HAMILTON CYCLE EMBEDDINGS OF COMPLETE TRIPARTITE GRAPHS AND THEIR APPLICATIONS

By

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To my parents, Jeff and Ruth, for their unending support

and

To Lauren, for her love and inspiration

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CHAPTER I

INTRODUCTION

Ever since Leonhard Euler solved the "Seven Bridges of Königsberg" problem in the early eighteenth century, graph theory has developed into an important tool in the study of natural sciences, computer science, and other areas of mathematics. Several characteristics of graphs are related to drawings of those graphs on surfaces. This thesis addresses the genus problem – one of the central problems in topological graph theory – as well as some additional embedding problems.

We begin by introducing the required terms and concepts.

I.1 Basic graph theory

A graph $G = (V(G), E(G))$ consists of a set $V(G)$ of *vertices* and a set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. Thus, our graphs are simple, with no loops or parallel edges. In this setting, if $e = \{u, v\}$, we will simply write $e = uv = vu \in E(G)$. If $e = uv$, then we say e *joins* u and v and the vertices u and v are called the *ends* of e. If v is an end of e, then we say e is *incident* with v. For distinct $u, v \in V(G)$, we say u and v are *adjacent* (or u and v are *neighbors*), denoted $u \sim v$, if $uv \in E(G)$. A subset $X \subset V(G)$ is called an *independent set* if $u \not\sim v$ for all $u, v \in X$. The *degree* of a vertex v is the number of edges incident with v; equivalently, the number of neighbors of v. If every vertex in G has degree d, we say G is *d-regular*.

For $k \geq 3$, a *cycle of length k* in a graph G is a sequence of distinct vertices $(v_1 v_2 \cdots v_k)$ such that $v_i \sim v_{i+1}$ for $i = 1, 2, ..., k-1$ and $v_k \sim v_1$. A cycle that contains every vertex in G is called a *hamilton cycle*; if G contains n vertices, then a hamilton cycle is simply a cycle of length n. A *walk* of length k in a graph G is an alternating sequence of (not necessarily distinct) vertices and distinct edges $(v_1 e_1 v_2 e_2 v_3 ... v_k e_k v_{k+1})$ such that each edge e_i is incident

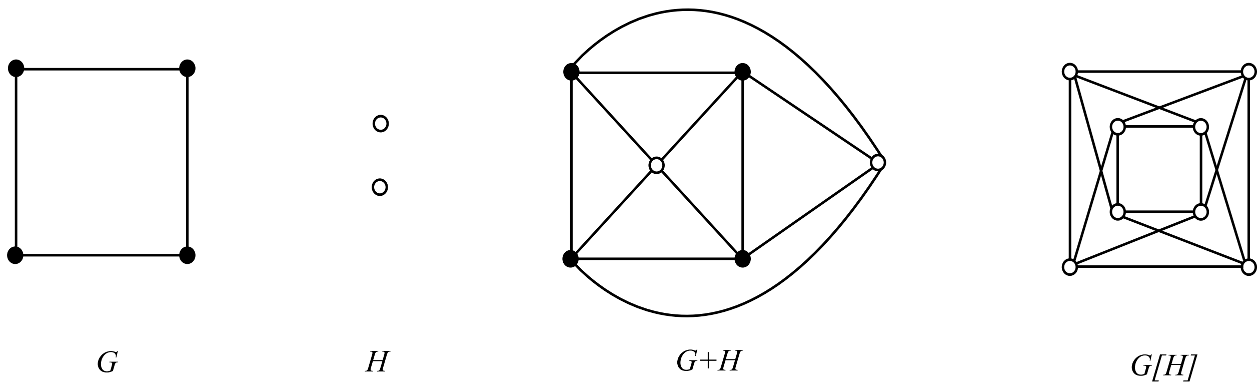

Figure I.1: Examples of a join graph and a lexicographic product.

with both v_i and v_{i+1} for $i = 1, 2, ..., k$. If $v_1 = v_{k+1}$, then the walk is *closed*. A cycle is necessarily a closed walk, but the converse does not hold. A closed walk that contains every edge in G is called an *euler circuit*, and a graph that contains an euler circuit is called *eulerian*.

We will be studying hamilton cycles in some special classes of graphs. The *complete graph* K_n is an n-vertex graph such that every vertex is adjacent to every other vertex. For $k \geq 2$, the *complete multipartite graph* $K_{n_1,n_2,...,n_k}$ consists of k independent sets of vertices $X_1, X_2, ..., X_k$ such that $|X_i| = n_i$ and two vertices $u \in X_i$ and $v \in X_j$ are adjacent if and only if $i \neq j$. If G is a multipartite graph with $k = 2, 3$ or 4, then G is a *complete bipartite graph*, *complete tripartite graph* or *complete quadripartite graph*, respectively. The *complement* of a graph G, denoted $\overline{G}$, is a graph on the same vertex set such that $u \sim v$ in $\overline{G}$ if and only if $u \not\sim v$ in G. The complement of a complete graph K_n is the edgeless graph $\overline{K_n}$.

At times we consider ways of building larger graphs from smaller ones. The *join* of G and H, denoted $G + H$, is the graph with vertex set $V(G) \cup V(H)$ such that two vertices u and v are adjacent if and only if $uv \in E(G) \cup E(H)$ or $u \in V(G)$ and $v \in V(H)$. In other words, $G + H$ contains all the original edges of G and H, as well as an edge from every vertex in G to every vertex in H. The *lexicographic product* of G and H, denoted $G[H]$, is the graph with vertex set $V(G) \times V(H)$ such that two vertices (u, x) and (v, y) are adjacent if and only if $u \sim v$ in G or $u = v$ and $x \sim y$ in H. In other words, $G[H]$ is obtained by replacing each vertex v in G with a copy H_v of H and placing an edge between every vertex in H_u and H_v whenever $u \sim v$ in G. Figure I.1 shows graphs G and H together with their join $G + H$ and the lexicographic product $G[H]$. Note that if $G = K_{n_1,n_2,...,n_k}$, then $G[\overline{K_m}] = K_{mn_1,mn_2,...,mn_k}$.

Let G be a graph and let $X = \{x_h \mid h \in H\} \subset V(G)$ and $Y = \{y_h \mid h \in H\} \subset V(G)$ be disjoint subsets of $V(G)$ indexed by the group H. We call $x_{h_1}y_{h_1h_2} \in E(G)$ an XY-edge of *slope* h_2. Unless otherwise noted, we will assume $H = \mathbb{Z}_n$ and the edge x_iy_j will be the XY-edge of slope $j - i$. For the majority of this thesis, we will focus on complete tripartite graphs. To that end, let $A = \{a_0, ..., a_{n-1}\}$, $B = \{b_0, ..., b_{n-1}\}$ and $C = \{c_0, ..., c_{n-1}\}$ be the vertices of $K_{n,n,n}$ so that A, B and C are the maximal independent sets. A hamilton cycle of the form $(a_{j_0}b_{k_0}c_{\ell_0}a_{j_1}b_{k_1}c_{\ell_1} \cdots a_{j_{n-1}}b_{k_{n-1}}c_{\ell_{n-1}})$ is called an *ABC cycle*.

For further information on graphs, see [3].

I.2 Topological graph theory

I.2.1 Background

Topological graph theory is the study of graphs drawn on surfaces. A *surface* in this case is a compact 2-manifold without boundary. There are two families of surfaces: the *orientable* surfaces and the *nonorientable* surfaces. The orientable surface of *genus* h, denoted S_h, can be interpreted as a sphere with h handles attached; equivalently, it is a torus with h holes. The nonorientable surface of *genus* k, denoted N_k, can be interpreted as a sphere with k *crosscaps* attached, where a crosscap is essentially a Möbius strip. The *euler characteristic* of S_h is $2 - 2h$, while the *euler characteristic* of N_k is $2 - k$; we denote the euler characteristic of an arbitrary surface Σ by $\chi(\Sigma)$. The *orientability characteristic* of an orientable or nonorientable surface is $+1$ or -1, respectively.

An *embedding* of a graph G, denoted $G \hookrightarrow \Sigma$, consists of an injective map $\nu : V(G) \to \Sigma$ and a continuous injective map $\varepsilon_i : [0, 1] \to \Sigma$ for each edge $e_i \in E(G)$ satisfying the following properties: 1) if $e_i = uv$, then $\{\varepsilon_i(0), \varepsilon_i(1)\} = \{\nu(u), \nu(v)\}$; 2) $\varepsilon_i(t) \notin \nu(V(G))$

for all $t \in (0,1)$; and 3) for $i \neq j$, $\varepsilon_i(s) \neq \varepsilon_j(t)$ for all $s, t \in (0,1)$. An embedding of G can be thought of as a drawing of G on the surface Σ such that no edges cross. Let $C = \bigcup_{e_i \in e(G)} \varepsilon_i([0,1])$ be the image of all the maps of an embedding; the connected components of $\Sigma \setminus C$ are called *faces* and denoted $F(G)$. If every face is homeomorphic to an open disk, then the embedding is an *open 2-cell embedding*; if the closure of every face is homeomorphic to a closed disk, then the embedding is a *closed 2-cell embedding*. The faces of an embedding satisfies the relation $|V(G)| - |E(G)| + |F(G)| = \chi(\Sigma)$; this is known as the *Euler formula*. In an open 2-cell embedding, a face is usually characterized by the ordering of vertices as you walk around the boundary of the face; we will simply use "face" to indicate the boundary walk of a face. For a closed 2-cell embedding, each face is simply a cycle in the original graph.

A given graph G can often be embedded on many different surfaces; therefore, we usually desire embeddings with certain properties. The *genus* of G, denoted $g(G)$, is the minimum h such that $G \hookrightarrow S_h$. Likewise the *nonorientable genus* of G, denoted $\tilde{g}(G)$, is the minimum k such that $G \hookrightarrow N_k$. The embeddings that achieve these minima – hereafter referred to simply as *genus embeddings* – generally have very small faces, such as triangles. There are related notions of *maximum genus* and *nonorientable maximum genus*; the embeddings that achieve these maxima generally have very large faces, the largest possible of which – when restricting to closed 2-cell embeddings – is a hamilton cycle. An embedding where every face is a triangle is called a *triangulation* or a *triangular embedding*, while an embedding where every face is a hamilton cycle is called a *hamilton cycle embedding*. If the faces of an embedding can be colored with k colors so that any two faces that share an edge are assigned different colors, then we say that embedding is *face k-colorable*. Some of the embeddings we construct will be face 2-colorable.

Finding these special embeddings for general graphs can be quite difficult. In fact, Thomassen [50] showed that determining the genus of a general graph is NP-complete. However, it is possible to determine the genus exactly for certain families of graphs. Ringel used some graphs with known genus to solve most cases of the Map Coloring Theorem [41], a generalization of the Four Color Theorem.

After Ringel and Youngs solved the final case of the Map Coloring Theorem in 1968, there was little progress made on the graph genus problem for the next three decades. But with the recent development of new techniques, interest in determining the genus of special graphs has been renewed. In particular, Ellingham, Stephens and Zha [14] used nonorientable embeddings of complete bipartite graphs with several large faces to prove that, with a few small exceptions, $\tilde{g}(K_{\ell,m,n}) = \tilde{g}(K_{\ell,m+n})$. This essentially means that we can add the required edges to a genus embedding of $K_{\ell,m+n}$ to obtain an embedding of $K_{\ell,m,n}$ without raising the genus of the surface. In a similar fashion, Ellingham and Stephens [12] used nonorientable embeddings of complete graphs whose faces were all hamilton cycles to prove that, with a few small exceptions, $\tilde{g}(\overline{K_m} + K_n) = \tilde{g}(K_{m,n})$ when $m \geq n - 1$. This again means we can add the required edges to a genus embedding of $K_{m,n}$ to obtain an embedding of $\overline{K_m} + K_n$ without raising the genus of the surface. The pursuit of a similar result for the genus of $\overline{K_m} + K_n$ has proven to be much more challenging [13]; one of the goals of this thesis is to make progress towards resolution of the orientable case.

Another goal of this research is to find embeddings of certain complete multipartite graphs that satisfy given properties. In a series of papers, Ringel and Youngs, et al., [37–40, 43–48] found genus embeddings of K_n and $K_{m,n}$ on both orientable and nonorientable surfaces. In [42] they also found hamilton cycle embeddings of $K_{n,n}$ on orientable surfaces. As mentioned in the previous paragraph, Ellingham, Stephens, and Zha [14] found genus embeddings of $K_{\ell,m,n}$ on nonorientable surfaces; this work included hamilton cycle embeddings of $K_{n,n}$ on nonorientable surfaces for all $n \geq 4$. Moreover, Ellingham and Stephens found genus embeddings of $\overline{K_m} + K_n$ from hamilton cycle embeddings of K_n on nonorientable surfaces [12] and on orientable surfaces for certain values of n [13]. In this thesis we extend these results to include hamilton cycle embeddings of $K_{n,n,n}$ and genus embeddings of $K_{t,n,n,n}$ for $t \geq 2n$ on both orientable and nonorientable surfaces.

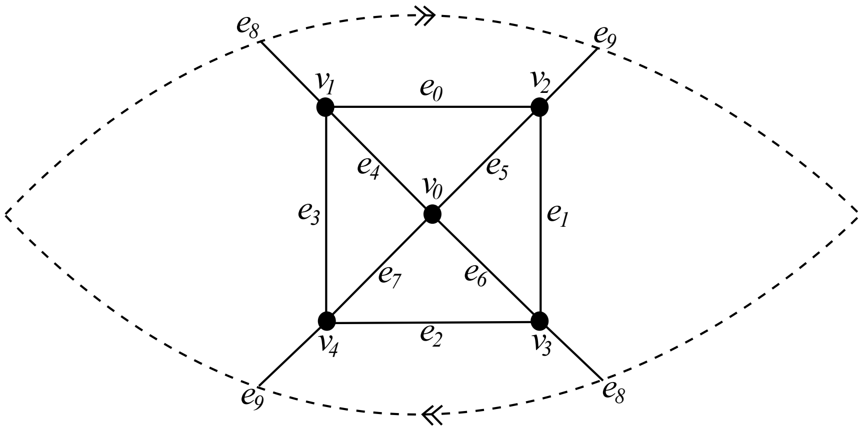

Figure I.2: An embedding of K_5 on the projective plane.

For more background on topological graph theory, see [27] or [36].

I.2.2 Combinatorial descriptions of embeddings

It is possible to represent an embedding completely combinatorially. The most common way to do this for open 2-cell embeddings is by specifying the *rotation system*, which includes a cyclic ordering $R_v = (e_0 e_1 \cdots e_k)$ of the ends of edges incident with each vertex v; the permutation R_v is called the *rotation* around v. The faces of the embedding are given by the collection of all closed walks $W = (v_0 e_0 v_1 e_1 v_2 ... v_{k-1} e_{k-1} v_0)$ satisfying the condition that $R_{v_i}(e_{i-1}) = e_i$ for every $i \in \mathbb{Z}_k$. The embedding resulting from a rotation system is always in an orientable surface.

To extend the definition of rotation systems to get nonorientable embeddings, we consider embedding schemes. An *embedding scheme* consists of a rotation system together with a map $\lambda : E(G) \to \{-1, +1\}$ that associates to each edge e the *edge signature* $\lambda(e)$. For any open walk $W = (v_0 e_0 v_1 e_1 v_2 ... v_{k-1} e_{k-1} v_k)$, define $\lambda(W) = \lambda(e_0)\lambda(e_1) \cdots \lambda(e_{k-1})$ to be the *net walk signature*. The faces of the resulting embedding are traced out in the following manner. Choose any edge e_0 with ends v_0 and v_1, and let $W_0 = (v_0 e_0 v_1)$. The next edge is given by $e_1 = R_{v_1}^{\lambda(W_0)}(e_0)$; assuming the other end of e_1 is v_2, we set $W_1 = (v_0 e_0 v_1 e_1 v_2)$. Now to each open walk W_j we continue adding the edge $e_{j+1} = R_{v_{j+1}}^{\lambda(W_j)}(e_j)$ until we encounter the edge e_0 again with a net walk signature of $+1$. The collection of all closed walks determined by this process correspond to the faces of the resulting embedding. If all of the edge signatures are $+1$, then the resulting embedding will be identical to the orientable embedding obtained from the rotation system alone. The following example gives the embedding scheme for the embedding $K_5 \hookrightarrow N_1$ shown in Figure I.2. The nonorientable surface N_1 represented in Figure I.2 is the projective plane, obtained by identifying the upper dashed curve with the lower dashed curve such that the arrows point in the same direction.

Example I.2.1.

$$
\begin{aligned}
&R_{v_0} : (e_4 e_5 e_6 e_7)\\
&R_{v_1} : (e_0 e_4 e_3 e_8)\\
&R_{v_2} : (e_0 e_9 e_1 e_5)\\
&R_{v_3} : (e_1 e_8 e_2 e_6)\\
&R_{v_4} : (e_2 e_9 e_3 e_7)\\
&\lambda(e_0) = \lambda(e_1) = \lambda(e_2) = \lambda(e_3) = \lambda(e_4) = \lambda(e_5) = \lambda(e_6) = \lambda(e_7) = +1\\
&\lambda(e_8) = \lambda(e_9) = -1
\end{aligned}
$$

The benefit of using rotation schemes is that you know you have an embedding; the detriment is that tracing out the faces can be quite difficult. Since we are trying to construct embeddings with specific facial boundaries, this is a problem. To remedy this, we use an alternate representation of an embedding. Rather than specifying the cycles that make up each rotation, we give the cycles that make up the boundary of each face. This obviously gives us control over the sizes of each face; however, it comes at a cost. While it is known that a collection of faces that double covers the edges of a graph G can be "sewn" together along common edges to get a drawing of G, it is possible that this drawing is on a *pseudosurface* rather than a surface. This happens if the neighborhood of any vertex in the resulting topological space is not homeomorphic to a disk. To ensure that we do have an embedding on a surface, it must be shown that there is a well-defined cyclic rotation around each vertex. Because all of the graphs we will encounter in this thesis are simple graphs, we will adopt the convention of presenting the rotation R_v around the vertex v as a *rotation graph* on the neighbors of v. If a face contains the subsequence $(\cdots uvw \cdots)$, then the vertices u and w are adjacent in R_v, representing the fact that the edges uv and vw occur consecutively in the rotation around v. Every rotation graph will be 2-regular; a rotation graph is *proper* if it consists of a single cycle. A collection of faces determines an embedding of a graph in a surface if the rotation graph around each vertex is proper. Example I.2.2 provides the collection of faces that corresponds to the embedding $K_5 \hookrightarrow N_1$ shown in Figure I.2. Following each face is the edge that appears in the rotation graph R_{v_1}. Considering all these edges, we conclude that R_{v_1} is given by the cycle $(v_2v_0v_4v_3)$. Note that this is equivalent to the rotation R_{v_1} given in Example I.2.1.

Example I.2.2.

$$\begin{array}{lll} F_0: & (v_0v_1v_2); & v_0v_2 \\ F_1: & (v_0v_2v_3) & \\ F_2: & (v_0v_3v_4) & \\ F_3: & (v_0v_4v_1); & v_4v_0 \\ F_4: & (v_1v_3v_4v_2); & v_2v_3 \\ F_5: & (v_2v_4v_1v_3); & v_4v_3 \end{array}$$

For further information on representing embeddings combinatorially, the reader is again referred to [27] or [36].

The benefit of having combinatorial descriptions of embeddings is that it makes it possible to present purely combinatorial methods of construction. We develop two new methods for building embeddings of $K_{n,n,n}$. In Chapter II, we present a cyclic construction that generates an entire embedding from two sequences of edge slopes. The facial boundaries in the resulting embedding are highly structured. While this construction can be used to create both nonorientable and orientable embeddings, we will focus on the nonorientable case. In Chapter III, a correspondence between an orientable hamilton cycle embedding of $K_{n,n,n}$ with certain properties and two pairs of orthogonal latin squares of order n with certain properties is demonstrated. We further show that an orientable hamilton cycle embedding of $K_{n,n,n}$ can be generated from a single pair of orthogonal latin squares provided one of those squares exhibits an additional property.

I.2.3 Voltage graphs

In addition to creating new methods for constructing embeddings, we will take advantage of some known techniques. The first of these – the voltage graph – was introduced by Gross in 1974 [25]. We will only need voltage graphs in Chapter III; therefore, we restrict our focus in this section to voltage graphs that yield orientable embeddings. For information on voltage graphs yielding nonorientable embeddings, see [27]. Let $G \hookrightarrow \Sigma$ be an embedding of a graph G, possibly with loops and parallel edges, in some orientable surface Σ. Assign a direction to each edge in G, and let $\alpha : E(G) \to \Gamma$ be a function from the edges of G to the group Γ. The map α is called a *voltage assignment*, the group Γ is called the *voltage group*, and the pair $(G \hookrightarrow \Sigma, \alpha)$ is called a *voltage graph.*

From a voltage graph $(G \hookrightarrow \Sigma, \alpha)$ we obtain an embedding of the *derived graph* G^α in the surface Σ^α as follows. The vertex set of G^α is $V(G) \times \Gamma$, and the edge set of G^α is $E(G) \times \Gamma$. We adopt the notation v_a and e_b for the vertex (v, a) and the edge (e, b), respectively. If $e \in E(G)$ is directed from u to v and $\alpha(e) = b$, then there is an (undirected) edge e_a in G^α connecting u_a and v_{ab} for all $a \in \Gamma$. Each vertex $v_a \in G^\alpha$, for every $a \in \Gamma$, inherits the rotation around v in the original embedding; this produces a rotation system that yields an embedding in the derived surface Σ^α. In Figure I.3, the graph G is shown on the left, embedded in the plane with voltage assignments from $\mathbb{Z}_3$, along with the derived graph G^α on the right, also embedded in the plane.

Using this construction, the resulting surface Σ^α will always be orientable. We do, however, want to determine the structure of each face in the embedding $G^\alpha \hookrightarrow \Sigma^\alpha$. To determine the faces of the derived embedding, one must trace out the faces of the embedded voltage graph. Let $W = e_1^{\varepsilon_1} e_2^{\varepsilon_2} \cdots e_k^{\varepsilon_k}$ be the closed walk bounding a face in the embedding $G \hookrightarrow \Sigma$,

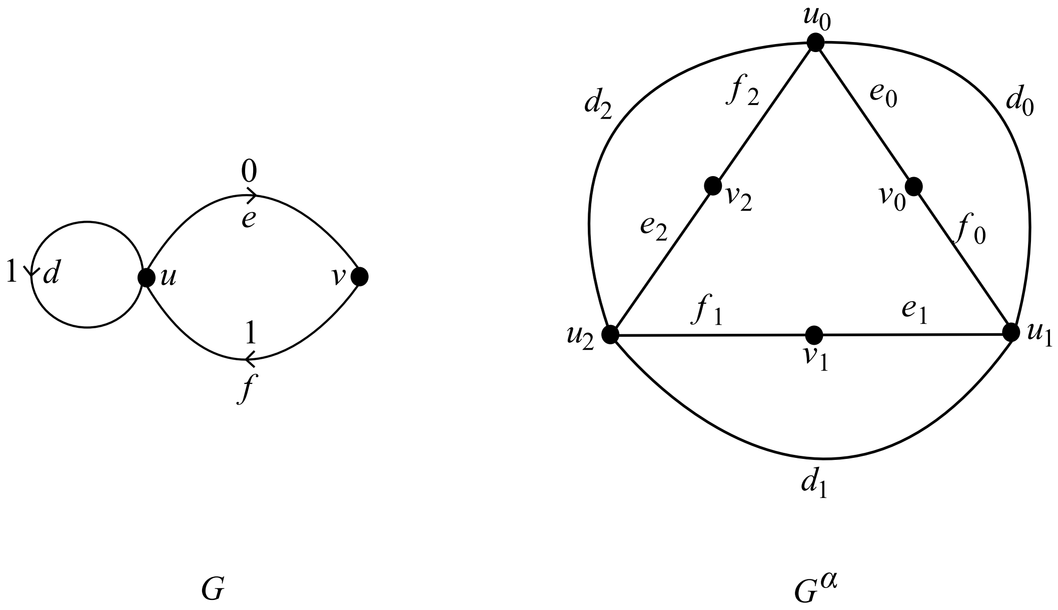

Figure I.3: Example of a voltage graph and its derived graph.

where $\varepsilon_i = +1$ if e_i is traced in the forward direction and $\varepsilon_i = -1$ if e_i is traced in the reverse direction. We define the *net voltage* of W to be $|W| = \alpha(e_1)^{\varepsilon_1}\alpha(e_2)^{\varepsilon_2}\cdots\alpha(e_k)^{\varepsilon_k}$. We want to be able to determine whether a closed walk W yields a hamilton cycle in the derived embedding without actually constructing the embedding. The following theorem aids in this determination.

Theorem I.2.3 (Gross and Tucker, Theorem 2.1.3 [27]). *Let W be a closed walk of length k bounding a face in the voltage graph $(G \hookrightarrow \Sigma, \alpha)$, and let the net voltage $|W|$ have order n in Γ. Then W yields $\frac{|\Gamma|}{n}$ faces of size kn in the derived embedding of G^α.*

Because we are only considering orientable graphs without edge signatures, we can trace the faces of the voltage graph without actually considering the embedding. To do so, we assign a consistent orientation, say clockwise, to each vertex; this induces a cyclic ordering of the edges, or rotation, around each vertex. Say, for example, that R_v is the rotation around v. If we enter v on the edge e as we are tracing a facial boundary, then we leave v on the edge $R_v(e)$. Following this procedure, every closed walk traced out corresponds to a face of the voltage graph. We consider the voltage graph in Figure I.3 as an example. We use the notation $e^{+1} \in R_v$ to indicate the edge is traced in the forward direction as we leave v and $e^{-1} \in R_v$ to indicate the edge is traced in the reverse direction. This is equivalent to e^{+1} and e^{-1} representing the tail and head of e, respectively. The rotation around each vertex is given below.

$$R_u : \quad (d^{+1}e^{+1}f^{-1}d^{-1})$$
$$R_v : \quad (e^{-1}f^{+1}).$$

We now choose any edge, say d, to use as the starting point for our trace. If we follow d in the forward direction from its tail at u to its head at u, we see that $R_u(d^{-1}) = d^{+1}$, so we have a closed walk, call it $W_0 = (d^{+1})$. Next, we start with d and trace it in the reverse direction from its head at u to its tail at u. Since $R_u(d^{+1}) = e^{+1}$, we next trace e in the forward direction from its tail at u to its head at v. Since $R_v(e^{-1}) = f^{+1}$, we next trace f in the forward direction from its tail at v to its head at u. Here we find that $R_u(f^{-1}) = d^{-1}$, so our walk is closed, call it $W_1 = (d^{-1}e^{+1}f^{+1})$. Repeating this process with any unused edge as our starting point, we obtain the closed walk $W_2 = (e^{-1}f^{-1})$. Since each edge has been used exactly once in both the forward and reverse directions, we have found all possible closed walks. We compute the net voltage of each closed walk below. Because we are working in the abelian group $\mathbb{Z}_3$, we use addition to represent the group operation.

$$|W_0| = \alpha(d)^{+1} = 1$$
$$|W_1| = \alpha(d)^{-1} + \alpha(e)^{+1} + \alpha(f)^{+1} = -1 + 0 + 1 = 0$$
$$|W_2| = \alpha(e)^{-1} + \alpha(f)^{-1} = -0 - 1 = 2$$

Applying Theorem I.2.3, we learn that W_0 yields 1 face of size 3, W_1 yields 3 faces of size 3, and W_2 yields 1 face of size 6. Thus, the derived embedding $G^\alpha \hookrightarrow \Sigma^\alpha$ has 6 vertices, 9 edges and 5 faces. Plugging this information into the Euler formula, we learn that $\chi(\Sigma^\alpha) = 0$, confirming what we knew from Figure I.3.

In Chapter III, we will build voltage graphs for hamilton cycle embeddings of $K_{n,n,n}$. The required voltage graph will have vertices a, b and c corresponding to the independent sets A, B and C. There will be n edges directed from a to b, with each voltage from an abelian group of order n being assigned to one of these edges. If $\alpha(e) = i$, then e represents all AB-edges of slope i. Similarly, there will be n edges directed from b to c and n edges from c to a, with each possible slope appearing once in each collection. Since the vertices and edges of our voltage graph are known ahead of time, all we will need to specify is the rotation around each vertex. It will suffice, then, to show that all of the faces in the derived embedding are hamilton cycles.

I.2.4 Surgical techniques

Some of the methods we will employ require performing surgery on some preexisting embeddings. One of these methods is quite simple. Let $G \hookrightarrow \Sigma$ be a hamilton cycle embedding of G with m hamilton cycle faces. By placing a new vertex in the center of each face and connecting it by new edges to each vertex on the boundary of that face, we create an embedding $\overline{K_m} + G \hookrightarrow \Sigma$. Moreover, since this new embedding is necessarily a triangulation, we actually have a genus embedding of $\overline{K_m} + G$. In particular, we can derive triangulations of $K_{2n,n,n,n}$ from hamilton cycle embeddings of $K_{n,n,n}$ and triangulations of $\overline{K_{n-1}} + K_n$ from hamilton cycle embeddings of K_n.

Another, more complex method we will employ is the so-called "diamond sum" construction. This surgical technique was introduced in dual form by Bouchet [8], reinterpreted by Magajna, Mohar and Pisanski [32], developed further by Mohar, Parsons, and Pisanski [35], and generalized by Kawarabayashi, Stephens and Zha [31]. In particular, the diamond sum construction allows us to combine genus embeddings of $K_{t_1,n,n,n}$ and $K_{t_2,3n}$ to get a genus embedding of $K_{t_1+t_2-2,n,n,n}$. This is achieved by removing a disk containing a vertex of degree $3n$ and all of its incident edges from each embedding and identifying the boundaries of the resulting holes in a suitable fashion. For similar applications of the diamond sum, see [12–14], and for more information on this technique, see [36, pages 117–118].

In Chapter IV, we extend the doubling construction in [13] to a tripling construction for hamilton cycle embeddings of complete graphs. By taking three copies of a hamilton cycle embedding of K_n, removing a point from each and joining them via a hamilton cycle embedding of $K_{n-1,n-1,n-1}$, we are able to obtain a hamilton cycle embedding of K_{3n-3}.

I.2.5 Bouchet covering triangulations

In a series of papers [2, 7, 9] in the 1970's and 1980's, Bouchet – together with Bénard and Fouquet – developed several methods for lifting triangulations of a graph G to triangulations of $G[\overline{K_m}]$. These methods, which Bouchet calls *covering triangulations*, are especially useful when G is the complete multipartite graph $K_{n_1,\ldots,n_q}$, because the lexicographic product $G[\overline{K_m}]$ is the complete multipartite graph $K_{mn_1,\ldots,mn_q}$. Thus, these covering triangulations yield a product construction for genus embeddings of complete multipartite graphs. There are three of the methods employed by Bouchet that will be useful in Chapter IV. A brief description of each follows, along with the relevant result.

The first method is a generative m-valuation [7]. Let $\tau : G \hookrightarrow \Sigma$ be a triangulation of G, and let T be the collection of triangle faces in τ. A map $\varphi : T \to \mathbb{Z}_m$ is called an *m-valuation* of τ. Let $V(G[\overline{K_m}]) = V(G) \times \mathbb{Z}_m$; for every triangle $t = uvw \in T$, set $C_t = \{(u,i)(v,j)(w,k) \mid i+j+k = \varphi(t)\}$. Under certain conditions, the collection of triangles $\{C_t \mid t \in T\}$ defines a triangulation of $G[\overline{K_m}]$ in a surface with the same orientability characteristic as Σ; this is called the *expansion* of τ. For an eulerian graph, the required condition is straightforward. For each vertex $v \in V(G)$, let $(t_1 t_2 \cdots t_{2k})$ be the permutation of triangle faces incident with v as they appear in clockwise order around v in the triangulation $G \hookrightarrow \Sigma$. Define

$$\dot{\varphi}(v) = \sum_{i=1}^{2k} (-1)^i \varphi(t_i).$$

If $\dot{\varphi}(v)$ generates the entire group $\mathbb{Z}_m$ for every vertex $v \in V(G)$, then φ is a *generative m-valuation* of τ.

