

# The Quadratic Approximation for Quintessence with Arbitrary Initial Conditions

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We examine models of quintessence in which a minimally-coupled scalar field  $\phi$  evolves near a local extremum of its potential  $V(\phi)$  at  $\phi_*$ . Assuming that  $(1/V)(\partial_\phi V)$  is small and  $w \approx -1$ , we Taylor expand the potential about  $\phi_*$  and derive a general expression for  $w(a)$ . The dynamics of this field are determined by the initial and final equation of state parameters  $w_i$  and  $w_0$ , the quantity  $V'''(\phi_*)/V(\phi_*)$ , and the direction of  $\dot{\phi}_i$  in relation to  $\dot{\phi}_0$ . This approximation is then tested for six values of  $V'''(\phi_*)/V(\phi_*)$  and shown to lie within 2% of the exact solution for five of these cases. However, the model becomes less precise near certain values of  $V'''(\phi_*)/V(\phi_*)$  where  $\dot{\phi}$  becomes very large.

## I. INTRODUCTION

Dark energy, the mysterious component of the universe responsible for its accelerated expansion, has been shown by experiment to constitute approximately 70% of its total energy density (see Refs. [1–9]). Although there has been no shortage of possible explanations for this late-time acceleration, we still know almost nothing of dark energy’s true nature. This startling premise—that 70% of our universe is still completely unknown to us—reveals that although we have made great strides in understanding the cosmos, there may still be much more to learn.

In this thesis, I will explore a particular model of dark energy known as quintessence. After providing a brief background on the physics of dark energy and the motivation behind quintessence, I will present the findings that Prof. Scherrer and I have made over the past year: that the plethora of potential curves for quintessence result in similar behavior which can be categorized by a much smaller class of models. This result has been previously shown for special cases, but this thesis will demonstrate that the same holds true for arbitrary initial conditions. I will first derive the class of models that we are proposing, which will then be compared to numerical solutions for various different potential curves. These results will then hopefully allow us to place meaningful limits on quintessence models as we continue to gather data on the accelerating universe.

### A. The Evolution of the Cosmos

The story of dark energy begins with Edwin Hubble’s discovery in 1929 that distant galaxies are receding away from us. He observed that these galaxies’ redshifts—and hence, velocities—were directly proportional to their distance from us, a formula known as Hubble’s Law. In light of Einstein’s theory of general relativity, this observation could best be understood as a homogeneous, isotropic expansion of space itself. The distance between any two points in space, then, is given by the formula

$$d_p = a(t)r \quad (1)$$

where  $d_p$  is the proper distance,  $r$  is a co-moving coordinate, unchanging for two objects at rest with respect to the expansion of space, and  $a(t)$  is the scale factor, a quantity representing the size of the universe at any given moment, generally normalized to 1 at the present time. Taking the time derivative yields

$$v_p = \dot{d}_p = \dot{a}r = \frac{\dot{a}}{a}d_p \quad (2)$$

which is Hubble’s Law. The quantity  $\dot{a}/a$  is called the Hubble parameter and is usually written as  $H$ , while its current value  $H_0$  is called the Hubble constant.

In 1922, Alexander Friedmann solved the equations of general relativity to relate the scale factor  $a$  to the energy density  $\rho$  and the pressure  $p$  of the universe in the following system of equations:

$$H^2 = \frac{\rho}{3} - \frac{k}{R_0^2 a(t)^2} \quad (3)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \quad (4)$$

in units where  $\hbar = c = 8\pi G = 1$ . The quantity  $k$  describes the curvature of the universe and takes on values of  $+1$ ,  $0$ , or  $-1$  for a universe that is positively curved, flat, or negatively curved, respectively.  $R_0$  is the radius of this curvature, and because the universe has been measured to be essentially flat [7, 8], we will assume that either  $k = 0$  or  $R_0$  is large enough to make this curvature term negligible. From here onward, we will therefore assume a flat universe. It can also be useful to express (3) in the form

$$\Omega(a) - 1 = \frac{k}{R_0^2 a(t)^2 H_0^2} \approx 0 \quad (5)$$

where  $\Omega(a)$  is the density parameter defined by  $\Omega(a) = \rho/3H^2$ . Since the right side of (5) is approximately 0, we can deduce that  $\Omega \approx 1$ .

The Friedmann equations in a flat universe can then be solved if the relationship between density and pressure is determined. The simplest kind of barotropic fluid with equation of state

$$p = w\rho \quad (6)$$

yields the solution

$$\rho(a) = \rho_0 a^{3(1+w)} \quad (7)$$

and hence

$$a(t) \propto t^{2/3(1+w)}, w \neq -1 \quad (8)$$

Furthermore, we see very clearly from (4) and (8) that only a fluid with  $w < -1/3$  leads to accelerated expansion. Since nonrelativistic matter has  $w = 0$  and radiation has  $w = 1/3$ , the late-time acceleration observed is only explainable through a more exotic type of energy that possesses this unusual quality of a negative pressure.

## B. The Cosmological Constant

Because the substances thought to contribute the vast majority of the universe's energy density—matter and radiation—both have  $w > 0$ , it had been taken for granted that the expansion of the universe was decelerating. Naturally, experiments were conducted to precisely measure the rate of this deceleration as given by the deceleration parameter,  $q = -\frac{\ddot{a}a}{\dot{a}^2}$ , with the surprising result that  $q$  is negative. In 1998, a comparison of the redshift and apparent luminosity of Type Ia supernovae showed with over 99% confidence that the universe is currently accelerating [1, 3], indicating that something was missing from our cosmological picture. The enigmatic culprit for this late-time acceleration has since been dubbed “dark energy.”

One of the first suspects for dark energy was a concept first introduced—and later discarded—by Einstein: a parameter known as the cosmological constant. When Einstein developed his theory of general relativity, he attempted to solve his equations for the case of a static universe ( $H = \dot{H} = 0$ ) and realized this was only possible with the inclusion of an additional term  $\Lambda$ , set precisely at the value  $\frac{1}{2\rho}$ . With this extra term, the Friedmann equations (3) and (4) then become

$$H^2 = \frac{\rho}{3} + \frac{\Lambda}{3} \quad (9)$$

$$\frac{\ddot{a}}{a} = -\frac{\rho}{6}(\rho + 3p) + \frac{\Lambda}{3} \quad (10)$$

It can be seen from these equations that the cosmological constant  $\Lambda$  functions as an additional component of the energy density with  $w = -1$ , which we would expect from (7). Furthermore, the supernova data show that, since  $\Omega \approx 1$ , the best fit for a universe composed of matter and a cosmological constant has the partial densities  $\Omega_{m0} \approx 0.3$  and  $\Omega_{\Lambda0} \approx 0.7$ . With only these two components in a flat universe, the Friedmann equation can be rearranged as

$$\frac{H^2}{H_0^2} = \Omega_{m0} a^{-3} + \Omega_{\Lambda0} \quad (11)$$

where (7) has been substituted for  $\rho(a)$ . This differential equation then has the solution

$$a(t) = \sqrt[3]{\Omega_{\Lambda0}^{-1} - 1} \sinh^{2/3}(t/t_\Lambda) \quad (12)$$

where  $t_\Lambda = 2/\sqrt{3\Lambda}$

### C. Beyond Lambda: Quintessence Models

While all of the current data are consistent with a universe dominated by a cosmological constant and nonrelativistic matter—often called the  $\Lambda$ CDM model—we nonetheless have no reason to restrict ourselves to such a model. If this exotic form of energy is not constant but allowed to vary through time, the simplest model that can be ascribed to it is a minimally-coupled scalar field. Such a proposed scalar field has been given the name ‘quintessence’ after the fifth element of the ancient Greeks; now, it represents the hypothesized fifth element of modern physics in addition to baryons, leptons, radiation, and dark matter. From quantum field theory, a scalar field  $\phi$  moving in a potential  $V(\phi)$  enjoys the following properties:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (13)$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad (14)$$

With this density and pressure, the Friedmann equations can be modified to the following form:

$$H^2 = \frac{1}{3} \left[ \rho_{m0} a^{-3} + \frac{1}{2}\dot{\phi}^2 + V(\phi) \right] \quad (15)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \quad (16)$$

In addition, (13) and (14) also determine the equation-of-state parameter

$$w_\phi = \frac{p_\phi}{\rho_\phi} = -1 + \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2V(\phi)} \quad (17)$$

Thus, quintessence produces an accelerated expansion when  $\dot{\phi}^2 < V(\phi)$ , which occurs only for a relatively flat potential. This acceleration is guaranteed when the following slow-roll conditions are satisfied:

$$\left( \frac{1}{V} \frac{dV}{d\phi} \right)^2 \ll 1 \quad (18)$$

$$\left| \frac{1}{V} \frac{d^2V}{d\phi^2} \right| \ll 1 \quad (19)$$

yet this is only a sufficient condition, not a necessary one. As a result, we wish to investigate solutions which extend beyond the narrow scope of slow-roll quintessence and relax this second condition in (19).

## II. MOTIVATION

After the theory of quintessence was developed, an enormous number of scalar-field potentials have since been proposed which agree with cosmological observations [11]. Rather than investigate each of these alternatives separately, we wish to categorize those quintessence models which yield similar behavior for  $w(a)$ , and hence for  $a(t)$ . The main objective of this thesis is to continue the work begun in Refs. [12–15], in which the trajectory of  $w(a)$  was modeled for a large class of potentials with only a few free parameters. Recent observational evidence has shown that  $w_0 \approx -1$  (where the subscript ‘0’ will herein denote the present value), and further assuming that  $w$  is close to  $-1$  at all times in the past allows (15) and (16) to be simplified considerably.

Ref. [12] examined the case of “thawing” potentials, in which the field is constrained to an initial velocity  $\dot{\phi} = 0$ . In addition, the potential was assumed to satisfy both slow-roll conditions (18) and (19) and have a roughly constant first slow-roll parameter  $(1/V)(\partial_\phi V)$ . Such potentials were shown to evolve according to a single class of models depending only on the parameters  $w_0$  and  $\Omega_{\phi 0}$ . Ref. [13] extended these results the more general case where  $w_i \neq -1$ , introducing the a third adjustable parameter  $w_i$  into the expression for  $w(a)$ .

In later work, these assumptions were further relaxed to permit (19) to be violated, as in the case of a field evolving near a local maximum [14] or minimum [15] of the potential at  $\phi = \phi_*$ . By Taylor expanding around this point and assuming  $w_i = -1$ , an expression was derived for  $w(a)$  which depended on the additional parameter  $V''(\phi_*)/V(\phi_*)$ . This result was further generalized in Ref. [16] where the Taylor expansion was not limited to a point at which  $V'(\phi) = 0$ .

This thesis attempts to introduce an additional degree of freedom to the results of Refs. [14] and [15] by allowing for  $\dot{\phi}_i \neq 0$ . As we will see, this leads to an extra parameter  $w_0$  in the expression for  $w(a)$  along with a variable ‘ $\pm$ ’ determined by the signs of  $\dot{\phi}_i$  and  $\dot{\phi}_0$ . In Section III, we present a derivation of the behavior of  $w(a)$  from the Friedmann equations, and in Section IV, this result is extended to special cases of the parameter  $V''(\phi_*)/V(\phi_*)$ . Then, in Section V, we present a method to generate numerical solutions which match these parameters and test the validity of our approximations.

