TOPOLOGICAL PROPERTIES OF ASYMPTOTIC CONES

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*To my wife,*

*JaNae*

*and our kids*

*Joshua, Anna, Thomas, and Rachel*

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# CHAPTER I

## INTRODUCTION

In his proof of the Polynomial Growth Theorem, Gromov associated to each group $G$, with polynomial growth, a locally compact metric space and an action of a finite index subgroup of $G$ on the space. The associated space was the Gromov-Hausdorff limit of the sequence of spaces $\{(G, \mathrm{dist}_n)\}$ where $\mathrm{dist}_n$ is a fixed word metric on $G$ rescaled by $\frac{1}{n}$ and is called on an *asymptotic cone of* $G$ [25]. However, this limit only makes sense when the sequence is uniformly locally compact, as is the case for a sequence of rescaled copies of a group with polynomial growth, and the limit is locally compact. A more general definition was given by van der Dries and Wilkie which involved the use of an ultrafilter but had the convenience of working for any sequence of metric spaces [52].

In *Asymptotic invariants of infinite groups*, Gromov [27, Section 5.F] observed a connection between the homotopic properties of the asymptotic cones of a finitely generated group and algorithmic properties of the group: if all asymptotic cones of a finitely generated group are simply connected, then the group is finitely presented, its Dehn function is bounded by a polynomial (hence its word problem is in NP) and its isodiametric function is linear. An analogous result for higher homotopy groups was proved by Riley [47]. The converse statement does not hold: there are finitely presented groups with non-simply connected asymptotic cones and polynomial Dehn functions [3], [48], and even with polynomial Dehn functions and linear isodiametric functions [39]. A partial converse statement was proved by Papasoglu [44]: a group with quadratic Dehn function has all asymptotic cones simply connected (for groups with subquadratic Dehn functions, i.e. hyperbolic groups, the statement was previously proved by Gromov [26]: all asymptotic cones in that case are $\mathbb{R}$-trees). An example of Thomas and Velickovic [51] shows that a finitely generated group can have one asymptotic cone which is a tree (and hence simply connected) while another has non-trivial $\pi_1$. Thomas and Velickovic's example can be modified to obtain a finitely generated group with one asymptotic cone which is an $\mathbb{R}$-tree and one asymptotic cone which is not locally simply connected [12]. Thus finitely generated groups can have asymptotic cones which are not locally bi-lipschitz.

Gromov asked what were the possible isomorphism types of fundamental groups for asymptotic cones of finitely generated groups [27]. In particular, he asked whether the following dichotomy is true: the fundamental group of an asymptotic cone of a finitely generated group is always either trivial or of order continuum. One motivation for this question was that asymptotic cones of nilpotent groups are simply connected (Pansu, [43]), the same is true for hyperbolic groups since all cones in that case are $\mathbb{R}$-trees, but asymptotic cones of many solvable non-nilpotent groups (say, the Baumslag-Solitar group $BS(2,1)$ or Sol) contain $\pi_1$-embedded Hawaiian earrings which seems to be a common property of many groups [7],[12].

Answering Gromov's question about fundamental groups of asymptotic cones, Erschler and Osin showed that every countable group is a subgroup of the fundamental group of an asymptotic cone of a finitely generated group [21]. Druţu and Sapir proved that, moreover, for every countable group $C$, there exists an asymptotic cone of a finitely generated group $G$ whose fundamental group is the free product of uncountably many copies of $C$ [18]. (Note that for finitely presented groups $G$, analogs of the results of Erschler-Osin and Druţu-Sapir are still unknown.)

It turned out that Gromov's dichotomy is false: there exists an asymptotic cone of a finitely generated group whose fundamental group is $\mathbb{Z}$ since the cone is homeomorphic to the direct product of a tree and a circle [38]. Cornulier and Tessera have produced additional counter examples by showing that solvable groups can have asymptotic cones with finite non-trivial fundamental groups [13].

If a group is finitely presented and one asymptotic cone is an $\mathbb{R}$-tree, then the group is hyperbolic, so all asymptotic cones are simply connected (it essentially follows from Gromov's version of the Cartan-Hadamard theorem for hyperbolic groups, see the appendix of [38]). Nevertheless in [41], a finitely presented group (a multiple HNN extension of a free group) with both simply connected and non-simply connected asymptotic cones was constructed.

## I.1 Main results and methods

A group all of whose asymptotic cones are simply connected will be called a *prairie group*. We previously noted that all asymptotic cones of hyperbolic groups and nilpotent groups are simply connected. Hence they are prairie groups. Papasoglu showed that groups with quadratic Dehn function are prairie groups [44]. A group is *constricted* if all of its asymptotic cones have (global) cut-points and *wide* if none of its asymptotic cones have cut-points. We show that for constricted groups Gromov's dichotomy does hold and that a modified version of Gromov's dichotomy holds for groups which are not wide.

**Theorem A** (Corollary III.1.23)**.** *Let G be a finitely generated group.*

*If G is constricted, then the fundamental group of an asymptotic cone of G is either trivial or contains an uncountably generated free subgroup.*

*If G is not wide, then G has an asymptotic cone whose fundamental group is trivial or contains an uncountably generated free subgroup.*

If an asymptotic cone of a group contains a cut-point, then it is tree-graded with respect to subsets called *pieces* which are maximal subsets without their own cut-points. For a definition of tree-graded spaces, see [18]. A naive attempt to prove Theorem A is to say that if the fundamental group is non-trivial then some piece contains an essential loop. However, the one-point wedge of two contractible spaces can have uncountable fundamental group (see the remark following Proposition III.2.9). Hence, it is possible that all pieces have trivial fundamental group while the asymptotic cone itself has non-trivial fundamental group.

As an initial step to circumvent this difficulty, we show the following proposition which is the main ingredient in the proof of Proposition C.

**Proposition B.** *[Proposition III.1.19] Suppose that $X$ is an unbounded homogeneous geodesic metric space and $C_i$ is a sequence of finite point sets from* $\mathrm{Con}^{\omega}(X,e,d)$. *Then* $\lim_e^{\omega} C_i$ *embeds isometrically into* $\mathrm{Con}^{\omega}(X,e,d)$.

A stronger version of this statement is proved, assuming the Continuum Hypothesis, in [42]. It is also related to work of Sisto in [49] and is the main ingredient in the proof of the following proposition.

**Proposition C.** *[Lemma III.1.20] Suppose that G is a non-virtually cyclic finitely generated group. If* $\mathrm{Con}^{\omega}(G,d)$ *has a cut-point, then every maximal transversal tree in* $\mathrm{Con}^{\omega}(G,d)$ *is a universal* $\mathbb{R}$*-tree.*

A tree $T \subset X$ is *transversal* in $X$, if the connected components of $T\backslash\{t\}$ are contained in distinct connected components of $X\backslash\{t\}$ for every $t \in T$. This maximal transversal tree is then used to find an uncountable set of points such that any two points are separated by a cut-point. Homogeneity together with this uncountable set of points, allows us to show that any essential loop has uncountable many translates such that any two are separated by a cut-point. These translates serve as generators for a subgroup of the fundamental group which is an uncountable product of cyclic groups.

In the process, we also obtained the following result which is interesting in its own right.

**Proposition D** (Proposition III.1.12)**.** *Let X be a homogeneous geodesic metric space. Every asymptotic cone of X is one-ended if and only if X is wide if and only if no asymptotic cone of X has a local cut-point.*

By extending these methods to unbounded sets, we were able to show that Gromov's dichotomy holds for HNN-extensions and amalgamated products with nicely embedded associated subgroups.

**Theorem E** (Theorem III.2.16)**.** *Suppose that G is an HNN-extension or amalgamated product where the associated subgroups are proper, quasi-isometrically embedded, prairie groups. Then every asymptotic cone of G is either simply connected or has uncountable fundamental group.*

Another weaker version of Gromov's dichotomy holds for multiple HNN extensions of free groups:

**Theorem F** (Theorem III.3.16)**.** *If G is a multiple HNN extensions of a free group, then every asymptotic cone of G is simply connected or G has an asymptotic cone with uncountable fundamental group.*

Multiple HNN extensions of free groups can have unusual asymptotic properties. Olshanskii and Sapir constructed a multiple HNN extension of a free group which has $\pi_1$-non-equivalent asymptotic cones [41] and another one which has all cones not simply connected and $n^2\log(n)$ Dehn function [39]. Conner and Kent gives prove that the cones from this last example actually have uncountable fundamental group [12]. As noted earlier, Burillo in [7] showed that Baumslag-Solitar groups can have $\pi_1$-embedded Hawaiian earring groups.

When Gromov's dichotomy was formulated, examples of groups with several non-homeomorphic (or moreover $\pi_1$-non-equivalent) asymptotic cones were not known. Now we know that a finitely generated group can have uncountably many pairwise $\pi_1$-non-equivalent asymptotic cones [18] (or much more pairwise non-homeomorphic cones, if the Continuum Hypothesis is assumed false [33]).

In [27], Gromov defined a loop division property and outlined a proof that a metric space has the loop division property if and only if all of its asymptotic cones are simply connected. Papasoglu presented a proof of the *only if* direction in [44]. Druţu gave a proof of the *if* direction in [16]. A version of the loop division property which guarantees that a particular asymptotic cone is simply connected was presented and used by Olshanskii and Sapir in [41]. We will define an analogue to Gromov's loop division property ($\varepsilon$-coarsely loop divisible) which we will use to understand the local topological structure of asymptotic cones. Coarse loop divisibly also allows us to understand some general algebraic properties of the fundamental group of an asymptotic cone.

**Theorem G.** *Let G be a finitely generated group and fix a pair* $(\omega, d)$.

*1) If G is uniformly* $\varepsilon$*-coarsely loop divisible, then* $\mathrm{Con}^{\omega}(G,d)$ *is uniformly locally simply connected and G has an asymptotic cone which is simply connected.*

*2) If* $\mathrm{Con}^{\omega}(G,d)$ *is semi-locally simply connected, then G is* $\varepsilon$*-coarsely loop divisible.*

*3) If a finitely generated G is not* $\varepsilon$*-coarsely divisible with respect to* $(\omega, d)$ *for any* $\varepsilon > 0$*, then the fundamental group of* $\mathrm{Con}^{\omega}(G,d)$ *is uncountable, not free, and not simple.*

These theorems hold for all complete homogenous geodesic metric spaces. In Section IV.1.1, we give a necessary condition for every asymptotic cone of a complete homogenous geodesic metric space to satisfy the conditions of part 3 of Theorem G. It turns out that many important groups such as $SL_3(\mathbb{Z})$ and other groups that have previously appeared in the literature related to asymptotic cones satisfy this condition, see Section IV.2.

## I.2 Further plans

Papasoglu (see Proposition IV.1.7) showed that if one requires $G$ to be uniformly $\varepsilon$-coarsely loop divisible with respect to $(\omega, d)$ for every $\varepsilon > 0$, then one obtains that $\mathrm{Con}^{\omega}(G,d)$ is actually simply connected. However; it is not clear if uniformly coarsely divisible is actually a necessary condition. Hence, the following questions are open.

Let $G$ be a finitely generated group.

**Question 1.** *If* $\mathrm{Con}^{\omega}(G,d)$ *is locally simply connected, is G uniformly* $\varepsilon$*-coarsely loop divisible?*

**Question 2.** *If* $\mathrm{Con}^{\omega}(G,d)$ *is simply connected, is G uniformly* $\varepsilon$*-coarsely loop divisible for every* $\varepsilon$*?*

Remark IV.1.11 gives examples of metric spaces which are not asymptotic cones where the answer to both of these question is no. There are no known examples of finitely generated groups which are coarsely loop divisible but not uniformly coarsely loop divisible which leaves the following question open.

**Question 3.** *Are uniformly coarsely loop divisible and coarsely loop divisible equivalent conditions for finitely generated groups?*

A positive answer to Question 3 would imply a positive answer to Question 1 and show that "locally simply connected" and "semi-locally simply connected" are equivalent properties for asymptotic cones of finitely generated groups.

Proposition IV.2.6 shows that a finitely generated group can have cones which are not locally isometric.

**Question 4.** *Can a finitely presented group have non-locally isometric asymptotic cones?*

Riley showed that if all the asymptotic cones of a finitely generated group $G$ are $n$-connected, then $G$ has an EilenbergMacLane space with finite $k$-skeleton ($G$ is of type $\mathscr{F}_{n+1}$) [47]. His proof uses an analogue to Gromov's loop division property for higher order spheres. Brady, Bridson, Forester and Shankar have studied possible higher order Dehn functions of the from $x^{\alpha}$ [1], [2].

**Question 5.** *Do results analogous to Theorem G hold for higher homotopy groups?*

**Question 6.** *Does Gromov's dichotomy hold for the higher homotopy groups, i.e. is* $\pi_n(\mathrm{Con}^{\omega}(G,d))$ *always trivial or uncountable for finitely generated groups* $G$*?*

# CHAPTER II

# PRELIMINARIES

## II.1 Conventions

When $\tau$ is a path in a metric space, we will use $|\tau|$ to denote its arc length. Then $|\cdot|$ maps the set of paths into the extended real line and is finite for rectifiable paths and $+\infty$ for non-rectifiable paths. We will assume that rectifiable paths are parameterized proportional to arc length.

## II.2 Geometric structure of groups

**Definition II.2.1** (Group presentation)**.** A group $G$ is *generated* by a subset $S$, if each element of $G$ is equal to a finite product of elements from $S \cup S^{-1}$ where $S^{-1} = \{s^{-1} \mid s \in S\}$. We will write $G = \langle S \rangle$. Let $u, v$ be two words in the alphabet $S \cup S^{-1}$. We will write $u \equiv v$ when $u$ and $v$ coincide letter by letter and $u =_G v$ if $u$ and $v$ are equal in $G$.

We will say that $\langle S \,|\, R \rangle$ is a presentation for $G$ if $G$ is generated by $S$ and whenever $u =_G v$ then $uv^{-1}$ is in the normal closure of $R$.

**Definition II.2.2** (Geometry of a group)**.** Given a presentation $\langle S \,|\, R \rangle$ of $G$ we can define a metric on $G$. For each element of $G$, let

$$|g|_S = \min\big\{k \mid g =_G s_{i_1}^{\varepsilon_1} \cdots s_{i_k}^{\varepsilon_k} \text{ where } \varepsilon_i \in \{-1,1\} \text{ and } s_{i_j} \in S\big\}.$$

Let $\mathrm{dist}_s(g,h) = |g^{-1}h|_S$. One can easily check that this defines a metric on $G$ which we will refer to as a *word metric* on $G$.

**Remark II.2.3.** This metric clearly depends on $S$. For example: If $G = \mathbb{Z}$ and $S = \{1\}$, then $\mathrm{dist}_S$ is the Euclidean metric on $\mathbb{Z}$ and if $S = \mathbb{Z}$ then $d_S(g,h)$ either 0 or 1. However, for finitely generated groups word metrics corresponding to finite generating sets are invariant in the following sense.

**Definition II.2.4.** A function $f : (X, \mathrm{dist}_X) \to (Y, \mathrm{dist}_Y)$ between metric spaces is a quasi-isometry if there exists $(\lambda, C)$ such that

1) $\frac{1}{\lambda}\,\mathrm{dist}_X(x,x') - C \leq \mathrm{dist}_Y\big(f(x), f(x')\big) \leq \lambda\,\mathrm{dist}_X(x,x') + C$ and

2) for every $y \in Y$ there exists and $x \in X$ such that $\mathrm{dist}_y\big(y, f(x)\big) \leq C$

**Lemma II.2.5.** *If $S$ and $T$ are finite generating sets for a group $G$, then $(G, \mathrm{dist}_S)$ is quasi-isometric to $(G, \mathrm{dist}_T)$.*

*Proof.* Let $\lambda = \max\big\{|s|_T, |t|_S \mid s \in S, t \in T\big\}$. Let $f : (G, \mathrm{dist}_S) \to (G, \mathrm{dist}_T)$ be the identity map and fix $g, h \in G$. Choose $s_{i_1}^{\varepsilon_1}, \cdots, s_{i_k}^{\varepsilon_k} \in S \cup S^{-1}$ such that $g^{-1}h =_G s_{i_1}^{\varepsilon_1} \cdots s_{i_k}^{\varepsilon_k}$ and $|g^{-1}h|_S = k = \mathrm{dist}_S(g,h)$. Similarly, choose $t_{i_1}^{\delta_1}, \cdots, t_{i_l}^{\delta_l} \in T \cup T^{-1}$ such that $g^{-1}h =_G t_{i_1}^{\delta_1} \cdots t_{i_l}^{\delta_l}$ and $|g^{-1}h|_T = l = \mathrm{dist}_T(g,h)$.

Then $u_{i_1}^{\varepsilon_1}\cdots u_{i_k}^{\varepsilon_k} =_G g^{-1}h =_G v_{i_1}^{\delta_1}\cdots v_{i_l}^{\delta_k}$ where $u_{i_j}$ is a word in $T\cup T^{-1}$ and $v_{i_j}$ is a word in $S\cup S^{-1}$ such that $u_{i_j} =_G s_{i_j}, v_{i_j} =_G t_{i_j}$ and $|u_{i_j}|_T, |v_{i_j}|_S \leq \lambda$. Hence $k \leq \lambda l$ and $l \leq \lambda k$ which completes the proof with $C = 0$.

□

**Definition II.2.6** (Cayley Graph and Cayley Complex)**.** Given a group $G$ with presentation $\langle S \,|\, R\rangle$, the *presentation complex* $K(S,R)$ is the CW complex with a single vertex, a 1-cell for each element of $S$, and 2-cells corresponding to elements of $R$ such that $\pi_1\big(K(S,R)\big) = G$. In general, we will consider $K(S,R)$ as a labeled CW complex by labeling each 1-cell of $K(S,R)$ with a letter from $S$ such that the attaching map of the 2-cell corresponding to an element $r \in R$ is the edge path determined by the spelling of $r$. Let $\tilde{K}(S,R)$ be the universal cover of $K(S,R)$.

We will give an explicit construction for $\tilde{K}(S,R)$. Let $\tilde{K}^{(1)}(S,R)$ be considered as a oriented graph with vertices labeled by elements of $G$ and oriented edges labeled by pairs $(g,s)$ where $g \in G$ and $s \in S$ with the convention that the edge $(g,s)$ has initial vertex $g$ and terminal vertex $gs$. The inverse edge for $(g,s)$ is formally $(gs,s^{-1})$. Since $\tilde{K}^{(1)}(S,R)$ doesn't depend on $R$ and to maintain standard notation, we will generally denote $\tilde{K}^{(1)}(S,R)$ by $\Gamma(G,S)$ and call it the *Cayley graph* of $G$ (with respect to the generating set $S$). We will endow $\Gamma(G,S)$ with the edge metric. Notice this metric restricted to the vertex set, which is labeled by $G$, agrees with the word metric $\mathrm{dist}_S$ on $G$. Since $\mathrm{dist}_S$ was left-invariant, $G$ acts isometrically on $\Gamma(G,S)$ by $h\cdot x = hx$ for $x$ a vertex of $\Gamma(G,S)$ and $h\cdot(x,s) = (hx,s)$ for an edge $(x,s)$.

To complete the construction of $\tilde{K}(S,R)$, we attach a 2-cell $D_{g,r}$ for each $g \in G$ and $r \in R$ via the edge path $(g_1,s_1^{\varepsilon_1})(g_2,s_2^{\varepsilon_2})\cdots(g_n,s_n^{\varepsilon_n})$ where $r \equiv s_1^{\varepsilon_1}s_2^{\varepsilon_2}\cdots s_n^{\varepsilon_n}$ for $\varepsilon_i = \pm 1$, $g_1 = g$, and $g_{i+1} = g_i s_i^{\varepsilon_i}$ for $i = 1,\cdots,n-1$. The Cayley complex $\Gamma^2(G,S,R)$ is the CW complex obtained from $\tilde{K}(S,R)$ by identifying all faces which correspond to the pairs $\{g,r\},\{gs,r\},\cdots\{gs^{m-1},r\}$ where $r \equiv s^m$ and $s$ is not a proper power in the free group on $S$.

**Definition II.2.7** (van Kampen diagrams)**.** Let $\Delta$ be a finite, oriented, connected, simply connected, planar 2-complex endowed with a labeling function $\mathbf{Lab}: E(\Delta) \to S\cup S^{-1}$ where $E(\Delta)$ denotes the set of oriented edges of $\Delta$, such that $\mathbf{Lab}\,(e^{-1}) = \mathbf{Lab}\,(e)^{-1}$.

Given a 2-cell $\pi$ of a diagram $\Delta$, we denote by $\partial\pi$, $\partial\Delta$ the boundary of $\pi$, $\Delta$ respectively. The labels of $\partial\pi$,$\partial\Delta$ are defined up to a cyclic permutation by traversing the loops $\partial\pi$,$\partial\Delta$ in the clockwise direction and will be denoted by $\mathbf{Lab}\,(\pi)$, $\mathbf{Lab}\,(\Delta)$ respectively. We will say that $\Delta$ is a *van Kampen diagram over a presentation* $\langle S \,|\, R\rangle$; if every 2-cell in $\Delta$ is labeled by a cyclic permutation of $r^{\pm 1}$ for some $r \in R$. Unless otherwise noted, we will always consider van Kampen diagrams as metric spaces with the edge metric.

In general, we will use **Lab** to represent the function from the set of edge paths in a labeled oriented CW complex to the set of words in the alphabet obtained by reading the label of a path.

At times it will be convenient to refer to diagrams on surfaces possible with boundary which are not planar or not simply connected. A diagram is *circular* if the underlying complex is homeomorphic to a planar disk, *spherical* if the underlying complex is homeomorphic to a 2-sphere, and *annular* if the underlying complex is homeomorphic to a planar annulus.

We will use $\theta$ to denote the canonical map taking a van Kampen diagram into the Cayley complex which restricts to a label preserving map on the 1-skeleton of the diagram. Explicitly, let $\Delta$ be a van Kampen diagram with a distinguished vertex $o$ and $g_o$ a vertex of $\Gamma(G,S)$. For $v$ a vertex of $\Delta$, let $\theta(v) = g_0 w_v$ where $w_v$ is the label of any path in $\Delta$ from $o$ to $v$. Whenever $\Delta$ is a simply connected diagram, this map is independent of the choice of $w_v$ and extends to a map on all of $\Delta$ as follows. For $e$ an edge of $\Delta$ labeled by $s$ with initial vertex $v$, let $\theta(e) = (\theta(v), s)$ where $\big(\theta(v), s\big)$ is the edge in $\Gamma(G,S)$ with initial vertex $\theta(v)$ and labeled by $s$. For $\pi$ a 2-cell of $\Delta$, we may choose a vertex $v$ on $\partial\pi$ such that $\mathbf{Lab}\,(\pi) \equiv r^{\pm 1}$ with this choose of base point. Then $\theta(\pi) = D_{\theta(v), r^{\pm 1}}$ where $D_{\theta(v), r^{\pm 1}}$ is the two cell in $\Gamma^2(G,S)$ with boundary, read from $\theta(v)$, labeled by $r^{\pm 1}$. The map $\theta$ is unique up to our choice of $g_0$ and $o$.

**Definition II.2.8** (Isoperimetric functions)**.** Suppose that $\langle S \mid R \rangle$ is a finite presentation for a group $G$. Let $\mathrm{Area}(\Delta)$ denote the number of $R$-cells in a van Kampen diagram $\Delta$. If $w$ is a word in $S \cup S^{-1}$, then $\mathrm{Area}(w) = \min\{\mathrm{Area}(\Delta) \mid \mathbf{Lab}\,(\partial\Delta) \equiv w\}$. If $\gamma$ is a loop in $\Gamma(G,S)$, then $\mathrm{Area}(\gamma) = \mathrm{Area}(\mathbf{Lab}\,(\gamma))$.

An *isoperimetric function* for the presentation $\langle S \mid R \rangle$ of $G$ is a non-decreasing function $\delta : \mathbb{N} \to [0,\infty)$ such that $\delta(|\partial\Delta|) \geq \mathrm{Area}(\mathbf{Lab}\,(\partial\Delta))$ for all van Kampen diagrams $\Delta$ over $\langle S \mid R \rangle$.

Two non-decreasing functions $f, g : \mathbb{N} \to [0,\infty)$ are *equivalent*, if there exists constants $B, C > 0$ such that $f(n) \leq Bg(Bn+B) + Bn + B$ and $g(n) \leq Cf(Cn+C) + Cn + C$.

Up to this equivalence, the Dehn function of a finitely presented group $G$ is independent of the finite presentation. Hence, we will call a minimal isoperimetric function for a presentation $\langle S \mid R \rangle$ of $G$ a *Dehn function for G*.

**Definition II.2.9** (Ultrafilters)**.** Let $I$ be a set. We will use $\mathscr{P}(I)$ to denote the power set of $I$. An *ultrafilter on I* is a non-trivial finitely additive probability measure $\omega$ defined on $\mathscr{P}(I)$ which takes values in $\{0,1\}$, i.e.

1. $\omega(A) \in \{0,1\}$ for $A \in \mathscr{P}(I)$
2. $\omega(X) = 1$, and
3. $\omega\big(\sqcup_{i=1}^{k} A_i\big) = \sum_{i=1}^{k} \omega(A_i)$ for pairwise disjoint $A_i$ in $\mathscr{P}(I)$.

An ultrafiler is *principal* if $\omega\big(\{i\}\big) = 1$ for any $i \in I$. A ultrafilter is *non-principal* if $\omega\big(\{i\}\big) = 0$ for all $i \in I$.

We will only consider non-principal ultrafilters on $\mathbb{N}$ and generally refer to them as simply ultrafilters.

**Definition II.2.10** (Ultralimits of real numbers)**.** Let $\omega$ be an ultrafilter on $\mathbb{N}$ and $c_n$ be a sequence of real numbers. The sequence $c_n$ is *bounded $\omega$-almost surely* or *$\omega$-bounded*, if there exists a number $M$ such that $\omega\big(\{n \mid |c_n| < M\}\big) = 1$.

**Lemma II.2.11.** *If $c_n$ is $\omega$-bounded, then there exists a unique number, which we will denote by* $\lim^{\omega} c_n$*, such that $\omega\big(\{n \mid |c_n - lim^{\omega} c_n| < \varepsilon\}\big) = 1$ for every $\varepsilon > 0$.*

*Proof.* Choose $M \in \mathbb{N}$ such that $\omega\big(\{n \mid |c_n| < M\}\big) = 1$.

Let $\mathscr{A}_i = \Big\{ \big(\frac{k}{i}, \frac{k+1}{i}\big] \mid k \in \{-Mi, -Mi+1, \cdots, Mi-1\} \Big\}$. Since $\mathscr{A}_i$ is a finite collection of disjoint sets; for each $i$, there exists an interval $A_i \in \mathscr{A}_i$ such that $\omega\big(\{n \mid c_n \in A_i\}\big) = 1$.

Let $\overline{A}_i$ be the topological closure of $A_i$. Notice $\omega\big(\bigcap_{i \in J} \{n \mid c_n \in \overline{A}_i\}\big) = 1$ for any finite set $J$. Thus $\bigcap_{i \in J} \overline{A}_i$ is non-empty for any finite set $J$ which implies, since each $\overline{A}_i$ is compact, that $\bigcap_{i=1}^{\infty} \overline{A}_i$ is non-empty. Since $\operatorname{diam}(\overline{A}_i)$ converges to 0, $\bigcap_{i=1}^{\infty} \overline{A}_i = c$ for a unique $c \in [-M, M]$ and $\omega\big(\{n \mid |c_n - c| < \varepsilon\}\big) = 1$ for every $\varepsilon > 0$. $\square$

If $c_n$ is a sequence numbers which is not $\omega$-bounded, then $\omega\big(\{n \mid |c_n| > M\}\big) = 1$ for every $M$. If $\omega\big(\{n \mid c_n > M\}\big) = 1$, we will say that $c_n$ *diverges $\omega$-almost surely* or is *$\omega$-divergent* and let $\lim^{\omega} c_n = \infty$.

**Lemma II.2.12.** *Ultralimits satisfy the same properties as standard limits, i.e. if $lim^{\omega} c_n, lim^{\omega} b_n$ both exist then*

1. $lim^{\omega}\big(c_n \pm b_n\big) = lim^{\omega} c_n \pm lim^{\omega} b_n$;
2. $lim^{\omega}\big(c_n b_n\big) = lim^{\omega} c_n lim^{\omega} b_n$;
3. $lim^{\omega}\big(c_n / b_n\big) = lim^{\omega} c_n / lim^{\omega} b_n$, *if* $lim^{\omega} b_n \neq 0$; *and*
4. $lim^{\omega}\big(c c_n\big) = c\, lim^{\omega} c_n$, *for* $c \in \mathbb{R}$.

The proof are exactly that same as for standard limits.

**Definition II.2.13** (Asymptotic cones)**.** Let $(X_n, \operatorname{dist}_n)$ be a sequence of metric spaces and $\omega$ an ultrafilter on $\mathbb{N}$. Consider a sequence of points $e = (e_n)$ such that $e_n \in X_n$ called an *observation sequence*.

Given two elements $\tilde{x} = (x_n), \tilde{y} = (y_n) \in \prod X_n$, set $\operatorname{dist}(\tilde{x}, \tilde{y}) = \lim^{\omega} \operatorname{dist}_n(x_n, y_n)$. We will say $\tilde{x} \sim \tilde{y}$, if $\operatorname{dist}(\tilde{x}, \tilde{y}) = 0$. This defines an equivalence relation on $\prod X_n$.

The *ultralimit* of $X_n$ relative to the observation sequence $e$ is

$$\lim_e^{\omega} X_n = \Big\{ \tilde{x} = (x_n) \in \prod X_n \mid \ \operatorname{dist}(\tilde{x}, e) < \infty \Big\} / \sim .$$

Now consider an $\omega$-divergent sequence of numbers $d = (d_n)$ called a *scaling sequence* and a metric space $(X, \operatorname{dist})$.

The asymptotic cone of $X$ with respect to $e$, $d$, and $\omega$ is

$$\operatorname{Con}^{\omega}\big(X, e, d\big) = \lim_e^{\omega}\big(X, \operatorname{dist}/d_n\big)$$

where $\operatorname{dist}/d_n$ is the metric on $X$ scaled by $\frac{1}{d_n}$.

Suppose that $\{X_n\}$ is a sequence of subsets of a metric space $(X, \operatorname{dist})$. At times it will be convenient to talk about the subset of $\operatorname{Con}^{\omega}\big(X, e, d\big)$ with representatives in $\prod X_n$. When it is clear from the text, we will denote this subset by $\lim^{\omega} X_n$ instead of, the more precise, $\lim_e^{\omega}(X_n, \operatorname{dist}/d_n)$. When used in this context, we will not require that $e_n$ be an element of $X_n$.

**Observation II.2.14.** $\mathrm{Con}^{\omega}\big(\mathbb{Z},(1),(d_n)\big)=\mathbb{R}$

*Proof.* Fix an ultrafilter $\omega$ and a scaling sequence $d=(d_n)$. Define a map $\varphi:\mathbb{R}\to\mathrm{Con}^{\omega}\big(\mathbb{Z},(1),(d_n)\big)$ by $\varphi(x)=(xd_n)$. Notice that $\varphi$ is well-defined since $\mathrm{dist}\big((0),\varphi(x)\big)=\lim{}^{\omega}\dfrac{|0-xd_n|}{d_n}=|x|<\infty$. Then

$$\mathrm{dist}\big(\varphi(x),\varphi(y)\big)=\lim{}^{\omega}\frac{|xd_n-yd_n|}{d_n}=|x-y|.$$

Hence $\varphi$ is an isometric embedding of $\mathbb{R}$ into $\mathrm{Con}^{\omega}\big(\mathbb{Z},(1),(d_n)\big)$.