Theorem I.2.4 (Bouchet, Theorem 1 [7]). *Let $\tau : G \hookrightarrow \Sigma$ be a triangulation of an*

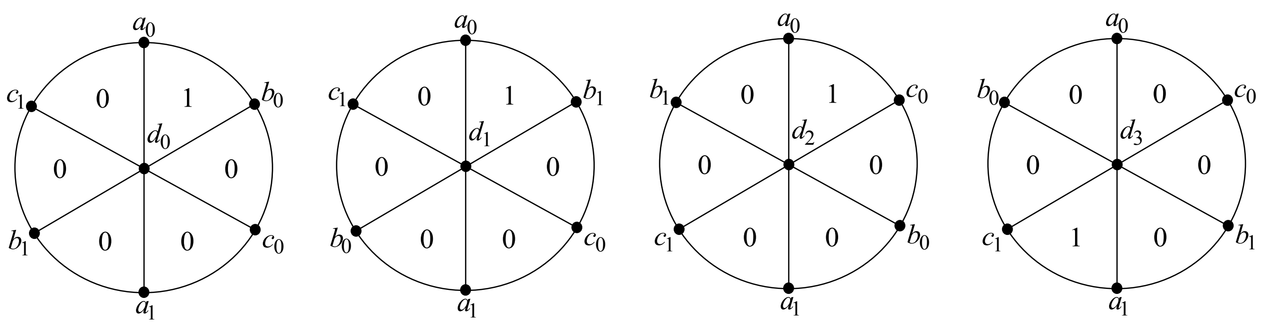

Figure I.4: Select neighborhoods from triangulation τ.

eulerian graph G, and let φ be a generative m-valuation of τ for some integer $m \geq 2$. Then the expansion of τ is a triangulation of $G[\overline{K_m}]$ in a surface with the same orientability characteristic as Σ.

Thus, it suffices to show that the triangulations we want to expand admit a generative m-valuation. For complete multipartite graphs, this is addressed in another theorem of Bouchet.

Theorem I.2.5 (Bouchet, Theorem 4 [7]). *Let G be an eulerian complete multipartite graph. If there exists a triangulation $\tau : G \hookrightarrow \Sigma$, then there exists a triangulation of $G[\overline{K_m}]$ in a surface with the same orientability characteristic as Σ.*

We are now able to prove the following result.

Corollary I.2.6. *If there exists a nonorientable triangulation of $K_{2n,n,n,n}$ with n even, then there exists a nonorientable triangulation of $K_{2mn,mn,mn,mn}$ for every integer $m \geq 1$.*

Proof. Every vertex in $K_{2n,n,n,n}$ has even degree, thus it is an eulerian graph. The result follows from Theorem I.2.5. □

While we omit the proofs of Theorems I.2.4 and I.2.5, we provide the following example for clarification.

Example I.2.7. Figure I.4 shows the neighborhoods of the D vertices for a nonorientable triangulation $\tau : K_{4,2,2,2} \hookrightarrow N_4$, where $A = \{a_0, a_1\}$, $B = \{b_0, b_1\}$, $C = \{c_0, c_1\}$ and $D =$

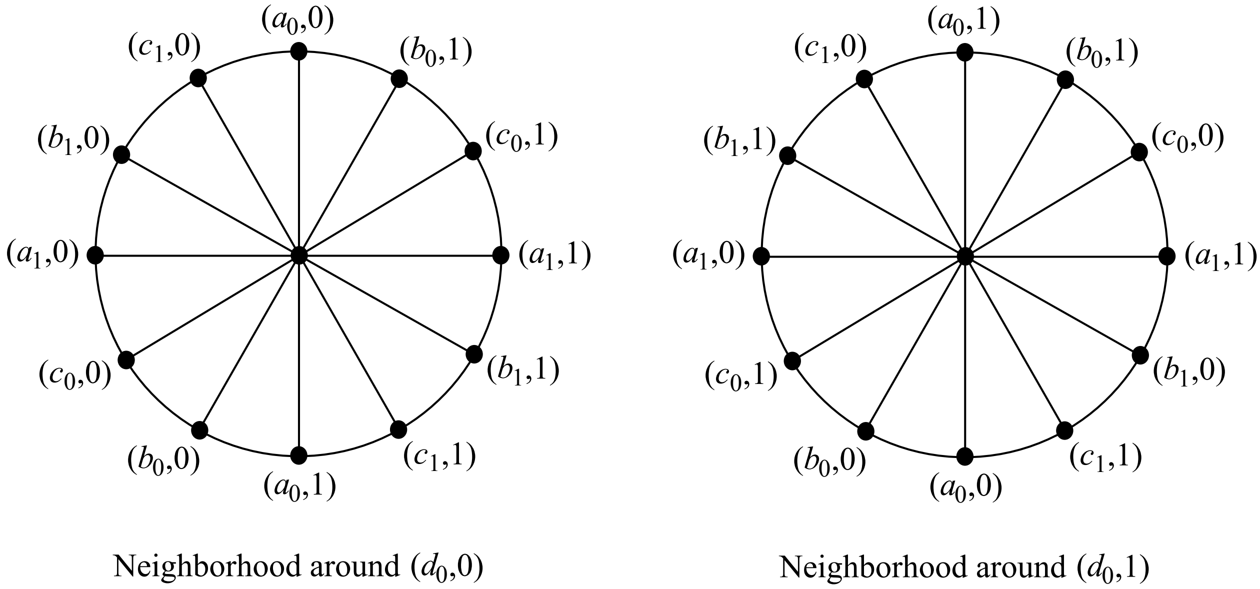

Figure I.5: Select neighborhoods from expansion of τ.

$\{d_0, d_1, d_2, d_3\}$ are the sets of the partition. The value $\varphi(t)$ is displayed in each triangle t; each vertex is contained in an odd number of triangles t with $\varphi(t) = 1$. Thus, $\dot{\varphi}(v) = 1$ for every v, and φ is a generative 2-valuation. The neighborhoods for the two vertices that result from d_0 in the expansion of τ are shown in Figure I.5.

The second method is similar to generative m-valuations, except that instead of requiring triangles to satisfy a condition based on φ, triangles must satisfy a condition that comes from a nowhere-zero p-flow on the dual of the original triangulation (p is prime). Using induction and Seymour's proof that every graph G without an isthmus admits a nowhere-zero 6-flow [49], Bouchet proved the following theorem.

Theorem I.2.8 (Bouchet, Corollary, p. 234 [9]). *If there exists a triangulation $G \hookrightarrow \Sigma$ and m is an integer such that $2, 3, 5 \nmid m$, then there exists a triangulation of $G[\overline{K_m}]$ in a surface with the same orientability characteristic as Σ.*

Remark I.2.9. If Tutte's conjecture that every isthmus-free graph admits a nowhere-zero 5-flow is proven to be true, then we could remove the requirement that $5 \nmid m$ from the statement of Theorem I.2.8.

This leads directly to the following corollary.

Corollary I.2.10. *If there exists a nonorientable triangulation of $K_{2n,n,n,n}$, then there exists a nonorientable triangulation of $K_{2mn,mn,mn,mn}$ for every integer $m \geq 1$ such that $2, 3, 5 \nmid m$.*

The last method that we will utilize is based on the existence of a well-separating cycle in G. We will leave the details to Bénard, Bouchet and Fouquet [2], but we present the following theorem.

Theorem I.2.11 (Bénard, Bouchet and Fouquet, Corollary 4.3 [2]). *Let G be a 4-colorable graph different from K_4, and let $m = 3^p$ for some integer $p \geq 1$. If there exists a triangulation $G \hookrightarrow \Sigma$, then there exists a triangulation of $G[\overline{K_m}]$ in a surface with the same orientability characteristic as Σ.*

We can now obtain the final result we need from Bouchet's covering triangulations.

Corollary I.2.12. *If there exists a nonorientable triangulation of $K_{2n,n,n,n}$, then for every integer $p \geq 0$ there exists a nonorientable triangulation of $K_{2mn,mn,mn,mn}$, where $m = 3^p$.*

Proof. Providing each independent set with a different color, it is easy to see that $K_{2n,n,n,n}$ is 4-colorable. The result follows from Theorem I.2.11. □

I.3 Latin squares

I.3.1 Definitions

Our terminology agrees with that set forth by Wanless in [51, 52]. A *latin square* of order n is an $n \times n$ matrix on some n-set E such that every row and every column contain exactly one copy of each element of E. Assume the rows and columns of L are labeled using the n-sets R and C, respectively; if the entry in row $r \in R$ and column $c \in C$ contains the entry $e \in E$, we say that L contains the ordered triple (r, c, e), or $L_{rc} = e$. A latin square is thus equivalent to a set of ordered triples. Unless otherwise noted, we will assume $R = C = E = \mathbb{Z}_n$. Two latin squares L_1 and L_2 on the sets E_1 and E_2, respectively, are called *orthogonal*, denoted

$L_1 \perp L_2$, if the ordered pairs obtained by overlapping the two squares cover every element of $E_1 \times E_2$ exactly once. If $L_1 \perp L_2$ for some L_2, we say L_1 has an *orthogonal mate.*

Example I.3.1. A and B are orthogonal latin squares of order 3, as evidenced by the overlapping entries shown in the matrix C. The center entry of A would be denoted $(1, 1, 2)$.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}, C = \begin{pmatrix} (0,0) & (1,1) & (2,2) \\ (1,2) & (2,0) & (0,1) \\ (2,1) & (0,2) & (1,0) \end{pmatrix}$$

Given an $n \times n$ latin square L, a *transversal* is a set of n ordered triples $\{(r_i, c_i, e_i) \in L \mid i \in \mathbb{Z}_n\}$ such that $\{r_0, ..., r_{n-1}\} = \{c_0, ..., c_{n-1}\} = \{e_0, ..., e_{n-1}\} = \mathbb{Z}_n$. In other words, a transversal is a collection of cells from L such that every row, column, and entry is covered exactly once. In Example I.3.1, note that the 0 entries in B correspond to a transversal in A; likewise for the 1 and 2 entries. It is well known that a matrix has an orthogonal mate if and only if it can be decomposed into disjoint transversals.

Theorem I.3.2. *A latin square L has an orthogonal mate if and only if L can be decomposed into disjoint transversals.*

For this thesis, we will utilize a generalization of a transversal known as a k-plex. If L is a latin square of order n, a *k-plex* is a set of kn ordered triples $\{(r_i, c_i, e_i) \in L \mid i \in \mathbb{Z}_{kn}\}$ such that the collections $\{r_0, ..., r_{kn-1}\}$, $\{c_0, ..., c_{kn-1}\}$ and $\{e_0, ..., e_{kn-1}\}$ each cover $\mathbb{Z}_n$ k times. In other words, a k-plex is a collection of cells from L such that every row, column, and entry is covered exactly k times. Thus, a transversal is a 1-plex, and an example of a 2-plex is given in Example I.3.3.

Example I.3.3. The starred entries in L form a 2-plex.

$$L = \begin{pmatrix} \mathit{0}^* & \mathit{1}^* & 2 & 3 & 4 & 5 \\ 1 & \mathit{2}^* & \mathit{3}^* & 4 & 5 & 0 \\ 2 & 3 & \mathit{4}^* & \mathit{5}^* & 0 & 1 \\ 3 & 4 & 5 & \mathit{0}^* & \mathit{1}^* & 2 \\ 4 & 5 & 0 & 1 & \mathit{2}^* & \mathit{3}^* \\ \mathit{5}^* & 0 & 1 & 2 & 3 & \mathit{4}^* \end{pmatrix}$$

If L can be decomposed into disjoint parts $K_1, K_2, ..., K_d$, where each K_i is an k_i-plex, then we call this a $(k_1, k_2, ..., k_d)$*-partition* of L. If all the parts have the same size k, then we simply call this a k-partition. A decomposition into transversals is a 1-partition; thus, we get the following result.

Theorem I.3.4. *A latin square L has an orthogonal mate if and only if there exists a 1-partition of L.*

While transversals can be difficult or impossible to locate in latin squares, k-plexes are generally much easier to find. In fact, it is conjectured that every latin square contains the maximum number of disjoint 2-plexes.

Conjecture I.3.5 (Wanless, Conjectures 2 and 3 [51]). *If L is a latin square of even order, then L admits a 2-partition. If L is a latin square of odd order, then L admits a $(2, 2, ..., 2, 1)$-partition.*

A latin square of order nm is said to be of *m-step type* if it can be divided into $m \times m$ latin subsquares A_{ij} as follows

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} and $A_{i'j'}$ contain the same entries if and only if $i + j \equiv i' + j' \pmod{n}$. These squares were first presented by Euler [16]; it was later shown that in certain cases m-step type latin squares are resistant to transversals.

Theorem I.3.6 (Maillet [33]). *Suppose that m is odd and n is even. If L is a m-step type latin square of order nm, then L contains no transversals.*

This result was later extended to include any odd k-plex.

Theorem I.3.7 (Wanless, Theorem 6 [51]). *Suppose that m and k are odd and n is even. If L is a m-step type latin square of order nm, then L contains no k-plexes.*

For more information and results concerning transversals and k-plexes in latin squares, see [51, 52].

For this thesis, we will use a slight generalization of Euler's m-step type latin square. Let L be a latin square of order n; a latin square of order nm is said to be a *m-step type latin square based on L* if it can be divided into $m \times m$ latin subsquares A_{ij} as follows

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} and $A_{i'j'}$ contain the same entries if and only if $L_{ij} = L_{i'j'}$. Thus, Euler's m-step type latin square is simply a m-step type latin square based on $L = \mathbb{Z}_n$, where we write $L = \mathbb{Z}_n$ throughout this paper to denote that L is the latin square formed by the addition table for $\mathbb{Z}_n$.

In Chapter III, we will construct m-step type latin squares based on a latin square L that actually admit a 1-partition. One idea that will aid in the construction of such squares is that of a turn-square. A *turn-square* is a latin square obtained by starting with the Cayley table of a group and "turning" some number of order 2 latin subsquares. This is equivalent to permuting the rows of the order 2 subsquares. In the example below, the starred entries in L are "turned" to form the turn-square L_t.

Example I.3.8.

$$L = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2^* & 3 & 4 & 5^* & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5^* & 0 & 1 & 2^* & 3 \\ 5 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}, \; L_t = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 5^* & 3 & 4 & 2^* & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 2^* & 0 & 1 & 5^* & 3 \\ 5 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

We generalize this concept to include more than just Cayley tables and allow turns on larger subsquares. An *m-turn-square* is a latin square obtained by taking a given latin square and permuting the rows of some latin subsquares of order m. If K is a m-turn-square obtained from a m-step type latin square based on L, we call K a *turned m-step type latin square based on L*. The latin square L_t given in Example I.3.9 is a turned 4-step type latin square based on $\mathbb{Z}_2$ that is obtained from L by permuting the rows of A_{00} and A_{11} down three rows, permuting the rows of A_{10} down two rows, and leaving the rows of A_{01} unchanged.

Example I.3.9.

$$L = \left(\begin{array}{cccc|cccc} 0 & 2 & 4 & 6 & 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 0 & 3 & 5 & 7 & 1 \\ 4 & 6 & 0 & 2 & 5 & 7 & 1 & 3 \\ 6 & 0 & 2 & 4 & 7 & 1 & 3 & 5 \\ \hline 1 & 3 & 5 & 7 & 0 & 2 & 4 & 6 \\ 3 & 5 & 7 & 1 & 2 & 4 & 6 & 0 \\ 5 & 7 & 1 & 3 & 4 & 6 & 0 & 2 \\ 7 & 1 & 3 & 5 & 6 & 0 & 2 & 4 \end{array}\right), L_t = \left(\begin{array}{cccc|cccc} 2 & 4 & 6 & 0 & 1 & 3 & 5 & 7 \\ 4 & 6 & 0 & 2 & 3 & 5 & 7 & 1 \\ 6 & 0 & 2 & 4 & 5 & 7 & 1 & 3 \\ 0 & 2 & 4 & 6 & 7 & 1 & 3 & 5 \\ \hline 5 & 7 & 1 & 3 & 2 & 4 & 6 & 0 \\ 7 & 1 & 3 & 5 & 4 & 6 & 0 & 2 \\ 1 & 3 & 5 & 7 & 6 & 0 & 2 & 4 \\ 3 & 5 & 7 & 1 & 0 & 2 & 4 & 6 \end{array}\right)$$

In Chapter III, we will also need a property of latin squares that describes the relative position of consecutive entries. Let L be a latin square of order n. For each $i \in \mathbb{Z}_n$, we form the graph G_i with n vertices $c_0, ..., c_{n-1}$ corresponding to the columns of L and an edge for each row between the columns that contain i and $i+1$. In other words, $c_j \sim c_{j'}$ if and only if $(r, c_j, i), (r, c_{j'}, i+1) \in L$ for some r. Since each column contains both i and $i+1$ exactly once, G_i is 2-regular; if G_i is a single cycle of length n for all $i \in \mathbb{Z}_n$, then we say L is *consecutively entry hamiltonian*, or *ce-hamiltonian* for short. An example of a ce-hamiltonian latin square is $\mathbb{Z}_n$ for every $n \geq 2$.

I.3.2 Biembeddings of latin squares

A *Steiner triple system* of order n, denoted STS(n), is a pair $(V, \mathcal{B})$, where V is a set of n elements and $\mathcal{B}$ is a set of 3-element subsets of V called *blocks* such that every distinct pair of elements in V is contained in exactly one block. If every distinct pair of elements of V is instead contained in exactly two blocks, then $(V, \mathcal{B})$ is a *twofold triple system*, denoted TTS(n). A *transversal design* of order n and block size 3, denoted TD$(3, n)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a set of $3n$ elements, $\mathcal{G}$ is a partition of V into 3 subsets of size n called *groups*, and $\mathcal{B}$ is again a set of blocks of size 3 such that every distinct pair of elements is contained in exactly one group or one block (but not both). A latin square of order n is equivalent to a TD$(3, n)$, where the rows, columns and entries form the 3 groups and each cell corresponds to a block. A *parallel class* in a STS(n), TTS(n) or TD$(3, n)$ is a subset of $\mathcal{B}$ that covers every element of V exactly once. For a TD$(3, n)$, this corresponds to a transversal in the associated latin square.

The connection between combinatorial designs and graph embeddings has been known for over a century. In [29], Heffter observed that a triangular embedding of K_n gives rise to a TTS(n). Later, Alpert proved that twofold triple systems are in one-to-one correspondence with triangulations of complete graphs (possibly in pseudosurfaces). Given a triangulation of K_n, the faces correspond to the blocks of a TTS(n); moreover, if the triangulation is face 2-colorable, each color class corresponds to an STS(n). Conversely, we can construct face 2-colorable triangulations of K_n from pairs of STS(n); under certain conditions, this triangulation will be in a surface. This is called a *biembedding of STS(n)*, and has been used, for example, to develop recursive constructions for triangulations of complete graphs [21]. We note that biembeddings of STS(n) can be in either a nonorientable or an orientable surface.

In a similar fashion, a face 2-colorable triangulation of $K_{n,n,n}$ yields a pair of latin squares, and this process can be reversed. If the pair of latin squares satisfies certain conditions, the resulting triangulation will be in a surface; this is known as a *biembedding of latin squares.* Contrary to its STS counterpart, these biembeddings are always in orientable surfaces; in fact, they comprise all orientable triangulations of $K_{n,n,n}$.

Theorem I.3.10 (Grannell, Griggs and Knor, Proposition 1 [18]). *Let $\tau : K_{n,n,n} \hookrightarrow \Sigma$ be a triangulation of $K_{n,n,n}$ in some surface Σ. The following are equivalent.*

(1) Σ is orientable.

(2) τ is face 2-colorable.

(3) τ is a biembedding of latin squares.

Our goal is to establish conditions on latin squares that ensure those squares are biembeddable. To describe one such condition, we need the following definition from [17].

Definition I.3.11. Suppose that $L = (L_{i,j})$ is a latin square of order n. If the permutation given by

$$\begin{pmatrix} L_{i,0} & L_{i,1} & \cdots & L_{i,n-1} \\ L_{i+1,0} & L_{i+1,1} & \cdots & L_{i+1,n-1} \end{pmatrix}$$

is a single cycle of length n for all $i \in \mathbb{Z}_n$, then we say L is *consecutively row hamiltonian*, or *cr-hamiltonian* for short.

The usefulness of this property is shown in the following result.

Theorem I.3.12 (Grannell and Griggs, Lemma 2.1 [17]). *If L is cr-hamiltonian, then there exists a biembedding of L with a copy of itself.*

Grannell and Griggs used this construction, along with some surgical techniques, to create large families of nonisomorphic triangulations of complete tripartite graphs, such as in the following theorem.

Theorem I.3.13 (Grannell and Griggs, Corollary 2.1.1 [17]). *For $n = 3(2^s)$ and s sufficiently large, there are at least $n^{\frac{n^2}{288}}$ nonisomorphic face 2-colorable triangulations of $K_{n,n,n}$, each of which has a monochromatic parallel class.*

Using this result on triangulations of complete tripartite graphs and additional surgical techniques, Grannell and Griggs were also able to obtain large families of nonisomorphic triangulations of complete graphs.

Theorem I.3.14 (Grannell and Griggs, Corollary 3.1.2 [17]). *Suppose there is a face 2-colorable triangulation of $K_{q,q,q}$ with a monochromatic parallel class. Suppose further that $n = 2^s q(m-1) + 1$, where $m \equiv 3$ or 7 (mod 12) and $m \geq 7$, and set $a = \frac{m-3}{192q^2(m-1)}$. Then, as $s \to \infty$, there are at least $n^{n^2(a-o(1))}$ nonisomorphic face 2-colorable triangulations of K_n in a nonorientable surface.*

Similar results for different families of n were obtained by Grannell and Knor in [24]. In Chapter V, we show that ce-hamiltonian latin squares are conjugate to cr-hamiltonian latin squares. A latin square L is *conjugate* to L' if L' can be obtained from L by permuting the roles of the rows, columns and entries.

I.4 Statement of Main Results

The main results of this paper fall into two categories: hamilton cycle embeddings and minimum genus embeddings. The following result addresses nonorientable hamilton cycle embeddings of complete tripartite graphs and is proved in Chapters II and IV using a cyclic construction and some Bouchet covering triangulations.

Theorem I.4.1. *There exists a nonorientable hamilton cycle embedding of $K_{n,n,n}$ for all $n \geq 2$.*

The orientable counterpart to this result is proved in Chapter III using latin squares and voltage graphs.

Theorem I.4.2. *There exists an orientable hamilton cycle embedding of $K_{n,n,n}$ for all $n \geq 1$, $n \neq 2$. Moreover, at least one face in this embedding is bounded by an ABC cycle.*

Using these hamilton cycle embeddings, we are able to construct minimum genus embeddings of some large families of graphs. The following results are proved in Chapter IV. We first address the orientable genus of joins of edgeless graphs with complete graphs.

Theorem I.4.3. *If $n = 3^q\left(2^p + \frac{1}{2}\right) + \frac{3}{2}$ for some $p \geq 3$ and $q \geq 0$, then $g(\overline{K_m} + K_n) = g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ for all $m \geq n-1$.*

Because $K_{m,n} \subseteq \overline{K_m} + G \subseteq \overline{K_m} + K_n$ for any n-vertex simple graph G, Theorem I.4.3 can easily be extended to the following result.

Corollary I.4.4. *Let $n = 3^q\left(2^p + \frac{1}{2}\right) + \frac{3}{2}$ for some $p \geq 3$ and $q \geq 0$. If G is any n-vertex simple graph, then $g(\overline{K_m} + G) = g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ for all $m \geq n-1$.*

In fact, we can obtain even stronger results by using both the doubling and tripling constructions provided in Section IV.1. This process is discussed further in Section V.1.

The remaining theorems below address the genus of some quadripartite graphs. The first theorem and corollary cover the nonorientable case.

Theorem I.4.5. *For all $n \geq 1$, $\tilde{g}(K_{2n,n,n,n}) = \tilde{g}(K_{2n,3n}) = (n-1)(3n-2)$.*

Using the diamond sum, we can extend Theorem I.4.5 to include complete quadripartite graphs where the largest independent set has size greater than $2n$.

Corollary I.4.6. *For all $n \geq 1$ and all $t \geq 2n$, $\tilde{g}(K_{t,n,n,n}) = \tilde{g}(K_{t,3n}) = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$.*

This nonorientable genus result can be extended to an even larger family by using a "sandwiching" effect as we did to obtain Corollary I.4.4; this is covered in Remark IV.2.2.

We obtain orientable genus results for some quadripartite graphs as well; however, there is a special case when $n = 2$.

Theorem I.4.7. *For all* $n \neq 2$, $g(K_{2n,n,n,n}) = g(K_{2n,3n}) = \left\lceil \frac{(n-1)(3n-2)}{2} \right\rceil$.

As before, this is extended using the diamond sum construction.

Corollary I.4.8. *For all* $n \geq 1$ *and all* $t \geq 2n$, *except* $(n,t) = (2,4)$, $g(K_{t,n,n,n}) = g(K_{t,3n}) = \left\lceil \frac{(t-2)(3n-2)}{4} \right\rceil$. *Also,* $g(K_{4,2,2,2}) = 3$.

Remark IV.3.2 explains how this can be extended even further by using a "sandwiching" effect.

CHAPTER II

NONORIENTABLE HAMILTON CYCLE EMBEDDINGS

The work in this chapter appears in [11].

II.1 Slope sequence construction

In this section we describe the general construction on which the proofs in Section II.3 are based. Some preliminary definitions are required. Let $S = ((s_0, t_0), (s_1, t_1), ..., (s_{n-1}, t_{n-1}))$. If $s_j \neq t_j$ for all $j \in \mathbb{Z}_n$ and the collection $\{s_0, ..., s_{n-1}, t_0, ..., t_{n-1}\}$ covers every element of $\mathbb{Z}_n$ twice, we say S is a *slope sequence* (because s_j and t_j will specify the *slope* $\ell - k$ of each edge $b_k c_\ell$ in the cycles X_i and Y_i below). Form the graph G_S with vertices $\{v_0, v_1, ..., v_{n-1}\}$ and m edges joining distinct vertices v_{j_1} and v_{j_2}, where $m = |\{s_{j_1}, t_{j_1}\} \cap \{s_{j_2}, t_{j_2}\}|$. We call G_S the *induced pair graph* for the slope sequence S. This graph is 2-regular, so G_S decomposes into a union of cycles. As Theorem II.1.1 shows, it will be desirable to have induced pair graphs that consist of a single cycle.

Theorem II.1.1. *Suppose $S = ((s_0, t_0), (s_1, t_1), ..., (s_{n-1}, t_{n-1}))$ is a slope sequence such that the following hold:*

(1) $\{j + s_j \mid j \in \mathbb{Z}_n\} = \{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$;

(2) $t_j - s_j$ is relatively prime to n for all $j \in \mathbb{Z}_n$;

(3) the induced pair graph G_S consists of a single cycle of length n.

Then the collection of cycles $\mathcal{X} = \{X_i \mid i \in \mathbb{Z}_n\}$ and $\mathcal{Y} = \{Y_i \mid i \in \mathbb{Z}_n\}$, given by

$$X_i: \quad (a_0 b_i c_{i+s_0} a_1 b_{i+1} c_{i+1+s_1} \cdots a_j b_{i+j} c_{i+j+s_j} \cdots a_{n-1} b_{i+n-1} c_{i+n-1+s_{n-1}}),$$
$$Y_i: \quad (a_0 b_i c_{i+t_0} a_1 b_{i+1} c_{i+1+t_1} \cdots a_j b_{i+j} c_{i+j+t_j} \cdots a_{n-1} b_{i+n-1} c_{i+n-1+t_{n-1}}),$$

form a hamilton cycle embedding of $K_{n,n,n}$ with all faces bounded by ABC cycles.

Proof. First, we must show that X_i and Y_i are indeed hamilton cycles. It is clear that every A and B vertex appears in every X_i and Y_i. Since $j + s_j$ covers $\mathbb{Z}_n$, it follows that $i + j + s_j$ also covers $\mathbb{Z}_n$, so every C vertex appears in X_i. The same argument with $j + t_j$ shows that every C vertex also appears in Y_i. By construction, these cycles are all ABC cycles.

Next, we show that these hamilton cycles form a double cover of $K_{n,n,n}$. The cycles X_{k-j} and Y_{k-j} both cover the edge $a_j b_k$ for all $j, k \in \mathbb{Z}_n$. Similarly the cycles $X_{\ell-(j-1)-s_{j-1}}$ and $Y_{\ell-(j-1)-t_{j-1}}$ both cover the edge $c_\ell a_j$ for all $j, \ell \in \mathbb{Z}_n$. Finally, consider an edge $b_k c_\ell$. We know from S being a slope sequence that there exist j' and j'' such that one of the following holds: (1) $s_{j'} = t_{j''} = \ell - k$, (2) $s_{j'} = s_{j''} = \ell - k$, or (3) $t_{j'} = t_{j''} = \ell - k$. These cases correspond to the following: (1) the cycles $X_{k-j'}$ and $Y_{k-j''}$ both cover the edge $b_k c_\ell$, (2) the cycles $X_{k-j'}$ and $X_{k-j''}$ both cover the edge $b_k c_\ell$, or (3) the cycles $Y_{k-j'}$ and $Y_{k-j''}$ both cover the edge $b_k c_\ell$. This holds for all $k, \ell \in \mathbb{Z}_n$; therefore, $\mathcal{X} \cup \mathcal{Y}$ forms a double cover of $K_{n,n,n}$.

To show that these hamilton cycles can be sewn together along common edges to yield an embedding of $K_{n,n,n}$, it remains to prove that the rotation graph around each vertex is a single cycle of length $2n$. Since this collection consists of all ABC faces, we know that the rotation graph around a vertex $a_j \in A$ will alternate between B and C vertices. If all of the C vertices appear in the same component of R_{a_j}, then all of the B vertices must be in the same component as well. Thus, it will suffice to prove that the C vertices are contained in the same cycle in the rotation graph around every A vertex. Similarly, it will suffice to prove that the A vertices are contained in the same cycle in the rotation graph around every B and C vertex.

Consider the vertex a_j. We know the cycle $X_{\ell-(j-1)-s_{j-1}}$ contains the sequence

$$(\cdots c_\ell a_j b_{\ell+1-s_{j-1}} \cdots)$$

and the cycle $Y_{\ell-(j-1)-s_{j-1}}$ contains the sequence

$$(\cdots c_{\ell-s_{j-1}+t_{j-1}} a_j b_{\ell+1-s_{j-1}} \cdots).$$

Thus, the vertex $c_{\ell-s_{j-1}+t_{j-1}}$ follows the vertex c_ℓ in the rotation graph around a_j. Continuing this argument, we find the C vertices form the cyclic sequence

$$\big(c_k c_{k+(t_{j-1}-s_{j-1})} c_{k+2(t_{j-1}-s_{j-1})} \cdots c_{k+(n-1)(t_{j-1}-s_{j-1})}\big)$$

in the rotation graph around a_j. Since $t_{j-1}-s_{j-1}$ is relatively prime to n, this includes every C vertex.

Consider the vertex b_k. We know the cycle X_{k-j} contains the sequence

$$(\cdots a_j b_k c_{k+s_j} \cdots).$$

Since S double covers $\mathbb{Z}_n$, there exists j' such that either (1) $s_{j'} = s_j$ or (2) $t_{j'} = s_j$. In either case we know the vertex v_j arising from the pair (s_j, t_j) is adjacent in the slope graph G_S to the vertex $v_{j'}$ arising from the pair $(s_{j'}, t_{j'})$. Since G_S is a single cycle of length n, we write

$$G_S = \big(v_j v_{\delta(j)} v_{\delta^2(j)} \cdots v_{\delta^{n-1}(j)}\big),$$

where $\delta(j) = j'$. In case (1), the cycle $X_{k-j'}$ contains the sequence

$$(\cdots a_{j'} b_k c_{k+s_{j'}} \cdots).$$

Likewise in case (2), the cycle $Y_{k-j'}$ contains the sequence

$$(\cdots a_{j'} b_k c_{k+t_{j'}} \cdots).$$

Since either (1) $k+s_{j'} = k+s_j$ or (2) $k+t_{j'} = k+s_j$, we have that $a_{j'} = a_{\delta(j)}$ follows a_j in the rotation graph around b_k. Repeating this argument, we see that the A vertices form the cyclic sequence

$$(a_j a_{\delta(j)} a_{\delta^2(j)} \cdots a_{\delta^{n-1}(j)})$$

in the rotation graph around b_k, which includes every A vertex. An analogous argument shows that the A vertices form the cyclic sequence

$$(a_{j+1} a_{\delta(j)+1} a_{\delta^2(j)+1} \cdots a_{\delta^{n-1}(j)+1})$$

lying in a single component in the rotation graph around c_ℓ. $\square$

II.2 Special case constructions

We begin by presenting the required nonorientable hamilton cycle embeddings of $K_{n,n,n}$ when $n \in \{3, 5, 7, 11, 13\}$.

