AN INVESTIGATION OF RESIDUATED LATTICES WITH MODAL OPERATORS

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To my loving partner,

Jonathan

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CHAPTER I

INTRODUCTION

Residuated structures appear in many different areas of mathematics, from Galois theory and algebraic topology to category theory and logic (cf. the survey [19]). As an example, in order to investigate the ideal theory of commutative rings, Ward and Dilworth introduced the concept of a residuated lattice, a lattice-ordered structure with a residuated multiplication [44]. Over the years, people have discovered the usefulness of this structure, in particular as a generalization of Boolean algebras and lattice-ordered groups. Recently, and with differing motivations, universal algebraists [10, 30] and non-classical logicians [28, 41] have begun to study residuated lattices more extensively. These two areas of research were eventually merged into a common subfield of algebraic logic, with the aid of the observation that residuated lattices are the algebraic counterpart of so-called substructural logics [24].

While the precise definition of a residuated lattice with a modal operator (specifically, a nucleus or conucleus) has only been recently introduced, mathematicians have been investigating specific instances of this concept for years. The first purely algebraic treatment of a residuated lattice with a modal operator can be traced back to McKinsey and Tarski's work in the 1940s [35, 36]. One of the goals of their work was to generalize the notion of a topological space so that it can be treated algebraically. Thus, power sets are replaced by Boolean algebras, and topologies - viewed as Kuratowski-style operators [31] - replaced by closure operators. To achieve a rather different aim, they also use these structures (closure algebras) to prove a conjecture due to Gödel that intuitionistic logic can be embedded into the modal logic $S4$ [27]. This latter result has been extensively investigated and expanded [9, 17, 20, 21, 32]. Instead of using McKinsey and Tarski's original closure algebras, however, researchers in this field have found it more intuitive to work with the dual notion of an interior algebra, which is precisely a Boolean algebra with a conucleus.

In another example of residuated lattices with modal operators, Montagna and Tsinakis show that the categories $\mathcal{CC}an\mathcal{RL}$ (of commutative, cancellative residuated lattices) and $\mathcal{A}_\sigma^*$ (a certain subclass of Abelian ℓ-groups with conuclei) are categorically equivalent [39]. In the last example we will consider, while it was known that all MV-algebras are intervals in Abelian ℓ-groups [40], Galatos and Tsinakis were the first to include this situation as one about residuated lattices (the negative cones of ℓ-groups) with nuclei. Indeed, similar to the work of Montagna and Tsinakis, Galatos and Tsinakis show that the categories $\mathcal{IGMV}$ (of integral GMV-algebras) and $(\mathcal{LG}_\gamma^-)^*$ (a certain subclass of negative cones of ℓ-groups with nuclei) are categorically equivalent [25].

One of the principal purposes of this dissertation is to show the similarities between these three situations. Thus, we will replicate some of the results obtained in the aforementioned papers. However, the organization of the proofs that we choose to use here has many distinct differences to how they were structured originally. First and foremost, the bulk of the work is done at the level of reducts, rather than with the images under the modal operators. The usefulness of Lemma III.3.3 is then seen by the ability to use the exact same proof of categorical equivalence in all three of these cases (Theorem IV.2.2). The author feels that the application of Lemma III.3.3 to these situations is the primary reason that such interesting results can be shown.

In order to establish these categorical equivalences, the author must first define some basic terminology, which is done in Chapter II. Then, in Chapter III, the author establishes Lemma III.3.3 and shows that it applies to all of these situations. Once established, that lemma can be used to show the categorical equivalences that were already known, as well as establish a categorical equivalence in the interior algebra case (which the author has never seen explicitly mentioned). Also in Chapter IV, the subvariety lattice of modal residuated lattices is compared to the subvariety lattice of residuated lattices, restricted to these cases. Theorem IV.3.3 states that there are intervals in the subvariety lattices of these modal residuated lattices that correspond to subvarieties of their images. One of the main results in the interior algebra case is that by restricting to the minimal varieties in these intervals, an isomorphism of subvariety lattices results [9, 20, 21]. In Chapter V, the author proves a similar result in the case of pointed Abelian ℓ-groups and MV-algebras (which can be viewed as a restriction of the situation from [25]). In fact, in that case, all but one of the intervals in the "modal" subvariety lattice consist of only one variety (Theorem V.2.1). Since these modal residuated lattices were first considered to achieve logical aims, it seems appropriate to have at least one section devoted to the logical aspects of these structures, and this is precisely the purpose of Section V.3. The penultimate chapter is concerned with using these connections to construct free algebras in certain classes of residuated lattices. This is done in a very general setting in Theorem VI.3.2, but it is also used to establish results about MV-algebras and negative cones of Abelian ℓ-groups in Theorems VI.1.3 and VI.2.5. The last chapter discusses future research topics to pursue related to these structures.

While a concerted effort was made to have this paper be as self-contained as possible, oversights do happen. For a nice survey of (cancellative) residuated lattices, see [2, 10, 24, 30], as well as the recent survey article [38]. Also, some basic understanding of the ideas of universal algebra is assumed throughout this dissertation; the standard texts on this topic are [12, 34].

CHAPTER II

RESIDUATED LATTICES

II.1 Background

This section is a quick introduction to the study of residuated lattices. Some good surveys of this topic include [10, 24, 30].

Definition II.1.1. *A* **residuated lattice** *is an algebraic structure* $\mathbf{A}=\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ *such that* $\langle A, \wedge, \vee\rangle$ *is a lattice,* $\langle A, \cdot, 1\rangle$ *is a monoid, and* $\backslash$ *and* $/$ *are right and left residuals, respectively, of* $\cdot$*; i.e., for all* $x, y, z \in A$*,* $x \cdot y \leq z$ *iff* $y \leq x\backslash z$ *iff* $x \leq z/y$*.*

As usual, the monoid operation $\cdot$ will be written as juxtaposition, so xy instead of $x \cdot y$. Also, to avoid the over-abundance of parentheses, the monoid operation should always be performed before the lattice operations. For example, $xy \wedge xz$ is shorthand for $(x \cdot y) \wedge (x \cdot z)$.

Note that this definition of a residuated lattice implies that the monoid multiplication, as well as the two "division" operations ($x\backslash$ and $/x$), are order-preserving. To see this, assume $a \leq b$ in a residuated lattice $\mathbf{A}$ and let $x \in A$. Then, since $xb \leq xb$, $a \leq b \leq x\backslash xb$, so $xa \leq xb$. Similarly, $a \leq b \leq bx/x$, so $ax \leq bx$. Since $a/x \leq a/x$, $(a/x)x \leq a \leq b$, and $a/x \leq b/x$. Similarly, $x(x\backslash a) \leq a \leq b$, so $x\backslash a \leq x\backslash b$.

Proposition II.1.2. *For a residuated lattice* $\mathbf{A}=\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ *and elements* $x, y, z \in A$*, the following (and their mirror images) hold in* $\mathbf{A}$*:*

(1) $x(y/z) \leq xy/z$

(2) $x/y \leq xz/yz$

(3) $(x/y)(y/z) \leq x/z$

(4) $x/yz = (x/z)/y$

(5) $x\backslash(y/z) = (x\backslash y)/z$

(6) $(1/x)(1/y) \leq 1/yx$

(7) $(x/x)x = x$

(8) $(x/x)^2 = x/x$

Proof.

(1) Since $y/z \leq y/z$, $(y/z)z \leq y$. Thus, $x(y/z)z \leq xy$, and $x(y/z) \leq xy/z$.

(2) $(x/y)y \leq x$, so $(x/y)yz \leq xz$, and $x/y \leq xz/yz$

(3) $(x/y)(y/z)z \leq (x/y)y \leq x$, so $(x/y)(y/z) \leq x/z$

(4) Since $(x/yz)yz \leq x$, $(x/yz)y \leq x/z$, and $x/yz \leq (x/z)/y$. For the other inequality, $[(x/z)/y]yz \leq (x/z)z \leq x$, so $(x/z)/y \leq x/yz$.

(5) By the mirror image of (1), $[x\backslash(y/z)]z \leq x\backslash[(y/z)z] \leq x\backslash y$, so $x\backslash(y/z) \leq (x\backslash y)/z$. Similarly, $x[(x\backslash y)/z] \leq [x(x\backslash y)]/z \leq y/z$, so $(x\backslash y)/z \leq x\backslash(y/z)$.

(6) $(1/x)(1/y)yx \leq (1/x)(1x) = (1/x)x \leq 1$, so $(1/x)(1/y) \leq 1/yx$

(7) Since $1 \cdot x = x \leq x$, $1 \leq x/x$, and $x \leq (x/x)x \leq x$, which establishes the equality.

(8) By (3), $(x/x)^2 = (x/x)(x/x) \leq x/x$. Since $1 \leq x/x$, $x/x \leq (x/x)^2$, which establishes the equality.

$\square$

The next proposition is from [10].

Proposition II.1.3. *For any residuated lattice* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ *and any subsets* $X, Y \subseteq A$,

(1) If $\bigvee X$ *and* $\bigvee Y$ *exist, then* $\bigvee_{x\in X, y\in Y} xy$ *exists and*

$$\left(\bigvee X\right)\left(\bigvee Y\right) = \bigvee_{x\in X, y\in Y} xy.$$

(2) If $\bigvee X$ *and* $\bigwedge Y$ *exist, then for any* $z \in A$, $\bigwedge_{x\in X} z/x$ *and* $\bigwedge_{y\in Y} y/z$ *exist and*

$$z/\bigvee X = \bigwedge_{x\in X} z/x \text{ and } \left(\bigwedge Y\right)/z = \bigwedge_{y\in Y} y/z.$$

Proof.

(1) For $x \in X$ and $y \in Y$, $xy \leq \left(\bigvee X\right) y \leq \left(\bigvee X\right)\left(\bigvee Y\right)$. Thus, $\left(\bigvee X\right)\left(\bigvee Y\right)$ is an upper bound of the set $\{xy \mid x \in X,\ y \in Y\}$. If a is an upper bound of $\{xy \mid x \in X,\ y \in Y\}$, then for an $x \in X$, and $y \in Y$, $xy \leq a$. So, $x \leq a/y$. Since this inequality holds for all $x \in X$, $\bigvee X \leq a/y$, and $\left(\bigvee X\right) y \leq a$. Thus, $y \leq \left(\bigvee X\right)\backslash a$. Since this is true for all $y \in Y$, $\bigvee Y \leq \left(\bigvee X\right)\backslash a$, and $\left(\bigvee X\right)\left(\bigvee Y\right) \leq a$. Therefore, the set $\{xy \mid x \in X,\ y \in Y\}$ has a least upper bound and $\left(\bigvee X\right)\left(\bigvee Y\right) = \bigvee_{x\in X, y\in Y} xy$.

(2) Since $\left(z/\bigvee X\right) x \leq \left(z/\bigvee X\right)\bigvee X \leq z$, $z/\bigvee X \leq z/x$. This shows that $z/\bigvee X$ is a lower bound of the set $\{z/x \mid x \in X\}$. If a is a lower bound of $\{z/x \mid x \in X\}$, then for every $x \in X$, $a \leq z/x$, so $ax \leq z$, and $x \leq a\backslash z$. Since this inequality holds for every $x \in X$, $\bigvee X \leq a\backslash z$, so $a\left(\bigvee X\right) \leq z$, and $a \leq z/\bigvee X$. Therefore, $\{z/x \mid x \in X\}$ has a greatest lower bound, and $z/\bigvee X = \bigwedge_{x\in X} z/x$.

For each $y \in Y$, $\left(\bigwedge Y\right)/z \leq y/z$. So, we see that $\left(\bigwedge Y\right)/z$ is a lower bound of the set $\{y/z \mid y \in Y\}$. If b is a lower bound of $\{y/z \mid y \in Y\}$, then for any $y \in Y$, $b \leq y/z$, and bz/y. Since this holds for every $y \in Y$, $bz \leq \bigwedge Y$, so $b \leq \left(\bigwedge Y\right)/z$. This shows that $\{y/z \mid y \in Y\}$ has a greatest lower bound, and $\left(\bigwedge Y\right)/z = \bigwedge_{y\in Y} y/z$.

$\square$

Notice that if $X = \{x\}$ and $Y = \{y_1, y_2\}$ in part (1) of the previous proposition, then it shows that $x(y_1 \vee y_2) = xy_1 \vee xy_2$, so that multiplication distributes over joins (on the left) in a residuated lattice. A similar argument shows that multiplication distributes over joins on the right as well.

The equations in the proof of the following proposition are from [23], where $\mathcal{RL}$ denotes the class of residuated lattice.

Proposition II.1.4. *The class $\mathcal{RL}$ is a variety.*

Proof. $\mathcal{RL}$ is an equational class, with the following equations:

(a) equations for a lattice

(b) equations for a monoid

(c) $x \approx x \wedge (xy \vee z)/y, \quad x(y \vee z) \approx xy \vee xz, \quad (x/y)y \vee x \approx x,$

$y \approx y \wedge x\backslash(xy \vee z), \quad (y \vee z)x \approx yx \vee zx, \quad y(y\backslash x) \vee x \approx x$

Assume $\mathbf{A}$ is a residuated lattice. By definition, $\mathbf{A}$ satisfies the equations of (a) and (b). For the remaining equations, let $x, y, z \in A$. Since $x \leq xy/y \leq (xy \vee z)/y$, we see that $x = x \wedge (xy \vee z)/y$. Similarly, $y \leq x\backslash xy \leq x\backslash(xy \vee z)$, so $y = y \wedge x\backslash(yx \vee z)$. The observation after the previous proposition verifies that $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$. As we have observed multiple times, $(x/y)y \leq x$ and $y(y\backslash x) \leq x$, which shows that $(x/y)y \vee x = x$ and $y(y\backslash x) \vee x = x$.

Assume some algebra $\mathbf{A}=\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ satisfies the equations specified in (a), (b), and (c). We must show that $\mathbf{A}$ is in fact a residuated lattice. Certainly, $\langle A, \wedge, \vee\rangle$ is a lattice, and $\langle A, \cdot, 1\rangle$ is a monoid. Next, we will show that multiplication in $\mathbf{A}$ is order-preserving. Assume $a, b \in A$ are such that $a \leq b$, and let $c \in A$. Then, $ca \vee cb = c(a \vee b) = cb$, and $ac \vee bc = (a \vee b)c = bc$, so $ca \leq cb$ and $ac \leq bc$.

To show that $\mathbf{A}$ satisfies the residuation law, let $x, y, z \in A$. First, assume $xy \leq z$. Then, $xy \vee z = z$. Since $x = x \wedge (xy \vee z)/y$, $x \leq (xy \vee z)/y = z/y$. Since multiplication is order-preserving, if $x \leq z/y$, then $xy \leq (z/y)y \leq z$ (since $(z/y)y \vee z = z$). So, $y \leq x\backslash(xy \vee z) \leq x\backslash z$. Lastly, assume $y \leq x\backslash z$. Then, $xy \leq x(x\backslash z) \leq z$. These three implications show that for all $x, y, z \in A$, $x \cdot y \leq z$ iff $x \leq z/y$ iff $y \leq x\backslash z$. Therefore, $\mathbf{A}$ is a residuated lattice. $\square$

II.2 Examples of Residuated Lattices

An **FL-algebra** $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is an algebra such that $\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a residuated lattice, and 0 is a distinguished element. Since the class of FL-algebras, denoted $\mathcal{FL}$, satisfies the same equations as the class of residuated lattices, $\mathcal{FL}$ is also a variety. By including 0 in the signature of $\mathcal{RL}$, $\mathcal{RL}$ can be thought of as the subvariety of $\mathcal{FL}$ corresponding to the additional equation $0 \approx 1$.

A **commutative** residuated lattice (FL-algebra) is a residuated lattice (FL-algebra) such that the monoid reduct is commutative. Thus, the class of commutative residuated lattices (FL-algebras), denoted by $\mathcal{CRL}$ ($\mathcal{CFL}$), is a variety. Notice that adding either $xy \approx yx$ or $x\backslash y \approx y/x$ to the other equations for the variety of residuated lattices would yield the variety $\mathcal{CRL}$. Although it is a slight abuse of notation, the symbol $\to$ will be used for both residuals in a commuative residuated lattice; i.e., $x \to y = x\backslash y = y/x$.

The details in the following examples come from a Vanderbilt University Qualifying Exam paper [22].

Example II.2.1. *A* **Heyting algebra** *is an algebra* $\langle A, \wedge, \vee, \to, 1, 0\rangle$ *such that* $\langle A, \wedge, \vee, 1, 0\rangle$ *is a bounded lattice, and for every* $a, b \in A$*,* $a \to b$ *is the relative pseudo-complement of* a *and* b*; that is,* $a \to b$ *is the greatest element in the set* $\{x \in A \mid x \wedge a \leq b\}$*.*

Notice that this condition is equivalent to saying that $c \wedge a \leq b$ iff $c \leq a \to b$. The variety $\mathcal{HA}$ of Heyting algebras is term-equivalent to the subvariety of $\mathcal{CFL}$ corresponding to the additional equations $xy \approx x \wedge y$ and $x \wedge 0 \approx 0$.

Let $\mathbf{A} = \langle A, \wedge, \vee, \to, 1, 0\rangle$ be a Heyting algebra. Define $\mathbf{A}' = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ by $x \cdot y = x \wedge y$ and $x/y = y\backslash x = y \to x$, for all $x, y \in A$. Certainly, $\langle A, \wedge, \vee\rangle$ is a lattice. Since $\wedge$ is commutative and associative, $\cdot$ is commutative and associative. For $x \in A$, $x \cdot 1 = x \wedge 1 = x$ and $1 \cdot x = 1 \wedge x = x$. Thus, $\langle A, \cdot, 1\rangle$ is a commutative monoid. Let $x, y, z \in A$. Since $x \wedge y = y \wedge x \leq z$ iff $x \leq y \to z$ iff $y \leq x \to z$ (in $\mathbf{A}$), $x \cdot y \leq z$ iff $x \leq z/y$ iff $y \leq x\backslash z$ (in $\mathbf{A}'$). Since 0 is still the least element, $\mathbf{A}'$ is in the specified subvariety of $\mathcal{CFL}$.

Given an algebra $\mathbf{B}=\langle B, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ in the aforementioned subvariety of $\mathcal{CFL}$, define $\mathbf{B}^*=\langle B, \wedge, \vee, \to, 1, 0\rangle$ by $x \to y = y/x$, for every $x, y \in B$. Clearly, $\langle B, \wedge, \vee\rangle$ is a lattice with least element 0 (in $\mathbf{B}^*$). Since, for every $x \in B$, $x = x \cdot 1 = x \wedge 1$, 1 is the top element of $\mathbf{B}^*$. Let $x, y, z \in B$. Then, $x \wedge y = x \cdot y \leq z$ iff $x \leq z/y$ (in $\mathbf{B}$), so $x \wedge y \leq z$ iff $x \leq y \to z$ (in $\mathbf{B}^*$). Therefore, relative pseudo-complements exist in $\mathbf{B}^*$, and $\mathbf{B}^*$ is a Heyting algebra.

Since the residuals $(\backslash, /, \to)$ were defined to be interchanged, the multiplication was defined to be compatible with the equation $x \cdot y \approx x \wedge y$, and none of the lattice operations $(\wedge, \vee, 1, 0)$ was altered, it is clear that $(\mathbf{A}')^* = \mathbf{A}$ and $(\mathbf{B}^*)'=\mathbf{B}$.

With this term-equivalence and the knowledge that multiplication distributes over joins in a residuated lattice, one can conclude that every Heyting algebra has, in fact, a distributive lattice reduct.

Example II.2.2. *A* **Boolean algebra** *is a bounded distributive lattice such that every element has a (necessarily unique) complement.*

The variety $\mathcal{BA}$ of Boolean algebras is term-equivalent to the subvariety of $\mathcal{CFL}$ corresponding to the additional equations $xy \approx x \wedge y$, $x \wedge 0 \approx 0$, and $(x \to y) \to y \approx x \vee y$.

Let $\mathbf{A}=\langle A, \wedge, \vee, \neg, 1, 0\rangle$ be a Boolean algebra, where $\neg x$ is the complement of $x \in A$. Define $\mathbf{A}'=\langle A, \wedge, \vee, \cdot, \to, 1, 0\rangle$ by $x \cdot y = x \wedge y$ and $x \to y = \neg x \vee y$, for every $x, y \in A$. Since a Boolean algebra is a Heyting algebra ($\neg a \vee b$ is a relative pseudo-complement of a and b), the only equation that needs to be checked is $(x \to y) \to y \approx x \vee y$.

First, observe that in any bounded distributive lattice, complements (when they exist) are unique. Assume that $a \wedge b = a \wedge c = 0$ and $a \vee b = a \vee c = 1$. Then, $b = b \vee (a \wedge b) = b \vee (a \wedge c) = (b \vee a) \wedge (b \vee c) = (a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c = (a \wedge c) \vee c = c$.

Next, observe that every Boolean algebra satisfies both the "law of double negation" and "DeMorgan's law." For the first, $\neg x$ is the complement of x as well as of $\neg\neg x$ (since the property 'being the complement of' is symmetric). Since complements are unique, $\neg\neg x = x$, for every $x \in A$. By distributivity, $(x \vee y) \wedge (\neg x \wedge \neg y) = (0 \wedge \neg y) \vee (\neg x \wedge 0) = 0$, and $(x \vee y) \vee (\neg x \wedge \neg y) = (1 \vee y) \wedge (x \vee 1) = 1$, so $\neg(x \vee y)=\neg x \wedge \neg y$.

Finally, for $x, y \in A$, $(x \to y) \to y = \neg(x \to y) \vee y = \neg(\neg x \vee y) \vee y = (\neg\neg x \wedge \neg y) \vee y$ $= (x \wedge \neg y) \vee y = (x \vee y) \wedge (\neg y \vee y) = (x \vee y) \wedge 1 = x \vee y$. Therefore, $\mathbf{A}'$ is in the desired subvariety of $\mathcal{CFL}$.

Now, let $\mathbf{B}$=$\langle B, \wedge, \vee, \cdot, \to, 1, 0\rangle$ be an algebra in the specified subvariety of $\mathcal{CFL}$. Define $\mathbf{B}^*$=$\langle B, \wedge, \vee, \neg, 1, 0\rangle$ by $\neg x = x \to 0$, for every $x \in B$. Since $\mathbf{B}$ is a bounded distributive lattice (in particular, a Heyting algebra) and $\mathbf{B}^*$ has the same lattice reduct as $\mathbf{B}$, $\mathbf{B}^*$ is also a bounded distributive lattice. It remains to be shown that complements exist in $\mathbf{B}^*$. For any $x \in B$, $x \wedge \neg x = x \wedge (x \to 0) \leq 0$. Since 0 is the bottom element of $\mathbf{B}^*$, this implies that $x \wedge \neg x = 0$. For the other condition of a complement, first consider $a, b \in B$ such that $a \wedge (b \to 0) \leq b$. Then, "meeting" both sides of the inequality by $b \to 0$, we see that $a \wedge (b \to 0) \leq b \wedge (b \to 0) \leq 0$, so $a \leq (b \to 0) \to 0 = b \vee 0 = b$. In particular, the previous argument shows that $(x \to 0) \to x \leq x$, for every $x \in B$. Thus, $1 \leq [(x \to 0) \to x] \to x = (x \to 0) \vee x = \neg x \vee x$. Therefore, $x \vee \neg x = 1$, and $\neg x$ is the complement of x in $\mathbf{B}^*$.

Lastly, it remains to show that $(\mathbf{A}')^* = \mathbf{A}$ and $(\mathbf{B}^*)'$=$\mathbf{B}$. As in the case of Heyting algebras, since the lattice operations were unaltered in the constructions and multiplication was defined to be compatible with the equation $xy \approx x \wedge y$, one need only check that $\to$ and $\neg$ were defined appropriately. To show that $(\mathbf{B}^*)'$=$\mathbf{B}$, let $\neg_1$ and $\neg_2$ be the complementation operations of $\mathbf{B}$ and $(\mathbf{B}^*)'$, respectively. Then, for any $x \in B$, $\neg_2 x = x \to 0 = \neg_1 x \vee 0 = \neg_1 x$.

To show that $(\mathbf{A}')^* = \mathbf{A}$, let $\to_1$ and $\to_2$ be the residuals of $\mathbf{A}$ and $(\mathbf{A}')^*$, respectively. Then, we must show that for any $x, y \in A$, $x {\to_1} y = x {\to_2} y = \neg x \vee y = (x {\to_1} 0) \vee y$. Since we have already observed that $x {\to_1} 0$ is the complement of x in $\mathbf{A}$ and that $\neg a \vee b$ is the relative complement of a and b (i.e., $a {\to_1} b$), it follows that $x {\to_1} y = x {\to_2} y$. Thus, $(\mathbf{A}')^* = \mathbf{A}$.

A **cancellative** residuated lattice is one that is cancellative as a monoid. While cancellative monoids are a quasivariety, cancellative residuated lattices form a variety (with additional identities $xy/y \approx x \approx y\backslash yx$). If $\mathbf{A}$=$\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a cancellatice residuated lattice, then for any $x, y \in A$, since $x \leq xy/y$, we see that $xy \leq (xy/y)y \leq xy$ and similarly, $yx \leq y(y\backslash yx) \leq yx$. By cancellativity, $x = xy/y$ and $x = y\backslash yx$. Now, if **A** satisfies the equations $xy/y \approx x \approx y\backslash yx$, then assume $xy = xz$, for $x, y, z \in A$. Then, $y = x\backslash xy = x\backslash xz = z$. Similarly, if $yx = zx$, then $y = yx/x = zx/x = z$. Thus, **A** is cancellative. Two varieties of cancellative residuated lattices that we will consider extensively in this dissertation are lattice-ordered groups and their negative cones.

Example II.2.3. *A* **lattice-ordered group** *is an algebra $\langle A, \wedge, \vee, \cdot, ^{-1}, 1, \rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, ^{-1}, 1\rangle$ is a group, and $\cdot$ is order-preserving in both arguments.*

The variety $\mathcal{LG}$ of ℓ-groups is term-equivalent to the subvariety of $\mathcal{RL}$ defined by the addition of the equations $(1/x)x \approx 1 \approx x(x\backslash 1)$.

Given an ℓ-group $\mathbf{A}$=$\langle A, \wedge, \vee, \cdot, ^{-1}, 1, \rangle$, define $\mathbf{A}'$=$\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ by $y\backslash x = y^{-1}x$ and $x/y = xy^{-1}$, for every $x, y \in A$. Clearly, $\langle A, \wedge, \vee\rangle$ is a lattice and $\langle A, \cdot, 1\rangle$ is a monoid. For $x, y, z \in A$, $xy \leq z$ iff $x \leq zy^{-1} = z/y$ iff $y \leq x^{-1}z = x\backslash z$. Thus $\mathbf{A}'$ is a residuated lattice. Also, for $x \in A$, $1/x = 1 \cdot x^{-1} = x^{-1}$ and $x\backslash 1 = x^{-1}$. Thus, $(1/x)x = x^{-1}x = 1$ and $x(x\backslash 1) = xx^{-1} = 1$. Therefore, $\mathbf{A}'$ is in the desired subvariety of $\mathcal{RL}$.

