GRAPH PLANAR ALGEBRA EMBEDDINGS AND NEW A_∞-SUBFACTORS

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Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August 9, 2024

Nashville, Tennessee

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ACKNOWLEDGMENTS

First, I would like to thank my advisor, Dietmar Bisch, for suggesting the problem solved in this dissertation and for his academic and professional guidance. I am especially grateful for his support during times when progress for this dissertation was slow. His patience and mentoring have helped me grow into the mathematician I am today.

I am very thankful to my parents Eduardo Cáceres and Edith Gonzales. Their unwavering support has provided me with countless academic and professional opportunities. I am also very grateful to my future wife Kelley for her unconditional love and support. She brought so much happiness (and our lovely dog Artemis) to my life during my PhD. A special mention goes to my Peruvian friends. Their friendship has been the perfect connection back home.

Finally, I am grateful to the faculty, staff, fellow graduate students, and students I taught. I have learned so much from all of them during my life at Vanderbilt.

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CHAPTER 1

Introduction

In [Jon83], Jones introduced the notion of index $[M:N]$ for an inclusion of II_1 factors $N \subset M$. For such inclusions, he defined the basic construction $N \subset M \overset{e_N}{\subset} \langle M, e_N \rangle$, where e_N is the *Jones projection* for the inclusion $N \subset M$. Using the basic construction he proved the remarkable fact that $[M:N] \in \{4\cos^2(\frac{\pi}{n}), n \geq 3\} \cup [4, \infty]$. If $[M:N] < \infty$, iterating this construction results in a tower of II_1 factors

$$M_0 = N \subset M = M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} \cdots .$$

The relative commutants $\{M_i' \cap M_j\ , i = 0, 1,\ i \leq j\}$, together with their inclusions

$$M_1' \cap M_i \subset M_0' \cap M_i,\ M_i' \cap M_j \subset M_i' \cap M_{j+1}$$

and the Jones projections $\{e_n\}_{n\geq 1}$ define the *standard invariant* of the subfactor $N \subset M$.

The sequence of inclusions $M_0' \cap M_i \subset M_0' \cap M_{i+1}$ are described by a single graph called the *principal graph* of $N \subset M$. It is a bipartite graph that describes the standard representation theory of the subfactor. Whenever this graph is finite we say the subfactor has *finite depth*, otherwise, it has *infinite depth*. Subfactors whose principal graph is the Coxeter-Dynkin graph A_∞

play a special role, and we will refer to them as A_∞-*subfactors*.

One of the main challenges in subfactor theory is classification. Popa proved in [Pop94] that hyperfinite amenable subfactors (e.g. finite depth subfactors) are completely classified by their standard invariant. Consequently, the first step in classification is to determine all possible standard invariants for subfactors whose indices are contained in a certain interval. Then, one attempts to understand the hyperfinite subfactors that have these standard invariants. Jones in [Jon22] invented *planar algebras* to give an algebraic and topological description of the standard invariant. In particular, A_∞-subfactors with index δ^2 have the Temperley-Lieb-Jones planar algebra $\mathrm{TLJ}(\delta)$ as their standard invariant, which is the "smallest" planar algebra there is. In this case, we say that the subfactor has *trivial standard invariant*.

Index	# of subfactors	Name
$\frac{1}{2}(5+\sqrt{13})$	2	Haagerup
≈ 4.37720	2	Extended Haagerup
$\frac{1}{2}(5+\sqrt{17})$	2	Asaeda-Haagerup
$3+\sqrt{3}$	2	3311
$\frac{1}{2}(5+\sqrt{21})$	2	2221
5	7	-
≈ 5.04892	2	$\mathfrak{su}(2)_5$ and $\mathfrak{su}(3)_4$
$3+\sqrt{5}$	11	-

Table 1.1: Indices of hyperfinite finite depth subfactors between 4 and 5.25

A complete classification of hyperfinite subfactors with index less than 4 was developed by Jones [Jon87], Ocneanu [Ocn88] and many others. In this case the principal graphs are ADE Coxeter-Dynkin diagrams. At index 4 the classification was given by Popa [Pop94]; the principal graphs are all *affine* Coxeter-Dynkin diagrams, including the infinite graphs A_∞ and D_∞. All hyperfinite subfactors with index ≤ 4 are amenable. There are countably many distinct ones with Jones index in $\{4\cos^2(\frac{\pi}{n}), n \geq 3\} \cup \{4\}$. The classification was then extended to subfactors with index in $(4,5]$. This was accomplished over a long period of time and involved many different people, a survey can be found in [JMS14]. Finally, the classification was extended to subfactors with index in $(5,5.25]$ in [AMP23]. Thus all finite depth subfactors of the hyperfinite II_1 factor with index in $(4,5.25]$ are known, see Table 1.1. However, *infinite depth* subfactors are far from being understood. In fact, any irreducible infinite depth subfactor with index in $(4,3+\sqrt{5})$ has trivial standard invariant, and hence is an A_∞-subfactor. A subfactor is called *irreducible* if $\dim N' \cap M = 1$.

We construct in this thesis A_∞-subfactors for all the indices ≤ 5 in Table 1.1. To accomplish this, we construct new commuting squares, a notion that we recall in Chapter 2. A *commuting square* is an inclusion of four finite dimensional von Neumann algebras

$$\begin{array}{ccc} A_{1,0} & \overset{\text{L}}{\subset} & A_{1,1} \\ \cup_K & & \cup_H \\ A_{0,0} & \overset{\text{G}}{\subset} & A_{0,1} \end{array}$$

as above, with a faithful trace on $A_{1,1}$ so that $A_{1,0}$ and $A_{0,1}$ are orthogonal modulo their intersection $A_{0,0}$ with respect to the inner product defined by the trace. We iterate the basic construction for the inclusions $A_{i,j} \subset A_{i,j+1}$ and $A_{i,j} \subset A_{i+1,j}$. Then, if every inclusion is connected, the commuting square is non-degenerate and the trace has the Markov property, $A_{0,\infty} \subset A_{1,\infty}$ will be an irreducible hyperfinite subfactor with index $\|K\|^2$, where $A_{n,\infty} = \left(\cup_k A_{n,k}\right)''$ $(n=0,1)$ and K is the inclusion graph of $A_{0,0} \subset A_{1,0}$. We call subfactors constructed in this manner *commuting square subfactors*. Most of the results regarding commuting squares in Chapter 2 can be found in [JS97] and [GdlHJ89]. In this chapter, we also give a brief overview of fusion rings and modules. These will be needed to rule out the possibility of our subfactors having finite depth.

In [JP11], the authors prove that the planar algebra of a finite depth subfactor embeds into the bipartite graph planar algebra of its principal graph. They describe inclusions of finite dimensional C*-algebras in terms of loop algebras, particularly those coming from the basic construction. Jones and Penneys present formulas for the Jones projections, conditional expectations and the Pimsner-Popa basis for these inclusions. We give a brief overview of this work in Chapter 3 and a short introduction to planar algebras in Chapter 4.

In Chapter 5 we prove the following graph planar algebra embedding theorem, which generalizes the main theorem in [JP11]:

Theorem 1.1. *Let $P_\bullet$ be the subfactor planar algebra associated to the commuting square subfactor $A_{0,\infty} \subset A_{1,\infty}$ and let $G_\bullet$ the graph planar algebra associated to the inclusion graph of $A_{0,0} \subset A_{1,0}$. There is an embedding of planar algebras $\psi : P_\bullet \to G_\bullet$.*

This theorem applies to any commuting square subfactor, regardless of whether it has finite or infinite depth.

We aim to combine our embedding theorem with Theorem 1.2 in [GMP$^+$23] to deduce that the commuting square subfactors we construct have infinite depth. Theorem 1.2 of [GMP$^+$23] states that finite depth subfactor planar algebras only embed into the graph planar algebra of their *module graphs*. All module graphs for the Haagerup and Extended Haagerup subfactors are known and can be found in [GMP$^+$23]. In Chapter 6 we outline how to compute the module graphs associated to the Asaeda-Haagerup subfactor based on the combinatorial data of the Asaeda-Haagerup fusion category. A commuting square based on a graph with norm $\sqrt{\frac{5+\sqrt{13}}{2}}$, that is not a module graph for the Haagerup subfactor, was constructed in [Sch90]. Theorem 1.1 and the main theorem in [GMP$^+$23] then imply that this subfactor has infinite depth. By classification, it must have trivial standard invariant.

In Chapter 7 we construct several new commuting squares, two of which are based on graphs that are not module graphs with norms $\sqrt{4.37720\ldots}$ and $\sqrt{\frac{5+\sqrt{17}}{2}}$. Theorem 1.1 implies that the subfactor planar algebra of these subfactors embeds into the graph planar algebra of these graphs. Therefore, the subfactors must have infinite depth and, by classification, they must be A_∞-subfactors. In Chapter 8, we present an alternative method to construct infinite depth subfactors by proving the existence of an explicit 1-parameter family of non-equivalent commuting squares at indices $\frac{5+\sqrt{17}}{2}$, $3+\sqrt{3}$, $\frac{5+\sqrt{21}}{2}$, 5 and $3+\sqrt{5}$. Kawahigashi showed in [Kaw23] that there can only be countably many commuting squares $\{A_{i,j}\}_{i,j=0}^1$ which realize a fixed finite depth subfactor $N \subset M$ as $A_{0,\infty} \subset A_{1,\infty}$. The classification of small index subfactors shows that there are only finitely many finite depth subfactors with indices in $(4, 5.25]$. Consequently, our 1-parameter families will produce at least one infinite depth irreducible hyperfinite subfactor at each of these indices. Using the classification of small index subfactors again, we can deduce that all of them, but the one with index $3+\sqrt{5}$, must have trivial standard invariant. Thus, we have constructed several new A_∞ subfactors. The results in Chapters 5 to 8 were obtained in joint work with Dietmar Bisch.

CHAPTER 2

Preliminaries

2.1 Subfactors and commuting squares

In this section we present the construction of hyperfinite subfactors with finite Jones index via so-called *commuting squares*. Commuting squares allow us to compute the index of a subfactor $N \subset M$ when it is approximated via finite dimensional C*-algebras in the following way

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ A_{n+1} & \subset & B_{n+1} \\ \cup & & \cup \\ A_n & \subset & B_n \end{array}$$

where $N = \left(\bigcup_{n\geq 0} A_n\right)''$ and $M = \left(\bigcup_{n\geq 0} B_n\right)''$, see [PP86]. We refer to [JS97] for proofs of most the results in this section. For the sake of brevity we assume the reader is familiar with the concept of *Jones index* for a subfactor, see [Jon83].

Definition 2.1. Let $A_0 \subset B_0$, $A_1 \subset B_1$ be finite von Neumann algebras such that

$$\begin{array}{ccc} A_1 & \subset & B_1 \\ \cup & & \cup \\ A_0 & \subset & B_0 \end{array} \tag{2.1}$$

and tr is a faithful normal trace in B_1. Then (2.1) is called a *commuting square* if $E_{B_0}E_{A_1} = E_{A_0}$ where $E_{B_0}, E_{A_1}, E_{A_0}$ are the unique trace-preserving conditional expectations from B_1 to B_0, A_1 and A_0 respectively.

Definition 2.2. Given an inclusion $B_0 \subset B_1$ of finite von Neumann algebras, we say that a tracial state tr_1 has the *Markov property* if it extends to a tracial state tr_2 on $\langle B_1, e_{B_0}\rangle$, the basic construction of $B_0 \subset B_1$, such that $E_{B_0}(e_{B_0}) = \lambda 1$ for some scalar λ. In this case, we say that tr_1 is a λ*-Markov trace* or simply a *Markov trace*.

Both, finite index subfactors $N \subset M$ and connected inclusions $A \subset B$ of finite dimensional C*-algebras, have a *unique* Markov trace. In the latter case, if $\Lambda = \Lambda_A^B$ is the inclusion matrix for $A \subset B$ (also referred to as the *Bratteli diagram* of the inclusion), then the Markov trace tr is given by a Perron-Frobenius eigenvector of

$\Lambda^t \Lambda$ (appropriately normalized such that $\mathrm{tr}(1) = 1$).

Proposition 2.3. *Consider a commuting square as in (2.1) of finite dimensional C*-algebras and let* tr *be a Markov trace for* $B_0 \subset B_1$. *Let* $B_2 = \langle B_1, e_{B_0} \rangle$ *be the basic construction and let* $A_2 = \{A_1, e_{B_0}\}'' \subset B(L^2(B_2, \mathrm{tr}_2))$*, where* tr_2 *is the extension of the Markov trace to* B_2*. Then*

$$\begin{array}{ccc} A_2 & \subset & B_2 \\ \cup & & \cup \\ A_1 & \subset & B_1 \end{array}$$

is also a commuting square.

Definition 2.4. Consider a commuting square

$$\begin{array}{ccc} A_{1,0} & \overset{L}{\subset} & A_{1,1} \\ \cup_K & & \cup_H \\ A_{0,0} & \overset{G}{\subset} & A_{0,1} \end{array} \tag{2.2}$$

of finite dimensional C*-algebras with a faithful normal trace on $A_{1,1}$ and inclusion graphs G, H, K and L. We say that (2.2) is a *nondegenerate* or *symmetric* commuting square if $GH = KL$ and $HL^t = G^t K$. This is equivalent to the condition that $\mathrm{span} A_{0,1} A_{1,0} = A_{1,1}$.

It can be shown that whenever we have a nondegenerate commuting square, the inclusion $A_0 \subset A_1 \subset A_2$ from the previous proposition is isomorphic to the basic construction $A_0 \subset A_1 \subset \langle A_1, e_{A_0} \rangle$.

Definition 2.5. For a graph G with adjacency matrix X, $\|G\|$ denotes the root of the largest eigenvalue of $X^t X$. We refer to $\|G\|$ as the *norm of the graph* G.

Proposition 2.6. *Suppose (2.2) is a nondegenerate commuting square with connected graphs* G*,* H*,* K *and* L*. Then* $\|H\| = \|K\|$ *and* $\|G\| = \|L\|$*. Moreover,* tr *is the Markov trace for* $A_{1,0} \subset A_{1,1}$*,* $\mathrm{tr}|_{A_{0,1}}$ *is the Markov trace for* $A_{0,0} \subset A_{0,1}$*,* tr *is the Markov trace for* $A_{0,1} \subset A_{1,1}$ *and* $\mathrm{tr}|_{A_{1,0}}$ *is the Markov trace for* $A_{0,0} \subset A_{1,0}$*.*

Proof. See Proposition 1.11 in [Sch90] or Corollary 5.3.4 part (c) in [JS97]. □

Thus, whenever we have a nondegenerate commuting square of finite dimensional C*-algebras we can iterate the basic construction in the horizontal and vertical directions. From this we obtain a lattice of the form

$$\begin{array}{ccccccccc}
A_{\infty,0} & \subset & A_{\infty,1} & \subset & A_{\infty,2} & \subset & \cdots & \subset & A_{\infty,\infty} \\
\cup & & \cup & & \cup & & & & \cup \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\cup & & \cup & & \cup & & & & \cup \\
A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset & \cdots & \subset & A_{2,\infty} \\
\cup & & \cup & & \cup & & & & \\
A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & A_{1,\infty} \\
\cup & & \cup & & \cup & & & & \cup \\
A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & A_{0,\infty}
\end{array} \tag{2.3}$$

where $A_{n,\infty} = \left(\bigcup_k A_{n,k}\right)'' \subset B(\mathscr{H}_{n,\infty})$ and $A_{\infty,k} = \left(\bigcup_n A_{n,k}\right)'' \subset B(\mathscr{H}_{\infty,k})$ are hyperfinite von Neumann algebras. Here, $\mathscr{H}_{n,\infty}$ (respectively $\mathscr{H}_{\infty,k}$) denotes the Hilbert space obtained by completing $\bigcup_k A_{n,k}$ (respectively $\bigcup_n A_{n,k}$) with respect to the inner product given by the Markov trace (this is also known as the *GNS construction*). If we assume all inclusion matrices to be connected, then $A_{n,\infty}$ and $A_{\infty,k}$, are factors. The subfactors $A_{0,\infty} \subset A_{1,\infty}$ or $A_{\infty,0} \subset A_{\infty,1}$ constructed in this manner will be called *commuting square subfactors*.

We wish to use the finite dimensional approximation to compute invariants of these subfactors.

Definition 2.7. Let $A \subset (B, \mathrm{tr})$ be a unital inclusion of finite von Neumann algebras with faithful normal trace tr on B. A *Pimsner-Popa* basis for B over A is a finite set $S = \{s\} \subset B$ which satisfies any of the following equivalent conditions:

1. $1 = \sum_{s\in S} s e_A s^*$,
2. $x = \sum_{s\in S} s E_A(s^* x)$ for all $x \in B$, and
3. $x = \sum_{s\in S} E_A(xs)s^*$ for all $x \in B$,

where $E_A : B \to A$ is the conditional expectation determined by tr and e_A is the Jones projection associated to this inclusion (see Proposition 4.3.3 in [JS97]).

Lemma 2.8. *Suppose we have a nondegenerate commuting square as in (2.2) with respect to the normalized Markov trace. Then there exists a finite set I, and $\{s_i :\ i \in I\} \subset A_{1,0}$, $\{f_i :\ i \in I\} \subset A_{0,0}$ such that*

(a) each f_i is a projection,

(b) $E_{A_{0,0}}(s_i^* s_j) = \delta_{ij} f_i$,

(c) $\sum_{i\in I} \operatorname{tr}(f_i) = \|K\|^2$*, and*

(d) $x = \sum_{i\in I} E_{A_{0,1}}(xs_i)s_i^*$*, for all* $x \in A_{1,1}$.

Proof. See Lemma 5.7.3 in [JS97]. □

Remark 2.9. Note that $\{s_i : i \in I\} \subset A_{1,0}$ is a Pimsner-Popa basis for $A_{1,1}$ over $A_{0,1}$ *and* for $A_{1,0}$ over $A_{0,0}$.

Corollary 2.10. *Suppose* $\{s_i : i \in I\}$ *is as in the previous lemma, then*

(a) $\{s_i : i \in I\}$ *is a Pimsner-Popa basis for* $A_{0,\infty} \subset A_{1,\infty}$.

(b) $[A_{1,\infty} : A_{0,\infty}] = \|K\|^2$.

Proof. It can be shown that for a symmetric commuting square $A_{0,2}A_{1,1}$ linearly spans $A_{1,2}$. From part (d) of the previous lemma, we have that $\bigcup_{i\in I} A_{0,1}s_i^*$ spans $A_{1,1}$, therefore $A_{1,2}$ is spanned by $\bigcup_{i\in I} A_{0,2}s_i^*$. Since $E_{A_{0,1}}(s_i^* s_j) = \delta_{ij} f_i$, we have that $\{A_{0,2}s_i^* : i \in I\}$ (with respect to the inner product given by the trace) is a pairwise orthogonal collection of subspaces of $A_{1,2}$. Hence, if $x \in A_{1,2}$, then $x = \sum_{i\in I} a_i s_i^*$ for some $a_i \in A_{0,2}$. Now, using part (b) from the previous lemma, and the fact that we have a commuting square, we obtain

$$E_{A_{0,1}}(xs_i) = a_i f_i,\ i \in I.$$

Note that from part (d) of the previous lemma we also have $s_i^* = f_i s_i^*$ and therefore

$$\sum_{i\in I} E_{A_{0,1}}(xs_i)s_i^* = \sum_{i\in I} a_i f_i s_i^* = x.$$

Repeating this process, we get for any n and $x \in A_{1,n}$ the identity

$$x = \sum_{i\in I} E_{A_{0,n}}(xs_i)s_i^*.$$

Moreover, since

$$\begin{array}{ccc} A_{1,n} & \subset & A_{1,\infty} \\ \cup & & \cup \\ A_{0,n} & \subset & A_{0,\infty} \end{array}$$

is clearly a commuting square we also have

$$x = \sum_{i\in I} E_{A_{0,\infty}}(xs_i)s_i^*, \quad \text{for all } x \in \bigcup_{n\geq 0} A_{1,n}.$$

Since the formula above is continuous in the weak operator topology with respect to x, we conclude

$$x = \sum_{i \in I} E_{A_{0,\infty}}(x s_i) s_i^*, \quad \text{for all } x \in A_{1,\infty}.$$

Therefore $\{s_i : i \in I\}$ is a Pimsner-Popa basis for $A_{0,\infty} \subset A_{1,\infty}$. Finally, using the Markov property of the trace on $A_{1,\infty}$, we have that

$$\begin{aligned} 1 = \mathrm{tr}(1) = \mathrm{tr}\left(\sum_{i \in I} s_i e_{A_{0,\infty}} s_i^*\right) \\ 1 = [A_{1,\infty} : A_{0,\infty}]^{-1} \sum_{i \in I} \mathrm{tr}(s_i s_i^*) \\ 1 = [A_{1,\infty} : A_{0,\infty}]^{-1} \sum_{i \in I} \mathrm{tr}(E_{A_{0,0}}(s_i s_i^*)) \\ 1 = [A_{1,\infty} : A_{0,\infty}]^{-1} \sum_{i \in I} \mathrm{tr}(f_i) \\ 1 = [A_{1,\infty} : A_{0,\infty}]^{-1} \|K\|^2. \end{aligned}$$

□

Remark 2.11. We can show $[A_{\infty,1} : A_{\infty,0}] = \|G\|^2$ with a similar argument.

Connected unital inclusions of C*-algebras and finite index inclusions of II_1 factors are both examples of *strongly Markov* inclusions (see [JP11]). In this case we have the following formula for the conditional expectation on the relative commutants.

Proposition 2.12. *If $A \subset B$ is strongly Markov and $[B : A] < \infty$, the conditional expectation $E_{B'} : A' \cap B(L^2(B_n, \mathrm{tr})) \to B' \cap B(L^2(B_n, \mathrm{tr}))$ is given by*

$$E_{B'}(x) = \frac{1}{[B : A]} \sum_{s \in S} s x s^*$$

where S is a Pimsner-Popa basis for B over A. In particular, the map is independent of the choice of a basis.

Proof. See Proposition 2.23 in [JP11] or Proposition 2.7 in [Bis97]. □

The next result shows that, in principle, one can compute the standard invariant of a commuting square subfactor from the finite dimensional commuting squares from which it is constructed.

Theorem 2.13 (Ocneanu compacntess)**.** *Given a lattice of inclusions arising from a nondegenerate commut-*

ing square as in (2.3), we have

$$A'_{0,\infty} \cap A_{n,\infty} = A'_{0,1} \cap A_{n,0}, \quad A'_{\infty,0} \cap A_{\infty,n} = A'_{1,0} \cap A_{0,n}, \textit{ for all } n \geq 0.$$

Proof. See Theorem 5.7.6 in [JS97]. □

The following result by Schou [Sch90], which is a generalization of [Wen88, Theorem 1.6], will allow us to conclude that the subfactors we construct are irreducible.

Theorem 2.14 (Wenzl's criterion)**.** $\dim(A'_{0,\infty} \cap A_{1,\infty}) \leq (\min\{\textit{1-norm of rows of } K \textit{ and } H\})^2$.

2.2 Fusion rings and modules

In this section we give a brief overview of fusion rings and fusion modules using the definitions found in [GS16]. We will assume the reader is familiar with the concept of tensor categories, see e.g. [EGNO15] for a thorough treatment of this subject.

A *multi-fusion* category $\mathscr{C}$ is a rigid semisimple $\mathbb{C}$-linear tensor category with finitely many isomorphism classes of simple objects and finite dimensional spaces of morphisms. If the unit object 1 of $\mathscr{C}$ is simple, then the category $\mathscr{C}$ is said to be a *fusion category*. Let $\mathrm{Gr}(\mathscr{C})$ be the Grothendieck ring of $\mathscr{C}$, that is the ring whose elements are isomorphism classes of objects in $\mathscr{C}$ and operations defined by $[X]+[Y] := [X \oplus Y]$ and $[X][Y] := [X \otimes Y]$, where X, Y are simple objects in $\mathscr{C}$ and $[X]$ denotes the equivalence class of X.

If $\mathrm{Irr}(\mathscr{C})$ is the set of isomorphism classes of simple objects in $\mathscr{C}$, since every object is isomorphic to a direct sum of simple objects, $\mathrm{Irr}(\mathscr{C})$ will be a $\mathbb{Z}_+$-basis for $\mathrm{Gr}(\mathscr{C})$. This implies that for every $\xi, \eta \in \mathrm{Irr}(\mathscr{C})$ there exist constants $N_{\xi\eta}^{\mu} \in \mathbb{Z}_+$ such that

$$\xi\eta = \sum_{\mu \in \mathrm{Irr}(\mathscr{C})} N_{\xi\eta}^{\mu} \mu.$$

We call $N_{\xi\eta}^{\mu}$ the *fusion rules* for $\mathrm{Gr}(\mathscr{C})$. If $\mathscr{C}$ is fusion, then $\mathrm{Gr}(\mathscr{C})$ will be a unital $\mathbb{Z}_+$ ring and we say $\mathrm{Gr}(\mathscr{C})$ is the *fusion ring* associated to $\mathscr{C}$. The rigidity structure on $\mathscr{C}$ induces an involution, that is we have a map $\xi \in \mathrm{Gr}(\mathscr{C}) \mapsto \bar{\xi} \in \mathrm{Gr}(\mathscr{C})$ for which $N_{\xi\bar{\xi}}^{1} = N_{\bar{\xi}\xi}^{1} = 1$ for all $\xi \in \mathrm{Irr}(\mathscr{C})$.

Definition 2.15. A *fusion ring* (F,S) is a unital ring F whose additive group is the free Abelian group on a finite set S containing 1, which is endowed with an involution denoted by $\xi \mapsto \bar{\xi}$, $\xi \in S$ which extends to an anti-involution in F. In addition $N_{\xi\bar{\xi}}^{1} = N_{\bar{\xi}\xi}^{1} = 1$, where $N_{\xi\eta}^{\mu}$ are the fusion rules for F with respect to S. We will refer to S as the *basis* for the fusion ring.

Remark 2.16. Every fusion category has a fusion ring associated to it. The converse is not true, not every fusion ring is the Grothendieck ring of a fusion category. Fusion rings that are Grothendieck rings are called *categorifiable*. There are also examples of non-isomorphic fusion categories with the same fusion ring. In general there can only be finitely many fusion categories which categorify a fixed fusion ring F. This is known as *Ocneanu rigidity*, see Section 2.7 in [ENO05].

Let (F,S) be a fusion ring with rank r, that is $r=|S|$, and a fixed element $\xi \in S$. The $r \times r$ matrix $N_\xi = (N_{\xi\eta}^{\mu})_{(\eta,\mu)\in S\times S}$ is called the *fusion matrix* associated to ξ. It describes the left action of ξ on F. The *Frobenius Perron dimension* of ξ, denoted by $d(\xi)$, is the largest eigenvalue of N_ξ. It can be shown that d extends to a ring homomorphism $d: F \to \mathbb{R}$ which is positive on the basis.

Example 2.17. Let $N \subset M$ be a finite index subfactor and consider the bimodules $\kappa = {}_N L^2(M)_M$, $\bar{\kappa} = {}_M L^2(M)_N$. Then $\kappa \otimes_M \bar{\kappa}$ and $\bar{\kappa} \otimes_N \kappa$ generate C*-tensor categories of $M-M$ and $N-N$ bimodules called the *principal and dual even parts* of the subfactor, which we will denote by $\mathscr{C}$ and $\mathscr{D}$ respectively. The subfactor is said to have *finite depth* if each of these categories has finitely many simple objects. In this case, they are both *unitary fusion categories*. The category of $N-M$ bimodules generated by κ will be a $\mathscr{C}$-$\mathscr{D}$ bimodule category (see Chapter 7 in [EGNO15]), which is a *Morita equivalence* between $\mathscr{C}$ and $\mathscr{D}$. We can assemble all of this into a unitary pivotal 2-category $\mathscr{C}_{N\subset M}$ where the objects are N and M, the 1-morphisms are the bimodules generated by κ and $\bar{\kappa}$ and the 2-morphisms are bimodule interwinners. The category $\mathscr{C}_{N\subset M}$ is known as the *standard invariant* of the subfactor and coincides with the classical notion of the standard invariant in terms of relative commutants as defined above (see [Bis97]).

Definition 2.18. A *(left) fusion module* (K,T) over a fusion ring (F,S) is a finite set T along with an indecomposable (left) representation of the fusion ring as endomorphisms of the free Abelian group K on T, such that the action defined by the representation satisfies

$$\xi \cdot \eta = \sum_{\mu \in T} N_{\xi,\eta}^{\mu} \mu$$

for all $\xi \in S, \eta, \mu \in T$, where $N_{\xi,\eta}^{\mu}$ are non-negative integers that satisfy $N_{\xi,\eta}^{\mu} = N_{\bar{\xi},\eta}^{\mu}$ for all $\xi \in S$, $\eta, \mu \in T$. Similarly, we can define a *right fusion module* and a *fusion bimodule*.

Let $\mathscr{K}$ be a $\mathscr{C}-\mathscr{D}$ bimodule category and $K = \mathrm{Gr}(\mathscr{K})$ its Grothendieck group. The $\mathscr{C}-\mathscr{D}$ bimodule structure of $\mathscr{K}$ induces a $C-D$ fusion bimodule structure on K, where C and D are the fusion rings for $\mathscr{C}$ and $\mathscr{D}$ respectively. We call K the *fusion bimodule* associated to $\mathscr{K}$. If $\mathscr{K}$ is instead a $\mathscr{C}$ module category we can define its *fusion module* similarly.

Example 2.19. Let $\mathscr{C}$ and $\mathscr{D}$ be as in Example 2.17 and $\mathscr{K}$ the category of $N-M$ bimodules. The *principal* *graph* of the subfactor is the bipartite graph with even vertices indexed by $\mathrm{Irr}(\mathscr{C})$, odd vertices indexed by $\mathrm{Irr}(\mathscr{K})$, and $N_{\xi,\kappa}^{\lambda}$ edges between $\xi \in \mathrm{Irr}(\mathscr{C})$ and $\lambda \in \mathrm{Irr}(\mathscr{K})$. The *dual principal graph* is defined similarly, using the right action from $\mathscr{D}$ on $\mathscr{K}$.

Remark 2.20. There is also a notion of Frobenius Perron dimension for a fusion bimodule (K,T). Let (K,T) be an $F-G$ fusion bimodule, where (F,S) and $(G,R))$ are fusion rings. We will use d to denote the Perron Frobenius dimensions in F, G and d_T to denote the Perron Frobenius dimension in K. Then, for any $\xi \in F$, $\mu \in G$ and $\eta \in K$ we have

$$d_T(\xi \cdot \eta \cdot \mu) = d(\xi)d_T(\eta)d(\mu).$$

Definition 2.21. Given a left fusion module (K,T) over (F,S), we denote by (μ,η) the dot product of two elements of K with respect to the basis T, and similarly for two elements of F (with respect to the basis S). We can also define *right multiplication by duals* for a left fusion module as follows: $\mu\bar{\lambda} := \sum_{\xi \in S}(\mu, \xi\lambda)\xi$ for $\mu, \lambda \in K$. We may define left multiplication by duals for right fusion modules in a similar way.

Note that $\bar{\lambda}$ is not an element in K; the expression is just a formal argument for multiplication. It can be shown that if d_T is Frobenius Perron dimension for M, we have $d(\mu\bar{\lambda}) = d_T(\mu)d_T(\lambda)$ for all $\mu, \lambda \in T$. See Lemma 5.6 in [GS16].

CHAPTER 3

Finite dimensional inclusions and loops

In this chapter we will use loop algebras to describe the iterated basic construction of a finite dimensional inclusion of C*-algebras. Moreover, we will be able to describe the commutants as loop algebras on the inclusion graphs. Most of the results in this section can be found in Section 3 of [JP11].

3.1 Loop algebras

We recall the notation used in [JP11].

Notation 3.1. Let Γ be a finite, connected, bipartite multigraph (i.e. a graph with possibly multiple edges between two vertices).