By checking all possible cases, we know there does not exist a slope sequence construction for a nonorientable embedding of $K_{3,3,3}$. The desired embedding is given by the following facial boundaries:

$$\begin{aligned}
&(a_0b_0c_0a_1b_1c_1a_2b_2c_2), \quad (a_0b_0c_1a_1b_1c_2a_2b_2c_0),\\
&(a_0b_1c_1a_1b_2c_2a_2b_0c_0), \quad (a_0b_2c_0a_2b_1c_2a_1b_0c_1),\\
&(a_0b_2c_1a_2b_1c_0a_1b_0c_2), \quad (a_0b_1c_0a_2b_0c_2a_1b_2c_1).
\end{aligned}$$

For $n \in \{5, 7, 11, 13\}$, Table II.1 provides a slope sequence that yields a nonorientable hamilton cycle embedding of $K_{n,n,n}$. To show that these embeddings are indeed nonorientable, in the same way as in the proof of Lemma II.3.1, consider the following sequences of faces and edges, where $F\ e\ F'$ implies F and F' share the edge e:

$$
\begin{array}{llllllll}
n = 5: & X_2 & a_0b_2 & Y_2 & b_3c_1 & Y_1 & c_4a_4 & X_2; \\
n = 7: & X_0 & b_0c_1 & X_4 & a_0b_4 & Y_4 & c_3a_5 & X_0; \\
n = 11: & X_0 & a_0b_0 & Y_0 & c_8a_6 & X_9 & b_0c_1 & X_0; \\
n = 13: & X_0 & a_0b_0 & Y_0 & c_8a_5 & X_{11} & b_0c_1 & X_0.
\end{array}
$$

n	j	0	1	2	3	4	5	6	7	8	9	10	11	12
5	s_j	1	4	1	4	0								
	t_j	2	3	3	0	2								
	$t_j - s_j$	1	4	2	1	2								
7	s_j	1	6	4	1	6	0	3						
	t_j	2	4	5	0	2	3	5						
	$t_j - s_j$	1	5	1	6	3	3	2						
11	s_j	1	10	1	10	4	5	3	0	7	8	6		
	t_j	2	9	9	4	5	3	0	7	8	6	2		
	$t_j - s_j$	1	10	8	5	1	9	8	7	1	9	7		
13	s_j	1	12	1	12	6	4	0	5	3	9	7	10	8
	t_j	2	11	11	6	4	0	5	3	9	7	10	8	2
	$t_j - s_j$	1	12	10	7	11	9	5	11	6	11	3	11	7

Table II.1: Slope sequences for $n \in \{5, 7, 11, 13\}$.

II.3 Applications of slope sequence construction

Lemma II.3.1. *There exists a nonorientable hamilton cycle embedding of $K_{n,n,n}$ with all faces bounded by ABC cycles for all $n \equiv 1 \pmod 4$ such that $n \geq 5$ and $3, 7 \nmid n$.*

Proof. Table II.1 in Section II.2 gives the necessary slope sequences for $n = 5$ and 13. It is a straightforward exercise to show that these sequences meet all the required conditions of Theorem II.1.1, and that the resulting embeddings are nonorientable.

Table II.2 gives the necessary slope sequences for $n = 4r + 1$, $r \geq 4$. It is easy to see that the collection $\{s_0, ..., s_{n-1}, t_0, ..., t_{n-1}\}$ double covers $\mathbb{Z}_n$. The slope graph G_S consists

j	s_j	t_j	$t_j - s_j$	j	s_j	t_j	$t_j - s_j$
0	1	2	1	$2r+3$	$-2r+2$	$-2r$	-2
1	-1	-2	-1	$2r+4$	$-2r$	$2r-5$	-6
2	1	-2	-3	$2r+5$	$2r-5$	$2r-7$	-2
3	-1	$2r$	$2r+1$	$2r+6$	$2r-7$	$2r-9$	-2
4	$2r$	$2r-2$	-2	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	$2r-2$	$2r-4$	-2	$3r$	5	3	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$3r+1$	3	$-2r+3$	$2r+1$
$r+1$	6	4	-2	$3r+2$	$-2r+3$	$-2r+1$	-2
$r+2$	4	0	-4	$3r+3$	$-2r+1$	-3	$2r-4$
$r+3$	0	$2r-1$	$2r-1$	$3r+4$	-3	-5	-2
$r+4$	$2r-1$	$2r-3$	-2	$3r+5$	-5	-7	-2
$r+5$	$2r-3$	-4	$2r$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r+6$	-4	-6	-2	$4r-2$	$-2r+9$	$-2r+7$	-2
$r+7$	-6	-8	-2	$4r-1$	$-2r+7$	$-2r+5$	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$4r$	$-2r+5$	2	$2r-3$

Table II.2: Slope sequence for $n = 4r+1$, $r \geq 4$.

of edges $v_j v_{j+1}$ for all $3 \leq j \leq n-1$, along with the edges v_0v_2, v_2v_1, and v_1v_3. This is a cycle of length n, as seen in Figure II.1. Let $D = \{t_j - s_j \mid j \in \mathbb{Z}_n\}$. From the table we see that

$$\begin{aligned} D &= \{-6,-4,-3,-2,-1,1,2r-4,2r-3,2r-1,2r,2r+1\} \\ &= \left\{-6,-4,-3,-2,-1,1,\tfrac{n-9}{2},\tfrac{n-7}{2},\tfrac{n-3}{2},\tfrac{n-1}{2},\tfrac{n+1}{2}\right\}. \end{aligned}$$

Since $2,3,7 \nmid n$, we know n is relatively prime to every element of D. The last condition we must prove is that $\{j + s_j \mid j \in \mathbb{Z}_n\} = \{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Note that for every j we have $s_j = k \Leftrightarrow s_{j+k} = -k$ and $t_j = k \Leftrightarrow t_{j+k} = -k$. Let $i \in \mathbb{Z}_n$, and set $k = s_i$ and $j = i+k$. It follows that $j + s_j = i + k + s_{i+k} = i + k - k = i$. Since i was arbitrary, we know $\{j + s_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. The same argument shows that $\{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Applying Theorem II.1.1 yields a hamilton cycle embedding of $K_{n,n,n}$. To determine the orientability

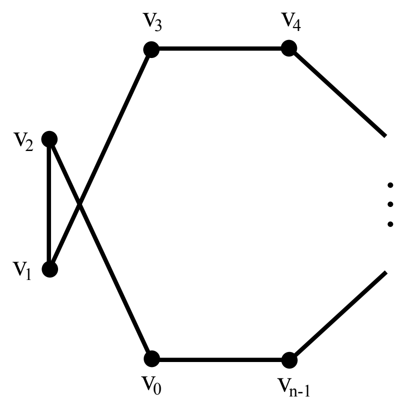

Figure II.1: The slope graph G_S for the slope sequences given in Tables II.2 and II.3.

of this embedding, consider the following three cycles:

$$
\begin{aligned}
X_1: &\quad (a_0b_1c_2a_1b_2c_1a_2b_3c_4\cdots),\\
Y_0: &\quad (a_0b_0c_2a_1b_1c_{n-1}a_2b_2c_0\cdots),\\
Y_1: &\quad (a_0b_1c_3a_1b_2c_0a_2b_3c_1\cdots).
\end{aligned}
$$

Assume this embedding admits an orientation, with X_1 oriented forwards. Note that Y_0 and X_1 share the edge c_2a_1 and Y_1 and X_1 share the edge a_0b_1, so both Y_0 and Y_1 must be oriented backwards. However, Y_0 and Y_1 share the edge b_2c_0, so they must have different orientations. This is a contradiction, so this embedding is nonorientable. □

Lemma II.3.2. *There exists a nonorientable hamilton cycle embedding of $K_{n,n,n}$ with all faces bounded by ABC cycles for all $n \equiv 3 \pmod 4$ such that $3, 7 \nmid n$.*

Proof. Table II.1 in Section II.2 gives the necessary slope sequence for $n = 11$. It is a straightforward exercise to show that this sequence meets all the required conditions of Theorem II.1.1, and that the resulting embedding is nonorientable.

Table II.3 gives the necessary slope sequences for $n = 4r+3$, $r \geq 3$. It is again easy to see that the collection $\{s_0, ..., s_{n-1}, t_0, ..., t_{n-1}\}$ double covers $\mathbb{Z}_n$. The slope graph G_S (Figure II.1) is identical to the slope graph constructed for the slope sequence in Table II.2. Let D

j	s_j	t_j	$t_j - s_j$	j	s_j	t_j	$t_j - s_j$
0	1	2	1	$2r+4$	$-2r+1$	$-2r-1$	-2
1	-1	-2	-1	$2r+5$	$-2r-1$	$2r-4$	-6
2	1	-2	-3	$2r+6$	$2r-4$	$2r-6$	-2
3	-1	$2r+1$	$2r+2$	$2r+7$	$2r-6$	$2r-8$	-2
4	$2r+1$	$2r-1$	-2	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	$2r-1$	$2r-3$	-2	$3r+1$	6	4	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$3r+2$	4	$-2r+2$	$2r+1$
$r+1$	7	5	-2	$3r+3$	$-2r+2$	$-2r$	-2
$r+2$	5	3	-2	$3r+4$	$-2r$	0	$2r$
$r+3$	3	$2r$	$2r-3$	$3r+5$	0	-4	-4
$r+4$	$2r$	$2r-2$	-2	$3r+6$	-4	-6	-2
$r+5$	$2r-2$	-3	$2r+2$	$3r+7$	-6	-8	-2
$r+6$	-3	-5	-2	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r+7$	-5	-7	-2	$n-2$	$-2r+6$	$-2r+4$	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$n-1$	$-2r+4$	2	$2r-2$

Table II.3: Slope sequence for $n = 4r+3$, $r \geq 4$.

again be the set of differences; from the table we see that

$$\begin{aligned} D &= \{-6, -4, -3, -2, -1, 1, 2r-3, 2r-2, 2r, 2r+1, 2r+2\} \\ &= \left\{-6, -4, -3, -2, -1, 1, \tfrac{n-9}{2}, \tfrac{n-7}{2}, \tfrac{n-3}{2}, \tfrac{n-1}{2}, \tfrac{n+1}{2}\right\}. \end{aligned}$$

This is the same D as in the proof of Lemma II.3.1, so again we know n is relatively prime to every element of D. We also have $s_j = k \Leftrightarrow s_{j+k} = -k$ and $t_j = k \Leftrightarrow t_{j+k} = -k$ as in the proof of Lemma II.3.1, which implies that $\{j + s_j \mid j \in \mathbb{Z}_n\} = \{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Applying Theorem II.1.1 yields a hamilton cycle embedding of $K_{n,n,n}$. Because s_0, s_1, s_2, t_0, t_1, and t_2 are the same in Tables II.2 and II.3, analyzing X_1, Y_0 and Y_1 in the same way as in the proof of Lemma II.3.1 shows that this embedding is nonorientable. $\square$

j	0	1	2	3	4	$\cdots$	$n-3$	$n-2$	$n-1$
s_j	1	1	3	3	5	$\cdots$	$n-3$	$n-1$	$n-1$
t_j	0	2	2	4	4	$\cdots$	$n-2$	$n-2$	0
$t_j - s_j$	-1	1	-1	1	-1	$\cdots$	1	-1	1

Table II.4: Slope sequence for $n = 4r+2$, $r \geq 0$.

Lemma II.3.3. *There exists a nonorientable hamilton cycle embedding of $K_{n,n,n}$ with all faces bounded by ABC cycles for all $n \equiv 2 \pmod 4$.*

Proof. Table II.4 gives the necessary slope sequences for $n \equiv 2 \pmod 4$. Since $t_j - s_j = (-1)^{j+1}$, we know $t_j - s_j$ is relatively prime to n for all $j \in \mathbb{Z}_n$. Since G_S consists of the edges $v_j v_{j+1}$ for all $j \in \mathbb{Z}_n$, it is clearly a single cycle of length n. Finally, note that $j+s_j = 2j+1$ if j is even and $j+s_j = 2j$ if j is odd. Since $n \equiv 2 \pmod 4$, this implies $\{j+s_j \mid j \in \mathbb{Z}_n, j \text{ even}\}$ covers all the odd values of $\mathbb{Z}_n$ and $\{j+s_j \mid j \in \mathbb{Z}_n, j \text{ odd}\}$ covers all the even values of $\mathbb{Z}_n$. Thus, $\{j+s_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Using the fact that $j+t_j = 2j$ if j is even and $j+t_j = 2j+1$ if j is odd, we derive that $\{j+t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$ as well. Applying Theorem II.1.1 provides a hamilton cycle embedding of $K_{n,n,n}$. To determine the orientability of this embedding, consider the following three cycles:

$$\begin{aligned} X_0 &: \quad (a_0 b_0 c_1 a_1 b_1 c_2 a_2 b_2 c_5 \cdots), \\ Y_0 &: \quad (a_0 b_0 c_0 a_1 b_1 c_3 a_2 b_2 c_4 \cdots), \\ Y_1 &: \quad (a_0 b_1 c_1 a_1 b_2 c_4 a_2 b_3 c_5 \cdots). \end{aligned}$$

Assume this embedding admits an orientation, with X_0 oriented forwards. Note that Y_0 and X_0 share the edge $a_0 b_0$ and Y_1 and X_0 share the edge $c_1 a_1$, so both Y_0 and Y_1 must be oriented backwards. However, Y_0 and Y_1 share the edge $b_2 c_4$, so they must have different orientations. This is a contradiction, so this embedding is nonorientable. (This argument works even in the case $n = 2$, reducing subscripts modulo 2.) □

II.4 Nonorientable results

To complete the proof of Theorem I.4.1, we need to use the connection between these embeddings and triangulations of $K_{2n,n,n,n}$. To these triangulations we must apply some of the covering triangulations developed in Section I.2.5. While the details of this procedure are saved for Chapter IV, the result is restated here for completeness.

Theorem I.4.1. *There exists a nonorientable hamilton cycle embedding of* $K_{n,n,n}$ *for all* $n \geq 2$.

CHAPTER III

ORIENTABLE HAMILTON CYCLE EMBEDDINGS

III.1 Preliminaries

We will use induced pair graphs again in this chapter, albeit in a different context. To facilitate the application of Corollary III.2.3, we present a collection of pairs for which the induced pair graph is a hamilton cycle. To show this, we present an ordering of the pairs such that v_i is adjacent to both v_{i-1} and v_{i+1} for all $i \in \mathbb{Z}_n$. We will denote by $\gcd(m,n)$ the greatest common divisor of integers m and n; as usual, we say m and n are *relatively prime* if $\gcd(m,n) = 1$.

Lemma III.1.1. *Let p be an integer relatively prime to n, and define the collection of pairs $P = \{(j, j+p) | j \in \mathbb{Z}_n\}$. Then the induced pair graph G_P is a hamilton cycle.*

Proof. Noting that $P = \{(0,p),(p,2p),(2p,3p),...,((n-2)p,(n-1)p),((n-1)p,0)\}$ it is readily seen that G_P is a hamilton cycle. □

The addition table for the group $\mathbb{Z}_n$ will be a key ingredient in all of the constructions presented in this chapter. In particular, the following property will be useful.

Lemma III.1.2. *If n is odd, then $\mathbb{Z}_n$ admits a 1-partition. If n is even, then $\mathbb{Z}_n$ admits a 2-partition.*

Proof. For all $j \in \mathbb{Z}_n$ let $T_j = \{(i, i+j, 2i+j) \mid i \in \mathbb{Z}_n\}$. T_j clearly covers every row and column of $\mathbb{Z}_n$ exactly once. If n is odd, the set of entries $\{2i+j \mid i \in \mathbb{Z}_n\} = \mathbb{Z}_n$ as well, so T_j is a transversal. Moreover, T_j and T_k are clearly disjoint for any $j \neq k$. The collection $\mathcal{T} = \{T_k \mid k \in \mathbb{Z}_n\}$ provides the desired 1-partition.

If n is even, then for $j = 0, 2, ..., n-2$ let $S_j = T_j \cup T_{j+1}$. S_j clearly covers every row and column of $\mathbb{Z}_n$ exactly twice. The set of entries covered by T_j is given by $\{2i+j \mid i \in \mathbb{Z}_n\}$;

since j and n are even this set covers every even element of $\mathbb{Z}_n$ exactly twice. Similarly, the set of entries covered by T_{j+1} is given by $\{2i+j+1 \mid i \in \mathbb{Z}_n\}$; since $j+1$ is odd and n is even this set covers every odd element of $\mathbb{Z}_n$ exactly twice. It follows that S_j is a 2-plex. Moreover, S_j and S_k are clearly disjoint for any $j \neq k$. The collection $\mathcal{S} = \{S_j \mid j = 0, 2, ..., n-2\}$ provides the desired 2-partition. □

We will refer to an orientable face 2-colorable hamilton cycle embedding as an *O2HC-embedding.*

III.2 O2HC-embeddings from Latin squares

Lemma III.2.1. *Let $\mathcal{Z}$ be the collection of facial walks obtained from a hamilton cycle embedding of $K_{n,n,n}$ such that $\mathcal{Z}$ consists of all ABC faces. The following conditions are equivalent:*

(1) There exist collections $\mathcal{X}, \mathcal{Y} \subset \mathcal{Z}$ such that $\mathcal{X} \cup \mathcal{Y} = \mathcal{Z}$, $\mathcal{X} \cap \mathcal{Y} = \emptyset$, and every edge of G appears in a face from both $\mathcal{X}$ and $\mathcal{Y}$.

(2) The embedding is orientable.

(3) The embedding is face 2-colorable.

Proof. (1)⇒(2) Since every edge appears once in a $\mathcal{X}$ face and once in a $\mathcal{Y}$ face, the faces admit a proper orientation (e.g. orient the $\mathcal{X}$ faces forwards as written in ABC order, and orient the $\mathcal{Y}$ faces backwards as written in ABC order).

(2)⇒(1) Let $\mathcal{X}$ be the faces oriented forwards as written in ABC order and $\mathcal{Y}$ be the faces oriented backwards as written in ABC order. If any distinct faces $X_1, X_2 \in \mathcal{X}$ share an edge, then they cannot both be oriented forwards. Thus, no two $\mathcal{X}$ faces share an edge, so each edge is appears in at most one face from $\mathcal{X}$. An analogous argument shows that each edge appears in at most one face from $\mathcal{Y}$, and the result follows.

The equivalence (1)⇔(3) is straightforward, so the proof is complete. □

Assume we have an O2HC-embedding of $K_{n,n,n}$ that consists of all ABC faces. Furthermore, assume that the A vertices appear in the same fixed order in each cycle. Partition the cycles into $\mathcal{X}$ and $\mathcal{Y}$ as in Lemma III.2.1, and let $\mathcal{X} = \{X_0, X_1, ..., X_{n-1}\}$. We can form a latin square L_X of order n by taking each subsequence $(\cdots a_j b_k c_\ell \cdots) \in X_i$ and letting ℓ be the entry in row j of column k. Following this process, it is readily seen that the entries arising from X_i form a transversal for all $0 \leq i \leq n-1$. Thus, L_X admits a 1-partition, which is equivalent to L_X having an orthogonal mate. Following a similar procedure, we can form a latin square L_Y that corresponds to the cycles in $\mathcal{Y}$.

Consider the following O2HC-embedding of $K_{5,5,5}$:

$$
\begin{aligned}
X_0 &: \ (a_0b_0c_0a_1b_1c_2a_2b_2c_4a_3b_3c_1a_4b_4c_3) \\
X_1 &: \ (a_0b_1c_1a_1b_2c_3a_2b_3c_0a_3b_4c_2a_4b_0c_4) \\
X_2 &: \ (a_0b_2c_2a_1b_3c_4a_2b_4c_1a_3b_0c_3a_4b_1c_0) \\
X_3 &: \ (a_0b_3c_3a_1b_4c_0a_2b_0c_2a_3b_1c_4a_4b_2c_1) \\
X_4 &: \ (a_0b_4c_4a_1b_0c_1a_2b_1c_3a_3b_2c_0a_4b_3c_2)
\end{aligned}
$$

$$
\begin{aligned}
Y_0 &: \ (a_0b_0c_1a_1b_1c_3a_2b_2c_0a_3b_3c_2a_4b_4c_4) \\
Y_1 &: \ (a_0b_1c_2a_1b_2c_4a_2b_3c_1a_3b_4c_3a_4b_0c_0) \\
Y_2 &: \ (a_0b_2c_3a_1b_3c_0a_2b_4c_2a_3b_0c_4a_4b_1c_1) \\
Y_3 &: \ (a_0b_3c_4a_1b_4c_1a_2b_0c_3a_3b_1c_0a_4b_2c_2) \\
Y_4 &: \ (a_0b_4c_0a_1b_0c_2a_2b_1c_4a_3b_2c_1a_4b_3c_3)
\end{aligned}
$$

From this we obtain the following latin squares, where L'_X and L'_Y provide the transversals.

$$L_X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}, L'_X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$$

$$L_Y = \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}, L'_Y = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$$

Since we can form two latin squares of order n from an O2HC-embedding of $K_{n,n,n}$ with certain properties, it is natural to determine the conditions under which a pair of order n latin squares give rise to an O2HC-embedding of $K_{n,n,n}$. To that end, we present the following theorem. The following notation is used, where T is a transversal of a latin square L.

$$\begin{aligned} E(L,r,c) &= \text{entry in } L \text{ that appears in row } r \text{ of column } c; \\ C(L,r,e) &= \text{column in } L \text{ that contains entry } e \text{ in row } r; \\ E(T,r) &= \text{entry in } T \text{ that appears in row } r; \\ C(T,r) &= \text{column in } T \text{ that contains entry in row } r. \end{aligned}$$

In other words, $(r,c,E(L,r,c)),(r,C(L,r,e),e) \in L$ and $(r,C(T,r),E(T,r)) \in T$.

Theorem III.2.2. *Let L_X and L_Y be latin squares of order n that each admit a 1-partition. For each j, label the transversals $S_0^j, S_1^j, ..., S_{n-1}^j$ in order as they appear in row j of L_X. In other words, S_k^j is the transversal in L_X that contains the entry in row j of column k. Similarly, label the transversals $T_0^j, T_1^j, ..., T_{n-1}^j$ in order as they appear in row j of L_Y. Thus, T_k^j is the transversal in L_Y that contains the entry in row j of column k. Define the following collections of pairs:*

(1) $P_A^j = \{(E(S_k^j, j-1), E(T_k^j, j-1)) | k \in \mathbb{Z}_n\}$ *for all* $j \in \mathbb{Z}_n$*;*

(2) $P_B^k = \{(E(L_X, j, k), E(L_Y, j, k)) | j \in \mathbb{Z}_n\}$ *for all* $k \in \mathbb{Z}_n$*;*

(3) $P_C^\ell = \{(C(L_X, j, \ell), C(L_Y, j, \ell)) | j \in \mathbb{Z}_n\}$ *for all* $\ell \in \mathbb{Z}_n$*.*

If the induced pair graphs $G_{P_A^j}$*,* $G_{P_B^k}$*, and* $G_{P_C^\ell}$ *form hamilton cycles for all* $j, k, \ell \in \mathbb{Z}_n$*, then there exists an O2HC-embedding of* $K_{n,n,n}$*.*

Proof. Form the following cycles:

$$\begin{aligned} X_i: &\quad (a_0 b_{C(S_i^0,0)} c_{E(S_i^0,0)} \cdots a_j b_{C(S_i^0,j)} c_{E(S_i^0,j)} \cdots a_{n-1} b_{C(S_i^0,n-1)} c_{E(S_i^0,n-1)}); \\ Y_i: &\quad (a_0 b_{C(T_i^0,0)} c_{E(T_i^0,0)} \cdots a_j b_{C(T_i^0,j)} c_{E(T_i^0,j)} \cdots a_{n-1} b_{C(T_i^0,n-1)} c_{E(T_i^0,n-1)}). \end{aligned}$$

Note that each X_i corresponds to the transversal S_i^0. If the entry (j, k, ℓ) appears in S_i^0, then the cycle X_i contains the sequence $a_j b_k c_\ell$. Moreover, these sequences of length 3 are assembled row by row so that the A vertices appear in increasing order. In a similar fashion, each Y_i corresponds to the transversal T_i^0. We will prove that the collections $\mathcal{X} = \{X_0, X_1, ..., X_{n-1}\}$ and $\mathcal{Y} = \{Y_0, Y_1, ..., Y_{n-1}\}$ together form an O2HC-embedding of $K_{n,n,n}$. It is not hard to show from the properties of latin squares that every AB edge and every BC edge is covered once by a cycle from $\mathcal{X}$ and once by a cycle from $\mathcal{Y}$. The fact that the A vertices appear in the same fixed order in each cycle implies that every CA edge is covered once by a cycle from $\mathcal{X}$ and once by a cycle from $\mathcal{Y}$ as well. To prove that this double cycle cover is in fact an O2HC-embedding, it remains to show that the rotation around each vertex is a single cycle of length $2n$.

Consider first the vertex a_j. For every $k \in \mathbb{Z}_n$, there exist i_1 and i_2 such that $k = C(S_{i_1}^0, j) = C(T_{i_2}^0, j)$. The cycle X_{i_1} contains the sequence $c_{E(S_{i_1}^0, j-1)} a_j b_k$. But we also know that $E(S_{i_1}^0, j-1)$ is the entry in row $j-1$ of the transversal that contains the entry in column k of row j; in other words, $E(S_{i_1}^0, j-1) = E(S_k^j, j-1)$. Similarly, Y_{i_2} contains the sequence $c_{E(T_{i_2}^0, j-1)} a_j b_k = c_{E(T_k^j, j-1)} a_j b_k$. Thus, for each k, the rotation around a_j contains the sequence $c_{E(S_k^j, j-1)} b_k c_{E(T_k^j, j-1)}$. To determine the complete rotation around a_j we need to determine how the endpoints of these sequences match up. But the subscripts on these endpoints are exactly the pairs in P_A^j, which we know match up to form a hamilton cycle for every j. Thus the rotation around a_j is a single cycle of length $2n$ for every j.

Next, consider the vertex b_k. For every $j \in \mathbb{Z}_n$, we know the sequence $a_j b_k c_{E(L_X,j,k)}$ appears in some cycle of $\mathcal{X}$. Similarly, the sequence $a_j b_k c_{E(L_Y,j,k)}$ appears in some cycle of $\mathcal{Y}$. Thus, for each j, the rotation around b_k contains the sequence $c_{E(L_X,j,k)} a_j c_{E(L_Y,j,k)}$. To determine the complete rotation around b_k, we again need to determine how the endpoints of these sequences match up. But the subscripts on these endpoints are exactly the pairs in P_B^k, which we know match up to form a hamilton cycle for every k. Thus the rotation around b_k is a single cycle of length $2n$ for every k.

Finally, consider the vertex c_ℓ. For every $j \in \mathbb{Z}_n$, we know the sequence $a_j b_{C(L_X,j,\ell)} c_\ell a_{j+1}$ appears in some cycle of $\mathcal{X}$. Similarly, the sequence $a_j b_{C(L_Y,j,\ell)} c_\ell a_{j+1}$ appears in some cycle of $\mathcal{Y}$. Thus, for each j, the rotation around c_ℓ contains the sequence $b_{C(L_X,j,\ell)} a_{j+1} b_{C(L_Y,j,\ell)}$. Just like the two preceding paragraphs, this corresponds exactly to the pairs in P_C^ℓ, so the rotation around c_ℓ is a single cycle of length $2n$ for every ℓ.

We have shown that the rotation around every vertex is indeed a single cycle of length $2n$. Combining this with Lemma III.2.1 proves that $\mathcal{X} \cup \mathcal{Y}$ forms an O2HC-embedding of $K_{n,n,n}$.

□

While the preceding construction is powerful, it can be quite difficult to find two latin squares that meet the required conditions. To simplify this matter, the following corollary describes a situation when it is sufficient to find a single latin square with certain properties.

Corollary III.2.3. *Let L be a latin square of order n that admits a 1-partition. Let p be an integer relatively prime to n, and define the collection of pairs*

$$P_\ell = \{(C(L,j,\ell), C(L,j,\ell - p)) | j \in \mathbb{Z}_n\} \textit{ for all } \ell \in \mathbb{Z}_n.$$

If the induced pair graph G_{P_ℓ} is a hamilton cycle for all $\ell \in \mathbb{Z}_n$, then there exists an O2HC-embedding of $K_{n,n,n}$.

Proof. Let $L_X = L$ and let $L_Y = L + p$ be the latin square obtained by adding p to every entry in L (and reducing modulo n). Since the transversals in L are maintained by adding p to each entry, L_X and L_Y both admit a 1-partition. We will show that these two latin squares meet the conditions of Theorem III.2.2.

Since the transversals in L_Y are simply the transversals in L_X with p added to each entry, we have that

$$\begin{aligned} P_A^j &= \{(E(S_k^j, j-1), E(T_k^j, j-1)) | k \in \mathbb{Z}_n\} \\ &= \{(E(S_k^j, j-1), E(S_k^j, j-1) + p) | k \in \mathbb{Z}_n\} \\ &= \{(r, r+p) | r \in \mathbb{Z}_n\} \end{aligned}$$

for all $j \in \mathbb{Z}_n$. By Lemma III.1.1 the induced graph for these pairs is a hamilton cycle.

It is also clear that $E(L_Y, j, k) = E(L_X, j, k) + p$, so

$$P_B^k = \{(E(L_X, j, k), E(L_Y, j, k)) | j \in \mathbb{Z}_n\} = \{(r, r+p) | r \in \mathbb{Z}_n\}$$

for all $k \in \mathbb{Z}_n$. Again by Lemma III.1.1 the induced graph for these pairs is a hamilton cycle.

Finally, we have that $C(L_X, j, \ell) = C(L, j, \ell)$ and $C(L_Y, j, \ell) = C(L, j, \ell - p)$. Thus,

$$P_C^\ell = \{(C(L_X, j, \ell), C(L_Y, j, \ell)) | j \in \mathbb{Z}_n\} = \{(C(L, j, \ell), C(L, j, \ell - p)) | j \in \mathbb{Z}_n\} = P_\ell.$$

By assumption, we know the induced graph for these pairs forms a hamilton cycle. By Theorem III.2.2, we have an O2HC-embedding of $K_{n,n,n}$. □

If a Latin square satisfies the conditions of Corollary III.2.3 for some p, then an appropriate permutation of the entries leads to a latin square that satisfies the conditions of Corollary III.2.3 for $p = 1$. The induced pair graph condition for $p = 1$ is simply a restatement of the definition of a ce-hamiltonian latin square.

The following construction for odd n is straightforward and illustrates the usefulness of Corollary III.2.3.

Theorem III.2.4. *If n is odd, then there exists an O2HC-embedding of $K_{n,n,n}$ obtained from a latin square.*

Proof. Consider the square given by $\mathbb{Z}_n$; we know $\mathbb{Z}_n$ has a 1-partition by Lemma III.1.2. It remains to show that the induced pair graph G_{P_ℓ} is a hamilton cycle for all $\ell \in \mathbb{Z}_n$. We have that

$$P_\ell = \{(C(L, j, \ell), C(L, j, \ell-1)) | j \in \mathbb{Z}_n\} = \{(\ell - j, \ell - j - 1) | j \in \mathbb{Z}_n\} = \{(r, r-1) | r \in \mathbb{Z}_n\}.$$

By Lemma III.1.1, G_{P_ℓ} is a hamilton cycle. Thus, $\mathbb{Z}_n$ is ce-hamiltonian, and by Corollary III.2.3 there exists the desired embedding of $K_{n,n,n}$. □

It is worth noting that we can also obtain orientable hamilton cycle embeddings from slope sequences. In fact, the slope sequence in the following theorem yields an embedding identical to the one obtained from the latin square in Theorem III.2.4.

Theorem III.2.5. *If n is odd, then there exists an O2HC-embedding of $K_{n,n,n}$ obtained from a slope sequence.*

Proof. It is a straightforward exercise to show that $S = \{(j, j+1) \mid j \in \mathbb{Z}_n\}$ satisfies the conditions of Theorem II.1.1. Moreover, since the first and second coordinates each cover $\mathbb{Z}_n$ once, it follows that the collections $\mathcal{X}$ and $\mathcal{Y}$ cover each edge of $K_{n,n,n}$ exactly once. By Lemma III.2.1, the resulting hamilton cycle embedding of $K_{n,n,n}$ is orientable. □

III.3 Step product construction

The construction of the required latin squares when n is even is considerably more complicated. In fact, Euler famously conjectured that for $n \equiv 2 \pmod 4$, no latin square of order n had a 1-partition [16]. Although Euler was wrong, it took nearly two centuries to construct a counterexample of order $4k+2$ for all $k \geq 2$ [5, 6]. We now seek to impose further structure on these squares. To accomplish this goal, we introduce a new construction called a *step product construction.* If L and M are latin squares of order n and m, respectively, then the step product of L and M will be a turned m-step type latin square based on L.

