A DYNAMIC FACTOR APPROACH TO NONLINEAR STABILITY ANALYSIS

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Abstract

A method of principal components is employed to investigate nonlinear dynamic factor structure using a large panel data. The evidence suggests the possibility of nonlinearity in the U.S. while it excludes the class of nonlinearity that can generate endogenous fluctuation or chaos.

Keywords: Chaos; Dynamic Factor Model; Lyapunov Exponents; Nonparametric Regression; Principal Components.

JEL classification: C14; C33

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1   Introduction

Traditionally, we treat expansions and contractions of the economy as a result of exogenous random shocks explained by a change in policy, a change in demand, technological change, and other supply shocks. An alternative view is to consider the endogenous aggregate fluctuation via a chaotic system, or a simple nonlinear deterministic system that can have stochastic-like unpredictable behavior. One of the most convenient empirical methods for considering two competing views of the business cycle is to compute a stability measure called the largest Lyapunov exponent. It measures the sensitive dependence on initial conditions and is often employed to define a chaotic behavior in either a deterministic or stochastic nonlinear system (Eckmann and Ruelle, 1985, and Nychka, Ellner, Gallant, and McCaffrey, 1992). A chaotic system has a positive Lyapunov exponent while an exogenous system with a unique and globally stable steady state has a negative Lyapunov exponent.

In practice, the nonparametric regression method is often employed to compute Lyapunov exponents of an unknown nonlinear autoregressive (AR) model. Such a nonparametric approach was first considered by Eckmann and Ruelle (1985) and Eckmann, Kamphorst, Ruelle, and Ciliberto (1986) and its statistical properties are later considered by McCaffrey, Ellner, Gallant, and Nychka (1992), Whang and Linton (1999), and Shintani and Linton (2004). By applying this nonparametric method to the GDP series from OECD countries, Shintani and Linton (2003) found that the Lyapunov exponents were significantly negative for most cases, which supported the exogenous view of a business cycle as opposed to the chaotic view. This approach is also applied to foreign exchange rates by Dechert and Gençay (1992), monetary

aggregates by Serletis (1995) and Barnett, Gallant, Hinich, Jungeilges, Kaplan, and Jensen (1995), and the stock return series by Abhyankar, Copeland, and Wong (1997) and Shintani and Linton (2004). However, one potential drawback of the approach employed in these studies is that only a univariate time series could be considered in each estimation. If the true system consists of N equations, the theoretical result known as Takens embedding implies that $2N + 1$ lags are required in the nonlinear AR model for this univariate method to be valid. Therefore, when N is large, the nonparametric estimation method is known to be subject to computational difficulties (the curse of dimensionality).

The main objective of this paper is to consider an alternative approach to conducting a test regarding the stability of the multivariate nonlinear system with a large N. Instead of running a nonparametric regression using each individual economic time series, we consider a multiple time series generated from a dynamic factor model with a common factor following a nonlinear process. Such a common factor approach achieves dimensional reduction by construction and thus is less subject to a high dimensionality problem, unlike the nonparametric regression applied to a single series. For this reason, introducing an assumption of the common factor structure in a system of equations seems to be advantageous in a stability analysis of business cycles.

The dynamic factor model, originally considered by Sargent and Sims (1977) and Geweke (1977), has long been employed in macroeconomic analysis, including construction of business cycle indexes, analysis of stock price dynamics and international business cycle analysis. The dynamic factor model also fits well within the framework of the dynamic stochastic general equilibrium (DSGE) models, the workhorse in

2

the modern macroeconomics literature, since they predict that a small set of driving forces is responsible for covariation in macroeconomic variables. Furthermore, recent theoretical studies revealed that chaotic endogenous fluctuation can be derived under a DSGE-type framework (e.g., Brock and Hommes, 1998). In this paper, we consider the possibility of applying the Lyapunov exponent-based test to the estimated common factors for the purpose of determining whether these sources of business cycles are better explained using endogenous or exogenous models.

Our testing procedure consists of two steps. We first estimate the common factors in the dynamic factor model by employing the method of principal components and then test the nonlinear factor structure and the chaotic behavior using the estimated factors. The estimator based on principal components has become increasingly popular in the recent literature of dynamic factor models (e.g., Stock and Watson, 2002a,b, Bai, 2003, Diebold, 2003, and Shintani, 2004). We emphasize that the principal component approach is indeed useful in a high dimensional nonlinear framework because it remains theoretically valid under a very flexible nonlinear dynamic factor structure when both the number of the series (N) and the time series observations (T) are large. To investigate this theoretical prediction, we conduct a Monte Carlo simulation using the data generated from both chaotic and stable nonlinear models. The results of the simulation suggest that the method works well for many combinations of N and T typically available in practice. To provide empirical evidence regarding the nonlinear dynamics and the stability, we utilize the full balanced panel of 159 monthly U.S. macroeconomic time series ($N = 159$) with the sample period from 1959:3 to 1998:12 ($T = 478$), and estimate a large dimensional dynamic factor model. The evidence suggests the possibility of nonlinearity in the U.S. while it excludes the

3

class of nonlinearity that can generate endogenous fluctuation or chaos. This evidence suggests the justification of using the nonlinear impulse response analysis proposed by Gallant, Rossi, and Tauchen (1993), and Potter (2000); thus, we also report the nonlinear impulse response functions of the common factors.

The remainder of the paper is organized as follows: Section 2 introduces a dynamic factor model and the Lyapunov exponent of a common factor. Simulation results to evaluate the finite sample performance of the method employed in this paper are provided in Section 3. Section 4 presents the empirical results of a nonlinear stability analysis using the large U.S. data. Some concluding remarks are made in Section 5.

2 Model

Let x_{it} be an i-th component of N-dimensional multiple time series $X_t = (x_{1t}, \ldots, x_{Nt})'$ and $t = 1, ..., T$. A simple dynamic factor model associates each x_{it} with a scalar common factor f_t in equations

$$x_{it} = b_i f_t + e_{it}, \quad i = 1, ..., N, \tag{1}$$

where b_i's are factor loadings with respect to i-th series, e_{it}'s are idiosyncratic shocks. A conditional mean function of the common factor f_t is assumed to have a p-th order nonlinear AR form,

$$f_t = m \left(f_{t-1}, ..., f_{t-p} \right) + u_t \tag{2}$$

where u_t is a martingale difference sequence with $E(u_t|\mathcal{F}_{t-1}) = 0$ and $E(u_t^2|\mathcal{F}_{t-1}) = \sigma_t^2$ where $\mathcal{F}_{t-1} = \{f_{t-1}, f_{t-2}, ...\}$. Depending on the shape of the nonlinear function $m \left(f_{t-1}, ..., f_{t-p} \right)$, the dynamics of f_t can be either chaos or stable. The largest Lya-

punov exponent of the common factor f_t is defined as

$$\lambda \equiv \lim_{M \to \infty} \frac{1}{2M} \ln |\nu_1(M)| \tag{3}$$

where $\nu_1(M)$ is the largest eigenvalue of $\mathbf{T}'_M \mathbf{T}_M$, and $\mathbf{T}_M = J_{M-1} \cdot J_{M-2} \cdot \dots \cdot J_0$

where

$$J_t = \begin{bmatrix} \Delta m_{1t} & \Delta m_{2t} & \cdots & \Delta m_{p-1,t} & \Delta m_{pt} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{4}$$

for $t = 0, 1, \ldots, M - 1$, where $\Delta m_{jt} = \partial m(\mathbf{f}_t)/\partial f_{t-j}$, for $j = 1, \ldots, p$, are partial derivatives of conditional mean function evaluated at $\mathbf{f}_t = (f_{t-1}, ..., f_{t-p})$. A chaotic system has a positive Lyapunov exponent while a exogenous system with a unique and globally stable steady state has a negative Lyapunov exponent.

When factor f_t is observable, the conditional mean function in equation (2) can be directly estimated using a nonparametric regression method applied to f_t. The result can then be used to construct a consistent estimator of λ following the approach used by Nychka, Ellner, Gallant, and McCaffrey (1992) and others. However, since f_t is not observable in the dynamic factor model, the direct estimation is not feasible.

One may want to employ a maximum likelihood estimator (MLE) that incorporates latent variables under some distributional assumptions on e_{it}'s and u_t, the approach used by Stock and Watson (1989), Chauvet (1998), and Kim and Nelson (1998) in their analysis of the business cycle index. However, the MLE is not suitable in the current context because of the following two reasons. First, the MLE approach allows nonlinearity but the functional form in equation (2) needs to be specified. The parametric approach of Lyapunov exponent estimation has been considered in some cases (e.g., Bask and de Luna, 2002). However, in practice, information on the

5

functional form is usually not available, and therefore the nonparametric approach is generally preferable in the estimation of Lyapunov exponent. Second, the MLE approach becomes computationally difficult when the number of the series (N); thus, the number of parameters becomes large.

In this paper, we first estimate the unobservable common factor f_t nonparametrically based on the principal components of N variables instead of relying on the MLE. We then run a nonparametric regression of equation (2) using the estimated factor and compute λ just as in the case when f_t is observable. Estimation of the latent factor based on principal components has become increasingly popular in the recent literature of dynamic factor models (e.g., Stock and Watson, 2002a,b, Bai, 2003, Diebold, 2003, and Shintani, 2004). We emphasize that this approach is advantageous for the nonlinear stability analysis in a dynamic factor framework since the estimator of the factor remains consistent under a very flexible nonlinear dynamic factor structure when the number of the series (N) and the time series observations (T) are large.

Our approach contrasts with the case of N dimensional system without assuming a common factor structure. In such a case, embedding is required to estimate the Lyapunov exponents (see Gençay and Dechert, 1992, for example). Suppose the dynamics of $X_t = (x_{1t}, \ldots, x_{Nt})'$ are given by $X_t = G(X_{t-1})$ where $G : R^N \rightarrow R^N$. In addition, suppose a single time series $y_t = w(X_t)$ where $w : R^N \rightarrow R$ will be used for the nonlinear stability analysis. The Lyapunov exponent of G is known to be preserved in the embedded dynamics of y_t given by $y_t = m(y_{t-1}, ..., y_{t-p^*})$ where p^* is an embedding dimension that satisfies $p^* \geq 2N + 1$. Therefore, computing λ of N dimensional system based on nonlinear AR model of a single series y_t is justifiable as

long as p^* lags are included. However, one limitation is that N needs to be small to employ a nonparametric method. As N becomes large, the nonparametric regression quickly becomes difficult since p^* depends on N (the curse of dimensionality). Our dynamic factor approach, in contrast, is not subject to this problem since the lag length p in equation (2) is assumed to be fixed regardless of the size of N.

The remaining issue is to see whether we can justify replacing the true factor with the estimated factor in the nonparametric regression. A related issue in the nonparametric kernel estimator for time series was also considered by Andrews (1995). He derived the conditions required for the generated regressor that depends on the common $\sqrt{T}$-consistent estimator to become negligible in the limiting distribution of nonparametric regression. In the rest of this section, we discuss that a similar argument can be also applied to the nonparametric estimation of factor dynamics.

Our estimator of a common factor, $\widetilde{f}_t$, is the first principal component of N series which corresponds to the first eigenvector of the $T \times T$ matrix XX' with normalization $T^{-1} \sum_{t=1}^{T} \widetilde{f}_t^2 = 1$, where X is the $T \times N$ data matrix with t-th row given by $X_t' = (x_{1t}, \cdots, x_{Nt})$. The asymptotic theory of $\widetilde{f}_t$ under $N, T \to \infty$ is developed by Stock and Watson (2002b) and Bai (2003) and others. Under mild conditions on moments and memory, the estimator $\widetilde{f}_t$ has been shown to be a $\sqrt{N}$-consistent estimator of f_t (up to a scaling constant). Below is a set of assumptions typically employed in these theoretical studies. Note that *const.* implies 'for some finite positive constant.'

Assumption F: (i) $E(f_t) = \mu_f = 0$, $E((f_t)^2) = \Sigma_f = 1$, $E((f_t)^4) \leq const.$ and $\sqrt{T}(F'F/T - \Sigma_f) = O_p(1)$ where $F = [f_1, \cdots, f_T]'$.

