FIBERED UNIVERSAL ALGEBRA FOR FIRST-ORDER LOGICS

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To my wife Jeannette and our son Abel.

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Chapter 1

Introduction

There is a vast and ever-growing expanse of nonclassical logics and with it, a need to develop a unified theoretical framework to classify, prove metalogical results, and develop semantics for *classes* of logics. For propositional logics, the field of abstract algebraic logic (AAL) provides a unified theoretical framework rooted in universal algebra. However, for first-order logics, there is not a clear consensus on what the unifying algebraic perspective should be (Font, 2016, p. 104). Two main challenges mentioned in the literature are (1) a lack of *general* algebraic approaches to providing semantics for first-order logics and (2) a lack of a notion of formula substitution which preserves logical consequence (Font, 2016, p. 106).

To address (1), we use Lawvere's theory of hyperdoctrines Lawvere (2006) to develop a general approach to providing "algebraic" semantics for nonclassical first-order logics. The key insight to this approach is to consider terms and formulas *in context*, that is, with a list of variable/sort pairs that at least include those occurring in the formula. The use of contexts is ubiquitous in classical first-order model theory where it allows one to unambiguously interpret terms as term functions and formulas as definable sets. However, it is scarcely formally presented there, which is unfortunate because contexts allow one to give an elegant point-free/valuation-free definition of satisfaction.[1]

Since we model truth and not proof, we use poset-valued instead of more general category-valued hyperdoctrines for our semantics called prop-categories in Pitts (2000). A prop-category is a pair $(\mathcal{C}, P)$, where $\mathcal{C}$ is a category with designated finite products and $P\colon \mathcal{C} \to \mathbf{Pos}$ is a contravariant functor to $\mathbf{Pos}$, the category of partially ordered sets and monotone maps. The prop-categories in Pitts (2000) are assumed to satisfy additional adjointness conditions so that they interpret the quantifiers and connectives of intuitionistic predicate logic. To model a broad variety of logics, we allow arbitrarily many quantifier and operation symbols in a first-order language $\mathscr{L}$ and we replace the adjointness condition with weaker algebraic ones. Despite this generalization, the prop-categorical semantics of a logic can still be encoded in a 2-category of prop-categories we denote $\mathbf{FA}_{\mathscr{L}}$ (or just $\mathbf{FA}$), where roughly theories are objects, structures are morphisms, and structure-preserving maps are 2-cells.

Moreover, the morphisms in $\mathbf{FA}$ determine a natural action of formula substitution which also addresses issue (2). A footnote in a standard reference on AAL (Font, 2016, p. 106) says:

> In first-order languages we replace an individual (free) variable by a term; we do not replace an

[1] Lawvere's point-free semantics also resolves a philosophical issue with assigning objects to variables discussed in (Button and Walsh, 2018, p. 10).

atomic formula, inside a more complex one, by an arbitrary formula, which is what is done in a sentential language.

However, if one unpacks the action of morphisms on theories it essentially does that. A restricted form of this notion of formula substitution appears in (Church, 1956, p. 191), though it seems to be missing from many contemporary textbooks on mathematical logic. This is not surprising, since the account in Church (1956) for classical first-order logic without function symbols is already complicated without the help of contexts.

Much like the study of algebraic semantics for first-order logic, currently, the study of prop-categorical semantics for logics (in contrast with type theories) appears to be focused on studying particular logics and not on establishing a general framework in the spirit of AAL. An early exception is the work of Hiroyuki Shirasu, who in Shirasu (1995) provides complete prop-categorical semantics for substructural predicate logics, and uses this semantics to prove their disjunctive and existence properties. In Shirasu (1998), prop-categorical and a general metaframe semantics are developed for first-order modal logics and it is proved that they are dual to each other. More recently, in Maruyama (2021), complete hyperdoctrine semantics for substructural predicate logics is also given and used to present a unified account of logical translations, including Kolmogorov's double negation translation of classical logic into intuitionistic logic and Girard's exponential translation of intuitionistic logic into linear logic. In contrast to these existing works on general prop-categorical semantics we model nonclassical quantifiers and focus on (1) establishing minimal conditions on prop-categories and first-order logics so that structures are morphisms in a 2-category of prop-categories and (2) developing a toolkit of general algebraic/categorical results on the resulting prop-categorical semantics.

Towards (1), we define a weak logic $\mathcal{L}^m$ such that whenever a logic $\mathcal{L}$ is stronger than $\mathcal{L}^m$, each $\mathcal{L}$-theory T defines a classifying prop-category $(\mathcal{C}_T, P_T)$ in $\mathbf{FA}$, which contains a generic T-model G in $(\mathcal{C}_T, P_T)$. The classifying prop-categories are the fibrational, first-order analogues of the Lindenbaum-Tarski algebras of propositional logics. Such a logic $\mathcal{L}$ is called *adequate* and $\mathcal{L} \supseteq \mathcal{L}^m$ is also necessary for $\mathcal{L}$ to be adequate. Moreover, $\mathcal{L}^m$ has a complete semantics with respect to the class of all of its classifying prop-categories (Theorem 3), as do many natural extensions of $\mathcal{L}^m$ by the addition of various sequent rules (Theorem 4). We then develop an algebraic view of the prop-categorical semantics of a logic $\mathcal{L}$ in analogy with the algebraic approach to the semantics of propositional logics. In this view, a full sub-2-category $\mathbf{FA}_{\mathcal{L}}$ of $\mathbf{FA}$ replaces the usual quasivariety of algebras forming the algebraic semantics of a propositional logic and we define the kernel of a morphism (Definition 3.3.1) to give an algebraic/morphic view of logical entailment. These results are directly applied to formulate and prove the later "fibered" universal algebraic results.

After singling out the class of first-order logics which have nice categorical semantics, in the last chapter, we develop a "fibered universal algebra" for $\mathbf{FA}$. The internal logic of a prop-category $(\mathcal{C}, P)$ in $\mathbf{FA}$ is defined and shown to be an $\mathcal{L}^m$-theory T, such that $(\mathcal{C}_T, P_T) \equiv (\mathcal{C}, P)$ in $\mathbf{FA}$. sub-prop-categories and morphic images of prop-categories are defined and $\mathbf{FA}$ is shown to have arbitrary products. The logical properties of products and sub-prop-categories are considered which mirror the logical properties of submatrices and products of matrices respectively in the matrix semantics of AAL (Font, 2016, p. 183).

We then prove a "fibered" analogue of the homomorphism theorem from universal algebra in two parts (Theorem 7). The first part, says that each morphism $F\colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ in $\mathbf{FA}_{\mathcal{L}}$ factors through a "quotient" prop-category which is also in $\mathbf{FA}_{\mathcal{L}}$. The second part characterizes when, given morphisms $F\colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ and $K\colon (\mathcal{C}, P) \to (\mathcal{E}, R)$, there exists a unique morphism $H\colon (\mathcal{E}, R) \to (\mathcal{D}, Q)$ such that $H \circ K = F$. Part 2 says that under reasonable "surjectivity" assumptions on K, there is a unique completion H if and only if $\ker K \leq \ker F$. Our results on morphic images and the internal logic are used to give "algebraic" proofs of these results. Moreover, the fibered homomorphism theorem determines an orthogonal factorization system for $\mathbf{FA}_{\mathcal{L}}$.

In the final section, we use the fibered homomorphism theorem to characterize two natural closure operators. The first operator characterizes the closure of a class of structures under the satisfaction of their common first-order theory. Taking submodels, homomorphic images and products does not preserve the satisfaction of first-order theories in the Tarskian semantics for classical first-order logic. However, for logics $\mathcal{L}$ given prop-categorical semantics in $\mathbf{FA}_{\mathcal{L}}$, the closure operator is characterized by $\mathbb{HSP}$ where $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$ close a collection of structures under natural notions of homomorphic image, submodel and product of structures in the prop-categorical semantics (Theorem 8). For example, if $\mathcal{L}$ is classical first-order logic, then the Tarskian Sg-structures are essentially the morphisms $F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathbf{Set}, \mathscr{P})$, where $\mathscr{P}$ is the preimage functor, and $(\mathcal{C}_{Sg}, P_{Sg})$ is the classifying prop-category of the minimal Sg-theory. Then $\mathbf{FA}_{\mathcal{L}}((\mathcal{C}_{Sg}, P_{Sg}), (\mathbf{Set}, \mathscr{P}))$ provides complete semantics for $\mathcal{L}$ (restricted to the signature Sg). Just as the two element Boolean algebra and the class of all Boolean algebras forms complete semantics for classical propositional logic, $\mathbf{FA}_{\mathcal{L}}$ also provides complete semantics for $\mathcal{L}$, where an Sg-structure is a morphism $F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}, P)$, for some $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})$. In this extended semantics, it is natural to look at the *coslice category* $\mathcal{C}_{Sg}\!\downarrow\!\mathbf{FA}$ of the underlying 1-category of $\mathbf{FA}$ instead of $\mathbf{FA}_{\mathcal{L}}((\mathcal{C}_{Sg}, P_{Sg}), (\mathbf{Set}, \mathscr{P}))$. It is in this category $\mathcal{C}_{Sg}\!\downarrow\!\mathbf{FA}$ that we define the operations of $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$, and obtain the fibered HSP-result.

The second operator closes a collection $\mathcal{X} \subseteq \mathrm{Ob}(\mathbf{FA})$ under logical consequence: that is, the closure of $\mathcal{X}$ is the largest $\mathcal{Y} \supseteq \mathcal{X}$ such that $\vDash_{\mathcal{Y}} = \vDash_{\mathcal{X}}$, i.e. they define the same logic. Since $\mathbf{FA}_{\mathcal{L}}$ is stable under products and sub-prop-categories, $\mathcal{Y} \supseteq \mathbb{SP}(\mathcal{X})$, where here $\mathbb{S}$ is closure under sub-prop-categories and $\mathbb{P}$ is closure under products of prop-categories. It turns out that we need another operation which we call $\mathbb{U}$, and Theorem 10 shows that $\mathcal{Y} = \mathbb{USP}(\mathcal{X})$. This result is a natural extension of a corresponding result by Blok and Jónsson in Blok and Jónsson (2006), which shows that for a collection of algebras $\mathcal{V}$, closure under the equational consequence is given by $\mathbb{U}_\lambda\mathbb{SP}(\mathcal{V})$, where λ is the cardinality of the set of variables over which equations are defined, and $B \in \mathbb{U}_\lambda(\mathcal{V})$ if every λ-generated subalgebra $A \leq B$ is in $\mathcal{V}$. (See Theorem 9 for the directly analogous "fibered" result which considers a fixed first-order signature and uses an operator $\mathbb{U}_{Sg}$.)

Chapter 2

Syntactic Concerns

This chapter introduces the syntax of formulas-in-context and provides an example of the benefits of contexts by giving a simple account of α-equivalence and equivalence up to change in free variables. Term substitution on (equivalence classes of) formulas is then defined and used to provide a natural notion of formula substitution that is structural for many logics. Later, in Section 3.3, the formula substitution maps are shown to correspond to the endomorphisms on a prop-category encoding the syntax of the logic.

we will put these formula substitution maps in bijective correspondence with the collection of endomorphisms on a prop-category encoding the syntax of the logic. The chapter concludes with a few basic definitions. Only sections 2.1 and 2.5 will be necessary for subsequent chapters.

For simplicity, consider classical first-order logic with its usual Tarskian semantics. There are two natural notions of entailment of a formula ψ from a set of formulas Φ. We say $\Phi \vDash \psi$ **globally** if, whenever $\mathcal{M}$ satisfies Φ, $\mathcal{M}$ satisfies ψ. Recall $\mathcal{M}$ satisfies a formula ϕ if and only if $\mathcal{M}, \nu$ satisfies ϕ for each valuation (variable assignment) ν. We say $\Phi \vDash \psi$ **locally** if, whenever $\mathcal{M}, \nu$ satisfies Φ, $\mathcal{M}, \nu$ satisfies ψ. By definition, every local entailment is a global entailment and they agree on sentences (formulas *without* free variables). If Φ is finite, then for some $n \in \mathbb{N}$, and each structure $\mathcal{M}$, every formula $\gamma \in \Phi \cup \{\psi\}$ may be interpreted as an n-ary relation $\mathcal{M}[\![\gamma]\!]$ on the underlying set M of $\mathcal{M}$. Then, $\Phi \vDash \psi$ globally if for all structures $\mathcal{M}$,

$$1 \subseteq \bigcap \mathcal{M}[\![\Phi]\!] \implies 1 \subseteq \bigcap \mathcal{M}[\![\psi]\!],$$

where and $1 = M^n$. And $\Phi \vDash \psi$ locally if for all $\mathcal{M}$,

$$\bigcap \mathcal{M}[\![\Phi]\!] \subseteq \mathcal{M}[\![\psi]\!].$$

Note that term substitution does not preserve all global entailments. Consider the valid global entailment:

$$p(x) \vDash \forall y\, p(y), \tag{2.1}$$

which is clearly not locally valid. Then the following substitution instance of (2.1):

$$p(f(x)) \vDash \forall y\, p(y), \tag{2.2}$$

is not globally valid. Naively understood, substitution also does not preserve local entailments:

$$\exists x\, R(x, y) \vDash \exists z\, R(z, y),$$

is clearly, locally valid, but after substituting x for y we get

$$\exists x\, R(x, x) \vDash \exists z\, R(z, x),$$

which is not globally or locally valid. The issue is that the variable x being substituted gets captured by the quantifier $\exists x$ on the left. If we want substitution to preserve local entailments, we must define substitution in a way that avoids variable capture. We also have another reason for wanting to avoid variable capture: such a notion of term substitution can be used to define a consequence preserving notion of formula substitution for first-order logics.

Recall for classical propositional logic, given any valid entailment such as

$$A \wedge B \vDash A,$$

one may uniformly substitute arbitrary formulas for the atomic formulas and still be left with a valid entailment. For example, if we let $\zeta(A) = A \vee C$ and $\zeta(B) = (C \to D) \to E$, we get the valid entailment

$$(A \vee C) \wedge ((C \to D) \to E) \vDash A \vee C.$$

In abstract algebraic logic (AAL), entailment is modeled as a *structural* consequence relation, i.e. one which is preserved by some action of "sustitution". Structurality seeks to capture the idea that a valid argument should depend only on its form and not on its content. Moreover, structurality is a key ingredient to the main theory of AAL, and it is mentioned in Font (2016) that a lack of a suitable notion of consequence preserving formula substitution for first-order logic is a main inhibitor to extending the theory to include first-order logics.

However, there is such a notion of formula substitution which in a restricted form appears in Church (1956). In first-order logic, the atomic formulas are of the form

$$R(M_1, \ldots, M_n) \quad \text{and} \quad M_1 = M_2,$$

where M_i for $i \in \{1, \ldots, n\}$ are terms built from the variables and function symbols. We cannot map an arbitrary atomic formula to any formula as with propositional logics since

$$P(x) \vDash P(f(x)),$$

is locally valid, but

$$P(x) \vDash Q(x),$$

is not even globally valid. Instead of mapping atomic formulas to arbitrary formulas, we map each n-ary relation symbol R in our signature to an arbitrary *formula-in-context* $\phi^R(x_1, \ldots, x_n)$. That is, a *context* for ϕ^R is a list of variables that at least includes those occurring free in the expression. Then one extends this assignment to an operation on formulas ζ as follows: ζ is the identity on equalities and for an atomic relation $R(M_1, \ldots, M_n)$,

$$\zeta(R(M_1, \ldots, M_n)) \coloneqq \phi^R(M_1, \ldots, M_n).$$

Notice that there are many ways substitute back the terms $M_1, \ldots M_n$ into ϕ^R, and the context specifies a unique choice.

Then one would like to extend this to an operation on formulas ζ by insisting that it commutes with the quantifiers and propositional connectives. However, variable capture is an issue. For example,

$$\forall x\, Q(x) \vDash \forall y\, Q(y),$$

is valid, but if $\zeta(Q) = \forall x\, R(x, y)$, then with naive term substitution, applying ζ to both the antecedent and consequent we get

$$\forall x\, \forall x\, R(x, x) \vDash \forall y\, \forall x\, R(x, y),$$

which is clearly not valid. However with capture avoiding substitution it becomes

$$\forall x\, \forall y\, R(y, x) \vDash \forall y\, \forall x\, R(x, y),$$

which is trivially valid. We will show that many logics, and in particular those defined syntactically, are structural with a more general notion of formula substitution where we can choose a natural number m and map each n-ary relation to a formula-in-context with nm free variables, map function symbols to m-lists of terms, and map equations to sets of m-equations.

Defining capture avoiding term substitution is maddening. After thinking about bound variables a bit, it becomes clear that their only semantic role is to point to the appropriate quantifier. This observation is implemented in De Bruijn indices, where

$$\forall x \, (\exists y \, R(x, w) \wedge \forall z \, S(x, y, z)),$$

can be represented as

$$\forall \, (\exists \, R(2, 3) \wedge \forall \, S(2, 4, 1)).$$

In the representation, variables are natural numbers, a bound variable is the number of quantifiers it is in the scope of up to its binding quantifier, and a free variable is a number which exceed this value for each quantifier it is in the scope of.

Thus it seems that the names of the bound variables are immaterial and so before substitution we can swap the bound variables out with “fresh” ones. However, this requires a notion of variable substitution! Luckily, we can avoid circularity by defining substitution inductively on the complexity of formulas. However, we now have to make a choice of which fresh variables to swap in and any choice will either break the associativity of the substitution operations or break the identity substitutions. Moreover, since semantically, bound variables are just pointers, we would like to not distinguish between different choices. Thus we must identify formulas up to α-equivalence, i.e. up to having the same representation in De Bruijn notation, and make sure substitution respects α-equivalence.

For most introductory books on mathematical logic, it seems the technical details of presenting the full construction of capture avoiding substitution outweighs the perceived benefits. This is even case with introductory texts on Lambda Calculus such as (Hindley and Seldin, 2008, p. 5), where key parts of the proof are left to the appendix or to a cited source.

First, we show with contexts, a simple “follow your nose” presentation of capture avoiding substitution can be given. The reasons are twofold: First, contexts eliminate the need for many tedious definitions and lemmas which essentially recover contexts from formulas without them. Secondly, contexts help avoid casework[1], because they stratify the terms and formulas, so that only meaningful substitutions are defined.