It only remains to show that $\varphi$ is surjective. Suppose that $\tilde{x}=(x_n)\in\mathrm{Con}^{\omega}\big(\mathbb{Z},(1),(d_n)\big)$. Then $\left\{\dfrac{x_n}{d_n}\right\}$ is $\omega$-bounded. Thus there exist $x\in\mathbb{R}$ such that $\lim^{\omega}\dfrac{x_n}{d_n}=x$. Hence $\mathrm{dist}\big(\tilde{x},\varphi(x)\big)=\lim^{\omega}\dfrac{|xd_n-x_n|}{d_n}=0$ which completes the proof. □

**Lemma II.2.15.** $\mathrm{Con}^{\omega}\big(X,e,d\big)$ *is a complete metric space. If* $X$ *is geodesic, then* $\mathrm{Con}^{\omega}\big(X,e,d\big)$ *is also geodesic.*

*Proof.* Let $\tilde{x}_i=(x_n^i)$ be a Cauchy sequence in $\mathrm{Con}^{\omega}\big(X,e,d\big)$ and

$$C_i=\Big\{n\mid\Big|\frac{\mathrm{dist}(x_n^j,x_n^i)}{d_n}-\mathrm{dist}(\tilde{x}_j,\tilde{x}_i)\Big|\le\frac{1}{2^i}\text{ for all }j\le i\Big\}.$$

Then $C_i$ is $\omega$-large. Let $D_n=\{i\mid n\in C_i\text{ and }i\le n\}$ and $m_n=\max D_n$, if $D_n$ is non-empty and $m_n=1$ otherwise. Notice that $\lim^{\omega}m_n=\infty$ (for the details, see the corresponding claim in the proof of Proposition III.1.2).

Let $B_k=\{n\mid m_n>k\}$ which is $\omega$-large. Notice that by construction $n\in C_{m_n}$.

**Claim 1.** $\tilde{y}=(x_n^{m_n})$ *is a well-defined point in* $\mathrm{Con}^{\omega}\big(X,e,d\big)$ *and* $\tilde{x}_i$ *converges to* $\tilde{y}$

*Proof.* Since $\tilde{x}_i$ is a cauchy sequence, there exists a $C$ such that $\mathrm{dist}(e,\tilde{x}_i),\mathrm{dist}(\tilde{x}_1,\tilde{x}_i)<C$ for all $i$. Then for $n\in B_1$,

$$\frac{\mathrm{dist}(x_n^{m_n},e_n)}{d_n}\le\frac{\mathrm{dist}(x_n^{m_n},x_n^1)}{d_n}+\frac{\mathrm{dist}(x_n^1,e_n)}{d_n}\le\mathrm{dist}(\tilde{x}_{m_n},\tilde{x}_1)+\frac{1}{2^{m_n}}+\frac{\mathrm{dist}(x_n^1,e_n)}{d_n}.$$

Taking the $\omega$-limit of both sides, we see that $\mathrm{dist}(e,\tilde{y})\le 2C$.

Fix $\varepsilon>0$ and $N$ such that $\mathrm{dist}(\tilde{x}_i,\tilde{x}_j)\le\dfrac{\varepsilon}{2}$ for $i,j\ge N$. For $j\ge N$ and $n\in B_j\cap\big\{N,N+1,\cdots\big\}\cap\big\{n\mid\frac{1}{2^{m_n}}\le\frac{\varepsilon}{2}\big\}$,

$$\frac{\mathrm{dist}(x_n^{m_n},x_n^j)}{d_n}\le\mathrm{dist}(\tilde{x}_{m_n},\tilde{x}_j)+\frac{1}{2^{m_n}}\le\varepsilon.$$

Since $B_j\cap\big\{N,N+1,\cdots\big\}\cap\big\{n\mid\frac{1}{2^{m_n}}\le\frac{\varepsilon}{2}\big\}$ is $\omega$-large, taking the $\omega$-limit we obtain $\mathrm{dist}(\tilde{y},\tilde{x}_j)\le\varepsilon$ for all $j\ge N$.

□

This finishes the proof that $\mathrm{Con}^{\omega}\big(X,e,d\big)$ is complete.

Suppose $\tilde{x}, \tilde{y} \in \mathrm{Con}^{\omega}\big(X, e, d\big)$. Choose $x_n, y_n \in X$ such that $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$. Let $l_n = \dfrac{\mathrm{dist}(x_n, y_n)}{d_n}$ and $\gamma_n : [0, l_n d_n] \to X$ be a geodesic from $x_n$ to $y_n$. Then $\lim^{\omega} l_n = l = \mathrm{dist}(\tilde{x}, \tilde{y})$. Let $l'_n = \max\{l, l_n\}$ and define $\gamma_n : [0, l'_n d_n] \to X$ by $\gamma'_n(t) = \gamma_n(t)$ for $t \leq l_n$ and $\gamma'_n(td_n) = \gamma_n(l_n d_n)$. Then $l'_n$ still $\omega$-converges to $l$.

Define $\gamma_n : [0, l] \to \mathrm{Con}^{\omega}\big(X, e, d\big)$ by $\gamma(t) = \big(\gamma'_n(td_n)\big)$.

Suppose that $t < l$. Then on an $\omega$-large set $td_n < l_n d_n$ which implies that $\gamma'_n(td_n) = \gamma_n(td_n)$. If $0 \leq s < t < 1$, then

$$\mathrm{dist}\big(\gamma(s), \gamma(t)\big) = \lim{}^{\omega} \frac{\mathrm{dist}\big(\gamma'_n(sd_n), \gamma'_n(td_n)\big)}{d_n} = \lim{}^{\omega} \frac{\mathrm{dist}\big(\gamma_n(sd_n), \gamma_n(td_n)\big)}{d_n} = \lim{}^{\omega} \frac{|sd_n - td_n|}{d_n} = |s - t|.$$

If $0 \leq s < l$, then

$$\begin{aligned}
\mathrm{dist}\big(\gamma(s), \gamma(l)\big) &= \lim{}^{\omega} \frac{\mathrm{dist}\big(\gamma'_n(sd_n), \gamma'_n(ld_n)\big)}{d_n} \\
&= \begin{cases} \lim^{\omega} \dfrac{\mathrm{dist}\big(\gamma_n(sd_n), \gamma_n(ld_n)\big)}{d_n} & \text{if } l_n > l \ \omega - almostsurely \\ \lim^{\omega} \dfrac{\mathrm{dist}\big(\gamma_n(sd_n), \gamma_n(l_n d_n)\big)}{d_n} & \text{if } l_n \leq l \ \omega - almostsurely \end{cases} \\
&= \begin{cases} \lim^{\omega} \dfrac{|sd_n - ld_n|}{d_n} & \text{if } l_n > l \ \omega - almostsurely \\ \lim^{\omega} \dfrac{|sd_n - l_n d_n|}{d_n} & \text{if } l_n \leq l \ \omega - almostsurely \end{cases} \\
&= |s - l|.
\end{aligned}$$

Thus $\gamma$ is a geodesic. Notice that $\mathrm{dist}(\gamma_n(l_n d_n), \gamma'_n(ld_n) \leq |l_n - l| d_n$. Hence $\gamma(l) = \tilde{y}$ and $\gamma$ is a geodesic from $\tilde{x}$ to $\tilde{y}$.

$\square$

**Lemma II.2.16.** *If $X$ is a homogeneous metric space, then the isometry type of* $\mathrm{Con}^{\omega}\big(X, e, d\big)$ *is independent of $e$.*

*Proof.* Fix two sequence $e_n, e'_n$ in $X$. Since $X$ is homogeneous there exists an isometry $\varphi_n : X \to X$ such that $\varphi_n(e_n) = e'_n$.

Let $\varphi : \mathrm{Con}^{\omega}\big(X, e, d\big) \to \mathrm{Con}^{\omega}\big(X, e, (d_n)\big)$ by $\varphi\big((x_n)\big) = \big(\varphi_n(x_n)\big)$. Then for any pair of points $x, y$, we have $\mathrm{dist}\big(\varphi_n(x), \varphi_n(y)\big) = \mathrm{dist}(x, y)$. This implies that $\varphi$ is well-defined surjective isometry

$\square$

Since the cone is independent of the observation sequence for homogenous spaces, we will frequently denote the asymptotic cone simply by $\mathrm{Con}^{\omega}\big(X, d\big)$ when $X$ is homogenous.

CHAPTER III

# GROMOV'S DICHOTOMY

## III.1 Wide groups and ends of asymptotic cones

The proof of the following lemma is the same as the proof that an asymptotic cone of a geodesic metric space is geodesic.

**Lemma III.1.1.** *Let $\omega$ be an ultrafilter on $\mathbb{N}$ and $d = (d_n)$ a scaling sequence. Suppose that $\{\gamma_n\}$ is a sequence of loops parameterized by arc length in a geodesic metric space $(X, \mathrm{dist})$ such that $|\gamma_n| = O(d_n)$. Then $\gamma(t) = \big(\gamma_n(t)\big)$ is a continuous map of $S^1$ into $\mathrm{Con}^\omega\big(X,e,d\big)$.*

The converse also holds.

**Proposition III.1.2.** *Let $X$ be a geodesic metric space. For every path $\gamma$ in $\mathrm{Con}^\omega\big(X,e,d\big)$, there exist paths $\gamma_n$ in $X$ such that $\gamma(t) = \big(\gamma_n(t)\big)$.*

Recall that there exists geodesics in a cone which are not limits of geodesics. However, here we do not put any restraints on the paths $\gamma_n$ (the proof shows that $\gamma_n$ can be chosen to be a $2^{m_n}$-gon where $m_n$ is an $\omega$-divergent sequence).

*Proof.* Suppose that $\gamma\colon [0,1] \to \mathrm{Con}^\omega\big(X,e,d\big)$ is a path. Let $\xi$ be a modulus of continuity for $\gamma$ (see Definition III.2.6).

For each diadic rational $r$, fix a representative $\big(a_n(r)\big)$ of $\gamma(r)$. Let $A_i = \Big\{0, \frac{1}{2^i}, \cdots, \frac{2^i-1}{2^i}, 1\Big\}$ and

$$C_i = \Big\{ n \mid \mathrm{dist}\big(\gamma(r),\gamma(s)\big) - \frac{1}{i} \le \frac{\mathrm{dist}\big(a_n(r),a_n(s)\big)}{d_n} \le \mathrm{dist}\big(\gamma(r),\gamma(s)\big) + \frac{1}{i} \text{ for all } s,r \in A_i \Big\}.$$

Then $C_i$ is $\omega$-large, since $|A_i|$ is finite and $\big(a_n(r)\big)$ is a representative of $\gamma(r)$. As well, $C_i \subset C_{i-1} \subset \cdots \subset C_1$ is nested.

Let $D_n = \{i \mid n \in C_i \text{ and } i \le n\}$ and $m_n = \max D_n$, if $D_n$ is non-empty and $m_n = 1$ otherwise.

**Claim.** $\lim^\omega m_n = \infty$.

*Proof of Claim.* Suppose that $m_n$ was bounded by $L$ on some $\omega$-large set $C$. Fix $n_0 \in C \cap \{2L, 2L+1, 2L+2, \cdots\} \cap C_{2L}$ (the intersection is non-empty since all three are $\omega$-large). Then $n_0 \in C_{2L}$ and $2L \le n_0$. This implies that $2L \in D_{n_0}$. Hence $m_{n_0} \ge 2L$ which contradicts our assumption that $m_n$ was bounded by $L$ on $C$ since $n_0 \in C$.

By the above argument $\omega\big(\{n \mid m_n \le L\}\big) = 0$. Hence $\omega\big(\{n \mid m_n > L\}\big) = 1$. Since this holds for every $L$, the claim is proved. □

Define $\gamma_n\colon [0,1] \to X$ by

$$\gamma_n(r) = a_n(r) \text{ for } r \in A_{m_n}$$

and extend $\gamma_n$ geodesically.

We can then define $\gamma'(t) = \big(\gamma_n(t)\big)$.

Let $B_k = \{n \mid m_n > k\}$ which is $\omega$-large by the claim. Fix $t_0$ in the diadic rationales and $n_0$ such that $t_0 \in A_{n_0}$. Then for all $n \in B_{n_0}$, $t_0 \in A_{m_n}$. This implies that $\gamma_n(t_0) = a_n(t_0)$ for $n \in B_{n_0}$. Since $B_{n_0}$ is $\omega$-large, $\gamma'(t_0) = \big(\gamma_n(t_0)\big) = \big(a_n(t_0)\big) = \gamma(t_0)$. Hence $\gamma'(t) = \gamma(t)$ on the diadic rationales.

Notice by our choice of $m_n$, we have $n \in C_{m_n}$ if $m_n \neq 1$. Thus for $n \in B_1$ and $r,s \in A_{m_n}$, we have

$$\operatorname{dist}\big(\gamma(r),\gamma(s)\big) - \frac{1}{m_n} \leq \frac{\operatorname{dist}\big(\gamma_n(r),\gamma_n(s)\big)}{d_n} \leq \operatorname{dist}\big(\gamma(r),\gamma(s)\big) + \frac{1}{m_n}.$$

Fix $x,y \in [0,1]$. Choose $r_x^1, r_x^2, r_y^1, r_y^2 \in A_{m_n}$ such that $x \in [r_x^1, r_x^2]$, $y \in [r_y^1, r_y^2]$ and $d(r_x^1, r_x^2) = d(r_y^1, r_y^2) = \frac{1}{2^{m_n}}$. Then for $n \in B_1$, $\gamma_n\big|_{[r_x^1,r_x^2]}$, $\gamma_n\big|_{[r_y^1,r_y^2]}$ are geodesics of length at most $d_n\left[\xi\left(\frac{1}{2^{m_n}}\right) + \frac{1}{m_n}\right]$. Thus

$$\operatorname{dist}\big(\gamma_n(x),\gamma_n(r_x^1)\big), \operatorname{dist}\big(\gamma_n(r_y^1),\gamma_n(y)\big) \leq d_n\left[\xi\left(\frac{1}{2^{m_n}}\right) + \frac{1}{m_n}\right].$$

Then combining this with the triangle inequality, we obtain

$$\begin{aligned}
\operatorname{dist}\big(\gamma_n(x),\gamma_n(y)\big) &\leq \operatorname{dist}\big(\gamma_n(x),\gamma_n(r_x^1)\big) + \operatorname{dist}\big(\gamma_n(r_x^1),\gamma_n(r_y^1)\big) + \operatorname{dist}\big(\gamma_n(r_y^1),\gamma_n(y)\big) \\
&\leq d_n\left[\xi\left(\frac{1}{2^{m_n}}\right) + \frac{1}{m_n}\right] + d_n\left[\xi\left(\operatorname{dist}(r_x^1,r_y^1)\right) + \frac{1}{m_n}\right] + d_n\left[\xi\left(\frac{1}{2^{m_n}}\right) + \frac{1}{m_n}\right] \\
&= 2d_n\xi\left(\frac{1}{2^{m_n}}\right) + \frac{3d_n}{m_n} + d_n\xi\left(\operatorname{dist}(x,y) + \frac{2}{2^{m_n}}\right)
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{dist}\big(\gamma'(x),\gamma'(y)\big) &\leq \lim_n{}^{\omega} \frac{\operatorname{dist}\big(\gamma_n(x),\gamma_n(y)\big)}{d_n} \\
&\leq \lim_n{}^{\omega} 2\xi\left(\frac{1}{2^{m_n}}\right) + \frac{3}{m_n} + \xi\left(d(x,y) + \frac{2}{2^{m_n}}\right) \\
&= \xi\Big(d(x,y)\Big)
\end{aligned}$$

which implies that $\gamma'$ is continuous. (Note that we used that fact that $\xi$ was a continuous modulus of continuity.) Hence $\gamma(t) = \gamma'(t)$ for all $t$.

$\square$

**Proposition III.1.3.** *Let $X$ be a simply connected geodesic metric space which has a linear isodiametric function. Suppose that $h\colon \mathbb{D} \to \operatorname{Con}^{\omega}\big(X,e,d\big)$ is a continuous map of the unit disc. Then there exist continuous maps $h_n\colon \mathbb{D} \to X$ such that $\gamma(t) = \big(\gamma_n(t)\big)$.*

The proof is very similar to that of Proposition III.1.2. However, we present it here to illustrate how to modify the proof of Proposition III.1.2 for discs of higher dimension.

*Proof.* For simplicity of notation, we will assume $\mathbb{D} = [0,1]^2$. Fix $M$ such that every loop of length at most $n$ bounds a disc of diameter at most $Mn$.

Suppose that $h\colon [0,1]^2 \to \mathrm{Con}^{\omega}\big(X,e,d\big)$ is a continuous map and let $\xi$ be a modulus of continuity for $h$.

For each diadic rational pair $\vec{r} = (r,s)$, fix a representative $\big(a_n(\vec{r})\big)$ of $h(\vec{r})$. Let

$$A_i = \Big\{\vec{r} = (r,s) \mid r,s \in \{0, \frac{1}{2^i}, \cdots, \frac{2^i - 1}{2^i}, 1\}\Big\}.$$

We can consider $A_i$ as the vertices of a cellular decomposition of $\mathbb{D}$ into squares with side length $\frac{1}{2^i}$ and denote the $j$-skeleton of this decomposition by $A_i^{(j)}$ for $j = 1,2$.

$$C_i = \Big\{n \;\Big|\; \mathrm{dist}\big(h(\vec{r}),h(\vec{s})\big) - \frac{1}{i} \le \frac{\mathrm{dist}\big(a_n(\vec{r}),a_n(\vec{s})\big)}{d_n} \le \mathrm{dist}\big(h(\vec{r}),h(\vec{s})\big) + \frac{1}{i} \text{ for all } \vec{r},\vec{s} \in A_i\Big\}.$$

Then $C_i$ is $\omega$-large, since $|A_i|$ is finite and $\big(a_n(\vec{r})\big)$ is a representative of $h(\vec{r})$. As well, $C_i \subset C_{i-1} \subset \cdots \subset C_1$ is nested.

Let $D_n = \{i \mid n \in C_i \text{ and } i \le n\}$ and $m_n = \max D_n$, if $D_n$ is non-empty and $m_n = 1$ otherwise.

**Claim 1.** $\lim^{\omega} m_n = \infty$.

*Proof of Claim 1.* The proof is the identical to that of Claim 1 from the previous proposition. $\square$

Define $h_n\colon A_{m_n}^{(1)} \to X$ by

$$h_n(\vec{r}) = a_n(\vec{r}) \text{ for } \vec{r} \in A_{m_n}$$

and extend $f_n$ geodesically to all of $A_{m_n}^{(1)}$. Let $e$ be a 2-cell of $A_{m_n}$. Then $|h_n(\partial e)|$ is at most $4\,\mathrm{diam}(h_n(\partial e))$ and we can extend $h_n$ to $\mathbb{D}$ by mapping each 2-cell $e$ of $A_{m_n}$ to a disc of diameter at most $4M\,\mathrm{diam}(h_n(\partial e))$.

We can now define $h'\colon \mathbb{D} \to \mathrm{Con}^{\omega}\big(X,e,d\big)$ by $h'(\vec{t}) = \big(h_n(\vec{t})\big)$ for all $\vec{t} \in \mathbb{D}$.

Let $B_k = \{n \mid m_n > k\}$ which is $\omega$-large by the claim. Fix a diadic rational pair $\vec{r}_0 = (r_0,s_0)$ and $n_0$ such that $\vec{r}_0 \in A_{n_0}$. Then for all $n \in B_{n_0}$, $\vec{r}_0 \in A_{m_n}$. This implies that $h_n(\vec{r}_0) = a_n(\vec{r}_0)$ for $n \in B_{n_0}$. Since $B_{n_0}$ is $\omega$-large, $h'(\vec{r}_0) = \big(h_n(\vec{r}_0)\big) = \big(a_n(\vec{r}_0)\big) = h(\vec{r}_0)$. Hence $h'(\vec{r}) = h(\vec{r})$ for any diadic pair $\vec{r} = (r,s)$.

Notice by our choice of $m_n$, we have $n \in C_{m_n}$ if $m_n \ne 1$. Thus for $n \in B_1$ and $\vec{r},\vec{s} \in A_{m_n}$, we have

$$\mathrm{dist}\big(h(\vec{r}),h(\vec{s})\big) - \frac{1}{m_n} \le \frac{\mathrm{dist}\big(h_n(\vec{r}),h_n(\vec{s})\big)}{d_n} \le \mathrm{dist}\big(h(\vec{r}),h(\vec{s})\big) + \frac{1}{m_n}. \tag{III.1}$$

Fix $\vec{x}_1,\vec{x}_2 \in \mathbb{D}$. Choose 2-cells $e_1,e_2$ of $A_{m_n}$ such that $\vec{x}_i \in e_i$ for $i = 1,2$. As well, choose a vertex $\vec{r}_i$ of $e_i$ for $i = 1,2$.

Then for $n \in B_1$, $\mathrm{diam}(h_n(e_i)) \le 4M\,\mathrm{diam}(h_n(\partial e_i))$. Since $h_n(\partial e_i)$ is a geodesic 4-gon, $\mathrm{diam}(h_n(\partial e_i))$ is at most twice the maximum distance between adjacent vertices. This with equation (III.1) implies that

$$\mathrm{diam}(h_n(e_i)) \le d_n M 8\Big[\xi\Big(\frac{1}{2^{m_n}}\Big) + \frac{1}{m_n}\Big].$$

Thus for $i = 1,2$ we have

$$\operatorname{dist}\big(h_n(\vec{x}_i), h_n(\vec{r}_i)\big) \leq d_n M 8 \Big[\xi\left(\frac{1}{2^{m_n}}\right) + \frac{1}{m_n}\Big].$$

Then combining this with the triangle inequality, we obtain

$$\begin{aligned}
\operatorname{dist}\big(h_n(\vec{x}_1), h_n(\vec{x}_2)\big) &\leq \operatorname{dist}\big(h_n(\vec{x}_1), h_n(\vec{r}_1)\big) + \operatorname{dist}\big(h_n(\vec{r}_1), h_n(\vec{r}_2)\big) + \operatorname{dist}\big(h_n(\vec{r}_2), h_n(\vec{x}_2)\big) \\
&\leq d_n M 8 \Big[\xi\left(\frac{1}{2^{m_n}}\right) + \frac{1}{m_n}\Big] + d_n \Big[\xi\left(\operatorname{dist}(\vec{r}_1, \vec{r}_2)\right) + \frac{1}{m_n}\Big] + d_n M 8 \Big[\xi\left(\frac{1}{2^{m_n}}\right) + \frac{1}{m_n}\Big] \\
&= 16 d_n M \xi\left(\frac{1}{2^{m_n}}\right) + \frac{d_n(16M+1)}{m_n} + d_n \xi\left(\operatorname{dist}(\vec{x}_1, \vec{x}_2) + \frac{2}{2^{m_n}}\right)
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{dist}\big(h'(\vec{x}_1), h'(\vec{x}_2)\big) &\leq \lim_n{}^{\omega} \frac{\operatorname{dist}\big(h_n(x_1), h_n(\vec{x}_2)\big)}{d_n} \\
&\leq \lim_n{}^{\omega} 16 M \xi\left(\frac{1}{2^{m_n}}\right) + \frac{(16M+1)}{m_n} + \xi\left(\operatorname{dist}(\vec{x}_1, \vec{x}_2) + \frac{2}{2^{m_n}}\right) \\
&= \xi\Big(d(\vec{x}_1, \vec{x}_2)\Big)
\end{aligned}$$

which implies that $h'$ is continuous. (Note that we used that fact that $\xi$ was a continuous modulus of continuity.) Hence $h(\vec{x}) = h'(\vec{x})$ for all $\vec{x} \in \mathbb{D}$.

□

We will use $\mathcal{N}_s(B)$ to represent the $s$-neighborhood of $B$.

**Lemma III.1.4.** *Let $X$ be a homogeneous geodesic metric space. For any pair $(\omega, d)$, the non-empty components of $\operatorname{Con}^{\omega}\big(X, e, d\big) \setminus \big\{(x_n)\big\}$ are unbounded for all $(x_n) \in \operatorname{Con}^{\omega}\big(X, e, d\big)$.*

*Proof.* The lemma is trivial if $X$ is bounded.

**Claim.** *Every asymptotic cone of an unbounded homogeneous geodesic metric space contains a bi-infinite geodesic.*

*Proof of Claim.* Since $X$ is unbounded and geodesic, there exist a geodesic $\gamma_n : [-nd_n, nd_n] \to X$ every $n$. By homogeneity, we may assume that $\gamma_n(0) = e_n$. We can extend $\gamma_n$ to all of $\mathbb{R}$ by $\gamma_n(t) = \gamma_n(nd_n)$ for $t \geq nd_n$ and $\gamma_n(t) = \gamma_n(-nd_n)$ for $t \leq -nd_n$. Define $\gamma : \mathbb{R} \to \operatorname{Con}^{\omega}\big(X, e, d\big)$ by $\gamma(t) = \big(\gamma_n(td_n)\big)$. Then

$$\operatorname{dist}\big(\gamma(s), \gamma(t)\big) = \lim{}^{\omega} \frac{\operatorname{dist}_n\big(\gamma_n(sd_n), \gamma_n(sd_n)\big)}{d_n} = \lim{}^{\omega} \frac{|sd_n - td_n|}{d_n} = |s - t|$$

which completes the proof. □

Since $\operatorname{Con}^{\omega}\big(X, e, d\big)$ is also homogeneous, it contains a bi-infinite geodesic through every point. Suppose that $A$ is a non-empty connected component of $\operatorname{Con}^{\omega}\big(X, e, d\big) \setminus \{x\}$ for some $x \in \operatorname{Con}^{\omega}\big(X, e, d\big)$. Let $a \in A$.

Then there exists a bi-infinite geodesic $\alpha\colon \mathbb{R} \to \mathrm{Con}^\omega\big(X,e,d\big)$ such $\alpha(0) = a$. Only one of $\alpha\big((-\infty,0]\big)$, $\alpha\big([0,\infty)\big)$ can intersect $x$. Hence $A$ must contain an unbounded ray. $\square$

This lemma also follows from [17, Lemma 3.12].

**Lemma III.1.5.** *Let $B_n$ be a sequence of uniformly bounded subsets of a geodesic metric space $X$ and $\kappa\colon \mathbb{N} \to \mathbb{R}$ be a sublinear function. If $X \backslash \mathscr{N}_{k(d_n)}(B_n)$ has more than one unbounded connected component; then, for $e_n \in B_n$, $\mathrm{Con}^\omega\big(X,e,d\big) \backslash \lim_e^\omega B_n$ has more than one unbounded connected component.*

*Proof.* Let $\{U_{n,1},\cdots,U_{n,i_n}\}$ be the set of unbounded connected components of $X \backslash \mathscr{N}_{k(d_n)}(B_n)$. Let $B = \lim_e^\omega B_n$, $Z = \lim_e^\omega U_{n,s_n}$, and $Y = \lim_e^\omega U_{n,t_n}$ where $s_n,t_n$ are distinct elements of $\{1,\cdots,i_n\}$ $\omega$-almost surely. Since $B_n$ is uniformly bounded and $U_{n,i}$ is unbounded, both $Y \backslash B$ and $Z \backslash B$ are nonempty and hence unbounded.

Suppose that $x \in Z \cap Y$. Then $x = (z_n) = (y_n)$ where $z_n \in U_{n,s_n}$ and $y_n \in U_{n,t_n}$. Since $U_{n,s_n}$ and $U_{n,t_n}$ are in distinct connected components of $X \backslash \mathscr{N}_{k(d_n)}(B_n)$, every path originating in $U_{n,s_n}$ and terminating in $U_{n,t_n}$ passes through $\mathscr{N}_{k(d_n)}(B_n)$. By considering a geodesic from $z_n$ to $y_n$, we can find $b_n' \in \mathscr{N}_{k(d_n)}(B_n)$ such that $\mathrm{dist}(z_n,b_n') + \mathrm{dist}(b_n',y_n) = \mathrm{dist}(z_n,y_n)$ which implies that $x = (b_n')$. As well, there exists $b_n \in B_n$ such that $\mathrm{dist}(b_n,b_n') \leq \kappa(d_n)$. Hence $x = (b_n') = (b_n)$.

Thus $Z \cap Y \subset B$ and the components of $Z \backslash B, Y \backslash B$ are unbounded components of $\mathrm{Con}^\omega\big(X,e,d\big) \backslash B$.

$\square$

At times it will be convenient to consider separating sets which are unbounded.

**Lemma III.1.6.** *Let $(B_n,e_n)$ be a sequence of pointed subsets of a geodesic metric space $X$, $\omega$ an ultrafilter, and $d = (d_n)$ an $\omega$-divergent sequence. Suppose that $a = (a_n), b = (b_n) \in \mathrm{Con}^\omega\big(X,e,d\big)$ are points such that there exists a sublinear function $\kappa\colon \mathbb{N} \to \mathbb{R}$ such that $a_n, b_n$ are in distinct components of $X \backslash \mathscr{N}_{\kappa(d_n)}(B_n)$ $\omega$-almost surely.*

*Then $\lim_e^\omega B_n$ separates $\mathrm{Con}^\omega\big(X,e,d\big)$ into at least two connected components and a, b are in distinct components of $\mathrm{Con}^\omega\big(X,e,d\big) \backslash \lim_e^\omega B_n$.*

The proof is the same as for bounded sets; the only difference is that we are not able to conclude that the components are unbounded since we cannot apply Lemma III.1.4.

**Definition III.1.7.** Let $X$ be a connected, locally connected topological space. A point $x \in X$ is a *local cut-point* if there exists an open connected neighborhood $U$ of $x$ such that $U \backslash \{x\}$ has at least two connected components. A point $x \in X$ is a *global cut-point* if $X \backslash \{x\}$ has at least two connected components. $X$ is *wide* if none of its asymptotic cones has a global cut-point. $X$ is *unconstricted* if one of its asymptotic cones has no global cut-points. $X$ is *constricted* if all of its asymptotic cones have global cut-points.

Let $B_1 \subset B_2 \subset \cdots$ be an ascending sequence of bounded sets in a metric space $X$ such that every set of bounded diameter is eventually contained in $B_n$ for some $n$. This implies that $\cup_n B_n = X$.

Two descending sequences $U_1 \supset U_2 \supset \cdots$ and $V_1 \supset V_2 \supset \cdots$ of subsets of $X$ are *equivalent* if for every $n$ there exists integers $m,k$ such that $V_m \subset U_n$ and $U_k \subset V_n$.

An *end of* $X$ is a descending sequence $U_1 \supset U_2 \supset \cdots$ where $U_i$ is an unbounded component of $X \backslash B_i$. It can be shown that up to the given equivalence on descending sequences of subsets of $X$ the set of ends of $X$ does not depend on $\{B_n\}$.

A metric space $X$ is *one-ended*, if $X \backslash B$ has a unique unbounded connected component for every bounded subset $B$ of $X$.

We will use the following definition and lemma from [17].