### III. THE DERIVATION

#### A. Initial Assumptions

We begin as in previous work by assuming a flat universe filled with matter and a scalar field. Also, we restrict our attention to quintessence models in which  $w \approx -1$  so that we can make use of the relation

$$\rho_\phi \approx \rho_{\phi 0} \approx -p_\phi \quad (20)$$

This condition is satisfied when the potential is relatively flat, i.e. when (18) is satisfied, as in the case of a hilltop or a valley. In reduced Planck units, such a universe is described by (15) and (16). Equation (16) can be greatly simplified by the substitution

$$u = (\phi - \phi_*)a^{3/2} \quad (21)$$

where  $\phi_*$  is, as in [14] and [15], the local maximum or minimum of the potential. Substituted into (16), this yields

$$\ddot{u} + \frac{3}{4}p_T u + a^{3/2} \frac{dV}{d\phi} = 0 \quad (22)$$

Furthermore, we can expand such a potential about its maximum or minimum at  $\phi_*$  to obtain the approximation

$$V(\phi) \approx V(\phi_*) + \frac{1}{2}V''(\phi_*)(\phi - \phi_*)^2 \quad (23)$$

Using this Taylor expansion, along with the assumption that  $p_T \approx -\rho_{\phi 0} \approx V(\phi_*)$ , (22) becomes

$$\ddot{u} + [V''(\phi_*) - \frac{3}{4}V(\phi_*)]u = 0 \quad (24)$$

which can be easily solved for  $u(t)$ .

#### B. Solving for $\phi$

Equation (24) yields the general solution:

$$u = Ce^{kt} + De^{-kt} \quad (25)$$

where

$$k = \sqrt{\frac{3}{4}V(\phi_*) - V''(\phi_*)} \quad (26)$$

Then, taking the derivative and dividing by  $k$ , we have

$$\frac{\dot{u}}{k} = Ce^{kt} - De^{-kt} \quad (27)$$

From here, we solve for  $C$  and  $D$  as follows:

$$C = \frac{ku + \dot{u}}{2k} e^{-kt} \quad (28)$$

$$D = \frac{ku - \dot{u}}{2k} e^{kt} \quad (29)$$

Then, using (21) and its derivative

$$\dot{u} = \dot{\phi} a^{3/2} + \frac{3}{2}(\phi - \phi_*) H a^{3/2} \quad (30)$$

we can express  $C$  and  $D$  in terms of the initial conditions  $\phi_i$  and  $\dot{\phi}_i$  at some  $t_i$  (and  $a_i$ ) in the distant past.

To get rid of the hanging  $\phi_*$  term, we first shift  $\phi$  so that  $\phi_* = 0$ . Then substituting (21) and (30) into (28) and (29) and evaluating at  $t_i$ , we obtain

$$C = \frac{k\phi_i + \dot{\phi}_i + \frac{3}{2}\phi_i H_i}{2k} a_i^{3/2} e^{-kt_i} \quad (31)$$

and

$$D = \frac{k\phi_i - \dot{\phi}_i - \frac{3}{2}\phi_i H_i}{2k} a_i^{3/2} e^{kt_i} \quad (32)$$

When these values are inserted into (25), we arrive at our first useful result

$$\phi = \frac{1}{2k} \left( \frac{a}{a_i} \right)^{-3/2} \left\{ \left[ \dot{\phi}_i + \left( \frac{3}{2} H_i + k \right) \phi_i \right] e^{k(t-t_i)} - \left[ \dot{\phi}_i + \left( \frac{3}{2} H_i - k \right) \phi_i \right] e^{-k(t-t_i)} \right\} \quad (33)$$

### C. Using the $\Lambda$ CDM approximation

Next, we attempt to express  $\phi$  solely as a function of time. Since we assume that  $w \approx -1$  and hence that  $\rho_\phi$  is approximately constant, we can use the solution to the Friedmann equations in a universe with only matter and a cosmological constant  $\Lambda$ . This is given by the following modified version of (12):

$$a(t) = \sqrt[3]{\Omega_{\phi 0}^{-1} - 1} \sinh^{2/3}(t/t_\Lambda) \quad (34)$$

where now  $t_\Lambda = 2/\sqrt{3\rho_{\phi 0}}$ . The Hubble parameter then can be found by the relation

$$H(t) = \frac{d}{dt} \ln(a) = \frac{2}{3t_\Lambda} \coth(t/t_\Lambda) \quad (35)$$

Substituting back into (33), this gives us

$$\phi(t) = \frac{1}{2k} \frac{\sinh(t_i/t_\Lambda)}{\sinh(t/t_\Lambda)} \left\{ \left[ \dot{\phi}_i + \left( \frac{\coth(t_i/t_\Lambda)}{t_\Lambda} + k \right) \phi_i \right] e^{k(t-t_i)} - \left[ \dot{\phi}_i + \left( \frac{\coth(t_i/t_\Lambda)}{t_\Lambda} - k \right) \phi_i \right] e^{-k(t-t_i)} \right\} \quad (36)$$

As a check, we note that as  $t_i \rightarrow 0$ :

$$\begin{aligned} \phi(t) &= \frac{1}{2k \sinh(t/t_\Lambda)} \left[ \left( \frac{\phi_i \cosh(t_i/t_\Lambda)}{t_\Lambda} \right) e^{kt} - \left( \frac{\phi_i \cosh(t_i/t_\Lambda)}{t_\Lambda} \right) e^{-kt} \right] \\ &= \frac{\phi_i}{kt_\Lambda} \frac{\sinh(kt)}{\sinh(t/t_\Lambda)} \end{aligned} \quad (37)$$

in agreement with Ref. [14]. Now we can distribute the  $\sinh(t_i/t_\Lambda)$  and take the time derivative to find

$$\begin{aligned} \dot{\phi}(t) &= \left[ \frac{\phi_i}{kt_\Lambda} \cosh(t_i/t_\Lambda) + \frac{\dot{\phi}_i}{k} \sinh(t_i/t_\Lambda) \right] \left[ \frac{k \cosh(k(t-t_i)) \sinh(t/t_\Lambda) - t_\Lambda^{-1} \sinh(k(t-t_i)) \cosh(t/t_\Lambda)}{\sinh^2(t/t_\Lambda)} \right] \\ &\quad + \phi_i \sinh(t_i/t_\Lambda) \left[ \frac{k \sinh(k(t-t_i)) \sinh(t/t_\Lambda) - t_\Lambda^{-1} \cosh(k(t-t_i)) \cosh(t/t_\Lambda)}{\sinh^2(t/t_\Lambda)} \right] \end{aligned} \quad (38)$$

#### D. Converting to $w(a)$

From here, we can solve for the parameter of state  $w$  via the relation

$$1 + w = \frac{\dot{\phi}^2}{\rho_{\phi 0}} = \frac{3}{4} \dot{\phi}^2 t_\Lambda^2 \quad (39)$$

Using (38) for the value of  $\dot{\phi}$ , we have

$$1 + w(t) = \frac{3}{4} \left\{ \left[ \frac{\phi_i}{kt_\Lambda} \cosh(t_i/t_\Lambda) + \frac{\dot{\phi}_i}{k} \sinh(t_i/t_\Lambda) \right] \left[ \frac{kt_\Lambda \cosh(k(t-t_i)) \sinh(t/t_\Lambda) - \sinh(k(t-t_i)) \cosh(t/t_\Lambda)}{\sinh^2(t/t_\Lambda)} \right] \right. \\ \left. + \phi_i \sinh(t_i/t_\Lambda) \left[ \frac{kt_\Lambda \sinh(k(t-t_i)) \sinh(t/t_\Lambda) - \cosh(k(t-t_i)) \cosh(t/t_\Lambda)}{\sinh^4(t/t_\Lambda)} \right] \right\}^2 \quad (40)$$

Then, in order to normalize  $1 + w$  to the present day value, i.e. for  $a = 1$ , we must change the variable back from  $t$  to  $a$  with the  $\Lambda$ CDM approximation in (12). The hyperbolic functions in (40) then simplify to

$$\sinh(t/t_\Lambda) = \frac{a^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} \quad (41)$$

$$\cosh(t/t_\Lambda) = \sqrt{1 + \sinh^2(t/t_\Lambda)} = \sqrt{1 + \frac{a^3}{\Omega_{\phi 0}^{-1} - 1}} \quad (42)$$

$$\sinh(k(t-t_i)) = \sinh \left[ kt_\Lambda \left( \sinh^{-1} \left( \frac{a^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} \right) - \sinh^{-1} \left( \frac{a_i^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} \right) \right) \right] \\ = \sinh \left[ kt_\Lambda \ln \left( \frac{\frac{a^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a^3}{\Omega_{\phi 0}^{-1} - 1}}}{\frac{a_i^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a_i^3}{\Omega_{\phi 0}^{-1} - 1}}} \right) \right] \quad (43)$$

$$= \frac{1}{2} \left[ \left( \frac{\frac{a^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a^3}{\Omega_{\phi 0}^{-1} - 1}}}{\frac{a_i^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a_i^3}{\Omega_{\phi 0}^{-1} - 1}}} \right)^{kt_\Lambda} - \left( \frac{\frac{a^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a^3}{\Omega_{\phi 0}^{-1} - 1}}}{\frac{a_i^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a_i^3}{\Omega_{\phi 0}^{-1} - 1}}} \right)^{-kt_\Lambda} \right] \\ \cosh(k(t-t_i)) = \frac{1}{2} \left[ \left( \frac{\frac{a^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a^3}{\Omega_{\phi 0}^{-1} - 1}}}{\frac{a_i^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a_i^3}{\Omega_{\phi 0}^{-1} - 1}}} \right)^{kt_\Lambda} + \left( \frac{\frac{a^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a^3}{\Omega_{\phi 0}^{-1} - 1}}}{\frac{a_i^{3/2}}{\sqrt{\Omega_{\phi 0}^{-1} - 1}} + \sqrt{1 + \frac{a_i^3}{\Omega_{\phi 0}^{-1} - 1}}} \right)^{-kt_\Lambda} \right] \quad (44)$$