Contexts non-trivially add to the syntax, in the sense that to each formula there are many valid formulas-in-context and so one may wonder if we are merely simplifying a proof at the expense of complicating the syntax. However, in classical first-order model theory contexts are unavoidable since they allow structures to unambiguously interpret terms as term functions and formulas as definable sets. In most textbooks on the subject, a context for a formula ϕ is specified by $\phi(\bar{x})$, where $\bar{x}$ is assumed to be a list of variables which includes those occurring free in ϕ. In addition, contexts allow one to define satisfaction of a formula by a structure without variable assigments (valuations) or flooding the signature with additional constants naming

[1] In (Hindley and Seldin, 2008, p. 7), substitution is defined for the Lambda calculus in seven cases.

the elements of S. This provides an elegant resolution to a philosophical problem discussed in (Button and Walsh, 2018, p. 10). Moreover, this notion of satisfaction can be defined without explicit reference to elements of sets, allowing one abstract away from sets to define categorical (hyperdoctrine) semantics for first-order logic.

2.1 Well-formed Expressions-in-Context

Throughout the paper, we fix a **first-order language** $\mathscr{L}$, consisting of a collection $\mathscr{L}_q$ of quantifier symbols and a collection $\mathscr{L}_\omega$ of propositional connectives each with a designated arity in ω. For each $n \in \omega$, let $\mathscr{L}_n$ denote the n-ary propositional connectives in $\mathscr{L}_\omega$. We consider multi-sorted first-order logic, and so a **signature** Sg is a collection of sort symbols $\sigma, \tau, \gamma, \ldots$, typed function symbols $f\colon \sigma_1, \ldots, \sigma_n \to \tau$, (we write $f : \tau$ for $n = 0$), and typed relation symbols $R \subseteq \sigma_1, \ldots, \sigma_n$.[2] (we write $R \subseteq \emptyset$ for $n = 0$)

Let V be a countably-infinite well-ordered set of "variables". To provide prop-categorical semantics, well-formed terms and formulas must be "in context". A **context** $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ is a list of variable–sort pairs, where the collection of variables $\{x_1, \ldots, x_n\}$ are pairwise distinct. We define $\mathrm{Var}(\Gamma) \coloneqq \{x_1, \ldots, x_n\}$. We denote the concatenation of two contexts Γ and Γ' by Γ, Γ'. Whenever we use this notation, we presuppose $\mathrm{Var}(\Gamma) \cap \mathrm{Var}(\Gamma') = \emptyset$, so Γ, Γ' is a valid context.

We let $M : \tau\ [\Gamma]$ be the assertion that the term M is of type τ and Γ is a valid context for M. The (well-formed) terms-in-context are defined inductively via the following two typing rules:

$$\frac{}{x : \sigma\ [\Gamma, x : \sigma, \Gamma']} \tag{2.3}$$

$$\frac{M_1 : \sigma_1\ [\Gamma] \quad \ldots \quad M_n : \sigma_n\ [\Gamma]}{f(M_1, \ldots, M_n) : \tau\ [\Gamma]} \tag{2.4}$$

for each variable x, each sort symbol σ, and function symbol $f\colon \sigma_1, \ldots, \sigma_n \to \tau$ in Sg. When $n = 0$ in which case f is a constant $c : \tau$, (2.4) becomes

$$\frac{}{c : \tau\ [\Gamma]} \tag{2.5}$$

These are the only typing rules for terms, and from these rules alone, one may prove that the typing rule

$$\frac{M : \sigma\ [\Delta] \qquad N : \tau\ [\Gamma, x : \sigma, \Gamma']}{N[M/x] : \tau\ [\Gamma, \Gamma']} \tag{2.6}$$

is admissible, where $N[M/x]$ denotes the operation of substituting each occurrence of x in N with M and

[2] $f\colon \sigma_1, \ldots, \sigma_n \to \tau$ and $R \subseteq \sigma_1, \ldots, \sigma_n$ are merely typing assignments and do not assert that f is a morphism or R is a subset of some set.

the variable sort pairs in Δ are contained in the set of those in Γ, Γ'. We denote this by $\mathrm{VS}(\Delta) \subseteq \mathrm{VS}(\Gamma, \Gamma')$.

We define the **equations-in-context** by the rule:

$$\frac{M_1 : \tau\ [\Gamma] \quad M_2 : \tau\ [\Gamma]}{M_1 = M_2 : \tau\ [\Gamma]}\ . \tag{2.7}$$

Let p be the designated type for well-formed formulas-in-context, which are specified by the following rules: For each sort $\tau \in \mathrm{Sg}$,

$$\frac{M_1 : \tau\ [\Gamma] \quad M_2 : \tau\ [\Gamma]}{M_1 =_\tau M_2 : \mathrm{p}\ [\Gamma]} \tag{2.8}$$

and for each $R \subseteq \sigma_1, \ldots, \sigma_n \in Sg$,

$$\frac{M_1 : \sigma_1\ [\Gamma] \quad \ldots \quad M_n : \sigma_n\ [\Gamma]}{R(M_1, \ldots, M_n) : \mathrm{p}\ [\Gamma]}\ . \tag{2.9}$$

These rules define the **atomic** formulas-in-context. Note that we have two notions of equality: typically the former is interpreted as strict equality, and the latter is often a coarser notion of equivalence or coherence. Additionally, we have the following rules:

$$\frac{\phi_1 : \mathrm{p}\ [\Gamma] \quad \ldots \quad \phi_n : \mathrm{p}\ [\Gamma]}{\diamond(\phi_1, \ldots, \phi_n) : \mathrm{p}\ [\Gamma]}\ \diamond \in \mathscr{L}_\omega \tag{2.10}$$

$$\frac{\phi : \mathrm{p}\ [\Gamma, x : \sigma]}{\Omega_{x:\sigma}(\phi) : \mathrm{p}\ [\Gamma]}\ \Omega \in \mathscr{L}_q \tag{2.11}$$

For each n-ary connective $\diamond \in \mathscr{L}_\omega$ and quantifier $\Omega \in \mathscr{L}_q$. Note that Rule 2.11 is the only rule which modifies the context. The rules for terms and formulas in context define a multisorted term/formula algebra over the variables V. The sorts of the algebra are all pairs (Γ, η), where η is a sort τ in Sg or $\eta = \mathrm{p}$ and Γ is a context. The elements of type (Γ, τ) are the terms-in-context $M : \tau\ [\Gamma]$ and the elements of type (Γ, p) are the formulas-in-context $\phi : \mathrm{p}\ [\Gamma]$. Then one takes the previous term and formula construction rules relative to a context Γ, as the operations of the algebra. We call this algebra $\mathrm{Fm}_{\mathscr{L}}^{Sg}$.

Notice that each term-in-context and formula-in-context has a unique derivation and that the type τ associated to a term-in-context $M : \tau\ [\Gamma]$ is wholly determined by M and Γ. Thus in the sequel we will often drop the types, and denote $M : \tau\ [\Gamma]$ as $M\ [\Gamma]$ and formulas-in-context $\phi : \mathrm{p}\ [\Gamma]$ as $\phi\ [\Gamma]$.

2.2 Term-substitution and α-equivalence

For $X \subsetneq V$, since V is well-ordered, we define $\mathrm{mex}(X)$ to be the minimal excluded variable. That is,

$$\mathrm{mex}(X) \coloneqq \min(V \setminus X).$$

Let $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ be a context, and $[M_1 : \sigma_1\ [\Gamma'], \ldots, M_n : \sigma_n\ [\Gamma']] = \bar{m}$ be a list of terms in the same context Γ'. We define the operation $\bar{m}^* \colon (\Gamma, \tau) \to (\Gamma', \tau)$ as follows:

$$\bar{m}^*(x_k : \sigma_k\ [x_1 : \sigma_1, \ldots, x_n : \sigma_n]) \coloneqq M_k : \sigma_k\ [\Gamma'], \quad k \in \{1, \ldots, n\}. \tag{2.12}$$

And for $f(N_1, \ldots, N_m) : \tau\ [\Gamma]$, where $f \colon \tau_1, \ldots, \tau_m \to \tau$, and $N_i : \tau_i\ [\Gamma]$ for $i \in \{1, \ldots, m\}$,

$$\bar{m}^*(f(N_1, \ldots, N_m) : \tau\ [\Gamma]) \coloneqq f(\bar{m}^*(N_1 : \tau_1\ [\Gamma]), \ldots, \bar{m}^*(N_m : \tau_m\ [\Gamma])) : \tau\ [\Gamma'], \tag{2.13}$$

where

$$f(\cdot, \ldots, \cdot) : \tau\ [\Gamma'] \colon (\Gamma', \tau_1) \times \ldots \times (\Gamma', \tau_m) \to (\Gamma', \tau),$$

is the operation in $\mathrm{Fm}^{Sg}_{\mathscr{L}}$. As a special case of (2.13), if $m = 0$, in which case f is a constant $c : \tau$, we have

$$\bar{m}^*(c : \tau\ [\Gamma]) \coloneqq c : \tau\ [\Gamma']. \tag{2.14}$$

Note that we specify which variable in Γ gets substituted by which term $M_i : \sigma_i\ [\Gamma']$, By the ordering of the variables in Γ and the ordering of the terms-in-context in $\bar{m}$. If we did not use contexts, $\bar{m}^*$ would be written as

$$[M_1 : \sigma_1\ [\Gamma']/x_1, \ldots, M_n : \sigma_i\ [\Gamma']/x_n]^*.$$

Consider $\bar{n} = [N_1 : \tau_1\ [\Gamma''], \ldots, N_m : \tau_m\ [\Gamma'']]$, where $\Gamma' = y_1 : \tau_1, \ldots, y_m : \tau_m$. Even though there are many contexts besides Γ' with the list of sorts $\tau_1, \ldots, \tau_m$, we will think of $\bar{n}$ as a morphism[3] $\bar{n} \colon \Gamma'' \to \Gamma'$. We define the composition by

$$\bar{m} \circ \bar{n} \coloneqq [\bar{n}^*(M_1 : \sigma_1\ [\Gamma']), \ldots, \bar{n}^*(M_n : \sigma_n\ [\Gamma'])].$$

Lemma 1. *Let $M : \sigma\ [\Gamma]$ be a term-in-context, $\bar{n} \colon \Gamma' \to \Gamma$ and $\bar{o} \colon \Gamma'' \to \Gamma'$. Then $\bar{o}^*\bar{n}^*(M : \sigma\ [\Gamma]) = (\bar{n} \circ \bar{o})^*(M : \sigma\ [\Gamma])$.*

Proof. If $M : \sigma\ [\Gamma] = x_k : \sigma_k\ [\Gamma]$, then

$$\bar{o}^*\bar{n}^*(x_k : \sigma_i\ [\Gamma]) = \bar{o}^*(N_k : \sigma_i\ [\Gamma]) = (\bar{n} \circ \bar{o})^*(x_k : \sigma_i\ [\Gamma]).$$

[3] To be precise $\bar{n}$ must have a specified codomain and so we should somehow include Γ' in the data of $\bar{n}$. This will no longer be necessary when we identify formulas and terms in context up to changes in the name of free variables.

And if $M : \sigma\ [\Gamma] = f(B_1, \ldots, B_k) : \sigma\ [\Gamma]$, then,

$$\bar{o}^*\bar{n}^*(M : \sigma\ [\Gamma]) = f(\bar{o}^*\bar{n}^* B_1\ [\Gamma], \ldots, \bar{o}^*\bar{n}^* B_k\ [\Gamma])\ [\Gamma'']$$
$$= f((\bar{n} \circ \bar{o})^* B_1\ [\Gamma], \ldots, (\bar{n} \circ \bar{o})^* B_1\ [\Gamma])\ [\Gamma''] = (\bar{n} \circ \bar{o})^*(M : \tau\ [\Gamma]). \qquad \square$$

Proposition 1. *For all* $\bar{m} \colon \Gamma' \to \Gamma$, $\bar{n} \colon \Gamma'' \to \Gamma'$ *and* $\bar{o} \colon \Gamma''' \to \Gamma''$, $(\bar{m} \circ \bar{n}) \circ \bar{o} = \bar{m} \circ (\bar{n} \circ \bar{o})$.

Proof. Let $\bar{m} = [M_1 : \sigma_1\ [\Gamma'], \ldots, M_n : \sigma_n\ [\Gamma']]$. From Lemma 1,

$$\bar{m} \circ (\bar{n} \circ \bar{o}) = [(\bar{n} \circ \bar{o})^*(M_1 : \sigma_1\ [\Gamma']), \ldots, (\bar{n} \circ \bar{o})^*(M_n : \sigma_n\ [\Gamma'])]$$
$$[\bar{o}^*\bar{n}^*(M_1 : \sigma_1\ [\Gamma']), \ldots, \bar{o}^*\bar{n}^*(M_n : \sigma\ [\Gamma'])] = (\bar{m} \circ \bar{n}) \circ \bar{o}. \qquad \square$$

For a context $\Gamma_2 = y_1 : \tau_1, \ldots, y_m : \tau_m$, we define id_{Γ_2} by

$$\mathrm{id}_{\Gamma_2} \coloneqq [y_1 : \tau_1\ [\Gamma_2], \ldots, y_m : \tau_m\ [\Gamma_2]].$$

Note that, if $\bar{m} \colon \Gamma_2 \to \Gamma_1$, then $\bar{m} \circ \mathrm{id}_{\Gamma_2} = \bar{m}$ and if $\bar{n} \colon \Gamma_3 \to \Gamma_2$, then $\mathrm{id}_{\Gamma_2} \circ \bar{n} = \bar{n}$. It follows that the contexts and lists of terms in a context (with specified codomains) form a category which we denote $\mathcal{C}_{Sg}$.

If $\Gamma_1 = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and $\mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2) = \emptyset$, we define,

$$\pi_1^{\Gamma_1, \Gamma_2} \coloneqq [x_1 : \sigma_1\ [\Gamma_1, \Gamma_2], \ldots, x_n : \sigma_n\ [\Gamma_1, \Gamma_2]]$$
$$\pi_2^{\Gamma_1, \Gamma_2} \coloneqq [y_1 : \tau_1\ [\Gamma_1, \Gamma_2], \ldots, y_m : \tau_m\ [\Gamma_1, \Gamma_2]].$$

If $\bar{m} \colon \Gamma_3 \to \Gamma_1$ and $\bar{n} \colon \Gamma_3 \to \Gamma_2$, then $\bar{m} = [M_1 : \sigma_1\ [\Gamma_3], \ldots, M_n : \sigma_n\ [\Gamma_3]]$ and $\bar{n} = [N_1 : \tau_1\ [\Gamma_3], \ldots, N_m : \tau_m\ [\Gamma_3]]$. We define

$$\langle \bar{m}, \bar{n} \rangle \coloneqq [M_1 : \sigma_1\ [\Gamma_3], \ldots, M_n : \sigma_n\ [\Gamma_3], N_1 : \tau_1\ [\Gamma_3], \ldots, N_m : \tau_m\ [\Gamma_3]].$$

Then

$$\pi_1^{\Gamma_1, \Gamma_2} \circ \langle \bar{m}, \bar{n} \rangle = \bar{m} \quad \text{and} \quad \pi_2^{\Gamma_1, \Gamma_2} \circ \langle \bar{m}, \bar{n} \rangle = \bar{n}.$$

Moreover, one verifies $\langle \bar{m}, \bar{n} \rangle$ is the unique such morphism. Thus Γ_1, Γ_2 is a product of Γ_1 and Γ_2. The category $\mathcal{C}_{Sg}$ also has a terminal object, which is the empty context $[\,]$, since for each context Γ_1, the empty list of terms in context Γ_1, which we denote $[\,[\Gamma_1]]$, is a morphism from Γ_1 to $[\,]$. It follows that $\mathcal{C}_{Sg}$ is a category with finite products.

Now given $\bar{m}\colon \Gamma' \to \Gamma$, we define substitution $\bar{m}^*\colon (\mathrm{p}, \Gamma) \to (\mathrm{p}, \Gamma')$ on atomic formulas-in-context:

$$\bar{m}^*(N_1 =_\tau N_2 : \mathrm{p}\,[\Gamma]) \coloneqq \bar{m}^*(N_1 : \tau\,[\Gamma]) =_\tau \bar{m}^*(N_2 : \tau\,[\Gamma]) : \mathrm{p}\,[\Gamma'] \tag{2.15}$$

and for $R \subseteq \tau_1, \ldots, \tau_m$, $N_i : \tau_i\,[\Gamma]$ for $i \in \{1, \ldots, m\}$

$$\bar{m}^*(R(N_1, \ldots, N_m) : \mathrm{p}\,[\Gamma]) \coloneqq R(\bar{m}^*(N_1 : \tau_1\,[\Gamma]), \ldots, \bar{m}^*(N_m : \tau_m\,[\Gamma])) : \mathrm{p}\,[\Gamma'] \tag{2.16}$$

Then we extend $\bar{m}^*\colon (\Gamma, \mathrm{p}) \to (\Gamma', \mathrm{p})$ to all formulas-in-context by

$$\bar{m}^*(\Diamond(\phi_1, \ldots, \phi_m) : \mathrm{p}\,[\Gamma]) \coloneqq \Diamond(\bar{m}^*(\phi_1 : \mathrm{p}\,[\Gamma]), \ldots, \bar{m}^*(\phi_m : \mathrm{p}\,[\Gamma])) : \mathrm{p}\,[\Gamma'] \tag{2.17}$$

Now the rule for quantifiers is trickier. For $z : \sigma$ where $z \notin \mathrm{Var}(\Gamma')$, we define:

$$\bar{m} \times \mathrm{id}_{z:\sigma} \coloneqq [M_1 : \sigma_1\,[\Gamma', z : \sigma], \ldots, M_n : \sigma_n\,[\Gamma', z : \sigma], z : \sigma\,[\Gamma', z : \sigma]], \tag{2.18}$$

which is a product of $\bar{m}$ and $\mathrm{id}_{z:\sigma}$. Then we define,

$$\bar{m}^*(\Omega_{x_k:\sigma}(\phi) : \mathrm{p}\,[\Gamma]) \coloneqq \Omega_{x_j:\sigma}(\bar{m} \times \mathrm{id}^*_{x_j:\sigma}(\phi : \mathrm{p}\,[\Gamma, x_k : \sigma])) : \mathrm{p}\,[\Gamma'], \tag{2.19}$$

where $x_j = \mathrm{mex}(\mathrm{Var}(\Gamma'))$. By induction, $\bar{m} \times \mathrm{id}^*_{x_j:\sigma}(\phi : \mathrm{p}\,[\Gamma, x_k : \sigma])$ is a well-formed formula-in-context $\Gamma', x_j : \sigma$, and so $\bar{m}^*(\Omega_{x_k:\sigma}(\phi) : \mathrm{p}\,[\Gamma])$ is a well-formed formula-in-context Γ'.

