of Theorem 49 as Theorem 53.

Theorem 53. *Let φ be a Blaschke product of the form $z^2 \varphi_\alpha \varphi_\beta$ for two nonzero points α and β in $\mathbb{D}$, $\alpha \neq \beta$. If α does not equal $-\beta$, then $\varphi(\mathcal{B})$ has only two nontrivial reducing subspaces $\mathcal{M}_0$ and $\mathcal{M}_0^\perp$.*

Proof. By Theorem 34, we only need to show that $\mathcal{M}_0^\perp$ is a minimal reducing subspace for $\varphi(\mathcal{B})$ unless $\alpha = -\beta$.

Assume that $\mathcal{M}_0^\perp$ is not a minimal reducing subspace for $\varphi(\mathcal{B})$. Then by the same reason as in Section II.8 we may assume

$$\mathcal{H} = \bigoplus_{i=0}^{2} M_i$$

such that each M_i is a nontrivial reducing subspace for $\varphi(\mathcal{B})$, $M_0 = \mathcal{M}_0$ is the distinguished reducing subspace for $\varphi(\mathcal{B})$ and

$$M_1 \oplus M_2 = \mathcal{M}_0^\perp.$$

Recall that

$$\varphi_0 = z \varphi_\alpha \varphi_\beta,$$

$$L_0 = span\{1, p_1, e_\alpha, e_\beta\},$$

with $e_\alpha = k_\alpha(z) k_\alpha(w), e_\beta = k_\beta(z) k_\beta(w)$ and

$$L_0 = (L_0 \cap M_0) \oplus (L_0 \cap M_1) \oplus (L_0 \cap M_2).$$

So we further assume that the dimension of $M_1 \cap L_0$ is one and the dimension of $M_2 \cap L_0$ is two. Take a nonzero element e_1 in $M_1 \cap L_0$, then by Theorem 30, there are numbers μ_1, λ_1

such that

$$d_{e_1}^1 = d_{e_1}^0 + \mu_1 e_1 + \lambda_1 e_0. \tag{II.24}$$

We only need to consider two possibilities, μ_1 is zero or nonzero.

If $\mu_1 = 0$, then (II.24) becomes

$$d_{e_1}^1 = d_{e_1}^0 + \lambda_1 e_0. \tag{II.25}$$

In this case, simple calculations give

$$
\begin{aligned}
\langle d_{e_1}^0, p_1 \rangle &= \langle we_1(0,w)e_0(z,w) - w\varphi_0(w)e_1(z,w), p_1(z,w) \rangle \\
&= \langle we_1(0,w)e_0(w,w) - w\varphi_0(w)e_1(w,w), p_1(z,w) \rangle \\
&= \langle we_1(0,w)e_0(w,w) - w\varphi_0(w)e_1(w,w), p_1(0,w) \rangle \\
&= \langle we_1(0,w)e_0(w,w) - w\varphi_0(w)e_1(w,w), w \rangle \\
&= \langle e_1(0,w)e_0(w,w) - \varphi_0(w)e_1(w,w), 1 \rangle \\
&= e_1(0,0)e_0(0,0) - \varphi_0(0)e_1(0,0) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle e_0, p_1 \rangle &= \langle e_0(z,w), p_1(z,w) \rangle \\
&= \langle e_0(z,w), p_1(w,w) \rangle \\
&= \langle e_0(0,w), 2w \rangle \\
&= \langle \varphi_0(w), 2w \rangle \\
&= 2\langle w\varphi_\alpha(w)\varphi_\beta(w), w \rangle \\
&= 2\varphi_\alpha(0)\varphi_\beta(0) = 2\alpha\beta \neq 0.
\end{aligned}
$$

Noting that $d_{e_1}^1$ is orthogonal to L_0, by (II.25) we have that $\lambda_1 = 0$, and hence

$$d_{e_1}^0 = d_{e_1}^1 \perp L_0.$$

So
$$\langle d^0_{e_1}, e_\alpha \rangle = 0 = \langle d^0_{e_1}, e_\beta \rangle.$$

On the other hand,

$$
\begin{aligned}
\langle d^0_{e_1}, e_\alpha \rangle &= \alpha e_1(0, \alpha) e_0(\alpha, \alpha) - \alpha \varphi_0(\alpha) e_1(\alpha, \alpha) \\
&= \alpha e_1(0, \alpha) e_0(\alpha, \alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle d^0_{e_1}, e_\beta \rangle &= \beta e_1(0, \beta) e_0(\beta, \beta) - \beta \varphi_0(\beta) e_1(\beta, \beta) \\
&= \beta e_1(0, \beta) e_0(\beta, \beta).
\end{aligned}
$$

Consequently

$$e_1(0, \alpha) = e_1(0, \beta) = 0. \tag{II.26}$$

Observe that e_0, e_1 and 1 are linearly independent. If this is not so, then $1 = ae_0 + be_1$ for some numbers a, b. But $e_1(0, \alpha) = 0$ and $e_0(0, \alpha) = 0$. This forces that $1 = 0$ and leads to a contradiction.

By Theorem 43, we can take an element $e \in M_2 \cap L_0$ such that

$$d^1_e = d^0_e + e_2 + \mu e_0$$

with $e_2 \neq 0$ and $e_2 \in M_2 \cap L_0$. Thus we have that e_2 is orthogonal to 1 and so e_2 is in $\{1, e_0, e_1\}^\perp$ and $\{1, e_0, e_1, e_2\}$ form a basis for L_0. Moreover for any $f \in M_2 \cap L_0$,

$$d^1_f = d^0_f + g + \lambda e_0$$

for some number λ and $g \in M_2 \cap L_0$. If g does not equal 0 then g is orthogonal to 1. Thus g is in $\{1, e_0, e_1\}^\perp$ and hence

$$g = ce_2$$

for some number c. Therefore taking a nonzero element $e_3 \in M_2 \cap L_0$ which is orthogonal to e_2, we have

$$d^1_{e_2} = d^0_{e_2} + \mu_2 e_2 + \lambda_2 e_0,$$

$$d^1_{e_3} = d^0_{e_3} + \mu_3 e_2 + \lambda_3 e_0,$$

and $\{e_0, e_1, e_2, e_3\}$ is an orthogonal basis for L_0.

If $\mu_2 = 0$, then by the same reason as before we get

$$
\begin{aligned}
\lambda_2 &= 0, \\
d^0_{e_2} &= d^1_{e_2} \perp L_0 \\
e_2(0, \alpha) &= e_2(0, \beta) \\
&= 0.
\end{aligned}
$$

So using

$$p_1 \in L_0 = span\{1, e_0, e_1, e_2\}$$

we have

$$\alpha = p_1(0, \alpha) = p_1(0, \beta) = \beta,$$

which contradicts our assumption that $\alpha \neq \beta$. Hence $\mu_2 \neq 0$.

Observe that 1 is in $L_0 = span\{e_0, e_1, e_2, e_3\}$ and orthogonal to both e_0 and e_2. Thus

$$1 = c_1 e_1 + c_3 e_3$$

for some numbers c_1 and c_3. So

$$
\begin{aligned}
1 &= c_1 e_1(0, \alpha) + c_3 e_3(0, \alpha) \\
&= c_1 e_1(0, \beta) + c_3 e_3(0, \beta).
\end{aligned}
$$

By (II.26), we have

$$1 = c_3 e_3(0, \alpha) = c_3 e_3(0, \beta),$$

to obtain that $c_3 \neq 0$ and

$$e_3(0, \alpha) = e_3(0, \beta) = 1/c_3.$$

If $\mu_3 = 0$, then by the same reason as before we get $e_3(0, \alpha) = e_3(0, \beta) = 0$. Hence $\mu_3 \neq 0$.

Now by the linearality of $d^1_{(\cdot)}$ and $d^0_{(\cdot)}$ we have

$$d^1_{\mu_3 e_2 - \mu_2 e_3} = d^0_{\mu_3 e_2 - \mu_2 e_3} + (\mu_3 \lambda_2 - \mu_2 \lambda_3) e_0.$$

By the same reason as before we get

$$\mu_3 \lambda_2 - \mu_2 \lambda_3 = 0$$

and

$$d^0_{\mu_3 e_2 - \mu_2 e_3} = d^1_{\mu_3 e_2 - \mu_2 e_3} \perp L_0$$

and therefore

$$\begin{aligned}
\mu_3 e_2(0, \alpha) - \mu_2 e_3(0, \alpha) &= \mu_3 e_2(0, \beta) - \mu_2 e_3(0, \beta) \\
&= 0.
\end{aligned}$$

So we get

$$e_2(0, \alpha) = \mu_2/\mu_3 c_3 = e_2(0, \beta).$$

Hence

$$p_1 \in L_0 = span\{1, e_0, e_1, e_2\}.$$

This implies that

$$\alpha = p_1(0, \alpha) = p_1(0, \beta) = \beta$$

which again contradicts our assumption that $\alpha \neq \beta$.

Another possibility is that $\mu_1 \neq 0$. In this case, (II.24) can be rewritten as

$$e_1 = \frac{1}{\mu_1} d_{e_1}^1 - \frac{1}{\mu_1} d_{e_1}^0 - \frac{\lambda_1}{\mu_1} e_0,$$

and we have that e_1 is orthogonal to 1 since $d_{e_1}^1$, $d_{e_1}^0$ and e_0 are orthogonal to 1. Thus 1 is in $M_2 \cap L_0$.

By Theorem 30, there is an element $e \in M_2 \cap L_0$ and a number λ_0 such that

$$d_1^1 = d_1^0 + e + \lambda_0 e_0. \tag{II.27}$$

If $e = 0$ then $\lambda_0 = 0$, and hence $d_1^0 \perp L_0$ and

$$1 = 1(0, \alpha) = 1(0, \beta) = 0.$$

So $e \neq 0$.

Since d_1^1 is in $L_0^\perp$, d_1^1 is orthogonal to 1. Noting that d_1^0 and e_0 are orthogonal to 1, we have that $e \perp 1$. Hence we get an orthogonal basis $\{e_0, e_1, 1, e\}$ of L_0.

Claim.

$$e(0, \alpha) - e(0, \beta) = 0.$$

Proof of the claim. Using Theorem 30 again, we have that

$$d_e^1 = d_e^0 + g + \lambda e_0$$

for some $g \in L_0 \cap M_2$. If $g \neq 0$, we have that $g \perp 1$ since d_e^1, d_e^0, and e_0 are orthogonal to 1. Thus we have that $g = \mu e$ for some number μ to obtain

$$d_e^1 = d_e^0 + \mu e + \lambda e_0.$$

Furthermore by the linearality of $d_{(\cdot)}^1$ and $d_{(\cdot)}^0$ we have that

$$d_{e-\mu 1}^1 = d_{e-\mu 1}^0 + (\lambda - \mu \lambda_0) e_0.$$

By the same reason (namely $d^1_{e-\mu 1} \perp L_0$, $d^0_{e-\mu 1} \perp 1$ and $\langle e_0, 1 \rangle \neq 0$) we have that

$$\lambda - \mu \lambda_0 = 0,$$

$$d^0_{e-\mu 1} = d^1_{e-\mu 1} \perp L_0$$

and

$$(e - \mu 1)(0, \alpha) = (e - \mu 1)(0, \beta) = 0.$$

Hence we have

$$e(0, \alpha) - e(0, \beta) = \mu - \mu = 0,$$

to complete the proof of the claim.