(ii) $|b_i| \leq const.$ and $\sqrt{N}\left(B'B/N - \Sigma_b\right) = O_p\left(1\right)$ where $B = [b_1, \cdots, b_N]'$.

(iii) $E\left(e_{it}\right) = 0$, $E\left|e_{it}^8\right| \leq const.$, $E(e_{is}e_{it}) = 0$ for all $t \neq s$, $\overline{\sigma}_e^2 = N^{-1}\sum_{i=1}^{N}\sigma_{ei}^2 \leq const.$, and $E(e_{it}e_{jt}) = \tau_{ij} \leq const.$ for all t, i and j.

(iv) $\{f_t\}$, $\{b_i\}$ and $\{e_{it}\}$ are mutually independent.

(v) $N/T^2 \to \infty$ as $T \to \infty$.

(i) and (ii) are the moment conditions for the factor f_t and factor loading b_i, respectively. (iii) is a condition on the cross-sectional correlation of the idiosyncratic error e_{it}. This allows the dynamic factor model to have an approximate factor structure instead of an exact factor structure. Serial correlation of e_{it} can be also incorporated but is not allowed here. (iv) and (v) are stronger than necessary but are introduced here to simplify the derivation of the theoretical result to be followed.

In principle, any nonparametric estimator can be used to estimate equation (2). Here, we follow Andrews (1995) and restrict our attention to Nadaraya-Watson type kernel regression estimator with $p = 1$ given by

$$\widehat{m}\left(f\right) = \sum_{t=1}^{T} f_t K\left(\frac{f_{t-1} - f}{h}\right) \Big/ \sum_{t=1}^{T} K\left(\frac{f_{t-1} - f}{h}\right) \tag{5}$$

where K is a kernel function and h is a bandwidth such that $h \to 0$ as $T \to \infty$. This is an infeasible estimator since f_t is not observable. The feasible estimator that replaces f_t with $\widetilde{f}_t$ will be denoted by $\widetilde{m}(f)$. To derive the asymptotic properties of the nonparametric estimator, we introduce the following assumptions.

Assumption K: (i) $m(f)$ is twice continuously differentiable for all f.

(ii) The kernel function K is symmetric around zero, $\int K(u)du = 1$, $\int |K(u)| \, du < \infty$, $|u| \, |K(u)| \to 0$ as $u \to 0$, $\sup|K(u)| < \infty$, $\int K^2(u)du < \infty$ and $\int K'^2(u)du < \infty$.

(iii) f_t is strictly stationary with twice continuously differentiable marginal density $g(f)(> 0)$ and is strong-mixing with mixing coefficients $\alpha(j)$ satisfying $\sum_{j=T}^{\infty} \alpha(j)^{1/2} = O(T^{-1})$, and $\max_{1 \leq t \leq T} |f_t| = O_p(\ln T)$.

(iv) $h = const. \times T^{-\beta}$ for $0 < \beta < \frac{1}{3}$.

(i) and (ii) are the standard conditions for the nonparametric kernel regression estimators. The last condition in (iii) is from extreme value theory and is expected to hold for a very broad class of distribution. The rate of bandwidth in (iv) allows an optimal rate that minimizes the MSE ($\beta = 1/5$).

Proposition 1. *Let x_{it} and f_t be generated from (1) and (2), respectively, and suppose that assumptions F and K are satisfied. Then $\widehat{m}(f) - \widetilde{m}(f) = o_p(1/\sqrt{Th})$ for all f.*

The results above provide the validity of the two-step estimation applied to the estimated factor. When both N and T tend to infinity, the effect of the estimation error of the factor becomes negligible in the asymptotic distribution of the final estimator, since the variance of the nonparametric kernel estimator $\widehat{m}(f)$ is $O_p(1/(Th))$. While we only provide the proof for $p = 1$ case in the Appendix, the result can easily be generalized to the case of higher order models with $p > 1$. Furthermore, the result of kernel autoregression in Proposition 1 can be applied to other nonparametric methods, such as neural networks, by replacing the assumptions to the ones suitable for each estimator.

9

3 Simulation Results

In this section, we study the finite sample performance of the Lyapunov exponent estimator of the common factor for the various combination of N and T. In particular, we consider two nonlinear AR(1) processes to generate f_t, the logistic map with a positive λ and the STAR model with a negative λ.[1] The first model is the logistic map with a system error given by

$$f_t = af_{t-1}(1 - f_{t-1}) + \sigma_t \varepsilon_t \tag{6}$$

where $\varepsilon_t \sim$iid $U(-1/2, 1/2)$ and $\sigma_t = \sigma \times \min\{af_{t-1}(1 - f_{t-1}), 1 - af_{t-1}(1 - f_{t-1})\}$. This particular form of conditional heteroskedasticity ensures that the process f_t is restricted to the unit interval as long as $0 \leq \sigma \leq 2$. In the case of $\sigma = 0$, the dynamics of the logistic map is deterministic chaos when the parameter a is in the range of $3.57 < a \leq 4$. In this simulation, we focus on the chaotic case with $a = 4$. Note that when $p = 1$, the definition of the Lyapunov exponent in equation (3) simplifies to

$$\lambda \equiv \lim_{M \to \infty} M^{-1} \sum_{t=1}^{M} \ln|m'(f_{t-1})| \tag{7}$$

where $m'(f)$ is the first derivative of the function $m(f)$. When $\sigma = 0$, the value of λ for equation (6) is positive and is known to be $\ln 2 \approx 0.693$. For moderate $\sigma > 0$, equation (6) becomes a stochastic system but its dynamics can be considered as noisy chaos when $\lambda > 0$. True λ's of equation (6) for various values of non-zero σ are computed using simulation method and provided in Table 1. It shows that λ approaches 0.693 as σ becomes smaller.

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[1] We consider these two nonlinear models because they are often used in economic analysis. For example, a logistic model is used in Day (1982) and a STAR model is used in Michael, Nobay and Peel (1997).

As a second nonlinear model, a smooth transition AR (STAR) model is considered. For a linear AR(1) model with $m(f_{t-1}) = \rho f_{t-1}$, the Lyapunov exponent is $\lambda = \ln|\rho|$ since $m'(f) = \rho$ for all f. Therefore, the stationary condition $|\rho| < 1$ corresponds to a negative Lyapunov exponent. For example, in the case of $\rho = 0.5$, λ is $-\ln 2 \approx -0.693$. In the case of a unit root with $\rho = 1$, λ is zero. Therefore, in the linear case with positive ρ, the faster speed of adjustment implies smaller λ ($\lambda \to -\infty$) and the higher persistence implies larger λ ($\lambda \to 0$). However, in economic analysis, the speed of adjustment is often believed to be faster when deviation from the steady state is larger. The STAR model can be used to incorporate such a nonlinear adjustment. In simulation, we use

$$f_t = \begin{cases} \mu + \rho f_{t-1} + \sigma \varepsilon_t & f_{t-1} > c \\ S(f_{t-1}) + \sigma \varepsilon_t & -c \leq f_{t-1} \leq c \\ -\mu + \rho f_{t-1} + \sigma \varepsilon_t & f_{t-1} < -c \end{cases} \tag{8}$$

where $S(f_{t-1}) = f_{t-1} - f_{t-1}\{1 - \exp(-f_{t-1}^2)\}$, $\varepsilon_t \sim$ iid $N(0,1)$, $\rho = 0.5$, $\mu = S(c) - \rho c$ and $c(> 0)$ is a threshold value that satisfies $S'(c) = \rho$. The function $S(f_{t-1})$ provides a simple STAR structure with the slowest speed of adjustment around the steady state level $f = 0$. The linear AR structure outside the $(-c, c)$ band is introduced here to ensure that the speed of adjustment is always positive. In this model, $m'(f) \to 1$ as f approaches 0 and $m'(f) \to \rho = 0.5$ as $|f|$ increases. Therefore, we expect that λ approaches 0 as σ becomes smaller and λ approaches $\ln|\rho| = -\ln 2 \approx -0.693$ as σ becomes larger. True λ's for various values of σ obtained from simulation in Table 1 lie between -0.693 and 0 and thus confirm this conjecture.

We first evaluate the performance of a nonparametric estimator of the Lyapunov exponent in the case of observable f_t. Such an estimator is not feasible in practice, but can be obtained in the simulation study. In this section, to estimate m function

11

in equation (2), we employ a local polynomial estimator which is a type of kernel estimator. Note that the Nadaraya-Watson estimator considered in the previous section is a local polynomial estimator of order zero (local constant estimator). Here we employ the local polynomial estimator of order two (local quadratic estimator) since this choice of order is known to be most appropriate for the first derivative estimation required for the Lyapunov exponent computation (See Fan and Gijbels, 1996).[2]

Table 1 reports the results for $T =$50, 100, 200, 400 and 800 for both the logistic map and the STAR model based on 1,000 iteration.[3] The set of values of σ under consideration is $\sigma \in \{0.00, 0.05, 0.10, 0.20, 0.50\}$ for the logistic map and $\sigma \in \{0.05, 0.10, 0.20, 0.50, 1.00\}$ for the STAR model. For the logistic map with positive true λ, the Lyapunov exponent estimator performs surprisingly well for all the cases. The mean of the estimators shows that small sample bias is almost negligible even with $T =$50. Decreasing standard deviation with increasing T confirms the consistency of the estimator. In contrast, the small sample bias seems to be present for the STAR model with negative true λ. This bias, however, diminishes quickly as sample size grows. The standard deviation is somewhat larger than that of logistic map case for the corresponding T, but is decreasing with T confirming the adequacy of asymptotic theory. The difference between mean and median implies some skewness to the left for small T but it disappears for large T. Because we cannot expect the feasible estimator based on the estimated factor to perform better than the infeasible estimator, this

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[2]Since the nonlinear estimation is not invariant to scale, we always normalize the factors in the (0,1) unit interval before running a nonparametric regression for this paper.

[3]We use standard normal density as a kernel function (Gaussian kernel) with several different choices of bandwidth h. In the table, the results based on $h = 0.3 \times$range are reported.

result serves as a benchmark in the following analysis.

Let us now consider the Lyapunov exponent estimation applied to the factor estimated using principal components. Individual series x_{it} is generated from equation (1) with e_{it} and b_i being iid $N(0,1)$. The common factor f_t generated from both the logistic map and the STAR model is normalized to have unit variance before generating x_{it} so that the common component $b_i f_t$ and the idiosyncratic component e_{it} each has the same contribution to the variance of x_{it}.

We evaluate the performance of our estimator for all the combinations of $N =25$, 50, 100, 200, and 400 and $T =100$, 200, and 400. Tables 2 to 4 report the results for $T =100$, 200, and 400, respectively, using various values of N. For each table, the results from the infeasible estimator for corresponding T are also reported for the purpose of comparison with our proposed estimator. For each table with fixed T, the estimator applied to the principal components of x_{it} approaches the infeasible estimator as N grows. This evidence combined with the results from Table 1 suggests convergence of the estimator predicted by the asymptotic theory under $N, T \to \infty$. The bias from small N turns out to be negative for both the logistic map and the STAR model. In addition, the skewness with small N can be now seen in the case of the logistic map, which was not present in the infeasible estimators with small T. This asymmetry in distribution, however, is not observed when N becomes as large as 100. In contrast, some skewness is present for large N in the case of the STAR model with small T because of the skewness in the infeasible estimator.

Overall, the asymptotic approximation seems to work well for many combinations of N and T typically available in practice. Most importantly, the small standard deviation in the simulation implies the successful detection of correct sign of λ for

both the logistic map and the STAR model despite the presence of some finite sample bias from small N. Since the positive λ is the key to distinguishing the chaotic process from the stable system, we expect that conducting the inference regarding the sign of λ of the principal components will work well in the next empirical section.

4   Empirical Results

4.1   Testing for Neglected Nonlinearity in Principal Components

In this section, we investigate the nonlinear stability of the common factors extracted from the 159 monthly macroeconomic variables from the U.S. We use the same data source (and transformations) employed in Stock and Watson (2002a) in their analysis of macroeconomic forecasting using estimated factors from a large number of predictors (diffusion index forecast), but we focus only on balanced panels. The series in a list provided in Appendix B of Stock and Watson (2002a) are divided into 14 categories, each of which represents an important facet of the macroeconomic activities (e.g., production, consumption, employment, inflation, interest rates). The sample period is from 1959:3 to 1998:12, giving a maximum number of time series observations $T = 478$. All the series are standardized to have sample mean zero and unit sample variance since principal components are not scale-invariant.