- $\mathscr{V}_+$ denotes the set of even vertices.
- $\mathscr{V}_-$ denotes the set of odd vertices.
- $\mathscr{E}$ denotes the set of edges, and its elements are denoted by ε, ξ and ζ.
- All edges are directed from even to odd vertices indicated by the source and target functions $s : \mathscr{E} \to \mathscr{V}_+$ and $t : \mathscr{E} \to \mathscr{V}_-$.
- The set of all edges traversed from odd to even vertices are $\mathscr{E}^* = \{\varepsilon^*,\ \varepsilon \in \mathscr{E}\}$ with source and target functions $s : \mathscr{E}^* \to \mathscr{V}_-$ and $t : \mathscr{E}^* \to \mathscr{V}_+$ such that $s(\varepsilon^*) = t(\varepsilon)$ and $t(\varepsilon^*) = s(\varepsilon)$.
- The dimension vector for the even vertices is a function $m_+ : \mathscr{V}_+ \to \mathbb{N}$.
- The dimension vector for odd vertices $m_- : \mathscr{V}_- \to \mathbb{N}$ is defined by

$$m_-(v) = \sum_{t(\varepsilon)=v} m_+(s(\varepsilon)).$$

Definition 3.2. Let $G_{0,\pm}$ be the complex vector space with basis $\mathscr{V}_\pm$ respectively. For $n \in \mathbb{N}$, $G_{n,\pm}$ will denote the complex vector space with basis loops of length $2n$ on Γ based at a vertex in $\mathscr{V}_\pm$ respectively.

We now describe the algebraic structure of $G_{n,+}$. In a similar manner, we obtain the algebraic structure of $G_{n,-}$.

Notation 3.3. Loops in $G_{n,+}$ are denoted by $[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*]$. Any time we write this expression, it is implied that

- $t(\varepsilon_i) = s(\varepsilon_{i+1}^*) = t(\varepsilon_{i+1})$ for all odd $i < 2n$,
- $t(\varepsilon_i^*) = s(\varepsilon_i) = s(\varepsilon_{i+1})$ for all even $i < 2n$, and
- $t(\varepsilon_{2n}^*) = s(\varepsilon_{2n}) = s(\varepsilon_1)$.

For a loop $x = [\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*] \in G_{n,+}$, we define the truncated path $x_{[j,k]}$ to be the j^{th} to k^{th} entries of x for $1 \le j \le k \le 2n$. For example if j and k are odd then $x_{[j,k]} = [\varepsilon_j\varepsilon_{j+1}^* \cdots \varepsilon_{k-1}^*\varepsilon_k]$. In particular $x_{[1,2n]} = x$.

Definition 3.4. We define an antilinear map $*$ on $G_{n,+}$ by the antilinear extension of the map

$$[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*]^* = [\varepsilon_{2n}\varepsilon_{2n-1}^* \cdots \varepsilon_2\varepsilon_1^*].$$

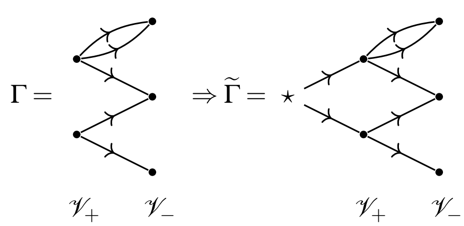
Note that this is just traversing the loop in the opposite direction. We can extend this notion for truncated paths $x_{[j,k]}$. We can also define a multiplication on $G_{n,+}$ by

$$x \cdot y = \delta_{x^*_{[n+1,2n]}, y_{[1,n]}}[x_{[1,n]}y_{[n+1,2n]}].$$

This is 0 if the second half of x and the first half of y do not coincide, otherwise we concatenate the first half of x and the second half of y. Clearly, $*$ is an involution for $G_{n,+}$ under this multiplication.

Definition 3.5. Let $\widetilde{\Gamma}$ be the augmentation of the bipartite graph Γ by adding a distinguished vertex $\star$ which is connected to each $v \in \mathscr{V}_+$ by $m_+(v)$ distinct edges, all oriented so they begin at $\star$. We will denote these added edges by η's, κ's or ρ's.

Example 3.6. Suppose the dimension vector for $\mathscr{V}_+$ only takes the value 1. Then augmenting a graph Γ is given by



Definition 3.7. For $n \in \mathbb{Z}_{\ge 0}$, let B_n be the complex vector space with basis given by loops of length $2n+2$ in $\widetilde{\Gamma}$ that are of the form

$$[\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*].$$

This means that the loop starts and ends at $\star$ and remains in Γ otherwise. We define the involution and multiplication for B_n similarly to our definition for $G_{n,+}$.

Remark 3.8. Sometimes it is convenient to write an element $[\eta_1\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*]$ in B_n as $[\ell_1(\ell_2)^*]$ where ℓ_i are paths of length $n+1$ that start at $\star$ but remain in Γ otherwise. With this notation, $t(\ell_1)=t(\ell_2)$ is $t(\varepsilon_n)$ when n is odd and $s(\varepsilon_n)$ when n is even.

Definition 3.9. Consider the map $\iota : B_n \to B_{n+1}$ defined by the linear extension of

$$[\ell_1(\ell_2)^*] \mapsto \begin{cases} \sum\limits_{s(\varepsilon)=s(\varepsilon_n)} [\ell_1\varepsilon\varepsilon^*(\ell_2)^*] & n \text{ even} \\ \sum\limits_{s(\varepsilon)=t(\varepsilon_n)} [\ell_1\varepsilon^*\varepsilon(\ell_2)^*] & n \text{ odd} \end{cases}$$

It is not hard to see that ι is an injective $*$-homomorphism and therefore we can identify B_n with its image $\iota(B_n)\subset B_{n+1}$. With this identification, we obtain a tower of algebras

$$B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots$$

Moreover, this allows us to define the product of elements in B_m and B_n by including both in $B_{\max\{m,n\}}$. More precisely, if $x\in B_m$ and $y\in B_n$ with $m\leq n$ then

$$x\cdot y = \delta_{x^*_{[m+2,2m+2]},y_{[1,m+1]}}[x_{[1,m+1]}y_{[m+2,2n+2]}].$$

This multiplication is consistent with the embeddings from Definition 3.9.

3.2 Isomorphism of towers

Let $A_0\subset A_1$ be a unital inclusion of finite dimensional C*-algebras with Bratelli diagram Γ and inclusion matrix Λ. Thus, if $A_0\simeq\bigoplus_{s=1}^k M_{\nu_i}(\mathbb{C})$ and $A_1\simeq\bigoplus_{r=1}^l M_{\mu_i}(\mathbb{C})$ then Λ is a $k\times l$ matrix where Λ_{sr} is the number of times the s-th summand of A_0 sits in the r-th summand of A_1. Recall that the Bratteli diagram is a bipartite graph with vertices $\mathscr{V}_+=\{v_1^+,\ldots,v_k^+\}$ which correspond to the simple summands of A_0 and vertices $\mathscr{V}_-=\{v_1^-,\ldots,v_l^-\}$ which correspond to the simple summands of A_1. By definition of the Bratteli diagram, we have Λ_{ij} edges going from v_i^+ to v_j^-. Since our inclusion is unital, the dimension vectors $\vec{\nu}$ and $\vec{\mu}$ associated to A_0 and A_1, respectively, satisfy the equation $\vec{\mu}=\Lambda^t\vec{\nu}$. Suppose that tr is a d-Markov trace on

A_1. We can iterate the basic construction to obtain a tower of algebras

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$$

It is a well-known fact (see Section 2.4 in [GdlHJ89]) that if we perform the basic construction to obtain $A_1 \subset A_2$, then A_2 will also be a finite dimensional C*-algebra and the inclusion matrix will be Λ^t. Hence its Bratteli diagram is just a reflection of Γ. Consequently, if we iterate the basic construction, the inclusion matrix for $A_m \subset A_{m+1}$ is Λ if m is even and Λ^t if m is odd. Moreover, the dimension vector for A_n, denoted $\vec{v}^n$, is given inductively by $\Lambda\vec{v}^{n-1}$ if n is even or $\Lambda^t\vec{v}^{n-1}$ if n is odd. Thus

$$v_s^n = \begin{cases} \sum_{r=1}^{k} v_r^{n-1}\Lambda_{sr} & \text{, n even} \\ \sum_{r=1}^{l} v_r^{n-1}\Lambda_{rs} & \text{, n odd} \end{cases}$$

Note that we can think of $\vec{v}^n$ as a function from $\mathscr{V}_+$ to $\mathbb{N}$ if n is even and from $\mathscr{V}_-$ to $\mathbb{N}$ if n is odd. Hence $\vec{v}^0$ will play the same role as m_+ in 3.1, which allows us to define $\widetilde{\Gamma}$. We will establish an isomorphism between the tower of algebras $(A_n)_{n\geq 0}$ and the tower of algebras $(B_n)_{n\geq 0}$ arising from the bipartite graph Γ as defined in 3.7.

Proposition 3.10. *If $\vec{v}^n = (v_s^n)$ is the dimension vector of A_n, then there are exactly v_s^n paths of length $n+1$ from $\star$ to v_s^+ if n is even (or to v_s^- if n is odd) in $\widetilde{\Gamma}$ that start in $\star$ but remain in Γ otherwise.*

Proof. By definition of $\widetilde{\Gamma}$ this is true for $n = 0$. Suppose it is true for $n = m$ odd, then we have

$$\begin{aligned} v_s^{m+1} &= \sum_{r=1}^{k} v_r^m \Lambda_{sr} \\ &= \sum_{r=1}^{k} (\#\text{ paths of length } m+1 \text{ from } \star \text{ to } v_r^-)(\#\text{ edges from } v_s^+ \text{ to } v_r^-) \\ &= \#\text{ paths of length } m+2 \text{ from } \star \text{ to } v_s^+ . \end{aligned}$$

If $n = m$ is even we proceed in a similar manner. □

Proposition 3.11. *For any $n \geq 0$ we have $B_n \simeq A_n$.*

Proof. We will prove the case for n even. The case for n odd is similar. Note the basis for B_n is comprised of elements of the form $[\ell_1(\ell_2)^*]$, where ℓ_1 and ℓ_2 are paths of length $n+1$ that start in $\star$ but otherwise remain in Γ.

From the previous proposition, we know that there are v_s^n paths of length $n+1$ that start in $\star$, remain in Γ otherwise, and end in v_s^+. This means we can label all these paths as $\ell_1^s, \ldots, \ell_{v_s^n}^s$. Now let $e_{ij}^s = [\ell_i^s(\ell_j^s)^*]$ for $1 \leq i, j \leq v_s^n$. By definition $\{e_{ij}^s,\ 1 \leq i, j \leq v_s^n\}_{s=1}^k$ is a basis for B_n that consists of matrix units since

$$\begin{aligned} e_{ij}^s e_{i'j'}^s &= [\ell_i^s(\ell_j^s)^*] \cdot [\ell_{i'}^s(\ell_{j'}^s)^*] \\ &= \delta_{\ell_j^s, \ell_{i'}^s}[\ell_i^s(\ell_{j'}^s)^*] \\ &= \delta_{j,i'} e_{ij'}^s. \end{aligned}$$

More precisely, for every s, $\{e_{ij}^s,\ 1 \leq i, j \leq v_s^n\}$ is a $v_s^n \times v_s^n$ system of matrix units and $e_{ij}^s e_{i'j'}^{s'} = 0$ for $s \neq s'$. This implies

$$B_n \simeq \bigoplus_{s=1}^k M_{v_s^n}(\mathbb{C}) \simeq A_n.$$

□

So far we have proved that at every level both towers are isomorphic. We proceed to prove that this isomorphism preserves the inclusion structure of the tower.

Proposition 3.12. *The inclusion matrix for $B_n \subset B_{n+1}$ is the same as the one for $A_n \subset A_{n+1}$.*

Proof. We will show the result for n even, the result for n odd is similar. Suppose n is even then $B_n \simeq \bigoplus_{s=1}^k M_{v_s^n}(\mathbb{C})$ where the s-th summand is generated by $\{e_{ij}^s,\ 1 \leq i, j \leq v_s^n\}$. To see the multiplicity of the s-th summand in the r-th summand of B_{n+1} we need to look at the image of a rank one projection for the s-th summand in the r-th summand. A rank one projection in the s-th summand is given by $e_{11}^s = [\ell_1^s(\ell_1^s)^*]$ and its image under the inclusion map is

$$\begin{aligned} \iota(e_{11}^s) &= \iota\left([\ell_1^s(\ell_1^s)^*]\right) \\ &= \sum_{s(\varepsilon)=v_s^+} [\ell_1^s \varepsilon \varepsilon^*(\ell_1^s)^*]. \end{aligned}$$

The part that is contained in the r-th summand in B_{n+1} is given by loops of the form $[\ell_1(\ell_2)^*]$ where $t(\ell_i) = v_r^-$. Thus, the r-th summand part of $\iota(e_{11}^s)$ is

$$\sum_{s(\varepsilon)=v_s^+, t(\varepsilon)=v_r^-} [\ell_1^s \varepsilon \varepsilon^*(\ell_1^s)^*],$$

which means it has precisely one rank projection 1 for every edge going from v_s^+ to v_r^-. Since there are Λ_{sr} many such edges, we conclude that the multiplicity of the s-th summand of B_n into the r-th summand of B_{n+1}

is exactly Λ_{sr}. From this we conclude that $B_n \subset B_{n+1}$ has inclusion matrix Λ. □

Recall that the tower $\{A_n\}_{n\geq 0}$ has additional structure given by the Markov traces, conditional expectations and Jones projections. The next step is to determine how these are expressed in the loop algebra representations $\{B_n\}_{n\geq 0}$.

Definition 3.13. Let $\lambda^i = (\lambda^i_s)$ be the Markov trace (column) vector for A_i for $i = 0,1$ such that

$$\sum_s v^0_s \lambda^0_s = 1 = \sum_s v^1_s \lambda^1_s$$

λ^i_s denotes the trace of a minimal projection in the s-th simple summand of A_i for $i = 0,1$. Since the trace on A_1 restricts to the trace on A_0 we must have $\Lambda\lambda^1 = \lambda^0$.

Remark 3.14. Since tr is a d-Markov trace, we also have $\Lambda\Lambda^t\lambda^0 = d^{-2}\lambda^0$ and $\Lambda^t\Lambda\lambda^1 = d^{-2}\lambda^1$, and $d = \sqrt{\|\Lambda^t\Lambda\|} = \sqrt{\|\Lambda\Lambda^t\|}$. This implies that the trace vector λ^n for A_n is given by $\lambda^n = d^{-2}\lambda^{n-2}$ for all $n \geq 2$. In particular we have

$$\lambda^n = \begin{cases} d^{-n}\lambda^0 & , n \text{ even} \\ d^{-n+1}\lambda^1 & , n \text{ odd} \end{cases}$$

Since $\mathscr{V}_+ = \{v^+_1, \ldots, v^+_k\}$ and $\mathscr{V}_- = \{v^-_1, \ldots, v^-_l\}$, if n is even, we can regard λ^n as a function from $\mathscr{V}_+$ to $\mathbb{N}$ by setting $\lambda^n(v^+_i) := \lambda^n_i$. Similarly, if n is odd, we regard λ^n as a function from $\mathscr{V}_-$ to $\mathbb{N}$.

Definition 3.15 (Traces)**.** We define a normalized trace on B_n by

$$\mathrm{tr}_n([\ell_1(\ell_2)^*]) = \delta_{\ell_1,\ell_2}\lambda^n(t(\ell_1)).$$

Remark 3.16. Note that the traces above are precisely the ones preserved by the isomorphism from 3.11. This is because the loops with $\ell_1 = \ell_2$ represent the rank one projections in A_n and the entries of λ^n are the traces of rank one projections in the corresponding simple summand. This also implies that $\mathrm{tr}_{n+1}|_{B_n} = \mathrm{tr}_n$.

Proposition 3.17. *(Conditional Expectations) If $x = [\eta_1\varepsilon_1\varepsilon^*_2\cdots\varepsilon_{2n-1}\varepsilon^*_{2n}\eta^*_2] \in B_n$, the tr_n-preserving conditional expectation $B_n \to B_{n-1}$ is given by*

$$E_{B_{n-1}}(x) = \begin{cases} d^{-2}\delta_{\varepsilon_n,\varepsilon_{n+1}}\left(\frac{\lambda^0(s(\varepsilon_n))}{\lambda^1(t(\varepsilon_n))}\right)[\eta_1\varepsilon_1\varepsilon^*_2\cdots\varepsilon_{n-1}\varepsilon^*_{n+2}\cdots\varepsilon_{2n-1}\varepsilon^*_{2n}\eta^*_2] & , n \text{ even} \\ \delta_{\varepsilon_n,\varepsilon_{n+1}}\left(\frac{\lambda^1(t(\varepsilon_n))}{\lambda^0(s(\varepsilon_n))}\right)[\eta_1\varepsilon_1\varepsilon^*_2\cdots\varepsilon^*_{n-1}\varepsilon_{n+2}\cdots\varepsilon_{2n-1}\varepsilon^*_{2n}\eta^*_2] & , n \text{ odd.} \end{cases}$$

Remark 3.18. If n is even and we write $x = [\ell_1 \varepsilon_n^* \varepsilon_{n+1} (\ell_2)^*] \in B_n$ (this means $t(\ell_1) = t(\varepsilon_n)$ and $t(\ell_2) = t(\varepsilon_{n+1})$) where ℓ_i are paths of length n starting at $\star$ and remaining in Γ otherwise, we have

$$E_{B_{n-1}}(x) = d^{-2} \delta_{\varepsilon_n, \varepsilon_{n+1}} \left(\frac{\lambda^0(s(\varepsilon_n))}{\lambda^1(t(\varepsilon_n))} \right) [\ell_1(\ell_2)^*].$$

We obtain a similar expression in the case of n odd.

Definition 3.19. (Jones Projections) For $n \geq 1$, consider the following elements in B_{n+1}: If n is odd, define

$$F_n = \sum_{\ell} \sum_{\substack{s(\varepsilon_n)=t(\ell) \\ s(\varepsilon_{n+1})=t(\ell)}} \frac{d[\lambda^1(t(\varepsilon_n))\lambda^1(t(\varepsilon_{n+1}))]^{1/2}}{\lambda^0(s(\varepsilon_n))} [\ell \varepsilon_n \varepsilon_n^* \varepsilon_{n+1} \varepsilon_{n+1}^* \ell^*],$$

and if n is even, define

$$F_n = \sum_{\ell} \sum_{\substack{t(\varepsilon_n)=t(\ell) \\ t(\varepsilon_{n+1})=t(\ell)}} \frac{[\lambda^0(s(\varepsilon_n))\lambda^0(s(\varepsilon_{n+1}))]^{1/2}}{d\lambda^1(t(\varepsilon_n))} [\ell \varepsilon_n^* \varepsilon_n \varepsilon_{n+1}^* \varepsilon_{n+1} \ell^*],$$

where the sum, in both cases, is taken over all paths ℓ of length n starting at $\star$ but remaining in Γ otherwise.

Proposition 3.20 (Basic Construction). *For $n \in \mathbb{N}$, the inclusion*

$$B_{n-1} \subset B_n \subset (B_{n+1}, \mathrm{tr}_{n+1}, d^{-1}F_n)$$

is standard (see Lemma 2.15 and Definition 2.16 in [JP11]). Hence for all $k > 0$, there are isomorphisms $\varphi_k : B_k \to A_k$ preserving the trace such that $\varphi_{k+1}|_{B_k} = \varphi_k$ and $\varphi_m(F_n) = df_n$ for all $m > n$, where f_n is the Jones projection for the inclusion $A_{n-1} \subset A_n$.

3.3 Relative commutants of loop algebras

We will now provide isomorphisms between the relative commutants of the tower $(B_n)_{n\geq 0}$ and the algebras $G_{n,\pm}$. To simplify notation we will represent elements of $G_{n,\pm}$ as $[\ell_1^{\pm}(\ell_2^{\pm})^*]$ where $\ell_i^{\pm}$ is a path of length n in Γ starting from $\mathscr{V}_{\pm}$.

Proposition 3.21. *A basis for $B_0' \cap B_n$ is given by*

$$S_{0,n} = \left\{ \sum_{t(\eta)=s(\ell_1^+)} [\eta \ell_1^+ (\ell_2^+)^* \eta^*] \in B_n;\ [\ell_1^+(\ell_2^+)^*] \in G_{n,+} \right\}.$$

A basis for $B_1' \cap B_{n+1}$ is given by

$$S_{1,n+1} = \left\{ \sum_{\substack{t(\eta)=s(\varepsilon) \\ t(\varepsilon)=s(\ell_1^-)}} [\eta\varepsilon\ell_1^-(\ell_2^-)^*\varepsilon^*\eta^*] \in B_n;\ [\ell_1^-(\ell_2^-)^*] \in G_{n,-} \right\}.$$

Definition 3.22. For $n \in \mathbb{Z}_{\geq 0}$, let $H_{n,+} = B_0' \cap B_n$, $H_{n,-} = B_1' \cap B_{n+1}$, $Q_{n,+} = A_0' \cap A_n$, and $Q_{n,-} = A_1' \cap A_{n+1}$.

Corollary 3.23. *There are canonical algebra $*$-isomorphisms $\phi_{n,\pm} : G_{n,\pm} \to H_{n,\pm}$ for all $n \geq 0$. If $n = 0$ the isomorphisms are given by*

$$\phi_{0,+}(v_+) = \sum_{t(\eta)=v_+} [\eta\eta^*] \text{ and } \phi_{0,-}(v_-) = \sum_{\substack{t(\eta)=s(\varepsilon) \\ t(\varepsilon)=v_-}} [\eta\varepsilon\varepsilon^*\eta^*].$$

For $n > 0$, the isomorphisms are given by

$$\phi_{n,+}(\ell_1^+(\ell_2^+)^*) = \sum_{t(\eta)=s(\ell_1^+)} [\eta\ell_1^+(\ell_2^+)^*\eta^*] \text{ and } \phi_{n,-}([\ell_1^-(\ell_2^-)^*]) = \sum_{\substack{t(\eta)=s(\varepsilon) \\ t(\varepsilon)=s(\ell_1^-)}} [\eta\varepsilon\ell_1^-(\ell_2^-)^*\varepsilon^*\eta^*].$$

Remark 3.24. For $n \geq 0$, $\psi_{n,\pm} = (\varphi_n|_{H_{n,\pm}} \circ \phi_{n,\pm})^{-1} : Q_{n,\pm} \to G_{n,\pm}$ are isomorphisms.

We describe next the conditional expectation on the relative commutants using an explicit Pimsner-Popa basis for B_1 over B_0.

Proposition 3.25 (Pimsner-Popa Basis)**.** *For each $v_+ \in \mathcal{V}_+$, pick a distinguished η_{v_+} with $t(\eta_{v_+}) = v_+$. Set*

$$S_1 = \left\{ \left(\frac{\lambda^0(s(\varepsilon_2))}{\lambda^1(t(\varepsilon_2))}\right)^{1/2} \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^*\eta^*] :\ [\varepsilon_1\varepsilon_2^*] \in G_{1,+} \right\},$$
$$S_2 = \left\{ \left(\frac{\lambda^0(s(\varepsilon_2))}{\lambda^1(t(\varepsilon_2))}\right)^{1/2} [\eta_{s(\varepsilon_1)}\varepsilon_1\varepsilon_2^*\eta_{s(\varepsilon_2)}^*] :\ s(\varepsilon_1) \neq s(\varepsilon_2) \right\}.$$

Then $S = S_1 \sqcup S_2$ is a Pimsner-Popa basis for B_1 over B_0.

Corollary 3.26 (Commutant Conditional Expectations)**.** *If*

$$x = \sum_{t(\rho)=s(\xi_1)} [\rho\xi_1\ell_1^-(\ell_2^-)^*\xi_{2n}^*\rho^*] \in B_0' \cap B_n$$

the tr_n*-preserving conditional expectation* $B_0' \cap B_n \to B_1' \cap B_n$ *is given by*

$$E_{B_1'}(x) = d^{-2}\delta_{\xi_1,\xi_{2n}}\left(\frac{\lambda^0(s(\xi_1))}{\lambda^1(t(\xi_1))}\right)\sum_{\substack{t(\rho)=s(\varepsilon)\\ t(\varepsilon)=t(\xi_1)}}[\rho\varepsilon\ell_1^-(\ell_2^-)^*\varepsilon^*\rho^*].$$

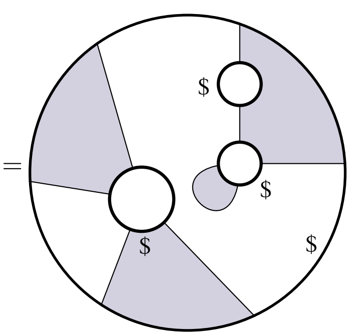
CHAPTER 4

Planar algebras

Planar algebras were introduced by Jones in [Jon99, Jon22] to describe the algebraic structure of the standard invariant of subfactors. In this section, we will give a broad description of a planar algebra. Our main interest lies in the definition of a *subfactor planar algebra* and a *graph planar algebra*.

Definition 4.1. A *planar n-tangle T* consists of the unit disk D $(= D_0)$ in $\mathbb{R}^2$ together with a finite (possibly empty) set of disjoint subdisks $D_1, D_2, \ldots, D_k$ in the interior of D. Each disk D_i, $i \geq 0$, will have an even number $2n_i \geq 0$ of marked points on its boundary (with $n = n_0$). Inside D there is also a finite set of disjoint smoothly embedded curves called *strings* which are either closed curves or whose boundaries are marked points of the D_i's. Each marked point is the boundary of some string, which meets the boundary of the corresponding disk transversally. The strings all lie in the complement of the interiors $\text{int}(D_i)$ of the D_i, $i > 0$. The connected components of the complements of the strings in $\text{int}(D) \setminus \bigcup_{i=1}^{k} D_i$ are called *regions* and are shaded black and white so that the regions whose closures meet have different shadings. The shading is part of the data of the tangle, as is the choice, at every D_i, $i \geq 0$, of a region whose closure meets that disk which is marked with a $.

Here is an example of a planar 3-tangle:

$T =$ 

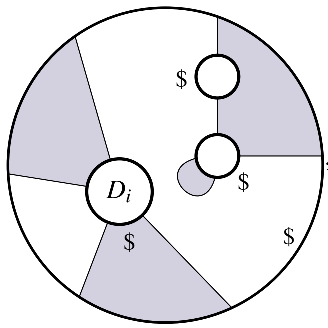
Notation 4.2. For every disk D_i in a tangle, we let $\partial(D_i) = (n_i, +)$ if the region corresponding to the marked region is white and $\partial(D_i) = (n_i, -)$ if the shading is black. In particular, if we set $\partial(T) = \partial(D_0)$ then for the example above we have $\partial(T) = (3, +)$. We will say that a k-tangle T is *positive* if $\partial(T) = (n, +)$ and *negative* if $\partial(T) = (n, -)$.

Remark 4.3. We are only interested in the isotopy class of a planar tangle. Later on we will see that it is useful to deform all the circles in our diagrams into rectangles to get a “standard form” for a tangle.

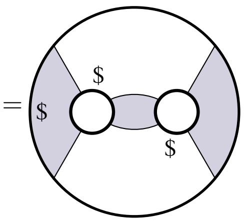
Definition 4.4 (Composition of tangles)**.** Consider T a n-tangle, S a n'-tangle and D_i a disk in T such that $n_i = n'$. If the shading associated to the marked interval of D_i coincides with the shading of the marked interval of the boundary of S then we define $T \circ_i S$ to be the n-tangle obtained by “gluing” S into D_i such that the marked interval and shading of D_i and S coincide. Below is an example that illustrates this process.

Example 4.5. Suppose we are given the following tangles:

$T =$

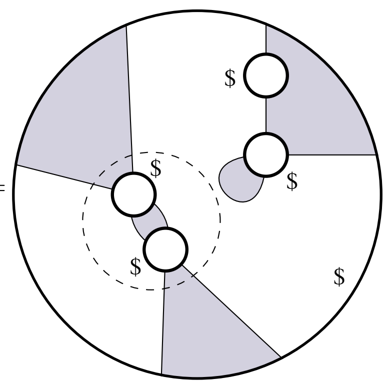


, $S =$



Then we can glue S into D as described below.

$T \circ_i S =$



where the dashed circle represents the disk boundary that was deleted. Note that we had to rotate S so its marked interval and shading matched the ones for D_i.

Definition 4.6 (Planar algebra)**.** A *planar algebra* is a family $\{P_{n,\pm}\}_{n\in\mathbb{N}}$ of complex vector spaces together with linear maps

$$Z_T : \bigotimes_{i=1}^{k} P_{\partial(D_i)} \to P_{\partial(T)}$$

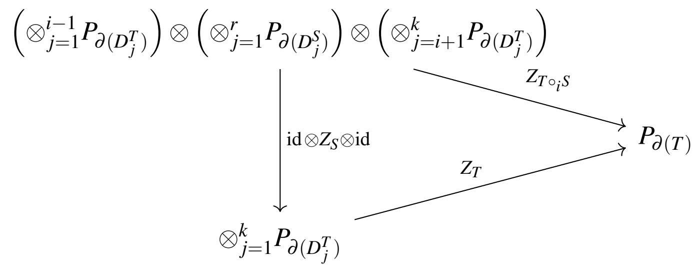
for every k-tangle T.

We require the maps Z_T to only depend on the isotopy class of T, be independent of the ordering of the internal disks of T and be “compatible with composition of tangles” in the following manner. If T is a

n-tangle with k internal disks $\{D_i^T\}_{i=1}^k$ and S is a n'-tangle with r internal disks $\{D_i^S\}_{i=1}^r$, then $T \circ_i S$ is an n-tangle with $k+r-1$ internal disks given by

$$D_j = \begin{cases} D_j^T & 1 \leq j < i \\ D_{j-i+1}^S & i \leq j \leq i+r-1 \\ D_{j-r+1}^T & i+r \leq j \leq r+k-1 \end{cases}$$

We say Z is compatible with the composition of tangles if the following diagram commutes:

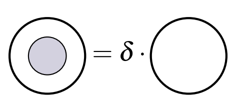


When the tangle T has no internal disks we follow the convention that an empty tensor product is the underlying field, in this case, $\mathbb{C}$. This means that each subspace $P_{n,\pm}$ has a distinguished subset $\{Z_T(1) :$ T a n-tangle without internal disks$\}$. We call these *Temperley-Lieb tangles*.

We are interested in the case where $P_{n,\pm}$ is finite-dimensional for all $n \geq 0$. If $P_{0,\pm}$ is one-dimensional, there is a unique way to identify $P_{0,+}$ with $\mathbb{C}$ such that $Z_{\bigcirc}(1) = 1$ and $P_{0,-}$ with $\mathbb{C}$ such that $Z_{\bigcirc}(1) = 1$, where both tangles have no internal disks nor strings but have different shading. There are two scalars associated to such a planar algebra,

$$\delta_1 = Z_{\circledcirc}(1), \quad \delta_2 = Z_{\circledcirc}(1),$$

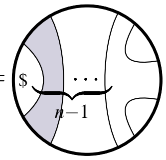
where the inner circles are strings, not disks. We call δ_1 and δ_2 the *loop parameters* of the planar algebra. In fact, any such planar algebra can be altered so $\delta_1 = \delta_2$ as explained in [Jon19, Lemma 2.3.9]. From now on we will assume our planar algebras satisfy $\dim P_{0,\pm} = 1$ and $\delta = \delta_1 = \delta_2$. Then $Z_{\circledcirc} = \delta \cdot Z_{\bigcirc}$, which diagrammatically can be expressed as:



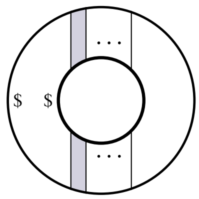
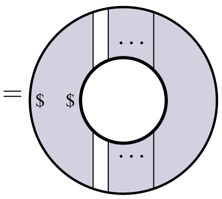
We obtain a similar relation for the diagrams with the other shading.

Remark 4.7. The relation above allows us to eliminate loops from tangles. Using composition of tangles we can show that if T is a tangle with a loop and T' is the same tangle but without the loop, then $Z_T = \delta \cdot Z_{T'}$.

Definition 4.8 (Jones projections)**.** For $n \geq 1$, the *Jones projections* in $P_{n+1,+}$ are $\delta^{-1} E_n$, where E_n is the distinguished element associated to:

$E_n =$ 

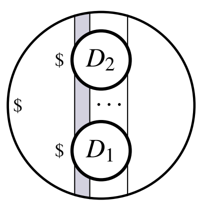
For every $n \geq 0$ consider the following n-tangles:

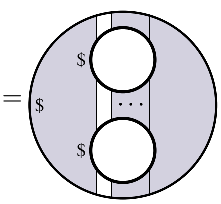
$I_n^+ =$  , $I_n^- =$ 

Clearly these tangles work as the identity under tangle composition. For example if T is any positive n-tangle then $I_n^+ \circ_1 T = T$ and consequently $Z_{I_n^+} \circ Z_T = Z_T$, where $Z_{I_n^+} : P_{n,+} \to P_{n,+}$. This however does not imply that $Z_{I_n^+}$ has to be the identity map on $P_{n,+}$.