Let us find the value of λ_0 in (II.27) which will be used to make the coefficients symmetric with respect to α and β. To do this, we first state a technical lemma which will be used in several other places in the sequel.

Lemma 54. *If $g = g(w) \in H^2(\mathbb{T})$, then*

$$\langle wg\varphi'_0, \varphi_0 \rangle = g(0) + g(\alpha) + g(\beta).$$

Proof. Since $\varphi_0 = z\varphi_\alpha\varphi_\beta$, simple calculations give

$$
\begin{aligned}
\langle wg\varphi'_0, \varphi_0 \rangle &= \langle wg(w\varphi_\alpha\varphi_\beta)', w\varphi_\alpha\varphi_\beta \rangle \\
&= \langle g(w\varphi_\alpha\varphi_\beta)', \varphi_\alpha\varphi_\beta \rangle \\
&= \langle g(\varphi_\alpha\varphi_\beta + w\varphi'_\alpha\varphi_\beta + w\varphi_\alpha\varphi'_\beta), \varphi_\alpha\varphi_\beta \rangle \\
&= \langle g, 1 \rangle + \langle wg\varphi'_\alpha, \varphi_\alpha \rangle + \langle wg\varphi'_\beta, \varphi_\beta \rangle \\
&= g(0) + \langle wg\varphi'_\alpha, \varphi_\alpha \rangle + \langle wg\varphi'_\beta, \varphi_\beta \rangle
\end{aligned}
$$

Writing φ_α as

$$
\begin{aligned}
\varphi_\alpha &= -\frac{1}{\bar{\alpha}} + \frac{\frac{1}{\bar{\alpha}} - \alpha}{1 - \bar{\alpha}w} \\
&= -\frac{1}{\bar{\alpha}} + \frac{1 - |\alpha|^2}{\bar{\alpha}} k_\alpha(w),
\end{aligned}
$$

we have

$$
\begin{aligned}
\langle wg\varphi_\alpha', \varphi_\alpha \rangle &= \frac{1 - |\alpha|^2}{\alpha}(wg\varphi_\alpha')(\alpha) \\
&= g(\alpha).
\end{aligned}
$$

The first equality follows from $\langle wg\varphi_\alpha', 1 \rangle = 0$ and the second equality follows from $\varphi_\alpha'(\alpha) = \frac{1}{1-|\alpha|^2}$.

By the symmetry of α and β, similar computations lead to

$$
\langle wg\varphi_\beta', \varphi_\beta \rangle = g(\beta)
$$

and the proof is finished.

We state the values of λ_0 and $\langle e_0, e_0 \rangle$ as a lemma.

Lemma 55.

$$
\lambda_0 = -\frac{\alpha + \beta}{4} \tag{II.28}
$$

$$
\langle e_0, e_0 \rangle = 4 \tag{II.29}
$$

Proof. Since d_1^1 is orthogonal to L_0, e_0 is in L_0, and e is orthogonal to e_0, (II.27) gives

$$
\begin{aligned}
0 &= \langle d_1^1, e_0 \rangle \\
&= \langle d_1^0 + e + \lambda_0 e_0, e_0 \rangle \\
&= \langle d_1^0, e_0 \rangle + \lambda_0 \langle e_0, e_0 \rangle.
\end{aligned}
$$

We need to compute $\langle d_1^0, e_0 \rangle$ and $\langle e_0, e_0 \rangle$ respectively.

$$
\begin{aligned}
\langle d_1^0, e_0 \rangle &= \langle -\varphi(w) + we_0, e_0 \rangle \\
&= \langle we_0, e_0 \rangle \\
&= \langle we_0(w, w), e_0(0, w) \rangle \\
&= \langle w(w\varphi_0' + \varphi_0), \varphi_0 \rangle \\
&= \langle w^2 \varphi_0', \varphi_0 \rangle + \langle w\varphi_0, \varphi_0 \rangle \\
&= \langle w^2 \varphi_0', \varphi_0 \rangle \\
&= \alpha + \beta.
\end{aligned}
$$

The last equality follows from Lemma 54 with $g = w$.

$$
\begin{aligned}
\langle e_0, e_0 \rangle &= \langle e_0(w, w), e_0(0, w) \rangle \\
&= \langle w\varphi_0' + \varphi_0, \varphi_0 \rangle \\
&= \langle w\varphi_0', \varphi_0 \rangle + \langle \varphi_0, \varphi_0 \rangle \\
&= \langle w\varphi_0', \varphi_0 \rangle + 1 \\
&= 4,
\end{aligned}
$$

where the last equality follows from Lemma 54 with $g = 1$. Hence

$$
\alpha + \beta + 4\lambda_0 = 0
$$

and

$$
\lambda_0 = -\frac{\alpha + \beta}{4}.
$$

Let P_{L_0} denote the projection of $H^2(\mathbb{T}^2)$ onto L_0. The element $P_{L_0}(k_\alpha(w) - k_\beta(w))$ has the property that for any $g \in L_0$,

$$
\begin{aligned}
\langle g, P_{L_0}(k_\alpha(w) - k_\beta(w)) \rangle &= \langle g, k_\alpha(w) - k_\beta(w) \rangle \\
&= g(0, \alpha) - g(0, \beta).
\end{aligned}
$$

Thus $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is orthogonal to g for $g \in L_0$ with

$$g(0, \alpha) = g(0, \beta).$$

So $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is orthogonal to $e_0, 1, e$. On the other hand,

$$\langle p_1, P_{L_0}(k_\alpha(w) - k_\beta(w)) \rangle \quad = \quad \alpha - \beta$$
$$\neq \quad 0.$$

This gives that the element $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is a nonzero element. Therefore there exists a nonzero number b such that

$$P_{L_0}(k_\alpha(w) - k_\beta(w)) = be_1.$$

Without loss of generality we assume that

$$e_1 = P_{L_0}(k_\alpha(w) - k_\beta(w)).$$

Observe that

$$p_1(\varphi(z), \varphi(w))e_1 + d_{e_1}^1 \quad \in \quad M_1,$$
$$p_1(\varphi(z), \varphi(w)) + d_1^1 \quad \in \quad M_2,$$
$$M_1 \quad \perp \quad M_2,$$

to get

$$\langle p_1(\varphi(z), \varphi(w))e_1 + d_{e_1}^1, p_1(\varphi(z), \varphi(w)) + d_1^1 \rangle = 0.$$

Thus we have

$$0 \quad = \quad \langle p_1(\varphi(z), \varphi(w))e_1 + d_{e_1}^1, p_1(\varphi(z), \varphi(w)) + d_1^1 \rangle$$
$$= \quad \langle (\varphi(z) + \varphi(w))e_1, \varphi(z) + \varphi(w) \rangle + \langle d_{e_1}^1, d_1^1 \rangle$$
$$= \quad \langle d_{e_1}^1, d_1^1 \rangle. \hspace{4cm} \text{(II.30)}$$

The second equality follows from

$$d_{e_1}^1, d_1^1 \in \ker T_{\varphi(z)}^* \cap \ker T_{\varphi(z)}^*.$$

The last equality follows from

$$e_1 \perp 1$$

and

$$e_1, 1 \in \ker T_{\varphi(z)}^* \cap \ker T_{\varphi(z)}^*.$$

Substituting (II.27) into Equation (II.30), we have

$$
\begin{aligned}
0 &= \langle d_{e_1}^1, d_1^0 + e + \lambda_0 e_0 \rangle \\
&= \langle d_{e_1}^1, d_1^0 \rangle \\
&= \langle d_{e_1}^1, -\varphi(w) + w e_0 \rangle \\
&= \langle d_{e_1}^1, w e_0 \rangle \\
&= \langle d_{e_1}^0 + \mu_1 e_1 + \lambda_1 e_0, w e_0 \rangle \\
&= \langle d_{e_1}^0, w e_0 \rangle + \mu_1 \langle e_1, w e_0 \rangle + \lambda_1 \langle e_0, w e_0 \rangle.
\end{aligned}
$$

The second equation comes from that $d_{e_1}^1$ is orthogonal to L_0 and both e and e_0 are in L_0. The third equation follows from the definition of d_1^0 and the forth equation follows from that $d_{e_1}^1$ is in $\ker T_{\varphi(z)}^* \cap \ker T_{\varphi(w)}^*$. We need to calculate $\langle d_{e_1}^0, w e_0 \rangle$, $\langle e_1, w e_0 \rangle$, and $\langle e_0, w e_0 \rangle$ separately.

To get $\langle d_{e_1}^0, w e_0 \rangle$, by the definition of $d_{e_1}^0$, we have

$$
\begin{aligned}
\langle d_{e_1}^0, w e_0 \rangle &= \langle -\varphi(w) e_1 + w e_1(0, w) e_0, w e_0 \rangle \\
&= \langle -\varphi(w) e_1, w e_0 \rangle + \langle w e_1(0, w) e_0, w e_0 \rangle
\end{aligned}
$$

Thus we need to compute $\langle -\varphi(w) e_1, w e_0 \rangle$ and $\langle w e_1(0, w) e_0, w e_0 \rangle$ one by one. The equality

$$\langle -\varphi(w) e_1, w e_0 \rangle = 0$$

follows from the following computations.

$$
\begin{aligned}
\langle -\varphi(w)e_1, we_0 \rangle &= \langle -w\varphi_0(w)e_1, we_0 \rangle \\
&= -\langle \varphi_0(w)e_1, e_0 \rangle \\
&= -\langle \varphi_0(w)e_1(w, w), e_0(0, w) \rangle \\
&= -\langle \varphi_0(w)e_1(w, w), \varphi_0(w) \rangle \\
&= -\langle e_1(w, w), 1 \rangle \\
&= -\langle e_1, 1 \rangle \\
&= 0.
\end{aligned}
$$

To get $\langle we_1(0, w)e_0, we_0 \rangle$, we continue as follows.

$$
\begin{aligned}
\langle we_1(0, w)e_0, we_0 \rangle &= \langle e_1(0, w)e_0, e_0 \rangle \\
&= \langle e_1(0, w)e_0(w, w), e_0(0, w) \rangle \\
&= \langle e_1(0, w)e_0(w, w), \varphi_0(w) \rangle \\
&= \langle e_1(0, w)(\varphi_0(w) + w\varphi_0'(w)), \varphi_0(w) \rangle \\
&= \langle e_1(0, w)\varphi_0(w), \varphi_0(w) \rangle + \langle e_1(0, w)w\varphi_0'(w), \varphi_0(w) \rangle \\
&= \langle e_1(0, w), 1 \rangle + \langle e_1(0, w)w\varphi_0'(w), \varphi_0(w) \rangle \\
&= e_1(0, 0) + \langle e_1(0, w)w\varphi_0'(w), \varphi_0(w) \rangle \\
&= \langle e_1, 1 \rangle + \langle e_1(0, w)w\varphi_0'(w), \varphi_0(w) \rangle \\
&= \langle e_1(0, w)w\varphi_0'(w), \varphi_0(w) \rangle \\
&= e_1(0, \alpha) + e_1(0, \beta).
\end{aligned}
$$

The last equality follows from Lemma 54 and

$$
e_1(0, 0) = \langle e_1, 1 \rangle = 0.
$$

Hence

$$\langle d^0_{e_1}, we_0 \rangle = e_1(0, \alpha) + e_1(0, \beta)$$

Recall that

$$d^1_1 = d^0_1 + e + \lambda_0 e_0$$

is orthogonal to L_0 and e_1 is orthogonal to both e, and e_0. Thus

$$
\begin{aligned}
0 &= \langle e_1, d^0_1 + e + \lambda_0 e_0 \rangle \\
&= \langle e_1, -\varphi(w) + we_0 \rangle \\
&= \langle e_1, we_0 \rangle.
\end{aligned}
$$