The estimated series of $\widetilde{f}_t$ based on the first principal component of the full balanced panel series x_{it} with $N = 159$ is shown in Figure 1. In the figure, $\widetilde{f}_t$ is rescaled to have the same drift and variance as the (logarithm of) industrial production.[4] The comparison of $\widetilde{f}_t$ with the NBER recessionary episodes, shown in the shaded area,

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[4]Therefore, the series in the figure is equivalent to the cummulated sum of the first principal component rather than the original series.

14

shows the clear procyclical property of the common factor. The asymptotic theory predicts the $\widetilde{f}_t$ to be a consistent estimator of f_t as $N \to \infty$. To investigate this prediction using the data, we take first five and two series from each category in Appendix B of Stock and Watson (2002a) and compute $\widetilde{f}_t$ based on two subsamples with $N = 63$ and 27. Two additional series of $\widetilde{f}_t$ are also shown in the same figure. Consistent with the theory, it shows a larger difference in $\widetilde{f}_t$ between $N = 27$ and $N = 159$ compared to the difference between $N = 63$ and $N = 159$.

As a preliminary analysis, we first investigate the nonlinearity in the factor dynamics in equation (2). In applications, a linear structure is often assumed in equation (2) to simplify the analysis (e.g., Stock and Watson, 1989). Since the chaotic dynamics with positive λ is only possible with nonlinear dynamics, we employ several tests for the null hypothesis of linearity (or neglected nonlinearity) that has a power against a wide range of nonlinear alternatives. When $p = 1$, our testing hypothesis is $m(f_{t-1}) = \rho f_{t-1}$, or $E\left[u_t | f_{t-1}\right] = 0$, where $u_t = f_t - \rho f_{t-1}$. This hypothesis can be rewritten as the unconditional moment restriction of the form $E\left[\Psi(f_{t-1})u_t\right] = 0$ with any vector of measurable functions $\Psi(f_{t-1})$; thus a number of tests can be constructed with the different choice of $\Psi(f_{t-1})$.

Ramsey's (1969) regression specification error test (RESET), which is one of the most well-known tests in the specification testing literature, uses a $s \times 1$ vector of polynomial functions of fitted value from linear regression, $\Psi(f_{t-1}) = ((\widehat{\rho}f_{t-1})^2, \ldots, (\widehat{\rho}f_{t-1})^s)'$. The test statistic is defined as $RESET = T\left(\sum_{t=1}^{T} \widehat{u}_t^2 - \sum_{t=1}^{T} \widehat{v}_t^2\right) / \sum_{t=1}^{T} \widehat{u}_t^2$, where $\widehat{u}_t = f_t - \widehat{\rho}f_{t-1}$ and $\widehat{v}_t$ are the residuals from the regression of $\widehat{u}_t$ on auxiliary regressors $\Psi(f_{t-1})$ (and f_{t-1}), and asymptotically follows χ^2 distribution with s degree of freedom.

For White's (1989) neural network test, $\Psi(f_{t-1}) = \left(\psi\left(\gamma_1 f_{t-1}\right), \ldots, \psi\left(\gamma_q f_{t-1}\right)\right)'$ is a $q \times 1$ vector of logistic activation functions ψ with the coefficients γ_j's being randomly drawn independent of f_{t-1}. The test statistic (NN) can be similarly constructed by using auxiliary regressors and its limit distribution is χ^2 distribution with q degree of freedom.

One drawback of the White's neural network test is the unidentifiability of γ_j's under the null hypothesis. Instead of using random γ_j's, Teräsvirta, Lin, and Granger (1993) replaced the activation functions with their Volterra expansion up to the third order under the null. This LM type neural network test (NN-LM) can be constructed by using auxiliary regressors based on quadratic and cubic terms from the Volterra expansion of the nonlinear AR model ($\Psi(f_{t-1}) = ((f_{t-1})^2, (f_{t-1})^3)'$ for AR(1) case). The test statistic asymptotically follows χ^2 distribution with $p(p+1)/2 + p(p+1)(p+2)/6$ degree of freedom where p is lag order of AR model.

The last test we consider is the kernel-based consistent specification test for AR models proposed by Fan and Li (1997). It utilizes the $\Psi(f_{t-1}) = E\left(u_t | f_{t-1}\right) g\left(f_{t-1}\right)$ where $g\left(f_{t-1}\right)$ is a density function of f_{t-1}. The test statistic (KERNEL) is based on kernel estimator of $E\left[\Psi(f_{t-1})u_t\right]$ and follows asymptotically normal with an appropriate standardization.

We apply these four tests to $\widetilde{f_t}$, which is estimated by the principal components method since f_t is not available. In addition to the first principal component, we apply the same tests to the k-th principal component for $k = 2$ to 6. Table 5 reports the results of all four tests applied to each principal component for $p = 1$ to 4. For RESET, the results based on $s = 4$ are reported. For NN, we use three (excluding the first) principal components of $\Psi(f_{t-1})$ with $q = 10$ to avoid collinearity of f_{t-1} and

$\Psi(f_{t-1})$. Improved Bonferroni procedures from five draws is then used to construct p-values (see Lee, White, and Granger, 1993, for this procedure in detail). The RESET, NN, and NN-LM tests reject the linear hypothesis of the common factor for many cases at the conventional significance level. In contrast, based on the KERNEL test, the same hypothesis is not rejected for all cases. One possible reason of this mixed outcome is the difference in the power among the specification tests.[5] However, since at least three tests provided evidence against linearity, we proceed to the estimation and test of the Lyapunov exponent of the common factor using the nonparametric regression method.

4.2   Lyapunov Exponents of Principal Components

Here we compute the Lyapunov exponent, λ, of the common factor obtained from the principal components. As an estimation method of the nonlinear AR model of equation (2), we employ a kernel-based nonparametric method – the local polynomial estimator of order two – the method used in the simulation section. In addition to the kernel-based method, we also consider the neural networks as an alternative method for estimating the nonlinear function. The first and latter methods can be considered as the local and global nonparametric methods, respectively, and thus reporting both results should provide useful information regarding the robustness of our approach to the choice of the estimator.[6] The stability of the nonlinear system can be directly investigated by conducting a hypothesis test regarding the sign of λ using the standard error of the estimator of λ (see Shintani and Linton, 2004, for this procedure in detail).

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[5]Lee's (2001) simulation study compared the performance of NN and KERNEL and reported that KERNEL was less powerful than NN unless the bootstrapped critical value was used.

[6]These two methods are also employed in Shintani and Linton (2003) in their study of nonlinear stability in the international output series.

17

Table 6 shows the Lyapunov exponent estimates of the principal components of 159 series based on the nonlinear AR model with $p = 1$ to 4. In addition to λ of the first principal component, those of the k-th principal component from $k = 2$ to 6 are also reported. Both full sample estimates $(M = T)$ and block estimates $(M < T)$ are presented.[7] For the kernel-based estimation, Gaussian kernel function, K, is used with the bandwidth $h = 0.2 \times$ range. For the neural network estimation, the logistic activation function, ψ, is used with the number of hidden units $q = 4$.[8] For all cases, the Lyapunov exponents of the first principal component are significantly negative, implying evidence against a chaotic explanation of business cycles. This empirical finding is robust to the choice of nonparametric methods – kernel regression and neural networks.

The second to sixth principal components also provide evidence against chaos while the second principal component has a somewhat larger exponent in comparison with the other principal components.

These results are consistent with the former studies that applied the nonparametric Lyapunov exponent test to a single time series. It is worth pointing out that we are considering a system of a large number of variables and still obtaining evidence of nonlinear stability.

---

[7]For block estimation, we use $M = 70$ and report the median based on six blocks. Here, M is approximately $8 \times (T/\ln T)^{1/2}$.

[8]To solve the minimization problem required for the neural network estimation, the criterion function is modified to have the weight decay identical to the one employed in Franses and van Dijk (2000).

4.3 Nonlinear Impulse Response Function of Principal Components

The negative Lyapunov exponents can be considered an empirical justification of the impulse response analysis that is commonly used among macroeconomists since it requires the assumption of the exogenous shocks and a stable steady state in the system.

For the nonlinear system, the nonlinear impulse response analysis was proposed by Gallant, Rossi, and Tauchen (1993), and Potter (2000). Here we compute the nonlinear impulse response function of the common factors based on the nonparametric method. One of the most frequently used definitions of the n-period ahead nonlinear impulse response is

$$
\begin{aligned}
IR_n(\delta) &= E(f_{t+n-1}|\mathbf{f}_{t-1} = \mathbf{f}^*) - E(f_{t+n-1}|\mathbf{f}_{t-1} = \mathbf{f}) \\
&= m_n(\mathbf{f}^*) - m_n(\mathbf{f})
\end{aligned}
$$

where $\mathbf{f}_{t-1} = (f_{t-1}, ..., f_{t-p})$, $\mathbf{f}^* = (\mu_f + \delta, \mu_f, ..., \mu_f)$ and $\mathbf{f} = (\mu_f, \mu_f, ..., \mu_f)$. Unlike the linear impulse response function, the nonlinear impulse response function is known to depend on the size of shock δ and the initial condition μ_f. Note that the mean μ_f is identical to the steady state level in the absence of shocks for a globally stable system with negative λ. In contrast, in the case of chaos, μ_f cannot be considered as the steady state. In addition, a small difference in δ results in completely different shapes of the impulse responses. Therefore, the nonlinear impulse response analysis is inappropriate for a system with positive λ since it cannot represent the typical shock propagation process in the system.

The nonlinear impulse response function can be obtained by estimating the n-

period ahead conditional mean function $m_n(f_{t-1}, ..., f_{t-p})$ using the same nonpara-

metric method used in the estimation of 1-period ahead conditional mean function

$m(f_{t-1}, ..., f_{t-p}) = m_1(f_{t-1}, ..., f_{t-p})$ in equation (2). As in the 1-period ahead case,

nonparametric regression $\widehat{m}_n(\mathbf{f})$ based on f_t is not feasible and $\widetilde{m}_n(\mathbf{f})$ based on $\widetilde{f}_t$

need to be employed.

Figures 2A to 2D show the nonlinear impulse response function for the case of

$p = 1$ to 4, respectively. For the size of the shock, we use $\delta \in \{+2\sigma, +\sigma, -\sigma, -2\sigma\}$ where σ is the standard deviation of the shocks estimated using the residuals

of the nonparametric regression of equation (2). The linear impulse responses from

the linear AR models are also shown in the figures for comparison. The shapes of

impulse responses in figures show a clear difference between the linear and nonlinear

cases.

Recall that the larger Lyapunov exponent was obtained for the second principal

component $(k = 2)$ compared to the first principal component $(k = 1)$ in the previous

section. The nonlinear impulse responses of the second principal component are

shown in Figures 3A to 3D for the case of $p = 1$ to 4, respectively. Comparisons with

Figures 2A to 2D reveal a much higher persistence of common shocks in the second

factor, which is consistent with a less stable result from the Lyapunov exponent. Half-

lives computed from these impulse responses can be used to measure the persistence

in common shocks. For example, when $p = 2$, half-lives from the linear impulse

responses are 1.20 months for $k = 1$ and 3.54 months for $k = 2$. When nonlinear

impulse responses are used, the half-lives of the $+2\sigma$, $+\sigma$, $-\sigma$, and -2σ shocks in the

first principal component $(k = 1)$ are 0.87, 0.93, 0.97, and 1.00 months, respectively.

In contrast, the half-lives of corresponding shocks in the second principal component

20

$(k = 2)$ are 5.05, 5.06, 5.27, and 5.32 months, respectively. According to Stock and Watson (2002a, p.153), the first factor loads primarily on output and employment and the second factor, on interest rate spreads, unemployment rates, and capacity utilization rates. While identification of each shock in two factors is not our intention here, the results of our nonlinear analysis seem to provide some useful information regarding the characteristics of the source of business cycles.