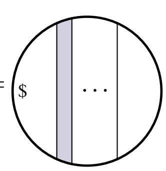
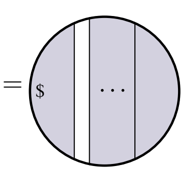
Definition 4.9. We say that a planar algebra is *non-degenerate* if for all $n \geq 0$ we have that $Z_{I_n^\pm} = \mathrm{id}_{P_{n,\pm}}$.

It is also important to note that we have an algebra structure on every $P_{n,\pm}$, where the multiplication is determined by the multiplication tangles:

$m_k^+ =$ 

, $m_k^- =$ 

For $x_1, x_2 \in P_{n,\pm}$ we set $x_1 x_2 := Z_{m_n^\pm}(x_1 \otimes x_2)$. The associativity of the product is obtained by composing the multiplication tangle with itself in the two possible ways and verifying that one obtains isotopic tangles. With this algebraic structure, we have that $P_{n,\pm}$ is a unital algebra for all $n \geq 0$ where the units are given by the distinguished elements defined by

$1_n^+ =$  , $1_n^- =$ 

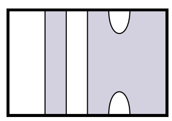
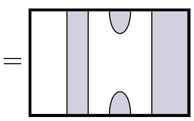
Definition 4.10 (Planar $*$-algebra)**.** We say that a planar algebra P is a $*$*-planar algebra* if each $P_{n,\pm}$ possesses a linear involution $*$ so that if θ is an orientation reversing diffeomorphism of $\mathbb{R}^2$ and T a tangle with k internal disks, then for any $x_1 \otimes \cdots \otimes x_k \in \bigotimes_{i=1}^{k} P_{\partial(D_i)}$ we have

$$[Z_{\theta(T)}(x_1 \otimes \cdots x_k)]^* = Z_T(x_1^* \otimes \cdots x_k^*).$$

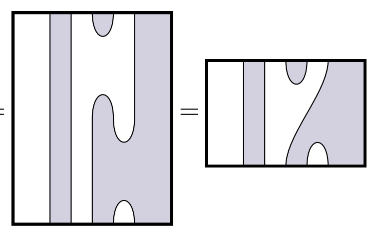
Remark 4.11. In a $*$-planar algebra, $\delta^{-1}E_n$ is a projection in the classical sense.

Definition 4.12 (C^*-planar algebra)**.** We say that a planar $*$-algebra is a C^**-planar algebra* if every $P_{n,\pm}$ has a norm making into a C^*-algebra with the algebra structure given by the multiplication tangle.

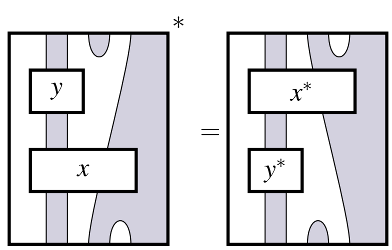
Since tangles only care about the isotopy class of the diagram, we can deform the disks to rectangles where half of the boundary points are at the top, the other half at the bottom and the marked interval is always on the left of the rectangle. Following this convention, composing two tangles with the multiplication tangle amounts to stacking them on top of each other. For example, consider the following 5-tangles

$x =$  , $y =$ 

which determine elements in $P_{5,\pm}$, then xy is

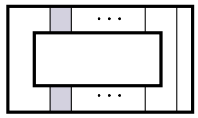
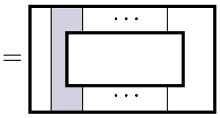
$xy =$ 

Remark 4.13. Using the convention from above, the involution from 4.10 can be diagrammatically represented as

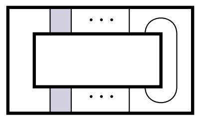
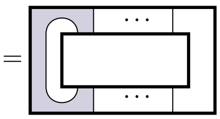


which amounts to turning the tangle upside down and taking the involution of the elements on which we are evaluating.

The planar algebra structure defined above implied that every $P_{n,\pm}$ is a unital associative algebra. There are however many tangles that induce maps between $P_{n,\pm}$ and $P_{n',\pm}$ where $n \neq n'$. One might wonder what extra structure these maps induce. There are two classes of tangles which begin to answer that question, the first being the *inclusion tangles*:

$\iota_n^+ =$  , $\iota_n^- =$  ,

where $\iota_n^{\pm}$ is a $n+1$-tangle. The other class of tangles are the *conditional expectations*:

$\alpha_n =$  , $\beta_n =$ 

where both are $n-1$-tangles. Note that these tangles only have one internal disk. Such tangles are called *annular tangles*. When composing an annular tangle T with any other tangle S we will denote the composition simply by $T \circ S$ as there is no confusion as to where to place S. For such tangles T, Z_T will simply be a linear map from $P_{\partial D}$ (where D is the only internal disk of T) to $P_{\partial T}$. Therefore we can justify treating T as a map $T : P_{\partial D} \to P_{\partial T}$. With this convention, the tangles previously introduced define maps

$$\iota_n^+ : P_{n,+} \to P_{n+1,+}$$

$$\iota_n^- : P_{n,-} \to P_{n+1,+}$$

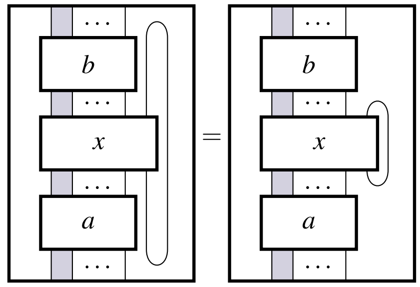
$$\alpha_n : P_{n,+} \to P_{n-1,+}$$

$$\beta_n : P_{n,+} \to P_{n-1,-}$$

Observe that $\alpha_{n+1} \circ \iota_n^+ = \delta I_n^+$ and $\beta_{n+1} \circ \iota_n^+ = \delta I_n^-$. Therefore, if our planar algebra is non-degenerate, we have that $\iota_n^{\pm}$ is an injective map. This means that we can regard $P_{n,+}$ as a subspace of $P_{n+1,+}$ and $P_{n,-}$ as a subspace of $P_{n+1,+}$, both invariant under the respective conditional expectation tangles. In particular, we can think of every $P_{n,+}$ as a $P_{n-1,+}$ or $P_{n-1,-}$-bimodule where the action is simply given by the product.

Proposition 4.14. *α_n and β_n are $P_{n-1,+}$ and $P_{n-1,-}$ bimodule homomorphism, respectively. By this we mean that if $a,b \in P_{n-1,+}$ and $x \in P_{n,+}$ then $\alpha_n(a \cdot x \cdot b) = a \cdot \alpha_n(x) \cdot b$, and similarly for β_n.*

Proof. Since the product in $P_{n,+}$ is given by the multiplication tangle and a,b sit in $P_{n,+}$ via the inclusion tangle, we have

$\alpha_n(a \cdot x \cdot b) =$  $= a \cdot \alpha_n(x) \cdot b$

.

□

In principle, there are infinitely many tangles that will induce some structure on the subspaces $P_{n,\pm}$, and one might wonder if there is a nice set of tangles that generates all of them via composition. The next theorem, [KS04, Theorem 3.5], answers that question.

Theorem 4.15. *Let $\mathscr{T}$ denote the set of all shaded tangles, and suppose $\mathscr{T}_1$ is a subclass of $\mathscr{T}$ which satisfies:*

(a) $1_n^+, 1_n^- \in \mathscr{T}_1$ *for all* $n \geq 0$,

(b) $E_n \in \mathscr{T}_1$ *for all* $n \geq 1$,

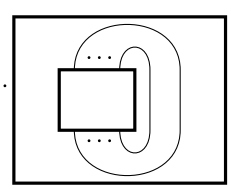
(c) $\alpha_n, \beta_n \in \mathscr{T}_1$ *for all* $n \geq 0$,

(d) $\iota_n^+, \iota_n^- \in \mathscr{T}_1$ *for all* $n \geq 0$,

(e) $m_n^+ \in \mathscr{T}_1$ *for all* $n \geq 0$,

and is closed under composition of tangles, then $\mathscr{T} = \mathscr{T}_1$.

Recall that $P_{0,\pm}$ can be identified with $\mathbb{C}$, hence α_1 and β_1 define linear functionals. In fact, letting $\mathrm{tr}_n = \delta^{-n}\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n : P_{n,+} \to P_{0,+} \simeq \mathbb{C}$ we obtain a linear functional on every space $P_{n,+}$. This linear functional is given by the following tangle:

$\mathrm{tr}_n = \delta^{-n} \cdot$ 

Remark 4.16. It is clear that $\mathrm{tr}_n(xy) = \mathrm{tr}_n(yx)$ for all $x, y \in P_{n,+}$, as one can *slide* x across the strings from the bottom to the top. Moreover $\mathrm{tr}_n(1_n^+) = 1$, hence tr_n is a normalized trace on $P_{n,+}$. However, if we draw a tangle as the one above but instead of capping the strings to the right we cap them to the left, we obtain another normalized trace which may not be equal to tr_n.

Definition 4.17 (Spherical planar algebra)**.** We say a planar algebra is *spherical* if the two traces defined in the previous remark coincide for all n.

Definition 4.18 (Positive planar algebra)**.** Let P be a C*-planar algebra, we say it is *positive* if $\delta^n \mathrm{tr}_n : P_n \to \mathbb{C}$ is a faithful positive map for all n.

Remark 4.19. The loop parameter of a positive planar algebra is positive.

Definition 4.20. Let $P_\bullet$ and $Q_\bullet$ be two planar algebras. A *morphism* ψ between $P_\bullet$ and $Q_\bullet$ is a family of linear maps $\psi_{n,\pm} : P_{n,\pm} \to Q_{n,\pm}$, $n \geq 0$, that is compatible with the action of tangles. This means for any n-tangle T with k internal discs, $x_i \in P_{n_i,\pm}$ we have

$$Z_T(\psi_{n_1,\pm}(x_1) \otimes \cdots \otimes \psi_{n_k,\pm}(x_k)) = \psi_{n,\pm}(Z_T(x_1 \otimes \cdots \otimes x_k)).$$

We say $\psi : P_\bullet \to Q_\bullet$ is an *embedding* if every $\psi_{n,\pm}$ is injective.

4.1 Subfactor planar algebra

Definition 4.21. A positive spherical planar algebra P for which $\dim P_{n,\pm} < \infty$ and $\dim P_{0,\pm} = 1$ is called a *subfactor planar algebra*.

Suppose $M_0 \subset (M_1, \mathrm{tr}_1)$ is an extremal inclusion of II_1 factors, set $\delta = \sqrt{[M_1 : M_0]}$ and let

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$$

be the tower obtained from iterating the basic construction, this means $M_{n+1} = \langle M_n, e_n \rangle$ where e_n is the Jones projection with range in $L^2(M_{n-1}, \mathrm{tr}_{n-1})$. Set $P_{n,+} = M_n \cap M_0'$ and $P_{n,-} = M_{n+1} \cap M_1'$ for $n \geq 0$. It is proved in Theorem 4.2.1 of [Jon22] that $\{P_{n,\pm}\}$ has a planar algebra structure. As seen in Theorem 4.15, it is enough to describe how the generating tangles act on these spaces to understand its planar algebra structure.

1. The distinguished element associated to $1_n^\pm$ is the unit 1 in M_n.
2. The distinguished element associated to E_n is $\delta e_n \in P_{n+1,+}$.
3. m_n^+ is given by the usual multiplication in $P_{n,+}$,
4. $\alpha_n : P_{n,+} \to P_{n-1,+}$ is given by $\delta E_{M_{n-1}}$ restricted to $P_{n,+}$, where $E_{M_{n-1}} : M_n \to M_{n-1}$ is the tr_{M_n}-

preserving conditional expectation,

5. $\beta_n : P_{n,+} \to P_{n-1,-}$ is given by $\delta E_{M_1'}$ restricted to $P_{n,+}$, where $E_{M_1'} : M_0' \to M_1'$ is the $\mathrm{tr}_{M_0'}$-preserving conditional expectation, where $\mathrm{tr}_{M_0'}|_{M_n} = \mathrm{tr}_{M_n}|_{M_0'}$ since $M_0 \subset M_1$ is extremal,

6. $\iota_n^+ : P_{n,+} \to P_{n+1,+}$ is given by the inclusion map $M_n \hookrightarrow M_{n+1}$ restricted to $P_{n,+}$,

7. $\iota_n^- : P_{n,-} \to P_{n+1,+}$ is given by the inclusion map $\iota : M_1' \hookrightarrow M_0'$ restricted to $P_{n,-}$.

We denote this abstract planar algebra coming from the subfactor $M_0 \subset M_1$ by $P^{M_0 \subset M_1}$. A surprising fact is that all subfactor planar algebras will arise from an extremal subfactor in this way, as seen in [Pop95], [KS09] or [GJS10].

4.2 Graph planar algebra

In this section we will focus on constructing a planar algebra from a bipartite graph. This construction is due to Jones [Jon00]. The construction for a general graph is similar but requires a more general definition of planar algebra which can be found in [Jon19]. It will not be needed here.

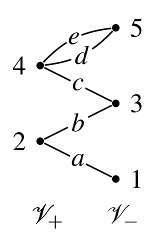
We will use the notation introduced in Chapter 3. Let Γ be a finite, connected, bipartite multigraph with adjacency matrix Λ.

Definition 4.22. If T is a planar tangle, a *state* σ of T is a function

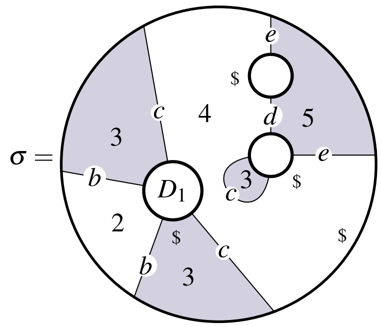
$$\sigma : \{\text{regions of } T\} \cup \{\text{strings of T}\} \to \mathscr{V}_+ \cup \mathscr{V}_- \cup \mathscr{E}$$

that sends unshaded regions to vertices in $\mathscr{V}_+$, shaded regions to vertices in $\mathscr{V}_-$ and strings to edges in $\mathscr{E}$ such that if R_1 and R_2 are two regions having a string S as part of their boundary, then $\sigma(S)$ is an edge connecting $\sigma(R_1)$ and $\sigma(R_2)$.

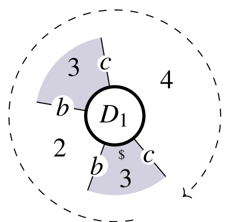
Example 4.23. Consider the following bipartite graph Γ,



then we can represent a state σ on a tangle T in the following way:



Remark 4.24. For every disk in T, a state σ on T induces a pointed loop in Γ by reading the values of the state clockwise around the disk starting from the marked interval. Reading around D_1 from the previous example we have:



$$\Longrightarrow \ell_{D_1}(\sigma) = 3b2b3c4c3$$

Given a disk D in a tangle T, we denote by $\ell_D(\sigma)$ the loop associated to it, in particular, we denote by ℓ_σ the loop associated to D^T.

Now let $\lambda : \mathscr{V}_+ \cup \mathscr{V}_- \to \mathbb{R}$ be defined as $\lambda|_{\mathscr{V}_+} = \lambda^0$ and $\lambda|_{\mathscr{V}_-} = d\lambda^1$ where λ^0 and λ^1 are the Markov trace vectors introduced in definition 3.13 and $d = \sqrt{\|\Lambda\Lambda^t\|}$.

Definition 4.25. Given a state σ on a tangle T we define the *rotation* $\mathrm{Rot}(\sigma)$ as follows:

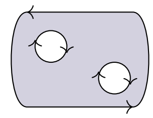
For every region r in T consider the positive orientation on it, then collapse every internal disk to a point so that each r has as its boundary a union of oriented closed piecewise smooth curves. Let $\mathrm{Rot}(r)$ be the rotation number of the boundary of r. Then we set

$$\mathrm{Rot}(\sigma) = \prod_{\text{regions } r \text{ of } T} \lambda(\sigma(r))^{\mathrm{Rot}(r)}.$$

Remark 4.26. To compute the rotation number of a region r, we orient r positively and look at the orientation it induces on its boundary. Note that the boundary of r is a finite collection of closed piecewise smooth curves. We define

$$\mathrm{Rot}(r) = \#\{\text{positively oriented curves}\} - \#\{\text{negatively oriented curves}\}.$$

For example, if r is



then $\text{Rot}(r) = 1 - 2 = -1$.

Recall from Section 2.1, that given a bipartite graph Γ, one defines the loop algebras $\{G_{n,\pm}\}$ whose elements are linear combinations of loops of length $2n$ starting in $\mathscr{V}_{\pm}$. Given $[\ell] \in G_{n,\pm}$ we define $\mu([\ell]) = \lambda(v)\lambda(w)$ where v and w are the first (or last) and middle vertex of the loop ℓ and

$$\lambda([\ell]) = \prod_{v \text{ vertex in } \ell} \lambda(v).$$

To obtain the graph planar algebra we need to determine how planar tangles act on these spaces. Let T be a tangle with k internal disks $\{D_i\}_{i=1}^{k}$ and consider $[\ell_i] \in G_{\partial(D_i)}$ then we define

$$Z_T([\ell_1] \otimes \cdots \otimes [\ell_k]) = \prod_i \sqrt{\frac{\mu([\ell_i])}{\lambda([\ell_i])}} \sum_{\substack{\sigma \text{ state on } T \\ \ell_i = \ell_{D_i(\sigma)}, \forall i}} \frac{\text{Rot}(\sigma)}{\sqrt{\lambda(\ell_\sigma)\mu([\ell_\sigma])}} [\ell_\sigma]$$

and extend Z_T by linearity. Therefore to evaluate T we need to consider all states σ that induce precisely the loop ℓ_i around the disk D_i, then read the loop it induces around the output disk and multiply it by a correction factor. With this action, Jones shows that $\{G_{n,\pm}\}$ has a planar algebra structure [Jon00].

Theorem 4.27. *With the above action of planar tangles, $\{G_{n,\pm}\}$ is a C^*-planar algebra.*

Remark 4.28. Observe that for this planar algebra we do not necessarily have $\dim G_{0,\pm} \neq 1$, in fact $G_{0,\pm} \simeq \mathbb{C}^{|\mathscr{V}_{\pm}|}$. This means that if $|\mathscr{V}_{\pm}| > 1$ then $\{G_{n,\pm}\}$ cannot be a subfactor planar algebra as the notion of sphericality does not make sense in this case.

Example 4.29. Recall that tangles T with no internal disks induce a distinguished element $Z_T(1)$. Hence, the tangle Z corresponds to an element in $G_{0,+}$. To compute this element we need to consider all possible states on this tangle. Let $\mathscr{V}_+ = \{v_1^+, \dots, v_k^+\}$ and $\mathscr{V}_- = \{v_1^-, \dots, v_l^-\}$, every state on the tangle is given by a pair of vertices (v_i^+, v_j^-) and an edge between them, hence:

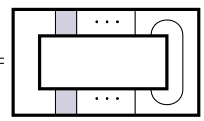
Z
$$(1) = \sum_{i=1}^{k} \sum_{j=1}^{l} \sum_{\substack{\varepsilon \text{ edge} \\ s(\varepsilon) = v_i^+, t(\varepsilon) = v_j^-}} \frac{d\lambda^1(v_j^-)}{\sqrt{\lambda^0(v_i^+)\lambda^0(v_i^+)}} [v_i^+].$$

Here we are using the convention that $\mu([v_i^+]) = \lambda(v_i^+)$. Recall that we have Λ_{ij} edges from v_i^+ to v_j^- and that $\Lambda\lambda^1 = \lambda^0$. Therefore

$$\begin{aligned} Z_{\circledcirc}(1) &= \sum_{i=1}^{k}\sum_{j=1}^{l} d\Lambda_{ij}\frac{\lambda^1(v_j^-)}{\lambda^0(v_i^+)}[v_i^+] \\ &= \sum_{i=1}^{k} d\frac{\lambda^0(v_i^+)}{\lambda^0(v_i^+)}[v_i^+] \\ &= d\sum_{i=1}^{k}[v_i^+] = dZ_{\bigcirc}(1). \end{aligned}$$

Similarly we have $Z_{\circledcirc} = d \cdot Z_{\bigcirc}$. Therefore the graph planar algebra has loop parameter d, where $d = \sqrt{\|\Lambda\Lambda^t\|}$.

Example 4.30. We illustrate how some of the generating tangles act on $\{G_{n,\pm}\}$. Consider the generating tangle

$\alpha_n =$ 

acting on $[\ell] = [\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*] \in G_{n,+}$, where n is even. Observe that if $\varepsilon_n \neq \varepsilon_{n+1}$ there exists no state on α_n which will induce the loop ℓ around the internal disk. On the other hand, if $\varepsilon_n = \varepsilon_{n+1}$ the only possible state σ will induce the loop $[\ell_1(\ell_2)^*]$ on the external disk. Thus

$$\alpha_n([\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*]) = \delta_{\varepsilon_n,\varepsilon_{n+1}}\sqrt{\frac{\mu([\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*])}{\lambda([\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*])}}\frac{\mathrm{Rot}(\sigma)}{\sqrt{\lambda([\ell_1(\ell_2)^*])\mu([\ell_1(\ell_2)^*])}}[\ell_1(\ell_2)^*]$$

Since every region in α_n has rotation number 1, we have that

$$\mathrm{Rot}(\sigma) = \lambda([\ell_1(\ell_2)^*])\lambda(s(\varepsilon_n)).$$

Note that $\lambda([\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*]) = \lambda([\ell_1(\ell_2)^*])\lambda(s(\varepsilon_n))\lambda(t(\varepsilon_n))$. Therefore

$$\alpha_n([\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*]) = \delta_{\varepsilon_n,\varepsilon_{n+1}}\sqrt{\frac{\mu([\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*])}{\mu([\ell_1(\ell_2)^*])}}\frac{\sqrt{\lambda(s(\varepsilon_n))}}{\sqrt{\lambda(t(\varepsilon_n))}}[\ell_1(\ell_2)^*].$$

Finally observe that the first vertex of $\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*$ and $\ell_1(\ell_2)^*$ are the same, whereas their middle vertices are $s(\varepsilon_n)$ and $t(\varepsilon_n)$, respectively. Hence we have

$$\alpha_n([\ell_1\varepsilon_n^*\varepsilon_{n+1}(\ell_2)^*]) = \delta_{\varepsilon_n,\varepsilon_{n+1}}\frac{\lambda(s(\varepsilon_n))}{\lambda(t(\varepsilon_n))}[\ell_1(\ell_2)^*].$$

Note that the formula above is very similar to the one we obtained in 3.17. In fact, we have

$$\alpha_n = (\phi_{n-1,+})^{-1} \circ (dE_{B_{n-1}}) \circ \phi_{n,+}$$

where $\phi_{n,+}$ is the isomorphism between $G_{n,+}$ and $B_0' \cap B_n$ described in Corollary 3.23.

In a similar manner, we compute the actions of other generating tangles on $G_{n,\pm}$:

1. m_n^+ will coincide with the multiplication in $G_{n,+}$,
2. $\beta_n = (\phi_{n-1,-})^{-1} \circ (dE_{B_1'}) \circ (\phi_{n,+})$, where $E_{B_1'}$ is the conditional expectation from $B_0' \cap B_n$ to $B_1' \cap B_n$ shown in 3.26,
3. $\iota_n^+ = (\phi_{n+1,+})^{-1} \circ \iota \circ \phi_{n,+}$ where ι is the inclusion of B_n into B_{n+1},
4. $\iota_n^- = (\phi_{n+1,+})^{-1} \circ \iota \circ \phi_{n,-}$ where ι is the inclusion of $B_1' \cap B_{n+1}$ into $B_0' \cap B_n$.
5. $E_n = (\phi_{n+1,+})^{-1}(F_n)$.

We deduce that the planar algebra structure on $\{G_{n,\pm}\}$ extends the *-algebra structure we defined on Chapter 3.

CHAPTER 5

The embedding theorem

Consider a symmetric commuting square of finite dimensional C*-algebras as in 2.2:

$$\begin{array}{ccc} A_{1,0} & \subset & A_{1,1} \\ \cup & & \cup \\ A_{0,0} & \subset & A_{0,1} \end{array} \tag{5.1}$$

Iterating the basic construction vertically and horizontally, we obtain the following lattice of commuting squares:

$$\begin{array}{ccccccccc} A_{\infty,0} & \subset & A_{\infty,1} & \subset & A_{\infty,2} & \subset & \cdots & \subset & A_{\infty,\infty} \\ \cup & & \cup & & \cup & & & & \cup \\ \vdots & & \vdots & & \vdots & & & & \vdots \\ \cup & & \cup & & \cup & & & & \cup \\ A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset & \cdots & \subset & A_{2,\infty} \\ \cup & & \cup & & \cup & & & & \\ A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & A_{1,\infty} \\ \cup & & \cup & & \cup & & & & \cup \\ A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & A_{0,\infty} \end{array}$$

where $A_{n,\infty} = \left(\bigcup_k A_{n,k}\right)''$ and $A_{\infty,k} = \left(\bigcup_n A_{n,k}\right)''$ as outlined in Section 2.1.

In this section we will show that the subfactor planar algebra associated to $A_{0,\infty} \subset A_{1,\infty}$ embeds into the graph planar algebra associated to the Bratteli diagram Γ of the first vertical inclusion $A_{0,0} \subset A_{1,0}$. There are three important ideas used to prove this fact. The first is that the relative commutants from the first vertical tower are isomorphic to the loop algebras associated to Γ as shown in Section 3.3. The second idea is to use Ocneanu compactness (Theorem 2.13)

$$A'_{0,\infty} \cap A_{n,\infty} = A'_{0,1} \cap A_{n,0} \subset A'_{0,0} \cap A_{n,0}, \text{ for all } n \geq 0.$$

Consequently, the subfactor planar algebra's vector spaces are included in the graph planar algebra's vector spaces. The final idea is to use explicit computations for the actions of generating tangles for the graph planar algebra, like the ones in example 4.30, to verify that these coincide with the action arising from the subfactor planar algebra.

Let $P_\bullet = P^{A_{0,\infty} \subset A_{1,\infty}}$ be the subfactor planar algebra associated to $A_{0,\infty} \subset A_{1,\infty}$ and $P_{n,\pm}$, $n \geq 0$, be the corresponding vector spaces. Thus $P_{n,+} = A'_{0,\infty} \cap A_{n,\infty}$ and $P_{n,-} = A'_{1,\infty} \cap A_{n+1,\infty}$. Moreover, the loop parameter for $P_\bullet$ is $\sqrt{[A_{1,\infty} : A_{0,\infty}]}$.

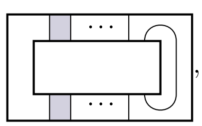
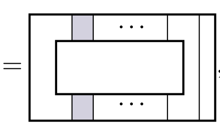
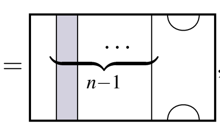
Now, consider $G_\bullet$ to be the graph planar algebra associated to Γ. Notice that the loop parameter is equal to $\sqrt{\|\Lambda\Lambda^t\|}$, where Λ is the inclusion matrix for $A_{0,0} \subset A_{1,0}$. Since our commuting square is symmetric, we have

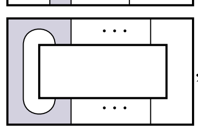
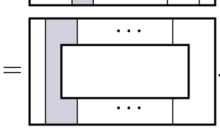
$$d^2 = \|\Lambda\Lambda^t\| = \|\Lambda\|^2 = [A_{1,\infty} : A_{0,\infty}],$$

and therefore both planar algebras have the same loop parameter d. By Ocneanu compactness, we have

$$A'_{0,\infty} \cap A_{n,\infty} \subset A'_{0,0} \cap A_{n,0} = Q_{n,+}, \quad A'_{1,\infty} \cap A_{n+1,\infty} \subset A'_{1,0} \cap A_{n+1,0} = Q_{n,-}.$$

Thus, the isomorphisms $\psi_{n,\pm} : Q_{n,\pm} \to G_{n,\pm}$ from Remark 3.24 define a map $\psi : P_\bullet \to G_\bullet$. We will show that ψ is a planar algebra embedding. Since $\psi_{n,\pm}$ is a *-algebra homomorphism for every n, we only need to verify that it preserves the actions of the following generating tangles:

$\alpha_n =$ , $\iota_n^+ =$ , $E_n =$ 
,

$\beta_n =$ , $\iota_n^- =$ .

In the case of the subfactor planar algebra $P_\bullet$, as stated in section 3.1, we have:

(a) $\alpha_n : P_{n,+} \to P_{n-1,+}$ is given by the trace-preserving conditional expectation from $A_{n,\infty}$ to $A_{n-1,\infty}$.

(b) $\beta_n : P_{n,+} \to P_{n-1,-}$ is given by the trace-preserving conditional expectation from $A'_{0,\infty}$ to $A'_{1,\infty}$.

(c) $\iota_n^+ : P_{n,+} \to P_{n+1,+}$ is given by the inclusion map from $A_{n,\infty}$ to $A_{n+1,\infty}$.

(d) $\iota_n^- : P_{n,-} \to P_{n+1,+}$ is given by the inclusion map from $A'_{1,\infty}$ to $A'_{0,\infty}$.

(e) E_n is given by the Jones projection in $A_{n+1,\infty}$ multiplied by the loop parameter d, that is $A_{n+1,\infty} = \langle A_{n,\infty}, E_n \rangle$

From example 4.30, for the graph planar algebra $G_\bullet$ we have:

(a) $\alpha_n = (\phi_{n-1,+})^{-1} \circ (dE_{B_{n-1}}) \circ \phi_{n,+}$,

(b) $\beta_n = (\phi_{n-1,-})^{-1} \circ (dE_{B_1'}) \circ (\phi_{n,+})$,

(c) $\iota_n^+ = (\phi_{n+1,+})^{-1} \circ \iota \circ \phi_{n,+}$ where ι is the inclusion of B_n into B_{n+1},

(d) $\iota_n^- = (\phi_{n+1,+})^{-1} \circ \iota \circ \phi_{n,-}$ where ι is the inclusion of $B_1' \cap B_{n+1}$ into $B_0' \cap B_n$.

(e) $E_n = (\phi_{n+1,+})^{-1}(F_n)$ where F_n is as in definition 3.19.

Here $(B_n)_{n\geq 0}$ is the loop algebra associated to the first vertical inclusion as in definition 3.7. We need to verify that all of these maps coincide when restricted to $P_\bullet$.

It is clear that the inclusion map from $A_{n,\infty}$ to $A_{n+1,\infty}$ coincides with the inclusion map from $A_{n,0}$ to $A_{n+1,0}$ when restricted. From proposition 3.20, we get that if ι is the inclusion from B_n into B_{n+1} then $\varphi_{n+1} \circ \iota \circ (\varphi_n)^{-1}$ is the inclusion from $A_{n,0}$ into $A_{n+1,0}$ and therefore $\varphi_{n+1} \circ \iota \circ (\varphi_n)^{-1} = \iota_n^+ : P_{n,+} \to P_{n+1,+}$. From this we have

$$\begin{aligned}
\iota_n^+ \circ \psi_{n,+} &= ((\phi_{n+1,+})^{-1} \circ \iota \circ \phi_{n,+}) \circ (\phi_{n,+})^{-1} \circ (\varphi_n)^{-1} \\
&= (\phi_{n+1,+})^{-1} \circ \iota \circ (\varphi_n)^{-1} \\
&= (\phi_{n+1,+})^{-1} \circ (\varphi_{n+1})^{-1} \circ \iota_n^+ \\
&= \psi_{n+1,+} \circ \iota_n^+
\end{aligned}$$

which means ι_n^+ acts the same using the action defined via the subfactor planar algebra and using the action defined via the graph planar algebra. A similar computation shows that the actions of i_n^- coincide.