Now, let $\mathbf{B}$=$\langle B, \wedge, \vee, \cdot, /, \backslash, 1\rangle$ be in the specified subvariety of $\mathcal{RL}$, and define the structure $\mathbf{B}^*$=$\langle B, \wedge, \vee, \cdot, ^{-1}, 1, \rangle$ by $x^{-1} = 1/x$, for every $x \in B$. Clearly, $\langle B, \wedge, \vee\rangle$ is a lattice, $\langle B, \cdot, 1\rangle$ is a monoid, and $\cdot$ preserves the order (in both arguments). It remains to show that every element of $\mathbf{B}^*$ has an inverse. Certainly, $x^{-1}x = (1/x)x = 1$, for every $x \in B$. For the other condition of an inverse, $xx^{-1} = x(1/x) = x/x = (x/x)\cdot 1 = (x/x)x(x\backslash 1) = x(x\backslash 1) = 1$. Therefore, $\mathbf{B}^*$ is a lattice-ordered group.

Since the lattice and monoid operations were unchanged and inverses are unique, it is clear that $(\mathbf{A}')^*$ =$\mathbf{A}$. In order to show that $(\mathbf{B}^*)'$=$\mathbf{B}$, we need only show that the residuals of the two algebras coincide. Let $/_1$ and $\backslash_1$ be the residuals of $\mathbf{B}$, and let $/_2$ and $\backslash_2$ be the residuals of $\mathbf{B}^*$. Then, $x/_2y = xy^{-1} = x(1/_1y) = x/_1y$. For the other residual, first notice that for any $a \in \mathbf{B}$, $(a\backslash 1)\cdot a = a\backslash a = 1\cdot(a\backslash a) = (1/a)\cdot a\cdot(a\backslash a) = (1/a)\cdot a = 1$, so $a\backslash 1 \leq 1/a$. Similarly, $1/a \leq a\backslash 1$. Thus, $y\backslash_2 x = y^{-1}x = (1/_1y)x = (y\backslash_1 1)x = y\backslash_1 x$. Therefore, $(\mathbf{B}^*)'$=$\mathbf{B}$.

Example II.2.4. *Given a lattice-ordered group* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, ^{-1}, 1, \rangle$*, define the algebra* $\mathbf{A}^- = \langle A^-, \wedge, \vee, \cdot, \backslash, /, 1, \rangle$*, where* $A^- = \{x \in A \mid x \leq 1\}$*, and for any* $x, y \in A^-$*,* $x\backslash y = x^{-1}y \wedge 1$ *and* $y/x = yx^{-1} \wedge 1$*. This algebra* $\mathbf{A}^-$ *is called the* **negative cone** *of* $\mathbf{A}$ *and is a residuated lattice.*

First, note that A^- is closed under both the monoid and the lattice operations of $\mathbf{A}$, so it is only remains to show that $\mathbf{A}^-$ satisfies the residuation law. Let $x, y, z \in A^-$. Then, $xy \leq z$ iff $x \leq zy^{-1}$ iff $x \leq zy^{-1} \wedge 1 = z/y$. Similarly, $xy \leq z$ iff $y \leq x\backslash z$. Thus, $\mathbf{A}^-$ is a residuated lattice. In fact, we have the following theorem from [2], where by **integral** we mean that 1 is the greatest element and the GMV-equation is $x/((x \vee y)\backslash x) \approx x \vee y \approx (x/(x \vee y))\backslash x$.

Theorem II.2.1. *A residuated lattice is a negative cone of an ℓ-group iff it is an integral, cancellative GMV-algebra.*

In particular, this theorem implies that the class of negative cones of ℓ-groups is a variety. Another variety of integral GMV-algebras that we will discuss are MV-algebras.

Example II.2.5. *An* **MV-algebra** *is an algebra* $\mathbf{A} = \langle A, \oplus, \neg, 0\rangle$ *that satisfies the following equations:*

(MV1) $\quad x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$

(MV2) $\quad x \oplus y \approx y \oplus x$

(MV3) $\quad x \oplus 0 \approx x$

(MV4) $\quad \neg\neg x \approx x$

(MV5) $\quad x \oplus \neg 0 \approx \neg 0$

(MV6) $\quad \neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$

The variety $\mathcal{MV}$ of MV-algebras is term-equivalent to the subvariety of $\mathcal{CFL}$ satisfying the extra equations $x \vee y \approx (x \to y) \to y$ and $x \wedge 0 \approx 0$. The main details of this term-equivalence come from [15]. First, given an MV-algebra $\mathbf{A} = \langle A, \oplus, \neg, 0\rangle$, define the structure $\mathbf{A}' = \langle A, \wedge, \vee, \cdot, \to 1, 0\rangle$ by $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$, $x \cdot y = \neg(\neg x \oplus \neg y)$, $x \to y = \neg x \oplus y$, and $1 = \neg 0$, for every $x, y \in A$. This term-equivalence is slightly more involved than the previous ones, and consequently it will require more work to establish. First, we will construct a partial order on A such that the least upper bound and greatest lower bound of x and y are $x \vee y$ and $x \wedge y$, respectively.

Lemma II.2.2. *Assume $\mathbf{A} = \langle A, \oplus, \neg, 0\rangle$ is an MV-algebra, with $\cdot$ and 1 defined as above, and let $x, y \in A$. Then, the following are equivalent:*

(1) $\neg x \oplus y = 1$

(2) $x \cdot \neg y = 0$

(3) $y = x \oplus (y \cdot \neg x)$

(4) There exists $z \in A$ such that $x \oplus z = y$

Proof. We will prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

Assume (1). Then, $x \cdot \neg y = \neg(\neg x \oplus \neg\neg y) = \neg(\neg x \oplus y) = \neg 1 = \neg\neg 0 = 0$.

Assume (2). Then, $x \oplus (y \cdot \neg x) = x \oplus (\neg(\neg y \oplus x)) = \neg(\neg y \oplus x) \oplus x = \neg(\neg x \oplus y) \oplus y = (x \cdot \neg y) \oplus y = 0 \oplus y = y$.

Assume (3). Then, take $z = y \cdot \neg x$.

Assume (4). Let z be such that $x \oplus z = y$. Then, first observe that $1 = \neg(\neg x \oplus 1) \oplus 1 = \neg(\neg 1 \oplus x) \oplus x = \neg(0 \oplus x) \oplus x = \neg x \oplus x$. So, $\neg x \oplus y = \neg x \oplus (x \oplus z) = (\neg x \oplus x) \oplus z = 1 \oplus z = 1$. $\square$

Define on A, $x \leq y$ iff any one (and therefore all) of the above conditions (1)-(4) holds. Since we already established that for every $x \in A$, $\neg x \oplus x = 1$, we see that $x \leq x$ (condition (1)), and $\leq$ is reflexive. If $x, y \in A$ are such that $x \leq y$ and $y \leq x$, then by conditions (2) and (3), $y = x \oplus (y \cdot \neg x) = x \oplus 0 = x$. Thus, $\leq$ is anti-symmetric. If $x, y, z \in A$ are such that $x \leq y$ and $y \leq z$, then by condition (4), there exists $u, v \in A$ such that $x \oplus u = y$ and $y \oplus v = z$. So, $x \oplus (u \oplus v) = (x \oplus u) \oplus v = y \oplus v = z$, and $x \leq z$. Thus, $\leq$ is transitive, and therefore a partial order on A.

Before we can show that the previously defined operations $\vee$ and $\wedge$ do in fact give the least upper bound and greatest lower bound in $\langle A, \leq\rangle$, we will need the following lemma.

Lemma II.2.3. *For $x, y, z \in A$,*

(1) $x \leq y$ *iff* $\neg y \leq \neg x$

(2) $x \cdot y \leq z$ *iff* $x \leq y \to z$

Proof.

(1) $x \leq y$ iff $\neg x \oplus y = 1$ iff $\neg\neg y \oplus \neg x = 1$ iff $\neg y \leq \neg x$

(2) $x \cdot y \leq z$ iff $\neg(x \cdot y) \oplus z = 1$ iff $\neg x \oplus \neg y \oplus z = 1$ iff $\neg x \oplus (y \to z) = 1$ iff $x \leq y \to z$

□

Remember that for $x, y \in A$, $x \vee y = \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. By condition (4), it is clear that $x \leq x \vee y$ and $y \leq x \vee y$. Now, assume that there exists $z \in A$ such that $x \leq z$ and $y \leq z$. We must show that $x \vee y \leq z$, or in particular that $\neg(x \vee y) \oplus z = 1$. First, notice that $\neg x \oplus z = 1$ (condition (1)) and $z = y \oplus (z \cdot \neg y)$ (condition (3)). So, we can arrive at the following computation:

$$\neg(x \vee y) \oplus z =$$

$$\neg(\neg(\neg x \oplus y) \oplus y) \oplus z =$$

$$\neg(\neg(\neg x \oplus y) \oplus y) \oplus (y \oplus (z \cdot \neg y)) =$$

$$\neg(\neg(\neg x \oplus y) \oplus y) \oplus (\neg(\neg z \oplus y) \oplus y) =$$

$$[\neg(\neg(\neg x \oplus y) \oplus y) \oplus y] \oplus \neg(\neg z \oplus y) =$$

$$\neg(\neg y \oplus (\neg x \oplus y)) \oplus (\neg x \oplus y) \oplus \neg(\neg z \oplus y) =$$

$$\neg(\neg y \oplus (\neg x \oplus y)) \oplus \neg x \oplus (\neg(\neg z \oplus y) \oplus y) =$$

$$\neg(\neg y \oplus (\neg x \oplus y)) \oplus \neg x \oplus z =$$

$$\neg(\neg y \oplus (\neg x \oplus y)) \oplus 1 =$$

$$1.$$

For $x, y \in A$, $x \wedge y = \neg(\neg x \vee \neg y)$. By the first part of the previous lemma, since $\neg x \leq \neg x \vee \neg y$ and $\neg y \leq \neg x \vee \neg y$, $x \wedge y$ is a lower bound of $\{x, y\}$ in $\langle A, \leq\rangle$. If $z \in A$ is a lower bound of $\{x, y\}$, then $\neg x \leq \neg z$ and $\neg y \leq \neg z$, so $\neg x \vee \neg y \leq \neg z$, and $z \leq x \wedge y$. Thus, $x \vee y$ is the least upper bound of x and y, and $x \wedge y$ is the greatest lower bound of x and y in $\langle A, \leq\rangle$. Therefore, $\langle A, \leq\rangle$ and so also $\langle A, \wedge, \vee\rangle$ are lattices.

For $x, y, z \in A$, observe that $$(x \cdot y) \cdot z =$$

$$\neg(\neg(x \cdot y) \oplus \neg z) =$$

$$\neg((\neg x \oplus \neg y) \oplus \neg z) =$$

$$\neg(\neg x \oplus (\neg y \oplus \neg z)) =$$

$$\neg(\neg x \oplus \neg(y \cdot z)) =$$

$$x \cdot (y \cdot z).$$

Also, $x \cdot y = \neg(\neg x \oplus \neg y) = \neg(\neg y \oplus \neg x) = y \cdot x$, $x \cdot 1 = \neg(\neg x \oplus \neg 1) = \neg(\neg x \oplus 0) = \neg\neg x = x$, and $1 \cdot x = \neg(\neg 1 \oplus \neg x) = \neg(0 \oplus \neg x) = \neg\neg x = x$. Therefore, $\langle A, \cdot, 1\rangle$ is a commutative monoid. With the aid of the second part of the previous lemma, we can conclude that $\mathbf{A}'$ is in $\mathcal{CFL}$. Lastly, we need to verify that $\mathbf{A}'$ satisfies the suggested equations. For $x \in A$, $\neg 0 \oplus x = 1 \oplus x = 1$, so $0 \leq x$. This shows that 0 is the bottom element of $\mathbf{A}'$, and thus $\mathbf{A}'$ satisfies the equation $x \wedge 0 \approx 0$. For $x, y \in A$, $(x \to y) \to y = (\neg x \oplus y) \to y = \neg(\neg x \oplus y) \oplus y = x \vee y$.

Now, assume $\mathbf{B}$=$\langle B, \wedge, \vee, \cdot, \to, 1, 0\rangle$ is an algebra in the specified subvariety of $\mathcal{CFL}$. Define $\mathbf{B}^* = \langle B, \oplus, \neg, 0\rangle$ by $\neg x = x \to 0$ and $x \oplus y = \neg(\neg x \cdot \neg y)$, for all $x, y \in B$. We wish to show that $\mathbf{B}^*$ is an MV-algebra. First, for $x, y \in B$, $x \oplus y = \neg(\neg x \cdot \neg y) = \neg(\neg y \cdot \neg x) = y \oplus x$. Before we show the associativity of $\oplus$, observe that for all $x \in B$, $\neg\neg x = (x \to 0) \to 0 =$

$x \vee 0 = x$. Now, for $x, y, z \in B$, $(x \oplus y) \oplus z = \neg(\neg(x \oplus y) \cdot \neg z) = \neg((\neg x \cdot \neg y) \cdot \neg z) = \neg(\neg x \cdot (\neg y \cdot \neg z)) = \neg(\neg x \cdot \neg(y \oplus z)) = x \oplus (y \oplus z)$.

Since $1 \to 0 = 1 \cdot (1 \to 0) \leq 0$, $1 \to 0 = 0$. So, $\neg 0 = 0 \to 0 = (1 \to 0) \to 0 = 1 \vee 0 = 1$. For $x \in B$, $x \oplus 0 = \neg(\neg x \cdot \neg 0) = \neg(\neg x \cdot 1) = \neg\neg x = x$. Since $0 \leq x \to 0$, $x \cdot 0 \leq x \cdot (x \to 0) \leq 0$, so $x \cdot 0 = 0$. Also, $x \oplus \neg 0 = \neg(\neg x \cdot 0) = \neg 0$. Before we show that equation (MV6) holds in $\mathbf{B}^*$, remember that if a,b, and c are elements of some commutative residuated lattice, then $a \to (b \to c) = ab \to c$.

For $x, y \in B$,

$$\neg(\neg x \oplus y) \oplus y =$$

$$(x \cdot \neg y) \oplus y =$$

$$\neg(\neg(x \cdot \neg y) \cdot \neg y) =$$

$$[x \cdot (y \to 0) \to 0] \cdot [y \to 0] \to 0 =$$

$$[x \to ((y \to 0) \to 0)] \cdot [y \to 0] \to 0 =$$

$$(x \to y)(y \to 0) \to 0 =$$

$$(x \to y) \to ((y \to 0) \to 0) =$$

$$(x \to y) \to y =$$

$$x \vee y.$$

By symmetry, we observe that $\neg(\neg y \oplus x) \oplus x = y \vee x = x \vee y = \neg(\neg x \oplus y) \oplus y$. Therefore, $\mathbf{B}^*$ is an MV-algebra.

To finish the proof of the term-equivalence, it remains to show that $(\mathbf{A}')^* = \mathbf{A}$ and $(\mathbf{B}^*)' = \mathbf{B}$. To show the first, let 0_1, $\neg_1$, and $\oplus_1$ be the operations of $\mathbf{A}$ and let 0_2, $\neg_2$ and $\oplus_2$ be the operations of $(\mathbf{A}')^*$. Clearly, $0_1 = 0_2$. For any $x \in A$, $\neg_2 x = x \to 0 = \neg_1 x \oplus_1 0 = \neg_1 x$. For $x, y \in A$, $x \oplus_2 y = \neg_2(\neg_2 x \cdot \neg_2 y) = \neg_1(\neg_1 x \cdot \neg_1 y) = \neg_1(\neg_1(\neg_1\neg_1 x \oplus_1 \neg_1\neg_1 y)) = x \oplus_1 y$. Therefore, $(\mathbf{A}')^* = \mathbf{A}$.

Let $\mathbf{B}=\langle B,\wedge_1,\vee_1,\cdot_1,\to_1,1_1,0_1\rangle$ and let $(\mathbf{B}^*)'$ be the algebra $\langle B,\wedge_2,\vee_2,\cdot_2,\to_2,1_2,0_2\rangle$. Clearly, $0_1=0_2$. (We will dispense with the subscripts, and just refer to 0.) For $x,y\in B$, $x\cdot_2 y=\neg(\neg x\oplus\neg y)=\neg\neg(\neg\neg x\cdot_1\neg\neg y)=x\cdot_1 y$. For any $x,y\in A$, $x\to_2 y=\neg x\oplus y=\neg(x\cdot_1\neg y)=(x\cdot_1(y\to_1 0)\to_1 0=x\to_1((y\to_1 0)\to_1 0)=x\to_1 y$. As we previously noticed, $1_1=0\to_1 0=\neg 0=1_2$. In order to show that $\wedge_1=\wedge_2$ and $\vee_1=\vee_2$, it suffices to show that the corresponding partial orders ($\leq_1$ and $\leq_2$) are the same. For $x,y\in B$, $x\leq_2 y$ iff $x\cdot_2\neg y=0$ iff $x\cdot_1\neg y=0$ iff $x\cdot_1(y\to_1 0)=0$, which implies that $x\leq_1(y\to_1 0)\to_1 0=y$. If $x\leq_1 y$, then $x\cdot_1(y\to_1 0)\leq_1 y\cdot_1(y\to_1 0)\leq_1 0$, so $x\cdot_1(y\to_1 0)=0$. Thus, $x\leq_2 y$ iff $x\leq_1 y$. Therefore, $(\mathbf{B}^*)'=\mathbf{B}$.

We see then that the class of MV-algebras (classically defined) is precisely the class of bounded integral commutative GMV-algebras. Another description of MV-algebras that is useful was given by Mundici in [40], where a pointed Abelian ℓ-group $\langle\mathbf{G},a\rangle$ is called **unital** if for every $x\in G$, there exists a natural number n such that $a^n\leq x\leq a^{-n}$ (or, equivalently, the convex subalgebra generated by the negative element a is the entire ℓ-group G).

Theorem II.2.4. *The categories of MV-algebras and unital Abelian ℓ-groups are categorically equivalent.*

In fact, for any negative element $a\in G$, the interval $[a,1]$ in $\mathbf{G}$ is an MV-algebra, where for any $x,y\in[a,1]$, $x\odot y=(x\cdot y)\vee a$, and $x\to y=(x^{-1}\cdot y)\wedge 1$.

II.3 Modal Operators on Residuated Lattices

In this section, we define and discuss the basic properties of modal operators (nuclei and conuclei) on residuated lattices. The results of this section can be found in [39].

Definition II.3.1. *An* **interior operator** *on a poset $\mathbf{P}=\langle P,\leq\rangle$ is a map $\sigma:P\to P$ such that σ is order-preserving, $\sigma(x)\leq x$, and $\sigma(\sigma(x))=\sigma(x)$, for all $x\in P$.*

Definition II.3.2. *A subposet $\mathbf{Q}$ of a poset $\mathbf{P}$ is called an* **interior system** *of $\mathbf{P}$ if it satisfies the condition:*

($*$) for all $x\in P$, the set $\{a\in Q\mid a\leq x\}$ has a greatest element.

Proposition II.3.3. *There is a bijective correspondence between interior operators σ on a poset $\mathbf{P}$ and interior systems $\mathbf{Q}$ of $\mathbf{P}$.*

Proof. Let σ be an interior operator on a poset $\mathbf{P}$. Let $\mathbf{Q} = P_\sigma = \sigma[P]$, endowed with the partial order given by $\mathbf{P}$. For $x \in P$, consider the set $S = \{a \in Q \mid a \leq x\}$. We wish to show that $\sigma(x)$ is the greatest element in S. Since $\sigma(x) \leq x$ and $\sigma(x) \in Q$, $\sigma(x) \in S$. For any other element $b \in S$, there exists $c \in P$ such that $b = \sigma(c)$, so $\sigma(b) = \sigma(\sigma(c)) = \sigma(c) = b$. Thus, since $b \leq x$, $b = \sigma(b) \leq \sigma(x)$. Therefore, $\sigma(x)$ is the greatest element in S.

Let $\mathbf{Q}$ be a subposet of $\mathbf{P}$ which satisfies the condition $(*)$. Define $\sigma : P \to P$ by $\sigma(x) = max\{a \in Q \mid a \leq x\}$, for all $x \in P$. Condition $(*)$ ensures that σ is well-defined. For each $x \in P$, let $S_x = \{a \in Q \mid a \leq x\}$. If $x, y \in P$ are such that $x \leq y$, then $S_x \subseteq S_y$ since $a \leq x$ implies $a \leq y$. Thus, $\sigma(x) = max(S_x) \leq max(S_y) = \sigma(y)$. Since $\sigma(x)$ is the greatest element in S_x, it is in particular in S_x. Thus, $\sigma(x) \leq x$. This argument also shows that $\sigma(\sigma(x)) \leq \sigma(x)$. Also, for $a \in Q$, if $a \leq x$, then $a = \sigma(a) \leq \sigma(x)$, so $\sigma(x) = max(S_x) \leq max(S_{\sigma(x)}) = \sigma(\sigma(x))$. Therefore, σ is an interior operator on $\mathbf{P}$.

If σ is an interior operator on $\mathbf{P}$, let $\mathbf{Q} = P_\sigma$, and define $\sigma'(x) = max\{a \in Q \mid a \leq x\}$, for each $x \in P$. We wish to show that $\sigma = \sigma'$. For $x \in P$, $\sigma'(x) = max\{a \in Q \mid a \leq x\} = \sigma(x)$, since we showed that $\{a \in Q \mid a \leq x\}$ has a greatest element by proving that it is $\sigma(x)$.

If $\mathbf{Q}$ is a subposet of $\mathbf{P}$ which satisfies $(*)$, let $\sigma(x)$ be the greatest element in the set $\{a \in Q \mid a \leq x\}$, and let $\mathbf{Q}' = P_\sigma$ endowed with the partial order of $\mathbf{P}$ restricted to Q. We wish to show that $Q = Q'$. If $x \in Q$, then $\sigma(x) = x$, since x is certainly the greatest element of the set $\{a \in Q \mid a \leq x\}$. Thus, $x \in Q'$. For the reverse inclusion, let $x \in Q'$. Then, there exists $y \in P$ such that $x = \sigma(y)$. Since $\sigma(y) = max\{a \in Q \mid a \leq y\}$, $x = \sigma(y) \in Q$. Therefore, $Q = Q'$. □

Definition II.3.4. *An interior operator σ on a residuated lattice $\mathbf{A}$ is a* **conucleus** *if $\sigma(1)$=1 and $\sigma(x)\sigma(y) \leq \sigma(xy)$, for all $x, y \in A$.*

Note that an interior operator on $\mathbf{A}$ satisfies the inequality $\sigma(x)\sigma(y) \leq \sigma(xy)$ if and only if it satisfies the equation $\sigma(\sigma(x)\sigma(y)) = \sigma(x)\sigma(y)$, for all $x, y \in A$. First, assume

$\sigma(x)\sigma(y) \leq \sigma(xy)$, for all $x, y \in A$. Then, since $\sigma(x) = \sigma(\sigma(x))$ and $\sigma(y)=\sigma(\sigma(y))$, $\sigma(x)\sigma(y)= \sigma(\sigma(x))\sigma(\sigma(y)) \leq \sigma(\sigma(x)\sigma(y)) \leq \sigma(x)\sigma(y)$, which establishes the desired equality. Next, assume $\sigma(\sigma(x)\sigma(y))= \sigma(x)\sigma(y)$, for all $x, y \in A$. Then, for $x, y \in A$, $\sigma(x)\sigma(y)= \sigma(\sigma(x)\sigma(y)) \leq \sigma(xy)$, since $\sigma(x) \leq x$, $\sigma(y) \leq y$, and multiplication and σ are order-preserving.

Also note that if a unary operation were added to the signature of residuated lattices, then the class of residuated lattices with a conucleus is an equational class and so also a variety. The variety of residuated lattices (commutative FL-algebras) with a conucleus will be denoted $\mathcal{RL}_\sigma$ ($\mathcal{CFL}_\sigma$).

Definition II.3.5. *An* **interior retract** $\mathbf{Q}$ *of a residuated lattice* $\mathbf{A}$ *is a subposet and submonoid of* $\mathbf{A}$ *that satisfies condition* $(*)$ *above.*

Proposition II.3.6. *There is a bijective correspondence between interior retracts of* $\mathbf{A}$ *and conuclei on* $\mathbf{A}$*.*

Proof. Let $\mathbf{Q}$ be an interior retract of $\mathbf{A}$. Define $\sigma : A \to A$ by $\sigma(x) = max\{a \in Q \mid a \leq x\}$, for each $x \in A$. Since $\mathbf{Q}$ is already a subposet of $\mathbf{A}$ which satisfies $(*)$, we know that σ is an interior operator of $\mathbf{A}$. It remains to show that $\sigma(1)=1$ and $\sigma(x)\sigma(y) \leq \sigma(xy)$, for all $x, y \in A$. Since $\mathbf{Q}$ is a submonoid of $\mathbf{A}$, in particular, $1 \in Q$, and $\sigma(1) = 1$. For $x, y \in A$, $\sigma(x), \sigma(y) \in Q$, $\sigma(x) \leq x$, and $\sigma(y) \leq y$ (since σ is an interior operator). Since $\mathbf{Q}$ is a submonoid of $\mathbf{A}$, $\sigma(x)\sigma(y) \in Q$, and since multiplication is order-preserving, $\sigma(x)\sigma(y) \leq xy$. Since $\sigma(xy) = max\{a \in Q \mid a \leq xy\}$, $\sigma(x)\sigma(y) \leq \sigma(xy)$. Therefore, σ is a conucleus on $\mathbf{A}$.