**Definition III.1.8.** Let $X$ be a geodesic metric space, and let $0 < \delta < 1$ and $\gamma \geq 0$. Let $a, b, c \in X$ with $\operatorname{dist}\big(c, \{a,b\}\big) = r > 0$, where $\operatorname{dist}(c, \{a,b\})$ is the minimum of $\operatorname{dist}(c,a)$ and $\operatorname{dist}(c,b)$. Define $\operatorname{div}_\gamma(a,b,c;\delta)$ as the infimum of the lengths of paths $a, b$ that avoid the ball $\mathrm{B}(c, \delta r - \gamma)$ (note that by definition a ball of non-positive radius is empty). If no such path exists, take $\operatorname{div}_\gamma(a,b,c;\delta) = \infty$.

**Lemma III.1.9** ([17, Lemma 3.14])**.** *Let $X$ be a geodesic metric space. Let $\omega$ be any ultrafilter and $d = (d_n)$ be an $\omega$-divergent sequence. Let $a = (a_n), b = (b_n), c = (c_n) \in \operatorname{Con}^\omega\big(X, e, d\big)$. Let $r = \operatorname{dist}\big(c, \{a,b\}\big)$. The following conditions are equivalent for any $0 \leq \delta < 1$.*

*(i) The closed ball $\overline{\mathrm{B}}(c, \delta)$ in $\operatorname{Con}^\omega\big(X, e, d\big)$ separates $a$ from $b$.*

*(ii) For every $\delta' > \delta$ and every (some) $\gamma \geq 0$ the limit $\lim^\omega \frac{\operatorname{div}_\gamma(a_n, b_n, c_n; \frac{\delta'}{r})}{d_n}$ is $\infty$.*

The following proposition is immediate, as it holds for all homogeneous geodesic metric spaces, see [22].

**Proposition III.1.10.** *An asymptotic cone of a finitely generated group can have 0,1,2 or uncountable many ends.*

**Lemma III.1.11.** *Let $X$ be a homogenous geodesic metric space. If $\operatorname{Con}^\omega\big(X, e, d\big)$ has a local cut-point, then there exists a cone of $X$ with a global cut-point.*

*Proof.* Suppose that $\operatorname{Con}^\omega\big(X, e, d\big)$ has a local cut-point. By homogeneity, $\tilde{x} = (x_n)$ is a local cut-point. Suppose that $U$ is an open connnected neighborhood of $\tilde{x}$ such that $U \backslash \{\tilde{x}\}$ has two components.

**Claim.** *There exists an $\varepsilon > 0$ such that $\tilde{x}$ separates every ball about $\tilde{x}$ with radius at most $\varepsilon$.*

Let $B_\varepsilon$ be the ball in $\operatorname{Con}^\omega\big(X, e, d\big)$ about $\tilde{x}$ of radius $\varepsilon$.

Fix $\varepsilon > 0$ such that $B_\varepsilon$ is a subset of $U$. Let $u, v$ be elements of $U$ which are in different components of $U \backslash \{\tilde{x}\}$. Any path in $U$ from $u$ to $v$ passes through $\tilde{x}$. (Since $\operatorname{Con}^\omega\big(X, e, d\big)$ is locally path connected and $U$ is open and connected, $U$ is path connected.) Hence, we can find a path $f\colon\ [0,1] \to U$ such that $f^{-1}(\tilde{x}) = \{\frac{1}{2}\}$ and $f(0), f(1)$ are in different components of $U \backslash \{\tilde{x}\}$. This implies that the inclusion map from $B_{\varepsilon'} \backslash \{\tilde{x}\}$ to $U \backslash \{\tilde{x}\}$ is not contain in a single component for any $\varepsilon' \leq \varepsilon$. Thus $B_{\varepsilon'} \backslash \{\tilde{x}\}$ is also not connected for any $\varepsilon' \leq \varepsilon$ which completes the proof of the claim.

We can now consider the cones $X_k^\omega = \operatorname{Con}^\omega\big(X, (e_n), (\frac{d_n}{k})\big)$. It is easy to see that $\tilde{x}$ is a cut-point of the ball of radius $k\varepsilon$ in $X_k$. Hence, $\lim^\omega X_k$ has a global cut point and by [18, Corollary 3.24] $\lim^\omega X_k$ is again an asymptotic cone of $X$.

$\square$

**Proposition III.1.12.** *Let $X$ be a homogeneous geodesic metric space. Every asymptotic cone of $X$ is one-ended if and only if $X$ is wide if and only if no asymptotic cone of $X$ has a local cut-point.*

*Proof.* $X$ is wide if and only if no asymptotic cone of $X$ has a local cut-point follows immediately from the previous lemma. Thus we need only prove that every asymptotic cone of $X$ is one-ended if and only if $X$ is wide.

The only if direction of this equivalence is trivial. We must show that if no asymptotic cone of $X$ has a cut-point, then every asymptotic cone of $X$ is one-ended. Suppose that no asymptotic cone of $X$ has a cut-point but $\mathrm{Con}^{\omega}\left(X,e,d\right)$ is not one-ended for some choice of $\omega, e, d$. Hence, there exists a bounded subset $\tilde{B}$ of $\mathrm{Con}^{\omega}\left(X,e,d\right)$ such that $\mathrm{Con}^{\omega}\left(X,e,d\right)\backslash\tilde{B}$ has at least two unbounded components. By homogeneity, we may assume that $\tilde{x} = (x_n) \in \tilde{B}$.

By Lemma III.1.5, $\widetilde{Y} = \mathrm{Con}^{\omega}\left(\mathrm{Con}^{\omega}\left(X,e,d\right),(\tilde{x}),(n)\right) \setminus \lim^{\omega} B$ has more than one connected component. Since $\tilde{B}$ is bounded, $\lim^{\omega}\tilde{B}$ is a point in $\widetilde{Y}$ which separates. Thus it is a cut-point of $\widetilde{Y}$. $\widetilde{Y}$ is again an asymptotic cone of $X$ [18, Corollary 3.24]. This contradicts the hypothesis that no cone of $X$ has a cut-point. □

In [17, Theorem 1.4]; Drutu, Mozes, and Sapir show that certain semisimple Lie groups (namely those specified in the theorem below) are wide. Hence, we can apply Proposition III.1.12 to obtain the following result.

**Theorem III.1.13.** *Let $\Gamma$ be an irreducible lattice in a semisimple Lie group of $\mathbb{R}$-rank 2. Suppose that $\Gamma$ is either of $\mathbb{Q}$-rank 1 or is of the form $SL_n(\mathscr{O}_{\mathscr{S}})$ where $n \geq 3$, $\mathscr{S}$ is a finite set of valuations of a number field $K$ including all infinite valuations, and $\mathscr{O}_{\mathscr{S}}$ is the corresponding ring of $\mathscr{S}$-integers. Then every asymptotic cone of $\Gamma$ is one-ended.*

Proposition III.1.12 together with [18, Corollary 6.13] give us the following.

**Theorem III.1.14.** *Let $G$ be a finitely generated non-virtually cyclic group satisfying a law. Then all asymptotic cones of $G$ are one-ended.*

The following proposition is a well know. We present it here only for comparison with Proposition III.1.16.

**Proposition III.1.15.** *Let $G$ be a finitely generated group. The following are equivalent:*

*a) $G$ is finite.*

*b) $G$ has an asymptotic cone which is a point.*

*c) $G$ has an asymptotic cone with 0 ends.*

*Proof.* If $G$ is an infinite finitely generated group, then $\Gamma(G,S)$ contains a bi-infinite geodesic for every finite generating set $S$. Thus $\mathrm{Con}^{\omega}\left(G,d\right)$ contains a bi-infinite geodesic for every infinite group $G$. If $G$ is finite then $\Gamma(G,S)$ is bounded for every generating set $S$ and $\mathrm{Con}^{\omega}\left(G,d\right)$ is a point for every pair $\left(\omega,d\right)$. Thus (a) and (b) ar equivalent. Clearly, (b) implies (c). If $\mathrm{Con}^{\omega}\big(G,d\big)$ has 0 ends for some pair $\big(\omega,d\big)$, then it doesn't contain a bi-infinite geodesic. Hence (c) implies (a).

$\square$

**Proposition III.1.16.** *Let G be a finitely generated group. The following are equivalent:*

*a) G is infinite and virtually cyclic.*

*b) G has an asymptotic cone which is a line.*

*c) G has an asymptotic cone with exactly 2 ends.*

*Proof.* If $G$ is infinite and virtually cyclic, then $\mathrm{Con}^{\omega}\big(G,d\big)$ is a line for every pair $\big(\omega,d\big)$. Thus (a) implies (b). The implication $(b) \Rightarrow (a)$ is Corollary 6.2 in [18]; as well, it also follows from [46], since a line has finite Minkowski dimension.

Thus we need only show that if $\mathrm{Con}^{\omega}\big(G,d\big)$ has exactly two ends for some pair $\big(\omega,d\big)$ than $G$ has an asymptotic cone which is a line.

Suppose that $\mathrm{Con}^{\omega}\big(G,d\big)\backslash B_{\rho}(x_0)$ has exactly two unbounded components for some $\rho > 0$ and $x_0 \in \mathrm{Con}^{\omega}\big(G,d\big)$. For each $i$, let $U_i$ and $V_i$ be the two unbounded components of $\mathrm{Con}^{\omega}\big(G,d\big)\backslash B_{i\rho}(x_0)$. We may assume that we have chosen $U_i, V_i$ such that $U_i \supset U_{i+1}$ and $V_i \supset V_{i+1}$ for all $i$. Fix $x_i \in U_i$ and $x_{-i} \in V_i$ such that $\mathrm{dist}(x_0,x_{\pm i}) = i\rho$ for all $i \in \mathbb{N}$.

Define a path $\alpha\colon \mathbb{R} \to \mathrm{Con}^{\omega}\big(G,d\big)$ by $\alpha(i) = x_i$, for $i \in \mathbb{Z}$, and for every $i \in \mathbb{Z}$ extend $\alpha$ to $[i,i+1]$ by sending the interval to a geodesic joining its endpoints.

**Claim 1.** *$\alpha$ is a quasi-geodesic with constants depending only on $\rho$ and $\mathrm{Con}^{\omega}\big(G,d\big)$ is contained in the $2\rho$-neighborhood of the image of $\alpha$.*

Notice that Claim 1 implies that $G$ has an asymptotic cone which is a line since any asymptotic cone of $\mathrm{Con}^{\omega}\big(G,d\big)$ is a line and an asymptotic cone of $G$.

Let $\alpha_i^- = \alpha\big((-\infty,i-4]\big)$, $\alpha_i^+ = \alpha\big([i+4,\infty)\big)$ and $Y_i = \mathrm{Con}^{\omega}\big(G,d\big)\backslash B_{\rho}(x_i)$ for all $i$. By homogeneity, $Y_i$ has exactly 2 unbounded connected components

**Subclaim 1.1.** *For all $i,j \in \mathbb{Z}$, $\mathrm{dist}(x_i,x_j) \geq \big|j-i\big|\rho - 2\rho$ and hence $\alpha_i^{\pm} \subset Y_i$.*

*Proof of Subclaim 1.1.* If $i,j$ have the same sign then by applying the triangle inequality to a geodesic triangle with vertices $x_0,x_i,x_j$, we obtain $\mathrm{dist}(x_i,x_j) \geq |j-i|\rho$.

Suppose that $i \leq 0 \leq j$. By construction, every geodesic from $x_i$ to $x_j$ passes within $\rho$ of $x_0$. Fix a geodesic from $x_i$ to $x_j$ and let $x_0'$ be a point on the geodesic such that $\mathrm{dist}(x_0,x_0') \leq \rho$. Then $-i\rho = \mathrm{dist}(x_i,x_0) \leq \mathrm{dist}(x_i,x_0') + \rho$ and $j\rho = \mathrm{dist}(x_0,x_j) \leq \mathrm{dist}(x_0',x_j) + \rho$ which gives us that $(j-i)\rho \leq \mathrm{dist}(x_i,x_0') + \mathrm{dist}(x_0',x_j) + 2\rho = \mathrm{dist}(x_i,x_j) + 2\rho$. Thus $\mathrm{dist}(x_i,x_j) \geq |j-i|\rho - 2\rho$. If $j \in (\infty,i-4] \cup [i+4,\infty)$, then $\mathrm{dist}(x_i,x_j) \geq 2\rho$. Since every point on $\alpha_i^{\pm}$ is with in $\rho$ of some $x_j$ for $j \in (\infty,i-4] \cup [i+4,\infty)$; $\alpha_i^{\pm} \subset Y_i$. $\square$

**Subclaim 1.2.** *$\alpha_i^+, \alpha_i^-$ are contained in distinct unbounded components of $Y_i$ for all $i$.*

*Proof of Subclaim 1.2.* We will show the subclaim for $i \geq 0$. The other case is similiar. Let $U, V$ be the two disjoint unbounded components of $Y_i$. By way of contradiction, we will assume that $\alpha_i^{\pm}$ are both contained in $U$. Choose $\tilde{g} \in \prod G$ such that $\tilde{g} \cdot x_0 = x_i$. For each $j \geq 1$, let $\beta_j^- = \alpha\big((-\infty, -2j-4]\big)$, $\beta_j^+ = \alpha\big([2j+4, \infty)\big)$. Since $\tilde{g}$ acts by isometries on $\mathrm{Con}^{\omega}\big(G, d\big)$, we obtain that $Y_i = \tilde{g} \cdot Y_0$ and $\tilde{g} \cdot \beta_j^{\pm}$ are in distinct unbounded components of $Y_i$ for any $j \geq 1$.

Fix $j \geq i$. Since $\alpha_i^-, \alpha_i^+$ are contained in the same connected component of $Y_i$ and $\tilde{g} \cdot \beta_j^-, \tilde{g} \cdot \beta_j^+$ are contained in distinct connected components of $Y_i$, one of $\tilde{g} \cdot \beta_j^{\pm}$ is contained in $V$. Suppose that $\tilde{g} \cdot \beta_j^+ \subset V$. (Again the other case is similar.) Notice that $\beta_j^{\pm} \subset \alpha_i^{\pm}$ which implies that $\beta_j^- \cup \beta_j^+ \subset U$.

By Subclaim 1.1, $\mathrm{dist}(x_i, \tilde{g} \cdot \beta_j^{\pm}) \geq (2j+4)\rho - 2\rho$ which implies that $\mathrm{dist}(x_0, \tilde{g} \cdot \beta_j^{\pm}) \geq (j+2)\rho$. Thus $\tilde{g} \cdot \beta_j^{\pm} \subset \mathrm{Con}^{\omega}\big(G, d\big) \backslash B_{(j+1)\rho}(x_0)$. Again by Subclaim 1.1, $\beta_j^{\pm} \subset \mathrm{Con}^{\omega}\big(G, d\big) \backslash B_{(j+1)\rho}(x_0)$. By construction $B_\rho(x_0), B_\rho(x_i) \subset B_{(j+1)\rho}(x_0)$ which implies that each of the three unbounded sets $\beta_j^{\pm}, \tilde{g} \cdot \beta_j^+$ must be contained in a distinct connected component of $\mathrm{Con}^{\omega}\big(G, d\big) \backslash B_{(j+1)\rho}(x_0)$. Since this holds for any $j \geq i$, $\mathrm{Con}^{\omega}\big(G, d\big)$ must have at least 3 ends which contradicts our assumption that $\mathrm{Con}^{\omega}\big(G, d\big)$ has exactly 2 ends. ☐

*Proof of Claim 1.* If $i, j \in \mathbb{Z}$ have different signs, then $\mathrm{dist}(x_i, x_j) \leq \mathrm{dist}(x_i, x_0) + \mathrm{dist}(x_j, x_0) = |i|\rho + |j|\rho = |i-j|\rho$

For $4 \leq i \leq j-4$, any geodesic from $x_0$ to $x_j$ is passes within $\rho$ of $x_i$ by Subclaim 1.2. Hence, we may find a point $x_i'$ on a geodesic from $x_0$ to $x_j$ such that $\mathrm{dist}(x_i, x_i') \leq \rho$. Then $i\rho \leq \mathrm{dist}(x_0, x_i') + \rho$ and $\mathrm{dist}(x_i, x_j) \leq \mathrm{dist}(x_i', x_j) + \rho$ which implies that $\mathrm{dist}(x_i, x_j) \leq (j-i)\rho + 2\rho = |j-i|\rho + 2\rho$. Similarly, we can obtain the inequality $\mathrm{dist}(x_i, x_j) \leq |j-i|\rho + 2\rho$ for $j+4 \leq i \leq -4$. It follows that $\alpha$ is a quasi-geodesic.

Suppose that there exists $x \in \mathrm{Con}^{\omega}\big(G, d\big)$ such that $\mathrm{dist}(x, \mathrm{im}\,\alpha) \geq 2\rho$. $\mathrm{Con}^{\omega}\big(G, d\big) \backslash B_\rho(x)$ has two unbounded components one of which contains $\mathrm{im}\,\alpha$. As in the proof of Subclaim 1.2, this would imply that $\mathrm{Con}^{\omega}\big(G, d\big)$ would have at least three ends. ☐

Thus any asymptotic cone of $\mathrm{Con}^{\omega}\big(G, d\big)$ is a line and also an asymptotic cone of $G$ which completes the proof of the proposition. ☐

**Lemma III.1.17.** *Suppose that $X$ is an unbounded homogeneous geodesic metric space and $T$ is a vertex homogeneous three valence tree with fixed edge length $\rho$. If $\mathrm{Con}^{\omega}\big(X, e, d\big)$ has more than two ends and a global cut-point, then there exists an isometry $f\colon T \to \mathrm{Con}^{\omega}\big(X, e, d\big)$ such that the components of $T \backslash \{v\}$ map to distinct components of $\mathrm{Con}^{\omega}\big(X, e, d\big) \backslash \big\{f(v)\big\}$ for every vertex $v$ of $T$.*

*Proof.* Fix $\rho > 0$. Let $T$ be a vertex homogeneous 3-valence tree with fix edge length $\rho$. We will now build an isometry $f\colon T \to \mathrm{Con}^{\omega}\big(X, e, d\big)$ such that the three components of $T \backslash \{v\}$ map into distinct components of $\mathrm{Con}^{\omega}\big(X, e, d\big) \backslash \big\{f(v)\big\}$ for every vertex $v$ of $T$. Fix a vertex $v_0$ of $T$.

Let $T_i$ be a sequence of connected subtrees of $T$ such that $v_0 = T_1$; $T_i \subset T_{i+1}$; $\cup_i T_i = T$; and $T_{i+1}$ has exactly one vertex not contained in $T_i$. This implies that $T_{i+1}$ can be obtained from $T_i$ be adding exactly one edge and one vertex.

Let $f(v_0) = x_0$ for some $x_0 \in \mathrm{Con}^{\omega}\big(X, e, d\big)$. By induction, assume that we have defined $f$ on $T_i$ such that $f|_{T_i}$ is an isometry and $f$ maps the components of $T_i \backslash \{v\}$ to distinct components of $\mathrm{Con}^{\omega}\big(X, e, d\big) \backslash \{f(v)\}$

for each vertex $v$ of $T_i$. Let $e$ be the edge of $T$ which is added to $T_i$ to obtain $T_{i+1}$. Then $e$ has exactly one vertex $e^-$ in $T_i$ and one vertex $e^+$ in $T_{i+1} \backslash T_i$. Notice that $T_i$ has valence 1 or 2 at $e^-$. This implies that $T_i \backslash \{e^-\}$ and hence $f\big(T_i \backslash \{e^-\}\big)$ has at most 2 components. Let $C$ be a component of $\mathrm{Con}^{\omega}\big(X,e,d\big) \backslash \{f(e^-)\}$ which is disjoint from $f\big(T_i \backslash \{e^-\}\big)$. Since all components are unbounded, we may choose a point $x \in C$ such that $\mathrm{dist}\big(x, f(e^-)\big) = \rho$. Let $f(e^+) = x$ and $f(e)$ be a geodesic from $f(e^-)$ to $f(e^+)$. It is immediate that the components of $T_{i+1} \backslash \{v\}$ map to distinct components of $\mathrm{Con}^{\omega}\big(X,e,d\big) \backslash \{f(v)\}$ for all vertices $v$ in $T_{i+1}$. It only remains to show that $f$ restricted to $T_{i+1}$ is still an isometry. This follows trivially from the fact that if $x,y$ are in distinct components of $\mathrm{Con}^{\omega}\big(X,e,d\big) \backslash \{z\}$, then $\mathrm{dist}(x,y) = \mathrm{dist}(x,z) + \mathrm{dist}(z,y)$.

This defines a map $f\colon\ T \to \mathrm{Con}^{\omega}\big(X,e,d\big)$. Since any two points lie in some $T_i$, $f$ is an isometry. We must show that the separation condition is preserved in the limit. Suppose that $v$ is a vertex of $T$ and $T_i$ contains the $2\rho$-neighborhood of $v$. By construction, $f$ takes the components of $T_i \backslash \{v\}$ into distinct components of $\mathrm{Con}^{\omega}\big(X,e,d\big) \backslash \{f(v)\}$. Notice that each component of $T \backslash \{v\}$ intersects a component of $T_i \backslash \{v\}$ nontrivially which implies that the separation condition still holds.

□

**Corollary III.1.18.** *In addition, $f$ can be chosen such that $f(t) = \big(f_n(t)\big)$ for all $t \in T$ where $f_n\colon\ T \to X$ takes edges of $T$ to geodesics in $X$.*

*Proof.* We will show how to modify the proof of Lemma III.1.17. Using the notation from above, we will inductively defining $f, f_n$ simultaneously. Suppose that $f, f_n$ are defined as desired on $T_i$. When choosing $x \in C$ we will also fix a representative $(x_n)$ of $x$. Let $f_n(e^+) = x_n$ which implies that $f(e^+) = \big(f_n(e^+)\big) = x$. Let $f_n$ map $e$ to any geodesic from $f_n(e^-)$ to $f_n(e^+)$ which implies that $f(e) = \big(f_n(e)\big)$ is a geodesic from $f(e^-)$ to $f(e^+)$. The rest of the proof remains unchanged. □

**Proposition III.1.19.** *Suppose that $X$ is an unbounded homogeneous geodesic metric space and $C_i$ is a sequence of finite point sets from $\mathrm{Con}^{\omega}\big(X,e,d\big)$. Then $\lim_e^{\omega} C_i$ embeds isometrically into $\mathrm{Con}^{\omega}\big(X,e,d\big)$. In addition; if $C_i$ is nested, then the canonical copy of $C_i$ in $\lim_e^{\omega} C_i$ is mapped to $C_i$.*

This proposition was previously shown under the Continuum Hypothesis by Osin and Sapir and for groups by Sisto [49].

*Proof.* Let $\iota_i\colon\ C_i \to \mathrm{Con}^{\omega}\big(X,e,d\big)$ be the inclusion induced map. Fix a representative for each element of $C = \cup_i C_i$. We can now define a double indexed sequence of maps $\iota_n^i\colon\ C_i \to X$ by letting $\iota_n^i(c)$ be the $n$-th coordinate of our chosen representative for $c \in C$. Thus, if the $C_i$ are nested and $c \in C_i$; then $\iota_n^j(c) = \iota_n^i(c)$ for all $j \geq i$. Hence, $c = \big(\iota_n^{k_i}(c)\big)$ for any sequence $k_i$. This will imply that the map defined below takes the canonical copy of $C_i$ in $\lim^{\omega} C_i$ to $C_i$. Let

$$A_i = \Big\{ n \mid \mathrm{dist}(c,c') - \frac{1}{i} \leq \frac{\mathrm{dist}\big(\iota_n^j(c), \iota_n^j(c')\big)}{d_n} \leq \mathrm{dist}(c,c') + \frac{1}{i} \text{ for all } c,c' \in C_j \text{ where } j \leq i \Big\}.$$

Since $\Big|\bigcup_{j \leq i} C_j\Big|$ is finite and $\iota_j$ is an isometry for every $j$, $A_i$ is $\omega$-large. Let $m_n = \max\{i \mid n \in A_i \text{ and } i \leq n\}$, if this set is non-empty and $m_n = 1$ otherwise. Suppose that $m_n$ was bounded by $L$ on some $\omega$-large set $A$.

Then $A_{2L} \cap A \subset \{1, \cdots, 2L-1\}$, which is a contradiction since $\omega(A_{2L}) = \omega(A) = 1$ and $\omega\big(\{1, \cdots, 2L-1\}\big) = 0$. Thus $\lim^{\omega} m_n = \infty$.

Define $\tilde{\iota}\colon \lim_e^{\omega} C_i \to \mathrm{Con}^{\omega}\big(X, e, d\big)$ by $\tilde{\iota}\big((c_n)\big) = \big(\iota_n^{m_n}(c_n)\big)$.

**Claim:** $\tilde{\iota}$ is a well-defined isometric embedding of $\lim_e^{\omega} C_n$ into $\mathrm{Con}^{\omega}\big(X, e, d\big)$.

Fix $c, c' \in \lim^{\omega} C_n$. We may choose representatives $c_n, c'_n \in C_n$ such that $c = (c_n)$ and $c = (c'_n)$. By construction, $\mathrm{dist}(c_n, c'_n) - \frac{1}{m_n} \le \frac{\mathrm{dist}\left(\iota_n^{m_n}(c_n), \iota_n^{m_n}(c'_n)\right)}{d_n} \le \mathrm{dist}(c_n, c'_n) + \frac{1}{m_n}$ for all $n$ such that $m_n \neq 1$. Since $m_n$ is $\omega$-divergent, this set is $\omega$-large and

$$\begin{aligned} \mathrm{dist}(c, c') = \lim{}^{\omega}\Big[\mathrm{dist}(c_n, c'_n) - \frac{1}{m_n}\Big] &\le \lim{}^{\omega}\Big[\frac{\mathrm{dist}\big(\iota_n^{m_n}(c_n), \iota_n^{m_n}(c'_n)\big)}{d_n}\Big] \\ &\le \lim{}^{\omega}\Big[\mathrm{dist}(c_n, c'_n) + \frac{1}{m_n}\Big] = \mathrm{dist}(c, c'). \end{aligned}$$

Thus $\tilde{\iota}$ is independent of the chosen representative and is an isometry.

$\square$

We can now use Lemma III.1.17 to prove that $\mathbb{R}$-trees can also be transversally embedded into cones with cut-points.

**Lemma III.1.20.** *Suppose that $X$ is a unbounded homogeneous geodesic metric space and $T$ is a universal $\mathbb{R}$-tree with continuum branching at every point. If $\mathrm{Con}^{\omega}\big(X, e, d\big)$ has more than two ends and a global cut-point, then there exists an isometry $f\colon T \to \mathrm{Con}^{\omega}\big(X, e, d\big)$ such that the components of $T \setminus \{v\}$ map to distinct components of $\mathrm{Con}^{\omega}\big(X, e, d\big) \setminus \big\{f(v)\big\}$ for every $v$ in $T$.*

*Proof.* Let $T_i$ be a three valence tree with edge length $\frac{1}{2^i}$ such that $T_i \subset T_{i+1}$ for all $i \in \mathbb{N}$ and $t_0$ a fixed vertex in $T_1$. We will assume that $T_i$ is endowed with the edge metric. We will use $[v, w]$ to denote the geodesic from $v$ to $w$ in $T_i$ and $(v, w) = [v, w] \setminus \{v, w\}$. If $v, w \in T_i \cap T_j$, then $[v, w]$ is independent of whether the geodesic is taken in $T_i$ or in $T_j$.

By Lemma III.1.17, there exist isometries $f_i\colon T_i \to \mathrm{Con}^{\omega}\big(X, e, d\big)$ which satisfy the separation condition of Lemma III.1.17. By homogeneity, we may assume $f_i(t_0) = f_j(t_0)$ for all $i, j$. By Corollary III.1.18, there exists a sequence of maps $f_n^i\colon T_i \to X$ such that $f_i(t) = \big(f_n^i(t)\big)$ for all $t \in T_i$. We will also require that $f_n^i(t_0) = f_n^j(t_0)$ for all $i, j$.

Let $V_i$ be the vertices of the ball of radius $i$ about $t_0$ in $T_i$. Then $\big|f_i(V_i)\big|$ is finite set and Proposition III.1.19 implies $\lim_e^{\omega} f_i(V_i)$ embeds isometrically. While $\lim_e^{\omega} f_i(V_i)$ is a universal $\mathbb{R}$-tree, we must still guarantee that the embedding preserves the separation property. To do this we will show how to modify the proof of Proposition III.1.19 so as to guarantee that the embedding preserves the desired separation property. Let

$$A_i = \Big\{n \;\Big|\; \mathrm{dist}(v, w) - \frac{1}{i} \le \frac{\mathrm{dist}\big(f_n^j(v), f_n^j(w)\big)}{d_n} \le \mathrm{dist}(v, w) + \frac{1}{i} \text{ for all } v, w \in V_j \text{ where } j \le i\Big\}.$$

For $r = \mathrm{dist}\big(f_n^j(v_0), \{f_n^j(v_1), f_n^j(v_2)\}\big)$ and $v_0, v_1, v_2 \in T_j$ such that $v_0$ separates $v_1$ from $v_2$ in $T_j$, let

$\rho_n^i(j,v_0,v_1,v_2) = \mathrm{div}_1\big(f_n^j(v_1), f_n^j(v_2), f_n^j(v_0); \frac{1}{ir}\big)$. Let

$$B_i = \{n \mid \rho_n^i(j,v_0,v_1,v_2) > id_n \mid j \leq i;\ v_0,v_1,v_2 \in V_j;\ \text{and } v_0 \in (v_1,v_2)\}.$$

As before $A_i$ is $\omega$-large for each $i$.

**Claim:** *$B_i$ is an $\omega$-large set.*

For each $j$ and each triple $v_0,v_1,v_2 \in V_j$ such that $v_0 \in (v_1,v_2)$, we have that

$$\lim{}^{\omega} \frac{\mathrm{div}_1\big(f_n^j(v_1), f_n^j(v_2), f_n^j(v_0); \frac{1}{2ir}\big)}{d_n} = \infty$$

by Lemma III.1.9 where $r = \mathrm{dist}\big(f_j(v_0), \{f_j(v_1), f_j(v_2)\}\big)$. Thus $\frac{\mathrm{div}_1\left(f_n^j(v_1), f_n^j(v_2), f_n^j(v_0); \frac{1}{ir_n}\right)}{d_n} > i$ on an $\omega$-large set where $r_n = \mathrm{dist}\big(f_n^j(v_0), \{f_n^j(v_1), f_n^j(v_2)\}\big)$. Since $V_j$ is finite, $B_i$ is the finite intersection of $\omega$-large sets which completes the proof of the claim.

Let $m_n = \max\{i \mid n \in B_i \cap A_i \text{ and } i \leq n\}$, if the intersection is non-empty for some $i \leq n$ and $m_n = 1$ otherwise.

Define $\tilde{t} = (t_0)$ and $\tilde{f}\colon\ \lim_i^{\omega} T_i \to \mathrm{Con}^{\omega}\big(X,e,d\big)$ by $\tilde{f}(t) = \big(f_n^{m_n}(t)\big)$.

Notice that $\lim_i^{\omega} T_i = \lim_i^{\omega} V_i$ and $\tilde{f}\big((t_o)\big) = f_i(t_0)$ for all $i$. As in the proof of Proposition III.1.19, $\lim^{\omega} m_n = \infty$ and $\tilde{f}$ is a well-defined isometric embedding of $\lim^{\omega} T_i$ into $\mathrm{Con}^{\omega}\big(X,e,d\big)$.