The commuting square relation implies $E_{A_{n-1,0}}(x) = E_{A_{n-1,\infty}}(x)$ for $x \in A_{n,0}$. Since the maps φ_n are trace-preserving, we have $(\varphi_{n-1})^{-1} \circ E_{A_{n-1,0}} = E_{B_{n-1}} \circ (\varphi_n)^{-1}$. This implies that for $x \in A_{0,\infty}' \cap A_{n,\infty} \subset A_{n,0}$, we have

$$\begin{aligned}
(\alpha_n \circ \psi_{n,+})(x) &= (((\phi_{n-1,+})^{-1} \circ (dE_{B_{n-1}}) \circ \phi_{n,+}) \circ (\phi_{n,+})^{-1} \circ (\varphi_n)^{-1})(x) \\
&= ((\phi_{n-1,+})^{-1} \circ (dE_{B_{n-1}}) \circ (\varphi_n)^{-1})(x) \\
&= ((\phi_{n-1,+})^{-1} \circ (\varphi_{n-1})^{-1} \circ dE_{A_{n-1,0}})(x) \\
&= ((\phi_{n-1,+})^{-1} \circ (\varphi_{n-1})^{-1} \circ dE_{A_{n-1,\infty}})(x) \\
&= (\psi_{n-1,+} \circ \alpha_n)(x)
\end{aligned}$$

and therefore the actions of α_n coincide. Since the Jones projection that implements the conditional expectation from $A_{n,\infty}$ to $A_{n-1,\infty}$ is the same as the Jones projection that implements the conditional expectation from $A_{n,0}$ to $A_{n-1,0}$, a similar computation shows that the actions by E_n also coincide.

Since the commuting square is symmetric, due to corollary 2.10, we obtain a Pimsner-Popa basis S that works for the inclusions $A_{0,0} \subset A_{1,0}$ and $A_{0,\infty} \subset A_{1,\infty}$. Using the formula from proposition 2.12, for any $x \in A_{n,\infty}$ we have

$$E_{A'_{1,\infty}}(x) = \sum_{s \in S} sxs^* = E_{A'_{1,0}}(x).$$

Observe that, since $(\varphi_1)^{-1}(S)$ is also a Pimnser-Popa basis for $B_0 \subset B_1$, for any $x \in A_{n,0}$ we have

$$\begin{aligned} E_{B'_1}((\varphi_n)^{-1}(x)) &= \sum_{s \in S} (\varphi_1)^{-1}(s)(\varphi_n)^{-1}(x)(\varphi_1)^{-1}(s^*) \\ &= (\varphi_n)^{-1}\left(\sum_{s \in S} sxs^*\right) = (\varphi_{n-1})^{-1}\left(E_{A'_{1,0}}(x)\right). \end{aligned}$$

Therefore, in $P_{n,+}$ we have

$$\begin{aligned} \beta_n \circ \psi_{n,+} &= \left((\phi_{n-1,-})^{-1} \circ (dE_{B'_1}) \circ \phi_{n,+}\right) \circ (\phi_{n,+})^{-1} \circ (\varphi_n)^{-1} \\ &= (\phi_{n-1,-})^{-1} \circ (dE_{B'_1}) \circ (\varphi_n)^{-1} \\ &= (\phi_{n-1,-})^{-1} \circ (\varphi_{n-1})^{-1} \circ dE_{A'_{1,0}} \\ &= (\psi_{n-1,-}) \circ dE_{A'_{1,\infty}} = (\psi_{n-1,-}) \circ \beta_n. \end{aligned}$$

From this, we conclude that the action by β_n on $P_{n,+}$ is the same for the subfactor planar algebra structure or the graph planar algebra structure. Thus we have proved:

Theorem 5.1. *Consider commuting square as in (5.1). Let $P_\bullet$ be the subfactor planar algebra associated to $A_{0,\infty} \subset A_{1,\infty}$ and $G_\bullet$ the graph planar algebra associated to the Bratteli diagram of $A_{0,0} \subset A_{1,0}$. The map $\psi : P_\bullet \to G_\bullet$ defined above is a planar algebra embedding.*

CHAPTER 6

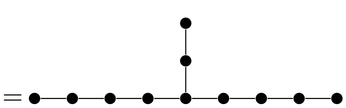
Module graphs and embeddings

Our main objective is to use the embedding theorem 5.1 to construct irreducible hyperfinite subfactors with trivial standard invariant for any index that is possible. In [Pop93], it is shown that for any $\lambda > 4$, one can construct irreducible non-hyperfinite subfactors with that index and trivial standard invariant. It is a major open problem in the theory of subfactors to determine which numbers > 4 are indices of irreducible hyperfinite subfactors [Jon83]. Hyperfinite subfactors constructed from commuting squares have Jones index of the form $\|\Gamma\|^2$ where Γ is a graph. From the classification of small index subfactors (see [JMS14] or [AMP23]) we know that there are five $\lambda \in (4,5)$ for which there exist irreducible hyperfinite finite depth subfactors with index λ. More precisely, we list the subfactors in the following table:

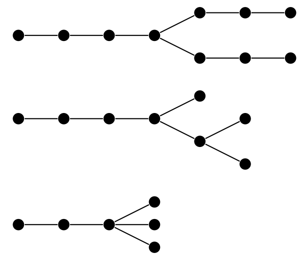
Index	Subfactors
$\frac{1}{2}(5+\sqrt{13})$	Haagerup subfactor and its dual
≈ 4.37720	Extended Haagerup subfactor and its dual
$\frac{1}{2}(5+\sqrt{17})$	Asaeda-Haagerup subfactor and its dual
$3+\sqrt{3}$	3311 subfactor and its dual
$\frac{1}{2}(5+\sqrt{21})$	2221 subfactor and its complex conjugate

By classification, we also know that all irreducible infinite depth subfactors with index in $(4, 3+\sqrt{5})$ have trivial standard invariant, but not much else is known about these infinite depth subfactors.

An irreducible hyperfinite subfactor $N \subset M$ with index $\frac{5+\sqrt{13}}{2}$ was constructed using a commuting square by Schou [Sch90, Chapter 7], which in principle could have finite depth. The Bratteli diagram for the first vertical inclusion is:

$\Gamma =$ 

The embedding theorem 5.1 implies that the subfactor planar algebra $P^{N\subset M}$ must embed into the graph planar algebra of Γ. On the other hand, Peters shows that the Haagerup subfactor planar algebra does not embed in the graph planar algebra of Γ [Pet10, Theorem 6.8]. In fact, corollary 1.4 in [GMP$^+$23] states that the Haagerup subfactor planar algebra only embeds in the graph planar algebra associated to the following graphs:



Note that the first two graphs are the principal and dual principal graphs of the Haagerup subfactor. This allows us to conclude that $N \subset M$ is not the Haagerup subfactor and therefore, by the classification of small index subfactor planar algebras, it must be an A_∞-subfactor, i.e. it must have trivial standard invariant.

This is not the only index where this technique can be applied. From theorem 1.3 in [GMP$^+$23] we know that the Extended Haagerup subfactor embeds in the graph planar algebras of the following graphs:

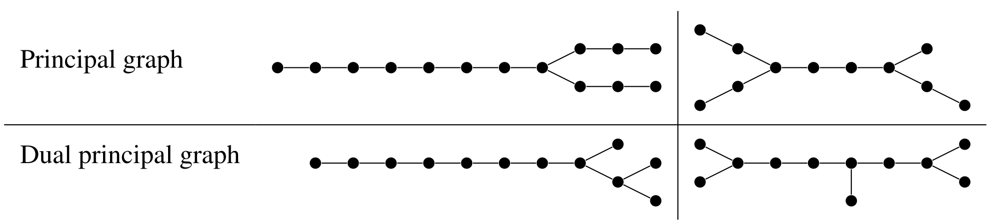


Table 6.1: Extended Haagerup module graphs

More generally, theorem 1.2 in [GMP$^+$23] states the following:

Theorem 6.1. *Suppose $P_\bullet$ is a finite depth subfactor planar algebra. Let $\mathscr{C}$ denote the unitary multifusion category of projections in $P_\bullet$, with distinguished object $X = \mathrm{id}_{1,+} \in P_{1,+}$, and the standard unitary pivotal structure with respect to X. There is an equivalence between:*

1. *Planar algebra embeddings $P_\bullet \to G_\bullet$, where $G_\bullet$ is the graph planar algebra associated to a finite connected bipartite graph Γ, and*
2. *indecomposable finitely semisimple pivotal left $\mathscr{C}$-module C^* categories $\mathscr{M}$ whose fusion graph with respect to X is Γ.*

We will refer to the fusion graphs of $\mathscr{M}$ with respect to X as *module graphs*. This means that if we can find a subfactor whose subfactor planar algebra embeds into the graph planar algebra of a graph that is not a module graph for any finite depth subfactor with the same index, then this subfactor has to have infinite depth. We use this idea to construct irreducible hyperfinite subfactors with trivial standard invariant and indices ~ 4.37720

and $\frac{5+\sqrt{17}}{2}$. To this end, we construct commuting squares whose Bratteli diagram of the first vertical inclusion has norm ~ 4.37720 (respectively $\frac{5+\sqrt{17}}{2}$) and is not one of the module graphs for the Extended Haagerup (respectively Asaeda-Haagerup) subfactor. In the case of the Asaeda-Haagerup subfactor we use the results from [GIS18] and [GS16] together with the techniques outlined in Section 4 of [GMP$^+$23] to compute a finite list of all potential module graphs.

6.1 Combinatorics of potential module graphs

Consider $\mathscr{C}$ a fusion category and $\{\mathscr{C}_i\}_i$ its Morita equivalence class. We want to determine all possible left $\mathscr{C}$-module C^*-categories $\mathscr{M}$ for a fixed unitary multifusion category $\mathscr{C}$ coming from a subfactor. In our case, $\mathscr{C}$ will be such that $1 = 1_+ \oplus 1_-$ and consequently we can write

$$\mathscr{C} = \begin{pmatrix} \mathscr{C}_{11} & \mathscr{C}_{12} \\ \mathscr{C}_{21} & \mathscr{C}_{22} \end{pmatrix}$$

where $\mathscr{C}_{ij} = 1_\pm \otimes \mathscr{C} \otimes 1_\pm$. In fact, for $\mathscr{C}$ coming from an irreducible finite depth subfactor $\mathscr{C}_{12}$ will be a Morita equivalence between the fusion categories $\mathscr{C}_{11}$ and $\mathscr{C}_{22}$ (see example 2.17). Now, if $\mathscr{M}$ is an indecomposable left $\mathscr{C}$-module category then $\mathscr{M}_1 := 1_+ \otimes \mathscr{M}$ and $\mathscr{M}_2 := 1_- \otimes \mathscr{M}$ will be indecomposable left $\mathscr{C}_{11}$-module and $\mathscr{C}_{22}$-module categories respectively. Theorem 3.3 in [GMP$^+$23] states the following

Theorem 6.2 ([ENO10, EGNO15])**.** *If $\mathscr{A}$ is a fusion category and $\mathscr{N}$ is a semisimple $\mathscr{A}$-module category, then $\mathscr{A} - \mathscr{B}$ bimodule category structures on $\mathscr{N}$ which extend the $\mathscr{A}$-module structure correspond exactly to functors $\mathscr{F} : \mathscr{A} \to \mathrm{End}_{\mathscr{A}}(\mathscr{N})$, and such a bimodule is a Morita equivalence if and only if $\mathscr{F}$ is an equivalence of multitensor categories. Two such bimodule categories are equivalent if and only if the functors differ by an inner autoequivalence. Furthermore,* $\mathrm{End}_{\mathscr{A}}(\mathscr{N})$ *is a tensor category (with simple unit object) if and only if $\mathscr{N}$ is indecomposable.*

This implies there is a bijection between indecomposable left $\mathscr{A}$-modules and Morita equivalences between $\mathscr{A}$ and fusion categories $\mathscr{B}$. Therefore, to determine all possible indecomposable left $\mathscr{C}$-module categories $\mathscr{M}$, it is enough to understand the Morita equivalence class of $\mathscr{C}_{11}$ (or $\mathscr{C}_{22}$). This information is captured by the Brauer-Picard groupoid (see definition 4.5 in [ENO10]).

For the Extended Haagerup fusion categories, the Brauer-Picard groupoid is computed in [GMP$^+$23]. The combinatorial information for all these Morita equivalences is then used to compute the four module graphs, depicted in Table 6.1, associated to the Extended Haagerup subfactor.

6.2 Asaeda-Haagerup module graphs

Let $\mathscr{A}\mathscr{H}_1$ and $\mathscr{A}\mathscr{H}_2$ be fusion categories corresponding to the even and odd part of the Asaeda-Haagerup subfactor. In [GS16], the authors identify a fusion category $\mathscr{A}\mathscr{H}_3$ which is also in the Morita equivalence class of $\mathscr{A}\mathscr{H}_1$ and $\mathscr{A}\mathscr{H}_3$. Moreover, they determine all the Morita equivalences between them. Denote the fusion rings of $\mathscr{A}\mathscr{H}_i$ by AH_i.

Theorem 6.3 ([GS16])**.** *There are exactly* 4 *invertible bimodule categories over each not-necessarily-distinct pair $\mathscr{A}\mathscr{H}_i$-$\mathscr{A}\mathscr{H}_j$, up to equivalence. These realize the following fusion bimodules, which are each realized uniquely:*

- $10_{11}, 12_{11}, 13_{11}, 14_{11}, 2_{12}, 5_{12}, 8_{12}, 9_{12}, 2_{13}, 3_{13}, 6_{13}, 7_{13},$
- $2_{21}, 5_{21}, 8_{21}, 9_{21}, 8_{22}, 11_{22}, 12_{22}, 13_{22}, 1_{23}, 3_{23}, 4_{23}, 6_{23},$
- $2_{31}, 3_{31}, 6_{31}, 7_{31}, 1_{32}, 2_{32}, 4_{32}, 6_{32}, 8_{33}, 11_{33}, 12_{33}, 13_{33}.$

Here a_{ij} denotes the a^{th} fusion bimodule on a list of potential AH_i-AH_j fusion bimodules. Using this notation, 9_{12} is the fusion bimodule realized by the Asaeda-Haagerup subfactor. It is generated by a single element X which corresponds to $\mathrm{id}_{1,+} \in P_{1,+}$ in the subfactor planar algebra.

Definition 6.4 ([GS16])**.** A *multiplication map* on a triple of fusion bimodules $({}_AK_B, {}_BL_C, {}_AM_C)$ is a homomorphism from ${}_AK \otimes_B L_C$ to ${}_AM_C$ which takes tensor products of basis elements in K and L to non-negative combinations of basis elements of M and preserves dimension and multiplication by duals. The triple $({}_AK_B, {}_BL_C, {}_AM_C)$ is said to be *multiplicatively compatible* if there exists such a multiplication map.

The authors, in a supplementary file, provide a list of multiplicative compatible bimodules, in particular:

$$9_{12} \cdot 2_{21} = 10_{11}, \qquad 9_{12} \cdot 5_{21} = 13_{11}, \qquad 9_{12} \cdot 8_{21} = 12_{11}, \qquad 9_{12} \cdot 9_{21} = 14_{11}$$
$$9_{12} \cdot 8_{22} = 2_{12}, \qquad 9_{12} \cdot 11_{22} = 5_{12}, \qquad 9_{12} \cdot 12_{22} = 8_{12}, \qquad 9_{12} \cdot 13_{22} = 9_{12}$$
$$9_{12} \cdot 1_{23} = 2_{13}, \qquad 9_{12} \cdot 3_{23} = 6_{13}, \qquad 9_{12} \cdot 4_{23} = 3_{13}, \qquad 9_{12} \cdot 6_{23} = 7_{13}.$$

To determine all the module graphs with respect to X we need to determine the multiplication maps for the triples above.

Let $({}_AK_B, {}_BL_C, {}_AM_C)$ be a triple of fusion bimodules, with bases ξ_i for $1 \le i \le l$, η_j for $1 \le j \le m$, and μ_k for $1 \le k \le n$, respectively. We will consider the lexicographic order for ordered pairs (p,q), $1 \le p \le l$, $1 \le q \le m$ and denote by $(p,q)'$ its successor in this order.

Definition 6.5. A (p,q)-partial multiplication map for $(p,q) \le (l,m)$ is an assignment of a vector of integers of length n, v^{ij}, for each pair $(i,j) \leq (p,q)$ such that

(a) $d(\xi_i)d(\eta_j) = \sum_{k=1}^{n} v_k^{ij} d(\mu_k)$,

(b) $(\bar{\xi_{i_1}}\xi_{i_2}, \eta_{j_1}\bar{\eta_{j_2}}) = \sum_{k=1}^{n} v_k^{i_1 j_1} v_k^{i_2 j_2}$,

(c) for all basis elements $\lambda \in A$ and $(i_1, j_1), (i_2, j_2) \leq (p,q)$ we have

$$((\xi_{i_1}(\eta_{j_1}\bar{\eta}_{j_2})\bar{\bar{\xi}}_{i_2}), \lambda) = \sum v_{k_1}^{i_1 j_1} v_{k_2}^{i_2 j_2} (\mu_{k_1}\bar{\mu}_{k_2}, \lambda),$$

(c') for all basis elements $\kappa \in C$ and $(i_1, j_1), (i_2, j_2) \leq (p,q)$ we have

$$((\bar{\bar{\xi}}_{i_1}(\bar{\eta}_{j_1}\eta_{j_2})\xi_{i_2}), \lambda) = \sum v_{k_1}^{i_1 j_1} v_{k_2}^{i_2 j_2} (\bar{\mu}_{k_1}\mu_{k_2}, \kappa),$$

(d) for all $(i_1, j_1), (i_2, j_2) \leq (p,q)$ we have

$$v^{i_1 j_1} \cdot v^{i_2 j_2} = (\bar{\bar{\xi}}_{i_1}\xi_{i_2}, \eta_{j_2}\bar{\eta}_{j_1}).$$

We define a $(0,0)$-partial multiplication map to be the empty map, and let $(0,0)' = (1,1)$. Using Algorithm 5.2 in [GS16] we inductively build all (p,q)-partial multiplication maps for each triple, in particular, the (l,m)-partial multiplication map will be the multiplication map for the triple.

Algorithm 6.6 ([GS16])**.** *Step 1: Start with the $(0,0)$-partial multiplication map. Then inductively find all extensions of a given (p,q)-partial multiplication map, $(p,q) < (l,m)$, to a $(p,q)'$-partial multiplication map as follows:*

1a) Let $(p,q)' = (p',q')$. Find candidates for $v^{p'q'}$ by checking conditions (a) and (b) above as follows: for each sum of squares decomposition (a_i, b_i) of $(\bar{\bar{\xi}}_{p'}\xi_{p'}, \eta_{q'}\bar{\eta}_{q'})$ such that $\sum b_i \leq n$, form the vector v of size n given by b_i copies of each a_i with the rest of the entries equal to 0. Then find all distinct vectors v' which arise as permutations of v.

1b) For each candidate v' for $v^{p'q'}$ found in (1a), check whether $v' \cdot d_M = d(\xi_i)d(\eta_j)$, where d_M is the dimension vector of the bimodule M. Finally, if the dimension condition is satisfied for v', check conditions (c), (c'), and (d) for $(i_1, j_1) = (p',q')$ and all $(i_2, j_2) \leq (p',q')$, using $v^{p'q'} = v'$.

Step 2: For each (l,m)-partial multiplication map found in Step 1, check whether

$$(\xi\rho)\eta = \xi(\rho\eta), \qquad \lambda(\xi\eta) = (\lambda\xi)\eta, \qquad (\xi\eta)\kappa = \xi(\eta\kappa),$$

for all basis vectors

$$\lambda \in A,\ \rho \in B,\ \kappa \in C,\ \xi \in K,\ \eta \in L,$$

where multiplication between elements of K and L is defined on basis elements by the partial multiplication map and extended biadditively.

Using the combinatorial data for the fusion modules and the algorithm above we compute the fusion graph of each fusion module with respect to X:

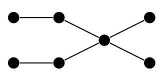
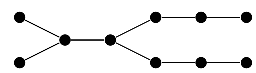
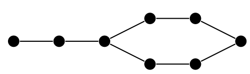
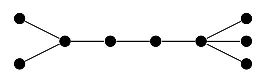
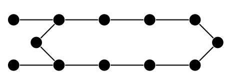
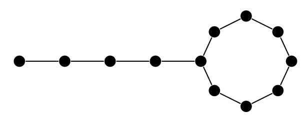
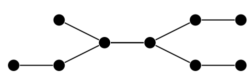
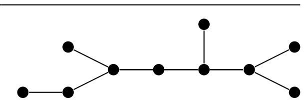
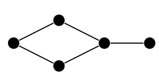
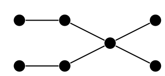
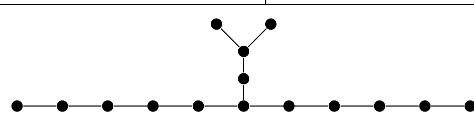
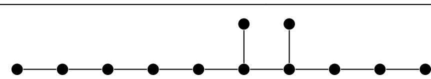
$9_{12} \cdot 8_{22} = 2_{12}$		$9_{12} \cdot 3_{23} = 6_{13}$	
$9_{12} \cdot 11_{22} = 5_{12}$		$9_{12} \cdot 4_{23} = 3_{13}$	
$9_{12} \cdot 12_{22} = 8_{12}$		$9_{12} \cdot 6_{23} = 7_{13}$	
$9_{12} \cdot 5_{21} = 13_{11}$		$9_{12} \cdot 8_{21} = 12_{11}$	
$9_{12} \cdot 1_{23} = 2_{13}$		$9_{12} \cdot 2_{21} = 10_{11}$	
$9_{12} \cdot 13_{22} = 9_{12}$		Principal graph	
$9_{12} \cdot 9_{21} = 14_{11}$		Dual Principal graph	

Table 6.2: Asaeda-Haagerup module graphs

Unlike the Extended-Haagerup case, there might be other fusion categories $\mathscr{C}$ in the Morita equivalence class of $\mathscr{AH}_i$. Consequently, we cannot give a full description of the Brauer-Picard groupoid for the Asaeda-Haagerup fusion category. However, in [GS16] the authors prove the following

Theorem 6.7. *Let $\mathscr{C}$ be a fusion category which is Morita equivalent to the $\mathscr{AH}_i$, $i = 1,2,3$, but not isomorphic as fusion categories to any of them. Then exactly one of the following four cases holds:*

(a) Every $\mathscr{A}\mathscr{H}_1$*-*$\mathscr{C}$ *Morita equivalence realizes* 9_1*, every* $\mathscr{A}\mathscr{H}_2$*-*$\mathscr{C}$ *Morita equivalence realizes* 19_2*, and every* $\mathscr{A}\mathscr{H}_3$*-*$\mathscr{C}$ *Morita equivalence realizes* 16_3.

(b) Every $\mathscr{A}\mathscr{H}_1$*-*$\mathscr{C}$ *Morita equivalence realizes* 16_1*, every* $\mathscr{A}\mathscr{H}_2$*-*$\mathscr{C}$ *Morita equivalence realizes* 4_2*, and every* $\mathscr{A}\mathscr{H}_3$*-*$\mathscr{C}$ *Morita equivalence realizes* 18_3.

(c) Every $\mathscr{A}\mathscr{H}_1$*-*$\mathscr{C}$ *Morita equivalence realizes* 21_1*, every* $\mathscr{A}\mathscr{H}_2$*-*$\mathscr{C}$ *Morita equivalence realizes* 17_2*, and every* $\mathscr{A}\mathscr{H}_3$*-*$\mathscr{C}$ *Morita equivalence realizes* 2_3.

(d) Every $\mathscr{A}\mathscr{H}_1$*-*$\mathscr{C}$ *Morita equivalence realizes* 1_1*, every* $\mathscr{A}\mathscr{H}_2$*-*$\mathscr{C}$ *Morita equivalence realizes* 3_2*, and every* $\mathscr{A}\mathscr{H}_3$*-*$\mathscr{C}$ *Morita equivalence realizes* 2_3.

Here a_i denotes the a^{th} fusion bimodule on a list of potential AH_i fusion bimodules. Verifying the compatibility of these modules with the bimodule that realizes 9_{12} we get

$$9_{12} \cdot 19_2 = 9_1 \qquad 9_{12} \cdot 4_2 = 16_1 \qquad 9_{12} \cdot 17_2 = 21_1 \qquad 9_{12} \cdot 3_2 = 1_1$$

Using Algorithm 6.6 without checking for condition (1c') in step 1b) we recover the multiplication maps for the above modules. We then obtain the following fusion graphs with respect to X:

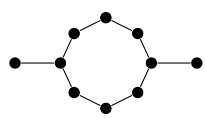
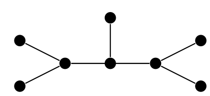
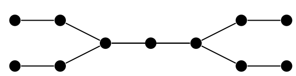
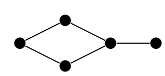
$9_{12} \cdot 19_2 = 9_1$		$9_{12} \cdot 4_2 = 16_1$	
$9_12 \cdot 17_2 = 21_1$		$9_{12} \cdot 3_2 = 1_1$	

Table 6.3: Asaeda-Haagerup potential module graphs

Thus, there are a total of 14 *different* potential module graphs in Table 6.2 and Table 6.3 that are associated to the Asaeda-Haagerup subfactor.

CHAPTER 7

New commuting squares

We construct non-degenerate commuting squares of the form

$$\begin{array}{ccc} A_{1,0} & \overset{\text{K}}{\subset} & A_{1,1} \\ \cup_G & & \cup_G \\ A_{0,0} & \overset{\text{H}}{\subset} & A_{0,1} \end{array} \tag{7.1}$$

where G is one of the following four graphs:

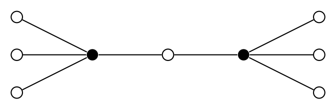


Figure 7.1: Small double broom

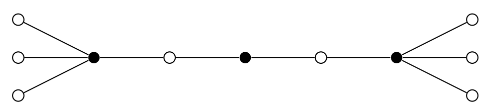


Figure 7.2: Medium double broom

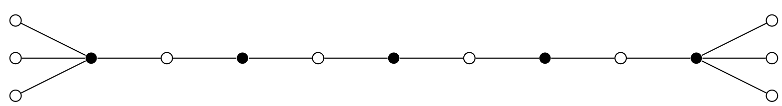


Figure 7.3: Large double broom

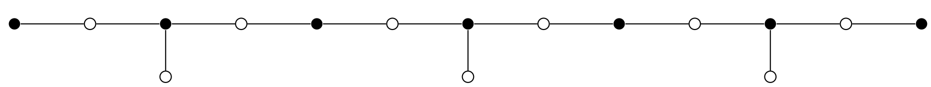


Figure 7.4: Quipu

The graphs have norms $\sqrt{5}$, $\sqrt{3+\sqrt{3}}$, $\sqrt{\frac{5+\sqrt{17}}{2}}$ and $\sqrt{4.37720\ldots}$ respectively. We will follow the set up in Chapter 8 of [Sch90] to present the bi-unitary connections that yield the commuting squares.

Let $\mathscr{Z}_{0,0}$ (respective $\mathscr{Z}_{0,1}$, $\mathscr{Z}_{1,0}$ and $\mathscr{Z}_{1,1}$) denote the set of minimal central projections in $A_{0,0}$ (respectively $A_{0,1}, A_{1,0}$ and $A_{1,1}$). Using the notation from (3.1), the elements of $\mathscr{Z}_{0,0}$ and $\mathscr{Z}_{0,1}$ are labelled by $\mathscr{V}_+$ (denoted by black vertices) and the elements of $\mathscr{Z}_{1,0}$ and $\mathscr{Z}_{1,1}$ are labelled by $\mathscr{V}_-$ (denoted by white vertices).

It is shown in [Sch90, Chapter 1] that the existence of a commuting square of finite dimensional C*-algebras of the form 7.1 is equivalent to constructing a unitary matrix u, satisfying the so-called bi-unitary condition.

u is a block-diagonal matrix of the form

$$u = \bigoplus_{(p,s)} u^{(p,s)}$$

where the labels of the blocks (p,s) runs over all $p \in \mathscr{Z}_{0,0}$ and all $s \in \mathscr{Z}_{1,1}$ that are connected by a path on H and G through $\mathscr{Z}_{0,1}$ (respectively, a path on G and K through $\mathscr{Z}_{1,0}$). Each direct summand $u^{(p,s)}$ is a $n(p,s) \times n(p,s)$-matrix, where $n(p,s)$ is the number of paths on H and G from p to s through $\mathscr{Z}_{0,1}$ (respectively, the number of paths on G and K through $\mathscr{Z}_{1,0}$), so each block $u^{(p,s)}$ is a matrix indexed as follows

$$u^{(p,s)} = \left(u^{(p,s)}_{q,r}\right)_{q,r},$$

where (q,r) runs over all possible $q \in \mathscr{Z}_{1,0}$ and $r \in \mathscr{Z}_{0,1}$, that a path from p to s can go through. Since the vertical inclusions, $A_{0,0} \subset A_{1,0}$ and $A_{0,1} \subset A_{1,1}$, do not have multiple edges, each $u^{(p,s)}_{q,r}$ is a $m \times n$-matrix, where m is the multiplicity of the edge qs in the inclusion $A_{1,0} \subset A_{1,1}$ and n is the multiplicity of the edge pr in the inclusion $A_{0,0} \subset A_{0,1}$. We may assume $A_{0,0}$ is Abelian, i.e. of the form $A_{0,0} = \mathbb{C}^{|\mathscr{Z}_{0,0}|}$.

Let $\lambda(\cdot)$ (respectively $\eta(\cdot)$) denote a fixed Perron-Frobenius vector for the inclusion graph of $A_{0,0} \subset A_{1,0}$ (respectively $A_{0,1} \subset A_{1,1}$). Set

$$w(p,q,r,s) = \sqrt{\frac{\lambda(p)\eta(s)}{\lambda(q)\eta(r)}}. \tag{7.2}$$

Define a matrix v by

$$v = \bigoplus_{(q,r)} v^{(q,r)}.$$

$v^{(q,r)}$ is a square matrix, which can be written as a block matrix $v^{(q,r)} = \left(v^{(q,r)}_{p,s}\right)_{p,s}$ with each block given by

$$v^{(q,r)}_{p,s} = w(p,q,r,s)(u^{(p,s)}_{q,r})^*, \tag{7.3}$$

where (p,q,r,s) runs through all quadruples in $\mathscr{Z}_{0,0} \times \mathscr{Z}_{1,0} \times \mathscr{Z}_{0,1} \times \mathscr{Z}_{1,1}$, which can be completed to a cycle $p-r-s-q-p$. Here X^* denotes the conjugate transpose of the matrix X.

The existence of a commuting square as in 7.1 is then equivalent to the existence of u and v as above such that both are unitaries (this is referred to as the *bi-unitary condition*). In this case, we say u and v are a *bi-unitary connection* for the inclusions 7.1.

Remark 7.1. In our case, both vertical inclusions are described by the same graph G hence we can use $\lambda(\cdot)$ to denote a Perron-Frobenius vector of the inclusion graph for $A_{0,0} \subset A_{1,0}$ (or $A_{0,1} \subset A_{1,1}$).

Surprisingly, we only needed to consider real unitaries u and v to determine the bi-unitary connections for the commuting squares based on the double brooms.

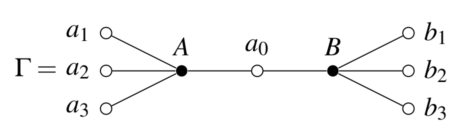
7.1 Horizontal inclusions

In [Sch90] a large family of commuting squares is constructed where $H = GG^t - I$ and $K = G^tG - I$. Ocneanu constructed a commuting square of the form (7.1) where $G = E_{10}$ and the horizontal inclusions are of the form $H = p(GG^t)$ and $K = p(G^tG)$ where p is a polynomial with integer coefficients. These commuting squares are particularly nice as they are non-degenerate, hence, they can be used to construct a subfactor with index $\|G\|^2$. Ocneanu's example produces an irreducible hyperfinite subfactor with the smallest known index greater than 4. It has index $\|E_{10}\|^2 \approx 4.026418$, see [Sch90, Chapter 8].

In the case where G is one of the double brooms above, we could show that there is no non-degenerate commuting square of the form (7.1) if K is a polynomial of G^tG. In this case, the eigenspaces of GG^t are all 1-dimensional. The non-degeneracy of the commuting square implies that $HGG^t = GG^tH$. By simultaneously diagonalizing both H and GG^t one can show that $H = p(GG^t)$ where p is a polynomial with potentially non-integer coefficients.

To obtain a non-degenerate commuting square based on one of these double brooms we only need to consider $H = p(GG^t)$ and $K \neq p(G^tG)$ such that both H and K are incidence matrices for connected bipartite graphs. To simplify this search (and future computations) we assume K to be symmetric about both its diagonals. Under these assumptions, given a fixed polynomial p one obtains relations for the entries of K. To further simplify the computations we choose p to be monic as this guarantees that HG will have small entries. We then proceed to perturb $p(G^tG)$ according to these relations and use the bi-unitary condition to test if our choice is compatible with the blocks of u and v being unitary. Most choices of p fail this test very quickly as they generally imply that one row or column of some block has norm greater than 1.