5 Conclusion

The largest Lyapunov exponent is a useful nonlinear stability measure in distinguishing chaos from an exogenous shock-driven aggregate fluctuation in an economy. In practice, this measure is nonparametrically estimated from a single economic time series. Such an approach, however, cannot be applied to test the nonlinear stability of a large system of equations because of the dimensionality problem of the nonparametric methods. To circumvent this problem, we proposed computing the nonlinear stability measure of the common component of multiple time series under the framework of a dynamic factor model.

We employed a method of principal components to estimate the unobservable common factors since the method is known to provide a consistent estimator when a large panel data is available. Furthermore, the nonparametric regression based on the estimated factor was shown to be a valid approach to investigating the nonlinear dynamic factor structure. The simulation evidence also suggested that our procedure worked well with a sample size typically available in practice.

The empirical evidence from a large U.S. panel data suggested the possibility of nonlinearity in the system while it excluded the class of nonlinearity that could

generate endogenous fluctuation or chaos. This result was consistent with the former studies that applied a nonparametric Lyapunov exponent test to a single time series (e.g., Shintani and Linton, 2003). Therefore, we now have evidence on nonlinear stability not only within a univariate or a small system of equations but also within a system of a large number of variables.

Appendix A: Proofs

Lemma A.1. *Suppose assumptions F and K are satisfied. Then,*

(i) $\quad \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \left(\widetilde{f}_t - H f_t \right) K \left(\frac{f_{t-1} - f}{h} \right) = o_p(1),$

(ii) $\quad \frac{1}{\sqrt{Th^3}} \sum_{t=1}^{T} \left(\widetilde{f}_{t-1} - H f_{t-1} \right) K' \left(\frac{f_{t-1} - f}{h} \right) = o_p(1),$

(iii) $\quad \frac{1}{\sqrt{Th^3}} \sum_{t=1}^{T} \left(\widetilde{f}_{t-1} - H f_{t-1} \right) f_t K' \left(\frac{f_{t-1} - f}{h} \right) = o_p(1),$ and

(iv) $\quad \frac{1}{\sqrt{Th^3}} \sum_{t=1}^{T} \left(\widetilde{f}_t - H f_t \right) \left(\widetilde{f}_{t-1} - H f_{t-1} \right) K' \left(\frac{f_{t-1} - f}{h} \right) = o_p(1)$

where $H = v^{-1}(\widetilde{F}'F/T)(B'B/N)$, $\widetilde{F} = \left[\widetilde{f}_1, \cdots, \widetilde{f}_T \right]'$ and v is the largest eigenvalue of XX'/TN.

Proof of Lemma A.1.

From Proposition 2 of Bai (2003), $\max_{1 \le t \le T} \left| \widetilde{f}_t - H f_t \right| = O_p(T^{-1/2})$ given $N/T^2 \to \infty$. The result for Lemma A.1(i) follows from

$$
\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \left(\widetilde{f}_t - H f_t \right) K \left(\frac{f_{t-1} - f}{h} \right)
$$

$$
\le \sqrt{Th} \left(\frac{1}{Th} \sum_{t=1}^{T} K \left(\frac{f_{t-1} - f}{h} \right) \right) \max_{1 \le t \le T} \left| \widetilde{f}_t - H f_t \right|
$$

$$
= O_p \left(\sqrt{Th} \right) \cdot O_p(1) \cdot O_p(T^{-1/2}) = O_p(T^{-\beta/2}).
$$

Similarly, for (ii),

$$
\frac{1}{\sqrt{Th^3}} \sum_{t=1}^{T} \left(\widetilde{f}_{t-1} - H f_{t-1} \right) K' \left(\frac{f_{t-1} - f}{h} \right)
$$

$$
\le \sqrt{Th} \left(\frac{1}{Th^2} \sum_{t=1}^{T} K' \left(\frac{f_{t-1} - f}{h} \right) \right) \max_{1 \le t \le T} \left| \widetilde{f}_t - H f_t \right|
$$

$$
= O_p \left(\sqrt{Th} \right) \cdot O_p(1) \cdot O_p(T^{-1/2}) = O_p(T^{-\beta/2}).
$$

The results for (iii) and (iv) follow from

$$
\frac{1}{\sqrt{Th^3}} \sum_{t=1}^{T} \left(\widetilde{f}_{t-1} - H f_{t-1} \right) f_t K' \left(\frac{f_{t-1} - f}{h} \right)
$$

$$
\le \sqrt{Th} \left(\frac{1}{Th^2} \sum_{t=1}^{T} K' \left(\frac{f_{t-1} - f}{h} \right) \right) \max_{1 \le t \le T} \left| \widetilde{f}_t - H f_t \right| \max_{1 \le t \le T} |f_t|
$$

$$
= O_p \left(\sqrt{Th} \right) \cdot O_p(1) \cdot O_p(T^{-1/2}) \cdot O_p(\ln T) = O_p \left(\frac{\ln T}{T^{\beta/2}} \right).
$$

and

$$\frac{1}{\sqrt{Th^3}} \sum_{t=1}^{T} \left(\widetilde{f}_t - Hf_t \right) \left(\widetilde{f}_{t-1} - Hf_{t-1} \right) K' \left(\frac{f_{t-1} - f}{h} \right)$$

$$\leq \quad \sqrt{Th} \left(\frac{1}{Th^2} \sum_{t=1}^{T} K' \left(\frac{f_{t-1} - f}{h} \right) \right) \left(\max_{1 \leq t \leq T} \left| \widetilde{f}_t - Hf_t \right| \right)^2$$

$$= \quad O_p \left(\sqrt{Th} \right) \cdot O_p(1) \cdot O_p(T^{-1}) = O_p(T^{-(\beta+1)/2}).$$

■

Proof of Proposition 1.

$$\sqrt{Th} \left(\widetilde{m}(f) - \widehat{m}(f) \right)$$

$$= \left[\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \widetilde{f}_t K \left(\frac{\widetilde{f}_{t-1} - f}{h} \right) \right] \Big/ \widetilde{g}(f) - \left[\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} f_t K \left(\frac{f_{t-1} - f}{h} \right) \right] \Big/ \widehat{g}(f)$$

$$= \left\{ \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \widetilde{f}_t K \left(\frac{\widetilde{f}_{t-1} - f}{h} \right) - \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} f_t K \left(\frac{f_{t-1} - f}{h} \right) \right\} \Big/ \widetilde{g}(f)$$

$$+ \left(\frac{1}{\widetilde{g}(f)} - \frac{1}{\widehat{g}(f)} \right) \left[\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} f_t K \left(\frac{f_{t-1} - f}{h} \right) \right]$$

$$= \left\{ \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \widetilde{f}_t K \left(\frac{f_{t-1} - f}{h} \right) - \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} f_t K \left(\frac{f_{t-1} - f}{h} \right) \right\} \Big/ \widetilde{g}(f)$$

$$+ \left\{ \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \widetilde{f}_t K \left(\frac{\widetilde{f}_{t-1} - f}{h} \right) - \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \widetilde{f}_t K \left(\frac{f_{t-1} - f}{h} \right) \right\} \Big/ \widetilde{g}(f)$$

$$+ \sqrt{Th} \left(\frac{1}{\widetilde{g}(f)} - \frac{1}{\widehat{g}(f)} \right) \left[\frac{1}{Th} \sum_{t=1}^{T} f_t K \left(\frac{f_{t-1} - f}{h} \right) \right]$$

$$= \quad a_T^* + b_T^* + c_T^*, \text{ say,}$$

where

$$\widehat{g}(f) = \frac{1}{Th} \sum_{t=1}^{T} K \left(\frac{f_{t-1} - f}{h} \right) \quad \text{and} \quad \widetilde{g}(f) = \frac{1}{Th} \sum_{t=1}^{T} K \left(\frac{\widetilde{f}_{t-1} - f}{h} \right).$$

For the numerator of a_T^*, we have

$$\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \left(\widetilde{f}_t - f_t \right) K \left(\frac{f_{t-1} - f}{h} \right)$$

$$= \quad \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} \left(\widetilde{f}_t - Hf_t \right) K \left(\frac{f_{t-1} - f}{h} \right) + \frac{1}{\sqrt{Th}} (H - 1) \sum_{t=1}^{T} f_t K \left(\frac{f_{t-1} - f}{h} \right)$$

$$= \quad o_p(1) + O_p(\sqrt{h}) = o_p(1).$$

24

The convergence of the first term follows from Lemma A.1(i). For the second term, we used

$$\sqrt{Th}\,(H-1)\frac{1}{Th}\sum_{t=1}^{T} f_t K\left(\frac{f_{t-1}-f}{h}\right)$$

$$= O_p(\sqrt{Th})O_p(T^{-1/2}) = O_p(\sqrt{h})$$

where $H-1 = O_p\left(\min(\sqrt{T},\sqrt{N})^{-1}\right) = O_p(T^{-1/2})$ follows from Assumption F(i) and (ii). For the numerator of b_T^*, from a mean value expansion, it can be rewritten as

$$\frac{1}{\sqrt{Th}}\sum_{t=1}^{T}\widetilde{f}_t K'\left(\frac{f_{t-1}-f}{h}\right)\left(\frac{\widetilde{f}_{t-1}-f_{t-1}}{h}\right)+\frac{1}{\sqrt{Th}}\sum_{t=1}^{T}\widetilde{f}_t K''\left(\frac{f_{t-1}^*-f}{h}\right)\left(\frac{\widetilde{f}_{t-1}-f_{t-1}}{h}\right)^2$$

where f_{t-1}^* lies between $\widetilde{f}_{t-1}$ and f_{t-1}. Since the first term dominates the second term, it suffices to show the convergence of the first term.

$$\frac{1}{\sqrt{Th^3}}\sum_{t=1}^{T}\widetilde{f}_t K'\left(\frac{f_{t-1}-f}{h}\right)\left(\widetilde{f}_{t-1}-f_{t-1}\right)$$

$$= \frac{1}{\sqrt{Th^3}}\sum_{t=1}^{T}\widetilde{f}_t\left(\widetilde{f}_{t-1}-Hf_{t-1}\right)K'\left(\frac{f_{t-1}-f}{h}\right)+\frac{1}{\sqrt{Th^3}}(H-1)\sum_{t=1}^{T}\widetilde{f}_t f_{t-1}K'\left(\frac{f_{t-1}-f}{h}\right)$$

$$= \frac{1}{\sqrt{Th^3}}\sum_{t=1}^{T}\left(\widetilde{f}_t-Hf_t\right)\left(\widetilde{f}_{t-1}-Hf_{t-1}\right)K'\left(\frac{f_{t-1}-f}{h}\right)$$

$$+H\frac{1}{\sqrt{Th^3}}\sum_{t=1}^{T}\left(\widetilde{f}_{t-1}-Hf_{t-1}\right)f_{t-1}K'\left(\frac{f_{t-1}-f}{h}\right)$$

$$+(H-1)\frac{1}{\sqrt{Th^3}}\sum_{t=1}^{T}\left(\widetilde{f}_t-Hf_t\right)f_{t-1}K'\left(\frac{f_{t-1}-f}{h}\right)$$

$$+\sqrt{Th}H(H-1)\frac{1}{Th^2}\sum_{t=1}^{T}f_t f_{t-1}K'\left(\frac{f_{t-1}-f}{h}\right)$$

$$= o_p(1).$$

For the first to third terms, convergence follows from Lemma A.1(iii) and (iv). Similarly, from a mean value expansion, we have

$$\sqrt{Th}\left(\widetilde{g}\left(f\right)-\widehat{g}\left(f\right)\right)$$

$$= \frac{1}{\sqrt{Th}}\sum_{t=1}^{T}K'\left(\frac{f_{t-1}-f}{h}\right)\left(\frac{\widetilde{f}_{t-1}-f_{t-1}}{h}\right)+\frac{1}{\sqrt{Th}}\sum_{t=1}^{T}K''\left(\frac{f_{t-1}^*-f}{h}\right)\left(\frac{\widetilde{f}_{t-1}-f_{t-1}}{h}\right)^2$$

with the dominant first term being $o_p(1)$ from Lemma A.1(ii). Since $\widehat{g}\left(f\right)^{-1} = O_p(1)$, we have $\sqrt{Th}\left(\widetilde{g}\left(f\right)^{-1}-\widehat{g}\left(f\right)^{-1}\right) = o_p(1)$ and $c_T^* = o_p(1)$. Furthermore, $\widetilde{g}\left(f\right)^{-1} = O_p(1)$ implies both a_T^* and b_T^* to be $o_p(1)$ since the numerators are shown to be $o_p(1)$. ■