All that remains is to show that $\tilde{f}$ satisfies the desired separation condition. Suppose that $v_0,v_1,v_2$ are points on $\lim_i^{\omega} T_i$ such that $v_1,v_2$ are in different components of $\lim_i^{\omega} T_i \backslash \{v_0\}$. Then there exist representatives $(v_n^0),(v_n^1),(v_n^2)$ of $v_1,v_2,v_3$ respectively such that $v_n^1,v_n^2$ are in distinct components of $T_n \backslash \{v_n^0\}$ $\omega$-almost surely. Thus

$$\frac{\mathrm{div}_1\big(f_n^j(v_n^1), f_n^j(v_n^1), f_n^j(v_n^0); \frac{1}{m_n r_n}\big)}{d_n} > m_n$$

on an $\omega$-large set where $r_n = \mathrm{dist}\big(f_n^j(v_0), \{f_n^j(v_1), f_n^j(v_2)\}\big)$ and $j \leq m_n$.

Lemma III.1.9 implies that $\tilde{f}(v_1), \tilde{f}(v_2)$ are in distinct components of $\mathrm{Con}^{\omega}\big(X,e,d\big) \backslash \{\tilde{f}(v_0)\}$ which completes the proof.

□

**Proposition III.1.21.** *Let $G$ be a finitely generated group. If $\mathrm{Con}^{\omega}\big(G,d\big)$ has a global cut-point, then $\mathrm{Con}^{\omega}\big(G,d\big)$ is simply connected or has uncountable fundamental group.*

*Proof.* We may assume that $G$ is not virtually cyclic, since the theorem is trivial in that case. Then $G$ has an asymptotic cone $\mathrm{Con}^{\omega}\big(G,d\big)$ with a global cut-point and more than two ends. By Lemma III.1.20, $\mathrm{Con}^{\omega}\big(G,d\big)$ contains an isometrically embedded universal $\mathbb{R}$-tree $T$ such that the components of $T \backslash \{v\}$ map to distinct components of $\mathrm{Con}^{\omega}\big(G,d\big) \backslash \big\{f(v)\big\}$ where $f$ is the isometric embedding of $T$ into $\mathrm{Con}^{\omega}\big(G,d\big)$.

Suppose that $\gamma\colon\ S^1 \to \mathrm{Con}^{\omega}\big(G,d\big)$ is an essential loop and fix $x_0 \in f(T)$ which we may assume is a base point of $\gamma$. Let $\rho = 2\,\mathrm{diam}(\gamma)$ and $S = \{x \in f(T) \mid \mathrm{dist}(x,x_0) = \rho\}$. Then $S$ has cardinality continuum and $\mathrm{dist}(x,y) = 2\rho$ for all $x,y \in S$. For $x \in S$, choose $g_x \in \prod G$ such that $g_x \cdot x_0 = x$. Let $S_\gamma = \{g_x \cdot \gamma \mid x \in S\}$ which is an uncountable set of essential loops in $\mathrm{Con}^{\omega}\big(G,d\big)$.

**Claim:** *No two loops from $S_\gamma$ are homotopic.*

Suppose that $g_x \cdot \gamma$ is homotopic $g_y \cdot \gamma$. Then there exists a continuous map $h\colon A \to \mathrm{Con}^\omega(G,d)$ of a planar annulus which takes one boundary component to $g_x \cdot \gamma$ and the other to $g_y \cdot \gamma$. Since $\mathrm{dist}(g_x \cdot \gamma, x_0) > 0$, $g_x \cdot \gamma$ and $g_y \cdot \gamma$ are in distinct components of $\mathrm{Con}^\omega(G,d) \setminus \{x_0\}$. Thus $h^{-1}(\{x_0\})$ separates the two boundary components of the annulus $A$. Then there exists a single component $C$ of $h^{-1}(\{x_0\})$ which separates the boundary components of $A$. This is a consequence of the Phragmén-Bower properties (see [31]). We can then modify $h$ by mapping the component of the plane bounded by $C$ to $x_0$. This is a null homotopy of $g_x \cdot \gamma$ which contradicts our choice of $\gamma$ and completes the proof of the claim and theorem.

□

**Corollary III.1.22.** *Let $G$ be a finitely generated group. If $\mathrm{Con}^\omega(G,d)$ has a global cut-point, then $\mathrm{Con}^\omega(G,d)$ is simply connected or its fundamental group contains an uncountably generated free subgroup.*

*Proof.* Suppose that we have constructed $f\colon T \to \mathrm{Con}^\omega(G,d)$, $\gamma$, $S = \{x \in f(T) \mid \mathrm{dist}(x,x_0) = \rho\}$, and $S_\gamma = \{g_x \cdot \gamma \mid x \in S\}$ as in the proof of Proposition III.1.21. Let $p_x\colon [0,1] \to f(T)$ be the unique geodesic in $f(T)$ from $x_0$ to $x \in S$.

Then $S'_\gamma = \{\mathrm{x} = p_x * g_x \cdot \gamma * \overline{p}_x \mid x \in S\}$ is a set of loops based at $x_0$ $\big($ where $\overline{p}_x(t) = p_x(1-t)\big)$.

**Claim:** *$S'_\gamma$ generates a free product of cyclic groups.*

Suppose that $\mathrm{x}_1^{n_1} * \cdots * \mathrm{x}_k^{n_k}$ is a null homotopic loop in $\mathrm{Con}^\omega(G,d)$ where $\mathrm{x}_i \neq \mathrm{x}_{i+1}$, $\mathrm{x}_1 \neq \mathrm{x}_k$ and $\mathrm{x}_i^{n_i}$ is an essential loop. Then there exists $h\colon \mathbb{D} \to \mathrm{Con}^\omega(G,d)$ a map from the unit disc in the plane such that $h(\partial\mathbb{D})$ is a parameterization of the curve $\mathrm{x}_1^{n_1} * \cdots * \mathrm{x}_k^{n_k}$. Let $C$ be the closure of the connected component of $h^{-1}\big(\mathrm{Con}^\omega(G,d) \setminus \{x_0\}\big)$ containing the subpath $p$ of $\partial\mathbb{D}^2$ which maps to $\mathrm{x}_1^{n_1}$. By construction, $\partial\mathbb{D} \cap C = p$ and $h(\partial C \setminus \{p\}) = x_0$. Define $h'\colon \mathbb{D} \to \mathrm{Con}^\omega(G,d)$ by $h'(y) = h(y)$ for $y \in C$ and $h'(y) = x_0$ for $y \notin C$. Then $h'$ is continuous and $\mathrm{x}_1^{n_1}$ is null homotopic which contradicts our choice of $x_1^{n_1}$. This completes the proof of the claim.

While the subgroup generated by $S'_\gamma$ may not by a free group ($\gamma$ might have finite order in the fundamental group), it is the free product of cyclic groups. Thus it is easy to find an uncountably generated free subgroup.

□

**Corollary III.1.23.** *Let $G$ be a finitely generated group. If $G$ is constricted, then every asymptotic cone of $G$ is simply connected or has uncountable fundamental group. If $G$ is not wide, then $G$ has an asymptotic cone which is simply connected or has uncountable fundamental group.*

## III.2 Groups with quasi-isometrically embedded subgroups

**Definition III.2.1.** A group is a *prairie group* if all of its asymptotic cones are simply connected.

**Lemma III.2.2.** *The following groups are prairie groups.*

1. *Nilpotent groups;*

2. *Hyperbolic groups; and*

3. *Groups with quadratic Dehn functions*

   (a) $SL_n(\mathbb{Z})$ *for* $n \geq 5$,

   (b) *Thompsons group* $F$,

   (c) *Mapping class groups,*

   (d) $CAT(0)$ *groups,*

   (e) *Automatic groups,*

   (f) *Baumslag-Solitar groups* $BS_{pp}$*, and many many others.*

*Proof.* In [43], Pansu shows that nilpotent groups have a unique asymptotic cone which is homeomorphic to $\mathbb{R}^n$ for some $n$. Gromov showed that non-elementary hyperbolic groups have cones which are isometric to a universal $\mathbb{R}$-tree with uncountable branching at every point. Papasolgu in [44] showed that if a group has a quadratic Dehn function then all of its asymptotic cones are simply connected.

Young showed that $SL_n(\mathbb{Z})$ for $n \geq 5$ has a quadratic Dehn function [53]. Guba showed that Thopson's group $F$ h as a quadratic Dehn function [28]. Mosher showed that the mapping class groups are automatic [34]. It is shown in *Metric spaces of non-positive curvature* that $CAT(0)$ groups have quadratic Dehn functions [6] and in *Word processing in groups* that automatic groups have quadratic Dehn functions [20].

It is a straight forward exercise using van Kampen diagrams to show that $BS_{pp}$ has a quadratic Dehn function. □

**Remark III.2.3.** In [12], the author with Greg Conner note that such groups are uniformly locally simply connected; specifically, every loop of length $r$ bounds a disc of diameter at most $Kr$ where $K$ only depends on the group. However, the discs are not necessarily Lipschitz.

**Lemma III.2.4.** *There exists a finitely presented prairie group such that all of its asymptotic cones have uncountable Lipschitz fundamental group.*

*Proof.* The discrete Heisenberg group $\langle x,y,z \mid z=[x,y],[x,z]=[y,z]=1 \rangle$ is a nilpotent group and hence a prairie group. In fact every asymptotic cone is homeomorphic to $\mathbb{R}^3$. However, it is shown in [15, Theorem 4.10] that the Lipschitz fundamental group of the real Heisenberg group isn't countable generated.

□

The key to Proposition III.1.21 was that the homotopy between the two loops passed through a cut-point so we could "cut" the homotopy off to build a null homotopy for one of the loops. We will show that the same idea holds if the separating set is a highly connected set instead of a point. To do this we will require the following well known covering lemma for open sets in the plane. We provide a proof for completeness and to fix notation.

**Lemma III.2.5.** *Every bounded open set* $U$ *of* $\mathbb{R}^2$ *is the union of a null sequence of diadic squares with disjoint interiors. In addition, the squares can be chosen such that if* $A_i$ *is the union of squares with side length at least* $\frac{1}{2^i}$*, then* $U \backslash A_i \subset \mathcal{N}_{\frac{\sqrt{2}}{2^{i-1}}}(\partial U)$.

*Proof.* Let $Q_i$ be a sequence of partitions of the plane with the Euclidean metric into closed square discs with side length $\frac{1}{2^i}$ such that $Q_i$ refines $Q_{i-1}$. $Q_i$ can be chosen to be the set of squares with vertices $\big\{(\frac{j}{2^i},\frac{k}{2^i}),(\frac{j+1}{2^i},\frac{k}{2^i}),(\frac{j+1}{2^i},\frac{k+1}{2^i}),(\frac{j}{2^i},\frac{k+1}{2^i}) \mid j,k\in\mathbb{Z}\big\}$.

Let $D_0$ be the maximal subset of $Q_0$ such that $A_0 \subset U$ where $A_0 = \bigcup\limits_{s\in D_0} s$. Then $U\backslash A_0 \subset \mathcal{N}_{\frac{\sqrt{2}}{2^{-1}}}(\partial U)$.

We will inductively define $D_i$ and $A_i$ as follows. Let $D_i$ be the maximal subset of $Q_i$ such that $\bigcup\limits_{s\in D_i} s \subset \overline{U\backslash A_{i-1}}$ where $\overline{U\backslash A_{i-1}}$ is the closure of $U\backslash A_{i-1}$. Let $A_i = \big(\bigcup\limits_{s\in D_i} s\big)\cup A_{i-1}$. We immediately have $U\backslash A_i \subset \mathcal{N}_{\frac{\sqrt{2}}{2^{i-1}}}(\partial U)$. Then $\bigcup\limits_{i=1}^{\infty} A_i = U$.

$\square$

**Definition III.2.6.** Let $\xi\colon \mathbb{R}^+ \to \mathbb{R}^+\cup\{\infty\}$ be a continuous function which vanishes at 0. Then $\xi$ is a *modulus of continuity* for $g\colon (X,\mathrm{dist}_X)\to(Y,\mathrm{dist}_Y)$, if $\mathrm{dist}_Y\big(g(x),g(y)\big)\le \xi\big(\mathrm{dist}_X(x,y)\big)$ for all $x,y\in X$.

Let $(X,\mathrm{dist})$ be a path connected metric space and $\zeta\colon \mathbb{R}^+\to\mathbb{R}^+\cup\{\infty\}$ be an increasing function. We will say that $\zeta$ is a *modulus of path-connectivity* for $(X,\mathrm{dist})$; if every pair of points $x,y\in X$ there exists a path $\alpha$ from $x$ to $y$ such that $\mathrm{diam}(\alpha)\le\zeta\big(\mathrm{dist}(x,y)\big)$. If $X$ is geodesic than the identity function is a modulus of path-connectivity for $(X,\mathrm{dist})$.

**Remark III.2.7.** Let $g\colon (X,\mathrm{dist}_X)\to(Y,\mathrm{dist}_Y)$ be a continuous function on a compact metric space $X$. Then $\xi(r)=\sup\big\{\mathrm{dist}_Y\big(g(x),g(y)\big) \mid \mathrm{dist}_X(x,y)\le r\big\}$ is a modulus of continuity which is finite for every $r$. If $\xi'$ is another modulus of continuity for $g$, then $\xi'(r)\ge\xi(r)$.

Let $(X,\mathrm{dist})$ be a path connected space. Then there exists a modulus of path-connectivity for $X$ which vanishes at 0 if and only if $X$ is uniformly locally path connected.

**Lemma III.2.8.** *Suppose that $X$ is a metric space containing a closed, simply connected, uniformly locally path connected and uniformly locally simply connected subset $E$. If $h\colon A\to X$ is a continuous map from a planar annulus such that $h^{-1}(E)$ separates the boundary components of $A$, then $h$ takes the boundary components of $A$ to null homotopic loops in $X$.*

*Proof.* Let $A=\{(x,y)\in\mathbb{R}^2 \mid \frac{1}{4}\le x^2+y^2\le 1\}$ and $\mathbb{D}$ be the unit disc in the plane. It is enough to show that the outer boundary of $A$ maps to a null homotopic loop. Since $h^{-1}(E)$ separates the boundary components of $A$, a component $C$ of $h^{-1}(E)$ separates the boundaries components of $A$. This follows from the Phragmén-Brouwer properties, see[31]. Let $U$ be the component of $\mathbb{D}\backslash C$ which contains the circle of radius $\frac{1}{2}$. Thus $\partial U\subset A$ and $h(\partial U)\subset E$. Let $\xi$ be a modulus of continuity for $h$.

We can decompose $U$ as a null sequence of diadic squares with disjoint interiors, as in Lemma III.2.5. As before, let $A_i$ be the union of squares with side length at least $\frac{1}{2^i}$ which are contained in $U$ and $D_i$ the set of squares in $A_i$ of side length $\frac{1}{2^i}$. Then $\bigcup\limits_{i=1}^{\infty} D_i$ induces a cellular structure on $U$. We will use $U^{(i)}$ to denote the $i$-skeleton of this cellular structure on $U$. Note this implies that a side of a square in $D_i$ is not necessarily an edge but is an edge path.

We will now define a continuous map $g\colon \mathbb{D}\to X$ such that $g|_{\mathbb{D}\backslash U}=h$. If the boundary of $U$ is a loop, then this is obvious. However, the boundary does not have to be a loop. It can be very complicated (consider the Warsaw circle).

Let $\iota\colon U \to \partial U$ be a closest point projection map (which in general will be discontinuous), i.e. any map such that $\operatorname{dist}(x,\iota(x)) \le \operatorname{dist}(x,z)$ for all $z' \in \partial U$. For every $x \in U^{(0)}$, let $g(x) = h(\iota(x))$.

**Claim.** *If $x \in U^{(0)} \backslash A_i$ and $y \in \partial U$, then* $\operatorname{dist}\big(g(x),g(y)\big) \le \xi\left(\operatorname{dist}(x,y) + \frac{\sqrt{2}}{2^{i-1}}\right)$.

If $x \in U^{(0)} \backslash A_i$ and $y \in \partial U$, then $\operatorname{dist}\big(x,\iota(x)\big) \le \frac{\sqrt{2}}{2^{i-1}}$. Thus $\operatorname{dist}\big(\iota(x),y\big) \le \operatorname{dist}(x,y) + \frac{\sqrt{2}}{2^{i-1}}$ and the claim follows.

We now wish to extend $g$ continuously to $\mathbb{D}\backslash U \cup U^{(1)}$. Let $\zeta\colon \mathbb{R}^+ \to R^+ \cup \{\infty\}$ be a modulus of path-connectivity of $X$ which vanishes at 0. Then there exists a $\eta > 0$ such that $\zeta(t) < \infty$ for all $t < \eta$. Suppose that $e$ is an edge of $U^{(1)}$ with vertices $x,y$ such that $\operatorname{dist}\big(g(x),g(y)\big) < \eta$. Then there exists a path $\alpha_{x,y}$ in $X$ from $g(x)$ to $g(y)$ such that $\operatorname{diam}(\alpha_{x,y}) \le \zeta\left(\operatorname{dist}(g(x),g(y)\right)$. We may extend $g$ by sending $e$ to $\alpha_{x,y}$. Repeating this for all sufficiently short edges of $U^{(1)}$ and sending the other edges to any path between their end points, we can extend $g$ to $\mathbb{D}\backslash U \cup U^{(1)}$.

**Claim.** *$g\colon \mathbb{D}\backslash U \cup U^{(1)} \to X$ is continuous.*

Suppose that $x_n$ is a sequence of points in $U^{(1)}$ such that $x_n \to x_0$. If $x_0 \notin \partial U$, then $x_n$ is eventually contained in $A_i$ for some $i$ and $g(x_n) \to g(x_0)$ by the Pasting Lemma for continuous functions (see [35]).

If $x_0 \in \partial U$, then we can choose $x_n'$ such that $x_n, x_n'$ are contained in a single edge of $D^{(1)}$ and $x_n' \in U^{(0)}$. As well we may assume that, $x_n$ is contained in a sufficiently short edge (so as to assume the length condition holds on the edge). Since $x_n$ converges to $\partial U$, for every $i$ there exists an $N_i$ such that $x_n \in U^{(1)} \backslash A_i$ for all $n > N_i$. Then $\operatorname{dist}(x_n,x_n') \le \frac{1}{2^i}$ for all $n > N_i$. Thus $\operatorname{dist}\big(g(x_n),g(x_n')\big) \le \zeta\left(\xi(\frac{1}{2^i})\right)$ for all $n > N_i$. As well, $\operatorname{dist}(x_n,x_n') \le \frac{1}{2^i}$ for all $n > N_i$ implies that $x_n'$ converges to $x_0$.

Then

$$\begin{aligned}\operatorname{dist}(g(x_0),g(x_n)) &\le \operatorname{dist}(g(x_0),g(x_n')) + \operatorname{dist}(g(x_n'),g(x_n))\\ &\le \xi\left(\operatorname{dist}(x_0,x_n') + \sqrt{2}/2^{i-1}\right) + \zeta\left(\xi(1/2^i)\right)\end{aligned}$$

for all $n \ge N_i$. Thus $g|_{\mathbb{D}\backslash U \cup U^{(1)}}$ is continuous which completes the second claim.

Let $\varepsilon_i = \max\limits_{s \in D_i}\{\operatorname{diam}(g(\partial s))\}$ which is necessarily finite for all $i$. Since $g|_{\mathbb{D}\backslash U \cup U^{(1)}}$ is continuous, $\varepsilon_i$ converges to 0. Since $E$ is simply connected and uniformly locally simply connected, there exists $\delta_i$ such that for every $s \in D_i$ $g(\partial s)$ bounds a disc with diameter at most $\delta_i$ where $\delta_i \to 0$ as $i \to \infty$.

Fix $i > 0$ and $s \in D_i$. Then we can extend $g$ to all of $s$ by extended $g|_{\partial s}$ to a disc with diameter at most $\delta_i$.

By doing this process for all $s \in \bigcup_{i \ge 0} D_i$, we can extend $g$ to all of $\mathbb{D}$. Repeating the argument from the second claim and using the fact that $\delta_i \to 0$, we can see that this extension is continuous.

□

An interesting related proposition is the following van Kampen type result for fundamental groups.

**Proposition III.2.9.** *Suppose that $X = U \cup V$ is a connected metric space and $U \cap V$ is a non-empty, closed, simply connected, uniformly locally path connected and uniformly locally simply connected. Then for $x_0 \in U \cap V$, $\pi_1(V,x_0) * \pi_1(U,x_0)$ canonically embeds into $\pi_1(X,x_0)$.*

The homomorphism will not necessarily be a surjection. In fact, $\pi_1(X,x_0)\backslash\big(\pi_1(V,x_0)*\pi_1(U,x_0)\big)$ will often be uncountable, if both $U$ and $V$ are not locally simply connected at $x_0$.

The Griffith space is the wedge of two contractible spaces which has uncountable fundamental group [24]. The point is that free products only allow for finite products of loops but if the loops are getting small, the fundamental group allows for infinite products. For a rigorous definition of *infinite products* and further information on this type of phenomenon, see [9], [10].

*Proof of Proposition III.2.9.* Suppose that $f_i\colon\ (I,0,1)\to(V,x_0,x_0)$ and $g_j\colon\ (I,0,1)\to(U,x_0,x_0)$ are essential loops such that the loop $f_1*g_1*\cdots*f_n*g_n$ is null homotopic in $X$. Let $h\colon\ \mathbb{D}\to X$ be a null homotopy and $C$ a component of $h^{-1}(V)$ containing the portion of $\partial D$ which maps to $f_1$. Since $U\cap V$ is path connected and locally path connected, we can define a map $h'\colon\ C\cup\partial\mathbb{D}\to V$ such that $h'|_C=h$ and $h'(\partial\mathbb{D}\backslash C)\subset U\cap V$. Then, as in Lemma III.2.8, $h'$ can be extended to a null homotopy of $f_1$ which contradicts the assumption that $f_1$ was an essential loop. $\square$

We will use Olshanskiy's definitions from [36] for a *0-refinement of a van Kampen diagram*, *0-edges and 0-cells*, a *cancelable pair* in a van Kampen diagram, a *copy* of a cell under 0-refining, and *reduced diagrams*. Our definitions of $M$-bands, medians, and boundary paths of $M$-bands will follow that of [40].

**Definition III.2.10** ($M$-bands)**.** Let $M\subset S\cup\{1\}$ where 1 is the empty word is $S\cup S^{-1}$ and $\Delta$ be a van Kampen diagram over $\langle S\,|\,R\rangle$. An $M$-edge is an edge in $\Delta$ or $\Gamma(G,S)$ labeled by an element of $M$. An *$M$-band $\mathscr{T}$* is a sequence of cells $\pi_1,...,\pi_n$ in a van Kampen diagram over $\langle S\,|\,R\rangle$ such that

(i) every two consecutive cells $\pi_i$ and $\pi_{i+1}$ in this sequence have a common $M$-edge $e_i$ and

(ii) every cell $\pi_i$, $i=1,...,n$ has exactly two $M$-edges, $e_{i-1}$ and $e_i$.

Consider lines $l(\pi_i,e_i)$ and $l(\pi_i,e_{i-1})$ connecting a point inside the cell $\pi_i$ with midpoints of the $M$-edges of $\pi_i$. The broken line formed by the lines $l(\pi_1,e)$, $\cdots$, $l(\pi_i,e_i)$, $l(\pi_i,e_{i-1})$, $\cdots$, $l(\pi_n,e_n)$ is called the *median* of the band $\mathscr{T}$ and will be denoted by $m(\mathscr{T})$. It connects the midpoints of each $M$-edge and lies inside the union of $\pi_i$. We say that an $M$-band is an *$M$-annulus*, if $\pi_1$ and $\pi_n$ share an $M$-edge. If $\mathscr{T}$ is an $M$-annulus, then the edges $e_1$ and $e_n$ coincide and $m(\mathscr{T})$ is a simple closed curve. An $M$-band $\mathscr{T}$ will be *reduced* if no two consecutive cells are inverse images of each other.

Each cell $\pi_i$ of an $M$-band $\mathscr{T}$ can be viewed as an oriented 4-gon with edges $e_{i-1}$,$p_i$, $e_i$, $q_i$ where $e_{i-1},e_i$ are $M$-edges of $\pi_i$; $p_i$ begins at the initial vertex of $e_{i-1}$ and ends at the initial vertex of $e_i$; and $q_i$ begins at the terminal vertex of $e_{i-1}$ and ends at the terminal vertex of $e_i$. Then $p_1p_2\cdots p_n$ and $q_1q_2\cdots q_n$ edge paths in $\Delta$ which we will refer to as the *combinatorial boundary paths of $\mathscr{T}$* and denote by $\mathbf{top}_c(\mathscr{T})$, $\mathbf{bot}_c(\mathscr{T})$ respectively. However, the combinatorial boundary paths can have backtracking in the diagram. The *(topological) boundary paths of $\mathscr{T}$* are subpaths of $\mathbf{top}_c(\mathscr{T})$ and $\mathbf{bot}_c(\mathscr{T})$ obtained by removing all maximal subpaths consisting entirely of backtracking and will be denoted by $\mathbf{top}(\mathscr{T})$ and $\mathbf{bot}(\mathscr{T})$ respectively. While a topological boundary path has no backtracking, its label is not necessarily freely reduced. It is also possible that one of $\mathbf{top}(\mathscr{T})$ and $\mathbf{bot}(\mathscr{T})$ is empty.

Let $\mathscr{T}$ be a $M$-annulus in a circular diagram $\Delta$. $\mathscr{T}$ is a *minimal $M$-annuli*, if there are no $M$-annuli contained in the bounded component of $\mathbb{R}^2 \backslash m(\mathscr{T})$ where $\Delta$ is considered as a subset of $\mathbb{R}^2$. $\mathscr{T}$ is said to be a *maximal $M$-annulus* in $\Delta$ if it is not contained in the bounded component of $\mathbb{R}^2 \backslash m(\mathscr{T}')$ for any other $M$-annulus $\mathscr{T}'$ in $\Delta$. For a more complete description of $M$-bands and their boundaries see [40].

**Definition III.2.11.** Let $G_e$ be an HNN extension of a group $\langle A \,|\, R' \rangle$ with finitely generated associated subgroups. Then $G_e$ has a presentation

$$\langle A, t \,|\, R' \cup \{u_i^t = v_i\}_{i=1}^k \rangle$$

where $\{u_1, ..., u_k\}, \{v_1, ..., v_k\}$ are generating sets for the associated subgroups $H_e = \langle u_i \rangle, K_e = \langle v_i \rangle$.

Let $G_a$ be an amalgamated product of groups $\langle A_1 \,|\, R_1 \rangle$ and $\langle A_2 \,|\, R_2 \rangle$ along $\varphi \colon H_1 \to H_2$ where $H_i$ is a finitely generated subgroup of $\langle A_i \,|\, R_i \rangle$. Then $G_a$ has a presentation

$$\langle A_1, A_2 \,|\, R_1, R_2 \cup \{u_i = \varphi(u_i)\}_{i=1}^k \rangle$$

where $\{u_1, ..., u_k\}$ is a generating set for the associated subgroup $H_1$.

We will fix the groups $G_e$ and $G_a$ and their presentations for the remainder of Section III.2.

**Definition III.2.12.** Let $H$ be a subgroup of a group $G$ generated by $S$ and $Z, Z'$ be subsets of $\Gamma(G,S)$. We will say that $Z, Z'$ are *$H$-separated* if there exists $g \in G$ such that $Z, Z'$ are contained in distinct components of $\Gamma(G,S) \backslash gH$ where $gH$ is the set of vertices of $\Gamma(G,S)$ labeled by elements from the coset $gH$.

**Lemma III.2.13.** *Let $H$ be a subgroup of a group $G$ generated by $S$. The property of being $H$-separated is invariant under the left action of $G$ on $\Gamma(G,S)$.*

**Lemma III.2.14.** *Suppose that $H_e$ or $K_e$ is a proper subgroup of $\langle A \,|\, R' \rangle$. Let $\gamma$ be a loop in $\Gamma(G_e, S_e)$ and $N > \operatorname{diam}(\gamma)$. Then there exists elements $\{g_1, \cdots, g_N\}$ in $G_e$ such that*

*(i)* *$g_i \cdot \gamma, g_j \cdot \gamma$ are $H$-separated for $H \in \{H_e, K_e\}$ and*

*(ii)* *$|g_i g_j^{-1}| \geq 2N$ and $|g_i| \leq 4N$ for all $i \neq j$.*

*Proof.* Without loss of generality, we will assume $K_e$ is a proper subgroup. Let $\gamma$ and $N$ be as in the statement of the lemma. Choose $a \in \langle A \,|\, R' \rangle \backslash K_e$ and let $g_i \equiv t^N (ta)^i t^{-N}$. Notice that $g_i$ has no pinches for any $i \in \mathbb{Z}$ and $g_i g_j^{-1} = g_{i-j}$. For $i \neq j$, $|g_i g_j^{-1}|$ is at least $2N$ since $t^N (ta)^{i-j} t^{-N}$ has no pinches. Being $K_e$-separated is invariant under the action of $G_e$ on $\Gamma(G_e, S_e)$; hence, it is enough to show that $\gamma$ and $g_i \cdot \gamma$ are $K_e$-separated.

Let $x$ be the vertex of $\Gamma(G,S)$ with label $g_i$ and $x_0$ the vertex with label 1.

Since $t^N (ta)^i t^{-N}$ has no pinches, $g_i$ and 1 are in different components of $\Gamma(G_e, S_e) \backslash T^{N+1} K_e$ where 1 is the identity element of $G_e$. As well, $\operatorname{dist}(g_i, T^{N+1} K_e) \geq N$ and $\operatorname{dist}(1, T^{N+1} K_e) \geq N$. Then $N > \operatorname{diam}(\gamma)$ implies that $\gamma$, $g_i \cdot \gamma$ are in distinct components of $\Gamma(G_e, S_e) \backslash T^{N+1} K_e$.

□

An analogous proof gives us the following result for $G_a$ where $g_j = a_1^N (a_1 a_2)^j a_1^{-N}$ for $a_i \in A_i \backslash H_i$.

**Lemma III.2.15.** *Suppose that $H_i$ is a proper subgroup of $G_i$ for $i = 1,2$. Let $\gamma$ be a loop in $\Gamma(G_a, S_a)$ and $N > \operatorname{diam}(\gamma)$. Then there exists elements $\{g_1, \cdots, g_N\}$ in $G_a$ such that*

*(i)* $g_i \cdot \gamma, g_j \cdot \gamma$ *are $H_1$-separated and*

*(ii)* $|g_i g_j^{-1}| \geq 2N$ *and* $|g_i| \leq 4N$ *for all $i \neq j$.*

**Theorem III.2.16.** *Suppose that $G$ is an HNN-extension or amalgamated product where the associated subgroups are proper, quasi-isometrically embedded, prairie groups. Then every asymptotic cone of $G$ is either simply connected or has uncountable fundamental group.*

*Proof.* Let $G \in \{G_e, G_a\}$ and $S$ be the corresponding generating set for $G$. Suppose that $\operatorname{Con}^{\omega}\big(G,d\big)$ is not simply connected. Then there exists $\gamma$ an essential loop in $\operatorname{Con}^{\omega}\big(G,d\big)$ and we may choose loops $\gamma_n$ in $\Gamma(G,S)$ such that $\big(\gamma_n(t)\big) = \gamma(t)$. Let $c_n = 2\operatorname{diam}(\gamma_n)$. Let $S_n$ be the set of elements of $G$ given by Lemma III.2.14 or Lemma III.2.15. For every two distinct elements $g_n, h_n$ of $S_n$, $g_n \cdot \gamma_n$ and $h_n \cdot \gamma_n$ are $H$-separated for some quasi-isometrically embedded prairie subgroup $H$ of $G$.

