Improving Algebra Students' Number and Operations Sense, Conceptual Knowledge, Executive Control, and Algebraic Thinking Skills: A Supplementary Curriculum

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June 15, 2008

Capstone Project

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Table of Contents

Abstract

Since the National Council of Teachers of Mathematics (NCTM) wrote their *Curriculum and Evaluation Standards for School Mathematics* in 1989, the mathematics education community has focused on developing students' mathematical proficiency. In recent years, mathematics education has continued to receive a great deal of attention due to concerns about students' weak mathematics skills and their effects on the United States' ability to compete globally. In the spring of 2008, the National Mathematics Advisory Panel (NMAP) recommended policies and practices necessary for the improvement of mathematical proficiency of American students. The NMAP chose to focus those recommendations on the implications for the preschool through middle school curriculum in preparing students for Algebra.

Although those recommendations may help future generations of American students, they do little to help the students who are currently taking Algebra and lack proficiency with prerequisite skills. The goal of this supplementary curriculum is to improve Algebra students' conceptual understanding of number and quantity and to use that improved understanding to enrich students' learning of basic Algebra concepts. The backbone of the curriculum is the careful design and sequencing of mental mathematics problems to foster rich discussions. The problem solving and subsequent discussions in the classroom promote the development of number and operations sense, conceptual knowledge, executive control and algebraic thinking skills. All of this skill development is possible when undertaken in a safe and equitable learning environment favoring process over result and requiring sound mathematical justification while providing plentiful opportunities for assessment of teaching and learning.

This project report has two main components: (1) a companion paper providing research support for each of the key aspects of the supplementary curriculum, and (2) curriculum materials including an instructor's guide for the curriculum, sample problem sets, and other teacher support materials.

Part I. The Supporting Research

Research Companion for Supplementary Curriculum

During most of the 20th century, the United States was the world leader in many scientific and applied fields. Critical to that leadership was a pool of mathematically proficient citizens who contributed in many areas of society. As we look to the future, educators and policymakers alike are concerned about students' lack of mathematical proficiency and its effects on U.S. global competitiveness. In fact, between 1990 and 2000, the percentage of non-Americans in the U.S. scientific and engineering workforce increased from 14% to 22% (National Mathematics Advisory Panel [NMAP], 2008). In addition, recent mathematics scores on the National Assessment of Educational Progress (NAEP)--with 39% of students at or below the "proficient level" in Grade 8 and only 23% proficient in Grade 12--provide little hope for increasing numbers of scientists and engineers in the immediate future. To compete globally and improve American students' mathematics performance, the President created the National Mathematics Advisory Panel (NMAP) in 2006 (NMAP, 2008). The Panel was charged with recommending policies and practices that would better prepare students for the future.

The Panel chose to focus on the implications for the preschool through middle school mathematics curriculum in their quest for improving the mathematical proficiency of American students because of the correlation between success in Algebra and future success. They outlined three key sets of skills that form the "Critical Foundation of Algebra": fluency with whole numbers, fluency with fractions, and particular aspects of geometry and measurement (NMAP, 2008, p.17-18). In their 2008 report, the NMAP goes on to explain these skills in greater detail and provide benchmarks for mastery of these skills. If educators and policymakers take these suggestions to heart and find ways to address them, we may see improvements in students' mathematical proficiency in the future.

Although these recommendations may help future generations of American students, they do little to help the students who are currently taking Algebra and lack proficiency in one or more of these skill sets. In fact, high school students with poor mastery of basic arithmetic concepts (an aspect of fluency with whole numbers and fractions) struggle with algebra (NMAP, 2008). A close look at the errors made by middle or high school Algebra students reveals that too many Algebra students are in this category. They may have some procedural understanding of prerequisite mathematics content, but lack the conceptual understanding and number sense necessary for learning algebra (Reys et al., 1999).

For future generations, the solution to this problem is to incorporate the teaching of algebraic ideas into the curriculum at an earlier age and to ensure that the foundational skill sets are in place before students take Algebra (National Research Council [NRC], 2001). But for current Algebra students who lack conceptual understanding of mathematics, we need to find a way to fill in the gaps. The goal of this supplementary curriculum is to improve Algebra students' conceptual understanding of number and quantity and to use that improved understanding to enrich students' learning of basic Algebra concepts.

The Curriculum

The idea for this curriculum developed over time. As a high school math teacher for four years, I was struck by my students' lack of number sense and conceptual understanding and I saw that it impeded their study of algebra. While teaching, I attended a lecture by Ruth Parker in which she used the tool of “Number Talks” to convey her perspective on the teaching of mathematics. I left this lecture thinking about how well Number Talks could foster the development of number sense and conceptual understanding. In my first semester of graduate school, I took a class on math teaching in the elementary school from Paul Cobb and had more experience with classroom discussions of arithmetic problems. It was during those discussions that I really began to think about the connections between strategies for solving arithmetic problems and the algebra concepts that students struggle with in high school. I realized that a supplementary curriculum made up of mental arithmetic problems (like Number Talks) might help to develop students’ number sense and conceptual understanding of mathematics while facilitating their study of algebra.

This supplementary curriculum is intended to be used for 10 minutes daily in the form of a mental math problem followed by a discussion of students’ solution methods. Each day, one arithmetic problem will be used as the basis for discussion. Within the discussion, the answer is given a minor role and the solution methods are the focus. The curriculum materials include the set of problems along with common student solutions, connections to algebraic concepts, and summary problems. The problems are designed to be used in sequence to allow students to try out classmates’ solution methods from previous problems and assess their relative efficiency. More specific details about components of the curriculum are provided in the Instructor’s Guide. The motivation for the curriculum design with regard to learners and learning, curriculum and instructional strategies, learning environment, and assessment is given below.

Learners and Learning

Currently, many middle and high school students take Algebra without adequate conceptual understanding of number and quantity. As mentioned above, this makes it more difficult for them to learn new algebra concepts. In order to fill in gaps in students' mathematical understanding and facilitate their learning of algebra, this curriculum focuses on the development of several related skills: number and operation sense, conceptual knowledge, executive control, and algebraic thinking skills.

Number and operations sense. Number sense has a range of different definitions but generally refers to the ability to flexibly solve or estimate numerical problems based on knowledge about place value and quantity (Anghileri, 2000; NMAP, 2008). For example, students with number sense might solve the problem 45+23 differently from the way they solve the problem 99+23. With 45+23, the solution might resemble the standard algorithm (adding 5 and 3 to get 8 and 40 and 20 to get 60, for a result of 68), but for 99+23, the students might recognize that 99 is 1 less than 100 and solve by adding 100 to 23 to get 123 and then subtracting 1, for a result of 122. Also related to number sense is knowledge of the basic arithmetic operations and related properties, often referred to as operation sense (Slavit, 1999). A basic example of operation sense is the notion that 5+3 will yield the same result as 3+5 and that this works for any two numbers added together (i.e., the commutative property of addition).

This curriculum aims to develop number and operation sense because they are crucial for mathematics learning. Specifically, students who lack number sense are unable to learn algorithms and number facts and they therefore struggle to check for the reasonableness of their answers (NMAP, 2008). Not only does a lack of number sense impede the learning process, but also skills associated with number sense actually facilitate students' mathematical understanding and confidence. Flexibility with number and operations supports students' problem solving (NMAP, 2008) and develops students' mathematical confidence (Anghileri, 2000). These skills are all important for students' mathematical proficiency and students' preparation for their study of algebra. In particular, activities that promote number sense provide opportunities for generalizing relationships and conceptually understanding operations, which lay the foundation for Algebra (NRC, 2001).

Many Algebra students lack basic number and operations sense and have difficulty with Algebra. For this reason, this curriculum strives to remediate this lack of number sense by working to develop students' number sense for 10 minutes each day. There are several key curriculum components that specifically foster the development of number sense. First, the focus of the discussion on varied solution methods and the fact that students are encouraged to come up with several different solution methods emphasize aspects of place value and the relationship between number and quantity. Also, the variety of problem types and the sequencing of the problems provide opportunities for students to practice new solution methods and generalize ideas about efficiency of particular solution methods. Both of these components are discussed in greater detail below.

Conceptual knowledge. By the time students enter Algebra, they have spent a number of years studying arithmetic. Unfortunately, most of their knowledge of arithmetic is procedural. For example, they definitely know how to subtract using a calculator and can probably subtract by hand if pushed to do so, but they do not think about the concepts behind the algorithms they are using. This is not surprising since this is exactly why algorithms were developed--to efficiently solve problems with little regard for context. Yet, without the conceptual understanding of why certain procedures work, students are unable to reason through slightly different problems or check to see that their answers make sense (Greeno, 2003). Also, although the standard algorithm is generally efficient, it is not necessarily the most efficient solution method for every problem. Conceptual understanding of the relationships between concepts makes it easier to solve problems in several different ways (Yackel & Hanna, 2003). Strong conceptual understanding also facilitates transfer of procedures and minimizes the number of procedures to be learned (Hiebert & Lefevre, 1986). Overall, research shows that conceptual understanding is an important aspect of mathematical proficiency (Bransford, Brown, & Cocking, 1999).

Conceptual understanding and number sense are clearly interrelated. In fact, without conceptual understanding of operations and number, students are unlikely to possess number sense. Again, within this curriculum, the focus on solution methods in the discussion targets conceptual understanding. In particular, probing questions and revoicing from the teacher draw out the students' conceptual knowledge. Although seemingly minor, the posing of the problem actually emphasizes conceptual understanding. Teachers are asked to pose problems horizontally (e.g., 45+23) instead of vertically (e.g., $\begin{array}{r}45\\ \underline{+23}\end{array}$) because the vertical form facilitates the procedural use of the standard algorithm and does not necessarily require students' conceptual understanding. Helping students to build conceptual understanding of arithmetic procedures is a goal that underlies many of the features of this curriculum.

Executive control. As students develop more conceptual knowledge, they will be better able to make informed decisions when problem solving. Executive control deals with decision-making and has two key functions in mathematics problem solving: choosing solution methods and checking work. With regard to choosing solution methods, students' executive control helps them to pick through known solution methods and choose an appropriate one (Hiebert & Lefevre, 1986). After students decide on solution methods and follow the corresponding procedures, they still need to make sure that they have not made a mistake. As executive control develops with conceptual knowledge, students become increasingly able to check their work efficiently (Hiebert & Lefevre, 1986). The ability to make good choices regarding solution methods and the ability to check one's work become increasingly important in Algebra courses and beyond as problems become more abstract and advanced.

In addition to developing executive control through the development of conceptual understanding, this curriculum provides several specific opportunities for students to practice their executive control. Discussions provide students with opportunities to learn about their peers' solution methods and to share their own. The design of the problems to build on solution methods and provide opportunities for students to practice new solution methods is one way that students can practice their executive control. Also, within the discussion, the teacher will probe students to encourage them to think about the efficiency of certain solutions. This collective discussion of efficiency prepares students for their internal decision-making on subsequent problems both within this curriculum and in other contexts.

Algebraic thinking skills. Although algebraic thinking skills are clearly related to algebra by name, their development starts well before students enter Algebra. Like number sense, algebraic thinking skills have a variety of definitions. Driscoll (1999) explains algebraic thinking as three habits of mind that that help with the learning of algebra: “doing-undoing”, “building rules to represent functions”, and “abstracting from computation” (p. 1-2). Critical to all three of these habits of mind is the ability to generalize. By promoting generalization, teachers can develop students’ operations sense (Driscoll, 1999) and algebraic thinking skills (Kaput, 1999). In particular, by highlighting algorithms used and explicitly comparing algorithms in class discussions, students’ algebraic thinking skills will improve.

