BEYOND CARDINALITY:
HOW CHILDREN LEARN TO REASON ABOUT EXACT NUMBERS

By

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CHAPTER 1

Introduction

Mathematical competence is essential for personal and societal growth. Arithmetic skills can predict a person's employment level, likelihood of incarceration, and psychological well-being (Bynner & Parsons, 2005). For example, men with low math ability are more at risk of depression than men with competent numeracy. Women with low math ability are less likely to have full-time employment than women with competent numeracy. Western society especially values precise calculation over estimation, making exact numerical reasoning an essential skill for academic and professional success. Given that precise mathematical competence is central to functioning in everyday life, important questions arise. How does one develop advanced approaches to manipulating exact numbers? What are the cognitive processes underlying a child's emerging ability for calculation?

Recent research has begun to answer these questions and identified a link between the development of exact numerical reasoning[1] and cardinal number knowledge. Cardinality, the ability to state the quantity of a set, is one of the first numerical skills a child learns (Gelman & Gallistel, 1978; Wynn, 1990, 1992) and may serve as a precursor to performing more advanced mathematics. The importance of cardinality is supported by cross-cultural studies that have found that cultures lacking cardinal number words are unable to reason about exact quantities (Gordon, 2004; Pica, Lemer, Izard, & Dehaene, 2004). Such research suggests that before a

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[1] This paper will use *numerical reasoning* to refer to non-symbolic calculation. Non-symbolic sets include object or groups of dots and will be devoid of symbols such as Arabic numerals or counting words, which are more advanced representations of quantities (Ansari, 2008).

person can reason about a set of objects, he or she must be able to state the exact quantity of the set.

This first chapter of this thesis lays the groundwork for examining how the emergence of exact number reasoning ability in children is specifically supported by the confluence of advances in understanding cardinality and the critical role that exact number words may play. The relevant literature on three central issues are reviewed. The first section of this introduction describes cardinality and how children learn to represent precise quantities. The following section investigates why representation must be verbal in nature. The final section of the introduction considers whether cardinality is sufficient for exact numerical reasoning and what other skills may be necessary.

Cardinality

Cardinality refers to the ability to verbally label an exact set of objects. Cognitive development studies typically associates cardinality with the last word in a count sequence, one procedure used to determine cardinality. While this is a common procedure, there are various other procedures used to verbally label a quantity. Nontraditional counting such as backward counts, subitizing, and estimation can also be used to determine cardinality, though some strategies provide for more accuracy than others. Regardless of the procedure used, to have cardinal number knowledge, a child must be able to verbally state the quantity of the set and understand that the verbal label represents that quantity.

According to Gelman (Gelman & Gallistel, 1978; Gelman & Meck, 1983), children first acquire a cardinality principle, as evidence by the early age by which children display cardinal number knowledge. This principle is then applied to different sets. More recent research contradicts the idea that cardinality comes from initial principles (Briars & Siegler, 1984; Le Corre, Van de Walle, Brannon & Carey, 2006). Children are able to count correctly before they are able to identify the accuracy of other peoples' counts (Briars & Siegler, 1984), suggesting that children apply a counting procedure before they extract more general principles. Similarly Le Corre, et al., found that children learn cardinality in stages. First, a child displays cardinal number knowledge of a set of one object (one-knower), then cardinal number knowledge of one and two (two-knower), and eventually learn the cardinal principle for all sets. This developmental trend suggests that one- and two-knowers represent quantities qualitatively different than cardinal principle knowers, concluding that cardinality principle knowledge is acquired later in a child's development of exact number knowledge. Children learn to count first and then develop cardinal number knowledge.

Counting procedures develop slowly. Through social interactions children learn to count between the ages of two and three (Mix, Sandhofer & Baroody, 2005); however, Wynn (1990, 1992) found that children take roughly one year to attribute meaning to a memorized list of counting words. Even if a child can count to 10, he/she may not know the cardinality of a set. For example, when a three-year-old counts a set of five items and is asked "how many are there?" he/she might not yet understand that the last counting word identifies the number of items in the set.

At first, children learn the meaning of "one" in the manner they learn the meanings of quantifiers such as "a" and apply the word to their exact representation of one. A few months later the child uses the same strategy for the word "two" and, later for the word "three." Quantities one, two and three are exactly represented as early as infancy, possibly due to parallel individuation (Feigenson, Carey & Spelke, 2002, Starkey & Cooper, 1980). Larger quantities are not exactly represented in infancy, and thus a new procedure is needed to assign meaning to counting words greater than three. Carey's bootstrapping theory argues that children use their understanding of one, two and three, combined with their knowledge of the counting chant to infer the successor function (Carey, 2001; 2004, 2009). Because three is one more than two and the word "four" comes after "three" in the counting chant, four must be one more than three. This knowledge of the successor function leads to the cardinality principle, which is then applied to all sets larger than four. Under this account, counting leads to the cardinality principle, which leads to cardinal number knowledge for all quantities within a child's counting range.

Recent research challenges Carey's view, Davidson, et al. (2012) find that children acquire cardinality before learning the successor principle and that they learn smaller quantities before larger ones, even when dealing with numbers greater than four. Instead of learning the successor function to conceptualize a cardinality principle, children learn cardinality one quantity at a time. Alternatively, counting may lead to cardinality by making the imprecise numerical representations more precise (Wynn, 1998; Dehaene, 2001). Counting words may force approximate representations of quantities into distinct categories the way that color terms help place colors into distinct categories (Davidoff et al. 1999). By assigning discrete verbal labels to non-discrete approximate representations, analogue representations are forced into exact categories.

Other research has challenged the very notion that children attain the cardinality principle as a whole insight. For example, Bermejo, et al. (1996) counters Gelman's cardinality principal by suggesting that cardinality develops in stages as opposed to being a foundational principle. According to Bermejo cardinality has six stages of development: in response to the questions "how many items are there?" (1) the child answers with a random number; (2) the child provides a count sequence without referring to the set; (3) the child counts the set; (4) the child uses the "last word" rule; (5) the child answers with the highest count word used, even when a nontraditional counting strategy results in a smaller last word; and (6) the child accurately identifies the quantity regardless of the counting strategy used. By encouraging children to use different count sequences (forward, backward, and erroneous counts), Bermejo was able uncover novel evidence that children ages three to five displayed a variety of the six stages of cardinality. Thus cardinality, as defined by Gelman is only the fourth stage in cardinality understanding and more advanced concepts of cardinality exist. Importantly, most of the stages rely on counting, which serves as essential skill for developing a full understanding of cardinality.

Although theoretical accounts of the rise of cardinality differ on the role of counting, the counting procedure is present in each. Cardinality may stem from the successor function, approximate representations of numbers, or procedural stages, but each of the major theoretical accounts reviewed above stress the importance of counting. Number words impose meaning on previously vague concepts of exact quantity. In other words, in order to represent exact sets, one needs to verbally label the sets. The next section addresses the importance of these verbal representations.

Cardinality and the Sapir-Whorf hypothesis

Cardinality is a verbal representation of exact quantity. The verbal nature of this representation is especially interesting given that exact numerical processing has been heavily linked to linguistic processing. Precise numerical reasoning may share the same underlying mechanisms as the ability to verbalize thoughts. For example, when bilinguals are taught to give exact or approximate answers to two-digit addition problems in one language and then switch to their second language, the language swap impacts the reaction time for solving exact problems, but not approximate problems (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999). The language swap and exact calculation may place similar demands on mental resources, resulting in a delay. Neuroscience, including lesion and neuroimaging data, similarly points to a relationship between language and exact math. In one investigation, a patient with left fronto-temporal damage and severe language limitations was impaired in exact addition but not approximation, whereas another patient with left parietal damage and Gerstmann's syndrome showed slowness in approximating but preserved exact addition abilities (Lemer, Dehaene, Spelke, & Cohen, 2003). The fronto-temporal damage resulted in impairments in both language and exact calculation. An fMRI study of healthy participant also finds similarity between the brain regions associated with exact numerical reasoning and language processing. The bilateral parietal lobes showed greater activation during approximate calculation, and left temporal regions typically associated with language – such as the angular gyrus – were more activated during exact calculation (Dehaene et al., 1999). These neuroscience findings further support the relationship between exact math and language functions.

Cross-cultural research suggests language and exact math are not only linked, but that linguistic ability drives numeric thought and that cardinal number knowledge impacts one's precise numerical reasoning skills. This causal relationship is demonstrated by two studies that investigated the mathematical abilities of Amazonian indigenous tribes that lack a language for exact numbers. Members of the Prahã tribe use words for "one," "two," and "many", yet even the word for "one" is approximate and signifies "roughly one" or "small" (Gordon, 2004). A series of tests evaluated Prahã members' numerical abilities and found them unable to identify exact quantities greater than three. Complementing this research, another study investigated the

Mundurukú, a tribe that uses specific number words consistently for quantities “one” and “two”, occasionally for “three” and “four,” and rarely for larger quantities (Pica, Lemer, Izard, & Dehaene, 2004). Mundurukú and French participants completed an approximate addition and comparison task and an exact subtraction task. On the approximation task, Mundurukú performed similarly to the French controls (Figure 1A), indicating that language differences between the groups did not impact approximate math abilities. On the exact subtraction task French controls succeeded at subtraction with all numerosities and showed a minimal problems size effect, characteristic of advanced exact subtraction ability. In contrast, as the set sizes grew, Mundurukú performance degraded dramatically, showing a drastic problem size effect. (Figure 1B). Pica and colleagues concluded that non-symbolic exact arithmetic depends on a language for exact number. Because the Mundurukú lack the words for large quantities, they cannot perform exact calculations and display severe problem size effects typical of novices. For set sizes larger than three, Mundurukú could only perform approximate calculations, suggesting that counting words or a counting routine is necessary for comprehending large exact quantities. Language, or a lack of language, appears to greatly influence numeric thought.

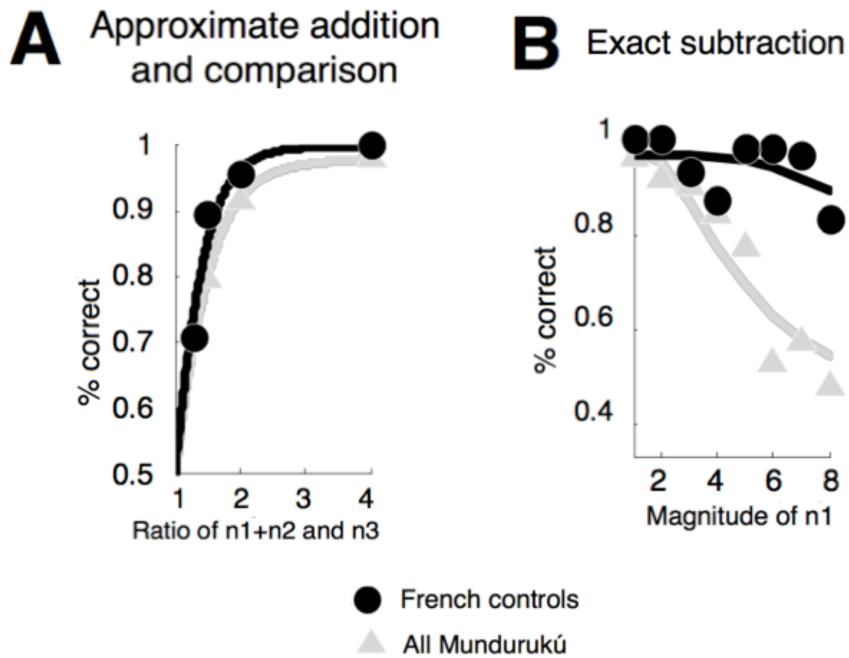


Figure 1. Mundurukú performance on non-symbolic approximate and exact arithmetic. A) Approximate addition and comparison. Indicate which is larger: quantity1+quantity2 or quantity3. French controls and Mundurukú performed similarly on this task. B) Exact subtraction. Point to the result of quantity1 – quantity2. Mundurukú showed a severe problem size effect in comparison to French controls' minimal problem size effect. Adapted from Pica, Lemer, Izard, & Dehaene, 2004.

The idea that language impact thought is known as the Sapir-Whorf hypothesis and has been debated for many years and in many domains, but only recently has become important for mathematical reasoning. According to the Sapir-Whorf hypothesis, language influences thought (Whorf, 1956). The strong version of the Sapir-Whorf hypothesis, called linguistic determinism, argues that language determines thought and that cognition is limited by the language one speaks. Recent research has focused on a weaker form of the hypothesis, called linguistic relativity, which argues that language influences but does not control how we see and behave in the world.

Those in support of the Sapir-Whorf hypothesis view language as a tool for both using mathematical knowledge and for constructing new knowledge. According to Ball and Bass (2003), mathematical language includes definitions, terminology and symbolic notation, and "decisions about what to name, when to name it, and how to specify that which is being named are important components of mathematical sensibility and discrimination central to the construction of mathematical knowledge." In this manner, using terminology such as cardinal number words builds a child's understanding of the underlying numerical concept.

Neuroscience studies are beginning to provide support for the notion that exact arithmetic is heavily dependent on brain regions critical for performing language tasks. Dehaene and colleagues (2003) propose separate brain circuits for processing numbers. According to the triple code model, the bilateral intraparietal sulcus mediates notation-independent quantity manipulation, including approximate judgments; and the left angular gyrus mediates the retrieval of verbal math facts and is important in exact representations of number. According to this model, exact number knowledge is stored as verbal math facts, and exact number knowledge is linguistic in nature. Without language, exact numerical thinking may not be possible.

Taken together, these various lines of research identify a clear link between language and exact numerical reasoning. Counting words and cardinality are especially important for understanding exact quantities. The next section examines how children move from precise numerical representations to numerical reasoning and proposes a new stage in cardinal number knowledge called advance cardinality.

Advanced Cardinality

Counting and cardinality help children precisely represent quantities, but representation and reasoning are not the same thing. Children may need additional skills beyond using words to label set sizes in order to manipulate exact sets. In order to add or subtract quantities, children may also need to draw on their knowledge of how a whole quantity is formed of parts.

Children learn part-whole knowledge at a young age. Three- and four-year-olds see parents cut up apples into smaller slices. Children take apart toys and put them together, learning that a whole is made of smaller parts. These experiences allow children to develop what Resnick (1989) calls protoquantitative schemas, vague part-whole knowledge. Protoquantitative schemas are not yet integrated with a measurement system or exact representation of numbers and thus, on their own protoquantitative schemas are very limited. For example, when young children rely on protoquantitative schema, they are often confused by unrelated perceptual cues. A child may see an apple on a small plate and by comparison think the apple is large. After the apple is cut up and the slices are placed on a large plate, the apple to plate ratio might cause the child to complain that he or she is getting less apple than before. Children's protoquantitative schmeas may be imprecise, but these schemas will serve as foundations for later math development.

Resnick describes how protoquantitative schemas develop into number concepts with the aid of language. Once children learn to count they combine their cardinal number knowledge with their part-whole protoquantitative schemas, resulting in stable, precise part-whole number concepts that are less susceptible to perceptual cues. In this manner counting and language form the measurement system that makes the part-whole protoquantitative schema exact. How the cardinality/protoquantitative schema integration occurs is poorly understood.

Here the term *advanced cardinality* refers to cardinal number knowledge that goes beyond the last-word rule. A child with basic cardinality fails to have any ability to reason about change to that set size (i.e. if it's split in half, or two are taken away). Yet a child with advanced cardinality has clear insight into how the whole cardinal value is comprised of subsets. With advanced cardinality, a child's semantic understanding of five includes the notion that the word five captures the cardinal value of the set of all 5 objects, as well as the knowledge that this set is comprised of subsets such as four objects and one object, and three objects and two objects. As with basic cardinality, the knowledge is centered on a target number word, but with advanced carnality the child has a more complex understanding of that word, which can support inferences about subsets and exact changes in the cardinal value of the set. Developmentally, children assign various levels of meaning to counting words. At first, the counting chant simply conveys ordinality. With experience, children learn basic cardinality, and assign quantity to the number words. Advanced cardinality represents a new level of meaning by including insights about the internal part structure of the whole set.

Verbally labeling the parts of sets may be critical to developing advanced cardinality and the ability to reason about the set. When number words are associated with part-whole knowledge, children can use language for describing sets and their subsets. The integration of language and part-whole insights about sets may form the basis of advanced cardinality.

The notion that children might form verbal associations between quantities and their parts is supported by the literature on math fact recall. Carpenter and Moser (1984) categorized children's acquisition of arithmetic concepts into three levels of problem solving strategies: direct modeling, counting, and recalling. As children develop, they shift from regularly employing counting strategies to retrieving math facts from memory (Ashcraft, 1982; Campbell

& Graham, 1985; Logan & Klapp, 1991). Young children learn that “two” goes with two items, but as they get older children learn other facts, such as “two plus two equals four”. With experience, addition facts and multiplication tables are memorized and internalized such that children can quickly and accurately recall this frequently used number knowledge. Such math facts are verbal in nature and often link a number word with its addends or its multiplicands and multipliers.