Let L be a latin square of order n with entries from $\mathbb{Z}_n$. For an integer $x \in \mathbb{Z}_n$, denote by $x \circ L$ the Latin square obtained by cyclically shifting the rows of L down x rows. Moreover, for an integer a and an integer $b \in \{0, 1, ..., a-1\}$, let $aL + b$ be the latin square obtained by multiplying every entry in L by a and then adding b to every resulting product, where the arithmetic is done in $\mathbb{Z}_{nm}$, not $\mathbb{Z}_n$.

Example III.3.1. Let

$$L = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

Then

$$1 \circ L = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{pmatrix} \text{ and } 5L + 2 = \begin{pmatrix} 2 & 7 & 12 & 17 \\ 7 & 2 & 17 & 12 \\ 12 & 17 & 2 & 7 \\ 17 & 12 & 7 & 2 \end{pmatrix}.$$

Let L and M be latin squares of order n and m, respectively, with rows, columns, and entries indexed by $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. Let $X = (x_{i,j})$ be an $n \times n$ matrix with entries from $\mathbb{Z}_m$. Define the *step product* $L \boxdot_X M$ to be the turned m-step type latin square of order nm given by

$$\left(\begin{array}{c|c|c|c}
x_{0,0} \circ (nM + L_{0,0}) & x_{0,1} \circ (nM + L_{0,1}) & \cdots & x_{0,n-1} \circ (nM + L_{0,n-1}) \\
\hline
x_{1,0} \circ (nM + L_{1,0}) & x_{1,1} \circ (nM + L_{1,1}) & \cdots & x_{1,n-1} \circ (nM + L_{1,n-1}) \\
\hline
 & & \vdots & \\
\hline
x_{n-1,0} \circ (nM + L_{n-1,0}) & x_{n-1,1} \circ (nM + L_{n-1,1}) & \cdots & x_{n-1,n-1} \circ (nM + L_{n-1,n-1})
\end{array}\right).$$

To help clarify this construction, we present a couple of examples.

Example III.3.2. Let $L = \mathbb{Z}_2$ and $M = \mathbb{Z}_3$, where we recall that this means L and M are the addition tables of the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively, and let X be the matrix of all 0's. Then

$$L \boxdot_X M = \left(\begin{array}{ccc|ccc}
0 & 2 & 4 & 1 & 3 & 5 \\
2 & 4 & 0 & 3 & 5 & 1 \\
4 & 0 & 2 & 5 & 1 & 3 \\
\hline
1 & 3 & 5 & 0 & 2 & 4 \\
3 & 5 & 1 & 2 & 4 & 0 \\
5 & 1 & 3 & 4 & 0 & 2
\end{array}\right).$$

Example III.3.3. Let $L = \mathbb{Z}_3$, $M = \mathbb{Z}_5$, and

$$X = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 0 & 2 \\ 4 & 1 & 4 \end{pmatrix}.$$

Then

$$
L \boxdot_X M = \left(\begin{array}{ccccc|ccccc|ccccc}
6 & 9 & 12 & 0 & 3 & 13 & 1 & 4 & 7 & 10 & 5 & 8 & 11 & 14 & 2 \\
9 & 12 & 0 & 3 & 6 & 1 & 4 & 7 & 10 & 13 & 8 & 11 & 14 & 2 & 5 \\
12 & 0 & 3 & 6 & 9 & 4 & 7 & 10 & 13 & 1 & 11 & 14 & 2 & 5 & 8 \\
0 & 3 & 6 & 9 & 12 & 7 & 10 & 13 & 1 & 4 & 14 & 2 & 5 & 8 & 11 \\
3 & 6 & 9 & 12 & 0 & 10 & 13 & 1 & 4 & 7 & 2 & 5 & 8 & 11 & 14 \\
\hline
1 & 4 & 7 & 10 & 13 & 2 & 5 & 8 & 11 & 14 & 9 & 12 & 0 & 3 & 6 \\
4 & 7 & 10 & 13 & 1 & 5 & 8 & 11 & 14 & 2 & 12 & 0 & 3 & 6 & 9 \\
7 & 10 & 13 & 1 & 4 & 8 & 11 & 14 & 2 & 5 & 0 & 3 & 6 & 9 & 12 \\
10 & 13 & 1 & 4 & 7 & 11 & 14 & 2 & 5 & 8 & 3 & 6 & 9 & 12 & 0 \\
13 & 1 & 4 & 7^* & 10 & 14 & 2 & 5 & 8 & 11 & 6 & 9 & 12 & 0 & 3 \\
\hline
5 & 8 & 11 & 14 & 2 & 12 & 0 & 3 & 6 & 9 & 4 & 7 & 10 & 13 & 1 \\
8 & 11 & 14 & 2 & 5 & 0 & 3 & 6 & 9 & 12 & 7 & 10 & 13 & 1 & 4 \\
11 & 14 & 2 & 5 & 8 & 3 & 6 & 9 & 12 & 0 & 10 & 13 & 1 & 4 & 7 \\
14 & 2 & 5 & 8 & 11 & 6 & 9 & 12 & 0 & 3 & 13 & 1 & 4 & 7 & 10 \\
2 & 5 & 8 & 11 & 14 & 9 & 12 & 0 & 3 & 6 & 1 & 4 & 7 & 10 & 13
\end{array}\right).
$$

Remark III.3.4. We use the set $\mathbb{Z}_n \times \mathbb{Z}_m$ to label the rows and columns of a step product square, while the entries are from $\mathbb{Z}_{nm}$. For example, the starred entry in Example III.3.3 corresponds to the ordered triple $((1, 4), (0, 3), 7)$.

Now that we have our construction, we need to use it to create O2HC-embeddings of complete tripartite graphs. By Corollary III.2.3, we simply need to find ce-hamiltonian latin squares that have a 1-partition. In Section III.4 we develop conditions for the step product of two latin squares to be ce-hamiltonian. In Section III.5 we determine when these squares have a 1-partition. Finally, the step product construction is used to construct embeddings in Sections III.6 and III.7.

III.4 Ce-hamiltonicity of step product squares

Requiring that L and M are ce-hamiltonian is not sufficient to ensure that the step product $L \boxdot_X M$ is ce-hamiltonian. The permutations of rows given by the entries in X affect the requisite induced pair graphs. Thus, to obtain a step product square that is ce-hamiltonian, we need to have some conditions on X. Let $X(k)$ be the set of cells that contain the entry k in L. In other words, $X(k) = \{(i,j) \mid L_{ij} = k\}$. Moreover, set $\sigma_k = \sum_{(i,j)\in X(k)} x_{ij}$ for all $k \in \mathbb{Z}_n$, where the addition is done in $\mathbb{Z}_m$. We call the vector $\nu(X, L) = (\sigma_0, \sigma_1, ..., \sigma_{n-1})$ the *representative vector for X over L.*

Remark III.4.1. Let L be a latin square of order n and let X be an $n \times n$ matrix with entries from $\mathbb{Z}_m$. We can permute the rows and columns of L without affecting $\nu(X, L)$ if we apply the same permutation of the rows and columns to X. Namely, if λ_r and λ_c are permutations of $\mathbb{Z}_n$ applied to the rows and columns, respectively, of L and X, then

$$\lambda_r\lambda_c(L) \boxdot_{\lambda_r\lambda_c(X)} \mathbb{Z}_m = (\lambda_r, 1_m)(\lambda_c, 1_m)(L \boxdot_X \mathbb{Z}_m)$$

where 1_m is the identity permutation on $\mathbb{Z}_m$, and

$$\nu(\lambda_r\lambda_c(X), \lambda_r\lambda_c(L)) = \nu(X, L).$$

Here and later in this chapter we will need the following observation about the square $x \circ \mathbb{Z}_n$.

Observation III.4.2. *If $M = \mathbb{Z}_m = \{(r, c, r + c) \mid r, c \in \mathbb{Z}_m\}$, then $x \circ M = \{(r, c, r + c - x) \mid r, c \in \mathbb{Z}_m\}$.*

We now prove sufficient conditions on X for the product $L \boxdot_X Z_m$ to be ce-hamiltonian.

Theorem III.4.3. *Let L be a ce-hamiltonian latin square of order n, let X be an $n \times n$ matrix with entries from $\mathbb{Z}_m$, and let $\nu(X, L) = (\sigma_0, ..., \sigma_{n-1})$ be the representative vector for*

X over L. Suppose that the following conditions hold:

(1) $\sigma_k - \sigma_{k+1}$ is relatively prime to m for all $0 \leq k \leq n-2$;

(2) $\sigma_{n-1} - \sigma_0 - n$ is relatively prime to m.

Then the latin square $L \boxdot_X \mathbb{Z}_m$ is ce-hamiltonian.

Proof. Let $K = L \boxdot_X \mathbb{Z}_m$. Define

$$P_{\ell+1} = \{(C(K,(i,j),\ell+1), C(K,(i,j),\ell)) | (i,j) \in \mathbb{Z}_n \times \mathbb{Z}_m\} \text{ for all } \ell \in \mathbb{Z}_{nm}$$

as in Corollary III.2.3. We need to show that the induced pair graph $G_{P_{\ell+1}}$ is a hamilton cycle for all $\ell \in \mathbb{Z}_{nm}$.

Write $\ell = an + b$, where $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$; we know every occurrence of $\ell \in K$ corresponds to the entries $a \in \mathbb{Z}_m$ and $b \in L$. By Remark III.4.1, permuting the rows and columns of L and X simultaneously does not affect $\nu(X, L)$. It also does not affect ce-hamiltonicity of the resulting step product square K. Thus, since L is ce-hamiltonian, we can permute the rows and columns of L and X simultaneously to obtain the following representation of L:

$$L = \begin{pmatrix} b & b+1 & & \\ & b & b+1 & \\ & & \ddots & \ddots \\ b+1 & & & b \end{pmatrix}.$$

For each $(i,j) \in X(b)$, write $\alpha_i = x_{ij}$. Similarly for each $(i,j) \in X(b+1)$, write $\beta_i = x_{ij}$.

We have the following representation for L:

$$\left(\begin{array}{c|c|c|c}
\alpha_0 \circ (n\mathbb{Z}_m + b) & \beta_0 \circ (n\mathbb{Z}_m + b + 1) & & \\
\hline
 & \alpha_1 \circ (n\mathbb{Z}_m + b) & \beta_1 \circ (n\mathbb{Z}_m + b + 1) & \\
\hline
 & & \ddots & \ddots \\
\hline
\beta_{n-1} \circ (n\mathbb{Z}_m + b + 1) & & & \alpha_{n-1} \circ (n\mathbb{Z}_m + b)
\end{array}\right).$$

We note here that $x \circ (aM + b) = a(x \circ M) + b$ for any latin square M and any integers a and b, and we will use the latter representation in the following arguments.

Assume first that $\ell \not\equiv -1 \pmod{n}$; we know that every occurrence of $\ell+1$ in L corresponds to the entries $a \in \mathbb{Z}_m$ and $b+1 \in L$. Assume that for some (i,j) we have $C(K,(i,j),\ell+1) = (y,z)$. Then the entry a appears in row j and column z of the square $\beta_i \circ \mathbb{Z}_m$; from Observation III.4.2 we know $a = j + z - \beta_i$. We want to determine a column γ such that a appears in row j and column γ of square $\alpha_i \circ \mathbb{Z}_m$. We know that $a = j + \gamma - \alpha_i = j + z - \beta_i$, which implies that $\gamma = z + \alpha_i - \beta_i$. From this we learn that $C(K,(i,j),\ell) = (y-1, z + \alpha_i - \beta_i)$. Thus, all of the pairs in $P_{\ell+1}$ have the form $((y,z),(y-1,z+\alpha_i-\beta_i))$. From this it is clear that we can partition $P_{\ell+1}$ into subsets of the form

$$\begin{aligned}
P(y) = \; & \{((y,z),(y-1,z+\alpha_{i_0}-\beta_{i_0})), \\
& ((y-1,z+\alpha_{i_0}-\beta_{i_0}),(y-2,z+\alpha_{i_0}+\alpha_{i_1}-\beta_{i_0}-\beta_{i_1})), ..., \\
& ((y+1,z+\textstyle\sum_{j=0}^{n-2}\alpha_{i_j}-\sum_{j=0}^{n-2}\beta_{i_j}),(y,z+\sum_{j=0}^{n-1}\alpha_{i_j}-\sum_{j=0}^{n-1}\beta_{i_j}))\}
\end{aligned}$$

for all $y \in \mathbb{Z}_n$. Using our representation of L described earlier, we know $b+1$ occurs in the row above b in each column of L, so we must have $i_{j+1} = i_j - 1$ and $\{i_0, i_1, ..., i_{n-1}\} = \mathbb{Z}_n$; therefore, $\sum_{j=0}^{n-1}\alpha_{i_j} = \sigma_b$ and $\sum_{j=0}^{n-1}\beta_{i_j} = \sigma_{b+1}$. Since $\sigma_b - \sigma_{b+1}$ is relatively prime to m, we know $z + \sigma_b - \sigma_{b+1} \neq z$, so each $P(y)$ induces a path on n edges. Moreover, the fact that $\sigma_b - \sigma_{b+1}$ is relatively prime to m implies that the endpoints of these paths match up to form a single cycle of length nm.

If $\ell \equiv -1 \pmod{n}$, then $\ell = an + (n-1)$ and $\ell + 1 = (a+1)n$, so we have $b = n - 1$

and $b+1=0$. Let $\alpha_i = x_{ij}$ for all $(i,j) \in X(n-1)$ and let $\beta_i = x_{ij}$ for all $(i,j) \in X(0)$. Assume that for some (i,j) we have $C(K,(i,j),\ell+1) = (y,z)$. Then the entry $a+1$ appears in row j and column z of the square $\beta_i \circ \mathbb{Z}_m$; again from Observation III.4.2 we know $a+1 = j+z-\beta_i$. We want to determine a column γ such that a appears in row j and column γ of square $\alpha_i \circ \mathbb{Z}_m$. We know that $a = j+\gamma-\alpha_i = j+z-\beta_i-1$, which implies that $\gamma = z+\alpha_i-\beta_i-1$. From this we learn that $C(K,(i,j),\ell) = (y-1, z+\alpha_i-\beta_i-1)$. Thus, all of the pairs in $P_{\ell+1}$ have the form $((y,z),(y-1,z+\alpha_i-\beta_i-1))$. From this it is clear that we can partition $P_{\ell+1}$ into subsets of the form

$$\begin{aligned} P(y) = \ & \{((y,z),(y-1,z+\alpha_{i_0}-\beta_{i_0}-1)), \\ & ((y-1,z+\alpha_{i_0}-\beta_{i_0}-1),(y-2,z+\alpha_{i_0}+\alpha_{i_1}-\beta_{i_0}-\beta_{i_1}-2)), ..., \\ & ((y+1,z+\textstyle\sum_{j=0}^{n-2}\alpha_{i_j}-\sum_{j=0}^{n-2}\beta_{i_j}-(n-1)),(y,z+\sum_{j=0}^{n-1}\alpha_{i_j}-\sum_{j=0}^{n-1}\beta_{i_j}-n))\} \end{aligned}$$

for all $y \in \mathbb{Z}_n$. From our representation of L described earlier, it is again clear that $i_{j+1} = i_j - 1$, so we must have $\{i_0, i_1, ..., i_{n-1}\} = \mathbb{Z}_n$; therefore, $\sum_{j=0}^{n-1}\alpha_{i_j} = \sigma_{n-1}$ and $\sum_{j=0}^{n-1}\beta_{i_j} = \sigma_0$. Since $\sigma_{n-1}-\sigma_0-n$ is relatively prime to m, we know $z+\sigma_{n-1}-\sigma_0-n \neq z$, so each $P(y)$ induces a path on n edges. Moreover, the fact that $\sigma_{n-1}-\sigma_0-n$ is relatively prime to m implies that the endpoints of these paths match up to form a single cycle of length nm.

Since $G_{P_{\ell+1}}$ is a hamilton cycle for all $\ell \in \mathbb{Z}_{nm}$, the step product square K is ce-hamiltonian. □

III.5 Decomposing step product squares into transversals

We want to present conditions for lifting k-plexes in L and $\mathbb{Z}_m$ to transversals in $L \boxdot_X \mathbb{Z}_m$. Before we proceed, a deeper exploration of the step product construction is required.

Observation III.5.1. *Let $(r_1,c_1,e_1) \in L$ and $(r_2,c_2,e_2) \in \mathbb{Z}_m$; furthermore, let $x_{r_1,c_1} \in X$ be the entry found in row r_1 and column c_1 of the matrix X. Utilizing Observation III.4.2, the step product square $K = L \boxdot_X \mathbb{Z}_m$ has the following form, where here and elsewhere* *in this chapter we write $[f(x)]_m$ to denote that the expression $f(x)$ is evaluated in $\mathbb{Z}_m$ and assume all other arithmetic for K is done in $\mathbb{Z}_{nm}$.*

$$\begin{array}{cc} & \begin{array}{ccc} & (c_1,c_2) & \end{array} \\ (r_1,r_2) & \left[\begin{array}{c|c|c} \ddots & \vdots & \iddots \\ \hline \cdots & n[e_2 - x_{r_1,c_1}]_m + e_1 & \cdots \\ \hline \iddots & \vdots & \ddots \end{array}\right] \end{array},$$

where we note that $e_2 = r_2 + c_2$. Thus, each pair of triples $(r_1, c_1, e_1) \in L$ and $(r_2, c_2, e_2) \in M$ defines a unique triple $((r_1, r_2), (c_1, c_2), n[e_2 - x_{r_1,c_1}]_m + e_1) \in L$. We use this to define a binary operation $\boxdot_X$ on entries of L and M such that $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) = ((r_1, r_2), (c_1, c_2), n[e_2 - x_{r_1,c_1}]_m + e_1)$. Furthermore, for any subset of ordered triples $T_1 \subseteq L$ and $T_2 \subseteq M$, we define $T_1 \boxdot_X T_2 = \{(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) \mid (r_i, c_i, e_i) \in T_i, i = 1, 2\}$. We extend this definition to include $(r_1, c_1, e_1) \boxdot_X T_2$ and $T_1 \boxdot_X (r_2, c_2, e_2)$ in the obvious way.

Observation III.5.2. *For every entry $((r_1, r_2), (c_1, c_2), e) \in L \boxdot_X M$, there exist unique $(r_1, c_1, e_1) \in L$ and $(r_2, c_2, e_2) \in \mathbb{Z}_m$ such that $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) = ((r_1, r_2), (c_1, c_2), e)$. Thus, if $S \cap S' = \emptyset$ or $T \cap T' = \emptyset$, then $(S \boxdot_X T) \cap (S' \boxdot_X T') = \emptyset$.*

The following lemma explains how transversals in L and $\mathbb{Z}_m$ are lifted to transversals in $L \boxdot_X \mathbb{Z}_m$.

Lemma III.5.3. *Let L be a latin square of order n, and let S be a transversal in L. Furthermore, let T be a transversal in $\mathbb{Z}_m$. For any $n \times n$ matrix X on $\mathbb{Z}_m$, the collection $S \boxdot_X T$ forms a transversal in $K = L \boxdot_X \mathbb{Z}_m$.*

Proof. Let $(r_1, r_2) \in \mathbb{Z}_n \times \mathbb{Z}_m$ be any row in K. Since S is a transversal in L, there is a unique triple $(r_1, c_1, e_1) \in S$ covering the row r_1. Since T is a transversal in $\mathbb{Z}_m$, there is a unique triple $(r_2, c_2, e_2) \in T$ covering the row r_2. The element $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) \in S \boxdot_X T$ covers the row (r_1, r_2).

Let $(c_1, c_2) \in \mathbb{Z}_n \times \mathbb{Z}_m$ be any column in K. Since S is a transversal in L, there is a unique triple $(r_1, c_1, e_1) \in S$ covering the column c_1. Since T is a transversal in $\mathbb{Z}_m$, there is a unique triple $(r_2, c_2, e_2) \in T$ covering the column c_2. The element $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) \in S \boxdot_X T$ covers the column (c_1, c_2).

Let $e \in \mathbb{Z}_n \times \mathbb{Z}_m$ be any entry in K. Define e_1 and e_2 to be the unique integers such that $e = e_2 n + e_1$, with $0 \leq e_2 \leq m - 1$ and $0 \leq e_1 \leq n - 1$. Since S is a transversal in L, there is a unique triple $(r_1, c_1, e_1) \in S$ covering the entry e_1. Let $x_{r_1, c_1} \in X$ be the entry found in row r_1 and column c_1 of the matrix X; since T is a transversal in $\mathbb{Z}_m$, there is a unique triple $(r_2, c_2, [e_2 + x]_m) \in T$ covering the entry $[e_2 + x]_m$. The element $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, [e_2 + x]_m) \in S \boxdot_X T$ covers the entry e.

We have shown that every row, column, and entry in K is covered *at least once* by $S \boxdot_X T$. Since $|S \boxdot_X T| = nm$, it must be true that every row, column, and entry in K is covered *exactly once* by $S \boxdot_X T$; therefore, $S \boxdot_X T$ is a transversal in K. $\square$

While the preceding lemma shows how to lift transversals in L to transversals in $L \boxdot_X \mathbb{Z}_m$ for *any* matrix X, the following results prove only that *there exists* a matrix X that allows m-plexes in L to be lifted to transversals in $L \boxdot_X \mathbb{Z}_m$.

Before we state and prove this lemma, however, some additional ideas are needed. Given an m-plex S in L, let π and τ be maps from all row and column pairs in S to the group $\mathbb{Z}_m$; with a slight abuse of notation, we write $\pi, \tau : S \to \mathbb{Z}_m$. For convenience we will use $\pi + g$ to denote the map given by $(r, c) \mapsto [\pi(r, c) + g]_m$; similar conventions will be used for τ. If we also have an $n \times n$ matrix X with entries from $\mathbb{Z}_m$, then we define

$$\begin{aligned} T(S,\pi,\tau,X) &= \{(r,c,e) \boxdot_X (\pi(r,c),\tau(r,c),[\pi(r,c)+\tau(r,c)]_m) \mid (r,c,e)\in S\} \\ &= \{((r,\pi(r,c)),(c,\tau(r,c)),n[\pi(r,c)+\tau(r,c)-x_{r,c}]_m+e \mid (r,c,e)\in S\} \\ &\subset S \boxdot_X \mathbb{Z}_m. \end{aligned}$$

Lemma III.5.4. *Let L be a latin square of order n, and let S be an m-plex in L. Suppose there exist functions $\pi,\tau : S \to \mathbb{Z}_m$ and an $n \times n$ matrix X with entries from $\mathbb{Z}_m$ such that:*

(1) for every fixed row r of L, $\Psi(\pi,r) = \{\pi(r,c) \mid (r,c,e)\in S\} = \mathbb{Z}_m$;

(2) for every fixed column c of L, $\Psi(\tau,c) = \{\tau(r,c) \mid (r,c,e)\in S\} = \mathbb{Z}_m$;

(3) for every fixed entry e of L, $\Psi(\pi,\tau,e) = \{[\pi(r,c)+\tau(r,c)-x_{r,c}]_m \mid (r,c,e)\in S\} = \mathbb{Z}_m$.

Then $T = T(S,\pi,\tau,X)$ is a transversal in $K = L \boxdot_X \mathbb{Z}_m$.

Proof. For each row r of L, condition (1) implies that the set $\{(r,\pi(r,c)) \mid (r,c,e)\in S\}$ covers every row of K with first coordinate r exactly once; since this holds for all r, T covers every row of K exactly once. An analogous argument based on condition (2) implies that T covers every column of K exactly once. Finally, for each entry e of L, condition (3) implies that the set $\{n[\pi(r,c)+\tau(r,c)-x_{r,c}]_m+e \mid (r,c,e)\in S\}$ covers every entry of K that is congruent to e modulo n exactly once; since this holds for all e, T covers every entry of K exactly once. Thus, T is a set of triples that covers every row, column and entry of K exactly once; we simply need to show all of these triples are actually in K. Note that the triples in T are of the form described in Observation III.5.1, so T is indeed a transversal in K. □

Corollary III.5.5. *Let L be a latin square of order n, and let S be an m-plex in L. Suppose the conditions of Lemma III.5.4 hold for some functions π and τ and some matrix X. Then there exists a family of m^2 disjoint transversals in $K = L \boxdot_X \mathbb{Z}_m$ covering all of $S \boxdot_X \mathbb{Z}_m$.*

Proof. Define $T_{g,h} = T(S,\pi+g,\tau+h,X)$ and set $\mathcal{T} = \{T_{g,h} \mid (g,h)\in \mathbb{Z}_m\times\mathbb{Z}_m\}$. Assume $(g,h) \neq (g',h')$; without loss of generality we can assume $g \neq g'$. Since $\pi+g$ differs from

$\pi + g'$ on all of S, it is clear that $T_{g,h}$ and $T_{g',h'}$ are disjoint. It remains to show that $T_{g,h}$ is a transversal for any $(g, h) \in \mathbb{Z}_m \times \mathbb{Z}_m$. By Lemma III.5.4, it will suffice to show that $\pi + g$ and $\tau + h$ satisfy conditions (1), (2) and (3). We obtain the sets $\Psi(\pi + g, r)$, $\Psi(\tau + h, c)$ and $\Psi(\pi + g, \tau + h, e)$ by adding the fixed amounts g, h and $g + h$ to the sets $\Psi(\pi, r)$, $\Psi(\tau, c)$ and $\Psi(\pi, \tau, e)$, respectively, and reducing modulo m. It is clear that these three sets also cover the entire group $\mathbb{Z}_m$, so each $T_{g,h}$ is a transversal. Thus, the collection $\mathcal{T}$ forms a collection of m^2 disjoint transversals contained in $S \boxdot_X \mathbb{Z}_m$; since $\mathcal{T}$ and $S \boxdot_X \mathbb{Z}_m$ both cover nm^3 entries, we must have that $\mathcal{T}$ covers all of $S \boxdot_X \mathbb{Z}_m$. $\square$

We now want to show that for any π and τ satisfying conditions (1) and (2) of the previous lemma, we can always find a matrix X such that condition (3) is satisfied as well.

Lemma III.5.6. *Let L be a latin square of order n, and let S be an m-plex in L. Suppose there exist functions $\pi, \tau : S \to \mathbb{Z}_m$ such that:*

(1) for every row r of S, $\Psi(\pi, r) = \{\pi(r, c) \mid (r, c, e) \in S\} = \mathbb{Z}_m$;

(2) for every column c of S, $\Psi(\tau, c) = \{\tau(r, c) \mid (r, c, e) \in S\} = \mathbb{Z}_m$.

Then we can assign values to the cells in an $n \times n$ matrix X that correspond to S such that for every entry e of S, $\Psi(\pi, \tau, e) = \{[\pi(r, c) + \tau(r, c) - x_{r,c}]_m \mid (r, c, e) \in S\} = \mathbb{Z}_m$.

Proof. Since any distinct $(r, c, e), (r', c', e') \in S$ must satisfy $(r, c) \neq (r', c')$, we can define $x_{r,c}$ independently for each $(r, c, e) \in S$ such that $\Psi(\pi, \tau, e) = \mathbb{Z}_m$ for every entry e of S. $\square$

If π is simply the projection of each pair $(r, c) \in S$ to its order by column amongst all pairs in row r of S, then we say π is the *canonical row projection*. Similarly, if τ is simply the projection of each pair $(r, c) \in S$ to its order by row amongst all pairs in column c of S, then we say τ is the *canonical column projection*. If π or τ is the projection of each pair to its reverse order by row or column, then we say π or τ is the *reverse canonical row projection* or *reverse canonical column projection*, respectively.

Remark III.5.7. The canonical projections and reverse canonical projections both satisfy the conditions of Lemma III.5.6.

Finally, we put the previous lemmas together to get a decomposition theorem.

Theorem III.5.8. *Let n and m be integers with m odd, and let L be a latin square of order n that admits an $(m, m, ..., m, 1, 1, ...1)$-partition. Then there exists an $n \times n$ matrix X on $\mathbb{Z}_m$ such that $K = L \boxdot_X \mathbb{Z}_m$ admits a 1-partition.*

Proof. Because m is odd, we know $\mathbb{Z}_m$ has a 1-partition by Lemma III.1.2. Assume L can be decomposed into p transversals and q labeled m-plexes, all of which are mutually disjoint. By counting the number of entries covered by each transversal or m-plex, we know $p + qm = n$. By Lemma III.5.3, each of the p transversals in L combines with each of the m transversals in $\mathbb{Z}_m$ to yield a transversal in K for any matrix X, providing pm total disjoint transversals in K. For each m-plex S, let π and τ be any maps that satisfy the conditions of Lemma III.5.6. By that result, we can assign values to the cells in an $n \times n$ matrix X that correspond to each m-plex S such that the conditions of Corollary III.5.5 are satisfied. Thus, each of the q m-plexes in L yields m^2 disjoint transversals in K, providing qm^2 total disjoint transversals in K. Because all of the underlying transversals and m-plexes in L are mutually disjoint, we know from Observation III.5.2 that the $pm + qm^2$ transversals in K are all mutually disjoint. Moreover, $pm + qm^2 = (p + qm)m = nm$, so we have the desired 1-partition of K. $\square$

We can relax the restriction that m is odd if L has a complete decomposition into m-plexes.

Lemma III.5.9. *Let L be a latin square of order m that admits an m-partition. Then there exists an $n \times n$ matrix X on $\mathbb{Z}_m$ such that $K = L \boxdot_X \mathbb{Z}_m$ admits a 1-partition.*

Proof. For each m-plex S, let π and τ be any maps that satisfy the conditions of Lemma III.5.6. By that result and Corollary III.5.5, there exists an $n \times n$ matrix X such that each of the $\frac{n}{m}$ m-plexes in L yields m^2 disjoint transversals in K, providing nm total disjoint transversals in K. $\square$

III.6 Construction for $n \equiv 2 \pmod 4$

Our goal is to build ce-hamiltonian latin squares that admit a 1-partition. In Section III.5, we proved the existence of a matrix X so that for an appropriate choice of L and M the step product $L \boxdot_X M$ has a 1-partition. Moreover, we showed in Section III.4 that if X satisfies certain properties, this step product square will be ce-hamiltonian. In this section, we determine some squares L and M for which the X from Theorem III.5.8 can be modified to also meet the conditions of Theorem III.4.3. The result will be a family of latin squares that correspond to O2HC-embeddings of complete tripartite graphs.

The first step will be defining appropriate latin squares to use as L and M in the step product. For the case when $n \equiv 2 \pmod 4$, we are going to find a matrix X so that $K_{2p} = \mathbb{Z}_p \boxdot_X \mathbb{Z}_2$ has the desired properties of L, and M will be $\mathbb{Z}_q$ for some odd q. In particular, we want K_{2p} to be ce-hamiltonian and to have a $(q, q, 1, 1, ..., 1)$-partition for any fixed odd q, $3 \le q \le p$.

Lemma III.6.1. *Let p and q be odd integers with q prime and $p \ge q \ge 3$. There exists a $p \times p$ matrix X on $\mathbb{Z}_2$ such that $K_{2p} = \mathbb{Z}_p \boxdot_X \mathbb{Z}_2$ is ce-hamiltonian and admits a $(q, q, 1, 1, ..., 1)$-partition.*

Proof. Let X initially be the $p \times p$ matrix of all 0's; we will modify X to get appropriate values. Since the union of two disjoint transversals is a 2-plex, Lemma III.1.2 implies that we can find $\frac{1}{2}(p-q)$ disjoint 2-plexes in $\mathbb{Z}_p$; call them $S_0, ..., S_{\frac{p-q}{2}-1}$. Using the canonical row and column projections together with Lemma III.5.6 and Corollary III.5.5, we can assign values to X so that the collection $\mathcal{S} = \{S_0 \boxdot_X \mathbb{Z}_2, ..., S_{\frac{p-q}{2}-1} \boxdot_X \mathbb{Z}_2\}$ can be partitioned into $2(p-q)$ disjoint transversals. Note that we only assigned values to the cells in X corresponding to the cells of $\mathbb{Z}_p$ covered by the S_i's.