Then, using the function  $F(a)$ , defined as

$$F(a) = \sqrt{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}} \quad (45)$$

we can greatly simplify these expressions as follows:

$$\sinh(t/t_\Lambda) = \frac{1}{\sqrt{F(a)^2 - 1}} \quad (46)$$

$$\cosh(t/t_\Lambda) = \frac{F(a)}{\sqrt{F(a)^2 - 1}} \quad (47)$$

$$\begin{aligned}
\sinh(k(t-t_i)) &= \frac{1}{2} \left[ \left( \frac{a}{a_i} \right)^{\frac{3kt_\Lambda}{2}} \left( \frac{F(a)+1}{F(a_i)+1} \right)^{kt_\Lambda} - \left( \frac{a}{a_i} \right)^{\frac{-3kt_\Lambda}{2}} \left( \frac{F(a)+1}{F(a_i)+1} \right)^{-kt_\Lambda} \right] \\
&= \frac{1}{2} \left[ \left( \frac{a}{a_i} \right)^{\frac{3kt_\Lambda}{2}} \left( \frac{F(a)+1}{F(a_i)+1} \right)^{kt_\Lambda} - \left( \frac{a}{a_i} \right)^{\frac{3kt_\Lambda}{2}} \left( \frac{F(a)^2-1}{F(a_i)^2-1} \right)^{kt_\Lambda} \left( \frac{F(a)+1}{F(a_i)+1} \right)^{-kt_\Lambda} \right] \\
&= \frac{1}{2} \left( \frac{a}{a_i} \right)^{\frac{3kt_\Lambda}{2}} \left[ \left( \frac{F(a)+1}{F(a_i)+1} \right)^{kt_\Lambda} - \left( \frac{F(a)-1}{F(a_i)-1} \right)^{kt_\Lambda} \right]
\end{aligned} \tag{48}$$

$$\cosh(k(t-t_i)) = \frac{1}{2} \left( \frac{a}{a_i} \right)^{\frac{3kt_\Lambda}{2}} \left[ \left( \frac{F(a)+1}{F(a_i)+1} \right)^{kt_\Lambda} + \left( \frac{F(a)-1}{F(a_i)-1} \right)^{kt_\Lambda} \right] \tag{49}$$

Thus, we can finally substitute these expressions into (40) to determine  $1+w(a)$ . After some factorization, we obtain

$$\begin{aligned}
1+w(a) &= \frac{3}{16K^2} \left( \frac{a}{a_i} \right)^{3(K-1)} \left\{ (K-F(a)) \left[ \dot{\phi}_i t_\Lambda + \phi_i(F(a_i)+K) \right] \left( \frac{F(a)+1}{F(a_i)+1} \right)^K \right. \\
&\quad \left. + (K+F(a)) \left[ \dot{\phi}_i t_\Lambda + \phi_i(F(a_i)-K) \right] \left( \frac{F(a)-1}{F(a_i)-1} \right)^K \right\}^2
\end{aligned} \tag{50}$$

where  $K = kt_\Lambda$ . Furthermore, the expression  $\dot{\phi}_i t_\Lambda$  is just  $\pm \sqrt{\frac{4}{3}(1+w_i)}$ , and so we can write the part of (50) inside braces as the sum of two terms—one of  $\phi_i$  and the other of  $1+w_i$ :

$$\begin{aligned}
1+w(a) &= \frac{3}{16K^2} \left( \frac{a}{a_i} \right)^{3(K-1)} \left\{ \phi_i \left[ (K-F(a))(F(a_i)+K) \left( \frac{F(a)+1}{F(a_i)+1} \right)^K + (K+F(a))(F(a_i)-K) \left( \frac{F(a)-1}{F(a_i)-1} \right)^K \right] \right. \\
&\quad \left. \pm \sqrt{\frac{4}{3}(1+w_i)} \left[ (K-F(a)) \left( \frac{F(a)+1}{F(a_i)+1} \right)^K + (K+F(a)) \left( \frac{F(a)-1}{F(a_i)-1} \right)^K \right] \right\}^2
\end{aligned} \tag{51}$$

In particular, we find that for  $a = a_0 = 1$ , (51) can be rearranged to give

$$\phi_i = \frac{2K \left( \frac{a_i}{a_0} \right)^{\frac{3}{2}(K-1)} \sqrt{\frac{4}{3}(1+w_0)} \mp \left[ (K-F(a_0)) \left( \frac{F(a_0)+1}{F(a_i)+1} \right)^K + (K+F(a_0)) \left( \frac{F(a_0)-1}{F(a_i)-1} \right)^K \right] \sqrt{\frac{4}{3}(1+w_i)}}{(K-F(a_0))(F(a_i)+K) \left( \frac{F(a_0)+1}{F(a_i)+1} \right)^K + (K+F(a_0))(F(a_i)-K) \left( \frac{F(a_0)-1}{F(a_i)-1} \right)^K} \tag{52}$$

Substitution into (51), after some rearrangement, yields our main result:

$$1+w(a) = \left( \frac{X_K(a)\sqrt{1+w_0} \mp Y_K(a)\sqrt{1+w_i}}{Z_K} \right)^2 \tag{53}$$

where

$$\begin{aligned}
X_K(a) &= \left( \frac{a}{a_i} \right)^{\frac{3}{2}K} \left( \frac{a}{a_0} \right)^{-\frac{3}{2}} \left[ (F(a_i)+K)(K-F(a)) \left( \frac{F(a)+1}{F(a_i)+1} \right)^K \right. \\
&\quad \left. + (F(a_i)-K)(K+F(a)) \left( \frac{F(a)-1}{F(a_i)-1} \right)^K \right]
\end{aligned} \tag{54}$$

$$\begin{aligned}
Y_K(a) &= \left( \frac{a}{a_0} \right)^{\frac{3}{2}K} \left( \frac{a}{a_i} \right)^{-\frac{3}{2}} \left[ (F(a_0)+K)(K-F(a)) \left( \frac{F(a)+1}{F(a_0)+1} \right)^K \right. \\
&\quad \left. + (F(a_0)-K)(K+F(a)) \left( \frac{F(a)-1}{F(a_0)-1} \right)^K \right]
\end{aligned} \tag{55}$$

and

$$Z_K = (K - F(a_0)) (F(a_i) + K) \left[ \frac{(F(a_0) + 1) (F(a_i) - 1)}{(F(a_i) + 1) (F(a_0) - 1)} \right]^{K/2} \\ + (K + F(a_0)) (F(a_i) - K) \left[ \frac{(F(a_i) + 1) (F(a_0) - 1)}{(F(a_0) + 1) (F(a_i) - 1)} \right]^{K/2} \quad (56)$$

There is a readily apparent symmetry between  $X_K(a)$  and  $Y_K(a)$ , which can be transformed into each other by replacing  $a_i$  with  $a_0$  and vice versa.

### E. Various Checks

A quick glance at (54) and (55) shows that

$$X_K(a_i) = Y_K(a_0) = 0 \quad (57)$$

but it is also useful to show that

$$X_K(a_0) = (K - F(a_0)) (F(a_i) + K) \left( \frac{a_0}{a_i} \right)^{3K/2} \left( \frac{F(a_0) + 1}{F(a_i) + 1} \right)^K \\ + (K + F(a_0)) (F(a_i) - K) \left( \frac{a_0}{a_i} \right)^{3K/2} \left( \frac{F(a_0) - 1}{F(a_i) - 1} \right)^K \\ = (K - F(a_0)) (F(a_i) + K) \sqrt{\frac{F(a_i)^2 - 1}{F(a_0)^2 - 1}}^{-K} \left( \frac{F(a_0) + 1}{F(a_i) + 1} \right)^K \\ + (K + F(a_0)) (F(a_i) - K) \sqrt{\frac{F(a_i)^2 - 1}{F(a_0)^2 - 1}}^{-K} \left( \frac{F(a_0) - 1}{F(a_i) - 1} \right)^K \quad (58) \\ = (K - F(a_0)) (F(a_i) + K) \left[ \frac{(F(a_0) + 1) (F(a_i) - 1)}{(F(a_i) + 1) (F(a_0) - 1)} \right]^{K/2} \\ + (K + F(a_0)) (F(a_i) - K) \left[ \frac{(F(a_i) + 1) (F(a_0) - 1)}{(F(a_0) + 1) (F(a_i) - 1)} \right]^{K/2} \\ = Z_K$$

And similarly,

$$Y_K(a_i) = -Z_K \quad (59)$$

This identity preserves the symmetry between  $X_K$  and  $Y_K$  and, taken together with (53) and (57), guarantees that  $w(a_i) = w_i$  and  $w(a_0) = w_0$ . This also allows us to write (53) in the simpler form:

$$1 + w(a) = \left( \frac{X_K(a)}{X_K(a_0)} \sqrt{1 + w_0} \pm \frac{Y_K(a)}{Y_K(a_i)} \sqrt{1 + w_i} \right)^2 \quad (60)$$

Another important check is the behavior of  $1 + w(a)$  as  $a_i \rightarrow 0$ . In this limit,  $F(a_i) \rightarrow \infty$ , and so the expressions  $F(a_i) \pm 1$  and  $F(a_i) \pm K$  approach  $F(a_i)$ . Thus, substituting 1 for  $a_0$ ,  $X_K/X_K(a_0)$  and  $Y_K/Y_K(a_i)$  become:

$$\frac{X_K(a)}{X_K(a_0)} = a^{\frac{3}{2}(K-1)} \left[ \frac{(K - F(a)) (F(a) + 1)^K + (K + F(a)) (F(a) - 1)^K}{\left( (K - \Omega_{\phi_0}^{-1/2}) (\Omega_{\phi_0}^{-1/2} + 1) \right)^K + \left( (K + \Omega_{\phi_0}^{-1/2}) (\Omega_{\phi_0}^{-1/2} - 1) \right)^K} \right] \quad (61)$$



$$\begin{aligned}
\frac{Y_K(a)}{Y_K(a_i)} &= \left(\frac{a}{a_i}\right)^{\frac{3}{2}(K-1)} \left[ \frac{\left(\Omega_{\phi_0}^{-1/2} + K\right) (K - F(a)) \left(\frac{F(a)+1}{\Omega_{\phi_0}^{-1/2}+1}\right)^K + \left(\Omega_{\phi_0}^{-1/2} - K\right) (K + F(a)) \left(\frac{F(a)-1}{\Omega_{\phi_0}^{-1/2}-1}\right)^K}{\left(\Omega_{\phi_0}^{-1/2} + K\right) (K - F(a_i)) \left(\frac{F(a_i)+1}{\Omega_{\phi_0}^{-1/2}+1}\right)^K + \left(\Omega_{\phi_0}^{-1/2} - K\right) (K + F(a_i)) \left(\frac{F(a_i)-1}{\Omega_{\phi_0}^{-1/2}-1}\right)^K} \right] \\
&\approx \left(\frac{a}{a_i}\right)^{\frac{3}{2}(K-1)} F(a_i)^{-(K+1)} \\
&\times \left[ \frac{\left(\Omega_{\phi_0}^{-1/2} + K\right) (K - F(a)) \left(\frac{F(a)+1}{\Omega_{\phi_0}^{-1/2}+1}\right)^K + \left(\Omega_{\phi_0}^{-1/2} - K\right) (K + F(a)) \left(\frac{F(a)-1}{\Omega_{\phi_0}^{-1/2}-1}\right)^K}{-\left(\Omega_{\phi_0}^{-1/2} + K\right) \left(\Omega_{\phi_0}^{-1/2} + 1\right)^{-K} + \left(\Omega_{\phi_0}^{-1/2} - K\right) \left(\Omega_{\phi_0}^{-1/2} - 1\right)^{-K}} \right] \\
&\approx a_i^3 \left(\Omega_{\phi_0}^{-1} - 1\right)^{-(K+1)/2} a^{\frac{3}{2}(K-1)} \\
&\times \left[ \frac{\left(\Omega_{\phi_0}^{-1/2} + K\right) (K - F(a)) \left(\frac{F(a)+1}{\Omega_{\phi_0}^{-1/2}+1}\right)^K + \left(\Omega_{\phi_0}^{-1/2} - K\right) (K + F(a)) \left(\frac{F(a)-1}{\Omega_{\phi_0}^{-1/2}-1}\right)^K}{-\left(\Omega_{\phi_0}^{-1/2} + K\right) \left(\Omega_{\phi_0}^{-1/2} + 1\right)^{-K} + \left(\Omega_{\phi_0}^{-1/2} - K\right) \left(\Omega_{\phi_0}^{-1/2} - 1\right)^{-K}} \right] \\
&= 0
\end{aligned} \tag{62}$$

Thus, we see that as  $a_i \rightarrow 0$ , (60) becomes the Scherrer-Dutta solution for ‘hilltop quintessence’ in Ref. [14].