Math fact retrieval and advanced cardinality may both rely on verbally labeled math knowledge, however advanced cardinality differs from math fact recall in a very important way. Advanced cardinality builds off children’s protoquantitative schemas and uses language to refine part-whole concepts. The general notion of quantity is expanded to include manipulable parts. Such ideas of number aid children as early as the direct modeling stage of problem solving, well before the math fact recall stage. By the time a child engages in math fact recall, he or she may already have well established advanced cardinality knowledge that is being applied to specific math problems.

Advanced cardinality is a new stage of cardinality that may help children transition from representing quantities to reasoning about them. The Sapir-Whorf hypothesis posits that children need access to number words for numerical reasoning, yet the acquisition of advanced cardinality goes one step further by suggesting that they need to verbally know how parts make up a whole to develop the ability to reason with exact quantities. Language may place constraints on imprecise protoquantitative schemas, driving children to develop advanced cardinality and refine their ideas of how sets are formed.

Summary & Aims

Scientific studies of mathematical reasoning provide important insights into the foundational skills necessary for children to thrive in the classroom. Developmental research identifies cardinality, the verbal labeling of a quantity, as an early skill that serves as the foundation for representing exact sets of objects. Cross-cultural and neuroscience findings support the Whorfian hypothesis that language influences mathematical thought. Finally, educational theories suggest that children build on protoquantitative schemas to learn essential part-whole knowledge required to precisely manipulate sets. While the previous research hints at skills useful for numerical reasoning, their specific influence has yet to be tested. Investigations of cardinality, instruction and advanced cardinality will narrow down the primary skills needed to make inferences about exact quantities.

The series of experiments has 2 main aims:

1. To isolate the essential skills necessary for exact numerical reasoning.
2. To examine the concepts of cardinality and advanced cardinality and their influence on numerical reasoning.

To address these aims, a series of developmental studies examines the numerical abilities as they form. Pre-kindergarten (pre-K) through 1st grade children will participate in a deep investigation of the intersection between language and mathematics. Such an approach will allow us to pinpoint when children learn to reason with exact sets and which linguistic knowledge contributes to that development.

Chapter two investigates the role of cardinality in numerical reasoning. The experiment is modeled after Pica's study of the Mundurukú, but with early elementary school children who, unlike the Mundurukú, all have access to number words. While cardinality may be necessary for precise mathematical manipulation, chapter two addresses whether cardinality is sufficient.

Chapter three builds on chapter two by addressing the role of instruction on numerical reasoning. Over the course of the kindergarten year, children acquire the skills needed to manipulate exact sets. Chapter three teases apart the importance of instruction and maturation for this development. By using a school cutoff design, children of roughly the same age but in different years of schooling will display their numerical reasoning abilities. Instruction and/or maturation may help children move beyond knowing the cardinality of a set to being able to make inferences about the set.

Chapter four examines a specific form of instruction with an advanced cardinality intervention. Pre-K children were trained on part-whole concepts with and without the accompanying number words. The combination of language and part-whole knowledge may extends children's' understanding of basic cardinality, and thus provide a more flexible concept of a number that allows children to reason about exact sets. Advanced cardinality may be an essential step between cardinality and precise numerical reasoning.

Chapter five summarizes the findings from chapters two through four. The significance of this collection of research is addressed.

References:

Ansari, D. (2008). Effects of development and enculturation on number representation in the brain. *Nature Reviews Neuroscience*, *9*(4), 278–291. doi:10.1038/nrn2334

Ashcraft, M. H. (1982). The development of mental arithmetic: A chronometric approach. *Developmental Review*, *2*(3), 213–236. Elsevier.

Ashcraft, M. H. (1992). Cognitive arithmetic: A review of data and theory. *Cognition*, *44*(1-2), 75–106. Elsevier.

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. *A research companion to principles and standards for school mathematics*, 27-44.

Baroody, A. J. (1983). The development of procedural knowledge: An alternative explanation for chronometric trends of mental arithmetic. *Developmental Review*, *3*(2), 225–230. Elsevier.

Bermejo, V. (1996). Cardinality development and counting. *Developmental Psychology*, *32*(2), 263–268.

Briars, D., & Siegler, R. (1984). A featural analysis of preschoolers' counting knowledge. *Developmental Psychology, 20*, 6007-618.

Butterworth, B., & Laurillard, D. (2010). Low numeracy and dyscalculia: identification and intervention. *Zdm*, *42*(6), 527–539. doi:10.1007/s11858-010-0267-4

Bynner, J. and Parsons, S. (2005) Does Numeracy Matter More? National Research and Development Centre for Adult Literacy and Numeracy, Institute of Education.

Campbell, J. I., & Graham, D. J. (1985). Mental multiplication skill: Structure, process, and acquisition. *Canadian Journal of Psychology/Revue canadienne de psychologie*, *39*(2), 338. Canadian Psychological Association.

Carey, S. (2001). Cognitive Foundations of Arithmetic: Evolution and Ontogenisis. *Mind &*

Language, *16*(1), 37–55.

Carey, S. (2004). Bootstrapping & the origin of concepts. *Daedalus*, *133*(1), 59–68.

Carey, S. (2009). Where our number concepts come from. *The Journal of Philosophy*, *106*(4), 220.

Davidoff, J., Davies, I., & Roberson, D. (1999). Colour categories in a stone-age tribe. *Nature*, *398*(6724), 203–204.

Davidson, K., Eng, K., & Barner, D. (2012). Does learning to count involve a semantic induction? *Cognition*, *123*(1), 162–173. doi:10.1016/j.cognition.2011.12.013

Dehaene, S. (2001). Précis of the number sense. *Mind & Language*, *16*(1), 16–36.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. *Cognitive Neuropsychology*, *20*(3-6), 487–506. doi:10.1080/02643290244000239

Dehaene, S., Spelke, E., Pinel, P., Stanescu, R., & Tsivkin, S. (1999). Sources of mathematical thinking: Behavioral and brain-imaging evidence. *Science*, *284*(5416), 970–974.

Feigenson, L., Carey, S., & Spelke, E. (2002). Infants' Discrimination of Number vs. Continuous Extent. *Cognitive Psychology*, *44*(1), 33–66. doi:10.1006/cogp.2001.0760

Gelman, R., & Gallistel, C. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Gelman, R., & Meck (1983). Preschoolers' counting: Principles before skill. *Cognition, 13*, 343-359.

Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*(5695), 496.

Le Corre, M., Van de Walle, G. A., Brannon, E., & Carey, S. (2006). Re-visiting the performance/competence debate in the acquisition of counting as a representation of the positive integers. *Cognitive Psychology, 52*(2), 130–169.

Lemer, C., Dehaene, S., Spelke, E., & Cohen, L. (2003). Approximate quantities and exact number words: dissociable systems. *Neuropsychologia*, *41*(14), 1942–1958. doi:10.1016/S0028-3932(03)00123-4

Logan, G. D., & Klapp, S. T. (1991). Automatizing alphabet arithmetic: I. Is extended practice necessary to produce automaticity? *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *17*(2), 179. American Psychological Association.

Mix, K. S., Sandhofer, C. M., & Baroody, A. J. (2005). Number words and number concepts: The interplay of verbal and nonverbal quantification in early childhood. *Advances in Child Development and Behaviour*, *33*, 305–346.

Pica, P., Lemer, C., Izard, V., & Dehaene, S. (2004). Exact and approximate arithmetic in an Amazonian indigene group. *Science*, *306*(5695), 499.

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologist*, *44*(2), 162.

Starkey, P., & Cooper, R. G., Jr. (1980). Perception of numbers by human infants. *Science*, *210*(4473), 1033–1035.

Whorf, B.L. (1956). *Language, thought, and reality: Selected writings of Benjamin Lee Whorf.* New York: Wiley.

Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology*, *24*(2), 220–251.

Wynn, K. (1990). Children's understanding of counting. *Cognition*, *36*(2), 155–193.

Wynn, K. (1998). Psychological foundations of number: numerical competence in human infants. *Trends in Cognitive Sciences*, *2*(8), 296–303.

CHAPTER 2

Reasoning about quantities: children reveal that words are not enough

Abstract

Cross-cultural studies suggest that reasoning about exact numerical changes in sets of objects relies on language. In particular, cardinal number – the knowledge of number words that refer to exact quantity – has been emphasized as the critical factor underlying exact numerical reasoning. In the current study, we show that cardinal number is not enough. Children in kindergarten though 3rd grade performed exact subtraction on sets of objects. Similar to indigenous tribes that lack a language for each number and despite access to cardinal number, Kindergarteners' accuracy on the non-symbolic number reasoning task was dramatically impacted by increases in set size. Finally, individual differences revealed that exact non-symbolic numerical reasoning is related to symbolic arithmetic and not cardinal number knowledge. Our findings suggest that cardinality is insufficient for reasoning about exact quantities and that the ability to make inferences about exact changes in the numerosity of sets emerges via enculturation practices during the early elementary school years.

Keywords: Mathematics; Cognitive Development; Subtraction; Cardinality; Language

Introduction

What role does culture play in bestowing numerical reasoning abilities, such as the ability to envision the number remaining if three things are taken away from a set of five? Cognitive anthropological studies reveal that such basic numerical cognitive abilities are not human universals, but are rather specific to cultures that have words signifying exact cardinal number values. For example, even healthy adult members of the Amazonian Prahã tribe, whose words for number are restricted to "one," "two," and "many", are surprisingly unable to make precise numerical decisions about sets of objects greater than three (Gordon, 2004), providing support for the Sapir-Whorf hypothesis (Whorf, 1956) that some forms of thought are quite dependent on language. Further insights into the relationship between basic cognitive reasoning about sets of concrete objects and the use of exact number words come from studies of the Mundurukú, whose small vocabulary for numbers are applied without categorical numerical boundaries when referring to sets of objects (see figure 2a) (Pica, Lemer, Izard, & Dehaene, 2004). To test the how the presence or absence of exact cardinal number words in a culture enables numerical cognition about concrete sets of objects, Pica and colleagues employed a non-symbolic number reasoning task that had no explicit requirements for language. After first seeing a small set of objects being placed in a can that occluded their view, then a smaller set of those objects being removed in plain sight, participants chose which of three pictures (0, 1 or 2), corresponded to the remaining contents of the can. Remarkably, Mundurukú performance degraded dramatically as the set size placed in the can exceeded four. French controls, in contrast, showed a minimal set-size effect, leading to a strong cultural-group by set size interaction (see figure 2b). This interaction lends support to the notion that a person's ability to envision exact transformations from one set to another may be enabled by number words that refer to exact values.

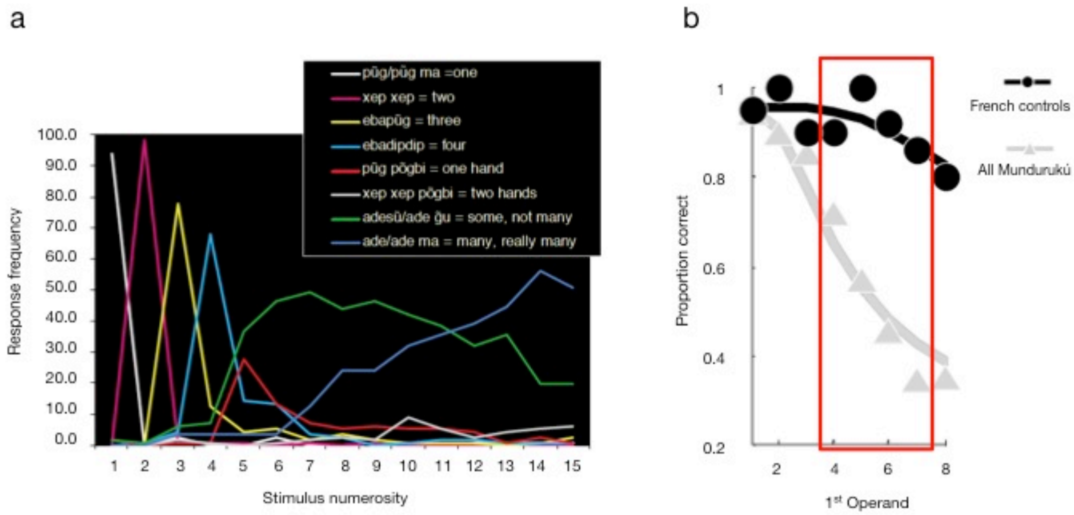


Figure. 2. Mundurukú numerical abilities, as appeared in Pica et al. 2004. On a cardinality task (a), for each numerosity on the x-axis, the graph shows the percent of the time that Mundurukú labeled it with a specific number word. On a non-symbolic number reasoning task (b), for each set size on the x-axis, accuracy is displayed for each group. Operands used in the current study are outlined in red.

Given the potential importance of understanding how enculturation processes enable numerical reasoning, we extend this specific line of research on the relationship between enculturation and cognition into the realm of cognitive development. Within a culture that uses number words to refer to exact cardinal values of sets of objects, we can examine the emergence of exact numerical reasoning for concrete sets of objects. This may provide a more direct opportunity to place additional constraints on inferences about which aspects of enculturation shape exact numerical reasoning, as in the case of reasoning about numerical changes in sets of objects. Studies of child development suggest that mastery of exact number word concepts is not a unitary process of acquiring and using cardinal number words. Rather, these studies point to a developmental [2]distinction between mastery of cardinal number concepts to signify the exact value of a set, and more advanced insights into exact number semantics that support the ability to reason about transformations between sets (Griffin, Case, & Capodilupo. 1995). For example, by pre-school age, U.S. children precisely associate counting words with exact sets of items, and demonstrate mastery of the cardinality principle -- the knowledge that the last word in a count series refers to the numerosity of the whole set (Wynn, 1992). However, even after mastering the count sequence and the cardinality principle, many preschool children often fail to grasp of the successor principle – that successive numbers in the count list differ by exactly 1 (Davidson, Eng, & Barner, 2012). Apparently, the ability to reason about the changes in the exact number of a set of concrete objects may not be bestowed merely by mastering the use of number words to refer to cardinal values of sets in an exact, categorical fashion, but rather may require additional enculturation practices such as those that occur during formal educational activities. In other words, numerical learning activities that characterize the first years of formal schooling may be crucial to the transformation in number reasoning ability that distinguish one culture from another, as in the contrast between the Mundurukú and the French described above.

These developmental considerations lead to a potentially provocative prediction for typically developing children in the U.S. If numerical reasoning about sets of concrete objects (e.g. the ability to envision the objects that remain after three are removed from a set of five) is not directly linked to insights about natural numbers, counting, and cardinality, but rather depend on additional enculturation processes unique to early schooling, then U.S. kindergartener's numerical reasoning about concrete sets should resemble those of the Mundurukú. This should

[2] with the possible exception of being able to apply a one-to-one correspondence strategy when two sets of objects are visible throughout, suggesting an understanding of the more general notion of exact value

hold true even for typically developing children who have mastered the precise use of number words to refer to the exact cardinal value of a set of objects, but have not yet been fully enculturated in formal mathematical training of early elementary school. In contrast, reasoning about exact transformations of concrete sets of objects should emerge over the early years of schooling, in tight correlation with learning about symbolic transformations of abstract arithmetic.

An examination of individual differences will reveal the aspects of enculturation most important for reasoning about exact sets and allow us to contrast children's fluency with cardinality versus other enculturation activities such as symbolic arithmetic. Are advances in reasoning about non-symbolic changes in number related to mastery of applying number words to sets an exact fashion as would be predicted by the Sapir-Whorfian hypothesis and the non-symbolic nature of each task? Or, in contrast, are advances in reasoning about non-symbolic changes in number more associated with the numerical relationships taught via symbolic arithmetic? This direct comparison seeks to clarify the importance of language and cardinality in reasoning about non-symbolic exact sets and investigate whether an alternative enculturated skill, symbolic numerical manipulation, can better account for reasoning about changes in exact sets

The current study examines a cross-sectional sample of typically developing U.S. children from the beginning of kindergarten through the beginning of 3rd grade. Adapting the general methods of Pica and colleagues (2004) we examine development of children's ability to represent exact number changes to non-symbolic concrete sets. In stark contrast to the Mundurukú, even the youngest children in this sample reliably provide the exact cardinal value for a target set 1 through 8 in a precise and categorical manner (see figure 2a), providing an opportunity to examine how such knowledge impacts numerical cognition at the onset of Kindergarten instruction, and how such abilities change over the course of the first several years of schooling.