Proposition 2. *Let $\bar{m}\colon \Gamma' \to \Gamma$ and $\bar{n}\colon \Gamma'' \to \Gamma'$. Then, for all formulas-in-context $\phi : \mathrm{p}\,[\Gamma]$,*

$$\bar{n}^*(\bar{m}^*(\phi\,[\Gamma])) = (\bar{m} \circ \bar{n})^*(\phi\,[\Gamma]).$$

Proof. The base cases are immediate from Proposition 1 and the only non-trivial inductive case is when $\phi\,[\Gamma]$

is of the form $\Omega_{x:\sigma}(\phi_1)\,[\Gamma]$. Let $y = \mathrm{mex}(\mathrm{Var}(\Gamma'))$ and $z = \mathrm{mex}(\mathrm{Var}(\Gamma''))$. Then,

$$
\begin{aligned}
&\bar{n}^*(\bar{m}^*(\Omega_{x:\sigma}(\phi_1)\,[\Gamma])) \\
&= \bar{n}^*(\Omega_{y:\sigma}(\bar{m} \times \mathrm{id}^*_{y:\sigma}(\phi_1\,[\Gamma, x : \sigma]))\,[\Gamma']) \\
&= \Omega_{z:\sigma}(\bar{n} \times \mathrm{id}^*_{z:\sigma}(\bar{m} \times \mathrm{id}^*_{y:\sigma}(\phi_1\,[\Gamma, x : \sigma])))\,[\Gamma''] \\
&= \Omega_{z:\sigma}((\bar{m} \times \mathrm{id}_{y:\sigma} \circ \bar{n} \times \mathrm{id}_{z:\sigma})^*(\phi_1\,[\Gamma, x : \sigma]))\,[\Gamma''] \\
&= \Omega_{z:\sigma}((\bar{m} \circ \bar{n}) \times \mathrm{id}^*_{z:\sigma}(\phi_1\,[\Gamma, x : \sigma]))\,[\Gamma''] \\
&= (\bar{m} \circ \bar{n})^*(\Omega_{x:\sigma}(\phi_1)\,[\Gamma]).
\end{aligned}
$$

□

Consider $\phi : \mathrm{p}\,[\Gamma]$, where $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$. Then we *want* $\mathrm{id}^*_\Gamma(\phi\,[\Gamma]) = \phi\,[\Gamma]$: But due to changes in bound variables $\mathrm{id}^*_\Gamma(\phi\,[\Gamma])$ may not equal $\phi\,[\Gamma]$. Moreover, our definition of substitution made use of an arbitrary ordering of the variables, and we think such choices shouldn't matter. More generally, in mathematical practice we do not distinguish between names of bound variables, for example, between $\int f(x)\,\mathrm{d}x$ and $\int f(y)\,\mathrm{d}y$.

Thus we define an equivalence on formulas-in-context up to a suitable change in bound variables, called **α-equivalence** recursively on the construction of formulas-in-context. Such an equivalence should identify formulas in-context so that each class corresponds to a unique formula in the De Bruijn notation.

We denote α-equivalence by $\sim_\alpha$ and say two terms-in-context are α-equivalent iff they are equal. We say two formulas in context $\phi\,[\Gamma]$, $\psi\,[\Gamma']$, are α-equivalent iff $\Gamma = \Gamma'$ and

$$\mathrm{id}^*_\Gamma(\phi\,[\Gamma]) = \mathrm{id}^*_\Gamma(\psi\,[\Gamma]).$$

Theorem 1. *α-equivalence defines a congruence on $\mathrm{Fm}^{Sg}_{\mathscr{L}}$ and term substitution determines a well-defined contravariant functor $P_{Sg} \colon \mathcal{C}_{Sg} \to \mathrm{Alg}_{\mathscr{L}_\omega}$, where $P_{Sg}(\Gamma)$ are the formulas in context Γ and $P_{Sg}(\bar{m}) = \bar{m}^*$.*

Proof. Then it is straightforward to verify that $\sim_\alpha$ is an equivalence which respects the operations on terms-in-context, and the operations on formulas-in-context determined by the propositional connectives. Suppose $\phi\,[\Gamma, x : \sigma] \sim_\alpha \psi\,[\Gamma, x : \sigma]$ and let $w = \mathrm{mex}(\mathrm{Var}(\Gamma))$. Then,

$$
\begin{aligned}
&\mathrm{id}^*_\Gamma(\Omega_{x:\sigma}(\phi)\,[\Gamma]) = \Omega_{w:\sigma}(\mathrm{id}^*_{\Gamma,w:\sigma}(\phi\,[\Gamma, x : \sigma])) \\
&= \Omega_{w:\sigma}((\mathrm{id}_{\Gamma,x:\sigma}\mathrm{id}_{\Gamma,w:\sigma})^*(\phi\,[\Gamma, x : \sigma])) = \Omega_{w:\sigma}(\mathrm{id}^*_{\Gamma,w:\sigma}\mathrm{id}^*_{\Gamma,x:\sigma}(\phi\,[\Gamma, x : \sigma])) \\
&= \Omega_{w:\sigma}(\mathrm{id}^*_{\Gamma,w:\sigma}\mathrm{id}^*_{\Gamma,x:\sigma}(\psi\,[\Gamma, x : \sigma])) = \Omega_{w:\sigma}(\mathrm{id}^*_{\Gamma,w:\sigma}(\psi\,[\Gamma, x : \sigma])) \\
&= \mathrm{id}^*_\Gamma\Omega_{x:\sigma}(\psi)\,[\Gamma].
\end{aligned}
$$

It follows that $\sim_\alpha$ is a congruence. For each formula in context $\phi\,[\Gamma]$,

$$id_\Gamma^* id_\Gamma^*(\phi\,[\Gamma]) = id_\Gamma^*(\phi\,[\Gamma]),$$

so $\mathrm{id}_\Gamma^*(\phi\,[\Gamma]) \sim_\alpha \phi\,[\Gamma]$. For each morphism $\bar{m}\colon \Gamma' \to \Gamma$ and formulas in context $\phi\,[\Gamma], \psi\,[\Gamma]$ such that $\phi\,[\Gamma] \sim_\alpha \psi\,[\Gamma]$,

$$\bar{m}^*(\phi\,[\Gamma]) = (\mathrm{id}_\Gamma \circ \bar{m})^*(\phi\,[\Gamma]) = \bar{m}^*\mathrm{id}_\Gamma^*(\phi\,[\Gamma]) = \bar{m}^*\mathrm{id}_\Gamma^*(\psi\,[\Gamma]) = \bar{m}^*(\psi\,[\Gamma]).$$

It follows that term substitution respects α-equivalence. From these observations and Proposition 2, it follows that P_{Sg} is a contravariant functor. □

We call the structure $(\mathcal{C}_{Sg}, P_{Sg})$ the **formula prop-category** (associated to Sg and $\mathscr{L}$), and consider it a first-order analogue of the formula algebra[4] of a propositional logic. An advantage of considering $(\mathcal{C}_{Sg}, P_{Sg})$ the "formula algebra" over $\mathrm{Fm}_{\mathscr{L}}^{Sg}$ is that it elegantly captures the structure of term-substitution. However, as of now, we have not encoded the operations quantifying over bound variables into the categorical perspective.

For Γ and $x : \sigma$, we define the binary product $\Gamma \times x : \sigma$ by

$$\Gamma \times x : \sigma \coloneqq \begin{cases} \Gamma, x : \sigma & x \notin \mathrm{Var}(\Gamma), \\ \Gamma, y : \sigma & x \in \mathrm{Var}(\Gamma), \end{cases}$$

where $y = \mathrm{mex}(\mathrm{Var}(\Gamma))$. Then for $\Gamma' = y_1 : \tau_1, \ldots, y_m : \tau_m$, we define

$$\Gamma \times \Gamma' \coloneqq \begin{cases} \Gamma & \Gamma' = [\,], \\ (\Gamma \times y_1 : \tau_1) \times (y_2 : \tau_2, \ldots, y_m : \tau_m) & \text{otherwise.} \end{cases}$$

Let $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and $\Gamma' = y_1 : \tau_1, \ldots, y_1 : \tau_m$ be contexts and let

$$\Gamma \times \Gamma' = x_1 : \sigma_1, \ldots, x_n : \sigma_n, z_1 : \tau_1, \ldots, z_m : \tau_m.$$

Then we define $\Omega_{\Gamma,\Gamma'}\colon P_{Sg}(\Gamma \times \Gamma') \to P_{Sg}(\Gamma)$ by

$$\Omega_{\Gamma,\Gamma'}(\phi\,[\Gamma \times \Gamma']) \coloneqq \Omega_{z_1:\tau_1} \ldots \Omega_{z_m:\tau_m}(\phi)\,[\Gamma].$$

[4]Sometimes called the term algebra, which causes confusion in the first-order setting.

In particular, if $\Gamma' = [\,]$, then $\Omega_{\Gamma,\Gamma'}$ is the identity function. $\Omega_{\Gamma,\Gamma'}$ is well-defined, since for $\phi\,[\Gamma \times \Gamma'] \sim_\alpha \psi\,[\Gamma \times \Gamma']$,

$$\begin{aligned}
\mathrm{id}^*_\Gamma \circ \Omega_{\Gamma,\Gamma'}(\phi\,[\Gamma \times \Gamma']) &= id^*_\Gamma(\Omega_{z_1:\tau_1} \ldots \Omega_{z_m:\tau_m}(\phi)[\Gamma]) \\
&= \Omega_{z_1:\tau_1} \ldots \Omega_{z_m:\tau_m}(\mathrm{id}^*_{\Gamma\times\Gamma'}(\phi\,[\Gamma \times \Gamma']))\,[\Gamma] \\
&\sim_\alpha \Omega_{z_1:\tau_1} \ldots \Omega_{z_m:\tau_m}(\mathrm{id}^*_{\Gamma\times\Gamma'}(\psi\,[\Gamma \times \Gamma']))\,[\Gamma] \\
&= \mathrm{id}^*_\Gamma \circ \Omega_{\Gamma,\Gamma'}(\psi\,[\Gamma \times \Gamma']).
\end{aligned}$$

Proposition 3. *For all* $\Omega \in \mathscr{L}_q$ *and contexts* Γ, $\Omega_{(\cdot),\Gamma} \colon UP_{Sg}(\_ \times \Gamma) \Rightarrow UP_{Sg}$ *is a natural transformation, where* $U \colon \mathrm{Alg}_{\mathscr{L}_\omega} \to \mathbf{Set}$ *is the forgetful functor.*

Proof. Let $\bar{m} \colon \Gamma_1 \to \Gamma_2$ and let Γ_3, Γ_4 be contexts such that $\Gamma_1, \Gamma_3 = \Gamma_1 \times \Gamma_3$ and $\Gamma_2, \Gamma_4 = \Gamma_2 \times \Gamma_4$. Let $\Gamma_3 = z_1 : \tau_1, \ldots, z_m : \tau_m$ and $\Gamma_4 = w_1 : \tau_1, \ldots w_m : \tau_m$ Then,

$$\begin{array}{ccc}
UP_{Sg}(\Gamma_2, \Gamma_4) & \xrightarrow{\Omega_{\Gamma_2,\Gamma_4}} & UP_{Sg}(\Gamma_2) \\
{\scriptstyle (\bar{m}\times\mathrm{id}_{\Gamma_3})^*}\downarrow & & {\scriptstyle \bar{m}^*}\downarrow \\
UP_{Sg}(\Gamma_1, \Gamma_3) & \xrightarrow{\Omega_{\Gamma_1,\Gamma_3}} & UP_{Sg}(\Gamma_1)
\end{array}$$

commutes since

$$\begin{aligned}
\bar{m}^* \circ \Omega_{\Gamma_2,\Gamma_4}(\phi\,[\Gamma_2, \Gamma_4]) &= \bar{m}^*(\Omega_{w_1:\tau_1} \ldots \Omega_{w_m:\tau_m}(\phi)\,[\Gamma_2]) \\
&= \Omega_{z_1:\tau_1} \ldots \Omega_{z_m:\tau_m}((\bar{m} \times \mathrm{id}_{\Gamma_3})^*(\phi\,[\Gamma_2, \Gamma_4]))\,[\Gamma_2] \\
&= \Omega_{\Gamma_1,\Gamma_3} \circ (\bar{m} \times \mathrm{id}_{\Gamma_3})^*(\phi\,[\Gamma_2, \Gamma_4]).
\end{aligned}$$

□

2.3 Equivalence up to Change in Free-variable Name

In mathematical practice, we also do not distinguish between two expressions equal up to a change in free variable names, for example between polynomials x^2+1 and y^2+1. Considering expressions also equivalent up to changes in free variables has some structural benefits. In particular, the category $\mathcal{C}_{Sg}$ then has strictly associative finite products.

We let $\mathrm{Srt}(Sg)$ be the collection of sorts in Sg, and for each context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, we let $\mathrm{Srt}(\Gamma) = \sigma_1, \ldots, \sigma_n$, be the associated list of sorts. Then we say $\Gamma' \sim \Gamma$ iff $\mathrm{Srt}(\Gamma) = \mathrm{Srt}(\Gamma')$.

Let $Q : \eta\,[\Gamma_1], P : \eta\,[\Gamma_2]$, be terms or formulas in context of the same type η. Then we say $Q : \eta\,[\Gamma_1] \sim$

$P : \eta\,[\Gamma_2]$ iff there exists a context Γ_3, such that $\Gamma_3 \sim \Gamma_1 \sim \Gamma_2$ and

$$id^*_{\Gamma_3}(Q : \eta\,[\Gamma_1]) = \mathrm{id}^*_{\Gamma_3}(P : \eta\,[\Gamma_2]). \tag{2.20}$$

Note that if Equation 2.20 holds, then for any context Γ such that $\mathrm{Srt}(\Gamma) = \mathrm{Srt}(\Gamma_3)$, we have

$$\begin{aligned} \mathrm{id}^*_{\Gamma}(Q : \eta\,[\Gamma_1]) = (\mathrm{id}_{\Gamma_3}\mathrm{id}_{\Gamma})^*(Q : \eta\,[\Gamma_1]) = \mathrm{id}^*_{\Gamma}\mathrm{id}^*_{\Gamma_3}(Q : \eta\,[\Gamma]) \\ \mathrm{id}^*_{\Gamma}\mathrm{id}^*_{\Gamma_3}(P : \eta\,[\Gamma_2]) = (\mathrm{id}_{\Gamma_3}\mathrm{id}_{\Gamma})^*(P : \eta\,[\Gamma_2]) = \mathrm{id}^*_{\Gamma}(P : \eta\,[\Gamma_2]). \end{aligned} \tag{2.21}$$

Proposition 4. *The relation $\sim$ defines a congruence on $\mathrm{Fm}^{Sg}_{\mathscr{L}}$.*

Proof. The equivalence on contexts induces an equivalence on sorts of $\mathrm{Fm}^{Sg}_{\mathscr{L}}$, by $\eta : [\Gamma_1] \sim \zeta : [\Gamma_2]$ iff $\eta = \zeta$ and $\Gamma_1 \sim \Gamma_2$. That $\sim$ is reflexive and symmetric is immediate. Transitivity follows from Equation 2.21. Consider $f : \sigma_1, \ldots, \sigma_n \to \tau$, and $M_i : \sigma_i\,[\Gamma_i]/\!\sim\, \in \sigma_i : [\Gamma]/\!\sim$. Then

$$\begin{aligned} & f(M_1 : \sigma_1\,[\Gamma_1]/\!\sim, \ldots, M_n : \sigma_n\,[\Gamma_n]/\!\sim) : \tau\,[\Gamma]/\!\sim \\ \coloneqq\; & f(\mathrm{id}^*_{\Gamma}(M_1 : \sigma_1\,[\Gamma_1]), \ldots, \mathrm{id}^*_{\Gamma}(M_n : \sigma_n\,[\Gamma_n])) : \tau\,[\Gamma]/\!\sim . \end{aligned}$$

Then $f(\cdot, \ldots, \cdot) : \tau\,[\Gamma]/\!\sim$ is well-defined, since if $\Gamma \sim \Gamma'$ and for each $i \in \{1, \ldots, n\}$, $M_i : \sigma_i\,[\Gamma_i] \sim M'_i : \sigma_i\,[\Gamma'_i]$, then

$$\begin{aligned} & \mathrm{id}^*_{\Gamma}\, f(\mathrm{id}^*_{\Gamma'}(M'_1 : \sigma_1\,[\Gamma'_1]), \ldots, \mathrm{id}^*_{\Gamma'}(M'_n : \sigma_n\,[\Gamma'_1])) : \tau\,[\Gamma'] \\ =\; & f(\mathrm{id}^*_{\Gamma}(M'_1 : \sigma_1[\Gamma'_1]), \ldots, \mathrm{id}^*_{\Gamma}(M'_n : \sigma_n\,[\Gamma'_n])) : \tau\,[\Gamma] \\ =\; & f(\mathrm{id}^*_{\Gamma}(M_1 : \sigma_1[\Gamma_1]), \ldots, \mathrm{id}^*_{\Gamma}(M_n : \sigma_n\,[\Gamma_n])) : \tau\,[\Gamma]. \end{aligned}$$

One similarly, shows that the operations defined by the formulation rules for the atomic formulas respect $\sim$. That the operations defined by the propositional connectives respect $\sim$ is immediate and so we only consider the operations defined by the quantifier symbols. For contexts Γ_1, Γ_2, and $\Gamma_1 \times \Gamma_2 \sim \Gamma_3, \Gamma_4$, we define

$$\Omega_{\Gamma_1,\Gamma_2}(\phi\,[\Gamma_3, \Gamma_4]/\!\sim) \coloneqq \Omega_{\Gamma_1,\Gamma_2}(\mathrm{id}^*_{\Gamma_1 \times \Gamma_2}(\phi\,[\Gamma_3, \Gamma_4]))/\!\sim .$$

Then if for $i \in \{1,2,3,4\}$, $\Gamma_i \sim \Gamma_i'$ and $\phi\,[\Gamma_3, \Gamma_4] \sim \psi\,[\Gamma_3', \Gamma_4']$,

$$\begin{aligned}
&\mathrm{id}^*_{\Gamma_1}(\Omega_{\Gamma_1', \Gamma_2'} \circ \mathrm{id}^*_{\Gamma_1' \times \Gamma_2'}(\psi\,[\Gamma_3', \Gamma_4'])) \\
&= \Omega_{\Gamma_1, \Gamma_2'} \circ id^*_{\Gamma_1 \times \Gamma_2'} \circ \mathrm{id}^*_{\Gamma_1' \times \Gamma_2'}(\psi\,[\Gamma_3', \Gamma_4']) \\
&= \Omega_{\Gamma_1, \Gamma_2} \circ \mathrm{id}^*_{\Gamma_1 \times \Gamma_2}(\psi\,[\Gamma_3', \Gamma_4']) \\
&= \Omega_{\Gamma_1, \Gamma_2} \circ \mathrm{id}^*_{\Gamma_1 \times \Gamma_2}(\phi\,[\Gamma_3, \Gamma_4]) \\
&= \mathrm{id}^*_{\Gamma_1}(\Omega_{\Gamma_1, \Gamma_2} \circ \mathrm{id}^*_{\Gamma_1 \times \Gamma_2}(\psi\,[\Gamma_3, \Gamma_4])),
\end{aligned}$$

and so

$$\Omega_{\Gamma_1, \Gamma_2}(\phi\,[\Gamma_3, \Gamma_4]/\!\sim) = \Omega_{\Gamma_1', \Gamma_2'}(\psi\,[\Gamma_3', \Gamma_4']/\!\sim).$$

□

Corollary 1. *We may now consider $(\mathcal{C}_{Sg}, P_{Sg})$ up to $\sim$. As before, $\mathcal{C}_{Sg}$ is a category with (now strictly associative) finite products, $P_{Sg}\colon \mathbb{C} \to \mathrm{Alg}_{\mathscr{L}_\omega}$ is a contravariant functor and each $\Omega_{(\cdot),\Gamma}\colon UP_{Sg}(\_ \times \Gamma) \to UP_{Sg}$ is a natural transformation. In addition, for contexts $\Gamma_1, \Gamma_2, \Gamma_3$,*

$$\Omega_{\Gamma_1,[]} = \mathrm{id}_{\Gamma_1} \quad \textit{and} \quad \Omega_{\Gamma_1, \Gamma_2 \times \Gamma_3} = \Omega_{\Gamma_1, \Gamma_2} \circ \Omega_{\Gamma_1 \times \Gamma_2, \Gamma_3}.$$

2.4 Formula Substitution.

We now define a very general form of formula substitution for which it will be straightforward to see that many first-order logics specified by derivation systems are structural with respect to the substitution action. This is because, one only needs to verify that the substitution action preserves the derivation rules and the substitution action commutes with the quantifier and propositional connective operations.