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Table 1. Simulation Results: Lyapunov Exponent Estimates of Observed Variables

(A) Logistic Map

σ	0.00	0.05	0.10	0.20	0.50
true λ	0.693	0.693	0.693	0.694	0.699
$T=50$	0.691/0.692	0.693/0.693	0.694/0.693	0.695/0.695	0.699/0.699
	(0.023)	(0.025)	(0.027)	(0.026)	(0.032)
100	0.694/0.693	0.693/0.693	0.693/0.693	0.695/0.694	0.699/0.699
	(0.014)	(0.013)	(0.013)	(0.014)	(0.018)
200	0.693/0.693	0.693/0.693	0.693/0.693	0.694/0.694	0.699/0.699
	(0.006)	(0.007)	(0.007)	(0.008)	(0.011)
400	0.693/0.693	0.693/0.693	0.693/0.693	0.694/0.694	0.699/0.699
	(0.003)	(0.004)	(0.004)	(0.004)	(0.006)
800	0.693/0.693	0.693/0.693	0.693/0.693	0.694/0.694	0.699/0.699
	(0.002)	(0.002)	(0.002)	(0.002)	(0.004)

(B) STAR Model

σ	0.05	0.10	0.20	0.50	1.00
true λ	-0.077	-0.159	-0.286	-0.460	-0.561
$T=50$	-0.289/-0.251	-0.367/-0.328	-0.501/-0.467	-0.686/-0.659	-0.803/-0.758
	(0.169)	(0.187)	(0.202)	(0.244)	(0.295)
100	-0.168/-0.154	-0.249/-0.232	-0.393/-0.372	-0.586/-0.565	-0.694/-0.671
	(0.076)	(0.093)	(0.124)	(0.165)	(0.194)
200	-0.114/-0.107	-0.193/-0.186	-0.329/-0.315	-0.516/-0.499	-0.621/-0.603
	(0.036)	(0.047)	(0.071)	(0.106)	(0.129)
400	-0.092/-0.090	-0.169/-0.167	-0.297/-0.295	-0.479/-0.474	-0.583/-0.577
	(0.019)	(0.026)	(0.042)	(0.066)	(0.080)
800	-0.081/-0.080	-0.157/-0.155	-0.281/-0.279	-0.459/-0.455	-0.562/-0.559
	(0.011)	(0.016)	(0.026)	(0.042)	(0.052)

Notes: The results are based on 1,000 Monte Carlo draws using the design described in the main text. Both sample mean and median of the kernel estimates of the largest Lyapunov exponents are reported (mean/median). Numbers in parentheses are sample standard deviations.

**Table 2. Simulation Results: Lyapunov Exponent Estimates of
Unobserved Common Factors (T=100)**

(A) Logistic Map

σ	0.00	0.05	0.10	0.20	0.50
true λ	0.693	0.693	0.693	0.694	0.699
$N = 25$	0.355/0.364	0.350/0.361	0.355/0.364	0.353/0.366	0.354/0.358
	(0.113)	(0.112)	(0.116)	(0.115)	(0.115)
50	0.514/0.517	0.517/0.520	0.516/0.516	0.513/0.519	0.516/0.521
	(0.076)	(0.075)	(0.074)	(0.076)	(0.076)
100	0.598/0.599	0.598/0.601	0.597/0.599	0.596/0.598	0.608/0.610
	(0.057)	(0.056)	(0.055)	(0.054)	(0.056)
200	0.645/0.645	0.643/0.644	0.643/0.644	0.643/0.645	0.649/0.648
	(0.044)	(0.043)	(0.046)	(0.044)	(0.045)
400	0.668/0.668	0.668/0.668	0.666/0.667	0.670/0.671	0.674/0.675
	(0.036)	(0.038)	(0.037)	(0.036)	(0.037)
∞	0.694/0.693	0.693/0.693	0.693/0.693	0.695/0.694	0.699/0.699
	(0.014)	(0.013)	(0.013)	(0.014)	(0.018)

(B) STAR Model

σ	0.05	0.10	0.20	0.50	1.00
true λ	-0.077	-0.159	-0.286	-0.460	-0.561
$N = 25$	-0.296/-0.264	-0.339/-0.313	-0.453/-0.433	-0.635/-0.619	-0.746/-0.719
	(0.136)	(0.123)	(0.133)	(0.171)	(0.201)
50	-0.231/-0.209	-0.291/-0.273	-0.422/-0.406	-0.612/-0.588	-0.719/-0.697
	(0.102)	(0.103)	(0.126)	(0.167)	(0.194)
100	-0.196/-0.177	-0.268/-0.251	-0.404/-0.385	-0.593/-0.568	-0.700/-0.672
	(0.093)	(0.101)	(0.122)	(0.163)	(0.196)
200	-0.184/-0.166	-0.259/-0.237	-0.401/-0.380	-0.587/-0.567	-0.694/-0.677
	(0.089)	(0.101)	(0.127)	(0.163)	(0.190)
400	-0.169/-0.154	-0.247/-0.232	-0.386/-0.370	-0.579/-0.569	-0.690/-0.676
	(0.077)	(0.090)	(0.114)	(0.157)	(0.189)
∞	-0.168/-0.154	-0.249/-0.232	-0.393/-0.372	-0.586/-0.565	-0.694/-0.671
	(0.076)	(0.093)	(0.124)	(0.165)	(0.194)

Notes: The results are based on 1,000 Monte Carlo draws using the design described in the main text. Infeasible estimates from Table 1 are also shown as $N = \infty$ as a benchmark case. Both sample mean and median of the kernel estimates of the largest Lyapunov exponents are reported (mean/median). Numbers in parentheses are sample standard deviations.

Table 3. Simulation Results: Lyapunov Exponent Estimates of Unobserved Common Factors (T=200)

(A) Logistic Map

σ	0.00	0.05	0.10	0.20	0.50
true λ	0.693	0.693	0.693	0.694	0.699
$N = 25$	0.355/0.364	0.355/0.365	0.358/0.362	0.354/0.364	0.356/0.364
	(0.103)	(0.102)	(0.098)	(0.102)	(0.100)
50	0.514/0.517	0.514/0.515	0.511/0.515	0.516/0.521	0.520/0.521
	(0.057)	(0.013)	(0.013)	(0.014)	(0.018)
100	0.599/0.600	0.599/0.600	0.601/0.603	0.599/0.600	0.606/0.606
	(0.038)	(0.039)	(0.040)	(0.040)	(0.040)
200	0.644/0.645	0.645/0.646	0.645/0.645	0.646/0.648	0.649/0.650
	(0.031)	(0.031)	(0.032)	(0.031)	(0.030)
400	0.668/0.668	0.669/0.669	0.668/0.669	0.669/0.668	0.674/0.674
	(0.025)	(0.026)	(0.027)	(0.025)	(0.026)
∞	0.693/0.693	0.693/0.693	0.693/0.693	0.694/0.694	0.699/0.699
	(0.006)	(0.007)	(0.007)	(0.008)	(0.011)

(B) STAR Model

σ	0.05	0.10	0.20	0.50	1.00
true λ	-0.077	-0.159	-0.286	-0.460	-0.561
$N = 25$	-0.214/-0.201	-0.271/-0.262	-0.394/-0.380	-0.573/-0.558	-0.679/-0.661
	(0.070)	(0.069)	(0.088)	(0.118)	(0.139)
50	-0.161/-0.151	-0.230/-0.222	-0.359/-0.348	-0.542/-0.529	-0.647/-0.632
	(0.052)	(0.058)	(0.080)	(0.113)	(0.133)
100	-0.141/-0.135	-0.215/-0.207	-0.345/-0.336	-0.529/-0.515	-0.634/-0.617
	(0.044)	(0.049)	(0.070)	(0.105)	(0.125)
200	-0.127/-0.119	-0.203/-0.197	-0.338/-0.327	-0.524/-0.511	-0.628/-0.615
	(0.041)	(0.049)	(0.073)	(0.105)	(0.125)
400	-0.119/-0.113	-0.196/-0.188	-0.330/-0.322	-0.517/-0.504	-0.621/-0.603
	(0.037)	(0.045)	(0.069)	(0.103)	(0.122)
∞	-0.114/-0.107	-0.193/-0.186	-0.329/-0.315	-0.516/-0.499	-0.621/-0.603
	(0.036)	(0.047)	(0.071)	(0.106)	(0.129)

Notes: The results are based on 1,000 Monte Carlo draws using the design described in the main text. Infeasible estimates from Table 1 are also shown as $N = \infty$ as a benchmark case. Both sample mean and median of the kernel estimates of the largest Lyapunov exponents are reported (mean/median). Numbers in parentheses are sample standard deviations.

Table 4. Simulation Results: Lyapunov Exponent Estimates of Unobserved Common Factors (T=400)

(A) Logistic Map

σ	0.00	0.05	0.10	0.20	0.50
true λ	0.693	0.693	0.693	0.694	0.699
$N = 25$	0.358/0.370	0.357/0.372	0.358/0.369	0.357/0.369	0.359/0.369
	(0.087)	(0.085)	(0.088)	(0.088)	(0.088)
50	0.516/0.519	0.515/0.518	0.513/0.516	0.514/0.515	0.519/0.523
	(0.044)	(0.045)	(0.044)	(0.045)	(0.045)
100	0.602/0.602	0.601/0.602	0.603/0.604	0.600/0.602	0.605/0.605
	(0.028)	(0.029)	(0.027)	(0.029)	(0.028)
200	0.646/0.646	0.646/0.647	0.646/0.646	0.646/0.646	0.651/0.651
	(0.022)	(0.022)	(0.022)	(0.022)	(0.021)
400	0.668/0.669	0.670/0.670	0.668/0.668	0.671/0.671	0.675/0.674
	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)
∞	0.693/0.693	0.693/0.693	0.693/0.693	0.694/0.694	0.699/0.699
	(0.003)	(0.004)	(0.004)	(0.004)	(0.006)

(B) STAR Model

σ	0.05	0.10	0.20	0.50	1.00
true λ	-0.077	-0.159	-0.286	-0.460	-0.561
$N = 25$	-0.170/-0.164	-0.236/-0.231	-0.356/-0.350	-0.534/-0.529	-0.639/-0.633
	(0.039)	(0.040)	(0.051)	(0.073)	(0.087)
50	-0.134/-0.130	-0.205/-0.202	-0.328/-0.323	-0.508/-0.501	-0.613/-0.606
	(0.029)	(0.032)	(0.046)	(0.069)	(0.082)
100	-0.112/-0.109	-0.187/-0.185	-0.313/-0.309	-0.495/-0.489	-0.600/-0.591
	(0.023)	(0.029)	(0.045)	(0.070)	(0.085)
200	-0.102/-0.099	-0.178/-0.176	-0.305/-0.302	-0.488/-0.483	-0.593/-0.586
	(0.022)	(0.027)	(0.041)	(0.064)	(0.078)
400	-0.098/-0.094	-0.174/-0.171	-0.302/-0.299	-0.482/-0.476	-0.585/-0.580
	(0.022)	(0.028)	(0.044)	(0.068)	(0.082)
∞	-0.092/-0.090	-0.169/-0.167	-0.297/-0.295	-0.479/-0.474	-0.583/-0.577
	(0.019)	(0.026)	(0.042)	(0.066)	(0.080)

Notes: The results are based on 1,000 Monte Carlo draws using the design described in the main text. Infeasible estimates from Table 1 are also shown as $N = \infty$ as a benchmark case. Both sample mean and median of the kernel estimates of the largest Lyapunov exponents are reported (mean/median). Numbers in parentheses are sample standard deviations.