This cross-sectional study seeks to test the prediction that comparing children who have all mastered the use of number words to refer to exact categorical values of sets of objects, yet have different degrees of experience with the enculturation practices of elementary school, will recapitulate the group by set size interaction previously reported between contrasting cultures with and without exact number words. Such a finding may underscore an important conceptual distinction in the development of numerical cognition between mastery of exact number words and the cardinality principle, and the emergence of the ability to reason about numerical changes in sets of concrete objects. Such findings may lend support to an alternative to the Sapir-Whorfian hypothesis, suggesting a central role for educational processes, rather than vocabulary, in enabling the enculturation of numerical reasoning.

Methods

Participants

A sample of 329 children were recruited from private schools within Davidson County, TN, and participated in two one-hour sessions, as part of a larger study of children's developing number, reading and attention skills. The inclusion criteria were a) English speaking, b) normal or corrected-to-normal vision, c) full completion and compliance with the instructions of the Non-symbolic Number Reasoning (NNR) task and d) achievement of at least 75 % accuracy on an assessment of cardinality skills for number words 1 through 8 (Starkey & McCandliss, 2014), to ensure that children had indeed mastered the use number words in English to refer to cardinal values of objects in a precise and categorical fashion. All data was collected early in the school year (mean date = October 9th, std = 18 days) in each of the grades kindergarten through 3rd grade. Of the 329 children who initially participated in the study, 29 failed to reach inclusion criteria. Five participants did not reach 75% accuracy in the cardinality task. Twenty-four did not complete the NNR task due to time constraints (n=5), a request by the child to end early (n=11) or some other reason (n=3). In addition, five children provided response patterns clearly indicating a misunderstanding of the task instructions. The sample of 300 participants that passed the inclusion criteria are described in table 1.

Grade	**Beginning of the school year**		**IQ**	**SES**			
	Mean Age (SD)	N	Score (SD)	Average of Daycare +PreSchool (SD)	N	Average of Parent Educ (SD)	N
K	5.93 (.40)	71 (33M, 38F)	106.86 (16.14)	19.20 (10.38)	21	17.15 (1.45)	21
1	6.94 (.37)	73 (36M, 37F)	103.37 (19.52)	22.21 (10.24)	18	16.92 (1.27)	18
2	7.90 (.35)	71 (36M, 35F)	111.13 (14.91)	18.13 (9.56)	15	17.25 (1.63)	14
3	8.92 (.42)	85 (42M, 43F)	109.19 (14.84)	20.32 (11.74)	17	16.91 (1.73)	17

Table 1. Description of the participants including IQ and socioeconomic status.

Procedures

Children participated in the computerized NNR task and cardinality tasks on one day and completed the Woodcock-Johnson III Math Fluency subtest during a testing session on a second day, no longer than 2 week before or after the computerized testing session. To ensure that children mastered the words for exact sets and the cardinality principle, each child completed an exact enumeration task in which they viewed dots representing pieces of fish food in displays ranging from 1-8 dots, randomly displayed 8 times per set size (see Starkey & McCandliss, 2014). In this task, children were instructed to produce only the final exact value of the set, ensuring that the answer given was their assessment of the set's cardinal value.

Each trial of the NNR computerized task presented children with a short video clip of a woman who first peers into a bucket containing 4 to 8 discs, then removes a subset by hand, and shows the viewer the subset she removed. The child's task is to press a button underneath one of three pictures representing the possible number of disks remaining: one, two, or three. As in Pica et al., (2004) the task required no use of number words, and involved watching simple concrete transformations of objects, as might happen in the real world. Children were tested individually on a Dell Latitude E6500 computer with a 15.4" display running E-prime 2.0 (Psychology Software Tools, Pittsburgh, PA, USA). Two practice trials were provided with the option of repeating the trials once if needed, followed by 30 test trials. Timing of the progression of each trial was controlled by the child at three points. The child initiated the start of each trial with a button press, which started a wide-view video showing a woman looking into a bucket (3200ms), followed by an overhead image of the 4 to 8 discs within the bucket. This image remained on screen until the child advanced the sequence with a second button press (or a 7000 msec time-out was reached). Next, a dynamic wide-view video showed the woman reaching into the bucket (1200ms), followed by a close-up of her palm holding 1 to 7 discs, which remained onscreen until the child pressed one of three buttons indicating the number of discs remaining. (see figure 3). Finally, an image of the correct number of items remaining in the bucket appeared (1500 ms) along with a sound indicating whether the response was correct or incorrect.

Figure 3. NNR task sequence of events

Stimuli

Stimuli were created to produce four levels of the factor “Set Size” defined by the number of discs of the first operand (first image of the bottom of the bucket). Set sizes included 4, 5, 6, and 7. 6 trial sequences were created for each set size, consisting of two trials each of the set size image followed by a second operand (n-1, n-2, n-3) as displayed by the image of discs on a hand. In addition, a set of six “filler trials” were created for first operand set size 8 paired with second operand 7, 6, 5, which served as a boundary condition, to avoid the influence of strategic biases that may be applied to the largest value in a repeated trials design.

Images of black disks were generated using MATLAB 7 (MathWorks, Natick, MA, USA) and were superimposed using Adobe Fireworks onto images of a bucket (4 to 8 disks) or the open palm of a hand (1-7 disks). The disks were visually manipulated to remain the same size throughout all trials, yet the spatial arrangements of the disks in the bucket's bottom surface was manipulated to ensure that no one low-level perceptual feature would correspond transparently to the numerosity of the first set. On half the trials, the size of the smallest invisible rectangular enclosure that could bound the set of disks was designed to increase (by 67%) for each additional item beyond 4, while overall density of each set decreased by between 25% and 31% per additional item over 4. In the other half of trials, enclosure size decreased by 40% for each additional item over 4, and density increased by between 90% and 104%. The median enclosure size was used for the second operand for all trials.

Woodcock-Johnson III Math Fluency subtest

Woodcock Johnson III is a standardized achievement test appropriate for subjects ages 5 and older, and the Math Fluency subtest measures how many symbolic arithmetic problems a child can complete in 3 minutes. The 160-item test begins with 60 addition and subtraction problems followed by a mix of addition, subtraction and multiplication problems. Raw scores were attained and consist of the number of correct answers given before the participant missed 6 consecutive problems.

Results

Accuracy scores and median reaction times were computed for each of four levels of the Set Size factor: initial set size (i.e. number of disks in the bucket) of 4, 5, 6, and 7.

Of the 300 participants passing the inclusion criteria, 25 were identified as outliers for having accuracy (n=13) or response times (n=12) more than 2 standard deviations away from their grade's mean. To ensure that the extreme cases did not drive effects, analyses are reported for the sample of 275 subjects after outliers were removed, however each of the significant findings reported below remained significant in the analysis of the larger sample.

Participants' accuracy data were submitted to a separate mixed model, repeated measures ANOVA designed to examine the factors of Grade (K,1,2,3), Set Size (first operand value 4,5,6,7) and Envelope Congruence (Congruent, Incongruent). According to Mauchly's test, the assumption of sphericity was violated for set-size effects, chi-square = 55.02, $p < .001$, so degrees of freedom were corrected with Greenhouse-Geisser estimates, epsilon = 0.89. Significant main effects emerged for Grade, $F(3,296) = 73.39$, $p<.001$, and Set Size, $F(2.67, 791.20) = 42.00$, $p<.001$. In contrast, no main effect nor any interactions emerged for the Envelope Congruence factor, suggesting that the spatial extent of the first set had no influence on children's decisions about exact non-symbolic subtraction. Crucially, Grade and Set Size produced a significant interaction, $F(8.02, 791.20) = 4.23$, $p<.001$ (see figure 4). Post-hoc planned comparisons revealed significant differences between all grades, $p < .03$) except between 2nd and 3rd grades.

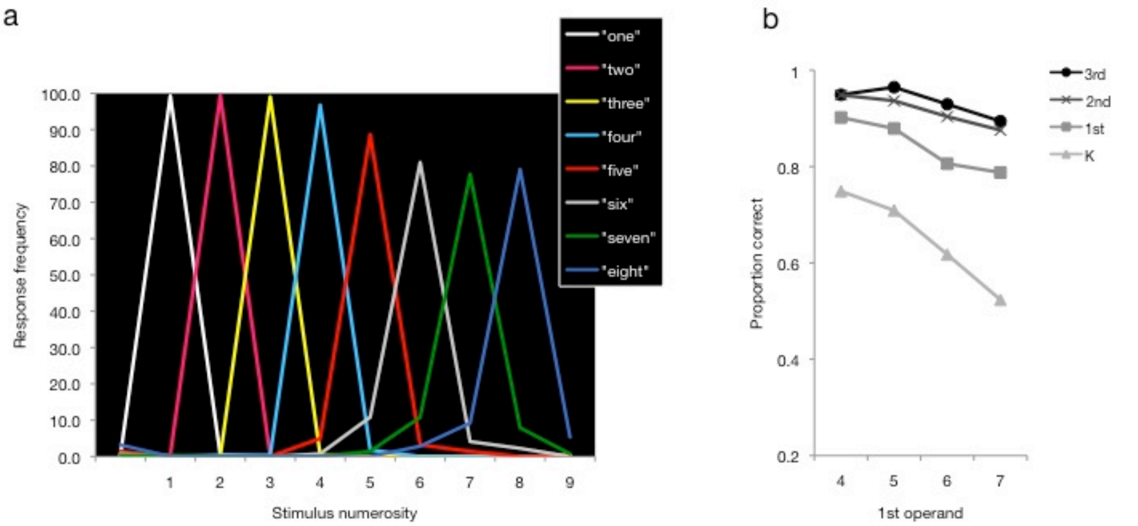


Figure 4. School children's numerical abilities. On the EE task (a), for each numerosity on the x-axis, the graph shows the percent of the time that Kindergarteners labeled it with a specific number word. On the ENS task (b), for each set size on the x-axis, accuracy is displayed for each grade.

Second and third graders' accuracies approached ceiling, potentially driving the group by set size interaction. Thus, response times, which have no measurement-imposed floor or ceiling, were submitted to a mixed model, repeated measures ANOVA with Grade and Set Size as factors. According to Mauchly's test, the assumption of sphericity was violated for set size effects, chi-square = 153.80, $p < .001$, so degrees of freedom were corrected with Greenhouse-Geisser estimates, epsilon = 0.78. Significant main effects emerged for Grade, $F(3, 288) = 35.38$, $p < .001$ and Set Size, $F(2.33, 670.03) = 301.19$, $p < .001$. Grade and Set Size produced a significant interaction $F(6.98, 670.03) = 3.78$, $p < .001$, indicating that the grade by set-size interaction is not simply due to a ceiling effect in the accuracy data.

Individual differences in NNR accuracy were analyzed to explore how this emerging skill is linked to two other processes that are concurrently developing: fluency in providing the exact cardinal value of non-symbolic sets, and fluency in symbolic arithmetic. When enumerating, larger sets result in slower response times, causing a linear relationship with a large positive slope. As children develop, they become faster at enumerating sets of objects and the set size x response time slope decreases (Svenson, & Sjöberg, 1983). Therefore, cardinality fluency was characterized by response time slopes for random sets of 5-7 dots (see Starkey & McCandliss, 2014 for details). Smaller sets in the subitizing range were excluded for not requiring a counting routine. One outlier was removed for having a slope more than 3 standard deviations from the mean. Individual differences in symbolic arithmetic were characterized by Woodcock Johnson III Math Fluency raw scores and individual differences in ENS ability were characterized by the accuracy slopes for set sizes 4-7. Nine children who did not score above chance on the easiest set size of 4 were excluded as outliers. Set size slopes in the cardinality task and Math Fluency scores both show significant grade effects, $F(3,228) = 5.77$, $p < .01$, and $F(3,228) = 231.25$, $p < .001$, respectively, in a manner similar to NNR accuracy slopes, motivating an investigation into how they may contribute to the development of NNR abilities. A linear regression model was used to predict kindergarteners' NNR accuracy slopes from cardinality fluency and no correlation emerged. A linear regression model was then used to predict kindergarteners' NNR accuracy slopes from symbolic math fluency. Math Fluency significantly predicted NNR slopes ($r^2 = 0.10$, $p < .05$), suggesting that symbolic arithmetic, not cardinality fluency, is linked to emerging NNR ability. When both cardinality fluency and Math Fluency were entered into a linear step-wise regression model to predict kindergarteners' NNR accuracy slopes, Math Fluency entered the model first and significantly predicted NNR slopes ($r^2 = 0.10$, $p < .05$) and cardinality fluency did not significantly account for variance in NNR.

Discussion

Reasoning about changes in sets of objects develops in early elementary school children. This cross-sectional investigation revealed that kindergarteners' performance was most impacted by increases in set sizes. All grades reached 70% accuracy when the first operand was 4, but for first operands of 5 or greater, kindergarteners' accuracy scores drastically deteriorated. Importantly, all of the kindergarteners could verbally identify even the largest sets included in the task, indicating that increases in set size during the NNR task impacted kindergarteners' numerical reasoning abilities, not simply their ability to determine a set's numerosity.

The kindergarteners performance declined as set size increased, in a manner very similar to that of the Mundurukú reported by Pica et al (2004). Though Mundurukú's limitations in this task have been attributed to a lack of language for exact number, such an account is at odds with the current findings. All children in the current study could successfully assign counting words to exact sets in a precise and categorical fashion, yet showed severe set size effects, indicating that cardinality is not sufficient for reasoning about changes in exact quantities. Therefore, the gap in Mundurukú and French numerical reasoning abilities likely emerged from cultural differences beyond access to cardinality and counting routines.

Cardinal number knowledge does not allow for precise numerical reasoning, contradicting the language hypothesis. Previous critiques of the language hypothesis note that some of the Mundurukú knew the Portuguese counting words yet failed to use them to complete the exact subtraction task, discounting the theory that verbal labels allow for exact numerical reasoning (Gelman & Butterworth, 2005). The current study builds on this critique by showing that even those with fluent enumeration skills still struggle to reason about exact quantities.

Fluency of applying an exact number word to a non-symbolic set develops across grades, yet individual differences in this ability bear no relation to differences in non-symbolic numerical reasoning. Children that displayed high fluency in providing the exact cardinal value of non-symbolic sets were no better on the non-symbolic numerical reasoning task than children that displayed low fluency in the cardinality task, further indicating that language is not the key to exact numerical reasoning. Instead, symbolic arithmetic ability predicted exact, non-symbolic numerical reasoning set size slope. The emergence of precise numerical reasoning, while likely driven by enculturation practices, seems to be more related to enculturation practices linked to symbolic arithmetic than those linked cardinal number knowledge and enumeration skills.

Access to counting words is not sufficient to make inferences about changes in exact sets of objects, however, it may remain a necessary step in the acquisition of non-symbolic numerical reasoning skills. For example, Brazilian homesigners who live in a culture that values exact number but lack signs for numbers cannot reason about exact quantities, suggesting that enculturation is insufficient and that language remains important for making such numerical inferences (Spaepen, Coppola, Spelke, Carey, & Goldin-Meadow, 2011). In the U.S., children acquire cardinality, at around age 4, meaning that they know that the last word in a count list correspond to the numerosity of a set. Counting words provide the first symbolic representations of numerosities and may help categorize approximate notions of number into exact amounts. The current study finds that exact number knowledge, represented by children's knowledge of cardinality, is not enough to manipulate exact sets, but that a second enculturation process may take place during the first formal year of schooling. The importance of formal instruction is further highlighted by the relationship between exact numerical reasoning and symbolic arithmetic ability, a common topic taught in early elementary school. Whether one ability drives the development of the other or whether both develop via a third skill learned in kindergarten still needs to be determined and futures studies should further examine the development of this relationship.

We theorize that non-symbolic numerical reasoning is acquired through instruction and that the poor performance among American kindergartners and Mundurkú is due to their limited experience with numbers as opposed to their age. Adult Munduruků were as restricted in their ability to make inferences about exact sets as children, eliminating maturation as the explanation for the acquisition of this skill. Future studies should directly measure schooling effects with a school cutoff design, which would better isolate the importance of schooling from maturation (Morrison, Smith, & Dowehrensberger, 1995).

This study demonstrates that basic cardinality is inadequate for precisely reasoning about numbers. Instead, children develop exact numerical reasoning abilities during the first formal years of schooling, well after mastering cardinality. The relationship between symbolic math fluency and non-symbolic numerical reasoning further points to the importance of formal math instruction, providing an alternative to the language hypothesis. Closer investigation of this essential time period will reveal the most effective cultural practices that lead to non-symbolic numerical reasoning abilities, which will, in turn, influence early elementary school instruction.

<u>References:</u>

Davidson, K., Eng, K., & Barner, D. (2012). Does learning to count involve a semantic induction? *Cognition*, *123*(1), 162–173. doi:10.1016/j.cognition.2011.12.013

Gelman, R., & Butterworth, B. (2005). Number and language: how are they related? *Trends in Cognitive Sciences*, *9*(1), 6–10. doi:10.1016/j.tics.2004.11.004 Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*(5695), 496.