From the computation of $\langle d^0_1, e_0 \rangle$ in the proof of Lemma 55 we have showed that

$$\langle we_0, e_0 \rangle = \alpha + \beta.$$

Therefore we have that

$$e_1(0, \alpha) + e_1(0, \beta) + \lambda_1(\bar{\alpha} + \bar{\beta}) = 0. \tag{II.31}$$

On the other hand,

$$
\begin{aligned}
0 &= \langle d^1_{e_1}, e_0 \rangle \\
&= \langle d^0_{e_1} + \mu_1 e_1 + \lambda_1 e_0, e_0 \rangle \\
&= \langle d^0_{e_1}, e_0 \rangle + 4\lambda_1
\end{aligned}
$$

and

$$\begin{aligned}
\langle d^0_{e_1}, e_0 \rangle &= \langle -\varphi(w)e_1 + we_1(0,w)e_0, e_0 \rangle \\
&= \langle we_1(0,w)e_0, e_0 \rangle \\
&= \langle we_1(0,w)e_0(w,w), e_0(0,w) \rangle \\
&= \langle we_1(0,w)(\varphi_0(w) + w\varphi'_0), \varphi_0(w) \rangle \\
&= \langle w^2 e_1(0,w)\varphi'_0, \varphi_0(w) \rangle \\
&= \alpha e_1(0,\alpha) + \beta e_1(0,\beta).
\end{aligned}$$

The last equality follows from Lemma 54 with $g = we_1(0,w)$. Thus

$$\alpha e_1(0,\alpha) + \beta e_1(0,\beta) + 4\lambda_1 = 0.$$

So

$$\lambda_1 = -\frac{\alpha}{4}e_1(0,\alpha) - \frac{\beta}{4}e_1(0,\beta). \tag{II.32}$$

Substituting (II.32) into (II.31), we have

$$[1 - \frac{\alpha(\bar{\alpha} + \bar{\beta})}{4}]e_1(0,\alpha) + [1 - \frac{\beta(\bar{\alpha} + \bar{\beta})}{4}]e_1(0,\beta) = 0.$$

Recall that

$$\lambda_0 = -\frac{\alpha + \beta}{4},$$

to get

$$(1 + \bar{\lambda}_0\alpha)e_1(0,\alpha) + (1 + \bar{\lambda}_0\beta)e_1(0,\beta) = 0. \tag{II.33}$$

We are going to draw another equation about $e_1(0,\alpha)$ and $e_1(0,\beta)$ from the property

that $d^1_{e_1}$ is orthogonal to L_0. To do this, recall that

$$
\begin{aligned}
e_1 &= P_{L_0}(k_\alpha(w) - k_\beta(w)) \in M_1 \cap L_0, \\
d^1_{e_1} &= d^0_{e_1} + \mu_1 e_1 + \lambda_1 e_0 \perp L_0, \\
L_0 &= span\{1, p_1, e_\alpha, e_\beta\}, \\
e_\alpha &= k_\alpha(z)k_\alpha(w), e_\beta = k_\beta(z)k_\beta(w).
\end{aligned}
$$

Thus $d^1_{e_1}$ is orthogonal to p_1, e_α and e_β.

Since $d^1_{e_1}$ is orthogonal to p_1 we have

$$
\langle d^0_{e_1}, p_1 \rangle + \mu_1 \langle e_1, p_1 \rangle + \lambda_1 \langle e_0, p_1 \rangle = 0.
$$

Noting

$$
\begin{aligned}
\langle d^0_{e_1}, p_1 \rangle &= \langle -\varphi(w)e_1 + we_1(0, w)e_0, p_1 \rangle \\
&= \langle we_1(0, w)e_0, p_1 \rangle \\
&= \langle we_1(0, w)e_0(w, w), w \rangle \\
&= \langle e_1(0, w)e_0(w, w), 1 \rangle \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\langle e_1, p_1 \rangle &= \langle P_{L_0}(K_\alpha(w) - K_\beta(w)), p_1 \rangle \\
&= \langle K_\alpha(w) - K_\beta(w), p_1 \rangle \\
&= \bar{\alpha} - \bar{\beta},
\end{aligned}
$$

and

$$\begin{aligned}
\langle e_0, p_1 \rangle &= \langle e_0(0, w), p_1(w, w) \rangle \\
&= \langle \varphi_0(w), 2w \rangle \\
&= \langle w\varphi_\alpha\varphi_\beta, 2w \rangle \\
&= 2\langle \varphi_\alpha\varphi_\beta, 1 \rangle \\
&= 2\varphi_\alpha(0)\varphi_\beta(0) \\
&= 2\alpha\beta,
\end{aligned}$$

we have

$$(\bar{\alpha} - \bar{\beta})\mu_1 + 2\alpha\beta\lambda_1 = 0,$$

to obtain

$$\lambda_1 = -\mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta}. \tag{II.34}$$

Since $d^1_{e_1} \perp e_\alpha$, we have

$$\langle d^0_{e_1}, e_\alpha \rangle + \mu_1\langle e_1, e_\alpha \rangle + \lambda_1\langle e_0, e_\alpha \rangle = 0,$$

to get

$$\langle d^0_{e_1}, e_\alpha \rangle + \mu_1\langle e_1, e_\alpha \rangle - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta}\langle e_0, e_\alpha \rangle = 0. \tag{II.35}$$

We need to calculate $\langle d^0_{e_1}, e_\alpha \rangle$, $\langle e_1, e_\alpha \rangle$ and $\langle e_0, e_\alpha \rangle$. Simple calculations show that

$$\begin{aligned}
\langle d^0_{e_1}, e_\alpha \rangle &= \langle -\varphi(w)e_1 + we_1(0, w)e_0, e_\alpha \rangle \\
&= \langle we_1(0, w)e_0, e_\alpha \rangle \\
&= \alpha e_1(0, \alpha)e_0(\alpha, \alpha),
\end{aligned}$$

$$\langle e_1, e_\alpha \rangle = e_1(\alpha, \alpha)$$

$$= \langle P_{L_0}(k_\alpha(w) - k_\beta(w)), e_\alpha \rangle$$

$$= \langle k_\alpha(w) - k_\beta(w), e_\alpha \rangle$$

$$= \frac{1}{1 - |\alpha|^2} - \frac{1}{1 - \alpha\bar{\beta}}$$

$$= \frac{\alpha(\bar{\alpha} - \bar{\beta})}{(1 - |\alpha|^2)(1 - \alpha\bar{\beta})}, \tag{II.36}$$

and

$$\langle e_0, e_\alpha \rangle = e_0(\alpha, \alpha) = \alpha\varphi_0'(\alpha) + \varphi_0(\alpha)$$

$$= \alpha^2 \frac{1}{1 - |\alpha|^2} \frac{\alpha - \beta}{1 - \alpha\bar{\beta}}. \tag{II.37}$$

Thus (II.36) and (II.37) give

$$\frac{e_1(\alpha, \alpha)}{e_0(\alpha, \alpha)} = \frac{\bar{\alpha} - \bar{\beta}}{\alpha(\alpha - \beta)}.$$

Substituting the above equality in Equation (II.35) leads to

$$\alpha e_1(0, \alpha) e_0(\alpha, \alpha) + \mu_1 e_1(\alpha, \alpha) - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} e_0(\alpha, \alpha) = 0.$$

Dividing the both sides of the above equality by $e_0(\alpha, \alpha)$ gives

$$\alpha e_1(0, \alpha) + \mu_1 \frac{e_1(\alpha, \alpha)}{e_0(\alpha, \alpha)} - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} = 0.$$

Hence we have

$$\alpha e_1(0, \alpha) + \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{\alpha(\alpha - \beta)} - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} = 0,$$

to obtain

$$\alpha e_1(0, \alpha) + (\beta + \lambda_0) \frac{2\mu_1(\bar{\alpha} - \bar{\beta})}{\alpha\beta(\alpha - \beta)} = 0. \tag{II.38}$$

Similarly, since $d^1_{e_1}$ is orthogonal to e_β, we have

$$\langle d^0_{e_1}, e_\beta \rangle + \mu_1 \langle e_1, e_\beta \rangle + \lambda_1 \langle e_0, e_\beta \rangle = 0,$$

to obtain

$$\langle d^0_{e_1}, e_\beta \rangle + \mu_1 \langle e_1, e_\beta \rangle - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} \langle e_0, e_\beta \rangle = 0. \tag{II.39}$$

We need to calculate $\langle d^0_{e_1}, e_\beta \rangle$, $\langle e_1, e_\beta \rangle$ and $\langle e_0, e_\beta \rangle$. Simple calculations as above show that

$$
\begin{aligned}
\langle d^0_{e_1}, e_\beta \rangle &= \langle -\varphi(w)e_1 + we_1(0,w)e_0, e_\beta \rangle \\
&= \langle we_1(0,w)e_0, e_\beta \rangle \\
&= \beta e_1(0,\beta)e_0(\beta,\beta), \\
\langle e_1, e_\beta \rangle &= e_1(\beta,\beta) \\
&= \langle P_{L_0}(k_\alpha(w) - k_\beta(w)), e_\beta \rangle \\
&= \langle k_\alpha(w) - k_\beta(w), e_\beta \rangle \\
&= \frac{1}{1 - \bar{\alpha}\beta} - \frac{1}{1 - |\beta|^2} \\
&= \frac{\beta(\bar{\alpha} - \bar{\beta})}{(1 - \bar{\alpha}\beta)(1 - |\beta|^2)} \tag{II.40}
\end{aligned}
$$

$$
\begin{aligned}
\langle e_0, e_\beta \rangle = e_0(\beta,\beta) &= \beta\varphi'_0(\beta) + \varphi_0(\beta) \\
&= \beta^2 \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \frac{1}{1 - |\beta|^2} \tag{II.41}
\end{aligned}
$$

Combining (II.40) with (II.41) gives

$$\frac{e_1(\beta,\beta)}{e_0(\beta,\beta)} = -\frac{\bar{\alpha} - \bar{\beta}}{\beta(\alpha - \beta)}.$$

Substituting the above equality in (II.39) gives

$$\beta e_1(0, \beta) e_0(\beta, \beta) + \mu_1 e_1(\beta, \beta) - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} e_0(\beta, \beta) = 0.$$

Dividing both sides of the above equality by $e_0(\beta, \beta)$ gives

$$\beta e_1(0, \beta) + \mu_1 \frac{e_1(\beta, \beta)}{e_0(\beta, \beta)} - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} = 0$$

Hence we have

$$\beta e_1(0, \beta) - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{\beta(\alpha - \beta)} - \mu_1 \frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} = 0,$$

to get

$$\beta e_1(0, \beta) - (\alpha + \lambda_0) \frac{2\mu_1(\bar{\alpha} - \bar{\beta})}{\alpha\beta(\alpha - \beta)} = 0. \tag{II.42}$$

Eliminating $\frac{2\mu_1(\bar{\alpha} - \bar{\beta})}{\alpha\beta(\alpha - \beta)}$ from (II.38) and (II.42) gives

$$\alpha(\alpha + \lambda_0) e_1(0, \alpha) + \beta(\beta + \lambda_0) e_1(0, \beta) = 0. \tag{II.43}$$