Table 5. Test for Neglected Nonlinearity in Principal Components

|	$p=1$	$p=2$	$p=3$	$p=4$	$p=1$	$p=2$	$p=3$	$p=4$
	RESET				NN			
	(Ramsey, 1969)				(White, 1989)			
$k=1$	0.012	0.025	0.053	0.054	0.016	0.003	0.007	<0.001
$k=2$	0.191	0.101	0.361	0.350	0.210	<0.001	<0.001	0.202
$k=3$	0.002	<0.001	<0.001	<0.001	0.001	0.002	<0.001	<0.001
$k=4$	0.001	0.111	0.257	0.117	0.002	0.003	0.002	<0.001
$k=5$	<0.001	<0.001	<0.001	<0.001	<0.001	0.001	0.586	0.004
$k=6$	0.008	0.020	0.047	0.125	0.007	0.038	0.080	0.086
	NN-LM				KERNEL			
	(Teräsvirta, Lin, and Granger, 1993)				(Fan and Li, 1997)			
$k=1$	0.006	0.002	<0.001	<0.001	0.745	0.777	0.775	0.763
$k=2$	0.108	0.001	<0.001	<0.001	0.804	0.814	0.816	0.832
$k=3$	0.001	<0.001	<0.001	<0.001	0.775	0.762	0.776	0.797
$k=4$	0.001	0.007	0.002	<0.001	0.675	0.759	0.782	0.764
$k=5$	<0.001	<0.001	<0.001	<0.001	0.760	0.770	0.783	0.790
$k=6$	0.005	0.017	0.085	0.060	0.729	0.779	0.797	0.823

Notes: Numbers are p-values of the tests for the null hypothesis of linearity applied to k-th principal components of 159 macroeconomic variables ($N = 159$). The sample period is 1959:3-1998:12 ($T = 478$).

Table 6. Lyapunov Exponent Estimates of Principal Components

	(1) Full Sample				(2) Block			
	$p=1$	$p=2$	$p=3$	$p=4$	$p=1$	$p=2$	$p=3$	$p=4$
				(A) Kernel Estimation				
$k=1$	-0.527	-0.321	-0.223	-0.160	-0.540	-0.311	-0.235	-0.139
	(0.018)	(0.027)	(0.017)	(0.015)	(0.038)	(0.047)	(0.046)	(0.028)
$k=2$	-0.239	-0.120	-0.077	-0.082	-0.239	-0.095	-0.069	-0.071
	(0.010)	(0.015)	(0.008)	(0.007)	(0.017)	(0.008)	(0.009)	(0.013)
$k=3$	-0.998	-0.837	-0.512	-0.391	-1.019	-0.763	-0.515	-0.393
	(0.020)	(0.028)	(0.021)	(0.019)	(0.052)	(0.058)	(0.047)	(0.052)
$k=4$	-2.283	-0.906	-0.315	-0.184	-2.252	-0.868	-0.327	-0.182
	(0.063)	(0.017)	(0.011)	(0.009)	(0.159)	(0.054)	(0.025)	(0.019)
$k=5$	-1.423	-1.003	-0.770	-0.481	-1.413	-0.983	-0.753	-0.447
	(0.011)	(0.021)	(0.020)	(0.017)	(0.024)	(0.058)	(0.047)	(0.052)
$k=6$	-0.722	-0.290	-0.146	-0.105	-0.715	-0.311	-0.150	-0.101
	(0.034)	(0.018)	(0.009)	(0.007)	(0.077)	(0.057)	(0.020)	(0.020)
				(B) Neural Network Estimation				
$k=1$	-0.515	-0.267	-0.218	-0.094	-0.527	-0.275	-0.220	-0.088
	(0.020)	(0.024)	(0.026)	(0.020)	(0.055)	(0.069)	(0.053)	(0.042)
$k=2$	-0.235	-0.092	-0.061	-0.046	-0.240	-0.088	-0.053	-0.041
	(0.018)	(0.009)	(0.006)	(0.005)	(0.037)	(0.007)	(0.007)	(0.013)
$k=3$	-0.888	-0.504	-0.415	-0.312	-0.953	-0.552	-0.415	-0.275
	(0.041)	(0.036)	(0.017)	(0.018)	(0.055)	(0.054)	(0.011)	(0.049)
$k=4$	-2.385	-0.890	-0.302	-0.210	-2.364	-0.924	-0.298	-0.189
	(0.079)	(0.026)	(0.019)	(0.013)	(0.246)	(0.032)	(0.040)	(0.025)
$k=5$	-1.987	-0.881	-0.761	-0.367	-1.940	-0.816	-0.701	-0.333
	(0.039)	(0.026)	(0.024)	(0.022)	(0.099)	(0.055)	(0.062)	(0.057)
$k=6$	-0.628	-0.275	-0.111	-0.081	-0.593	-0.274	-0.116	-0.084
	(0.032)	(0.023)	(0.011)	(0.009)	(0.082)	(0.049)	(0.025)	(0.020)

Notes: The largest Lyapunov exponents of k-th principal components of 159 macroeconomic variables ($N = 159$). The sample period is 1959:3-1998:12 ($T = 478$). Numbers in parentheses are HAC standard errors based on QS kernel with optimal bandwidth (see Andrews, 1991). Median values are reported for the subsample estimates (Block).