If σ is a conucleus on $\mathbf{A}$, define $\mathbf{Q}$ as the image on A under σ, endowed with the order in $\mathbf{A}$. Since σ is an interior operator, $\mathbf{Q}$ is a subposet of $\mathbf{A}$, which satisfies $(*)$. Since $\sigma(1) = 1$, $1 \in Q$. For $\sigma(x), \sigma(y) \in Q$, $\sigma(x)\sigma(y) = \sigma(\sigma(x)\sigma(y)) \in Q$ (since σ is a conucleus). This shows that $\mathbf{Q}$ is a submonoid of $\mathbf{A}$, and so it is also an interior retract. This bijective correspondence then follows from the previous bijective correspondence. □

Proposition II.3.7. *If* $\mathbf{A}{=}\langle A, \wedge, \vee, \cdot, \backslash, /, 1, \sigma\rangle$ *is a residuated lattice with a conucleus* σ*, then the algebra* $\mathbf{A}_\sigma{=}\langle A_\sigma, \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1\rangle$ *is a residuated lattice, where* $A_\sigma{=}\sigma[A]$ *and for all* $x, y \in A_\sigma$*,* $x \wedge_\sigma y{=}\sigma(x \wedge y)$*,* $x\backslash_\sigma y{=}\sigma(x\backslash y)$*, and* $x/_\sigma y{=}\sigma(x/y)$*.*

Proof. Since σ is a conucleus on $\mathbf{A}$, we know that $\mathbf{A}_\sigma$ is an interior retract of $\mathbf{A}$, and in particular, it is a subposet and submonoid of $\mathbf{A}$. Thus, $\langle A_\sigma, \cdot, 1\rangle$ is a monoid. Now, let $x, y \in A_\sigma$. Then, in $\mathbf{A}$, $x \leq x \vee y$ and $y \leq x \vee y$, so (since σ is idempotent) $x = \sigma(x) \leq \sigma(x \vee y)$ and $y = \sigma(y) \leq \sigma(x \vee y)$. Since $\sigma(x \vee y) \in A_\sigma$, $\sigma(x \vee y)$ is an upper bound of x and y (in $\mathbf{A}_\sigma$ and so also in $\mathbf{A}$). Since $x \vee y$ is the least upper bound of x and y (in $\mathbf{A}$), $x \vee y \leq \sigma(x \vee y) \leq x \vee y$. Thus, $\sigma(x \vee y) = x \vee y$ is the least upper bound of x and y (in $\mathbf{A}_\sigma$). In order to show that $\langle A_\sigma, \wedge_\sigma, \vee\rangle$ is a lattice, it remains to show that for all $x, y \in A_\sigma$, $x \wedge_\sigma y$ is the greatest lower bound of x and y in $\mathbf{A}_\sigma$. Let $x, y \in A_\sigma$. Since $x \wedge y$ is a lower bound (in $\mathbf{A}$) of x and y and σ is idempotent and order-preserving, $x \wedge_\sigma y = \sigma(x \wedge y)$ is a lower bound (in $\mathbf{A}_\sigma$) of x and y. If $z \in A_\sigma$ is a lower bound of x and y, then $z \leq x \wedge y$, so $z = \sigma(z) \leq \sigma(x \wedge y) = x \wedge_\sigma y$. Thus, $x \wedge_\sigma y$ is the greatest lower bound (in $\mathbf{A}_\sigma$) of x and y. Therefore, $\langle A_\sigma, \wedge_\sigma, \vee\rangle$ is a lattice.

It remains to show that for all $x, y, z \in A_\sigma$, $x \cdot y \leq z$ iff $y \leq x\backslash_\sigma z$ iff $x \leq z/_\sigma y$. Assume $x, y, z \in A_\sigma$. First, observe that $x \leq z/_\sigma y$ iff $x \leq z/y$. If $x \leq z/y$, then we see that $x = \sigma(x) \leq \sigma(z/y) = z/_\sigma y$. If $x \leq z/_\sigma y$, then $x \leq \sigma(z/y) \leq z/y$. Similarly, $y \leq x\backslash_\sigma z$ iff $y \leq x\backslash z$. Therefore, $x \leq z/_\sigma y$ iff $x \leq z/y$ iff $x \cdot y \leq z$ iff $y \leq x\backslash z$ iff $y \leq x\backslash_\sigma z$, which is what was required to prove. □

The conuclear image functor Γ sends a modal residuated lattice $\langle \mathbf{A}, \sigma\rangle$ to its image $\mathbf{A}_\sigma$, and a modal residuated lattice homomorphism $f : \langle \mathbf{A}, \sigma\rangle \to \langle \mathbf{B}, \tau\rangle$ to its restriction $\Gamma(f)$ on the image $\sigma[A]$. We need to show that this restriction maps into $\tau[B]$ and is a residuated lattice homomorphism. First, since f preserves conuclei, for every $x \in A$, $\Gamma(f)(\sigma(x)) = f(\sigma(x)) = \tau(f(x))$, so $\Gamma(f)$ maps $\sigma[A]$ into $\tau[B]$. Also, $\Gamma(f)(1) = f(1) = 1$. Consider $\circ \in \{\vee, \cdot\}$. Then, for $a, b \in \sigma[A]$, $\Gamma(f)(a \circ b) = f(a \circ b) = f(a) \circ f(b) = \Gamma(f)(a) \circ \Gamma(f)(b)$. Now, consider $\circ \in \{\wedge, \backslash, /\}$. For $x, y \in A$,

$$\Gamma(f)(\sigma(x) \circ_\sigma \sigma(y)) =$$

$$\Gamma(f)(\sigma(\sigma(x) \circ \sigma(y))) =$$

$$f(\sigma(\sigma(x) \circ \sigma(y))) =$$

$$\tau(f(\sigma(x) \circ \sigma(y))) =$$

$$\tau(f(\sigma(x)) \circ f(\sigma(y))) =$$

$$\tau(\tau(f(x)) \circ \tau(f(y))) =$$

$$\tau(f(x)) \circ_\tau \tau(f(y)) =$$

$$f(\sigma(x)) \circ_\tau f(\sigma(y)) =$$

$$\Gamma(f)(\sigma(x)) \circ_\tau \Gamma(f)(\sigma(y)).$$

Therefore, $\Gamma(f)$ is in fact a residuated lattice homomorphism.

Example II.3.1. *An* **interior algebra** *is a Boolean algebra* $\mathbf{A} = \langle A, \wedge, \vee, \neg, 1, 0\rangle$ *with an interior operator* σ *such that* $\sigma(1) = 1$ *and for every* $x, y \in A$, $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$.

The variety $\mathcal{BA}_\sigma$ of interior algebras is term-equivalent to the subvariety of $\mathcal{CFL}_\sigma$ defined by adding the equations corresponding to Boolean algebras. With the previously illustrated term-equivalence between Boolean algebras and a subvariety of $\mathcal{CFL}$ already in place, it only remains to show that for a Boolean algebra $\mathbf{A}$ with an interior operator σ, the identity $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$ holds for all $x, y \in A$ iff the inequality $\sigma(x)\sigma(y) \leq \sigma(xy)$ holds for all $x, y \in A$. First, notice that the inequality $\sigma(x \wedge y) \leq \sigma(x) \wedge \sigma(y)$ holding for all $x, y \in A$ is actually equivalent to the assertion that σ is order-preserving. If σ is order-preserving, then since $x \wedge y \leq x$ and $x \wedge y \leq y$, $\sigma(x \wedge y) \leq \sigma(x)$ and $\sigma(x \wedge y) \leq \sigma(y)$, so $\sigma(x \wedge y) \leq \sigma(x) \wedge \sigma(y)$. For the other direction, assume $a \leq b$, for $a, b \in A$. Then, $a = a \wedge b$, so $\sigma(a) = \sigma(a \wedge b) \leq \sigma(a) \wedge \sigma(b) \leq \sigma(a)$, and $\sigma(a) \leq \sigma(b)$. Thus, σ is order-preserving. Now, since $\wedge = \cdot$ in $\mathbf{A}$, it follows that $\sigma(x) \wedge \sigma(y) \leq \sigma(x \wedge y)$ for all $x, y \in A$ iff $\sigma(x)\sigma(y) \leq \sigma(xy)$ for all $x, y \in A$. This establishes the suggested term-equivalence.

The previous concepts were motivated by interior operators in topological spaces. Sometimes, as is also true in the field of topology, it is more convenient to consider the dual concept of closure operators.

Definition II.3.8. *A* **closure operator** *on a poset* $\mathbf{P} = \langle P, \leq \rangle$ *is a map* $\gamma : P \to P$ *such that* γ *is order-preserving,* $x \leq \gamma(x)$*, and* $\gamma(\gamma(x))=\gamma(x)$*, for all* $x \in P$*.*

Notice that γ is a closure operator on a poset $\mathbf{P} = \langle P, \leq \rangle$ if and only if γ is an interior operator on the dual poset $\mathbf{P}^\partial = \langle P, \leq^\partial \rangle$, where $x \leq^\partial y$ in $\mathbf{P}^\partial$ iff $y \leq x$ in $\mathbf{P}$.

Definition II.3.9. *A subposet* $\mathbf{Q}$ *of a poset* $\mathbf{P}$ *is called a* **closure system** *of* $\mathbf{P}$ *if it satisfies the condition:*

(∗∗) for all $x \in P$*, the set* $\{a \in Q \mid x \leq a\}$ *has a least element.*

Proposition II.3.10. *There is a bijective correspondence between closure operators* γ *on a poset* $\mathbf{P}$ *and closure systems* $\mathbf{Q}$ *of* $\mathbf{P}$*.*

Proof. The proposition follows from the previous note, the bijective correspondence between interior operators on $\mathbf{P}^\partial$ and interior systems of $\mathbf{P}^\partial$, and the fact that for any subset $R \subseteq P$ and $x \in P$, whenever one of $max_{\mathbf{P}^\partial}\{a \in R \mid a \leq^\partial x\}$ and $min_{\mathbf{P}}\{a \in R \mid x \leq a\}$ exists, the other does, as well, and they are equal. □

Definition II.3.11. *A closure operator* γ *on a residuated lattice* $\mathbf{A}$ *is a* **nucleus** *if* $\gamma(x)\gamma(y) \leq \gamma(xy)$*, for all* $x, y \in A$*.*

Definition II.3.12. *A* **closure retract** $\mathbf{Q}$ *of a residuated lattice* $\mathbf{A}$ *is a subposet of* $\mathbf{A}$ *that satisfies condition* $(**)$ *above and for all* $x \in A$ *and* $y \in Q$*,* $x\backslash y, y/x \in Q$*.*

Proposition II.3.13. *There is a bijective correspondence between closure retracts of* $\mathbf{A}$ *and nuclei on* $\mathbf{A}$*.*

Proof. We already know that there is a bijective correspondence between closure operators on $\mathbf{A}$ and subposets $\mathbf{Q}$ of $\mathbf{A}$ which satisfy $(**)$, and that this correspondence is given by $\gamma \mapsto \gamma(A)$ and $\mathbf{Q} \mapsto \gamma_{\mathbf{Q}}$, where $\gamma_{\mathbf{Q}}(x) = min\{a \in Q \mid x \leq a\}$, for all $x \in A$. We must show that whenever γ is a nucleus, $\gamma(A)$ is a closure retract, and whenever $\mathbf{Q}$ is a closure retract, $\gamma_{\mathbf{Q}}$ is a nucleus.

First, assume γ is a nucleus on $\mathbf{A}$. Let $\mathbf{Q}$ be the associated subposet of $\mathbf{A}$. Let $x \in P$ and $y \in Q = \gamma(A)$. Then, $\gamma(y/x)x \leq \gamma(y/x)\gamma(x) \leq \gamma((y/x)x) \leq \gamma(y) = y$ since γ is idempotent and $y \in \gamma(A)$. Thus, $\gamma(y/x) \leq y/x \leq \gamma(y/x)$, so $y/x = \gamma(y/x) \in Q$. Similarly, $x\gamma(x\backslash y) \leq \gamma(x)\gamma(x\backslash y) \leq \gamma(x(x\backslash y)) \leq \gamma(y) = y$, so $\gamma(x\backslash y) \leq x\backslash y \leq \gamma(x\backslash y)$, and $x\backslash y = \gamma(x\backslash y) \in Q$. Therefore, $\mathbf{Q}$ is a closure retract of $\mathbf{A}$.

Now, assume $\mathbf{Q}$ is a closure retract of $\mathbf{A}$. Let $\gamma(x)$ be the least element in $\{a \in Q \mid x \leq a\}$, for each $x \in A$. Since we already know that γ is a closure operator on $\mathbf{A}$, it only remains to show that γ satisfies $\gamma(x)\gamma(y) \leq \gamma(xy)$, for all $x, y \in A$. Clearly, for every $b \in Q$, b is the least element of the set $\{a \in Q \mid b \leq a\}$, so $\gamma(b) = b$. Also, for any $x \in A$, since $\gamma(x)$ is the minimum of the set $\{a \in Q \mid x \leq a\}$, $\gamma(x) \in Q$. Now, assume $x, y \in A$. Then, since γ is extensive, $xy \leq \gamma(xy)$, so $x \leq \gamma(xy)/y$. Since γ preserves order, $\gamma(x) \leq \gamma(\gamma(xy)/y)$. Since $\gamma(xy) \in Q$ and $\mathbf{Q}$ is a closure retraction, $\gamma(xy)/y \in Q$. Thus, $\gamma(x) \leq \gamma(xy)/y$ since γ fixes Q. By the residuation law, $y \leq \gamma(x)\backslash\gamma(xy) \in Q$, so $\gamma(y) \leq \gamma(x)\backslash\gamma(xy)$, and $\gamma(x)\gamma(y) \leq \gamma(xy)$. Therefore, γ is a nucleus on $\mathbf{A}$. □

Proposition II.3.14. *If* $\mathbf{A}{=}\langle A, \wedge, \vee, \cdot, \backslash, /, 1, \gamma\rangle$ *is a residuated lattice with a nucleus* γ*, then the algebra* $\mathbf{A}_\gamma{=}\langle A_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1)\rangle$ *is a residuated lattice, where* $A_\gamma{=}\gamma[A]$ *and for all* $x, y \in A_\gamma$*,* $x \vee_\gamma y{=}\gamma(x \vee y)$ *and* $x \cdot_\gamma y{=}\gamma(x \cdot y)$*.*

Proof. In the proof that $\mathbf{A}_\gamma$ is a closure retract of $\mathbf{A}$, we showed that for any $x \in A$ and $y \in A_\gamma$, $x\backslash y, y/x \in A_\gamma$, by showing that $x\backslash y = \gamma(x\backslash y)$ and $y/x = \gamma(y/x)$. So, in particular, for any $a, b \in A$, since $\gamma(a) \in A_\gamma$ and $\gamma(b) \in A$, we see that $\gamma(b)\backslash\gamma(a) = \gamma(\gamma(b)\backslash\gamma(a))$ and $\gamma(a)/\gamma(b) = \gamma(\gamma(a)/\gamma(b))$. Therefore, $\mathbf{A}_\gamma$ is closed under $\backslash$ and $/$.

For $x, y \in A_\gamma$, $x \wedge y \leq x$ and $x \wedge y \leq y$, so $\gamma(x \wedge y) \leq \gamma(x) = x$ and $\gamma(x \wedge y) \leq \gamma(y) = y$.

Thus, $\gamma(x \wedge y) \leq x \wedge y \leq \gamma(x \wedge y)$, so $x \wedge y = \gamma(x \wedge y)$ is the greatest lower bound of x and y (in $\mathbf{A}_\gamma$). In order to show that $\langle A_\gamma, \wedge, \vee_\gamma \rangle$ is a lattice, it remains to show that for every $x, y \in A_\gamma$, $x \vee_\gamma y$ is the least upper bound (in $\mathbf{A}_\gamma$) of x and y. Let $x, y \in A_\gamma$. Since $x \vee y$ is an upper bound of x and y, $\gamma(x \vee y) = x \vee_\gamma y$ is also an upper bound of x and y. If $z \in A_\gamma$ is an upper bound of x and y, then $x \vee y \leq z$, so $x \vee_\gamma y = \gamma(x \vee y) \leq \gamma(z) = z$. Thus, $x \vee_\gamma y$ is the least upper bound of x and y in $\mathbf{A}_\gamma$. Therefore, $\langle A_\gamma, \wedge, \vee_\gamma \rangle$ is indeed a lattice.

Next, we will show that $\langle A_\gamma, \cdot_\gamma, \gamma(1) \rangle$ is a monoid. First, observe that for any $a, b \in A$, $\gamma(\gamma(a)\gamma(b)) = \gamma(ab)$. To show this, let $a, b \in A$. Then, $\gamma(ab) \leq \gamma(\gamma(a)\gamma(b)) \leq \gamma(\gamma(ab)) = \gamma(ab)$, which establishes the desired equality. Now, take $x, y, z \in A_\gamma$. Then, $x \cdot_\gamma (y \cdot_\gamma z) = \gamma(x\gamma(yz)) = \gamma(\gamma(x)\gamma(yz)) = \gamma(x(yz)) = \gamma((xy)z) = \gamma(\gamma(xy)\gamma(z)) = \gamma(\gamma(x \cdot y) \cdot z) = (x \cdot_\gamma y) \cdot_\gamma z$, which shows that $\cdot_\gamma$ is associative. Also, $x \cdot_\gamma \gamma(1) = \gamma(x\gamma(1)) = \gamma(\gamma(x)\gamma(1)) = \gamma(x \cdot 1) = \gamma(x) = x = \gamma(x) = \gamma(1 \cdot x) = \gamma(\gamma(1)\gamma(x)) = \gamma(\gamma(1) \cdot x) = \gamma(1) \cdot_\gamma x$, showing that $\gamma(1)$ is the identity of $\cdot_\gamma$. Therefore, $\langle A_\gamma, \cdot_\gamma, \gamma(1) \rangle$ is a monoid.

To show that $\cdot_\gamma$ is a residuated map (with residuals $\backslash$ and $/$), first observe that for all $x, y, z \in A_\gamma$, $xy \leq z$ iff $x \cdot_\gamma y \leq z$. Assume $xy \leq z$. Then, $x \cdot_\gamma y = \gamma(xy) \leq \gamma(z) = z$. Now, assume $x \cdot_\gamma y \leq z$. Then, $xy \leq \gamma(xy) = x \cdot_\gamma y \leq z$. Thus, $x \cdot_\gamma y \leq z$ iff $xy \leq z$ iff $y \leq x \backslash z$ iff $x \leq z/y$. Therefore, we have shown that $\langle A_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1) \rangle$ is a residuated lattice. $\square$

As with the conuclei, there is a nuclear image functor sending modal residuated lattices to their images and homomorphisms to their restrictions on these images.

The usefulness of these modal operators is that sometimes more complicated algebras can arise as the images of simpler ones (or at least better understood ones) with a modal operator. For example, the variety of Heyting algebras is precisely the class of images of interior algebras [36]. Also, commutative, cancellative residuated lattices are the images of Abelian ℓ-groups with a conucleus (denoted $\mathcal{A}_\sigma$) [39]. Lastly, integral GMV-algebras are the images of negative cones of ℓ-groups with nuclei (denoted $\mathcal{LG}^-_\gamma$) [25]. Considering that for any element a in an integral residuated lattice, $\gamma_a(x) = x \vee a$ defines a nucleus, we see that MV-algebras are precisely the images of pointed negative cones of Abelian ℓ-groups.

Throughout the rest of the dissertation, it may prove helpful to refer to the following table:

Table II.1: Examples of the (Co)nuclear Image Functor

Modal Variety	(Co)nuclear Images	Subreducts
$\mathcal{BA}_\sigma$	$\mathcal{HA}$	$\mathcal{DL}_{0,1}$
$\mathcal{A}_\sigma$	$\mathcal{CC}an\mathcal{RL}$	$\mathcal{JCM}$
$\mathcal{LG}_\gamma^-$	$\mathcal{IGMV}$	$\mathcal{CA}$

The purpose of the next couple chapters is to reconstruct the results about the connections between these varieties. First, since the image of a residuated lattice with a conucleus is a $\{\vee, \cdot, 1\}$-subreduct of the original residuated lattice and the image of a residuated lattice with a nucleus is a $\{\wedge, \backslash, /\}$-subreduct, it is worth investigating the appropriate classes of subreducts of $\mathcal{BA}$, $\mathcal{A}$, and $\mathcal{LG}^-$. This is done in the next chapter.

CHAPTER III

REDUCTS

In this chapter, we will begin the construction of the free objects with respect to the modal image functors, by constructing free objects with respect to forgetful functors. We will need to consider the following (quasi)varieties. First, let $\mathcal{DL}_{0,1}$ be the variety of bounded, distributive lattices. Let $\mathcal{JCM}$ be the quasivariety of join-semilattice-ordered, commutative, cancellative monoids that satisfy the equation $x(y \vee z) \approx xy \vee xz$. Lastly, let $\mathcal{CA}$ be the variety of cone algebras, which are structures $\langle C, \backslash, /, 1\rangle$ that satisfy the following equations:

(1) $1\backslash x \approx x$ and $x/1 \approx x$

(2) $x\backslash x \approx 1$ and $x/x \approx 1$

(3) $(x\backslash y)\backslash(x\backslash z) \approx (y\backslash x)\backslash(y\backslash z)$ and $(z/x)/(y/x) \approx (z/y)/(x/y)$

(4) $x\backslash(y/z) \approx (x\backslash y)/z$ and $x/(y\backslash x) \approx y/(x\backslash y)$

III.1 Subreducts

By definition, every $\{\wedge, \vee, 0, 1\}$-reduct of a Boolean algebra is a bounded, distributive lattice. It is clear that every $\{\vee, \cdot, 1\}$-reduct of an Abelian ℓ-group is in $\mathcal{JCM}$. Lastly, one can show that the $\{\backslash, /, 1\}$-reduct of a negative cone of an ℓ-group $\mathbf{A}$ is a cone algebra. For example, for $x, y, z \in A$,

$$(x\backslash y)\backslash(x\backslash z) =$$

$$(x^{-1}y \wedge 1)^{-1}(x^{-1}z \wedge 1) \wedge 1 =$$

$$(y^{-1}x \vee 1)(x^{-1}z \wedge 1) \wedge 1 =$$

$$(y^{-1}x \vee 1)x^{-1}z \wedge 1 =$$

$$(y^{-1}z \vee x^{-1}z) \wedge 1 =$$

$$(x^{-1}z \vee y^{-1}z) \wedge 1 =$$

$$(x^{-1}y \vee 1)y^{-1}z \wedge 1 =$$

$$(x^{-1}y \vee 1)(y^{-1}z \wedge 1) \wedge 1 =$$

$$(y^{-1}x \wedge 1)^{-1}(y^{-1}z \wedge 1) \wedge 1 =$$

$$(y\backslash x)\backslash(y\backslash z),$$

and $x\backslash(y/z) = x^{-1}(yz^{-1} \wedge 1) \wedge 1 = x^{-1}yz^{-1} \wedge x^{-1} \wedge 1 = x^{-1}yz^{-1} \wedge 1 = x^{-1}yz^{-1} \wedge z^{-1} \wedge 1 = (x^{-1}y \wedge 1)z^{-1} \wedge 1 = (x\backslash y)/z$.

Showing that every algebra in these (quasi)varieties arises as a subreduct of an algebra in the corresponding residuated lattice variety is much harder to show. However, in the first and third cases it is already known. We now go through the details for the second case.

Let $\mathbf{L} \in \mathcal{JCM}$. Since the monoid reduct of $\mathbf{L}$ is commutative and cancellative, there exists a group of quotients $\mathbf{G}(\mathbf{L})$ of this monoid. Next, we will show that a lattice order can be put on $\mathbf{G}(\mathbf{L})$ such that $\mathbf{L}$ embeds into the $\{\vee, \cdot, 1\}$-reduct of $\mathbf{G}(\mathbf{L})$. Clearly, the only possible order that can be put on $\mathbf{G}(\mathbf{L})$ that extends the order on $\mathbf{L}$ is the one given by $a^{-1}b \preceq c^{-1}d$ iff $cb \leq ad$, where $\leq$ is the order on $\mathbf{L}$. We will now show that this in fact defines a lattice order on $\mathbf{G}(\mathbf{L})$.

Theorem III.1.1. *Let $\mathbf{L} \in \mathcal{JCM}$. Let $\mathbf{G}(\mathbf{L})$ be the group of quotients of the monoid reduct of $\mathbf{L}$. Then we have the following:*

(1) The $\preceq$ defined above gives a partial order on $\mathbf{G}(\mathbf{L})$.

(2) The join of $a^{-1}b$ and $c^{-1}d$ exists in $\langle \mathbf{G}(\mathbf{L}), \preceq \rangle$ and is given by the formula

$$(a^{-1}b) \vee_{\mathbf{G}} (c^{-1}d) = (ac)^{-1}(cb \vee_{\mathbf{L}} ad).$$

(3) $\mathbf{L}$ is a $\{\vee, \cdot, 1\}$-subreduct of the Abelian ℓ-group $\mathbf{G}(\mathbf{L})$.

Proof.

(1) First, we need to show that $\preceq$ is well-defined. Assume $a^{-1}b = x^{-1}y$ and $c^{-1}d = z^{-1}w$. We need to show that $cb \leq ad$ iff $zy \leq xw$. By the cancellativity of $\mathbf{L}$, we see that $cb \leq ad$ iff $cbzy \leq adzy$. Since $xb = ay$ and $zd = cw$, this is true if and only if $cbzy \leq xbcw$, which again by cancellativity, is equivalent to $zy \leq xw$.

Clearly, $\preceq$ is reflexive, since $\mathbf{L}$ is commutative. Now, if $a^{-1}b \preceq c^{-1}d \preceq x^{-1}y$, then $cb \leq ad$ and $xd \leq cy$. So, $cbx \leq adx \leq acy$, and $bx \leq ay$. Thus, $a^{-1}b \preceq x^{-1}y$. Lastly, if $a^{-1}b \preceq c^{-1}d \preceq a^{-1}b$, then $cb \leq ad \leq cb$, so $ad = cb$, and $a^{-1}b = c^{-1}d$. This shows that $\preceq$ defines a partial order on $\mathbf{G}(\mathbf{L})$.

(2) We need to show that $(ac)^{-1}(cb \vee_{\mathbf{L}} ad)$ is an upper bound of $a^{-1}b$ and $c^{-1}d$. Since $cb \leq cb \vee_{\mathbf{L}} ad$, $acb \leq a(cb \vee_{\mathbf{L}} ad)$, and $a^{-1}b \preceq (ac)^{-1}(cb \vee_{\mathbf{L}} ad)$. Similarly, $c^{-1}d \preceq (ac)^{-1}(cb \vee_{\mathbf{L}} ad)$. If $x^{-1}y$ is an upper bound of $a^{-1}b$ and $c^{-1}d$, then $xb \leq ay$ and $xd \leq cy$. So, acy is an upper bound in $\mathbf{L}$ of cxb and axd. Thus, $x(cb \vee_{\mathbf{L}} ad) = xcb \vee_{\mathbf{L}} xad \leq acy$. Therefore, $(ac)^{-1}(cb \vee_{\mathbf{L}} ad) \preceq x^{-1}y$, which completes the proof that $(a^{-1}b) \vee_{\mathbf{G}} (c^{-1}d) = (ac)^{-1}(cb \vee_{\mathbf{L}} ad)$.

(3) Observe that $\mathbf{G}(\mathbf{L})$ is an Abelian ℓ-group since for $x, y \in G(\mathbf{L})$, $x \wedge_{\mathbf{G}} y = (x^{-1} \vee_{\mathbf{G}} y^{-1})^{-1}$. Also, if $a^{-1}b \preceq c^{-1}d$, then $cb \leq ad$, so $cbxy \leq adxy$, and $(ax)^{-1}by \preceq (cx)^{-1}dy$, which can also be written as $a^{-1}bx^{-1}y \preceq c^{-1}dx^{-1}y$.

Since $\mathbf{G}(\mathbf{L})$ is the group of quotients of the monoid reduct of $\mathbf{L}$, we know that $\mathbf{L}$ embeds into the monoid reduct of $\mathbf{G}(\mathbf{L})$. To show that this embedding preserves joins, let $a, b \in L$. Then, $a \vee_{\mathbf{G}} b = 1^{-1}a \vee_{\mathbf{G}} 1^{-1}b = (1 \cdot 1)^{-1}(1 \cdot a \vee_{\mathbf{L}} 1 \cdot b) = a \vee_{\mathbf{L}} b$.