Let $g = (g_n), h = (h_n) \in \prod^{\omega} S_n$.

**Claim.** *Then $g \cdot \gamma$, $h \cdot \gamma$ are well-defined loops in $\operatorname{Con}^{\omega}\big(G,d\big)$ and $g \cdot \gamma$ is not homotopic to $h \cdot \gamma$ if $g,h$ are distinct elements of $\prod^{\omega} S_n$.*

The first assertion follows from the fact that $g_n$ grows big O of the scaling sequence.

Suppose that $g \cdot \gamma$ is homotopic to $h \cdot \gamma$ for distinct $h,g$. Then $\omega$-almost surely $g_n \neq h_n$ and there exists $k_n$ such that $g_n \cdot \gamma_n$ and $h_n \cdot \gamma_n$ are in distinct components of $\Gamma(G,S) \backslash k_n H$.

Thus $g \cdot \gamma$, $h \cdot \gamma$ are in distinct components of $\operatorname{Con}^{\omega}\big(G,d\big) \backslash \lim^{\omega} k_n H$ by Lemma III.1.6. Since $H$ is quasi-isometrically embedded; $\lim^{\omega} k_n H$ is bi-Lipschitz to $\operatorname{Con}^{\omega}\big(H,d\big)$ which is simply connected, uniformly locally simply connected, and geodesic.

Thus $\lim^{\omega} k_n H$ is simply connected, uniformly locally simply connected, and uniformly locally path connected. Hence, Lemma III.2.8 implies that $g \cdot \gamma$ and $h \cdot \gamma$ are null-homotopic which contradicts our choice of $\gamma$.

This completes the proof of the claim. The theorem follows since $\prod^{\omega} S_n$ is uncountable. ☐

**Corollary III.2.17.** *If $G$ is has more than one end, then every asymptotic cone of $G$ is either simply connected or has uncountable fundamental group.*

*Proof.* If $G$ has more than one end, then it has a graph of groups decomposition with finite edge groups and hence is an HNN extension or an amalgamated product with finite associated subgroups and finite subgroups are always quasi-isometrically embedded prairie groups. ☐

This corollary was also shown in [18] since groups with more than one end are relatively hyperbolic.

A lemma due to Burillo.

**Lemma III.2.18** ([7])**.** *If $X$ has is quasi-isometric to a metric space with a log metric then every asymptotic cone of $X$ is totally disconnected.*

**Corollary III.2.19.** *Suppose that $G$ is an HNN-extension or amalgamated product where the associated subgroups are exponentially distorted. Then every asymptotic cone of $G$ is either simply connected or has uncountable fundamental group.*

*Proof.* We will proceed as in the proof of Theorem III.2.16. We only need to show how to circumvent the use of Lemma III.2.8.

We can construct $S_n$ as before and let $g = (g_n), h = (h_n)$ for $g_n, h_n \in S_n$.

If $g_n \neq h_n$ $\omega$-almost surely, then there exists $X = \lim^{\omega} k_n H$ such that $g \cdot \gamma$, $h \cdot \gamma$ are in distinct components of $\mathrm{Con}^{\omega}\big(G,d\big)\backslash X$. Since $H$ is exponentially distorted, it is totally disconnected by Lemma III.2.18.

Suppose that $h\colon A \to \mathrm{Con}^{\omega}\big(G,d\big)$ is a homotopy from $g \cdot \gamma$ to $h \cdot \gamma$. Then there exists a component $C$ of $h^{-1}(X)$ which separates the boundary components of $A$. Since $X$ is totally disconnected, $h(C)$ must be a point. Hence $h$ can be modified to a map on the disc by sending the component of the disc bounded by $C$ to $h(C)$. Thus $g \cdot \gamma$ must be null-homotopic, which contradicts our choice of $\gamma$.

□

**Corollary III.2.20.** *Let $G = \langle a, t \,|\, (a^p)^t = a^q \rangle$ be the Baumslag-Solitar group where $|p| \neq |q|$. For every $(\omega, d)$, $\mathrm{Con}^{\omega}\big(G,d\big)$ has the following properties.*

*(i)* $\mathrm{Con}^{\omega}\big(G,d\big)$ *is not semilocally simply connected.*

*(ii)* $\pi_1(\mathrm{Con}^{\omega}\big(G,d\big), x_0)$ *is not simple.*

*(iii) Every decomposition of* $\pi_1(\mathrm{Con}^{\omega}\big(G,d\big), x_0)$ *into a free product of subgroups has a factor which is a not free and uncountable.*

*(iv)* $\pi_1(\mathrm{Con}^{\omega}\big(G,d\big), x_0)$ *contains an uncountable free subgroup.*

*Proof.* Let $G = \langle a, t \,|\, (a^p)^t = a^q \rangle$ be the Baumslag-Solitar group where $|p| \neq |q|$. Properties $(i) - (iii)$ are proved in Corollary 3.2 of [12]. So we need only prove $(iv)$. The proof is an adaptation of the proof of Corollary III.1.22.

Since $\mathrm{Con}^{\omega}\big(G,d\big)$ is not semilocally simply connected, it is not simply connected. Thus is contains an essential loop $\gamma$. Theorem III.2.16 shows how to find an uncountable set of essential loops all of which are in distinct components of $\mathrm{Con}^{\omega}\big(G,d\big) \setminus \lim_e^{\omega} g_n \langle a^q \rangle$ for some choice of $g_n \in G$.

Using this uncountable set of loops, we can find $S'_{\gamma}$ as in Corollary III.1.22. We will now use the notation from Corollary III.1.22 and show how to modify the proof.

Suppose that $\mathrm{x}_1^{n_1} * \cdots * \mathrm{x}_k^{n_k}$ is a null homotopic loop in $\mathrm{Con}^{\omega}\big(G,d\big)$ where $\mathrm{x}_i \neq \mathrm{x}_{i+1}$, $\mathrm{x}_1 \neq \mathrm{x}_k$ and $\mathrm{x}_i^{n_i}$ is an essential loop. Then there exists $h\colon \mathbb{D} \to \mathrm{Con}^{\omega}\big(G,d\big)$ a map from the unit disc in the plane such that $h(\partial\mathbb{D})$ is a parameterization of the curve $\mathrm{x}_1^{n_1} * \cdots * \mathrm{x}_k^{n_k}$. Let $C$ be the closure of the connected component of $\mathbb{D} \backslash h^{-1}\{\lim_e^{\omega} g_n \langle a^q \rangle\}$ containing the subpath $p$ of $\partial\mathbb{D}$ which maps to $\mathrm{x}_1^{n_1}$.

Recall that $\langle a^q \rangle$ is exponential distorted in $G$. Thus $\lim_e^{\omega} g_n \langle a^q \rangle$ is totally disconnected by Lemma III.2.18.

Since $C$ is the closure of a component of $\mathbb{D} \backslash h^{-1}\Big(\{\lim_e^{\omega} g_n \langle a^q \rangle\}\Big)$, $\partial C \backslash \{p\}$ is connected and maps into $\lim_e^{\omega} g_n \langle a^q \rangle$. Hence $h(\partial C \backslash \{p\})$ is a point $b$.

Define $h'\colon \mathbb{D} \to \mathrm{Con}^{\omega}(G,d)$ by $h'(y) = h(y)$ for $y \in C$ and $h'(y) = b$ for $y \notin C$. Then $h'$ is continuous and $\mathrm{x}_1^{n_1}$ is null homotopic which contradicts our choice of $x_1^{n_1}$.

Again, the subgroup generated by $S'_\gamma$ may not by a free group but it is the free product of cyclic groups. Thus it is easy to find an uncountably generated free subgroup. This completes the proof of the corollary.

□

### III.2.1 Partitions of van Kampen diagrams

**Definition III.2.21.** Suppose that $\beta$ is a simple closed curve contained in the interior of a planar disc $D$. Then $D \backslash \beta$ has exactly two components. The component of $D \backslash \beta$ whose closure contains $\partial D$ will be called the unbounded component of $D \backslash \beta$. The other component will be called the bounded component. A point $v \in D$ is *interior (or exterior)* to $\beta$, if it is contained in the bounded (or unbounded) component of $D \backslash \beta$.

The following definition of partitions are due to Papasoglu in [44].

**Partitions of the unit disc in the plane:** Let $D$ be the unit disk in $\mathbb{R}^2$ or the planar annulus $\{(x,y) | x^2 + y^2 \in [\frac{1}{4}, 1]\}$. A *partition $P$ of $D$* is a finite collection of closed discs $D_1, \cdots, D_k$ in the plane with pairwise disjoint interiors such that $D = \cup_i D_i$, $\partial D = \partial(D_1 \cup \cdots \cup D_k)$, and $D_i \cap D_j = \partial D_i \cap \partial D_j$ when $i \neq j$. A point $p$ on $\partial D_1 \cup \cdots \cup \partial D_k$ is called a *vertex of the partition* if for every open set $U$ containing $p$, $U \cap (\partial D_1 \cup \cdots \cup \partial D_k)$ is not homeomorphic to an interval. An *edge of a partition* is a pair of adjacent vertices of a disc in the partition. A *piece of a partition* is the set of the vertices of a disc in the partition. A partition is then a cellular decomposition of the underline space of $P$ where each vertex has degree at least 3; so we will use the standard notation, $P^{(i)}$, to denote the $i$-th skeleton of a partition.

**Geodesic $n$-gons in a metric space X:** An *$n$-gon* in $X$ is a map from the set of vertices of the standard regular $n$-gon in the plane into $X$, i.e. an ordered set of $n$ points in $X$. If $X$ is a geodesic metric space, we can extend the $n$-gon to edges by mapping the edge between adjacent vertices of the standard regular $n$-gon in the plane to a geodesics segment joining the corresponding vertices of the $n$-gon in $X$. We will say that such an extension is a *geodesic $n$-gon* in $X$.

**Partitions of loops in a geodesic metric space X:** Let $\mathbb{D}$ be the unit disc in the plane and $\gamma\colon \partial\mathbb{D} \to X$ be a continuous map. A *partition of $\gamma$* is a map $\Pi$ from the set of vertices of a partition $P$ of $\mathbb{D}$ to $X$ such that $\Pi\big|_{\partial P \cap P^{(0)}} = \gamma\big|_{\partial P \cap P^{(0)}}$. The *vertices/edges/pieces* of $\Pi$ are the images of vertices/edges/pieces of $P$. We will write $\Pi(\partial D_i)$ for the pieces of $\Pi$, where $D_i$ are the 2-cells of the partition $P$.

**Remark III.2.22.** Suppose that $\Pi\colon P^{(0)} \to X$ is a partition of a loop $\gamma$ in a geodesic metric space. We can extend $\Pi$ to $P^{(1)}$ by mapping every edge contained in $\partial P^{(2)}$ to the corresponding subpath of $\gamma$ and every edge not contained in $\partial P^{(2)}$ to a geodesic segment joining its end points. Then the *length of a piece* is the arc length of the loop $\Pi(\partial D_i)$. We will write $|\Pi(\partial D_i)|$ for the length of the piece $\Pi(\partial D_i)$. We define the *mesh of* $\Pi$ by

$$\mathrm{mesh}(\Pi) = \max_{1 \le i \le k} \{|\Pi(\partial D_i)|\}.$$

At times it will be convenient to ignore some pieces of a partition. If $Z$ is a subset of the pieces of $P$, then the *relative mesh of* $\Pi$ is

$$\mathrm{rmesh}_Z(\Pi) = \max_{D_i \in Z} \{|\Pi(\partial D_i)|\}.$$

When $X$ is a Cayley graph of a group, we will also assume that the partition takes vertices of $P$ to vertices in the Cayley graph. A partition $\Pi$ is called a $\delta$-*partition*, if $\operatorname{mesh}\Pi < \delta$. A loop of length $k$ in a geodesic metric space is *partitionable* if it has a $\frac{k}{2}$-partition.

Let $P(\gamma,\delta)$ be the minimal number of pieces in a $\delta$-partition of $\gamma$ if a $\delta$-partition exist and infinity otherwise.

If $P$ is a partition of the unit disc in the plane, then $P^{(1)}$ can be considered as a planar graph where every vertex has degree at least 3. Then one can use the fact that the Euler characteristic of a planar graph is 1 to obtain the following.

**Lemma III.2.23.** *Let $\Pi\colon\ P^{(1)} \to \Gamma(G,S)$ be a partition of a loop in the Cayley graph of $G$. If $\Pi$ has $F$ pieces, then $\Pi$ has at most $3F$ edges and at most $2F$ vertices.*

A straightforward inductive argument gives us the following lemma.

**Lemma III.2.24.** *Suppose that $T$ is a finite simplicial tree with at most $j$ vertices of degree 1. Then $T$ has at most $j-1$ vertices with degree greater than 2.*

Our goal for the remainder of Section III.2.1 and Section III.3 is to define partitions of van Kampen diagrams and show how to use the standard techniques for reducing van Kampen diagrams to build nice partitions of loops in the Cayley graph.

**Definition III.2.25.** Suppose that $\langle A,t\,|\,R\rangle$ is an HNN-extension with stable letter $t$. Let $w$ be a word in the alphabet $S\cup S^{-1}$. We will use $|w|_F$ to denote the freely reduced word length of $w$, $|w|_G$ to denote the minimal word length of $w$ in $\langle A,t\,|\,R\rangle$ and $|w|_t$ to denote the number of $t$-letters in $w$.

A word $w$ is a *$t$-shortest word* if $|w|_t \le |w'|_t$ for all $w' =_G w$ and

$$|w|_G = |w|_t + \sum |v_i|_G \tag{III.2}$$

where $v_i$ ranges over maximal $a$-subwords of $w$. To avoid trivialities, we will also require that every $a$-subword of a $t$-shortest word be freely reduced.

We will say that $w$ is an *almost $t$-shortest word* if $|w|_t \le |w'|_t$ for all $w' =_G w$.

A path $\gamma$ in the Cayley graph of $G$ is a *$t$-shortest path* (or an *almost $t$-shortest path*) if $\mathbf{Lab}\,(\gamma)$ is a $t$-shortest word (or an almost $t$-shortest word).

The equality in (III.2) implies that if we replace each maximal $a$-subword of a $t$-shortest path with a geodesic, then the whole path is geodesic. This gives us the following result.

**Lemma III.2.26.** *Every edge in $\Gamma(G,S)$ labeled by a $t$-letter on a $t$-shortest path from $g$ to $h$ is also an edge of a geodesic from $g$ to $h$.*

**Definition III.2.27.** Let $P$ be a partition of the unit disc $\mathbb{D}^2$ or the unit annulus in the plane and $\Delta$ a van Kampen diagram over $\langle S\,|\,R\rangle$. A continuous map $\Psi\colon\ P^{(2)} \to \Delta$ is a *partition of* $\Delta$ if it satisfies the following conditions.

(i) $\Psi(P^{(0)}) \subset \Delta^{(0)}$

(ii) $\Psi$ takes edges of $P$ to edge paths in $\Delta^{(1)}$

(iii) For each closed 2-cell $D$ of $P$, $\Psi(D)$ is a reduced subdiagram of $\Delta$

If we consider $\Delta$ as a metric space with the edge metric, then $\Psi|_{P^{(0)}}$ is a partition of the loop $\partial\Delta$ under our previous definition.

As before, the *edges/vertices/pieces of* $\Psi$ are the image under $\Psi$ of edges/vertices/pieces of $P$ in $\Delta$.

Define the *mesh of* $\Psi$ by $\text{mesh}(\Psi) = \text{mesh}(\theta \circ \Psi)$ where $\theta$ is the canonical map into the Cayley complex.

$\Psi$ is an *h-partition* of $\Delta$, if $\Psi$ is partition of $\Delta$ and a homeomorphism. If $\Psi$ is a $h$-partition of $\Delta$ and $\theta \circ \Psi$ takes edges of $P$ to geodesic paths ($t$-shortest paths), then we will say $\Psi$ is a *geodesic partition ($t$-shortest partition)* of $\Delta$.

This gives the underling space of $\Delta$ two cell structures, the cell structure inherited as a van Kampen diagram and the cell structure inherited from the partition. When there is a chance of confusion, we will specify if we are considering a vertex/edge in the underling space as a $\Psi$-vertex/$\Psi$-edge or a $\Delta$-vertex/$\Delta$-edge.

The following lemma follows trivially by considering each of the three types of 0-refinements.

**Lemma III.2.28.** *Suppose that $\Psi\colon\ P^{(2)} \to \Delta$ is a partition (or a geodesic partition) and $\Delta'$ is a 0-refinement of $\Delta$. Then there exists a partition (or geodesic partition) $\Psi\colon\ P^{(2)} \to \Delta'$ which preserves the number of pieces, edges, and vertices; the mesh of the partition; and the labels of edges (after removing any possible 1's).*

## III.3 HNN extensions with free associated subgroups

Let $G$ be a multiple HNN extension of a free group $F$ with free associated subgroups. Then $G$ has a presentation

$$\langle A \cup \{t_i\} \mid \{u_{i,s}^{t_i} = v_{i,s}\} \text{ for } i = 1,\dots,k \text{ and } s = 1,\dots,j_i \rangle$$

where $U_i = \langle u_{i,1},\dots,u_{i,j_i}\rangle, V_i = \langle v_{i,1},\dots,v_{i,j_i}\rangle$ are free subgroups with free generating sets $\{u_{i,j}\}$, $\{v_{i,j}\}$ respectively and $t_i$ are stable letters. We will use $\langle S \mid R \rangle$ to denote this presentation for $G$ which we will fix throughout Section III.3. Let

$$K = \max\{|u_{i,1}|_F,\dots,|u_{i,j_i}|_F, |v_{i,1|_F},\dots,|v_{i,j_i}|_F\}.$$

We will also fix the constant $K$ throughout this section. To simplify notation, we will frequently refer to $t_i$-bands in diagrams over $\langle S \mid R \rangle$ as just $t$-bands when the specific $i$ is inconsequential.

**Lemma III.3.1.** *Let $\mathscr{T}$ be a $t$-band in a van Kampen diagram $\Delta$. Then $\Delta$ can be modified while preserving the numbers of cells and the boundary label of $\Delta$ such that the label of **top**$(\mathscr{T})$ and **bot**$(\mathscr{T})$ are freely reduced words.*

*Proof.* If $\mathbf{Lab}\,(\mathbf{bot}(\mathscr{T})) = w_1uu^{-1}w_2$, then we may cut $\Delta$ along the subpath of $\mathbf{bot}(\mathscr{T})$ labeled by $uu^{-1}$ and re-identify them as in Figure III.1. This is the so called *diamond move* (see [11]). A similar process can be performed for $\mathbf{top}(\mathscr{T})$. □

Edges in gray are labeled by $u$ or $u^{-1}$ respectively

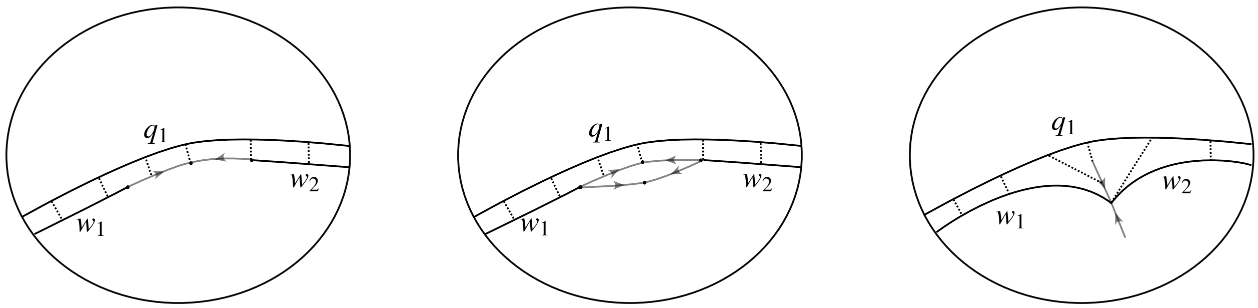


Figure III.1: Modifying $\Delta$ to insure that the label of the boundary of a $t$-band is freely reduced.

**Lemma III.3.2.** *Suppose that $\mathscr{T}$ is a reduced $t$-band in a van Kampen diagram $\Delta$ over $\langle S \mid R \rangle$ endowed with the edge metric. Then there exist an $L$ such that $\mathbf{top}(\mathscr{T})$ is in the $L$-neighborhood of $\mathbf{bot}(\mathscr{T})$ where $L$ is a constant depending only on the associated subgroups.*

*Proof.* The lemma is trivial if you are considering $\mathbf{top}_c(\mathscr{T})$ and $\mathbf{bot}_c(\mathscr{T})$ in placy of $\mathbf{top}(\mathscr{T})$ and $\mathbf{bot}(\mathscr{T})$. So we will prove the lemma by finding a bound on the diameter of the backtracking that was removed to obtain $\mathbf{top}(\mathscr{T})$.

Recall that $\{u_{i,1},...,u_{i,j_i}\},\{v_{i,1},...,v_{i,j_i}\}$ are free generating sets for the associated subgroups where $u_{i,k}, v_{i,k}$ are words in the alphabet $A$. For the purposes of this lemma; let $U$ be the disjoint union of $\langle u_{i,1},...,u_{i,j_i}\rangle$ and $\langle v_{i,1},...,v_{i,j_i}\rangle$ and if $g \in U$, let $|g|_s$ denote the length in the associated subgroup. Let

$$L' = \max\{|g|_s \mid g \in U \text{ and } |g|_G \leq 2K\}.$$

Fix $\mathscr{T}$ a reduced $t$-band in $\Delta$ and $v$ a vertex on $\mathbf{top}(\mathscr{T})$. Then there exists a vertex $v'$ on $\mathbf{bot}_c(\mathscr{T})$ such that $\operatorname{dist}(v,v') \leq K+1$. Suppose that $p$ is a maximal subpath of $\mathbf{bot}_c(\mathscr{T})$ which contains the vertex $v'$ and has freely trivial label in $F(A)$. We will assume (without loss of generality) that $\mathbf{bot}_c(\mathscr{T})$ is labeled by words from $\{u_{i,1},...,u_{i,j_i}\}$. Then for some $j$, $\mathbf{Lab}(p) = w_1 u_{j,s_1}^{\varepsilon_1} \cdots u_{j,s_r}^{\varepsilon_r} w_2$ where $w_1$ is a terminal segment of $u_{j,s_0}^{\varepsilon_0}$, $w_2$ is an initial segment of $u_{j,s_{r+1}}^{\varepsilon_{r+1}}$, and $\varepsilon_i = \pm 1$. Let $g = u_{j,s_0}^{\varepsilon_0} \cdots u_{j,s_{r+1}}^{\varepsilon_{r+1}}$. By construction $|g|_G$ is at most $2K$ and in $U$. Thus $|g|_s \leq L'$. This implies that $v'$ is at most $L'K$ from a vertex of $\mathbf{bot}(\mathscr{T})$.

Thus $v$ is at most $L = L'K + K + 1$ from a vertex of $\mathbf{bot}(\mathscr{T})$ which completes the lemma.

□

The following lemma is a correction of a lemma by Olshanskii and Sapir in [39].

**Lemma III.3.3.** *There exists a constant $L$ such that every diagram over $\langle S \mid R \rangle$ which has no $t$-annuli and all $t$-bands are reduced has diameter no greater than $\frac{3L|\partial\Delta|}{2}$.*

*Proof.* Let $L$ be the constant from Lemma III.3.2.

Let $s$ be the number of $t$-bands in $\Delta$ and $n = |\partial\Delta|$. Then $s \leq \frac{n}{2}$. There exists a $t$-band $\mathscr{T}$ such that (without loss of generality) $\mathbf{top}_c(\mathscr{T})$ is contained in $\partial\Delta$ (see Lemma 2.1 of [39]). Then $\Delta$ is obtained by gluing $\mathscr{T}$ and a diagram $\Delta_1$ with $s-1$ $t$-bands which satisfies the same hypothesis. Every vertex on a $\mathbf{bot}(\mathscr{T})$ can be connected to the boundary of $\Delta$ by a path of length at most $L$. By induction on $s$, we can deduce that every vertex inside $\Delta$ can be connected to the boundary of $\Delta$ by a path of length at most $Ls \leq \frac{Ln}{2}$. Hence the diameter of $\Delta$ is at most $\frac{3Ln}{2}$. $\square$

**Lemma III.3.4.** *Let $\Delta$ be a van Kampen diagram with no $t$-annuli, every $t$-band reduced, and $\gamma\colon [0,1] \to \Delta$ be a parametrization of $\partial\Delta$. Suppose that $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ is a partition of the unit interval and $I$ a subset of $\{0, \cdots, k\}$ such that $\gamma$ restricted to $[t_i, t_{i+1}]$ is a $t$-shortest path for $i \in I$. Then $\theta(\Delta)$ has diameter no greater than*

$$\frac{5L}{2}\Big(\sum_{i \notin I}\Big|\gamma|_{[t_i,t_{i+1}]}\Big| + \sum_{i \in I}\operatorname{dist}\big(\theta\circ\gamma(t_i), \theta\circ\gamma(t_{i+1})\big)\Big)$$

*where $L$ is the constant for Lemma III.3.2 and $\theta$ is the canonical map into the Cayley graph.*

*Proof.* Let $C = \sum_{i \notin I}\big|\gamma|_{[t_i,t_{i+1}]}\big| + \sum_{i \in I}\operatorname{dist}\big(\theta\circ\gamma(t_i), \theta\circ\gamma(t_{i+1})\big)$. By the same argument as in Lemma III.3.3, every vertex of $\Delta$ can be connected to a vertex on $\partial\Delta$ by a path of length at most $Ls$ where $s$ is the number of $t$-bands in $\Delta$.

For $i \in I$, let $w_i = \mathbf{Lab}\,(\gamma|_{[t_i,t_{i+1}]})$ and $\tilde{w}_i$ be a geodesic word obtained by replacing each maximal $a$-subpath of $w_i$ by a geodesic word. For $i \notin I$, let $w_i = \mathbf{Lab}\,(\gamma|_{[t_i,t_{i+1}]}) = \tilde{w}_i$. Then $C = |\tilde{w}_0\tilde{w}_1\cdots\tilde{w}_k|$. Fix $\Delta_i$ a reduced van Kampen diagram with $\partial\Delta_i = p_i\tilde{p}_i$ where $\mathbf{Lab}\,(p_i) = w_i$ and $\mathbf{Lab}\,(\tilde{p}_i) = \tilde{w}_i^{-1}$. Let $s_i$ be the number of $t$-bands in $\Delta_i$. Since no $t$-band of $\Delta_i$ can start and stop on $p_i$; hence, $s, s_i \leq \frac{C}{2}$. By repeating the arguments from Lemma III.3.3, we can see that any point in $\Delta_i$ is at most $Ls_i$ from a point on $\tilde{p}_i$. Hence, if $x, y$ are two points on $\partial\Delta$, then $\operatorname{dist}\big(\theta(x), \theta(y)\big) \leq Ls_i + Ls_j + \frac{C}{2} \leq \frac{LC}{2} + \frac{LC}{2} + \frac{C}{2}$.

If $x, y$ are two points in $\Delta$; $\operatorname{dist}\big(\theta(x), \theta(y)\big) \leq 2(Ls) + (Ls_i + Ls_j + \frac{C}{2})$. Therefore $\theta(\Delta) \leq LC + (LC + \frac{C}{2}) \leq \frac{5LC}{2}$.

$\square$

**Remark III.3.5.** Let $\Pi\colon P^{(0)} \to \Gamma(G,S)$ be a partition of a loop $\gamma$ in $\Gamma(G,S)$. We can extend $\Pi$ to $P^{(1)}$ as in Remark IV.0.18; but instead of mapping the interior edges of $P$ to geodesics, we will map the interior edges to $t$-shortest paths in $\Gamma(G,S)$. We can label the edges of $P^{(1)}$ with the label of their image. Then we can fill each piece with a reduced circular van Kampen diagram. This produces a van Kampen diagram with boundary label equal to the $\mathbf{Lab}\,(\gamma)$ and $\Pi$ induces a canonical homeomorphism from $P^{(2)}$ onto this van Kampen diagram. Thus every partition $\Pi$ of $\gamma$ induces a $t$-shortest partition $\Psi$ of a diagram such that $\Pi = \theta\circ\Psi$. Then by Lemma III.3.4, each subdiagram corresponding to a piece has diameter at most $\frac{5Ln}{2}$.

### III.3.1 Removing $t$-bands from partitions

**Definition III.3.6.** Suppose $\Psi\colon P^{(2)} \to \Delta$ is a $t$-shortest partition of a van Kampen diagram $\Delta$. A $t$-band $\mathscr{T}$ *crosses* a $\Psi$-edge $e$, if $e$ contains a $t$-edge from $\mathscr{T}$. If $\mathscr{T}$ is a $t$-annulus which crosses a $\Psi$-edge $e$, we will call the end points of the corresponding $t$-edge, the *crossing vertices* of $\mathscr{T}$.

**Lemma III.3.7.** *If $\Psi$ is a $t$-shortest partition of $\Delta$ and $\mathscr{T}$ is a $t$-band in $\Delta$, then $\mathscr{T}$ crosses each $\Psi$-edge at most once.*

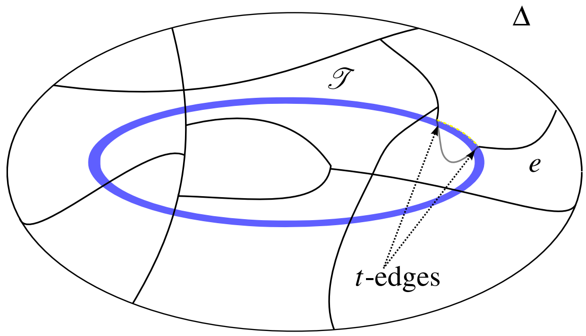


Figure III.2: A $\Psi$-edge which crosses $\mathscr{T}$ twice cannot be $t$-shortest.

*Proof.* If $\mathscr{T}$ crossed a $\Psi$-edge $e$ twice, then $e$ would contain two $t$-edges and the subword of $e$ beginning and ending with these $t$-edges would be equal to a subword of $\mathbf{top}_c(\mathscr{T})$ or $\mathbf{bot}_c(\mathscr{T})$. See Figure III.2. Thus $e$ was not $t$-shortest. (Note we are using the fact the diagrams are planar.) $\square$

**Corollary III.3.8.** *Let $\Psi\colon P^{(2)} \to \Delta$ be a $t$-shortest partition of $\Delta$ and $\mathscr{T}$ be a $t$-annulus in $\Delta$. Then the bounded component of $P^{(2)} \backslash \Psi^{-1}(m(\mathscr{T}))$ contains a vertex of $P$.*

**Corollary III.3.9.** *Let $\Psi\colon P^{(2)} \to \Delta$ be a $t$-shortest partition of $\Delta$. Then $\Delta$ can have at most $V$ maximal $\mathscr{T}$-annuli where $V$ is the number of vertices of the partition $\Psi$.*

**Lemma III.3.10.** *Let $\Psi\colon P^{(2)} \to \Delta$ be a h-partition of $\Delta$ with $F$ pieces where $\Delta$ is an annular diagram where the boundary components have labels which are trivial in $G$. Suppose that $\mathscr{T}$ is a $t$-annulus in $\Delta$ such that $\mathscr{T}$ crosses each edge at most once and if $v$ is a crossing vertex of a $\Psi$-edge with vertices $e_-, e_+$, then* $\mathrm{dist}(e_-,e_+) \leq \mathrm{dist}(e_-,v) + \mathrm{dist}(v,e_+)$. *Let $B = \max_{D\in P}\{\mathrm{diam}(\theta\circ\Psi(D))\}$.*

*Then there exists a partition $\widetilde{\Psi}\colon \widetilde{P}^{(2)} \to \Delta'$ where $\Delta'$ is obtained by removing $\mathscr{T}$ such that*

*(i)* $\widetilde{\Psi}$ *has no more than $9F^2 + 4F$ pieces, and*

*(ii)* $\mathrm{mesh}\,(\widetilde{\Psi}) \leq \max\{3(B+2K), \mathrm{mesh}\,(\Psi)\}$,

*where $K$ is the max of the word length of the generators of the associated subgroups.*

*Proof.* Let $\Psi\colon P^{(2)} \to \Delta$ be a partition of $\Delta$ as in the statement of the lemma and let $A$ be the underline space of $P$. Let $\Delta_A$ be the subdiagram of $\Delta$ obtained by removing all cells interior to $\mathbf{top}_c(\mathscr{T})$.