Griffin, S., Case, R., & Capodilupo, A. (1995). Teaching for understanding: The importance of the central conceptual structures in the elementary mathematics curriculum. *Teaching for transfer: Fostering generalization in learning*, 121-151.

Morrison, F. J., Smith, L., & Dow-Ehrensberger, M. (1995). Education and cognitive development: A natural experiment. *Developmental Psychology*, *31*(5), 789–799.

Pica, P., Lemer, C., Izard, V., & Dehaene, S. (2004). Exact and approximate arithmetic in an Amazonian indigene group. *Science*, *306*(5695), 499.

Spaepen E., Coppola M., Spelke E. S., Carey S. E., Goldin-Meadow S. (2011). Number without a language model. Proc. Natl. Acad. Sci. U.S.A. 108, 3163–3168. doi: 10.1073/pnas.1015975108

Starkey, G. S., & McCandliss, B. D. (2014). Journal of Experimental Child Psychology. *Journal of Experimental Child Psychology*, *126*(C), 120–137. doi:10.1016/j.jecp.2014.03.006

Svenson, O., & Sjöberg, K. (1983). Speeds of subitizing and counting processes in different age groups. *The Journal of Genetic Psychology: Research and Theory on Human Development*.

Whorf, B.L. (1956). *Language, thought, and reality: Selected writings of Benjamin Lee Whorf.* New York: Wiley.

Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology*, *24*(2), 220–251.

CHAPTER 3

The impact of one year of schooling on exact numerical reasoning

Abstract

In early elementary school, children learn to reason about exact sets of objects. Education, maturation, and fluent use of mathematical language are all suggested to be the key to this new and important skill. The current study teases the factors apart by directly contrasting the effects of maturation versus schooling within the same language community. Using a school cutoff design, old kindergarteners are compared to young first graders. Result show a large discrepancy in numerical reasoning ability at the beginning of the school year suggesting that the first graders benefited from the previous year's worth of formal instruction. A follow up longitudinal study examined how the kindergarten group changed over the school year. At the end of the year, grade no longer impacted numerical reasoning skills suggesting that the kindergarten experience is responsible to the development of numerical reasoning skills.

Keywords: Mathematics; Cognitive Development; School Cutoff, Reasoning

Introduction

Exact numerical reasoning is essential for children to succeed in the classroom and thrive as adults, however the cognitive and developmental basis for such skills remains poorly understood. Language (Pica, P., Lemer, C., Izard, V., & Dehaene, 2004; Gordon, 2004; Spaepen, Coppola, Spelke, Carey & Goldein-Meadow, 2011), counting (Carey, 2001), general enculturation practices (Gelman & Butterworth, 2004; Butterworth, Reeve, Reynolds & Lloyd 2008) and maturation (Piaget, 1960) are all proposed to be critical for reasoning about exact numbers. Teasing these factors apart is the challenge. Studies comparing speakers of different languages are often confounded by cultural differences, and schooling effects are often confounded by maturation. Isolating the influence of enculturation practices, such as schooling, from language, counting, and age will help to explain the development of exact mental arithmetic.

Cross-cultural studies have suggested that language is essential for understanding and manipulating exact quantities. The Amazonian Mundurukú tribe has no words for numbers beyond “a,” “few,” “some,” and “many” (Pica, P., Lemer, C., Izard, V., & Dehaene, 2004). In an exact subtraction task, adults and children viewed videos that contained no language or symbols. They merely depicted a small set of dots lingering above a can for a while, before entering the can so they could no longer be seen, followed by a smaller set of dots leaving the can and lingering for a while. To assess when the viewer knew the exact number of dots left in the can, participants were asked to select one of three pictures, which always showed zero, one or two dots remaining in a can. As the set size of the dots entering and leaving the can grew beyond the smallest numbers, Mundurukú child and adult performance degraded dramatically, whereas the French controls showed a minimal set-size effect. Pica and colleagues concluded that non-symbolic exact numerical reasoning depends on a language for exact quantities. However, many things differ between the Mundurukú and Western subjects, including the importance of exact quantity in their culture, their social interactions such as barter and exchange versus selling and buying, and even presumably irrelevant things like their diet. Therefore, although the conclusion that culture plays a strong role in number concept development gains important support, our understanding of which cultural mechanisms of transmission are necessary, and which are sufficient, remains an open, but crucial question. Does just the act of mastering number names for cardinal values enable one to reason about number, or are other cultural practices enabling such numerical cognition?

We can observe the enculturation process more directly by examining the development of non-symbolic numerical reasoning abilities in Western children, who all live in similar cultures. In Western societies, much of the enculturation process occurs in school. Therefore, a recent study employed a technique similar to those used by Pica and colleagues (2004) to examine the emergence of numerical reasoning skills early in elementary school - the first years of the enculturation process (Moneta-Koehler & McCandliss, in preparation). By the start of kindergarten, all the children in the study had mastered the counting routine and had acquired cardinality for exact number, displaying that they were not verbally limited like the Mundurukú. On a non-symbolic exact subtraction task, kindergarteners were impacted by set size in a manner similar to Mundurukú, whereas second and third graders showed minimal set-size effects. Both young and old children presented similar language abilities, suggesting that language was not sufficient to perform exact numerical reasoning, but that instruction and/or maturation are necessary for precise mental calculation.

Early elementary school children range from ages five to seven, a stage associated with improved cognitive control (Rueda, Rothbart, McCandliss, Saccomanno & Posner, 2005). Children at this age also develop basic reasoning skills including concepts of conservation and class inclusion (Piaget, 1960). This age-related cognitive development coincides with the start of formal instruction, confounding the individual roles of age and schooling. Thus, the vast differences between the kindergarten and 1st grade children's numerical reasoning skills could be mainly associated with a maturation driven change in cognition or kindergarten instruction.

The current study investigates the foundations of precise numerical reasoning by disentangling the effects of age and schooling with a school cutoff design. In a school cutoff design, participants are selected based on how close their birthdays are to the cutoff date for school entry. For example, a child with a birthday that fell a week before the cutoff date would be placed into a grade higher than a child with a birthday that fell a week after the cutoff date. Such children would differ by two weeks in age, but experientially, they would differ by an entire school year with regard to the educational experiences that are specific to these grade assignments. Thus a school cutoff design allows researchers to investigate schooling effects while minimizing age effects (Morrison, Smith, & Dowehrensberger, 1995).

The current question focuses on how exact number cognition is supported by general language differences between cultures, versus other cultural practices involving number. Thus, it would be highly informative to compare children of very similar ages who were segregated by age-cutoff into a group that was just about to embark on kindergarten versus a group that just completed kindergarten. By testing near the school cutoff age at the beginning of the school year, we can contrast "young 1st graders" (i.e. those who just recently completed kindergarten) and "old Kindergarteners" (i.e. those just about to embark on kindergarten instruction).

These children are part of a culture that uses exact number labels pervasively and teaches such skills at a very young age. If exact numerical reasoning relies more heavily on specific kindergarten instruction than on maturation, exact subtraction performance will be significantly better in the young 1st grader group than in the group of old kindergartners. Experiment 2 follows up with these children at the end of the school year to further assess the impact of a kindergarten education. If numerical reasoning skills are more dependent on instruction than maturation, than kindergartners' performance at the end of the school year will resemble that of 1st graders at the beginning of the school year.

The current investigation is part of a larger study and the numerical reasoning paradigm from chapter two was adapted to capture more nuanced reasoning abilities that are of interest for the larger study. The previous design focused on subtraction problems with solutions ranging from one to three. The new design consists of problems with solutions ranging from zero to six, such that performance can be evaluated based on accuracy and precision - incorrect answers can be evaluated for how close they are to correct solutions. The new paradigm is important for comparing children's performance on this exact numerical reasoning task to a similar approximate numerical reasoning task. However, for the purposes of this cutoff study, the experiments focuses on aggregate task accuracy, to maintain consistency with previous cutoff analyses (Bisanz, Morrison & Dunn, 1995; Morrison, et al., 1995).

Experiment 1

Experiment 1 examines the numerical reasoning abilities of old kindergarteners and young first graders at the beginning of the school year. At the beginning of the school year, kindergarteners will have experienced minimal formal math instruction, whereas first graders will have an entire year of kindergarten education to draw from. Thus, these similarly aged children will reveal the importance of that kindergarten year for exact numerical reasoning.

Methods

Participants

A sample of 58 elementary school children were selected from 12 Nashville area public schools: 35 kindergarteners (ranging from five years and nine months to six years old at the cutoff date for school entry) and 24 first graders (ranging from six years to six years and three month old at the cutoff date). In contrast to chapter two, the current study was run in public school programs. The school district used Pearson's enVisionMATH curriculum that focused on problem-based learning and relied heavily on the use of visual for conceptual learning. Lessons related to numerical reasoning included comparing sets, decomposing sets up to 10 into addend pairs, acting out situations related to addition and subtraction, and composing and decomposing number into 10s and ones. Data were collected in the first three months of the school year. Letters were sent home to the parents of children in an afterschool program. The two groups (kindergarten and 1st grade) have very similar scores on level of parental education (see table 2).

Grade	Mean Age (SD)	N	Average of Parent Educ (SD)	N
K	5.80 (.10)	*35 (16F, 19M)*	*16.25 (1.80)*	*24*
1	6.24 (.09)	*24 (13F, 11M)*	*16.81 (1.70)*	*16*

Table 2. Description of the participants and socioeconomic status regarding the years of parents' education (collected on a subset of the participants).

Procedure

Children completed a computerized non-symbolic numerical reasoning task. In this task, children viewed short videos of a woman removing discs from a bucket and determined how many discs remained. The task avoided the use of Arabic numerals and number words, and instead tested how children manipulate visual objects presented in a real-world scenario. Children were given two practice trails and if they failed on both, the experimenter repeated the practice trials. They then completed 28 self-initiated subtraction problems where they saw a women looking into a bucket (3200ms), discs in a bucket (viewable for 7000ms with the option to advance sooner), the woman removing discs from a bucket (1200ms) and discs on a hand (viewable for an unlimited amount of time). Children responded by pressing a key with an image of zero to six discs in a bucket and then saw an image of the correct number of items remaining in the bucket (1500) with a sound indicating their success on the trial. Trials consisted of first operands ranging from four to nine and second operands ranging from zero to seven. The difference between operands ranged from zero through six and each occurred four times, resulting in each response option being equiprobable throughout the study (see table 3).

Operand 1	Operand 2	Difference
4	4	0
5	5	0
5	4	1
5	3	2
5	2	3
5	1	4
5	0	5
6	6	0
6	5	1
6	4	2
6	3	3
6	2	4
6	1	5
6	0	6
7	7	0
7	6	1
7	5	2
7	4	3
7	3	4
7	2	5
7	1	6
8	7	1
8	6	2
8	5	3
8	4	4
8	3	5
8	2	6
9	3	6

Table 3. 28 subtraction problems

Stimuli

Images were created with MATLAB 7 (MathWorks, Natick, MA, USA). To ensure that participants relied on numerosity instead of low-level perceptual features, spatial arrangement of the discs varied. For half the trials the area occupied by the first operand increased by 67% for each additional item beyond four, causing the range of densities to decrease by between 25% and 31% relative to the other half of trials. In these trails the areas of the dot display was congruent with set size (area congruent trials). For the other half of the trials area for the first operand decreased by 40% for each additional item beyond 4, causing density of each set decreased by between 90% and 104% (area incongruent trials). The median enclosure size was consistently used for the second operand. Disk size remained constant throughout all phases of the experiment.

2.2 Results

Accuracy scores were computed for each participant. Old kindergarteners performed significantly worse than young first graders on the exact, non-symbolic subtraction task, $t(57) = 5.65$, $p < .001$. (see table 4). A linear regression model was used to predict children's numerical reasoning accuracy scores from children's age, entered as the number of days between the child's 6^{th} birthday and the school cutoff date. Age significantly predicted numerical reasoning accuracy ($r^2 = 0.27$, $p < .001$). Grade was then entered into the model to assess schooling effects above and beyond age effects. In a step-wise linear regression model with age and grade as predictors, grade entered the model first and significantly predicted numerical reasoning accuracy ($r^2 = 0.36$, $p < .001$). When age and grade were both entered into the model the age effect became non-significant ($r^2 = 0.00$, $p = .93$) and only grade uniquely predicted numerical reasoning ability ($r^2 = 0.12$, $p < .001$).

With a school cutoff design, the key test to estimate the effect of grade is a regression discontinuity analysis. A regression discontinuity analysis compared the two regression lines for age and numerical reasoning accuracy scores for each group and tested whether the two regression lines significantly differed from one another (see figure 5). The univariate general linear model revealed a significant difference in the mean centered intercepts for each grade, $F(1,56) = 7.51$, $p < .01$. Across the whole model, there was no unique effect of age, nor was there a difference in the slope of the regression lines for each grade. The estimated grade effect is 0.35, $t(1) = 2.75$, $p < .01$, suggesting that the difference in grade accounts for a 35% increase in accuracy across the two grade groups. These findings suggest that whatever experiences are differentiated by assigning children to kindergarten for a year versus assigning them to other experiences for a year significantly improves a child's ability to perform exact, non-symbolic calculation within a task that has no explicit demands or stimuli associated with linguistic or symbolic number.

Group	Mean Percent Correct (SD)
Old K	0.39 (.28)
Young 1st	0.74 (.15)

Table 4. Descriptive statistics of children's overall accuracy

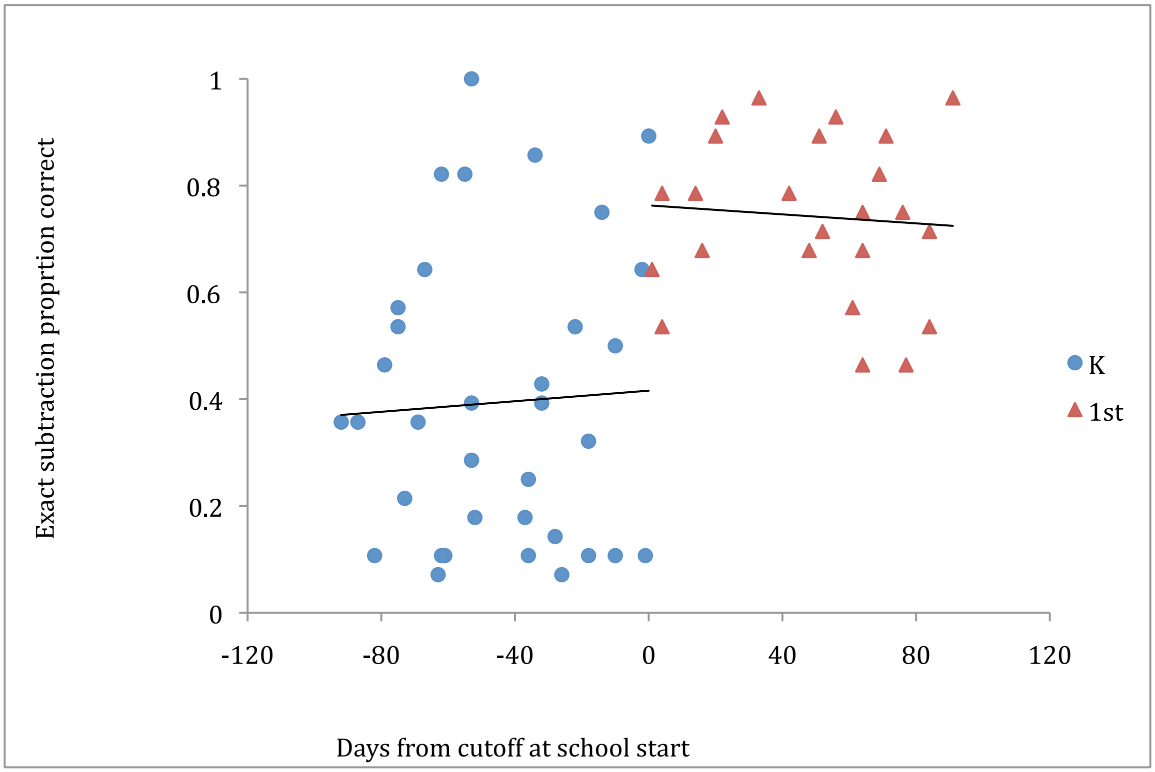


Figure 5. Time point 1 subtraction accuracy plotted by age. The discontinuity between regression lines for each group reveals an effect of grade at the beginning of the school year.