$\square$

Lemma III.1.2. *We have the following:*

(1) The class of $\{\wedge, \vee, 0, 1\}$*-subreducts of Boolean algebras is precisely the class of bounded, distributive lattices.*

(2) The class of $\{\vee, \cdot, 1\}$-subreducts of Abelian ℓ-groups is precisely $\mathcal{JCM}$.

(3) The class of $\{\backslash, /, 1\}$-subreducts of negative cones of ℓ-groups is precisely the class of cone algebras.

Proof. The details for (1) and (3) can be found in [4] and [11], respectively. Lastly, (2) follows from Theorem III.1.1 (3). $\square$

III.2 Adjunction

We wish to show that in the above cases, the constructed objects are free with respect to the appropriate forgetful functors. First, however, we will show that such a free object always exists. Let $\mathcal{V}$ be a variety in signature τ. Consider the variety $\mathcal{W}$ of all algebras of signature $\tau' \subseteq \tau$. Then, there is a forgetful functor $G : \mathcal{V} \to \mathcal{W}$ that sends each algebra in $\mathcal{V}$ to its τ'-reduct. We will show that this functor has a left adjoint.

Theorem III.2.1. *Let $\mathbf{A}$ be an algebra in $\mathcal{W}$. Then, there exists a free object in $\mathcal{V}$ with respect to G.*

Proof. First, let $\mathbf{F}$ be the free algebra in $\mathcal{V}$ generated by the set A, the underlying set of $\mathbf{A}$. Let X be the set of ordered pairs of the form $(f^{\mathbf{A}}(a_1, ..., a_n), f^{\mathbf{F}}(a_1, ..., a_n))$, where $f \in \tau'$ has arity n and $a_1, ..., a_n \in A$. Let θ be the congruence relation of $\mathbf{F}$ that is generated by X. We wish to show that $\bar{\mathbf{F}} = \mathbf{F}/\theta$ is the suggested free object.

Consider the map $g : \mathbf{A} \to G(\bar{\mathbf{F}})$ that is the restriction of the canonical epimorphism $\pi : \mathbf{F} \to \bar{\mathbf{F}}$ to A. We wish to show that g is a homomorphism. Take $f \in \tau'$, with arity n, and $a_1, ..., a_n \in A$. Then, $g(f^{\mathbf{A}}(a_1, ..., a_n)) = \pi(f^{\mathbf{A}}(a_1, ..., a_n)) = \pi(f^{\mathbf{F}}(a_1, ..., a_n)) = f^{\bar{\mathbf{F}}}(\pi(a_1), ..., \pi(a_n)) = f^{\bar{\mathbf{F}}}(g(a_1), ..., g(a_n))$.

Now, assume that $h : \mathbf{A} \to G(\mathbf{B})$ is a homomorphism, where $\mathbf{B}$ is in $\mathcal{V}$. Since $\mathbf{F}$ is the free algebra generated by A, there exists a unique homomorphism $\bar{h} : \mathbf{F} \to \mathbf{B}$ that extends h. If we could show that $ker(\pi) \subseteq ker(\bar{h})$, there would exist a unique homomorphism

$h' : \bar{\mathbf{F}} \to \mathbf{B}$ such that $h' \circ \pi = \bar{h}$. To show this, take $f \in \tau'$, with arity n, and $a_1, ..., a_n \in A$. Then, $\bar{h}(f^{\mathbf{A}}(a_1, ..., a_n)) = h(f^{\mathbf{A}}(a_1, ..., a_n)) = f^{\mathbf{B}}(h(a_1), ..., h(a_n)) = f^{\mathbf{B}}(\bar{h}(a_1), ..., \bar{h}(a_n)) = \bar{h}(f^{\mathbf{F}}(a_1, ..., a_n))$. Since $X \subseteq ker(\bar{h})$, $ker(\pi) = \theta \subseteq ker(\bar{h})$. We see then that for $a \in A$, $(G(h') \circ g)(a) = h'(\pi(a)) = \bar{h}(a) = h(a)$. It only remains to show that this h' is unique with respect to satisfying $G(h') \circ g = h$. To that end, suppose $k : \bar{\mathbf{F}} \to \mathbf{B}$ satisfies $G(k) \circ g = h$. Then, for $a \in A$, $(k \circ \pi)(a) = (G(k) \circ g)(a) = h(a)$. By the uniqueness of $\bar{h}$, $k \circ \pi = \bar{h}$. Now, by the initial uniqueness of h', $k = h'$. $\square$

It is worth noting that although this free object exists, $\mathbf{A}$ will only embed into $\bar{\mathbf{F}}$ if it satisfies the quasi-equational theory of $\mathcal{V}$ in the language τ'. In particular, the class of τ' - subreducts of $\mathcal{V}$ always forms a quasi-variety.

III.3 Proof of Lemma III.3.3

Having these free objects will be important for our constructions in the next section, but we will need to say more in order to achieve the categorical equivalences. Thus, consider the following definitions.

Definition III.3.1. *A set T of terms (in the signature τ) is called* **representative** *with respect to a subvariety $\mathcal{W}'$ of $\mathcal{W}$ if for every algebra $\mathbf{A}$ in $\mathcal{W}'$, every element of $F(\mathbf{A})$ can be represented by $t(a_1, ..., a_n)$, for some term $t \in T$ and some $a_1, ..., a_n \in A$, where we are using a to stand for both the element in A as well as its equivalence class in $F(\mathbf{A})$.*

Obviously, for any subvariety $\mathcal{W}'$, the set of all terms in the signature τ is the largest representative set. As a more interesting example, in the situation of commutative groups and commutative, cancellative monoids, the set $\{x^{-1}y\}$ is a representative set.

Definition III.3.2. *A pair of terms (t_1, t_2) (in the signature τ and with no common variables) is called τ'-* **reducible** *if there exists pairs of terms (s_1^i, s_2^i) (in the signature τ' and variables among the variables in t_1 and t_2) such that for every algebra $\mathbf{A}$ in $\mathcal{V}$ and every*

$a_1, ..., a_n, b_1, ..., b_m \in A$, $t_1(a_1, ..., a_n) = t_2(b_1, ..., b_m)$ *iff for every* i, $s_1^i(a_1, ..., a_n, b_1, ..., b_m) = s_2^i(a_1, ..., a_n, b_1, ..., b_m)$.

Referring back to the commutative group example, the pair $(x^{-1}y, z^{-1}w)$ is monoid-reducible (by the pair (yz, xw)) since for every commutative group $\mathbf{G}$ and $a, b, c, d \in G$, $a^{-1}b = c^{-1}d$ iff $bc = ad$.

Lemma III.3.3. *If there exists a set of terms T that is representative with respect to a subvariety $\mathcal{W}'$ of $\mathcal{W}$ and every pair (t_1, t_2) of terms of T (where if $t_1 = t_2$, they are taken to have no common variables) is τ'-reducible, then whenever an algebra $\mathbf{A}$ in $\mathcal{W}'$ can be embedded into an algebra $G(\mathbf{B})$, where $\mathbf{B}$ is in $\mathcal{V}$, the subalgebra of $\mathbf{B}$ generated by the image of $\mathbf{A}$ is (isomorphic to) the free object $F(\mathbf{A})$.*

Proof. We can assume, without loss of generality, that the embedding $i : \mathbf{A} \to G(\mathbf{B})$ is in fact an inclusion function. Then, by the freeness of $F(\mathbf{A})$, there exists a unique homomorphism $h : F(\mathbf{A}) \to \mathbf{B}$ such that $G(h) \circ g = i$, where g is the restriction of the canonical epimorphism π to A. We need only show that h is an embedding (since the image of h is the subalgebra of $\mathbf{B}$ generated by A.)

Assume $h(x) = h(y)$, for $x, y \in F(\mathbf{A})$. By the representativity of T, there exists terms $t_1, t_2 \in T$ (not necessarily distinct) and elements $a_1, ..., a_n, b_1, ..., b_m \in A$ such that $x = t_1(a_1, ..., a_n)$ and $y = t_2(b_1, ..., b_m)$. Thus, ${t_1}^{\mathbf{B}}(a_1, ..., a_n) = {t_2}^{\mathbf{B}}(b_1, ..., b_m)$, since h is the identity on A. Let (s_1^i, s_2^i) be the pairs of τ' terms guaranteed to exist by the τ'-reducibility of the pairs of terms (t_1, t_2). Then, we see that for each i, ${s_1^i}^{\mathbf{B}}(a_1, ..., a_n, b_1, ..., b_m) = {s_2^i}^{\mathbf{B}}(a_1, ..., a_n, b_1, ..., b_m)$. So, in $F(\mathbf{A})$, $h(s_1^i(a_1, ..., a_n, b_1, ..., b_m)) = h(s_2^i(a_1, ..., a_n, b_1, ..., b_m))$. Since $\mathbf{A}$ can be embedded into $F(\mathbf{A})$, and s_1^i, s_2^i are τ' terms, this shows that for each i, $s_1^i(a_1, ..., a_n, b_1, ..., b_m) = s_2^i(a_1, ..., a_n, b_1, ..., b_m)$. Using τ'-reducibility again, we see that $x = t_1(a_1, ..., a_n) = t_2(b_1, ..., b_m) = y$. $\square$

As an example, consider the group of quotients of an algebra $\mathbf{L} \in \mathcal{JCM}$.

Lemma III.3.4. *If* $\mathbf{L} \in \mathcal{JCM}$ *generates an Abelian* ℓ*-group* $\mathbf{G}$*, then every element of* G *has the form* $a^{-1}b$*, for some* $a, b \in L$*.*

Proof. We merely need to show that the set $S = \{a^{-1}b \mid a, b \in L\}$ is a subalgebra of $\mathbf{G}$. Clearly, S is a subgroup of $\mathbf{G}$. Since $a^{-1}b \vee_{\mathbf{G}} c^{-1}d = (ac)^{-1}(bc \vee_{\mathbf{L}} ad)$ and for $x, y \in G$, $x \wedge y = (x^{-1} \vee y^{-1})^{-1}$, S is a subalgebra of $\mathbf{G}$. □

By Lemmas III.3.3 and III.3.4 and the fact that $a^{-1}b = c^{-1}d$ if and only if $bc = ad$, we obtain the following corollary.

Corollary III.3.5. *Whenever* $\mathbf{L} \in \mathcal{JCM}$ *is embedded into an Abelian* ℓ*-group* $\mathbf{G}$*, the subalgebra generated by* L *in* $\mathbf{G}$ *is the free object* $F(\mathbf{L})$ *with respect to the forgetful functor.*

In [39], Montagna and Tsinakis discuss arbitrary ℓ-groups, but must restrict their attention to Ore residuated lattices, because these are precisely the ones that can be embedded into ℓ-groups in a representable way (so they would then be entitled to use Lemma III.3.3). In this dissertation, we restrict to the commutative case because we will want to discuss varieties later that are not present in general.

III.4 Boolean Algebras Case

We will now show an analogous result of Corollary III.3.5 in the Boolean algebra case.

Lemma III.4.1. *Assume that the bounded distributive lattice* $\mathbf{L}$ *is a* $\{\wedge, \vee, 0, 1\}$*-subreduct of the Boolean algebra* $\mathbf{B}$*. Let* $\bar{\mathbf{L}}$ *be the Boolean subalgebra of* $\mathbf{B}$ *generated by* L*. Then, we have the following:*

(1) Every element of $\bar{\mathbf{L}}$ *can be written in the form* $\bigwedge_{i=1}^{n}(a_i \vee b_i')$*, where* n *is a natural number and for* $i = 1, ..., n$*,* $a_i, b_i \in L$*, and* x' *denotes the complement of* x*.*

(2) Let n *and* m *be natural numbers, and let* $a_i, b_i, c_j, d_j \in L$*, for* $i = 1, ..., n$ *and* $j = 1, ..., m$*. Let* $I = \{1, ..., n\}$ *and* $J = \{1, 2\}$*. Then,* $\bigwedge_{i=1}^{n}(a_i \vee b_i') \leq \bigwedge_{j=1}^{m}(c_j \vee d_j')$ *(in* $\mathbf{B}$*) iff for every* $j = 1, ..., m$ *and every* $f \in J^I$*,* $d_j \wedge \bigwedge_{i \in f^{-1}(1)} a_i \leq c_j \vee \bigvee_{i \in f^{-1}(2)} b_i$ *(in* $\mathbf{L}$*).*

Proof.

(1) Let L' be the set of elements of $\bar{\mathbf{L}}$ of the form $\bigwedge_{i=1}^{n}(a_i \vee b_i{}')$, where n is a natural number and for each $i = 1, ..., n$, $a_i, b_i \in L$. For each $a \in L$, since $1 \in L$, $a = a \vee 1' \in L'$. Since $L \subseteq L' \subseteq \bar{L}$, it suffices to show that L' is a Boolean algebra.

In order to do this, we need to show that L' is closed under the operations of $\mathbf{B}$. First, $\{0, 1\} \subseteq L \subseteq L'$. Clearly, L' is closed under meets. Now, consider $\bigwedge_{j=1}^{n}(a_j \vee b_j{}')$ and $\bigwedge_{j=1}^{m}(c_j \vee d_j{}')$. Let $x_{1j} = a_j \vee b_j{}'$ and $x_{2j} = c_j \vee d_j{}'$, and notice that by adding $1 \vee 0'$ wherever needed, we may assume that $n = m$. So, $\left(\bigwedge_{j=1}^{n}(a_j \vee b_j{}')\right) \vee \left(\bigwedge_{j=1}^{n}(c_j \vee d_j{}')\right)$ $= \bigvee_{i=1}^{2} \bigwedge_{j=1}^{n} x_{ij}$, which by the distributivity of the underlying lattice of $\mathbf{B}$ with $I = \{1, 2\}$ and $J = \{1, ..., n\}$, equals $\bigwedge_{f \in J^I} \bigvee_{i \in I} x_{if(i)}$. For each $f \in J^I$, $\bigvee_{i \in I} x_{if(i)} = x_{1f(1)} \vee x_{2f(2)} = (a_{f(1)} \vee b'_{f(1)}) \vee (c_{f(2)} \vee d'_{f(2)}) = (a_{f(1)} \vee c_{f(2)}) \vee (b_{f(1)} \wedge d_{f(2)})'$. Since $\mathbf{L}$ is a sublattice of $\mathbf{B}$, $a_{f(1)} \vee c_{f(2)}, b_{f(1)} \wedge d_{f(2)} \in L$. Thus, $\left(\bigwedge_{j=1}^{n}(a_j \vee b_j{}')\right) \vee \left(\bigwedge_{j=1}^{n}(c_j \vee d_j{}')\right)$ is a meet of the appropriate type of elements and so is in L'.

Lastly, $\left(\bigwedge_{i=1}^{n}(a_i \vee b_i{}')\right)' = \bigvee_{i=1}^{n}(b_i \wedge a_i')$. Letting $x_{i1} = b_i$, $x_{i2} = a_i'$, $I = \{1, ..., n\}$ and $J = \{1, 2\}$, this becomes $\bigvee_{i \in I} \bigwedge_{j \in J} x_{ij}$. Again by distributivity, this equals $\bigwedge_{f \in J^I} \bigvee_{i \in I} x_{if(i)}$. Fix $f \in J^I$ and let $I_1 = f^{-1}(1)$ and $I_2 = f^{-1}(2)$. Then, $\bigvee_{i \in I} x_{if(i)} = \left(\bigvee_{i \in I_1} x_{i1}\right) \vee \left(\bigvee_{i \in I_2} x_{i2}\right)$ $= \left(\bigvee_{i \in I_1} b_i\right) \vee \left(\bigvee_{i \in I_2} a_i'\right) = \left(\bigvee_{i \in I_1} b_i\right) \vee \left(\bigwedge_{i \in I_2} a_i\right)'$. (Note that for some f, I_1 or I_2 will be empty, but this is not a problem since the empty join is $0 \in L$.) Since $\bigvee_{i \in I_1} b_i$, $\bigwedge_{i \in I_2} a_i \in L$, $\left(\bigwedge_{i=1}^{n}(a_i \vee b_i{}')\right)'$ is a meet of the appropriate type of elements and so is in L'. This completes the proof that $L' = \bar{L}$, which establishes the result.

(2) First, $\bigwedge_{i=1}^{n}(a_i \vee b_i{}') \leq \bigwedge_{j=1}^{m}(c_j \vee d_j{}')$ iff for every $j = 1, ..., m$, $\bigwedge_{i=1}^{n}(a_i \vee b_i{}') \leq c_j \vee d_j{}'$. Now,

these inequalities are true iff for every j,

$$d_j \wedge \bigwedge_{i=1}^{n}(a_i \vee b_i{}') \leq c_j.$$

Letting $x_{i1} = a_i$ and $x_{i2} = b_i'$, we see that if $I = \{1, ..., n\}$ and $J = \{1, 2\}$,

$$d_j \wedge \bigwedge_{i=1}^{n}(a_i \vee b_i{}') =$$

$$d_j \wedge \bigwedge_{i \in I} \bigvee_{j \in J} x_{ij} =$$

$$d_j \wedge \bigvee_{f \in J^I} \bigwedge_{i \in I} x_{if(i)} =$$

$$\bigvee_{f \in J^I} \left(d_j \wedge \bigwedge_{i \in I} x_{if(i)} \right).$$

Thus, we see that for every j,

$$d_j \wedge \bigwedge_{i=1}^{n}(a_i \vee b_i{}') \leq c_j$$

iff for every j and every $f \in J^I$,

$$d_j \wedge \bigwedge_{i \in I} x_{if(i)} \leq c_j.$$

Since we observe that $$d_j \wedge \bigwedge_{i \in I} x_{if(i)} =$$

$$d_j \wedge \left(\bigwedge_{i \in f^{-1}(1)} x_{i1} \right) \wedge \left(\bigwedge_{i \in f^{-1}(2)} x_{i2} \right) =$$

$$d_j \wedge \left(\bigwedge_{i \in f^{-1}(1)} a_i \right) \wedge \left(\bigwedge_{i \in f^{-1}(2)} b_i' \right) =$$

$$d_j \wedge \left(\bigwedge_{i \in f^{-1}(1)} a_i \right) \wedge \left(\bigvee_{i \in f^{-1}(2)} b_i \right)',$$

these inequalities are true iff for every $j = 1, ..., m$ and every $f \in J^I$,

$$d_j \wedge \bigwedge_{i \in f^{-1}(1)} a_i \leq c_j \vee \bigvee_{i \in f^{-1}(2)} b_i.$$

This establishes the result.

$\square$

Therefore, by Lemma III.3.3, we obtain the following corollary.

Corollary III.4.2. *Whenever the bounded, distributive lattice* $\mathbf{L}$ *is embedded into the Boolean algebra* $\mathbf{B}$*, the subalgebra generated by* L *in* $\mathbf{B}$ *is the free object* $F(\mathbf{L})$ *with respect to the forgetful functor.*

III.5 Negative Cones Case

Lastly, we will show an analogous result in the case of the negative cones of ℓ-groups. Recall that in this case, $x \backslash y = x^{-1}y \wedge 1$ and $y/x = yx^{-1} \wedge 1$.

Lemma III.5.1. *Let* $\mathbf{A} \in \mathcal{LG}^-$*. Let* $a, b, c \in A$*. Then, the following equations (and their opposites) hold in* $\mathbf{A}$*:*

(1) $b \backslash bc = c$

(2) $a(a \backslash b) = b \wedge a$

(3) $ab \backslash c = b \backslash (a \backslash c)$

(4) $a \backslash bc = (a \backslash b)((b \backslash a) \backslash c)$

(5) $b/(a \backslash b) = b \vee a$

Proof.

(1) $b\backslash bc = b^{-1}(bc) \wedge 1 = c \wedge 1 = c$

(2) $a(a\backslash b) = a(a^{-1}b \wedge e) = b \wedge a$

(3) $b\backslash(a\backslash c) = b\backslash(a^{-1}c \wedge 1) = b^{-1}(a^{-1}c \wedge 1) \wedge 1 = b^{-1}a^{-1}c \wedge b^{-1} \wedge 1 = b^{-1}a^{-1}c \wedge 1 = (ab)^{-1}c \wedge 1 = ab\backslash c$

(4) Observe that $a(a\backslash b) = b \wedge a = a \wedge b = b(b\backslash a)$. So, $a(a\backslash b)((b\backslash a)\backslash c) = b(b\backslash a)((b\backslash a)\backslash c) \leq bc$. Thus, $(a\backslash b)((b\backslash a)\backslash c) \leq a\backslash bc$. For the other inequality, assume $x \in A$ and $ax \leq bc$. Then, by integrality, $ax \leq b$, so $x \leq a\backslash b$. Letting $y = (a\backslash b)\backslash x$, we see that $(a\backslash b)y = x \wedge (a\backslash b) = x$. Thus, $b(b\backslash a)y = a(a\backslash b)y = ax \leq bc$. This implies that $(b\backslash a)y = b\backslash(b(b\backslash a)y) \leq b\backslash bc = c$, so $y \leq (b\backslash a)\backslash c$, and $x = (a\backslash b)y \leq (a\backslash b)((b\backslash a)\backslash c)$. In particular, this implies that $a\backslash bc \leq (a\backslash b)((b\backslash a)\backslash c)$, which yields the desired equation.

(5) $b/(a\backslash b) = b/(a^{-1}b \wedge 1) = b(a^{-1}b \wedge 1)^{-1} \wedge 1 = b(1 \vee b^{-1}a) \wedge 1 = (b \vee a) \wedge 1 = b \vee a$.

□

Lemma III.5.2. *For every pair (n, m) of positive integers, there exist m cone algebra terms $f_1^{n,m}, ..., f_m^{n,m}$ (in $n+m$ variables) such that for every $\mathbf{A} \in \mathcal{LG}^-$ and $a_1, ..., a_n, b_1, ..., b_m \in A$, $a_1 \cdots a_n\backslash b_1 \cdots b_m = f_1^{n,m}(a_1, ..., a_n, b_1, ..., b_m) \cdots f_m^{n,m}(a_1, ..., a_n, b_1, ..., b_m)$.*

Proof. The proof will proceed by a double induction on n and m. For $m = n = 1$, $f_1^{1,1}(x_1, y_1) = x_1\backslash y_1$. Now, assume that the conclusion holds for the pair $(k, 1)$. We wish to show that the conclusion holds for the pair $(k+1, 1)$. Let $f_1^{k+1,1}(x_1, ..., x_k, x_{k+1}, y_1) = x_{k+1}\backslash f_1^{k,1}(x_1, ..., x_k, y_1)$. Let $\mathbf{A} \in \mathcal{LG}^-$ and let $a_1, ..., a_{k+1}, b_1 \in A$. Then, by (3) in the previous lemma, we see that $a_1 \cdots a_k a_{k+1}\backslash b_1 = a_{k+1}\backslash(a_1 \cdots a_k\backslash b_1) = a_{k+1}\backslash f_1^{k,1}(a_1, ..., a_k, b_1) = f_1^{k+1,1}(a_1, ..., a_k, a_{k+1}, b_1)$. This shows that for every n, the pair $(n, 1)$ satisfies the conclusion.

Now, assume that for every n, the pair (n, p) satisfies the conclusion. First, we must show that the pair $(1, p+1)$ satisfies the conclusion. For $i = 1, ..., p$, let $f_i^{1,p+1}(x_1, y_1, ..., y_p, y_{p+1}) =$

$f_i^{1,p}(x_1, y_1, ..., y_p)$. Also, let $f_{p+1}^{1,p+1}(x_1, y_1, ..., y_p, y_{p+1}) = f_1^{p,1}(y_1, ..., y_p, x_1)\backslash y_{p+1}$. Let $\mathbf{A} \in \mathcal{LG}^-$ and $a_1, b_1, ..., b_{p+1} \in A$. Then, by (4) of the previous lemma,

$$a_1\backslash b_1 \cdots b_p b_{p+1} =$$

$$(a_1\backslash b_1 \cdots b_p)((b_1 \cdots b_p\backslash a_1)\backslash b_{p+1}) =$$

$$f_1^{1,p}(a_1, b_1, ..., b_p) \cdots f_p^{1,p}(a_1, b_1, ..., b_p)(f_1^{p,1}(b_1, ..., b_p, a_1)\backslash b_{p+1}) =$$

$$f_1^{1,p+1}(a_1, b_1, ..., b_p, b_{p+1}) \cdots f_p^{1,p+1}(a_1, b_1, ..., b_p, b_{p+1}) f_{p+1}^{1,p+1}(a_1, b_1, ..., b_p, b_{p+1}),$$

which yields the conclusion of the lemma for the pair $(1, p+1)$.

Now, assume that the pair $(k, p+1)$ satisfies the conclusion of the lemma. We wish to show that the pair $(k+1, p+1)$ does as well. For $i = 1, ..., p+1$, let $t_i = f_i^{k,p+1}(x_1, ..., x_k, y_1, ..., y_{p+1})$. Now, for $j = 1, ..., p+1$, let $f_j^{k+1,p+1}(x_1, ..., x_k, x_{k+1}, y_1, ..., y_{p+1}) = f_j^{1,p+1}(x_{k+1}, t_1, ..., t_{p+1})$. Now, let $\mathbf{A} \in \mathcal{LG}^-$ and let $a_1, ..., a_{k+1}, b_1, ..., b_{p+1} \in A$. Then, by (3) of the previous lemma,

$$a_1 \cdots a_k a_{k+1}\backslash b_1 \cdots b_{p+1} =$$

$$a_{k+1}\backslash(a_1 \cdots a_k\backslash b_1 \cdots b_{p+1}) =$$

$$a_{k+1}\backslash f_1^{k,p+1}(a_1, ..., a_k, b_1, ..., b_{p+1}) \cdots f_{p+1}^{k,p+1}(a_1, ..., a_k, b_1, ..., b_{p+1}) =$$

$$f_1^{1,p+1}(a_{k+1}, t_1, ..., t_{p+1}) \cdots f_{p+1}^{1,p+1}(a_{k+1}, t_1, ..., t_{p+1}) =$$

$$f_1^{k+1,p+1}(a_1, ..., a_k, a_{k+1}, b_1, ..., b_{p+1}) \cdots f_{p+1}^{k+1,p+1}(a_1, ..., a_k, a_{k+1}, b_1, ..., b_{p+1}),$$

which yields the conclusion of the lemma for the pair $(k+1, p+1)$. □

Note that it is also possible to rewrite a left residual of products as a product of terms that each only involve the left residual.