Let $V = \{v_1, v_2, \cdots, v_k\}$ be the set of crossing vertices of $\mathscr{T}$ which are contained in $\mathbf{top}_c(\mathscr{T})$ where the ordering is obtained by traversing $\mathbf{top}_c(\mathscr{T})$ in the clockwise direction. Let $q_i$ be a subpath of $\mathbf{top}_c(\mathscr{T})$ between $v_i$ and $v_{i+1}$ without backtracking (where the indices are taken modulo $k$) which intersects $V$ only at $v_i, v_{i+1}$ and $m(q_i)$ the corresponding subpath of $m(\mathscr{T})$. Since $\mathscr{T}$ crosses each vertex at most once, $k \leq 3F$.

By construction $m(q_i)$ is contained inside of $\Psi(D)$ for some piece $D$ of $P$. Thus $q_i$ is in the $K$-neighborhood of $\Psi(D)$ and $\mathrm{diam}(\theta\circ\Psi(q_i)) \leq B + 2K$

**Claim 1.** *There exists a refinement $P'$ of $P$ and a partition $\Psi'\colon P'^{(2)} \to \Delta$ with $\Psi'(x) = \Psi(x)$ for all $x \in P^{(1)}$ such that*

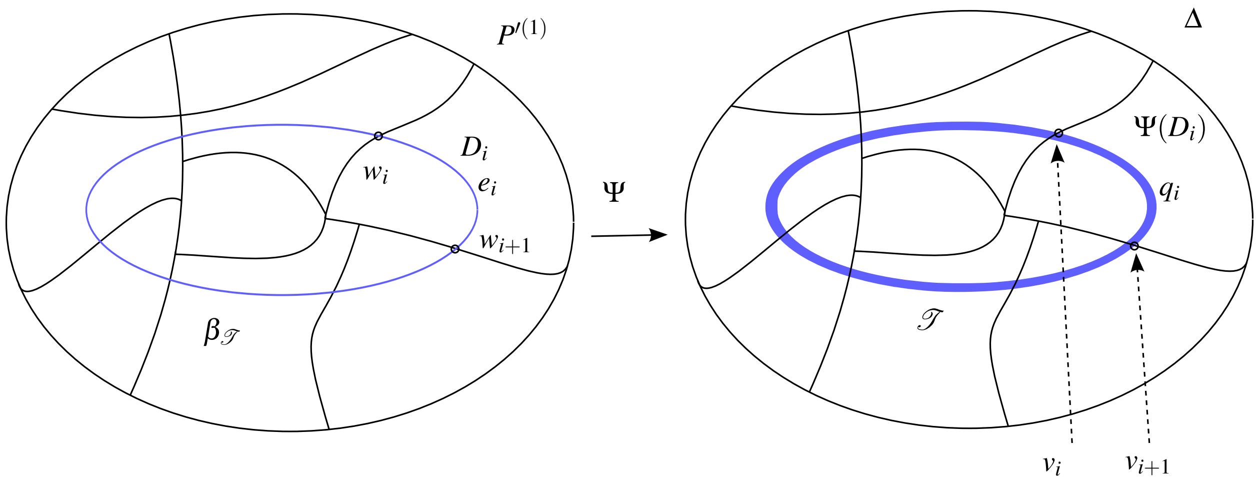


Figure III.3: Constructing $P'$

*(i) the number of pieces of $P'$ is less than $4F$;*

*(ii)* $\operatorname{mesh}(\Psi') \leq \operatorname{mesh}(\Psi)$*; and*

*(iii) there is a simple closed curve $\beta_{\mathscr{T}}$ in $P'^{(1)}$ such that*

*(a)* $\Psi(\beta_{\mathscr{T}}) \subset \boldsymbol{top}_c(\mathscr{T})$*,*

*(b) $\beta_{\mathscr{T}}$ has at most $3F$ edges, and*

*(c) if $\Psi'(x)$ is interior to $m(\mathscr{T})$, then $x$ is interior to $\beta_{\mathscr{T}}$.*

*Proof of Claim 1.* Let $w_i = \Psi^{-1}(v_i)$ and $W = \{w_i\}$. For each pair $i$, there exist a unique cell $D_i$ of $P$ such that $\Psi^{-1}\big(m(q_i)\big) \subset D_i$. Let $e_i$ be an arc in $D_i$ from $w_i$ to $w_{i+1}$ such that $e_i \cap P^{(1)} = \{w_i, w_{i+1}\}$. In addition, we may assume that the arcs $e_i$ have disjoint interiors. Then $\beta_{\mathscr{T}} = e_1 * e_2 * \cdots * e_k$ is a simple closed curve.

Let $P'^{(0)} = P^{(0)} \cup W$. The edges of $P'$ are the closure of the connected subsets of $P^{(1)} \cup \beta_{\mathscr{T}} \setminus P'^{(0)}$. This gives $\beta_{\mathscr{T}}$ a cellular structure. Each vertex of $\beta_{\mathscr{T}}$ corresponds to a crossing vertex of $\partial_o \mathscr{T}$. Since $\beta_{\mathscr{T}}$ has at most $3F$ vertices and each edge cuts a piece of $P$ into two pieces, $P'$ has at most $4F$ pieces.

We can define $\Psi'|_{P^{(1)}} = \Psi$ and map $e_i$ to $q_i$. By Lemma III.3.1, we may also assume that $\Psi'(e_i)$ has freely reduced label. We can extend $\Psi'$ to the 2-cells of $P'$ in the natural way. Then $\Psi' \colon P'^{(2)} \to \Delta$ is a partition of $\Delta$ which satisfies the first and third conditions of the claim.

The geodesic condition on crossing vertices guarantees that the mesh does not increase as we add the vertices $v_i$ and the edges $e_i$.

□

Claim 1 gives us that $\Psi(\beta_{\mathscr{T}})$ bounds a subdiagram of $\Delta$ with freely trivial boundary label and $\beta_{\mathscr{T}}$ bounds a subcomplex of $P'^{(1)}$. There exist a simplicial tree $L_{\mathscr{T}}$ labeled by $a$-letters and a map $\Upsilon \colon \beta_{\mathscr{T}} \to L_{\mathscr{T}}$ such that $\theta \circ \Psi'|_{\beta_{\mathscr{T}}} = \theta' \circ \Upsilon$ where $\theta'$ is a label preserving map from $L_{\mathscr{T}}$ into $\Gamma(G,S)$. $L_{\mathscr{T}}$ is constructed by choosing a free reduction of $\mathbf{Lab}\,(\Psi'(\beta_{\mathscr{T}}))$.

We can replace the subdiagram in $\Delta$ bounded by $\Psi(\beta_{\mathscr{T}})$ with $L_{\mathscr{T}}$. This creates a pairing of $\Delta$-edges in $\Delta$. What we want to be able to do is mirror this identification of edges on $\beta_{\mathscr{T}}$. The problem is that this identification can pair proper segments of edges in $\beta_{\mathscr{T}}$. To correct this we will need to add new vertices to $P'$ to insure that this identification respects $\Psi'$-edges. In general, this will cause the mesh to increase since edges of $\beta_{\mathscr{T}}$ do not map to geodesics. So we will subdivide pieces to get a useful bound on our new mesh. This is where the bound $B$ on the diameter of each piece comes into play.

We will say that a subpath of $\beta_{\mathscr{T}}$ is an *$L_{\mathscr{T}}$-segment*, if all vertices of the edge path except possible the initial and terminal vertices have degree 2 in $\Upsilon(\beta_{\mathscr{T}})$.

**Claim 2.** *There exists a refinement $P''$ of $P'$ and a partition $\Psi'' \colon P''^{(2)} \to \Delta$ with $\Psi''(x) = \Psi'(x)$ for all $x \in P''^{(2)} = P'^{(2)}$ such that*

*(i) the number of pieces of $\Psi''$ is no more than $9F^2 + 4F$,*

*(ii) $\beta_{\mathscr{T}}$ is subdivided into at most $9F^2$ edges and each edge is an $L_{\mathscr{T}}$-segment, and*

*(iii) $\operatorname{rmesh}_Z(\Psi'') \leq \max\{3(B+K), \operatorname{mesh}(\Psi)\}$ where $Z$ is the set of pieces of $P''$ which are not interior to $\beta_{\mathscr{T}}$.*

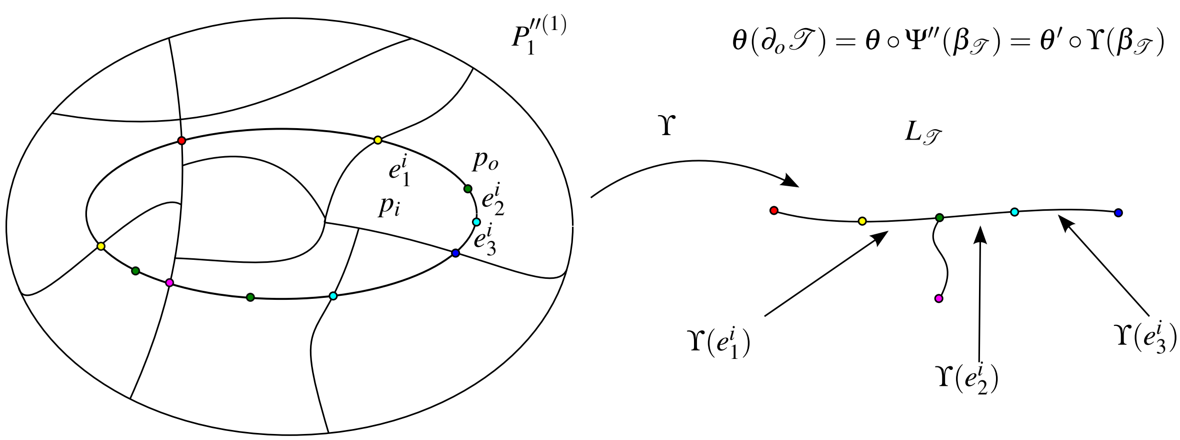


Figure III.4: $P_1''^{(1)}$ and $L_{\mathscr{T}}$

*Proof of Claim 2.* $\Upsilon$ must map each $e_i$ injectively into $L_{\mathscr{T}}$, since $\mathbf{Lab}(\Psi'(e_i))$ is freely reduced. Thus a vertex of $L_{\mathscr{T}}$ with degree 1 must be the image of a vertex of $e_i$ for some $i$ and $\Upsilon(\beta_{\mathscr{T}})$ has at most $3F$ vertices of degree 1. Then Lemma III.2.24 implies that it has at most $3F$ vertices of degree greater than 2. For each $i$, we can add new vertices to $e_i$ which are the unique $\Upsilon$-preimage of vertices of $L_{\mathscr{T}}$ with degree greater than 2 or the unique $\Upsilon$-preimage of a point of $\Upsilon(W)$ (see Figure III.4). Doing this subdivides $e_i$ into at most $3F$ edges which we will label by $e_j^i$ with their ordering induced by $e_i$. This divides $\beta_{\mathscr{T}}$ into at most $9F^2$ edges.

Let $P_1''$ be the cellular decomposition obtained by adding $\{e_i^j\}$ to $P'$. Notice the $P_1''$ is not a partition of $A$ since it has vertices of degree 2.

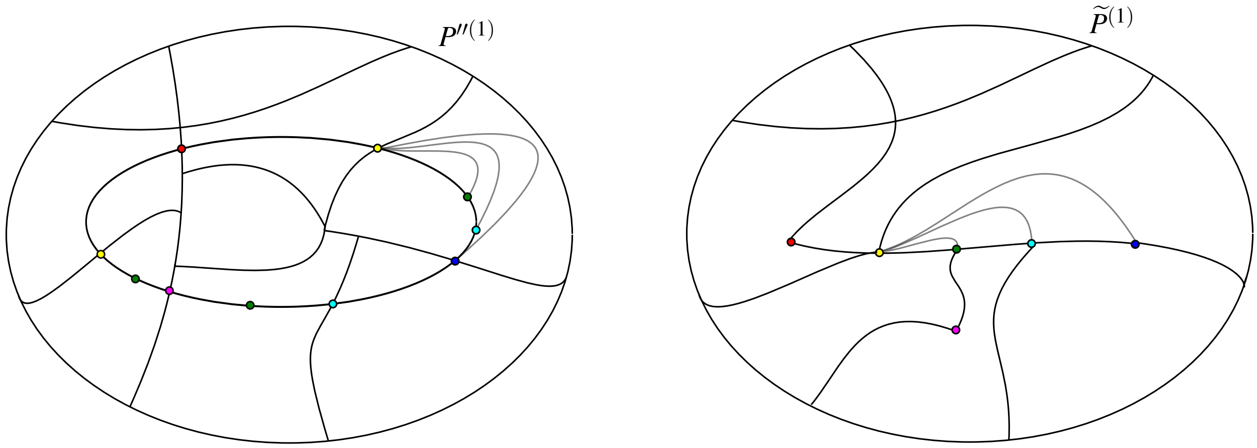


Figure III.5: Constructing $\widetilde{P}$

In $P'$ there existed exactly two pieces which share $e_i$ as a common edge, $p_i$ which is contained in the bounded component of $\mathbb{R}^2 \backslash \beta_{\mathscr{T}}$ and $p_o$ which is contained in the unbounded component (see Figure III.4).

We will now subdivide the piece $p_o$ to obtain pieces with bounded mesh (see Figure III.5). Let $f^i_j$ be an arc in $p_o$ from the initial vertex of $e^i_1$ to the terminal vertex of $e^i_j$ for all $j > 1$. We also will require that the new edges have disjoint interiors contained in $p_o$. This subdivides $p_o$ into at most $3F + 1$ pieces, i.e. we add $3F$ pieces to our count. Repeating this process for each $i$, gives us a partition $P''$ of $A$.

We must now explain how to map these edges into $\Delta$. Each new edge connects points with image on the $\Psi(\beta_{\mathscr{T}})$. Thus we can send each edge to the reduced subpath of $\Psi(\beta_{\mathscr{T}})$ connecting the images of their vertices and map the 2-cells in the natural way. Let $\Psi'' \colon P''^{(2)} \to \Delta$ be this new partition.

The distance between $e^i_j$ and $e^i_{j'}$ is at most $B + 2K$ for all $j$ and $j'$. This implies that the requirement on the mesh is then satisfied.

□

We can replace the subdiagram of $\Delta$ bounded by $\Psi'(\beta_{\mathscr{T}})$ with $L_{\mathscr{T}}$, creating a new van Kampen diagram $\Delta'$. This also induces a paring of edges on $\beta_{\mathscr{T}}$ such that after removing the disc bounded by $\beta_{\mathscr{T}}$ and identifying edges of $\beta_{\mathscr{T}}$ according to this pairing, we obtain a new partition $\widetilde{P}$ of the quotient space $A'$. If $m(\mathscr{T})$ separates the boundary components of $A$, then $A'$ is a planar disc. If $m(\mathscr{T})$ doesn't separate the boundary components of $A$, then $A'$ is an annulus. Then $\Psi''$ induces a map $\widetilde{\Psi} \colon \widetilde{P}^{(2)} \to \Delta'$ with the desired properties, see Figure III.5.

□

**Definition III.3.11.** Recall that $G$ has a presentation

$$\langle A \cup \{t_i\} \,|\, \{u_{i,s}^{t_i} = v_{i,s}\} \text{ for } i = 1, ..., k \text{ and } s = 1, ..., j_i \rangle$$

where $U_i = \langle u_{i,1}, ..., u_{i,j_i} \rangle, V_i = \langle v_{i,1}, ..., v_{i,j_i} \rangle$ are free subgroups with free generating sets $\{u_{i,j}\}$, $\{v_{i,j}\}$ respectively and $t_i$ are stable letters.

Let $X_i$ be the midpoints of the set of edges $\{(g, t_i) \mid g \in U_i\}$ in $\Gamma(G, S)$ .

By Britton's lemma, $gX_i$ separates $\Gamma(G,S)$ for every $g \in G$. Let $x_1, x_2$ be two points in $X_i$ such that $x_2 =_G x_1 u_{i,j}$. Then in $\Gamma^2(G,S)$ we can find an arc joining $x_1$ to $x_2$ which intersects $\Gamma(G,S)$ only at $x_1$ and $x_2$. Let $T_i$ be the subset of $\Gamma^2(G,S)$ obtained by connecting all such points of $X_i$ by arcs which intersect $\Gamma(G,S)$ only at their endpoints. Since $U_i$ is free, $T_i$ is a tree. Then $T_i$ separates $\Gamma^2(G,S)$ and will be called the *median tree* for $X_i$. Notice that $X_i$, $T_i$ are not cellular subset of $\Gamma(G,S)$ or $\Gamma^2(G,S)$, even thought they do have a natural cellular structure.

Let $Z, Z'$ be subsets of $\Gamma(G,S)$. We will say that $Z, Z'$ are *$t$-separated* if there exists $g \in G$ and $i$ such that $Z, Z'$ are in distinct components of $\Gamma(G,S) \backslash gX_i$. This is equivalent to saying that as subsets of $\Gamma^2(G,S)$; $Z, Z'$ are in distinct components of $\Gamma^2(G,S) \backslash gT_i$.

**Remark III.3.12.** Notice that $t$-separated does not imply $U_i$-separated or $V_i$-separated. Let $Z$ the set of vertices of $\Gamma(G,S)$ that have a label without pinches which begins with the letter $t_1$. Let $Z'$ be the remainder of the vertices of $\Gamma(G,S)$. Then $Z, Z'$ are in distinct components of $\Gamma(G,S) \backslash X_1$. Since $Z \cup Z'$ contains all the vertices of $G$, they cannot be $U_i$-separated or $V_i$ separated for any $i$. The point is that $X_i$ separates by removing midpoints of edges and $gV_i$ or $gU_i$ separates by removing vertices.

**Lemma III.3.13.** *Suppose $U_i$ is proper, $V_i$ is proper, or the number of stable letter in $S$ is greater than 1 . Let $\gamma$ be a loop in $\Gamma(G,S)$ and $N > \operatorname{diam}(\gamma)$. Then there exists elements $\{g_1, \cdots, g_N\}$ in $G$ such that $g_i \cdot \gamma, g_j \cdot \gamma$ are $t$-separated and $|g_i g_j^{-1}| \geq 2N$ for all $i \neq j$; and $|g_i| \leq 4N$.*

*Proof.* If $U_i$ or $V_i$ is proper, then $\{g_j\}$ can be constructed as in Lemma III.2.14. If $S$ has at least two stable letters, then let $g_i = t_1^N t_2^i t_1^{-N}$. In any of the three cases, the proof of Lemma III.2.14 also shows that the loops $\{g_j \cdot \gamma\}$ are pairwise $t$-separated. □

**Lemma III.3.14.** *Suppose that $\theta\colon\ \Delta^{(2)} \to \Gamma^2(G,S)$ is the canonical label preserving cellular map from a van Kampen diagram $\Delta$ over $\langle S \,|\, R \rangle$ to the Cayley complex. Then $\theta^{-1}(gT_i)$ is a set of medians of $t_i$-bands in $\Delta$.*

*Proof.* The only cells in $\Gamma^2(G,S)$ intersecting $gT_i$ are those corresponding to relations of the form $u_{i,j}^t = v_{i,j}$. The preimage of each edge of $gT_i$ is a median of such a cell in $\Delta$. □

**Lemma III.3.15.** *Suppose that $\Delta_A$ is an annular diagram such that the components of $\theta(\partial \Delta_A)$ are $t$-separated. Then there exist a $t$-annulus in $\Delta_A$ which separates the boundary components of $\Delta_A$.*

*Proof.* Since the components of $\theta(\partial \Delta_A)$ are $t$-separated, there exists $g \in G$ and $i$ such that they are in distinct components of $\Gamma^2(G,S) \backslash gT_i$. Then $\theta^{-1}(gT_i)$ separates the components of $\partial \Delta_A$ and the result follows from Lemma III.3.14. □

**Theorem III.3.16.** *Let $G$ be a multiple HNN of a free group with free associated subgroups. Then either all asymptotic cones of $G$ are simply connected or $G$ has an asymptotic cone with uncountable fundamental group.*

*Proof.* If $G$ has only one stable letter and both associated subgroups are not proper, then $G$ has a quadratic Dehn function (see [4]) and every asymptotic cone of $G$ is simply connected.

If there exists an asymptotic cone of $G$ which is not simply connected, then there exists a sequence of loops $\gamma_n$ in $\Gamma(G,S)$ such that $P\big(\gamma_n, \frac{|\gamma_n|}{2}\big) \geq n$ for all n. Let $d_n = |\gamma_n|$. Then $d_n$ diverges $\omega$-almost surely and $\gamma(t) = \big(\gamma_n(t)\big)$ is a loop which has no finite partition in $\mathrm{Con}^{\omega}\big(G,d\big)$.

Using Lemma III.3.13, we can choose $S_n = \{g_{n,1}, \cdots, g_{n,k_n}\}$ of element of $G$ such that

a) if $i \neq j$, then $g_{n,i} \cdot \gamma_n$ and $g_{n,j} \cdot \gamma_n$ are $t$-separated and

b) for all $i$, $2\,\mathrm{diam}(\gamma) d_n \leq |g_{n,i}| \leq 4\,\mathrm{diam}(\gamma) d_n$.

**Claim.** *Let $g = (g_n), h = (h_n)$ be distinct elements in $\prod^{\omega} S_n$. Then $g \cdot \gamma$ is a well-defined loop $\mathrm{Con}^{\omega}\big(G,d\big)$ and $g \cdot \gamma$ is not homotopic to $h \cdot \gamma$.*

The first assertion follows from the fact that $g_n$ grows big O of the scaling sequence.

Suppose that $g \cdot \gamma$ is homotopic to $h \cdot \gamma$. Then we have a homotopy $h\colon A \to \mathrm{Con}^{\omega}\big(G,d\big)$ between the two loops where $A$ is a planar annulus. Let $P$ be a partition of $A$ where each piece is a triangle such that $\mathrm{diam}(h(D)) \leq \frac{1}{84L}$ for each piece $D$ of $P$. Then we can chose partitions $\Pi_n\colon P^{(0)} \to \Gamma(G,S)$ such that $\big(\Pi_n(x)\big) = h(x)$ for all $x \in P^{(0)}$. As in Remark III.3.5, $\Pi_n$ induces a $t$-shortest partition $\Psi_n\colon P^{(2)} \to \Delta_n'$ where $\Delta_n'$ is an annular van Kampen diagram where both boundary paths are labeled by $\mathbf{Lab}\,(\gamma_n)$. The $\mathrm{mesh}\,(\Psi_n) \leq \frac{|\gamma_n|}{60L} + o(|\gamma_n|) < \frac{|\gamma_n|}{30L}$ $\omega$-almost surely. Lemma III.3.4 implies that the $\mathrm{diam}(\theta \circ \Psi(D)) \leq 5L\,\mathrm{mesh}\,(\Psi) < \frac{|\gamma_n|}{6}$ $\omega$-almost surely.

Since $g \neq h$, $g_n \neq h_n$ $\omega$-almost surely and the loops $g_n \cdot \gamma_n$ and $h_n \cdot \gamma_n$ are $t$-separated $\omega$-almost surely. Lemma III.3.15 implies that there exists a $t$-annulus in $\Delta_n$ which separates the two boundary components of $\Delta_n$ $\omega$-almost surely. Lemma III.3.10 implies we can remove this $t$-annulus to obtain a partition $\widetilde{\Psi}_n$ of a circular diagram $\Delta_n'$ with $\mathbf{Lab}\,(\partial\Delta_n') = \mathbf{Lab}\,(\gamma_n)$ $\omega$-almost surely. Notice that $\mathrm{mesh}\,(\widetilde{\Psi}_n) < 3(\frac{|\gamma_n|}{6} + K)$ and has at most $9F^2 + 4F$ where $F$ is the number of pieces of $P$. This then contradicts our choice of $\gamma_n$.

□

CHAPTER IV

# LOOP DIVISION PROPERTIES

The following definitions of locally connectivity properties are standard, see [29, Chapter 1].

**Definition IV.0.17.** A space $X$ is called *locally simply connected* if for every pair $(U,x)$ where $U$ is a neighborhood of $x \in X$, there exists $V$, a neighborhood of $x$ contained in $U$, such that the inclusion induced homomorphism from $\pi_1(V,x)$ to $\pi_1(U,x)$ is trivial; i.e. every loop in $V$ bounds a disc in $U$. A metric space $X$ is *uniformly simply connected* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every loop with diameter at most $\delta$ bounds a disc with diameter at most $\varepsilon$.

A space $X$ is called *semilocally simply connected* if every point $x \in X$ has a neighborhood $U$ such that the inclusion induced homomorphism from $\pi_1(U,x)$ to $\pi_1(X,x)$ is trivial, i.e. every loop in $U$ bounds a disc in the whole space.

**Remark.** *A space that is locally simply connected is semilocally simply connected. The converse is false, since the cone on any space that is not locally simply connected is semilocally simply connected but still not locally simply connected. See [29, Section 1.3].*

The following definition of a *partition* is due to Papasoglu [44].

**Partitions of the unit disc in the plane:** Let $\mathbb{D}$ be the unit disk in $\mathbb{R}^2$. A *partition $P$ of* $\mathbb{D}$ is a finite collection of closed discs $D_1, \cdots, D_k$ in the plane with pairwise disjoint interiors such that $\mathbb{D} = \cup_i D_i$, $\partial\mathbb{D} = \partial(D_1 \cup \cdots \cup D_k)$, and $D_i \cap D_j = \partial D_i \cap \partial D_j$ when $i \neq j$. A point $p$ on $\partial D_1 \cup \cdots \cup \partial D_k$ is called a *vertex of the partition* if for every open set $U$ containing $p$, $U \cap (\partial D_1 \cup \cdots \cup \partial D_k)$ is not homeomorphic to an interval. An *edge of a partition* is a pair of vertices which are joined by a path in $\partial D_1 \cup \cdots \cup \partial D_k$ that intersects the set of vertices only at its endpoints. We will say that such vertices are *adjacent*. A *piece of a partition* is a maximal set of vertices of the partition contained in a single disc of the partition. A partition is then a cellular decomposition of the unit disc where each vertex has degree at least 3; so we will use the standard notation, $P^{(i)}$, to denote the $i$-th skeleton of a partition for $i = 0,1,2$.

**Geodesic $n$-gons in a metric space X:** An *$n$-gon* in $X$ is a map from the set of vertices of the standard regular $n$-gon in the plane into $X$, i.e. an ordered set of $n$ points in $X$. If $X$ is a geodesic metric space, we can extend an $n$-gon to edges by mapping the edge between adjacent vertices of the standard regular $n$-gon in the plane to a geodesics segment joining the corresponding vertices of the $n$-gon in $X$. We will say that such an extension is a *geodesic $n$-gon* in $X$.

**Partitions of loops in a geodesic metric space X:** Let $\gamma : \partial\mathbb{D} \to X$ be a continuous map. A *partition of* $\gamma$ is a map $\Pi$ from the set of vertices of a partition $P$ to $X$ such that $\Pi|_{\partial\mathbb{D}\cap P^{(0)}} = \gamma|_{\partial\mathbb{D}\cap P^{(0)}}$. The *vertices/edges/pieces* of $\Pi$ are the images of vertices/edges/pieces of $P$. We will write $\Pi(\partial D_i)$ for the pieces of $\Pi$, where $D_i$ are the 2-cells of $P$.

**Remark IV.0.18.** Suppose that $\Pi : P^{(0)} \to X$ is a partition of a loop $\gamma$ in a geodesic metric space. We can extend $\Pi$ to $P^{(1)}$ by mapping each edge contained in $\partial\mathbb{D}$ to the corresponding subpath of $\gamma$ and every edge not contained in $\partial\mathbb{D}$ to a geodesic segment joining its end points. The *length of a piece* is the arc length of the loop $\Pi(\partial D_i)$. We will write $|\Pi(\partial D_i)|$ for the length of the piece $\Pi(\partial D_i)$. We define the *mesh of* $\Pi$ by

$$\mathrm{mesh}(\Pi) = \max_{1\le i\le k}\{|\Pi(\partial D_i)|\}.$$

When $X$ is a Cayley graph of a group, we will also assume that the partition takes vertices of $P$ to vertices in the Cayley graph. A partition $\Pi$ is called a $\delta$*-partition*, if $\mathrm{mesh}\,\Pi < \delta$. A loop of length $k$ in a geodesic metric space is *partitionable* if it has a $\frac{k}{2}$-partition.

Let $P(\gamma,\delta)$ be the minimal number of pieces in a $\delta$-partition of $\gamma$ if a $\delta$-partition exist and $+\infty$ otherwise.

## IV.1 Coarse Loop Division Property

**Definition IV.1.1.** Let $X$ be a geodesic metric space.

Define $\vartheta^i : \mathbb{N} \to \mathbb{N}\cup\{\infty\}$ by $\vartheta^i(n) = \sup\big\{P(\alpha, \frac{|\alpha|}{2^i}) \mid \alpha \text{ is a loop in X such that } n-1 < |\alpha| \le n\big\}$. We will call $\vartheta = \vartheta^1$ the *divisibility* function of $X$.

Suppose $\omega$ is an ultrafilter on $\mathbb{N}$, $(d_n)$ an $\omega$-divergent sequence of positive real numbers, and $\varepsilon$ a positive real number. We will say that $X$ is $\varepsilon$*-coarsely loop divisible*; if for every $\delta \in (0,\varepsilon)$ there exists an $A \subset \mathbb{N}$ with $\omega(A) = 1$ such that the divisibility function $\vartheta$ restricted to $\bigcup_{n\in A}[\delta d_n, \varepsilon d_n]$ is bounded by a constant $K = K(\delta,\varepsilon)$.

We will say that $X$ is *uniformly* $\varepsilon$*-coarsely loop divisible*; if the constant $K = K(\delta,\varepsilon)$ can be chosen independent of $\delta$.

We will say that a group $G$ is (uniformly) $\varepsilon$-coarsely loop divisible; if the Cayley graph $\Gamma(G,S)$ is (uniformly) $\varepsilon$-coarsely divisible.

The property of being $\varepsilon$-coarsely loop divisible depends on $(\omega,d)$. When there is a chance of confusion, we will say that $X$ is $\varepsilon$-coarsely loop divisible with respect to $(\omega,d)$.

If $X$ is $\varepsilon$-coarsely loop divisible for every $\varepsilon$ and the bound $K(\delta,\varepsilon)$ can be chosen independent of both $\delta$ and $\varepsilon$, then $\mathrm{Con}^\omega(X,e,d)$ has Olshanskii-Sapir's property LDC($K$) as defined in [39].