#### IV. SPECIAL CASES

Equations (54), (55), and (60) apply only when  $K^2 > 0$ , yet  $K^2$  can take on any real value. In this next section, we repeat this derivation for  $K^2 < 0$  and  $K^2 = 0$ .

##### A. $K^2 < 0$

Technically, if we write  $K$  as  $i\kappa t_\Lambda$ , where  $\kappa$  is a positive real number, (54), (55), and (60) give the correct formula for  $1 + w(a)$ . However, taking the real part of this formula can be quite complicated, and so we instead begin anew from the Friedmann equations. In the  $K^2 < 0$  case, equation (24) then gives:

$$u = A \sin(\kappa t) + B \cos(\kappa t) \tag{63}$$

From here, as before, we evaluate at  $t_i$  to solve for  $A$  and  $B$ . Since

$$\frac{\dot{u}}{\kappa} = A \cos(\kappa t) - B \sin(\kappa t) \tag{64}$$

we find that

$$\begin{aligned}
A &= u \sin(\kappa t) + \frac{\dot{u}}{\kappa} \cos(\kappa t) \\
&= \left( \phi_i \sin(\kappa t_i) + \frac{\dot{\phi}_i + \frac{3}{2}\phi_i H_i}{\kappa} \cos(\kappa t_i) \right) a_i^{3/2}
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
B &= u \cos(\kappa t) - \frac{\dot{u}}{\kappa} \sin(\kappa t) \\
&= \left( \phi_i \cos(\kappa t_i) - \frac{\dot{\phi}_i + \frac{3}{2}\phi_i H_i}{\kappa} \sin(\kappa t_i) \right) a_i^{3/2}
\end{aligned} \tag{66}$$

Thus, by the definition of  $u$ , we can write the following expression for  $\phi$ :

$$\phi(a, t) = \frac{1}{\kappa} \left( \frac{a}{a_i} \right)^{-3/2} \left\{ \left[ \phi_i \kappa \sin(\kappa t_i) + \left( \dot{\phi}_i + \frac{3}{2} \phi_i H_i \right) \cos(\kappa t_i) \right] \sin(\kappa t) + \left[ \phi_i \kappa \cos(\kappa t_i) - \left( \dot{\phi}_i + \frac{3}{2} \phi_i H_i \right) \sin(\kappa t_i) \right] \cos(\kappa t) \right\} \quad (67)$$

Then, by the  $\Lambda$ CDM approximation, we can write this as just a function of  $t$ :

$$\phi(t) = \frac{1}{\kappa} \left( \frac{\sinh(t_i/t_\Lambda)}{\sinh(t/t_\Lambda)} \right) \left\{ \left[ \phi_i \kappa \sin(\kappa t_i) + \left( \dot{\phi}_i + \frac{3}{2} \phi_i H_i \right) \cos(\kappa t_i) \right] \sin(\kappa t) + \left[ \phi_i \kappa \cos(\kappa t_i) - \left( \dot{\phi}_i + \frac{3}{2} \phi_i H_i \right) \sin(\kappa t_i) \right] \cos(\kappa t) \right\} \quad (68)$$

Now we can begin to work towards a formula for  $1+w$  by taking the time derivative of the equation above. This yields, after evaluating  $H(t)$  with the  $\Lambda$ CDM approximation,

$$\begin{aligned} \dot{\phi}(t) = & \left[ \dot{\phi}_i \sinh(t_i/t_\Lambda) \cos(\kappa t_i) + \phi_i \left( \kappa \sinh(t_i/t_\Lambda) \sin(\kappa t_i) + \frac{1}{t_\Lambda} \cosh(t_i/t_\Lambda) \cos(\kappa t_i) \right) \right] \left( \frac{\cos(\kappa t)}{\sinh(t/t_\Lambda)} - \frac{\sin(\kappa t) \cosh(t/t_\Lambda)}{\kappa t_\Lambda \sinh^2(t/t_\Lambda)} \right) \\ + & \left[ \dot{\phi}_i \sinh(t_i/t_\Lambda) \sin(\kappa t_i) - \phi_i \left( \kappa \sinh(t_i/t_\Lambda) \cos(\kappa t_i) - \frac{1}{t_\Lambda} \cosh(t_i/t_\Lambda) \sin(\kappa t_i) \right) \right] \left( \frac{\sin(\kappa t)}{\sinh(t/t_\Lambda)} - \frac{\cos(\kappa t) \cosh(t/t_\Lambda)}{\kappa t_\Lambda \sinh^2(t/t_\Lambda)} \right) \end{aligned} \quad (69)$$

After noting that  $\dot{\phi}_\Lambda \approx \pm \sqrt{\frac{4}{3}(1+w)}$  and converting the hyperbolic functions back to functions of  $a$ , this becomes

$$\begin{aligned} \sqrt{\frac{4}{3}(1+w)} = & \frac{1}{\kappa t_\Lambda} \left( \frac{a}{a_i} \right)^{-3/2} \left\{ \sqrt{\frac{4}{3}(1+w_i)} \left[ \cos(\kappa t_i) (\kappa t_\Lambda \cos(\kappa t) - F(a) \sin(\kappa t)) + \sin(\kappa t_i) (\kappa t_\Lambda \sin(\kappa t) + F(a) \cos(\kappa t)) \right] \right. \\ & + \phi_i \left[ (\kappa t_\Lambda \sin(\kappa t_i) + F(a_i) \cos(\kappa t_i)) (\kappa t_\Lambda \cos(\kappa t) - F(a) \sin(\kappa t)) \right. \\ & \left. \left. - (\kappa t_\Lambda \cos(\kappa t_i) - F(a_i) \sin(\kappa t_i)) (\kappa t_\Lambda \sin(\kappa t) + F(a) \cos(\kappa t)) \right] \right\} \quad (70) \end{aligned}$$

which can be evaluated at  $t_0$  and solved for  $\phi_i$  as such:

$$\begin{aligned} \phi_i = & \left\{ \kappa t_\Lambda \left( \frac{a_0}{a_i} \right)^{3/2} \sqrt{\frac{4}{3}(1+w_0)} \mp \left[ \cos(\kappa t_i) (\kappa t_\Lambda \cos(\kappa t_0) - F(a_0) \sin(\kappa t_0)) \right. \right. \\ & \left. \left. + \sin(\kappa t_i) (\kappa t_\Lambda \sin(\kappa t_0) + F(a_0) \cos(\kappa t_0)) \right] \sqrt{\frac{4}{3}(1+w_i)} \right\} \\ & \left[ (\kappa t_\Lambda \sin(\kappa t_i) + F(a_i) \cos(\kappa t_i)) (\kappa t_\Lambda \cos(\kappa t_0) - F(a_0) \sin(\kappa t_0)) \right. \\ & \left. - (\kappa t_\Lambda \cos(\kappa t_i) - F(a_i) \sin(\kappa t_i)) (\kappa t_\Lambda \sin(\kappa t_0) + F(a_0) \cos(\kappa t_0)) \right]^{-1} \quad (71) \end{aligned}$$

Then, substituting into (70) and converting the sines and cosines into functions of  $a$ , we obtain the previously found equation form:

$$\sqrt{1+w(a)} = \frac{X_K(a)}{X_K(a_0)} \sqrt{1+w_0} \pm \frac{Y_K(a)}{Y_K(a_i)} \sqrt{1+w_i} \quad (72)$$

where

$$\begin{aligned}
X_K(a) = \left(\frac{a}{a_0}\right)^{-3/2} & \left\{ \left[ |K| \sin \left( |K| \ln \sqrt{\frac{F(a_i)+1}{F(a_i)-1}} \right) + F(a_i) \cos \left( |K| \ln \sqrt{\frac{F(a_i)+1}{F(a_i)-1}} \right) \right] \right. \\
& \times \left[ |K| \cos \left( |K| \ln \sqrt{\frac{F(a)+1}{F(a)-1}} \right) - F(a) \sin \left( |K| \ln \sqrt{\frac{F(a)+1}{F(a)-1}} \right) \right] \\
& - \left[ |K| \cos \left( |K| \ln \sqrt{\frac{F(a_i)+1}{F(a_i)-1}} \right) - F(a_i) \sin \left( |K| \ln \sqrt{\frac{F(a_i)+1}{F(a_i)-1}} \right) \right] \\
& \left. \times \left[ |K| \sin \left( |K| \ln \sqrt{\frac{F(a)+1}{F(a)-1}} \right) + F(a) \cos \left( |K| \ln \sqrt{\frac{F(a)+1}{F(a)-1}} \right) \right] \right\}
\end{aligned} \tag{73}$$

and

$$\begin{aligned}
Y_K(a) = \left(\frac{a}{a_i}\right)^{-3/2} & \left\{ \left[ |K| \cos \left( |K| \ln \sqrt{\frac{F(a_0)+1}{F(a_0)-1}} \right) - F(a_0) \sin \left( |K| \ln \sqrt{\frac{F(a_0)+1}{F(a_0)-1}} \right) \right] \right. \\
& \times \left[ |K| \sin \left( |K| \ln \sqrt{\frac{F(a)+1}{F(a)-1}} \right) + F(a) \cos \left( |K| \ln \sqrt{\frac{F(a)+1}{F(a)-1}} \right) \right] \\
& - \left[ |K| \sin \left( |K| \ln \sqrt{\frac{F(a_0)+1}{F(a_0)-1}} \right) + F(a_0) \cos \left( |K| \ln \sqrt{\frac{F(a_0)+1}{F(a_0)-1}} \right) \right] \\
& \left. \times \left[ |K| \cos \left( |K| \ln \sqrt{\frac{F(a)+1}{F(a)-1}} \right) - F(a) \sin \left( |K| \ln \sqrt{\frac{F(a)+1}{F(a)-1}} \right) \right] \right\}
\end{aligned} \tag{74}$$

from which we can see that the same identities hold as in the  $K^2 > 0$  case, namely

$$X_K(a_i) = Y_K(a_0) = 0 \tag{75}$$

and

$$X_K(a_0) = -Y_K(a_i) \tag{76}$$

### B. $K^2 = 0$

We first solve (24) in the  $k = 0$  case to find that

$$u = \phi a^{3/2} = At + B \tag{77}$$

Then, evaluating at  $t_i$ , we solve for  $A$  and  $B$  to obtain

$$A = \dot{u} = \left( \dot{\phi}_i + \frac{3}{2} H_i \right) a_i^{3/2} \tag{78}$$

and

$$B = \left[ \phi_i + \left( \dot{\phi}_i + \frac{3}{2} H_i \right) t_i \right] a_i^{3/2} \tag{79}$$