Now combining (II.33) and (II.43), we have the following linear system of equations about $e_1(0, \alpha)$ and $e_1(0, \beta)$

$$(1 + \bar{\lambda}_0 \alpha) e_1(0, \alpha) + (1 + \bar{\lambda}_0 \beta) e_1(0, \beta) = 0$$
$$\alpha(\alpha + \lambda_0) e_1(0, \alpha) + \beta(\beta + \lambda_0) e_1(0, \beta) = 0. \tag{II.44}$$

If

$$e_1(0, \alpha) = e_1(0, \beta) = 0,$$

then p_1 is in $L_0 = span\{e_0, e_1, 1, e\}$. But noting

$$e_0(0, \alpha) = e_0(0, \beta)$$

and

$$e(0, \alpha) = e(0, \beta)$$

we have

$$p_1(0, \alpha) = p_1(0, \beta),$$

which contradicts the assumption that $\alpha \neq \beta$. So at least one of $e_1(0, \alpha)$ and $e_1(0, \beta)$ is nonzero. Then the determinant of the coefficient matrix of System (II.44) has to be zero. This implies

$$\begin{vmatrix} 1 + \bar{\lambda}_0\alpha & 1 + \bar{\lambda}_0\beta \\ \alpha(\alpha + \lambda_0) & \beta(\beta + \lambda_0) \end{vmatrix} = 0$$

Making elementary row reductions on the above determinant, we get

$$\begin{vmatrix} (\alpha - \beta)\bar{\lambda}_0 & 1 + \bar{\lambda}_0\beta \\ (\alpha - \beta)(\alpha + \beta + \lambda_0) & \beta(\beta + \lambda_0) \end{vmatrix} = 0.$$

Since

$$\alpha + \beta = -4\lambda_0$$

and

$$\alpha - \beta \neq 0,$$

we have

$$\begin{vmatrix} \bar{\lambda}_0 & 1 + \bar{\lambda}_0\beta \\ -3\lambda_0 & \beta(\beta + \lambda_0) \end{vmatrix} = 0.$$

Expanding this determinant we have

$$
\begin{aligned}
0 &= \bar{\lambda}_0(\beta^2 + \beta\lambda_0) + 3\lambda_0(1 + \bar{\lambda}_0\beta) \\
&= \bar{\lambda}_0(\beta^2 + \beta\lambda_0 + 3\beta\lambda_0) + 3\lambda_0 \\
&= \bar{\lambda}_0(\beta^2 + 4\beta\lambda_0) + 3\lambda_0 \\
&= \bar{\lambda}_0(-\alpha\beta) + 3\lambda_0
\end{aligned}
$$

Taking absolute value on both sides of the above equation, we have

$$
\begin{aligned}
0 &= |\bar{\lambda}_0(-\alpha\beta) + 3\lambda_0| \\
&\geq |\lambda_0|(3 - |\alpha\beta|) \\
&\geq 2|\lambda_0|,
\end{aligned}
$$

to get

$$
\lambda_0 = 0.
$$

This implies

$$
\alpha + \beta = 0,
$$

to complete the proof.

M-BEREZIN TRANSFORMS

III.1 m-Berezin transforms

In this section we obtain some useful properties of the m-Berezin transform. First we give

an integral representation of the m-Berezin transform $B_m(S)$. For $z \in B$ and a nonnegative

integer m, let

$$K_z^m(u) = \frac{1}{(1 - \langle u, z \rangle)^{m+n+1}}, \qquad u \in B.$$

For $u, \lambda \in B$, we can easily see that

$$\sum_{|k|=0}^{m} C_{m,k} u^k \overline{\lambda^k} = (1 - \langle u, \lambda \rangle)^m. \tag{III.1}$$

Proposition 56. *Let $S \in \mathfrak{L}(L_a^2)$, $m \geq 0$ and $z \in B$. Then*

$$B_m S(z) = C_n^{m+n} (1 - |z|^2)^{m+n+1} \times$$
$$\int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda.$$

Proof. For $\lambda \in B$, the definition of B_m implies

$$B_m S(z) = C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \left\langle S_z \lambda^k, \lambda^k \right\rangle$$

$$= C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \int_B S(\varphi_z^k k_z)(\lambda) \overline{\varphi_z^k(\lambda) k_z(\lambda)} d\lambda$$

$$= C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \int_B \int_B \varphi_z^k(u) k_z(u) \overline{\varphi_z^k(\lambda) k_z(\lambda) S^* K_\lambda(u)} du d\lambda \tag{III.2}$$

where the last equality holds by $S(\varphi_z^k k_z)(\lambda) = \left\langle S(\varphi_z^k k_z), K_\lambda \right\rangle = \left\langle \varphi_z^k k_z, S^* K_\lambda \right\rangle$. Using (III.1)

and (I.1), (III.2) equals

$$
C_n^{m+n} \int_B \int_B (1 - \langle \varphi_z(u), \varphi_z(\lambda) \rangle)^m k_z(u) \overline{k_z(\lambda) S^* K_\lambda(u)} \, du \, d\lambda
$$

$$
= C_n^{m+n} \int_B \int_B \left(\frac{k_z(u) \overline{k_z(\lambda)}}{K_\lambda(u)} \right)^{m/(n+1)} k_z(u) \overline{k_z(\lambda) S^* K_\lambda(u)} \, du \, d\lambda
$$

$$
= C_n^{m+n} (1 - |z|^2)^{m+n+1} \int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} \, du \, d\lambda
$$

as desired.

The next proposition gives another form of B_m.

Proposition 57. *Let $S \in \mathfrak{L}(L_a^2(B))$, $m \geq 0$ and $z \in B$. Then*

$$
B_m S(z) = C_n^{m+n} (1 - |z|^2)^{m+n+1} \sum_{|k|=0}^m C_{m,k} \left\langle S(u^k K_z^m), u^k K_z^m \right\rangle. \tag{III.3}
$$

Proof. Since

$$
\int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} \, du \, d\lambda
$$

$$
= \sum_{|k|=0}^m C_{m,k} \int_B \int_B u^k \overline{\lambda^k} K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} \, du \, d\lambda
$$

$$
= \sum_{|k|=0}^m C_{m,k} \int_B S(u^k K_z^m)(\lambda) \overline{\lambda^k} \overline{K_z^m(\lambda)} \, d\lambda,
$$

Now (III.3) follows from Proposition 56.

For $n = 1$, the right hand side of (III.3) was used by Suarez in [49] to define the m-Berezin transforms on the unit disk.

Recall that given $f \in L^\infty$, $B_m(f)(z)$ is defined as $B_m(T_f)(z)$. The following proposition gives a nice formula of $B_m(f)(z)$. Let $d\nu_m(u) = C_n^{m+n}(1 - |u|^2)^m du$.

Proposition 58. *Let $z \in B$ and $f \in L^\infty$. Then*

$$
B_m(f)(z) = \int_B f \circ \varphi_z(u) \, d\nu_m(u).
$$

Proof. By the change of variables, Theorem 2.2.2 in [38] and (III.3), we have

$$\int_B f \circ \varphi_z(u) d\nu_m(u)$$

$$= C_n^{m+n} \int_B f(u) \left(\frac{(1 - |z|^2)(1 - |u|^2)}{|1 - \langle u, z \rangle|^2} \right)^m \left(\frac{(1 - |z|^2)}{|1 - \langle u, z \rangle|^2} \right)^{n+1} du$$

$$= C_n^{m+n}(1 - |z|^2)^{m+n+1} \sum_{|k|=0}^m C_{m,k} \int_B f(u)|u^k|^2 |K_z^m(u)|^2 du$$

$$= C_n^{m+n}(1 - |z|^2)^{m+n+1} \sum_{|k|=0}^m C_{m,k} \left\langle T_f(u^k K_z^m), u^k K_z^m \right\rangle = B_m(T_f)(z).$$

The proof is complete.

The formula in the above proposition was used in [1] to define the m-Berezin transform of functions f.

Clearly, (I.2) gives $\|B_m S\|_\infty \leq C(m,n)\|S_z\| = C(m,n)\|S\|$ for $S \in \mathfrak{L}(L_a^2)$. Thus, $B_m : \mathfrak{L}(L_a^2) \to L^\infty$ is a bounded linear operator. The following theorem gives the norm of B_m.

Theorem 59. *Let $m \geq 0$. Then $\|B_m\| = C_n^{m+n} \sum_{|k|=0}^m |C_{m,k}| \frac{n!k!}{(n+|k|)!}$.*

Proof. From [15], we have the duality result $\mathfrak{L}(L_a^2) = T^*$. So, the definition of B_m gives the norm of B_m. In fact,

$$\|B_m\| = \left\| C_n^{m+n} \sum_{|k|=0}^m C_{m,k} \frac{n!k!}{(n+|k|)!} \frac{u^k}{\|u^k\|} \otimes \frac{u^k}{\|u^k\|} \right\|_{C_1}$$

$$= C_n^{m+n} \sum_{|k|=0}^m |C_{m,k}| \frac{n!k!}{(n+|k|)!}$$

as desired.

The Möbius map $\varphi_z(w)$ has the following property ([38]):

$$\varphi_z'(0) = -(1 - |z|^2)P_z - (1 - |z|^2)^{1/2}Q_z. \tag{III.4}$$

To show that m-Berezin transforms are Lipschitz with respect to the pseudo-hyperbolic distance we need the following lemmas.

For $z, w \in \mathbb{C}^n$, $z \hat{\otimes} w$ on $\mathbb{C}^n$ is defined by $(z \hat{\otimes} w)\lambda = \langle \lambda, w \rangle z$.

Lemma 60. *Let $z, w \in B$ and $\lambda = \varphi_z(w)$. Then*

$$\varphi_z'(w) = (1 - \langle \lambda, z \rangle)(I - \lambda \hat{\otimes} z)[\varphi_z'(0)]^{-1}.$$

Proof. Suppose that P_z and Q_z have the matrix representations as $((P_z)_{ij})$ and $((Q_z)_{ij})$ under the standard base of $\mathbb{C}^n$, respectively. In fact,

$$(P_z)_{ij} = \frac{z_i \bar{z}_j}{|z|^2} \qquad \text{if} \quad z \neq 0.$$

Let $(a_{ij}(w)) = \varphi_z'(w)$. Write $\varphi_z(w) = (f_1(w), \cdots, f_n(w))$. Then

$$a_{ij}(w) = \frac{\partial f_i}{\partial w_j}(w).$$

Noting that

$$f_i(w) = \frac{z_i - (P_z w)_i - (1 - |z|^2)^{1/2}(Q_z w)_i}{1 - \langle w, z \rangle},$$

we have

$$
\begin{aligned}
a_{ij}(w) &= \frac{(z_i - (P_z w)_i - (1 - |z|^2)^{1/2}(Q_z w)_i)\bar{z}_j}{(1 - \langle w, z \rangle)^2} - \frac{(P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij}}{1 - \langle w, z \rangle} \\
&= \frac{f_i(w)\bar{z}_j}{1 - \langle w, z \rangle} - \frac{(P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij}}{1 - \langle w, z \rangle}.
\end{aligned}
$$

Let $\lambda = \varphi_z(w)$. The above equality becomes

$$a_{ij}(w) = \frac{\lambda_i \bar{z}_j - ((P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij})}{1 - \langle w, z \rangle}$$

Thus

$$\varphi_z'(w) = \frac{\lambda \hat{\otimes} z - (P_z + (1 - |z|^2)^{1/2}Q_z)}{1 - \langle w, z \rangle}.$$

From Theorem 2.2.5 in [38], we have

$$\frac{1}{1 - \langle w, z \rangle} = \frac{1 - \langle \lambda, z \rangle}{1 - |z|^2}.$$

Thus (III.4) implies

$$
\begin{aligned}
\varphi_z'(w)\varphi_z'(0) &= \frac{-(1-|z|^2)\lambda\hat\otimes z + (1-|z|^2)P_z + (1-|z|^2)Q_z}{1-\langle w,z\rangle} \\
&= \frac{(1-|z|^2)(-\lambda\hat\otimes z + I)}{1-\langle w,z\rangle} \\
&= (1-\langle \lambda,z\rangle)(I - \lambda\hat\otimes z)
\end{aligned}
$$

where the first equality follows from $P_zQ_z = Q_zP_z = 0$, $P_zz = z$, and $Q_zz = 0$. The proof is complete.