Although the school cutoff study was designed to contrast development of aggregate task accuracy within an exact subtraction task, it is important to note that the design of the current task differs from the task previously reported in Chapter 2. To investigate whether the current study shared a key finding with the task version reported in chapter two – that task performance was dependent on set size – a follow-up analysis was conducted. Accuracy scores were computed for each of four levels of the Set Size factor: initial set size (i.e. number of disks in the bucket) of 5, 6, 7, and 8 (note set sizes 4 and 9 were filler trials to avoid boundary effects, and set sizes generally co-vary across initial and second sets, see table 3). Accuracy data were submitted to a separate mixed model, repeated measures ANOVA designed to examine the factors of Grade (old K, young 1st), Set Size (first operand value 5,6,7,8) and Area Congruence (Congruent, Incongruent). As in Chapter 2, significant main effect emerged for Set Size, $F(3, 171) = 14.35$, $p<.001$, suggesting that both tasks demonstrate that performance is dependent on set size. In addition, as in the previous analysis Grade demonstrated a significant main effect, $F(1,57) = 31.79$, p .001, and Area Congruence produced no effect. No significant interactions between these factors researched significance (see figure 6).

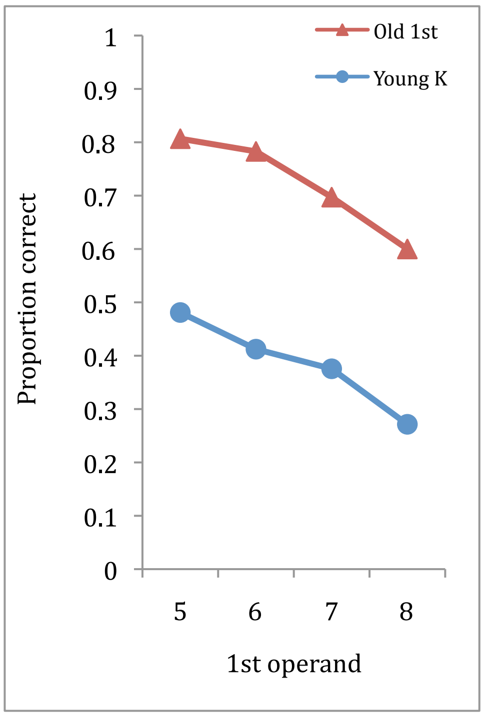


Figure 6. For each set size on the x-axis, accuracy is displayed for each grade.

Discussion

Overall, these results clearly indicate that children's numerical reasoning abilities are highly influenced by schooling. Age and grade each correlate with numerical reasoning skills and are highly correlated with each other. However, grade has a stronger relationship with numerical reasoning, and grade made unique contribution beyond that of age. The lack of an age effect is unsurprising given that by design children were selected from a narrow age range. This study design provided a highly informative contrast for a regression discontinuity analysis that further found an important effect of age. Regression lines comparing age to numerical reasoning skills showed drastically higher intercepts for the young 1st grade group than for the old kindergarten group, much higher than would be predicted by extending the regression line from the old kindergarten group all the way through the young 1st grade group (and vice versa). If children's numerical reasoning development was driven by increased overall exposure to the general culture or maturational factors, rather than factors unique to the kindergarten experience, then we should expect a general age effect and no difference in intercepts between the two groups. Instead, drastic difference between the intercepts suggests that the first graders' numerical reasoning abilities are driven by their school experience. At the beginning of the school year, the young first graders have the benefit of a kindergarten education, something that the old kindergarteners have barely experienced. Thus, for this sample of children, the kindergarten experience likely drives the improved performance of the first grade group.

The set size effect indentified in chapter two emerged in this experiment, confirming that performance accuracy in this task is indeed set-size dependent. Larger quantities elicited more errors and slower response times relative to smaller ones. Interestingly, the grade by set size interaction previously reported in a different variant of this task was not present in the current results. Such differences may be potentially explained by several differences across these studies, including the more restricted grade range examined (K-1st vs. K-3rd), the complexity of the array of forced-choice response options (7 versus 3), or the overall set sizes examined (5,6,7,8 versus 4,5,6,7). Such factors may diminish the emergence of Grade by Set Size interactions in younger grades. Moreover, the children in chapter two were drawn from private schools, whereas the children in the current study attended public school. As a result of differences in socioeconomic status or school curriculum, the first graders in the current study are more strongly impacted by set size than the first graders in chapter two. On comparable set sizes of five, six, and seven the first graders in the current study performed overall more like the kindergarteners from chapter two, and the kindergartners in the current study performed much worse than the kindergarteners from chapter two. These differences might suggest that set size by grade interactions are strongest as children approach ceiling performance on the easier sets, but do not emerge when all set sizes are far from ceiling as was the case for both groups in the current study.

Given the cross-sectional framework of this experiment, support for schooling effect rests heavily on the assumption that the old kindergarten and young first graders were otherwise similar, and the central differences between them are attributable to the age-based assignment of grade. One key piece of information may provide evidence on the degree to which this assumption is valid – a longitudinal follow-up study of the old kindergarteners in this sample.

Experiment 2

Experiment 2 examines the numerical reasoning abilities of old kindergarteners and young first graders at the end of the school year. At the end of the school year, kindergarteners will have experienced a year's worth of formal schooling, and display how that experience impacts their numerical reasoning abilities. If kindergarten education accounts for gains in numerical reasoning, the kindergartners' year-end performance will match that of first graders at the beginning of the school year. Furthermore, the longitudinal follow up will reveal how a first grade education further impacts numerical reasoning ability.

Methods

Twenty-eight kindergarteners and 17 first graders agreed to participate in a second round of data collection during the last two months of the school year. Children completed the same task from Experiment 1. One 1^{st} grader's year-end score was more than 3 standard deviations away from the mean and much lower than his or her time one score. The subject was removed as an outlier, resulting in 16 first graders for Experiment 2.

Results

After six months of schooling, kindergarteners' exact, numerical reasoning scores improved from their scores at the beginning of the year, $t(27) = 6.28$, $p < .001$ (see table 5). The first graders' similarly made gains from the beginning to the end of the year, $t(16) = 3.19$, $p <.01$. Comparing year end scores, first graders performed better than kindergarteners on the exact, non-symbolic subtraction task, $t(42) = 2.77$, $p < .05$. To test the value of a kindergarten education the important comparison is between the kindergarteners year-end performance and that of the first graders at the beginning of the year. Importantly, there was no significant difference between numerical reasoning scores for kindergarteners at the end of the school year and first graders at the beginning of their school year.

	Mean Percent Correct (SD)	
Group	Beginning of the year	End of the year
Old Kindergarteners	0.39 (0.28)	0.70 (0.23)
Young 1st Graders	0.75 (0.14)	0.87 (0.11)

Table 5. Descriptive statistics of children's overall accuracy by group and by time period.

To reexamine the impact of grade and age at the end of the year, a linear regression model was used to predict children's numerical reasoning accuracy scores from children's age. Age showed no significant effect of year-end numerical reasoning accuracy. Grade was then entered into the model. In a step-wise linear regression model with age and grade as predictors, grade entered the model first and significantly predicted numerical reasoning accuracy ($r^2 = 0.11$, $p < .001$). With both age and grade in the model, the relationship between grade and numerical reasoning ability showed a marginally significant trend to (age: $r^2 = 0.01$, $p = .53$; grade: $r^2 = 0.06$, $p =.08$). Moreover, a regression discontinuity analysis compared the two regression lines for age and numerical reasoning accuracy scores for each group at the end of the school year (see figure 7). The univariate general linear model revealed no effects of grade. The univariate general linear model also revealed no effect of age or a difference in the slope of the regression lines for each grade.

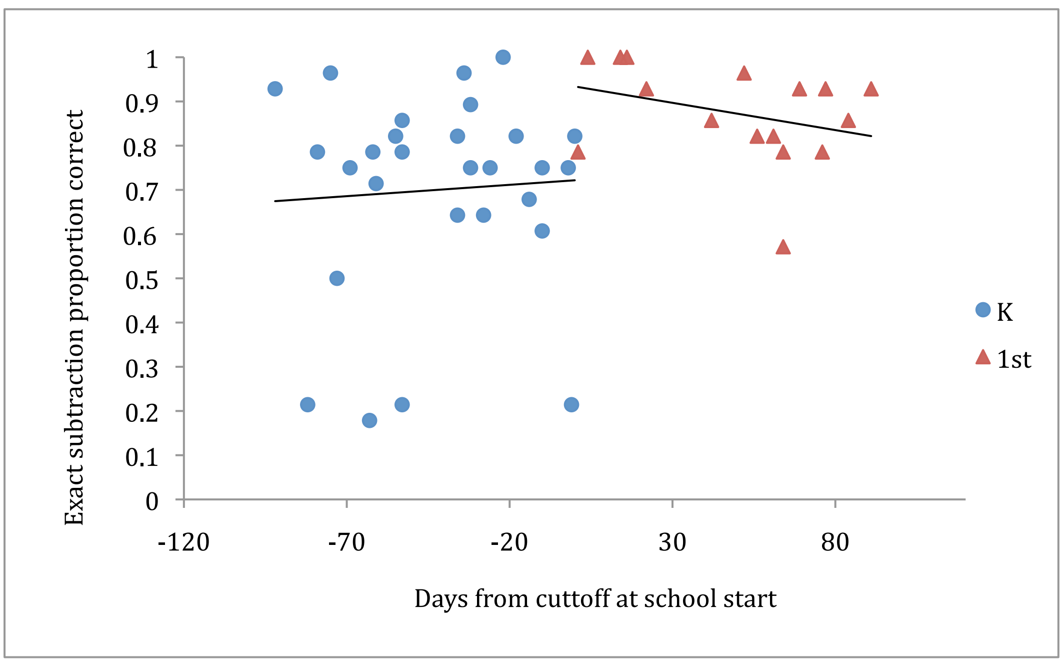


Figure 7. Time point 2 subtraction accuracy plotted by age. The regression lines for each group do not reveal an effect of grade at the end of the school year.

Discussion

Experiment 2 found that kindergarten children greatly improve their numerical reasoning abilities over the course of the school year. For this sample, impressive learning takes places during kindergarten, either because of an important maturational stage or educational experience. Importantly, the performance of the kindergarteners at the end of the school year, resemble that of first graders at the beginning of the school year. Such an effect further highlights the importance of school for numerical reasoning skills. If numerical reasoning developed with maturation, we would expect the older 1^{st} grade children to outperform the younger kindergarteners, but they do not.

By the end of the year, neither age nor grade uniquely contribute to precise numerical reasoning abilities, suggesting that for this sample, schooling effects are limited to kindergarten and may not continue in 1^{st} grade. The year-end numerical reasoning scores are difficult to interpret because the scores approached ceiling. Future studies should use a more challenging task with larger set sizes to reduce the ceiling effect and better capture how the first grade experience continues to contribute to numerical reasoning abilities.

General Discussion

The ability to exactly calculate has been thought to depend on language, cardinality, maturation, and/or schooling. The current study focused on isolating schooling effects from age by employing a school cutoff design. At the beginning of the school year, old kindergarteners performed significantly worse than young first graders. Regression analyses determined that the difference in performance is due to the different grades and not a result of the slight difference in ages between the groups.

Our findings are compatible with previous studies of age and schooling. A previous school-cutoff study found an impact of schooling, as opposed to age, on mental arithmetic abilities (Bisanz, Morrison, & Dunn, 1995). Cross-cultural data support the ideas that mental arithmetic ability is susceptible to cultural influence, as evident by the varied arithmetic abilities across countries (NCES, 2011). Additionally, both Mundurukú children and adults performed poorly on an exact subtraction task, suggesting that maturation alone does not lead to the development of exact arithmetic skills.

By the end of the school year, kindergartners' exact, non-symbolic subtraction abilities greatly improved. Most importantly, the kindergartners' year-end numerical reasoning abilities resembled those of the first graders' abilities at the beginning of the year. Over the course of the kindergarten year, these children improved more than would be predicted by maturation, suggesting that schooling strongly influences numerical reasoning ability.

The current study highlights the importance of schooling in the development of numerical reasoning skills, however, instruction alone may not be sufficient. Instead, the combination of instruction, age and language may guide the acquisition of exact arithmetic. For example, children may need to reach a particular age or acquire knowledge of cardinality before they are ready to receive the type of instruction offered in kindergarten. Future studies should examine the interactions between age, language and instruction to determine the essential combination of factors needed to exactly calculate. Additionally, kindergarten education consists of a series of lessons and experiences. A closer look at specific instructional techniques will reveal the minimal cognitive abilities necessary to exactly reason about quantities.

In sum, kindergarten education in U.S. middle income schools plays an important role in children's abilities to exactly subtract quantities. This study brings us one step closer to determining the combination of essential skills necessary for exact arithmetic. Establishing the foundation for exact calculation will lead to the creation of more efficient instructional practices and the development of early assessments and interventions for children at risk for developing math disabilities.

References:

Bisanz, J., Morrison, F. J., & Dunn, M. (1995). Effects of age and schooling on the acquisition of elementary quantitative skills. *Developmental Psychology*, *31*(2), 221.

Butterworth, B., Reeve, R., Reynolds, F., & Lloyd, D. (2008). Numerical thought with and without words: Evidence from indigenous Australian children. *Proceedings of the National Academy of Sciences*, *105*(35), 13179–13184. doi:10.1073/pnas.0806045105

Carey, S. (2001). Cognitive Foundations of Arithmetic: Evolution and Ontogenisis. *Mind & Language*, *16*(1), 37–55.

Gelman, R., & Butterworth, B. (2005). Number and language: how are they related? *Trends in Cognitive Sciences*, *9*(1), 6–10. doi:10.1016/j.tics.2004.11.004 Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*(5695), 496.

Morrison, F. J., Smith, L., & Dow-Ehrensberger, M. (1995). Education and cognitive development: A natural experiment. *Developmental Psychology*, *31*(5), 789–799.

Moneta-Koehler, L. & McCandliss, B. D. (in preparation). Reasoning about quantities: children reveal that words are not enough.

Piaget, J. (1960). *The psychology of intelligence.* Patterson, NJ: Littlefield, Adams.

Pica, P., Lemer, C., Izard, V., & Dehaene, S. (2004). Exact and approximate arithmetic in an Amazonian indigene group. *Science*, *306*(5695), 499.

Rueda, M. R., Rothbart, M. K., McCandliss, B. D., Saccomanno, L., & Posner, M. I. (2005). Training, maturation, and genetic influences on the development of executive attention. *Proceedings of the national Academy of Sciences of the United States of America*, *102*(41), 14931-14936.

Spaepen E., Coppola M., Spelke E. S., Carey S. E., Goldin-Meadow S. (2011). Number without a language model. Proc. Natl. Acad. Sci. U.S.A. 108, 3163–3168. doi: 10.1073/pnas.1015975108

National Center for Education Statistics (NCES). Trends in International Mathematics and Science Study: TIMSS 2011 results. Retrieved January 14, 2015, from http://nces.ed.gov/timss/results11.asp

CHAPTER 4

Advanced cardinality and children's exact numerical reasoning abilities

Abstract

Language, in the form of verbal counting and cardinality, has been identified as necessary for exact numerical reasoning. The current study further examines the role of language by teaching advanced cardinality, knowledge about cardinal number knowledge that goes beyond the last-word rule. Children trained on all the parts that form a whole with either number word (e.g. "five is made of the three block and the two block") or color words (e.g. "five is made of the gray block and the pink block"). Results show children's numerical reasoning skills improve only for the set sizes trained with number words, suggesting that language continues to impact precise numerical reasoning skills and should be emphasized during part-whole instruction. Moreover, advanced cardinality training resulted in numerical reasoning gains only for set sizes within one's zone of proximal development. Basic cardinal number knowledge differs among sets, contrasting the commonly held view that a general cardinality principle is similarly applied to all quantities.

Keywords: Mathematics; Cognitive Development; Cardinality; Language; Number Bonds, Part-whole

Introduction

Cardinality, the ability to state the quantity of a set, is a prerequisite for more advanced mathematics. Developmentally, children learn to count and use cardinality before they learn to add and subtraction quantities (Gelman & Gallistel, 1978). Cross-culturally we see that cultures lacking a language for exact quantities, such the Praha and the Mundurukú, are severely impaired in their abilities to reason about exact sets (Gordon, 2004; Pica, Lemer, Izard, & Dehaene, 2004). For example, members of the Amazonian Mundurukú tribe performed a non-symbolic subtraction task. As the set sizes increase, Mundurukú accuracy drastically decreases in a manner dissimilar from a group of French controls (Pica et al., 2004). The study concluded that one needs the words for a quantity before he or she can make mathematical inferences about that quantity. Such a dependence on language is supported by the Sapir-Whorf hypothesis (Whorf, 1956), and by neuroscience studies that show a relationship between language and exact number processing (Lemer, Dehaene, Spelke, & Cohen, 2003; Dehaene, Piazza, Pinel & Cohen, 2003; Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999). Combined, the literature strongly suggests that language and cardinality are necessary for precise numerical reasoning.