Substitution into (77) yields

$$\phi = \left[ \phi_i + \left( \dot{\phi}_i + \frac{3}{2} H_i \right) (t - t_i) \right] \left( \frac{a}{a_i} \right)^{-3/2} \tag{80}$$

Then using the  $\Lambda$ CDM approximation, we have

$$\phi(t) = \left[ \phi_i + \left( \dot{\phi}_i + \frac{3}{2} H_i \right) (t - t_i) \right] \left( \frac{\sinh(t_i/t_\Lambda)}{\sinh(t/t_\Lambda)} \right) \quad (81)$$

and taking the derivative gives us

$$\begin{aligned} \dot{\phi}(t) = & \left[ \dot{\phi}_i \sinh(t_i/t_\Lambda) + \frac{\phi_i}{t_\Lambda} \cosh(t_i/t_\Lambda) \right] \frac{1}{\sinh(t/t_\Lambda)} \\ & - \frac{1}{t_\Lambda} \left[ \phi_i \sinh(t_i/t_\Lambda) + \left( \dot{\phi}_i \sinh(t_i/t_\Lambda) + \frac{\phi_i}{t_\Lambda} \cosh(t_i/t_\Lambda) \right) (t - t_i) \right] \frac{\cosh(t/t_\Lambda)}{\sinh^2(t/t_\Lambda)} \end{aligned} \quad (82)$$

We can now convert back to  $a$  to find that

$$\begin{aligned} \dot{\phi}(a) = & \frac{1}{t_\Lambda} \left( \frac{a}{a_i} \right)^{-3/2} \left\{ \dot{\phi}_i t_\Lambda \left[ 1 - F(a) \ln \left[ \left( \frac{a}{a_i} \right)^{3/2} \left( \frac{F(a) + 1}{F(a_i) + 1} \right) \right] \right] \right. \\ & \left. + \phi_i \left[ F(a_i) - F(a) - F(a_i) F(a) \ln \left[ \left( \frac{a}{a_i} \right)^{3/2} \left( \frac{F(a) + 1}{F(a_i) + 1} \right) \right] \right] \right\} \end{aligned} \quad (83)$$

From here, we evaluate at  $t_0$  to find  $\phi_i$  and use the relation  $\pm \sqrt{\frac{4}{3}(1+w)} = \phi t_\Lambda$  to arrive at the solution:

$$\sqrt{1+w(a)} = \frac{X_0(a)}{X_0(a_0)} \sqrt{1+w_0} \pm \frac{Y_0(a)}{Y_0(a_i)} \sqrt{1+w_i} \quad (84)$$

where

$$X_0 = \left( \frac{a}{a_0} \right)^{-3/2} \left[ F(a_i) - F(a) - \frac{1}{2} F(a_i) F(a) \ln \left[ \left( \frac{F(a_i) - 1}{F(a_i) + 1} \right) \left( \frac{F(a) + 1}{F(a) - 1} \right) \right] \right] \quad (85)$$

and

$$Y_0 = \left( \frac{a}{a_i} \right)^{-3/2} \left[ F(a_0) - F(a) - \frac{1}{2} F(a_0) F(a) \ln \left[ \left( \frac{F(a_0) - 1}{F(a_0) + 1} \right) \left( \frac{F(a) + 1}{F(a) - 1} \right) \right] \right] \quad (86)$$

### C. Conclusions

Thus, we arrive at a general formula for  $1+w(a)$ , that is,

$$1+w(a) = \left( \frac{X_K(a)}{X_K(a_0)} \sqrt{1+w_0} \pm \frac{Y_K(a)}{Y_K(a_i)} \sqrt{1+w_i} \right)^2 \quad (87)$$

The  $X_K$ 's take on the following forms

$$X_{K^2>0}(a) = \left(\frac{a}{a_i}\right)^{\frac{3}{2}K} \left(\frac{a}{a_0}\right)^{-\frac{3}{2}} \left[ (F(a_i) + K)(K - F(a)) \left(\frac{F(a) + 1}{F(a_i) + 1}\right)^K + (F(a_i) - K)(K + F(a)) \left(\frac{F(a) - 1}{F(a_i) - 1}\right)^K \right] \quad (88)$$

$$X_{K^2=0}(a) = \left(\frac{a}{a_0}\right)^{-3/2} \left[ F(a_i) - F(a) - \frac{1}{2}F(a_i)F(a) \ln \left[ \left(\frac{F(a_i) - 1}{F(a_i) + 1}\right) \left(\frac{F(a) + 1}{F(a) - 1}\right) \right] \right] \quad (89)$$

$$X_{K^2<0} = \left(\frac{a}{a_0}\right)^{-3/2} \left\{ \left[ |K| \sin \left( |K| \ln \sqrt{\frac{F(a_i) + 1}{F(a_i) - 1}} \right) + F(a_i) \cos \left( |K| \ln \sqrt{\frac{F(a_i) + 1}{F(a_i) - 1}} \right) \right] \times \left[ |K| \cos \left( |K| \ln \sqrt{\frac{F(a) + 1}{F(a) - 1}} \right) - F(a) \sin \left( |K| \ln \sqrt{\frac{F(a) + 1}{F(a) - 1}} \right) \right] - \left[ |K| \cos \left( |K| \ln \sqrt{\frac{F(a_i) + 1}{F(a_i) - 1}} \right) - F(a_i) \sin \left( |K| \ln \sqrt{\frac{F(a_i) + 1}{F(a_i) - 1}} \right) \right] \times \left[ |K| \sin \left( |K| \ln \sqrt{\frac{F(a) + 1}{F(a) - 1}} \right) + F(a) \cos \left( |K| \ln \sqrt{\frac{F(a) + 1}{F(a) - 1}} \right) \right] \right\} \quad (90)$$

and the  $Y_K$ 's are just these with  $a_i$  and  $a_0$  interchanged. It should also be noted that these formulae are not unique; since the relevant quantity is  $X_K(a)/X_K(a_0)$ , any scalar multiple of these is equally valid. (88), for instance, is a factor of  $2i$  larger than (90) if an imaginary  $K$  is directly substituted.

In addition, the high degree of symmetry in these equations motivates a simpler way of expressing them. Indeed, we can also write this solution in the following form:

$$\sqrt{1+w(a)} = \left(\frac{a}{a_0}\right)^{-3/2} \left( \frac{f_K(a)g_K(a_i) - f_K(a_i)g_K(a)}{f_K(a_0)g_K(a_i) - f_K(a_i)g_K(a_0)} \right) \sqrt{1+w_0} \pm \left(\frac{a}{a_i}\right)^{-3/2} \left( \frac{f_K(a)g_K(a_0) - f_K(a_0)g_K(a)}{f_K(a_i)g_K(a_0) - f_K(a_0)g_K(a_i)} \right) \sqrt{1+w_i} \quad (91)$$

where  $f_K(a)$  and  $g_K(a)$  are the much more manageable functions:

$$f_K(a) = \begin{cases} (K + F(a)) \left(\frac{F(a)-1}{F(a)+1}\right)^{K/2} & (K^2 > 0) \\ F(a) & (K^2 = 0) \\ |K| \sin \ln \left(\frac{F(a)+1}{F(a)-1}\right)^{|K|/2} + F(a) \cos \ln \left(\frac{F(a)+1}{F(a)-1}\right)^{|K|/2} & (K^2 < 0) \end{cases} \quad (92)$$

$$g_K(a) = \begin{cases} (K - F(a)) \left(\frac{F(a)+1}{F(a)-1}\right)^{K/2} & (K^2 > 0) \\ 1 - F(a) \ln \left(\frac{F(a)+1}{F(a)-1}\right)^{1/2} & (K^2 = 0) \\ |K| \cos \ln \left(\frac{F(a)+1}{F(a)-1}\right)^{|K|/2} - F(a) \sin \ln \left(\frac{F(a)+1}{F(a)-1}\right)^{|K|/2} & (K^2 < 0) \end{cases} \quad (93)$$

## V. COMPARISON WITH NUMERICAL SOLUTIONS

### A. Method

In the following section, we compare these models for  $w(a)$  with numerical results for three different potentials: a quadratic

$$V(\phi) = V_0 + V_2(\phi - \phi_*)^2 \quad (94)$$

a Gaussian

$$V(\phi) = V_0 e^{-(\phi - \phi_*)^2 / \sigma^2} \quad (95)$$

and—perhaps the most important model of scalar field evolution near a potential maximum or minimum—the PNGB potential

$$V(\phi) = V_0 [1 + \cos((\phi - \phi_*)/f)] \quad (96)$$

These, of course, are not meant to be an exhaustive list of scalar field potentials, but we merely wish to show that these different potentials will produce similar curves for  $w(a)$  which converge upon our model in (91), (92), and (93). Our numerical solutions to Eqs. (15) and (16) are constrained by four boundary conditions— $w_i$ ,  $w_0$ ,  $\Omega_{\phi_0}$ , and  $K$ —which together determine the parameters in each of these potential forms. The boundary condition on  $K$  requires that

$$1 - K^2 = \frac{4V''(\phi_*)}{3V(\phi_*)} \quad (97)$$

and so

$$V''(\phi_*) = \frac{3}{4}(1 - K^2)V(\phi_*) \quad (98)$$

Furthermore, through the condition on  $\Omega_{\phi_0}$ , we have:

$$\rho_{\phi_0} = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0) \quad (99)$$

and since  $\dot{\phi}_0^2 = (1 + w_0)\rho_{\phi_0}$  and  $\rho_{\phi_0} = 3H_0^2\Omega_{\phi_0}$ , this expression can be rewritten as:

$$V(\phi_0) = \frac{3}{2}H_0^2\Omega_{\phi_0}(1 - w_0) \quad (100)$$

The conditions in (98) and (100) can then be applied to the various potential forms to determine the given parameters in each case. For a quadratic, shifting to  $\phi_* = 0$  these become

$$V_2 = \frac{3}{8}(1 - K^2)V_0 \quad (101)$$

and

$$V_0 + V_2\phi_0^2 = \frac{3}{2}H_0^2\Omega_{\phi_0}(1 - w_0) \quad (102)$$

respectively, which yield the solution

$$V_0 = \frac{\frac{3}{2}H_0^2\Omega_{\phi_0}(1 - w_0)}{1 + \frac{3}{8}(1 - K^2)\phi_0^2} \quad (103)$$

$$V_2 = \frac{3}{8}(1 - K^2)\frac{\frac{3}{2}H_0^2\Omega_{\phi_0}(1 - w_0)}{1 + \frac{3}{8}(1 - K^2)\phi_0^2} \quad (104)$$

In a similar fashion, we have for a Gaussian

$$\sigma^2 = \frac{8}{3(K^2 - 1)} \quad (105)$$

$$V_0 = \frac{3}{2}H_0^2\Omega_{\phi_0}(1 - w_0)e^{\frac{3}{8}(K^2 - 1)\phi_0^2} \quad (106)$$

which yields a negative value of  $\sigma^2$  for  $K^2 < 1$ . While this is no longer a Gaussian curve in the strict sense, the resulting potential—a sharply increasing well—is nonetheless valid for our purposes. For a PNGB potential, we find

$$f = \frac{2}{3\sqrt{K^2 - 1}} \quad (107)$$

$$V_0 = \frac{\frac{3}{2}H_0^2\Omega_{\phi_0}(1-w_0)}{1 + \cos(\frac{3}{2}\phi_0\sqrt{K^2-1})} \quad (108)$$

This potential undergoes a similar change around  $K^2 = 1$ , where  $f$  becomes imaginary, changing the cosine curve into a hyperbolic cosine well.