The previous paragraph required combining $p-q$ disjoint transversals in $\mathbb{Z}_p$ to get $\frac{1}{2}(p-q)$ disjoint 2-plexes. Thus, there are q disjoint transversals in $\mathbb{Z}_p$ remaining; call them $T_0, ..., T_{q-1}$. We need to assign values to X so the collection $\{T_0 \boxdot_X \mathbb{Z}_2, ..., T_{q-1} \boxdot_X \mathbb{Z}_2\}$ can be partitioned into 2 disjoint q-plexes. For all $(r,c,e) \in T_0$, we keep $x_{r,c} = 0$ and form the collection

$$\begin{aligned} W_1 &= \{(r,c,e) \boxdot_X (0,0,0), (r,c,e) \boxdot_X (0,1,1), (r,c,e) \boxdot_X (1,0,1) \mid (r,c,e) \in T_0\} \\ &= \{((r,0),(c,0),e), ((r,0),(c,1),p+e), ((r,1),(c,0),p+e) \mid (r,c,e) \in T_0\}. \end{aligned}$$

Note that W_1 covers every row of the form $(r,0)$ twice and every row of the form $(r,1)$ once, every column of the form $(c,0)$ twice and every column of the form $(c,1)$ once, and every entry of the form e once and every entry of the form $p+e$ twice, where $r,c,e \in \mathbb{Z}_p$. For all $(r,c,e) \in T_1$ we set $x_{r,c} = 1$ and form the collection

$$\begin{aligned} W_2 &= \{(r,c,e) \boxdot_X (1,1,0) \mid (r,c,e) \in T_1\} \\ &= \{((r,1),(c,1),p+e) \mid (r,c,e) \in T_1\}. \end{aligned}$$

Note that W_2 covers every row of the form $(r,1)$ once, every column of the form $(c,1)$ once, and every entry of the form $p+e$ once, where $r,c,e \in \mathbb{Z}_p$. The collection $W_1 \cup W_2$ covers every row and column of K_{2p} exactly twice, covers every entry of the form e once, and covers every entry of the form $p+e$ three times, where $e \in \mathbb{Z}_p$.

Thus far, we have assigned values to all entries $x_{r,c} \in X$ except if $(r,c,e) \in T_i$ for some $i = 2,3,...,q-1$. Let $\nu(X,L) = (\sigma_0, \sigma_1, ..., \sigma_{p-1})$ be the representative vector for X over L as defined in Section III.4. Enumerate $T_2 = \{(r_0,c_0,0), (r_1,c_1,1), ..., (r_{p-1},c_{p-1},p-1)\}$. We will assign values to x_{r_i,c_i} for all $i \in \mathbb{Z}_p$ such that $\nu(X,L)$ satisfies the conditions of Theorem III.4.3. Since every x_{r_i,c_i} and σ_i is simply 0 or 1, we assign values to x_{r_i,c_i} such that $\sigma_i = 1$ if i is even, and $\sigma_i = 0$ otherwise. Thus, for all $i = 0,1,...,p-2$ we have $\sigma_i - \sigma_{i+1} \equiv 1 \pmod 2$, and $\sigma_{p-1} - \sigma_0 - p = 1 - 1 - p \equiv 1 \pmod 2$ as well, so the conditions of Theorem III.4.3 are satisfied. It remains to show that we can find the partition into 2 disjoint q-plexes without changing any more entries in X.

Let $E_0 = \{(0,0,0),(1,1,0)\} \subset \mathbb{Z}_2$ and $E_1 = \{(0,1,1),(1,0,1)\} \subset \mathbb{Z}_2$. Form the collection $W_3 = \{(r,c,e) \boxdot_X E_{x_{r,c}} \mid (r,c,e) \in T_2\}$; W_3 can take two forms depending on the value of

$x_{r,c}$. If $x_{r,c} = 0$, then

$$W_3 = \{((r,0),(c,0),e),((r,1),(c,1),e) \mid (r,c,e) \in T_2\};$$

if $x_{r,c} = 1$, then

$$W_3 = \{((r,0),(c,1),e),((r,1),(c,0),e) \mid (r,c,e) \in T_2\}.$$

Note that in either case W_3 covers every row and column in K_{2p} exactly once; moreover, W_3 covers every entry $p[x_{r,c} - x_{r,c}]_2 + e = e$ exactly twice, where $e \in \mathbb{Z}_p$. Therefore, the collection $W_1 \cup W_2 \cup W_3$ forms a 3-plex. We now have $q-3$ remaining unused transversals, which are $T_3, ..., T_{q-1}$. Recalling that $x_{r,c} = 0$ for any $(r,c,e) \in T_i$ with $i = 3, ..., q-1$, let

$$\begin{aligned} W_{even} &= \{T_i \boxdot_X E_0 \mid i \text{ even}\} \\ &= \{((r,0),(c,0),e),((r,1),(c,1),e) \mid (r,c,e) \in T_i, i \text{ even}\} \end{aligned}$$

and

$$\begin{aligned} W_{odd} &= \{T_i \boxdot_X E_1 \mid i \text{ odd}\} \\ &= \{((r,0),(c,1),p+e),((r,1),(c,0),p+e) \mid (r,c,e) \in T_i, i \text{ odd}\}. \end{aligned}$$

Note that W_{even} covers every row and column of K_{2p} exactly $\frac{1}{2}(q-3)$ times. Additionally, W_{even} covers every entry e exactly $q-3$ times, where $e \in \mathbb{Z}_p$. Similarly, W_{odd} covers every row and column of K_{2p} exactly $\frac{1}{2}(q-3)$ times. Additionally, W_{odd} covers every entry $p+e$ exactly $q-3$ times. Thus, $W_{even} \cup W_{odd}$ forms a $(q-3)$-plex, and the collection $\mathcal{W} = W_1 \cup W_2 \cup W_3 \cup W_{even} \cup W_{odd}$ forms a q-plex. Let $\mathcal{V} = K_{2p} \setminus (\mathcal{S} \cup \mathcal{W})$; since $\mathcal{S}$ and $\mathcal{W}$ together cover every row, column, and entry of K_{2p} exactly $2(p-q)+q = 2p-q$ times, $\mathcal{V}$ must cover every row, column, and entry of K_{2p} exactly $2p-(2p-q) = q$ times. Therefore, $\mathcal{V}$ is also a q-plex, and we have our desired partition of K_{2p} into $2(p-q)$ transversals and 2 q-plexes, all of which are mutually disjoint. As mentioned before, the matrix X we constructed satisfies the conditions of Theorem III.4.3, so K_{2p} is also ce-hamiltonian. $\square$

We want to use K_{2p} as the first ingredient of a step product construction with $\mathbb{Z}_q$; to differentiate from the matrix X used to form K_{2p}, we will form the product $K_{2p} \boxdot_Y \mathbb{Z}_q$. We need to show that the matrix Y that we obtain from Theorem III.5.8 can be modified to satisfy the conditions of Theorem III.4.3 without destroying the partition into transversals. To do so, we will apply Theorem III.5.8 to K_{2p} and $\mathbb{Z}_q$ with one q-plex labeled with the canonical projections and the other q-plex labeled with the reverse canonical projections. This will yield a matrix Y such that $K_{2p} \boxdot_Y \mathbb{Z}_q$ has a 1-partition. We will then alter π and τ for each q-plex and show that making appropriate changes in Y yields a matrix such that $K_{2p} \boxdot_Y \mathbb{Z}_q$ still admits a 1-partition, but is ce-hamiltonian as well.

Before we accomplish this goal, it will now be helpful to identify some properties of the q-plexes created in Lemma III.6.1. According to the construction described by the collections W_1 and W_2, 3 of the 4 entries of K_{2p} obtained from $(r, c, e) \boxdot_X \mathbb{Z}_2$ are in $\mathcal{W}$, where $(r, c, e) \in T_0$. Similarly, 1 of the 4 entries of K_{2p} obtained from $(r, c, e) \boxdot_X \mathbb{Z}_2$ is in $\mathcal{W}$, where $(r, c, e) \in T_1$. We assume for the sake of presentation that T_0 and T_1 are found along the first two diagonals of $\mathbb{Z}_p$. We then have the following representation for K_{2p}, where the superscript denotes whether each entry is in the q-plex $\mathcal{W}$ or $\mathcal{V}$:

$$\left(\begin{array}{cc|cc|cc|cc}
0^{\mathcal{W}} & p^{\mathcal{W}} & p+1^{\mathcal{V}} & 1^{\mathcal{V}} & & & & \\
p^{\mathcal{W}} & 0^{\mathcal{V}} & 1^{\mathcal{V}} & P+1^{\mathcal{W}} & & & & \\
\hline
 & & 2^{\mathcal{W}} & p+2^{\mathcal{W}} & p+3^{\mathcal{V}} & 3^{\mathcal{V}} & & \\
 & & p+2^{\mathcal{W}} & 2^{\mathcal{V}} & 3^{\mathcal{V}} & p+3^{\mathcal{W}} & & \\
\hline
 & & & & & \ddots & & \ddots \\
\hline
2p-1^{\mathcal{V}} & p-1^{\mathcal{V}} & & & & & p-2^{\mathcal{W}} & 2p-2^{\mathcal{W}} \\
p-1^{\mathcal{V}} & 2p-1^{\mathcal{W}} & & & & & 2p-2^{\mathcal{W}} & p-2^{\mathcal{V}}
\end{array}\right).$$

Assume that $\mathcal{W}$ is labeled with the canonical projections π and τ. The triple $(r, c, e) \in T_0$ gives rise to the triples $((r, 0), (c, 0), e), ((r, 0), (c, 1), p + e), ((r, 1), (c, 0), p + e) \in \mathcal{W}$. Assume

π and τ take the following values:

$$\begin{aligned}
\pi((r,0),(c,0)) &= \rho_1 \\
\pi((r,0),(c,1)) &= \rho_2 \\
\pi((r,1),(c,0)) &= \rho_3 \\
\tau((r,0),(c,0)) &= \gamma_1 \\
\tau((r,0),(c,1)) &= \gamma_2 \\
\tau((r,1),(c,0)) &= \gamma_3
\end{aligned}.$$

From the way π and τ are defined, we know $\rho_2 = \rho_1 + 1$ and $\gamma_3 = \gamma_1 + 1$. We want to switch the values of π and τ for these triples while preserving condition (3) of Lemma III.5.4. To do so, we will simultaneously need to change the matrix Y. First, consider the elements $A_1 = [\rho_1 + \gamma_1 - x_{(r,0),(c,0)}]_q$ and $A_2 = [\rho_2 + \gamma_2 - x_{(r,0),(c,1)}]_q$. If we switch ρ_1 and ρ_2, then to ensure A_1 and A_2 cover the same elements of $\mathbb{Z}_q$, we also need to increase $x_{(r,0),(c,0)}$ by one and decrease $x_{(r,0),(c,1)}$ by one. This increases σ_e by one while decreasing σ_{p+e} by one. Such a switch is called a *$\mathcal{W}$-row switch on e*. Now consider $A_3 = [\rho_3 + \gamma_3 - x_{(r,1),(c,0)}]_q$. If we switch γ_1 and γ_3, then to ensure A_1 and A_3 cover the same elements of $\mathbb{Z}_q$, we also need to increase $x_{(r,0),(c,0)}$ by one and decrease $x_{(r,1),(c,0)}$ by one. This again increases σ_e by one while decreasing σ_{p+e} by one. Such a switch is called a *$\mathcal{W}$-column switch on e*.

Given any $e \in \mathbb{Z}_p$, we can perform just a $\mathcal{W}$-row switch on e or we can independently perform both a $\mathcal{W}$-row switch and a $\mathcal{W}$-column switch on e. Thus, we can increase σ_e by either one or two, respectively, while decreasing σ_{p+e} by that same amount. We want to define similar switches for the triples in $\mathcal{V}$ that are obtained from T_1; however, if we want the switches to affect σ_e and σ_{p+e} in the same way, we need to assume that $\mathcal{V}$ is initially labeled with the reverse canonical projections. It is not hard to show that $\mathcal{V}$-row and $\mathcal{V}$-column switches on e also increase σ_e by one or two while decreasing σ_{p+e} by that same amount. Thus, we can increase σ_e by up to four while decreasing σ_{p+e} by that same amount without affecting the transversals that $\mathcal{W}$ and $\mathcal{V}$ yield through Corollary III.5.5.

The following lemma supplies a special case in the proof of Theorem III.6.3 below, while also providing an example of both Lemma III.6.1 and switches.

Lemma III.6.2. *There exists a ce-hamiltonian latin square of order 18 that admits a 1-partition.*

Proof. A matrix X' guaranteed by Lemma III.6.1 so that $K_6 = \mathbb{Z}_3 \boxdot_{X'} \mathbb{Z}_2$ is ce-hamiltonian and can be partitioned into 2 disjoint 3-plexes is shown below, along with K_6. The starred entries in K_6 provide the 3-plex $\mathcal{W}$, while the unstarred entries provide the other 3-plex $\mathcal{V}$.

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, K_6 = \left(\begin{array}{cc|cc|cc} \mathit{0}^* & \mathit{3}^* & 4 & 1 & \mathit{2}^* & 5 \\ \mathit{3}^* & 0 & 1 & \mathit{4}^* & 5 & \mathit{2}^* \\ \hline 4 & \mathit{1}^* & \mathit{2}^* & \mathit{5}^* & 3 & 0 \\ \mathit{1}^* & 4 & \mathit{5}^* & 2 & 0 & \mathit{3}^* \\ \hline 5 & 2 & \mathit{0}^* & 3 & \mathit{1}^* & \mathit{4}^* \\ 2 & \mathit{5}^* & 3 & \mathit{0}^* & \mathit{4}^* & 1 \end{array}\right).$$

Let $\pi_{\mathcal{W}}$ and $\tau_{\mathcal{W}}$ be the canonical projections for $\mathcal{W}$, and let $\pi_{\mathcal{V}}$ and $\tau_{\mathcal{V}}$ be the reverse canonical projections for $\mathcal{V}$; a matrix X guaranteed by Theorem III.5.8 so that $K_6 \boxdot_X \mathbb{Z}_3$ has a 1-partition is shown below.

$$X = \left(\begin{array}{cc|cc|cc} 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 2 & 2 \end{array}\right)$$

At this point, we have $\nu(X, L) = (1, 1, 1, 2, 2, 2)$. This does not satisfy the conditions of Theorem III.4.3, so we perform the following $\mathcal{W}$-row switch on 1. The superscript on an entry $(r, c, e) \in K_6$ below is given by $\pi_{\mathcal{W}}(r, c), \tau_{\mathcal{W}}(r, c)$; the resulting changes in X are also shown, denoted by $*$.

$$K_6 : \left(\begin{array}{cc|cc|cc} 0 & 3 & 4 & 1 & 2 & 5 \\ 3 & 0 & 1 & 4 & 5 & 2 \\ \hline 4 & 1 & 2 & 5 & 3 & 0 \\ 1 & 4 & 5 & 2 & 0 & 3 \\ \hline 5 & 2 & 0 & 3 & \mathit{1}^{1,1} & \mathit{4}^{2,2} \\ 2 & 5 & 3 & 0 & 4 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|cc|cc} 0 & 3 & 4 & 1 & 2 & 5 \\ 3 & 0 & 1 & 4 & 5 & 2 \\ \hline 4 & 1 & 2 & 5 & 3 & 0 \\ 1 & 4 & 5 & 2 & 0 & 3 \\ \hline 5 & 2 & 0 & 3 & \mathit{1}^{2,1} & \mathit{4}^{1,2} \\ 2 & 5 & 3 & 0 & 4 & 1 \end{array}\right),$$

$$X : \left(\begin{array}{cc|cc|cc} 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathit{2}^{*} & \mathit{1}^{*} \\ 0 & 1 & 0 & 2 & 2 & 2 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|cc|cc} 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathit{0}^{*} & \mathit{0}^{*} \\ 0 & 1 & 0 & 2 & 2 & 2 \end{array}\right).$$

These switches increased σ_1 by one while decreasing σ_4 by one; we now have $\nu(X, L) = (1, 2, 1, 2, 1, 2)$. This satisfies the conditions of Theorem III.4.3, so the square $K_6 \boxdot_X \mathbb{Z}_3$ is ce-hamiltonian and has a 1-partition. □

We are now ready to provide the main construction for most values of n such that $n \equiv 2 \pmod 4$. The remaining cases of this residual class are covered in Section III.8.

Theorem III.6.3. *Let p and q be odd integers with q prime and $p \geq q \geq 3$. There exists a $2p \times 2p$ matrix X such that $K_{2p} \boxdot_X \mathbb{Z}_q$ is a ce-hamiltonian latin square that admits a 1-partition.*

Proof. Lemma III.6.1 gives us a partition of K_{2p} into $2(p - q)$ transversals and 2 q-plexes $\mathcal{W}$ and $\mathcal{V}$; we can assume $\mathcal{W}$ is labeled with the canonical projections and $\mathcal{V}$ is labeled with the inverse of the canonical projections. By Theorem III.5.8, there exists a $2p \times 2p$ matrix

X such that $K_{2p} \boxdot_X \mathbb{Z}_q$ has a 1-partition. We need to show that this X can be altered to meet the conditions of Theorem III.4.3.

If $p > q$, then K_{2p} contains at least one transversal, call it T. Since q is prime, $\gcd(y, q) = 1$ for all $0 \neq y \in \mathbb{Z}_q$. Moreover, since $q \geq 3$, there exists some $\delta \in \mathbb{Z}_q$ such that $1 - \delta$ and $\delta - 1 - 2p$ are both nonzero. Since Lemma III.5.3 works for any X, and Theorem III.5.8 allows arbitrary values in entries of X corresponding to transversals in K_{2p}, we can assign arbitrary values to $x_{r,c}$ when $(r, c, e) \in T$. Enumerate $T = \{(r_0, c_0, 0), ..., (r_{2p-1}, c_{2p-1}, 2p - 1)\}$; just as in the proof of Lemma III.6.1, both x_{r_i,c_i} and σ_i are integers modulo q, so we simply assign values to x_{r_i,c_i} so that $\sigma_i = 1$ if i is even, $\sigma_i = 0$ if $i = 1, 3, ..., 2p - 3$, and $\sigma_{2p-1} = \delta$. Thus, for $i = 0, 2, ..., 2p - 4$ we have $\sigma_i - \sigma_{i+1} \equiv 1 \pmod q$, for $i = 1, 3, ..., 2p - 3$ we have $\sigma_i - \sigma_{i+1} \equiv -1 \pmod q$, $\sigma_{2p-2} - \sigma_{2p-1} = 1 - \delta$, and $\sigma_{2p-1} - \sigma_0 - 2p = \delta - 1 - 2p$. Since $-1, 1, 1 - \delta$ and $\delta - 1 - 2p$ are all relatively prime to q, the conditions of Theorem III.4.3 are satisfied.

Now assume $p = q \geq 5$, and for $e = 1, ..., p - 2$ define the following expressions, where all arithmetic is done modulo q:

$$\begin{aligned}
\varepsilon_1(e) &= \sigma_{e-1} - \sigma_e \\
\varepsilon_2(e) &= \sigma_e - \sigma_{e+1} \\
\varepsilon_3(e) &= \sigma_{p+e-1} - \sigma_{p+e} \\
\varepsilon_4(e) &= \sigma_{p+e} - \sigma_{p+e+1}
\end{aligned}.$$

If $e = 0$, then simply replace $\varepsilon_1(e)$ with $\sigma_{2p-1} - \sigma_0 - 2p$, while if $e = p - 1$, then simply replace $\varepsilon_4(e)$ with $\sigma_{2p-1} - \sigma_0 - 2p$.

Set $z_1 = -\varepsilon_1(e)$, $z_2 = \varepsilon_2(e)$, $z_3 = \varepsilon_3(3)$ and $z_4 = -\varepsilon_4(e)$. Let $I(e) = \{i \mid \varepsilon_i(e) \neq 0$; if $|I(e)| = 4$ then $\varepsilon_i(e)$ is relatively prime to q for all i. If $|I(e)| < 4$, then let $Z_I(e) = \{z_i \mid i \in I(e)\}$; there must exist some number $s \in \{1, 2, 3, 4\} \setminus Z_I(e)$. By definition of $Z_I(e)$, it must be true that $\varepsilon_1(e) + s$, $\varepsilon_2(e) - s$, $\varepsilon_3(e) - s$, and $\varepsilon_4(e) + s$ are all nonzero, and therefore relatively prime to q. Now, we simply perform enough switches so that σ_e is increased by s and σ_{p+e}

is decreased by s. Since switching on e in this way will never make any $\varepsilon_i(e') = 0$ for any other $e' \in \mathbb{Z}_p$, we can repeat this process for all $e \in \mathbb{Z}_p$ until $\nu(X, L)$ satisfies the conditions of Theorem III.4.3.

The final case $p = q = 3$ is covered by Lemma III.6.2. $\square$

III.7 Construction for $n \equiv 0 \pmod 4$

In the case when $n \equiv 0 \pmod 4$, we will use $L = \mathbb{Z}_{\frac{n}{2}}$ and $M = \mathbb{Z}_2$. Before we provide the general constructions, we modify the switching procedure described for K_{2p} to work on the square $\mathbb{Z}_{\frac{n}{2}}$. In the 2-partition guaranteed by Lemma III.1.2, every row r of each 2-plex S contains the entries (r, c, e) and $(r, c+1, e+1)$, where $e = r + c$ is even. Let π be defined so that $\pi(r, c) = 0$ and $\pi(r, c+1) = 1$ for each row r; this is equivalent to the canonical row projection except when $c = \frac{n}{2} - 1$. We again want to switch the values of π for these triples without changing the entries they yield in $S \boxdot_X \mathbb{Z}_2$. As before, this switch results in increasing $x_{r,c}$ by one and decreasing $x_{r,c+1}$ by one; in $\mathbb{Z}_2$ this is equivalent to flipping the values σ_e and σ_{e+1} from 0 to 1 or from 1 to 0. This will similarly be called a *S-row switch on e*. In an analogous fashion, we note that every column of S contains the entries $(r, c, e-1)$ and $(r+1, c, e)$, where again $e = r + c + 1$ is even. Let τ be defined so that $\tau(r, c) = 0$ and $\tau(r+1, c) = 1$ for each column c; this is equivalent to the canonical column projection except when $r = \frac{n}{2} - 1$. By switching the τ values of these triples, we flip the values σ_{e-1} and σ_e in $\mathbb{Z}_2$; this is a *S-column switch on e*. By performing both a S-row switch on e and a S-column switch on e, we leave the value of σ_e unchanged while flipping both σ_{e-1} and σ_{e+1} in $\mathbb{Z}_2$. The following lemma will be needed.

Lemma III.7.1. *Let p be even, and let S be a 2-plex in $\mathbb{Z}_p$. If π, τ and X satisfy the conditions of Lemma III.5.4, then the set $\{x_{r,c} \mid (r, c, e) \in S, x_{r,c} = 1\}$ contains an even number of elements.*

Proof. For each entry $e \in Z_p$, let $(r_1, c_1, e), (r_2, c_2, e) \in S$ be the two triples of S containing

e; condition (3) of Lemma III.5.4 implies that we must have

$$\{\pi(r_1, c_1) + \tau(r_1, c_1) - x_{r_1,c_1}, \pi(r_2, c_2) + \tau(r_2, c_2) - x_{r_2,c_2}\} = \{0, 1\}.$$

Thus, since p is even,

$$\sum_{(r,c,e)\in S} (\pi(r, c) + \tau(r, c) - x_{r,c}) = 0,$$

where the addition is taken in $\mathbb{Z}_2$. Again since p is even,

$$\sum_{(r,c,e)\in S} \pi(r, c) = \sum_{(r,c,e)\in S} \tau(r, c) = 0$$

in $\mathbb{Z}_2$, so we must have

$$\sum_{(r,c,e)\in S} x_{r,c} = 0$$

as well. It follows that the set $\{x_{r,c} \mid (r, c, e) \in S, x_{r,c} = 1\}$ has even order. $\square$

We are now ready to prove the main construction for $n \equiv 0 \pmod 8$.

Theorem III.7.2. *Let $n \equiv 0$ (mod 8). There exists an $\frac{n}{2} \times \frac{n}{2}$ matrix X such that $\mathbb{Z}_{\frac{n}{2}} \boxdot_X \mathbb{Z}_2$ is a ce-hamiltonian latin square that admits a 1-partition.*

Proof. Set $b = \frac{n}{4}$; note that b is even. By Lemma III.1.2 there exists a 2-partition $\mathcal{S}$ of $\mathbb{Z}_{2b}$. For each 2-plex we use π and τ as defined at the beginning of this section; by Lemma III.5.9 there exists a $2b \times 2b$ matrix X such that $\mathbb{Z}_{2b} \boxdot_X \mathbb{Z}_2$ has a 1-partition. We need to show that this X can be modified to meet the conditions of Theorem III.4.3. According to Lemma III.7.1, each 2-plex $S \in \mathcal{S}$ has an even number of corresponding 1's in X. Thus, if $\nu(X, \mathbb{Z}_{2b}) = (\sigma_0, \sigma_1, ..., \sigma_{2b-1})$, then

$$\sum_{i=0}^{2b-1} \sigma_i = \sum_{x_{r,c}\in X} x_{r,c} = \sum_{S\in\mathcal{S}} \left(\sum_{(r,c,e)\in S} x_{r,c} \right) = 0,$$

and the representative vector $\nu(X, \mathbb{Z}_{2b})$ has an even number of 1's. Let $v(i)$ be the vector of length $2b$ with 1's in position i and $i+1$ and 0's everywhere else; the set $\{v(i) \mid 0 \leq i \leq 2b-2\}$ forms a basis over $\mathbb{Z}_2$ for all length $2b$ vectors with an even number of 1's. Furthermore, any S-row or S-column switch on e will simply add $v(e)$ or $v(e-1)$, respectively, to $\nu(X, \mathbb{Z}_{2b})$. Since b is even, the vector $(0,1,0,1,...,0,1)$ contains an even number of 1's; we now perform the required S-row and S-column switches to ensure that $\nu(X, \mathbb{Z}_{2b}) = (0,1,0,1,...,0,1)$. It is now clear that $\nu(X, \mathbb{Z}_{2b})$ meets the conditions of Theorem III.4.3. $\square$

When $n \equiv 4 \pmod 8$, we require a special construction. Again set $b = \frac{n}{4}$, where b is now odd, and let X be the $2b \times 2b$ matrix with $x_{2b-1,c} = 1$ for all even c, and $x_{r,c} = 0$ otherwise. We form the square $J_n = \mathbb{Z}_{2b} \boxdot_X \mathbb{Z}_2$ and show that it has the required properties.

Theorem III.7.3. *Let $n \equiv 4$ (mod 8) with $n \geq 12$. The latin square J_n is ce-hamiltonian and admits a 1-partition.*

Proof. It is readily seen that $\nu(X, \mathbb{Z}_{2b}) = (0,1,0,1,...,0,1)$; thus, J_n is ce-hamiltonian by Theorem III.4.3. It remains to show that J_n admits a 1-partition.

For a 2-plex S in $\mathbb{Z}_{2b}$ and any maps π and τ, we say $(r,c,e) \in S$ has a *uniform label* if $\pi(r,c) = \tau(r,c)$; otherwise we say $(r,c,e) \in S$ has a *mixed label.* Let S be the following 2-plex in $\mathbb{Z}_{2b}$, where the superscript on each entry (r,c,e) defines $\pi(r,c), \tau(r,c)$.

$$S = \left(\begin{array}{cc|ccc|ccc}
0^{0,0} & 1^{1,1} & & & & & & \\ \hline
 & & 3^{0,0} & & & (b+2)^{1,0} & & \\
 & & & \ddots & & & \ddots & \\
 & & & & (2b-1)^{0,0} & & & (b-2)^{1,0} \\ \hline
b^{0,1} & (b+1)^{1,0} & & & & & & \\ \hline
 & & (b+3)^{0,1} & & & 2^{1,1} & & \\
 & & & \ddots & & & \ddots & \\
 & & & & (b-1)^{0,1} & & & (2b-2)^{1,1}
\end{array}\right)$$

Set $T_{g,h} = T(S, \pi + g, \tau + h, X)$; we will show that the conditions of Lemma III.5.4 are met. It is readily seen from their definition that π takes both values 0 and 1 in every row of S and τ takes both values 0 and 1 in every column of S, so conditions (1) and (2) are met for any $g, h \in \mathbb{Z}_2$. Finally, $x_{r,c} = 0$ for all $(r, c, e) \in S$, and each entry in S has one uniform label and one mixed label, so $\Psi(\pi + g, \tau + h, e) = \{0, 1\}$ for each entry $e \in S$, and condition (3) is met for any $g, h \in \mathbb{Z}_2$ as well. By Lemma III.5.4, $T_{0,0}$ is a transversal in J_n. As in the proof of Corollary III.5.5, $T_{0,1}$ is also a transversal in J_n. Additionally, let $S + y$ be the 2-plex with the same labels as S that is obtained by shifting the cells of S y columns to the right, and let $T_{g,h} + y = T(S + y, \pi + g, \tau + h, X)$. If y is even, then the entries in row $2b - 1$ of $S + y$ are all even, so the corresponding entries in X are 0 and it is readily seen that $T_{0,0} + y$ and $T_{0,1} + y$ are also transversals. We claim the collection $\mathcal{S} = \{T_{0,0}, T_{0,0} + 2, .., T_{0,0} + (2b-2), T_{0,1}, T_{0,1} + 2, ..., T_{0,1} + (2b-2)\}$ forms $2b$ mutually disjoint transversals in J_n. Indeed, the 2-plexes $S + y$ and $S + z$ are disjoint unless $z = y + b - 1$ (or $y = z + b - 1$). In that case, the 2-plexes $S + y$ and $S + (y + b - 1)$ overlap in every row except row 0 and b. However, for every overlapped entry, the transversals $T_{0,0} + y$ and $T_{0,1} + y$ use the entries in row 0 of the corresponding subsquare $\mathbb{Z}_2$, while the transversals $T_{0,0} + (y + b - 1)$ and $T_{0,1} + (y + b - 1)$ use the entries in row 1 of the corresponding subsquare $\mathbb{Z}_2$. Thus, the members of $\mathcal{S}$ are mutually disjoint, and $|\mathcal{S}| = 2b$.

We now present a similar 2-plex that we can use to cover the remaining entries in $\mathbb{Z}_{2b}$. Let S' be the following 2-plex, with superscripts again defining π, τ.

$$
S' = \left(\begin{array}{c|cc|cccc|ccc}
 & 1^{0,0} & 2^{1,1} & & & & & & & \\ \hline
 & & & 4^{0,0} & & & & (b+3)^{1,0} & & \\
 & & & & \ddots & & & & \ddots & \\
 & & & & & (2b-2)^{0,0} & & & & (b-3)^{1,0} \\
(b-1)^{1,0} & & & & & & 0^{0,0} & & & \\ \hline
 & (b+1)^{0,1} & (b+2)^{1,0} & & & & & & & \\ \hline
 & & & (b+4)^{0,1} & & & & 3^{1,1} & & \\
 & & & & \ddots & & & & \ddots & \\
 & & & & & (b-2)^{0,1} & & & & (2b-3)^{1,1} \\ \hline
(2b-1)^{0,1} & & & & & & b^{1,1} & & &
\end{array}\right)
$$

Note that b has two uniform labels and $2b-1$ has two mixed labels; these correspond to nonzero entries in the last row of X. Every other entry has both a uniform and a mixed label, and the corresponding entries in X are 0. Lemma III.5.4 again implies that $T' = T(S', \pi, \tau, X)$ is a transversal. By defining $S' + y$ and $T'_{g,h} + y$ in a manner similar to $S + y$ and $T_{g,h} + y$ in the previous paragraph and using an analogous argument, we learn that $\mathcal{S} = \{T'_{0,0}, T'_{0,0} + 2, .., T'_{0,0} + (2b-2), T'_{0,1}, T'_{0,1} + 2, ..., T'_{0,1} + (2b-2)\}$ forms $2b$ mutually disjoint transversals in J_n.

It remains to show that the transversals in $\mathcal{S}$ are disjoint from the transversals in $\mathcal{S}'$. For even y and z, it is clear that each 2-plex $S + y$ is disjoint from each 2-plex $S' + z$ in every row except row 0 or b, because $S + y$ and $S' + z$ use columns of different parity in all rows except 0 and b. But for any even overlapped entry in row 0, $T_{0,0} + y$ and $T_{0,1} + y$ use the entries found in row 0 of the corresponding subsquare $\mathbb{Z}_2$, while $T'_{0,0} + z$ and $T'_{0,1} + z$ use the entries found in row 1 of the corresponding subsquare $\mathbb{Z}_2$. In a similar fashion for any odd overlapped entry in row 0, $T_{0,0} + y$ and $T_{0,1} + y$ use the entries found in row 1 of the corresponding subsquare $\mathbb{Z}_2$, while $T'_{0,0} + z$ and $T'_{0,1} + z$ use the entries found in row 0 of the corresponding subsquare $\mathbb{Z}_2$. It is similarly shown that $T_{0,0} + y$, $T_{0,1} + y$, $T'_{0,0} + z$ and $T'_{0,1} + z$ are disjoint in row b. The collection $\mathcal{S} \cup \mathcal{S}'$ contains $4b = n$ mutually disjoint transversals and is the desired 1-partition of J_n. $\square$

An example of the process used to find the 1-partition of J_{12} guaranteed by Theorem III.7.3 can be found in Appendix A.