In what follows we consider all expressions equal up to changes in free and bound variable names. Let ζ be a map which: (1) for some $n \in \mathbb{N}$, sends each sort σ to a context Γ_σ, such that $n = |\operatorname{Var}(\Gamma_\sigma)|$ (2) sends each function symbol $f\colon \sigma_1, \ldots, \sigma_n \to \tau$, to a morphism

$$[M_1^f : \tau_1\,[\Gamma_{\sigma_1}, \ldots, \Gamma_{\sigma_n}], \ldots, M_{n_\tau}^f : \tau_{n_\tau}\,[\Gamma_{\sigma_1}, \ldots, \Gamma_{\sigma_n}]] =: \bar{m}^f,$$

where $[x_1 : \tau_1, \ldots, x_{n_\tau} : \tau_{n_\tau}] = \zeta(\tau)$ and (3) sends each relation symbol $R \subseteq \sigma_1, \ldots, \sigma_n$ to a formula-in-context

$$\phi^R\,[\Gamma_{\sigma_1}, \ldots, \Gamma_{\sigma_n}].$$

Then we extend ζ to all expressions-in-context as follows:

$$\zeta(x : \sigma\,[\Gamma]) \coloneqq [x_1 : \sigma_1\,[\zeta(\Gamma)], \ldots, , x_{n_\sigma} : \sigma_{n_\sigma}\,[\zeta(\Gamma)]],$$

where $\zeta(\sigma) = [x_1 : \sigma_1, \ldots, x_{n_\sigma} : \sigma_{n_\sigma}]$. For $f \colon \sigma_1, \ldots \sigma_n \to \tau$,

$$\zeta(f(M_1, \ldots, M_n) : \tau\,[\Gamma]) \coloneqq \bar{m}^f \circ \langle \zeta(M_1 : \sigma_1\,[\Gamma]), \ldots, \zeta(M_n : \sigma_n\,[\Gamma]) \rangle.$$

For each equation $M^1 = M^2 : \tau\,[\Gamma]$,

$$\zeta(M^1 = M^2 : \tau\,[\Gamma]) \coloneqq \{M_i^1 = M_i^2 : \tau_i\,[\zeta(\Gamma)]\}_{i=1}^{n_\tau},$$

where

$$\zeta(M^i : \tau\,[\Gamma]) = [M_1^i : \tau_i\,[\zeta(\Gamma)], \ldots, M_{n_\tau}^i : \tau_{n_\tau}\,[\zeta(\Gamma)]].$$

For each equality predicate,

$$\zeta(M^1 =_\tau M^2\,[\Gamma]) \coloneqq \bigwedge_{i=1}^{n_\tau} M_i^1 =_{\tau_i} M_i^2\,[\zeta(\Gamma)],$$

and for each relational atom,

$$\zeta(R(M_1, \ldots, M_n)\,[\Gamma]) \coloneqq \langle \zeta(M_1 : \sigma_1\,[\Gamma]), \ldots, \zeta(M_n : \sigma_n\,[\Gamma]) \rangle^*(\phi^R\,[\zeta(\Gamma)]).$$

For each formula-in-context of the form $\diamond(\phi_1, \ldots, \phi_n)\,[\Gamma]$,

$$\zeta(\diamond(\phi_1, \ldots, \phi_n)\,[\Gamma]) \coloneqq \diamond(\zeta(\phi_1\,[\Gamma]), \ldots, \zeta(\phi_n\,[\Gamma]))\,[\zeta(\Gamma)]$$

For formulas of the form $\Omega_{x:\sigma}(\phi)\,[\Gamma]$, where $\Omega \in \{\exists, \forall\}$,

$$\zeta(\Omega_{x:\sigma}(\phi)\,[\Gamma]) \coloneqq \Omega_{\zeta(\Gamma),\zeta(\sigma)}(\zeta(\phi\,[\Gamma, x : \sigma])).$$

Later we will define morphisms between prop-categories and see that the endomorphisms on $(\mathcal{C}_{Sg}, P_{Sg})$ are in bijective correspondence with these formula substitution maps. As a consequence, (it could also be shown directly) the formula substitution maps define an action on the formula algebra. Moreover, to see a first-order logic such as intuitionistic predicate logic with a complete derivation system is structural with

$$\frac{}{M = M : \sigma\ [\Gamma]} \qquad \frac{M = M' : \sigma\ [\Gamma]}{M' = M : \sigma\ [\Gamma]} \qquad \frac{M = M' : \sigma\ [\Gamma] \quad M' = M'' : \sigma\ [\Gamma]}{M = M'' : \sigma\ [\Gamma]}$$

$$\frac{M = M' : \sigma\ [\Delta] \quad N = N' : \tau\ [\Gamma, x : \sigma, \Gamma']}{N[M/x] = N'[M'/x] : \tau\ [\Gamma, \Gamma']}\ \mathrm{VS}(\Delta) \subseteq \mathrm{VS}(\Gamma, \Gamma').$$

Figure 2.1: Equational Logic.

respect to this notion of formula substitution, one need only verify that the substitution maps preserve the derivation rules. Thus first-order logics can be considered in the abstract algebraic logic framework where a logic is a closure operator on some abstract set of formulas, which is structural with respect to an action by a monoid on the formulas. This line of work will not be pursued here, since as we've seen, prop-categories elegantly capture the syntactic structure of first-order logics and locally (per fiber) restrict to the abstract algebraic perspective.

2.5 First-order Logics

In the following chapters, we consider formulas equal up α-equivalence, i.e. up to renaming of bound variables. As described in the prior sections, this allows us to define the action of simultaneous term substitution on (α-equivalence classes of) formulas that avoids variable capture which we denote by $\phi[M_1/x_1, \ldots, M_n/x_n]$.

The well-formed **sequents-in-context** are of the form $\phi_1, \ldots, \phi_n \vdash \phi_{n+1}\ [\Gamma]$, where $n \in \omega$ and for each $i \leq n+1$, $\phi_i : \mathrm{prop}\ [\Gamma]$. We call the sequents-in-context and equations-in-context **assertions** and let A_{Sg} denote the collection of all Sg-assertions. A **theory** T is an ordered pair $T = (\mathrm{Sg}(T), \mathrm{A}(T))$, where $\mathrm{Sg}(T)$ is a signature and $\mathrm{A}(T)$ is a collection of $\mathrm{Sg}(T)$-assertions. For each signature Sg, we let Th_{Sg} denote the complete lattice of Sg-theories ordered by $T_1 \leq T_2$ if and only if $\mathrm{A}(T_1) \subseteq \mathrm{A}(T_2)$.

Now that we have defined the well-formed expressions, a **(first-order) logic**[5] we define an action by substitution which can be used to define a (first-order) logic as *structural* closure operator. is a closure operator on the lattice of all Sg-theories for each signature Sg. In the sequel, all the logics we consider are built over the usual (typed) equational derivation system. The rules of this system are shown in Figure 2.1.

Given $T, T' \in \mathrm{Th}_{Sg}$, we let $T_{\mathcal{L}}$ denote the $\mathcal{L}$-closure of T and $T \vdash_{\mathcal{L}} T'$, denote the assertion that T' is **derivable** from T, i.e. that $T' \leq T_{\mathcal{L}}$.

[5]We keep the definition of a logic general here, though we could insist that the closure operator be structural with respect to some collection of formula substitution maps from the prior section.

Chapter 3

Semantics

3.1 Prop-Categorical Semantics

In this section, we define the class of prop-categories that provide semantics for first-order logics and provide examples. Structures in prop-categories and the notion of a structure satisfying a theory are then defined and it is shown how a class of prop-categories defines a first-order logic.

A **prop-category** $(\mathcal{C}, P)$, is a category $\mathcal{C}$ with designated finite products, and a contravariant functor $P\colon \mathcal{C} \to \mathbf{Pos}$, where $\mathbf{Pos}$ is the category of all partially ordered sets and monotone maps. The objects of $\mathcal{C}$ interpret the sorts, the morphisms interpret terms-in-context and the posets $P(c)$ for each $c \in \mathrm{Ob}(\mathcal{C})$ interpret the formulas-in-context. We define $\mathrm{Ob}(\mathbf{FA})$ to be the collection of all prop-categories which additionally satisfy:

1. For each $c \in \mathrm{Ob}(\mathcal{C})$, $P(c)$ is an $\mathscr{L}_\omega$-algebra[1] and for all $f \in \mathrm{Mor}(\mathcal{C})$, $P(f)$ is an $\mathscr{L}_\omega$-algebra homomorphism.

2. For each $c \in \mathrm{Ob}(\mathcal{C})$, there is a designated element $Eq_c \in P(c \times c)$.

3. For each $c \in \mathrm{Ob}(\mathcal{C})$, and each $\Omega \in \mathscr{L}_q$, there is a natural transformation $\Omega_{(\cdot),c}\colon UP(\_ \times c) \Rightarrow UP$, where $U\colon \mathbf{Pos} \to \mathbf{Set}$ is the forgetful functor.

4. For each $c \in \mathrm{Ob}(\mathcal{C})$, $P(c)$ has a designated binary operation $\otimes^{P(c)}$, and nullary operation $e_c \in P(c)$ so that $(P(c), \otimes^{P(c)}, e_c)$ is a monoid.

5. For all $\Omega \in \mathscr{L}_q$, and all $b, c, d \in \mathrm{Ob}(\mathcal{C})$,

$$\Omega_{b,1} \circ P(\pi_1^{b,1}) = id_{P(b)} \quad \text{and} \quad \Omega_{b,c\times d} \circ P(a_{b,c,d}) = \Omega_{b,c} \circ \Omega_{b\times c,d}, \tag{3.1}$$

   where $a_{b,c,d}\colon b \times (c \times d) \to (b \times c) \times d$ is the change-in-product isomorphism.

6. $Eq_1 = e_{1\times 1}$, and for all $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$, $c = c_1 \times c_2$,

$$Eq_c = P(\langle \pi_1^{c_1,c_2}\pi_1^{c,c}, \pi_1^{c_1,c_2}\pi_2^{c,c}\rangle)Eq_{c_1} \otimes P(\langle \pi_2^{c_1,c_2}\pi_1^{c,c}, \pi_2^{c_1,c_2}\pi_2^{c,c}\rangle)Eq_{c_2}. \tag{3.2}$$

[1] By $\mathscr{L}_\omega$-algebra we mean any algebra in the signature $\mathscr{L}_\omega$. We do not assume these algebras satisfy any particular collection of equations, nor do we assume their operations are monotone.

Conditions 1-3 are necessary for structures in $(\mathcal{C}, P)$ to interpret equations and formulas in context. Condition 4 and the requirement that the codomain of P is $\mathbf{Pos}$ are needed to interpret sequents[2]. Conditions 5 and 6 are properties satisfied by the classifying prop-categories associated to certain theories and are not strictly necessary to define the prop-categorical semantics. However, they must be assumed to interpret structures as morphisms out of classifying prop-categories, a key ingredient in the proofs of our main results.

Remark 1. There are adjointness conditions for the quantifiers $\{\forall, \exists\}$ (See Example 1) originally due to Lawvere (2006) (originally 1969) which asserts that for all objects $a, b \in \mathrm{Ob}(\mathcal{C})$, $\forall_{a,b}, \exists_{a,b}$ are the left and right adjoints of $P(\pi_1^{a,b})$ respectively. One can show that Condition 5 follows from these adjointness conditions. In Lawvere (1970), an adjoint interpretation of the equality predicate is also provided. For an account in the context of intuitionistic logic, see (Jacobs, 1999, p. 190). We use Conditions 5 and 6 instead of the corresponding adjointness conditions, because they are simple, comparatively weak and in a form that will be directly used to prove essential properties of the fibered semantics[3]. Moreover, in the spirit of AAL, we seek to provide general semantics for as broad of a variety of logics as possible and to use the property of *having this semantics* – here, a semantics with classifying prop-categories – as a means to classify nonclassical first-order logics. However, for many important examples $\mathscr{L}_q = \{\forall, \exists\}$ and the quantifiers and equality do satisfy these adjointness conditions. Thus, one may safely replace Conditions 5 and 6 with these adjointness conditions provided corresponding "adjoint" derivation rules are added to the minimal logic $\mathscr{L}^m$ in Section 3.2 to preserve the completeness results therein.

Example 1. Let L be an algebra of signature $\mathscr{L}_\omega$ which is also a complete lattice and let $\mathscr{L}_q = \{\forall, \exists\}$. Then the contravariant functor $\mathbf{Set}(\_, L)\colon \mathbf{Set} \to \mathbf{Pos}$ determines a prop-category in $\mathrm{Ob}(\mathbf{FA})$. The operations on L extend point-wise to logical operations on $\mathbf{Set}(A, L)$ for all sets A. For each set X, let $e_X = \top$, $\otimes := \wedge$ and

$$Eq_X(x_1, x_2) = \begin{cases} \top & x_1 = x_2, \\ \bot & x_1 \neq x_2. \end{cases}$$

For sets X, Y, the quantifiers $\exists_{X,Y}$, $\forall_{X,Y}$ are defined as the left and right adjoints of $\mathbf{Set}(\pi_1^{X,Y}, L)$ respectively. That is,

$$\exists_{X,Y}(R)(x) = \bigvee_{y \in Y} R(x, y), \quad \text{and} \quad \forall_{X,Y}(R)(x) = \bigwedge_{y \in Y} R(x, y).$$

In particular, $\mathbf{Set}(\_, 2)$ is the prop-category whose semantics corresponds to the usual Tarskian semantics for

[2]In AAL, a (logical) matrix $\langle A, F\rangle$ consists of an algebra A in the propositional language and a subset $F \subseteq A$ of "truth values"(Font, 2016, p. 183). One can develop a "fibered" matrix semantics by dropping Condition 4 on prop-categories and defining a fibered matrix as $\langle(\mathcal{C}, P), \mathscr{F}\rangle$ where $(\mathcal{C}, P)$ is a prop-category and $\mathscr{F} \subseteq \sqcup_{c \in \mathrm{Ob}(\mathcal{C})} P(c)$.

[3]Proving 6 from the adjointness conditions requires some work and is proved in (Jacobs, 1999, pg. 198) for a slightly different setup.

classical first-order logic[4].

Another significant example in this class is $\mathbf{Set}(\_, [0,1])$, where for each set A, the poset $\mathbf{Set}(A, [0,1])$ is known as the set of all **fuzzy sets on** A. There is a vast literature on fuzzy set theory (for an introduction see Hájek (1998)) which generalizes set theory by considering partial or probabilistic membership to sets.

An important class of logical operations on fuzzy sets are T-norms (Klement et al., 2000, pg. 4) and dually S-norms. A T-norm is an operation $\odot\colon [0,1]^2 \to [0,1]$ which makes $([0,1], \odot, 1)$ a partially ordered monoid. Canonical examples of T-norms include the minimum, product and the Łukasiewicz T-norm which is defined for $x, y \in [0,1]$ by $x \odot y := \max\{0, x+y-1\}$. In Mesiar and Thiele (2000), to each T-norm $\odot$, a quantifier $\Omega^{\odot}$, is defined on sets A, B and $R\colon A \times B \to [0,1]$, by

$$\Omega^{\odot}_{A,B}(R) := \inf_{U \in \mathrm{fin}(B)} \{\bigodot_{u \in U} R(\_, u)\},$$

where $\mathrm{fin}(B)$ is the set of all finite subsets of B. In particular, note that if $\odot(a,b) = \min\{a,b\}$ then $\Omega^{\odot} = \forall$. One may verify that for an arbitrary T-norm $\odot$, $\Omega^{\odot}$ satisfies Conditions 3 and 5. Thus $\mathbf{Set}(\_, [0,1])$ may be extended with additional quantifiers $\Omega^{\odot}$ for each T-norm $\odot$. Moreover, $\Omega^{\odot}$ may also be extended with S-quantifiers corresponding to S-norms as defined in Mesiar and Thiele (2000). If $\odot$ is the usual product, then for $R\colon 1 \times 2 \to [0,1]$, $R(0,0) = R(0,1) = 1/2$, $\forall_{1,B}(R)(0) = \exists_{1,B}(R)(0) = 1/2$, whereas $\Omega^{\odot}_{1,B}(R)(0) = 1/4$. Moreover, observe that in this example $\odot$ may be used for $\otimes$ instead of $\min$.