We will see (Proposition IV.1.21) that for finitely generated groups this definition is independent of the generating set in the sense that if $S, S'$ are two finite generating sets for $G$, then $\Gamma(G,S)$ is $\varepsilon$-coarsely loop divisible if and only if $\Gamma(G,S')$ is $\varepsilon'$-coarsely loop divisible for some $\varepsilon' > 0$.

**Remark IV.1.2.** Suppose that $\vartheta$ is bounded on $\left[\frac{n}{2^l}, n\right]$ by $K$. Let $\alpha$ be a loop of length $n$ and fix a partition of $\alpha$ into at most $\vartheta(n)$ pieces with mesh less than $\frac{n}{2}$. As in Remark IV.0.18, the partition can be extended to the 1-skeleton of the partition such that each loop has length less than $\frac{n}{2}$. We can then partition each piece with length at least $\frac{n}{4}$ into at most $K$ pieces of length less than $\frac{n}{4}$. This builds a $\frac{n}{4}$-partition of $\alpha$ with at most $K^2$ pieces. Hence $\vartheta^2(n) \le K^2$. Iterating this process, we obtain $\vartheta^l(n) \le K^l$.

**Lemma IV.1.3.** *Fix $l \in \mathbb{N}$. If $X$ is $\varepsilon$-coarsely loop divisible, then for every $\delta \in (0,\varepsilon)$ there exists an $A \subset \mathbb{N}$ with $\omega(A) = 1$ such that $\vartheta^l$ restricted to $\bigcup_{n\in A}[\delta d_n, \varepsilon d_n]$ is bounded by a constant $K = K(\delta,\varepsilon,l)$.*

Thus the coarse loop division property does not depend on which function $\vartheta^l$ is used in its definition.

*Proof.* Suppose $X$ is $\varepsilon$-coarsely loop divisible. Fix $\delta$ such that $0 < \delta < \varepsilon$. Choose a $K$ and an $\omega$-large $A$ such that $\vartheta$ restricted to $\bigcup_{n\in A}\left[\frac{\delta}{2^l}d_n, \varepsilon d_n\right]$ is bounded by $K$. By Remark IV.1.2, $\vartheta^l$ restricted to $\bigcup_{n\in A}\left[\delta d_n, \varepsilon d_n\right]$ is bounded by $K^l$.

□

**Definition IV.1.4.** Let $(\gamma_n)$ be a sequence of loops in a metric space $X$ and $d = (d_n)$ an $\omega$-divergent sequence of real numbers. Then $(\gamma_n)$ is *not $(m,d,\varepsilon,\delta)$-partitionable* if $\delta d_n \leq |\gamma_n| \leq \varepsilon d_n$ and $P(\gamma_n, |\gamma_n|/2) > m$ $\omega$-almost surely. When $d$ and $\varepsilon$ are fixed, we will say that $(\gamma_n)$ is *not $(\delta,m)$-partitionable*. Additionally; given a sequence of loops which is not $(\delta,m)$-partitionable, we will say that a fixed member $\gamma_n$ of the sequence is not $(\delta,m)$-partitionable if $\delta d_n \leq |\gamma_n| \leq \varepsilon d_n$ and $P(\gamma_n, |\gamma_n|/2) > m$.

**Remark IV.1.5.** Let $\gamma : \partial\mathbb{D} \to X$ be parameterized by arc length. Suppose that $8\,\mathrm{diam}(\gamma) < |\gamma|$. Let $P$ be the cellular decomposition of the unit disc $\mathbb{D}$ such that $P^{(1)}$ is $\partial\mathbb{D} \cup A$ where $A$ is a maximal square inscribed in $\mathbb{D}$. Then $\Pi : P^{(0)} \to X$ defined by $\Pi(t) = \gamma(t)$ is a partition of $\gamma$ with five pieces (four 2-gons and one 4-gon) and $\mathrm{mesh}(\Pi) \leq \max\{\frac{|\gamma|}{4} + \mathrm{diam}(\gamma),\ 4\,\mathrm{diam}(\gamma)\} < \frac{|\gamma|}{2}$.

Thus, if $(\gamma_n)$ is not $(m,d,\varepsilon,\delta)$-partitionable for some $m \geq 5$, then $|\gamma_n| \leq 8\,\mathrm{diam}(\gamma_n)$. Hence; if $(\gamma_n)$ is not $(\delta,m)$-partitionable, then $|\gamma_n| \leq O\big(\mathrm{diam}(\gamma_n)\big)$ where the big $O$ constant is independent of $(\gamma_n)$.

The following two propositions were proved by Papasoglu in [44, pages 792-793]. The formulations are slightly different here but the proofs are the same. The proofs are also outlined in [39].

**Proposition IV.1.6.** *Let $X$ be a metric space and $(\gamma_n)$ a sequence of loops in $X$ such that $|\gamma_n| = O(d_n)$. If each $\gamma_n$ has a $\delta_n$-partition with at most $k$ pieces, then the loop $\gamma(t) = \big(\gamma_n(t)\big)$ in $\mathrm{Con}^\omega\big(X,e,d\big)$ has a $\delta$-partition with at most $k$ pieces where $\delta = \lim^\omega \frac{\delta_n}{d_n}$.*

**Proposition IV.1.7.** *Let $X$ be a complete geodesic metric space. If $X$ is uniformly $\varepsilon$-coarsely loop divisible for every $\varepsilon > 0$ with respect to the pair $\big(\omega,d\big)$, then $\mathrm{Con}^\omega\big(X,e,d\big)$ is simply connected.*

To prove Proposition IV.1.7, Papasoglu uses Proposition IV.1.6 to show that every loop in $\mathrm{Con}^\omega\big(X,e,d\big)$ is partitionable and the number of pieces is independent of the loop. He then iterates the process of taking partitions and extending them to the 1-skeleton as in Remark IV.0.18. A consequence of this procedure is that the diameter of the constructed disc is proportional to the length of the loop (the proportionality constant can be chosen to be the bound on the number of pieces in the partitions).

**Lemma IV.1.8.** *Suppose that $X$ is a complete geodesic metric space which is uniformly $\varepsilon$-coarsely loop divisible with respect to the pair $\big(\omega,d\big)$. Then there exists a constant $K$ such that every loop in $\mathrm{Con}^\omega\big(X,e,d\big)$ with diameter less than $\frac{\varepsilon}{8}$ bounds a disc with diameter less than $K\varepsilon$.*

*Proof.* Since $X$ is uniformly $\varepsilon$-coarsely loop divisible every loop in $\mathrm{Con}^\omega\big(X,e,d\big)$ with length less than $\varepsilon$ is partitionable with a uniform bound on the number of pieces required. Suppose that a loop in $\mathrm{Con}^\omega\big(X,e,d\big)$ has length at least $\varepsilon$ and diameter less than $\frac{\varepsilon}{8}$. Then it has a partition with 5 pieces by Remark IV.1.5. Thus every loop in $\mathrm{Con}^\omega\big(X,e,d\big)$ with diameter less than $\frac{\varepsilon}{8}$ is partitionable and we can apply the proof of Proposition IV.1.7. □

Lemma IV.1.8 can be restated in the following way.

**Proposition IV.1.9.** *Let $X$ be a complete geodesic metric space. If $X$ is uniformly $\varepsilon$-coarsely loop divisible, then* $\mathrm{Con}^{\omega}\big(X,e,d\big)$ *is uniformly locally simply connected.*

**Proposition IV.1.10.** *Let $X$ be a complete geodesic metric space. If $X$ is uniformly $\varepsilon$-coarsely loop divisible, then $X$ has an asymptotic cone which is simply connected.*

*Proof.* Suppose that $X$ is uniformly $\varepsilon$-coarsely loop divisible for some $\big(\omega,d\big)$ and $\varepsilon>0$. We can consider an ultralimit of the metric spaces $X_k=\mathrm{Con}^{\omega}\big(X,e,(d_n/k)\big)$. By Corollary 3.24 in [18], $\lim^{\omega} X_k$ is again an asymptotic cone of $X$. Thus we can choose $\big(\mu,(p_n)\big)$ and $(x_n)$ such that $\mathrm{Con}^{\mu}\big(X,(x_n),(p_n)\big)$ and $\lim^{\omega} X_k$ are isometric.

The identity map $id$ from $\mathrm{Con}^{\omega}\big(X,e,d\big)$ to $X_k$ rescales distances by a fixed constant which implies that $P(\gamma,|\gamma|/2)=P(id(\gamma),id(|\gamma|)/2)$. Since $X$ is uniformly $\varepsilon$-coarsely loop divisible, there exists $\nu_0$ such that $P(\gamma,|\gamma|/2)<\nu_0$ for every loop $\gamma$ contained in a ball of radius $\frac{\varepsilon}{4}$ in $\mathrm{Con}^{\omega}\big(X,e,d\big)$. Hence; every loop $\gamma$ contained in a ball of radius $\frac{k\varepsilon}{4}$ in $X_k$ has the property that $P(\gamma,|\gamma|/2)<\nu_0$. Thus for any loop $\alpha$ in $\lim^{\omega} X_k$; $P(\alpha,|\alpha|/2)<\nu_0$. Hence $\lim^{\omega} X_k$ is uniformly $\varepsilon$-coarsely loop divisible for every $\varepsilon>0$ with respect to the pair $\big(\mu,(p_n)\big)$ and Proposition IV.1.7 implies that $\mathrm{Con}^{\mu}\big(X,(x_n),(p_n)\big)$ is simply connected. $\square$

Lemma IV.1.8 shows that a necessary condition for a group to be uniformly $\varepsilon$-coarsely loop divisible for every $\varepsilon>0$ is that all loops in $\mathrm{Con}^{\omega}\big(X,e,d\big)$ bound discs with diameters proportional to their length.

**Remark IV.1.11.** Let $X$ be a topological space. The *topological cone* of $X$ written $\hat{X}$ is the quotient space of $X\times[0,1]$ obtained by identifying all points $(x,1)$ for $x\in X$. $X$ canonically embeds in $\hat{X}$ by $x\mapsto(x,0)$ and we will generally identify $X$ with $X\times\{0\}$. The *Hawaiian earring* is the one-point compactification of a sequence of disjoint arcs and can be realized in the plane as the union of circles centered at $(0,\frac{1}{n})$ with radius $\frac{1}{n}$. We will use $\mathbf{E}$ to denote this subspace of the plane and $\mathbf{a}_n$ to denote the circle centered at $(0,\frac{1}{n})$ with radius $\frac{1}{n}$. The *Hawaiian earring group* is $\pi_1(\mathbf{E},(0,0))=\mathbb{H}$. Let $\mathbf{E}_n=\bigcup\limits_{i\geq n}\mathbf{a}_i$ and $\mathbb{H}_n=\pi_1(\mathbf{E}_n,(0,0))\leq\mathbb{H}$. Notice that $\mathbf{E}_n$ is homeomorphic to $\mathbf{E}$ which implies that $\mathbb{H}_n$ is isomorphic to $\mathbb{H}$.

$\hat{\mathbf{E}}$ is a space which is not uniformly $\varepsilon$-coarsely loop divisible but is simply connected and not locally simply connected. Suppose that instead of coning from a single point, we were to cone each circle individually. Then as long as we required that the sequence of cone points converged to the wedge point of $\mathbf{E}$ but at a rate slower than the radii of the loops, this space would be locally simply connected but not be uniformly $\varepsilon$-coarsely loop divisible for any $\varepsilon$. These two examples show that for general metric spaces being uniformly $\varepsilon$-coarsely loop divisible is not a necessary condition for a space to be simply connected or locally simply connected.

Erschler-Osin [21] and Druţu-Sapir [18] proved that many metric spaces $\pi_1$-embed into the asymptotic cones of finitely generated groups. In both papers, the spaces that were $\pi_1$-embedded into the asymptotic cones of finitely generated groups were uniformly locally simply connected.

A positive answer to either Question 1 or Question 2 would imply that the results of Erschler-Osin and Druţu-Sapir cannot be extended to spaces which are semilocally simply connected but not locally simply connected.

We will now prove some implications of the coarse loop division property.

The following lemma is an immediate consequence of Proposition IV.1.6 and Proposition III.1.2.

**Lemma IV.1.12.** *Suppose that $X$ is a complete geodesic metric space which is $\varepsilon$-coarsely loop divisible. Every loop in $\mathrm{Con}^{\omega}\big(X,e,d\big)$ with length less than $\varepsilon$ is partitionable.*

**Lemma IV.1.13.** *Suppose that $X$ is a complete homogeneous geodesic metric space. If every loop in $\mathrm{Con}^{\omega}\big(X,e,d\big)$ with length less than $\varepsilon$ is partitionable, then $X$ is $\varepsilon'$-coarsely loop divisible with respect to the pair $\big(\omega,d\big)$ for every $\varepsilon'<\varepsilon$.*

*Proof.* Suppose that $X$ is not $\varepsilon'$-coarsely loop divisible with respect to the pair $\big(\omega,d\big)$ for some $\varepsilon'$ with $0<\varepsilon'<\varepsilon$. Then there exists a $\delta>0$ such that for every $\omega$-large $A$, $\vartheta$ restricted to $\bigcup\limits_{n\in A}[\delta d_n,\varepsilon' d_n]$ is unbounded.

Let $\gamma_n$ be a loop based at $x_n$ such that $\delta d_n\le|\gamma_n|\le\varepsilon' d_n$ and satisfies at least one of the two following properties.

a) $P\Big(\gamma_n,\dfrac{|\gamma_n|}{2}\Big)>n$

b) $P\Big(\gamma_n,\dfrac{|\gamma_n|}{2}\Big)\ge P\Big(\alpha,\dfrac{|\alpha|}{2}\Big)$ for all $\alpha$ such that $\delta d_n\le|\alpha|\le\varepsilon' d_n$

Let $m_n=P\Big(\gamma_n,\dfrac{|\gamma_n|}{2}\Big)$. Since $\vartheta$ restricted to $\bigcup\limits_{n\in A}[\delta d_n,\varepsilon d_n]$ is unbounded for every $\omega$-large $A$; $\lim^{\omega} m_n=+\infty$. Thus for every $m$, $(\gamma_n)$ is not $(\delta,m)$-partitionable.

The path $\gamma(t)=\big(\gamma_n(t)\big)$ is a well-defined loop in $\mathrm{Con}^{\omega}\big(X,e,d\big)$ with positive diameter and arc length at most $\varepsilon'<\varepsilon$. By assumption, there exists a $\frac{|\gamma|}{2}$-partition of $\gamma$ with $L$ pieces. However, this induces a $|\gamma_n|/2$-partition of $\gamma_n$ with $L$ pieces $\omega$-almost surely. Hence $P\Big(\gamma_n,\dfrac{|\gamma_n|}{2}\Big)\le L$ $\omega$-almost surely, which contradicts our choice of $m_n$.

□

**Proposition IV.1.14.** *Suppose that $X$ is a complete homogenous geodesic metric space. If $\mathrm{Con}^{\omega}\big(X,e,d\big)$ is semilocally simply connected then $X$ is $\varepsilon$-coarsely loop divisible for some $\varepsilon>0$.*

It is not known whether the converse holds. The converse is Question 2 with the *uniform* hypothesis removed.

*Proof of Proposition IV.1.14.* Suppose that every loop in $\mathrm{Con}^{\omega}\big(X,e,d\big)$ contained in a ball of radius $\varepsilon$ is nulhomotopic in $\mathrm{Con}^{\omega}\big(X,e,d\big)$. Then for every $\gamma$ of length at most $\varepsilon$, there exists a continuous map of a disc into $\mathrm{Con}^{\omega}\big(X,e,d\big)$ which extends $\gamma$ and is necessarily uniformly continuous. For sufficiently small $\nu$, a $\nu$-partition of the disc gives us a finite $|\gamma|/2$-partition for $\gamma$. Then the result follows from Lemma IV.1.13

□

**Theorem IV.1.15.** *Let $X$ be a complete homogenous geodesic metric space. If $X$ is not $\varepsilon$-coarsely loop divisible with respect to $\big(\omega,d\big)$ for every $\varepsilon>0$, then $\mathrm{Con}^{\omega}\big(X,e,d\big)$ has uncountable fundamental group.*

The proof will require the following result of Cannon and Conner.

**Theorem IV.1.16** (Cannon, Conner [9]). *Let $X$ be a topological space, let $\varphi : \pi_1(X,x_0) \to L$ be a homomorphism to a group $L$, $U_1 \supset U_2 \supset \cdots$ be a countable local basis for $X$ at $x_0$, and $G_i$ be the image of the natural map from $\pi_1(U_i,x_0)$ into $\pi_1(X,x_0)$. If $L$ is countable, then the sequence $\varphi(G_1) \supset \varphi(G_2) \supset \cdots$ is eventually constant.*

*Proof of Theorem IV.1.15.* Let $X$ be a complete homogenous geodesic metric space. Suppose that $X$ is not $\varepsilon$-coarsely loop divisible for any $\varepsilon$ and $\mathrm{Con}^{\omega}\big(X,e,d\big) = X^{\omega}$ has countable fundamental group. Let $i_*$ be the identity map on $\pi_1(X^{\omega},e)$. Theorem IV.1.16 implies that $i_*(G_n)$ is eventually constant where $G_n$ is the image of the natural map from $\pi_1(B_{1/n}(e),e)$ into $\pi_1(X^{\omega},e)$.

Fix $N$ such that this sequence is constant for $m \geq N$, and let $\varepsilon = 1/N$. Therefore every loop in $B_{1/N}(\tilde{x})$ can be homotoped into $B_{1/m}(e)$ for any $m \geq N$. In general, this will not imply that the ball is simply connected. However, it does imply that every loop $\gamma$ of length less than $\varepsilon$ has a partition with finitely many pieces and mesh at most $\frac{|\gamma|}{2}$. Then Lemma IV.1.13 implies that $X$ is $\varepsilon$-coarsely loop divisible which is a contradiction.

□

**Theorem IV.1.17.** *Let $X$ be a complete homogenous geodesic metric space. If $X$ is not $\varepsilon$-coarsely loop divisible with respect to the pair $\big(\omega,d\big)$ for every $\varepsilon > 0$, then the fundamental group of $\mathrm{Con}^{\omega}\big(X,e,d\big)$ is not free. In particular, if $\pi_1\Big(\mathrm{Con}^{\omega}\big(X,e,d\big)\Big) = *_j G_j$ for some free product of groups $G_j$, then there exists a $j$ such that $G_j$ is uncountable and not free.*

We will use the following two results in the proof of Theorem IV.1.17.

**Theorem IV.1.18.** *Suppose that $\varphi : \mathbb{H} \to \mathbb{F}$ is a surjective homomorphism where $\mathbb{F}$ is a free group. Then $\mathbb{F}$ has finite rank.*

If we consider homomorphisms from the natural inverse limit containing $\mathbb{H}$ to free groups, then this is a theorem of Higman [30]. When we consider homomorphism from $\mathbb{H}$, this is a consequence of Theorem IV.1.16 and a proof can be found in [50].

**Theorem IV.1.19** ([19]). *Suppose that $\varphi : \mathbb{H} \to *_j G_j$ is a homomorphism. Then there exists an $n$ such that $\varphi(\mathbb{H}_n)$ is contained in a subgroup which is conjugate to $G_j$ for some $j$.*

*Proof of Theorem IV.1.17.* Since $X$ is not $\varepsilon$-coarsely loop divisible with respect to $\big(\omega,d\big)$ for every $\varepsilon$, we may find a null sequence of loops $\alpha_n$ in $\mathrm{Con}^{\omega}\big(X,e,d\big)$ such that $\alpha_n$ has no finite $\frac{|\alpha_n|}{2}$-partition. Since $\mathrm{Con}^{\omega}\big(X,e,d\big)$ is transitive by isometries, we may choose $\alpha_n$ such that $\alpha_i(0) = \alpha_j(0) = e$ for all $i,j$. By passing to a subsequence, we may assume that $|\alpha_n| < \frac{|\alpha_{n-1}|}{2}$. This implies that the ball of radius $|\alpha_n|$ does not contain a loop which is homotopic to $\alpha_i$ for $i < n$. Since $\alpha_n$ forms a null sequence of loops and $\alpha_i(0) = \alpha_j(0)$ for all $i,j$, there exists a continuous map $f$ from $\mathbf{E}$ to $\mathrm{Con}^{\omega}\big(X,e,d\big)$ such that $f(\mathbf{a}_n) = \alpha_n$.

Suppose that $\pi_1\big(\mathrm{Con}^{\omega}\big(X,e,d\big),(x_n)\big)$ was free. Then $f_*(\mathbb{H})$ would be free and Theorem IV.1.18 would then imply that it has finite rank. Hence $f_*(\mathbb{H})$ is countable which by Theorem IV.1.16 would imply that $f_*\Big(\pi_1\big(\mathbf{E}_n,(0,0)\big)\Big)$ as a sequence in $n$ is eventually constant. This contradicts our choice of $\alpha_n$.

Thus for every $n$, $f_*(\mathbb{H}_n)$ is uncountable and not free. The last claim of the theorem follows from Theorem IV.1.19.

□

**Proposition IV.1.20.** *Let $X$ be a complete homogenous geodesic metric space. If $X$ is not $\varepsilon$-coarsely loop divisible with respect to $(\omega, d)$ for every $\varepsilon > 0$, then the fundamental group of $\mathrm{Con}^{\omega}(X, e, d)$ is not simple.*

*Proof.* Let $X^{\omega} = \mathrm{Con}^{\omega}(X, e, d)$ and $\alpha_i$ be a null sequence of loops in $X^{\omega}$ constructed as in the proof of Theorem IV.1.17. Let $A_n$ be the union of the images of $\alpha_i$ for $i > n$. Let $Y_n$ be the topological cone of $A_n$ in $X^{\omega}$, i.e. the subset of $\hat{X}^{\omega}$ consisting of $\mathrm{Con}^{\omega}(X, e, d) \times \{0\}$ and the canonically embedded $\hat{A}_n$. The inclusion map $\iota_n : \mathrm{Con}^{\omega}(X, e, d) \to Y_i$ defined by $x \mapsto (x, 0)$ induces a map $\iota_{n*}$ on fundamental groups with non-trivial kernel. Hence, it is enough to show that the induced map on fundamental groups is non-trivial.

**Claim.** *For $i \leq n$, $\iota_n(\alpha_i)$ is homotopically essential in $Y_n$.*

*Proof of claim.* Suppose that $h : \mathbb{D} \to Y_n$ is a nullhomotopy of $\iota_n(\alpha_i)$ for some $i \leq n$ where $\mathbb{D}$ is the unit disk in the plane. Let $z$ be the cone point. Notice that $A_n$ separates $Y_n$. Hence the boundary of each component of $h^{-1}(\hat{A}_n)$ is contained in $h^{-1}(A_n)$. By possible modifying $h$, we may assume that each component of $h^{-1}(\hat{A}_n)$ which is not contained in $h^{-1}(A_n)$ intersects the cone point $z$. (Suppose $B$ is a component of $h^{-1}(\hat{A}_n)$ such that $h(B) \cap \{z\} = \emptyset$. Then we can push $h$ down along cone lines to insure that $h(B) \subset A_n$.)

Since each component of $h^{-1}(\hat{A}_n)$ which is not contained in $h^{-1}(A_n)$ intersects $h^{-1}(A_n)$ and $h^{-1}(z)$ (two disjoint closed sets), there are only finitely many components of $h^{-1}(\hat{A}_n)$ which are not contained in $h^{-1}(A_n)$.

Let $C$ be the component of $h^{-1}(X^{\omega})$ containing the unit circle in the plane. Then $C$ is a planar annulus of finite genus. (The genus is equal to the number of components of $h^{-1}(\hat{A}_n)$ which are not contained in $h^{-1}(A_n)$.) Since each boundary component of $C$ except the unit circle maps into $A_n$, the diameter of its image is at most $|\alpha_{n+1}| < \frac{|\alpha_n|}{2}$. This implies that $h : C \to X^{\omega}$ can be used to find a finite partition of $\alpha_i$ with mesh at most $\frac{|\alpha_n|}{2}$. Hence, $\alpha_i$ is partitionable which contradicts our choice of $\alpha_i$.

□

The property of being $\varepsilon$-coarsely loop divisible is a quasi-isometry invariant in the following sense.

**Proposition IV.1.21.** *If $X$ and $Y$ are two quasi-isometric homogenous geodesic metric spaces, then $X$ is $\varepsilon$-coarsely loop divisible if and only if $Y$ is $\varepsilon'$-coarsely loop divisible for some $\varepsilon' > 0$.*

*Proof.* If $X$ and $Y$ are are quasi-isometric, then their cones are bi-lipschitz. If $X$ is $\varepsilon$-coarsely loop divisible for some $\varepsilon > 0$, then Proposition IV.1.6 implies that every loop of length less than $\varepsilon$ in $\mathrm{Con}^{\omega}(X, e, d)$ is partitionable.

Let $f : \mathrm{Con}^{\omega}(X, e, d) \to \mathrm{Con}^{\omega}(Y, e', d)$ be a bi-lipschitz map with bi-lipschitz constant $C$. By iterating partitions as in Remark IV.1.2, we can see that every loop of length less than $\varepsilon$ in $\mathrm{Con}^{\omega}(X, e, d)$ has a partition with finitely many pieces and mesh at most $\frac{|\gamma|}{2C}$. Let $\gamma$ be a loop in $\mathrm{Con}^{\omega}(Y, e', d)$ with length less than $\frac{\varepsilon}{C}$. Then $f^{-1} \circ \gamma$ has length at most $\varepsilon$ and hence has a partition with mesh at most $\frac{|\gamma|}{2C}$. Then composing the partition with $f$ gives us a partition of $\gamma$ with finitely many pieces and mesh at most $\frac{|\gamma|}{2}$. Lemma IV.1.13 implies that $Y$ is $\varepsilon'$-coarsely loop divisible for every $\varepsilon' < \frac{\varepsilon}{C}$.

□

### IV.1.1 Absolutely non-divisible sequences

**Definition IV.1.22.** A sequence of loops $(\alpha_n)$ is *absolutely non-divisible* if there exists an $M$ such that the sequences $P\Big(\alpha_n, \dfrac{|\alpha_n|}{M}\Big)$ and $|\alpha_n|$ both tend to $+\infty$ and $\left\{\dfrac{|\alpha_{n+1}|}{|\alpha_n|}\right\}$ is bounded.

**Remark IV.1.23.** Suppose that $|\alpha_n|$ is unbounded and $\left\{\dfrac{|\alpha_{n+1}|}{|\alpha_n|}\right\}$ is bounded. To simplify our notation, we will let $|\alpha_n| = a_n$ and $B$ be a bound on $\left\{\dfrac{a_{n+1}}{a_n}\right\}$.

Let $n_0 = 1$. Then we can define $\{n_i\}$, inductively, by letting $n_{i+1} = \min\{n \in \mathbb{N} \mid a_n > a_{n_i} + 1 \text{ and } n > n_i\}$. If $n_{i+1} \neq n_i + 1$, then $a_k \leq a_{n_i} + 1$ for all $n_i \leq k < n_{i+1}$.

Thus $\dfrac{a_{n_{i+1}}}{a_{n_i}} = \dfrac{a_{n_{i+1}}}{a_{(n_{i+1})-1}} \cdot \dfrac{a_{(n_{i+1})-1}}{a_{n_i}} \leq B\dfrac{a_{n_i}+1}{a_{n_i}} \leq B\max\{2, \frac{2}{a_{n_0}}\}$.

Therefore $\{a_{n_i}\}_i$ is a subsequence which is absolutely non-divisible.

Thus, it is possible to loosen this definition slightly and only require that $|\alpha_n|$ be unbounded.

**Lemma IV.1.24.** *Fix $\omega$ an ultrafilter on $\mathbb{N}$, $d$ an $\omega$-divergent sequence, and $A$ an infinite subset of the natural numbers. Suppose that $A = \{b_1 < b_2 < b_3 < \cdots\}$ has the property that the set of ratios $\left\{\frac{b_{k+1}}{b_k}\right\}$ is bounded by $L$. Then for any $\varepsilon > 0$, there exists a sequence $(a_n)$ in $A$ such that* $\lim^\omega \frac{a_n}{d_n} \in \left[\frac{\varepsilon}{L}, \varepsilon\right]$.

We allow $a_n$ to have repeated terms; hence, $a_n$ is not necessarily a subsequence of $b_n$. However $a_n$ is not eventually constant, since $\lim^\omega d_n = +\infty$.

*Proof.* Let $L$ be an upper bound on the set $\left\{\frac{b_{n+1}}{b_n}\right\}$. For all $n$ such that $\frac{b_1}{d_n} \leq \varepsilon$, choose $(i_n)$ such that $\frac{b_{i_n}}{d_n} \leq \varepsilon < \frac{b_{i_n+1}}{d_n}$. Let $a_n = b_{i_n}$.

Then $\varepsilon d_n < b_{i_n+1}$ which implies that $\frac{\varepsilon}{L} < \frac{b_{i_n}}{d_n} = \frac{a_n}{d_n} \leq \varepsilon$. For all $n$ such that $\frac{b_1}{d_n} > \varepsilon$, let $a_n = b_1$. Then $\lim^\omega \frac{a_n}{d_n} \in \left[\frac{\varepsilon}{L}, \varepsilon\right]$. ☐

**Lemma IV.1.25.** *Let $X$ be a complete geodesic metric space. If there exists a sequence of absolutely non-divisible loops in $X$, then for every pair $(\omega, d)$ and $\varepsilon > 0$, $X$ is not $\varepsilon$-coarsely loop divisible.*

*Proof.* Fix $\varepsilon > 0$, $\omega$ an ultrafilter, and $d$ an $\omega$-divergent sequence of real numbers.

Let $(\gamma_n)$ be a sequence of loops in $X$ which is absolutely non-divisible. By passing to a subsequence as in Remark IV.1.23, we may assume that the lengths of $\gamma_n$ are nondecreasing. Let $A = \{|\gamma_n|\}$ and $L$ be an upper bound on $\left\{\frac{|\gamma_{n+1}|}{|\gamma_n|}\right\}$.

Let $(a_n) \subset A$ be a sequence constructed as in Lemma IV.1.24 where we replace $\varepsilon$ by $\frac{\varepsilon}{2}$. Consider the sequence of loops $\gamma_{k_n}$ where $\gamma_{k_n}$ has length $a_n$. Since $\lim^\omega \frac{a_n}{d_n} \in \left[\frac{\varepsilon}{2L}, \frac{\varepsilon}{2}\right]$, we have $|\gamma_{k_n}| \in \left[\frac{d_n\varepsilon}{L}, d_n\varepsilon\right]$ $\omega$-almost surely. However, $P\big(\gamma_{k_n}, \frac{|\gamma_{k_n}|}{M}\big)$ tends to $+\infty$. Hence, $\vartheta$ restricted to $\bigcup_{n\in A} [\frac{\varepsilon}{L}d_n, \varepsilon d_n]$ is unbounded for all $\omega$-large $A$. Hence Lemma IV.1.3 implies that $X$ is not $\varepsilon$-coarsely loop divisible. Since $\varepsilon$ was arbitrary, $X$ is not $\varepsilon$-coarsely loop divisible with respect to $(\omega, d)$ for any $\varepsilon > 0$. Since $(\omega, d)$ were also arbitrary, this completes the proof. ☐

Lemma IV.1.25 and Theorem IV.1.15 immediately imply the following corollary.