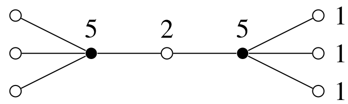
For example, let Γ be the small broom in Figure 7.1 with the following labelling of the vertices:



where $\mathscr{V}_+ = (A, B)$ and $\mathscr{V}_- = (a_3, a_2, a_1, a_0, b_1, b_2, b_3)$. Then G is the following $\mathscr{V}_+ \times \mathscr{V}_-$ matrix

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

We computed and fixed a Perron-Frobenius eigenvector for the graph Γ as follows:

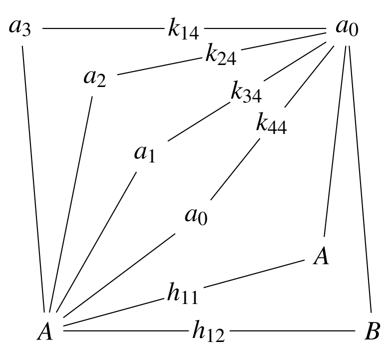
$\lambda(\cdot) =$ 

Since we are assuming the horizontal inclusions are described by matrices that are symmetric about both their diagonals, they are of the form

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{11} \end{pmatrix}, \quad K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} \\ k_{12} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{16} \\ k_{13} & k_{23} & k_{33} & k_{34} & k_{35} & k_{25} & k_{15} \\ k_{14} & k_{24} & k_{34} & k_{44} & k_{34} & k_{24} & k_{14} \\ k_{15} & k_{25} & k_{35} & k_{34} & k_{33} & k_{23} & k_{13} \\ k_{16} & k_{26} & k_{25} & k_{24} & k_{23} & k_{22} & k_{12} \\ k_{17} & k_{16} & k_{15} & k_{14} & k_{13} & k_{12} & k_{11} \end{pmatrix}$$

where the vertices are in the order (A,B) and $(a_3,a_2,a_1,a_0,b_1,b_2,b_3)$ respectively, i.e. H and K are $\mathscr{V}_+ \times \mathscr{V}_+$ and $\mathscr{V}_- \times \mathscr{V}_-$ matrices respectively.

To understand the block structure of the summand $u^{(A,a_0)}$ we look at the possible $A - r - a_0 - q - A$ cycles, where $r \in \mathscr{V}_+$ and $q \in \mathscr{V}_-$:



This implies that $u^{(A,a_0)}$ has the following block structure

$$u^{(A,a_0)} = \begin{array}{c} \\ a_3 \\ a_2 \\ a_1 \\ a_0 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left(\begin{array}{c|c} * & * \\ \hline * & * \\ \hline * & * \\ \hline * & * \end{array}\right) \\ \begin{array}{cc} h_{11} & h_{12} \end{array} \end{array} \begin{array}{c} \\ k_{14} \\ k_{24} \\ k_{34} \\ k_{44} \\ \\ \end{array}$$

where the labels on the bottom and the right denote the number of columns and rows for the blocks respectively. A similar analysis implies the following block structure for the following summands of v,

$$\begin{array}{rc} & \begin{array}{cccc} a_0 & b_1 & b_2 & b_3 \end{array} \\ v^{(a_3,B)} = A & \left(\begin{array}{c|c|c|c} \;*\; & \;*\; & \;*\; & \;*\; \end{array} \right) h_{12}, \\ & \begin{array}{cccc} k_{14} & k_{15} & k_{16} & k_{17} \end{array} \end{array}$$

$$\begin{array}{rc} & \begin{array}{cccc} a_0 & b_1 & b_2 & b_3 \end{array} \\ v^{(a_2,B)} = A & \left(\begin{array}{c|c|c|c} \;*\; & \;*\; & \;*\; & \;*\; \end{array} \right) h_{12}, \\ & \begin{array}{cccc} k_{24} & k_{25} & k_{26} & k_{27} \end{array} \end{array}$$

$$\begin{array}{rc} & \begin{array}{cccc} a_0 & b_1 & b_2 & b_3 \end{array} \\ v^{(a_1,B)} = A & \left(\begin{array}{c|c|c|c} \;*\; & \;*\; & \;*\; & \;*\; \end{array} \right) h_{12}. \\ & \begin{array}{cccc} k_{34} & k_{35} & k_{36} & k_{37} \end{array} \end{array}$$

In particular, since each block has to be unitary, the sum of the squares of the entries of each column must add up to 1. Therefore $\|v^{(a_3,B)}_{A,a_0}\|_2^2 = k_{14}$, $\|v^{(a_2,B)}_{A,a_0}\|_2^2 = k_{24}$ and $\|v^{(a_1,B)}_{A,a_0}\|_2^2 = k_{34}$, where $\|X\|_2^2$ denotes the sum of the squares of the absolute values of the entries of the block X. The bi-unitary condition (7.3) then implies that

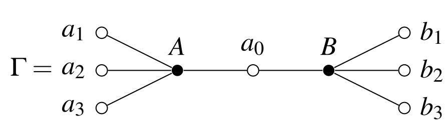
$$\|u^{(A,a_0)}_{a_3,B}\|_2^2 = \frac{1}{2}k_{14},\ \|u^{(A,a_0)}_{a_2,B}\|_2^2 = \frac{1}{2}k_{24},\ \|u^{(A,a_0)}_{a_1,B}\|_2^2 = \frac{1}{2}k_{34}.$$

Since $h_{12} = \sum_{i=0}^{3} \|u^{(A,a_0)}_{a_i,B}\|_2^2$, we have $2h_{12} \geq k_{14} + k_{24} + k_{34}$. Similarly, we obtain other inequalities that relate the entries of K with the entries of H. We then use these inequalities to perturb $p(G^tG)$ to obtain a suitable candidate for K.

A similar analysis is used to obtain the horizontal inclusions for commuting squares based on the "Quipu".

7.2 Small double broom

For the "small double broom" we will use the following labeling of the vertices:



where $\mathscr{V}_+ = (A,B)$ and $\mathscr{V}_- = (a_3,a_2,a_1,a_0,b_1,b_2,b_3)$.

Writing the vertices in the order $(A, B, a_3, a_2, a_1, a_0, b_1, b_2, b_3)$ the adjacency matrix of Γ is

$$\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}$$

where

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

describes the vertical inclusions of (7.1). In this case, we will try to build a commuting square with horizontal inclusion matrices given by

$$H = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

where the vertices are in the order (A, B) for H, respectively $(a_3, a_2, a_1, a_0, b_1, b_2, b_3)$ for K. Observe that H and K have some multiple edges.

Note that the (p, s)-entry of $GK = HG$ counts the number of paths on G and K going from p to s through $\mathscr{Z}_{1,0}$. Since

$$GK = \begin{pmatrix} 3 & 3 & 3 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 3 & 3 & 3 \end{pmatrix}$$

we deduce that u will have two 4×4 blocks, six 3×3 blocks and six 1×1 blocks. Figure 7.5 describes the block structure of each direct summand of u.

	s	a_3	a_2	a_1	a_0		b_1	b_2	b_3
p	q \ r	A	A	A	A	B	B	B	B
A	a_3		•••	•••	•••	•			
	a_2	•••		•••	•••	•			
	a_1	•••	•••	•••			•		
	a_0	•••	•••		••• •••	• •		•	•
B	a_0	•	•		• •	••• •••		•••	•••
	b_1			•			•••	•••	•••
	b_2				•	•••	•••		•••
	b_3				•	•••	•••	•••	

Figure 7.5: Small double broom - structure of u

In this table, the thick lines delimit each summand of u and the dots describe the block structure of each summand. For example the summand u^{A,a_3} is a 3×3 matrix made up of three blocks. Each block is a 1×3 arrangement correspoding to each of the following $p - r - s - q - p$ cycles:

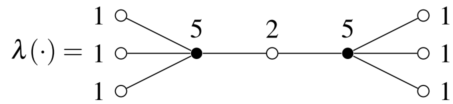
$$\begin{array}{ccc} a_2 - a_3 & a_1 - a_3 & a_0 - a_3 \\ \vert \quad\;\; \vert & \vert \quad\;\; \vert & \vert \quad\;\; \vert \\ A \equiv A & A \equiv A & A \equiv A \end{array}$$

Similarly, we can describe the block structure of each direct summand of v:

	r	A				B			
q	p \ s	a_3	a_2	a_1	a_0	a_0	b_1	b_2	b_3
a_3	A		• • •	• • •	• • •	•			
a_2	A	• • •		• • •	• • •	•			
a_1	A	• • •	• • •	• • •			•		
a_0	A	• • •	• • •		•• •• ••	••		•	•
	B	•	•		••	•• •• ••		• • •	• • •
b_1	B			•			• • •	• • •	• • •
b_2	B				•	• • •	• • •		• • •
b_3	B				•	• • •	• • •	• • •	

Figure 7.6: Small double broom - structure of v

Recall the values of the fixed Perron-Frobenius eigenvector for the graph Γ that we will use:



For both u and v we choose the 1×1 blocks to be 1 and using the bi-unitary condition (7.3) we obtain the following values for both u and v:

	s	a_3	a_2	a_1	a_0		b_1	b_2	b_3
p	q \ r	A	A	A	A	B	B	B	B
A	a_3		•••	•••	•••	$\frac{1}{\sqrt{2}}$			
	a_2	•••		•••	•••	$\frac{1}{\sqrt{2}}$			
	a_1	•••	•••	•••			1		
	a_0	•••	•••		••• •••	• •		1	1
B	a_0	1	1		• •	••• •••		•••	•••
	b_1			1			•••	•••	•••
	b_2				$\frac{1}{\sqrt{2}}$	•••	•••		•••
	b_3				$\frac{1}{\sqrt{2}}$	•••	•••	•••	

	r	A				B			
q	p \ s	a_3	a_2	a_1	a_0	a_0	b_1	b_2	b_3
a_3	A		⋮	⋮	⋮	1			
a_2	A	⋮		⋮	⋮	1			
a_1	A	⋮	⋮	⋮			1		
a_0	A	⋮	⋮		•• •• ••	••		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
	B	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$		••	•• •• ••		⋮	⋮
b_1	B			1			⋮	⋮	⋮
b_2	B				1	⋮	⋮		⋮
b_3	B				1	⋮	⋮	⋮	

Since the columns and rows of a unitary have norm 1 we get that the blocks $u^{(A,a_0)}_{a_0,B}$, $u^{(B,a_0)}_{a_0,A}$, $v^{(a_0,A)}_{B,a_0}$ and $v^{(a_0,B)}_{A,a_0}$

are all 0. At this point, we make some choices for u and v and set

$$\begin{pmatrix} u^{(A,a_0)}_{a_3,A} \\ \hline u^{(A,a_0)}_{a_2,A} \\ \hline u^{(A,a_0)}_{a_0,A} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/\sqrt{2} \\ \hline 0 & 0 & -1/\sqrt{2} \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} u^{(B,a_0)}_{a_0,B} \\ \hline u^{(B,a_0)}_{b_2,B} \\ \hline u^{(B,a_0)}_{b_3,B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1/\sqrt{2} & 0 & 0 \\ \hline 1/\sqrt{2} & 0 & 0 \end{pmatrix}$$

$$\left(v^{(a_0,A)}_{A,a_3} \;\middle|\; v^{(a_0,A)}_{A,a_2} \right) = \left(\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} \end{array}\right) \qquad \left(v^{(a_0,B)}_{B,b_2} \;\middle|\; v^{(a_0,B)}_{B,b_3} \right) = \left(\begin{array}{c|c} -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{array}\right)$$

Using the bi-unitary condition we obtain some of the entries of u and v listed in the next tables:

	s	a_3	a_2	a_1	a_0		b_1	b_2	b_3
p	q \ r	A	A	A	A	B	B	B	B
A	a_3		•••	•••	$0\ 0\ \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$			
	a_2	•••		•••	$0\ 0\ \frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$			
	a_1	•••	•••	•••			1		
	a_0	$0\ 0\ 1$	$0\ 0\ -1$		$\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}$	$\begin{smallmatrix}0\\0\end{smallmatrix}$		1	1
B	a_0	1	1		$\begin{smallmatrix}0\\0\end{smallmatrix}$	$\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}$		$-1\ 0\ 0$	$1\ 0\ 0$
	b_1			1			•••	•••	•••
	b_2				$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}\ 0\ 0$	•••		•••
	b_3				$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\ 0\ 0$	•••	•••	

Figure 7.7: Small double broom - partial entries of u

	r	A				B			
q	p \ s	a_3	a_2	a_1	a_0	a_0	b_1	b_2	b_3
a_3	A		⋮	⋮	$\begin{smallmatrix}0\\0\\1\end{smallmatrix}$	1			
a_2	A	⋮		⋮	$\begin{smallmatrix}0\\0\\-1\end{smallmatrix}$	1			
a_1	A	⋮	⋮	⋮			1		
a_0	A	$\begin{smallmatrix}0\\0\\\frac{1}{\sqrt{2}}\end{smallmatrix}$	$\begin{smallmatrix}0\\0\\\frac{-1}{\sqrt{2}}\end{smallmatrix}$		$\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}$	$0\ 0$		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
	B	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$		$0\ 0$	$\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}$		$\begin{smallmatrix}\frac{-1}{\sqrt{2}}\\0\\0\end{smallmatrix}$	$\begin{smallmatrix}\frac{1}{\sqrt{2}}\\0\\0\end{smallmatrix}$
b_1	B			1			⋮	⋮	⋮
b_2	B				1	$\begin{smallmatrix}-1\\0\\0\end{smallmatrix}$	⋮		⋮
b_3	B				1	$\begin{smallmatrix}1\\0\\0\end{smallmatrix}$	⋮	⋮	

Figure 7.8: Small double broom - partial entries of v

We now have multiple ways to select the remaining entries. We present one way to do this:

p	s / $q \diagdown r$	a_3	a_2	a_1	a_0		b_1	b_2	b_3
		A	A	A	A	B	B	B	B
A	a_3		$0\ 1\ 0$	$1\ 0\ 0$	$0\ 0\ \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$			
	a_2	$1\ 0\ 0$		$0\ 1\ 0$	$0\ 0\ \frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$			
	a_1	$0\ 1\ 0$	$1\ 0\ 0$	$0\ 0\ 1$			1		
	a_0	$0\ 0\ 1$	$0\ 0\ -1$		$\begin{matrix}1&0&0\\0&1&0\end{matrix}$	$\begin{matrix}0\\0\end{matrix}$		1	1
B	a_0	1	1		$\begin{matrix}0\\0\end{matrix}$	$\begin{matrix}0&1&0\\0&0&1\end{matrix}$		$-1\ 0\ 0$	$1\ 0\ 0$
	b_1			1			$1\ 0\ 0$	$0\ 0\ 1$	$0\ 1\ 0$
	b_2				$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}\ 0\ 0$	$0\ 1\ 0$		$0\ 0\ 1$
	b_3				$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\ 0\ 0$	$0\ 0\ 1$	$0\ 1\ 0$	

Figure 7.9: Small double broom - entries of *u*

q	r / $p \diagdown s$	A				B			
		a_3	a_2	a_1	a_0	a_0	b_1	b_2	b_3
a_3	A		$\begin{matrix}0\\1\\0\end{matrix}$	$\begin{matrix}1\\0\\0\end{matrix}$	$\begin{matrix}0\\0\\1\end{matrix}$	1			
a_2	A	$\begin{matrix}1\\0\\0\end{matrix}$		$\begin{matrix}0\\1\\0\end{matrix}$	$\begin{matrix}0\\0\\-1\end{matrix}$	1			
a_1	A	$\begin{matrix}0\\1\\0\end{matrix}$	$\begin{matrix}1\\0\\0\end{matrix}$	$\begin{matrix}0\\0\\1\end{matrix}$			1		
a_0	A	$\begin{matrix}0\\0\\\frac{1}{\sqrt{2}}\end{matrix}$	$\begin{matrix}0\\0\\\frac{-1}{\sqrt{2}}\end{matrix}$		$\begin{matrix}1&0\\0&1\\0&0\end{matrix}$	$0\ 0$		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
	B	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$		$0\ 0$	$\begin{matrix}0&0\\1&0\\0&1\end{matrix}$		$\begin{matrix}\frac{-1}{\sqrt{2}}\\0\\0\end{matrix}$	$\begin{matrix}\frac{1}{\sqrt{2}}\\0\\0\end{matrix}$
b_1	B			1			$\begin{matrix}1\\0\\0\end{matrix}$	$\begin{matrix}0\\0\\1\end{matrix}$	$\begin{matrix}0\\1\\0\end{matrix}$
b_2	B				1	$\begin{matrix}-1\\0\\0\end{matrix}$	$\begin{matrix}0\\1\\0\end{matrix}$		$\begin{matrix}0\\0\\1\end{matrix}$
b_3	B				1	$\begin{matrix}1\\0\\0\end{matrix}$	$\begin{matrix}0\\0\\1\end{matrix}$	$\begin{matrix}0\\1\\0\end{matrix}$	

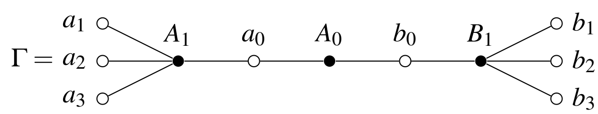
Figure 7.10: Small double broom - entries of *v*

Since both H and K correspond to connected bipartite graphs and since G satisfies Wenzl's criterion for irreducibility (see [Wen88]), we have constructed an irreducible subfactor of the hyperfinite II_1 factor with index 5.

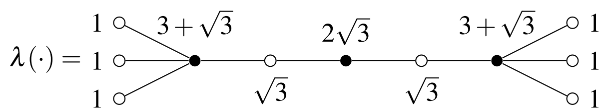
Remark 7.2. We made several choices to determine the entries of u and v. One can show that any other choice for u (and consequently for v) will be *gauge equivalent* to the one above, i.e. it will be of the form w_1uw_2 where w_i is a unitary matrix (see [JS97, Remark 5.3.3]). This means that any subfactor constructed using any of these choices will be isomorphic to the one constructed above.

7.3 Medium double broom

For the “medium double broom” we will use the following labeling of the vertices:



and the following Perron-Frobenius eigenvector for the graph Γ:



Using the same notation as in the previous section, we thus have

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

for the vertical inclusions. We computed

$$H = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

for the horizontal ones.

Recall that the (p,s)-entry of GK counts the number of paths from p to s. Since

$$GK = \begin{pmatrix} 2 & 2 & 2 & 3 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 & 2 & 2 & 2 \end{pmatrix}$$

we deduce that u has two 3×3 blocks, ten 2×2 blocks and twelve 1×1 blocks. The following tables describe the block structure of each direct summand of *u* and *v*:

p	s / q \ r	a_3	a_2	a_1	a_0		b_0		b_1	b_2	b_3
		A_1	A_1	A_1	A_1	A_0	A_0	B_1	B_1	B_1	B_1
A_1	a_3	••			••	•					•
	a_2		••		••	•				•	
	a_1			::			•	•			
	a_0	••	••		••	•	•	•	•		
A_0	a_0	•	•		•	•	•	•	•		
	b_0			•	•	•	•	•		•	•
B_1	b_0			•	•	•	•	••		••	••
	b_1				•	•			::		
	b_2		•				•	••		••	
	b_3	•					•	••			••

Figure 7.11: Medium double broom - structure of *u*

q	r / p \ s	A_1				A_0		B_1			
		a_3	a_2	a_1	a_0	a_0	b_0	b_0	b_1	b_2	b_3
a_3	A_1	:			:	•					•
a_2	A_1		:		:	•				•	
a_1	A_1			::			•	•			
a_0	A_1	:	:		:	•	•	•	•		
	A_0	•	•		•	•	•	•	•		
b_0	A_0			•	•	•	•	•		•	•
	B_1			•	•	•	•	:		:	:
b_1	B_1				•	•			::		
b_2	B_1		•				•	:		:	
b_3	B_1	•					•	:			:

Figure 7.12: Medium double broom - structure of *v*

We will outline how to determine the 3×3 blocks as the remaining entries can be determined using the bi-unitary condition. Consider the 3×3 blocks:

$$u^{(A_1,a_0)} = \begin{array}{c} \\ a_3 \\ a_2 \\ a_0 \end{array}\!\!\begin{array}{c} \begin{array}{cc} A_1 & A_0 \end{array} \\ \left(\begin{array}{c|c} \alpha_3 & x_3 \\ \hline \alpha_2 & x_2 \\ \hline \alpha_0 & x_0 \end{array}\right) \end{array}, \quad v^{(a_0,A_1)} = \begin{array}{c} \\ A_1 \\ A_0 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a_3 & a_2 & a_0 \end{array} \\ \left(\begin{array}{c|c|c} \beta_3^t & \beta_2^t & \beta_0^t \\ \hline y_3 & y_2 & y_0 \end{array}\right) \end{array}$$

where $\alpha_i,\beta_j \in M_{1\times 2}(\mathbb{R})$. and observe that $v^{(a_i,A_0)}$ and $u^{(A_0,a_i)}$ are 1×1 blocks for $i=2,3$. Hence we can assume they are both equal to 1. Using the bi-unitary condition we obtain $y_i = x_i = \sqrt{\frac{\lambda(a_i)\lambda(A_0)}{\lambda(A_1)\lambda(a_0)}}$ for $i=2,3$ and therefore

$$u^{(A_1,a_0)} = \begin{array}{c} \\ a_3 \\ a_2 \\ a_0 \end{array}\!\!\begin{array}{c} \begin{array}{cc} A_1 & A_0 \end{array} \\ \left(\begin{array}{c|c} \alpha_3 & \sqrt{\frac{3-\sqrt{3}}{3}} \\ \hline \alpha_2 & \sqrt{\frac{3-\sqrt{3}}{3}} \\ \hline \alpha_0 & x_0 \end{array}\right) \end{array}, \quad v^{(a_0,A_1)} = \begin{array}{c} \\ A_1 \\ A_0 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a_3 & a_2 & a_0 \end{array} \\ \left(\begin{array}{c|c|c} \beta_3^t & \beta_2^t & \beta_0^t \\ \hline \sqrt{\frac{3-\sqrt{3}}{3}} & \sqrt{\frac{3-\sqrt{3}}{3}} & y_0 \end{array}\right) \end{array}.$$

Since the last row (respective column) of $u^{(A_1,a_0)}$ (respective $v^{(a_0,A_1)}$) must have norm 1, we can assume $y_0 = x_0 = \sqrt{\frac{-3+2\sqrt{3}}{3}}$. Because the rows of $u^{(A_1,a_0)}$ form an orthonormal basis of $\mathbb{C}^3$ we have

$$\langle\alpha_3,\alpha_2\rangle = -\frac{3-\sqrt{3}}{3} \qquad \|\alpha_3\|^2 = \|\alpha_2\|^2 = \frac{\sqrt{3}}{3}$$

$$\langle\alpha_3,\alpha_0\rangle = \langle\alpha_3,\alpha_0\rangle = -\frac{-5+3\sqrt{3}}{3} \qquad \|\alpha_0\|^2 = \frac{8-3\sqrt{3}}{3}$$

We now look at the 2×2 blocks $v^{(a_i,A_1)}$ and $u^{(A_1,a_i)}$ for $i=2,3$. Using the bi-unitary condition we get

$$v^{(a_3,A_1)} = A_1\!\begin{array}{c} \begin{array}{cc} a_3 & a_0 \end{array} \\ \left(\begin{array}{c|c} * & \sqrt{\lambda(a_0)}\alpha_3^t \\ * & \end{array}\right) \end{array}, \quad v^{(a_2,A_1)} = A_1\!\begin{array}{c} \begin{array}{cc} a_2 & a_0 \end{array} \\ \left(\begin{array}{c|c} * & \sqrt{\lambda(a_0)}\alpha_2^t \\ * & \end{array}\right) \end{array}$$

Since both of these have to be unitary, they can be chosen to be of the form:

$$v^{(a_3,A_1)} = \sqrt{\lambda(a_0)}\left(\begin{array}{c|c} \alpha_{32} & \alpha_{31} \\ -\alpha_{31} & \alpha_{32} \end{array}\right), \quad v^{(a_2,A_1)} = \sqrt{\lambda(a_0)}\left(\begin{array}{c|c} \alpha_{22} & \alpha_{21} \\ -\alpha_{21} & \alpha_{22} \end{array}\right)$$

Once again, using the bi-unitary condition and the fact that $u^{(A_1,a_i)}$ for $i = 2,3$ are unitaries, we have

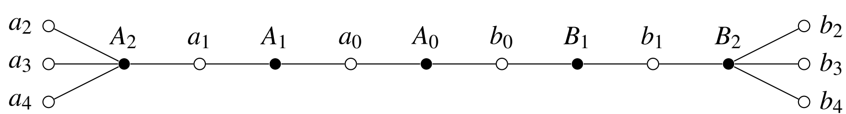
$$u^{(A_1,a_3)} = \sqrt{\lambda(a_0)}\left(\begin{array}{cc} \alpha_{32} & -\alpha_{31} \\ \hline \alpha_{31} & \alpha_{32} \end{array}\right), \quad u^{(A_1,a_2)} = \sqrt{\lambda(a_0)}\left(\begin{array}{cc} \alpha_{22} & -\alpha_{21} \\ \hline \alpha_{21} & \alpha_{22} \end{array}\right)$$

The bi-unitary condition applied to the blocks $u^{(A_1,a_i)}_{a_0,A_1}$ for $i = 0,2,3$ imply that $\beta_i = \alpha_i$. We can now conclude that any choice of α_i's which make $u^{(A_1,a_0)}$ unitary will do the same for $v^{(a_0,A_1)}$. By completing the last row of $u^{(A_1,a_0)}$ to an orthonormal basis we obtain such a choice. A similar reasoning works for the blocks $u^{(B_1,b_0)}$ and $v^{(b_0,B_1)}$.

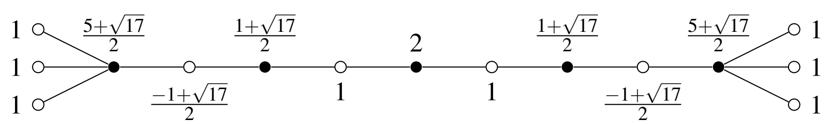
The remaining 2×2 blocks are easily obtained by using the bi-unitary condition. Hence we have a bi-unitary connection on the medium double broom.

7.4 Large double broom

For the "large double broom" we will use the following labeling of the vertices:

$\Gamma =$ 

and the following Perron-Frobenius eigenvector for the graph Γ:

$\lambda(\cdot) =$ 

Using the same notation as above, we have

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

for the vertical inclusions, and

$$H = \begin{pmatrix} 3 & 2 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 2 & 3 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 \end{pmatrix}$$

for the horizontal ones. Since

$$GK = \begin{pmatrix} 3 & 3 & 3 & 5 & 3 & 2 & 3 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 2 & 3 & 5 & 3 & 3 & 3 \end{pmatrix}$$

we deduce that u has two 5×5 blocks, ten 3×3 blocks, twenty-two 2×2 blocks and sixteen 1×1 blocks, same for v. Figure 7.16 and figure 7.17 (found at the end of this section) describe the block structure for both u and v. The main difficulty in computing the bi-unitary connection will be to determine the entries of the four 5×5 blocks: $u^{(A_2,a_1))}$, $u^{(B_2,b_1)}$, $v^{(a_1,A_2)}$ and $v^{(b_1,B_2)}$.

Note that in Figure 7.16, $u^{(A_2,a_1))}$ and $u^{(B_2,b_1)}$ are in a symmetric position, hence we will try to assume we can obtain the entries of $u^{(B_2,b_1)}$ by "‘reflecting" the entries of $u^{(A_2,a_1)}$.

Consider

$$u^{(A_2,a_1)} = \begin{array}{c} \\ a_4 \\ \\ a_3 \\ a_2 \\ a_1 \end{array} \begin{array}{c} \begin{array}{ccccc} & A_2 & & A_1 & \end{array} \\ \left(\begin{array}{ccc|cc} \gamma_{11} & \gamma_{12} & \gamma_{13} & w_{11} & w_{12} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & w_{21} & w_{22} \\ \hline \xi_{11} & \xi_{12} & \xi_{13} & \alpha_{11} & \alpha_{12} \\ \hline \xi_{21} & \xi_{22} & \xi_{23} & \alpha_{21} & \alpha_{22} \\ \hline \xi_{31} & \xi_{32} & \xi_{33} & \alpha_{31} & \alpha_{32} \end{array}\right) \end{array}$$

and set $\gamma_i = (\gamma_{i1}, \gamma_{i2}, \gamma_{i3})$, $\xi_i = (\xi_{i1}, \xi_{i2}, \xi_{i3})$, $w_i = (w_{i1}, w_{i2})$ and $\alpha_i = (\alpha_{i1}, \alpha_{i2})$. From the bi-unitary condition we have

$$v^{(a_4,A_1)} = \sqrt{\frac{8}{7-\sqrt{17}}}\begin{pmatrix} w_1^t & w_2^t \end{pmatrix}$$

which implies that $\|w_i\| = \sqrt{\frac{7-\sqrt{17}}{8}}$ and $\langle w_1, w_2\rangle = 0$ as $v^{(a_4,A_1)}$ has to be unitary. Since the first two rows of $u^{(A_2,a_1)}$ are orthonormal, we also get $\|\gamma_i\| = \sqrt{\frac{1+\sqrt{17}}{8}}$ and $\langle \gamma_1, \gamma_2\rangle = 0$. Without loss of generality we can suppose $v^{(a_4,A_1)} = \mathrm{id}_2$ and $\gamma_i = \sqrt{\frac{1+\sqrt{17}}{8}} e_i$, where e_i is the canonical i-th basis vector in $\mathbb{C}^3$. Thus, we have the following entries:

$$u^{(A_2,a_1)} = \begin{array}{c} \\ a_4 \\ \\ a_3 \\ a_2 \\ a_1 \end{array} \begin{array}{c} \begin{array}{ccc|cc} & A_2 & & A_1 & \end{array} \\ \left(\begin{array}{ccc|cc} \sqrt{\frac{1+\sqrt{17}}{8}} & 0 & 0 & \sqrt{\frac{7-\sqrt{17}}{8}} & 0 \\ 0 & \sqrt{\frac{1+\sqrt{17}}{8}} & 0 & 0 & \sqrt{\frac{7-\sqrt{17}}{8}} \\ \hline \xi_{31} & \xi_{32} & \xi_{33} & \alpha_{31} & \alpha_{32} \\ \hline \xi_{21} & \xi_{22} & \xi_{23} & \alpha_{21} & \alpha_{22} \\ \hline \xi_{11} & \xi_{12} & \xi_{13} & \alpha_{11} & \alpha_{12} \end{array}\right) \end{array}$$

Now, from the orthogonality between the rows of $u^{(A_2,a_1)}$ we obtain

$$\xi_{ij} = -\sqrt{\frac{7-\sqrt{17}}{1+\sqrt{17}}}\alpha_{ij} \tag{7.4}$$

for any $i = 1, 2, 3$ and $j = 1, 2$. Since every row must have norm 1 we have the identity

$$1 = \xi_{i3}^2 + \frac{8}{1+\sqrt{17}}\|\alpha_i\|^2. \tag{7.5}$$

From the bi-unitary condition we obtain

$$v^{(a_i,A_1)} = \sqrt{\frac{\lambda(A_2)\lambda(a_1)}{\lambda(a_i)\lambda(A_1)}}\left(\begin{array}{c|c} \alpha_{i1} & * \\ \alpha_{i2} & * \end{array}\right)$$

for $i = 1, 2, 3$. Since v^{a_i,A_1} has to be unitary, we can assume it is of the form

$$v^{(a_i,A_1)} = \sqrt{\frac{\lambda(A_2)\lambda(a_1)}{\lambda(a_i)\lambda(A_1)}}\left(\begin{array}{c|c} \alpha_{i1} & -\alpha_{i2} \\ \alpha_{i2} & \alpha_{i1} \end{array}\right)$$

and $\|\alpha_i\|^2 = \frac{\lambda(a_i)\lambda(A_1)}{\lambda(A_2)\lambda(a_1)}$. In particular $\|\alpha_2\|^2 = \|\alpha_3\|^2 = \frac{7-\sqrt{17}}{8}$ and $\|\alpha_1\|^2 = \frac{-3+\sqrt{17}}{2}$. Using this in equation 7.5 we obtain that $\xi_{23}^2 = \xi_{33}^2 = \frac{5-\sqrt{17}}{2}$ and $\xi_{13}^2 = -4+\sqrt{17}$.