In this curriculum, because of the focus on solution methods rather than answers, the teacher will have many opportunities to highlight algorithms and compare them. In particular, the teacher will revoice students’ solution methods (discussed in more detail below) and ask questions about generalizing solution methods during the class discussion. It is important to note that although all three of the aforementioned habits of mind are addressed in this curriculum, abstracting from computation is the most prevalent because of the computational nature of the problems posed.

Curriculum and Instructional Strategies

Within this curriculum, the problems and the subsequent discussion are carefully designed to support the development of number sense, conceptual knowledge, executive control, and algebraic thinking skills. The curricular choices of requiring mental mathematics and of sequencing problems so that they build on previous problems and skills are based on educational research and are discussed below. Built around those carefully designed problems and central to this curriculum, are meaningful class discussions. Discourse is an important part of effective mathematics classrooms (NCTM, 1989). A review of environmental supports for discourse follows in the next section, but instructional decisions related to discourse, namely soliciting multiple solution methods and revoicing students' solution methods, are also discussed below.

Mental mathematics. Traditional mathematics courses place heavy emphasis on written mathematics for practicing procedures learned in class. Practice with mental mathematics will build conceptual knowledge by highlighting aspects of the number system and encouraging flexible problem solving. In fact, as students work to solve mental math problems, they draw on their previously developed number sense while they continue to develop their number sense even further (Sowder, 1995). Also, when students are required to solve arithmetic problems mentally, their limited memory capacity provides a natural incentive for them to find an efficient solution method (NRC, 2001). In order to regularly find efficient problem solving methods, students must consider the mathematical structure of the numbers involved and flexibly adapt the solution methods used. Sowder (1995) calls this flexibility "adaptive expertise," and goes on to explain that it requires deep conceptual understanding of procedures (p.24). Mental mathematics helps students develop number sense, conceptual understanding, and executive control.

This curriculum is based on students' daily execution of mental mathematics problems. Students are generally not permitted to use paper and pencil or calculators and are asked to try to solve the presented problem several different ways. If they want to, students can try to perform the standard algorithm (usually designed for paper and pencil manipulation) in their heads, but when they try to find a second solution method they will have to try to exploit the structure of the number system and properties of operations. As they discover that alternative solution methods are more efficient and easier for mental math, students will be internally motivated to use their number sense and build on it as well.

Problem design and sequencing. Mental mathematics problems have the potential to improve students' number sense. By combining that potential with careful design and sequencing of the problems, students' skill development will be enhanced further. One important problem design consideration is that they must be appropriately challenging (Bransford et al., 2000). Problems that are too simple do not engage students, and problems that are too challenging have the potential to frustrate students. Also, they have to be flexible enough to conceptually engage an entire class of students with varying skill levels (Lampert, 1990a). With the goal of developing number and operations sense in mind, it is also important that the problems highlight the structure of mathematics (Hiebert et al., 1997). In addition to designing problems to appropriately challenge students, the sequencing of problems is important so that later problems give students opportunities to practice skills they acquire on earlier problems.

The problems in this curriculum are carefully chosen to develop students' number sense. They are arithmetic problems, which should not be particularly challenging for Algebra students, but the challenge is for students to find several different ways to solve the problems. Although students may not necessarily use number sense when using the standard algorithm, asking for additional methods does require students to use and build on their number sense. More implications of soliciting multiple solution methods are discussed below. In addition to having problems that are designed to encourage discussion of mathematical structure, the problems in this curriculum are carefully sequenced. By building on strategies used for previous problems and offering students opportunities to try out strategies that they learned in the discussion, the problems support students learning and promote students' development of number and operations sense.

Soliciting multiple solution methods. Carefully designed and well-sequenced mental mathematics prepare the class for productive discussion. By soliciting multiple solution methods within the discussion, the teacher can further enhance students' skill development. In particular, by asking for multiple solution methods, the teacher encourages students to develop or choose their own strategies, which will lead to better student learning (Siegler, 2003) and improved connections between concepts (NRC, 2001). Also, a discussion of various solution methods allows for comparison of strategies (Siegler, 2003), which can boost conceptual understanding (NRC, 2001). Lastly, because an important aspect of algebra is an understanding of equivalence, comparisons of solutions methods that deal with questions of equivalence will help with students' development of algebraic thinking skills (Chazan & Yerushalmy, 2003).

There are several key ways that this curriculum supports a discussion of multiple solution methods. First, as mentioned above, the teacher is asked to pose the problem horizontally. This arrangement will make it less straightforward for students to use the standard algorithm to solve the problem and will encourage students to develop or choose their own strategies. Another way that this curriculum supports a discussion of multiple solution methods is that the teacher waits to start the discussion until several students have solved the problem multiple ways. Students are asked to subtly indicate when they have a solution by putting up a finger close to their chest. They are expected to try to use other methods and put up additional fingers for every method they use so that the teacher has an indication of student progress, without making it a competition between students. Once they begin the discussion, the teacher will ask questions to probe students for different solution methods as well as asking questions about equivalence of methods.

To support the teachers' solicitation of multiple solution methods, this curriculum includes discussions of common student solutions for each problem. The importance of pedagogical content knowledge for effective discussion-based teaching is clear (Bransford et al., 2000). By providing the information about common student solutions and their links to more abstract algebraic concepts, the curriculum aims to better prepare teachers for rich mathematical discussion.

Revoicing. Revoicing is a technique used by participants in a discussion in which one participant may repeat, rephrase, explain or elaborate the comment of another person (Forman, 2003). In the case of classroom mathematical discussions, the teacher is often the most frequent employer of this technique. There are many good reasons for teachers to revoice students' ideas or comments during mathematical discussions. Revoicing can be used to clarify students' ideas, probe for students' reasoning, introduce new vocabulary or notation, steer the discussion, and connect students' ideas (Lampert & Cobb, 2003). The act of adding symbols or notation during the revoicing of students' ideas helps to formalize the mathematical concepts at hand (Lampert, 1990a) and improve students' conceptual knowledge (Hiebert & Lefevre, 1986). Also, by encouraging students to revoice other students' ideas, they have to think analytically about the ideas and fit them within their knowledge bases (Siegler, 2003). Overall, revoicing is a versatile instructional technique.

As a part of this curriculum, the teacher plays an important moderating role in discussions by using many different revoicing moves. Whether revoicing orally or in writing, the teacher is constantly working to clarify, formalize, connect, or direct the mathematical ideas that arise in the discussion. In particular, to help connect the problems used in this curriculum with the algebraic ideas being discussed within the Algebra curriculum, the teacher needs to use revoicing to formalize and point out generalizations. We want students to not only fill in gaps within their number and operations sense, but to connect their new conceptual understanding to abstract ideas they are learning in Algebra. The teacher does not have to be the only member of the discussion to revoice. As the classroom norms, discussed below, become more firmly established, the hope is for students to revoice their classmates' solution methods to make connections between methods and deepen their own conceptual understanding.

The Learning Environment

In many schools, decisions about the learning environment are separated from curricular decisions. Because of the impact of the learning environment on the learning that takes place in a classroom (Forman, 2003), decisions about the learning environment and the curriculum cannot be separated. This is especially true in classrooms where discussion is a focus because of the social interaction that takes place (National Council of Teachers of Mathematics [NCTM], 1991). By defining the classroom social and sociomathematical norms, teachers have the opportunity to design their classroom learning environments (Yackel & Cobb, 1996). Yet, defining the classroom norms will not guarantee that they are enacted. In fact, it is likely that students will need to be reminded about classroom norms throughout the course in order for the norms to be internalized (Lampert & Cobb, 2003). The key classroom norms for promoting productive mathematical discussion are: (1) ensuring that the classroom is a safe place for students to express themselves, (2) valuing all members of the classroom community, (3) focusing on solution methods rather than the answer, and (4) requiring sound mathematical justification to determine correctness.

Safety. In order for students to fully participate in mathematical discussions, the classroom must be a safe place for students to express themselves. When students reveal their personal ideas they increase their vulnerability (Lampert, 1990b), so it is important for the learning environment to be safe so that students are willing to take risks and share their ideas. Because of the social dynamics for adolescents, a classroom social norm that provides physical and emotional safety is not to be taken for granted. In fact, the teacher's role in developing this social norm is ongoing. By providing a safe space for students to express their ideas, the classroom discussion is likely to be enhanced (NCTM, 1991). When students provide faulty solution methods, those ideas are often great opportunities to deepen students' conceptual understanding (Hiebert et al., 1997). A safe learning environment does not mean that students' ideas will not be challenged. In fact, as is discussed below in greater detail, rich mathematical discussion is full of questioning followed by justification and resolution. Also, in a classroom with this norm, students come to understand that challenging each other's ideas should not be taken personally, but, instead, is a part of the class's quest for mathematical understanding.

Because of this curriculum's focus on discussion of students' solution methods, it is essential for students to feel safe enough to express their ideas. This starts from the beginning of the year with the teacher's explanation of the classroom norms of respecting classmates' ideas and recognizing mistakes as learning opportunities. Because this may not be standard for students, it must be followed by regular recommendations and corrections of inappropriate behavior. Within daily discussions, the teacher has the opportunity to reinforce the norms. When a student makes a mistake, the teacher might say "thank you for doing it that way and sharing your solution because we got to talk about some interesting mathematics" or the teacher might summarize what was learned from the particular error (Hiebert et al., 1997). These strategies will not only encourage that same student to continue contributing to the discussion, but they will also demonstrate to other students that they should feel free to share their ideas as well.

Equity. Very closely related to the social norm of making the learning environment safe for students is the norm of valuing *all* students. Teachers must believe that every child can learn mathematics if they want to achieve equity in their mathematics classrooms (Hiebert et al., 1997). To that end, they must ensure that all students learn how to participate in discussions. This does not mean that all students should be treated in exactly the same way, but that the teacher must find a way to ensure that everyone is involved. Secada and Berman (1999) observed the following about encouraging multiple solution paths:

> This practice, when managed with sensitivity, encourages widespread participation; expresses appreciation for diversity in thought; establishes value for these differences among students; allows the approval of the solution to come from the mathematics rather than from the teacher; and features the substance, content and logic of a procedure or task rather than its performer. (p. 37)

In fact, if learning environments are thoughtfully designed to promote equity, learning opportunities increase for all students (Hiebert et al., 1997). Consideration of equity is especially important in Algebra classrooms because of Algebra's role as a gatekeeper for future mathematics courses (Chazan, 2000).

Although all teachers must be aware of equity concerns within their classrooms and they need to handle those concerns on individual bases, this curriculum makes an effort to promote equity. The problems within the curriculum are designed to allow all students to struggle with them at some level. For some students, this will mean that they struggle to do the mental mathematics required. For others, they will struggle with the challenge of finding additional solution methods. Also, this curriculum's heavy emphasis on discussion and multiple solution methods clearly has the potential to promote an equitable learning environment. By placing value on students' different ways of thinking, all students are valued as members of the classroom mathematical community.

Process. A classroom built upon social norms of safety and equity is likely to support high quality mathematical discussion. By adding to those norms the sociomathematical norm of a focus on solution methods rather than the answer, the classroom mathematical discussions will reveal and build on students' conceptual understanding. The numerical answer to a problem reveals little about students' thinking processes. By making the focus of the discussion students' solution methods, students' conceptual knowledge becomes apparent (Lampert, 1990b). Even more of the students' reasoning is revealed when the teacher asks why students chose particular solution methods. When this occurs, the discussion becomes conceptual (rather than purely computational) in nature (Lampert & Cobb, 2003). Included in a focus on solution methods is the notion of mathematical difference between solution methods. As the teacher solicits different solution methods, students learn what is meant by *mathematically different* in their classroom and push themselves to make the appropriate distinctions in their solution methods. This process of comparing ideas and deciphering difference is cognitively challenging for students (Yackel & Cobb, 1996).