Lemma III.5.3. *If* **C** *generates a negative cone of an ℓ-group* **A**, *then every element of A has the form $a_1 \cdots a_n$, for some $a_1, ..., a_n \in C$.*

Proof. We need to show that the set $S = \{a_1 \cdots a_n \mid a_1, ..., a_n \in C\}$ is a subalgebra of $\mathbf{A}$. Clearly, S is a submonoid of A. Also, since $a(a\backslash b) = b \wedge a$ and $b/(a\backslash b) = b \vee a$, it only remains to show that S is closed under the division operations, but this follows from Lemma III.5.2 and the fact that $\mathbf{C}$ is a cone subalgebra of $\mathbf{A}$. $\square$

Lemma III.5.4. *Let* $\mathbf{A} \in \mathcal{LG}^-$*. Then,* $a_1 \cdots a_n \leq b_1 \cdots b_m$ *iff each* $f_i^{n,m}(a_1, ..., a_n, b_1, ..., b_m) = 1$*, for* $i = 1, ..., m$*.*

Proof. We see that $a_1 \cdots a_n \leq b_1 \cdots b_m$ iff $a_1 \cdots a_n \backslash b_1 \cdots b_m = 1$. By Lemma III.5.2, this is equivalent to $f_1^{n,m}(a_1, ..., a_n, b_1, ..., b_m) \cdots f_m^{n,m}(a_1, ..., a_n, b_1, ..., b_m) = 1$. But this product is equal to 1 if and only if each factor equals 1, which completes the proof. $\square$

Therefore, Lemmas III.3.3, III.5.3, and III.5.4 imply the following corollary.

Corollary III.5.5. *Whenever the cone algebra* $\mathbf{C}$ *is embedded into the negative cone of an* ℓ*-group* $\mathbf{A}$*, the subalgebra generated by* C *in* $\mathbf{A}$ *is the free object* $F(\mathbf{C})$ *with respect to the forgetful functor.*

CHAPTER IV

MAIN RESULTS ABOUT MODAL IMAGE FUNCTORS

The main purpose of this chapter is to restrict the nuclear and conuclear image functors to produce categorical equivalences. In the first section, we will actually construct the free objects with respect to these functors.

IV.1 Constructions

Observe that since the images of modal residuated lattices are residuated lattices as well as subreducts (in the appropriate language), Lemma III.1.2 gives us the following.

Lemma IV.1.1. *We have the following:*

(1) If $\langle \mathbf{B}, \sigma \rangle$ is a Boolean algebra with a conucleus, then $\mathbf{B}_\sigma$ is a Heyting algebra.

(2) If $\langle \mathbf{G}, \sigma \rangle$ is an Abelian ℓ-group with a conucleus, then $\mathbf{G}_\sigma$ is a commutative, cancellative residuated lattice.

(3) If $\langle \mathbf{A}, \gamma \rangle$ is a negative cone of an ℓ-group with a nucleus, then $\mathbf{A}_\gamma$ is an integral GMV-algebra.

As usual, showing the converse of the previous lemma is much more complicated. We not only want to show that these latter varieties are precisely the classes of images of the former ones, but also that they can be recovered in a very specific way. To explain this in the Boolean algebras case, consider Figure IV.1 below, where Γ is the conuclear image functor, G and H are the appropriate forgetful functors, and F is the left adjoint of G.

Given a Heyting algebra $\mathbf{A}$, we wish to construct $\Sigma(\mathbf{A})$ such that $\Gamma(\Sigma(\mathbf{A})) = \mathbf{A}$ and $H(\Sigma(\mathbf{A})) = F(G(\mathbf{A}))$. That is, we want $\mathbf{A}$ to be the conuclear image of an interior algebra whose underlying Boolean algebra is $F(G(\mathbf{A}))$. We will now show how to achieve such a Σ. First, we will need the following lemma.

$$\begin{tikzcd} \mathcal{BA}_\sigma \arrow[r, "\Gamma"'] \arrow[d, "H"'] & \mathcal{HA} \arrow[l, dotted, bend right, "\Sigma"'] \arrow[d, "G"] \\ \mathcal{BA} \arrow[r, "G"'] & \mathcal{DL}_{0,1} \arrow[l, dotted, bend right, "F"'] \end{tikzcd}$$

Figure IV.1: Free objects with respect to the forgetful and conuclear image functors

Lemma IV.1.2. *We have the following:*

(1) Let $\langle \mathbf{B}, \sigma \rangle$ be a Boolean algebra with a conucleus. For $a_i, b_i \in \sigma[B]$, $\sigma(\bigwedge_{i=1}^{n}(a_i \vee b_i')) = \bigwedge_{i=1}^{n}(b_i \to_\sigma a_i)$.

(2) Let $\langle \mathbf{G}, \sigma \rangle$ be an Abelian ℓ-group with a conucleus. For $a, b \in \sigma[G]$, $\sigma(a^{-1}b) = a \to_\sigma b$.

(3) Let $\langle \mathbf{A}, \gamma \rangle$ be a negative cone of an ℓ-group with a nucleus. For $a_1, ..., a_n \in \gamma[A]$, $\gamma(a_1 \cdots a_n) = a_1 \circ_\gamma \cdots \circ_\gamma a_n$.

Proof.

(1) By the correspondence between conuclei and interior retracts (Proposition II.3.6), we need to show that $\bigwedge_{i=1}^{n}(b_i \to_\sigma a_i)$ is the greatest element in $\sigma[B]$ that is less than or equal to $\bigwedge_{i=1}^{n}(a_i \vee b_i')$. First, it is clear that $\bigwedge_{i=1}^{n}(b_i \to_\sigma a_i) \leq \bigwedge_{i=1}^{n}(a_i \vee b_i')$, since for $i = 1, ..., n$, $b_i \wedge (b_i \to_\sigma a_i) \leq a_i$, so $b_i \to_\sigma a_i \leq a_i \vee b_i'$. Now, let $z \in \sigma[B]$ be such that $z \leq \bigwedge_{i=1}^{n}(a_i \vee b_i')$. Then, for every i, $z \leq a_i \vee b_i'$, so $b_i \wedge z \leq a_i$. Since $z \in \sigma[B]$, $z \leq b_i \to_\sigma a_i$. So, $z \leq \bigwedge_{i=1}^{n}(b_i \to_\sigma a_i)$.

(2) This follows from the definition of $\to_\sigma$ in conuclear images of ℓ-groups.

(3) Similar to (1), we need to show that $a_1 \circ_\gamma \cdots \circ_\gamma a_n$ is the least element in $\gamma[A]$ that is greater than or equal to $a_1 \cdots a_n$. First, since for $x, y \in \gamma[A]$, $xy \leq \gamma(xy) = x \circ_\gamma y$, a

simple induction argument shows that $a_1 \cdots a_n \leq a_1 \circ_\gamma \cdots \circ_\gamma a_n$. Now, suppose $a_1 \cdots a_n \leq z$, for $z \in \gamma[A]$. By the residuation law for $\mathbf{A}$, we see that $a_n \leq a_{n-1} \backslash (\cdots (a_2 \backslash (a_1 \backslash z)) \cdots)$. Since $\mathbf{A}_\gamma$ is a $\{\backslash, /, 1\}$-subreduct of $\mathbf{A}$, this inequality holds in $\mathbf{A}_\gamma$ as well. Thus, the residuated law for $\mathbf{A}_\gamma$ implies that $a_1 \circ_\gamma \cdots \circ_\gamma a_n \leq z$.

□

Since the previous lemma involved taking the image of the representative terms in all three cases, we see that if we are going to recover the original residuated lattice from a modal operator on its free object with respect to the forgetful functors, the only choice is to define the modal operator as above. It only remains to show that these modal operators are well-defined, and that the image is precisely the original residuated lattice.

Let $\mathbf{L}$ be a commutative, cancellative residuated lattice. Consider the $\{\vee, \cdot, 1\}$-reduct $\bar{\mathbf{L}}$ of $\mathbf{L}$, which is in $\mathcal{JCM}$. By Lemma III.1.2 (2), we can construct the free object $F(\bar{\mathbf{L}}) \in \mathcal{A}$. We wish to show that $\sigma_{\mathbf{L}}(a^{-1}b) = a \to_{\mathbf{L}} b$ is a well-defined conucleus on $F(\bar{\mathbf{L}})$ whose image is $\mathbf{L}$. To that end, let $a, b, c, d \in L$. Assume $a^{-1}b = c^{-1}d$. Then, since $bc = ad$, $a \to b = ac \to bc = ac \to ad = c \to d$. The next lemma completes the proof.

Lemma IV.1.3. *Let $\mathbf{L} \in \mathcal{CC}an\mathcal{RL}$. Let $\mathbf{F}(\bar{\mathbf{L}})$ and $\sigma_{\mathbf{L}}$ be defined as above.*

(1) $\langle \mathbf{F}(\bar{\mathbf{L}}), \sigma_{\mathbf{L}} \rangle \in \mathcal{A}_\sigma$

(2) $\mathbf{F}(\bar{\mathbf{L}})_{\sigma_{\mathbf{L}}} = \mathbf{L}$ *(as residuated lattices).*

Proof.

(1) We need to show that $\sigma_{\mathbf{L}}$ is in fact a conucleus on $\mathbf{F}(\bar{\mathbf{L}})$. Since $a(a \to b) \leq b$, we see that $\sigma_{\mathbf{L}}(a^{-1}b) = a \to b \leq a^{-1}b$. Also, observe that $\sigma_{\mathbf{L}}(\sigma_{\mathbf{L}}(a^{-1}b)) = \sigma_{\mathbf{L}}(a \to b) = \sigma_{\mathbf{L}}(1^{-1}(a \to b)) = 1 \to (a \to b) = a \to b = \sigma_{\mathbf{L}}(a^{-1}b)$. Now, assume that $a^{-1}b \leq c^{-1}d$. Then, $bc \leq ad$, so $\sigma_{\mathbf{L}}(a^{-1}b) = a \to b = ac \to bc \leq ac \to ad = c \to d = \sigma_{\mathbf{L}}(c^{-1}d)$. Clearly, $\sigma_{\mathbf{L}}(1) = 1$. It only remains to show that $\sigma_{\mathbf{L}}(a^{-1}b)\sigma_{\mathbf{L}}(c^{-1}d) \leq \sigma_{\mathbf{L}}(a^{-1}bc^{-1}d)$.

Since $ac(a \to b)(c \to d) = a(a \to b)c(c \to d) \leq bd$, $(a \to b)(c \to d) \leq ac \to bd$, which completes the proof.

(2) Since the underlying set of $\mathbf{F}(\bar{\mathbf{L}})_{\sigma_{\mathbf{L}}}$ is L, it only remains to be shown that all of the operations of $\mathbf{F}(\bar{\mathbf{L}})_{\sigma_{\mathbf{L}}}$ and $\mathbf{L}$ are identical. Recall that the join and monoid operations of $\mathbf{F}(\bar{\mathbf{L}})_{\sigma_{\mathbf{L}}}$ are the same as those of $\mathbf{F}(\bar{\mathbf{L}})$ as well as those of $\mathbf{L}$. Thus, we need to show that for every $a, b \in L$, $a \wedge_{\mathbf{L}} b = \sigma_{\mathbf{L}}(a \wedge b)$ and $a \to_{\mathbf{L}} b = \sigma_{\mathbf{L}}(a^{-1}b)$. This latter one is clear since this is how we defined $\sigma_{\mathbf{L}}$.

Now, let $a, b \in L$. Then, since $\sigma_{\mathbf{L}}(a \wedge b) \leq a \wedge b$, we see that $\sigma_{\mathbf{L}}(a \wedge b)$ is a lower bound of a and b. Assume $c \in L$ is a lower bound of a and b. Then, $c \leq a \wedge b$, so $c = \sigma_{\mathbf{L}}(c) \leq \sigma_{\mathbf{L}}(a \wedge b)$. This shows that $\sigma_{\mathbf{L}}(a \wedge b) = a \wedge_{\mathbf{L}} b$. This completes the proof that $\mathbf{F}(\bar{\mathbf{L}})_{\sigma_{\mathbf{L}}} = \mathbf{L}$ (as residuated lattices).

□

Let $\mathbf{A}$ be a Heyting algebra. Consider the $\{\wedge, \vee, 0, 1\}$-reduct $\bar{\mathbf{A}}$ of $\mathbf{A}$, which is in $\mathcal{DL}_{0,1}$. By Lemma III.1.2 (1), we can construct the free object $F(\bar{\mathbf{A}}) \in \mathcal{BA}$. We wish to show that $\sigma_{\mathbf{A}}(\bigwedge_{i=1}^{n}(a_i \vee b_i')) = \bigwedge_{i=1}^{n}(b_i \to_{\mathbf{A}} a_i)$ is a well-defined conucleus on $F(\bar{\mathbf{A}})$ whose image is $\mathbf{A}$.

Lemma IV.1.4. *Let $\mathbf{A} \in \mathcal{HA}$. Let $\mathbf{F}(\bar{\mathbf{A}})$ and $\sigma_{\mathbf{A}}$ be defined as above.*

(1) $\langle \mathbf{F}(\bar{\mathbf{A}}), \sigma_{\mathbf{A}} \rangle \in \mathcal{BA}_{\sigma}$

(2) $\mathbf{F}(\bar{\mathbf{A}})_{\sigma_{\mathbf{A}}} = \mathbf{A}$ (as Heyting algebras).

Proof.

(1) By Proposition II.3.6, such a conucleus exists iff $\mathbf{A}$ is an interior retract of $\mathbf{F}(\bar{\mathbf{A}})$. By construction, $\mathbf{A}$ is a subposet and a submonoid of $\mathbf{F}(\bar{\mathbf{A}})$ (remembering that the multiplication of a Boolean algebra is just the meet). It remains to show that for every $x \in F(\bar{\mathbf{A}})$, the set $\{z \in A \mid z \leq x\}$ has a greatest element.

Let $x \in F(\bar{\mathbf{A}})$. Then, $x = \bigwedge_{i=1}^{n}(a_i \vee b_i')$, for some $a_1, ..., a_n, b_1, ..., b_n \in A$. Consider $y = \bigwedge_{i=1}^{n}(b_i \to a_i)$. For $i = 1, ..., n$, $b_i \wedge (b_i \to a_i) \le a_i$, so $b_i \to a_i \le a_i \vee b_i'$. Thus, $y \in \{z \in A \mid z \le x\}$. Now, let $z \in A$ such that $z \le x$. Then, for every i, $z \le a_i \vee b_i'$, so $b_i \wedge z \le a_i$. Since $z \in A$, $z \le b_i \to a_i$. So, $z \le y$. Therefore, y is the greatest element in $\{z \in A \mid z \le x\}$. This implies that $\mathbf{A}$ is an interior retract of $\mathbf{F}(\bar{\mathbf{A}})$, whose corresponding conucleus is $\sigma_{\mathbf{A}}$.

(2) Since $\sigma_{\mathbf{A}}[F(\bar{\mathbf{A}})] = A$ and both $\mathbf{A}$ and $\sigma_{\mathbf{A}}[\mathbf{F}(\bar{\mathbf{A}})]$ are $\{0, 1\}$-sublattices of $\mathbf{F}(\bar{\mathbf{A}})$, they must be equal (as bounded distributive lattices). Lastly, for $x, y \in A$, $x \to_\sigma y = \sigma_{\mathbf{A}}(y \vee x') = x \to_{\mathbf{A}} y$.

$\square$

Let $\mathbf{A}$ be an integral GMV-algebra. Consider the $\{\backslash, /, 1\}$-reduct $\bar{\mathbf{A}}$ of $\mathbf{A}$, which is in $\mathcal{CA}$. By Lemma III.1.2 (3), we can construct the free object $F(\bar{\mathbf{A}}) \in \mathcal{LG}^-$. We wish to show that $\gamma_{\mathbf{A}}(a_1 \cdots a_n) = a_1 \circ_{\mathbf{A}} \cdots \circ_{\mathbf{A}} a_n$ is a well-defined nucleus on $F(\bar{\mathbf{A}})$ whose image is $\mathbf{A}$.

Lemma IV.1.5. *Let* $\mathbf{A} \in \mathcal{IGMV}$. *Let* $\mathbf{F}(\bar{\mathbf{A}})$ *and* $\gamma_{\mathbf{A}}$ *be defined as above.*

(1) $\langle \mathbf{F}(\bar{\mathbf{A}}), \gamma_{\mathbf{A}} \rangle \in \mathcal{LG}^-_\gamma$

(2) $\mathbf{F}(\bar{\mathbf{A}})_{\gamma_{\mathbf{A}}} = \mathbf{A}$ *(as GMV-algebras).*

Proof.

(1) By Proposition II.3.13, we need to show that $\mathbf{A}$ is a closure retract of $F(\bar{\mathbf{A}})$. Certainly, $\mathbf{A}$ is a subposet of $F(\bar{\mathbf{A}})$. Given $x \in F(\bar{\mathbf{A}})$ and $y \in A$, there exists $b_1, ..., b_n \in A$ such that $x = b_1 \cdots b_n$. So, $x \backslash y = b_1 \cdots b_n \backslash y = b_n \backslash (\cdots (b_1 \backslash y) \cdots) \in A$. Similarly, $y/x \in A$. It only remains to show that for every $x \in F(\bar{\mathbf{A}})$, there exists a least element $y \in A$ that is greater than or equal to x. If $x = b_1 \cdots b_n$, consider $y = b_1 \circ_{\mathbf{A}} \cdots \circ_{\mathbf{A}} b_n$. Since $d \le c \backslash c \circ_{\mathbf{A}} d$, we see that $cd \le c \circ_{\mathbf{A}} d$. A simple induction argument shows

that $b_1 \cdots b_n \leq b_1 \circ_{\mathbf{A}} \cdots \circ_{\mathbf{A}} b_n$. Now, assume that $b_1 \cdots b_n \leq z$, for $z \in A$. Then, $b_n \leq b_{n-1}\backslash(\cdots(b_2\backslash(b_1\backslash z))\cdots)$. Since $\mathbf{A}$ is a $\{\backslash, /, 1\}$-subreduct of $F(\bar{\mathbf{A}})$, this inequality holds in $\mathbf{A}$ as well. Thus, the residuated law for $\mathbf{A}$ implies that $b_1 \circ_{\mathbf{A}} \cdots \circ_{\mathbf{A}} b_n \leq z$, which completes the proof.

(2) Since $\gamma_{\mathbf{A}}[F(\bar{\mathbf{A}})] = A$ and both $\mathbf{A}$ and $\gamma_{\mathbf{A}}[\mathbf{F}(\bar{\mathbf{A}})]$ are $\{\backslash, /, 1\}$-subreducts of $\mathbf{F}(\bar{\mathbf{A}})$, it only remains to show that the multiplication and lattice operations are the same. Certainly, $x \circ_\gamma y = \gamma(xy) = x \circ_{\mathbf{A}} y$, for $x, y \in A$. Now, since $\gamma_{\mathbf{A}}(x \wedge y) \leq \gamma_{\mathbf{A}}(x) = x$, we see that $\gamma_{\mathbf{A}}(x \wedge y)$ is a lower bound of x and y. If $z \in A$ is a lower bound of x and y, then $x \wedge y \leq z$, so $\gamma_{\mathbf{A}}(x \wedge y) \leq \gamma_{\mathbf{A}}(z) = z$. Thus, $\gamma_{\mathbf{A}}(x \wedge y) = x \wedge_{\mathbf{A}} y$. Similarly, $\gamma_{\mathbf{A}}(x \vee y) = x \vee_{\mathbf{A}} y$.

□

The preceding constructions imply the following proposition.

Proposition IV.1.6. *We have the following:*

(1) The class of images of Boolean algebras with conuclei is precisely the variety of Heyting algebras.

(2) The class of images of Abelian ℓ-groups with conuclei is precisely the variety of commutative, cancellative residuated lattices.

(3) The class of images of negative cones of ℓ-groups with nuclei is precisely the variety of integral GMV-algebras.

IV.2 Categorical Equivalences

Now, we can turn our attention to the modal image functors. Specifically, we consider $\Gamma_1 : \mathcal{BA}_\sigma \to \mathcal{HA}$, $\Gamma_2 : \mathcal{A}_\sigma \to \mathcal{CCanRL}$, and $\Gamma_3 : \mathcal{LG}_\gamma^- \to \mathcal{IGMV}$. In this section, we will show that the previously constructed modal residuated lattices are in fact the free objects with respect to these functors. Observe that all of the defined modal operators are given by sending a residuated lattice term computed in $F(\mathbf{A})$ to the same term computed in $\mathbf{A}$. In fact, these terms are precisely the representability terms from the reduct chapter. Let $\Sigma(A) = \langle F(\mathbf{A}), \delta_{\mathbf{A}} \rangle$ be these free objects with their defined modal operator. In the following theorem, $\mathcal{V}$ is $\mathcal{BA}_\sigma$, $\mathcal{A}_\sigma$, or $\mathcal{LG}_\gamma^-$, $\mathcal{W}$ is the corresponding class of images, and $\bar{\mathcal{W}}$ is the corresponding class of reducts.

Theorem IV.2.1. *All three of the functors* Γ_1, Γ_2, *and* Γ_3 *have left adjoints:*

(1) For $\mathbf{A} \in \mathcal{HA}$, $\Sigma_1(\mathbf{A}) \in \mathcal{BA}_\sigma$ *is the free object over* $\mathbf{A}$ *with respect to* Γ_1.

(2) For $\mathbf{A} \in \mathcal{CC}an\mathcal{RL}$, $\Sigma_2(\mathbf{A}) \in \mathcal{A}_\sigma$ *is the free object over* $\mathbf{A}$ *with respect to* Γ_2.

(3) For $\mathbf{A} \in \mathcal{IGMV}$, $\Sigma_3(\mathbf{A}) \in \mathcal{LG}_\gamma^-$ *is the free object over* $\mathbf{A}$ *with respect to* Γ_3.

Proof. Since $\Gamma(\Sigma(\mathbf{A})) = \mathbf{A}$, we need to show that for any $\langle \mathbf{B}, \epsilon \rangle \in \mathcal{V}$ and any $\mathcal{W}$-homomorphism $f : \mathbf{A} \to \mathbf{B}_\epsilon$, there exists a unique $\mathcal{V}$-homomorphism $\bar{f} : \Sigma(\mathbf{A}) \to \langle \mathbf{B}, \epsilon \rangle$ such that $\bar{f}$ extends f. Since $\mathbf{B}_\epsilon$ is a $\bar{\mathcal{W}}$-subreduct of $\mathbf{B}$, we can view f as a $\bar{\mathcal{W}}$-homomorphism from $\bar{\mathbf{A}}$ into $\bar{\mathbf{B}}$. By the freeness of $F(\mathbf{A})$, there exists a unique homomorphism $\bar{f} : F(\mathbf{A}) \to \mathbf{B}$ that extends f. In order to prove that this $\bar{f}$ is a $\mathcal{V}$-homomorphism, we need to show that for any $x \in F(\mathbf{A})$, $\bar{f}(\delta_{\mathbf{A}}(x)) = \epsilon(\bar{f}(x))$.

To this end, for any $x \in F(\mathbf{A})$, there exists $a_1, ..., a_n \in A$ such that $x = t^{F(\mathbf{A})}(a_1, ..., a_n)$, where t is one of the appropriate representability terms. Thus,

$$\bar{f}(\delta_{\mathbf{A}}(x)) =$$

$$\bar{f}(\delta_{\mathbf{A}}(t^{F(\mathbf{A})}(a_1, ..., a_n))) =$$

$$\bar{f}(t^{\mathbf{A}}(a_1, ..., a_n)) =$$

$$f(t^{\mathbf{A}}(a_1, ..., a_n)) =$$

$$t^{\mathbf{B}_\epsilon}(f(a_1), ..., f(a_n)) =$$

$$\epsilon(t^{\mathbf{B}}(f(a_1), ..., f(a_n))) =$$

$$\epsilon(t^{\mathbf{B}}(\bar{f}(a_1), ..., \bar{f}(a_n))) =$$

$$\epsilon(\bar{f}(t^{F(\mathbf{A})}(a_1, ..., a_n))) =$$

$$\epsilon(\bar{f}(x)).$$

Therefore, since $\bar{f}$ extends f, and $F(\mathbf{A})$ is generated by A, $\bar{f}$ is the unique $\mathcal{V}$-homomorphism that extends f. This completes the proof. $\square$

By taking certain full subcategories, every adjunction restricts to a categorical equivalence. On the right-hand side, we will take all of $\mathcal{W}$, since for every $\mathbf{A} \in \mathcal{W}$, $\Gamma(\Sigma(\mathbf{A})) = \mathbf{A}$. On the left-hand side, consider $\mathcal{V}^*$, the full subcategory of $\mathcal{V}$ consisting of those pairs $\langle \mathbf{B}, \epsilon \rangle$ such that $\mathbf{B}_\epsilon$ generates $\mathbf{B}$ as a residuated lattice.

Theorem IV.2.2. *We have the following:*

(1) The functors Γ_1 and Σ_1 induce a categorical equivalence between $\mathcal{BA}^*_\sigma$ and $\mathcal{HA}$.

(2) The functors Γ_2 and Σ_2 induce a categorical equivalence between $\mathcal{A}^*_\sigma$ and $\mathcal{CC}an\mathcal{RL}$.

(3) The functors Γ_3 and Σ_3 induce a categorical equivalence between $(\mathcal{LG}^-_\gamma)^*$ and $\mathcal{IGMV}$.

Proof. We need to show that $\langle \mathbf{B}, \epsilon \rangle$ is in $\mathcal{V}^*$ if and only if $\Sigma(\Gamma(\langle \mathbf{B}, \epsilon \rangle))$ is isomorphic to $\langle \mathbf{B}, \epsilon \rangle$. Clearly, if $\Sigma(\Gamma(\langle \mathbf{B}, \epsilon \rangle))$ is isomorphic to $\langle \mathbf{B}, \epsilon \rangle$, then $\langle \mathbf{B}, \epsilon \rangle$ is in $\mathcal{V}^*$. On the other hand, assume $\langle \mathbf{B}, \epsilon \rangle$ is in $\mathcal{V}^*$. If $\mathbf{A} = \mathbf{B}_\epsilon$, then $\Sigma(\Gamma(\langle \mathbf{B}, \epsilon \rangle)) = \langle F(\mathbf{A}), \delta_{\mathbf{A}} \rangle$. Since $\mathbf{A}$ generates $\mathbf{B}$, we can use one of Corollaries III.3.5, III.4.2, or III.5.5 to show that $\mathbf{B}$ is isomorphic to $F(\mathbf{A})$. By the same calculation at end of the previous theorem, we see that this isomorphism in fact also establishes that $\langle \mathbf{B}, \epsilon \rangle$ is isomorphic to $\langle F(\mathbf{A}), \delta_{\mathbf{A}} \rangle$, which completes the proof. $\square$

In this section, we will discuss the relationship that Γ imposes between the subvariety lattices of $\mathcal{V}$ and $\mathcal{W}$. First, we have the following lemma.

Lemma IV.3.1. *If $\mathcal{V}'$ is a subvariety of $\mathcal{V}$, then $\Gamma[\mathcal{V}']$ is a subvariety of $\mathcal{W}$.*

Proof. First, observe that $\Gamma\left(\prod_{i\in I}\langle \mathbf{B}_i, \epsilon_i\rangle\right) = \prod_{i\in I}\Gamma(\langle \mathbf{B}_i, \epsilon_i\rangle)$, so $\Gamma[\mathcal{V}']$ is closed under products.