We now will determine the entries α_{ij}. Consider the block $u^{(A_2,a_0)}$. Using the bi-unitary condition we get

$$u^{(A_2,a_0)} = \begin{array}{c} \\ a_3 \\ a_2 \\ a_1 \end{array} \begin{array}{c} \begin{array}{cc} A_1 & A_0 \end{array} \\ \left(\begin{array}{c|c} \sqrt{\frac{\lambda(a_1)}{\lambda(a_0)}}\tilde{\alpha}_3 & * \\ \hline \sqrt{\frac{\lambda(a_1)}{\lambda(a_0)}}\tilde{\alpha}_2 & * \\ \hline \sqrt{\frac{\lambda(a_1)}{\lambda(a_0)}}\tilde{\alpha}_1 & * \end{array}\right) \end{array}$$

where $\tilde{\alpha}_i = (-\alpha_{i,2}, \alpha_{i,1})$. Since $v^{(a_i,A_0)}$ are 1×1 blocks for $i = 1,2,3$, we can suppose $v^{(a_2,A_0)} = v^{(a_3,A_0)} = 1$ and $v^{(a_1,A_0)} = -1$. The bi-unitary condition then implies

$$u^{(A_2,a_0)} = \begin{array}{c} \\ a_3 \\ a_2 \\ a_1 \end{array} \begin{array}{c} \begin{array}{cc} A_1 & A_0 \end{array} \\ \left(\begin{array}{c|c} \sqrt{\frac{\lambda(a_1)}{\lambda(a_0)}}\tilde{\alpha}_3 & \sqrt{\frac{5-\sqrt{17}}{2}} \\ \hline \sqrt{\frac{\lambda(a_1)}{\lambda(a_0)}}\tilde{\alpha}_2 & \sqrt{\frac{5-\sqrt{17}}{2}} \\ \hline \sqrt{\frac{\lambda(a_1)}{\lambda(a_0)}}\tilde{\alpha}_1 & -\sqrt{-4+\sqrt{17}} \end{array}\right) \end{array} \tag{7.6}$$

Since $u^{(A_2,a_0)}$ is unitary, its columns must form an orthonormal basis of $\mathbb{C}^3$. Given the last column, we can complete it to an orthonormal basis:

$$x_1 = \begin{pmatrix} \frac{\sqrt{5-\sqrt{17}}}{2} \\ 0 \\ \frac{\sqrt{-1+\sqrt{17}}}{2} \end{pmatrix}, \; x_2 = \begin{pmatrix} -\sqrt{\frac{-11+3\sqrt{17}}{4}} \\ \sqrt{\frac{-3+\sqrt{17}}{2}} \\ \frac{\sqrt{21-5\sqrt{17}}}{2} \end{pmatrix}, \; x_3 = \begin{pmatrix} \sqrt{\frac{5-\sqrt{17}}{2}} \\ \sqrt{\frac{5-\sqrt{17}}{2}} \\ -\sqrt{-4+\sqrt{17}} \end{pmatrix}$$

We can now assume that

$$u^{(A_2,a_0)} = \begin{array}{c} \\ a_3 \\ a_2 \\ a_1 \end{array} \begin{array}{c} \begin{array}{cc} A_1 & A_0 \end{array} \\ \left(\begin{array}{cc|c} & & \sqrt{\frac{5-\sqrt{17}}{2}} \\ \cline{3-3} q_1(s) & \vdots \; q_2(s) & \sqrt{\frac{5-\sqrt{17}}{2}} \\ \cline{3-3} & & -\sqrt{-4+\sqrt{17}} \end{array}\right) \end{array} \tag{7.7}$$

where $q_1(s) = \cos(s)x_1 + \sin(s)x_2$ and $q_2(s) = -\sin(s)x_1 + \cos(s)x_2$. Combining 7.6 and 7.7 we obtain α_{ij} as a function of s. From 7.4 we also obtain ξ_{ij} as a function of s for $i = 1,2,3$ and $j = 1,2$.

The fact that the rows of $u^{(A_2,a_1)}$ are orthogonal and 7.4 imply that

$$0 = \langle \xi_i, \xi_j \rangle + \langle \alpha_i, \alpha_j \rangle = \xi_{i3}\xi_{j3} + \frac{8}{1+\sqrt{17}} \langle \alpha_i, \alpha_j \rangle \tag{7.8}$$

for $i \neq j$. From the unitarity of 7.6 we get

$$\langle \alpha_2, \alpha_3 \rangle = \langle \tilde{\alpha}_2, \tilde{\alpha}_3 \rangle = -\frac{\lambda(a_0)}{\lambda(a_1)} \frac{5-\sqrt{17}}{2} \tag{7.9}$$

$$\langle \alpha_i, \alpha_1 \rangle = \langle \tilde{\alpha}_i, \tilde{\alpha}_1 \rangle = \frac{\lambda(a_0)}{\lambda(a_1)} \sqrt{\frac{5-\sqrt{17}}{2}} \sqrt{-4+\sqrt{17}}, \quad i = 2,3 \tag{7.10}$$

From 7.8, 7.9, 7.10, and the fact that $\xi_{23}^2 = \xi_{33}^2 = \frac{5-\sqrt{17}}{2}$ and $\xi_{13}^2 = -4+\sqrt{17}$ we compute

$$\xi_{23} = \xi_{33} = \pm \sqrt{\frac{5-\sqrt{17}}{2}}, \quad \xi_{13} = \mp \sqrt{-4+\sqrt{17}}.$$

We will work with $\xi_{13} = -\sqrt{-4+\sqrt{17}}$.

Note that all entries of $u^{(A_2,a_0)}$ depend on a single parameter s. Moreover, since we chose the entries so that the rows form an orthonormal basis of $\mathbb{C}^5$, we conclude that it has to be unitary. To determine the possible values s can take we need to investigate the blocks $v^{(a_i,A_2)}$ for $i = 1, \ldots, 4$ and make sure they can be completed to unitaries using the entries from $u^{(A_2,a_0)}$. It is important to note that $v^{(a_1,A_2)}$ is also a 5×5 block and hence will be the main source of obstructions for s.

Using the bi-unitary condition for the blocks $v^{(a_i,A_2)}$ with $i = 2,3,4$, we have the following:

$$v^{(a_4,A_2)} = A_2 \overset{\begin{matrix} a_2 & a_1 \end{matrix}}{\left(\begin{array}{c|cc} * & 1 & 0 \\ * & 0 & 1 \\ * & 0 & 0 \end{array}\right)}, \quad v^{(a_3,A_2)} = A_2 \overset{\begin{matrix} a_3 & a_2 & a_1 \end{matrix}}{\left(\begin{array}{c|c|c} g_3^t & g_2^t & \sqrt{\frac{\lambda(a_1)}{\lambda(a_3)}}\xi_3^t \end{array}\right)}, \quad v^{(a_2,A_2)} = A_2 \overset{\begin{matrix} a_4 & a_3 & a_1 \end{matrix}}{\left(\begin{array}{c|c|c} f_4^t & f_3^t & \sqrt{\frac{\lambda(a_1)}{\lambda(a_2)}}\xi_2^t \end{array}\right)}$$

where the f_i's and $g_i's$ are unit vectors in $\mathbb{C}^3$. Since each of the above matrices has to be unitary we can choose $v^{(a_4,A_2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. To determine the remaining entries we need to look at the blocks $u^{(A_2,a_i)}$ for $i = 2,3,4$.

Once again, using the bi-unitary condition we have

$$u^{(A_2,a_4)} = \begin{array}{c} \\ a_2 \\ a_1 \\ \\ \end{array} \overset{A_2}{\left(\begin{array}{ccc} f_{41} & f_{42} & f_{43} \\ \hline \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \end{array}\right)}, \qquad u^{(A_2,a_3)} = \begin{array}{c} a_3 \\ a_2 \\ a_1 \end{array} \overset{A_2}{\left(\begin{array}{c} g_3 \\ \hline f_3 \\ \hline h_1 \end{array}\right)}, \qquad u^{(A_2,a_2)} = \begin{array}{c} a_4 \\ a_3 \\ a_1 \end{array} \overset{A_2}{\left(\begin{array}{ccc} 0 & 0 & 1 \\ \hline & g_2 & \\ \hline & g_1 & \end{array}\right)}$$

Since $u^{(A_2,a_2)}$ is unitary, we can assume it is of the form

$$u^{(A_2,a_2)} = \begin{array}{c} a_4 \\ a_3 \\ a_1 \end{array} \overset{A_2}{\left(\begin{array}{ccc} 0 & 0 & 1 \\ \hline \cos(t) & \sin(t) & 0 \\ \hline -\sin(t) & \cos(t) & 0 \end{array}\right)}$$

and consequently g_2 is a function of t. Using the fact that $v^{(a_3,A_2)}$ is unitary, we have that $g_3 = \sqrt{\frac{\lambda(a_1)}{\lambda(a_3)}} g_2 \times \xi_3$ provided $g_2 \perp \xi_3$ (here $\times$ denotes the cross product). Similarly we have that $f_3 \perp g_3$ and $f_3 \perp \xi_2$, therefore

$$f_3 = \frac{g_3 \times \xi_2}{\|g_3 \times \xi_2\|}$$

and consequently $h_1 = g_3 \times f_3$. Note that g_2 depends on t and ξ_3 depends on s, in fact

$$\langle g_2, \xi_3 \rangle = r_1 \cos(t+s) + r_2 \sin(t+s)$$

where $r_1 = \sqrt{-4+\sqrt{17}}$ and $r_2 = \frac{\sqrt{13-3\sqrt{17}}}{2}$. We can plot this function:

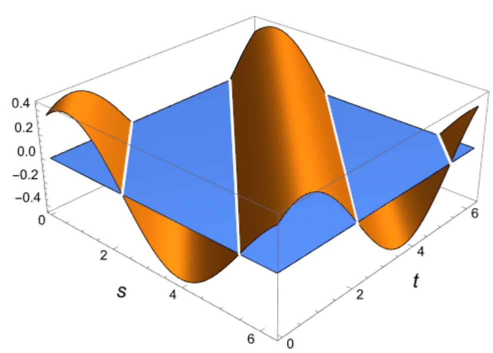


Figure 7.13: $\langle g_2, \xi_3 \rangle(t,s)$

Observer that there are straight lines on which $\langle g_2, \xi_3 \rangle = 0$.

The last set of restrictions will come from looking at the 5×5 block $v^{(a_1,A_2)}$. Using the bi-unitary condition we have

$$v^{(a_1,A_2)} = \begin{array}{c} \\ A_2 \\ \\ A_1 \end{array} \left(\begin{array}{cc|c|c|c} \multicolumn{2}{c}{a_4} & a_3 & a_2 & a_1 \\ \zeta_1^t & \zeta_2^t & \sqrt{\frac{1}{\lambda(a_1)}} h_1^t & \sqrt{\frac{1}{\lambda(a_1)}} g_1^t & \xi_1^t \\ \hline w_3^t & w_4^t & \beta_3^t & \beta_2^t & \beta_1^t \end{array} \right)$$

where $\zeta_i = \sqrt{\frac{1}{\lambda(a_1)}}\eta_i$. This block is similar to the block $u^{(A_2,a_1)}$, in particular the orthogonality of the last three columns implies

$$\frac{1}{\lambda(a_1)} \langle h_1, g_1 \rangle + \langle \beta_3, \beta_2 \rangle = 0, \tag{7.11}$$
$$\sqrt{\frac{1}{\lambda(a_1)}} \langle h_1, \xi_1 \rangle + \langle \beta_3, \beta_1 \rangle = 0, \tag{7.12}$$
$$\sqrt{\frac{1}{\lambda(a_1)}} \langle g_1, \xi_1 \rangle + \langle \beta_2, \beta_1 \rangle = 0. \tag{7.13}$$

If we now suppose $u^{(A_0,a_3)} = -1$, $u^{(A_0,a_2)} = 1$ and $u^{(A_0,a_1)}_{a_0,A2} > 0$ then we obtain

$$v^{(a_0,A_2)} = \begin{array}{c} \\ A_1 \\ \\ A_0 \end{array} \left(\begin{array}{c|c|c} a_3 & a_2 & a_1 \\ * & * & * \\ * & * & * \\ \hline \sqrt{\frac{5-\sqrt{17}}{2}} & -\sqrt{\frac{5-\sqrt{17}}{2}} & \sqrt{-4+\sqrt{17}} \end{array} \right)$$

and a reasoning similar to the one used in 7.9 and 7.10 implies that

$$\langle \beta_3, \beta_2 \rangle = \frac{-3+\sqrt{17}}{4}, \qquad \langle \beta_3, \beta_1 \rangle = \sqrt{\frac{-45+11\sqrt{17}}{16}}, \qquad \langle \beta_2, \beta_1 \rangle = -\sqrt{\frac{-45+11\sqrt{17}}{16}}$$

Plugging this into (7.11), (7.12) and (7.13) we obtain the following:

$$F_1(s,t) = \frac{1}{\lambda(a_1)}\langle h_1, g_1\rangle + \frac{-3+\sqrt{17}}{4} = 0, \tag{7.14}$$

$$F_2(s,t) = \sqrt{\frac{1}{\lambda(a_1)}}\langle h_1, \xi_1\rangle + \sqrt{\frac{-45+11\sqrt{17}}{16}} = 0, \tag{7.15}$$

$$F_3(s,t) = \sqrt{\frac{1}{\lambda(a_1)}}\langle g_1, \xi_1\rangle - \sqrt{\frac{-45+11\sqrt{17}}{16}} = 0. \tag{7.16}$$

On the other hand, orthonormality of the first three rows implies

$$\zeta_{1i}\zeta_{1j} + \zeta_{2i}\zeta_{2j} + \frac{h_{1i}h_{1j} + g_{1i}g_{1j}}{\lambda(a_1)} + \xi_{1i}\xi_{1j} = 0, \quad i \neq j \in \{1,2,3\}$$
$$\zeta_{1i}^2 + \zeta_{2i}^2 + \frac{h_{1i}^2 + g_{1i}^2}{\lambda(a_1)} + \xi_{1i}\xi_{1j} = 1, \quad i \in \{1,2,3\}$$

Since $u^{(A_2,a_4)}$ is also orthonormal, we have $f_{4i}f_{4j} + \eta_{1i}\eta_{2j} + \eta_{2i}\eta_{1j} = 0$ for $i \neq j$ and $f_{4i}^2 + \eta_{1i}^2 + \eta_{1i}^2 = 1$, consequently we obtain

$$G_{i,j}(s,t) = \frac{-f_{4i}f_{4j} + h_{1i}h_{1j} + g_{1i}g_{1j}}{\lambda(a_1)} + \xi_{1i}\xi_{1j} = 0, \quad i \neq j \in \{1,2,3\}$$
$$G_{i,i}(s,t) = \frac{1 - f_{4i}^2 + h_{1i}^2 + g_{1i}^2}{\lambda(a_1)} + \xi_{1i}^2 - 1 = 0, \quad i \in \{1,2,3\}$$

Finding a common root for all F_i's and $G_{i,j}$'s algebraically is very complicated. Below we plot F_1, F_2 and F_3:

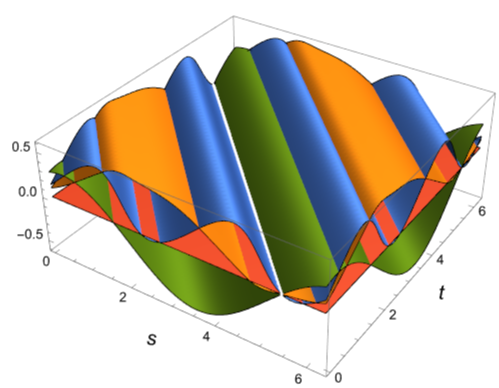


Figure 7.14: $F_i(s,t)$

Observe that the $F_i(s,t)$'s appear to be zero in one of the lines for which $\langle g_2, \xi_3\rangle = 0$. Moreover, the plot suggests that there is a solution for $s=0$. Solving for $\langle g_2, \xi_3\rangle(0,t)=0$ we obtain

$$t_0 = 2\pi - \arctan\left(\frac{\sqrt{-1+\sqrt{17}}}{2}\right) \approx 5.3873.$$

We now consider the plot of $F_i(0,t)$ and $G_{i,j}(0,t)$:

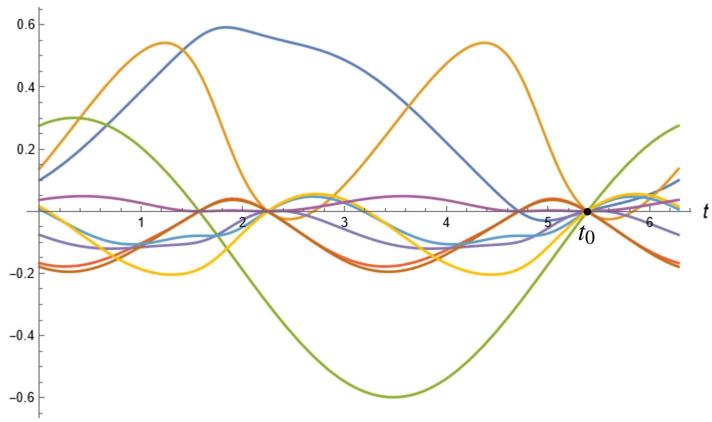


Figure 7.15: $F_i(0,t)$ and $G_{i,j}(0,t)$

Observe that they seem to share a single common root at t_0. Algebraically we can verify that this is indeed a common root and therefore

$$v^{(a_1,A_2)} = \begin{array}{c} \\ A_2 \\ \\ A_1 \end{array} \left(\begin{array}{cc|c|c|c} \multicolumn{2}{c|}{a_4} & a_3 & a_2 & a_1 \\ & & \sqrt{\frac{-169+41\sqrt{17}}{32}} & \sqrt{\frac{-1+\sqrt{17}}{8}} & -\frac{\sqrt{29-7\sqrt{17}}}{2} \\ \zeta_1^t & \zeta_2^t & \frac{-9+\sqrt{17}}{8} & \frac{1}{2} & \frac{\sqrt{-3+\sqrt{17}}}{2} \\ & & -\sqrt{\frac{31-7\sqrt{17}}{8}} & 0 & -\sqrt{-4+\sqrt{17}} \\ \hline w_3^t & w_4^t & \beta_3^t & \beta_2^t & \beta_1^t \end{array} \right).$$

To determine the ζ_i's we observe that any choice of η_i's that makes $u^{(A_2,a_4)}$ into a unitary will suffice, for example

$$u^{(A_2,a_4)} = \begin{array}{c} a_2 \\ \\ a_1 \\ \end{array} \overset{A_2}{\left(\begin{array}{ccc} \sqrt{\frac{-75+19\sqrt{17}}{32}} & \sqrt{\frac{23+\sqrt{17}}{32}} & \frac{5-\sqrt{17}}{4} \\ \hline \frac{3-\sqrt{17}}{2} & 0 & \sqrt{\frac{-11+3\sqrt{17}}{2}} \\ -\sqrt{\frac{-101+29\sqrt{17}}{32}} & \frac{-1+\sqrt{17}}{8} & -\sqrt{\frac{31-7\sqrt{17}}{8}} \end{array} \right)}.$$

To determine w_i's and the β_i's in the block $v^{(a_1,A_2)}$ we complete the first three rows of $v^{(a_1,A_2)}$ to an orthonormal basis of $\mathbb{C}^5$. Once again, the choice of these vectors does not matter as the inner products $\langle w_i, w_j\rangle$ and $\langle \beta_i, \beta_j\rangle$ are already determined by the rest of the matrix. Finally, the two 5×5 blocks we considered are $u^{(A_2,a_1)} =$

$$\begin{array}{c} \\ a_4 \\ \\ a_3 \\ a_2 \\ a_1 \end{array}
\begin{array}{c}
\begin{array}{ccccc} & A_2 & & A_1 & \end{array} \\
\left(\begin{array}{ccc|cc}
\sqrt{\frac{1+\sqrt{17}}{8}} & 0 & 0 & \sqrt{\frac{7-\sqrt{17}}{8}} & 0 \\
0 & \sqrt{\frac{1+\sqrt{17}}{8}} & 0 & 0 & \sqrt{\frac{7-\sqrt{17}}{8}} \\ \hline
\sqrt{-4+\sqrt{17}} & \frac{-3+\sqrt{17}}{4} & \sqrt{\frac{5-\sqrt{17}}{2}} & -\frac{\sqrt{5-\sqrt{17}}}{2} & -\sqrt{\frac{-3+\sqrt{17}}{8}} \\ \hline
-\sqrt{\frac{-19+5\sqrt{17}}{8}} & 0 & \sqrt{\frac{5-\sqrt{17}}{2}} & \sqrt{\frac{7-\sqrt{17}}{8}} & 0 \\ \hline
-\frac{\sqrt{29-7\sqrt{17}}}{2} & \frac{\sqrt{-3+\sqrt{17}}}{2} & -\sqrt{-4+\sqrt{17}} & \sqrt{\frac{-4+\sqrt{17}}{2}} & -\frac{1}{\sqrt{2}}
\end{array}\right)
\end{array}$$

and $v^{(a_1,A_2)} =$

$$\begin{array}{c} \\ \\ A_2 \\ \\ A_1 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} & a_4 & a_3 & a_2 & a_1 \end{array} \\
\left(\begin{array}{cc|c|c|c}
-\sqrt{\frac{-19+5\sqrt{17}}{8}} & \frac{-9+\sqrt{17}}{8} & \sqrt{\frac{-169+41\sqrt{17}}{32}} & \sqrt{\frac{-1+\sqrt{17}}{8}} & -\frac{\sqrt{29-7\sqrt{17}}}{2} \\
0 & \sqrt{\frac{-1+\sqrt{17}}{32}} & \frac{-9+\sqrt{17}}{8} & \frac{1}{2} & \frac{\sqrt{-3+\sqrt{17}}}{2} \\
\sqrt{\frac{5-\sqrt{17}}{2}} & -\sqrt{\frac{-11+3\sqrt{17}}{8}} & -\sqrt{\frac{31-7\sqrt{17}}{8}} & 0 & -\sqrt{-4+\sqrt{17}} \\ \hline
\sqrt{\frac{-3+\sqrt{17}}{8}} & -\frac{\sqrt{5-\sqrt{17}}}{2} & \sqrt{\frac{-3+\sqrt{17}}{8}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{\sqrt{5-\sqrt{17}}}{2} & \sqrt{\frac{-3+\sqrt{17}}{8}} & \frac{\sqrt{5-\sqrt{17}}}{2} & \sqrt{\frac{7-\sqrt{17}}{8}} & -\sqrt{\frac{-4+\sqrt{17}}{2}}
\end{array}\right)
\end{array}.$$

In a similar manner we obtain the 5×5 blocks $u^{(B_2,b_1)}$ and $v^{(b_1,B_2)}$. The remaining 3×3 blocks: $u^{(A_2,b_1)}$, $u^{(B_2,a_1)}$, $v^{(a_1,B_2)}$ and $v^{(b_1,A_2)}$ work exactly as the 3×3 blocks in the medium broom. The rest of u and v can be easily filled using the bi-unitary condition. In Appendix A we present all of the entries of u and v for the bi-unitary connection.

Remark 7.3. From figure (7.14) we have that $F_i(s,t) = 0$ on the line $s+t = t_0$. Moreover, we can verify that $G_{i,j}(s,t) = 0$ on this line and therefore we have a 1-parameter family of solutions for u and v.

Theorem 7.4. *There is an irreducible, hyperfinite A_∞-subfactor with index $\frac{5+\sqrt{17}}{2}$.*

Proof. Note that the large double broom is not one of the module graphs in Table 6.2 or Table 6.3. By the embedding theorem 5.1 we deduce that the subfactor obtained from this commuting square is not the Asaeda-Haagerup subfactor and consequently has infinite depth. Since the 1-norm of the first column of G is 1, Wenzl's criterion implies this subfactor is irreducible. Therefore we have constructed an irreducible hyperfinite subfactor with index $\frac{5+\sqrt{17}}{2}$ and infinite depth. By classification it has to have trivial standard invariant. $\square$

Next, we list tables that record the structure of the bi-unitary connection for the large double broom (see page 60).

p	s / r \ q	a_4	a_3	a_2	a_1		a_0		b_0		b_1		b_2	b_3	b_4
		A_2	A_2	A_2	A_2	A_1	A_1	A_0	A_0	B_1	B_1	B_2	B_2	B_2	B_2
A_2	a_4			•••	••••••	••••			•	•					••••
	a_3		•••	•••	•••	••	••	•			•	••		••	
	a_2	•••	•••		•••	••	••	•			•	••	••		
	a_1	••••••	•••	•••	•••	••	••	•	•	•	•	••	••	••	
A_1	a_1	••••	••	••	••			•	•	•	•	•	•	•	
	a_0		••	••	••				•	•	•	•			•
A_0	a_0		•	•	•	•				•	•	•			•
	b_0	•			•	•	•				•	•	•	•	
B_1	b_0	•			•	•	•	•				••	••	••	
	b_1		•	•	•	•	•	•	•			••	••	••	••••
B_2	b_1		••	••	••	•	•	•	•	••	••	•••	•••	•••	••••••
	b_2			••	••	•			•	••	••	•••		•••	•••
	b_3		••		••	•			•	••	••	•••	•••	•••	
	b_4	••••					•	•			••••	••••••	•••		

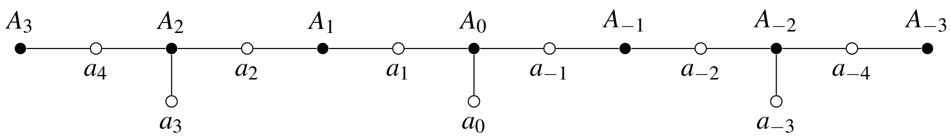
Figure 7.16: Large double broom - structure of *u*

q	r / s \ p	A_2				A_1		A_0		B_1		B_2			
		a_4	a_3	a_2	a_1	a_1	a_0	a_0	b_0	b_0	b_1	b_1	b_2	b_3	b_4
a_4	A_2			•••	••••••	••••			•	•					••••
a_3	A_2		•••	•••	•••	••	••	•			•	••		••	
a_2	A_2	•••	•••		•••	••	••	•			•	••	••		
a_1	A_2	••••••	•••	•••	•••	••	••	•	•	•	•	••	••	••	
	A_1	••••	••	••	••			•	•	•	•	•	•	•	
a_0	A_1		••	••	••				•	•	•	•			•
	A_0		•	•	•	•				•	•	•			•
b_0	A_0	•			•	•	•				•	•	•	•	
	B_1	•			•	•	•	•				••	••	••	
b_1	B_1		•	•	•	•	•	•	•			••	••	••	••••
	B_2		••	••	••	•	•	•	•	••	••	•••	•••	•••	••••••
b_2	B_2			••	••	•			•	••	••	•••		•••	•••
b_3	B_2		••		••	•			•	••	••	•••	•••	•••	
b_4	B_2	••••					•	•			••••	••••••	•••		

Figure 7.17: Large double broom - structure of *v*

7.5 Quipu

For the "quipu" we will use the following labeling of the vertices:

$\Gamma =$ 

Since the entries of the Perron-Frobenius eigenvector for the graph Γ do not have simple algebraic expressions, we will just use $\lambda(v)$ to refer to them, where v is a vertex of Γ. In particular, the symmetry of Γ implies $\lambda(A_i) = \lambda(A_{-i})$ for $i = 1,2,3$ and $\lambda(a_j) = \lambda(a_{-j})$ for $j = 1,2,3,4$. Since $(\lambda(A_i))_i$ is a Perron-Frobenius eigenvector of G^tG and $(\lambda(a_j))_j = G(\lambda(A_i))_i$, we also have the following relations between the $\lambda(A_i)'s$ and the $\lambda(a_j)'s$:

$$\begin{aligned}
&\lambda(a_0) = \lambda(A_0), && t\lambda(A_0) = 2\lambda(a_1) + \lambda(a_0),\\
&\lambda(a_1) = \lambda(A_1) + \lambda(A_0), && t\lambda(A_1) = \lambda(a_2) + \lambda(a_1),\\
&\lambda(a_2) = \lambda(A_2) + \lambda(A_1), && t\lambda(A_2) = \lambda(a_4) + \lambda(a_3) + \lambda(a_2),\\
&\lambda(a_3) = \lambda(A_2), && t\lambda(A_3) = \lambda(a_4),\\
&\lambda(a_4) = \lambda(A_3) + \lambda(A_2), &&
\end{aligned} \tag{7.17}$$

where t is the Perron-Frobenius eigenvalue of G^tG. Setting $\lambda(A_0) = 1$, we can use all but the last of the above relations to obtain

$$\begin{aligned}
&\lambda(a_0) = 1, && \lambda(A_0) = 1,\\
&\lambda(a_1) = \frac{t-1}{2}, && \lambda(A_1) = \frac{t-3}{2},\\
&\lambda(a_2) = \frac{t^2-4t+1}{2}, && \lambda(A_2) = \frac{t^2-5t+4}{2},\\
&\lambda(a_3) = \frac{t^2-5t+4}{2}, && \lambda(A_3) = \frac{t^3-7t^2+13t-5}{2t},\\
&\lambda(a_4) = \frac{t^3-7t^2+13t-5}{2}. &&
\end{aligned} \tag{7.18}$$

It can be shown that G^tG has minimal polynomial $p(x) = x^3 - 8x^2 + 17x - 5$, and hence

$$t^3 = 8t^2 - 17t + 5. \tag{7.19}$$

Using the same notation as in the previous section we have

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

for the vertical inclusions, and

$$H = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

for the horizontal ones. Since

$$GK = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1 \\ 2 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 \\ 2 & 2 & 4 & 4 & 2 & 4 & 4 & 2 & 2 \\ 2 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 \\ 1 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

we deduce that u has four 4×4 blocks, eight 3×3 blocks, twenty-one 2×2 blocks and twenty 1×1 blocks, same for v. Figure 7.18 and figure 7.19 describe the block structure for both u and v.

u \ rs		a_4		a_3	a_2		a_1		a_0	a_{-1}		a_{-2}		a_{-3}	a_{-4}	
pq		A_3	A_2	A_2	A_2	A_1	A_1	A_0	A_0	A_0	A_{-1}	A_{-1}	A_{-2}	A_{-2}	A_{-2}	A_{-3}
A_3	a_4					•	•				•	•				
A_2	a_4				•	•	•	••		••	•	•	•			
	a_3			•			•	••		••	•			•		
	a_2		•		•	•	•	••	::	••	•	•	•		•	
A_1	a_2	•	•		•	•	•	••	::	••	•	•	•		•	•
	a_1	•	•	•	•	•	:	::		::	:	•	•	•	•	•
A_0	a_1		••	••	••	••	::	::		::	::	••	••	••	••	
	a_0				::	::			::			::	::			
	a_{-1}		••	••	••	••	::	::		::	::	••	••	••	••	
A_{-1}	a_{-1}	•	•	•	•	•	:	::		::	:	•	•	•	•	•
	a_{-2}	•	•		•	•	•	••	::	••	•	•	•		•	•
A_{-2}	a_{-2}		•		•	•	•	••	::	••	•	•	•		•	
	a_{-3}			•			•	••		••	•			•		
	a_{-4}				•	•	•	••		••	•	•	•			
A_{-3}	a_{-4}					•	•				•	•				