In this curriculum the teacher will field the students' answers to the problems initially, but they will just be used as a starting point for discussion. From that point on, the discussion becomes focused on students' different solution methods. When the students provide information about their solution methods the teacher can probe for deeper conceptual understanding by asking about reasoning within the method as well as why the method was chosen at all. These types of questions make the discussion conceptual in nature and help students learn the internal questions that they should be using to make decisions about solution methods. Also, because students explain their reasoning, the teacher is able to assess students' conceptual understanding (discussed more in the following section). Because of the classroom focus on process over result, students may begin to put more of their focus on their solution methods and less emphasis on the answer as they adjust to this change in focus.

Justification. The norm of requiring sound mathematical justification to determine correctness of solutions rounds out the learning environment by emphasizing reasoning. In traditional mathematical instruction, solutions are often deemed correct by verification from the textbook or the teacher. Mathematics becomes a much more flexible subject when reasoning and mathematical justification determine a solution's correctness (Lampert, 1990b). This norm nicely supports the equity norm because it redistributes the mathematical authority from the textbook and teacher to all of the participants. In addition, the classroom norm of justification emphasizes reasoning and proof, which is a National Council of Teachers of Mathematics (NCTM) process standard at all grade levels (Yackel & Hanna, 2003). By emphasizing justification, students learn that mathematics should make sense (NCTM, 2000). The norms of requiring sound mathematical justification to determine correctness and focusing on solution methods rather than the answer supports the safety norm because it places the emphasis on the students' ideas rather than on the students themselves.

Justification takes a prominent role in this curriculum. Students develop number sense, conceptual understanding, and algebraic thinking through the focus on justification. Although students may have solved problems like the ones in this curriculum for many years, most have not thought much about the justification for their solution methods. Teachers create this focus on justification by deflecting any students' attempts to seek validation of correctness from them or from the textbook. Also, by asking probing questions about students' reasoning and the mathematical justification for the choices made, the teacher can emphasize mathematical justification. Eventually, students will see this as the way to judge the correctness of their own solutions.

Assessment

Assessment has long been a part of mathematics curriculum and instruction. In the past, assessment was primarily used in a summative manner to measure student progress. Recently, assessment has become more integrated into the teaching and learning process. In particular, "*formative assessment* is defined as assessment carried out during the instructional process for the purpose of improving teaching or learning"

(Shepherd et al., 2005, p.275). Studies show that teachers who use formative assessments to inform their teaching enhance the learning of their students (NCTM, 2000). If designed appropriately, formative assessment also provides opportunities for students to assess their own progress. By building opportunities for formative assessment into curricula, teachers and students are able to use the information gained to improve students' learning.

Teacher assessment. Teachers are regularly required to make instructional decisions. Formative assessment provides valuable information for those decisions. Because assessing student understanding and revising instruction is an ongoing process, it is beneficial for assessment to be integrated into instruction as a part of the classroom routine (NCTM, 2000). Teachers can assess students' knowledge by observing students as they work. Also, classroom discussions are good opportunities for teachers to assess individual students' knowledge and reasoning (Shafer & Romberg, 1999). If teachers ask probing questions that pull out student understanding and if they solicit responses from a variety of students, classroom discussions can yield considerable information about students' mathematical understanding.

Formative assessment is seamlessly integrated into this curriculum. The daily discussions of multiple solution methods provide opportunities for the teacher to gather information about the students' knowledge. In particular, the probing questions pull out the students' reasoning. As part of the discussion, the teacher might follow up a student's description of his or her solution method with the question "who solved the problem the same way?" This question not only allows more students to participate in the discussion, but it also provides information for the teacher about other students in the class. The summary problems provided as part of the curriculum are also formative assessments. They consist of pairs of problems to be worked on by pairs of students. Unlike the daily problems, solutions to summary problems are to be written down. But, like the daily problems, students are asked to use multiple solution methods. In fact, each student is asked to write out two different ways to solve each problem and why they chose those ways. The summary problems are designed to be used after a series of related problems have been solved and discussed in class. Written solutions collected from each student nicely complement the information about students' understanding obtained from the daily discussions.

Student self-assessment. As we strive to teach students to become independent problem solvers, we must teach them how to evaluate and revise their thinking. Metacognition is an individual's knowledge about himself and his cognitive abilities (Newman & Newman, 2007). Well-designed formative assessment not only provides opportunities for the teacher to assess the students' progress, but also improves the students' metacognitive abilities. Students' executive control is linked to their metacognitive awareness (Anghileri, 2000). Also, instruction that emphasizes metacognition has been shown to improve students' knowledge transfer (Bransford et al., 2000). One of the ways that formative assessment helps to develop students' self-awareness is by providing feedback. To be most effective, the teacher and the students must understand that the primary purpose of feedback is to facilitate learning (Shepard et al., 2005).

As mentioned above, formative assessment is built into this curriculum. There are two key ways that this curriculum aids students' self-assessment. First, by listening to the questions asked by the teacher, students learn the criteria they should be using to evaluate their solutions. For example, questions that ask about students' reasoning impress upon students that they should be using reasoning to decide the next step to take or which solution method to use. This information helps students build their metacognitive skills. The second way that this curriculum helps students with self-assessment is by providing feedback. Questions from the teacher or from classmates in response to a student's statement provide direct feedback. Some of the feedback that students get from discussions may be much more indirect. For example, students may learn that their solution methods are not the most efficient by hearing about their classmates' more efficient methods. Because this curriculum is designed to fill in gaps in students' number sense and to facilitate learning of algebra, the use of feedback for information is fitting.

Discussion

Since the NCTM wrote their *Curriculum and Evaluation Standards for School Mathematics* in 1989, the mathematics education community has focused on developing students' mathematical power. Mathematical power is defined as "an individual's abilities to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems" (NCTM, 1989, New Goals for Students section, para. 2). Related to this idea is the notion of agency given to or taken away from students of mathematics, which involves the ability and freedom to engage in independent thought (Boaler & Greeno, 2000). This supplementary curriculum seeks to give Algebra students agency in the world of mathematics to allow them to develop mathematical power. The backbone of the curriculum is the careful design and sequencing of mental mathematics problems to foster rich discussions. The problem solving and subsequent discussions promote the development of number and operations sense, conceptual knowledge, executive control and algebraic thinking skills. All of this skill development is possible when undertaken in a safe and equitable learning environment favoring process over result and requiring sound mathematical justification while providing plentiful opportunities for assessment of teaching and learning.

Although this curriculum aims to fill in gaps in students' skills, it is not free from potential implementation issues. The first of these issues relates to time, which is always in short supply for teachers and students. Although this curriculum only requires 10 minutes daily, those 10-minute segments of instructional time add up over the course of the year. Clearly, teachers will have to forego something else in the Algebra curriculum to make time for this supplementary curriculum. Another potential issue is related to the dependence of this curriculum on the establishment of set norms within the learning environment. If those norms differ from the norms expected for the remainder of the class time, they may not ever truly take shape as classroom norms. For example, if the justification of correctness falls to the teacher or the textbook within the rest of the class, students may have trouble internalizing the norm of providing mathematical justification to establish correctness as a part of this curriculum. Conflicting norms will greatly inhibit the implementation of the curriculum. In addition to implementation issues, there are also some student-related concerns. First, without the appropriate framing of the curriculum, students may be bothered by the seemingly simple nature of the problems. Also, this curriculum may need to be modified for students with learning differences involving a shortage of working memory because of the memory-intensive nature of the tasks.

Despite the fact that this curriculum has the potential to fill in gaps in Algebra students' knowledge and understanding, further research in this area is needed. In particular, the relationship between students' number sense skills and their effects on Algebra skill acquisition is not well understood. Also, this curriculum would eventually need to be tested to see if the gains in students understanding outweigh the costs associated with the loss of instructional time for daily implementation. Finally, more research should be done to find other ways for students who are under-prepared for Algebra to catch up on necessary skills.

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Part II. The Curriculum

Instructor's Guide

This supplementary curriculum is intended to be used for 10 minutes daily to improve students' number sense. Number sense is critical for students' conceptual understanding of algebra (see accompanying Research Companion for more details).

The purpose of this guide is to explain the following aspects of the curriculum:

- The curriculum materials
- The instructional strategies
- The learning environment
- Assessment

The Curriculum Materials

Central to this curriculum are the daily problems to be used at the beginning of class. Generally, one problem should be used at the start of every class period. The problems are designed to be used in sequence because later problems build on the solution methods used for earlier problems. Also, they can be used in conjunction with lessons pertaining to certain algebra topics. For the teacher's convenience, listed next to each problem are the common student solutions. Many solutions list the connections to algebraic concepts as well.

The Instructional Strategies

When examined outside of the context of the classroom instruction, the problems included in this curriculum supplement appear elementary. In fact, if given calculators or paper and pencil, the students should be able to solve the problems with ease. It is for this reason that these problems are designed to be done mentally (meaning no calculators or paper and pencil are permitted). Each problem has an answer that is numerical in nature but it is the solution methods that are most important for the students' skill development.

Posing the Problem

The problem should be displayed horizontally[1] and remain visible to the students. Students should then be reminded that when they have arrived at a solution they should put one finger up close to their chest[2] and they should continue to see if they can get the same answer other ways (and should put a finger up for each subsequent solution). Because students are asked to find multiple ways to solve the problem, this allows for a classroom with students of mixed abilities. Also, it is important to allow enough time for

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[1] The reason that the problem should be displayed horizontally (meaning in the form: $34 + 52 =$ and not: $\begin{array}{r} 34 \\ +\ 52 \\ \hline \end{array}$) is that we want to avoid encouraging students to only use the standard algorithm.

[2] We want students to show their fingers when they have a solution so that you can assess student progress. We do not want students to treat it like a race or for them to feel pressured when other students have solutions. It is for this reason that students should be asked to hold up their fingers close to their chests.

some students to generate multiple solution methods in order to guarantee that there will be a discussion of multiple methods.

The Discussion

After students have had time to generate mental solutions to the problem, those solutions should be meaningfully shared with the class. This part is key because it is how students will learn to be more flexible with numbers. The goal of the discussion is to pull out as many different solution methods as possible and to emphasize process over result as students thoughtfully explain their methods. It is likely that the majority of students will get the "right" answer but it is worth discussing the "wrong" answers as well. Start the discussion by soliciting answers that people got and write those down. Then, ask students to explain their methods. The following list of questions might help to generate discussion:

- "Can someone who got ___ explain how they got it?"
- "Did anyone else do it like that?"
- "Did anyone do it a different way?"

As students are providing solutions, you may have to probe for more information or to encourage students to explain why they performed certain steps. Probing questions include:

- "What did you do next?"
- "Why did you do that? Tell me more."

It is important that all students pay attention to other students' solution methods. Asking comparison questions or having other students rephrase should encourage students to pay attention and make links between their solutions and those of their classmates. Also, to facilitate group understanding, the teacher should record student solution methods on the board as they are described orally. It is important for the teacher to try to avoid jumping to any conclusions about set-up or reasoning, to force students to be explicit about their intentions. After recording the method, it is a good idea for the teacher to "revoice," or point out, key strategies used. Some common strategies include:

- Break apart numbers
- Make a 10
- Make it simpler
- Rearrange the numbers
- Standard algorithm
- Start with the 10's
- Think about money
- Think about multiples
- Use decimals
- Use factors
- Use fractions
- Use inverse operation
- Use landmark numbers (25, 50, 75, 200, etc)
- Use what you already know

To facilitate the revoicing of strategies, the strategies used are listed next to every common student solution method and examples of each strategy are provided. Because we want to tie the number concepts that are being used to algebraic concepts, it is important for the teacher to summarize the students' methods by showing how students using their understanding of quantities to solve the problems (see the sample script below for examples).