Lemma 61. *Suppose $|z| > 1/2$ and $|w| > 1/2$. If $|\varphi_z(w)| \le \epsilon < 1/2$, then*

$$
\|P_z - P_w\| \le 50\epsilon(1-|z|^2)^{1/2}.
$$

Proof. First we will get the estimate of the distance between z and w. Since $|\varphi_z(w)| \le \epsilon < 1/2$, w is in the ellipsoid:

$$
\varphi_z(\epsilon B) = \{w \in B : \frac{|P_zw - c|^2}{\epsilon^2\rho^2} + \frac{|Q_zw|^2}{\epsilon^2\rho} < 1\}
$$

with center $c = \frac{(1-\epsilon^2)z}{(1-\epsilon^2|z|^2)}$ and $\rho = \frac{1-|z|^2}{1-\epsilon^2|z|^2}$. Noting that $|z| > 1/2$ and $\epsilon < 1/2$, we have $\rho \le 2(1-|z|^2)$. Thus

$$
|Q_zw|^2 \le \epsilon^2\rho \le 2\epsilon^2(1-|z|^2), \qquad |P_zw - c| \le \epsilon\rho \le 2\epsilon(1-|z|^2)
$$

and

$$
|z - c| \le \frac{\epsilon^2(1-|z|^2)}{(1-\epsilon^2|z|^2)} \le 2\epsilon^2(1-|z|^2).
$$

So, we have

$$
|P_zw - z| \le |P_zw - c| + |z - c| \le 3\epsilon(1-|z|^2).
$$

Because $I = P_z + Q_z$ and $P_zQ_z = 0$, writing

$$
(z - w) = P_z(z - w) + Q_z(z - w),
$$

we have

$$|z - w|^2 = |P_z(z - w)|^2 + |Q_z(z - w)|^2$$
$$= |P_z w - z|^2 + |Q_z w|^2$$
$$\leq 11\epsilon^2(1 - |z|^2). \tag{III.5}$$

Noting that

$$\frac{z}{|z|} \hat{\otimes} \frac{z}{|z|} = \frac{(z - w)}{|z|} \hat{\otimes} \frac{z}{|z|} + \frac{w}{|z|} \hat{\otimes} \frac{(z - w)}{|z|} +$$
$$\left[\left(\frac{1}{|z|^2} - \frac{1}{|w|^2}\right) w\right] \hat{\otimes} w + \frac{w}{|w|} \hat{\otimes} \frac{w}{|w|},$$

we have

$$P_z - P_w = \frac{(z - w)}{|z|} \hat{\otimes} \frac{z}{|z|} + \frac{w}{|z|} \hat{\otimes} \frac{(z - w)}{|z|} + \left[\left(\frac{1}{|z|^2} - \frac{1}{|w|^2}\right) w\right] \hat{\otimes} w,$$

to obtain

$$\|P_z - P_w\| \leq \frac{|z - w|}{|z|} + \frac{2|z - w|}{|z|} + \frac{||z|^2 - |w|^2|}{|z|^2}$$
$$\leq 2|z - w| + 4|z - w| + 8|z - w|$$
$$\leq 14\sqrt{11}\epsilon(1 - |z|^2)^{1/2}$$
$$\leq 50\epsilon(1 - |z|^2)^{1/2}$$

where the last inequality holds by (III.5).

For given $z, w \in B$, set $A(z, w) = -(1 - |z|^2)P_w - (1 - |z|^2)^{1/2}Q_w$.

Lemma 62. *Suppose $|z| > 1/2$ and $|w| > 1/2$. If $|\varphi_z(w)| \leq \epsilon < 1/2$, then*

$$\|\varphi_z'(0) - A(z, w)\| \leq 150\epsilon(1 - |z|^2).$$

Proof. Using (III.4), we have

$$\|\varphi'_z(0) - A(z,w)\| \quad = \|(1 - |z|^2)(P_w - P_z) + (1 - |z|^2)^{1/2}(P_z - P_w)\|$$

$$\leq 3(1 - |z|^2)^{1/2}\|P_z - P_w\|$$

$$\leq 150\epsilon(1 - |z|^2)$$

where the last inequality follows from Lemma 61.

Let $\mathfrak{U}(n)$ be the group of $n \times n$ complex unitary matrices.

Lemma 63. *Let* $z, w \in B$. *Then* $U_z U_w = V_U U_{\varphi_w(z)}$ *where*

$$V_U f(u) = f(Uu) det U$$

for $f \in L_a^2$ *and* $U = \varphi_{\varphi_w(z)} \circ \varphi_w \circ \varphi_z$ *satisfying*

$$\|I + U\| \leq C(n)\rho(z, w).$$

Proof. The map $\varphi_{\varphi_w(z)} \circ \varphi_w \circ \varphi_z$ is an automorphism of B that fixes 0, hence it is unitary by the Cartan theorem in [38]. Thus $\varphi_w \circ \varphi_z = \varphi_{\varphi_w(z)} \circ U$ for some $U \in \mathfrak{U}(n)$. Since φ_w is an involution, we have

$$
\begin{aligned}
U_z U_w f(u) &= (f \circ \varphi_w \circ \varphi_z)(u) J\varphi_w(\varphi_z(u)) J\varphi_z(u) \\
&= (f \circ \varphi_{\varphi_w(z)})(Uu) J\varphi_w(\varphi_w \circ \varphi_{\varphi_w(z)}(Uu)) \\
&\quad \cdot \quad J\varphi_w(\varphi_{\varphi_w(z)}(Uu)) J\varphi_{\varphi_w(z)}(Uu) det U \\
&= (f \circ \varphi_{\varphi_w(z)})(Uu) J\varphi_{\varphi_w(z)}(Uu) det U \\
&= V_U U_{\varphi_w(z)} f(u).
\end{aligned}
$$

Now we will show that

$$\|I + U\| \leq C(n)\rho(z, w).$$

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Noting that U is continuous for $|z| \leq 1/2$ and $|w| \leq 1/2$, we need only to prove

$$\|I + U\| \leq 20000\rho(z, w),$$

for $|z| > 1/2$, $|w| > 1/2$ and $|\varphi_w(z)| < 1/2$. Let $\lambda = \varphi_w(z)$. Then $|\lambda| = \rho(z, w)$ and $z = \varphi_w(\lambda)$. Since

$$\varphi_w \circ \varphi_z(u) = \varphi_\lambda(Uu),$$

taking derivatives both sides of the above equations and using the chain rule give

$$\varphi'_w(\varphi_z(u)) \varphi'_z(u) = \varphi'_\lambda(Uu)U.$$

Set $u = 0$, the above equality becomes

$$U = [\varphi'_\lambda(0)]^{-1}\varphi'_w(z)\varphi'_z(0).$$

By Lemma 60, write

$$\begin{aligned}
U + I &= [\varphi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda\hat{\otimes}w)[\varphi'_w(0)]^{-1}\varphi'_z(0) + I \\
&= [\varphi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda\hat{\otimes}w)[\varphi'_w(0)]^{-1}[\varphi'_z(0) - A(z, w)] \\
&\quad + \left([\varphi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda\hat{\otimes}w)[\varphi'_w(0)]^{-1}A(z, w) + I\right) \\
&:= I_1 + I_2.
\end{aligned}$$

By Lemma 62, we have

$$\begin{aligned}
\|I_1\| &\leq \|[\varphi'_\lambda(0)]^{-1}\| \|1 - \langle \lambda, w \rangle\| \|I - \lambda\hat{\otimes}w\| \|[\varphi'_w(0)]^{-1}\| \|\varphi'_z(0) - A(z, w)\| \\
&\leq 4 \times 2 \times 2 \times \frac{3}{(1 - |w|^2)} \left[150|\lambda|(1 - |z|^2)\right].
\end{aligned}$$

Theorem 2.2.2 in [38] leads to

$$\frac{1 - |z|^2}{1 - |w|^2} = \frac{1 - |\lambda|^2}{|1 - \langle \lambda, w \rangle|^2}.$$

Thus

$$\|I_1\| \le 4 \times 2 \times 2 \times 3 \times 2 \times 150|\lambda| = 14400|\lambda|.$$

Also, we have

$$\left| 1 - \frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}} \right| \le \left| 1 - \frac{1-|z|^2}{1-|w|^2} \right| \le 32|\lambda|.$$

Hence, we get

$$\left\| I - \frac{1-|z|^2}{1-|w|^2}P_w - \frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}}Q_w \right\| \le 32|\lambda|.$$

On the other hand, clearly,

$$\|[\varphi_\lambda'(0)]^{-1} + I\| \le 4|\lambda|, \quad |(1 - \langle \lambda, w \rangle) - 1| \le |\lambda|$$

and

$$\|(I - \lambda \hat{\otimes} w) - I\| \le |\lambda|.$$

These give

$$\|I + [\varphi_\lambda'(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\| \le 16|\lambda|.$$

Hence, we have

$$\begin{aligned}
\|I_2\| &\le \|[\varphi_\lambda'(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)[\varphi_w'(0)]^{-1}A(z,w) \\
&\quad - [\varphi_\lambda'(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\| \\
&\quad + \|[\varphi_\lambda'(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w) + I\| \\
&\le \left\|[\varphi_\lambda'(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\right\| \left\| I - \frac{1-|z|^2}{1-|w|^2}P_w - \frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}}Q_w \right\| \\
&\quad + 16|\lambda| \\
&\le 4 \times 2 \times 2 \times 32|\lambda| + 16|\lambda| < 600|\lambda|.
\end{aligned}$$

Combining the above estimates we conclude that

$$\|U + I\| \le 14400|\lambda| + 600|\lambda| < 20000|\lambda|.$$

Theorem 64. *Let $S \in \mathfrak{L}(L_a^2(B))$, $m \geq 0$ and $z \in B$. Then $B_m S_z = (B_m S) \circ \varphi_z$.*

Proof. Proposition 57 and (I.2) give

$$B_m S_z(0) = C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \left\langle S_z u^k, u^k \right\rangle = B_m S(z) = (B_m S) \circ \varphi_z(0).$$

For any $w \in B$, Proposition 56 and Lemma 63 imply

$$
\begin{aligned}
(B_m S_z) \circ \varphi_w(0) &= B_m((S_z)_w)(0) \\
&= C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{U_w U_z S^* U_z U_w K_\lambda(u)} du d\lambda \\
&= C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{V_U U_{\varphi_z(w)} S^* U_{\varphi_z(w)} V_U^* K_\lambda(u)} du d\lambda \\
&= B_m S_{\varphi_z(w)}(0)
\end{aligned}
$$

where V_U is in Lemma 63. Thus, $B_m S_z(w) = (B_m S) \circ \varphi_z(w)$.