How children develop cardinal number knowledge has been debated for some time. Early theories suggest that children access a general cardinality principle and apply that principle to counting (Gelman & Gallistel, 1978). Contrary views suggest that children use the counting chant to learn the meaning of "one" then the meaning of "two" and finally the meaning of "three" before acquiring a general cardinality principle to apply to all larger numbers (Le Corre & Carey 2009; Le Corre, Van de Walle, Brannon & Carey, 2006; Sarnecka & Carey. 2008; Sarnecka & Lee, 2009; Wynn, 1990; Wynn 1992). First, children rely on an innate system of parallel individuation that precisely distinguishes between the small sets one, two and three (Feigenson, Carey & Spelke, 2002, Starkey & Cooper, 1980). Just as children learn the meaning of quantifiers such as “a” in speech, they learn the meanings of “one” and, after roughly six months, the meaning of “two” and eventually “three.” The precise words map fairly easily onto the similarly precise quantity representations. Larger numbers, however, initially have approximate representations and thus exact number words do not easily map onto them. Instead, according to Carey’s (2001; 2004; 2009) bootstrapping theory, practice reciting the counting list helps children learn that the counting words are ordered and that a word that appears later in the list represents a quantity greater than its immediate predecessor. In other words, because the word ‘four’ comes immediately after the word ‘three’ in the counting list, four items must be one greater than three. This knowledge of the successor function teaches children the cardinality principle, which generalizes to all quantities greater than three (or, in some cases, four). Baroody, et al. (1983) similarly found that mathematical rules and principles, such as the successor function, can be automatically applied to new numbers.

By contrast, Davidson et al. (2012) found that children learn basic cardinal number knowledge before they learn the successor function. Children that correctly label sets do not always know which of two counting words is associated with the larger quantity. These children also might not understand that successive numbers differ by one. In contrast to Carey’s view on a natural number insight that applies to all natural numbers greater than four, Davidson, et al. found that children continue to learn cardinality one quantity at a time, starting with small quantities and then learning larger quantities. For example, after acquiring cardinality for four, a child then learns five, followed by six. This evidence strongly supports the notion that children do not generalize a principle to all quantities greater than four.

Even after children acquire basic cardinal number knowledge, a recent investigation suggests that cardinality is not sufficient for exact numerical reasoning and cardinal number knowledge needs to be supplemented with instruction. In the study, elementary school children performed a subtraction task similar to that used with the Mundurukú (Moneta-Koehler & McCandliss, in preparation). The youngest subjects, kindergarteners, performed just like the Mundurukú, displaying deceasing accuracy as the problem sizes increase. Interestingly, unlike the Mundurukú, the kindergarteners all had cardinality for the quantities used in the task, demonstrating that cardinality is not the missing link to exact numerical reasoning. Language is not enough. After completing kindergarten, the first year of formal instruction, children made great gains and these gains are attributed to schooling, not maturation (Moneta-Koehler & McCandliss, in preparation). Therefore, language must be combined with instruction for children to accurately reason about exact sets.

Resnick (1989) suggests that instruction helps children integrate their cardinal number knowledge with protoquantitative schemas regarding how a whole is made up of smaller parts. Early in life children learn how a pie is sliced into pieces and how puzzle pieces combine to make a picture. These types of early experiences form imprecise part-whole schemas. Resnick theorized that when these protoquanitiative combined with language, children are able to precisely reason about sets. Formal instruction may help children integrate their cardinal number knowledge and protoquantitative part-whole schemas, and recent educational trends support this idea.

According to the Common Core, an educational activity called "number bonds" should be taught in kindergarten (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Number bonds refers to all the pairwise number combinations within a given number that add up to that number. For example, the set of five is comprised of the number bonds one and four, and the number bonds two and three. Educational activities designed to teach number bonds are often accompanied by visuals, which serve to emphasize part-whole schemas and encourage children to see how precise whole numbers are comprised of smaller precise sets. In this part-whole number instruction, a child will learn that, say, five is made up of the combination of one and four and the combination of two and three. In contrast to activities that teach children all the "two" addition facts (2 + 1 = 3, 2 + 2 = 4, 2 + 3 = 5, etc.), in number bonds activities children focus on the meaning of the sum and learning all the ways that the sum can be constructed by smaller precise numbers. This instructional shift was partially motivated by the high performance of Chinese and Singaporean students (Organisation for Economic Co-operation and Development, 2013) and their use of numerical regroupings. For example, Chinese teachers are more likely to point out the multiple ways one can regroup a number. They will encourage children to break, say, 26 in to 20 + 6, 10 + 10 + 6, 20 + 3 + 3, and 10 + 10 + 3 + 3, resulting in a stronger conceptual understanding of numbers (Ma, 1999). Eager to improve American students' international ranking, U.S. teacher began using Singapore and Chinese teaching styles, including the use of number bonds.

The current study builds on Resnick's idea that children combine cardinality and part-whole knowledge to advance their understanding of numerical cardinality and reason about exact sets. We will refer to this expanded view of cardinality as "advanced cardinality". The concept of cardinality, as it is currently used in the literature, does not capture the notion that the semantics of an exact number word must include essential part-whole knowledge about that quantity. Thus, in contrast to the well established definition of cardinality in the literature reviewed above, when achieving advanced cardinality, children come to learn that the meaning of the word “five” refers not just to an exact set of five objects, but also the combination of the precise subsets one and four, as well as the combination of the precise subsets two and three. The goal of this investigation is to test whether reasoning about precise numbers requires children to explicitly integrate their cardinal number words with part whole knowledge. Must the associations between quantities and their parts need to be verbal in nature for one to learn advanced cardinality? Must the word “five” be associated with the words “one and four” and “two and three” in order to reason about exactly five things? Thus, a central aim of this study is to test for a language effect, by contrasting part-whole training with number words versus part-whole training without number words. If advanced cardinality learning relies on language, integrating cardinality words with part whole knowledge will lead to different learning than simply introducing part whole knowledge via number bonds.

The study further examines the impact of advanced cardinality by assessing how insight gained about one number, such as five, transfers to other numbers, such as six, thus addressing the degree to which a specific intervention might change children’s general number knowledge and reasoning abilities, versus advance their understanding only of a specific number concept. If children learn general principles such as the successor function, cardinal number principles, and automatically accessed arithmetic rules (Carey, 2001; Baroody 1983), then we would expect advanced cardinality training on one set to transfer to another unlearned set, thus supporting gains in numerical reasoning for all set sizes. Alternatively, if children progress not through the induction of abstract principles, but rather through progressive development of specific number concepts through experience, children will only show numerical reasoning gains for the trained set size. Thus, a second major goal of this study is to test for a Set Size specificity training effect, by contrasting numerical reasoning gains with sets in the verbally mediated advanced cardinality training versus gains with the sets in the control condition.

In addition, the study is designed to examine an additional question about learning advanced cardinality that might highlight important differences between how learning could progress differently between the quantity five and the quantity six. Will children learn to reason with either quantity equally well, or will children learn to reason with the smaller set before advancing to larger ones? If learning progresses in a way that reflects insight into general, abstract principles into cardinality of natural numbers larger than four, as advanced by Le Corre & Carey (2009) children might be expected to respond equally well to training on five or six. However, if insights into principles of cardinality are learned by progressively extending knowledge and insight from smaller quantities to larger ones, as Davidson et al. (2012) suggests, children who demonstrate insight into advanced cardinality for four should be well better prepared to learn advanced cardinality for five, however advanced cardinality training on six should have little impact. Thus, the third and final goal of this study is to test for a training magnitude effect, by contrasting numerical reasoning gains for the quantity five versus six.

Methods

Participants

42 pre-kindergarten (pre-K) children from five Nashville area preschools participated in the study. Typically developing pre-K students know basic cardinality for numbers but have not been exposed to the formal math instruction introduced in kindergarten. Thus, the intervention was not confounded with other formal math instruction shown to improve children numerical reasoning abilities. Consent forms and questionnaires were sent home for interested parents to

complete and return to the participating child's preschool. The parental questionnaire probed the child's upbringing, including time spent in daycare, family socioeconomic status and any psychological or development problems.

Participants were in included in the study if they displayed typical development, had normal or corrected-to-normal vision, spoke English, and completed the assessment and training tasks. Additionally, children were only included if, during the prestest, they displayed basic cardinality: achievement of at least 75% accuracy on the enumeration task and, after 3 trials per set-size, could correctly forms all quantities in the give-a-number task. Of the initial 42 participants, six did not have basic cardinality, three requested to end the study early, and one was revealed to have a developmental disability, resulting in 32 participants.

Procedure

Children completed the pretests, intervention and posttests in four one-on-one sessions at their preschool with a visiting experimenter. On day one, children completed the cardinality and numerical reasoning tasks; on days two and three children received the intervention; and on day 4 children performed one round of the intervention and repeated the numerical reasoning tasks. In this manner, children were exposed to the intervention three of the four days.

Cardinality Tasks

The pretest included brief assessments of basic cardinality and advanced cardinality. The basic cardinality assessment included an enumeration task and the give-a-number task (Wynn, 1992). In the enumeration task, child viewed sets of randomly arranged black dots on a grey background. The children were instructed to say "how many dots" were on the page, and the experimenter noted his/her response. All sets between one and six were included four times, resulting in 24 trials. Wynn's Give-a-number task was also conducted. Children were asked to give sets of blocks to one of two stuffed animals. Sets started small with just one block and increased by one until children were asked to give six blocks to the animal toy.

Bermejo's jump forward task (Bermejo, 1996; Bermejo, Morales, & deOsuna, 2004) and has previously been used to assess whether a child can only use the last word in a count sequence to determine cardinality or if the child's understanding of the quantity exceeds the counting strategy. During the jump forward task, children were showed five blocks and were asked to count the set starting with the word "two." The experimenter then asked, "how many blocks are there all together?" and noted the child's response. The child then repeated the task with a set of six blocks.

Numerical Reasoning Tasks

The non-symbolic subtraction task was based off the task used with the Mundurukú and has previously been used to assess kindergartener's exact reasoning abilities. The experimenter introduced the task as a game in which children view videos of a women placing discs into a bucket and determine whether four, five, or six discs remained (see figure 7). By situating the task in a real-world scenario, avoiding symbols with which some children may not be fluent, and keeping answers in the positive range, the youngest participants were able to complete the task. The instructions purposefully avoided words such as "count" or "subtract" allowing children's own strategies to emerge.

Children were given two practice trials with the option of repeating the trials if the child failed both times. Children completed 18 self-initiated subtraction problems where they viewed videos of a women looking into a bucket (3200ms), an image of discs in a bucket (viewable for 7000ms with the option to advance sooner), a video of a woman removing discs from a bucket (1200ms) and an image of the removed discs on a hand (viewable for an unlimited amount of time) (see figure 8). The difference was always one, two, or three causing the first and second operands to covary. Children responded by pressing a key with an image of one, two, or three discs in a bucket and saw an image of the correct number of items remaining in the bucket (1500) with a sound indicating their success on the trial. While the child completed the task, the experimenter noted strategies used during each stage of each trial. Strategies consisted of verbal counting, finger counting, stating the quantity of a set, and performing a verbal or finger operation.

Trials consisted of four set-sizes, as determined by the first operand. Six trials of each set-size were included such that there all three answers (one, two, and three) were represented twice (e.g, for set-size four, there were two trials of 4-3, 4-2, and 4-1).

Images of dots were generated using MATLAB 7 (MathWorks, Natick, MA, USA). Dot size remained constant, but their arrangement varied in order to prevent children from relying on non-numerical properties. Low-level perceptual features, such as enclosure size and density have been shown to influence children's quantity decisions (Piaget, 1952). First operand discs appeared in rectangular invisible enclosures of three sizes. On half the trials enclosure size for the sum increased (by 67%) and density decreased (by between 25% and 31%) with quantity (Envelope Congruent trials), and on half of the trials enclosure size for the sum decreased (by 40%) and density increased (by between 90% and 104%) with quantity (Envelope Incongruent trials). The median enclosure size was used for the second operand.

Figure 8. Exact numerical reasoning subtraction task sequence of events

The addition task complemented the subtraction task by manipulating the same quantities (see table 6). Children viewed videos of a women looking into a bucket (3200ms), an image of discs in a bucket (viewable for 7000ms with the option to advance sooner), a video of a woman about to add discs to the bucket (1200ms), and an image of those additional discs on a hand (viewable for unlimited time until the child responded), Children responded by pressing a key with an image of four, five or six discs in a bucket. As with the stimuli in the subtraction task, enclosure size and density varied by the set size, defined by the sum in the addition task.

Subtraction Task			Addition Task		
Operand 1	Operand 2	Difference	Operand 1	Operand 2	Sum
4	3	1	3	1	4
4	2	2	2	2	4
4	1	3	1	3	4
5	4	1	4	1	5
5	3	2	3	2	5
5	2	3	2	3	5
6	5	1	5	1	6
6	4	2	4	2	6
6	3	3	3	3	6

Table 6. Trials for each of the numerical reasoning tasks

Intervention

The intervention consisted of a computer game that resembleed Tetris and was based off a game designed by the London Knowledge Lab (http://number-sense.co.uk/numberbonds, Butterworth & Laurillard, 2010). In the game children visually formed a target number by horizontally lining up digital Cuisenaire rods. The rods were segmented to best represent their corresponding quantities. If the correct rod was chosen, the entire width of the workspace was filled, indicating that the target quantity was correctly made. For example, if the target number was five, a gray rod representing three might have "fallen" from the top left side of the screen to the bottom left side of the screen (see figure 9). The subject then reviewed a set of colored rods representing all the numbers from one through four and verbally selected one by its number name, in this case "two." If the child chose correctly, the second rod fit perfectly between the first rod and the right hand side of the screen, forming the whole quantity five.

Figure 9**.** Intervention based off the Number Bonds game designed by the London Knowledge Lab (Butterworth & Laurillard, 2010). In order to make 5, children select the rod from the right side of the screen to complete the workspace on the left side of the screen.

The child heard the sentence, “Five is made of three and two” and repeated it to reinforce the verbal part-whole knowledge. The whole quanitity was displayed as a complete row along the bottom of the screen and remained in place for the duration of the round. If the child chose an incorrect quantity, the second rod either failed to touch the first rod or overlapped with the first rod. In both incorrect scenarios the first and second rod disappeared and the first rod “fell” again until the child chose the correct second rod. The goal of the game was to build a tower of number bonds representing all the numerical pairs that add up to the target quantity. Each child learned one quantity, counterbalanced across the quantities five and six.

The other quantity was used in the control condition, in which the subject completed the same intervention as the test group, but learned different verbal associations. Instead of calling the rods by their number names, they identified them by color. For example, if the control condition target number was five, a gray rod representing three might have "fallen" from the top of the screen. The child then selected the pink rod representing two by saying the word "pink," and the part-whole knowledge was reinforced with the verbal statement "five is made of gray and pink." The visual stimuli and part-whole concepts were controlled between conditions and the two groups only differed by the verbal associations children learned.

For the intervention, the quantity taught with number words varied as well as which quantity was taught first, resulting in four conditions which were counterbalanced across participants.

Results

Cardinality pretest

All participants revealed knowledge of the words up to "six" in the give-a-number task and achieved a mean accuracy score at or above 0.75 in the enumeration task (mean: 0.95 sd: 0.07). Alternatively, on the jump forward task, no children successfully stated both quantities. Four children correctly labeled the set of five blocks and two different children correctly named the set of six. Thus all children displayed cardinal number knowledge but did not show major advances beyond the limits of basic cardinality.

Subtraction pretest

Average accuracy scores by set size were calculated for each subject on the subtraction tasks. As in the Mundurukú study, set size for the subtraction task is determined by the first operand, such that accuracy for four includes the trials 4-1, 4-2 and 4-3. Accuracy data was submitted to a repeated-measures ANOVA to assess Sets Size (4, 5, 6) and Envelope Congruence (Congruent, Incongruent). Significant Set Size effects emerged $F(2,62) = 13.445$, $p<.001$, suggesting that larger sets were harder to reason with than smaller ones (see figure 10). Reaction time data similarly showed a Set-Size effect $F(2,54) = 9.80$, $p<.001$, ruling out a time/accuracy tradeoff. Pairwise comparisons revealed that accuracy for Set Size six was lower than that for Set Size five, $p <.05$. Because Set Size five was easier than Set Size six, children might make greater gains on the Set Size five posttest trials because they are ready to learn that set. Reasoning with five might be in pre-K children's zone of proximal development (Vygotsky, 1978). Alternatively, the training might show a greater impact on Set Size six given that there is greater room for growth. No main effects nor any interactions emerged for the Envelope Congruence factor, suggesting that the spatial extent of the first set had no influence on children's decisions about exact non-symbolic subtraction.