Although it may not always be possible to discuss all students' possible solution methods, the more that are discussed, the better it is for the students' development of number sense. Because these types of problems and subsequent discussions may be new to the students, at the outset it may be difficult for students to think beyond the initial algorithm. Here are several suggestions for generating more solution methods:

- Allow enough time during the solving phase for students to get multiple solutions
- Provide a solution of your own as a "different way to do it" (not "the only way")
- Ask questions that might spur thoughts about different strategies once students know the names of strategies (e.g. "Can anyone think of a way to do this using landmark numbers?")

Ideally, a wide variety of solution methods will come from the students, so the last two suggestions should be used sparingly.

In addition to increasing students' flexibility with numbers, we would like to improve their reasoning skills when it comes to choosing solution methods. Although there are many different ways to solve a given problem, students should value efficiency in their choices of solution methods. For example, with a problem like 99+23, they could opt to add 90 and 20 to get 110 and then 9 and 3 to get 12 so the sum would be 110+12=122. But, instead, they could see 99 as 1 less than 100 and realize that they can just compute 100+23=123 and then subtract 1 to get 122. Although both of these solution methods are valid and demonstrate an understanding of place value, the second is clearly more efficient. One of our goals is to have students realize the value of that particular strategy for that particular problem but that the first solution method might be the most efficient for a different type of problem.

It is likely that students are somewhat intrinsically motivated to find the most efficient method, but it is helpful for the teacher to reinforce that motivation. The teacher can ask "which of these strategies are efficient for this particular problem?" and then follow that up with "why do you think that?". Although some students might make note of the efficiency of particular methods, it is good to make it explicit for the rest of the class and have several students vocalize their thoughts about the efficiency.

Several other classroom characteristics are crucial for developing student understanding through productive discussions and are discussed below. For an example of a posed problem and the ensuing discussion, see the script of the sample following this guide.

The Learning Environment- Classroom Culture

Because the focus of this curriculum is the discussion of solution methods, the learning environment must support productive discussion. With that focus in mind, there are several critical classroom norms. First, the classroom must be a safe place for students to express their solutions without fear of social or academic consequences. The teacher can work toward this goal by explicitly demanding appropriate behavior and quickly correcting inappropriate behavior. Also, the teacher can encourage students to feel comfortable making mistakes by placing value on students' mistakes. For example, the teacher can say something such as "thank you for doing it that way and sharing your solution because we got to talk about some interesting mathematics."

Second, the teacher must place the focus of the discussion on solution methods instead of numerical answers. This is accomplished by asking questions that emphasize the students' processes (see above for specific examples). Students will eventually internalize this notion and turn their attention to solution methods when they are working independently. Also, by asking about reasons for students' choices of solution methods, the discussion becomes much more conceptual in nature.

Third, solution methods should be evaluated for correctness based on mathematical justification and not verification from the teacher or the curriculum guide. As the emphasis of the discussion turns to solution methods, the classroom norm of mathematical justification will support this emphasis. In discussions, students work to explain sound mathematical reasoning in support of their solutions to enable the group to judge its correctness. This norm will support students' development of self-evaluation skills as they work independently in other aspects of their mathematical training.

Lastly, all students must be valued members of the classroom community. In mathematical discussions, all students should be encouraged to share their thinking and they should know that their peers and their teacher respect them. The teacher should make efforts to encourage all students to voluntarily share their solution methods. If the classroom is indeed a safe space for discussion, then equity of participation is more likely to be accomplished.

Assessment

First and foremost, the class discussion serves as a formative assessment of student understanding. As students share their solution methods and judge ideas mathematically, the teacher is informed regarding student knowledge and skills. Also, students can self-assess as they verbalize their solution methods or compare their methods to those of their classmates. As mentioned above, during the discussion students should compare the efficiency of different strategies to help them make informed decisions regarding strategies to use on subsequent problems.

Although the discussions do serve as formative assessments, they may not provide complete information regarding the level of understanding of every student in the class. For that reason, a pair of summary problems is provided for each problem set. These summary problems are to be used slightly differently from the other problems. They are to be worked on by pairs of students and each student will submit a written explanation of two solution methods for each summary problem. The pair of students can agree on the methods to be employed, but each student must write down those methods in a manner that fully demonstrates their understanding as well as their reasoning for choosing those methods. While pairs of students are working on the summary problems, the teacher can circulate to gather some initial data regarding student understanding. After both circulating and looking at the solutions submitted by each student, the teacher has much more information regarding individual student knowledge and skills.

Sample Problem and Discussion

Teacher: Good morning. I am going to post the problem of the day and I want you to take a few minutes to solve the problem in your head. Remember, when you have a solution to the problem, hold up one finger close to your chest like this [teacher demonstrates]. Then, try to think of another way to solve the problem and if you do, hold up another finger. You can keep trying for more solution methods until we start the discussion. I also want to remind you that you will have time to discuss your solutions with your classmates in a few minutes but for this part please be silent and you don't need anything except for your brains (no calculators, paper or pencils) for this exercise. Are there any questions? Okay, well let's get started.
[Teacher posts the problem $34 + 52$]
[Some student hands go up when they have a solution]

Teacher: Remember to put your fingers up close to your chests.
[Teacher waits until almost the entire class has a finger up and some students have more than 1 finger up]

Teacher: Okay, it looks like we are ready to talk about our solutions. First, let's just collect some answers that people got. Who's willing to share their answer?

Student 1: 86

Student 2: 85

Teacher: Any other answers out there?

Student 3: 96

Teacher: Is that all of them? Okay, now, does someone want to explain how they got their answer?

Student 3: I did 4+2=6 and 3+5=8, oh whoops, that should be 86, not 96.

Teacher: Can you explain why you added 4 and 2 together and 3 and 5 together?

Student 3: That's how you add.

Teacher: Oh, I see, you are using the standard algorithm that you would normally do on paper, in your head… Did anyone else get the same answer but do it a different way?

Student 1: My way is kinda the same, but a little different….I added 30 and 50 and got 80 and then added 4 and 2 to get 6, so total is 86.

Teacher: What do you guys think…is that the same strategy as Student 3 or a different one?

Student 4: It think it is the same, but I like Student 1's way because it makes sense and is easier to keep track of in your head.

Teacher: Why is it the same?

Student 4: Well, the 3+5 that Student 3 talked about is the 30+50 that Student 1 explained. [Teacher gesture to written work] So the Student 3's 8 is Student 1's 80 and the other numbers were the same for both of them.

Teacher: So you are saying that the 3+5 means 30+50, what does everyone else think about that?

Student 5: I think Student 4's right, the 3+5 is just a shortcut.

Teacher: Okay, what I am hearing is that they are thinking of the problem as (30+4)+(50+2)=(30+50)+(4+2)=80+6=86. Can we do that?

Student 2: Yes, you can rearrange the numbers however you want.

Teacher: Can you always do that?

Student 1: Not always, but in this case it is all addition so you can.

Teacher: Okay, lets keep thinking about this as we see other solutions. Did anyone solve the problem a different way?

Student 6: I did 34+50 because I know that's 84 and then I needed to add 2 more, so that is 86.

Teacher: Okay, now why did you add 2 more here [point to place where Student said to add 2 more]?

Student 6: Well I was supposed to add 52 but I only added 50 to start with, so I had to add 2 more.

Teacher: So, what you are saying is that you though of 52 as 50+2. So you could say that the problem looked like this for you 34+(50+2)=(34+50)+2=84+2. Can we do this?

Student 2: Yes, just like the other one, you can group them however you want when they are all being added up.

Teacher: Okay, were there any more solution methods that we haven't talked about?

Teacher: No. Okay, as we continue to do problems like this I want you to try out different solutions methods and think about which methods work best in which situations.

Sample Problem Sets with Common Solution Methods

The following problem sets include 10 problems and 1 set of summary problems to be used in sequence. For each problem, common student solution methods are described along with potential connections to algebra concepts.

Problem Set: Addition and Subtraction with Integers

1) $34 + 52$ Answer = 86

Common student solution methods:
A) 4+2, and 3+5 so 86 [STANDARD ALGORITHM]

B) 34+50 so 84 and then 2 more, so 86 total [USE LANDMARK NUMBERS]
$34 + 52 = 34 + (50 + 2) = (34 + 50) + 2 = 84 + 2$
a+(b+c)= (a+b)+c **Associative property**
Algebra example:
$2 + (3 + x) = (2 + 3) + x = 5 + x$

C) 30 + 50, and then 4 + 2 [START WITH THE 10s]
$34 + 52 = (30 + 4) + (50 + 2) = (30 + 50) + (4 + 2) = 80 + 6 = 86$
(a+b)+(c+d)= (a+c)+(b+d) = (a + b + c + d) **Commutative Property**
Algebra example:
$(2x + 3y) + (3x + 5y) = (2x + 3x) + (3y + 5y)$
$= (5x) + (8y) = 5x + 8y$

2) $34 + 8$ Answer = 42

Common student solution methods:
A) $4 + 8 = 12$ so that's 2, carry the 1, to get 42 [STANDARD ALGORITHM]

B) $34 + 8 = (30 + 4) + 8 = 30 + (4 + 8) = 30 + 12 = 42$ [BREAK APART NUMBERS]
Associative property

C) $34 + (6 + 2) = (34 + 6) + 2 = 40 + 2 = 42$ [MAKE A 10]

3) $34 + 58$ Answer = 92

Common student solution methods:
A) 4+8=12 so that's 2, carry the 1, 8+1=9 so that's 92 [STANDARD ALGORITHM]

B) $34 + 58 = 34 + (50 + 8) = (34 + 50) + 8 = 84 + 8 = 92$. [USE LANDMARK NUMBERS]
or
$34 + 58 = 34 + (50 + 8) = (34 + 50) + 8 = 84 + 8 = 84 + (6 + 2) = (84 + 6) + 2 = 90 + 2 = 92$
[BREAK APART NUMBERS, MAKE A '10']

C) $34 + 58 = 34 + (6 + 52) = (34 + 6) + 52 = 40 + 52 = 92$ We need 6 to make 40 so that leaves 52 more. We add 40 and 52 and get 92. [MAKE A '10']

D) Similar, but start with the larger number $58 + (2 + 32) = (58 + 2) + 32 = 60 + 32 = 92$ [REARRANGE THE NUMBERS, MAKE A 10]

E) We know $34 + 8 = 42$ from yesterday, so $34 + 58$ is 50 more, so 92. [USE WHAT YOU ALREADY KNOW]

F) 58 is close to 60, so let's use 60. $34 + 60 = 94$, but we added 2 too many so we get 92 [MAKE IT SIMPLER]

3) $43 + 29$ Answer = 72

Common student solution methods:
A) $3 + 9 = 12$ so that's 2, carry the 1, $4 + 2 = 6$ and add 1 so you get 72. [STANDARD ALGORITHM]

B) $(40 + 3) + (20 + 9) = (40 + 20) + (3 + 9) = 60 + 12 = 72$ [START WITH 10s]
Commutative and associative properties

C) $43 + (7 + 22) = (43 + 7) + 22 = 50 + 22 = 72$ [BREAK APART NUMBERS, MAKE A 10]