The following is a toy example which suggests integral operators are a fruitful place to look for nonclassical quantifiers and metrics for different notions of equality:

Example 2. Let $\mathcal{K}$ be the collection of compact subsets of $\mathbb{R}^n$ for some $n < \omega$, each equipped with the Euclidean metric along with $1 = \{*\}$, where 1, has the discrete metric. Let $\mathcal{C}$ be the category whose objects are finite products of sets in $\mathcal{K}$ equipped with the associated product metric and continuous functions between them. For $A \in \mathrm{Ob}(\mathcal{C})$, let $P(A)$ be the collection of continuous real-valued functions from A to $\mathbb{R}$, which we denote $C(A, \mathbb{R})$, and for $f\colon A \to B$, let $P(f)\colon C(B,\mathbb{R}) \to C(A,\mathbb{R})$ be precomposition by f. Then, post-composition by any continuous map $h\colon \mathbb{R}^n \to \mathbb{R}$ can interpret each n-ary connective $\diamond \in \mathscr{L}_n$.

For $A \in \mathrm{Ob}(\mathcal{C})$, we let e_A be the constant zero function and $Eq_A\colon A \times A \to \mathbb{R}$ be defined by $Eq_A(a_1, a_2) := -|a_1 - a_2|$, where $|\cdot|$ is the Manhattan norm. For $f, g \in C(A, \mathbb{R})$, we define $f \otimes g := f + g$. Then $(A, \otimes^{P(A)}, e_A)$ is a monoid. Moreover, Condition 6 is satisfied.

For all objects A, B we have a map $\int_{A,B}\colon P(A \times B) \to P(A)$, which sends each continuous function $p\colon A \times B \to \mathbb{R}$ to $\int_B p(x,y)\,\mathrm{d}y$. One may verify that $\int_{(\cdot),C}$ is natural for each object C and since 1 was

[4] Except in the semantics we define, we will allow sorts to be interpreted as the emptyset.

given the discrete metric, $\int_{A,1} \circ P(\pi_1^{A,1}) = id_{P(A)}$. from Fubini's Theorem, $\int_{(\cdot),(\cdot)}$ also satisfies the second part of Condition 5. Thus $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA})$. Note, in general, $\int_{A,B}$ is not left or right adjoint to $P(\pi_1^{A,B})$. To see this, consider $A = B = [0,1]$, $r(x,y)\colon A \times B \to \mathbb{R}$, $r(x,y) = y$. Then $\int_{A,B}(r) = 1/2$ whereas $\exists_{A,B}(r) = 1$ and $\forall_{A,B}(r) = 0$.

Let Sg be a signature and $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA})$. An Sg**-structure** S **in** $(\mathcal{C}, P)$ is an assignment of an object $[\![\sigma]\!]$ to each sort symbol σ, a morphism $[\![f]\!]\colon [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \to [\![\tau]\!]$ to each function symbol $f\colon \sigma_1, \ldots, \sigma_n \to \tau$ and an element $[\![R]\!] \in P([\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!])$ to each relation symbol $R \subseteq \sigma_1, \ldots, \sigma_n$ in Sg. (To disambiguate between structures, we sometimes write $S[\![\cdot]\!]$ instead of $[\![\cdot]\!]$.) For a context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, we define $[\![\Gamma]\!] \coloneqq [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$, in particular if $n = 0$, $[\![\Gamma]\!] \coloneqq 1$. Each term-in-context $M : \tau\ [\Gamma]$ may be given a unique interpretation $[\![M : \tau\ [\Gamma]]\!]$ defined recursively as follows:

$$[\![x_i : \sigma_i\ [x_1 : \sigma_1, \ldots, x_n : \sigma_n]]\!] \coloneqq \pi_i\colon [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \to [\![\sigma_i]\!]$$

$$[\![c : \tau\ [\Gamma]]\!] \coloneqq [\![c]\!] \circ !_{[\![\Gamma]\!]}$$

$$[\![f(M_1, \ldots, M_n) : \tau\ [\Gamma]]\!] \coloneqq [\![f]\!] \circ \langle [\![M_1[\Gamma]]\!], \ldots, [\![M_n[\Gamma]]\!] \rangle,$$

where π_i is the i-th projection map and $!_{[\![\Gamma]\!]}$ denotes the unique morphism from $[\![\Gamma]\!]$ to 1. Note we often abbreviate $[\![M : \tau\ [\Gamma]]\!]$ as $[\![M[\Gamma]]\!]$ as the type τ can be inferred from the term M and context Γ.

S interprets formulas-in-context recursively as follows:

$$\begin{aligned}
[\![R(M_1, \ldots, M_n)\ [\Gamma]]\!] &\coloneqq P(\langle [\![M_1[\Gamma]]\!], \ldots, [\![M_n[\Gamma]]\!] \rangle)([\![R]\!]), \\
[\![M_1 =_\tau M_2\ [\Gamma]]\!] &\coloneqq P(\langle [\![M_1[\Gamma]]\!], [\![M_2[\Gamma]]\!] \rangle)([\![Eq_{[\![\tau]\!]}]\!]), \\
[\![\Diamond(\phi_1, \ldots, \phi_n)\ [\Gamma]]\!] &\coloneqq \Diamond^{P([\![\Gamma]\!])}([\![\phi_1\ [\Gamma]]\!], \ldots, [\![\phi_n\ [\Gamma]]\!]), \\
[\![\Omega_{x:\sigma}(\phi)\ [\Gamma]]\!] &\coloneqq \Omega_{[\![\Gamma]\!],[\![\sigma]\!]} \circ P(a^S_{\Gamma,x:\sigma})([\![\phi\ [\Gamma, x : \sigma]]\!]),
\end{aligned}$$

where $a^S_{\Gamma,x:\sigma}\colon [\![\Gamma]\!] \times [\![x : \sigma]\!] \to [\![\Gamma, x : \sigma]\!]$ is the change-in-product isomorphism[5].

Then, S **satisfies an equation-in-context** $M_1 = M_2 : \tau\ [\Gamma]$, if

$$[\![M_1 : \tau\ [\Gamma]]\!] = [\![M_2 : \tau\ [\Gamma]]\!]$$

and S **satisfies a sequent-in-context** $\phi_1, \ldots, \phi_n \vdash \phi_{n+1}\ [\Gamma]$, if for $n \geq 0$,

$$[\![\phi_1[\Gamma]]\!] \otimes \ldots \otimes [\![\phi_n[\Gamma]]\!] \leq [\![\phi_{n+1}[\Gamma]]\!].$$

[5]This handles the case where $\Gamma = \emptyset$ and $[\![\Gamma, x : \sigma]\!] = [\![x : \sigma]\!] \neq [\![\Gamma]\!] \times [\![x : \sigma]\!] = 1 \times [\![x : \sigma]\!]$.

In particular, for $n = 0$, if $e_{[\![\Gamma]\!]} \leq [\![\phi_{n+1}[\Gamma]]\!]$. We define the **theory of** S, denoted $\mathrm{Th}(S)$, to be the Sg-theory where $\mathrm{A}(\mathrm{Th}(S))$ is the collection of all assertions satisfied by S. We say S **satisfies** T or S **is a** T**-model** if $T \leq \mathrm{Th}(S)$.

Example 3. In Mostowski (1957), Mostowski introduced a general class of quantifiers and studied classical first-order logic extended with these quantifiers. Instead of $\mathbf{Set}(\_, 2)$, we consider the isomorphic powerset functor $\mathscr{P}\colon \mathbf{Set} \to \mathbf{Pos}$, which sends a set A to its powerset $\mathscr{P}(A)$ and a function $f\colon A \to B$ to its preimage operator. A **Mostowski quantifier**[6] Ω, is specified for each set A, by a subset $\Omega(A) \subseteq \mathscr{P}(A)$. Then for A, B sets, and $R \subseteq A \times B$, we define

$$\Omega_{A,B}(R) \coloneqq \{a \in A : \{b \in B : (a, b) \in R\} \in \Omega(B)\}.$$

Note that quantifiers $\forall$ and $\exists$ belong to the class, defined for a set A, by $\forall(A) \coloneqq \{A\}$ and $\exists(A) \coloneqq \mathscr{P}(A)\backslash\{\emptyset\}$ respectively. A nonstandard Mostowski quantifier of early interest Kaufmann (1985) is the "there exists uncountably many" quantifier Q_1, where for each set A, $Q_1(A)$ is the collection of all uncountable subsets of A.

One may verify that $(\mathbf{Set}, \mathscr{P})$ augmented with a Mostowski quantifier Ω satisfies Condition (3) and so structures in $(\mathbf{Set}, \mathscr{P})$ can interpret the language of classical logic augmented with Ω. However, in general, many such natural quantifiers, fail to satisfy Condition (5). The first part of Equation 3.1 requires that $Q_1(1) = \{1\}$, and the second part of Equation 3.1 requires for all sets A, B and $R \subseteq A \times B$, that

$$R \in Q(A \times B) \iff \{a \in A : R(a, y) \in Q(B)\} \in Q(A). \tag{3.3}$$

One sees that this condition also fails for Q_1, by considering $A = B = \mathbb{R}$ and $R = \mathbb{R} \times \{0\}$.

One can do away with Condition (5) by expanding the syntax to allow quantifiers to bind arbitrary contexts as follows

$$\frac{\phi : \mathrm{prop}\ [\Gamma, \Gamma']}{\Omega_{\Gamma'}(\phi) : \mathrm{prop}\ [\Gamma]}\ .$$

Then a structure S interprets $\Omega_{\Gamma'}(\phi)\,[\Gamma]$ as

$$[\![\Omega_{\Gamma'}(\phi)\,[\Gamma]]\!] \coloneqq \Omega_{[\![\Gamma]\!],[\![\Gamma']\!]} \circ P(a^S_{\Gamma,\Gamma'})([\![\phi\,[\Gamma, \Gamma']]\!]).$$

This extension of the syntax is conservative for existential and universal quantifiers, but not for the counting

[6] In Mostowski's original formulation is more restrictive and requires quantifiers to be cardinality invariant. Our presentation here is essentially that of Westerståhl (2019).

quantifiers. For example, if Ω^n is interpreted as "there exists exactly n." Then for each $\phi : \mathrm{prop}\,[x:\sigma, y:\tau]$,

$$\Omega^2_{x:\sigma,y:\tau}\,\phi(x,y)\,[\,] \equiv \Omega^1_{y:\tau}\Omega^2_{x:\sigma}\,\phi(x,y) \vee \Omega^2_{y:\tau}\Omega^1_{x:\sigma}\,\phi(x,y)\,[\,].$$

To recover the results in this paper, one must add the following isomorphism invariant condition on the quantifiers: For each isomorphism $f\colon c \to d$,

$$\Omega_{(\cdot),c} \cdot P(\_ \times f) = \Omega_{(\cdot),d},$$

which is a condition Mostowski originally imposed so that his quantifiers are bijective invariant.

We also note that similar considerations hold for the equality predicate and Condition 6.

A subcollection $\mathcal{X} \subseteq \mathrm{Ob}(\mathbf{FA})$, defines a logic $\vDash_{\mathcal{X}}$, where given Sg-theories T and T', $T \vDash_{\mathcal{X}} T'$ if every T-model $S \in (\mathcal{C}, P) \in \mathcal{X}$ is also a T'-model. Given a logic $\mathcal{L}$, we let $\mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})$ be the collection of all prop-categories in $\mathrm{Ob}(\mathbf{FA})$ such that $\mathcal{L}$ is sound with respect to $\vDash_{\mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})}$, that is, whenever $T \vdash_{\mathcal{L}} T'$, then $T \vDash_{\mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})} T'$.

3.2 Classifying Prop-Categories and General Completeness Theorems

In the subsequent sections, we suppose $\mathscr{L}$ has designated operation symbols $e \in \mathscr{L}_0$ and $\otimes \in \mathscr{L}_2$. Given a sufficiently rich theory T, we will construct the *classifying prop-category* $(\mathcal{C}_T, P_T) \in \mathrm{Ob}(\mathbf{FA})$, which is the first-order analogue of the Lindenbaum-Tarski (LT) algebras of propositional logic. As with LT-algebras, each classifying prop-category $(\mathcal{C}_T, P_T)$ contains a *generic* T-model $G \in (\mathcal{C}_T, P_T)$, i.e. $\mathrm{Th}(G) = T$, which interprets each term and formula in context as its associated equivalence class obtained by "quotienting out" by the theory T.

We call a logic $\mathcal{L}$ over $\mathscr{L}$ **adequate** if for each $\mathcal{L}$-theory T, $(\mathcal{C}_T, P_T)$ can be constructed and $(\mathcal{C}_T, P_T) \in \mathrm{Ob}(\mathbf{FA})$. Define $\mathcal{L}^m$ to be the first-order logic whose rules are listed in Figure 3.1.[7]

Theorem 2. *If $\mathcal{L} \supseteq \mathcal{L}^m$, then $\mathcal{L}$ is adequate.*

To construct $\mathcal{C}_T$, from an $\mathcal{L}$-theory T, we consider contexts, formulas and terms in context equal up to renaming of their free variables. Morphisms are sequences of equivalence classes of terms in a common context:

$$\gamma\colon \Gamma' \to \Gamma = [N_1 : \sigma_1\,[\Gamma']/\!\sim, \ldots, N_n : \sigma_n\,[\Gamma']/\!\sim],$$

[7] In AAL, propositional logics which satisfy $\diamond$-Cong for each connective are called selfextensional, which is the weakest class of logics in the Fregean hierarchy. Wójcicki (1982) (Font, 2016, pg. 419) Kleene's strong 3-valued logic $K3$ (Kleene, 1952, p. 332) is a simple example of a non selfextensional propositional logic. However, A first-order version of $K3$ can still be modeled as the fragment of an adequate logic by restricting to sequents with empty antecedent.

$$\frac{}{\phi \vdash \phi\ [\Gamma]}\ \text{Ax} \qquad \frac{\phi \vdash \psi\ [\Gamma] \quad \psi \vdash \theta\ [\Gamma]}{\phi \vdash \theta\ [\Gamma]}\ \text{Cut}$$

$$\frac{\Phi \vdash \psi\ [\Gamma]}{\Phi \vdash \psi\ [\Gamma, x : \sigma]}\ \text{Cwk} \qquad \frac{M = M' : \sigma\ [\Delta] \quad \Phi \vdash \psi\ [\Gamma, x : \sigma, \Gamma']}{\Phi[M/x] \vdash \psi[M'/x]\ [\Gamma, \Gamma']}\ \text{Sub}$$

$$\frac{\phi \dashv\vdash \psi\ [\Gamma, x : \sigma]}{\Omega_{x:\sigma}(\phi) \vdash \Omega_{x:\sigma}(\psi)\ [\Gamma]}\ \Omega\text{-Con} \qquad \frac{\phi_1 \dashv\vdash \phi_1'\ [\Gamma] \quad \ldots \quad \phi_n \dashv\vdash \phi_n'\ [\Gamma]}{\Diamond(\phi_1, \ldots, \phi_n) \vdash \Diamond(\phi_1', \ldots, \phi_n')\ [\Gamma]}\ \Diamond\text{-Cong}$$

$$\frac{\Phi, \alpha, \beta, \Psi \vdash \theta\ [\Gamma]}{\Phi, \alpha \otimes \beta, \Psi \vdash \theta\ [\Gamma]}\ \otimes\text{-Ref} \qquad \frac{\Phi, e, \Psi \vdash \phi\ [\Gamma]}{\Phi, \Psi \vdash \phi\ [\Gamma]}\ e\text{-Ref}$$

Figure 3.1: Rules for $\mathcal{L}^m$. (Where $\mathrm{VS}(\Delta) \subseteq \mathrm{VS}(\Gamma, \Gamma')$ and $\forall \Omega \in \mathscr{L}_q$, $\forall n \in \omega$ and $\forall \Diamond \in \mathscr{L}_n$).

where, $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and

$$N : \tau\ [\Gamma] \sim M : \tau\ [\Gamma] \iff N = M : \tau\ [\Gamma] \in T.$$

Composition of morphisms is by component-wise substitution and one may show using the rules of equational logic that $\mathcal{C}_T$ is a category with strictly associative finite products (Pitts, 2000, p. 30).

For each context Γ, we say $\phi\,[\Gamma] \sim \psi\,[\Gamma]$ if and only if $\phi \vdash \psi\,[\Gamma], \psi \vdash \phi\,[\Gamma] \in T$. From Ax and Cut, $\sim$ defines and equivalence relation on formulas-in-context Γ and we let $P_T(\Gamma)$ be the partial order of all such classes where $\phi\,[\Gamma]/\sim\ \leq\ \psi\,[\Gamma]/\sim$ if and only if $\phi \vdash \psi\,[\Gamma] \in T$. For each morphism $\gamma \colon \Gamma' \to \Gamma$, we define $P_T(\gamma) \colon P_T(\Gamma) \to P_T(\Gamma')$ by simultaneous substitution:

$$P_T([N_1 : \sigma_1\,[\Gamma']/\sim, \ldots, N_n : \sigma_n\,[\Gamma']/\sim])(\phi\,[\Gamma]/\sim) \coloneqq \phi[N_1/x_1, \ldots, N_n/x_n]\,[\Gamma']/\sim .$$

For each $n \in \omega$ and $\Diamond \in \mathscr{L}_n$, we define:

$$\Diamond^\Gamma(\phi_1[\Gamma]/\sim, \ldots, \phi_n[\Gamma]/\sim) \coloneqq \Diamond(\phi_1, \ldots, \phi_n)[\Gamma]/\sim .$$

For each context Γ, define $e_\Gamma \coloneqq e\,[\Gamma]/\sim$. For each quantifier symbol $\Omega \in \mathscr{L}_q$, and $\phi\,[\Gamma, \Gamma']$, where $\Gamma' = y_1 : \tau_1, \ldots, y_m : \tau_m$, we interpret $\Omega_{\Gamma, \Gamma'}$ as

$$\Omega_{\Gamma,\Gamma'}(\phi\,[\Gamma, \Gamma']/\sim) \coloneqq \Omega_{y_1:\tau_1} \ldots \Omega_{y_m:\tau_m}(\phi)\,[\Gamma]/\sim .$$

For each $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, let $\Gamma' = x_1' : \sigma_1, \ldots, x_n', \sigma_n$. We define

$$Eq_\Gamma \coloneqq x_1 =_{\sigma_1} x_1' \otimes \ldots \otimes x_n =_{\sigma_n} x_n' \, [\Gamma, \Gamma']/\!\sim .$$

In particular, when $\Gamma = []$, $Eq_{[]} = e[]/\!\sim = e_{[]\times[]}$. Using the rules in Figure 3.1 one shows that these operations are well-defined and $(\mathcal{C}_T, P_T) \in \mathrm{Ob}(\mathbf{FA})$.