**Corollary IV.1.26.** *Let $X$ be a complete homogenous geodesic metric space. If there exists a sequence of loops in $X$ which is absolutely non-divisible, then every asymptotic cone of $X$ has uncountable fundamental group and is not semi-locally simply connected at any point.*

### IV.1.2 Simply connected cones

When Papasoglu proved Proposition IV.1.7, he used the uniform bound on the number of pieces in a partition to construct discs. Being coarsely loop divisible implies that loops in the cone are partitionable but does not give a bound on the number of pieces which is independent of the loop. Thus Papasoglu's method is insufficient to build discs when a space is only coarsely loop divisible and not uniformly coarsely loop divisible. Here we will show that requiring a linear isodiametric function on partitions along with coarsely loop divisible is sufficient to build discs.

When considering subsets of $\mathbb{N}$, we will write $[a,b]$ for the set $\{n \in \mathbb{Z} \mid a \leq n \leq b\}$. For $A \subset \mathbb{N}$, we will let $A^c = \mathbb{N} \backslash A$. For $d \in \mathbb{R}^+$ and $A \subset \mathbb{N}$, let $\mathscr{M}_d(A) = \{x \in N \mid [\frac{x}{d}, xd] \cap A \neq \emptyset\}$.

**Proposition IV.1.27.** *Suppose that for every $(\mu,(p_n))$ there exists an $\varepsilon > 0$ such that $X$ is $\varepsilon$-coarsely loop divisible with respect to the pair $(\mu,(p_n))$. Then there exists a pair $(\omega,d)$ such that $X$ is $\varepsilon$-coarsely loop divisible for every $\varepsilon > 0$ with respect to $(\omega,d)$.*

Before we can prove Proposition IV.1.27, we will need a necessary condition for $X$ to be $\varepsilon$-coarsely loop divisible for every pair $(\omega,d)$.

**Lemma IV.1.28.** *Let $A_k = \vartheta^{-1}([1,k])$, $A'_k = \vartheta^{-1}(\{k\})$, and $B_k = \vartheta^{-1}([k+1,\infty))$. If $X$ is $\varepsilon$-coarsely loop divisible for every pair $(\omega,d)$, then for every $s \in \mathbb{N}$ there exists $b = b(s)$ such that*

*i) if $c_k^i = \sup\{\frac{y}{x} \mid i < x \text{ and } [x,y] \subset A_k\}$, then $c_k = \lim_{i\to\infty} c_k^i$ and $c_k \to \infty$,*

*ii) if $b_k = \sup\{\frac{y}{x} \mid [x,y] \subset \mathscr{M}_s(B_k)\}$, then $b_k < b$ for all sufficiently large $k$, and*

*iii) if $c'_k = \{\frac{y}{x} \mid [x,y] \subset \mathscr{M}_s(A'_k)\}$, then $c'_k < \alpha$ for all $k$.*

*Proof.* For fixed $k$, $c_k^i$ is a decreasing sequence in $i$. Hence, $c_k$ exists as an extended real number ($c_k^i$ might be infinite for all $i$). The sequence $c_k$ is increasing since the sets $A_k$ are nested.

*Proof of (i).* Suppose that there existed $L$ such that $c_k < L$ for all $k$. We may choose an increasing sequence $k_n$ such that $c_n^i < 2L$ for all $i > k_n$. Thus for every interval $[x,y]$ such that $k_n < x$ and $\frac{x}{y} \geq 2L$, $[x,y] \not\subset A_n$, i.e. $[x,y] \cap B_n \neq \emptyset$.

Fix an ultrafilter $\omega$ and let $d_n = (k_n)^2$. Suppose $X$ is $\varepsilon$-coarsely loop divisible for some $\varepsilon > 0$. Then $\bigcup_{n\in A} [\frac{\varepsilon d_n}{2L}, \varepsilon d_n] \subset A_t$ for some $t$ and $\omega$-large $A$. However; for all sufficiently large $n$, $k_n < \frac{\varepsilon d_n}{2L}$ which implies that $[\frac{\varepsilon d_n}{2L}, \varepsilon d_n] \cap B_n \neq \emptyset$ for all sufficiently large $n$. This contradictions our choice of $t$ such that $\bigcup_{n\in A} [\frac{\varepsilon d_n}{2L}, \varepsilon d_n] \subset A_t$.

*Proof of (ii).* Fix $s \in \mathbb{N}$. Suppose that $(ii)$ does not hold. Then there exists $[x_n,y_n] \subset \mathscr{M}_s(B_n)$ such that $\frac{y_n}{x_n} > n$. Fix an ultrafilter $\omega$ and let $d_n = (x_n y_n)^{\frac{1}{2}}$, the geometric center of the interval $[x_n,y_n]$. Then for every $n' \leq n$, $\mathscr{M}_{\sqrt{n}}(d_n) \subset [x_n,y_n] \subset \mathscr{M}_s(B_{n'})$. (The first inclusion follows by our choice of $d_n$ and the second holds since $\mathscr{M}_s(B_n) \subset \mathscr{M}_s(B_{n'})$ for $n' \leq n$.)

Suppose $X$ is $\varepsilon$-coarsely loop divisible with respect to $(\omega, d)$ for some $\varepsilon \in (0,1)$. For any $0 < \delta < \varepsilon$, $\bigcup_{n>m} [\delta d_n, \varepsilon d_n] \subset \mathscr{M}_s(B_m)$ for every $m > \frac{1}{\delta^2}$. If $\delta < \frac{\varepsilon}{2s}$ and $[\delta d_n, \varepsilon d_n] \subset \mathscr{M}_s(B_m)$, then $[\delta d_n, \varepsilon d_n] \cap B_m \neq \emptyset$. Since this hold for every sufficiently large $m$, we can derive a contradiction as in (i).

The proof of (*iii*) is the same as proof of (*ii*).

$\square$

*Proof of Proposition IV.1.27.* We will use the notation from Lemma IV.1.28. The lemma is trivial if some $c_k = \infty$. Thus we will assume that for every $k$, $c_k < \infty$.

Let $k'_1 = 1$ and $s_1 = \frac{c_{k'_1}}{3}$. We may choose $k_1 > k'_1$ and $b_1$ such that $\sup\left\{\frac{b}{a} \middle| [a,b] \subset \mathscr{M}_{s_1}(B_k)\right\} < b_1$ for all $k \geq k_1$.

Suppose that we have inductively define $s_i$, $k'_i$, $k_i$ and $b_i$ for all $i < n$.

Choose $k'_n \in \mathbb{N}$ such that $c_{k'_n} > b_{n-1}^3 \cdot c_{k'_{n-1}}^2$ and let $s_n = \frac{c_{k'_n}}{3}$. Again, we may choose $k_n > k'_n$ and $b_n$ such that $\sup\left\{\frac{b}{a} \middle| [a,b] \subset \mathscr{M}_{s_n}(B_k)\right\} < b_n$ for all $k \geq k_n$.

Choose $[a_{1,1}, b_{1,1}]$ a maximal interval in $A_{k_1}$ containing a point of $\left(\mathscr{M}_{s_1}(B_{k_1})\right)^c$. Suppose that for all $i < n$, we have chosen $[a_{i,i}, b_{i,i}]$.

Let $[a_{n,n}, b_{n,n}]$ be a maximal interval in $A_{k_n}$ containing a point of $\left(\mathscr{M}_{s_n}(B_{k_n})\right)^c$ such that $b_{n-1,n-1} < a_{n,n}$.

**Claim.** *Let $x \in \left(\mathscr{M}_{s_i}(B_{k_i})\right)^c$. Then $[\frac{x}{s_i}, xs_i] \subset A_{k_i}$ and there exist $x' \in \left(\mathscr{M}_{s_{i-1}}(B_{k_{i-1}})\right)^c \cap [\frac{x}{s_i}, xs_i]$ such that $\mathscr{M}_{s_{i-1}}\left([\frac{x'}{s_{i-1}}, x's_{i-1}]\right) \subset [\frac{x}{s_i}, xs_i]$.*

*Proof of claim.* Let $x \in \left(\mathscr{M}_{s_i}(B_{k_i})\right)^c$. Then $\mathscr{M}_{s_i}(x) \cap B_{k_i} = \emptyset$ which implies that $[\frac{x}{s_i}, xs_i]$ in $A_{k_i}$.

Let $a = \frac{x}{s_i}$ and $b = xs_i$. Then $\frac{x}{a}, \frac{b}{x} = s_i = \frac{c_{k'_i}}{3}$. This implies that $\frac{b}{a} > \left(\frac{c_{k'_i}}{3}\right)^2 > \left(\frac{b_{i-1}^9 \cdot c_{k'_{i-1}}^4}{9}\right)$.

Let $t = \max\{b_{i-1}, c_{k'_{i-1}}\}$. Since $\frac{b}{t^4 a} > b_{i-1}$, $[at^2, \frac{b^2}{t^2}]$ contains a point $x' \in \left(\mathscr{M}_{s_{i-1}}(B_{k_{i-1}})\right)^c$. Then the inequality $t \geq c_{k'_{i-1}} > s_{i-1}$, along with the inclusion $\mathscr{M}_t\left([\frac{x'}{t}, x't]\right) \subset [a,b]$ imply that $\mathscr{M}_{s_{i-1}}\left([\frac{x'}{s_{i-1}}, x's_{i-1}]\right) \subset [a,b]$. This completes the proof of the claim.

Fix $n$. The claim shows that we can find a nested sequence of intervals $[a_{1,n}, b_{1,n}] \subset [a_{2,n}, b_{2,n}] \subset \cdots \subset [a_{n,n}, b_{n,n}]$ such that $[a_{1,n}, b_{1,n}] \subset A_{k_i}$ and $\mathscr{M}_{s_{i-1}}\left([a_{i,n}, b_{i,n}]\right) \subset [a_{i+1,n}, b_{i+1,n}]$.

Let $d_n = (a_{1,n} b_{1,n})^{\frac{1}{2}}$. Then $\bigcup_{n>i} [\frac{d_n}{s_i}, s_i d_n] \subset A_{k_n}$. Therefore $X$ is $\varepsilon$-coarsely loop divisible with respect to the pair $(\omega, d)$ for all $\varepsilon > 0$, since $s_i$ diverges. $\square$

This gives us the following analogue to Proposition IV.1.7. Rather than require a bound on the number of pieces in a partition, we only require a linear bound on the diameter of partitions and $\varepsilon$-coarsely loop divisible for all $\varepsilon > 0$.

**Proposition IV.1.29.** *Suppose that for some fixed pair $(\omega, d)$, a complete geodesic metric space $X$ is $\varepsilon$-coarsely loop divisible for all $\varepsilon > 0$. If there exists an $l, L, N$ and an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that every loop $\gamma$ in $X$ with $|\gamma| \geq L$ has a partition $\Pi$ of $\gamma$ with the property that*

*(i) $\Pi$ has at most $f \circ \vartheta^l(|\gamma|)$ pieces,*

*(ii) $\Pi$ is a $\frac{|\gamma|}{2}$-partition of $\gamma$, and*

*(iii)* $\operatorname{diam}(\Pi) \leq N|\gamma|$

*then* $\operatorname{Con}^{\omega}\big(X,e,d\big)$ *is simply connected.*

*Proof.* Suppose that for some fixed pair $\big(\omega,d\big)$ and all $\varepsilon > 0$, $X$ is $\varepsilon$-coarsely loop divisible. Fix $l, L, N$ and $f : \mathbb{R} \to \mathbb{R}$ as in statement of the lemma.

We will break the proof into two parts. First we will show that every geodesic $n$-gon $\alpha$ in $\operatorname{Con}^{\omega}\big(X,e,d\big)$ which is a limit of geodesic $n$-gons from $X$ bounds a disc of diameter at most $2N|\alpha|$. We will then show that this is enough to imply that all loops are nullhomotopic.

**Step 1.** Let $\alpha$ be a geodesic $n$-gon in $\operatorname{Con}^{\omega}\big(X,e,d\big)$ such that $\alpha(t) = \big(\alpha_n(t)\big)$ where $\alpha_n$ is a geodesic $n$-gon in $X$. By hypothesis; for each $n$ such that $|\alpha_n| > L$, there exists a partition $\Pi_n$ of $\alpha_n$ which satisfy conditions $(i)-(iii)$ of the lemma.

By Lemma IV.1.3, there exists a $K$ and an $\omega$-large set $A$ such that $(\vartheta^l)^{-1}(\bigcup\limits_{n\in A}\big[|\frac{|\alpha|d_n}{2}, 2|\alpha|d_n\big])$ is bounded by $K$. We will assume that for all $n \in A$, $|\alpha_n| \in \bigcup\limits_{n\in A}\big[\frac{|\alpha|d_n}{2}, 2|\alpha|d_n\big]$. Thus $\Pi_n$ has at most $f(K)$ pieces $\omega$-almost surely.

Then Proposition IV.1.6 implies that the partitions $\Pi_n$ induce a partition $\Pi$ of $\alpha$ which satisfies conditions $(1)$ and $(2)$ of the lemma. In Papasoglu's proof of Proposition IV.1.6, $\Pi$ is just the $\omega$-limit of the partitions $\Pi_n$; thus, condition $(3)$ is also satisfied for $\Pi$.

Fix $\gamma$ a geodesic $n$-gon in $\operatorname{Con}^{\omega}\big(X,e,d\big)$ such that $\gamma(t) = \big(\gamma_n(t)\big)$ for $\gamma_n$ a geodesic $n$-gon in $X$.

We have shown that there exists a partition $\Pi_1 : P_1^{(0)} \to \operatorname{Con}^{\omega}\big(X,e,d\big)$ of $\gamma$ into pieces of length $|\gamma|/2$ with the diameter of the partition no greater than $N|\gamma|$.

Proceeding by induction, suppose that we have defined $\Pi_k : P_k^{(0)} \to \operatorname{Con}^{\omega}\big(X,e,d\big)$ a partition of $\gamma$ into pieces of length $\frac{|\gamma|}{2^k}$ for $k < i$ such that for all $1 < k \leq i-1$

- $\Pi_k$ extends $\Pi_{k-1}$
- for $x \in \operatorname{im}\Pi_k$ $\operatorname{dist}(x, \operatorname{im}\Pi_{k-1}) \leq \dfrac{N|\gamma|}{2^k}$.

The partition $\Pi_{i-1}$ extends to a map $\widetilde{\Pi}_{i-1}$ on the $(1)$-skeleton of $P_{i-1}$ as in Remark IV.0.18. Then we can partition each of the subloops into pieces of length less than $|\gamma|/2^i$ with the desired diameters. We can then use these partitions to extend $\Pi_{i-1}$ to $\Pi_i$ satisfying the two induction hypothesis.

For all $i > j$; if $x \in \operatorname{im}\Pi_i$, then $\operatorname{dist}(x, \operatorname{im}\Pi_j) \leq \sum\limits_{s=j}^{i} \dfrac{N|\gamma|}{2^s}$. Hence, $\Pi_i$ converges to a continuous function from the unit disc into $\operatorname{Con}^{\omega}\big(X,e,d\big)$ which extends $\gamma$. Therefore $\gamma$ bounds a disc of diameter $2N|\gamma|$. This completes Step 1.

**Step 2.** Let $Q_n$ be the convex hull of the regular $2^n$-gon inscribed in $S^1$, the unit circle in the plane with the standard Euclidean metric. Then $Q_n$ has a natural cell decomposition with $2^n$ vertices and $2^n$ edges and one 2-cell. Furthermore, we may assume that the 0-skeleton of $Q_n$ form a nested sequence of subsets of $S^1$. Let $A_1^2 = Q_2$ which has diameter 2. For $n > 2$, $Q_n \backslash (\operatorname{interior}(Q_{n-1}))$ is a set of $2^n$ triangles with vertices on $S^1$ each of which share a unique edge with $Q_{n-1}$ and have diameter less than $\frac{\pi}{2^{n-1}}$. Let $\{A_i^n\}_{i=1}^{2^n}$ be this set of triangles. Then $A = \bigcup\limits_{i,n} A_i^n$ covers the interior of the unit disc and a dense subset of its boundary.

Fix a loop $\gamma : S^1 \to \mathrm{Con}^\omega\big(X,e,d\big)$.

We may choose a geodesic 4-gon $\gamma_1^2 : \partial A_2^{(1)} \to \mathrm{Con}^\omega\big(X,e,d\big)$ such that $\gamma_1^2|_{Q_2\cap S^1} = \gamma|_{Q_2\cap S^1}$ and $\gamma_1^2$ is the limit of geodesic 4-gons from $X$. We can inductively define geodesic 3-gons $\{\gamma_i^n : \partial A_i^n \to \mathrm{Con}^\omega\big(X,e,d\big)\}$

i) $\gamma_i^n|_{\partial A_i^n \cap Q_{n-1}} = \gamma_j^{n-1}|_{\partial A_i^n \cap Q_{n-1}}$ for some $j$ and

ii) $\gamma_i^n|_{\partial A_i^n \cap S^1} = \gamma|_{\partial A_i^n \cap S^1}$.

Using Step 1, we can define maps $\{h_i^n : A_i^n \to \mathrm{Con}^\omega\big(X,e,d\big)\}$ such that $h_i^n$ is a nullhomotopy of $\gamma_i^n$ and $\mathrm{diam}(h_i^n)$ no greater than $2N\delta_i^n$ where $\delta_i^n$ is the sum of the distances between the image of adjacent vertices of $A_i^n$.

This defines a function $h : A \to \mathrm{Con}^\omega\big(X,e,d\big)$ by $h(a) = h_i^n(a)$ for some $i$ and $n$. This is well defined by Condition (i). Since $\gamma$ is continuous on a compact set; for ever $\varepsilon > 0$, there exists a $K$ such that $2N\delta_i^n < \varepsilon$ for all $n > K$. Thus $h$ is continuous on $A$. By Condition (ii), $h|_{A\cap S^1} = \gamma|_{A\cap S^1}$ which implies that $h$ extends to a nullhomotopy of $\gamma$.

□

**Corollary IV.1.30.** *Let $G$ be a group and $S$ a finite generating set for $G$. Suppose that there exists an $l,L,N$ and an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that every loop $\gamma$ in $\Gamma(G,S)$ with $|\gamma| \geq L$ has a partition $\Pi$ of with the property that*

*(i)* $\Pi$ *has at most* $f \circ \vartheta^l(|\gamma|)$ *pieces,*

*(ii)* $\Pi$ *is a* $\frac{|\gamma|}{2}$*-partition of* $\gamma$*, and*

*(iii)* $\mathrm{diam}(\Pi) \leq N|\gamma|$.

*Then at least one of the following occurs.*

*(A) G has an asymptotic cone which is not semilocally simply connected and has an uncountable fundamental group.*

*(B) Every asymptotic cone of G is locally simply connected and G has an asymptotic cone which is simply connected.*

*Proof.* If for some ultrafilter and scaling sequence $G$ is not $\varepsilon$-coarsely divisible for every $\varepsilon > 0$, then $G$ has an asymptotic cone which is not semilocally simply connected and has uncountable fundamental group.

Otherwise, for every pair $\big(\omega,d\big)$; $G$ is $\varepsilon$-coarsely divisible with respect to $\big(\omega,d\big)$ for some $\varepsilon > 0$. The proof of Proposition IV.1.29 implies that every asymptotic cone of $G$ is locally simply connected.

Proposition IV.1.27 implies that there exists a pair $\big(\omega,d\big)$ such that $G$ is $\varepsilon$-coarsely divisible for every $\varepsilon > 0$ with respect to $\big(\omega,d\big)$. Proposition IV.1.29 implies that $\mathrm{Con}^\omega\big(G,d\big)$ is simply connected. □

## IV.2 Examples

**Lemma IV.2.1.** *Suppose that $G$ is a group with a finite presentation $\langle S \mid R \rangle$ which has an exponential isoperimetric function. If there exists a sequence of loops $\gamma_n$ in $\Gamma(G,S)$ such that $|\gamma_n|$ grows at most linearly and* $\mathrm{Area}(\gamma_n)$ *has an exponential lower bound, then there exists a sequence of absolutely non-divisible loops in* $\Gamma(G,S)$.

*Proof.* Let $\gamma_n$ be a sequence of loops in $\Gamma(G,S)$ such that $|\gamma_n| \leq Ln$ and $Ab^n \leq \mathrm{Area}(\gamma_n) = \delta(|\gamma_n|) \leq Dc^{Ln}$ where $\delta$ is the Dehn function for the presentation $\langle S \mid R \rangle$ and $A, b, c, D, L$ are positive constants. Fix $M$ such that $c^{\frac{L}{M}} \leq b$.

Suppose that for some subsequence $n_i$, $\gamma_{n_i}$ has a $\frac{|\gamma_{n_i}|}{M}$-partition with at most $K$ pieces where $K$ is independent of $i$. Then

$$Ab^{n_i} \leq \mathrm{Area}(\gamma_{n_i}) = \delta(|\gamma_{n_i}|) \leq K\delta(\frac{|\gamma_{n_i}|}{M}) \leq KDc^{\frac{Ln_i}{M}}.$$

This implies that $\dfrac{b^{n_i}}{c^{\frac{Ln_i}{M}}} = \left(\dfrac{b}{c^{\frac{L}{M}}}\right)^{n_i}$ is bounded independent of $i$ which contradicts our choice of $M$. Thus $P\big(\gamma_n, \frac{|\gamma_n|}{M}\big)$ diverges and the lemma follows from Remark IV.1.23.

□

**Corollary IV.2.2.** *Every asymptotic cone of the following groups is not semilocally simply connected and has an uncountable fundamental group which is not free and not simple. In addition, any decomposition of the fundamental group of an asymptotic cone of one of the following groups as a free product has a factor which is not free and uncountable.*

1. *$SL_3(\mathbb{Z})$;*
2. *Baumslag-Solitar groups – $BS_{pq} = \langle a,t \mid t^{-1}a^pt = a^q \rangle$ for $|p| \neq |q|$;*
3. *the 3-manifold $Sol_3$, $\mathbb{R}^3$ endowed with the Riemannian metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$;*
4. *any extension of $\mathbb{R}^n$ by $\mathbb{R}$ via a matrix with all real eigenvalues of norm strictly greater than 1 and at least two eigenvalues with different sign;*
5. *Baumslag-Gertsen group – $\langle a,t \mid (t^{-1}a^{-1}t)a(t^{-1}at) = a^2 \rangle$;*
6. *$Out(F_n)$ and $Aut(F_n)$ for $n \geq 3$;*
7. *$G_7 = \langle a,s,t \mid [a,a^t] = [s,t] = 1, aa^t = a^s \rangle$; and*
8. *$G_8 = \langle \theta_1, \theta_2, a, k \mid a^{\theta_i} = a, k^{\theta_i} = ka, i = 1,2 \rangle$.*

$G_7$ is of interest since it is metabelian and not polycyclic. It is sometimes referred to as the Baumslag group. $G_8$ was constructed by Olshanskii and Sapir and has cubic Dehn function and linear isodiametric function.

*Proof.* Epstein and Thurson in [20] showed the existence of a sequence of loops in $SL_3(\mathbb{R})$ and $BS_{pq}$ for $|p| \neq |q|$ satisfying the conditions of Lemma IV.2.1.

The result for Baumslag-Solitar groups and $Sol_3$ was already known and is due to [8]. Cornulier showed the existence of a sequence of absolutely non-divisible loops for groups of the type (4) in [14].

Kassabov and Riley in [32] showed that the loops in the Cayley graph of $G_7$ with label $[a, a^{t^n}]$ have the desired properties.

For $Out(F_n)$ and $Aut(F_n)$ Bridson and Vogtmann exhibit the necessary sequence in [5].

That leaves only (5) and (8). Since $G_5 = \langle a, t \,|\, (t^{-1}a^{-1}t)a(t^{-1}at) = a^2 \rangle$ has a Dehn function which is greater than any tower of exponentials, we cannot apply Lemma IV.2.1. Gersten in [23] showed the existence of a sequence of loops $\gamma_k$ such that $\gamma_k$ has length $3 \cdot 2^{k+1}$ and area at least $2^{2^{2^{\cdots^{2}}}}\Big\}{\scriptstyle k\ times}$. Platonov in [45] showed that $\delta(n) = 2^{2^{2^{\cdots^{2}}}}\Big\}{\scriptstyle \log_2(n)\ times}$ is an isoperimetric function for $G_5$.

Suppose that for some subsequence $n_i$, $P\big(\gamma_{n_i}, \frac{|\gamma_{n_i}|}{6}\big) \leq K$. Then for all $n_i$

$$2^{2^{2^{\cdots^{2}}}}\Big\}{\scriptstyle n_i\ times} \leq \mathrm{Area}(\gamma_{n_i}) \leq K2^{2^{2^{\cdots^{2}}}}\Big\}{\scriptstyle \log_2(\frac{32^{n_i}}{6})\ times} = K2^{2^{2^{\cdots^{2}}}}\Big\}{\scriptstyle (n_i-1)\ times}$$

which is a contradiction. Hence $\gamma_k$ is an absolutely non-divisible sequence of loops and the result follows from Corollary IV.1.26.

Ol'shanskii and Sapir in [39] constructed a sequence of loops $\gamma_n$ in the Cayley complex of $G_8 = \langle \theta_1, \theta_2, a, k \,|\, a^{\theta_i} = a, k^{\theta_i} = ka, i = 1, 2 \rangle$ such that $\gamma_n$ has length $6n$. Additionally, they showed that $\gamma_n$ cannot bound a disc decomposed into at most $l$ subdiscs of perimeter $n$ where $l \leq \sqrt{n}$ and hence is an absolutely non-divisible sequence of loops.

□

**Remark IV.2.3.** Suppose that $X$ is a one-dimensional metric space and $Y$ is the support of any finite set of paths in $X$. Then $Y$ is a compact one-dimensional metric space and thus its fundamental group is locally free, residually free, and residually finite by Theorem 5.11 in [10]. Since $X$ is one-dimensional, the homomorphism from the fundamental group of $Y$ to the fundamental group of $X$ induced by set inclusion is injective (see Theorem 3.7 in [10]). Thus $\pi_1(X, x_0)$ is locally free.

Burillo in [8] shows that all asymptotic cones of solvable $BS_{p,q}$ with $|p| \neq |q|$ and $Sol_3$ have topological dimension 1. As well, any extension of $\mathbb{R}^n$ by $\mathbb{R}$ via a matrix with all real eigenvalues of norm strictly greater than 1 and at least two eigenvalues with different sign will have one-dimensional asymptotic cones (see [14]). Thus the fundamental groups of their cones are locally free.

To prove Corollary IV.2.2, we analysed mappings of Hawaiian earrings into asymptotic cones and showed that the induced homomorphism's image had the desired properties. However, this method doesn't give us much information concerning the structure of the rest of the fundamental group. When an asymptotic cones of a group is one-dimensional, one can apply standard techniques for one-dimensional space, as in Remark IV.2.3, to better understand the structure of the fundamental. Requiring a dimension constraint on the asymptotic cone is a strong condition and does not apply to many well studied groups.

### IV.2.1 A group with a locally simply connected cone which is not simply connected

In [37], it was shown that there exists a group with the following properties.

**Theorem IV.2.4.** *There is a finitely generated group $G$ whose Dehn function $f(n)$ satisfies the following properties:*

1. *there are sequences of positive numbers $d_i \to \infty$ and $\lambda_i \to \infty$ such that $f(n) \leq cn^2$ for arbitrary integer $n \in [\frac{d_i}{\lambda_i}, d_i\lambda_i]$ and some constant $c$ and*
2. *there is a positive constant $c'$ and an increasing sequence of numbers $n_i \to \infty$ such that $\frac{f(n_i)}{n_i^2} \to \infty$ but for every $i$, and for every integer $n$ with $n < c'n_i$, we have $f(n) \leq c'n_i^2$.*

Ol'shanskii construct $G$ as a multiple HNN extension of a free group using $S$-machines.

**Corollary IV.2.5.** *If $G$ is as in Theorem IV.2.4.*

*(A) There exists an $\omega$ such that $\mathrm{Con}^{\omega}(G,(n_i))$ has a nontrivial fundamental group.*

*(B) $\mathrm{Con}^{\omega}(G,(n_i))$ is locally simply connected for all $\omega$.*

*(C) $\mathrm{Con}^{\omega}(G,(d_i))$ has trivial fundamental group for all $\omega$.*

*Proof.* Ol'shanskii and Sapir in [41] showed that the second condition implies the existence of a $b < 1$ such that $\frac{f(n_i)}{f(bn_i)} \to \infty$. This was used to show that divisibility function restricted to $\bigcup\limits_i [bn_i, n_i]$ is unbounded. Then (A) follows.

The first condition implies that $G$ is uniformly $\varepsilon$-coarsely loop divisible for every $\varepsilon > 0$ with respect to the pair $\big(\omega,(d_i)\big)$ for any ultrafilter $\omega$. Therefore $\mathrm{Con}^{\omega}(G,(d_i))$ has trivial fundamental group.

The second condition implies $\big($by the same argument that was used to show $G$ is uniformly $\varepsilon$-coarsely loop divisible for every $\varepsilon > 0$ with respect to the pair $\big(\omega,(d_i)\big)$ that there exists an $\varepsilon > 0$ such that $G$ is uniformly $\varepsilon$-coarsely loop divisible for the pair $\big(\omega,(n_i)\big)$. Hence, $\mathrm{Con}^{\omega}(G,(n_i))$ is locally simply connected. □

**Question 7.** *Can this group have an asymptotic cone which is not locally simply connected?*

Thomas and Velicovick consider a group $G_I = \langle a, b \,|\, (a^n b^n)^7 = 1;\ n \in I \rangle$ which they show for an appropriate choice of $I$ has simply connected and non-simply connected asymptotic cones [51].

**Proposition IV.2.6.** *Let $I_0 = \{2^{2^n}\}$. Let $I_{0,k} = I_0 \cap [2^{2^k}, \infty)$ and $I = \bigcup_{k=1}^{\infty} 2^k \cdot I_{0,k}$. Then $G_I = \langle a, b \,|\, (a^n b^n)^7\ n \in I \rangle$ has a cone which is locally simply connected and a cone which is not semi-locally simply connected.*

*Proof.* Let $\gamma_n$ be the loop based at the identity with label $(a^n b^n)^7$ for $n \in I$. Thomas and Velicovick show using small cancelation that $P\big(\gamma_n, \frac{|\gamma_n|}{2}\big) = \infty$ [51, Lemma 1.1].

If we let $d_n = 8^{2^{n-1}}$, then the argument of Thomas and Velicovick shows that $\mathrm{Con}^{\omega}\big(G, d\big)$ is an $\mathbb{R}$-tree for any $\omega$.

Let $\rho_n = 2^n 2^{2^n} = 2^{2^n+n}$. Let $\gamma_{n,k}$ be the loop with label $(a^{2^{2^n+k}} b^{2^{2^n+k}})^7$ for $k \geq n$. Then $\frac{|\gamma_{n,k}|}{\rho_n} = \frac{14}{2^{n-k}}$. Hence $(\gamma_{n,n-i})$ is a loop of length $\frac{14}{2^i}$ in $\mathrm{Con}^{\omega}\big(G,(\rho_n)\big)$ which has no finite partition. This implies that $\mathrm{Con}^{\omega}\big(G,(\rho_n)\big)$ is not $\varepsilon$-coarsely loop divisible for any $\varepsilon > 0$. Thus $\mathrm{Con}^{\omega}\big(G,(\rho_n)\big)$ is not semi-locally simply connected and has uncountable fundamental group for any $\omega$.

$\square$

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