D) 29 is close to 30, so let's use 30. $43 + 30 = 73$ but we added 1 too many, so we have to subtract 1 and we get 72. [MAKE IT SIMPLER]

4) $56 + 67$ Answer = 123

Common student solution methods:
A) $6 + 7 = 13$, so that is 3 carry the 1, and $5 + 6 = 11$ so that's 12, for a total of 123. [STANDARD ALGORITHM]

B) $(50 + 6) + (60 + 7) = (50 + 60) + (6 + 7) = 110 + 13 = 110 + 10 + 3 = 120 + 3 = 123$
[START WITH 10s, BREAK APART NUMBERS]
Commutative and associative properties

C) $56 + (4 + 63) = (56 + 4) + 63 = 60 + 63 = (60 + 60) + 3 = 123$
[BREAK APART NUMBERS, MAKE A 10]

D) Similar, but starting with larger number [REARRANGE THE NUMBERS, BREAK APART NUMBERS, MAKE A 10]
$67 + (3 + 53) = (67 + 3) + 53 = 70 + 53 = 70 + (50 + 3) = 120 + 3 = 123$

E) 67 is close to 70, so let's use 70, $56 + 70 = 126$, but we added 3 too many, so we have to subtract 3 and get 123. [MAKE IT SIMPLER]

5) $99 + 23$ Answer = 122

Common student solution methods:
A) $9 + 3 = 12$ so that's 2, carry the 1, and $9 + 2 = 11$ and 1 more makes 12, so we get 122. [STANDARD ALGORITHM]

B) $(90 + 9) + (20 + 3) = (90 + 20) + (9 + 3) = 110 + 12 = 122$ [START WITH 10s] Commutative and associative properties

C) 99 is almost 100 and $100 + 23 = 123$, but we added 1 too many so we get 122. [MAKE IT SIMPLER]

D) $99 + (1 + 22) = (99 + 1) + 22 = 100 + 22 = 122$ [BREAK APART NUMBERS, MAKE A 10]

E) Almost like a dollar and a quarter, which would be $1.25. It is 3 less (1 for the 99 and 2 for the 23) so that is 123. [THINK ABOUT MONEY]

6) $36 + 49$ Answer = 85

Common student solution methods:
A) $6 + 9 = 15$ so that's 5, carry the 1. $3 + 4 = 7$ and 1 more is 8. So that's 85. [STANDARD ALGORITHM]

B) $(30 + 6) + (40 + 9) = (30 + 40) + (6 + 9) = 70 + 15 = 85$ [START WITH 10s] Commutative and associative properties

C) $36 + (4 + 45) = (36 + 4) + 45 = 40 + 45 = 85$ [BREAK APART NUMBERS, MAKE A 10]

D) $49 + (1 + 35) = (49 + 1) + 35 = 50 + 35 = 85$ [REARRANGE THE NUMBERS, BREAK APART NUMBERS, MAKE A 10]

E) 49 is close to 50 and $36 + 50 = 86$. But 50 is one more than 49 so we have to subtract 1, and we get 85. [MAKE IT SIMPLER]

7) $68 - 48$ Answer = 20

Common student solution methods:

A) $8-8=0$ and $6-4=2$, so we get 20. [STANDARD ALGORITHM]

B) $(60-40)+(8-8)=20$ [START WITH 10s]

C) $68-(8+40)=(68-8)-40=60-40=20$ [BREAK APART NUMBERS]

D) We can add a 2 to both numbers and get $70-50=20$. This is a good place to use a number line and see that this difference between the two numbers stays the same when you shift them by the same amount (in this case 2). [MAKE IT SIMPLER]
This is related to the notion of adding and subtracting a number on one side of the equation to keep it equal to the same amount.
$68-48=70-50=(68+2)-(48+2)=68+2-48-2=(68-48)+2-2$
Distributive property (explore in greater depth in the next problem)
For example: Completing the square.
$x^2+6x+1 \quad x^2+6x\underline{+9}+1\underline{-9} \quad (x+3)^2-8$

E) Think about how you get from 48 to 68. It takes 2 to get from 48 to 50, and 18 more to get to 68 so that is 20 total. [USE INVERSE OPERATION]

8) 85-32 Answer = 83

Common student solution methods:
A) $5-2=3$ and $8-3=5$so 53. [STANDARD ALGORITHM]

B) $85-32=85-(30+2)=85-30-2=55-2=53$ [USE LANDMARK NUMBERS]
This has a nice connection to the **distributive property** and distributing the minus sign (we want to subtract both things, not just one)
Might even want to look at what it means arithmetically if you don't distribute:
$85-32=85-(30+2)=85-30+2=55+2=57$
C) $85-32=(80+5)-(30+2)=(80-30)\overset{?}{\pm}(5\overset{?}{\pm}2)$ [START WITH THE 10s]
Interesting discussion because it makes sense conceptually, but is difficult when you try to put it down symbolically. Start with what makes sense conceptually: you want to subtract the 30 from 80 and the 2 from the 5 and then combine (which means add, this might be a place to question)
$85-32=(80+5)-(30+2)=(80-30)+(5-2)=50+3=53$
Distributive property
Algebra example:
$(5x+3y)-(2x+4y)=(5x+3y)-2x-4y$
$=(5x-2x)+(3y-4y)=3x-y$

9) 74-48 Answer = 26

Common student solution methods:

A) $4-8=4$ and $7-4=3$ so 34. Error!

B) Try to do $4-8$, but that doesn't work, so we have to borrow and get $14-8$ which is 6. Then we do $6-4=2$ so 26. [STANDARD ALGORITHM]

C) $74-48=74-(40+8)=74-40-8=34-8=34-4-4=30-4=26$
Distributive property [BREAK APART NUMBERS, LANDMARK NUMBERS, MAKE A 10]

D) $74-48=(70+4)-(40+8)=(70-40)+(4-8)=30+(-4)=26$ [BREAK NUMBERS APART, START WITH 10s]

E) Or slightly different: $74-48=(60+14)-(40+8)=(60-40)+(14-8)=20+6=26$
Why do it this way? You can't take 8 from 4, so we need that 4 to be 14 (but if we take 14 away from 74 we just have 60 left) [START WITH 10s]

F) To get from 48 to 50 is 2 units. Then from 50 to 74 is another 24 so that's a total of $24+2=26$ [LANDMARK NUMBERS, USE INVERSE OPERATION]

10) $101-25$ Answer = 76

Common student solution methods:
A) $1-5$ doesn't work so we have to borrow and we get $11-5=6$. That leaves us with $9-2=7$, so the answer is 76. [STANDARD ALGORITHM]

B) Like above, but actually dealing with quantities. So we have 101 and we can break that up as $90+11$ and subtract $20+5$, so we get $(90-20)+(11-5)=70+6=76$. [BREAK NUMBERS UP, START WITH THE 10s]

C) $100-25=75$ so we need to add 1 and that is 76. [USE LANDMARK NUMBERS, USE WHAT YOU ALREADY KNOW]

D) That's almost like one dollar minus a quarter so that would be 3 quarters or 75 cents. But we need to add 1 cent so we get 76. [THINK ABOUT MONEY]

Summary Problems:
Give two different solution methods to solve each of the following problems. You can work with your partner but you each need to write up and turn in an explanation of the solution methods you use and why you decided to use them.

1) $51+74$

2) $83-26$

Problem Set: Multiplication and Division with Integers

1) $34 \cdot 3$ Answer = 102

Common student solution methods:
A) $4 \cdot 3 = 12$ so that's 2, carry the 1. $3 \cdot 3 = 9$ and 1 more makes 10, so 102. [STANDARD ALGORITHM]

B) $(30 + 4) \cdot 3 = 3 \cdot (30 + 4) = 3 \cdot 30 + 3 \cdot 4 = 90 + 12 = 102$ [BREAK APART NUMBERS]
Distributive property
$a(b + c) = a \cdot b + a \cdot c$
Algebra example:
$2(x + 3y)$

$2x + 6y$

C) $34 + 34 + 34 = 68 + 34 = (60 + 8) + (30 + 4) = (60 + 30) + (8 + 4) = 90 + 12 = 102$
[DEFINITION]
Definition of Multiplication
$a \cdot b = b + b + b + b ... + b$ (a times)
Algebra example:
$5x = x + x + x + x + x$

D) Break it up into numbers that the multiplication facts are easy to remember
$(20 + 12 + 2) \cdot 3 = (20 \cdot 3) + (12 \cdot 3) + (2 \cdot 3) = 60 + 36 + 6 = (60 + 30) + (6 + 6) = 90 + 12 = 102$
Distributive property

2) $24 \cdot 6$ Answer = 144

Common student solution methods:
A) $4 \cdot 6 = 24$ so that's 4, carry the 2. $2 \cdot 6 = 12$ and 2 more is 14, so 144. [STANDARD ALGORITHM]

B) $(20 + 4) \cdot 6 = 20 \cdot 6 + 4 \cdot 6 = 120 + 24 = (120 + 20) + 4 = 140 + 4 = 144$ [BREAK APART NUMBERS]
Distributive property

C) $(12 \cdot 2) \cdot 6 = 12 \cdot (2 \cdot 6) = 12 \cdot 12 = 144$ [USE FACTORS, USE WHAT YOU ALREADY KNOW]
Associative property of multiplication
Algebra example:
$2(3x) = (2 \cdot 3)x = 6x$

$z(yz) = z(zy) = (z \cdot z) \cdot y = z^2 y = yz^2$

D) $(12 + 12) \cdot 6 = 12 \cdot 6 + 12 \cdot 6 = 72 + 72 = (70 + 2) + (70 + 2) = 140 + 4 = 144$ [USE FACTORS, USE WHAT YOU ALREADY KNOW]

Distributive property

E) 24 is close to 25. So, 25 quarters makes \$1.50. But, I did 1 too many, six times, so that's $150-6=144$. [THINK ABOUT MONEY]

3) $16\cdot 5$ Answer = 80

Common student solution methods:
A) $6\cdot 5=30$ so 0, carry the 3, and $1\cdot 5=5$and 3 more is 8, so 80. [STANDARD ALGORITHM]

B) $(10+6)\cdot 5=10\cdot 5+6\cdot 5=50+30=80$ [BREAK APART NUMBERS]
Distributive property

C) $(8\cdot 2)\cdot 5=8\cdot(2\cdot 5)=8\cdot 10=80$ or $(2\cdot 8)\cdot 5=2\cdot(8\cdot 5)=2\cdot 40=80$
[USE FACTORS]
Associative Property of Multiplication

D) $10\cdot 16=160$ so half of that is 80. [MAKE IT SIMPLER, USE DOUBLES]

4) $34\cdot 4$ Answer = 136

Common student solution methods:
A) $4\cdot 4=16$, so 6 carry the 1, and $3\cdot 4=12$ plus 1 is 13, so 136 [STANDARD ALGORITHM]

B) $(30+4)\cdot 4=30\cdot 4+4\cdot 4=120+16=136$ [BREAK APART NUMBERS]
Distributive property

C) $34\cdot(2\cdot 2)=(34\cdot 2)\cdot 2=68\cdot 2=(60+8)\cdot 2=60\cdot 2+8\cdot 2=120+16=136$ [USE FACTORS, BREAK APART NUMBERS]

D)
$(17\cdot 2)\cdot 4=17\cdot(2\cdot 4)=17\cdot 8=(10+7)\cdot 8=80+56=80+(50+6)=(80+50)+6=130+6=136$
[USE FACTORS, BREAK APART NUMBERS]
Distributive property