Lemma 65. *Let $S \in \mathfrak{L}(L_a^2(B))$, $m \geq 1$ and $z \in B$. Then*

$$B_m S(z) = \frac{m+n}{m} B_{m-1} \left(S - \sum_{i=1}^{n} T_{\overline{(\varphi_z)_i}} S T_{(\varphi_z)_i} \right)(z)$$

where $(\varphi_z)_i$ is i-th variable of φ_z.

Proof. By Theorem 64, we just need to show that

$$B_m S(0) = \frac{m+n}{m} B_{m-1} \left(S - \sum_{i=1}^{n} T_{\overline{u_i}} S T_{u_i} \right)(0).$$

Using Proposition 56 and (III.1), we get

$$B_m S(0) = C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{S^* K_\lambda(u)} \, du \, d\lambda$$

$$= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^{n} \sum_{|k|=0}^{m-1} C_{m-1,k} \int_B \int_B u_i \overline{\lambda_i} u^k \overline{\lambda^k S^* K_\lambda(u)} \, du \, d\lambda$$

$$= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^{n} \sum_{|k|=0}^{m-1} C_{m-1,k} \int_B S(u^k u_i)(\lambda) \overline{\lambda^k \lambda_i} \, d\lambda$$

$$= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^{n} \sum_{|k|=0}^{m-1} C_{m-1,k} \left\langle S T_{u_i}(u^k), T_{u_i}(u^k) \right\rangle$$

as desired.

For $m = 0$, the following result was obtained in [18].

Theorem 66. *Let $S \in \mathfrak{L}(L_a^2(B))$ and $m \geq 0$. Then there exists a constant $C(m, n) > 0$ such that*

$$|B_m S(z) - B_m S(w)| < C(m, n) \|S\| \rho(z, w).$$

Proof. We will prove this theorem by induction on m. If $m = 0$, (I.2) gives

$$|B_0 S(z) - B_0 S(w)| = |tr[S_z(1 \otimes 1)] - tr[S_w(1 \otimes 1)]|$$

$$= |tr[S_z(1 \otimes 1) - S U_w(1 \otimes 1) U_w]|$$

$$= |tr[S_z(1 \otimes 1) - S U_z(U_z U_w 1 \otimes U_z U_w 1) U_z]|$$

From Lemma 63, the last term equals

$$|tr[S_z(1 \otimes 1 - U_{\varphi_w(z)} 1 \otimes U_{\varphi_w(z)} 1)]| \leq \|S_z\| \|1 \otimes 1 - U_{\varphi_w(z)} 1 \otimes U_{\varphi_w(z)} 1\|_{C^1}$$

$$\leq \sqrt{2} \|S_z\| (2 - 2|\langle 1, k_{\varphi_w(z)} \rangle|^2)^{1/2}$$

$$= 2\|S\| [1 - (1 - |\varphi_w(z)|^2)^{n+1}]^{1/2}$$

$$\leq C(n) \|S\| |\varphi_w(z)|$$

where the second equality holds by $\|T\|_{C^1} \leq \sqrt{l} (tr[T^* T])^{1/2}$ where l is the rank of T.

Suppose $|B_{m-1}S(z) - B_{m-1}S(w)| < C(m,n)\|S\|\rho(z,w)$. By Lemma 65, we have

$$|B_m S(z) - B_m S(w)|$$
$$\leq \frac{m+n}{m}|B_{m-1}S(z) - B_{m-1}S(w)|$$
$$+ \frac{m+n}{m}\sum_{i=1}^{n}\left|B_{m-1}\left(T_{\overline{(\varphi_z)_i}}ST_{(\varphi_z)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\varphi_w)_i}}ST_{(\varphi_w)_i}\right)(w)\right|.$$

Since the term in the summation is less than or equals

$$\left|B_{m-1}\left(T_{\overline{(\varphi_z)_i}}ST_{(\varphi_z)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\varphi_w)_i}}ST_{(\varphi_z)_i}\right)(z)\right|$$
$$+ \left|B_{m-1}\left(T_{\overline{(\varphi_w)_i}}ST_{(\varphi_z)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\varphi_w)_i}}ST_{(\varphi_w)_i}\right)(z)\right|$$
$$+ \left|B_{m-1}\left(T_{\overline{(\varphi_w)_i}}ST_{(\varphi_w)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\varphi_w)_i}}ST_{(\varphi_w)_i}\right)(w)\right|,$$

it is sufficient to show that

$$\left|B_{m-1}\left(T_{\overline{(\varphi_z)_i}}ST_{(\varphi_z)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\varphi_w)_i}}ST_{(\varphi_z)_i}\right)(z)\right| < C(m,n)\|S\|\rho(z,w).$$

Lemma 63 gives

$$\left|B_{m-1}\left(T_{\overline{(\varphi_z)_i} - \overline{(\varphi_w)_i}}ST_{(\varphi_z)_i}\right)(z)\right|$$
$$= C_n^{m+n-1}\left|tr\left[\left(T_{\overline{(\varphi_z)_i} - \overline{(\varphi_w)_i}}ST_{(\varphi_z)_i}\right)_z\sum_{|k|=0}^{m-1}C_{m-1,k}\frac{n!k!}{(n+|k|)!}\frac{u^k}{\|u^k\|}\otimes\frac{u^k}{\|u^k\|}\right]\right|$$
$$\leq C_n^{m+n-1} \tag{III.6}$$
$$\cdot\sum_{|k|=0}^{m-1}|C_{m-1,k}|\frac{n!k!}{(n+|k|)!}\left|\left\langle S_z T_{(\varphi_z)_i\circ\varphi_z}\frac{u^k}{\|u^k\|}, T_{((\varphi_z)_i - (\varphi_w)_i)\circ\varphi_z}\frac{u^k}{\|u^k\|}\right\rangle\right|$$
$$\leq C(m,n)\|S_z\|\left\|T_{((\varphi_z)_i - (\varphi_w)_i)\circ\varphi_z}\frac{u^k}{\|u^k\|}\right\|_2. \tag{III.7}$$

Let $\lambda = \varphi_w(z)$. Then

$$\left\| T_{((\varphi_z)_i - (\varphi_w)_i) \circ \varphi_z} \frac{u^k}{\|u^k\|} \right\|_2^2 \leq \int_B |(\varphi_z \circ \varphi_z)_i(u) - (\varphi_w \circ \varphi_z)_i(u)|^2 du$$

$$= \int_B |(Uu)_i - (\varphi_\lambda(u))_i|^2 du$$

$$\leq 2 \int_B |(Uu)_i + u_i|^2 + |u_i + (\varphi_\lambda(u))_i|^2 du$$

where $\varphi_w \circ \varphi_z = \varphi_\lambda \circ U$ for some $U \in \mathfrak{U}(n)$.

Noting that

$$\varphi_\lambda(u) + u = \frac{\lambda - \langle u, \lambda \rangle u + [1 - (1 - |\lambda|^2)^{1/2}]Q_\lambda(u)}{1 - \langle u, \lambda \rangle},$$

we have that for $|\lambda| \leq 1/2$,

$$|\varphi_\lambda(u) + u| \leq 2(|\lambda| + |\lambda| + |\lambda|^2) \leq 6|\lambda|.$$

By Lemma 63 we also have

$$\int_B |(Uu)_i + u_i|^2 du = \int_B |((U + I)u)_i|^2 du \leq C\|U + I\|^2 \leq C|\lambda|^2.$$

Thus (III.6) is less than or equal to

$$C(m, n)\|S_z\|[36|\lambda|^2 + C|\lambda|^2]^{1/2} \leq C(m, n)\|S\||\lambda|.$$

The proof is complete.

Lemma 67. *Let* $S \in \mathfrak{L}(L_a^2(B))$ *and* $m, j \geq 0$. *If* $|S^* K_\lambda(z)| \leq C$ *for any* $z \in B$ *then* $(B_m B_j)(S) = (B_j B_m)(S)$.

Proof. By Theorem 64, it is enough to show that $(B_m B_j)S(0) = (B_j B_m)S(0)$. From Propo-

sition 58, Proposition 56 and Fubini's Theorem, we have

$$B_m(B_jS)(0) = B_m(T_{B_jS})(0)$$

$$= C_n^{m+n} \int_B B_jS(z)(1-|z|^2)^m dz$$

$$= C_n^{m+n} C_n^{j+n} \int_B \int_B \int_B (1-|z|^2)^{m+j+n+1}(1-\langle u,\lambda\rangle)^j \times$$

$$K_z^j(u)\overline{K_z^j(\lambda)}\,\overline{S^*K_\lambda(u)}\,dud\lambda dz$$

$$= C_n^{m+n} C_n^{j+n} \int_B \int_B (1-\langle u,\lambda\rangle)^j \int_B (1-|z|^2)^{m+j+n+1} \times$$

$$K_z^j(u)\overline{K_z^j(\lambda)}dz \overline{S^*K_\lambda(u)}\,dud\lambda.$$

Let

$$F_{m,j}(u,\lambda) = (1-\langle u,\lambda\rangle)^j \int_B (1-|z|^2)^{m+j+n+1} K_z^j(u)\overline{K_z^j(\lambda)}dz.$$

Then $F_{m,j}(u,\lambda) = \sum_{i=1}^{l} H_i(u)\overline{G_i}(\lambda)$ where H_i and G_i are holomorphic functions and for some $l \geq 0$. Thus, from Lemma 9 in [17], we just need to show $F_{m,j}(\lambda,\lambda) = F_{j,m}(\lambda,\lambda)$ for $\lambda \in B$. The change of variables implies

$$F_{m,j}(\lambda,\lambda) = (1-|\lambda|^2)^j \int_B (1-|z|^2)^{m+j+n+1}|K_\lambda^j(z)|^2 dz$$

$$= (1-|\lambda|^2)^j \int_B (1-|\varphi_\lambda(z)|^2)^{m+j+n+1}|K_\lambda^j(\varphi_\lambda(z))|^2|k_\lambda(z)|^2 dz$$

$$= (1-|\lambda|^2)^m \int_B (1-|z|^2)^{m+j+n+1}|K_\lambda^m(z)|^2 dz$$

$$= F_{j,m}(\lambda,\lambda)$$

as desired.

Lemma 68. *For any $S \in \mathfrak{L}(L_a^2(B))$, there exists sequences $\{S_\alpha\}$ satisfying*

$$|S_\alpha^* K_\lambda(u)| \leq C(\alpha)$$

such that $B_m(S_\alpha)$ converges to $B_m(S)$ pointwisely.

Proof. Since H^∞ is dense in L_a^2 and the set of finite rank operators is dense in the ideal K

of compact operators on L^2, the set $\{\sum_{i=1}^{l} f_i \otimes g_i : f_i, \ g_i \in H^\infty\}$ is dense in the ideal K in the norm topology. Since K is dense in the space of bounded operators on L_a^2 in strong operator topology, (III.3) gives that for any $S \in \mathfrak{L}(L_a^2)$, there exists a finite rank operator sequences $S_\alpha = \sum_{i=1}^{l} f_i \otimes g_i$ such that $B_m(S_\alpha)$ converges to $B_m(S)$ pointwisely for some $f_i, \ g_i$ in H^∞. Also, for $l \geq 0$, for such $S_\alpha = \sum_{i=1}^{l} f_i \otimes g_i$, we have

$$
\begin{aligned}
|S_\alpha^* K_\lambda(u)| &= \left| \sum_{i=1}^{l} (g_i \otimes f_i) K_\lambda(u) \right| \\
&= \left| \sum_{i=1}^{l} \langle K_\lambda(u), f_i(u) \rangle \, g_i(u) \right| \\
&\leq \sum_{i=1}^{l} |f_i(\lambda)||g_i(u)| \\
&\leq \sum_{i=1}^{l} \|f_i\|_\infty \|g_i\|_\infty < C.
\end{aligned}
$$

The proof is complete.