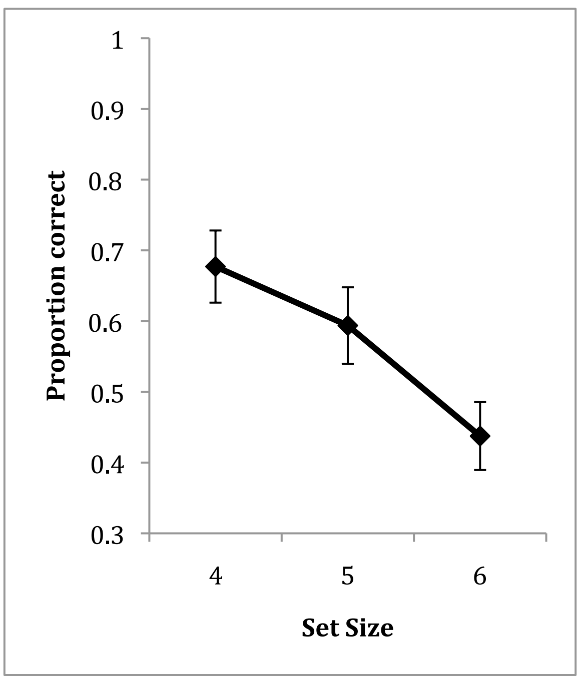


Figure 10. Pretest subtraction accuracy by Set Size.

Advanced cardinality training

Children engaged in the intervention for a set amount of time, but varied in the number of training rounds they completed. Paired sample t tests compared number of rounds played between quantity (mean of 19.75 and 17 rounds of training on 5 and 6, respecitvely) and training condition (mean of 18.94 and 17.81 rounds on the number and color term conditions, respectively) and revealed significant difference between quantity, $t(31) = 4.61$, $p < .001$. Children completed more rounds of the intervention when training on five than on six. One-way independent t-tests comparing the rounds of training on five and six for each condition revealed only a significant difference during the number condition $t(30) = 1.86$, $p < .05$. Children completed more rounds of training when learning five with number words than when learning six with number words.

To further examine the role of training, children's pretest subtraction scores were compared to children's training performance. A linear regression model was used to predict the number of rounds children played of the number condition from their subtraction pretest accuracy scores. Pretest subtraction performance significantly predicted the number-condition training rounds completed, $r^2 = 0.36$, $p < .001$. The number-condition was separated by quantity and reentered into new linear regression models. Pretest subtraction performance significantly predicted how many rounds children played when learning five with number words, $r^2 = 0.49$, $p < .01$, but did not predict how many rounds children played when learning six. Pretest subtraction abilities were related to children's advanced cardinality training of quantity five, but not six.

Impact of advanced cardinality training on exact subtraction

After training, children repeated the numerical reasoning assessments, and gains were characterized. Accuracy data was submitted to a repeated-measures ANOVA to assess Sets Size (4, 5, 6) and Time (Pretest, Posttest). Aside from a main effect of Set Size, $F(2,62) = 26.713$, $p<.001$, no other main effects or interactions were found. As a whole, children did no better on the subtraction posttest as they did on the pretest.

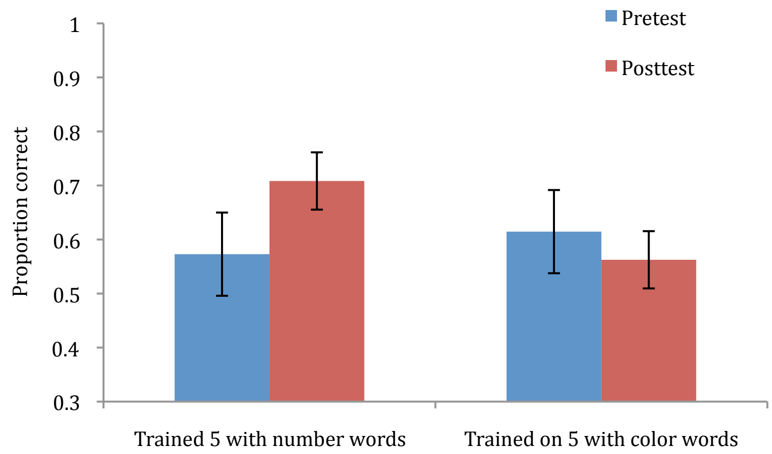
Does learning advanced cardinality rely on the use of number words?

In order to assess the impact of training condition, Set Size specificity was examined. Subjects' subtraction trials were coded according to their intervention group. For the intervention, children either learned the target number five or six with number labels. If a child learned five with number words, then the subtraction trials 5-4, 5-3 and 5-2 were all coded as having been trained in the intervention with number words and the trials 6-5, 6-4 and 6-5 were all coded as being trained with color words, with the reverse being true if a child learned six with number words. A paired sample t-test compared children's posttest accuracy scores for numerb word trained sets vs. color word trained sets and found no difference. To assess how number word training impacted gains on the subtraction task, accuracy data was submitted to a repeated-measures ANOVA assessing Training (Number Word Trained Set Size, Color Word Trained Set Size) and Time (Pretest, Posttest). The interaction between Training and Time on the subtraction task reached significance, $F(1,31) = 4.399$, $p<.05$, indicating that children improved from pretest to posttest only on the subtraction trials involving the quantity trained with number words. Children that trained on one quantity with number words show no improved numerical reasoning for the other quantity, suggesting that cardinality relied on the use of number words and did not transfer to quantities studied with color words. A paired sampled T-test on subtraction Number Word Trained Set Size and Color Word Set Size pretest scores revealed a significant difference, $T(31) = 2.095$, $p < .05$, suggesting that the Number Word Trained Set Size posttest gains may be influenced by differences between the groups' pretest scores.

Is there a zone of proximal development for learning advanced cardinality?

Trained magnitudes include both five and six. Separating the quantities reveals whether number word advanced cardinality training impacts numerical reasoning abilities for only one quantity. Accuracy data was submitted to a repeated-measures ANOVA assessing Training (Number Word Trained Set Size, Color Word Trained Set Size), Time (Pretest, Posttest) and Set Size (5,6). A 3-way interaction between Training, Time and Set Size reached significance, $F(1,30) = 4.32$, $p < 0.05$, suggesting that training impacts gains for magnitudes five and six differently. Pairwise comparisons assessed the impact of Time (Pretest, Posttest) for each magnitude and for each training condition. Only the children that trained on five withnumer words showed effects of time on subtraction trials of Set Size 5, $p < 0.05$ (see figure 11a). Verbally mediated advanced cardinality training only works for the quantity five. Children who trained on six with number words showed no significant gains (see figure 11b). Finally, for Set Size 5, there was no significant difference between the training groups' pretest scores suggesting that the gains were driven by the intervention.

a



b

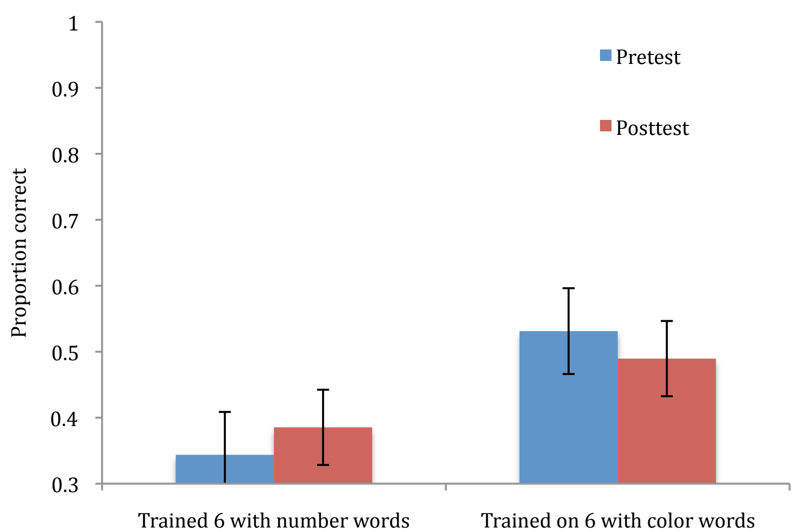


Figure 11. Pre and posttest numerical reasoning accuracy for untrained and trained quantities separated for subtraction a) Set Size 5 and b) Set Size 6.

Does advanced cardinality training assist in exact addition?

The addition task was newly developed for this study. An exploratory analysis was performed to examine the effectiveness of the task and the extent to which verbally mediated advanced cardinality training transferred to addition reasoning skills. Set size for the addition task was determined by the sum, such that accuracy for four includes the trials 3+1, 2+2, and 1+3. Pretest accuracy data was submitted to a repeated-measures ANOVA to assess Sets Size (4, 5, 6), however no Set Size effects emerged, suggesting that smaller addition problems were just as difficult to solve as the large ones. After training, posttest scores and gains were characterized. Accuracy data was submitted to a repeated-measures ANOVA to assess Sets Size (4, 5, 6) and Time (Pretest, Posttest), revealing no main effects nor interactions.

In order to investigate Set-Size specificity, a paired sample t-test compared children's posttest accuracy scores for quantities they were trained on with number words vs. color words and found no difference. To assess how training impacted gains on the addtion task, accuracy data was submitted to a repeated-measures ANOVA assessing Training (Number Word Trained Set Size, Color Word Trained Set Size) and Time (Pretest, Posttest) and again found no effects. As in the subtraction task, effects may be limited to a specific Set Size or magnitude.

To examine magnitude specificity, accuracy data was submitted to a repeated-measures ANOVA assessing Training (Number Word Trained Set Size, Color Word trained Set Size), Time (Pretest, Posttest) and Set Size (5,6) and no significant effects emerged. Finally, for Set Sizes 5 and 6, paired Sample T-Tests assessed the impact of Time (Pretest, Posttest) on numerical reasoning accuracy data for children who trained with number words that set and children who trained with number words on the opposite set. Only the children that trained on five with number words showed trending effects of time on numerical reasoning trials of Set Size 5, $T(15) = 1.84$, $p = 0.086$. Like with the subtraction findings, advanced cardinality training may only works for the quantity five. Children who trained on six with number words showed no gains. Finally, for Set Size 5, there was no significant difference between the training groups' pretest scores suggesting that the trending gains were driven by the intervention.

Discussion

Pre-K children had basic cardinal number knowledge, as displayed by their accurate responses to the enumeration and give-a-number tasks. Conversely, these children did not demonstrate strong advanced beyond the limits of basic cardinality, evident by their inability to determine cardinality with a nontraditional counting strategy. Thus, the participants were perfectly prepared to receive advanced cardinality training.

Prior to training, children completed numerical reasoning tasks: one subtraction task and one addition task. We focused on the subtraction task for two reasons. First, the task has been previously shown to work with young children. Second the subtraction schema more closely matches the training schema. During training, children learned a target quantity followed by its addends, and similarly, subtraction problems start with the target quantity followed by its parts. Addition problems, however, flip the order from training, starting with the parts and ending with the sum. Thus the advanced cardinality knowledge gained during training may more easily transfer to subtraction than addition.

On the nonsymbolic subtraction task children showed lower accuracy and longer response time when reasoning about larger quantities. This classic set size effect (Zbrodoff & Logan, 2005) suggests that larger quantities are harder to reason with than smaller ones and replicated previous findings that basic cardinality is insufficient for numerical reasoning. All the children had cardinal number knowledge of five and six, yet performed poorly subtracting with those quantities. Additional skills were necessary for numerical reasoning. Performance on this subtraction pretest predicted children's performance during the advanced cardinality intervention. Children with higher subtraction accuracy scores played more rounds of the number condition training game. Importantly, this relationship only held for those children that learned all the ways to make five. Subtracting accuracy was unrelated to the number of rounds children trained on six. This difference is the first piece of evidence suggesting that children learn the quantities five and six differently.

At the end of the training, children that learned part-whole knowledge with numerical labels improved their numerical reasoning skills, whereas children that learned part-whole knowledge with color labels did not. Number word mediated advanced cardinality training successfully shifted children from representing a quantity to reasoning with a quantity. Children needed more than part-whole number training, for the color condition training did not result in improved numerical reasoning. Instead, number terms extended children's part-whole schemas to something they could apply to new situations, namely non-symbolic subtraction. The role of language extends beyond counting and continues to impact children's mathematical competencies, further supporting the Sapir-Whorf hypothesis.

Number words mediated advanced cardinality did not transfer to the quantity learned with color words. Children that learned the number words for the subsets of five were unable to reason about six. Instead of learning a general part-whole procedure or advanced cardinality principle as suggested by Baroody (1983), children learned part-whole knowledge of a specific set that only improved reasoning with that set. This set size specific learning contradicts the idea that a cardinality principle is applied to all quantities greater than four. The number word advanced cardinality training provided children precise part-whole concepts and subsequent reasoning ability, but only for the trained quantity. The children failed to develop advance cardinality for the quantity in the color condition, limiting reasoning capabilities with that quantity.

Importantly, training on advanced cardinality improves subtraction ability but only for the quantity five. The children number word training on the quantity five strengthened their reasoning skills for trials of Set Size five, whereas children with number word training on six did not improve on subtraction trials of Set Size six. Children's inabilities to reason with six suggests that that children may learn cardinal number knowledge for smaller quantities before larger ones (Davidson, et al., 2012) and that six is beyond these children's zone of proximal development (Vygotsky, 1978). Children were prepared to learn about the magnitude five, and five only. Prior to completing the posttest, children showed a correlation between their pretest subtraction accuracy scores and the number of rounds they engaged with the intervention. Again this effect only held for children that trained with number words on five and not six, further suggesting that these children's prior knowledge of numerical reasoning prepared them to learn advanced cardinality for five. Children were evidently not ready to learn about six. Recent research has found that, after learning to mentally represent four, they then learn about all the other quantities in their count range (Le Corre & Carey 2009; Le Corre, et al., 2006; Sarnecka & Carey. 2008; Sarnecka & Lee, 2009; Wynn, 1990; Wynn 1992). The current study contradicts this concept by showing children capable of counting to six, but unable to represent six in the same manner in which they represented, and subsequently reasoned with, five. Instead children's magnitude specificity suggests that children learn quantities one at a time, and that children need to master five before they can accurately represent six, acquire advanced cardinality for six and finally reason with six.

Alternatively, the magnitude specificity may have arisen because children completed an average of 2.75 more rounds of number word training on the quantity five than on six. The five condition was shorter given that it only taught four combinations, compared to the five addend pairs used to make the quantity six, explaining the difference in rounds of play. Such a discrepancy may impact children's ability to reason with each magnitude. However, the discrepancy does not explain why there was a correlation between children's initial reasoning ability and rounds of number word training only for the quantity five. Nevertheless, future studies should hold rounds of training constant and allow training time to vary to compliment this study and clarify why children learned to reason with five and not six.

On the addition task, children showed a trend toward magnitude specificity, such that children that trained on five made gains on trials of Set Size five. The addition task may have required too far transfer from the advanced cardinality training in order for the gains to reach significance. During the intervention, children learned how a whole quantity breaks into parts, whereas the addition task requires a flipped schema starting with the parts and ending with the whole. Moreover, the addition task failed to display classic set-size effects suggesting that the task may not be ideally suited for capturing pre-K children's numerical reasoning abilities.

Importantly, these findings suggest that advanced cardinality results in a deep understanding of part-whole knowledge, something quite different than simply math fact recall. In both instanced, a child may learn to verbally associate a target quantity with its addends, but unlike math fact recall, advanced cardinality knowledge transfers to unfamiliar situations. After training on advanced cardinality, children applied their new part-whole knowledge to a novel task, novel stimuli, and without the need for any linguistic labels. Advanced cardinality practice of the form "5 is made of 3 and 2" led to improvement on seeing 5 things, seeing 3 taken away, and inferring that there are 2 left, suggesting a novel inference, rather than a fact look up, which would be heavily influenced by surface details.

In sum, number word mediated advanced cardinality training does bridge the gap between basic cardinality and numerical reasoning. Part-whole knowledge combined with numeric verbal labels allows children to precisely reason with quantities, further supporting the role of language in exact mathematical competency. Interestingly, this sample of pre-K children were prepared to learn only about the quantity five and not six, suggesting that only five was in their zone of proximal development. Children's initial cardinal number knowledge of five differed from that of six, countering the commonly held belief that children apply a cardinality principle to all quantities in their counting range. Thus, in order to improve mathematical reasoning, children would benefit from advanced cardinality training with smaller quantities, followed by larger ones.

References:

Baroody, A. J. (1983). The development of procedural knowledge: An alternative explanation for chronometric trends of mental arithmetic. *Developmental Review*, *3*(2), 225–230. Elsevier.

Bermejo, V. (1996). Cardinality development and counting. *Developmental Psychology*, *32*(2), 263–268.