Figure 1. First Principal Component

$N = 159$

$N = 27$

$N = 63$
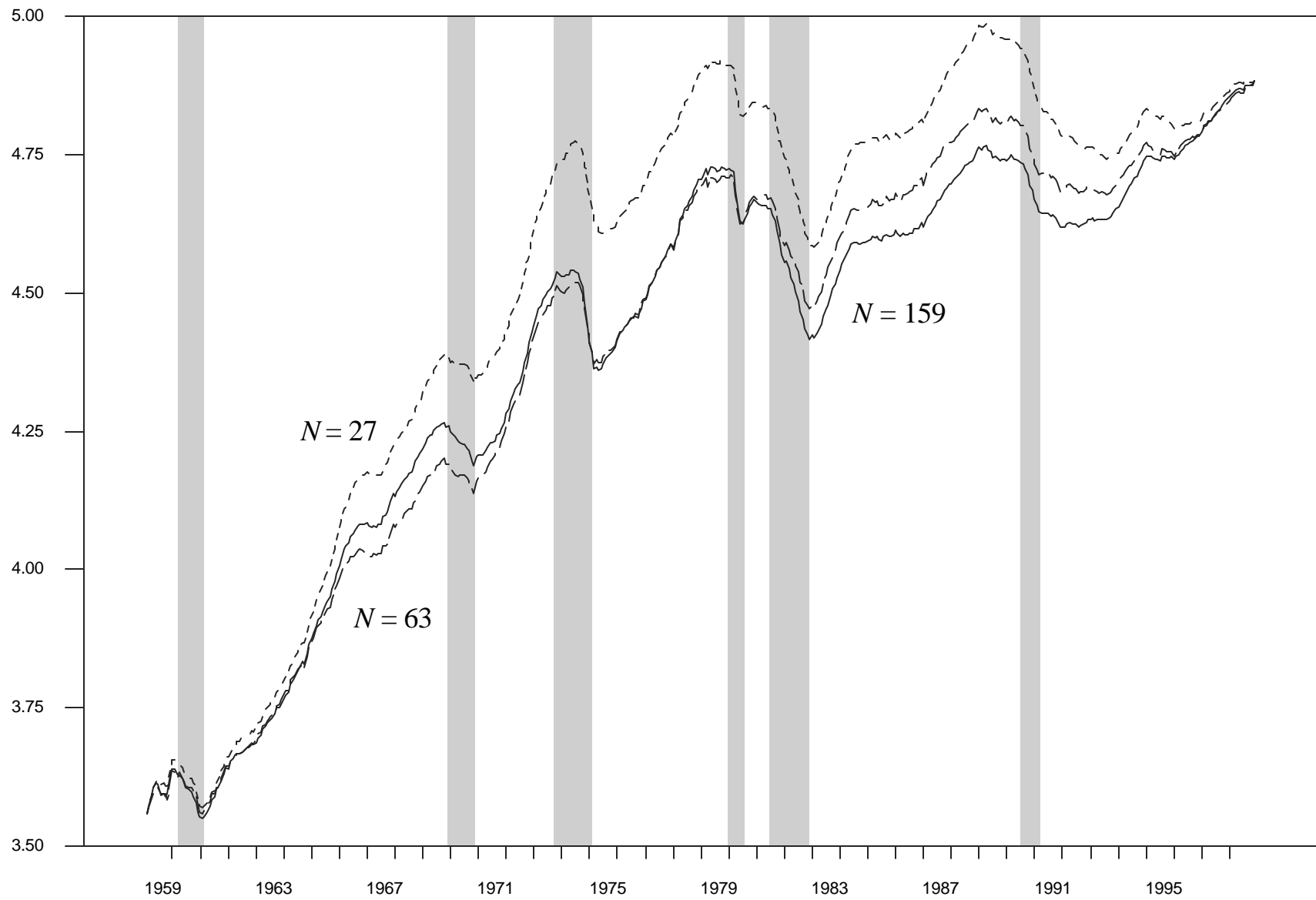
Note: Rescaled using industrial production.
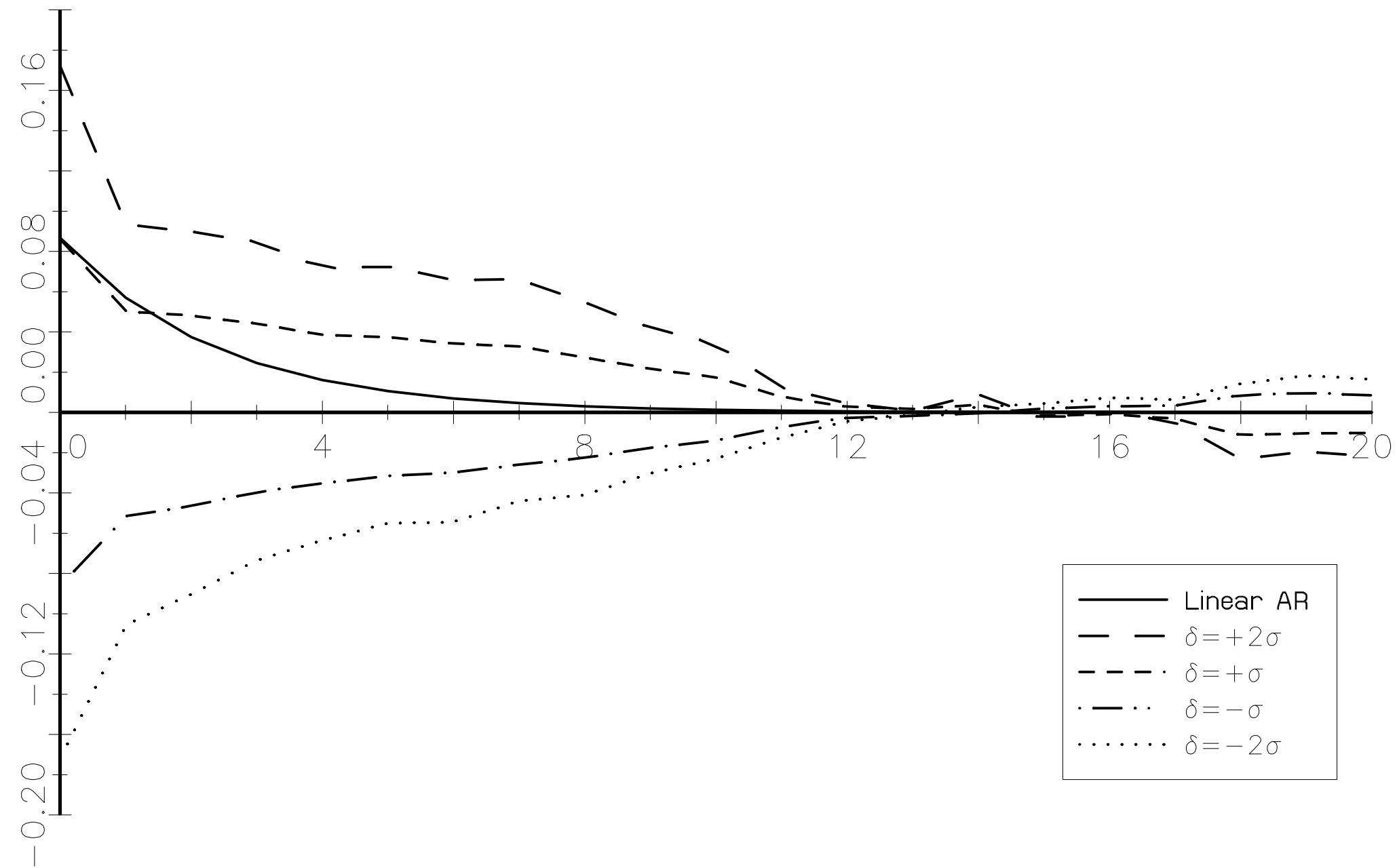
Figure 2A
Nonlinear Impulse Responses
(NLAR(1), k=1)
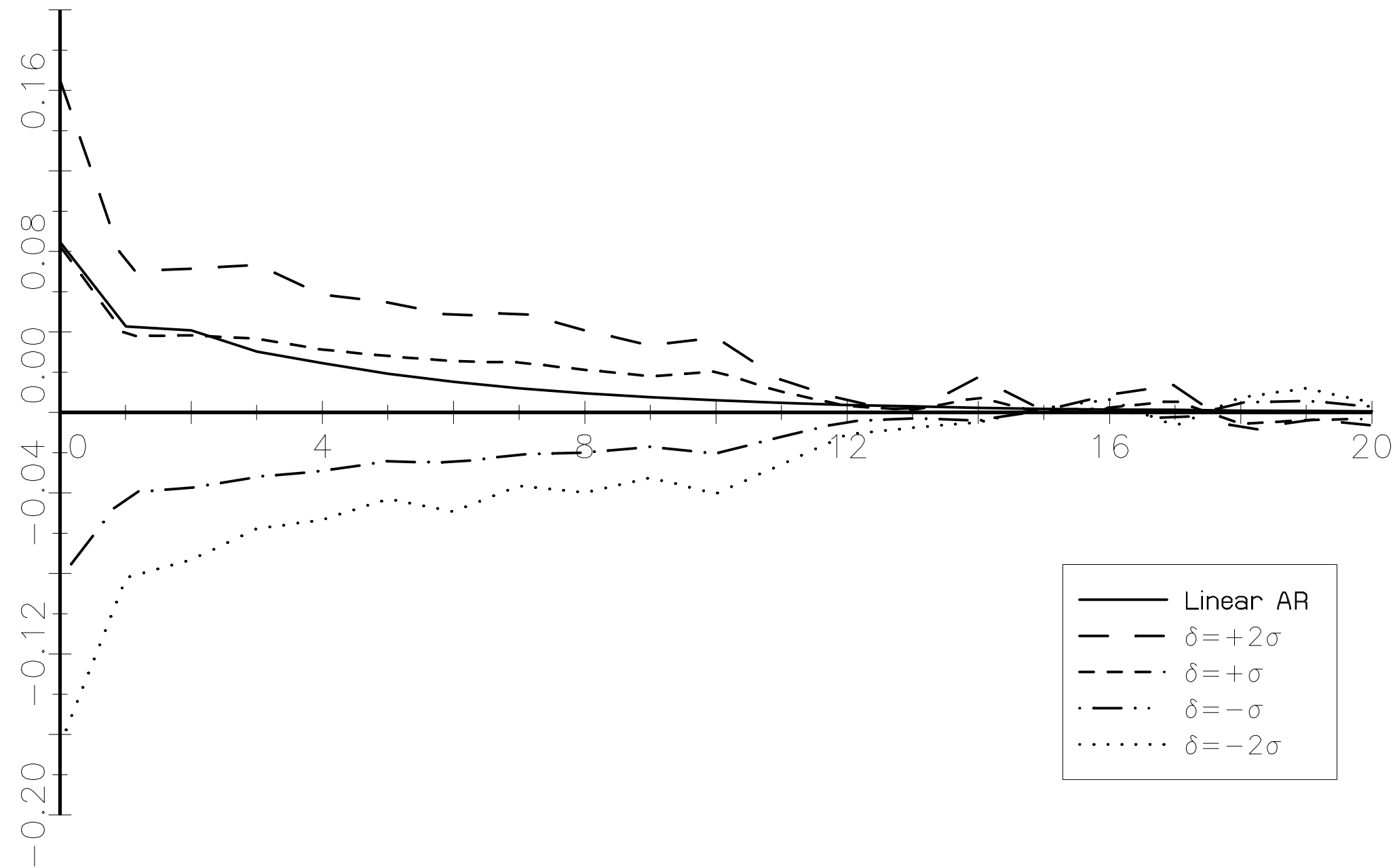
Figure 2B
Nonlinear Impulse Responses
(NLAR(2), k=1)
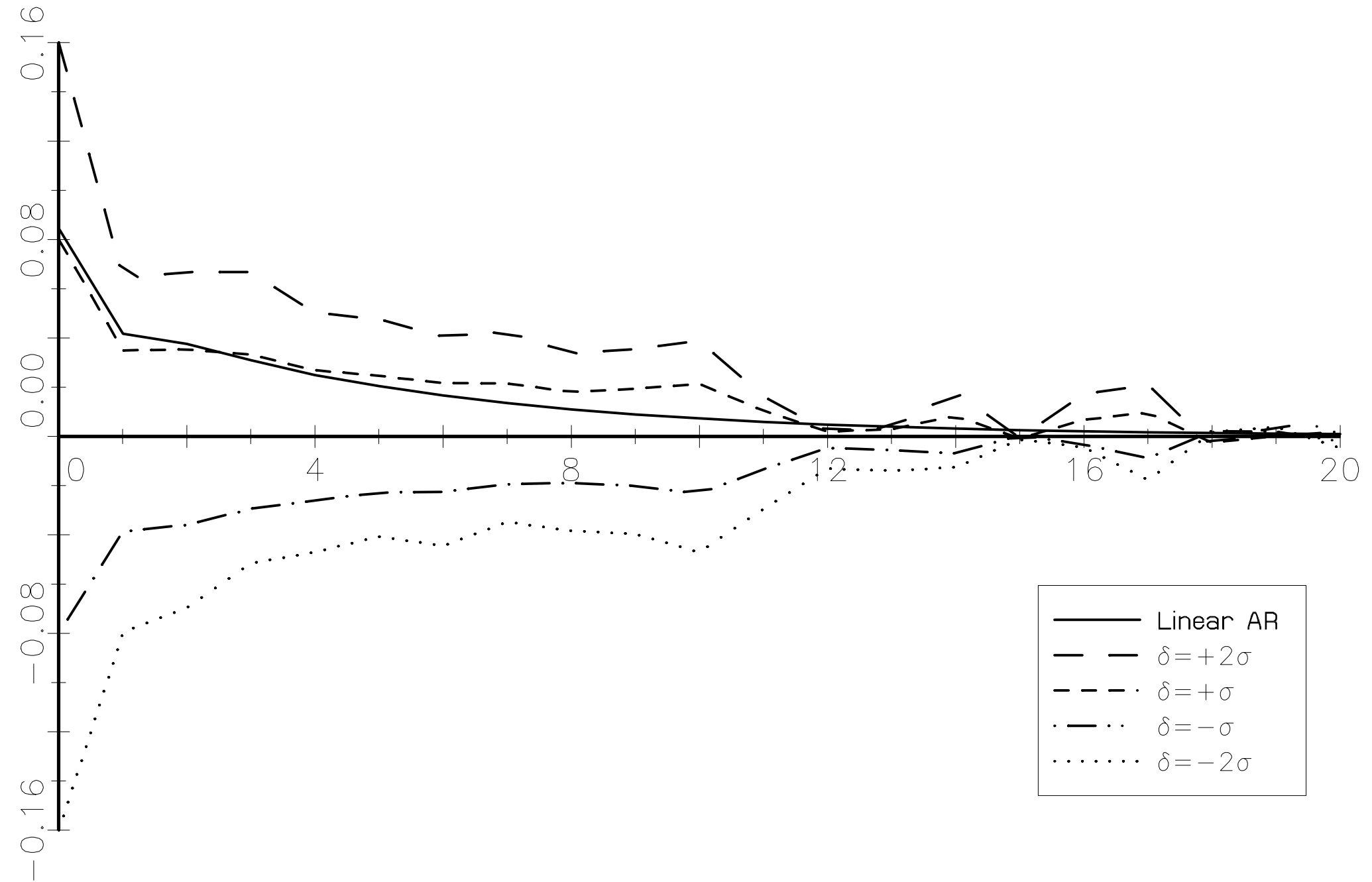
Figure 2C
Nonlinear Impulse Responses
(NLAR(3), k=1)
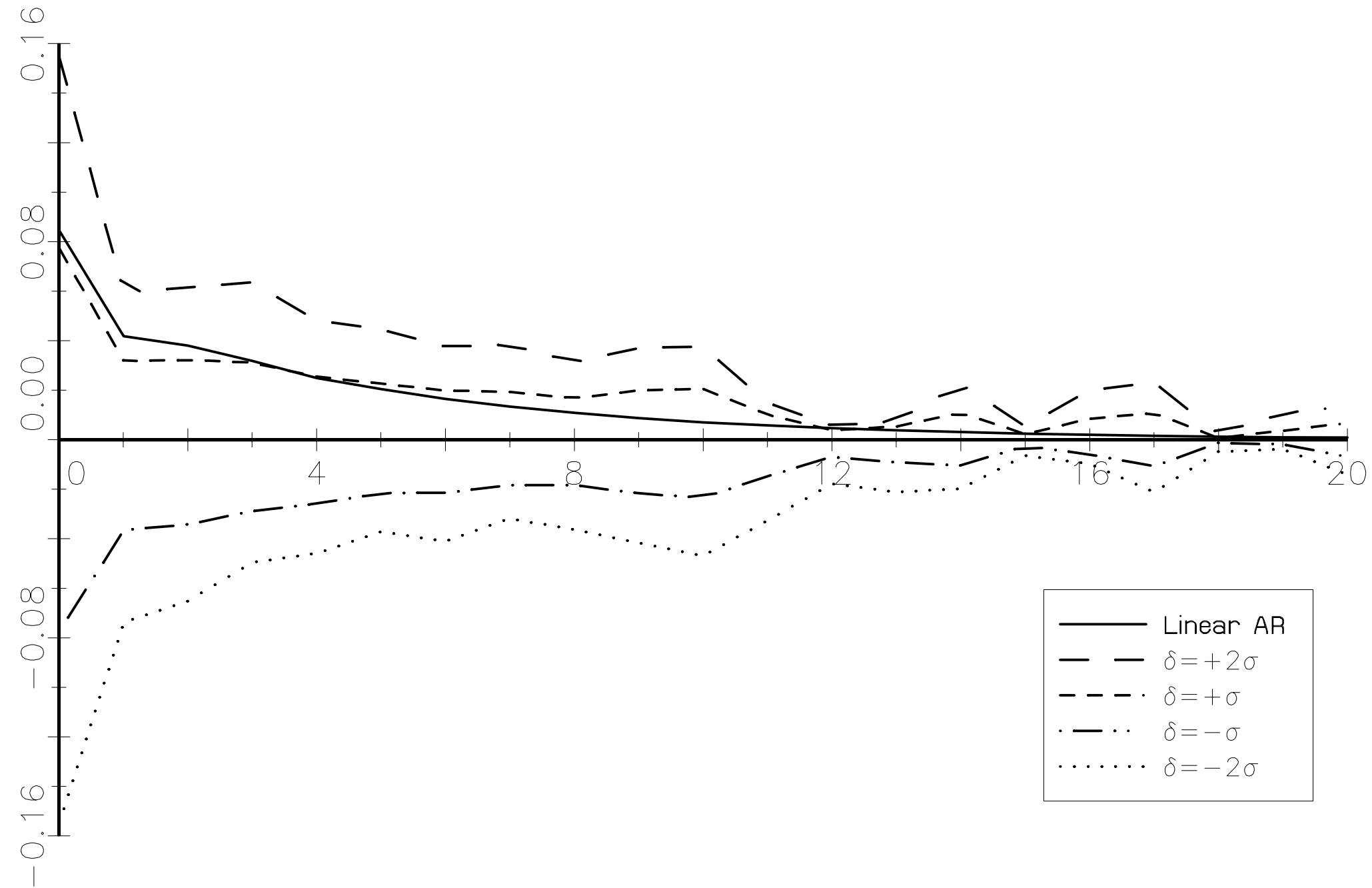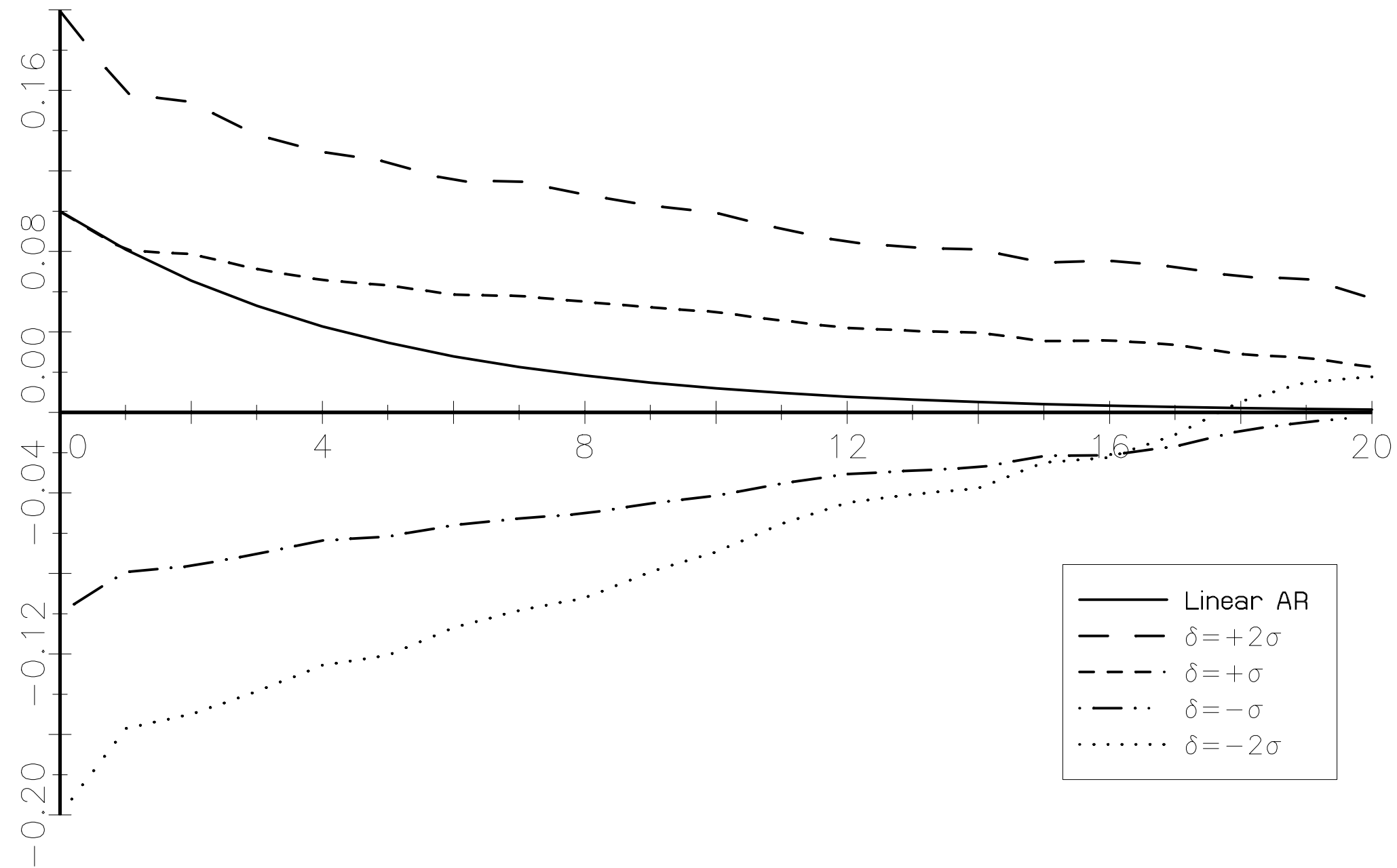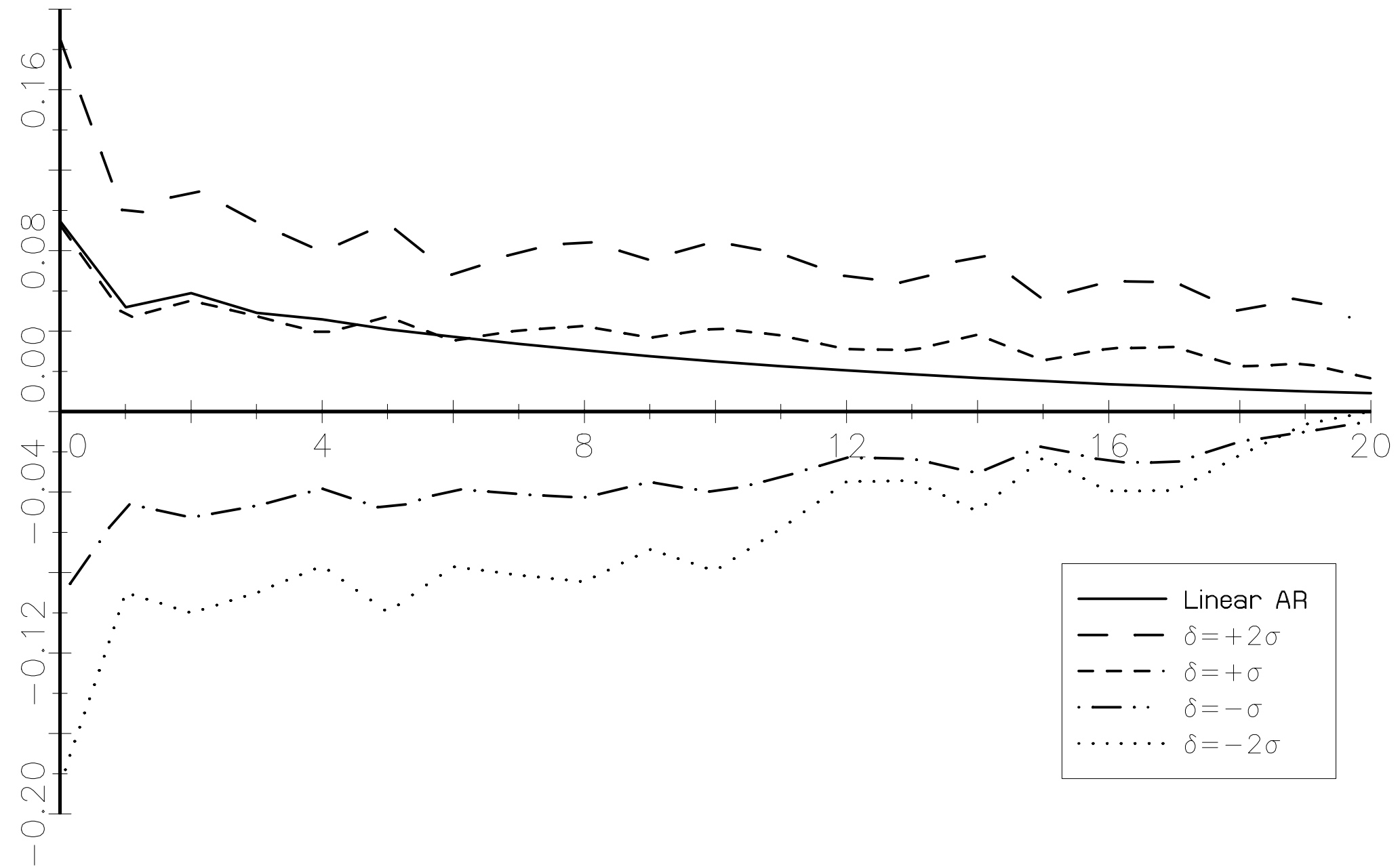
Figure 2D
Nonlinear Impulse Responses
(NLAR(4), k=1)