Thus, the appropriate parameters for a potential curve with a given  $K$  value can be determined by specifying  $w_0$  and  $\phi_0$ . Three of the four boundary conditions— $K$ ,  $\Omega_{\phi_0}$ , and  $w_0$ —therefore depend only on the present state, and hence it is appropriate to run the simulation backwards from  $a_0$  to  $a_i$  in order to find a value of  $\phi_0$  which satisfies the fourth boundary condition:  $w_i$ . The particular algorithm in these simulations first found values of  $\phi_0$  between which  $\dot{\phi}_i$  changes sign, and then pinpointed the  $\phi_0$  value for which  $w_i = -1$ . Then, by deviating from this point, appropriate  $\phi_0$  values could be found corresponding to the correct  $w_i$ .

## B. Graphical Comparisons

Using this method, numerical solutions were obtained for six values of  $K^2$ : 0,  $\pm 4$ ,  $\pm 40$ , and  $-60$ , with  $\Omega_{\phi_0} \approx 0.73$  (see pp. 16-21).  $w_i$  and  $w_0$  were chosen to allow reasonable deviations from our assumption that  $w \approx -1$ . An  $a_i$  value of 0.2 was chosen, corresponding with  $z = 4$ , to prevent  $w$  from freezing too rapidly.

All of these except the  $K^2 = -40$  case show remarkable agreement ( $< 2\%$  error) between our model and the numerical simulations with no systematic error in either direction. As expected, the model becomes progressively more accurate as  $w_0 \rightarrow -1$ , leading us to speculate that any potential with nonzero  $V(\phi_*)$  and  $V''(\phi_*)$  will produce a  $w(a)$  curve arbitrarily close to (91), (92), and (93) as  $w_i$  and  $w_0$  approach  $-1$ .

The  $K^2 = -40$  case is nonetheless surprising, but highlights a trend that is present in Figs. 1-8 as well. As  $K^2$  decreases,  $w_{avg}$  must become progressively closer to  $-1$  in order to generate suitable agreement between the model and numerical solutions; in other words, this model is more accurate for greater values of  $K^2$ . This phenomenon is unrelated to the accuracy of the Taylor expansion; indeed, the quadratic potential displays this same trend. Rather, the problem most likely lies in the definition of  $K$ . Our model in (91), (92), and (93) becomes more sensitive to slight changes in  $K$  as  $K$  becomes increasingly negative, and Fig. 13 (p. 22) shows one example of how a small adjustment in the value of  $K^2$  can produce almost perfect agreement in the  $-40$  case. Thus, the imprecision in some of our initial assumptions—namely, that  $p_T \approx -\rho_{\phi_0} \approx V(\phi_*)$ —are most likely responsible for the poor agreement in Figs. 9-10, and a more accurate method of approximation may perhaps be worthy of additional research.

Nonetheless, as evidenced by Figs. 11-12, the decreasing value of  $K^2$  is not the only factor leading to the irregularity in the  $K^2 = -40$  case. Since  $K^2 = -60$  shows excellent agreement between the numerical results and our model, there is clearly something unique about a  $K^2$  value of  $-40$  which creates such an anomaly. At  $K^2 = -37.96$ , our model for  $w_i = -0.9999$  and  $w_0 = -0.9998$  breaks down and exhibits a peak above 170,000, which is clearly unphysical but demonstrates that  $\phi$  grows extremely large in such a potential. This sort of resonance in the model for  $w(a)$  also exists around  $K^2$  values of  $-12.12$  and  $-78.19$  where the curve for  $1 + w(a)$  acquires an extra root. The graphs for  $K^2 = -40$  most likely show such poor agreement because of its proximity to one of these resonances. Since the maximum value of  $\phi$  is changing very rapidly around  $K^2 = -40$ , even a slight variation from the best-fitting  $K$  can cause our approximation to deviate significantly from numerical results. The source of this resonance phenomenon is still not fully understood, however, and may also deserve to be studied in further detail.

## VI. CONCLUSIONS

Continuing upon the methods of Refs. [14] and [15], we have derived a general expression for  $w(a)$  of a minimally-coupled scalar field evolving near a potential maximum or minimum. Unlike the references cited above, however, this model does not assume that  $\dot{\phi}_i = 0$ , thereby introducing the additional parameter  $w_0$  and a ‘ $\pm$ ’ sign into the expression for  $w(a)$ . Thus, under our initial assumptions that  $w \approx -1$  and  $(1/V)(\partial_\phi V) \ll 1$  (satisfied near a potential extremum at  $\phi_*$ ), the most general expression for  $w(a)$  has five free parameters which together determine a unique evolution of the field:  $\Omega_{\phi_0}$ ,  $V''(\phi_*)/V(\phi_*)$ ,  $w_i$ ,  $w_0$  and a ‘ $\pm$ ’ which depends upon the direction of  $\dot{\phi}_i$  relative to  $\dot{\phi}_0$ .

Varying these five parameters, we then compared our model to the exact solution as determined numerically. The model proved to be a good approximation ( $< 2\%$  error) for all but for one value of  $V''(\phi_*)/V(\phi_*)$ , which we speculate was due to a resonance-like effect for certain curvatures. Such a resonance peak in  $w(a)$  was predicted by our model for various values of the curvature parameter, yet this phenomenon is still not fully understood and may merit additional research.

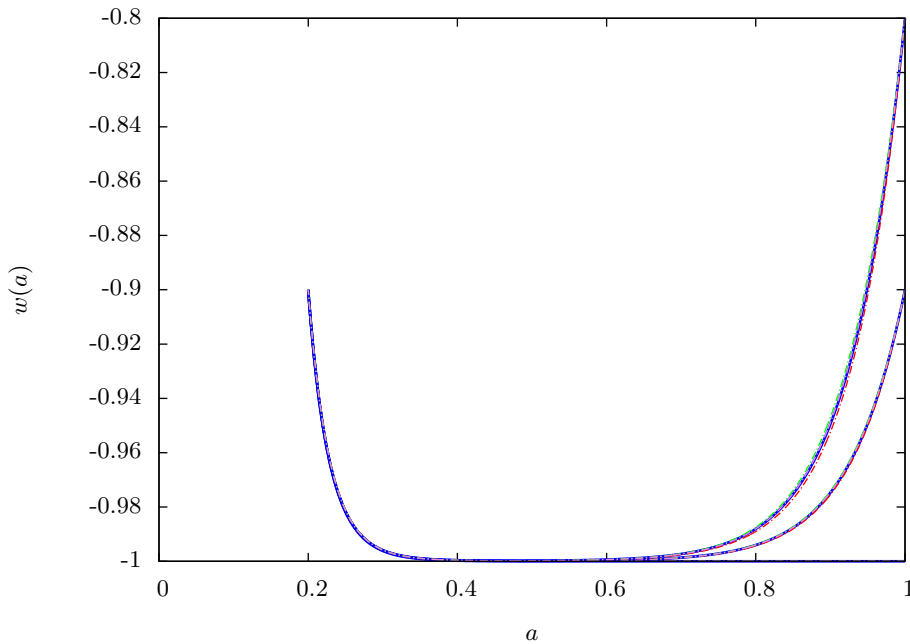


FIG. 1: The evolution of  $w(a)$  under a quadratic potential with  $K^2 = 40$ .  $w_i$  is fixed at  $-0.9$  with  $a_i = 0.2$ . Our model (blue) is shown along with numerical solutions with quadratic (red), Gaussian (green), and PNGB (violet) potentials.  $w_0$  values of  $-1$ ,  $-0.9$ , and  $-0.8$  are displayed in the case where the (+) solution of the ' $\pm$ ' in (91) has been chosen.

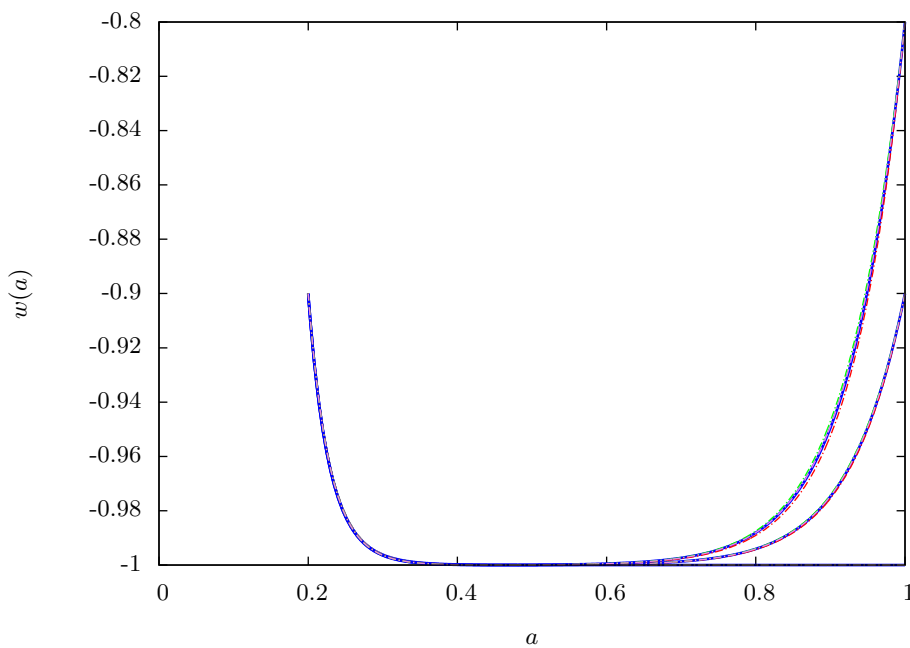


FIG. 2: As Fig 1, except the (-) solution has been chosen.