The generic T-model $G \in (\mathcal{C}_T, P_T)$, is defined on sorts by $G[\![\sigma]\!] \coloneqq x : \sigma$, on function symbols $f\colon \sigma_1, \ldots, \sigma_n \to \tau$, by $G[\![f]\!] \coloneqq [f(\bar{x}) : \tau\, [\Gamma]/\!\sim]$ and on relation symbols $R \subseteq \sigma_1, \ldots, \sigma_n$ by $G[\![R]\!] \coloneqq R(\bar{x})\, [\Gamma]/\!\sim$. By induction one shows that $G[\![M : \tau\, [\Gamma]]\!] = [M : \tau\, [\Gamma]/\!\sim]$ and $G[\![\phi\, [\Gamma]]\!] = \phi\, [\Gamma]/\!\sim$. From $\otimes$-Ref and e-Ref it follows that $\mathrm{Th}(G) = T$.

Theorem 3. *The logic $\vDash_\mathcal{K}$ provides a complete semantics for $\vdash_{\mathcal{L}^m}$, where $\mathcal{K}$ is any subclass of $\mathrm{Ob}(\mathbf{FA})$ containing $\{(\mathcal{C}_T, P_T) : T$ is an $\mathcal{L}^m$-theory$\}$.*

Proof. To show that $\vdash_{\mathcal{L}^m} \subseteq \vDash_{\mathrm{Ob}(\mathbf{FA})}$, one verifies that each $S \in (\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA})$ satisfies every interpretation of the rules defining $\mathcal{L}^m$. Now suppose $T \vDash_\mathcal{K} T'$. Then the generic $T_{\mathcal{L}^m}$-model satisfies T', and so $T \vdash_{\mathcal{L}^m} T'$. □

Corollary 2. *If $\mathcal{L}$ is adequate, then $\vdash_\mathcal{L} \supseteq \vdash_{\mathcal{L}^m}$.*

Since otherwise, the generic model of some $\mathcal{L}$-theory will fail to satisfy an instance of a rule of $\mathcal{L}^m$. As a consequence, we let $(\mathcal{C}_{Sg}, P_{Sg}) \coloneqq (\mathcal{C}_{Sg_{\mathcal{L}^m}}, P_{Sg_{\mathcal{L}^m}})$. In $(\mathcal{C}_{Sg}, P_{Sg})$, morphisms $\gamma\colon \Gamma \to \Gamma'$ are just lists of terms and the elements of $P_{Sg}(\Gamma)$ are just formulas, both up to α-equivalence and change in the name of free variables, which are equivalences considered in mathematical practice. Thus, $(\mathcal{C}_{Sg}, P_{Sg})$ is a suitable "fibered" analogue of the formula algebra from AAL.

Suppose $\mathcal{L}$ is adequate and let $\mathcal{K} = \{(\mathcal{C}_T, P_T) : T \text{ is an } \mathcal{L}\text{-theory}\}$. Then using generic models, one proves $\vDash_\mathcal{K} \subseteq \vdash_\mathcal{L}$. All the remains to prove completeness is to show soundness, i.e. $\vdash_\mathcal{L} \subseteq \vDash_\mathcal{K}$. If $\mathcal{L}$ is defined by a sequent calculus, we only need to show each structure in a classifying prop-category satisfies the defining rules of $\mathcal{L}$. We have the following general result:

Theorem 4. *Suppose $\mathscr{L}_q = \{\forall, \exists\}$ and $\mathcal{L}$ is an extension of $\mathcal{L}^m$ possibly by $=$-Adj, $\exists$-Adj, $\forall$-Adj shown in Figure 3.2 and any number of structural and propositional connective rules[8]. Then $\vdash_\mathcal{L} = \vDash_\mathcal{K}$, where $\mathcal{K} = \{(\mathcal{C}_T, P_T) : T$ is an $\mathcal{L}$-theory$\}$.*

Proof. Showing that $\vDash_\mathcal{K}$ satisfies the structural and propositional connective rules of $\mathcal{L}$ is straightforward.

[8]Propositional connective rules, are sequent rules whose meta-formulas do not include equality or any quantifiers such as modus ponens. Structural rules additionally do not include propositional connectives such as Cut.

Suppose $\mathcal{L}$ satisfies $\exists$-Adj and let

$$\phi_1, \ldots, \phi_n, \psi \vdash \theta \, [\Gamma, x : \sigma],$$

be an interpretation of the top line of $\exists$-Adj. Let $S \in (\mathcal{C}_T, P_T) \in \mathcal{K}$ and for each $i \in \{1, \ldots, n\}$, let $S[\![\phi_i \, [\Gamma]]\!] = \phi_i^S \, [\Gamma_S]/\sim$, $S[\![\theta \, [\Gamma]]\!] = \theta^S \, [\Gamma_S] \sim$ and $S[\![\psi \, [\Gamma, x : \sigma]]\!] = \psi^S \, [\Gamma_S, \Gamma_\sigma]/\sim$, where $S[\![\sigma]\!] = \Gamma_\sigma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$. Then,

$$\begin{aligned}
& \bigotimes_{i=1}^{n} S[\![\phi_i[\Gamma, x : \sigma]]\!] \otimes S[\![\psi[\Gamma, x : \sigma]]\!] \leq S[\![\theta[\Gamma, x : \sigma]]\!] \\
\iff \; & \bigotimes_{i=1}^{n} G[\![\phi_i^S[\Gamma_S, \Gamma_\sigma]]\!] \otimes G[\![\psi^S[\Gamma_S, \Gamma_\sigma]]\!] \leq G[\![\theta^S[\Gamma_S, \Gamma_\sigma]]\!] \\
\iff \; & T \vdash_{\mathcal{L}} \phi_1^S, \ldots, \phi_n^S, \psi^S \vdash \theta^S \, [\Gamma_S, \Gamma_\sigma] \\
\iff \; & T \vdash_{\mathcal{L}} \phi_1^S, \ldots, \phi_n^S, \exists_{x_1:\sigma_1} \ldots \exists_{x_n:\sigma_n}(\psi^S) \vdash \theta^S \, [\Gamma_S] \\
\iff \; & \bigotimes_{i=1}^{n} S[\![\phi_i[\Gamma]]\!] \otimes S[\![\exists_{x:\sigma}(\psi)[\Gamma]]\!] \vdash S[\![\theta[\Gamma]]\!].
\end{aligned}$$

It follows that $\vDash_{\mathcal{K}}$ satisfies $\exists$-Adj. The proof that $\vDash_{\mathcal{K}}$ satisfies $\forall$-Adj is similar.

Now suppose $\mathcal{L}$ satisfies $=$-Adj and let S be a structure in $(\mathcal{C}_T, P_T)$. Consider an interpretation of $=$-Adj:

$$\frac{\phi_1, \ldots, \phi_n \vdash \psi[x/x'] \, [\Gamma, x : \sigma]}{\phi_1, \ldots, \phi_n, x =_\sigma x' \vdash \psi \, [\Gamma, x : \sigma, x' : \sigma]} \, .$$

Let $S[\![\phi_i[\Gamma, x : \sigma]]\!] = \phi_i^S[\Gamma_S, \Gamma_\sigma]/\sim$, where $\Gamma_S \in S[\![\Gamma]\!]$, $\Gamma_\sigma \in S[\![\sigma]\!]$ and $S[\![\psi[\Gamma, x : \sigma, x', \sigma']]\!] = \psi^S[\Gamma_S, \Gamma_\sigma, \Gamma'_\sigma]/\sim$, where $\Gamma'_\sigma \in S[\![\sigma]\!]$. From Sub and Cwk, one may derive the rule

$$\frac{\Phi \vdash \psi \, [\Gamma, x : \sigma, y : \tau, \Gamma']}{\Phi \vdash \psi \, [\Gamma, y : \tau, x : \sigma, \Gamma']} \tag{3.4}$$

Let $\Gamma_\sigma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and $\Gamma'_\sigma = x'_1 : \sigma_1, \ldots, x'_n : \sigma_n$. Let $\bar{x} = x_1, \ldots, x_n$ and $\bar{x}' = x'_1, \ldots, x'_n$ be

$$\frac{\phi_1,\ldots,\phi_n \vdash \psi[x/x']\ [\Gamma, x:\sigma]}{\phi_1,\ldots,\phi_n, x =_\sigma x' \vdash \psi\ [\Gamma, x:\sigma, x':\sigma]} \text{=-Adj}$$

$$\frac{\Phi \vdash \psi\ [\Gamma, x:\sigma]}{\Phi, \vdash \forall_{x,\sigma}(\psi)\ [\Gamma]} \forall\text{-Adj} \qquad \frac{\Phi, \psi \vdash \theta\ [\Gamma, x:\sigma]}{\Phi, \exists_{x:\sigma}(\psi) \vdash \theta\ [\Gamma]} \exists\text{-Adj}$$

Figure 3.2: Adjoint rules for $\forall$, $\exists$ and $=$.

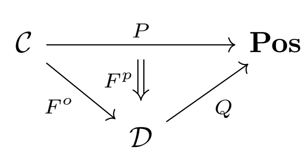
the lists of variables in Γ_σ and Γ'_σ respectively. Then $S[\![\psi[x/x'][\Gamma, x:\sigma]]\!] = \psi^S[\bar{x}/\bar{x}'][\Gamma_S, \Gamma_\sigma]/\!\sim$ and

$$\begin{aligned}
& S[\![\phi_1[\Gamma, x:\sigma]]\!] \otimes \ldots \otimes S[\![\phi_n[\Gamma, x:\sigma]]\!] \leq S[\![\psi[x/x'][\Gamma, x:\sigma]]\!] \\
\iff\ & G[\![\phi_1^S[\Gamma_S, \Gamma_\sigma]]\!] \otimes \ldots \otimes G[\![\phi_n^S[\Gamma_S, \Gamma_\sigma]]\!] \leq G[\![\psi^S[\bar{x}/\bar{x}'][\Gamma_S, \Gamma_\sigma]]\!] \\
\iff\ & T \vdash_{\mathcal{L}} \phi_1^S, \ldots, \phi_n^S \vdash \psi^S[\bar{x}/\bar{x}']\ [\Gamma_S, \Gamma_\sigma] \\
\iff\ & T \vdash_{\mathcal{L}} \phi_1^S, \ldots, \phi_n^S, x_1 =_{\sigma_1} x'_1, \ldots, x_n =_{\sigma_n} x'_n \vdash \psi^S\ [\Gamma_S, \Gamma_\sigma, \Gamma'_\sigma] && (\text{ =-Adj and 3.4}) \\
\iff\ & T \vdash_{\mathcal{L}} \phi_1^S, \ldots, \phi_n^S, \bigotimes_{i=1}^{n} x_i =_{\sigma_i} x'_i \vdash \psi^S\ [\Gamma_S, \Gamma_\sigma, \Gamma'_\sigma] && (\otimes\text{-Ref or } e\text{-Ref}) \\
\iff\ & S[\![\phi_1[\Gamma, x:\sigma, x':\sigma]]\!] \otimes \ldots \otimes S[\![\phi_n[\Gamma, x:\sigma, x':\sigma]]\!] \otimes S[\![x =_\sigma x'[\Gamma, x:\sigma, x':\sigma]]\!] \\
& \leq S[\![\psi[\Gamma, x:\sigma, x':\sigma]]\!].
\end{aligned}$$

Therefore, S satisfies all interpretations of =-Adj. $\square$

3.3 The 2-Categorical View of the Prop-Categorical Semantics

We may extend $\mathrm{Ob}(\mathbf{FA})$ to a 2-category by defining morphisms $F\colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ by the following data: (1) a product preserving functor $F^o\colon \mathcal{C} \to \mathcal{D}$ and (2) a natural transformation $F^p\colon P \Rightarrow Q \circ F^o$,



such that the following hold:

1. For each $c \in \mathrm{Ob}(\mathcal{C})$, $F^p_c\colon P(c) \to Q \circ F^o(c)$ is an $\mathscr{L}_\omega$-algebra homomorphism.

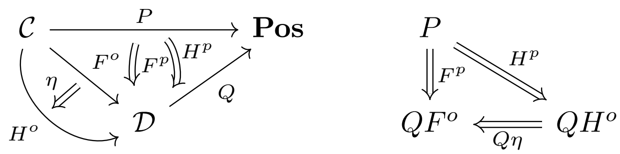
2. For all $\Omega \in \mathscr{L}_q$ and all $b, c \in \mathrm{Ob}(\mathcal{C})$,

$$F^p_b \circ \Omega_{b,c} = \Omega_{F^o b, F^o c} \circ Q(a^{-1}_{F,b,c}) \circ F^p_{b\times c}.$$

3. For all $c \in \mathrm{Ob}(\mathcal{C})$,

$$F^p_{c\times c}(Eq_c) = Q(a_{F,c,c})(Eq_{F^o c}),$$

where $a_{F,b,c}\colon F^o(b\times c)\to F^o b\times F^o c$ is the change-in-product isomorphism. Given $K\colon(\mathcal{D},Q)\to(\mathcal{E},R)$, we define $K\circ F$ by $(K\circ F)^o := K^o\circ F^o$ and $(K\circ F)^p := K^p_{F^o}\cdot F^p$. For parallel morphisms $F,H\colon(\mathcal{C},P)\to(\mathcal{D},Q)$, we define a 2-cell $\eta\colon F\Rightarrow H$ to be a natural transformation $\eta\colon F^o\Rightarrow H^o$ such that $F^p = Q\eta\cdot H^p$.



Compositions of 2-cells is just as it is in $\mathbf{Cat}$. If η is a natural isomorphism, we call η a 2-isomorphism. For a logic $\mathcal{L}$, we now let $\mathbf{FA}_{\mathcal{L}}$ be the corresponding full sub-2-category of $\mathbf{FA}$.

Theorem 5. *Let $(\mathcal{C},P),(\mathcal{D},Q)\in\mathrm{Ob}(\mathbf{FA})$ and $F\colon(\mathcal{C},P)\to(\mathcal{D},Q)$ be a morphism. Then for each theory T, each T-model S in $(\mathcal{C},P)$ gives rise to a T-model $F(S)$ in $(\mathcal{D},Q)$.*

Proof. First we define the structure $F(S)$. For each sort symbol σ, $F(S)[\![\sigma]\!] := F^o(S[\![\sigma]\!])$, for each function symbol $f\colon\sigma_1,\dots,\sigma_n\to\tau$, $F(S)[\![f]\!] := F^o(S[\![f]\!])\circ a_\Gamma^{-1}$, and for each relation symbol $R\subseteq\sigma_1,\dots,\sigma_n$, $F(S)[\![R]\!] := Q(a_\Gamma^{-1})\circ F^p_{S[\![\Gamma]\!]}(S[\![R]\!])$, where $a_\Gamma\colon F^o(S[\![\Gamma]\!])\to F(S)[\![\Gamma]\!]$ is the change-of-product morphism and $\Gamma = x_1:\sigma_1,\dots,x_n:\sigma_n$. A product preserving functor preserves the satisfaction of equations-in-context since by induction on the complexity of an arbitrary term-in-context $M:\tau\ [\Gamma]$,

$$F(S)[\![M:\tau\ [\Gamma]]\!] = F^o(S[\![M:\tau\ [\Gamma]]\!])\circ a_\Gamma^{-1}.$$

If, for each formula-in-context $\phi\,[\Gamma]$,

$$F(S)[\![\phi[\Gamma]]\!] = Q(a_\Gamma^{-1})(F^p_{S[\![\Gamma]\!]}(S[\![\phi[\Gamma]]\!])), \tag{3.5}$$

then if S satisfies a sequent-in-context $\phi_1,\dots,\phi_n\vdash\psi\ [\Gamma]$, since $F^p_{S[\![\Gamma]\!]}$ and $Q(a^{-1})$ are monotone $\mathscr{L}$-algebra homomorphisms,

$$\begin{aligned}
&\bigotimes_{i=1}^{n} S[\![\phi_i[\Gamma]]\!] \leq S[\![\psi[\Gamma]]\!] \\
\implies &\bigotimes_{i=1}^{n} Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(S[\![\phi_i[\Gamma]]\!]) \leq Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(S[\![\psi[\Gamma]]\!]) \\
\implies &\bigotimes_{i=1}^{n} F(S)[\![\phi_i[\Gamma]]\!] \leq F(S)[\![\psi[\Gamma]]\!],
\end{aligned}$$

where if $n = 0$, then $\bigotimes_{i=1}^{n} = e_{S[\![\Gamma]\!]}$. And so $F(S)$ satisfies $\phi_1, \ldots, \phi_n \vdash \psi\ [\Gamma]$. It follows that $F(S)$ is a T-model and so we only need to prove that Equation 3.5 holds.