Figure 7.18: Quipu - structure of u

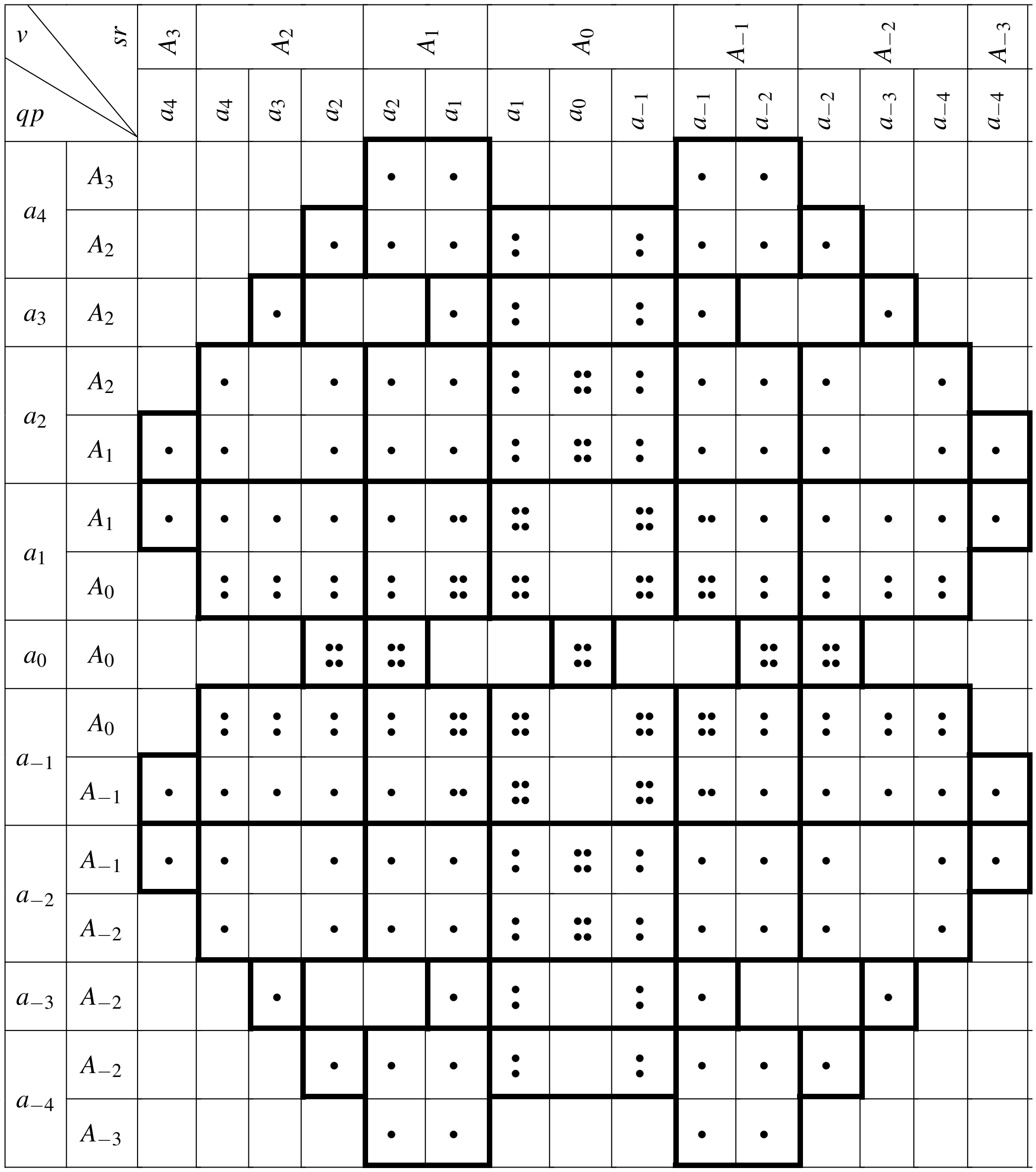


Figure 7.19: Quipu - structure of v

We make the following choices for the 1×1 blocks in u:

$$u^{(A_{\pm3},a_{\pm2})} = u^{(A_{\pm2},a_{\pm4})} = u^{(A_{\pm2},a_{\pm3})} = u^{(A_{\pm1},a_{\pm3})} = 1, \quad u^{A_{\pm3},a_{\pm1}} = -1.$$

Similarly for the 1×1 blocks in v, we choose:

$$v^{(a_{\pm2},A_{\pm3})} = v^{(a_{\pm4},A_{\pm2})} = v^{(a_{\pm3},A_{\pm2})} = v^{(a_{\pm3},A_{\pm1})} = 1, \quad v^{a_{\pm1},A_{\pm3}} = -1.$$

Using the bi-unitary condition we start filling in the entries for the 2×2 blocks that are connected to some 1×1 block. To simplify the notation we set

$$x_1=\sqrt{\frac{\lambda(A_3)\lambda(a_2)}{\lambda(a_4)\lambda(A_1)}},\quad y_1=\sqrt{\frac{\lambda(A_3)\lambda(a_1)}{\lambda(a_4)\lambda(A_1)}},\quad x_2=\sqrt{\frac{\lambda(a_4)}{\lambda(a_2)}},\quad y_2=\sqrt{\frac{\lambda(a_4)\lambda(A_1)}{\lambda(A_2)\lambda(a_2)}},\quad y_1=\sqrt{\frac{\lambda(A_3)\lambda(a_1)}{\lambda(A_2)\lambda(a_2)}}.$$

We then have the following for the blocks $v^{a_{\pm 4},A_{\pm 1}}$:

v \ sr		A_1		A_{-1}	
qp		a_2	a_1	a_{-1}	a_{-2}
a_4	A_3	x_1	$-y_1$	$-y_1$	x_1
	A_2	•	•	•	•
a_{-4}	A_{-2}	•	•	•	•
	A_{-3}	x_1	$-y_1$	$-y_1$	x_1

Note that using 7.17 we get

$$x_1^2+y_1^2=\frac{\lambda(A_3)(\lambda(a_1)+\lambda(a_2))}{\lambda(a_4)\lambda(A_1)}=1.$$

We get a similar result for the blocks $u^{(A_{\pm 1},a_{\pm 4})}$. Since every block has to be unitary, we determine the remaining entries in the following way:

v \ sr		A_1		A_{-1}	
qp		a_2	a_1	a_{-1}	a_{-2}
a_4	A_3	x_1	$-y_1$	$-y_1$	x_1
	A_2	y_1	x_1	x_1	y_1
a_{-4}	A_{-2}	y_1	x_1	x_1	y_1
	A_{-3}	x_1	$-y_1$	$-y_1$	x_1

We make similar choices for all the blocks $v^{(a_{\pm 4},A_{\pm 1})}$ and $u^{(A_{\pm 1},a_{\pm 4})}$. For the blocks $u^{(A_{\pm 2},a_{\pm 2})}$, the bi-unitary condition implies

u \ rs		a_2		a_{-2}	
pq		A_2	A_1	A_{-1}	A_{-2}
A_2	a_4	x_2	y_2	y_2	x_2
	a_2	•	•	•	•
A_{-2}	a_{-2}	•	•	•	•
	a_{-4}	x_2	y_2	y_2	x_2

Once again we choose the remaining entries so that every block is unitary. Thus

$$u^{(A_2,a_2)} = \begin{array}{c} \\ a_4 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cc} A_2 & A_1 \end{array} \\ \left(\begin{array}{c|c} x_2 & y_2 \\ \hline -y_2 & x_2 \end{array}\right) \end{array}.$$

The blocks $u^{(A_{\pm 2},a_{\pm 2})}$ and $v^{(a_{\pm 2},A_{\pm 2})}$ are filled in a similar manner. Using 7.18 and 7.19 we have

$$x_2^2 + y_2^2 = \frac{t^3 - 7t^2 + 13t - 5}{t^2 - 4t} = 1,$$

hence the blocks $u^{(A_{\pm 2},a_{\pm 2})}$ and $v^{(a_{\pm 2},A_{\pm 2})}$ are unitary.

We proceed to handle the blocks $u^{(A_{\pm 1},a_{\pm 2})}$ and $v^{(a_{\pm 2},A_{\pm 1})}$, which are connected to the previous 2×2 blocks. Set $x_4 = \sqrt{\frac{\lambda(A_2)}{\lambda(A_1)}} x_2 = \sqrt{\frac{\lambda(A_2)\lambda(a_4)}{\lambda(A_1)\lambda(a_2)}}$, since $u^{(A_{\pm 2},a_{\pm 2})}_{a_{\pm 2},A_{\pm 1}} = x_2$ and $v^{(a_{\pm 2},A_{\pm 2})}_{A_{\pm 1},a_{\pm 2}} = x_2$ then

u \ rs		a_2		a_{-2}	
pq		A_2	A_1	A_{-1}	A_{-2}
A_1	a_2	x_4	•	•	x_4
	a_1	•	•	•	•
A_{-1}	a_{-1}	•	•	•	•
	a_{-2}	x_4	•	•	x_4

v \ sr		A_1		A_{-1}	
qp		a_2	a_1	a_{-1}	a_{-2}
a_2	A_2	x_4	•	•	x_4
	A_1	•	•	•	•
a_{-2}	A_{-1}	•	•	•	•
	A_{-2}	x_4	•	•	x_4

Let $z_4 = \sqrt{1-x_4^2}$ and choose the remaining entries in $v^{(a_{\pm 2},A_{\pm 1})}$ in the following way

v \ sr		A_1		A_{-1}	
qp		a_2	a_1	a_{-1}	a_{-2}
a_2	A_2	x_4	z_4	z_4	x_4
	A_1	$-z_4$	x_4	x_4	$-z_4$
a_{-2}	A_{-1}	$-z_4$	x_4	x_4	$-z_4$
	A_{-2}	x_4	z_4	z_4	x_4

Since $v^{(a_{\pm 2},A_{\pm 1})}_{A_{\pm 1},a_{\pm 2}} = -z_4$, the bi-unitary condition implies that $u^{(A_{\pm 1},a_{\pm 2})}_{a_{\pm 2},A_{\pm 1}} = -z_4$ and consequently the blocks $u^{(A_{\pm 1},a_{\pm 2})}$ are obtained in the following way:

u \ rs		a_2		a_{-2}	
pq		A_2	A_1	A_{-1}	A_{-2}
A_1	a_2	x_4	$-z_4$	$-z_4$	x_4
	a_1	z_4	x_4	x_4	z_4
A_{-1}	a_{-1}	z_4	x_4	x_4	z_4
	a_{-2}	x_4	$-z_4$	$-z_4$	x_4

We proceed to determine the entries of the 3×3 blocks $u^{(A_{\pm 2},a_{\pm 1})}$ and $v(a_{\pm 1},A_{\pm 2})$. Let

$$x_3 = \sqrt{\frac{\lambda(a_4)\lambda(A_1)}{\lambda(A_2)\lambda(a_1)}}, \qquad x_1 = \sqrt{\frac{\lambda(a_2)\lambda(A_3)}{\lambda(A_2)\lambda(a_1)}}, \qquad y_3 = \sqrt{\frac{\lambda(a_3)\lambda(A_1)}{\lambda(A_2)\lambda(a_1)}}, \qquad z_3 = \sqrt{\frac{\lambda(a_2)\lambda(A_1)}{\lambda(A_2)\lambda(a_1)}}z_4.$$

Since

$$\begin{aligned} v^{(a_{\pm 4},A_{\pm 1})}_{A_{\pm 2},a_{\pm 1}} &= x_1 & \qquad v^{(a_{\pm 3},A_{\pm 1})}_{A_{\pm 2},a_{\pm 1}} &= 1 & \qquad v^{(a_{\pm 2},A_{\pm 1})}_{A_{\pm 2},a_{\pm 1}} &= z_4, \\ u^{(A_{\pm 1},a_{\pm 4})}_{a_{\pm 1},A_{\pm 2}} &= x_1 & \qquad u^{(A_{\pm 1},a_{\pm 3})}_{a_{\pm 1},A_{\pm 2}} &= 1 & \qquad u^{(A_{\pm 1},a_{\pm 2})}_{a_{\pm 1},A_{\pm 2}} &= z_4, \end{aligned}$$

the bi-unitary condition implies that the blocks are given by

u \ rs		a_1		a_{-1}	
pq		A_1	A_0	A_0	A_{-1}
A_2	a_4	x_3	••	••	x_3
	a_3	y_3	••	••	y_3
	a_2	z_3	••	••	z_3
A_{-2}	a_{-2}	z_3	••	••	z_3
	a_{-3}	y_3	••	••	y_3
	a_{-4}	x_3	••	••	x_3

v \ sr		A_2			A_{-2}		
qp		a_4	a_3	a_2	a_{-2}	a_{-3}	a_{-4}
a_1	A_1	x_3	y_3	z_3	z_3	y_3	x_3
	A_0	:	:	:	:	:	:
a_{-1}	A_0	:	:	:	:	:	:
	A_{-1}	x_3	y_3	z_3	z_3	y_3	x_3

Using 7.18 and 7.19 we get

$$x_3^2+y_3^2+z_3^2=\frac{-t^5+12t^4-49t^3+76t^2-33t+5}{t^3-5t^2+4t}=1.$$

Thus, the known column (or row) of every block has norm 1.

Let $\alpha_3,\beta_3,\gamma_3\in\mathbb{C}^2$ be such that
$$u^{(A_2;a_1)}=\begin{array}{c} \\ a_4 \\ a_3 \\ a_2 \end{array}\begin{array}{c} \begin{array}{cc} A_1 & \quad A_0 \end{array} \\ \left(\begin{array}{c|cc} x_3 & \alpha_{3,1} & \alpha_{3,2} \\ y_3 & \beta_{3,1} & \beta_{3,2} \\ \hline z_3 & \gamma_{3,1} & \gamma_{3,2} \end{array}\right)\end{array}$$
is unitary. We fill the remaining blocks in the following way

u \ rs		a_1		a_{-1}	
pq		A_1	A_0	A_0	A_{-1}
A_2	a_4	x_3	α_3	$\tilde{\alpha}_3$	x_3
	a_3	y_3	β_3	$\tilde{\beta}_3$	y_3
	a_2	z_3	γ_3	$\tilde{\gamma}_3$	z_3
A_{-2}	a_{-2}	z_3	γ_3	$\tilde{\gamma}_3$	z_3
	a_{-3}	y_3	β_3	$\tilde{\beta}_3$	x_3
	a_{-4}	x_3	α_3	$\tilde{\alpha}_3$	x_3

v \ sr		A_2			A_{-2}		
qp		a_4	a_3	a_2	a_{-2}	a_{-3}	a_{-4}
a_1	A_1	x_3	y_3	z_3	z_3	y_3	x_3
	A_0	α_3^t	β_3^t	γ_3^t	γ_3^t	β_3^t	α_3^t
a_{-1}	A_0	$\tilde{\alpha}_3{}^t$	$\tilde{\beta}_3{}^t$	$\tilde{\gamma}_3{}^t$	$\tilde{\gamma}_3{}^t$	$\tilde{\beta}_3{}^t$	$\tilde{\alpha}_3{}^t$
	A_{-1}	x_3	y_3	z_3	z_3	y_3	x_3

where $\tilde{\alpha}_i = (-\alpha_{i,2}, \alpha_{i,1})$. Observe that since each row and each column has norm 1 we have

$$\|\gamma_3\|^2 = x_3^2 + y_3^2 = \frac{\lambda(a_2)\lambda(A_3) + \lambda(a_3)\lambda(A_1)}{\lambda(A_2)\lambda(a_1)}. \tag{7.20}$$

This allows us to determine the entries of the 2×2 blocks $u^{(A_0, a_{\pm 4})}$, $u^{(A_0, a_{\pm 3})}$, $v^{(a_{\pm 3}, A_0)}$ and $v^{(a_{\pm 4}, A_0)}$. Set

$$\alpha_4 = \sqrt{\frac{\lambda(A_2)\lambda(a_1)}{\lambda(a_4)\lambda(A_0)}}\alpha_3, \quad \beta_4 = \sqrt{\frac{\lambda(A_2)\lambda(a_1)}{\lambda(a_3)\lambda(A_0)}}\alpha_3.$$

Then the bi-unitary condition implies

u \ rs / pq		a_4	a_3	a_{-3}	a_{-4}
		A_2	A_2	A_{-2}	A_{-2}
A_0	a_1	α_4	β_4	β_4	α_4
	a_{-1}	$\tilde{\alpha}_4$	$\tilde{\beta}_4$	$\tilde{\beta}_4$	$\tilde{\alpha}_4$

v \ sr / qp		A_0	
		a_1	a_{-1}
a_4	A_2	α_4^t	$\tilde{\alpha}_4{}^t$
a_3	A_2	β_4^t	$\tilde{\beta}_4{}^t$
a_{-3}	A_{-2}	β_4^t	$\tilde{\beta}_4{}^t$
a_{-4}	A_{-2}	α_4^t	$\tilde{\alpha}_4{}^t$

Note that $\|\alpha_3\|^2 = 1 - \frac{\lambda(a_2)\lambda(A_3)}{\lambda(A_2)\lambda(a_1)}$ and consequently $\|\alpha_4\|^2 = \frac{\lambda(A_2)\lambda(a_1) - \lambda(A_3)\lambda(a_2)}{\lambda(A_0)\lambda(a_4)}$. Using 7.18 and 7.19 we have

$$\|\alpha_4\|^2 = \frac{-t^5 + 12t^4 - 48t^3 + 73t^2 - 37t + 5}{2t^4 - 14t^3 + 26t^2 - 10t} = 1,$$

similarly we obtain $\|\beta_4\|^2 = 1$. Hence, all the blocks above are unitary.

For the remaining four 3×3 blocks of u and v, we set $x_5 = \sqrt{\frac{\lambda(a_2)}{\lambda(a_1)}}x_4 = \sqrt{\frac{\lambda(A_2)\lambda(a_4)}{\lambda(A_1)\lambda(a_1)}}$. Then, by the bi-unitary condition, we have

u (pq \ rs)		a_1		a_{-1}	
		A_1	A_0	A_0	A_{-1}
A_1	a_2	x_5	••	••	x_5
	a_1	:	::	::	:
A_{-1}	a_{-1}	:	::	::	:
	a_{-2}	x_5	••	••	x_5

v (qp \ sr)		A_1		A_{-1}	
		a_2	a_1	a_{-1}	a_{-2}
a_1	A_1	x_5	••	••	x_5
	A_0	:	::	::	:
a_{-1}	A_0	:	::	::	:
	A_{-1}	x_5	••	••	x_5

and to determine the remaining entries we make the following choices

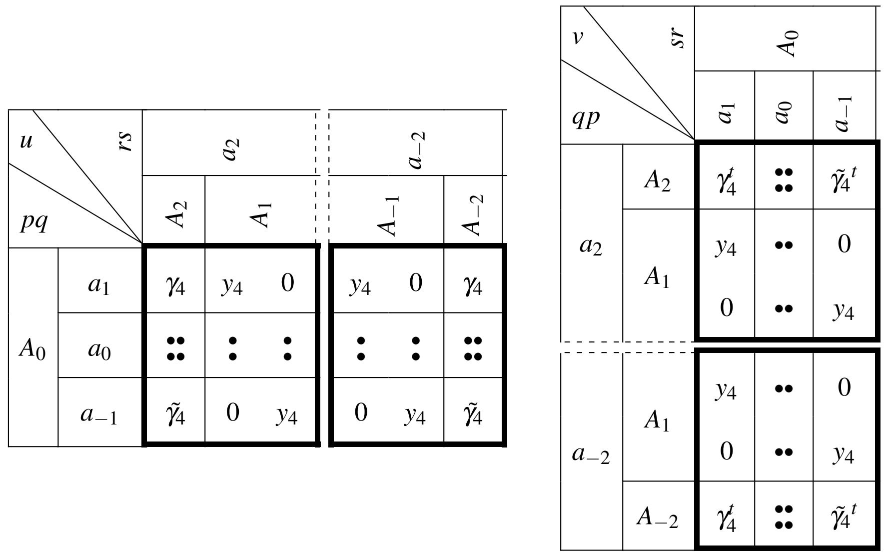
u (pq \ rs)		a_1			a_{-1}		
		A_1	A_0		A_0		A_{-1}
A_1	a_2	x_5	y_5	0	0	y_5	x_5
	a_1	y_5	$-x_5$	0	0	$-x_5$	y_5
		0	0	1	1	0	0
A_{-1}	a_{-1}	0	0	1	1	0	0
		y_5	$-x_5$	0	0	$-x_5$	y_5
	a_{-2}	x_5	y_5	0	0	y_5	x_5

v (qp \ sr)		A_1			A_{-1}		
		a_2	a_1		a_{-1}		a_{-2}
a_1	A_1	x_5	y_5	0	0	y_5	x_5
	A_0	y_5	$-x_5$	0	0	$-x_5$	y_5
		0	0	1	1	0	0
a_{-1}	A_0	0	0	1	1	0	0
		y_5	$-x_5$	0	0	$-x_5$	y_5
	A_{-1}	x_5	y_5	0	0	y_5	x_5

where $y_5 = \sqrt{1-x_5^2} = \sqrt{\frac{\lambda(A_1)\lambda(a_1)-\lambda(A_2)\lambda(a_4)}{\lambda(A_1)\lambda(a_1)}}$. All blocks are unitary with these choices. We can use these to obtain the entries of the 4×4 blocks $u^{(A_0,a_{\pm 2})}$ and $v^{(a_{\pm 2},A_0)}$. Set

$$y_4 = \sqrt{\frac{\lambda(a_1)\lambda(A_1)}{\lambda(A_0)\lambda(a_2)}}y_5 = \sqrt{\frac{\lambda(A_1)\lambda(a_1)-\lambda(A_2)\lambda(a_4)}{\lambda(A_0)\lambda(a_2)}}, \quad \gamma_4 = \sqrt{\frac{\lambda(A_2)\lambda(a_1)}{\lambda(a_2)\lambda(A_0)}}\gamma_3,$$

then the bi-unitary condition implies



Note that by 7.20 we have

$$\|\gamma_4\|^2 = \frac{\lambda(a_2)\lambda(A_3)+\lambda(a_3)\lambda(A_1)}{\lambda(a_2)\lambda(A_0)}. \tag{7.21}$$

and using 7.18 and 7.19 we obtain

$$y_4^2 + \|\gamma_4\|^2 = \frac{-t^4+9t^3-25t^2+24t-5}{2t} = 1.$$

Now, let $X, Y \in M_2(\mathbb{C})$ be two matrices such that

$$u^{(A_0,a_2)} = \begin{array}{c} \\ a_1 \\ a_0 \\ \\ a_{-1} \end{array} \begin{array}{c} \begin{array}{cccc} A_2 & & A_1 & \end{array} \\ \left(\begin{array}{cc|cc} \gamma_{4,1} & \gamma_{4,2} & y_4 & 0 \\ \hline X_{1,1} & X_{1,2} & Y_{1,1} & Y_{1,2} \\ X_{2,1} & X_{2,2} & Y_{2,1} & Y_{2,2} \\ \hline -\gamma_{4,2} & \gamma_{4,1} & 0 & y_4 \end{array}\right) \end{array}$$

is unitary. Note that such matrices exist as the first and last row of $u^{(A_0,a_2)}$ are orthonormal. Completing these two rows to an orthonormal basis produces such X and Y. The entries for $u^{(A_0,a_{\pm 2})}$ and $v^{(a_{\pm 2},A_0)}$ will be

<table>
<tr><td>u \ rs</td><td></td><td colspan="2">a_2</td><td colspan="2">a_{-2}</td></tr>
<tr><td>pq</td><td></td><td>A_2</td><td>A_1</td><td>A_{-1}</td><td>A_{-2}</td></tr>
<tr><td rowspan="3">A_0</td><td>a_1</td><td>γ_4</td><td>y_4 0</td><td>y_4 0</td><td>γ_4</td></tr>
<tr><td>a_0</td><td>X</td><td>Y</td><td>Y</td><td>X</td></tr>
<tr><td>a_{-1}</td><td>$\tilde{\gamma}_4$</td><td>0 y_4</td><td>0 y_4</td><td>$\tilde{\gamma}_4$</td></tr>
</table>

<table>
<tr><td>v \ sr</td><td></td><td colspan="3">A_0</td></tr>
<tr><td>qp</td><td></td><td>a_1</td><td>a_0</td><td>a_{-1}</td></tr>
<tr><td rowspan="2">a_2</td><td>A_2</td><td>γ_4^t</td><td>X^t</td><td>$\tilde{\gamma}_4{}^t$</td></tr>
<tr><td>A_1</td><td>y_4
0</td><td>Y^t</td><td>0
y_4</td></tr>
<tr><td rowspan="2">a_{-2}</td><td>A_{-1}</td><td>y_4
0</td><td>Y^t</td><td>0
y_4</td></tr>
<tr><td>A_{-2}</td><td>γ_4^t</td><td>X^t</td><td>$\tilde{\gamma}_4{}^t$</td></tr>
</table>

Since $u^{(A_0,a_2)}$ is unitary, its columns are orthonormal and therefore

$$0 = X_{1,1}X_{1,2} + X_{2,1}X_{2,2} + \gamma_{4,1}\gamma_{4,2} - \gamma_{4,1}\gamma_{4,2} = X_{1,1}X_{1,2} + X_{2,1}X_{2,2}$$
$$0 = Y_{1,1}Y_{1,2} + Y_{2,1}Y_{2,2}.$$

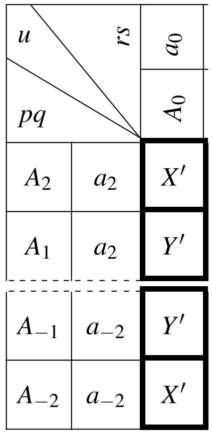
Hence, the columns of X and Y are orthogonal. Moreover, since every column has norm 1, we have

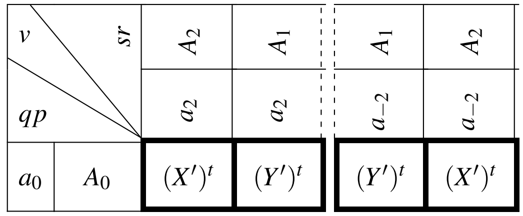
$$1 = \|\gamma_4\|^2 + X_{1,1}^2 + X_{2,1}^2 = \|\gamma_4\|^2 + X_{1,2}^2 + X_{2,2}^2$$

and hence

$$X_{1,1}^2 + X_{2,1}^2 = X_{1,2}^2 + X_{2,2}^2 = 1 - \|\gamma_4\|^2 = \frac{\lambda(a_2)\lambda(A_0) - \lambda(a_2)\lambda(A_3) - \lambda(a_3)\lambda(A_1)}{\lambda(a_2)\lambda(A_0)}. \tag{7.22}$$

Similarly, we get $1 - y_4^2 = Y_{1,1}^2 + Y_{2,1}^2 = Y_{1,2}^2 + Y_{2,2}^2$. Set $X' = \sqrt{\frac{\lambda(A_0)\lambda(a_2)}{\lambda(a_0)\lambda(A_2)}}X$ and $Y' = \sqrt{\frac{\lambda(A_0)\lambda(a_2)}{\lambda(a_0)\lambda(A_1)}}Y$. The last four 2×2 blocks for u and v will then be

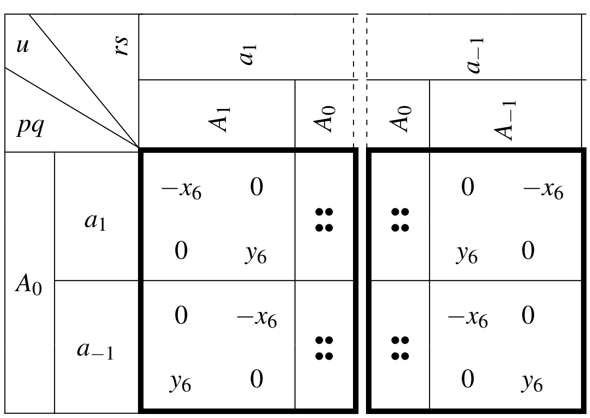


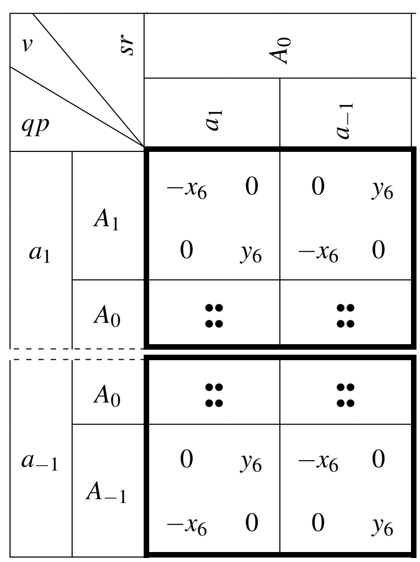


Using 7.18, 7.19 and 7.22 we get

$$
\begin{aligned}
(X'_{1,1})^2+(X'_{2,1})^2=(X'_{1,2})^2+(X'_{2,2})^2 &= \frac{\lambda(a_2)\lambda(A_0)-\lambda(a_2)\lambda(A_3)-\lambda(a_3)\lambda(A_1)}{\lambda(a_0)\lambda(A_2)}\\
&= \frac{t^4-9t^3+23t^2-14t+5}{-2t^2+8t}=1\\
(Y'_{1,1})^2+(Y'_{2,1})^2=(Y'_{1,2})^2+(Y'_{2,2})^2 &= \frac{\lambda(A_0)\lambda(a_2)-\lambda(A_1)\lambda(a_1)+\lambda(A_2)\lambda(a_4)}{\lambda(a_0)\lambda(A_1)}\\
&= \frac{t^4-9t^3+25t^2-22t+7}{2}=1
\end{aligned}
$$

and since the columns of X' and Y' are orthogonal to each other we conclude that both X' and Y' are unitary. We now have only two 4×4 blocks remaining for u and v: $u^{(A_0,a_{\pm1})}$ and $v^{(a_{\pm1},A_0)}$. Using the entries of $u^{(A_{\pm1},a_{\pm1})}_{a_{\pm1},A_0}$ and $v^{(a_{\pm1},A_{\pm1})}_{A_0,a_{\pm1}}$, the bi-unitary condition implies





where $x_6 = \sqrt{\frac{\lambda(A_1)}{\lambda(A_0)}}x_5 = \sqrt{\frac{\lambda(A_2)\lambda(a_4)}{\lambda(A_0)\lambda(a_1)}}$ and $y_6 = \sqrt{\frac{\lambda(A_1)}{\lambda(A_0)}}$. Using 7.18 and 7.19 we get

$$x_6^2 + y_6^2 = \frac{t^4 - 11t^3 + 41t^2 - 56t + 17}{2} = 1$$

Thus, the first two columns of $u^{(A_0,a_1)}$ have norm 1, and moreover, they are orthonormal. Hence there exists $Z, W \in M_2(\mathbb{C})$ such that

$$u^{(A_0,a_1)} = \begin{array}{c} \\ a_1 \\ \\ a_{-1} \end{array} \overset{\begin{array}{cc} A_1 & \quad A_0 \end{array}}{\left(\begin{array}{cc|c} -x_6 & 0 & \\ 0 & y_6 & Z \\ \hline 0 & -x_6 & \\ y_6 & 0 & W \end{array}\right)}$$

is unitary. Finally, we can finish the remaining entries of $u^{A_0,a_{\pm 1}}$ and $v^{(a_{\pm 1},A_0)}$ in the following way:

u	rs	a_1			a_{-1}		
pq		A_1		A_0	A_0	A_{-1}	
A_0	a_1	$-x_6$	0	Z	W	0	$-x_6$
		0	y_6			y_6	0
	a_{-1}	0	$-x_6$	W	Z	$-x_6$	0
		y_6	0			0	y_6

v	sr	A_0			
qp		a_1		a_{-1}	
a_1	A_1	$-x_6$	0	0	y_6
		0	y_6	$-x_6$	0
	A_0	Z^t		W^t	
a_{-1}	A_0	W^t		Z^t	
	A_{-1}	0	y_6	$-x_6$	0
		$-x_6$	0	0	y_6

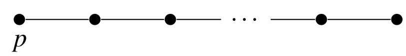
Observe that this makes all blocks unitary and is consistent with the bi-unitary condition. We conclude that there is a commuting square with vertical inclusions given by G and horizontal inclusions given by H and K.

Theorem 7.5. *There is an irreducible, hyperfinite A_∞-subfactor with index ≈ 4.3720.*

Proof. Note that Γ is not one of the graphs into whose graph planar algebra, the Extended-Haagerup subfactor planar algebra embeds. The embedding theorem 5.1 implies that the subfactor obtained with the commuting square constructed above is not the Extended-Haagerup subfactor and consequently has infinite depth. Since the 1-norm of the first row of G is 1, Wenzl's criterion implies that this subfactor is irreducible. Therefore we have constructed an irreducible hyperfinite subfactor with index ≈ 4.37720 and infinite depth. By classification it must have trivial standard invariant. $\square$

CHAPTER 8

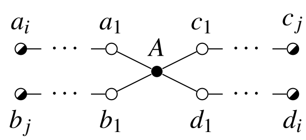
A 1-parameter family of commuting squares for 4-stars

Definition 8.1 ([Sch90])**.** We say that a graph G is an m-star, if G is connected, and has a "central" vertex p, of valency m, and m rays of the form  with k_i vertices (not counting p), $i = 1, \dots, m$. We will denote an m-star by $S(k_1, k_2, \dots, k_m)$, $k_1 \le k_2 \le \cdots \le k_m$.