III.8 Voltage graph constructions

The latin square constructions presented in this chapter yield hamilton cycle embeddings of $K_{n,n,n}$ for all $n \neq 2p$, where $p = 1$ or p is prime. A simple exhaustive search shows that every hamilton cycle embedding of $K_{2,2,2}$ must be nonorientable; in this section we present a voltage graph construction that covers all the remaining open cases.

As mentioned in Section I.2.3, the voltage group will be an abelian group of order n, and all edges are assumed to be directed from a to b, from b to c, and from c to a. We will use i_a, i_b and i_c to denote the edge with voltage i from a to b, from b to c and from c to a, respectively. Additionally, we will use $\overline{e}$ to denote that e is traced in the reverse direction. We do this to keep track of the directions in which each edge is traced, which will allow us to verify that the embeddings we construct are orientable. The following lemma will simplify the proofs in this section.

Lemma III.8.1. *Let $W_1 = (i_a\ j_b\ k_c)$ and $W_2 = (\overline{p_c}\ \overline{q_b}\ \overline{r_a})$ be closed walks in a voltage graph for $K_{n,n,n}$ with voltage group $\mathbb{Z}_n$. If $\gcd(i+j+k, n) = 1$ (resp. $\gcd(-p-q-r, n) = 1$), then W_1 (resp. W_2) yields a single hamilton cycle face in the derived embedding.*

Proof. Theorem I.2.3 implies that both W_1 and W_2 yield a single face of length $3n$ in the derived embedding. We must show that these faces are actually hamilton cycles. The resulting faces are shown below. For convenience, we set $\beta = i+j+k$ and $\gamma = p+q+r$.

$$W_1: \quad (a_0\ b_i\ c_{i+j}\ a_\beta\ b_{i+\beta}\ c_{i+j+\beta}\ a_{2\beta}\ b_{i+2\beta}\ c_{i+j+2\beta} \cdots a_{(n-1)\beta}\ b_{i+(n-1)\beta}\ c_{i+j+(n-1)\beta})$$

$$W_2: \quad (a_0\ c_{-p}\ b_{-p-q}\ a_{-\gamma}\ c_{-p-\gamma}\ b_{-p-q-\gamma}\ a_{-2\gamma}\ c_{-p-2\gamma}\ b_{-p-q-2\gamma}\ \cdots a_{-(n-1)\gamma}\ c_{-p-(n-1)\gamma}\ b_{-p-q-(n-1)\gamma})$$

Because β and γ are both of order n in $\mathbb{Z}_n$, these are hamilton cycles. $\square$

We begin by presenting some special case constructions for $p = 2$ and $p = 3$.

Lemma III.8.2. *For $p = 2$ or 3, there exists a voltage graph such that the derived embedding is an orientable hamilton cycle embedding of $K_{2p,2p,2p}$.*

Proof. For $p = 2$, consider the voltage graph with the following rotation system.

$$\begin{aligned} R_a :&\quad (0_a\ 1_a\ 2_a\ 3_a\ 0_c\ 3_c\ 2_c\ 1_c) \\ R_b :&\quad (0_a\ 0_b\ 3_a\ 2_b\ 2_a\ 1_b\ 1_a\ 3_b) \\ R_c :&\quad (0_c\ 0_b\ 1_c\ 1_b\ 2_c\ 3_b\ 3_c\ 2_b) \end{aligned}$$

A trace of the faces in this voltage graph yields the closed walks and net voltages below.

$$\begin{aligned} W_0 :&\quad (0_a\ 0_b\ 1_c); |W_0| = 1 \\ W_1 :&\quad (1_a\ 3_b\ 3_c\ \overline{2_c}\ \overline{3_b}\ \overline{0_a}); |W_1| = 2 \\ W_2 :&\quad (2_a\ 1_b\ 2_c\ \overline{1_c}\ \overline{1_b}\ \overline{1_a}); |W_2| = 2 \\ W_3 :&\quad (3_a\ 2_b\ 0_c\ \overline{3_c}\ \overline{2_b}\ \overline{2_a}); |W_3| = 2 \\ W_4 :&\quad (\overline{0_c}\ \overline{0_b}\ \overline{3_a}); |W_4| = 1 \end{aligned}$$

We know W_0 and W_4 each yield a hamilton cycle face from Lemma III.8.1. From Theorem I.2.3 we know W_i yields 2 faces of length 12 for $i = 1, 2, 3$. To prove they are indeed hamilton cycles, it will suffice to show that one of the faces obtained from each closed walk is a hamilton cycle. This is accomplished below.

$$\begin{aligned} W_1 :&\quad (a_0\ b_1\ c_0\ a_3\ c_1\ b_2\ a_2\ b_3\ c_2\ a_1\ c_3\ b_0) \\ W_2 :&\quad (a_0\ b_2\ c_3\ a_1\ c_0\ b_3\ a_2\ b_0\ c_1\ a_3\ c_2\ b_1) \\ W_3 :&\quad (a_0\ b_3\ c_1\ a_1\ c_2\ b_0\ a_2\ b_1\ c_3\ a_3\ c_0\ b_2) \end{aligned}$$

Thus, the embedding derived from this voltage graph is an orientable hamilton cycle embedding of $K_{4,4,4}$.

For $p = 3$, consider the voltage graph with the following rotation scheme.

$$\begin{aligned} R_a: &\quad (0_a\ 1_c\ 1_a\ 2_c\ 2_a\ 5_c\ 4_c\ 4_a\ 0_c\ 3_a\ 3_c\ 5_a) \\ R_b: &\quad (0_a\ 2_b\ 1_a\ 3_b\ 4_a\ 5_b\ 3_a\ 4_b\ 2_a\ 1_b\ 5_a\ 0_b) \\ R_c: &\quad (0_b\ 5_c\ 1_b\ 2_c\ 4_b\ 0_c\ 3_b\ 3_c\ 5_b\ 4_c\ 2_b\ 1_c) \end{aligned}$$

A trace of the faces in this voltage graph yields the closed walks and net voltages below.

$$\begin{aligned} W_0: &\quad (0_a\ 2_b\ 1_c\ 1_a\ 3_b\ 3_c\ 5_a\ 0_b\ 5_c\ \overline{4_c}\ \overline{2_b}\ \overline{1_a}\ \overline{2_c}\ \overline{4_b}\ \overline{2_a}\ \overline{5_c}\ \overline{1_b}\ \overline{5_a}); |W_0| = 0 \\ W_1: &\quad (2_a\ 1_b\ 2_c); |W_1| = 5 \\ W_2: &\quad (3_a\ 4_b\ 0_c); |W_2| = 1 \\ W_3: &\quad (4_a\ 5_b\ 4_c); |W_3| = 1 \\ W_4: &\quad (\overline{0_c}\ \overline{3_b}\ \overline{4_a}); |W_4| = 5 \\ W_5: &\quad (\overline{1_c}\ \overline{0_b}\ \overline{0_a}); |W_5| = 5 \\ W_6: &\quad (\overline{3_c}\ \overline{5_b}\ \overline{3_a}); |W_6| = 1 \end{aligned}$$

From Lemma III.8.1, W_i yields a single hamilton cycle face for $i = 1, 2, ..., 6$. From Theorem I.2.3 we know W_0 yields 6 faces of length 18. To prove that they are indeed hamilton cycles, it will suffice to show that one of the faces obtained from this closed walk is a hamilton cycle. This is accomplished below.

$$W_0: \quad (a_0\ b_0\ c_2\ a_3\ b_4\ c_1\ a_4\ b_3\ c_3\ a_2\ c_4\ b_2\ a_1\ c_5\ b_1\ a_5\ c_0\ b_5)$$

Thus, the embedding derived from this voltage graph is an orientable hamilton cycle embedding of $K_{6,6,6}$ □

We are now going to give a general construction for $n = 2p$, where $p \geq 5$ is prime. This voltage graph will be constructed in several steps. To start out, we will present the closed walks we want to be facial boundaries in our voltage graph. Then, we will show that these walks yield hamilton cycles in the derived embedding. Finally, we will verify our voltage graph is well-defined by showing that the rotation graph around every vertex is proper. The voltage group we will be using for these graphs is $\mathbb{Z}_p \times \mathbb{Z}_2$; this group is isomorphic to $\mathbb{Z}_{2p}$ but is preferred for notational convenience. For the remainder of this section, we simply write x for $(x, 0)$ and x' for $(x, 1)$.

Definition III.8.3. Let $p \geq 5$ be prime, and define the sequences $\omega_i = i_a\ (i+3)_b\ (p-2i-2)_c$ and $\theta_i = \overline{(p-2i)_c}\ \overline{(i-1)_b}\ \overline{i_a}$. Define Ω to be the following closed walk.

$$\begin{aligned}\Omega:\ & (1'_a\ (p-1)'_b\ 0'_c\ 0'_a\ 3_b\ (p-2)_c\ \omega_1\ \omega_2\ \cdots\ \omega_{p-3}\ \omega_{p-2} \\ & \overline{(p-1)'_c}\ \overline{2'_b}\ \overline{(p-3)'_a}\ \theta_1\ \theta_2\ \cdots\ \theta_{p-3}\ \theta_{p-2}\ \overline{2_c}\ \overline{(p-2)_b}\ \overline{(p-1)'_a})\end{aligned}$$

Lemma III.8.4. *For all prime $p \geq 5$, Ω yields $2p$ hamilton cycle faces in the derived embedding of $K_{2p,2p,2p}$.*

Proof. It will suffice to show that one of the resulting faces in the derived embedding is a hamilton cycle. Starting with the vertex a_0, we obtain the following facial boundary in the embedding of $K_{2p,2p,2p}$.

$$\begin{aligned}&(a_0\ b_{1'}\ c_0\ a_{0'}\ b_0\ c_3\ a_1\ b_2\ c_6\ a_2\ b_4\ c_9\ a_3\ b_6\ c_{12} \cdots \\ &a_{(p-4)}\ b_{(p-8)}\ c_{(p-9)}\ a_{(p-3)}\ b_{(p-6)}\ c_{(p-6)}\ a_{(p-2)}\ b_{(p-4)}\ c_{(p-3)} \\ &a_{(p-1)}\ c_{0'}\ b_{(p-2)}\ a_{1'}\ c_{3'}\ b_{3'}\ a_{2'}\ c_{6'}\ b_{5'}\ a_{3'}\ c_{9'}\ b_{7'} \cdots \\ &a_{(p-3)'}\ c_{(p-9)'}\ b_{(p-5)'}\ a_{(p-2)'}\ c_{(p-6)'}\ b_{(p-3)'}\ a_{(p-1)'}\ c_{(p-3)'}\ b_{(p-1)'})\end{aligned}$$

For the sake of clarity, we list the vertices below by the order in which they appear within each independent set. Note that the net voltages of ω_i and θ_i are both 1, the net voltages of the sequences $(i+3)_b\ (p-2i-2)_c\ (i+1)_a$ and $\overline{i_a}\ \overline{(p-2i-2)_c}\ \overline{i_b}$ are both 2, and the net voltages of the sequences $(p-2i-2)_c\ (i+1)_a\ (i+4)_b$ and $\overline{(i-1)_b}\ \overline{i_a}\ \overline{(p-2i-2)_c}$ are both

3. This is evident in the following sequences.

$$\begin{aligned}
A:&\quad (a_0\ a_{0'}\ a_1\ a_2 \cdots a_{(p-2)}\ a_{(p-1)}\ a_{1'}\ a_{2'} \cdots a_{(p-2)'}\ a_{(p-1)'})\\
B:&\quad (b_{1'}\ b_0\ b_2\ b_4 \cdots b_{(p-4)}\ b_{(p-2)}\ b_{3'}\ b_{5'} \cdots b_{(p-3)'}\ b_{(p-1)'})\\
C:&\quad (c_0\ c_3\ c_6\ c_9 \cdots c_{(p-6)}\ c_{(p-3)}\ c_{0'}\ c_{3'} \cdots c_{(p-6)'} c_{(p-3)'})
\end{aligned}$$

This cycle is clearly a hamilton cycle. Since Ω was a walk of length $6p$, it must be true that $|\Omega| = 0$. From Theorem I.2.3, we know Ω yields $2p$ faces of length $6p$, each of which must be a hamilton cycle. $\square$

The closed walk Ω provides half of our desired voltage graph. Before we build the remaining half, we want to construct the partial rotations at each vertex in the voltage graph as determined by Ω. In the observation that follows, we use the notation $[a\ b\ c \cdots d]$ to denote a path in the corresponding rotation (i.e. a is not adjacent to d in the rotation graph).

Lemma III.8.5. *The partial rotations determined by Ω consist of the following paths with the given endpoints. Each path is labeled for reference later in this section.*

$$\begin{aligned}
a:&\quad P_1^A = [(p-3)'_a\ \cdots\ 1'_a]\\
&\quad P_3^A = [(p-1)'_a\ 1'_a]\\
&\quad P_5^A = [0'_c\ 0'_a]\\
b:&\quad P_1^B = [2_b\ \cdots\ (p-1)_b]\\
&\quad P_3^B = [2'_b\ (p-3)'_a]\\
&\quad P_5^B = [0'_a\ \cdots\ (p-1)'_a]\\
&\quad P_7^B = [1'_a\ (p-1)'_b]\\
c:&\quad P_1^C = [(p-1)_b\ \cdots\ 2_b]\\
&\quad P_3^C = [(p-1)'_c\ 2'_b]\\
&\quad P_5^C = [(p-1)'_b\ 0'_c]
\end{aligned}$$

Proof. Let $\Omega_1 = (\omega_0\ \omega_1 \cdots \omega_{p-1})$ and $\Omega_2 = (\theta_0\ \theta_1 \cdots \theta_{p-1})$. The rotation around a determined by the closed walks Ω_1 and Ω_2 is given by

$$Q_1 = (0_a\ (p-2)_c\ 1_a\ (p-4)_c\ 2_a\ (p-6)_c \cdots (p-2)_a\ 2_c\ (p-1)_a 0_c).$$

To construct Ω from Ω_1 and Ω_2, we must first remove the subsequence $\omega_{p-1}\ \omega_0$ from Ω_1 and the subsequence $\theta_{p-1}\ \theta_0$ from Ω_2. By doing so, we lose the subsequence $(p-2)_a\ 2_c\ (p-1)_a\ 0_c\ 0_a\ (p-2)_c\ 1_a$ from Q_1, which results in a partial rotation around a given by

$$Q_2 = [1_a\ (p-4)_c\ 2_a\ (p-6)_c \cdots (p-2)_a].$$

Finally, we add the sequences $\theta_{p-2}\ \overline{2_c}\ \overline{(p-2)_b}\ \overline{(p-1)'_a}\ 1'_a\ (p-1)'_b\ 0'_c\ 0'_a\ 3_b\ (p-2)_c\ \omega_1$ and $\omega_{p-2}\ \overline{(p-1)'_c}\ \overline{2'_b}\ \overline{(p-3)'_a}\ \theta_1$, which induce the following partial rotations around a.

$$\begin{aligned}
P_1^A &= [(p-3)'_a\ (p-2)_c\ 1_a]\ Q_2\ [(p-2)_a\ 2_c\ (p-1)'_c] \\
P_3^A &= [(p-1)'_a\ 1'_a] \\
P_5^A &= [0'_c\ 0'_a]
\end{aligned}$$

For the partial rotation around b determined by Ω, we again consider first the rotation around b determined by Ω_1 and Ω_2, which is given by

$$R_1 = (0_a\ 3_b\ 4_a\ 7_b\ 8_a\ 11_b \cdots (p-8)_a\ (p-5)_b\ (p-4)_a\ (p-1)_b).$$

Removing $\omega_{p-1}\ \omega_0$ and $\theta_{p-1}\ \theta_0$ results in a loss of the subsequences $(p-1)_b\ 0_a\ 3_b$ and $(p-2)_b\ (p-1)_a\ 2_b$ from R_1; this splits R_1 into the two partial rotations R_2 and R_3 shown below.

$$\begin{aligned}
R_2 &= [3_b\ 4_a\ 7_b\ 8_a\ 11_b \cdots (p-2)_b] \\
R_3 &= [2_b \cdots (p-8)_a\ (p-5)_b\ (p-4)_a\ (p-1)_b]
\end{aligned}$$

Finally, we add in the remaining pieces of Ω to obtain the following partial rotations around

b.

$$
\begin{aligned}
P_1^B &= R_3 \\
P_3^B &= [2'_b\ (p-3)'_a] \\
P_5^B &= [0'_a\ 3_b]\ R_2\ [(p-2)_b\ (p-1)'_a] \\
P_7^B &= [1'_a\ (p-1)'_b]
\end{aligned}
$$

Using a similar process on c, we get an initial rotation from Ω_1 and Ω_2 given by

$$S_1 = (0_c\ (p-1)_b\ 6_c\ (p-4)_b\ 12_c\ (p-7)_b \cdots (p-12)_c\ 5_b\ (p-6)_c\ 2_b).$$

Removing $\omega_{p-1}\ \omega_0$ and $\theta_{p-1}\ \theta_0$ results in a loss of the subsequences $2_b\ 0_c\ (p-1)_b$, $3_b\ (p-2)_c$ and $2_c\ (p-2)_b$ from S_1; this splits S_1 into three partial rotations. Note, however, that the subsequences $3_b\ (p-2)_c$ and $2_c\ (p-2)_b$ are included in the remaining pieces of Ω, so the removal of the subsequence $2_b\ 0_c\ (p-1)_b$ yields a partial rotation around c given by

$$S_2 = [(p-1)_b\ 6_c\ (p-4)_b\ 12_c\ (p-7)_b \cdots (p-12)_c\ 5_b\ (p-6)_c\ 2_b].$$

Adding in the unused subsequences from Ω results in the following partial rotations around c.

$$
\begin{aligned}
P_1^C &= S_2 \\
P_3^C &= [(p-1)'_c\ 2'_b] \\
P_5^C &= [(p-1)'_b\ 0'_c]
\end{aligned}
$$

□

We now progress to the 3-cycles that will complete our voltage graph. Because we want to use each edge once as e and once as $\overline{e}$, we present p 3-cycles of the form $(i_a\ j_b\ k_c)$ and p 3-cycles of the form $(\overline{i_c}\ \overline{j_b}\ \overline{k_a})$. Cycles of the first form are presented in Table III.1, while cycles of the second form are presented in Table III.2. In both tables, we let $h = \frac{p-1}{2}$.

Before the main theorem is proved, we again make an observation about the partial rotations determined by the Δ_i's and Λ_i's.

Cycle $(i_a\ j_b\ k_c)$	i	j	k	Net Voltage
Δ_0	0	$2'$	0	$2'$
Δ_1	$3'$	$1'$	$(p-3)'$	$1'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Δ_ℓ	$(2\ell+1)'$	$(2\ell-1)'$	$(p-2\ell-1)'$	$(2\ell-1)'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Δ_{h-1}	$(p-2)'$	$(p-4)'$	$2'$	$(p-4)'$
Δ_h	$p-1$	2	$3'$	$4'$
Δ_{h+1}	$2'$	$4'$	$(p-2)'$	$4'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Δ_ℓ	$(2\ell+1)'$	$(2\ell+3)'$	$(p-2\ell-1)'$	$(2\ell+3)'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Δ_{p-3}	$(p-5)'$	$(p-3)'$	$5'$	$(p-3)'$
Δ_{p-2}	$(p-3)'$	$(p-2)'$	$1'$	$(p-4)'$
Δ_{p-1}	$(p-1)'$	$0'$	$(p-1)'$	$(p-2)'$

Table III.1: Required 3-cycles of the form $\Delta = (i_a\ j_b\ k_c)$, where $h = \frac{p-1}{2}$.

Cycle $(\overline{i_c}\ \overline{j_b}\ \overline{k_a})$	i	j	k	Net Voltage
Λ_0	0	$(p-1)'$	$p-1$	$2'$
Λ_1	$1'$	$(p-3)'$	$0'$	$2'$
Λ_2	$3'$	$(p-5)'$	$(p-5)'$	$7'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Λ_ℓ	$(2\ell-1)'$	$(p-2\ell-1)'$	$(p-2\ell-1)'$	$(2\ell+3)'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Λ_{h-2}	$(p-6)'$	$4'$	$4'$	$(p-2)'$
Λ_{h-1}	$(p-4)'$	$(p-4)'$	$2'$	$6'$
Λ_h	$(p-2)'$	$(p-6)'$	$(p-2)'$	$10'$
Λ_{h+1}	$0'$	$p-1$	0	$1'$
Λ_{h+2}	$2'$	$(p-8)'$	$(p-4)'$	$10'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Λ_ℓ	$(2\ell-1)'$	$(p-2\ell-5)'$	$(p-2\ell-1)'$	$(2\ell+7)'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Λ_{p-3}	$(p-7)'$	$1'$	$5'$	$1'$
Λ_{p-2}	$(p-5)'$	$(p-2)'$	$3'$	$4'$
Λ_{p-1}	$(p-3)'$	$0'$	$1'$	$2'$

Table III.2: Required 3-cycles of the form $\Lambda = (\overline{i_c}\ \overline{j_b}\ \overline{k_a})$, where $h = \frac{p-1}{2}$.

Lemma III.8.6. *Let $p \geq 11$. The partial rotations determined by the Δ_i's and Λ_j's consist of the following paths with the given endpoints. Each path is again labeled for future reference.*

$$\begin{aligned}
a:\quad & P_2^A = [(p-1)'_c\ (p-1)'_a] \\
& P_4^A = [1'_a\ \cdots\ 0'_c] \\
& P_6^A = [0'_a\ 1'_c\ (p-3)'_a] \\
b:\quad & P_2^B = [(p-1)_b\ 0_a\ 2'_b] \\
& P_4^B = [(p-3)'_a\ \cdots\ 0'_a] \\
& P_6^B = [(p-1)'_a\ 0'_b\ 1'_a] \\
& P_8^B = [(p-1)'_b\ (p-1)_a\ 2_b] \\
c:\quad & P_2^C = [2_b\ \cdots\ (p-1)'_c] \\
& P_4^C = [2'_b\ 0_c\ (p-1)'_b] \\
& P_6^C = [0'_c\ (p-1)_b]
\end{aligned}$$

Proof. For the rotation around a, observe that the families $\{\Delta_\ell \mid 1 \leq \ell \leq h-1\}$ and $\{\Lambda_\ell \mid h+2 \leq \ell \leq p-3\}$ yield the partial rotations

$$\begin{aligned}
Q_1 &= [(p-5)'_c\ 5'_a\ (p-7)'_c\ 7'_a\ (p-9)'_c\ 9'_a \cdots 4'_c\ (p-4)'_a\ 2'_c\ (p-2)'_a] \\
Q_2 &= [(p-3)'_c\ 3'_a]
\end{aligned}$$

and the families $\{\Delta_\ell \mid h+1 \leq \ell \leq p-3\}$ and $\{\Lambda_\ell \mid 2 \leq \ell \leq h-2\}$ yield the partial rotations

$$\begin{aligned}
Q_3 &= [(p-4)'_c\ 4'_a\ (p-6)'_c\ 6'_a\ (p-8)'_c\ 8'_a \cdots (p-7)'_a\ 5'_c\ (p-5)'_a\ 3'_c] \\
Q_4 &= [(p-2)'_c\ 2'_a].
\end{aligned}$$

By considering the remaining 3-cycles – namely Δ_0, Δ_h, Δ_{p-2}, Δ_{p-1}, Λ_0, Λ_1, Λ_{h-1}, Λ_h, Λ_{h+1}, Λ_{p-2} and Λ_{p-1}, where $h = \frac{p-1}{2}$ – we learn that the partial rotations around a are the following.

$$
\begin{aligned}
P_2^A &= [(p-1)'_c \ (p-1)'_a] \\
P_4^A &= [1'_a \ (p-3)'_c] \ Q_2 \ [3'_a \ (p-5)'_c] \ Q_1 \ [(p-2)'_a \ (p-2)'_c] \ Q_4 \ [2'_a \ (p-4)'_c] \ Q_3 \ [3'_c \ (p-1)_a \ 0_c \ 0_a \ 0'_c] \\
P_6^A &= [0'_a \ 1'_c \ (p-3)'_a]
\end{aligned}
$$

For the rotation around b, observe that the families $\{\Delta_\ell \mid 1 \le \ell \le h-1\}$ and $\{\Lambda_\ell \mid h+2 \le \ell \le p-3\}$ yield the partial rotations

$$
\begin{aligned}
R_1 &= [3'_a \ 1'_b \ 5'_a \ 3'_b \ 7'_a \ 5'_b \cdots (p-6)'_a \ (p-8)'_b \ (p-4)'_a \ (p-6)'_b] \\
R_2 &= [(p-2)'_a \ (p-4)'_b]
\end{aligned}
$$

and the families $\{\Delta_\ell \mid h+1 \le \ell \le p-3\}$ and $\{\Lambda_\ell \mid 2 \le \ell \le h-2\}$ yield the partial rotation

$$
R_3 = [2'_a \ 4'_b \ 4'_a \ 6'_b \ 6'_a \ 8'_b \cdots (p-7)'_a \ (p-5)'_b \ (p-5)'_a \ (p-3)'_b].
$$

By considering the remaining Δ and Λ cycles, we learn that the partial rotations around b are the following.

$$
\begin{aligned}
P_2^B &= [(p-1)_b \ 0_a \ 2'_b] \\
P_4^B &= [(p-3)'_a \ (p-2)'_b \ 3'_a] \ R_1 \ [(p-6)'_b \ (p-2)'_a] \ R_2 \ [(p-4)'_b \ 2'_a] \ R_3 \ (p-3)'_b 0'_a] \\
P_6^B &= [(p-1)'_a \ 0'_b \ 1'_a] \\
P_8^B &= [(p-1)'_b \ (p-1)_a \ 2_b]
\end{aligned}
$$

For the rotation around c, we consider two cases. If $p \equiv 1 \pmod 4$, then h is even. Observe that the families $\{\Delta_\ell \mid 1 \le \ell \le h-1\}$ and $\{\Lambda_\ell \mid h+2 \le \ell \le p-3\}$ yield the partial rotations

$$
\begin{aligned}
S_1 &= [(p-4)'_b \ 2'_c \ (p-8)'_b \ 6'_c \ (p-12)'_b \ 10'_c \cdots 5'_b \ (p-7)'_c \ 1'_b \ (p-3)'_c] \\
S_2 &= [(p-6)'_b \ 4'_c \ (p-10)'_b \ 8'_c \ (p-14)'_b \ 12'_c \cdots 7'_b \ (p-9)_c \ 3'_b \ (p-5)'_c]
\end{aligned}
$$

and the families $\{\Delta_\ell \mid h+1 \le \ell \le p-3\}$ and $\{\Lambda_\ell \mid 2 \le \ell \le h-2\}$ yield the partial rotations

$$S_3 = [(p-3)'_b\ 5'_c\ (p-7)'_b\ 9'_c\ (p-11)'_b\ 13'_c \cdots 10'_b\ (p-8)'_c\ 6'_b\ (p-4)'_c]$$
$$S_4 = [3'_c\ (p-5)'_b\ 7'_c\ (p-9)'_b\ 11'_c\ (p-13)'_b \cdots 8'_b\ (p-6)'_c\ 4'_b\ (p-2)'_c].$$

By considering the remaining Δ and Λ cycles, we learn that the partial rotations around c are the following.

$$\begin{aligned}
P_2^C = \;& [2_b\ 3'_c]\ S_4\ [(p-2)'_c\ (p-6)'_b]\ S_2\ [(p-5)'_c\ (p-2)'_b\ 1'_c\ (p-3)'_b]\ S_3 \\
& [(p-4)'_c\ (p-4)'_b]\ S_1\ [(p-3)'_c\ 0'_b\ (p-1)'_c] \\
P_4^C = \;& [2'_b\ 0_c\ (p-1)'_b] \\
P_6^C = \;& [0'_c\ (p-1)_b]
\end{aligned}$$

On the other hand, if $p \equiv 3 \pmod 4$, then h is odd. Observe that the families $\{\Delta_\ell \mid 1 \le \ell \le h-1\}$ and $\{\Lambda_\ell \mid h+2 \le \ell \le p-3\}$ yield the partial rotations

$$S_1 = [(p-4)'_b\ 2'_c\ (p-8)'_b\ 6'_c\ (p-12)'_b\ 10'_c \cdots 7'_b\ (p-9)'_c\ 3'_b\ (p-5)'_c]$$
$$S_2 = [(p-6)'_b\ 4'_c\ (p-10)'_b\ 8'_c\ (p-14)'_b\ 12'_c \cdots 5'_b\ (p-7)'_c\ 1'_b\ (p-3)'_c]$$

and the families $\{\Delta_\ell \mid h+1 \le \ell \le p-3\}$ and $\{\Lambda_\ell \mid 2 \le \ell \le h-2\}$ yield the partial rotations

$$S_3 = [(p-3)'_b\ 5'_c\ (p-7)'_b\ 9'_c\ (p-11)'_b\ 13'_c \cdots 8'_b\ (p-6)'_c\ 4'_b\ (p-2)'_c]$$
$$S_4 = [3'_c\ (p-5)'_b\ 7'_c\ (p-9)'_b\ 11'_c\ (p-13)'_b \cdots 10'_b\ (p-8)'_c\ 6'_b\ (p-4)'_c].$$

By considering the remaining Δ and Λ cycles, we learn that the partial rotations around c are the following.

$$\begin{aligned}
P_2^C = \;& [2_b\ 3'_c]\ S_4\ [(p-4)'_c\ (p-4)'_b]\ S_1\ [(p-5)'_c\ (p-2)'_b\ 1'_c\ (p-3)'_b]\ S_3 \\
& [(p-2)'_c\ (p-6)'_b]\ S_2\ [(p-3)'_c\ 0'_b\ (p-1)'_c] \\
P_4^C = \;& [2'_b\ 0_c\ (p-1)'_b] \\
P_6^C = \;& [0'_c\ (p-1)_b]
\end{aligned}$$

□

By concatenating the paths representing the partial rotations given by Lemmas III.8.5 and III.8.6, we get the following cycles which, as we will see later, represent the complete rotation graphs around the vertices a, b and c.

Lemma III.8.7. *Let $p \geq 5$ be prime. The following are cycles of length $4p$.*

$$\begin{aligned} R_a &: (P_1^A\ P_2^A\ P_3^A\ P_4^A\ P_5^A\ P_6^A) \\ R_b &: (P_1^B\ P_2^B\ P_3^B\ P_4^B\ P_5^B\ P_6^B\ P_7^B\ P_8^B) \\ R_c &: (P_1^C\ P_2^C\ P_3^C\ P_4^C\ P_5^C\ P_6^C) \end{aligned}$$

Proof. By concatenating the corresponding paths, it is clear that R_a is a closed walk. Moreover, each of the $2p$ edges from a to b and each of the $2p$ edges from c to a appears either exactly once in the interior of one of the partial rotation paths, or appears as the endpoint of two different partial rotation paths. Therefore each edge appears exactly once in R_a, so R_a is a cycle of length $4p$. Similar arguments apply for both R_b and R_c. □

We are now able to construct hamilton cycle embeddings of $K_{n,n,n}$ whenever $n = 2p$ for a prime p.