Now, assume that $\mathbf{A} \leq \Gamma(\langle \mathbf{B}, \epsilon\rangle)$. Then, $\bar{A}$ is a $\bar{\mathcal{W}}$-subreduct of $\bar{\mathbf{B}}$. Thus, by one of the corollaries, $F(\mathbf{A}) \leq \mathbf{B}$. To establish that $\Sigma(\mathbf{A}) \leq \langle \mathbf{B}, \epsilon\rangle$, we need to show that for $x \in F(\mathbf{A})$, $\delta_{\mathbf{A}}(x) = \epsilon(x)$. There exists $a_1, ..., a_n \in A$ such that $x = t(a_1, ..., a_n)$. Thus, $\delta_{\mathbf{A}}(x) = \delta_{\mathbf{A}}(t(a_1, ..., a_n)) = t^{\mathbf{A}}(a_1, ..., a_n) = t^{\mathbf{B}_\epsilon}(a_1, ..., a_n) = \epsilon(t^{\mathbf{B}}(a_1, ..., a_n) = \epsilon(t(a_1, ..., a_n)) = \epsilon(x)$. Since $\Gamma(\Sigma(\mathbf{A})) = \mathbf{A}$, this shows that $\mathbf{A} \in \Gamma[\mathcal{V}']$.

Lastly, assume that $f : \Gamma(\langle \mathbf{B}, \epsilon\rangle) \to \mathbf{A}$ is a surjective $\mathcal{W}$-homomorphism. Then, by the categorical equivalence, there exists a $\mathcal{V}$-homomorphism $\bar{f} : \Sigma(\Gamma(\langle \mathbf{B}, \epsilon\rangle)) \to \Sigma(\mathbf{A})$ that extends f. Since A generates $\Sigma(\mathbf{A})$, $\bar{f}$ is also surjective. By an argument similar to the one above, we see that $\Sigma(\Gamma(\langle \mathbf{B}, \epsilon\rangle)) \leq \langle \mathbf{B}, \epsilon\rangle$. Thus, $\Gamma[\mathcal{V}']$ is closed under homomorphic images. □

Now, for a subvariety $\mathcal{W}'$ of $\mathcal{W}$, define $\Gamma^{-1}(\mathcal{W}')$ as the subclass of $\mathcal{V}$ consisting of all $\langle \mathbf{B}, \epsilon\rangle$ such that $\mathbf{B}_\epsilon \in \mathcal{W}'$. Also, define $\Sigma(\mathcal{W}')$ as the subvariety of $\mathcal{V}$ generated by the class $\{\Sigma(\mathbf{A}) \mid \mathbf{A} \in \mathcal{W}'\}$.

Lemma IV.3.2. *For every subvariety $\mathcal{W}'$ of $\mathcal{W}$, $\Gamma^{-1}(\mathcal{W}')$ is a subvariety of $\mathcal{V}$. Moreover, $\Gamma[\Gamma^{-1}(\mathcal{W}')] = \Gamma[\Sigma(\mathcal{W}')] = \mathcal{W}'$.*

Proof. Since Γ preserves direct products, $\Gamma^{-1}(\mathcal{W}')$ is closed under products. Now, if $\langle \mathbf{A}, \eta\rangle \leq \langle \mathbf{B}, \epsilon\rangle$, then $\Gamma(\langle \mathbf{A}, \eta\rangle) \leq \Gamma(\langle \mathbf{B}, \epsilon\rangle)$, so $\Gamma^{-1}(\mathcal{W}')$ is closed under subalgebras. Lastly, if $\langle \mathbf{A}, \eta\rangle$ is a homomorphic image of $\langle \mathbf{B}, \epsilon\rangle$, then $\Gamma(\langle \mathbf{A}, \eta\rangle)$ is a homomorphic image of $\Gamma(\langle \mathbf{B}, \epsilon\rangle)$, so $\Gamma^{-1}(\mathcal{W}')$ is closed under homomorphic images.

Finally, observe that since $\Gamma(\Sigma(\mathbf{A})) = \mathbf{A}$, $\Sigma(\mathcal{W}') \subseteq \Gamma^{-1}(\mathcal{W}')$. Thus, we see that $\mathcal{W}' \subseteq \Gamma[\Sigma(\mathcal{W}')] \subseteq \Gamma[\Gamma^{-1}(\mathcal{W}')] \subseteq \mathcal{W}'$, which completes the proof. $\square$

Finally, we have the following theorem relating subvarieties $\mathcal{V}'$ and $\mathcal{W}'$ of $\mathcal{V}$ and $\mathcal{W}$, respectively.

Theorem IV.3.3. *We have the following:*

(1) Let $\mathcal{W}'$ be a subvariety of $\mathcal{HA}$. Then, there is an interval in the subvariety lattice of $\mathcal{BA}_\sigma$ that corresponds to $\mathcal{W}'$.

(2) Let $\mathcal{W}'$ be a subvariety of $\mathcal{CC}an\mathcal{RL}$. Then, there is an interval in the subvariety lattice of $\mathcal{A}_\sigma$ that corresponds to $\mathcal{W}'$.

(3) Let $\mathcal{W}'$ be a subvariety of $\mathcal{IGMV}$. Then, there is an interval in the subvariety lattice of $\mathcal{LG}^-_\gamma$ that corresponds to $\mathcal{W}'$.

Proof. For subvariety $\mathcal{V}'$ of $\mathcal{V}$, we need to show that $\Gamma[\mathcal{V}'] = \mathcal{W}'$ if and only if $\mathcal{V}'$ is in the interval $[\Sigma(\mathcal{W}'), \Gamma^{-1}(\mathcal{W}')]$. The "if" direction is clear. For the converse, assume $\Gamma[\mathcal{V}'] = \mathcal{W}'$. Then, by definition, $\mathcal{V}' \subseteq \Gamma^{-1}(\mathcal{W}')$. Now, take $\mathbf{A} \in \mathcal{W}'$. Then, there exists $\langle \mathbf{B}, \epsilon \rangle \in \mathcal{V}'$ such that $A = \Gamma(\langle \mathbf{B}, \epsilon \rangle)$. As we have previously shown, this implies that $\Sigma(\mathbf{A}) \leq \langle \mathbf{B}, \epsilon \rangle$. Therefore, $\Sigma(\mathcal{W}') \subseteq \mathcal{V}'$, completing the proof. $\square$

In the first case, it is known that if you restrict to the minimal variety corresponding to Heyting algebras, then there is a lattice isormophism between this subvariety lattice and the subvariety lattice of Heyting algberas [9, 20, 21]. In the next chapter, we show a similar result in a restricted setting of the third case.

CHAPTER V

BLOK-ESAKIA-TYPE THEOREM FOR MV-ALGEBRAS

Recall that if you consider only those nuclei of the form $\gamma_a(x) = x \vee a$ for some fixed element a, then the third case shows that the class of intervals in the negative cones of Abelian ℓ-groups is precisely the class of MV-algebras. We will choose to consider MV-algebras as intervals in Abelian ℓ-groups, rather than in their negative cones. Also, since the nuclei are determined by this fixed element a, we will wish to consider the variety of negatively-pointed Abelian ℓ-groups (denoted $p\mathcal{A}$), rather than the variety augmented with the nucleus. From the results of the previous chapter, for a subvariety $\mathcal{V}$ of MV-algebras, there is an interval $[\Sigma(\mathcal{V}), \Gamma^{-1}(\mathcal{V})]$ in the subvariety lattice of $p\mathcal{A}$ that corresponds to $\mathcal{V}$. The main result of this chapter is that for non-trivial $\mathcal{V}$, this interval contains only one variety, yielding a Blok-Esakia-type theorem for MV-algebras.

V.1 Subvariety Lattice of $p\mathcal{A}$

In this section, we will begin the investigation of the subvariety lattice of $p\mathcal{A}$.

Proposition V.1.1. *There are exactly two subvarieties of* $p\mathcal{A}$ *that map via* Γ *to the trivial variety of MV-algebras, namely* $\Sigma(\bot) = \bot$ *and* $\Gamma^{-1}(\bot) = \mathcal{V}(\langle \mathbb{Z}, 0\rangle)$.

Proof. Clearly, the unital Abelian ℓ-group corresponding to the trivial MV-algebra is the trivial ℓ-group. This shows that $\Sigma(\bot) = \bot$. We see that $[a, 1]$ is trivial iff $a = 1$. So, $\Gamma^{-1}(\bot)$ consists of all pairs $\langle \mathbf{G}, 1\rangle$, where $\mathbf{G}$ is an Abelian ℓ-group. Since $\mathcal{V}(\mathbb{Z}) = \mathcal{A}$ [45], it is clear that $\mathcal{V}(\langle \mathbb{Z}, 0\rangle) = \Gamma^{-1}(\bot)$. Since $\mathcal{A}$ has no proper, non-trivial subvariety, we see that these are the only two subvarieties of $p\mathcal{A}$ that map to $\bot$. $\square$

The next two propositions show that the atom $\mathcal{V}(\langle \mathbb{Z}, 0\rangle)$ has a unique cover in the subvariety lattice of $p\mathcal{A}$.

Proposition V.1.2. *Every subvariety of $p\mathcal{A}$ that has an algebra $\langle \mathbf{G}, a\rangle$ where $a \neq 1$ also contains the algebra $\langle \mathbb{Z}, -1\rangle$.*

Proof. Since a is strictly negative, the subalgebra generated by a, which consists of all powers of a, is isomorphic to $\langle \mathbb{Z}, -1\rangle$. $\square$

Proposition V.1.3. *$\langle \mathbb{Z}, 0\rangle$ is in the variety generated by $\langle \mathbb{Z}, -1\rangle$.*

Proof. Consider the product of ω copies of $\langle \mathbb{Z}, -1\rangle$. Let B be the set of all bounded elements of this product. Clearly, B is the convex subalgebra of the product generated by the distinguished element, all of whose entries are -1. Thus, B is the kernel of some homomorphism. Since there are certainly elements of the product that are not bounded, we can see that this homomorphic image is not trivial, although the distinguished element equals the identity. Therefore, $\langle \mathbb{Z}, 0\rangle$ is (isomorphic to) a subalgebra of this image. $\square$

While the next proposition is not strictly necessary (as it is covered by the results of the next section), the author includes it because the proof is illuminating. In particular, although both $\mathbb{Z}$ and $\mathbb{R}$ generate $\mathcal{A}$, the next result shows the differences between $\langle \mathbb{Z}, -1\rangle$ and $\langle \mathbb{R}, -1\rangle$. Also, observe that this next result gives an alternative proof of Chang's Completeness Theorem (see [13, 14]), without requiring the full generality of the next section.

Proposition V.1.4. *$\langle \mathbb{R}, -1\rangle$ generates the entire variety $p\mathcal{A}$.*

Proof. Let $\langle \mathbf{G}, a\rangle \in p\mathcal{A}$. Since $\mathcal{V}(\mathbb{R}) = \mathcal{A}$, we know that there exists a set I and an Abelian ℓ-group $\mathbf{H}$ such that $\mathbf{H} \leq \prod_{i\in I} \mathbb{R}$ and $\mathbf{G}$ is a homomorphic image (under f) of $\mathbf{H}$. Let $b \in H$ be such that $f(b) = a$. Then, $f(b \wedge 1) = f(b) \wedge 1 = a \wedge 1 = a$, so we can assume, without loss of generality, that b is negative. Since $H \subseteq \prod_{i\in I} \mathbb{R}$, there exists $c_i \in \mathbb{R}$ such that $b = (c_i)_{i\in I}$. Since b is negative, so too are all of the c_i. We see then that $\langle \mathbf{H}, b\rangle \leq \prod_{i\in I} \langle \mathbb{R}, c_i\rangle$ and $f : \langle \mathbf{H}, b\rangle \to \langle \mathbf{G}, a\rangle$ is a $p\mathcal{A}$-homomorphism (since $f(b) = a$).

Now, we observe that for $a < 0$, $\langle \mathbb{R}, a\rangle$ is isomorphic to $\langle \mathbb{R}, -1\rangle$, via the function $g(x) = \frac{x}{|a|}$. This shows that $\mathcal{V}(\langle \mathbb{R}, 0\rangle, \langle \mathbb{R}, -1\rangle) = p\mathcal{A}$. By the previous two propositions, we see that $\langle \mathbb{Z}, 0\rangle \in \mathcal{V}(\langle \mathbb{R}, -1\rangle)$, so also $\langle \mathbb{R}, 0\rangle \in \mathcal{V}(\langle \mathbb{R}, -1\rangle)$, which completes the proof. $\square$

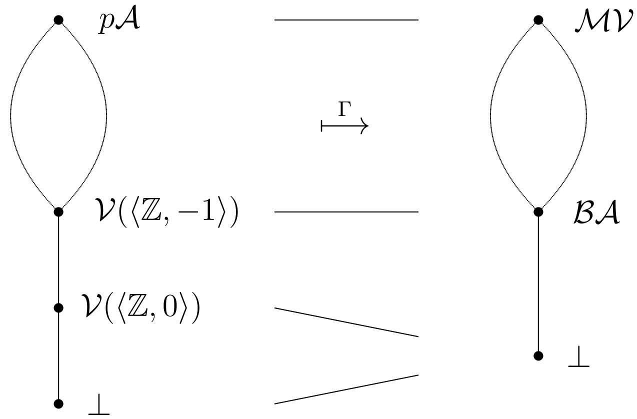

Figure V.1: The subvariety lattices of $p\mathcal{A}$ and $\mathcal{MV}$

The previous three propositions show that the subvariety lattice of $p\mathcal{A}$ consists of $\perp \subseteq \mathcal{V}(\langle \mathbb{Z}, 0 \rangle) \subseteq \mathcal{V}(\langle \mathbb{Z}, -1 \rangle) \subseteq \mathcal{W} \subseteq \mathcal{V}(\langle \mathbb{R}, -1 \rangle) = p\mathcal{A}$, where $\mathcal{W}$ is any subvariety other than the four listed. This information is depicted in Figure V.1 above.

We observe that $\mathcal{V}(\langle \mathbb{Z}, -1 \rangle)$ corresponds to $\mathcal{BA}$, the variety of Boolean algebras, and $\mathcal{V}(\langle \mathbb{R}, -1 \rangle)$ corresponds to $\mathcal{MV}$. Since $\langle \mathbb{R}, -1 \rangle$ is unital, we note that the interval corresponding to $\mathcal{MV}$ must consist of only one variety (namely, $p\mathcal{A}$). The goal of the next section is to show that the same result holds for every non-trivial variety of MV-algebras.

V.2 Varieties Generated by Unital ℓ-groups

The next theorem is the main result of this chapter.

Theorem V.2.1. *If $\mathcal{V}$ is a non-trivial variety of MV-algebras, then $\Gamma^{-1}(\mathcal{V}) = \Sigma(\mathcal{V})$. That is, there is exactly one subvariety of $p\mathcal{A}$ that maps via Γ to $\mathcal{V}$.*

The following proposition provides the first step of the proof.

Proposition V.2.2. *Every subvariety of $p\mathcal{A}$ is generated by its totally-ordered members.*

Proof. We only need to show that every subdirectly irreducible algebra in $p\mathcal{A}$ is totally ordered. Since the addition of the constant does not affect subdirect irreducibility, $\langle \mathbf{G}, a \rangle$

is subdirectly irreducible in $p\mathcal{A}$ if and only if $\mathbf{G}$ is subdirectly irreducible in $\mathcal{A}$. Since it is known that subdirectly irreducible Abelian ℓ-groups are totally ordered [7], this completes the proof. $\square$

We will prove Theorem V.2.1 by showing that if $\langle \mathbf{G}, a\rangle$ is such that $\mathbf{G}$ is totally ordered and $a \neq 1$, then $\langle \mathbf{G}, a\rangle$ is in the variety generated by $\langle \mathbf{H}, a\rangle$, where $\mathbf{H}$ is the convex subalgebra of $\mathbf{G}$ generated by a. Since $\langle \mathbf{H}, a\rangle \in \Sigma(\mathcal{V})$, so is $\langle \mathbf{G}, a\rangle$. Thus, $\Gamma^{-1}(\mathcal{V}) = \Sigma(\mathcal{V})$, for non-trivial $\mathcal{V}$.

The next two lemmas are pointed versions of results that originally appeared in [29], the Ph.D. thesis of Mary Elizabeth Huss. We will use the symbol $\mathbf{B}\overleftarrow{\times}\mathbf{A}$ to indicate the lexicographic order on $\mathbf{B}\times\mathbf{A}$ given by first checking the order in $\mathbf{A}$, then checking the order in $\mathbf{B}$.

Lemma V.2.3. *If $\langle \mathbf{H}, a\rangle$ is a non-trivial, unital Abelian ℓ-group, then $\langle \mathbf{H}\overleftarrow{\times}\mathbb{Z}, (a,0)\rangle$ is in the variety generated by $\langle \mathbf{H}, a\rangle$.*

Proof. First, consider the direct product $\mathbf{H}^{\mathbb{N}}$ of denumerably many copies of $\mathbf{H}$. Then, consider the convex subalgebra S of $\mathbf{H}^{\mathbb{N}}$ consisting of elements with finitely many non-zero entries. We wish to embed $\mathbf{H}\overleftarrow{\times}\mathbb{Z}$ into the quotient $\mathbf{H}^{\mathbb{N}}/S$.

For any $h \in H$, let $\bar{h}$ be the S-equivalence class of the element of $\mathbf{H}^{\mathbb{N}}$ all of whose entries are h. Also, let a_0 be the S-equivalence class of $(|a|, |a|^2, |a|^3, ...)$. Consider the function $f : \mathbf{H}\overleftarrow{\times}\mathbb{Z} \to \mathbf{H}^{\mathbb{N}}/S$ given by $f(h,n) = {a_0}^n\bar{h}$. If ${a_0}^n\bar{h} = {a_0}^m\bar{k}$, then ${a_0}^{n-m} = \overline{kh^{-1}}$. This implies that all but finitely many powers of a^{n-m} equal kh^{-1}, which is not possible unless $n-m=0$. Thus, we see that $(h,n) = (k,m)$. Therefore, f must be injective.

We must also show that f is in fact an ℓ-group homomorphism. Certainly, it is true that $f((h,n)\cdot(k,m)) = f(hk, n+m) = {a_0}^{n+m}\overline{hk} = ({a_0}^n\bar{h})({a_0}^m\bar{k}) = f(h,n)f(k,m)$. We also see that $f((h,n)\wedge(k,n)) = f(h\wedge k, n) = {a_0}^n\overline{h\wedge k} = ({a_0}^n\bar{h})\wedge({a_0}^n\bar{k}) = f(h,n)\wedge f(k,n)$. For $n < m$, observe that $f((h,n)\wedge(k,m)) = f(h,n) = {a_0}^n\bar{h}$.

On the other hand, since $\mathbf{H}$ is unital, there must exist a positive integer p such that

$hk^{-1} < |a|^p$. This means that all but finitely many powers of a^{m-n} are greater than hk^{-1}. This shows that $\overline{hk^{-1}} < {a_0}^{m-n}$, so ${a_0}^n\bar{h} < {a_0}^m\bar{k}$. Thus, $f(h,n) \wedge f(k,m) = {a_0}^n\bar{h} \wedge {a_0}^m\bar{k} = {a_0}^n\bar{h} = f((h,n) \wedge (k,m))$. This shows that f preserves meets. Since a similar argument holds for joins, f is an ℓ-group homomorphism.

The above argument explicitly shows how to obtain $\mathbf{H}\overleftarrow{\times}\mathbb{Z}$ in the variety generated by $\mathbf{H}$, but we need to get a similar result for the pointed objects. Starting with $\langle \mathbf{H}, a\rangle$, taking the direct product, and then the quotient by S, we obtain $\langle \mathbf{H}^{\mathbb{N}}/S, \bar{a}\rangle$. Since f sends $(0,a)$ to $\bar{a}$, we see that $\langle \mathbf{H}\overleftarrow{\times}\mathbb{Z}, (a,0)\rangle$ is in fact isomorphic to a subalgebra of $\langle \mathbf{H}^{\mathbb{N}}/S, \bar{a}\rangle$, completing the proof. $\square$

For the next lemma from Huss' thesis, we will require the following theorem, originally from [26].

Theorem V.2.4. *Let* $\mathbf{A}$ *be a totally-ordered Abelian* ℓ*-group. Let* $v_1(\bar{x}), ..., v_m(\bar{x})$ *be* n*-ary group words and let* $\bar{b} \in A^n$*. Then, there exists* $\bar{k} \in \mathbb{Z}^n$ *such that* $v_i(\bar{b}) < v_j(\bar{b})$ *if and only if* $v_i(\bar{k}) < v_j(\bar{k})$*, for* $i, j = 1, ..., m$*.*

Lemma V.2.5. *For any totally-ordered Abelian* ℓ*-group* $\mathbf{A}$ *and for any negatively-pointed Abelian* ℓ*-group* $\langle \mathbf{H}, a\rangle$*,* $\langle \mathbf{H}\overleftarrow{\times}\mathbf{A}, (a,1)\rangle$ *is in the variety generated by* $\langle \mathbf{H}\overleftarrow{\times}\mathbb{Z}, (a,0)\rangle$*.*

Proof. For the purpose of showing a contradiction, suppose there is a term $w(x_1, ..., x_n)$ such that $\langle \mathbf{H}\overleftarrow{\times}\mathbb{Z}, (a,0)\rangle \vDash w(\bar{x})$, but $\langle \mathbf{H}\overleftarrow{\times}\mathbf{A}, (a,1)\rangle \nvDash w(\bar{x})$. Then, there exists $b_1, ..., b_n \in A$ and $h_1, ..., h_n \in H$ such that in $\langle \mathbf{H}\overleftarrow{\times}\mathbf{A}, (a,1)\rangle$, $w((h_1,b_1), ..., (h_n,b_n)) \neq (1,1)$. By the properties of ℓ-groups, there has to exist group terms $w_{ij}(x_1, ..., x_n)$ (using the variables x_1, ..., x_n, as well as e and the added constant) such that $w(\bar{x}) = \bigwedge_{i\in I}\bigvee_{j\in J} w_{ij}(\bar{x})$, for finite sets I and J.

Thus, if $y_i = (h_i, b_i)$, $w(\bar{y}) = \bigwedge_{i\in I}\bigvee_{j\in J} w_{ij}(\bar{y}) = \bigwedge_{i\in I}\bigvee_{j\in J}(w_{ij}(\bar{h}), w_{ij}(\bar{b}))$, since the underlying group structure of $\mathbf{H}\overleftarrow{\times}\mathbf{A}$ is the same as in the direct product. To compute $w(\bar{y})$ further, fix $i \in I$, and look at $\bigvee_{j\in J} w_{ij}(\bar{y})$. Since the join is happening in $\mathbf{H}\overleftarrow{\times}\mathbf{A}$, we must first consider $\bigvee_{j\in J} w_{ij}(\bar{b}) = c_i$ in $\mathbf{A}$. Letting $J(i) = \{j \in J | w_{ij}(\bar{b}) = c_i\}$, which is non-empty since $\mathbf{A}$ is totally-ordered, we see that $\bigvee_{j\in J} w_{ij}(\bar{y}) = \bigvee_{j\in J}(w_{ij}(\bar{h}), w_{ij}(\bar{b})) = (\bigvee_{j\in J(i)} w_{ij}(\bar{h}), c_i)$. Then, to compute $w(\bar{y}) = \bigwedge_{i\in I}(\bigvee_{j\in J(i)} w_{ij}(\bar{h}), c_i)$, let $d = \bigwedge_{i\in I} c_i$ and $I_0 = \{i \in I | c_i = d\}$. Thus, we see that $w(\bar{y}) = (\bigwedge_{i\in I_0}\bigvee_{j\in J(i)} w_{ij}(\bar{h}), d)$.

Since $\langle \mathbb{Z}, 0\rangle$ is isomorphic to a quotient of $\langle \mathbf{H}\overleftarrow{\times}\mathbb{Z}, (a, 0)\rangle$, and $\langle \mathbf{A}, 1\rangle$ is in the variety generated by $\langle \mathbb{Z}, 0\rangle$, observe that $\langle \mathbf{A}, 1\rangle \vDash w(\bar{x})$. Thus, $d = \bigwedge_{i\in I} c_i = \bigwedge_{i\in I}\bigvee_{j\in J} w_{ij}(\bar{b}) = w(\bar{b}) = 1$. Since $\bar{y}$ was chosen so that $w(\bar{y}) \neq (1, 1)$, $\bigwedge_{i\in I_0}\bigvee_{j\in J(i)} w_{ij}(\bar{h}) \neq 1$.

Now, before we can use Theorem V.2.4, we need to have actual group terms (without the extra constant). However, since we are computing the $w_{ij}(\bar{b})$ in $\langle \mathbf{A}, 1\rangle$, we see that if we let $\widetilde{w_{ij}}$ be w_{ij} with all the added constants replaced by 1, then $\widetilde{w_{ij}}(\bar{b})$ computed in $\mathbf{A}$ is equal to $w_{ij}(\bar{b})$ computed in $\langle \mathbf{A}, 1\rangle$. Now, by Theorem V.2.4, we see that there exists $\bar{k} \in \mathbb{Z}^n$ such that $\widetilde{w_{ij}}(\bar{b}) < \widetilde{w_{i'j'}}(\bar{b})$ if and only if $\widetilde{w_{ij}}(\bar{k}) < \widetilde{w_{i'j'}}(\bar{k})$. In $\langle \mathbf{A}, e\rangle$ and $\langle \mathbb{Z}, 0\rangle$, this translates to $w_{ij}(\bar{b}) < w_{i'j'}(\bar{b})$ if and only if $w_{ij}(\bar{k}) < w_{i'j'}(\bar{k})$.

Observe that since $\langle \mathbb{Z}, 0\rangle \vDash w(\bar{x})$, $\bigwedge_{i\in I}\bigvee_{j\in J} w_{ij}(\bar{k}) = 0$. Fixing $i \in I$ and letting $\alpha_i = \bigvee_{j\in J} w_{ij}(\bar{k})$, the equivalence derived from Theorem V.2.4 implies that $\{j \in J | w_{ij}(\bar{k}) = \alpha_i\} = \{j \in J | w_{ij}(\bar{b}) = c_i\} = J(i)$. Similarly, $\{i \in I | \alpha_i = 0\} = \{i \in I | c_i = d\} = I_0$. So, now if $z_i = (h_i, k_i)$, we see that $w(\bar{z}) = \bigwedge_{i\in I}\bigvee_{j\in J} w_{ij}(\bar{z}) = \bigwedge_{i\in I}\bigvee_{j\in J}(w_{ij}(\bar{h}), w_{ij}(\bar{k})) = \bigwedge_{i\in I}(\bigvee_{j\in J(i)} w_{ij}(\bar{h}), \alpha_i) = (\bigwedge_{i\in I_0}\bigvee_{j\in J(i)} w_{ij}(\bar{h}), 0) \neq (1, 0)$, since when computing the $w_{ij}(\bar{h})$ in $\mathbf{H}$ we use the same constant a as before. However, it was assumed that $\langle \mathbf{H}\overleftarrow{\times}\mathbb{Z}, (a, 0)\rangle \vDash w(\bar{x})$. This completes the proof. □

Now, we are ready to prove Theorem V.2.1. Recall that since $\Gamma^{-1}(\mathcal{V})$ is generated by its totally-ordered members and whenever $\mathcal{V}$ is non-trivial, every positively-pointed Abelian ℓ-group where $a = 1$ is in $\Sigma(\mathcal{V})$, Theorem V.2.1 will be a corollary of the following lemma.