5) $34\cdot 14$ Answer = 476

Common student solution methods:
A) $4\cdot 4=16$, so 6 carry the 1, and $3\cdot 4=12$ so 13, so 136. Then, for the next row, $1\cdot 4=4$ and $1\cdot 3=3$ so 34. So 476 total. [STANDARD ALGORITHM]

B) $(34)\cdot(10+4)=34\cdot 10+34\cdot 4=340+136=476$ [BREAK APART NUMBERS, USE WHAT YOU ALREADY KNOW]
Distributive property
Connection to FOIL:
$(30+4)(10+4)=30\cdot 10+30\cdot 4+4\cdot 10+4\cdot 4=300+120+40+16=476$
Algebra example:
$(x+2)(x-3)$
$x\cdot x-3\cdot x+2\cdot x-2\cdot 3$
$x^2-3x+2x-6$
x^2-x-6

C)
$(17\cdot 2)(2\cdot 7)=(17\cdot 7)(2\cdot 2)=(10+7)\cdot 7\cdot 4=(10\cdot 7+7\cdot 7)\cdot 4=(70+49)\cdot 4$
$=70\cdot 4+49\cdot 4=280+(200-4)=(280+200)-4=480-4=476$
[USE FACTORS, BREAK APART NUMBERS, MAKE IT SIMPLER]
Distributive property

6) $34\cdot 15$ Answer = 510

Common student solution methods:
A) $4\cdot 5=20$, so 0, carry the 2, and $5\cdot 3=15$ plus 2 is 170. $1\cdot 4=4$ and $1\cdot 3=3$ so we add 7 and 4 and get 11, so carry the 1 and, 1 and 3 and 1 more is 5 so we get 510.
[STANDARD ALGORITHM]

B) We did $34\cdot 14$ and that was 476 so we need to add 34:
$476+34=476+(24+10)=(476+24)+10=500+10=510$ [USE WHAT YOU ALREADY KNOW, MAKE A 100]
Distributive property: $34\cdot 14+34\cdot 1=34(14+1)=34\cdot 15$
Algebra example, factoring:
$5x+10y=5\cdot x+5\cdot 2y=5(x+2y)$

C)
$34\cdot(10+5)=34\cdot 10+34\cdot 5=34\cdot 10+\frac{1}{2}(34\cdot 10)=340+\frac{1}{2}(340)=$
$340+170=(300+100)+(40+70)=400+110=510$
[BREAK APART NUMBERS, USE WHAT YOU ALREADY KNOW]

D) $(17\cdot 2)\cdot 15=17\cdot 30=(10+7)\cdot 30=10\cdot 30+7\cdot 30=300+210=510$
[USE FACTORS, BREAK APART NUMBERS]

E) $34\cdot 30=(30+4)\cdot 30=(30\cdot 30)+(4\cdot 30)=900+120=1020$ but 30 is 2 times 15, so
$34\cdot 15=\frac{1}{2}(1020)=\frac{1}{2}(1000+20)=\frac{1}{2}\cdot 1000+\frac{1}{2}\cdot 20=500+10=510$
[MAKE IT SIMPLER, USE DOUBLES]

7) $28 \div 8$ Answer = $\frac{7}{2} = 3\frac{1}{2}$

Common student solution methods:
A) 8 goes into 28 3 times because $3 \cdot 8 = 24$ and there is 4 left over so that is $3\frac{4}{8} = 3\frac{1}{2}$ [STANDARD ALGORITHM]

B) $8 \cdot 3 = 24$ so that leaves 4 extra, so we get $3\frac{4}{8} = 3\frac{1}{2}$ [USE INVERSE OPERATION]

C) $28 = 4 \cdot 7 = 2 \cdot 2 \cdot 7$ and $8 = 4 \cdot 2 = 2 \cdot 2 \cdot 2$ so they both have two 2s so we get $\frac{7}{2}$. [USE FACTORS]
Algebra example:
$$\frac{x^2 + 3x - 4}{x^2 - 16} = \frac{(x-1)(x+4)}{(x-4)(x+4)} = \frac{x-1}{x-4}$$

D) $8 + 8 = 16; 16 + 8 = 24$ so that was 3 8s and we were 4 short so that is a half of an 8, so the answer is $3\frac{1}{2}$. [THINK ABOUT MULTIPLES]

8) $300 \div 25$ Answer = 12

Common student solution methods:
A) 25 goes into 30 1 time with 5 left over. Bring down the 0 to make 50. 25 goes into 50 2 times exactly. [STANDARD ALGORITHM]

B) $25 \cdot 4 = 100$ so 3 times that is 300, so $3 \cdot 4 = 12$. [USE INVERSE OPERATION, USE WHAT YOU ALREADY KNOW]

C) 4 quarters makes a dollar, so 12 quarters is $3.00. So the answer is 12. [THINK ABOUT MONEY]

D) $300 = 3 \cdot 100 = 3 \cdot 10 \cdot 10 = 3 \cdot 2 \cdot 2 \cdot 5 \cdot 5$ and $25 = 5 \cdot 5$ so that leaves $3 \cdot 2 \cdot 2 = 12$ [USE FACTORS]

9) $75 \cdot 80$ Answer = 6000

Common student solution methods:
A) $0 \cdot 75 = 0$, and $8 \cdot 5 = 40$ so that is 0 carry the 4. $8 \cdot 7 = 56$ plus 4 more is 60. So 6000 is the answer. [STANDARD ALGORITHM]

B) $75 \cdot 8 = (70 + 5) \cdot 8 = 70 \cdot 8 + 5 \cdot 8 = 560 + 40 = 600$. So 6000. [USE WHAT YOU ALREADY KNOW, BREAK APART NUMBERS]

C) $75 \cdot 80 = (3 \cdot 25)(4 \cdot 20) = (3 \cdot 20)(25 \cdot 4) = 60 \cdot 100 = 6000$ [USE FACTORS]

D) 75 cents is 3 quarters so 80 times that is 240 quarters, which is 60 dollars (because that's divided by 4). So that makes 6000. [THINK ABOUT MONEY]

E) $.75 \cdot 80 = \frac{3}{4} \cdot 80 = 3 \cdot \left(\frac{1}{4} \cdot 80\right) = 3 \cdot 20 = 60$ so the answer is 6000. [USE FRACTIONS]

10) $24 \cdot 19$ Answer = 456

A) $4 \cdot 9 = 36$ so that's 6 carry the 3. $2 \cdot 9 = 18$ and 3 more is 21. Now, on to the next row: $1 \cdot 4 = 4$ and $1 \cdot 2 = 2$ so that's $2 + 2 = 4$ and $4 + 1 = 5$ so that's 456 total. [STANDARD ALGORITHM]

B) $24 \cdot 20 = 480$ but that means that 24 was counted twenty times and not just 19, so it should be 24 less $480 - 24 = 480 - (20 + 4) = 480 - 20 - 4 = 460 - 4 = 456$ [MAKE IT SIMPLER] **Distributive property**

C) 24 is close to 25 so we can think of that as a quarter. So, how much money do we have with 19 quarters? 20 quarters would make $5.00. But we need 1 less so that's $4.75, or 475. But, we don't actually have 25, we have 24, so that means we added 19 too many. 20 less would be 455, so 19 less is 456. [THINK ABOUT MONEY]

Summary Problems:
Give two different solution methods to solve each of the following problems. You can work with your partner but you each need to write up and turn in an explanation of the solution methods you use and why you decided to use them.

1) $26 \cdot 8$

2) $48 \div 28$

Problem Set: Multiplication and Division with Fractions and Decimals

1) $4 \cdot \frac{3}{2}$ Answer = 6

Common student solution methods:
A) We can think of 4 as $\frac{4}{1}$ so we just multiply across and get $\frac{12}{2} = 6$. [STANDARD ALGORITHM]

B) This is the same as $4 \cdot \frac{1}{2} \cdot 3 = (4 \cdot \frac{1}{2}) \cdot 3 = 2 \cdot 3 = 6$ [BREAK APART NUMBERS]
Commutative and Associative Properties of Multiplication
Algebra example:
$xz \cdot \frac{y}{x^2} = z \cdot x \cdot \left(\frac{1}{x} \cdot \frac{1}{x} \cdot y\right) = z \cdot \left(x \cdot \frac{1}{x} \cdot \frac{1}{x}\right) \cdot y = z \cdot y \cdot \frac{1}{x} = \frac{yz}{x}$

C) $4 \cdot 1.5 = 4 \cdot (1 + 0.5) = 4 \cdot 1 + 4 \cdot 0.5 = 4 + 4 \cdot \frac{1}{2} = 4 + 2 = 6$ [USE DECIMALS]

2) $24 \cdot \frac{5}{18}$ Answer = $\frac{20}{3}$

Common student solution methods:
A) We can think of 24 as $\frac{24}{1}$ so we just multiply across and get $\frac{24}{1} \cdot \frac{5}{18}$. $4 \cdot 5 = 20$ so that's 0 carry the 2 and $2 \cdot 5 = 10$ and 2 more is 12 so we get 120. So $\frac{24}{1} \cdot \frac{5}{18} = \frac{120}{18}$. 2 goes into both of them so we get $\frac{60}{9}$ and 3 goes into both of those, so $\frac{20}{3}$ is the reduced answer. [STANDARD ALGORITHM]

B) $(4 \cdot 6) \cdot \left(\frac{5}{3 \cdot 6}\right) = (4 \cdot 6) \cdot \left(\frac{1}{6} \cdot \frac{5}{3}\right) = \left(4 \cdot \frac{5}{3}\right) \cdot \left(6 \cdot \frac{1}{6}\right) = \frac{4 \cdot 5}{3} = \frac{20}{3}$. [USE FACTORS]
Commutative and Associative Properties of Multiplication

C) 24 and 18 are both divisible by 2 so we can reduce those to get $12 \cdot \frac{5}{9}$. Then, 12 and 9 are both divisible by 3 so we get $4 \cdot \frac{5}{3} = \frac{4}{1} \cdot \frac{5}{3} = \frac{20}{3}$. [USE FACTORS]

3) $18 \cdot 5 \cdot \frac{1}{3}$ Answer = 30

Common student solution methods:
A) Multiply across and compute $18 \cdot 5$. So $5 \cdot 8 = 40$ so that is 0, carry the 4. Next $5 \cdot 1 = 5$ and 4 more is 9. So we get 90 on the top and 3 on the bottom. $\frac{90}{3} = 30$. [STANDARD ALGORITHM]

B) $(18 \cdot \frac{1}{3}) \cdot 5 = 6 \cdot 5 = 30$ [REARRANGE THE NUMBERS]
Commutative and Associative Properties of Multiplication
Algebra example:
$6x \cdot 19y \cdot \frac{1}{3} = \left(6 \cdot \frac{1}{3}\right) \cdot x \cdot 19y = 2 \cdot 19xy = 38xy$

C) $\frac{18}{1}\cdot\frac{5}{1}\cdot\frac{1}{3}=\frac{18\cdot 5}{3}$ and 18 and 3 are both divisible by 3 so we get $\frac{6\cdot 5}{1}=30$. [USE FACTORS]

4) $36\cdot 0.5$ Answer = 18

Common student solution methods:
A) $5\cdot 6=30$ so that is 0 carry the 3. $3\cdot 5=15$ and 3 more is 18 so we get 180. But there was a decimal so we move the decimal over one place and get 18. [STANDARD ALGORITHM]

B) $(30+6)\cdot 0.5=30\cdot 0.5+6\cdot 0.5$. $30\cdot 5=150$ so $30\cdot 0.5$ is 15 and $6\cdot 5=30$ so $6\cdot 0.5=3$. So the answer is 15+3=18. [BREAK APART NUMBERS, USE WHAT YOU ALREADY KNOW]