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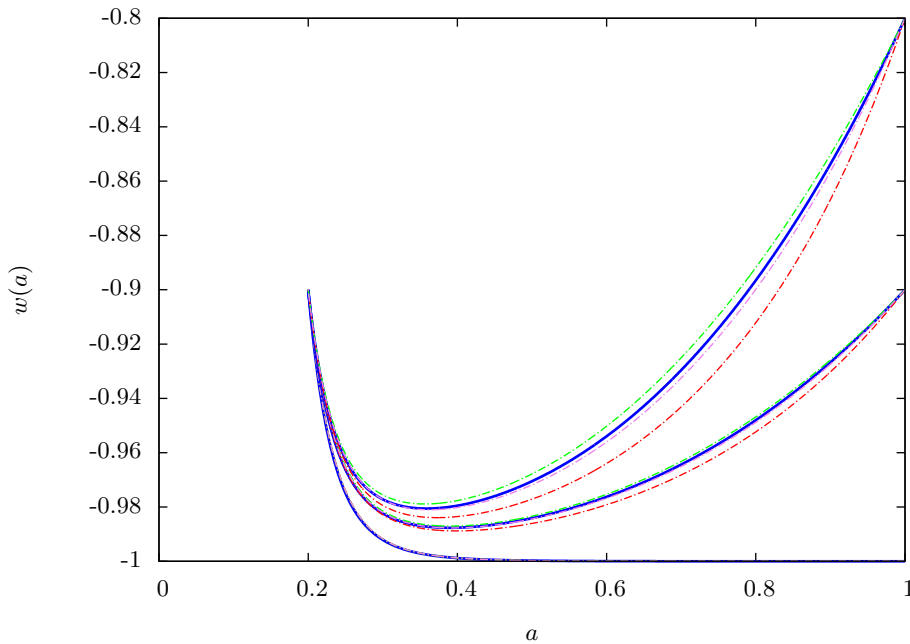


FIG. 3: An analog of Fig. 1 with the (+) solution for  $K^2 = 4$ .

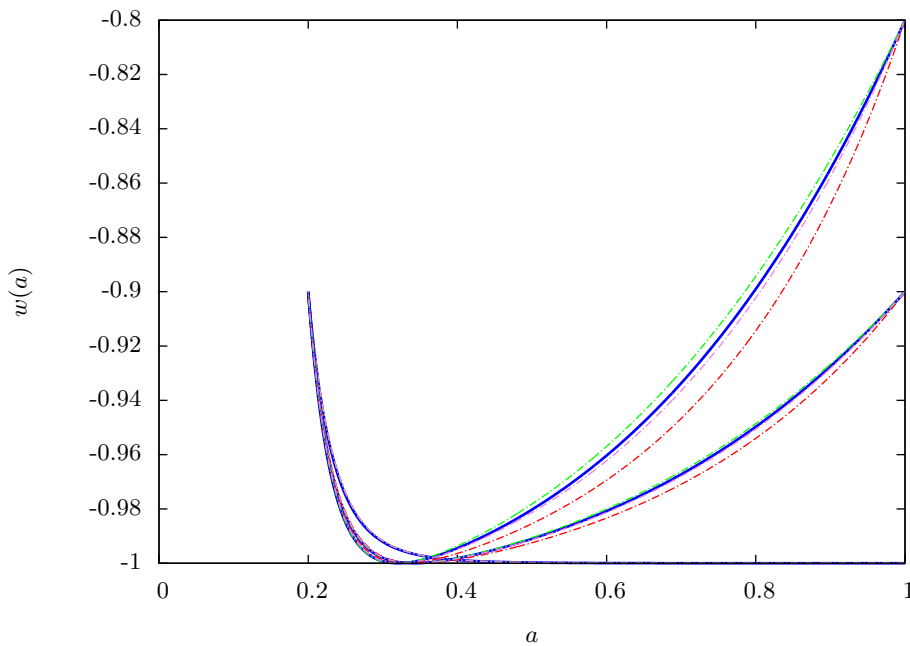


FIG. 4: As Fig. 3, except the (-) solution has been chosen.

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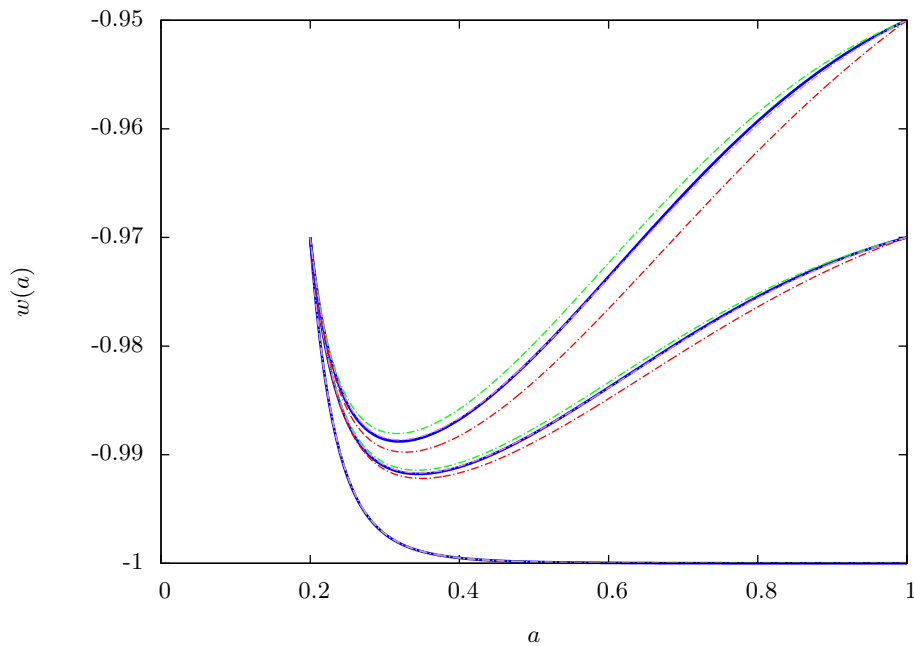


FIG. 5: An analog of Fig. 1 with the (+) solution for  $K^2 = 0$ .  $w_i$  is fixed at  $-0.97$  with solutions plotted for  $w_0 = -1, -0.97$ , and  $-0.95$ .

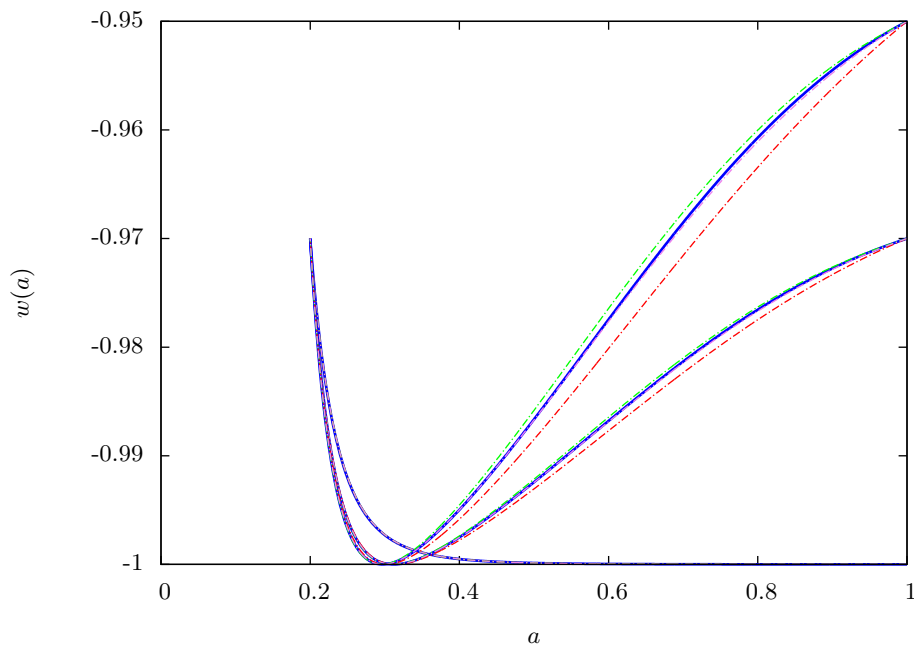


FIG. 6: As Fig. 5, except the (-) solution has been chosen.

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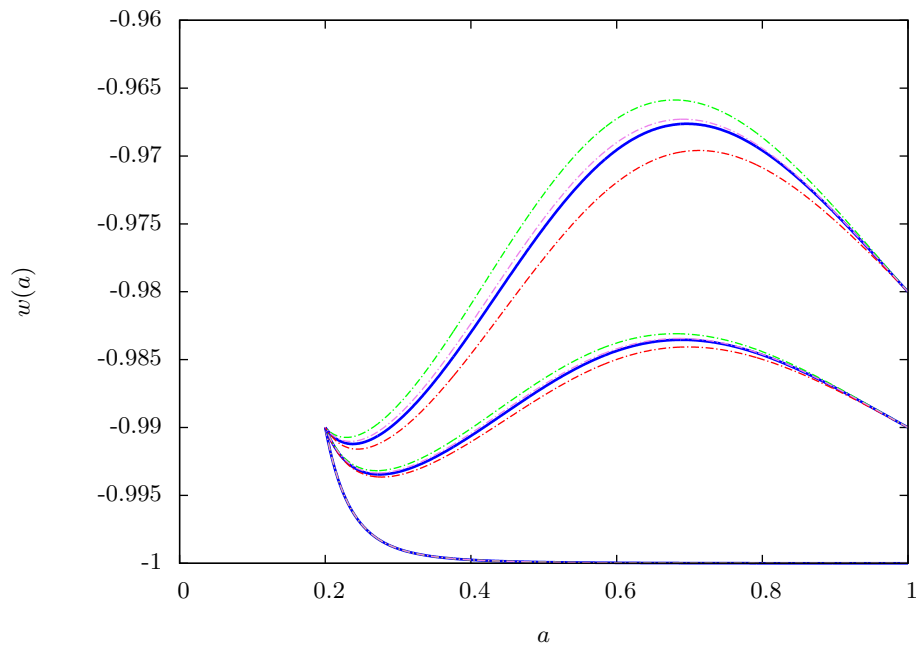


FIG. 7: An analog of Fig. 1 with the (+) solution for  $K^2 = -4$ .  $w_i$  is fixed at  $-0.99$  with solutions plotted for  $w_0 = -1, -0.99$ , and  $-0.98$ .

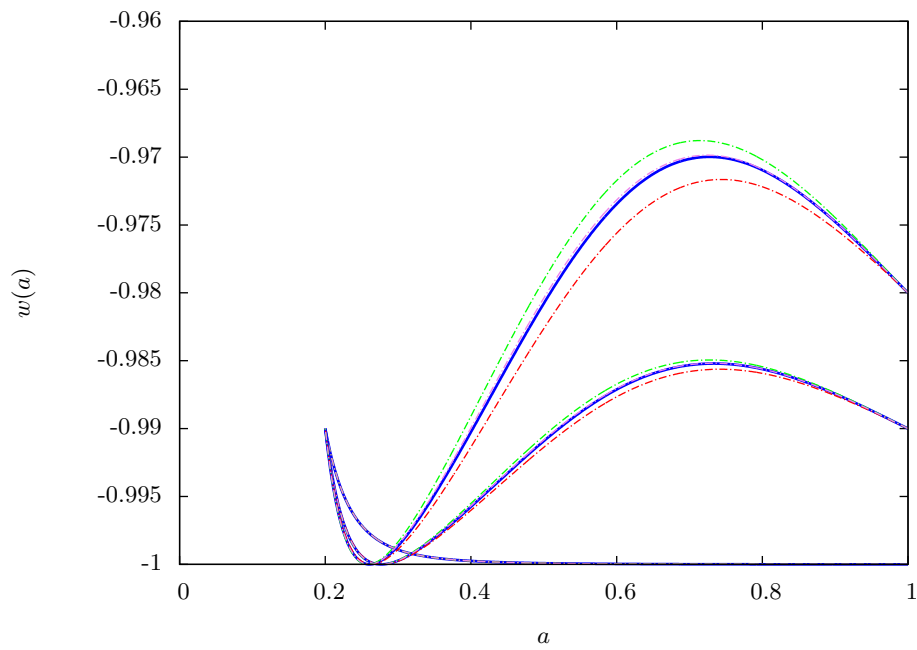


FIG. 8: As Fig. 7, except the (-) solution has been chosen.