Proposition 69. *Let $S \in \mathfrak{L}(L_a^2(B))$ and $m, j \geq 0$. Then*

$$
(B_m B_j)(S) = (B_j B_m)(S).
$$

Proof. Let $S \in \mathfrak{L}(L_a^2)$. Then Lemma 68 implies that there exists a sequence $\{S_\alpha\}$ satisfying $|S_\alpha^* K_\lambda(u)| \leq C(\alpha)$, hence $B_m(B_j S_\alpha)(z) = B_j(B_m S_\alpha)(z)$ by Lemma 67. From Proposition 58, we know

$$
B_m(B_j S_\alpha)(z) = \int_B (B_j S_\alpha) \circ \varphi_z(u) d\nu_m(u)
$$

and $\|(B_j S_\alpha) \circ \varphi_z\|_\infty \leq C(j,n)\|S\|$. Also, $(B_j S_\alpha) \circ \varphi_z(u)$ converges to $(B_j S) \circ \varphi_z(u)$. Therefore $B_m(B_j S_\alpha)(z)$ converges to $B_m(B_j S)(z)$. By the uniqueness of the limit, we have $(B_m B_j)(S) = (B_j B_m)(S)$.

Proposition 70. *Let $S \in \mathfrak{L}(L_a^2)$ and $m \geq 0$. If $B_0 S(z) \to 0$ as $z \to \partial B$ then $B_m S(z) \to 0$ as $z \to \partial B$.*

Proof. Suppose $B_0 S(z) \to 0$ as $z \to \partial B$. Then we will prove that $S_z \to 0$ in the T^*-norm as $z \to \partial B$. Suppose it is not true. Then for some net $\{w_\alpha\} \in B$ and an operator $V \neq 0$

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in $\mathfrak{L}(L_a^2)$, there exists a sequence $\{S_{w_\alpha}\}$ such that $S_{w_\alpha} \to V$ in the T^*-norm as $w_\alpha \to \partial B$, hence $tr[S_{w_\alpha}T] \to tr[VT]$ for any $T \in T$. Let $T = k_\lambda \otimes k_\lambda$ for fixed $\lambda \in B$. Then Theorem 64 implies

$$
\begin{aligned}
tr[S_{w_\alpha}T] &= tr[S_{w_\alpha}(k_\lambda \otimes k_\lambda)] \\
&= \langle S_{w_\alpha}k_\lambda, k_\lambda \rangle \\
&= B_0 S_{w_\alpha}(\lambda) \\
&= (B_0 S) \circ \varphi_{w_\alpha}(\lambda) \to 0
\end{aligned}
$$

as $w_\alpha \to \partial B$. Since $tr[VT] = B_0 V(\lambda)$ and B_0 is one-to-one mapping, $V = 0$. This is the contradiction. Thus $S_z \to 0$ as $z \to \partial B$ in the T^*-norm. (I.2) finishes the proof of this proposition.

III.2   Approximation by Toeplitz operators

In this section we will give a criterion for operators approximated by Toeplitz operators with symbol equal to their m-Berezin transforms. The main result in this section is Theorem 77. It extends and improves Theorem 2.4 in [50]. Even on the unit disk, we will show an example that the result in the theorem is sharp on the unit disk.

From Proposition 1.4.10 in [38], we have the following lemma

Lemma 71. *Suppose $a < 1$ and $a + b < n + 1$. Then*

$$
\sup_{z \in B} \int_B \frac{d\lambda}{(1 - |\lambda|^2)^a |1 - \langle \lambda, z \rangle|^b} < \infty.
$$

This lemma gives the following lemma which extends Lemma 4.2 in [33].

Let $1 < q < \infty$ and p be the conjugate exponent of q. If we take $p > n + 2$, then $q < (n + 2)/(n + 1)$.

Lemma 72. *Let $S \in \mathfrak{L}(L_a^2(B))$ and $p > n + 2$. Then there exists $C(n, p) > 0$ such that*

$h(z) = (1 - |z|^2)^{-a}$ *where* $a = (n+1)/(n+2)$ *satisfies*

$$\int_B |(SK_z)(w)|h(w)dw \leq C(n,p)\|S_z1\|_p h(z) \tag{III.8}$$

for all $z \in B$ *and*

$$\int_B |(SK_z)(w)|h(z)dz \leq C(n,p)\|S_w^*1\|_p h(w) \tag{III.9}$$

for all $w \in B$.

Proof. Fix $z \in B$. Since

$$U_z1 = (-1)^n(1 - |z|^2)^{(n+1)/2}K_z,$$

we have

$$SK_z = (-1)^n(1 - |z|^2)^{-(n+1)/2}SU_z1$$
$$= (-1)^n(1 - |z|^2)^{-(n+1)/2}U_zS_z1$$
$$= (1 - |z|^2)^{-(n+1)/2}(S_z1 \circ \varphi_z)k_z.$$

Thus, letting $\lambda = \varphi_z(w)$, the change of variables implies

$$\int_B \frac{|(SK_z)(w)|}{(1 - |w|^2)^a}dw = \frac{1}{(1 - |z|^2)^{(n+1)/2}}\int_B \frac{|(S_z1 \circ \varphi_z)(w)||k_z(w)|}{(1 - |w|^2)^a}dw$$
$$= \frac{1}{(1 - |z|^2)^a}\int_B \frac{|S_z1(\lambda)|}{(1 - |\lambda|^2)^a|1 - \langle \lambda, z\rangle|^{n+1-2a}}d\lambda$$
$$\leq \frac{\|S_z1\|_p}{(1 - |z|^2)^a}\left(\int_B \frac{1}{(1 - |\lambda|^2)^{aq}|1 - \langle \lambda, z\rangle|^{(n+1-2a)q}}d\lambda\right)^{\frac{1}{q}}.$$

The last inequality comes from Holder's inequality. Since $aq < 1$ and $aq + (n+1-2a)q < n+1$, Lemma 71 implies (III.8).

To prove (III.9), replace S by S^* in (III.8), interchange w and z in (III.8) and then use the equation

$$(S^*K_w)(z) = \langle S^*K_w, K_z\rangle = \langle K_w, SK_z\rangle = \overline{SK_z}(w) \tag{III.10}$$

to obtain the desired result.

Lemma 73. *Let $S \in \mathcal{L}(L_a^2(B))$ and $p > n + 2$. Then*

$$\|S\| \le C(n,p) \left(\sup_{z \in B} \|S_z 1\|_p \right)^{1/2} \left(\sup_{z \in B} \|S_z^* 1\|_p \right)^{1/2}$$

where $C(n,p)$ is the constant of Lemma 72.

Proof. (III.10) implies

$$
\begin{aligned}
(Sf)(w) &= \langle Sf, K_w \rangle \\
&= \int_B f(z) \overline{(S^* K_w)}(z) dz \\
&= \int_B f(z)(SK_z)(w) dz
\end{aligned}
$$

for $f \in L_a^2$ and $w \in B$. Thus, Lemma 72 and the classical Schur's theorem finish the proof.

Lemma 74. *Let S_m be a bounded sequence in $\mathcal{L}(L_a^2(B))$ such that $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$. Then*

$$\sup_{z \in B} |\langle (S_m)_z 1, f \rangle| \to 0 \qquad \text{(III.11)}$$

as $m \to \infty$ for any $f \in L_a^2(B)$ and

$$\sup_{z \in B} |(S_m)_z 1| \to 0 \qquad \text{(III.12)}$$

uniformly on compact subsets of B as $m \to \infty$.

Proof. To prove (III.11), we only need to have

$$\sup_{z \in B} \left| \left\langle (S_m)_z 1, w^k \right\rangle \right| \to 0 \qquad \text{(III.13)}$$

as $m \to \infty$ for any multi-index k.

Since

$$K_z(w) = \sum_{|\alpha|=0}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \overline{z}^\alpha w^\alpha, \tag{III.14}$$

we have

$$B_0 S_m(\varphi_z(\lambda)) = B_0(S_m)_z(\lambda)$$
$$= (1-|\lambda|^2)^{n+1} \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \frac{(n+|\beta|)!}{n!\beta!} \left\langle (S_m)_z w^\alpha, w^\beta \right\rangle \overline{\lambda}^\alpha \lambda^\beta$$

where α, β are multi-indices.

Then for any fixed k and $0 < r < 1$,

$$\int_{rB} \frac{B_0 S_m(\varphi_z(\lambda))\overline{\lambda}^k}{(1-|\lambda|^2)^{n+1}} d\lambda$$
$$= \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \frac{(n+|\beta|)!}{n!\beta!} \left\langle (S_m)_z w^\alpha, w^\beta \right\rangle \int_{rB} \overline{\lambda}^{\alpha+k} \lambda^\beta d\lambda$$
$$= r^{2n+2|k|} \left(\left\langle (S_m)_z 1, w^k \right\rangle + \sum_{|\alpha|=1}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \left\langle (S_m)_z w^\alpha, w^{\alpha+k} \right\rangle r^{2|\alpha|} \right).$$

Since S_m is bounded sequence, we have

$$\left| \left\langle (S_m)_z 1, w^k \right\rangle \right|$$
$$\leq r^{-2n-2|k|} \left| \int_{rB} \frac{B_0 S_m(\varphi_z(\lambda))\overline{\lambda}^k}{(1-|\lambda|^2)^{n+1}} d\lambda \right| +$$
$$\sum_{|\alpha|=1}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \|(S_m)_z\| \|w^\alpha\| \|w^{\alpha+k}\| r^{2|\alpha|}$$
$$\leq r^{-2n-2|k|} \|B_0 S_m\|_\infty \int_{rB} \frac{|\lambda^k|}{(1-|\lambda|^2)^{n+1}} d\lambda + C \sum_{|\alpha|=1}^{\infty} r^{2|\alpha|},$$

hence, by assumption

$$\limsup_{m\to\infty} \sup_{z\in B} | \left\langle (S_m)_z 1, w^k \right\rangle | \leq C \sum_{|\alpha|=1}^{\infty} r^{2|\alpha|}.$$

Letting $r \to 0$, we have (III.13).

Now we prove (III.12). From (III.14), we get

$$\begin{aligned}
|(S_m)_z 1(\lambda)| &= |\langle (S_m)_z 1, K_\lambda \rangle| \\
&\leq \sum_{|\alpha|=0}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} |\langle (S_m)_z 1, w^\alpha \rangle| |\lambda^\alpha| \\
&\leq \sum_{|\alpha|=0}^{l-1} \frac{(n+|\alpha|)!}{n!\alpha!} |\langle (S_m)_z 1, w^\alpha \rangle| + \sum_{|\alpha|=l}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \|S_m\| \|w^\alpha\| |\lambda^\alpha|
\end{aligned}$$

for $z \in B$, $\lambda \in rB$ and $l \geq 1$. Since the second summation is less than or equals to

$$\begin{aligned}
\sum_{j=l}^{\infty} \left(\frac{(n+j)!}{n!j!} \right)^{1/2} \sum_{|\alpha|=j} \left(\frac{j!}{\alpha!} \right)^{1/2} |\lambda^\alpha| &\leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n!j!} \left[\sum_{|\alpha|=j} \frac{j!}{\alpha!} |\lambda^\alpha|^2 \right]^{1/2} \\
&\leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n!j!} r^j,
\end{aligned}$$

for any $\epsilon > 0$, we can find sufficiently large l such that the second summation is less than ϵ. Thus, (III.13) imply $\sup_{z \in B} |(S_m)_z 1| \to 0$ uniformly on compact subsets of B as $m \to \infty$.