C) 0.5 is $\frac{1}{2}$ so we just have to find half of 36. Half of 30 is 15 and half of 6 is 3 so 15+3=18. [USE FRACTIONS, BREAK APART NUMBERS, USE WHAT YOU ALREADY KNOW]

5) $36\cdot 0.25$ Answer = 9

Common student solution methods:
A) $5\cdot 6=30$ so that is 0 carry the 3 and $5\cdot 3=15$ and 3 more is 18, so 180. $2\cdot 6=12$ so that's 2 carry the 1 and $2\cdot 3=6$ and 1 more is 7, so 72. So we get 0, and 0 and 9, which is 900. But there are 2 decimal places so the answer is 900. [STANDARD ALGORITHM]

B) $36\cdot\frac{1}{4}=9$ [USE FRACTIONS]

C) 0.25 is a quarter so if I have 36 quarters that is $9. So the answer is 9. [THINK ABOUT MONEY]

6) $60\cdot 1.5$ Answer = 90

Common student solution methods:
A) $5\cdot 0=0$ and $5\cdot 6=30$ so that is 300. $1\cdot 0=0$ and $1\cdot 6=6$ so that is 60. So we get 900 but there is 1 decimal place so we get 90. [STANDARD ALGORITHM]

B) 60 time $1.50. Well, 60 times $1 is $60 and 60 times $0.50 (half a dollar) is $30. So we get $60+$30=$90. So the answer is 90. [THINK ABOUT MONEY, BREAK APART NUMBERS]

C) $60\cdot(1+0.5)=(60\cdot 1)+\left(60\cdot\frac{1}{2}\right)=60+30=90$ [BREAK APART NUMBERS, USE FRACTIONS]

Distributive Property

D) $60 \cdot \frac{3}{2} = 60 \cdot \left(\frac{1}{2} \cdot 3\right) = \left(60 \cdot \frac{1}{2}\right) \cdot 3 = 30 \cdot 3 = 90$ [USE FRACTIONS]
Commutative and Associative Properties of Multiplication

7) $60 \div 1.5$ Answer = 40

Common student solution methods:
A) Think of this as 600 divided by 15. 15 goes into 60 4 times and then 15 goes into 0, 0 times so we get 40. [MAKE IT SIMPLER, STANDARD ALGORITHM]
Multiplicative Inverse: We multiplied and divided by 10
$(60 \div 1.5) \cdot (10 \div 10) = (60 \cdot 10) \div (1.5 \cdot 10) = 600 \div 15$ or it might be easier to visualize this way: $\frac{60}{1.5} \cdot \frac{10}{10} = \frac{600}{15}$
Algebra Example:
$$\frac{\frac{1}{2}x^2}{\frac{1}{4}y^3} \cdot \frac{4}{4} = \frac{2x^2}{y^3}$$

B) 1.5 is the same as $\frac{3}{2}$ so we have $60 \div \frac{3}{2} = 60 \cdot \frac{2}{3} = \frac{60}{3} \cdot 2 = 20 \cdot 2 = 40$ [USE FRACTIONS]
Algebra Fraction Division Example:
$$xy^2 \div \frac{y}{x^2} = xy^2 \cdot \frac{x^2}{y} = \left(x \cdot x^2\right) \cdot \left(y^2 \cdot \frac{1}{y}\right) = x^3 \cdot y = x^3 y$$

C) \$60 divided by \$1.50. Well I know that 10 times \$1.50 is \$15. Now, I just have to figure out how many \$15 go into \$60. $15 \cdot 2 = 30$ so $15 \cdot 4 = 60$. So that means that the answer is 4 times 10 which is 40. [THINK ABOUT MONEY, USE INVERSE OPERATION, BREAK APART NUMBERS]

D) $\frac{60}{1.5} = \frac{600}{15} = \frac{200}{5} = \frac{40}{1}$ [MAKE IT SIMPLER, USE FRACTIONS, USE FACTORS]

8) $60 \div 1.2$ Answer = 50

Common student solution methods:
A) Think of this as 600 divided by 12. 12 goes into 60 5 times and then 12 goes into 0, 0 times so we get 50. [MAKE IT SIMPLER, STANDARD ALGORITHM]

B) 1.2 is the same as $\frac{12}{10} = \frac{6}{5}$ so we get $60 \div \frac{6}{5} = 60 \cdot \frac{5}{6} = \frac{60}{6} \cdot 5 = 10 \cdot 5 = 50$ [USE FRACTIONS]

C) $\frac{60}{1.2} = \frac{600}{12} = \frac{100}{2} = \frac{50}{1} = 50$ [MAKE IT SIMPLER, STANDARD ALGORITHM]

9) $52 \cdot 2.1$ Answer = 109.2

Common student solution methods:
A) $1 \cdot 2 = 2$ and $1 \cdot 5 = 5$ so that is 52. $2 \cdot 2 = 4$ and $2 \cdot 5 = 10$ so we have 2 and then we add 5 and 4 to get 9. So our result is 1092, but we need to move the decimal place to get 109.2. [STANDARD ALGORITHM]

B) $52 \cdot (2 + 0.1) = 52 \cdot 2 + 52 \cdot 0.1 = 104 + 5.2 = 109.2$ [BREAK APART NUMBERS]
Distributive property

C) $52 \cdot \frac{21}{10}$ and 52 and 10 are both divisible by 2. So we get $26 \cdot \frac{21}{5}$. 26 is close to 25 so $25 \cdot 21 = 25 \cdot (20 + 1) = 25 \cdot 20 + 25 = 500 + 25 = 525$ but I need to add 21 1s so that is $525 + 21 = 525 + (20 + 1) = (525 + 20) + 1 = 545 + 1 = 546$. So we have $\frac{546}{5}$. [USE FRACTIONS, MAKE IT SIMPLER]

10) $14.7 \div \frac{7}{3}$ Answer = 6.3

Common student solution methods:
A) $14.7 \cdot \frac{3}{7} = \frac{14.7 \cdot 3}{7}$. $14.7 \cdot 3$: $7 \cdot 3 = 21$ so that is 1, carry the 2 and then $3 \cdot 4 = 12$ and 2 more is 14 so that is 4, carry the 1. $3 \cdot 1 = 3$ and 1 more is 4 so we get 44.1. Now we have to divide that by 7. 7 goes into 44, 6 times with 2 left over. Bring down the 1. 7 goes into 21 3 times so we get 6.3. [STANDARD ALGORITHM]

B)
$$14.7 \div \frac{7}{3} = 14.7 \cdot \frac{3}{7} = (14 + 0.7) \cdot \frac{3}{7} = \left(14 \cdot \frac{3}{7}\right) + \left(0.7 \cdot \frac{3}{7}\right) = \left(\frac{14}{7} \cdot 3\right) + \left(\frac{7}{10} \cdot \frac{3}{7}\right) = 2 \cdot 3 + \left(\frac{7}{7} \cdot \frac{3}{10}\right)$$
$$= 6 + \frac{3}{10} = 6.3$$
[BREAK APART NUMBERS]
Algebra example:
$$\left(3x + 2x^2\right) \cdot \frac{1}{x^3} = 3x \cdot \frac{1}{x^3} + 2x^2 \cdot \frac{1}{x^3} = \frac{3x}{x^3} + \frac{2x^2}{x^3} = \frac{3}{x^2} + \frac{2}{x}$$

C) $\frac{147}{10} \div \frac{7}{3} = \frac{147}{10} \cdot \frac{3}{7} = \frac{147}{7} \cdot \frac{3}{10} = \frac{(140+7)}{7} \cdot \frac{3}{10} = \left(\frac{140}{7} + \frac{7}{7}\right) \cdot \frac{3}{10} = (20 + 1) \cdot \frac{3}{10} = (21) \cdot \frac{3}{10} = \frac{21 \cdot 3}{10} = \frac{63}{10} = 6.3$
[USE FRACTIONS]

Summary Problems:
Give two different solution methods to solve each of the following problems. You can work with your partner but you each need to write up and turn in an explanation of the solution methods you use and why you decided to use them.

1) $\frac{2}{3} \cdot 8 \cdot 15$

2) $12 \div 0.75$

Sample Curricular Alignment
Algebra 1: Glencoe Mathematics

Problem Set: Addition and Subtraction of Integers

Related Algebra 1 Topic	Example of Problem and Solution with Algebra Connection
Ch. 1: The Language of Algebra 1-6: Commutative and Associative Properties	Problem: $34+52$ Solution: 34+50 so 84 and then 2 more, so 86 total $34+52=34+(50+2)=(34+50)+2=84+2$ Connection: Associative Property $2+(3+x)=(2+3)+x=5+x$
Ch. 2: Real Numbers 2-2: Adding and Subtracting Rational Numbers	Problem: $56+67$ Solution: $56+(4+63)=(56+4)+63=60+63$ $=(60+60)+3=123$ Connection: Simplify $-12-34=-12-(30+4)$ $=(-12-30)-4$ $=-42-4=-46$
Ch. 3: Solving Linear Equations 3-4: Solving Multi-Step Equations	Problem: $43+29$ Solution: $(40+3)+(20+9)=(40+20)+(3+9)$ $=60+12=72$ Connection: Solve $(2x+3)+(5x-4)=1$ $2x+3+5x-4=1$ $2x+5x+3-4=1$ $(2x+5x)+(3-4)=1$ $7x-1=1$
Ch. 6: Solving Linear Inequalities 6-3: Solving Multi-Step Inequalities	Problem: $56+67$ Solution: $(50+6)+(60+7)=(50+60)+(6+7)=110+13$ $=110+10+3=120+3=123$ Connection: Solve $3+(2x-1)<5$ $3+(-1+2x)<5$ $(3+-1)+2x<5$ $2+2x<5$
Ch. 8: Polynomials 8-4: Adding and Subtracting Polynomials	Problem: $36+49$ Solution: $(30+6)+(40+9)=(30+40)+(6+9)$ $=70+15=85$ Connection: Simplify $(6x^2+1)+(4x^2-3)=(6x^2+4x^2)+(1-3)$ $=10x^2-2$

Problem Set: Multiplication and Division of Integers

Related Algebra 1 Topic	Example of Problem and Solution with Algebra Connection
Ch. 1: The Language of Algebra 1-5: The Distributive Property 1-6: Commutative and Associative Properties	Problem: $34 \cdot 3$ Solution: $(30+4)\cdot 3 = 3\cdot(30+4) = 3\cdot 30 + 3\cdot 4$ $= 90 + 12 = 102$ Connection: Distributive Property $2(x+3y) = 2x+6y$
Ch. 2: Real Numbers 2-3: Multiplying Rational Numbers 2-3: Dividing Rational Numbers	Problem: $24 \cdot 6$ Solution: $(12\cdot 2)\cdot 6 = 12\cdot(2\cdot 6) = 12\cdot 12 = 144$ Connection: $-3\cdot -6 = (-1\cdot 3)(-1\cdot 6)$ $= (-1\cdot -1)(3\cdot 6) = 18$
Ch. 3: Solving Linear Equations 3-4: Solving Multi-Step Equations	Problem: $16\cdot 5$ Solution: $(10+6)\cdot 5 = 10\cdot 5 + 6\cdot 5 = 50 + 30 = 80$ Connection: $2(3x-1) = 5$ $6x - 2 = 5$
Ch. 6: Solving Linear Inequalities 6-3: Solving Multi-Step Inequalities	Problem: $34\cdot 4$ Solution: $(30+4)\cdot 4 = 30\cdot 4 + 4\cdot 4 = 120 + 16 = 136$ Connection: $4(x-3) > 5$ $4x - 12 > 5$
Ch. 8: Polynomials 8-1: Multiplying Monomials 8-2: Dividing Monomials 8-6: Multiplying a Polynomial by a Monomial 8-7: Multiplying Polynomials	Problem: $34\cdot 14$ Solution: $(30+4)(10+4) = 30\cdot 10 + 30\cdot 4 + 4\cdot 10 + 4\cdot 4$ $= 300 + 120 + 40 + 16 = 476$ Connection: $(x+2)(x-3) = x\cdot x - 3\cdot x + 2\cdot x - 2\cdot 3$ $= x^2 - 3x + 2x - 6 = x^2 - x - 6$
Ch. 9: Factoring 9-1: Factors and Greatest Common Factors 9-2: Factoring Using the Distributive Property	Problem: $75\cdot 80$ Solution: $75\cdot 80 = (3\cdot 25)(4\cdot 20) = (3\cdot 20)(25\cdot 4)$ $= 60\cdot 100 = 6000$ Connection: $2x + 4y = 2(x+2y)$
Ch. 12: Rational Expressions and Equations 12-2: Rational Expressions 12-3: Multiplying Rational Expressions	Problem: $28 \div 8$ Solution: $28 = 4\cdot 7 = 2\cdot 2\cdot 7$ and $8 = 4\cdot 2 = 2\cdot 2\cdot 2$ so they both have two 2s so we get $\frac{7}{2}$. Connection: $\frac{x^2+3x-4}{x^2-16} = \frac{(x-1)(x+4)}{(x-4)(x+4)} = \frac{x-1}{x-4}$