Theorem III.8.8. *Let $p \geq 11$ be prime. The embedding given by the faces $\Omega, \Delta_0, ..., \Delta_{p-1}$, $\Lambda_0, ..., \Lambda_{p-1}$ is an orientable voltage graph embedding whose derived embedding is an orientable hamilton cycle embedding of $K_{2p,2p,2p}$.*

Proof. From the way the faces $\Omega, \Delta_0, ..., \Delta_{p-1}, \Lambda_0, ..., \Lambda_{p-1}$ were constructed, we know each edge is used once as e and once as $\overline{e}$; thus, the embedding given by these faces is orientable. Moreover, the rotation graphs that we obtain from these faces are given by Lemma III.8.7. Since R_a, R_b and R_c consist of a single cycle, our voltage graph is embedded in some orientable surface. It follows that the derived embedding is an orientable embedding of $K_{2p,2p,2p}$; thus, it remains to show that the boundary of every face is a hamilton cycle. From Lemma III.8.4 we

	Cycle $(i_a\ j_b\ k_c)$	i	j	k	Net Voltage	Cycle $(\overline{i_c}\ \overline{j_b}\ \overline{k_a})$	i	j	k	Net Voltage
	Δ_0	0	$2'$	0	$2'$	Λ_0	0	$4'$	4	$3'$
	Δ_1	$3'$	$1'$	$2'$	$1'$	Λ_1	$1'$	$1'$	$0'$	$2'$
$p=5$	Δ_2	4	2	$3'$	$4'$	Λ_2	$3'$	$3'$	$3'$	$4'$
	Δ_3	$2'$	$3'$	$1'$	$1'$	Λ_3	$0'$	4	0	$4'$
	Δ_4	$4'$	$0'$	$4'$	$3'$	Λ_4	$2'$	$0'$	$1'$	$3'$
	Δ_0	0	$2'$	0	$2'$	Λ_0	0	$6'$	6	$5'$
	Δ_1	$3'$	$1'$	$4'$	$1'$	Λ_1	$1'$	$4'$	$0'$	$5'$
	Δ_2	$5'$	$3'$	$2'$	$3'$	Λ_2	$3'$	$3'$	$2'$	$1'$
$p=7$	Δ_3	6	2	$3'$	$4'$	Λ_3	$5'$	$1'$	$5'$	$4'$
	Δ_4	$2'$	$4'$	$5'$	$4'$	Λ_4	$0'$	6	0	$6'$
	Δ_5	$4'$	$5'$	$1'$	$3'$	Λ_5	$2'$	$5'$	$3'$	$3'$
	Δ_6	$6'$	$0'$	$6'$	$5'$	Λ_6	$4'$	$0'$	$1'$	$5'$

Table III.3: Required 3-cycles for $p = 5$ and 7.

know Ω yields $2p$ hamilton cycles in the derived embedding. To show that all of the 3-cycles yield hamilton cycles, we use the isomorphism from $\mathbb{Z}_p \times \mathbb{Z}_2$ to $\mathbb{Z}_{2p}$ induced by mapping the generator $1'$ to 1. Under this mapping, Lemma III.8.1 implies that it suffices to show $|\Delta_i|$ and $|\Lambda_i|$ are of order $2p$ in the group $\mathbb{Z}_p \times \mathbb{Z}_2$. This is true as long as $|\Delta_i| = x'$ and $|\Lambda_i| = y'$ for some $x, y \in \mathbb{Z}_p \setminus \{0\}$. From Tables III.1 and III.2 this condition is satisfied, so all of the 3-cycles yield hamilton cycles as well. Thus, the derived embedding from the voltage graph given by $\Omega, \Delta_0, ..., \Delta_{p-1}, \Lambda_0, ..., \Lambda_{p-1}$ is a hamilton cycle embedding of $K_{2p,2p,2p}$. □

The following lemma covers the remaining cases $p = 5$ and $p = 7$.

Lemma III.8.9. *For $p = 5$ or 7, there exists a voltage graph such that the derived embedding is an orientable hamilton cycle embedding of $K_{2p,2p,2p}$.*

Proof. The construction uses Ω together with the 3-cycles shown in Table III.3. The resulting rotations for $p = 5$ are

$$\begin{aligned}
a:&\ \ (0_a\ 0'_c\ 0'_a\ 1'_c\ 2'_a\ 3_c\ 1_a\ 1_c\ 2_a\ 4_c\ 3_a\ 2_c\ 4'_c\ 4'_a\ 1'_a\ 2'_c\ 3'_a\ 3'_c\ 4_a\ 0_c)\\
b:&\ \ (0_b\ 1_a\ 4_b\ 0_a\ 2'_b\ 2'_a\ 3'_b\ 3'_a\ 1'_b\ 0'_a\ 3_b\ 4'_a\ 0'_b\ 1'_a\ 4'_b\ 4_a\ 2_b\ 3_a\ 1_b\ 2_a)\\
c:&\ \ (0_c\ 4'_b\ 0'_c\ 4_b\ 1_c\ 1_b\ 2_c\ 3_b\ 3_c\ 0_b\ 4_c\ 2_b\ 3'_c\ 3'_b\ 1'_c\ 1'_b\ 2'_c\ 0'_b\ 4'_c\ 2'_b)
\end{aligned}$$

and for $p = 7$ are

$$
\begin{aligned}
a: &\quad (0_a\ 0'_c\ 0'_a\ 1'_c\ 4'_a\ 5_c\ 1_a\ 3_c\ 2_a\ 1_c\ 3_a\ 6_c\ 4_a\ 4_c\ 5_a\ 2_c\ 6'_c\ 6'_a\ 1'_a\ 4'_c\ 3'_a\ 2'_c\ 5'_a\ 5'_c\ 2'_a\ 3'_c\ 6_a\ 0_c)\\
b: &\quad (0_b\ 1_a\ 4_b\ 5_a\ 1_b\ 2_a\ 5_b\ 6'_a\ 0'_b\ 1'_a\ 6'_b\ 6_a\ 2_b\ 3_a\ 6_b\ 0_a\ 2'_b\ 4'_a\ 5'_b\ 3'_a\ 1'_b\ 5'_a\ 3'_b\ 2'_a\ 4'_b\ 0'_a\ 3_b\ 4_a)\\
c: &\quad (0_c\ 6'_b\ 0'_c\ 6_b\ 6_c\ 3_b\ 5_c\ 0_b\ 4_c\ 4_b\ 3_c\ 1_b\ 2_c\ 5_b\ 1_c\ 2_b\ 3'_c\ 3'_b\ 2'_c\ 5'_b\ 1'_c\ 4'_b\ 5'_c\ 1'_b\ 4'_c\ 0'_b\ 6'_c\ 2'_b).
\end{aligned}
$$

□

III.9 Orientable results

Combining the latin square construction with the voltage graph construction, we can prove Theorem I.4.2, which we restate.

Theorem I.4.2. *There exists an orientable hamilton cycle embedding of $K_{n,n,n}$ for all $n \geq 1$, $n \neq 2$. Moreover, at least one face in this embedding is bounded by an ABC cycle.*

Proof. If n is odd, then the desired embedding is given by Theorem III.2.4. If $n \equiv 0$ or 4 (mod 8) with $n \geq 8$, apply Theorem III.7.2 or III.7.3, respectively, to get a ce-hamiltonian latin square that has a 1-partition. The desired embedding follows from Corollary III.2.3. If $n = 2m$ for some nonprime odd m, $m > 1$, then we can write $n = 2pq$, where p and q are odd, q is prime, and $p \geq q \geq 3$. Apply Theorem III.6.3 to get a ce-hamiltonian latin square that has a 1-partition. Again, the desired embedding follows from Corollary III.2.3. If $n = 4$ or 6, then the desired embedding is given by Lemma III.8.2. If $n = 10$ or 14, the desired embedding is given by Lemma III.8.9. Finally, if $n = 2p$ for a prime $p \geq 11$, the desired embedding is given by Theorem III.8.8. □

CHAPTER IV

APPLICATIONS TO GENUS CALCULATIONS

IV.1 Genus of some joins of edgeless graphs with complete graphs

This section is an extension of the work of Ellingham and Stephens in [13]. We start by presenting two useful lemmas, which are Lemma 4.1 and Lemma 2.2, respectively, in [13]. We note here that Lemma IV.1.2 was proved using the diamond sum technique described briefly in Section I.2.4.

Lemma IV.1.1. *Let G be an m-regular simple graph on n vertices, with $m \geq 2$. The following are equivalent.*

(1) G has an orientable hamilton cycle embedding.

(2) $\overline{K_m} + G$ has an orientable triangulation.

(3) $g(\overline{K_m} + G) = g(K_{m,n})$ and $4 \mid (m-2)(n-2)$.

Lemma IV.1.2. *Let $n \geq 1$ and $m \geq n-1$ be integers. If $g(\overline{K_m} + K_n) = g(K_{m,n})$ and $4 \mid (m-2)(n-2)$, then $g(\overline{K_{m'}} + K_n) = g(K_{m',n})$ for all $m' \geq m$.*

Using the first lemma, we can determine the genus of $\overline{K_{n-1}} + K_n$ from orientable hamilton cycle embeddings of K_n. Using the second lemma, we can extend this result to $\overline{K_m} + K_n$ for all $m \geq n-1$. To that end, we present a recursive construction for orientable hamilton cycle embeddings of complete graphs. Our construction is a slight extension of the following result.

Theorem IV.1.3 (Ellingham and Stephens, Theorem 4.3 [13]). *Suppose $n \equiv 2 \pmod 4$ and $n \geq 6$. If K_n has an orientable hamilton cycle embedding, then K_{2n-2} also has an orientable hamilton cycle embedding.*

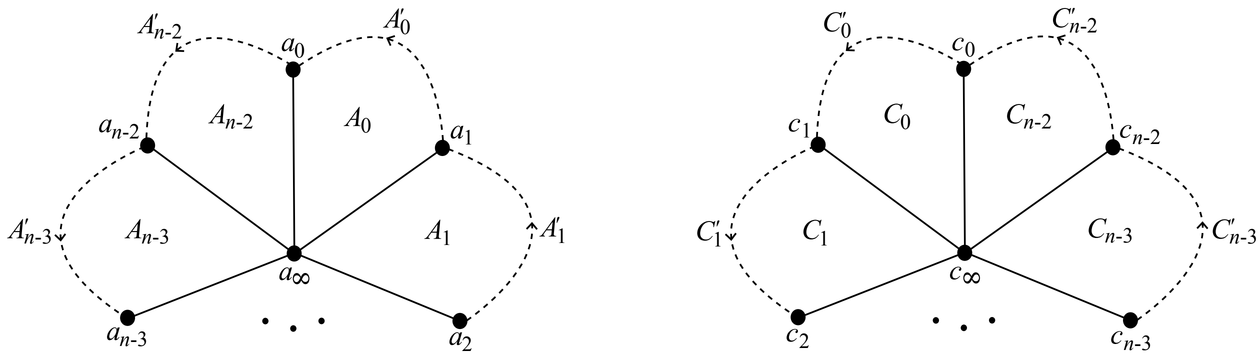

Figure IV.1: Rotations and faces for hamilton cycle embedding of K_n.

Instead of a recursive construction that roughly doubles the number of vertices, we will take an existing orientable hamilton cycle embedding of K_n and produce an orientable hamilton cycle embedding of K_{3n-3}.

Theorem IV.1.4. *Suppose $n \geq 4$ and K_n has an orientable hamilton cycle embedding. Then K_{3n-3} also has an orientable hamilton cycle embedding.*

Proof. Suppose K_n has an orientable hamilton cycle embedding, and provide each vertex with a clockwise rotation. This induces a counterclockwise direction on the boundary of each face.

Take one copy of the embedding, which we will denote by G_a, and label any vertex a_∞. Label the remaining vertices $a_0, a_1, ..., a_{n-2}$ in clockwise order as they appear in the rotation around a_∞. For each $i \in \mathbb{Z}_{n-1}$, let A_i denote the face that follows the path $a_i a_\infty a_{i+1}$ as it passes through a_∞. Let $G'_a = G_a - a_\infty$ be the graph on vertex set $V_a = \{a_i \mid i \in \mathbb{Z}_{n-1}\}$ obtained by removing a_∞ and all of its incident edges from G_a. Each face A_i now corresponds to a directed path from a_{i+1} to a_i in G'_a. This rotation scheme and the resulting paths can be seen in Figure IV.1. We take another copy of the embedding of K_n and construct the graph G'_b on vertex set $V_b = \{b_i \mid i \in \mathbb{Z}_{n-1}\}$ in an identical manner, replacing each a_i and A_i with b_i and B_i, respectively. We take a third copy of the embedding of K_n and construct the graph G'_c on vertex set $V_c = \{c_i \mid i \in \mathbb{Z}_{n-1}\}$ in a similar manner, only the vertices are labeled $c_0, c_{n-2}, c_{n-3}, ..., c_2, c_1$ in clockwise order as they appear in the rotation around

c_∞. The resulting C_i' is now a directed path from c_i to c_{i+1}. This rotation scheme and the resulting paths can also be seen in Figure IV.1.

Let F_∞ be the directed cycle $(c_{n-2}b_{n-2}a_{n-2}c_{n-3}b_{n-3}a_{n-3}\cdots c_1b_1a_1c_0b_0a_0)$, and let $\overline{F_\infty}$ be the underlying undirected cycle. For each $i \in \mathbb{Z}_{n-1}$, let F_i be the directed cycle $A_i' \cup B_{i-1}' \cup C_{i-1}' \cup \{a_ib_i, b_{i-1}c_{i-1}, c_ia_{i+1}\}$. These new directed edges a_ib_i, $b_{i-1}c_{i-1}$ and c_ia_{i+1} are the reverse of edges in F_∞. Therefore, the collection $\mathcal{F} = \{F_i \mid i \in \mathbb{Z}_{n-1}\} \cup \{F_\infty\}$ covers every edge of the graph $H_1 = G_a' \cup G_b' \cup G_c' \cup \overline{F_\infty}$ (on vertex set $V_a \cup V_b \cup V_c$) once in each direction. It is clear from construction that every face is actually a hamilton cycle in H_1; we claim the collection $\mathcal{F}$ determines an orientable hamilton cycle embedding of H_1. To do so, it suffices to show that the rotation around each vertex is a single cycle. We will prove this for an arbitrary vertex a_i. Assume the rotation around a_i in G_a is given by the cycle $(a_\infty a_{\pi(1)}a_{\pi(2)}\cdots a_{\pi(n-2)})$. This rotation stays the same except for the subsequence $(\cdots a_{\pi(n-2)}a_\infty a_{\pi(1)}\cdots)$. Instead of the paths $a_{\pi(n-2)}a_ia_\infty$ and $a_\infty a_ia_{\pi(1)}$ appearing in the cycles A_i and A_{i-1}, respectively, we have the paths $a_{\pi(n-2)}a_ib_i$ in F_i, $b_ia_ic_{i-1}$ in F_∞, and $c_{i-1}a_ia_{\pi(1)}$ in F_{i-1}. Thus, the rotation around a_i in H_1 is given by $(b_ic_{i-1}a_{\pi(1)}a_{\pi(2)}\cdots a_{\pi(n-2)})$, which is a single cycle. An analogous argument works for the rotations around b_i and c_i, so our claim is correct.

By Theorem I.4.2, there exists a hamilton cycle embedding of $H_2 = K_{n-1,n-1,n-1}$ with at least one ABC face, call it D. We can label the vertices of H_2 so that D is the reverse of F_∞; this forces V_a, V_b, and V_c to be the tripartition of H_2.

Delete the interior of the face F_∞ in H_1 to get an embedding with boundary curve $\overline{F_\infty}$. Also delete the interior of the face D in H_2 to get another embedding with boundary curve $\overline{F_\infty}$. The two embeddings share no edges except those in $\overline{F_\infty}$, so we can glue them together by identifying their boundary curves. The result is an orientable embedding of $H_1 \cup H_2$ such that every face is a hamilton cycle on $V_a \cup V_b \cup V_c$. Since G_a, G_b and G_c are complete graphs on V_a, V_b and V_c, respectively, and H_2 is the complete tripartite graph with independent sets V_a, V_b and V_c, $H_1 \cup H_2$ is simply the complete graph on vertex set $V_a \cup V_b \cup V_c$. Therefore, we have an orientable hamilton cycle embedding of K_{3n-3}. $\square$

We apply Theorem IV.1.4 to a family of embeddings obtained in [13]. The following result is actually a restatement of Theorem 4.4 in that paper.

Theorem IV.1.5 (Ellingham and Stephens [13]). *If $n = 2^p + 2$ for some $p \geq 3$, then there exists an orientable hamilton cycle embedding of K_n.*

The following theorem presents the first infinite family of values of n congruent to 3 modulo 4 for which the genus of $\overline{K_m} + K_n$ is known for all $m \geq n - 1$. Recall that the condition $m \geq n-1$ allows us to view the embedding of $\overline{K_m} + K_n$ as an embedding of $K_{m,n}$ with some edges added to form a complete graph on the partite set of size n.

Theorem IV.1.6. *If $n = 3(2^p + 1)$ for some $p \geq 3$, then $g(\overline{K_m} + K_n) = g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ for all $m \geq n - 1$.*

Proof. We apply Theorem IV.1.4 to an orientable hamilton cycle embedding of K_{2^p+2} guaranteed by Theorem IV.1.5 to obtain an orientable hamilton cycle embedding of K_n. Applying Lemmas IV.1.1 and IV.1.2 yields the desired result. □

One advantage of the tripling construction given by Theorem IV.1.4 over the doubling construction in [13] is that it can be applied to hamilton cycle embeddings of K_n for both $n \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$. By a repeated application of this construction to the hamilton cycle embeddings obtained in Theorem IV.1.6, we obtain Theorem I.4.3, which holds for an infinite family that includes values in both modulo classes.

Theorem I.4.3. *If $n = 3^q\left(2^p + \frac{1}{2}\right) + \frac{3}{2}$ for some $p \geq 3$ and $q \geq 0$, then $g(\overline{K_m} + K_n) = g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ for all $m \geq n - 1$.*

Proof. If $q = 0$, then this is equivalent to Theorem IV.1.5. Now for $q \geq 1$ and a fixed p, take the orientable hamilton cycle embedding of $K_{3(2^p+1)}$ found in the proof of Theorem IV.1.6. The result is obtained by induction on q using Theorem IV.1.4 together with Lemmas IV.1.1 and IV.1.2. □

As mentioned in Section I.4, this easily extends to the following result.

Corollary I.4.4. *Let* $n = 3^q \left(2^p + \frac{1}{2}\right) + \frac{3}{2}$ *for some* $p \geq 3$ *and* $q \geq 0$. *If* G *is any* n*-vertex simple graph, then* $g(\overline{K_m} + G) = g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ *for all* $m \geq n - 1$.

We can further extend these results using the following lemma of [13].

Lemma IV.1.7. *If* $g(\overline{K_m} + K_n) = g(K_{m,n})$ *for all* $m \geq n-1$, *then* $g(\overline{K_{m'}} + K_{n-1}) = K_{m',n-1}$ *for all* $m' \geq n$.

Corollary IV.1.8. *Let* $n = 3^q \left(2^p + \frac{1}{2}\right) + \frac{1}{2}$ *for some* $p \geq 3$ *and* $q \geq 0$. *If* G *is any* n*-vertex simple graph, then* $g(\overline{K_m} + G) = g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ *for all* $m \geq n + 1$.

All of the results in Theorems IV.1.5, IV.1.6 and I.4.3 and Corollaries I.4.4 and IV.1.8 were obtained by repeated applications of the doubling and tripling constructions to an orientable hamilton cycle embedding of $K_{10,10,10}$. Finding more embeddings to serve as starting points would greatly increase the usefulness of these recursive constructions.

IV.2 Nonorientable genus of some complete quadripartite graphs

Here we develop the connection between hamilton cycle embeddings of $K_{n,n,n}$ and triangulations of $K_{2n,n,n,n}$ and utilize the covering triangulations from Section I.2.5. The following result is the nonorientable counterpart to Lemma 4.1 in [13].

Lemma IV.2.1. *The following are equivalent.*

(1) There exists a nonorientable hamilton cycle embedding of G *with* p *faces.*

(2) There exists a nonorientable triangulation of $\overline{K_p} + G$.

Moreover, if either (1) or (2) holds, then G *is* p*-regular.*

Lemma IV.2.1 leads to a proof of Theorem I.4.5, which we restate; here we use the convention that the nonorientable genus of a planar graph is zero.

Theorem I.4.5. *For all* $n \geq 1$, $\tilde{g}(K_{2n,n,n,n}) = \tilde{g}(K_{2n,3n}) = (n-1)(3n-2)$.

Proof. $K_{2,1,1,1}$ is planar, so we will assume $n \geq 2$. We know from [39] that $\tilde{g}(K_{2n,3n}) = (n-1)(3n-2)$. Since $K_{2n,3n} \subset K_{2n,n,n,n}$, we have $\tilde{g}(K_{2n,n,n,n}) \geq (n-1)(3n-2)$. From Euler's formula, an embedding that achieves this genus must be a triangulation, so it will suffice to find a nonorientable triangulation of $K_{2n,n,n,n}$.

If n is odd, write $n = 3^p 7^q m$, where $3, 7 \nmid m$. If $m \neq 1$, then Lemmas II.3.1 and II.3.2 imply the existence of a nonorientable hamilton cycle embedding of $K_{m,m,m}$. Lemma IV.2.1 yields a triangulation of $K_{2m,m,m,m}$. Applying Corollary I.2.12 provides a triangulation of $K_{2(3^p m),3^p m,3^p m,3^p m}$, and applying Corollary I.2.10 gives us the desired triangulation of $K_{2n,n,n,n}$. If $m = 1$, then we use a nonorientable hamilton cycle embedding of either $K_{3,3,3}$ or $K_{7,7,7}$ from Section II.2 as our starting point before applying Lemma IV.2.1 and the results of Section I.2.5.

If n is even, write $n = 2^p 2m$, where m is odd. By Lemma II.3.3 there exists a nonorientable hamilton cycle embedding of $K_{2m,2m,2m}$. Lemma IV.2.1 yields a triangulation of $K_{4m,2m,2m,2m}$, and applying Corollary I.2.6 gives us the desired triangulation of $K_{2n,n,n,n}$. This completes the proof. □

The construction of the necessary triangulations for $n \geq 2$ in the proof of Theorem I.4.5 completes the proof of Theorem I.4.1. Unfortunately, the hamilton cycle faces in the embeddings of $K_{n,n,n}$ obtained from Bouchet's covering triangulations of $K_{2n,n,n,n}$ are not, in general, ABC cycles.

The following extension of Theorem I.4.5 is the application of the diamond sum technique alluded to in Section I.2.4.

Corollary I.4.6. *For all $n \geq 1$ and all $t \geq 2n$, $\tilde{g}(K_{t,n,n,n}) = \tilde{g}(K_{t,3n}) = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$.*

Proof. We know that $K_{t,3n} \subseteq K_{t,n,n,n}$, and from [39] we know $\tilde{g}(K_{t,3n}) = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$, so $\tilde{g}(K_{t,n,n,n}) \geq \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$. We now apply the diamond sum construction to nonorientable minimum genus embeddings of $K_{2n,n,n,n}$ and $K_{t-2n+2,3n}$. By Theorem I.4.5 we know $\tilde{g}(K_{2n,n,n,n}) = (n-1)(3n-2)$, and again by [39] we know $\tilde{g}(K_{t-2n+2,3n}) = \left\lceil \frac{(t-2n)(3n-2)}{2} \right\rceil$. Via the diamond sum construction, we learn that $\tilde{g}(K_{t,n,n,n}) \leq (n-1)(3n-2) + \left\lceil \frac{(t-2n)(3n-2)}{2} \right\rceil = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$, and the result follows. $\square$

Remark IV.2.2. Corollary I.4.6 implies that for any graph G satisfying $\overline{K_{3n}} \subseteq G \subseteq K_{n,n,n}$ and for all $t \geq 2n$, the nonorientable genus of $\overline{K_t} + G$ is the same as the nonorientable genus of $K_{t,3n}$. In other words, $\tilde{g}(\overline{K_t} + G) = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$. Moreover, in the special case $t = 2n$, we also get $\tilde{g}(G + H) = (n-1)(3n-2)$ for graphs G and H satisfying $\overline{K_{3n}} \subseteq G \subseteq K_{2n,n}$ and $\overline{K_{2n}} \subseteq H \subseteq K_{n,n}$.

IV.3 Orientable genus of some complete quadripartite graphs

We again use the connection between hamilton cycle embeddings of $K_{n,n,n}$ and triangulations of $K_{2n,n,n,n}$ to determine the orientable genus of some quadripartite graphs. The following result is a slight restatement of Lemma 4.1 in [13].

Lemma IV.3.1. *The following are equivalent.*

(1) There exists an orientable hamilton cycle embedding of G with p faces.

(2) There exists an orientable triangulation of $\overline{K_p} + G$.

Moreover, if either (1) or (2) holds, then G is p-regular.

Lemma IV.3.1 leads to a proof of Theorem I.4.7, which we restate.

Theorem I.4.7. *For all $n \neq 2$, $g(K_{2n,n,n,n}) = g(K_{2n,3n}) = \left\lceil \frac{(n-1)(3n-2)}{2} \right\rceil$.*

Proof. We know from [38] that $g(K_{2n,3n}) = \left\lceil \frac{(n-1)(3n-2)}{2} \right\rceil$. Since $K_{2n,3n} \subset K_{2n,n,n,n}$, we have $g(K_{2n,n,n,n}) \geq \left\lceil \frac{(n-1)(3n-2)}{2} \right\rceil$. From Euler's formula, an embedding that achieves this genus must be a triangulation, so it will suffice to find an orientable triangulation of $K_{2n,n,n,n}$. By Theorem I.4.2 there exists an orientable hamilton cycle embedding of $K_{n,n,n}$, and the desired triangulation follows from Lemma IV.3.1. $\square$

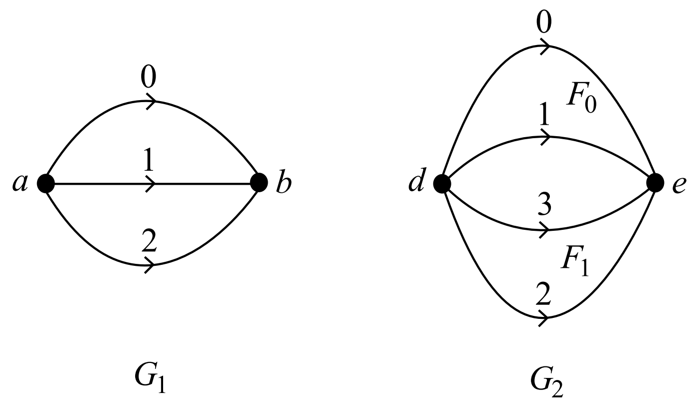

Figure IV.2: Voltage graphs for embeddings Ψ_1 and Ψ_3.

We would like to extend this theorem using the diamond sum in a manner similar to our extension of Theorem I.4.5 in the previous section. Before we can do that, however, we must address the case when $n = 2$. Because there is no orientable hamilton cycle embedding of $K_{2,2,2}$, no triangulation of $K_{4,2,2,2}$ exists either; thus, contrary to expectations, $g(K_{4,2,2,2}) > \left\lceil \frac{(2-1)(6-2)}{2} \right\rceil = 2$. To perform the diamond sum operation, we need to show that $g(K_{5,2,2,2}) = \left\lceil \frac{(5-2)(6-2)}{4} \right\rceil = 3$.

Let $\Psi_1 : K_{3,3} \hookrightarrow S_1$ be the embedding of $K_{3,3}$ with three hamilton cycle faces C_0, C_1 and C_2 that is derived from the voltage graph G_1 with voltage group $\mathbb{Z}_3$ that is shown in Figure IV.2. By placing a new vertex c_i in the center of each hamilton cycle face C_i and placing an edge between c_i and each vertex in C_i in the natural way, for $i \in \{0, 1, 2\}$, we obtain a triangulation $\Psi_2 : K_{3,3,3} \hookrightarrow S_1$. We can assume without loss of generality that the rotation graph around a_0 is given by the cycle $(b_0c_0b_1c_1b_2c_2)$.

Now let $\Psi_3 : K_{4,4} \hookrightarrow S_2$ be the embedding of $K_{4,4}$ with two hamilton cycle faces F_0' and F_1' (derived from F_0 and F_1 in Figure IV.2, respectively) and four 4-cycle faces that is derived from the voltage graph G_2 with voltage group $\mathbb{Z}_4$ that is shown in Figure IV.2. By placing a new vertex f_i in the center of each hamilton cycle face F_i' and placing an edge between f_i and each vertex in F_i' in the natural way, for $i \in \{0, 1\}$, we obtain an embedding $\Psi_4 : K_{4,4,2} \hookrightarrow S_2$. The rotation graph around d_0 is given by the cycle $(e_0f_0e_1e_3f_1e_2)$.

We now form the diamond sum of Ψ_2 and Ψ_4 by removing the vertex a_0 and its neigh-

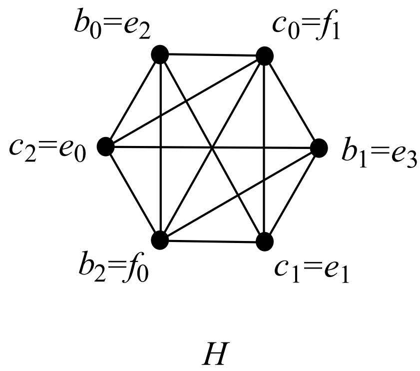

Figure IV.3: Graph K that arises from diamond sum operation.

borhood from Ψ_2, removing the vertex d_0 and its neighborhood from Ψ_4, and identifying the vertices around the boundaries of the holes as shown in Figure IV.3. Doing so yields an embedding $\overline{K_5} + H \hookrightarrow S_3$, where $V(\overline{K_5}) = \{a_1, a_2, d_1, d_2, d_3\}$ and H is the graph shown in Figure IV.3. Note that $H \cong K_{2,2,1,1}$; thus, we have an embedding of $K_{5,2,2,1,1}$ in the orientable surface S_3. Since $K_{5,6} \subset K_{5,2,2,2} \subset K_{5,2,2,1,1}$, we know $3 = g(K_{5,6}) \leq g(K_{5,2,2,2}) \leq 3$, as required.

We are now able to extend Theorem I.4.7 using the application of the diamond sum technique alluded to in Section I.2.4.

Corollary I.4.8. *For all $n \geq 1$ and all $t \geq 2n$, except $(n,t) = (2,4)$, $g(K_{t,n,n,n}) = g(K_{t,3n}) = \left\lceil \frac{(t-2)(3n-2)}{4} \right\rceil$. Also, $g(K_{4,2,2,2}) = 3$.*

Proof. We know that $K_{t,3n} \subseteq K_{t,n,n,n}$, and from [38] we know $g(K_{t,3n}) = \left\lceil \frac{(t-2)(3n-2)}{4} \right\rceil$, so $g(K_{t,n,n,n}) \geq \left\lceil \frac{(t-2)(3n-2)}{4} \right\rceil$. If $n \neq 2$, we apply the diamond sum construction to orientable minimum genus embeddings of $K_{2n,n,n,n}$ and $K_{t-2n+2,3n}$. By Theorem I.4.7 we know $g(K_{2n,n,n,n}) = \left\lceil \frac{(n-1)(3n-2)}{2} \right\rceil$, and again by [38] we know $g(K_{t-2n+2,3n}) = \left\lceil \frac{(t-2n)(3n-2)}{4} \right\rceil$. Via the diamond sum construction, we learn that $g(K_{t,n,n,n}) \leq \left\lceil \frac{(n-1)(3n-2)}{2} \right\rceil + \left\lceil \frac{(t-2n)(3n-2)}{4} \right\rceil = \left\lceil \frac{(t-2)(3n-2)}{4} \right\rceil$, and the result follows. If $n = 2$, we apply the diamond sum construction to orientable minimum genus embeddings of $K_{5,2,2,2}$ and $K_{t-3,6}$. As mentioned before, $g(K_{4,2,2,2}) > 2$; because $K_{4,2,2,2} \subset K_{5,2,2,2}$, we know $g(K_{4,2,2,2}) \leq g(K_{5,2,2,2}) = 3$ as well, so $g(K_{4,2,2,2}) = 3$. $\square$

Remark IV.3.2. Corollary I.4.8 implies that for all $n \geq 1$ and all $t \geq 2n$, except $(n,t) = (2,4)$, and for any graph G satisfying $\overline{K_{3n}} \subseteq G \subseteq K_{n,n,n}$, the genus of $\overline{K_t} + G$ is the same as the genus of $K_{t,3n}$. In other words, $g(\overline{K_t} + G) = \left\lceil \frac{(t-2)(3n-2)}{4} \right\rceil$. If $n = 2$ and $\overline{K_6} \subseteq G \subseteq K_{2,2,2}$, then $g(\overline{K_4} + G) \in \{2, 3\}$. Moreover, in the special case $t = 2n$ and $n \neq 2$, we also get $g(G + H) = \left\lceil \frac{(n-1)(3n-2)}{2} \right\rceil$ for graphs G and H satisfying $\overline{K_{3n}} \subseteq G \subseteq K_{2n,n}$ and $\overline{K_{2n}} \subseteq H \subseteq K_{n,n}$.

CHAPTER V

FUTURE RESEARCH

The topics in this thesis open the door to several future research endeavors. While progress was made on some difficult genus problems, there are still open cases to consider. Several of the concepts for latin squares introduced in this paper can be used to generalize some known results. Moreover, the constructions presented in these pages are often very flexible and could potentially be applied to solve other problems.

V.1 Genus of join graphs

The genus of the join graph $\overline{K_m} + G$, where $|V(G)| = n$ and $m \geq n-1$, has only been determined for a few infinite, yet sparse, families of n. If $n \equiv 2$ or $3 \pmod 4$, then this problem is equivalent to constructing orientable hamilton cycle embeddings of the complete graph K_n. The tripling construction given in Chapter IV can be applied to any hamilton cycle embedding of K_n. If we were to find more families of embeddings to serve as building blocks, this would greatly enhance the power of this construction. Of the 12 residual classes that need to be resolved modulo 24, the doubling and tripling constructions imply only 6 of these are needed, as shown in the following result.