Let G be a 4-star of the form $G = S(i,i,j,j)$. We show the existence of a 1-parameter family of non-equivalent bi-unitary connections for inclusions of the form

$$\begin{array}{ccc} A_{1,0} & \overset{G^t}{\subset} & A_{1,1} \\ \cup_G & & \cup_{G^t} \\ A_{0,0} & \overset{G}{\subset} & A_{0,1} \end{array} \tag{8.1}$$

We will use the following labeling of the vertices:



where black vertices denote minimal central projections in $\mathscr{Z}_{0,0}$, $\mathscr{Z}_{1,1}$, and white vertices denote minimal central projections in $\mathscr{Z}_{1,0}$, $\mathscr{Z}_{0,1}$. The size of the blocks of u is given by the entries of GG^t, which are equal to the number of paths of length 2 starting and ending on a black vertex. Similarly, the size of the summands of v is given by the number of paths of length 2 starting and ending on a white vertex. Consequently, we have the following summands for u and v:

Size	Summands
1×1	$u^{(A,x_2)}, u^{(x_2,A)}, u^{(x_{2k},x_{2(k\pm1)})}, v^{(x_{2k-1},x_{2(k\pm1)-1})}, v^{x_1,y_1}$
2×2	$u^{(x_{2k},x_{2k})}, v^{(x_{2k-1},x_{2k-1})}$
4×4	$u^{(A,A)}$

where $x \neq y \in \{a,b,c,d\}$, all other summands are 0. In particular, the only 4×4 summand has the following block structure

$$u^{(A,A)} = \begin{array}{c} \\ a_1 \\ b_1 \\ c_1 \\ d_1 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & b_1 & c_1 & d_1 \end{array} \\ \left(\begin{array}{c|c|c|c} * & * & * & * \\ \hline * & * & * & * \\ \hline * & * & * & * \\ \hline * & * & * & * \end{array}\right) \end{array}.$$

Note that $v^{(x_1,y_1)} \in \mathbb{T}$, for $x \neq y \in \{a,b,c,d\}$. Using the bi-unitary condition, we can find all the entries off the diagonal of $u^{(A,A)}$. In particular, $u^{(A,A)}$ is unitarily equivalent to a 4×4 unitary of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 & \alpha_3 \\ \alpha_2 & \beta z_1 & \beta z_2 & \alpha_2 z_3 \\ \alpha_2 & \beta x_1 & \beta x_2 & \alpha_2 x_3 \\ \alpha_3 & \alpha_2 y_1 & \alpha_2 y_2 & \alpha_1 y_3 \end{pmatrix}$$

where $x_i, y_i, z_i \in \mathbb{T}$ and the α_i, β are fixed positive real numbers of the form $w(A,x,y,A)^{-1}$ where $x \neq y \in \{a,b,c,d\}$ (see **??**). Since every row of u has to have norm 1 we have

$$\begin{cases} \alpha_1^2 + 2\alpha_2^2 + \alpha_3^2 & = 1, \\ 2\alpha_2^2 + 2\beta^2 & = 1. \end{cases} \tag{8.2}$$

Schou showed in [Sch90] that one can obtain such 4×4 unitaries for any (i,j). We will prove that one can actually obtain a 1-parameter family of non-equivalent unitaries for every fixed (i,j).

We first note that once we have the first three rows of u, these will determine three orthonormal vectors in $\mathbb{C}^4$. By picking an element in the orthogonal complement such that its first entry is α_3 we will obtain the values of the y_i's. Thus, we only need to determine z_i's and x_i's such that the first three rows of u are orthonormal.

We choose $x_3 = -z_3$. The orthogonality of the second and third row of u imply that (z_1, z_2) and (x_1, x_2) are orthogonal vectors in $\mathbb{C}^2$. Consequently, we have $x_1 = -w\bar{z}_2$ and $x_2 = w\bar{z}_1$ where $w \in \mathbb{T}$. Hence, we have the following for u:

$$u = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 & \alpha_3 \\ \alpha_2 & \beta z_1 & \beta z_2 & \alpha_2 z_3 \\ \alpha_2 & -\beta w\bar{z}_2 & \beta w\bar{z}_1 & -\alpha_2 z_3 \\ \alpha_3 & * & * & * \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

Using the orthogonality of the rows, we obtain

$$\alpha_1\alpha_2+\alpha_2\beta(z_1+z_2)+\alpha_2\alpha_3z_3=\langle u_2,u_1\rangle=0$$
$$\alpha_1\alpha_2+\alpha_2\beta w(\bar{z}_1-\bar{z}_2)-\alpha_2\alpha_3z_3=\langle u_3,u_1\rangle=0.$$

Dividing the equations by α_2 and multiplying the second one by $\overline{w}$ we obtain

$$\alpha_1+\beta(z_1+z_2)+\alpha_3z_3=0,$$
$$\alpha_1\overline{w}+\beta(\bar{z}_1-\bar{z}_2)-\alpha_3\overline{w}z_3=0.$$

By taking the complex conjugate of the second equation we obtain

$$z_1+z_2=-\frac{\alpha_1+\alpha_3z_3}{\beta},$$
$$z_1-z_2=-\frac{\alpha_1w-\alpha_3w\bar{z}_3}{\beta}.$$

We can solve for z_1 and z_2, namely:

$$z_1=-\frac{\alpha_1(1+w)+\alpha_3(z_3-w\bar{z}_3)}{2\beta},$$
$$z_2=-\frac{\alpha_1(1-w)+\alpha_3(z_3+w\bar{z}_3)}{2\beta}.$$

Note that the equations above are equivalent to $\langle u_2,u_1\rangle=\langle u_3,u_1\rangle=0$. Now, let $z_3=e^{it}$ and $w=e^{is}$ and determine for which pairs (t,s) the above formulas give $|z_1|=|z_2|=1$. We have

$$|z_1|^2=\frac{\alpha_1^2+\alpha_3^2+\alpha_1^2\cos(s)-\alpha_3^2\cos(s-2t)}{2\beta^2}$$
$$|z_2|^2=\frac{\alpha_1^2+\alpha_3^2-\alpha_1^2\cos(s)+\alpha_3^2\cos(s-2t)}{2\beta^2}.$$

From (8.2) we observe that $2\beta^2=\alpha_1^2+\alpha_3^2$ and therefore

$$|z_1|^2=1+\frac{\alpha_1^2\cos(s)-\alpha_3^2\cos(s-2t)}{2\beta^2}$$
$$|z_2|^2=1-\frac{\alpha_1^2\cos(s)-\alpha_3^2\cos(s-2t)}{2\beta^2}.$$

This means that $|z_1| = |z_2| = 1$ precisely when $\alpha_1^2 \cos(s) - \alpha_3^2 \cos(s-2t) = 0$. Therefore

$$t = \frac{1}{2}\left(s - \arccos\left(\left(\frac{\alpha_1}{\alpha_3}\right)^2 \cos(s)\right)\right)$$

and consequently $\arccos\left(\left(\frac{\alpha_3}{\alpha_1}\right)^2\right) \leq s \leq \arccos\left(-\left(\frac{\alpha_3}{\alpha_1}\right)^2\right)$. This shows there is a 1-parameter family of solutions for which u is a unitary.

Since every graph in the commuting square has simple edges and the first row of u is the same for all of these connections, we conclude that they are all non-equivalent connections. Below we have a table showing the indices for the subfactors obtained from a commuting square associated to $S(i,i,j,j)$.

$j \backslash i$	1	2	3	4	$\cdots$	∞
1	4					
2	$\frac{5+\sqrt{17}}{2}$	5				
3	$3+\sqrt{3}$	5.1249	$3+\sqrt{5}$			
4	$\frac{5+\sqrt{21}}{2}$	5.1642	5.2703	$\frac{7+\sqrt{13}}{2}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
∞	$2+2\sqrt{2}$	5.1844	5.2870	5.3184		$\frac{16}{3}$

Table 8.1: Indices of $S(i,i,j,j)$-commuting square subfactors

We will use this 1-parameter family of bi-unitary connections to show the existence of A_∞-subfactors at the remaining Jones indices between 4 and 5 at which there exist finite depth subfactors. This will follow from a result of Kawahigashi that we describe next.

Let $N \subset M$ be a hyperfinite subfactor with finite index and finite depth. The proposition and corollary are due to Kawahigashi [Kaw23, Corollary 3.6]:

Proposition 8.2. *Any finite-dimensional commuting squares $\{A_{0,l} \subset A_{1,l}\}_l$ giving a hyperfinite subfactor*

$A_{0,\infty} \subset A_{1,\infty}$ that is isomorphic to $N \subset M$ is of the form

$$A_{k,l} = \begin{cases} \mathrm{End}({}_S S_1 \otimes_S \cdots \otimes_S S_1 \otimes_S X \otimes_P Q \otimes_P \cdots \otimes_P Q_P), \\ k/2 \text{ copies of } S_1 \text{ and } l/2 \text{ copies of } Q, & \text{if } k \text{ and } l \text{ are even,} \\ \mathrm{End}({}_{S_1} S_1 \otimes_S \cdots \otimes_S S_1 \otimes_S X \otimes_P Q \otimes_P \cdots \otimes_P Q_P), \\ (k+1)/2 \text{ copies of } S_1 \text{ and } l/2 \text{ copies of } Q, & \text{if } k \text{ is odd and } l \text{ is even,} \\ \mathrm{End}({}_S S_1 \otimes_S \cdots \otimes_S S_1 \otimes_S X \otimes_P Q \otimes_P \cdots \otimes_P Q_Q), \\ k/2 \text{ copies of } S_1 \text{ and } (l+1)/2 \text{ copies of } Q, & \text{if } k \text{ is even and } l \text{ is odd,} \\ \mathrm{End}({}_{S_1} S_1 \otimes_S \cdots \otimes_S S_1 \otimes_S X \otimes_P Q \otimes_P \cdots \otimes_P Q_Q), \\ (k+1)/2 \text{ copies of } S_1 \text{ and } (l+1)/2 \text{ copies of } Q, & \text{if } k \text{ and } l \text{ are odd,} \end{cases}$$

where $P \subset Q$ is a subfactor Morita equivalent to $R = M^{\mathrm{opp}} \subset N^{\mathrm{opp}} = S$, S_1 is the basic construction for $R \subset S$ and ${}_S X_P$ is the $S-P$ bimodule giving the Morita equivalence between the category of $S-S$ bimodules arising from $S \subset S_1$ and $P-P$ bimodules arising from $P \subset Q$.

Corollary 8.3. *There are only countably many non-equivalent commuting squares $\{A_{i,j}, 0 \leq i,j \leq 1\}$ such that $A_{0,\infty} \subset A_{1,\infty}$ is isomorphic to $N \subset M$.*

Proof. Given a fusion category $\mathscr{C}$, there are only finitely many fusion categories Morita equivalent to $\mathscr{C}$ (see [Lon94], Section 6). Let $R \subset S$ be as in Proposition 8.2. Fix a fusion category of $P-P$ bimodules Morita equivalent to the fusion category of S-S bimodules arising from $S \subset S_1$. Since there are countably many subfactors $P \subset Q$ and countably many bimodules ${}_S X_P$, by Proposition 8.2, we have countably many non-equivalent commuting squares that produce $N \subset M$. □

Finally, the existence of our 1-parameter family of non-equivalent commuting squares implies

Theorem 8.4. *There are irreducible hyperfinite A_∞-subfactors with index $\frac{5+\sqrt{17}}{2}$, $3+\sqrt{3}$, $\frac{5+\sqrt{21}}{2}$ and 5. There is an irreducible hyperfinite subfactor with index $3+\sqrt{5}$ that either has principal graph A_∞ or is an $A_3 * A_4$ subfactor of Bisch-Jones [BJ97].*

Proof. By classification of small index subfactors, we have finitely many finite depth subfactors at the indices $\frac{5+\sqrt{17}}{2}$, $3+\sqrt{3}$, $\frac{5+\sqrt{21}}{2}$, 5 and $3+\sqrt{5}$. Corollary 8.3 then implies that our 1-parameter families of non-equivalent bi-unitary connections will produce at least one irreducible infinite depth subfactor at each of these indices. Using the classification of small index subfactors again, we have that all of them but the one with index $3+\sqrt{5}$ must have trivial standard invariant. In the case of index $3+\sqrt{5}$, since the subfactor is irreducible, by classification it has to be either an A_∞-subfactor or an $A_3 * A_4$-subfactor, which is a special case of a Fuss-Catalan subfactor from [BJ97]. □

Appendix A

Bi-unitary connection for Large broom

A.1 Large double broom - unitary u ($d = \sqrt{17}$

p	s / $q \backslash r$	a_4 / A_2	a_3 / A_2	a_2 / A_2	a_1 / A_2	a_1 / A_1	a_0 / A_1	a_0 / A_0
A_2	a_4			$0 \quad 0 \quad 1$	$\begin{matrix} \sqrt{\frac{1+d}{8}} & 0 & 0 \\ 0 & \sqrt{\frac{1+d}{8}} & 0 \end{matrix}$	$\begin{matrix} \sqrt{\frac{7-d}{8}} & 0 \\ 0 & \sqrt{\frac{7-d}{8}} \end{matrix}$		
	a_3		$-\sqrt{\frac{-75+19d}{8}} \quad -\sqrt{\frac{31-7d}{8}} \quad \frac{-3+d}{2}$	$\sqrt{\frac{-1+d}{8}} \quad \frac{1-d}{4} \quad 0$	$\sqrt{-4+d} \quad \frac{-3+d}{4} \quad \sqrt{\frac{5-d}{2}}$	$-\frac{\sqrt{5-d}}{2} \quad -\sqrt{\frac{-3+d}{8}}$	$\frac{\sqrt{5-d}}{2} \quad -\sqrt{\frac{-11+3d}{4}}$	$\sqrt{\frac{5-d}{2}}$
	a_2	$\sqrt{\frac{-75+19d}{32}} \quad \sqrt{\frac{23+d}{32}} \quad \frac{5-d}{4}$	$-\sqrt{\frac{-101+29d}{32}} \quad \frac{-1+d}{8} \quad -\sqrt{\frac{31-7d}{8}}$		$-\sqrt{\frac{-19+5d}{8}} \quad 0 \quad \sqrt{\frac{5-d}{2}}$	$\sqrt{\frac{7-d}{8}} \quad 0$	$0 \quad \sqrt{\frac{-3+d}{2}}$	$\sqrt{\frac{5-d}{2}}$
	a_1	$\begin{matrix} \frac{3-d}{2} & 0 & \sqrt{\frac{-11+3d}{2}} \\ -\sqrt{\frac{-101+29d}{32}} & \frac{-1+d}{8} & -\sqrt{\frac{31-7d}{8}} \end{matrix}$	$\frac{21-5d}{8} \quad -\sqrt{\frac{-101+29d}{32}} \quad -\sqrt{\frac{-75+19d}{8}}$	$\frac{-1+d}{4} \quad \sqrt{\frac{-1+d}{8}} \quad 0$	$-\frac{\sqrt{29-7d}}{2} \quad \frac{\sqrt{-3+d}}{2} \quad -\sqrt{-4+d}$	$\sqrt{\frac{-4+d}{2}} \quad -\frac{1}{\sqrt{2}}$	$\frac{\sqrt{-1+d}}{2} \quad \frac{\sqrt{21-5d}}{2}$	$-\sqrt{-4+d}$
A_1	a_1	$\begin{matrix} \sqrt{\frac{-1+d}{8}} & \frac{-1+d}{4} \\ \frac{1-d}{4} & \sqrt{\frac{-1+d}{8}} \end{matrix}$	$\sqrt{\frac{-1+d}{8}} \quad \frac{-1+d}{4}$	$0 \quad 1$	$\sqrt{\frac{3+d}{8}} \quad -\sqrt{\frac{5-d}{8}}$			1
	a_0		$\frac{1-d}{4} \quad \sqrt{\frac{-1+d}{8}}$	$-1 \quad 0$	$\sqrt{\frac{5-d}{8}} \quad \sqrt{\frac{3+d}{8}}$			
A_0	a_0		1	-1	$\frac{\sqrt{7-d}}{4}$	$-\frac{\sqrt{9+d}}{4}$		
	b_0	1			$\frac{\sqrt{9+d}}{4}$	$\frac{\sqrt{7-d}}{4}$	-1	
B_1	b_0	1			$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5-d}}{2}$	$-\frac{\sqrt{-1+d}}{2}$
	b_1		1	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{-1+d}}{2}$	$\frac{\sqrt{5-d}}{2}$
B_2	b_1		$-\sqrt{\frac{-11+3d}{2}} \quad \frac{3-d}{2}$	$0 \quad 1$	$\sqrt{\frac{-3+d}{2}} \quad -\frac{\sqrt{13-3d}}{2}$	$\frac{\sqrt{-3+d}}{2}$	$-\sqrt{\frac{5-d}{2}}$	$\sqrt{\frac{-3+d}{2}}$
	b_2			$-1 \quad 0$	$0 \quad \sqrt{\frac{1+d}{8}}$	$\sqrt{\frac{7-d}{8}}$		
	b_3		$\frac{-3+d}{2} \quad -\sqrt{\frac{-11+3d}{2}}$		$-\sqrt{\frac{5-d}{2}} \quad -\sqrt{\frac{-19+5d}{8}}$	$\sqrt{\frac{7-d}{8}}$		
	b_4	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$					$\sqrt{\frac{-3+d}{2}}$	$\sqrt{\frac{5-d}{2}}$

p	q \ r, s	b_0: B_0	b_0: B_1	b_1: B_1	b_1: B_2	b_2: B_2	b_3: B_2	b_4: B_2
A_2	a_4	$\sqrt{\frac{5-d}{2}}$	$\sqrt{\frac{-3+d}{2}}$					$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
	a_3			$\sqrt{\frac{7-d}{8}}$	$-\sqrt{\frac{-19+5d}{8}}\quad -\sqrt{\frac{5-d}{2}}$		$-\sqrt{\frac{-11+3d}{2}}\quad \frac{-3+d}{2}$	
	a_2			$\sqrt{\frac{7-d}{8}}$	$\sqrt{\frac{1+d}{8}}\quad 0$	$0\quad -1$		
	a_1	$\sqrt{\frac{-3+d}{2}}$	$-\sqrt{\frac{5-d}{2}}$	$\frac{\sqrt{-3+d}}{2}$	$-\frac{\sqrt{13-3d}}{2}\quad \sqrt{\frac{-3+d}{2}}$	$1\quad 0$	$\frac{3-d}{2}\quad -\sqrt{\frac{-11+3d}{2}}$	
A_1	a_1	$\frac{\sqrt{5-d}}{2}$	$\frac{\sqrt{-1+d}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	
	a_0	$-\frac{\sqrt{-1+d}}{2}$	$\frac{\sqrt{5-d}}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$			1
A_0	a_0		-1	$\frac{\sqrt{7-d}}{4}$	$\frac{\sqrt{9+d}}{4}$			1
	b_0			$-\frac{\sqrt{9+d}}{4}$	$\frac{\sqrt{7-d}}{4}$	-1	1	
B_1	b_0				$\sqrt{\frac{3+d}{8}}\quad \sqrt{\frac{5-d}{8}}$	$0\quad -1$	$\sqrt{\frac{-1+d}{8}}\quad \frac{1-d}{4}$	
	b_1	1			$-\sqrt{\frac{5-d}{8}}\quad \sqrt{\frac{3+d}{8}}$	$1\quad 0$	$\frac{-1+d}{4}\quad \sqrt{\frac{-1+d}{8}}$	$\begin{matrix} \sqrt{\frac{-1+d}{8}} & \frac{1-d}{4} \\ \frac{-1+d}{4} & \sqrt{\frac{-1+d}{8}} \end{matrix}$
B_2	b_1	$-\sqrt{-4+d}$	$\frac{\sqrt{21-5d}}{2}\quad \frac{\sqrt{-1+d}}{2}$	$-\frac{1}{\sqrt{2}}\quad \sqrt{\frac{-4+d}{2}}$	$-\sqrt{-4+d}\quad \frac{\sqrt{-3+d}}{2}\quad -\frac{\sqrt{29-7d}}{2}$	$0\quad \sqrt{\frac{-1+d}{8}}\quad \frac{-1+d}{4}$	$-\sqrt{\frac{-75+19d}{8}}\quad -\sqrt{\frac{-101+29d}{32}}\quad \frac{21-5d}{8}$	$\begin{matrix} -\sqrt{\frac{31-7d}{8}} & \frac{-1+d}{8} & -\sqrt{\frac{-101+29d}{32}} \\ \sqrt{\frac{-11+3d}{2}} & 0 & \frac{3-d}{2} \end{matrix}$
	b_2	$\sqrt{\frac{5-d}{2}}$	$\sqrt{\frac{-3+d}{2}}\quad 0$	$0\quad \sqrt{\frac{7-d}{8}}$	$\sqrt{\frac{5-d}{2}}\quad 0\quad -\sqrt{\frac{-19+5d}{8}}$		$-\sqrt{\frac{31-7d}{8}}\quad \frac{-1+d}{8}\quad -\sqrt{\frac{-101+29d}{32}}$	$\frac{5-d}{4}\quad \sqrt{\frac{23+d}{32}}\quad \sqrt{\frac{-75+19d}{32}}$
	b_3	$\sqrt{\frac{5-d}{2}}$	$-\sqrt{\frac{-11+3d}{4}}\quad \frac{\sqrt{5-d}}{2}$	$-\sqrt{\frac{-3+d}{8}}\quad -\frac{\sqrt{5-d}}{2}$	$\sqrt{\frac{5-d}{2}}\quad \frac{-3+d}{4}\quad \sqrt{-4+d}$	$0\quad \frac{1-d}{4}\quad \sqrt{\frac{-1+d}{8}}$	$\frac{-3+d}{2}\quad -\sqrt{\frac{31-7d}{8}}\quad -\sqrt{\frac{-75+19d}{8}}$	
	b_4			$\begin{matrix} \sqrt{\frac{7-d}{8}} & 0 \\ 0 & \sqrt{\frac{7-d}{8}} \end{matrix}$	$\begin{matrix} 0 & \sqrt{\frac{1+d}{8}} & 0 \\ 0 & 0 & \sqrt{\frac{1+d}{8}} \end{matrix}$	$1\quad 0\quad 0$		

A.2 Large double broom - unitary v ($d = \sqrt{17}$)

p	s / r / q	A_2				A_1		A_0		B_1		B_2			
		a_4	a_3	a_2	a_1	a_1	a_0	a_0	b_0	b_0	b_1	b_1	b_2	b_3	b_4
a_4	A_2			$\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$			1	1					$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
a_3	A_2		$\begin{matrix} -\sqrt{\frac{-75+19d}{8}} \\ -\sqrt{\frac{31-7d}{8}} \\ \frac{-3+d}{2} \end{matrix}$	$\begin{matrix} \sqrt{\frac{-1+d}{8}} \\ \frac{1-d}{4} \\ 0 \end{matrix}$	$\begin{matrix} \frac{5-d}{2} \\ \sqrt{-4+d} \\ \sqrt{\frac{-11+3d}{2}} \end{matrix}$	$\begin{matrix} \frac{1-d}{4} \\ -\sqrt{\frac{-1+d}{8}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{-1+d}{8}} \\ \frac{1-d}{4} \end{matrix}$	1			1		$\begin{matrix} \frac{3-d}{2} \\ -\sqrt{\frac{-11+3d}{2}} \end{matrix}$		$\begin{matrix} -\sqrt{\frac{-11+3d}{2}} \\ \frac{-3+d}{2} \end{matrix}$
a_2	A_2	$\begin{matrix} \sqrt{\frac{-75+19d}{32}} \\ \sqrt{\frac{23+d}{32}} \\ \frac{5-d}{4} \end{matrix}$	$\begin{matrix} -\sqrt{\frac{-101+29d}{32}} \\ \frac{-1+d}{8} \\ -\sqrt{\frac{31-7d}{8}} \end{matrix}$		$\begin{matrix} \frac{3-d}{2} \\ 0 \\ \sqrt{\frac{-11+3d}{2}} \end{matrix}$	$\begin{matrix} 1 \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ 1 \end{matrix}$	1			1	$\begin{matrix} 1 \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ -1 \end{matrix}$		
a_1	A_2	$\begin{matrix} -\sqrt{\frac{-19+5d}{8}} & \frac{-9+d}{8} \\ 0 & \sqrt{\frac{-1+d}{32}} \\ \sqrt{\frac{5-d}{2}} & -\sqrt{\frac{-11+3d}{8}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{-169+41d}{32}} \\ \frac{-9+d}{8} \\ -\sqrt{\frac{31-7d}{8}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{-1+d}{8}} \\ \frac{1}{2} \\ 0 \end{matrix}$	$\begin{matrix} -\sqrt{\frac{29-7d}{4}} \\ \frac{\sqrt{-3+d}}{2} \\ -\sqrt{-4+d} \end{matrix}$	$\begin{matrix} \sqrt{\frac{5-d}{8}} \\ -\sqrt{\frac{3+d}{8}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{3+d}{8}} \\ \sqrt{\frac{5-d}{8}} \end{matrix}$	$-\frac{\sqrt{7-d}}{4}$	$\frac{\sqrt{9+d}}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$		$\begin{matrix} -\frac{\sqrt{13-3d}}{2} \\ \sqrt{\frac{-3+d}{2}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{1+d}{8}} \\ 0 \end{matrix}$	$\begin{matrix} -\sqrt{\frac{-19+5d}{8}} \\ -\sqrt{\frac{5-d}{2}} \end{matrix}$
	A_1	$\begin{matrix} \sqrt{\frac{-3+d}{8}} & -\frac{\sqrt{5-d}}{2} \\ \frac{\sqrt{5-d}}{2} & \sqrt{\frac{-3+d}{8}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{-3+d}{8}} \\ \frac{\sqrt{5-d}}{2} \end{matrix}$	$\begin{matrix} 0 \\ \sqrt{\frac{7-d}{8}} \end{matrix}$	$\begin{matrix} \frac{1}{\sqrt{2}} \\ -\sqrt{\frac{-4+d}{2}} \end{matrix}$			$\frac{\sqrt{9+d}}{4}$	$\frac{\sqrt{7-d}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$		$\frac{\sqrt{-3+d}}{2}$	$\sqrt{\frac{7-d}{8}}$	$\sqrt{\frac{7-d}{8}}$
a_0	A_1		$\begin{matrix} -\sqrt{\frac{-11+3d}{4}} \\ \frac{\sqrt{5-d}}{2} \end{matrix}$	$\begin{matrix} -\sqrt{\frac{-3+d}{2}} \\ 0 \end{matrix}$	$\begin{matrix} \frac{\sqrt{21-5d}}{2} \\ \frac{\sqrt{-1+d}}{2} \end{matrix}$				-1	$\frac{\sqrt{5-d}}{2}$	$\frac{\sqrt{-1+d}}{2}$	$-\sqrt{\frac{5-d}{2}}$			$\sqrt{\frac{-3+d}{2}}$
	A_0		$-\sqrt{\frac{5-d}{2}}$	$\sqrt{\frac{5-d}{2}}$	$\sqrt{-4+d}$	-1				$-\frac{\sqrt{-1+d}}{2}$	$\frac{\sqrt{5-d}}{2}$	$\sqrt{\frac{-3+d}{2}}$			$\sqrt{\frac{5-d}{2}}$

p	q \ s, r	A_2: a_4	A_2: a_3	A_2: a_2	A_2: a_1	A_1: a_1	A_1: a_0	A_0: a_0	A_0: b_0	B_1: b_0	B_1: b_1	B_2: b_1	B_2: b_2	B_2: b_3	B_2: b_4
b_0	A_0	$\sqrt{\frac{5-d}{2}}$			$\sqrt{\frac{-3+d}{2}}$	$\frac{\sqrt{5-d}}{2}$	$-\frac{\sqrt{-1+d}}{2}$				-1	$\sqrt{-4+d}$	$\sqrt{\frac{5-d}{2}}$	$-\sqrt{\frac{5-d}{2}}$	
b_0	B_1	$\sqrt{\frac{-3+d}{2}}$			$-\sqrt{\frac{5-d}{2}}$	$\frac{\sqrt{-1+d}}{2}$	$\frac{\sqrt{5-d}}{2}$	-1				$\begin{matrix} \frac{\sqrt{-1+d}}{2} \\ \frac{\sqrt{21-5d}}{2} \end{matrix}$	$\begin{matrix} 0 \\ -\sqrt{\frac{-3+d}{2}} \end{matrix}$	$\begin{matrix} \frac{\sqrt{5-d}}{2} \\ -\sqrt{\frac{-11+3d}{4}} \end{matrix}$	
b_1	B_1		$\sqrt{\frac{7-d}{8}}$	$\sqrt{\frac{7-d}{8}}$	$\frac{\sqrt{-3+d}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{7-d}}{4}$	$\frac{\sqrt{9+d}}{4}$			$\begin{matrix} -\sqrt{\frac{-4+d}{2}} \\ \frac{1}{\sqrt{2}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{7-d}{8}} \\ 0 \end{matrix}$	$\begin{matrix} \frac{\sqrt{5-d}}{2} \\ \sqrt{\frac{-3+d}{8}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{-3+d}{8}} & \frac{\sqrt{5-d}}{2} \\ -\frac{\sqrt{5-d}}{2} & \sqrt{\frac{-3+d}{8}} \end{matrix}$
b_1	B_2		$\begin{matrix} -\sqrt{\frac{5-d}{2}} \\ -\sqrt{\frac{-19+5d}{8}} \end{matrix}$	$\begin{matrix} 0 \\ \sqrt{\frac{1+d}{8}} \end{matrix}$	$\begin{matrix} \sqrt{\frac{-3+d}{2}} \\ -\frac{\sqrt{13-3d}}{2} \end{matrix}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{\sqrt{9+d}}{4}$	$-\frac{\sqrt{7-d}}{4}$	$\begin{matrix} \sqrt{\frac{5-d}{8}} \\ \sqrt{\frac{3+d}{8}} \end{matrix}$	$\begin{matrix} -\sqrt{\frac{3+d}{8}} \\ \sqrt{\frac{5-d}{8}} \end{matrix}$	$\begin{matrix} -\sqrt{-4+d} \\ \frac{\sqrt{-3+d}}{2} \\ -\sqrt{\frac{29-7d}{4}} \end{matrix}$	$\begin{matrix} 0 \\ \frac{1}{2} \\ \sqrt{\frac{-1+d}{8}} \end{matrix}$	$\begin{matrix} -\sqrt{\frac{31-7d}{8}} \\ \frac{-9+d}{8} \\ \sqrt{\frac{-169+41d}{32}} \end{matrix}$	$\begin{matrix} -\sqrt{\frac{-11+3d}{8}} & \sqrt{\frac{5-d}{2}} \\ \sqrt{\frac{-1+d}{32}} & 0 \\ \frac{-9+d}{8} & -\sqrt{\frac{-19+5d}{8}} \end{matrix}$
b_2	B_2			$\begin{matrix} -1 \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ 1 \end{matrix}$	1			1	$\begin{matrix} 1 \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ 1 \end{matrix}$	$\begin{matrix} \sqrt{\frac{-11+3d}{2}} \\ 0 \\ \frac{3-d}{2} \end{matrix}$		$\begin{matrix} -\sqrt{\frac{31-7d}{8}} \\ \frac{-1+d}{8} \\ -\sqrt{\frac{-101+29d}{32}} \end{matrix}$	$\begin{matrix} \frac{5-d}{4} \\ \sqrt{\frac{23+d}{32}} \\ \sqrt{\frac{-75+19d}{32}} \end{matrix}$
b_3	B_2		$\begin{matrix} \frac{-3+d}{2} \\ -\sqrt{\frac{-11+3d}{2}} \end{matrix}$		$\begin{matrix} -\sqrt{\frac{-11+3d}{2}} \\ \frac{3-d}{2} \end{matrix}$	1			1	$\begin{matrix} \frac{1-d}{4} \\ \sqrt{\frac{-1+d}{8}} \end{matrix}$	$\begin{matrix} -\sqrt{\frac{-1+d}{8}} \\ \frac{1-d}{4} \end{matrix}$	$\begin{matrix} \sqrt{\frac{-11+3d}{2}} \\ \sqrt{-4+d} \\ \frac{5-d}{2} \end{matrix}$	$\begin{matrix} 0 \\ \frac{1-d}{4} \\ \sqrt{\frac{-1+d}{8}} \end{matrix}$	$\begin{matrix} \frac{-3+d}{2} \\ -\sqrt{\frac{31-7d}{8}} \\ -\sqrt{\frac{-75+19d}{8}} \end{matrix}$	
b_4	B_2	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$					1	1			$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$		

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