Bermejo, V., Morales, S., & deOsuna, J. G. (2004). Supporting children’s development of cardinality understanding. *Learning and Instruction*, *14*(4), 381–398. doi:10.1016/j.learninstruc.2004.06.010

Butterworth, B., & Laurillard, D. (2010). Low numeracy and dyscalculia: identification and intervention. *Zdm*, *42*(6), 527–539. doi:10.1007/s11858-010-0267-4

Carey, S. (2001). Cognitive Foundations of Arithmetic: Evolution and Ontogenisis. *Mind & Language*, *16*(1), 37–55.

Carey, S. (2004). Bootstrapping & the origin of concepts. *Daedalus*, *133*(1), 59–68.

Carey, S. (2009). Where our number concepts come from. *The Journal of Philosophy*, *106*(4), 220.

Davidson, K., Eng, K., & Barner, D. (2012). Does learning to count involve a semantic induction? *Cognition*, *123*(1), 162–173. doi:10.1016/j.cognition.2011.12.013

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. *Cognitive Neuropsychology*, *20*(3-6), 487–506. doi:10.1080/02643290244000239

Dehaene, S., Spelke, E., Pinel, P., Stanescu, R., & Tsivkin, S. (1999). Sources of mathematical thinking: Behavioral and brain-imaging evidence. *Science*, *284*(5416), 970–974.

Feigenson, L., Carey, S., & Spelke, E. (2002). Infants' Discrimination of Number vs. Continuous

Extent. *Cognitive Psychology*, *44*(1), 33–66. doi:10.1006/cogp.2001.0760

Gelman, R., & Gallistel, C. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*(5695), 496.

Le Corre, M., & Carey, S. (2007). One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles. *Cognition*, *105*(2), 395–438. doi:10.1016/j.cognition.2006.10.005

Le Corre, M., Vanderwalle, G., Brannon, E., & Carey, S. (2006). Re-visiting the competence/performance debate in the acquisition of the counting principles. *Cognitive Psychology*, *52*(2), 130–169. doi:10.1016/j.cogpsych.2005.07.002

Lemer, C., Dehaene, S., Spelke, E., & Cohen, L. (2003). Approximate quantities and exact number words: dissociable systems. *Neuropsychologia*, *41*(14), 1942–1958. doi:10.1016/S0028-3932(03)00123-4

Ma, L. (1999). Knowing and teaching elementary mathematics: Teachers understanding of fundamental mathematics in China and the United States. Hillsdale, NJ: Erlbaum.

Moneta-Koehler, L. & McCandliss, B. D. (in preparation). Reasoning about quantities: children reveal that words are not enough.

Moneta-Koehler, L. & McCandliss, B. D. (in preparation). The impact of one year of schooling on exact numerical reasoning.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors.

Organisation for Economic Co-operation and Development. (2013). PISA 2012 Results in focus: What 15-year olds know and what they can do with what they know. Paris: OECD.

Pica, P., Lemer, C., Izard, V., & Dehaene, S. (2004). Exact and approximate arithmetic in an Amazonian indigene group. *Science*, *306*(5695), 499.

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologist*, *44*(2), 162.

Sarnecka, B. W., & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. *Cognition*, *108*(3), 662–674. doi:10.1016/j.cognition.2008.05.007

Sarnecka, B. W., & Lee, M. D. (2009). Journal of Experimental Child Psychology. *Journal of Experimental Child Psychology*, *103*(3), 325–337. doi:10.1016/j.jecp.2009.02.007

Starkey, P., & Cooper, R. G., Jr. (1980). Perception of numbers by human infants. *Science*, *210*(4473), 1033–1035.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Whorf, B.L. (1956). *Language, thought, and reality: Selected writings of Benjamin Lee Whorf.* New York: Wiley.

Wynn, K. (1990). Children's understanding of counting. *Cognition*, *36*(2), 155–193.

Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology*, *24*(2), 220–251.

Zbrodoff, N. J., & Logan, G. D. (2005). What everyone finds: The problem-size effect. In J. I. D. Campbell (Ed.), Handbook of mathematical cognition (pp. 331-345). New York: Psychology Press.

CHAPTER 5

Discussion and integration of findings

Summary of findings

Broadly, this research probes the mechanisms that underlie numerical reasoning. Language, maturation and instruction all work together to provide the representations and strategies necessary for children to manipulate exact quantities. **Chapter two** looked at the role of cardinality in reasoning about exact sets. Children that have cardinality still failed to precisely reason about quantities, challenging the previously proposed theory that language is the key to numerical reasoning. While language may be necessary, access to counting words is not sufficient for manipulating quantities.

Chapter three looked at potential missing links to numerical reasoning: maturation and education. As opposed to the kindergartners, the private school first graders in chapter two had the ability to precisely reason about quantities, suggesting that either age or schooling supports this development. Using a school cut-off design, young first graders and old kindergarteners completed the exact numerical reasoning task. These children were almost the same age, yet the first graders again out-performed the kindergarteners. Using a regression discontinuity analysis, instruction was identified as the primary reason for the first grader's superior performance. By the end of the school year kindergartens made significant gains in their reasoning abilities and their year-end performance resembled that of first graders at the beginning of the year. Thus, the kindergarten experience in middle income schools allowed children to precisely reason about quantities.

Chapter four pushed the investigation further by proposing an instructional technique that contributes to children's acquisition of precise numerical reasoning skills. Children may learn part-whole number knowledge in kindergarten. The study investigated a concept called advanced cardinality and tested the importance of language for acquiring this sophisticated cardinal number knowledge. The training study taught children to verbally associate target quantities with their parts and assessed how this verbally mediated training led to advanced cardinality knowledge and numerical reasoning. Verbally mediated advanced cardinality training did improve children's numerical reasoning ability, but only for the quantities studied with number words. For example, if a child learned the number words for the parts that form the quantity five, the child showed gains when reasoning with five, but not when reasoning with six. Furthermore, number word mediated advanced cardinality training only appeared to work for the quantity five. Children that trained with six did not gain numerical reasoning abilities, suggesting that five, not six, is within a middle income pre-K child's zone of proximal development.

Instruction and the Sapir-Whorphian hypothesis

Cross-cultural studies and neuroscience findings similarly point to the importance of language for precise mathematical reasoning. Amazonian tribes that do not have exact counting words are limited in their numerical reasoning skills (Gordon, 2004; Pica, Lemer, Izard, & Dehaene, 2004). Neuroscience data concludes that the brain regions associated with language are also associated with exact numerical reasoning, suggesting that the two abilities may be processed similarly (Dehaene, Spelke, Pinel, Stanescu & Tsivkin, 1999). Thus, counting and cardinality are likely early steps to reasoning about quantities, supporting the Whorfian hypothesis that language influence thought (Whorf, 1956).

Contrasting the Whorfian hypothesis, chapter 2 found that children with basic cardinal number knowledge were still poor at exact numerical reasoning. Language did not allow for children to precisely manipulate sets. Similarly, Butterworth and colleagues (2008) found that language does not account for mathematical reasoning skills. They conducted a study with indigenous Australian children who spoke either Warlpiri, a language with the words for “one,” “two,” and “many,” Anindilyakwa, a language with the words for “one,” “two,” “three/four,” and “many,” or English. Children were asked to match the number of counters to a previously shown (and then hidden) set of counters, to match a set of counters to a set of auditory taps, or to match a set of counters to the sum of two sets that were placed under a mat. All four- and five-year-old children’s accuracies were low on these tasks (less than 40% for large sets), and no language effects were found between children, leading to the conclusion that language does not mediate exact number knowledge. Butterworth rejects the language hypothesis and instead claims that other cultural difference such as schooling, not language, account for precise mathematical reasoning ability (Gelman & Butterworth, 2009). As chapter three determined, kindergarten instruction develops one’s ability to reason about quantities, supporting the role of additional cultural experiences.

Neither language nor schooling solely explains precise numerical skills, but instead work together. Instruction focus children’s attention on language, and language categorizes and refines abstract concepts. While a child may acquire basic cardinality via general interactions with the world, carefully designed instruction, such as that used in chapter four, help children learn how a whole is broken up into its parts, and importantly, how to label those parts. These verbally labeled parts refine vague protoquantitative schemas allowing a child to reason with a quantity. In this manner, instruction and language work in tandem to move a child from having basic cardinal number knowledge to being able to reason with those numbers. First, language and cardinality allow for precise representations of a quantity. Then instruction builds on this basic cardinal number knowledge to allow for more advanced cardinal number knowledge to develop.

Stages of cardinal number knowledge

Cardinal number knowledge is often determined by one's ability to state the last word in a count sequence, however this strategy does not always work. As found in chapter 4, a child may be able to determine cardinality when using a standard counting strategy, but fail when forced to use a nontraditional counting strategy, such as counting starting with the number word "two". Thus, the last-word rule is just one stage on cardinal number knowledge. Other counting rules, such stating the largest count word also fall in this strategy-depended stage of cardinality.

Next a child learns to accurately assign a number word to a quantity regardless of strategy. Previously, this accurate mapping between a number word and a set was considered to be the last stage of number knowledge (Bermejo & Lago, 1990; Fuson & Hall, 1983). Once a child can state the quantity of a set, the child has cardinal number knowledge.

Chapter 4 proposed a later stage called advanced cardinality that allows children to make part-whole inferences about sets. Counting words can be associated with much more that just the quantity of a set and advanced cardinality captures these additional associations. For example, when one hears the word "five" it might activate a set of five things and all the parts associated with five. Children's previous imprecise part-whole knowledge of five gains precision with the use of number words. Advanced cardinality develops with the aid of number words and serves as an important step for developing the ability to exactly reason about quantities.

Development of numerical reasoning skills

Based on the literature and the three previous chapters, I propose a developmental trajectory for learning to reason about quantities. In order to exactly reason about quantities, a child needs to precisely represent a quantity. At first, quantities, such a five and six are approximately represented (Halberda & Feigenson, 2008; Xu & Spelke; 2000), but once a child learns to maps the word “five” onto the set of 5 objects, the concept becomes more precise. Following the Whorfian hypothesis, the word places the concepts into a category, such that the quantity five becomes distinct from the quantities four and six.

This basic cardinality develops one quantity at a time. Just after age 2 a child will learning the counting chant, but will not assign those words to any sets (Wynn, 1992). Then, in the same manner in which a child learns quantifiers such as “a” or “some”, a child will map the word “one” onto their precise concept of one. Roughly six months later the child will learn the meaning of “two” and after another six months learn the meaning of “three” (Le Corre & Carey, 2009). Quantities one, two and three are exactly represented as early as infancy (Feigenson, Carey & Spelke, 2002, Starkey & Cooper, 1980), thus, the mapping of the precise number word to the precise quantity representation is fairly straightforward. Quantities larger than three fall outside the subitizing range for a typical child and only exist as approximate representations. Previous theories argue that mapping a precise counting word, such as “four” onto an imprecise representation of the set is very difficult, but once a child masters it, can apply the mapping to all the sets within his or her counting range (Le Corre & Carey 2009; Le Corre, Van de Walle, Brannon & Carey, 2006; Sarnecka & Carey. 2008; Sarnecka & Lee, 2009; Wynn, 1990; Wynn 1992). Study four counters that theory by finding that children’s knowledge of the word “five” is very different from their knowledge of the word “six.” Children’s ability to reason with five correlated with their performance during advanced cardinality training for five, yet this relationship did not exist for the quantity six. Children's prior knowledge of five differed than that for six such that the children were prepared to learn more about five, not six. Children do not learn to represent sets four and larger all at once, but continue to master smaller sets before mastering larger ones.

Once a child masters basic cardinality for a quantity, he or she is still not ready to reason with that quantity. While cross-cultural studies have suggested that number words and the subsequent exact representations of quantities are the keys to exact numerical reasoning (Pica, et al,, 2004), chapter two found that children with fluent enumeration skills were still poor at numerical reasoning. Thus, basic cardinal number knowledge is not enough for to reason about quantities. Instead, children need additional experiences with numbers, which, as chapter 3 found, occurred during kindergarten. Experiences such as number word mediated advanced cardinality instruction (chapter 4), focuses children's attention on the parts that form a whole number and the number words used to characterize these part-whole relationships. Once a child learns to verbally associate the parts that form the target quantity, he or she is prepared to reason with that quantity (chapter 4). Advanced cardinality is the final step before numerical reasoning.

Interestingly, as with basic cardinality, advanced cardinality and numerical reasoning may develop one quantity at a time beyond the number four. A child will apply advanced cardinality knowledge to reasoning with five before he or she masters and applies advanced cardinality to reasoning about six (chapter 4). Instruction must continue with larger and larger sets for children to build their numerical reasoning skills.

Implications for education

Chapter three showed how middle income children learn to reason with exact sets over the course of kindergarten. Impressively, chapter four showed that these reasoning skills can be acquired in just three days with a carefully designed intervention. By identifying and training the essential skills required for numerical reasoning, we can teach children to make great gains in a very short amount of time. Advanced cardinality is one such skill that can rapidly allow children to reason about exact sets and should be seriously considered when designing preschool and early elementary school curriculum.

The role of verbally mediated advanced cardinality follows the recent research on the importance of mapping number concepts to symbols. Recent studies have found that children's abilities to map quantities to Arabic numerals correlates with math achievement (Holloway & Ansari, 2009; Mundy & Gilmore, 2009) and may even predict mathematics learning disabilities (Rousselle & Noël, 2007). Similarly, advanced cardinality focuses on the link between number concepts (how parts form a whole) and symbols (verbal number labels) and this mapping improves children's numerical reasoning skills. Recent educational practices further support the concept of advanced cardinality. The Common Core recommends teaching number bonds in kindergarten, and some teachers and administrates have integrated Singapore math into the classroom. Both of these initiatives emphasize teaching the many ways a whole number can be broken up into parts. Chapter four provides empirical evidence that these practices, when combined with language, help children reason about quantities.

We have already made great strides in determining the cognitive foundations of precise numerical reasoning. This set of studies builds off the current findings regarding language and instruction and attempts to determine the missing link between exact numerical representations and exact numerical reasoning. Cardinality is an important step to numerical reasoning, but only after learning advanced cardinality can children accurately reasoning about quantities. Chapter four proposed one means of teaching advanced cardinality, but other instructional approaches should be evaluated to determine the most effective means for teaching young children and providing them with a strong foundation for future math learning.

<u>References:</u>

Bermejo, V. (1996). Cardinality development and counting. *Developmental Psychology*, *32*(2), 263–268.

Butterworth, B., Reeve, R., Reynolds, F., & Lloyd, D. (2008). Numerical thought with and without words: Evidence from indigenous Australian children. *Proceedings of the National Academy of Sciences*, *105*(35), 13179–13184. doi:10.1073/pnas.0806045105

Dehaene, S., Spelke, E., Pinel, P., Stanescu, R., & Tsivkin, S. (1999). Sources of mathematical thinking: Behavioral and brain-imaging evidence. *Science*, *284*(5416), 970–974.

Feigenson, L., Carey, S., & Spelke, E. (2002). Infants' Discrimination of Number vs. Continuous Extent. *Cognitive Psychology*, *44*(1), 33–66. doi:10.1006/cogp.2001.0760

Gelman, R., & Butterworth, B. (2005). Number and language: how are they related? *Trends in Cognitive Sciences*, *9*(1), 6–10. doi:10.1016/j.tics.2004.11.004

Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*(5695), 496.

Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology*, *44*(5), 1457–1465. doi:10.1037/a0012682

Le Corre, M., & Carey, S. (2007). One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles. *Cognition*, *105*(2), 395–438. doi:10.1016/j.cognition.2006.10.005

Le Corre, M., Vanderwalle, G., Brannon, E., & Carey, S. (2006). Re-visiting the competence/performance debate in the acquisition of the counting principles. *Cognitive Psychology*, *52*(2), 130–169. doi:10.1016/j.cogpsych.2005.07.002

Sarnecka, B. W., & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. *Cognition*, *108*(3), 662–674. doi:10.1016/j.cognition.2008.05.007

Sarnecka, B. W., & Lee, M. D. (2009). Journal of Experimental Child Psychology. *Journal of Experimental Child Psychology*, *103*(3), 325–337. doi:10.1016/j.jecp.2009.02.007

Starkey, P., & Cooper, R. G., Jr. (1980). Perception of numbers by human infants. *Science*, *210*(4473), 1033–1035.

Whorf, B.L. (1956). *Language, thought, and reality: Selected writings of Benjamin Lee Whorf.* New York: Wiley.

Wynn, K. (1990). Children's understanding of counting. *Cognition*, *36*(2), 155–193.

Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology*, *24*(2), 220–251.

Xu, F., & Spelke, E. S. (2000). Large number discrimination in 6-month-old infants. *Cognition*, *74*(1), B1–B11.