Consider a formula-in-context $\phi\ [\Gamma]$. Suppose $\phi\ [\Gamma]$ is atomic. Case 1: $\phi\ [\Gamma]$ is of the form $R(M_1, \ldots, M_n)[\Gamma]$, where $R \subseteq \tau_1, \ldots, \tau_n$, and each $M_i : \tau_i\ [\Gamma]$ is well-formed. If γ is the list $\gamma = [M_1[\Gamma], \ldots, M_n[\Gamma]]$, we define

$$S[\![\gamma]\!] \coloneqq \langle S[\![M_1[\Gamma]]\!], \ldots, S[\![M_n[\Gamma]]\!]\rangle.$$

Note that

$$F(S)[\![\gamma]\!] = a_{\Gamma'} \circ F^o(S[\![\gamma]\!]) \circ a_\Gamma^{-1}$$

Then,

$$\begin{aligned}
&F(S)[\![R(M_1, \ldots, M_n)[\Gamma]]\!] \\
&= Q(F(S)[\![\gamma]\!])(F(S)[\![R]\!]) \\
&= Q(F(S)[\![\gamma]\!]) \circ Q(a_{\Gamma'}^{-1})(F^p_{S[\![\Gamma']\!]}(S[\![R]\!])) \\
&= Q(a_\Gamma^{-1}) \circ Q(F^o(S[\![\gamma]\!])) \circ F^p_{S[\![\Gamma']\!]}(S[\![R]\!]) \\
&= Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]} \circ P(S[\![\gamma]\!])(S[\![R]\!]) \\
&= Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(S[\![R(M_1, \ldots, M_n)[\Gamma]]\!]).
\end{aligned}$$

Otherwise, $\phi\ [\Gamma]$ is of the form $M_1 =_\tau M_2\ [\Gamma]$. Let $\Gamma' = [x_1 : \tau, x_2 : \tau]$ and $\gamma\colon \Gamma \to \Gamma'$ be the context morphism represented by $[M_1[\Gamma], M_2[\Gamma]]$. Then,

$$\begin{aligned}
&F(S)[\![M_1 =_\tau M_2[\Gamma]]\!] \\
&= Q(F(S)[\![\gamma]\!])(Eq_{F(S)[\![\tau]\!]}) \\
&= Q(a_\Gamma^{-1}) \circ Q(F^o(S[\![\gamma]\!])) \circ Q(a_{\Gamma'}) \circ Eq_{F(S)[\![\tau]\!]} \\
&= Q(a_\Gamma^{-1}) \circ Q(F^o(S[\![\gamma]\!])) \circ F^p_{S[\![\tau]\!]\times S[\![\tau]\!]}(Eq_{S[\![\tau]\!]}) \\
&= Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]} \circ P(S[\![\gamma]\!])(Eq_{S[\![\tau]\!]}) \\
&= Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(S[\![M_1 =_\tau M_2[\Gamma]]\!]).
\end{aligned}$$

Now suppose $\phi\,[\Gamma]$ is of the form $\diamond(\phi_1, \ldots, \phi_n)[\Gamma]$, where $\diamond$ is some n-ary operation in $\mathscr{L}_\omega$ and for each $i \in \{1, \ldots, n\}$, $F(S)[\![\phi_i[\Gamma]]\!] = Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(S[\![\phi_i[\Gamma]]\!])$. Then,

$$\begin{aligned}
&F(S)[\![\diamond(\phi_1, \ldots, \phi_n)[\Gamma]]\!] \\
&= \diamond^{Q(F(S)[\![\Gamma]\!])}(F(S)[\![\phi_1[\Gamma]]\!], \ldots, F(S)[\![\phi_n[\Gamma]]\!]) \\
&= \diamond^{Q(F(S)[\![\Gamma]\!])}(Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(S[\![\phi_1[\Gamma]]\!]), \ldots, Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(S[\![\phi_n[\Gamma]]\!])) \\
&= Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(\diamond^{P(S[\![\Gamma]\!])}(S[\![\phi_1[\Gamma]]\!], \ldots, S[\![\phi_n[\Gamma]]\!])) \\
&= Q(a_\Gamma^{-1}) \circ F^p_{S[\![\Gamma]\!]}(S[\![\diamond(\phi_1, \ldots, \phi_n)[\Gamma]]\!]).
\end{aligned}$$

Suppose $\phi\,[\Gamma]$ is of the form $\Omega_{x:\sigma}(\psi)\,[\Gamma]$ for $\Omega \in \mathscr{L}_q$. Let $I = S[\![\Gamma]\!]$ and $X = S[\![\sigma]\!]$. For $c = c_1 \times \cdots \times c_n$, we let $a_c\colon F^o(c) \to F^o c_1 \times \cdots \times F^o c_n$, and $a_{c_1,c_2}\colon F^o(c_1 \times c_2) \to F^o c_1 \times F^o c_2$ be the change in product isomorphisms. Then

$$\begin{aligned}
&F(S)[\![\Omega_{x:\sigma}(\psi)[\Gamma]]\!] \\
&= \Omega_{F(S)[\![\Gamma]\!],F(S)[\![\sigma]\!]} \circ Q(a^{F(S)}_{\Gamma,x:\sigma})(F(S)[\![\psi[\Gamma, x:\sigma]]\!]) \\
&= Q(a_I^{-1}) \circ \Omega_{F^o(I),X} \circ Q(a_I \times id_{F(X)}) \circ Q(a^{F(S)}_{\Gamma,x:\sigma})(F(S)[\![\psi[\Gamma, x:\sigma]]\!]) \\
&= Q(a_I^{-1}) \circ \Omega_{F^o(I),X} \circ Q(a_I \times id_{F(X)}) \circ Q(a^{F(S)}_{\Gamma,x:\sigma}) \circ Q(a^{-1}_{S[\![\Gamma,x:\sigma]\!]}) \circ F^p_{S[\![\Gamma,x:\sigma]\!]}(S[\![\psi[\Gamma, x:\sigma]]\!]) \\
&= Q(a_I^{-1}) \circ \Omega_{F^o(I),X} \circ Q(a^{-1}_{I,X}) \circ Q(F^o(a^S_{\Gamma,x:\sigma})) \circ F^p_{S[\![\Gamma,x:\sigma]\!]}(S[\![\psi[\Gamma, x:\sigma]]\!]) \\
&= Q(a_I^{-1}) \circ \Omega_{F^o(I),X} \circ Q(a^{-1}_{I,X}) \circ F^p_{I\times X} \circ P(a^S_{\Gamma,x:\sigma})(S[\![\psi[\Gamma, x:\sigma]]\!]) \\
&= Q(a_I^{-1}) \circ F^p_I \circ \Omega_{I,X} \circ P(a^S_{\Gamma,x:\sigma})(S[\![\psi[\Gamma, x:\sigma]]\!]) \\
&= Q(a^{-1}_{S[\![\Gamma]\!]}) \circ F^p_{S[\![\Gamma]\!]}(S[\![\Omega_{x:\sigma}(\psi)[\Gamma]]\!].
\end{aligned}$$

$\square$

The next result says we can identify T-models $S \in (\mathcal{C}, P)$ with morphisms $\overline{S}\colon (\mathcal{C}_T, P_T) \to (\mathcal{C}, P)$ which we use to develop an "algebraic" view of entailment.

Theorem 6. *Let $\mathcal{L}$ be adequate and T an $\mathcal{L}$-theory. For each $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA})$ and T-model S in $(\mathcal{C}, P)$ there is a morphism $\overline{S}\colon (\mathcal{C}_T, P_T) \to (\mathcal{C}, P)$ in $\mathbf{FA}$, unique up to a 2-isomorphism such that $\overline{S}(G) = S$, where G is the generic T-model in $(\mathcal{C}_T, P_T)$.*

Proof. We define $\overline{S}^o$ on objects by $\overline{S}^o(\Gamma) \coloneqq S[\![\Gamma]\!]$, and on morphisms by

$$\overline{S}^o([M_1[\Gamma]/\sim, \ldots, M_n[\Gamma]/\sim]) \coloneqq \langle S[\![M_1[\Gamma]]\!], \ldots, S[\![M_n[\Gamma]]\!]\rangle.$$

Since S is a T-model, $\overline{S}^o$ is well defined on morphisms and it is straightforward to show that $\overline{S}^o : \mathcal{C}_T \to \mathcal{C}$ is a product preserving functor such that $\overline{S}(G) = S$, where G is the generic T-algebra in $\mathcal{C}_T$. We extend $\overline{S}^o$ by defining $\overline{S}^p_\Gamma(\phi[\Gamma]/\sim) \coloneqq S[\![\phi[\Gamma]]\!]$, and $\overline{S}^p_\Gamma$ is well-defined and monotone because S is a T-model. By definition, $\overline{S}(G)$ agrees with S on relation symbols. It is also straightforward to verify that $\overline{S}^p : P_T \to P \circ \overline{S}^o$ is a natural transformation, and that for each context Γ, $\overline{S}^P_\Gamma$ is an $\mathscr{L}$-homomorphism.

We now verify Condition 2. For $m > 0$, let $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, $\Gamma' = x_{n+1} : \sigma_{n+1}, \ldots, x_{n+m} : \sigma_{n+m}$, and for each $i \in \{1, \ldots, n+m\}$, let $\pi_i\colon S[\![\Gamma, \Gamma']\!] \to S[\![\sigma_i]\!]$ be the ith projection map. Let $S[\![\Gamma]\!] = I$, $S[\![\Gamma']\!] = I'$ and for $i \in \{1, \ldots, n+m\}$, let $S[\![\sigma_i]\!] = X_i$. Let $a\colon S[\![\Gamma, \Gamma']\!] \to S[\![\Gamma]\!] \times S[\![\Gamma']\!]$ be the change-of-product isomorphism. Then for each $\Omega \in \mathscr{L}_q$, if $m > 0$,

$$\begin{aligned}
&\overline{S}^p_\Gamma \circ \Omega_{\Gamma,\Gamma'}(\phi[\Gamma, \Gamma']/\sim) \\
&= \overline{S}^p_\Gamma(\Omega_{x_{n+1}:\sigma_{n+1}} \cdots \Omega_{x_{n+m}:\sigma_{n+m}}(\phi)[\Gamma]/\sim) \\
&= \Omega_{I,X_{n+1}} \circ \ldots \circ \Omega_{I\times X_{n+1}\ldots\times X_{n+m-1},X_{n+m}}(S[\![\phi[\Gamma, \Gamma']]\!]) \\
&= \Omega_{S[\![\Gamma]\!],S[\![\Gamma']\!]} \circ a^{-1*}(S[\![\phi[\Gamma, \Gamma']]\!]) && \text{(by 5)} \\
&= \Omega_{\overline{S}^o(\Gamma),\overline{S}^o(\Gamma')} \circ a^{-1*} \circ \overline{S}^p_{\Gamma\times\Gamma'}(\phi[\Gamma, \Gamma']/\sim).
\end{aligned}$$

Otherwise, $\Gamma' = [\,]$ and

$$\begin{aligned}
\overline{S}^p_\Gamma \circ \Omega_{\Gamma,[\,]}(\phi\,[\Gamma]/\sim) &= \overline{S}^p_\Gamma(\phi\,[\Gamma]/\sim) = S[\![\phi\,[\Gamma]]\!] \\
&= \Omega_{S[\![\Gamma]\!],S[\![\,]\!]} \circ P(\pi_1^{S[\![\Gamma]\!],S[\![\,]\!]})(S[\![\phi[\Gamma]]\!]) && \text{(by 5)} \\
&= \Omega_{\overline{S}^o\Gamma,\overline{S}^o[\,]} \circ P(a^{-1}_{\overline{S},\Gamma,[\,]}) \circ S^p_{\Gamma\times[\,]}(\phi\,[\Gamma]/\sim).
\end{aligned}$$

Let $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$, $\Gamma' = x_{n+1} : \sigma_1, \dots, x_{2n} : \sigma_n$. If $n > 0$,

$$\begin{aligned}
&\overline{S}^p_{\Gamma\times\Gamma'}(Eq_\Gamma) \\
&= \bigotimes_{i=1}^{n} S[\![x_i =_{\sigma_i} x_{n+i}[\Gamma,\Gamma']]\!] \\
&= \bigotimes_{i=1}^{n} P(\langle \pi_i^{S[\![\Gamma,\Gamma']\!]}, \pi_{i+n}^{S[\![\Gamma,\Gamma']\!]}\rangle) Eq_{S[\![\sigma_i]\!]} \\
&= \bigotimes_{i=1}^{n} P(\langle \pi_i^{S[\![\Gamma]\!]}\pi_1^{S[\![\Gamma]\!],S[\![\Gamma']\!]}, \pi_i^{S[\![\Gamma']\!]}\pi_2^{S[\![\Gamma]\!],S[\![\Gamma']\!]}\rangle \circ a) Eq_{S[\![\sigma_i]\!]} \\
&= P(a) \circ \bigotimes_{i=1}^{n} P(\langle \pi_i^{S[\![\Gamma]\!]}\pi_1^{S[\![\Gamma]\!],S[\![\Gamma']\!]}, \pi_i^{S[\![\Gamma']\!]}\pi_2^{S[\![\Gamma]\!],S[\![\Gamma']\!]}\rangle) Eq_{S[\![\sigma_i]\!]} \\
&= P(a) \circ Eq_{\overline{S}^o(\Gamma)}. && \text{(by 6)}
\end{aligned}$$

And if $n = 0$,

$$\begin{aligned}
\overline{S}^p_{[\,]}(Eq_{[\,]}) = \overline{S}^p_{[\,]}(e[\,]/\sim) = S[\![e[\,]]\!] = e_1 && \text{(by 6)} \\
= P(a_{\overline{S},[\,],[\,]})(e_{1\times 1}) = P(a_{\overline{S},[\,],[\,]})(Eq_{\overline{S}^o[\,]}).
\end{aligned}$$

Now all that remains to show is that if $F\colon (\mathcal{C}_T, P_T) \to (\mathcal{C}, P)$ is another morphism such that $F(G) = \overline{S}(G) = S$ then they are 2-isomorphic.

Let $a_{F,\Gamma}\colon F^o(\Gamma) \to \overline{F(G)}^o(\Gamma) = \overline{S}^o(\Gamma)$ be the change-of-product isomorphism. Then $a_F\colon F^o \Rightarrow \overline{S}^o$ is a natural isomorphism, and $Pa_F \cdot \overline{S}^p = Pa_F \cdot \overline{F(G)}^p = F^p$, from Equation 3.5. Thus a_F is a 2-isomorphism from F to $\overline{S}$. □

Definition 3.3.1. Let $F\colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ be a morphism in **FA**. We define the **kernel of** F, denoted $\ker F$ to consist of the following data:

1. A relation on $\mathrm{Ob}(\mathcal{C})$, such that $c_1 \sim c_2$ iff $F^o c_1 = F^o c_2$.

2. A relation on $\mathrm{Mor}(\mathcal{C})$ such that $f_1 \sim f_2$ iff $F^o f_1 = F^o f_2$.

3. A relation on $\sqcup_{c\in\mathrm{Ob}(\mathcal{C})} P(c)$ such that $r_1 \prec r_2$ iff $F^p_{c_1}(r_1) \le F^p_{c_2}(r_2)$, where $r_i \in P(c_i)$.

If $K\colon (\mathcal{C}, P) \to (\mathcal{E}, R)$ is another morphism, we say $\ker K \le \ker F$, if and only if the relations of $\ker K$ are contained in the corresponding relations of $\ker F$.

Let S be an Sg-structure in $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA})$ and let $a \in \mathrm{A}_{Sg}$. If $a = \phi_1, \ldots, \phi_n \vdash \phi\ [\Gamma]$, define

$$S[\![a]\!] \coloneqq (\bigotimes_{i=1}^{n} S[\![\phi_i[\Gamma]]\!], S[\![\phi[\Gamma]]\!]),$$

and if a is and equation $M_1 = M_2 : \tau\ [\Gamma]$, define

$$S[\![a]\!] \coloneqq (S[\![M_1 : \tau\ [\Gamma]]\!], S[\![M_2 : \tau\ [\Gamma]]\!]).$$

For each Sg-theory T, we define $S[\![T]\!] \coloneqq \{S[\![a]\!] : a \in \mathrm{A}(T)\}$ and for $F\colon (\mathcal{C}_T, P_T) \to (\mathcal{C}, P)$ where T is an $\mathcal{L}^m$-theory, let $\mathrm{Th}(F)$ be the $\mathrm{Sg}(T)$-theory such that

$$\mathrm{A}(\mathrm{Th}(F)) \coloneqq \{a \in \mathrm{A}_{Sg} : G[\![a]\!] \in \ker F\}.$$

It is straightforward to verify that $\mathrm{Th}(F) = \mathrm{Th}(F(G))$. Thus, for each structure $S \in (\mathcal{C}, P)$, $\mathrm{Th}(S) = \mathrm{Th}(\overline{S}(G)) = \mathrm{Th}(\overline{S})$. Moreover, whenever F is 2-isomorphic to a parallel 2-cell K, then $\mathrm{Th}(F) = \mathrm{Th}(K)$.

Let T, and T' be Sg-theories and $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA})$. From our prior observations the following are equivalent:

1. $T \vDash_{(\mathcal{C},P)} T'$.
2. $\forall F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}, P)$ such that $T \leq \mathrm{Th}(F)$, then $T' \leq \mathrm{Th}(F)$.
3. $\forall F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}, P)$ such that $G[\![T]\!] \subseteq \ker F$, then $G[\![T']\!] \subseteq \ker F$.

Thus in the sequel, we take Sg-structures in $(\mathcal{C}, P)$ to be morphisms $F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}, P)$. In the remainder of the section, we show how the prop-categorical semantics provide a natural notion of structural action on theories. This material is not necessary for the development of subsequent sections.

Let $H\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (C_{Sg'}, P_{Sg'})$ be a morphism and let T_1, T_2 be Sg-theories. We can define an action of H on Sg equations and sequents in context which can be extended to T_1 by taking the union: that is $H \cdot T_1$ is the Sg' theory whose assertions are $\bigcup_{a \in A(T_1)} H \cdot a$. For equations-in-context $M^1 = M^2 : \tau\ [\Gamma]$,

$$H \cdot (M^1 = M^2 : \tau\ [\Gamma]) \coloneqq \{M_i^1 = M_i^2 : \tau_i\ [\Gamma_H]\}_{i=1}^{n},$$

where $H^o(M^i : \tau\ [\Gamma]) = [M_1^i : \tau_1\, [\Gamma_H], \ldots, M_n^i : \tau_n\, [\Gamma_H]]$ for $i \in \{1, 2\}$. And for sequents-in-context $\phi_1, \ldots, \phi_n \vdash \phi_{n+1}\ [\Gamma]$,

$$H \cdot (\phi_1, \ldots, \phi_n \vdash \phi_{n+1}\ [\Gamma]) = \phi_1^H, \ldots, \phi_n^H \vdash \phi_{n+1}^H\ [\Gamma_H],$$

where $H^p_\Gamma(\phi_i\,[\Gamma]) = \phi_i^H\,[\Gamma_H]$, for all $i \in \{1,\dots,n+1\}$. Since products in $(\mathcal{C}_{Sg}, P_{Sg})$ are unique up to permutation of the list of their variable sort pairs, if $K\colon (\mathcal{C}_{Sg'}, P_{Sg'}) \to (\mathcal{C}_{Sg''}, P_{Sg''})$, then $(K \circ F)\cdot T_1 = K \cdot (F \cdot T_1)$ and clearly, $id_{(\mathcal{C}_{Sg}, P_{Sg})} \cdot T_1 = T_1$. Moreover, suppose $\mathcal{V} \subseteq \mathrm{Ob}(\mathbf{FA}_{\mathscr{L}})$ and that $T_1 \vDash_{\mathcal{V}} T_2$. Let $(\mathcal{C}, P) \in \mathcal{V}$ and $F\colon (\mathcal{C}_{Sg'}, P_{Sg'}) \to (\mathcal{C}, P)$ such that $G[\![H \cdot T_1]\!] \subseteq \ker F$. Then $G[\![T_1]\!] \subseteq \ker F \circ H$ and since $T_1 \vDash_{\mathcal{V}} T_2$, $G[\![T_2]\!] \subseteq \ker F \circ H$. It follows that $G[\![H \cdot T_2]\!] \subseteq \ker F$ and so $H \cdot T_1 \vDash_{\mathcal{V}} H \cdot T_2$. Therefore, all logics defined semantically by subcollections of $\mathrm{Ob}(\mathbf{FA}_{\mathscr{L}})$ are structural with respect the actions by morphisms.