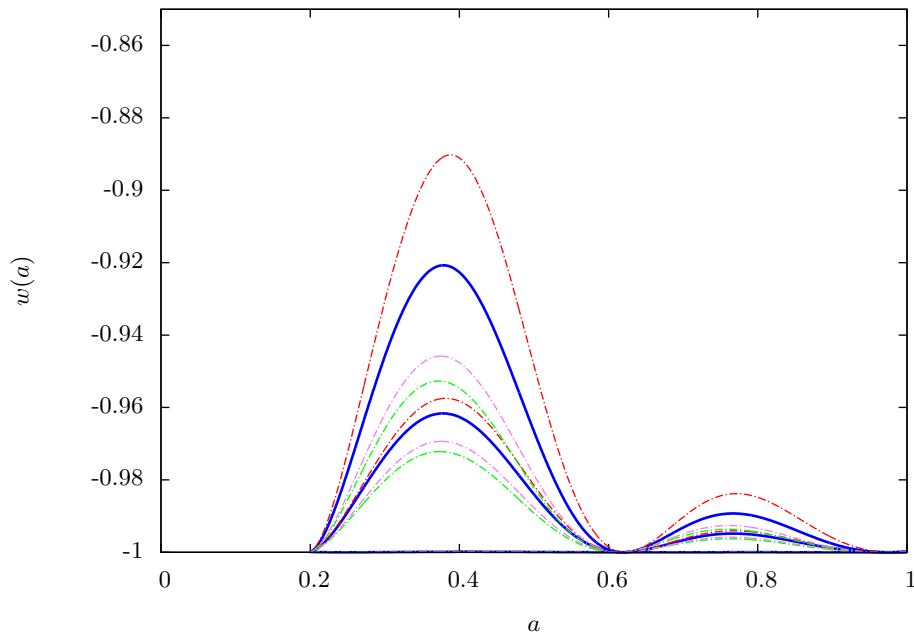


FIG. 9: An analog of Fig. 1 with the (+) solution for  $K^2 = -40$ .  $w_i$  is fixed at  $-0.9999$  with solutions plotted for  $w_0 = -1, -0.9999$ , and  $-0.9998$ .

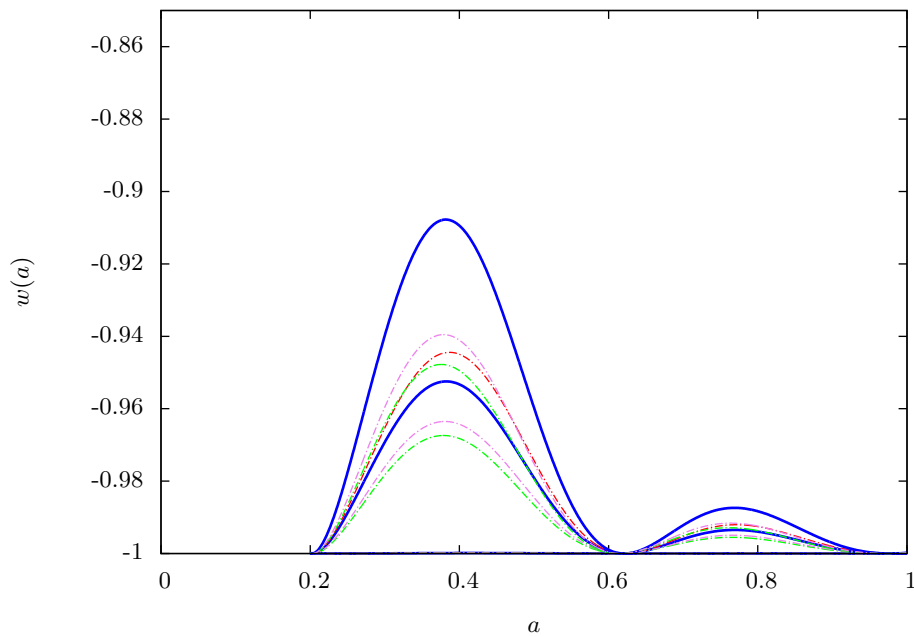


FIG. 10: As Fig. 9, except the (-) solution has been chosen. No solution was found for the quadratic potential in the  $w_0 = -0.9998$  case.

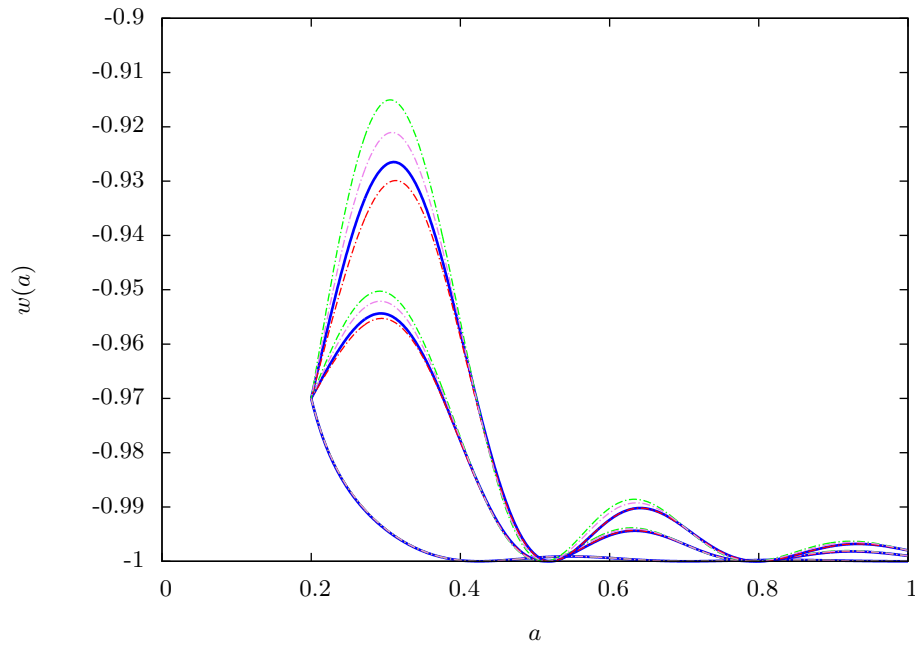


FIG. 11: An analog of Fig. 1 with the (+) solution for  $K^2 = -60$ .  $w_i$  is fixed at  $-0.97$  with solutions plotted for  $w_0 = -1, -0.999$ , and  $-0.998$ .

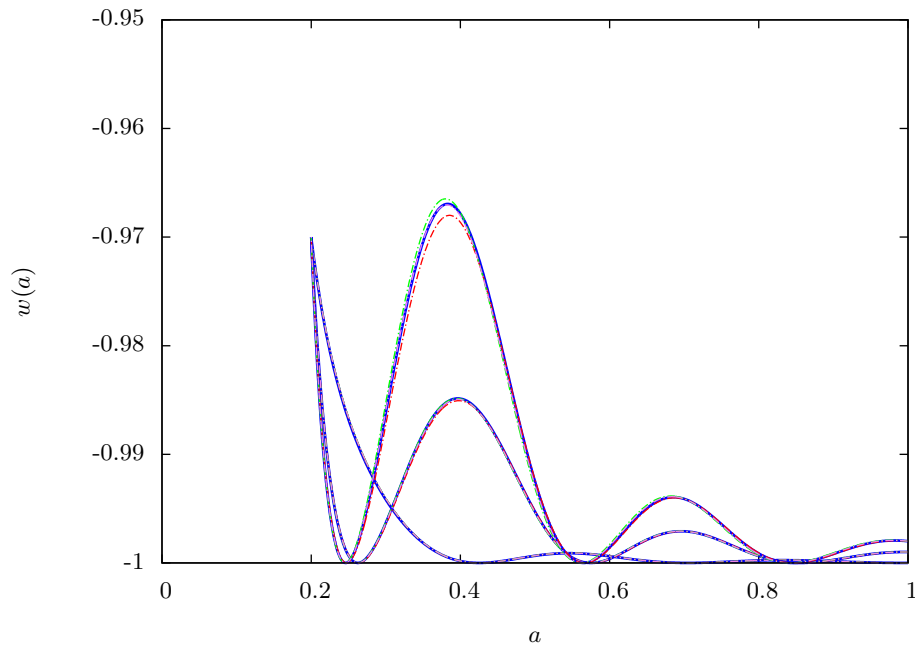


FIG. 12: As Fig. 11, except the (-) solution has been chosen.

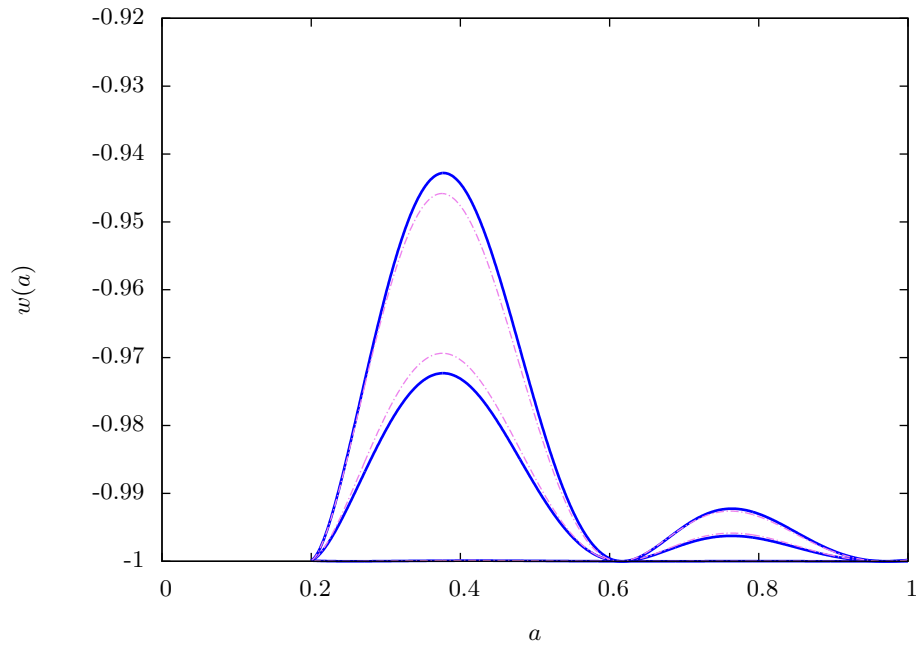


FIG. 13: The numerical solutions for a PANGB potential in Fig. 9 overlaid with the  $K^2 = -40.39$  model.