Problem Set: Multiplication and Division of Fractions and Decimals

Related Algebra 1 Topic	Example of Problem and Solution with Algebra Connection
Ch. 1: The Language of Algebra 1-5: The Distributive Property 1-6: Commutative and Associative Properties	Problem: $4\cdot\frac{3}{2}$ Solution: $4\cdot\frac{1}{2}\cdot 3=(4\cdot\frac{1}{2})\cdot 3=2\cdot 3=6$ Connection: $xz\cdot\frac{y}{x^2}=z\cdot x\cdot\left(\frac{1}{x}\cdot\frac{1}{x}\cdot y\right)=z\cdot\left(x\cdot\frac{1}{x}\cdot\frac{1}{x}\right)\cdot y=z\cdot y\cdot\frac{1}{x}=\frac{yz}{x}$
Ch. 3: Solving Linear Equations 3-4: Solving Multi-Step Equations	Problem: $60\cdot 1.5$ Solution: $60\cdot(1+0.5)=(60\cdot 1)+\left(60\cdot\frac{1}{2}\right)=60+30=90$ Connection: $6(x-2)=12$ $6x-12=12$
Ch. 6: Solving Linear Inequalities 6-3: Solving Multi-Step Inequalities	Problem: $52\cdot 2.1$ Solution: $52\cdot(2+0.1)=52\cdot 2+52\cdot 0.1=104+5.2=109.2$ Connection: $\frac{1}{2}\left(x-4\right)\le 6$ $\frac{1}{2}x-2\le 6$
Ch. 8: Polynomials 8-1: Multiplying Monomials 8-2: Dividing Monomials 8-6: Multiplying a Polynomial by a Monomial 8-7: Multiplying Polynomials	Problem: $14.7\div\frac{7}{3}$ $14.7\div\frac{7}{3}=14.7\cdot\frac{3}{7}=(14+0.7)\cdot\frac{3}{7}$ Solution: $=\left(14\cdot\frac{3}{7}\right)+\left(0.7\cdot\frac{3}{7}\right)=\left(\frac{14}{7}\cdot 3\right)+\left(\frac{7}{10}\cdot\frac{3}{7}\right)$ $=2\cdot 3+\left(\frac{7}{7}\cdot\frac{3}{10}\right)=6+\frac{3}{10}=6.3$ Connection: $\left(3x+2x^2\right)\cdot\frac{1}{x^3}=3x\cdot\frac{1}{x^3}+2x^2\cdot\frac{1}{x^3}$ $=\frac{3x}{x^3}+\frac{2x^2}{x^3}=\frac{3}{x^2}+\frac{2}{x}$
Ch. 9: Factoring 9-1: Factors and Greatest Common Factors 9-2: Factoring Using the Distributive Property	Problem: $14.7\div\frac{7}{3}$ $\frac{147}{10}\div\frac{7}{3}=\frac{147}{10}\cdot\frac{3}{7}=\frac{147}{7}\cdot\frac{3}{10}=\frac{(140+7)}{7}\cdot\frac{3}{10}$ Solution: $=\left(\frac{140}{7}+\frac{7}{7}\right)\cdot\frac{3}{10}=(20+1)\cdot\frac{3}{10}$ $=(21)\cdot\frac{3}{10}=\frac{21\cdot 3}{10}=\frac{63}{10}=6.3$ Connection: $\frac{x}{3}-\frac{5}{9}=\frac{1}{9}(3x-5)$
Ch. 12: Rational Expressions and Equations 12-2: Rational Expressions 12-3: Multiplying Rational Expressions 12-4: Dividing Rational Expressions 12-5: Dividing Polynomials 12-6/7: Rational Expressions with Like/Unlike Denominators	Problem: $60\div 1.5$ Solution: 1.5 is the same as $\frac{3}{2}$ so we have $60\div\frac{3}{2}=60\cdot\frac{2}{3}=\frac{60}{3}\cdot 2=20\cdot 2=40$ Connection: $xy^2\div\frac{y}{x^2}=xy^2\cdot\frac{x^2}{y}=\left(x\cdot x^2\right)\cdot\left(y^2\cdot\frac{1}{y}\right)$ $=x^3\cdot y=x^3y$

Examples of Solution Methods

BREAK APART NUMBERS
Problem: $34 + 8$
Solution: $34 + 8 = (30 + 4) + 8 = 30 + (4 + 8) = 30 + 12 = 42$

Problem: $24 \cdot 6$
Solution: $(20 + 4) \cdot 6 = 20 \cdot 6 + 4 \cdot 6 = 120 + 24 = (120 + 20) + 4 = 140 + 4 = 144$

MAKE A 10
Problem: $34 + 8$
Solution: $34 + (6 + 2) = (34 + 6) + 2 = 40 + 2 = 42$

Problem: $34 + 58$
Solution: $34 + 58 = 34 + (6 + 52) = (34 + 6) + 52 = 40 + 52 = 92$ We need 6 to make 40 so that leaves 52 more. We add 40 and 52 and get 92.

MAKE IT SIMPLER
Problem: $34 + 58$
Solution: 58 is close to 60, so let's use 60. $34 + 60 = 94$, but we added 2 too many so we get 92.

Problem: $24 \cdot 19$
Solution: $24 \cdot 20 = 480$ but that means that 24 was counted twenty times and not just 19, so it should be 24 less $480 - 24 = 480 - (20 + 4) = 480 - 20 - 4 = 460 - 4 = 456$.

REARRANGE THE NUMBERS
Problem: $34 + 58$
Solution: $58 + (2 + 32) = (58 + 2) + 32 = 60 + 32 = 92$

Problem: $18 \cdot 5 \cdot \frac{1}{3}$
Solution: $(18 \cdot \frac{1}{3}) \cdot 5 = 6 \cdot 5 = 30$.

STANDARD ALGORITHM
Problem: $34 + 52$
Solution: 4+2, and 3+5 so 86

Problem: $34 \cdot 14$
Solution: $4 \cdot 4 = 16$, so 6 carry the 1, and $3 \cdot 4 = 12$ so 13, so 136. Then, for the next row, $1 \cdot 4 = 4$ and $1 \cdot 3 = 3$ so 34. So 476 total.

START WITH THE 10s
Problem: $34 + 52$
Solution: $34 + 52 = (30 + 4) + (50 + 2) = (30 + 50) + (4 + 2) = 80 + 6 = 86$

Problem: $68-48$
Solution: $(60-40)+(8-8)=20$.

THINK ABOUT MONEY
Problem: $99+23$
Solution: Almost like a dollar and a quarter, which would be \$1.25. It is 3 less (1 for the 99 and 2 for the 23) so that is 123.

Problem: $24 \cdot 6$
Solution: 24 is close to 25. So, 25 quarters makes \$1.50. But, I did 1 too many, six times, so that's $150-6=144$.

THINK ABOUT MULTIPLES
Problem: $28 \div 8$
Solution: $8+8=16;16+8=24$ so that was 3 8s and we were 4 short so that is a half of an 8, so the answer is $3\frac{1}{2}$.

USE DECIMALS
Problem: $4 \cdot \frac{3}{2}$
Solution: $4 \cdot 1.5 = 4 \cdot (1+0.5) = 4 \cdot 1 + 4 \cdot 0.5 = 4 + 4 \cdot \frac{1}{2} = 4 + 2 = 6$.

USE DOUBLES
Problem: $34 \cdot 15$
Solution: $34 \cdot 30 = (30+4) \cdot 30 = (30 \cdot 30) + (4 \cdot 30) = 900 + 120 = 1020$ but 30 is 2 times 15, so $34 \cdot 15 = \frac{1}{2}(1020) = \frac{1}{2}(1000+20) = \frac{1}{2} \cdot 1000 + \frac{1}{2} \cdot 20 = 500 + 10 = 510$.

Problem: $16 \cdot 5$
Solution: $10 \cdot 16 = 160$ so half of that is 80.

USE FACTORS
Problem: $34 \cdot 4$
Solution: $34 \cdot (2 \cdot 2) = (34 \cdot 2) \cdot 2 = 68 \cdot 2 = (60+8) \cdot 2 = 60 \cdot 2 + 8 \cdot 2 = 120 + 16 = 136$

Problem: $24 \cdot \frac{5}{18}$
Solution: $(4 \cdot 6) \cdot \left(\frac{5}{3 \cdot 6}\right) = (4 \cdot 6) \cdot \left(\frac{1}{6} \cdot \frac{5}{3}\right) = \left(4 \cdot \frac{5}{3}\right) \cdot \left(6 \cdot \frac{1}{6}\right) = \frac{4 \cdot 5}{3} = \frac{20}{3}$.

USE FRACTIONS
Problem: $75 \cdot 80$
Solution: $.75 \cdot 80 = \frac{3}{4} \cdot 80 = 3 \cdot \left(\frac{1}{4} \cdot 80\right) = 3 \cdot 20 = 60$ so the answer is 6000.

Problem: $36 \cdot 0.25$
Solution: $36 \cdot \frac{1}{4} = 9$.

USE INVERSE OPERATION
Problem: $68 - 48$
Solution: Think about how you get from 48 to 68. It takes 2 to get from 48 to 50, and 18 more to get to 68 so that is 20 total.

Problem: $28 \div 8$
Solution: $8 \cdot 3 = 24$ so that leaves 4 extra, so we get $3\frac{4}{8} = 3\frac{1}{2}$

USE LANDMARK NUMBERS (25, 50, 75, 200, ETC)
Problem: $34 + 52$
Solution: 34+50 so 84 and then 2 more, so 86 total
$34 + 52 = 34 + (50 + 2) = (34 + 50) + 2 = 84 + 2$

Problem: $101 - 25$
Solution: $100 - 25 = 75$ so we need to add 1 and that is 76.

USE WHAT YOU ALREADY KNOW
Problem: $101 - 25$
Solution: $100 - 25 = 75$ so we need to add 1 and that is 76.

Problem: $75 \cdot 80$
Solution: $75 \cdot 8 = (70 + 5) \cdot 8 = 70 \cdot 8 + 5 \cdot 8 = 560 + 40 = 600$. So 6000.

References for the curriculum:

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