Linear AR
$\delta = +2\sigma$
$\delta = +\sigma$
$\delta = -\sigma$
$\delta = -2\sigma$

Figure 3A
Nonlinear Impulse Responses
(NLAR(1), k=2)

Legend:
- Linear AR
- $\delta = +2\sigma$
- $\delta = +\sigma$
- $\delta = -\sigma$
- $\delta = -2\sigma$

Figure 3B
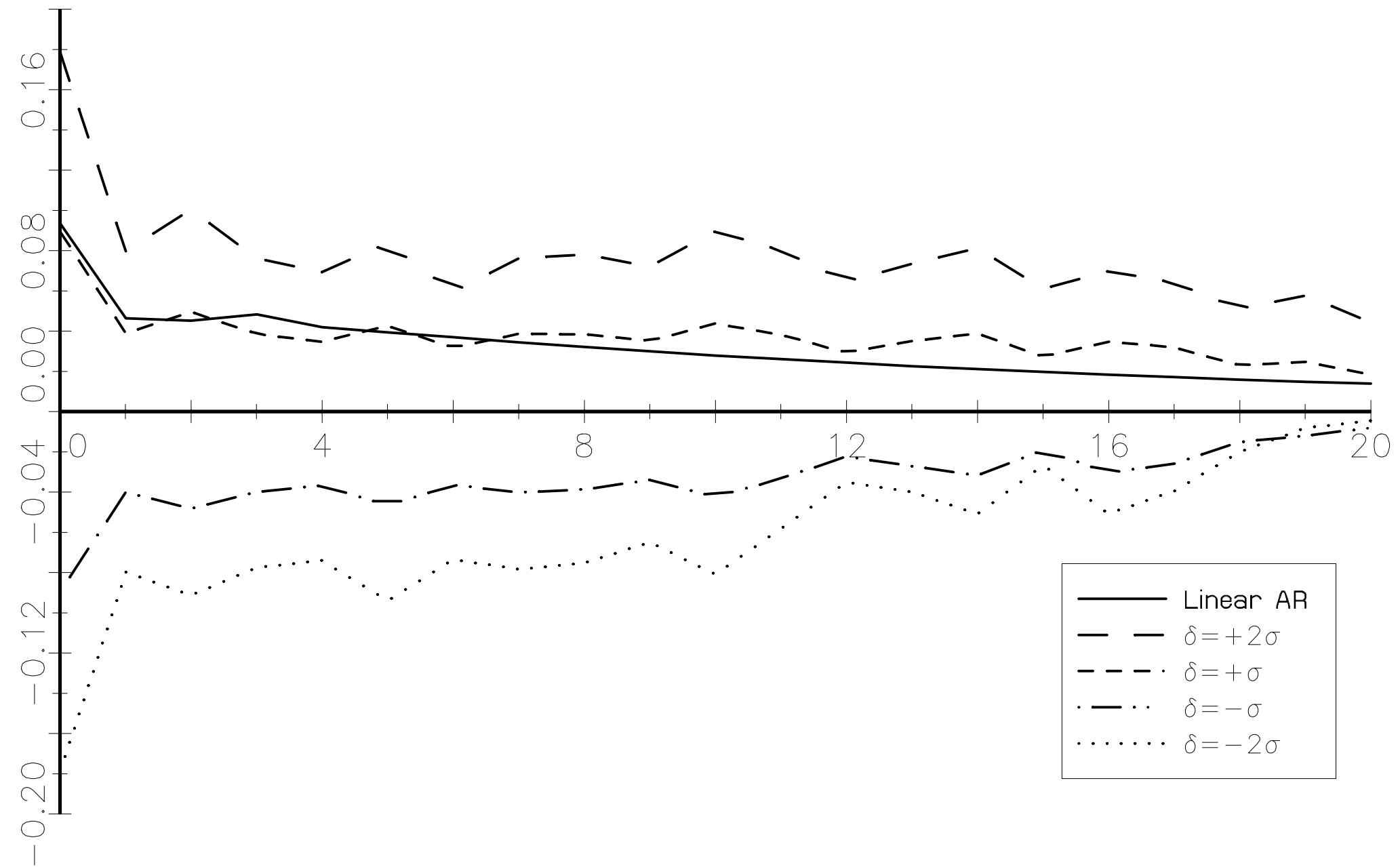Nonlinear Impulse Responses
(NLAR(2), k=2)

Figure 3C
Nonlinear Impulse Responses
(NLAR(3), k=2)

Figure 3D
Nonlinear Impulse Responses
(NLAR(4), k=2)