Proposition V.1.1. *Suppose there exists an orientable hamilton cycle embedding of K_{15} and of K_n for all $n \geq 11$ such that $n \equiv 7, 11, 14, 19, 22$ or 23 (mod 24). Then there exists an orientable hamilton cycle embedding of K_n for all $n \equiv 2$ or 3 (mod 4), $n \notin \{2, 6, 7\}$.*

Proof. There is trivially no such embedding when $n = 2$, and Jungerman [30] showed that there are no orientable hamilton cycle embeddings of K_6 or K_7. We show how to cover the remaining residual classes, proceeding by induction on n. The graph K_3 has an obvious hamilton cycle embedding in the sphere, and we know such an embedding exists for K_{10} from Theorem IV.1.5, so the proposition holds for $n \leq 10$.

Assume the proposition holds for all $n' < n$, where $n \equiv 2$ or $3 \pmod 4$ and $n \geq 11$. If $n \equiv 7, 11, 14, 19, 22$ or $23 \pmod{24}$, then an orientable hamilton cycle embedding of K_n exists by assumption. If $n \equiv 2, 3, 6, 10, 15$ or $18 \pmod{24}$, then either $n \equiv 2 \pmod 8$, or $n \equiv 3$ or $6 \pmod{12}$.

Suppose first that $n \equiv 2 \pmod 8$, so $n \geq 18$. Then $n = 8p + 2 = 2(4p + 2) - 2$, where $4p + 2 \geq 10$. By induction K_{4p+2} has the required embedding, so by Theorem IV.1.3 K_n has the required embedding as well.

Suppose now that $n \equiv 3 \pmod{12}$. The required embedding exists for $n = 15$ by assumption, so we may suppose that $n \geq 27$. Then $n = 12p + 3 = 3(4p + 2) - 3$, where $4p + 2 \geq 10$. By induction K_{4p+2} has the required embedding, so by Theorem IV.1.4 K_n has the required embedding as well.

Finally, suppose that $n \equiv 6 \pmod{12}$. Since $n = 18$ is covered by the case of $n \equiv 2$ (mod 8), we may assume that $n \geq 30$. Then $n = 12p + 6 = 3(4p + 3) - 3$, where $4p + 3 \geq 11$. By induction K_{4p+3} has the required embedding, so by Theorem IV.1.4 K_n has the required embedding as well, and the proof is complete. $\square$

Therefore, we will seek alternative methods for building orientable hamilton cycle embeddings of complete graphs for the values specified in Proposition V.1.1. Along with new construction methods, a better understanding of applications of the doubling and tripling construction is desired. From any value n for which an orientable hamilton cycle embedding of K_n is known to exist, we can construct an infinite set of values $T(n)$ such that an orientable hamilton cycle embedding of K_m exists for all $m \in T(n)$. The set is constructed recursively as follows: for any value $m \in T(n)$, if $m \equiv 2 \pmod 4$, then $2m - 2$ and $3m - 3$ are also in $T(n)$ by virtue of the doubling construction given in [13] and the tripling construction given by Theorem IV.1.4, respectively; if $m \equiv 3 \pmod 4$, then only $3m - 3$ is also in $T(n)$. A tree depicting several values in $T(10)$ and how they were obtained is shown in Figure V.1. An edge labeled by d represents a link formed by virtue of the doubling construction, while an edge labeled t represents a link formed by virtue of the tripling construction.

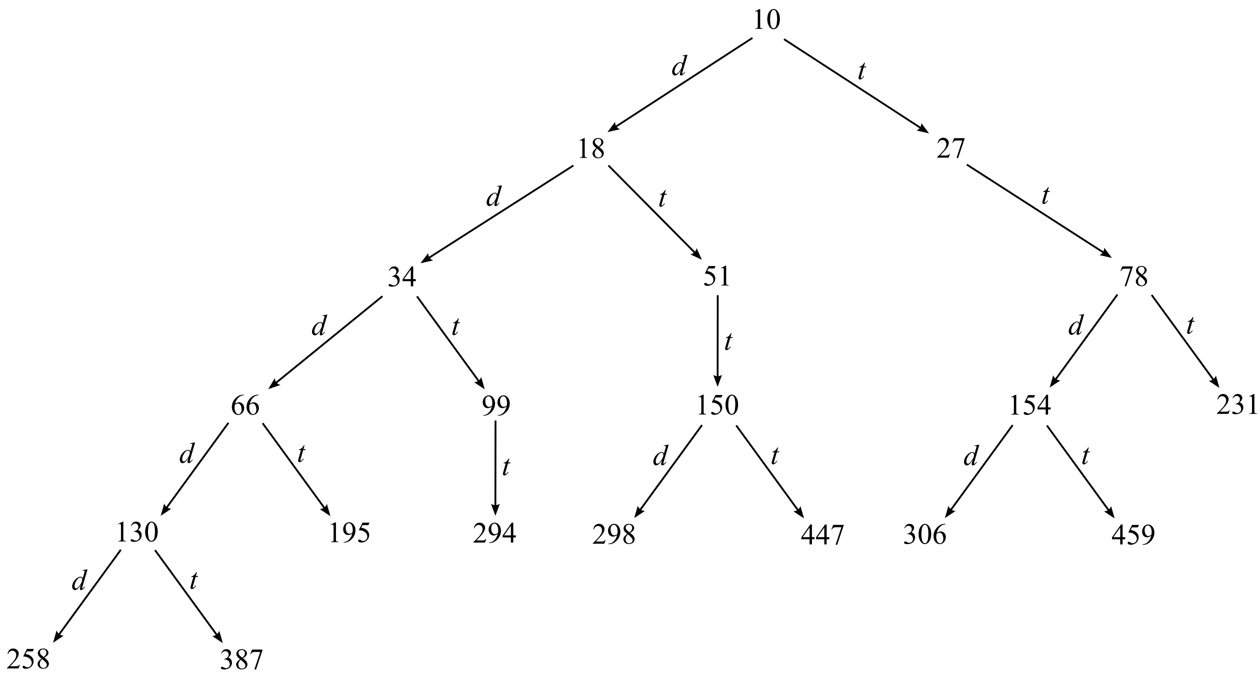

Figure V.1: A tree showing $m \in T(10)$ with $m \leq 500$.

We want to determine how many of the possible values m for which an orientable hamilton cycle embedding of K_m could possibly exist that are covered by a given $T(n)$. Let $X \subset \mathbb{Z}$, and define the *nth partial density of* X to be

$$\mathrm{PD}(X, n) = \frac{|\{x \in X \mid x \leq n\}|}{n}$$

and the *density* of X to be

$$\rho(X) = \lim_{n \to \infty} \mathrm{PD}(X, n).$$

From Figure V.1 we learn that

$$\begin{aligned}
\mathrm{PD}(T(10), 100) &= \tfrac{8}{100} &=& \ 0.080 \\
\mathrm{PD}(T(10), 200) &= \tfrac{12}{200} &=& \ 0.060 \\
\mathrm{PD}(T(10), 300) &= \tfrac{16}{300} &\approx& \ 0.053 \\
\mathrm{PD}(T(10), 400) &= \tfrac{18}{400} &=& \ 0.045 \\
\mathrm{PD}(T(10), 500) &= \tfrac{20}{500} &=& \ 0.040.
\end{aligned}$$

We would like to find an n such that $\rho(T(n)) > 0$ for any n, but evidence suggests this is not possible.

Conjecture V.1.2. $\rho(T(n)) = 0$ *for every integer* n.

If Conjecture V.1.2 is true, then we would like to find a set N of zero density such that

$$\rho\left(\bigcup_{n \in N} T(n)\right) > 0.$$

V.2 Properties of latin squares

With the generalization of Euler's q-step type latin square that was given in Section I.3, we would like to determine conditions on L such that generalizations of Theorem I.3.6 or I.3.7 hold true for a q-step type latin square based on L. In particular neither theorem is true for all L, because the square $K_6 \boxdot \mathbb{Z}_3$ constructed in Lemma III.6.2 has $m = 3$ and $n = 6$ yet has a 1-partition.

While we were able to obtain orientable hamilton cycle embeddings of $K_{n,n,n}$ for all possible values of n, some of these were obtained using voltage graph constructions. For the case $n = 10$, the following ce-hamiltonian latin square with a 1-partition yields an O2HC-embedding of $K_{10,10,10}$.

Example V.2.1. The following latin square L, found by a computer search on data collected by McKay [34] and manipulated using the sum composition method of Hedayat and Seiden [28], meets the conditions of Corollary III.2.3 with $p = 1$ and provides an orientable hamilton cycle embedding of $K_{10,10,10}$; L' provides the 1-partition.

$$L = \begin{pmatrix} 4 & 2 & 7 & 5 & 3 & 6 & 9 & 0 & 8 & 1 \\ 3 & 4 & 6 & 0 & 8 & 9 & 7 & 1 & 2 & 5 \\ 6 & 9 & 4 & 2 & 1 & 7 & 3 & 5 & 0 & 8 \\ 9 & 8 & 2 & 3 & 7 & 1 & 4 & 6 & 5 & 0 \\ 1 & 7 & 0 & 4 & 9 & 5 & 2 & 8 & 6 & 3 \\ 8 & 5 & 9 & 7 & 4 & 0 & 1 & 2 & 3 & 6 \\ 7 & 6 & 8 & 9 & 5 & 4 & 0 & 3 & 1 & 2 \\ 2 & 3 & 1 & 6 & 0 & 8 & 5 & 4 & 7 & 9 \\ 0 & 1 & 5 & 8 & 2 & 3 & 6 & 7 & 9 & 4 \\ 5 & 0 & 3 & 1 & 6 & 2 & 8 & 9 & 4 & 7 \end{pmatrix}, L' = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 1 & 4 & 0 & 3 & 8 & 2 & 9 & 7 \\ 4 & 8 & 9 & 5 & 1 & 7 & 2 & 6 & 0 & 3 \\ 7 & 5 & 6 & 1 & 9 & 4 & 3 & 0 & 2 & 8 \\ 8 & 0 & 5 & 7 & 2 & 1 & 4 & 9 & 3 & 6 \\ 1 & 4 & 0 & 6 & 8 & 9 & 5 & 3 & 7 & 2 \\ 3 & 7 & 4 & 9 & 5 & 2 & 1 & 8 & 6 & 0 \\ 2 & 9 & 7 & 8 & 3 & 6 & 0 & 1 & 4 & 5 \\ 6 & 3 & 8 & 2 & 7 & 0 & 9 & 5 & 1 & 4 \\ 9 & 2 & 3 & 0 & 6 & 8 & 7 & 4 & 5 & 1 \end{pmatrix}.$$

We believe that such a latin square exists for all $n = 2p$, where $p \geq 5$ is prime, and we plan to conduct a further analysis of the sum composition method to see if it can be extended to all such values.

Conjecture V.2.2. *If $n = 2p$ for a prime $p \geq 5$, then there exists a ce-hamiltonian latin square of order n with a 1-partition.*

V.3 Enumeration results

A promising research direction offered by the results of this thesis is the enumeration of some special structures. In particular, the following property of ce-hamiltonian latin squares could prove to be very useful.

Proposition V.3.1. *If L is ce-hamiltonian, then L is conjugate to a square L_c that is cr-hamiltonian.*

Proof. If L is ce-hamiltonian, then the induced pair graph G_P for

$$P_\ell = \{(C(L,j,\ell), C(L,j,\ell-1)) | j \in \mathbb{Z}_n\} \text{ for all } \ell \in \mathbb{Z}_n$$

is a hamilton cycle. Put another way, the permutation given by

$$\pi_\ell = \begin{pmatrix} C(L,0,\ell) & C(L,1,\ell) & \cdots & C(L,n-1,\ell) \\ C(L,0,\ell-1) & C(L,1,\ell-1) & \cdots & C(L,n-1,\ell-1) \end{pmatrix}$$

consists of a single cycle of length n for each $\ell \in \mathbb{Z}_n$. We form the conjugate square L_c by permuting the roles of the rows, columns and entries in L. Specifically, we map the row i in L to the column i in L_c, the column j in L to the entry j in L_c, and the entry k in L to the row k in L_c. Applying this map, we obtain

$$L_c = \begin{pmatrix} C(L,0,0) & C(L,1,0) & \cdots & C(L,n-1,0) \\ C(L,0,1) & C(L,1,1) & \cdots & C(L,n-1,1) \\ \vdots & \vdots & \ddots & \vdots \\ C(L,0,n-1) & C(L,1,n-1) & \cdots & C(L,n-1,n-1) \end{pmatrix}.$$

It is clear that any two consecutive rows i and $i+1$ yield the permutation π_{i+1}^{-1}, which is a single cycle of length n. Thus, the square L_c is cr-hamiltonian. □

Theorem I.3.12 tells us that any cr-hamiltonian square has a biembedding with a copy of itself. Since the flexibility of the latin square construction in Chapter III allows us to build many ce-hamiltonian latin squares (thus, cr-hamiltonian latin squares), we may obtain many nonisomorphic triangulations of complete tripartite graphs. Using this connection, we might be able to extend the results of Grannell, Griggs and Knor [17, 24] concerning lower bounds on the number of nonisomorphic orientable triangulations of $K_{n,n,n}$ and K_n. Moreover, the step product construction may also allow for the creation of many nonisomorphic pairs of orthogonal latin squares, which would improve the known lower bound on the number of such squares of each order. Enumeration attempts could also be made in the nonorientable case using the slope sequence construction.

APPENDIX A

EXAMPLE OF THEOREM III.7.3

In this appendix, we construct the latin square J_{12} and show how we obtain the 1-partition guaranteed by Theorem III.7.3. We will address each 2-plex in $\mathbb{Z}_6$ and the resulting transversals in J_{12} separately. Recall that X is the 6×6 matrix of all 0's except that the final row is given by

$$1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0,$$

and J_{12} is the step product

$$J_{12} = \mathbb{Z}_6 \boxdot_X \mathbb{Z}_2 = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0 & 6 & 1 & 7 & 2 & 8 & 3 & 9 & 4 & 10 & 5 & 11 \\
6 & 0 & 7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 \\
\hline
1 & 7 & 2 & 8 & 3 & 9 & 4 & 10 & 5 & 11 & 0 & 6 \\
7 & 1 & 8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 & 6 & 0 \\
\hline
2 & 8 & 3 & 9 & 4 & 10 & 5 & 11 & 0 & 6 & 1 & 7 \\
8 & 2 & 9 & 3 & 10 & 4 & 11 & 5 & 6 & 0 & 7 & 1 \\
\hline
3 & 9 & 4 & 10 & 5 & 11 & 0 & 6 & 1 & 7 & 2 & 8 \\
9 & 3 & 10 & 4 & 11 & 5 & 6 & 0 & 7 & 1 & 8 & 2 \\
\hline
4 & 10 & 5 & 11 & 0 & 6 & 1 & 7 & 2 & 8 & 3 & 9 \\
10 & 4 & 11 & 5 & 6 & 0 & 7 & 1 & 8 & 2 & 9 & 3 \\
\hline
11 & 5 & 0 & 6 & 7 & 1 & 2 & 8 & 9 & 3 & 4 & 10 \\
5 & 11 & 6 & 0 & 1 & 7 & 8 & 2 & 3 & 9 & 10 & 4
\end{array} \right).$$

We start with the 2-plex S labeled with π, τ as shown below:

$$
S = \begin{pmatrix}
\mathit{0}^{\mathit{0,0}} & \mathit{1}^{\mathit{1,1}} & 2 & 3 & 4 & 5 \\
1 & 2 & \mathit{3}^{\mathit{0,0}} & 4 & \mathit{5}^{\mathit{1,0}} & 0 \\
2 & 3 & 4 & \mathit{5}^{\mathit{0,0}} & 0 & \mathit{1}^{\mathit{1,0}} \\
\mathit{3}^{\mathit{0,1}} & \mathit{4}^{\mathit{1,0}} & 5 & 0 & 1 & 2 \\
4 & 5 & \mathit{0}^{\mathit{0,1}} & 1 & \mathit{2}^{\mathit{1,1}} & 3 \\
5 & 0 & 1 & \mathit{2}^{\mathit{0,1}} & 3 & \mathit{4}^{\mathit{1,1}}
\end{pmatrix},
$$

which yields the underlined transversal $T_{0,0}$ in J_{12} marked by the superscript a below:

$$
J_{12} = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
\underline{0}^a & & & & & & & & & & & \\
 & & & \underline{1}^a & & & & & & & & \\
\hline
 & & & & \underline{3}^a & & & & & & & \\
 & & & & & & & & \underline{11}^a & & & \\
\hline
 & & & & & & \underline{5}^a & & & & & \\
 & & & & & & & & & & \underline{7}^a & \\
\hline
 & \underline{9}^a & & & & & & & & & & \\
 & & \underline{10}^a & & & & & & & & & \\
\hline
 & & & & & \underline{6}^a & & & & & & \\
 & & & & & & & & & \underline{2}^a & & \\
\hline
 & & & & & & & \underline{8}^a & & & & \\
 & & & & & & & & & & & \underline{4}^a
\end{array}\right).
$$

We now give the underlined transversal $T_{0,1}$ in J_{12} marked by the superscript b below; note that all of the entries are obtained by shifting the a transversal across the row of its corresponding size-2 subsquare:

$$J_{12} = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0^a & \underline{6^b} & & & & & & & & & & \\
 & & \underline{7^b} & 1^a & & & & & & & & \\
\hline
 & & & & 3^a & \underline{9^b} & & & & & & \\
 & & & & & & & & 11^a & \underline{5^b} & & \\
\hline
 & & & & & & 5^a & \underline{11^b} & & & & \\
 & & & & & & & & & & 7^a & \underline{1^b} \\
\hline
\underline{3^b} & 9^a & & & & & & & & & & \\
 & & 10^a & \underline{4^b} & & & & & & & & \\
\hline
 & & & & \underline{0^b} & 6^a & & & & & & \\
 & & & & & & & & \underline{8^b} & 2^a & & \\
\hline
 & & & & & & \underline{2^b} & 8^a & & & & \\
 & & & & & & & & & & \underline{10^b} & 4^a
\end{array}\right).$$

The next 2-plex to consider is $S+2$; the unlabeled version is shown below:

$$S+2 = \begin{pmatrix}
0 & 1 & \mathit{2}^* & \mathit{3}^* & 4 & 5 \\
\mathit{1}^* & 2 & 3 & 4 & \mathit{5}^* & 0 \\
2 & \mathit{3}^* & 4 & 5 & 0 & \mathit{1}^* \\
3 & 4 & \mathit{5}^* & \mathit{0}^* & 1 & 2 \\
\mathit{4}^* & 5 & 0 & 1 & \mathit{2}^* & 3 \\
5 & \mathit{0}^* & 1 & 2 & 3 & \mathit{4}^*
\end{pmatrix}.$$

The 2-plex $S+2$ together with π and τ yields the underlined transversals $T_{0,0}+2$ and $T_{0,1}+2$ shown below in J_{12} marked by the superscripts c and d, respectively. Note that c and d are obtained by shifting the transversals a and b, respectively, four cells to the right:

$$J_{12} = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0^a & 6^b & & & \underline{2^c} & \underline{8^d} & & & & & & \\
& & 7^b & 1^a & & & \underline{9^d} & \underline{3^c} & & & & \\
\hline
& & & & 3^a & 9^b & & & \underline{5^c} & \underline{11^d} & & \\
\underline{7^c} & \underline{1^d} & & & & & & & 11^a & 5^b & & \\
\hline
& & & & & & 5^a & 11^b & & & \underline{1^c} & \underline{7^d} \\
& & \underline{9^c} & \underline{3^d} & & & & & & & 7^a & 1^b \\
\hline
3^b & 9^a & & & \underline{5^d} & \underline{11^c} & & & & & & \\
& & 10^a & 4^b & & & \underline{6^c} & \underline{0^d} & & & & \\
\hline
& & & & 0^b & 6^a & & & \underline{2^d} & \underline{8^c} & & \\
\underline{10^d} & \underline{4^c} & & & & & & & 8^b & 2^a & & \\
\hline
& & & & & & 2^b & 8^a & & & \underline{4^d} & \underline{10^c} \\
& & \underline{6^d} & \underline{0^c} & & & & & & & 10^b & 4^a
\end{array}\right).$$

The final shift of S to consider is $S+4$; the unlabeled version is shown below:

$$S+4 = \begin{pmatrix}
0 & 1 & 2 & 3 & \mathit{4}^* & \mathit{5}^* \\
\mathit{1}^* & 2 & \mathit{3}^* & 4 & 5 & 0 \\
2 & \mathit{3}^* & 4 & \mathit{5}^* & 0 & 1 \\
3 & 4 & 5 & 0 & \mathit{1}^* & \mathit{2}^* \\
\mathit{4}^* & 5 & \mathit{0}^* & 1 & 2 & 3 \\
5 & \mathit{0}^* & 1 & \mathit{2}^* & 3 & 4
\end{pmatrix}.$$

The 2-plex $S+4$ together with π and τ yields the underlined transversals $T_{0,0}+4$ and $T_{0,1}+4$ shown below in J_{12} marked by the superscripts e and f, respectively. Note that e and f are again obtained by shifting the transversals c and d, respectively, four cells to the right:

$$J_{12} = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0^a & 6^b & & & 2^c & 8^d & & & \underline{4^e} & \underline{10^f} & & \\
& & 7^b & 1^a & & & 9^d & 3^c & & & \underline{11^f} & \underline{5^e} \\ \hline
\underline{1^e} & \underline{7^f} & & & 3^a & 9^b & & & 5^c & 11^d & & \\
7^c & 1^d & & & \underline{9^e} & \underline{3^f} & & & 11^a & 5^b & & \\ \hline
& & \underline{3^e} & \underline{9^f} & & & 5^a & 11^b & & & 1^c & 7^d \\
& & 9^c & 3^d & & & \underline{11^e} & \underline{5^f} & & & 7^a & 1^b \\ \hline
3^b & 9^a & & & 5^d & 11^c & & & \underline{1^f} & \underline{7^e} & & \\
& & 10^a & 4^b & & & 6^c & 0^d & & & \underline{8^e} & \underline{2^f} \\ \hline
\underline{4^f} & \underline{10^e} & & & 0^b & 6^a & & & 2^d & 8^c & & \\
10^d & 4^c & & & \underline{6^f} & \underline{0^e} & & & 8^b & 2^a & & \\ \hline
& & \underline{0^f} & \underline{6^e} & & & 2^b & 8^a & & & 4^d & 10^c \\
& & 6^d & 0^c & & & \underline{8^f} & \underline{2^e} & & & 10^b & 4^a
\end{array}\right).$$

So far we have found 6 transversals in J_{12}; the remaining transversals come from shifts of S'. The 2-plex S' labeled with π, τ is shown below:

$$T = \begin{pmatrix}
0 & 1^{0,0} & 2^{1,1} & 3 & 4 & 5 \\
1 & 2 & 3 & 4^{0,0} & 5 & 0^{1,0} \\
2^{1,0} & 3 & 4 & 5 & 0^{0,0} & 1 \\
3 & 4^{0,1} & 5^{1,0} & 0 & 1 & 2 \\
4 & 5 & 0 & 1^{0,1} & 2 & 3^{1,1} \\
5^{0,1} & 0 & 1 & 2 & 3^{1,1} & 4
\end{pmatrix},$$

which yields the underlined transversal $T'_{0,0}$ in J_{12} marked by the superscript g below:

$$J_{12} = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0^a & 6^b & \underline{1^g} & & 2^c & 8^d & & & 4^e & 10^f & & \\
& & 7^b & 1^a & & \underline{2^g} & 9^d & 3^c & & & 11^f & 5^e \\
\hline
1^e & 7^f & & & 3^a & 9^b & \underline{4^g} & & 5^c & 11^d & & \\
7^c & 1^d & & & 9^e & 3^f & & & 11^a & 5^b & \underline{6^g} & \\
\hline
& & 3^e & 9^f & & & 5^a & 11^b & \underline{0^g} & & 1^c & 7^d \\
\underline{8^g} & & 9^c & 3^d & & & 11^e & 5^f & & & 7^a & 1^b \\
\hline
3^b & 9^a & & \underline{10^g} & 5^d & 11^c & & & 1^f & 7^e & & \\
& & 10^a & 4^b & \underline{11^g} & & 6^c & 0^d & & & 8^e & 2^f \\
\hline
4^f & 10^e & & & 0^b & 6^a & & \underline{7^g} & 2^d & 8^c & & \\
10^d & 4^c & & & 6^f & 0^e & & & 8^b & 2^a & & \underline{3^g} \\
\hline
& \underline{5^g} & 0^f & 6^e & & & 2^b & 8^a & & & 4^d & 10^c \\
& & 6^d & 0^c & & & 8^f & 2^e & & \underline{9^g} & 10^b & 4^a
\end{array}\right).$$

We now give the underlined transversal $T'_{0,1}$ in J_{12} marked by the superscript h below; note that all of the entries are obtained by shifting the g transversal across the row of its corresponding size-2 subsquare:

$$
J_{12} = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0^a & 6^b & 1^g & \underline{7^h} & 2^c & 8^d & & & 4^e & 10^f & & \\
 & & 7^b & 1^a & \underline{8^h} & 2^g & 9^d & 3^c & & & 11^f & 5^e \\
\hline
1^e & 7^f & & & 3^a & 9^b & 4^g & \underline{10^h} & 5^c & 11^d & & \\
7^c & 1^d & & & 9^e & 3^f & & & 11^a & 5^b & 6^g & \underline{0^h} \\
\hline
 & & 3^e & 9^f & & & 5^a & 11^b & 0^g & \underline{6^h} & 1^c & 7^d \\
8^g & \underline{2^h} & 9^c & 3^d & & & 11^e & 5^f & & & 7^a & 1^b \\
\hline
3^b & 9^a & \underline{4^h} & 10^g & 5^d & 11^c & & & 1^f & 7^e & & \\
 & & 10^a & 4^b & 11^g & \underline{5^h} & 6^c & 0^d & & & 8^e & 2^f \\
\hline
4^f & 10^e & & & 0^b & 6^a & \underline{1^h} & 7^g & 2^d & 8^c & & \\
10^d & 4^c & & & 6^f & 0^e & & & 8^b & 2^a & \underline{9^h} & 3^g \\
\hline
\underline{11^h} & 5^g & 0^f & 6^e & & & 2^b & 8^a & & & 4^d & 10^c \\
 & & 6^d & 0^c & & & 8^f & 2^e & \underline{3^h} & 9^g & 10^b & 4^a
\end{array}\right).
$$

The next 2-plex to consider is $T+2$; the unlabeled version is shown below:

$$
T+2 = \begin{pmatrix}
0 & 1 & 2 & \mathit{3}^* & \mathit{4}^* & 5 \\
1 & \mathit{2}^* & 3 & 4 & 5 & \mathit{0}^* \\
\mathit{2}^* & 3 & \mathit{4}^* & 5 & 0 & 1 \\
3 & 4 & 5 & \mathit{0}^* & \mathit{1}^* & 2 \\
4 & \mathit{5}^* & 0 & 1 & 2 & \mathit{3}^* \\
\mathit{5}^* & 0 & \mathit{1}^* & 2 & 3 & 4
\end{pmatrix}.
$$

The 2-plex $T+2$ together with π and τ yields the underlined transversals $T'_{0,0}+2$ and $T'_{0,1}+2$ shown below in J_{12} marked by the superscripts i and j, respectively. Note that i and j are obtained by shifting the transversals g and h, respectively, four cells to the right:

$$J_{12} = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0^a & 6^b & 1^g & 7^h & 2^c & 8^d & \underline{3^i} & \underline{9^j} & 4^e & 10^f & & \\
 & & 7^b & 1^a & 8^h & 2^g & 9^d & 3^c & \underline{10^j} & \underline{4^i} & 11^f & 5^e \\
\hline
1^e & 7^f & & & 3^a & 9^b & 4^g & 10^h & 5^c & 11^d & \underline{0^i} & \underline{6^j} \\
7^c & 1^d & \underline{8^i} & \underline{2^j} & 9^e & 3^f & & & 11^a & 5^b & 6^g & 0^h \\
\hline
\underline{2^i} & \underline{8^j} & 3^e & 9^f & & & 5^a & 11^b & 0^g & 6^h & 1^c & 7^d \\
8^g & 2^h & 9^c & 3^d & \underline{10^i} & \underline{4^j} & 11^e & 5^f & & & 7^a & 1^b \\
\hline
3^b & 9^a & 4^h & 10^g & 5^d & 11^c & \underline{0^j} & \underline{6^i} & 1^f & 7^e & & \\
 & & 10^a & 4^b & 11^g & 5^h & 6^c & 0^d & \underline{7^i} & \underline{1^j} & 8^e & 2^f \\
\hline
4^f & 10^e & & & 0^b & 6^a & 1^h & 7^g & 2^d & 8^c & \underline{3^j} & \underline{9^i} \\
10^d & 4^c & \underline{11^j} & \underline{5^i} & 6^f & 0^e & & & 8^b & 2^a & 9^h & 3^g \\
\hline
11^h & 5^g & 0^f & 6^e & \underline{7^j} & \underline{1^i} & 2^b & 8^a & & & 4^d & 10^c \\
\underline{5^j} & \underline{11^i} & 6^d & 0^c & & & 8^f & 2^e & 3^h & 9^g & 10^b & 4^a
\end{array}\right).$$

The final 2-plex to consider is $T+4$; the unlabeled version is shown below:

$$T+4 = \begin{pmatrix}
\mathit{0}^* & 1 & 2 & 3 & 4 & \mathit{5}^* \\
1 & \mathit{2}^* & 3 & \mathit{4}^* & 5 & 0 \\
2 & 3 & \mathit{4}^* & 5 & \mathit{0}^* & 1 \\
\mathit{3}^* & 4 & 5 & 0 & 1 & \mathit{2}^* \\
4 & \mathit{5}^* & 0 & \mathit{1}^* & 2 & 3 \\
5 & 0 & \mathit{1}^* & 2 & \mathit{3}^* & 4
\end{pmatrix}.$$

The 2-plex $T+4$ together with π and τ yields the final two underlined transversals $T'_{0,0}+4$ and $T'_{0,1}+4$ shown below in J_{12} marked by the superscripts k and ℓ, respectively. Note that k and ℓ are again obtained by shifting the transversals i and j, respectively, four cells to the right. The complete decomposition of J_{12} is given below:

$$J_{12} = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0^a & 6^b & 1^g & 7^h & 2^c & 8^d & 3^i & 9^j & 4^e & 10^f & \underline{5^k} & \underline{11^\ell} \\
\underline{6^\ell} & \underline{0^k} & 7^b & 1^a & 8^h & 2^g & 9^d & 3^c & 10^j & 4^i & 11^f & 5^e \\
\hline
1^e & 7^f & \underline{2^k} & \underline{8^\ell} & 3^a & 9^b & 4^g & 10^h & 5^c & 11^d & 0^i & 6^j \\
7^c & 1^d & 8^i & 2^j & 9^e & 3^f & \underline{10^k} & \underline{4^\ell} & 11^a & 5^b & 6^g & 0^h \\
\hline
2^i & 8^j & 3^e & 9^f & \underline{4^k} & \underline{10^\ell} & 5^a & 11^b & 0^g & 6^h & 1^c & 7^d \\
8^g & 2^h & 9^c & 3^d & 10^i & 4^j & 11^e & 5^f & \underline{6^k} & \underline{0^\ell} & 7^a & 1^b \\
\hline
3^b & 9^a & 4^h & 10^g & 5^d & 11^c & 0^j & 6^i & 1^f & 7^e & \underline{2^\ell} & \underline{8^k} \\
\underline{9^k} & \underline{3^\ell} & 10^a & 4^b & 11^g & 5^h & 6^c & 0^d & 7^i & 1^j & 8^e & 2^f \\
\hline
4^f & 10^e & \underline{5^\ell} & \underline{11^k} & 0^b & 6^a & 1^h & 7^g & 2^d & 8^c & 3^j & 9^i \\
10^d & 4^c & 11^j & 5^i & 6^f & 0^e & \underline{7^\ell} & \underline{1^k} & 8^b & 2^a & 9^h & 3^g \\
\hline
11^h & 5^g & 0^f & 6^e & 7^j & 1^i & 2^b & 8^a & \underline{9^\ell} & \underline{3^k} & 4^d & 10^c \\
5^j & 11^i & 6^d & 0^c & \underline{1^\ell} & \underline{7^k} & 8^f & 2^e & 3^h & 9^g & 10^b & 4^a
\end{array}\right).$$

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