Lemma V.2.6. *For any totally-ordered Abelian ℓ-group $\mathbf{G}$ and $a < 1$, $\langle \mathbf{G}, a\rangle$ is in the variety generated by $\langle \mathbf{H}, a\rangle$, where $\mathbf{H}$ is the convex subalgebra of $\mathbf{G}$ generated by a.*

Proof. Suppose that $\langle \mathbf{G}, a\rangle$ is not in the variety generated by $\langle \mathbf{H}, a\rangle$. Then, there must exist a term $w(\bar{x}) = \bigwedge_{i\in I}\bigvee_{j\in J_i} w_{ij}(\bar{x})$, where the $w_{ij}(\bar{x})$ are groups terms involving the variables, the identity, and the added constant, such that $\langle \mathbf{H}, a\rangle \vDash w(\bar{x})$, but $\langle \mathbf{G}, a\rangle \nvDash w(\bar{x})$. We can assume (without loss of generality) that $w(\bar{x})$ is minimal, in the sense that if any conjunct or disjunct is removed, the resulting term is either satisfied by $\langle \mathbf{G}, a\rangle$ or not satisfied by $\langle \mathbf{H}, a\rangle$.

Since $\langle \mathbf{G}, a\rangle \nvDash w(\bar{x})$, there exists $g_1, ..., g_n \in G$ such that in $\langle \mathbf{G}, a\rangle$, $w(\bar{g}) \neq 1$. Consider the H-equivalence classes ${g_1}', ..., {g_n}'$ of $g_1, ..., g_n$. Also, take any $k_1, ..., k_n \in H$. Then, if $y_i = (k_i, {g_i}')$, observe that $w(\bar{y}) = \bigwedge_{i\in I}\bigvee_{j\in J_i} w_{ij}(\bar{y}) = \bigwedge_{i\ inI}\bigvee_{j\in J_i}(w_{ij}(\bar{k}), w_{ij}(\bar{g}'))$ in $\langle \mathbf{H}\overleftarrow{\times}(\mathbf{G}/\mathbf{H}), (a, 1')\rangle$. Let $J(i) = \{j \in J_i | w_{ij}(\bar{g}') = \bigvee_{j\in J_i} w_{ij}(\bar{g}')\}$ and $I_0 = \{i \in I | \bigvee_{j\in J_i} w_{ij}(\bar{g}') = \bigwedge_{i\in I}\bigvee_{j\in J_i} w_{ij}(\bar{g}')\}$. We see that $w(\bar{y}) = (\bigwedge_{i\in I_0}\bigvee_{j\in J(i)} w_{ij}(\bar{k}), \bigwedge_{i\in I}\bigvee_{j\in J_i} w_{ij}(\bar{g}'))$.

By the previous two lemmas, $\langle \mathbf{H}\overleftarrow{\times}(\mathbf{G}/\mathbf{H}), (a, 1')\rangle$ is in the variety generated by $\langle \mathbf{H}, a\rangle$, so $\langle \mathbf{H}\overleftarrow{\times}(\mathbf{G}/\mathbf{H}), (a, 1')\rangle \vDash w(\bar{x})$. Thus, we know that $\bigwedge_{i\in I}\bigvee_{j\in J_i} w_{ij}(\bar{g}') = 1'$ and $\bigwedge_{i\in I_0}\bigvee_{j\in J(i)} w_{ij}(\bar{k}) = 1$. Since this holds for any $\bar{k} \in H^n$, $\langle \mathbf{H}, a\rangle \vDash \bigwedge_{i\in I_0}\bigvee_{j\in J(i)} w_{ij}(\bar{x})$. For a fixed $i \in I$, let $j \in J_i$ be such that $w_{ij}(\bar{g}) = \bigvee_{j\in J_i} w_{ij}(\bar{g})$. Then, $w_{ij}(\bar{g}') = \bigvee_{j\in J_i} w_{ij}(\bar{g}')$, so $j \in J(i)$. This means that $\bigvee_{j\in J_i} w_{ij}(\bar{g}) = \bigvee_{j\in J(i)} w_{ij}(\bar{g})$. Now, let $i \in I$ be such that $\bigvee_{j\in J_i} w_{ij}(\bar{g}) = \bigwedge_{i\in I}\bigvee_{j\in J_i} w_{ij}(\bar{g})$. Just as before, we see that $i \in I_0$. Thus, $\bigwedge_{i\in I}\bigvee_{j\in J_i} w_{ij}(\bar{g}) = \bigwedge_{i\in I_0}\bigvee_{j\in J(i)} w_{ij}(\bar{g})$. Therefore, $\langle \mathbf{G}, a\rangle \nvDash \bigwedge_{i\in I_0}\bigvee_{j\in J(i)} w_{ij}(\bar{x})$.

Since $w(\bar{x})$ was chosen to be minimal, we see that $I_0 = I$, and for every $i \in I$, $J(i) = J_i$. This means that for any $i \in I$ and any $j \in J_i$, $w_{ij}(\bar{g}') = \bigvee_{j\in J_i} w_{ij}(\bar{g}') = \bigwedge_{i\in I}\bigvee_{j\in J_i} w_{ij}(\bar{g}') = 1'$. So, each $w_{ij}(\bar{g}) \in H$.

Now, let C be any set of least cardinality such that $\{g_1, .., g_n\} \subseteq \langle C \cup H\rangle_{\mathbf{G}}$, where for any $X \subseteq G$, $\langle X\rangle_{\mathbf{G}}$ is the subgroup generated by X in $\mathbf{G}$. Since $\{g_1, ..., g_n\}$ itself satisfies this condition, we see that C must be finite. If $C = \emptyset$, then $\{g_1, .., g_n\} \subseteq H$, but $\langle \mathbf{H}, a\rangle \vDash w(\bar{x})$ and $w(\bar{g}) \neq 1$. So, if the elements of C are $c_1, ..., c_k$, for $i = 1, ..., n$, there must be integers

$p_1^i, ..., p_k^i$ and $h_i \in H$ such that $g_i = {c_1}^{p_1^i} \cdots {c_k}^{p_k^i} h_i$.

Let $v(\bar{x})$ be any of the group terms $w_{ij}(\bar{x})$. Then, there exists integers $q_0, q_1, ..., q_n$ such that $v(x_1, ..., x_n) = {x_1}^{q_1} \cdots {x_n}^{q_n} a^{q_0}$. Thus, we see that $v(\bar{g}) = {g_1}^{q_1} \cdots {g_n}^{q_n} a^{q_0} = ({c_1}^{p_1^1} \cdots {c_k}^{p_k^1} h_1)^{q_1} \cdots ({c_1}^{p_1^n} \cdots {c_k}^{p_k^n} h_n)^{q_n} a^{q_0} = {c_1}^{(p_1^1 q_1 + ... + p_1^n q_n)} \cdots {c_k}^{(p_k^1 q_1 + ... + p_k^n q_n)} ({h_1}^{q_1} \cdots {h_n}^{q_n} a^{q_0})$. If all of the powers of the elements of C are 0, then letting $\bar{h} = (h_1, ..., h_n)$, we observe that $v(\bar{h}) = {h_1}^{q_1} \cdots {h_n}^{q_n} a^{q_0} = v(\bar{g})$. If this happened for every w_{ij}, then $w(\bar{h}) = w(\bar{g}) \neq 1$. Thus, there exists at least one w_{ij} such that some of those powers are non-zero. Since $w_{ij}(\bar{g}) \in H$, we see that there exists $\{d_1, ..., d_r\} \subseteq C$ and non-zero integers $t_1, ..., t_r$ such that ${d_1}^{t_1} \cdots {d_r}^{t_r} \in H$.

Now, in $\mathbf{G}/\mathbf{H}$, we get that $({d_1}')^{t_1} \cdots ({d_r}')^{t_r} = 1'$. Since every non-identity element of a totally-ordered ℓ-group has infinite order, we can assume that $gcd(t_1, ..., t_r) = 1$, where $gcd(a, b)$ stands for the greatest common divisor of a and b . By Lemma V.2.7 following this proof, there exists elements ${u_1}', ..., {u'}_{r-1}$ such that $\{{d_1}', ..., {d_r}'\} \subseteq \langle {u_1}', ..., {u'}_{r-1} \rangle_{\mathbf{G}/\mathbf{H}}$. But, this means that $\{d_1, ..., d_r\} \subseteq \langle \{u_1, ..., u_{r-1}\} \cup H \rangle_{\mathbf{G}}$. Finally, we observe that if $C' = (C - \{d_1, ..., d_r\}) \cup \{u_1, ..., u_{r-1}\}$, $\{g_1, ..., g_n\} \subseteq \langle C \cup H \rangle_{\mathbf{G}} \subseteq \langle C' \cup H \rangle_{\mathbf{G}}$. Since the cardinality of C' is less than the cardinality of C, this is a contradiction, which completes the proof. $\square$

Lemma V.2.7. *In any commutative group, if* ${a_1}^{n_1} \cdots {a_r}^{n_r} = b$*, where each* $n_i \neq 0$ *and* $gcd(n_1, ..., n_r) = 1$*, then there exists* $u_1, ..., u_{r-1}$ *such that* $\{a_1, ..., a_r\} \subseteq \langle u_1, ..., u_{r-1}, b \rangle$.

Proof. The proof will proceed by induction on r. If $r = 1$, then we have $a^n = b$, but n must equal 1 or -1, so $a \in \langle b \rangle$. If $r = 2$, we have ${a_1}^n {a_2}^m = b$, with $gcd(n, m) = 1$. Thus, there exists integers s and t such that $ns + mt = 1$. Letting $u = {a_1}^{-t} {a_2}^s$, we see that $u^n = {a_1}^{-nt} {a_2}^{ns} = ({a_2}^m b^{-1})^t {a_2}^{ns} = {a_2}^{(mt+ns)} b^{-t} = a_2 b^{-t}$ and $u^m = {a_1}^{-mt} {a_2}^{ms} = {a_1}^{-mt-ns} b^s = {a_1}^{-1} b^s$. So, $\{a_1, a_2\} \subseteq \langle u, b \rangle$.

Now, assume the result holds for $k \geq 2$; we wish to show that it holds for $k + 1$. To that end, suppose we have ${a_1}^{n_1} \cdots {a_k}^{n_k} {a_{k+1}}^{n_{k+1}} = b$, with $gcd(n_1, ..., n_k, n_{k+1}) = 1$. If $gcd(n_1, ..., n_k) = 1$, then we see that ${a_1}^{n_1} \cdots {a_k}^{n_k} = b {a_{k+1}}^{-n_{k+1}}$. By the inductive hypothesis, there exists $u_1, ..., u_{k-1}$ such that $\{a_1, ..., a_k\} \subseteq \langle u_1, ..., u_{k-1}, b {a_{k+1}}^{-n_{k+1}} \rangle$. Thus,

$\{a_1, ..., a_k, a_{k+1}\} \subseteq \langle u_1, ..., u_{k-1}, a_{k+1}, b\rangle$.

On the other hand, if $gcd(n_1, ..., n_k) = d > 1$, then there exists $m_1, ..., m_k$ such that $({a_1}^{m_1} \cdots {a_k}^{m_k})^d {a_{k+1}}^{n_{k+1}} = b$. Since it is assumed that $gcd(n_1, ..., n_k, n_{k+1}) = 1$, it must also be true that $gcd(d, n_{k+1}) = 1$. By the result for $r = 2$, we see that there exists u such that $\{{a_1}^{m_1} \cdots {a_k}^{m_k}, a_{k+1}\} \subseteq \langle u, b\rangle$. This means, in particular, that there exists integers α and β such that ${a_1}^{m_1} \cdots {a_k}^{m_k} = u^\alpha b^\beta$. Now, since $gcd(m_1, ..., m_k) = 1$, by the inductive hypothesis, there must exist $v_1, ..., v_{k-1}$ such that $\{a_1, ..., a_k\} \subseteq \langle v_1, ..., v_{k-1}, u^\alpha b^\beta\rangle$. Finally, we observe that $\{a_1, ..., a_k, a_{k+1}\} \subseteq \langle v_1, ..., v_{k-1}, u, b\rangle$, which establishes the result. $\square$

Finally, we observe that from Theorem V.2.1 we immediately get the following corollaries, the first of which is a modification of the lattice isomorphism between the lattice of so-called equational classes of unital Abelian ℓ-groups and the subvariety lattice of $\mathcal{MV}$ (originally established in [18]).

Theorem V.2.8. *The functor Γ induces a lattice isomorphism between the subvariety lattice of $p\mathcal{A}$ (excluding the trivial variety) and the subvariety lattice of $\mathcal{MV}$.*

Consider the following (recursive) definition of the translation T from formulas in the language of MV-algebras to formulas in the language of negatively-pointed ℓ-groups. First, define $T(0) = a$, $T(1) = 1$, and $T(x) = (x \wedge 1) \vee a$, for any variable x. For formulas α and β, define $T(\alpha \wedge \beta) = T(\alpha) \wedge T(\beta)$ and $T(\alpha \vee \beta) = T(\alpha) \vee T(\beta)$. Lastly, define $T(\alpha \oplus \beta) = (T(\alpha) \cdot T(\beta)) \vee a$ and $T(\alpha \to \beta) = (T(\alpha)^{-1} \cdot T(\beta)) \wedge 1$. It is straightforward to show that if $\langle \mathbf{G}, a\rangle$ is a negatively-pointed Abelian ℓ-group and ϕ is an MV-formula, then $\langle \mathbf{G}, a\rangle \vDash T(\phi)$ iff $[a, 1] \vDash \phi$.

Theorem V.2.9. *For every non-trivial variety $\mathcal{V}$ of MV-algebras, its corresponding variety $\sigma(\mathcal{V})$ of negatively-pointed Abelian ℓ-groups is axiomatized by the formulas $\{T(\phi) \mid \mathcal{V} \vDash \phi\}$, relative to $p\mathcal{A}$.*

Proof. It is clear that $\Gamma^{-1}(\mathcal{V})$ is axiomatized by these formulas, relative to $p\mathcal{A}$, and by Theorem V.2.1, $\Sigma(\mathcal{V}) = \Gamma^{-1}(\mathcal{V})$. $\square$

V.3 Gödel-McKinsey-Tarski Translation

It seems pertinent to go through the details of the aforementioned translation more thoroughly. Consider the following (recursive) definition of the translation D of formulas in the language of FL-algebras into the language of FL-algebras with a conucleus $\Box$:

(1) $D(0) = \Box 0$

(2) $D(1) = \Box 1$

(3) $D(p) = \Box p$, for every propositional variable p

(4) For formulas α and β, $D(\alpha \cdot \beta) = D(\alpha) \cdot D(\beta)$.

(5) For formulas α and β, $D(\alpha \vee \beta) = D(\alpha) \vee D(\beta)$.

(6) For formulas α and β, $D(\alpha \wedge \beta) = \Box(D(\alpha) \wedge D(\beta))$.

(7) For formulas α and β, $D(\alpha \backslash \beta) = \Box(D(\alpha) \backslash D(\beta))$.

(8) For formulas α and β, $D(\alpha / \beta) = \Box(D(\alpha) / D(\beta))$.

One possible concern is that when $\wedge = \cdot$, there appears to be an inconsistency depending on if one chooses to define $D(\alpha \wedge \beta)$ by $D(\alpha \wedge \beta) = \Box(D(\alpha) \wedge D(\beta))$ or by $D(\alpha \cdot \beta) = D(\alpha) \cdot D(\beta) = D(\alpha) \wedge D(\beta)$. The next lemma, however, resolves this conflict.

Lemma V.3.1. *Let ϕ be a formula in the language of residuated lattices and let $\langle \mathbf{A}, \Box \rangle$ be a residuated lattice with a conucleus. Then, $D(\phi) = \Box D(\phi)$ in $\langle \mathbf{A}, \Box \rangle$.*

Proof. The proof will proceed by induction on the length of the formula ϕ. If ϕ is just 0, 1, or p, then $D(\phi)$ is $\Box 0$, $\Box 1$, or $\Box p$, respectively, in which case $\Box D(\phi) = D(\phi)$ since conuclei satisfy the equation $\Box\Box x = \Box x$.

Now, assume that $\phi = \alpha \circ \beta$, where $\circ \in \{\vee, \cdot\}$. Referring to the comment after Definition II.3.4 and the proof of Proposition II.3.7, we recall that conuclei satisfy the following equations: $\Box(\Box x \cdot \Box y) = \Box x \cdot \Box y$ and $\Box(\Box x \vee \Box y) = \Box x \vee \Box y$. Thus, $D(\phi) = D(\alpha \circ \beta) = D(\alpha) \circ D(\beta) = \Box D(\alpha) \circ \Box D(\beta) = \Box(\Box D(\alpha) \circ \Box D(\beta)) = \Box(D(\alpha) \circ D(\beta)) = \Box D(\phi)$.

Lastly, assume that $\phi = \alpha \circ \beta$, where $\circ \in \{\wedge, \backslash, /\}$. Then, we see that $D(\phi) = D(\alpha \circ \beta) = \Box(D(\alpha) \circ D(\beta)) = \Box\Box(D(\alpha) \circ D(\beta)) = \Box D(\alpha \circ \beta) = \Box D(\phi)$. $\Box$

Lemma V.3.2. *Let $\phi(p_1, ..., p_n)$ be a formula in the language of residuated lattices and let $\langle \mathbf{A}, \Box \rangle$ be a residuated lattice with a conucleus. Then, for every $a_1, ..., a_n \in A$, $\phi^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n)$ $= D(\phi)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n)$.*

Proof. Again, the proof will proceed by induction on the length of the formula ϕ. If ϕ is 0 or 1, then $\phi^{\mathbf{A}_\Box}$ is $\Box 0$ or $\Box 1$, which is precisely $D(\phi)$. Now, assume $\phi = p$. Let $a \in A$. Then, $\phi^{\mathbf{A}_\Box}(\Box a) = \Box a$ and $D(\phi) = \Box p$, so $D(\phi)^{\langle \mathbf{A}, \Box \rangle}(a) = \Box a$.

Now, assume that $\phi = \alpha \circ \beta$, where $\circ \in \{\vee, \cdot\}$. Then,

$$\phi^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) =$$

$$(\alpha \circ \beta)^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) =$$

$$\alpha^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) \circ \beta^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) =$$

$$D(\alpha)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n) \circ D(\beta)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n) =$$

$$D(\alpha \circ \beta)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n) =$$

$$D(\phi)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n).$$

Lastly, assume that $\phi = \alpha \circ \beta$, where $\circ \in \{\wedge, \backslash, /\}$. Then,

$$\phi^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) =$$

$$(\alpha \circ \beta)^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) =$$

$$\Box(\alpha^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) \circ \beta^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) =$$

$$\Box(D(\alpha)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n) \circ D(\beta)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n) =$$

$$D(\alpha \circ \beta)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n) =$$

$$D(\phi)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n).$$

$\Box$

Proposition V.3.3. *Let $\phi(p_1, ..., p_n)$ be a formula in the language of residuated lattices and let $\langle \mathbf{A}, \Box \rangle$ be a residuated lattice with a conucleus. Then, $\mathbf{A}_\Box \models \phi$ if and only if $\langle \mathbf{A}, \Box \rangle \models D(\phi)$.*

Proof. First, $\mathbf{A}_\Box \models \phi$ iff for every $a_1, ..., a_n \in A$, $\phi^{\mathbf{A}_\Box}(\Box a_1, ..., \Box a_n) \geq \Box 1$. By the previous lemma, this is the case iff for every $a_1, ..., a_n \in A$, $D(\phi)^{\langle \mathbf{A}, \Box \rangle}(a_1, ..., a_n) \geq \Box 1 = 1$, which is true iff $\langle \mathbf{A}, \Box \rangle \models D(\phi)$. $\Box$

Definition V.3.4. *For varieties $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{RL}_\sigma$ and $\mathcal{RL}$, respectively, $\mathcal{V}$ is called a* **modal companion** *of $\mathcal{W}$ if for every residuated lattice formula ϕ, $\mathcal{V} \models D(\phi)$ iff $\mathcal{W} \models \phi$.*

Proposition V.3.5. *For varieties $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{RL}_\sigma$ and $\mathcal{RL}$, respectively, $\Gamma[\mathcal{V}] = \mathcal{W}$ iff $\mathcal{V}$ is a modal companion of $\mathcal{W}$.*

Proof. Observe that $\Gamma[\mathcal{V}] = \mathcal{W}$ iff $\mathcal{W}$ and $\Gamma[\mathcal{V}]$ satisfy exactly the same residuated lattice formulas. Since $\mathcal{V} \models D(\phi)$ iff $\Gamma[\mathcal{V}] \models \phi$, we derive the desired equivalence. $\Box$

Intuitionistic logic INT corresponds to the variety of Heyting algebras, and the modal logic $S4$ corresponds to the variety of interior algebras. So, we get the following, originally from [36].

Theorem V.3.6. *Let ϕ be a formula in the language of intuitionistic logic. Then, $INT \vdash \phi$ if and only if $S4 \vdash D(\phi)$.*

As we observed in Theorem V.2.9, the modal varieties $\Gamma^{-1}(\mathcal{W})$ are axiomatized by the additional equations $\{D(\phi) \mid \phi \in L\}$, where L is the logic corresponding to $\mathcal{W}$. Also, $\Gamma[\mathcal{V}]$ is axiomzatized by $\{\phi \mid D(\phi) \in M\}$, where M is the logic corresponding to $\mathcal{V}$. These logical considerations provide an alternative way of investigating these intervals of modal companions.

CHAPTER VI

CONSTRUCTING FREE ALGEBRAS

VI.1 Free MV-algebras

The purpose of this section is to construct free MV-algebras from free Abelian ℓ-groups:

Theorem VI.1.1. *Let X be any set of cardinality κ. Consider $Y = X \cup \{y\}$, where $y \notin X$, and let $\mathbf{F}$ be the free Abelian ℓ-group over Y. Now, consider the MV-algebra $\Gamma(\langle \mathbf{F}, y \wedge 1\rangle)$. For any $x \in X$, let $\bar{x} = (x \wedge 1) \vee (y \wedge 1)$. If $\bar{X} = \{\bar{x} \mid x \in X\}$, let $\mathbf{A}$ be the MV-subalgebra of $\Gamma(\langle \mathbf{F}, y \wedge 1\rangle)$ generated by $\bar{X}$. Then, we have the following:*

(1) The cardinality of $\bar{X}$ is κ.

(2) $\mathbf{A}$ is freely generated (in $\mathcal{MV}$) by $\bar{X}$.

Proof.

(1) Consider the map that sends x to $\bar{x}$. If this map is injective, this would imply that the cardinality of $\bar{X}$ is the same as the cardinality of X, namely κ. For distinct $x, z \in X$, consider the function f from Y into $\mathbb{Z}$ that sends x to 0 and all other elements of Y (including z) to -1. Then, this map extends to a homomorphism $\bar{f} : \mathbf{F} \to \mathbb{Z}$. We see then that $\bar{f}(\bar{x}) = \bar{f}((x \wedge 1) \vee (y \wedge 1)) = (0 \wedge 0) \vee (-1 \wedge 0) = 0$, while $\bar{f}(\bar{z}) = \bar{f}((z \wedge 1) \vee (y \wedge 1)) = (-1 \wedge 0) \vee (-1 \wedge 0) = -1$. Thus, $\bar{x}$ and $\bar{z}$ must be distinct elements of $\mathbf{F}$.

(2) We must show that for any MV-algebra $\mathbf{B}$ (represented as $\Gamma(\langle \mathbf{G}, b\rangle)$, for some Abelian ℓ-group $\mathbf{G}$ and some $b \in G^-$) and any function $g : \bar{X} \to B$, there is a homomorphism $\bar{g} : \mathbf{A} \to \mathbf{B}$ that extends g (see Figure VI.1 below).

Consider the map $h : Y \to G$ that sends y to b and each $x \in X$ to $g(\bar{x}) \in B \subseteq G$. Then, this function h extends to a homomorphism $\bar{h} : \mathbf{F} \to \mathbf{G}$. Notice also that

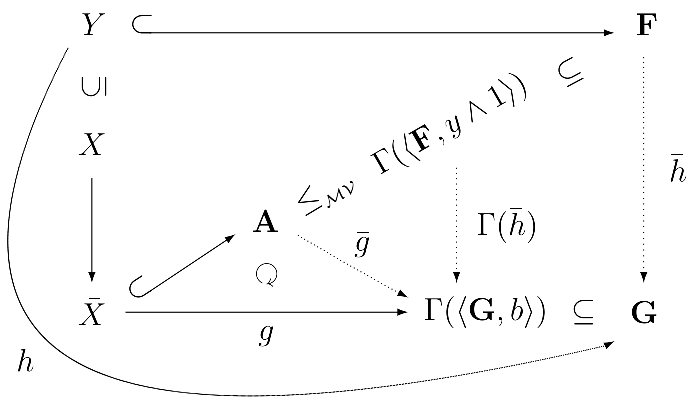

Figure VI.1: Diagram for the proof of Theorem VI.1.1

$\bar{h}(y \wedge 1) = b \wedge 1 = b$. Thus, $\bar{h}$ can be restricted to an MV-algebra homomorphism $\Gamma(\bar{h}) : \Gamma(\langle \mathbf{F}, y \wedge 1\rangle) \to \Gamma(\langle \mathbf{G}, b\rangle)$. We may now define $\bar{g} : \mathbf{A} \to \mathbf{B}$ as the restriction of $\Gamma(\bar{h})$ to $\mathbf{A}$. It only remains to show that $\bar{g}$ extends g. For $x \in X$, $\bar{g}(\bar{x}) = \bar{h}(\bar{x}) = \bar{h}((x \wedge 1) \vee (y \wedge 1)) = (h(x) \wedge 1) \vee (h(y) \wedge 1) = (g(\bar{x}) \wedge 1) \vee (b \wedge 1) = g(\bar{x})$, since $g(\bar{x}) \in B = [b, 1]$. This completes the proof.