Lemma 75. *Let $\{S_m\}$ be a sequence in $\mathfrak{L}(L_a^2(B))$ such that for some $p > n+2$, $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$,*

$$\sup_{z \in B} \|(S_m)_z 1\|_p \leq C \qquad \text{and} \qquad \sup_{z \in B} \|(S_m^*)_z 1\|_p \leq C$$

where $C > 0$ is independent of m, then $S_m \to 0$ as $m \to \infty$ in operator norm.

Proof. Lemma 73 implies

$$\|S_m\| \leq C(n,p) \left(\sup_{z \in B} \|(S_m)_z 1\|_p \right)^{1/2} \left(\sup_{z \in B} \|(S_m^*)_z 1\|_p \right)^{1/2} \leq C(n,p),$$

hence, Lemma 74 gives

$$\sup_{z \in B} |(S_m)_z 1| \to 0 \tag{III.15}$$

uniformly on compact subsets of B as $m \to \infty$.

Here, for $n + 2 < s < p$, Holder's inequality gives

$$\sup_{z \in B} \|(S_m)_z 1\|_s^s \leq \sup_{z \in B} \int_{B \setminus r\overline{B}} |(S_m)_z 1(w)|^s dw + \sup_{z \in B} \int_{r\overline{B}} |(S_m)_z 1(w)|^s dw$$

$$\leq C \sup_{z \in B} \|(S_m)_z 1\|_p^s (1 - r)^{1 - s/p} + \sup_{z \in B} \int_{r\overline{B}} |(S_m)_z 1(w)|^s dw$$

and (III.15) implies the second term tends to 0 as $m \to \infty$. Also, the first term is less than or equals to $C^s (1-r)^{1-s/p}$ which can be small by taking r close to 1. Consequently, Lemma 73 gives

$$\|S_m\| \leq C(n, s) \left(\sup_{z \in B} \|(S_m)_z 1\|_s \right)^{1/2} \left(\sup_{z \in B} \|(S_m^*)_z 1\|_s \right)^{1/2}.$$

$$\leq C(n, s) \left(\sup_{z \in B} \|(S_m)_z 1\|_s \right)^{1/2} \to 0$$

Corollary 76. *Let $S \in \mathfrak{L}(L_a^2(B))$ such that for some $p > n + 2$,*

$$\sup_{z \in B} \|S_z 1 - (T_{B_m S})_z 1\|_p \leq C \qquad \text{and} \qquad \sup_{z \in B} \|S_z^* 1 - (T_{B_m(S^*)})_z 1\|_p \leq C, \qquad \text{(III.16)}$$

where $C > 0$ is independent of m. Then $T_{B_m S} \to S$ as $m \to \infty$ in operator norm.

Proof. Let $S_m = S - T_{B_m S}$. Then Proposition 69 and Theorem 66 imply

$$B_0(S_m) = B_0 S - B_0(T_{B_m S})$$

$$= B_0 S - B_0(B_m S)$$

$$= B_0 S - B_m(B_0 S)$$

which tends uniformly to 0 as $m \to \infty$, hence $\|B_0(S_m)\|_\infty \to 0$. Consequently, by Lemma 75 we complete the proof.

Theorem 77. *Let $S \in \mathfrak{L}(L_a^2(B))$. If there is $p > n + 2$ such that*

$$\sup_{z \in B} \|T_{(B_m S) \circ \varphi_z} 1\|_p < C \quad \text{and} \quad \sup_{z \in B} \|T^*_{(B_m S) \circ \varphi_z} 1\|_p < C \qquad \text{(III.17)}$$

where $C > 0$ is independent of m, then $T_{B_m S} \to S$ as $m \to \infty$ in operator norm.

Proof. By Corollary 76, we only need to show that (III.17) implies (III.16). Since $T_{(B_m S)\circ\varphi_z} = (T_{B_m S})_z$ and

$$T^*_{(B_m S)\circ\varphi_z} = T_{\overline{B_m S_z}} = T_{B_m(S_z^*)} = T_{(B_m(S^*))\circ\varphi_z},$$

it is sufficient to show that

$$\sup_{z\in B} \|S_z 1\|_p < \infty.$$

By Lemma 73, we get

$$\|T_{B_m S}\| \leq C(n,p) \left(\sup_{z\in B} \|T_{B_m S\circ\varphi_z} 1\|_p \right)^{1/2} \left(\sup_{z\in B} \|T^*_{B_m S\circ\varphi_z} 1\|_p \right)^{1/2}$$

$$< C$$

where C is independent of m, hence writing $S_m = S - T_{B_m S}$, we have $\|S_m\| \leq C$ where C is independent of m. Also, the proof of Corollary 76 implies

$$\|B_0 S_m\|_\infty \to 0$$

as $m \to \infty$.

Let f be a polynomial with $\|f\|_q = 1$. Then Lemma 74 implies

$$\sup_{z\in B} |\langle (S_m)_z 1, f \rangle| \to 0$$

as $m \to \infty$. Thus, for any $\epsilon > 0$ and $z_0 \in B$, we have

$$|\langle S_{z_0} 1, f \rangle| \leq \sup_{z\in B} |\langle (S_m)_z 1, f \rangle| + |\langle (T_{B_m S})_{z_0} 1, f \rangle| \leq \epsilon + C$$

for sufficiently large m, where C is independent of m. Since ϵ is arbitrary, we get

$$\sup_{z\in B} \|S_z 1\|_p < \infty$$

as desired.

III.3 Compact operators

Given $U \in \mathfrak{U}(n)$, define $V_U f(w) = f(Uw) det U$ for $f \in L_a^2(B)$. Then V_U is a unitary operator on $L_a^2(B)$. We say that $S \in \mathfrak{L}(L_a^2(B))$ is a radial operator if $SV_U = V_U S$ for any $U \in \mathfrak{U}(n)$.

If $S \in \mathfrak{L}(L_a^2(B))$, the radialization of S is defined by

$$S^\sharp = \int_{\mathfrak{U}} V_U {}^* S V_U dU$$

where dU is the Haar measure on the compact group $\mathfrak{U}(n)$ and the integral is taken in the weak sense. Then $S^\sharp = S$ if S is radial and $\mathfrak{U}$-invariance of dU shows that $S^\sharp$ is indeed a radial operator.

If $f \in L^\infty$ and $g, h \in L_a^2$ then

$$\langle V_U {}^* T_f V_U g, h \rangle = \int_B f(w) V_U g(w) \overline{V_U h(w)} dw$$
$$= \int_B f(U^* w) g(w) \overline{h(w)} dw.$$

Thus $V_U {}^* T_f V_U = T_{f \circ U^*}$ and

$$V_U {}^* T_{f_1} \cdots T_{f_l} V_U = T_{f_1 \circ U^*} \cdots T_{f_l \circ U^*}$$

for $f_1, \ldots, f_l \in L^\infty$, $l \geq 0$.

Lemma 78. *Let $S \in \mathfrak{L}(L_a^2(B))$ be a radial operator. Then*

$$T_{B_m(S)} = \int_B S_w d\nu_m(w).$$

Proof. Let $z \in B$. By (III.3) and Lemma 63, we obtain

$$B_0 \left(\int_B S_w d\nu_m(w) \right)(z) = \left\langle \left(\int_B S_w d\nu_m(w) \right)_z 1, 1 \right\rangle$$
$$= \int_B \langle U_z U_w S U_w U_z 1, 1 \rangle \, d\nu_m(w)$$
$$= \int_B \left\langle U_{\varphi_z(w)} V_U^* S V_U U_{\varphi_z(w)} 1, 1 \right\rangle d\nu_m(w)$$

where V_U is in Lemma 63. Since S is a radial operator, Theorem 64, Proposition 58 and Proposition 69 imply that the last integral equals

$$\int_B \left\langle U_{\varphi_z(w)} S U_{\varphi_z(w)} 1, 1 \right\rangle d\nu_m(w) = \int_B B_0 S \circ \varphi_z(w) d\nu_m(w)$$
$$= B_m B_0 S(z)$$
$$= B_0 B_m S(z)$$
$$= B_0(T_{B_m(S)})(z).$$

Since B_0 is one-to-one mapping, the proof is complete.

Theorem 79. *Let $S \in \mathfrak{T}(L^\infty)$ be a radial operator. Then S is compact if and only if $B_0 S \equiv 0$ on ∂B.*

Proof. It is obvious that if S is compact then $B_0 S(z) \longrightarrow 0$ as $z \longrightarrow \partial B$. So we only need to show the if part.

Suppose $B_0 S \equiv 0$ on ∂B. Then $B_m S \equiv 0$ on ∂B by Proposition 70, hence $T_{B_m S}$ is compact for all $m \geq 0$.

Let

$$Q = \int_{\mathfrak{U}} T_{f_1 \circ U^*} \cdots T_{f_l \circ U^*} dU$$

with $f_1, \ldots, f_l \in L^\infty$ for some $l \geq 0$. Then $Q \in \mathfrak{L}(L_a^2)$. By Lemma 78, for any $z \in B$, we

have

$$T_{(B_m(Q))\circ\varphi_z} = \int_B ((Q)_z)_w d\nu_m(w)$$

$$= \int_B \int_{\mathfrak{U}} T_{f_1\circ U^*\circ\varphi_z\circ\varphi_w} \cdots T_{f_l\circ U^*\circ\varphi_z\circ\varphi_w} dU d\nu_m(w).$$

Consequently,

$$\|T_{(B_m(Q))\circ\varphi_z}\| \leq C(l)\|f_1 \circ U^* \circ \varphi_z \circ \varphi_w\|_\infty \cdots \|f_l \circ U^* \circ \varphi_z \circ \varphi_w\|_\infty$$

$$= C(l)\|f_1\|_\infty \cdots \|f_l\|_\infty.$$

Similarly, we have

$$\|T^*_{(B_m(Q))\circ\varphi_z}\| \leq C(l)\|f_1\|_\infty \cdots \|f_l\|_\infty.$$

Thus, Theorem 77 gives that

$$T_{B_m(Q)} \to Q \tag{III.18}$$

in $\mathfrak{L}(L_a^2)$-norm.

Since $S \in \mathfrak{T}(L^\infty)$, there exists a sequence $\{S_k\}$ such that $S_k \to S$ in the operator norm where each S_k is a finite sum of finite products of Toeplitz operators. Since the radialization is continuous and S is radial, $S_k^\sharp \to S^\sharp = S$. From Lemma 78, we have

$$\|T_{B_m S}\| = \left\|\int_B S_w d\nu_m(w)\right\| \leq \int_B \|S_w\| d\nu_m(w) = \|S\|.$$

Thus

$$\|S - T_{B_m S}\| \leq \|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\| + \|T_{B_m(S_k^\sharp)} - T_{B_m S}\|$$

$$\leq 2\|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\|$$

and (III.18) imply $T_{B_m(S)} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm, hence S is compact.

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