$\square$

We explicitly state that the construction above shows that every free MV-algebra is a subalgebra of an interval of a free Abelian ℓ-group. We then achieve the following alternative proof of Chang's Completeness Theorem [13, 14]:

Corollary VI.1.2. *The MV-algebra $[-1, 0] \subseteq \mathbb{R}$ generates the variety of MV-algebras.*

Proof. For every free MV-algebra $\mathbf{F}$, there exists a free Abelian ℓ-group $\mathbf{G}$ and an element $a \in G^-$ such that $\mathbf{F}$ is a subalgebra of the MV-algebra defined on the interval $[a, 1]$ of $\mathbf{G}$. By standard universal algebraic considerations, since $\mathbb{Z}$ generates the variety of Abelian ℓ-groups

[45], there exists a set I such that $\mathbf{G} \leq \prod_{i \in I} \mathbb{Z}$. Thus, there exist $a_i \in \mathbb{Z}^-$, for $i \in I$, such that $[a, e] \leq_{\mathcal{MV}} \prod_{i \in I} [a_i, 0]$. Since the MV-algebra $[-1, 0]$ contains every finite MV-chain as a subalgebra, this completes the proof. $\square$

While it is certainly not the case that the unital Abelian ℓ-group corresponding to a free MV-algebra is a free Abelian ℓ-group, the previous construction does show that it is a subalgebra of a free Abelian ℓ-group. We can then ask if there is anything more that can be said about these Abelian ℓ-groups, and in fact, the construction does allow us to prove that these unital Abelian ℓ-groups are projective as Abelian ℓ-groups. More can be said about this situation, however, and this is the content of the next theorem, which while known in the finitely generated case is certainly not known in this generality. Note that an algebra $\mathbf{A}$ in a variety $\mathcal{V}$ is said to be projective if whenever there are homomorphisms $f : \mathbf{A} \to \mathbf{C}$ and $g : \mathbf{B} \to \mathbf{C}$ with g surjective, then there exists a homomorphism $h : \mathbf{A} \to \mathbf{B}$ such that $g \circ h = f$.

Theorem VI.1.3. *The unital Abelian ℓ-group corresponding to a projective MV-algebra is projective as an Abelian ℓ-group.*

Proof. Let $\langle \mathbf{G}, a \rangle$ be the unital Abelian ℓ-group corresponding to a projective MV-algebra $\mathbf{A}$. We wish to show that $\mathbf{G}$ is projective. To that end, assume that there are Abelian ℓ-groups $\mathbf{H}$ and $\mathbf{K}$ and homomorphisms $f : \mathbf{G} \to \mathbf{H}$ and $g : \mathbf{K} \to \mathbf{H}$, with g surjective. We need to show that there is a homomorphism $h : \mathbf{G} \to \mathbf{K}$ such that $g \circ h = f$ (see Figure VI.2 below).

First, if $f(a) = 1$, then f must be the trivial homomorphism. So, by taking h as the trivial homomorphism from $\mathbf{G}$ to $\mathbf{K}$, it is clear that $g \circ h = f$, and we are done. Thus, we may assume that $f(a) < 1$.

If $\bar{\mathbf{H}}$ is the convex subalgebra of $\mathbf{H}$ generated by $f(a)$, we see that f is a unital ℓ-group homomorphism from $\langle \mathbf{G}, a \rangle$ to $\langle \bar{\mathbf{H}}, f(a) \rangle$ (noting that since a is mapped into $\bar{\mathbf{H}}$, all of G must be as well). Now, since g is surjective, there exists $b \in K^-$ such that $g(b) = f(a)$. Letting $\bar{\mathbf{K}}$

$$
\begin{tikzcd}
\mathbf{G} \arrow[rrd, "f"] \arrow[dd, dotted, "h"'] & & \\
{} \arrow[r, phantom, "\circlearrowleft"] & {} & \mathbf{H} \\
\mathbf{K} \arrow[rru, "g"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle \mathbf{G}, a\rangle \arrow[rrd, "f"] \arrow[dd, dotted, "h"'] & & \\
{} \arrow[r, phantom, "\circlearrowleft"] & {} & \langle \bar{\mathbf{H}}, f(a)\rangle \\
\langle \bar{\mathbf{K}}, b\rangle \arrow[rru, "\bar{g}"'] & &
\end{tikzcd}
$$

Figure VI.2: Diagram for the proof of Theorem VI.1.3

be the convex subalgebra of $\mathbf{K}$ generated by b and letting $\bar{g}$ be the restriction of g to $\bar{\mathbf{K}}$, we observe that $\bar{g}$ is a unital ℓ-group homomorphism from $\langle \bar{\mathbf{K}}, b\rangle$ to $\langle \bar{\mathbf{H}}, f(a)\rangle$. We claim that $\bar{g}$ is surjective. For any $x \in \bar{H}$, there exists an integer n such that $f(a)^n \leq x \leq f(a)^{-n}$. Since g is surjective, there must exist $y \in K$ such that $g(y) = x$. Letting $z = (y \wedge b^{-n}) \vee b^n \in \bar{K}$, we see that $\bar{g}(z) = g(z) = (x \wedge f(a)^{-n}) \vee f(a)^n = x$, so $\bar{g}$ is surjective.

By the categorical equivalence between unital Abelian ℓ-groups and MV-algebras (as well as the correspondence of onto maps in those categories), we know that $\langle \mathbf{G}, a\rangle$ is projective in the class of unital Abelian ℓ-groups. Thus, there must exist a homomorphism h from $\langle \mathbf{G}, a\rangle$ to $\langle \bar{\mathbf{K}}, b\rangle$ such that $\bar{g} \circ h = f$. Since h maps $\mathbf{G}$ into a subalgebra of $\mathbf{K}$, h may also be viewed as a homomorphism from $\mathbf{G}$ into $\mathbf{K}$, and certainly $g \circ h = f$, which completes the proof. $\square$

VI.2 Free Negative Cones

As we have noted, the class of negative cones of Abelian ℓ-groups is actually a variety, which we denote $\mathcal{A}^-$. Using methods similar to the ones in the previous section, we now describe how to obtain free algebras in the variety $\mathcal{A}^-$ from free Abelian ℓ-groups:

Theorem VI.2.1. *Let X be any set of cardinality κ, and let $\mathbf{F}$ be the free Abelian ℓ-group over X. For any $x \in X$, let $x^- = x \wedge 1$. If $X^- = \{x^- \mid x \in X\}$, let $\mathbf{A}$ be the subalgebra of the negative cone $\mathbf{F}^-$ of $\mathbf{F}$ generated by X^-. Then, we have the following:*

(1) The cardinality of X^- is κ.

(2) $\mathbf{A}$ is freely generated (in $\mathcal{A}^-$) by X^-.

Proof.

(1) We claim that the map that sends x to x^- is injective. For distinct $x, y \in X$, consider the function f from X into $\mathbb{Z}$ that sends x to 0 and all other elements of X (including y) to -1. Then, this map extends to a homomorphism $\bar{f} : \mathbf{F} \to \mathbb{Z}$. We see then that $\bar{f}(x^-) = 0$, while $\bar{f}(y^-) = -1$. Thus, x^- and y^- must be distinct elements of $\mathbf{F}$.

(2) Assume that there is a function $g : X^- \to G^-$, for some Abelian ℓ-group $\mathbf{G}$. We need to show that this map can be extended to a homomorphism $\bar{g} : \mathbf{A} \to \mathbf{G}^-$. To that end, consider the function $h : X \to G$ given by $h(x) = g(x^-) \in G^- \subseteq G$. Then, this map can be extended to a homomorphism $\bar{h} : \mathbf{F} \to \mathbf{G}$. Restricting first to the negative cones, then to $\mathbf{A}$, we get an $\mathcal{A}^-$– homomorphism $\bar{g} : \mathbf{A} \to \mathbf{G}^-$. To show that $\bar{g}$ extends g, let $x \in X$, and observe that $\bar{g}(x^-) = \bar{h}(x^-) = \bar{h}(x \wedge 1) = h(x) \wedge 1 = g(x^-) \wedge 1 = g(x^-)$ since $g(x^-) \in G^-$. This completes the proof.

$\square$

It should be noted that, in general, this subalgebra $\mathbf{A}$ described above is a proper subset of the negative cone $\mathbf{F}^-$ of the free Abelian ℓ-group $\mathbf{F}$. For example, the Abelian ℓ-group $\mathbb{Z}^2$ is freely generated by the set $X = \{(-1, 1)\}$, but the resulting set $X^- = \{(-1, 0)\}$ generates a subalgebra isomorphic to $\mathbb{Z}^-$ in the negative cone $(\mathbb{Z}^-)^2$.

We now give a description of the finitely generated free algebras in the variety $\mathcal{A}^-$, based on the following characterization of finitely generated free algebras in the variety $\mathcal{A}$ of Abelian ℓ-groups. By standard universal algebraic techniques, using the fact that $\mathbb{R}$ generates the variety $\mathcal{A}$, we see that for any natural number n, the free algebra in $\mathcal{A}$ on n generators, denoted $\mathbf{F}_{\mathcal{A}}(n)$, is a subalgebra of $\mathbb{R}^{\mathbb{R}^n}$. Since the free algebra is generated by the n projections, it is also clear that every function in $\mathbf{F}_{\mathcal{A}}(n)$ is continuous and piecewise-linear functions with integer coefficients. A function $f : \mathbb{R}^n \to \mathbb{R}$ is linear if there exist coefficients $a_1, ..., a_n$ such that $f(x_1, ..., x_n) = a_1x_1 + \cdots + a_nx_n$. A function f is piecewise-linear if there exist linear functions $f_1, ..., f_m$ such that for every $x \in \mathbb{R}^n$, there exists an $i \in \{1, ..., m\}$ such that $f(x) = f_i(x)$. Now, we can state the most difficult part of the Baker-Beynon Theorem [3, 5, 6]:

Theorem VI.2.2. *Let n be a natural number. Every continuous, piecewise-linear function with integer coefficients from $\mathbb{R}^n$ to $\mathbb{R}$ is in $\mathbf{F}_{\mathcal{A}}(n)$.*

Observe the similarity between the previous theorem and the following one, which is a slight variation (into our context) of a result due to McNaughton [37]. Again, since $[-1, 0]$ generates the variety $\mathcal{MV}$, it is known that every function in $\mathbf{F}_{\mathcal{MV}}(n)$, the free MV-algebra on n generators, viewed as the subalgebra of $[-1, 0]^{[-1,0]^n}$ generated by the n projections, is continuous and piecewise-affine with integer coefficients. A piecewise-affine function is defined analogously to a piecewise-linear one except that an affine function f is one such that there exist coefficients $a_0, a_1, ..., a_n$ such that $f(x_1, ..., x_n) = a_0 + a_1x_1 + \cdots + a_nx_n$.

Theorem VI.2.3. *Let n be a natural number. Every continuous, piecewise-affine function with integer coefficients from $[-1, 0]^n$ to $[-1, 0]$ is in $\mathbf{F}_{\mathcal{MV}}(n)$.*

Since MV-algebras are intervals of negative cones of Abelian ℓ-groups, in some sense, the variety $\mathcal{A}^-$ is between the varieties $\mathcal{A}$ and $\mathcal{MV}$. It seems natural then to ask whether a Baker-Beynon-McNaughton-type theorem holds for $\mathcal{A}^-$. Using a similar definition of piecewise-linear functions from $(\mathbb{R}^-)^n$ to $\mathbb{R}^-$ as in the ℓ-group case, we get Theorem VI.2.5 below about $\mathbf{F}_{\mathcal{A}^-}(n)$, the free algebra in the variety $\mathcal{A}^-$ on n generators. First, however, we need the following lemma:

Lemma VI.2.4. *Every continuous, piecewise-linear function with integer coefficients from $(\mathbb{R}^-)^n$ to $\mathbb{R}^-$ can be extended to a continuous piecewise-linear function with integer coefficients from $\mathbb{R}^n$ to $\mathbb{R}^-$.*

Proof. Let f be a continuous, piecewise-linear function with integer coefficients from $(\mathbb{R}^-)^n$ to $\mathbb{R}^-$. Define $\Sigma(f) : \mathbb{R}^n \to \mathbb{R}^-$ by $\Sigma(f)(x) = f(x \wedge e)$, for $x \in \mathbb{R}^n$, where e is the identity of $\mathbb{R}^n$. We wish to show that this $\Sigma(f)$ is in fact continuous and piecewise-linear with integer coefficients. Since f and the function that sends x to $x \wedge e$ are both continuous, $\Sigma(f)$ is also continuous. Now, for each of the m linear functions that constitute f, there are at most 2^n linear functions that can be obtained from it by deleting some of its summands (including deleting all of them and obtaining the constant 0 function). We claim that there are no more than $2^n m$ linear functions (with integer coefficients) that constitute $\Sigma(f)$. To see this, let $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Then, there exists $i \in \{1, ..., m\}$ such that $\Sigma(f)(x) = f(x \wedge e) = f_i(x \wedge e) = a_1^i(x_1 \wedge 0) + \cdots + a_n^i(x_n \wedge 0)$, for some integers $a_1^i, ..., a_n^i$. By considering the linear function g obtained from f_i by deleting all of the summands corresponding to when $x_j \wedge 0 = 0$, we see that $\Sigma(f)(x) = g(x)$. Thus, $\Sigma(f)$ is a continuous, piecewise-linear function with integer coefficients from $\mathbb{R}^n$ to $\mathbb{R}^-$ that extends f. $\square$

We now prove the Baker-Beynon-McNaughton-type theorem for $\mathcal{A}^-$:

Theorem VI.2.5. *Let n be a natural number. Then, $\mathbf{F}_{\mathcal{A}^-}(n)$ is the subalgebra of $(\mathbb{R}^-)^{(\mathbb{R}^-)^n}$ made up of the continuous, piecewise-linear functions with integer coefficients.*

Proof. In order to construct $\mathbf{F}_{\mathcal{A}^-}(n)$, we will start with $\mathbf{F}$, the free Abelian ℓ-group on n generators. With the aid of Theorem VI.2.2, observe that the negative cone $\mathbf{F}^-$ of $\mathbf{F}$ consists precisely of the continuous piecewise-linear functions with integer coefficients from $\mathbb{R}^n$ to $\mathbb{R}$ whose image is contained in $\mathbb{R}^-$. Since the generating set of $\mathbf{F}$ is the n projections $\pi_1, ..., \pi_n$, we need to consider $\tau_i = \pi_i \wedge \bar{0}$, for $i = 1, ..., n$, where $\bar{0}$ is the constant 0 function. Then, by Theorem VI.2.1, we see that $\mathbf{F}_{\mathcal{A}^-}(n) = \langle \tau_1, ..., \tau_n \rangle_{\mathbf{F}^-}$.

Now, let C be the set of continuous, piecewise-linear functions with integer coefficients from $(\mathbb{R}^-)^n$ to $\mathbb{R}^-$. We wish to show that the map $\Sigma : \mathbf{C} \to \mathbf{F}^-$ (from Lemma VI.2.4) is actually an embedding (noting that straight-forward verifications show that $\mathbf{C}$ is in fact a subalgebra of $(\mathbb{R}^-)^{(\mathbb{R}^-)^n}$). Clearly, $\Sigma(\bar{0}) = \bar{0}$. Let $* \in \{\wedge, \vee, \cdot, \to\}$. Then, for $x \in \mathbb{R}^n$,

$\Sigma(f*g)(x) = (f*g)(x\wedge e) = f(x\wedge e)*g(x\wedge e) = \Sigma(f)(x)*\Sigma(g)(x) = (\Sigma(f)*\Sigma(g))(x)$, so $\Sigma(f*g) = \Sigma(f)*\Sigma(g)$. Lastly, since $\Sigma(f)$ extends f, Σ is clearly injective.

Observe that if $\rho_i \in C$ is the i-th projection, then $\Sigma(\rho_i) = \tau_i$, since $\Sigma(\rho_i)(x) = \rho_i(x\wedge e) = x_i \wedge 0 = \tau_i(x)$. Thus, $\langle \rho_1, ..., \rho_n\rangle_{\mathbf{C}}$ is isomorphic to $\langle \tau_1, ..., \tau_n\rangle_{\mathbf{F}^-} = \mathbf{F}_{\mathcal{A}^-}(n)$. The final thing we need to show is that $\langle \rho_1, ..., \rho_n\rangle_{\mathbf{C}} = C$.

To that end, let $h \in C$. Consider $\Sigma(h) \in F^-$. In addition to Theorem VI.2.2, it is known that every function in $\mathbf{F}$ can be represented as a join of meets of linear functions [1]. So, $\Sigma(h) = \bigvee_{i\in I}\bigwedge_{j\in J} h_{ij}$, for linear functions h_{ij}. For a linear function k given by $k(x_1, ..., x_n) = k_1x_1 + \cdots + k_nx_n$, define the negative cone term k' by $k'(x_1, ..., x_n) = \sum |k_i|x_i \to \sum k_jx_j$, for $(x_1, ..., x_n) \in (\mathbb{R}^-)^n$, where the sum in the denominator is taken over the negative k_i and the sum in the numerator is taken over the positive k_j. We claim that $h = \bigvee_{i\in I}\bigwedge_{j\in J} h'_{ij}$, so that $h \in \langle \rho_1, ..., \rho_n\rangle_{\mathbf{C}}$.

For any $x \in (\mathbb{R}^-)^n$, $h(x) = \Sigma(h)(x) = \bigvee_{i\in I}\bigwedge_{j\in J} h_{ij}(x)$. We note that $h'_{ij}(x) = h_{ij}(x) \wedge 0$. Since $\Sigma(h)$ only takes negative values, we see that each disjunct must be negative, and so at least one of each of the disjunct's conjuncts must be negative. We then see that replacing $h_{ij}(x)$ by $h_{ij}(x)\wedge 0$ has no effect on the computation, so $h(x) = \bigvee_{i\in I}\bigwedge_{j\in J} h_{ij}(x) = \bigvee_{i\in I}\bigwedge_{j\in J} h'_{ij}(x)$. This completes the proof that $\mathbf{C} = \mathbf{F}_{\mathcal{A}^-}(n)$. $\square$

We explicitly note that the proof above shows that every continuous, piecewise-linear function from $(\mathbb{R}^-)^n$ to $\mathbb{R}^-$ can be represented as a join of meets of basic functions (in the language of [16]), where a basic function is one of the form $f \wedge \bar{0}$, for a linear function f. In fact, from [16], it can be derived that the n-generated free algebra in $\mathcal{A}^-$ is the algebra of all such joins of meets of basic functions with n variables. Since they chose to consider $\mathbb{Z}^-$ instead of $\mathbb{R}^-$, however, they could not discuss continuity, and therefore could not even consider the possibility of a Baker-Beynon-McNaughton-type theorem.

VI.3 Algebras Induced by a Unary Term

The purpose of this section is to generalize the constructions used in the previous two sections. To that end, let $\mathcal{V}$ be a variety of algebras in the signature τ. Let t be a unary term in the signature τ. Fix a subsignature $\tau' \subseteq \tau$. For an algebra $\mathbf{A} \in \mathcal{V}$, define $A_t = \{t(a) \mid a \in A\}$. Also, for any operation $f \in \tau'$, define f_t on A_t by $f_t(x_1, ..., x_n) = t^{\mathbf{A}}(f^{\mathbf{A}}(x_1, ..., x_n))$.

Definition VI.3.1. *Let $\mathbf{A} \in \mathcal{V}$. The algebra $\mathbf{A}_t$ induced by t is defined as the set A_t together with the operations f_t, for $f \in \tau'$.*

Consider the class $\mathcal{W} = \mathcal{S}(\{\mathbf{A}_t \mid \mathbf{A} \in \mathcal{V}\})$, where $\mathcal{S}$ is the subalgebra operator (in the signature τ'). Consider the map Λ that sends an algebra $\mathbf{A} \in \mathcal{V}$ to its induced algebra $\mathbf{A}_t \in \mathcal{W}$. We wish to show that Λ is in fact a functor from $\mathcal{V}$ to $\mathcal{W}$. In order to do this, we need to show that if $g : \mathbf{A} \to \mathbf{B}$ is an τ-homomorphism, for $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, then the restriction of g to A_t is an τ'-homomorphism into B_t.

First, since g is a homomorphism, $g(t^{\mathbf{A}}(a)) = t^{\mathbf{B}}(g(a))$, so g maps A_t into B_t. Now, take $f \in \tau'$. Then, we see that

$$g(f^{\mathbf{A}_t}(t^{\mathbf{A}}(a_1), ..., t^{\mathbf{A}}(a_n))) =$$

$$g(t^{\mathbf{A}}(f^{\mathbf{A}}(t^{\mathbf{A}}(a_1), ..., t^{\mathbf{A}}(a_n)))) =$$

$$t^{\mathbf{B}}(f^{\mathbf{B}}(t^{\mathbf{B}}(g(a_1)), ..., t^{\mathbf{B}}(g(a_n)))) =$$

$$f^{\mathbf{B}_t}(t^{\mathbf{B}}(g(a_1)), ..., t^{\mathbf{B}}(g(a_n))) =$$

$$f^{\mathbf{B}_t}(g(t^{\mathbf{A}}(a_1)), ..., g(t^{\mathbf{A}}(a_n))).$$

Since Λ sends homomorphisms to certain restrictions, it clearly preserves the identity and composition. Thus, Λ is a (covariant) functor from $\mathcal{V}$ to $\mathcal{W}$.

In addition to the examples in the previous sections (where $t(x) = (x \wedge 1) \vee (a \wedge 1)$ or $t(x) = x \wedge 1$), another motivating example for this generalization is reducts (where $t(x) = x$, and the language is appropriately reduced). Since Λ preserves products and $\mathcal{W}$ was defined to be closed under subalgebras, we know that free algebras exist in $\mathcal{W}$ [8]. We can ask what kind of relationship (if any) exists between free algebras in $\mathcal{W}$ and free algebras in $\mathcal{V}$. As it turns out, whenever t is idempotent, we always have the following situation, which generalizes the similar results that we obtained in the previous sections. While this theorem is related to results in [33], the proof we give here is completely independent of that work.

Theorem VI.3.2. *Let $\Lambda : \mathcal{V} \to \mathcal{W}$ be the functor corresponding to the idempotent term t. Also, assume that $\mathcal{W}$ contains a non-trivial algebra. Let X be any set of cardinality κ, and let $\mathbf{F}$ be the free algebra in $\mathcal{V}$ with generating set X. If $\bar{X} = \{t(x) \mid x \in X\}$, let $\mathbf{A}$ be the τ'-subalgebra of $\Lambda(\mathbf{F})$ generated by $\bar{X}$. Then, we have the following:*

(1) The cardinality of $\bar{X}$ is κ.

(2) $\mathbf{A}$ is freely generated (in $\mathcal{W}$) by $\bar{X}$.

Proof.

(1) We wish to show that the function that sends $x \in X$ to $t(x)$ is injective. Let $\mathbf{D}$ be an algebra in $\mathcal{V}$ such that $\mathbf{D}_t$ has at least two elements, say $t(a)$ and $t(b)$. For distinct $x, y \in X$, consider any function $g : X \to D$ that sends x to a and y to b. Then, g must extend to a homomorphism $\bar{g} : \mathbf{F} \to \mathbf{D}$. Thus, we see that $\bar{g}(t^{\mathbf{F}}(x)) = t^{\mathbf{D}}(a) \neq t^{\mathbf{D}}(b) = \bar{g}(t^{\mathbf{F}}(y))$. Therefore, $t(x)$ and $t(y)$ must be distinct elements of $\mathbf{F}$. This shows that $\bar{X}$ has cardinality κ.

(2) We need to show that for any algebra $\mathbf{B} \in \mathcal{W}$, every function $h : \bar{X} \to B$ extends to a homomorphism $\bar{h} : \mathbf{A} \to \mathbf{B}$.

To that end, let $h : \bar{X} \to B$ be given, with $\mathbf{B}$ an τ'-subalgebra of $\mathbf{C}_t$, for some $\mathbf{C} \in \mathcal{V}$. Consider $k : X \to C$ defined by $k(x) = h(t(x))$. Then, there exists a homomorphism $\bar{k} : \mathbf{F} \to \mathbf{C}$ that extends k. Let $\bar{h}$ be the restriction of $\Lambda(\bar{k})$ to $\mathbf{A}$. Clearly, $\bar{h}$ is an L'-homomorphism from $\mathbf{A}$ into $\mathbf{C}_t$. It remains to show that $\bar{h}$ maps into B and that it extends h. Since $\mathbf{B}$ is a subalgebra of $\mathbf{C}_t$, if we show that $\bar{h}$ extends h, then $\bar{h}$ must map into B.

In order to show that $\bar{h}$ extends h, let $x \in X$. Then, $\bar{h}(t(x)) = \bar{k}(t^{\mathbf{F}}(x)) = t^{\mathbf{C}}(\bar{k}(x)) = t^{\mathbf{C}}(k(x)) = t^{\mathbf{C}}(h(t(x))) = h(t(x))$, since t is idempotent and $h(t(x)) \in B \subseteq C_t$. This completes the proof that $\mathbf{A}$ is freely generated by $\bar{X}$.

$\square$

CHAPTER VII

FUTURE RESEARCH

As far as the intervals of modal companions are concerned, I would like to answer the following questions:

(1) What relationships exist between properties (e.g., decidability, finite embeddability property) of varieties of commutative, cancellative residuated lattices and any (or all) of the subvarieties in the corresponding interval in the subvariety lattice of Abelian ℓ-groups with conuclei? And similarly for varieties of integral GMV-algebras and their corresponding intervals in the subvariety lattice of negative cones of Abelian ℓ-groups with nuclei?

(2) Is there a nice axiomatization (or algebraic description) for the least subvariety in each of these intervals (similar to how Grzegorczyk logic is the greatest modal companion of intuitionistic logic)?

(3) Does a Blok-Esakia-type theorem hold in these contexts?

Even in the restricted case of MV-algebras, where the latter two questions have already been answered, there are still many open questions to consider about various logical properties of varieties of MV-algebras.

As far as the categorical equivalences are concerned, a natural next step would be the generalization of the aforementioned categorical equivalences. I anticipate that the variety of GMV-algebras with conuclei will be a reasonable place to begin when looking for a categorical equivalence that includes the others as special cases. If this fails, we will then consider the join of the varieties of Boolean algebras and Abelian ℓ-groups in the subvariety lattice of GMV-algebras. Recently, a relatively simple axiomatization for this join has been shown [43], and it is fairly clear that the equivalence will extend to the join. The problem with this, however, is that we would also need to consider the join of Heyting algebras and commutative, cancellative residuated lattices in the subvariety lattice of residuated lattices. While it certainly exists, as of now its description is unsatisfying, and so we would need to find an axiomatization for this join as well.

While categorical equivalences relate individual algebras in different classes, the usefulness of the GMT translation is that it is about the correspondence between subvarieties of (modal) residuated lattices, and so by extension, it is also about logics. I, jointly with Professor Hiroakira Ono at the Japan Advanced Institute for Science and Technology, have begun the logical investigation of modal substructural logics, the counterpart of residuated lattices with a conucleus, by developing a sequent calculus for the most general modal substructural logic [42]. While every standard modal companion of intuitionistic logic is (by definition) conservative over classical logic, this condition is relaxed in this substructural setting. The question then becomes the investigation of modal companions of a specific substructural logic L that are also conservative over another specified substructural logic K; the collection of such logics is denoted $MC(L, K)$. Preliminary results, like the theorem below, have been established, but many more open questions still remain.

Theorem VII.0.3. *For any substructural logics L and K, we have the following:*

(1) If $MC(L, K) \neq \emptyset$, it has a least element and a maximal element,

(2) $MC(L, L) \neq \emptyset$, and

(3) $MC(FL, CL) = \emptyset$.

A classification of those pairs of substructural logics L and K such that $MC(L, K)$ is non-empty is a primary goal of this sub-project. Also, for the non-empty $MC(L, K)$, we would like to have a characterization of those that have a greatest element (which includes, in particular, all of the cases that have been previously considered).

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