This allows one to define a (first-order) logic as a closure operator $\vdash^{Sg}_{\mathcal{L}}$ on Th_{Sg}, for each signature Sg that is structural with respect to the action of some subcollection of morphisms $\mathcal{G}$ of $\mathbf{FA}_{\mathscr{L}}$ between formula prop-categories. Particularly interesting candidates are when $\mathcal{G}$ includes all morphisms between formula prop-categories and when $\mathcal{G}$ consists of all morphisms $H\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}_{Sg'}, P_{Sg'})$ which map single sort contexts to single sort contexts.

Chapter 4

Fibered Universal Algebra

4.1 Basic Constructions in FA

Let $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA})$ and consider the signature Sg whose sorts are $\mathrm{Ob}(\mathcal{C})$, whose function symbols are $f : c_1, \ldots, c_n \to c$, for each $f : c_1 \times \cdots \times c_n \to c \in \mathrm{Mor}(\mathcal{C})$ and whose relation symbols are $R \subseteq c_1, \ldots, c_n$ for each $R \in P(c_1 \times \cdots \times c_n)$. Note that $f : c_1 \times \cdots \times c_n \to c \in \mathrm{Ob}(\mathcal{C})$ is included both as an n-ary operation symbol $f : c_1, \ldots, c_n \to c$ and as a unary operation symbol $f : c_1 \times \cdots \times c_n \to c$ in Sg and similarly for relation symbols. There is a canonical Sg-structure S in $(\mathcal{C}, P)$ called the **internal structure of** $(\mathcal{C}, P)$, where $[\![c]\!] = c$, $[\![f]\!] = f$ and $[\![R]\!] = R$ for each sort, function symbol and relation symbol respectively. We define $\mathrm{Th}(\mathcal{C}, P) \coloneqq \mathrm{Th}(S)$.

Proposition 5. *Let $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA})$ and let S be the internal structure of $(\mathcal{C}, P)$, and T the theory of S. Then $\overline{S}\colon (\mathcal{C}_T, P_T) \to (\mathcal{C}, P)$ determines a 2-equivalence.*

Proof. Define $\iota\colon (\mathcal{C}, P) \to (\mathcal{C}_T, P_T)$ by $\iota^o(c) \coloneqq x : c$ for $c \in \mathrm{Ob}(\mathcal{C})$, $\iota^o(f\colon c_1 \to c_2) \coloneqq f(x) : c_2\ [x : c_1]/\sim$ for $f \in \mathrm{Mor}(\mathcal{C})$ and $\iota^p(R) \coloneqq R(x)\ [x : c]/\sim$ for each $c \in \mathrm{Ob}(\mathcal{C})$ and each $R \in P(c)$. It is straightforward to verify that ι^o is a finite product preserving functor. Let $f\colon c_1 \to c_2 \in \mathrm{Mor}(\mathcal{C})$. Then for $R \in P(c_2)$,

$$\begin{aligned}
\iota^p_{c_1} \circ P(f)(R) &= P(f)(R)(x)[x : c_1]/\sim \\
&= R(f(x))[x : c_1]/\sim \\
&= P_T(f(x) : c_2[x : c_1]/\sim)(R(x)[x : c_2]/\sim) \\
&= P_T(\iota^o(f)) \circ \iota^p_{c_2}(R).
\end{aligned}$$

Thus ι^p is a natural transformation. It is also straightforward to verify, for each $c \in \mathrm{Ob}(\mathcal{C})$ that ι^p_c is an monotone $\mathscr{L}$-algebra homomorphism. Let $\Omega \in \mathscr{L}_q$, $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$ and $R \in P(c_1 \times c_2)$. Then

$$\begin{aligned}
&\Omega_{\iota^o c_1, \iota^o c_2} \circ P_T(a^{-1}) \circ \iota^p_{c_1 \times c_2}(R) \\
&= \Omega_{x_2 : c_2}(R(\langle \pi_1, \pi_2 \rangle (x_1, x_2)))[x_1 : c_1]/\sim \\
&= \Omega_{c_1, c_2}(R)(x)[x : c_1]/\sim \\
&= \iota^p_{c_1} \circ \Omega_{c_1, c_2}(R).
\end{aligned}$$

Let $c \in \mathrm{Ob}(\mathcal{C})$. Then

$$\begin{aligned}
& P_T(a^{-1}) \circ \iota^p_{c\times c}(Eq_c) \\
&= Eq_c(\langle \pi_1, \pi_2 \rangle (x_1, x_2))[x_1 : c, x_2 : c]/\sim \\
&= Eq_c(x_1, x_2)[x_1 : c, x_2 : c]/\sim \\
&= x_1 =_c x_2 \; [x_1 : c, x_2 : c]/\sim \\
&= Eq_{x:c}.
\end{aligned}$$

It follows that ι is a morphism.

Now $\overline{S}^o \circ \iota^o = id_{\mathcal{C}}$, and for each $c \in \mathrm{Ob}(\mathcal{C})$ and $R \in P(c)$,

$$(\overline{S}^p_{\iota^o} \cdot \iota^p)_c(R) = \overline{S}^p_{\iota^o c} \circ \iota^p_c(R) = \overline{S}^p_{\iota^o c}(R(x)[x : c]/\sim) = R.$$

It follows that $\overline{S} \circ \iota = \mathrm{id}_{(\mathcal{C},P)}$.

In the other direction,

$$\iota^o \circ \overline{S}^o([x_1 : c_1, \ldots, x_n : c_n]) = [x : c_1 \times \cdots \times c_n],$$

$$\begin{aligned}
& \iota^o \circ \overline{S}^o([M_1 : c_1[\Gamma]/\sim, \ldots, M_m : c_m[\Gamma]/\sim]) \\
&= \langle S[\![M_1 : c_1[\Gamma]]\!], \ldots S[\![M_m : c_m[\Gamma]]\!]\rangle(x) : c_1 \times \cdots \times c_m \; [x : S[\![\Gamma]\!]]/\sim
\end{aligned}$$

and

$$(\iota^p_{\overline{S}^o} \cdot \overline{S}^p)_\Gamma(\phi[\Gamma]/\sim) = S[\![\phi[\Gamma]]\!](x)[x : S[\![\Gamma]\!]]/\sim .$$

For each context $\Gamma = x_1 : c_1, \ldots, x_n : c_n$, define $\eta \colon \iota^o \cdot \overline{S}^o \Rightarrow \mathrm{id}_{C_T}$ by

$$\eta_\Gamma = [\pi_1(x) : c_1[x : S[\![\Gamma]\!]]/\sim, \ldots, \pi_n(x) : c_n[x : S[\![\Gamma]\!]]/\sim].$$

One may verify that η is a natural isomorphism and that $P_T\eta \cdot \mathrm{id}^p_{(\mathcal{C}_T,P_T)} = \iota^p_{\overline{S}^o} \cdot \overline{S}^p$. It follows that $\eta \colon \iota \circ \overline{S} \Rightarrow \mathrm{id}_{(\mathcal{C}_T,P_T)}$ is a 2-isomorphism and so $(\mathcal{C}_T, P_T)$ and $(\mathcal{C}, P)$ are equivalent. □

Thus $(\mathcal{C}_T, P_T) \equiv (\mathcal{C}, P)$ and so $(\mathcal{C}_T, P_T)$ is a syntactic representation of $(\mathcal{C}, P)$ similar to the representation of an algebra as the free algebra over its elements quotiented out by is equational theory. Moreover, it can be shown that if $(\mathcal{C}, P), (\mathcal{D}, Q) \in \mathrm{Ob}(\mathbf{FA})$ are equivalent in $\mathbf{FA}$, then $\vDash_{(\mathcal{C},P)} = \vDash_{(\mathcal{D},Q)}$ and so

$\vDash_{(\mathcal{C},P)} = \vDash_{(\mathcal{C}_T,P_T)}$.

Let $(\mathcal{C},P),(\mathcal{D},Q) \in \mathrm{Ob}(\mathbf{FA})$. We say $(\mathcal{D},Q)$ **is a sub-prop-category of** $(\mathcal{C},P)$, if there exists a morphism $\iota\colon (\mathcal{D},Q) \to (\mathcal{C},P)$, such that ι^o is faithful and for each $c \in \mathrm{Ob}(\mathcal{C})$, ι^p_c is an order embedding. We call the morphism ι a **subprop-morphism**.

Proposition 6. *If $(\mathcal{D},Q)$ is a sub-prop-category of $(\mathcal{C},P)$, then $\vDash_{(\mathcal{C},P)} \subseteq \vDash_{(\mathcal{D},Q)}$.*

Proof. Suppose $T \vDash_{(\mathcal{C},P)} T'$ and consider $F\colon (\mathcal{C}_{Sg},P_{Sg}) \to (\mathcal{D},Q)$, such that $G[\![T]\!] \subseteq \ker F$. Then $G[\![T]\!] \subseteq \ker(\iota \circ F)$, and since $T \vDash_{(\mathcal{C},P)} T'$, $G[\![T']\!] \subseteq \ker(\iota \circ F)$. Since ι is a subprop-morphism, $G[\![T']\!] \subseteq \ker(F)$ and so $T \vDash_{(D,Q)} T'$. □

Let $F\colon (\mathcal{C},P) \to (\mathcal{D},Q)$ be a morphism in **FA**. In general, $(F(\mathcal{C}),F(P))$, is not a sub-prop-category of $(\mathcal{D},Q)$ where $F(\mathcal{C})$ is the image of $\mathcal{C}$ under F^o, and for all $g\colon d_1 \to d_2 \in \mathrm{Mor}(F(\mathcal{C}))$, $F(P)(g) \coloneqq Q(g)|_{F^p_{c_2}P(c_2)}\colon F^p_{c_2}P(c_2) \to F^p_{c_1}P(c_1)$, where $F^o c_i = d_i$ for $i \in \{1,2\}$. When $(F(\mathcal{C}),F(P))$ does define a sub-prop-category of $(\mathcal{D},Q)$ we call it the **image of** F.

Lemma 2. *Let $F\colon (\mathcal{C},P) \to (\mathcal{D},Q)$ be a morphism in* **FA** *such that F^o is injective on objects. Then $(F(\mathcal{C}),F(P))$ is a sub-prop-category of $(\mathcal{D},Q)$ and there exists a unique morphism $H\colon (\mathcal{C},P) \to (F(\mathcal{C}),F(P))$ such that $\iota \circ H = F$, where, $\iota\colon (F(\mathcal{C}),F(P)) \hookrightarrow (\mathcal{D},Q)$ is the inclusion morphism. We call H, the* ***corestriction of** F **to*** *$(F(\mathcal{C}),F(P))$. Furthermore, H is strictly finite product preserving, full, surjective on objects, and for each $c \in \mathrm{Ob}(\mathcal{C})$, H^p_c is surjective.*

Proof. Since F^o is injective on objects, $F(\mathcal{C})$ is a sub-category of $\mathcal{D}$. Let $d_1,d_2 \in \mathrm{Ob}(F(\mathcal{C}))$ and $F^o(c_i) = d_i$. Then the designated product diagram of d_1 and d_2 in $F(\mathcal{C})$ which we denote $(d_1 \hat{\times} d_2, \hat{\pi}_1, \hat{\pi}_2)$ (to distinguish it from the designated diagram of $d_1 \times d_2$ in $\mathcal{D}$) is $(F^o(c_1 \times c_2), F^o(\pi_1), F^o(\pi_2))$ where $(c_1 \times c_2, \pi_1, \pi_2)$ is the designated product diagram of $c_1 \times c_2$ in $\mathcal{C}$. Given $g_1\colon d \to d_1$ and $g_2\colon d \to d_2$ we let $< g_1, g_2 > \coloneqq F^o(\langle f_1, f_2 \rangle)$, which is the unique morphism such that $\hat{\pi}_i \circ < g_1, g_2 > = g_i$ for $i \in \{1,2\}$. We also define the designated terminal object in $F(\mathcal{C})$ to be $1_{F(\mathcal{C})} \coloneqq F^o(1_{\mathcal{C}})$. Define $F(P)(d_1) \coloneqq F^p(P(c_1)) \leq Q(d_1)$ and for $g \in F(\mathcal{C})(d_1,d_2)$, if $F^o(f) = g$, define $F(P)(g) \coloneqq Q(g)|_{F(P)(d_2)} = Q(F^o f)|_{F^p_{c_2}P(c_2)}$. Since $F^p\colon P \Rightarrow Q \circ F^o$, we have $F(P)(g)(F(P)(d_2)) \subseteq F(P)(d_1)$. For $\Omega \in \mathscr{L}_q$ we denote the interpretation of Ω in $(F(\mathcal{C}),F(P))$ by $\Omega^{F(P)}$, which we define for $r \in F(P)(d_1 \hat{\times} d_2)$ as

$$\Omega^{F(P)}_{F^o c_1, F^o c_2}(r) \coloneqq \Omega_{F^o c_1, F^o c_2} \circ Q(a^{-1}_{F,c_1,c_2})(r).$$

Let $r' \in P(c_1 \times c_2)$ such that $F^p_{c_1 \times c_2}(r') = r$. Then,

$$\Omega^{F(P)}_{d_1,d_2}(r) = \Omega_{d_1,d_2} \circ Q(a^{-1}) \circ F^P_{c_1 \times c_2}(r) = F^p_{c_1}(\Omega_{c_1,c_2}(r')) \in F(P)(d_1).$$

Equality $Eq^{F(P)}$ is defined by

$$Eq^{F(P)}_{F^o c} := Q(a_{F,c,c})(Eq_{F^o c}) = F^p_{c\times c}(Eq_c) \in F(P)(F^o c \mathbin{\hat{\times}} F^o c).$$

First we show that $(F(\mathcal{C}), F(P)) \in \mathbf{FA}$. Conditions 1 and 2 are immediate. Condition 3 follows from the fact that for each $F^o c \in \mathrm{Ob}(F(\mathcal{C}))$,

$$\Omega^{F(P)}_{(\cdot),d} = \Omega_{(\cdot),F^o c} \cdot Q(a^{-1}_{F,(\cdot),c})|_{F(P)((\cdot)\hat{\times} F^o c)}.$$

Condition 4 is satisfied as we take $\otimes^{F(P)(d_1)} := \otimes^{Q(d_1)}|_{F(P)(d_1)}$ and $e^{F(P)}_{d_1} := F^p_{c_1}(e_{c_1}) = e_{d_1}$.

Next we consider Condition 5. Let $d = F^o c \in F(\mathcal{C})$ and $\Omega \in \mathscr{L}_q$. Then

$$\Omega^{F(P)}_{d,F^o(1)} \circ F(P)(\hat{\pi}_1^{d,F^o(1)}) = \Omega_{d,F^o(1)} \circ Q(a^{-1}_{F,c,1}) \circ Q(\hat{\pi}_1^{d,F^o(1)})|_{F(P)(d)}$$
$$\Omega_{d,F^o(1)} \circ Q(\pi_1^{d,F^o(1)})|_{F(P)(d)} = id_{F(P)(d)}.$$

Let $d_1, d_2, d_3 \in F(\mathcal{C})$, and $F^o(c_i) = d_i$. Let $r \in F(P)((d_1 \hat{\times} d_2)\hat{\times} d_3)$ and $r' \in P((c_1 \times c_2) \times c_3)$ such that $F^p_{(c_1\times c_2)\times c_3}(r') = r$. Then

$$
\begin{aligned}
&= \Omega^{F(P)}_{d_1,d_2\hat{\times} d_3} \circ Q(\hat{a}_{d_1,d_2,d_3})(r)\\
&= \Omega_{d_1,d_2\hat{\times} d_3} \circ Q(a^{-1}_{F,c_1,c_2\times c_3}) \circ Q(F^o(a_{c_1,c_2,c_3})) \circ F^p_{(c_1\times c_2)\times c_3}(r')\\
&= \Omega_{F^o c_1, F^o(c_2\times c_3)} \circ Q(a^{-1}_{F,c_1,c_2\times c_3}) \circ F^p_{c_1\times(c_2\times c_3)} \circ P(a_{c_1,c_2,c_3})(r')\\
&= F^p \circ \Omega_{c_1,c_2\times c_3} \circ P(a_{c_1,c_2,c_3})(r')\\
&= F^p_{c_1} \circ \Omega_{c_1,c_2} \circ \Omega_{c_1\times c_2,c_3}(r')\\
&= \Omega_{F^o c_1,F^o c_2} \circ Q(a^{-1}_{F,c_1,c_2}) \circ F^p_{c_1\times c_2} \circ \Omega_{c_1\times c_2,c_3}(r')\\
&= \Omega_{F^o c_1,F^o c_2} \circ Q(a^{-1}_{F,c_1,c_2}) \circ \Omega_{F^o(c_1\times c_2),F^o c_3} \circ Q(a^{-1}_{F,c_1\times c_2,c_3}) \circ F^p_{(c_1\times c_2)\times c_3}(r')\\
&= \Omega^{F(P)}_{d_1,d_2} \circ \Omega^{F(P)}_{(d_1\hat{\times} d_2),d_3}(r).
\end{aligned}
$$

Thus Condition 5 is satisfied.

We now consider Condition 6.

$$Eq^{F(p)}_{F^o(1)} = F^p_{1\times 1}(Eq_1) = F^p_{1\times 1}(e_{1\times 1}) = e_{F^o 1 \hat{\times} F^o 1}.$$

Let $d_1, d_2 \in \mathrm{Ob}(F(\mathcal{C}))$ and $F^o c_i = d_i$. Let $c = c_1 \times c_2$ and $d = d_1 \hat{\times} d_2$. Then,

$$\begin{aligned}
Eq_d^{F(P)} &= F^p_{c\times c}(Eq_c) \\
&= F^p_{c\times c}(\bigotimes_{i=1}^{2} P(\langle \pi_i^c \pi_1^{c,c}, \pi_i^c \pi_2^{c,c}\rangle) Eq_{c_i}) \\
&= \bigotimes_{i=1}^{2} F^p_{c\times c} \circ P(\langle \pi_i^c \pi_1^{c,c}, \pi_i^c \pi_2^{c,c}\rangle) Eq_{c_i} \\
&= \bigotimes_{i=1}^{2} Q(F^o\langle \pi_i^c \pi_1^{c,c}, \pi_i^c \pi_2^{c,c}\rangle) \circ F^p_{c_i\times c_i} Eq_{c_i} \\
&= \bigotimes_{i=1}^{2} F(P)(<\hat{\pi}_i^d \hat{\pi}_1^{d,d}, \hat{\pi}_i^d \hat{\pi}_2^{d,d}>) Eq_{d_i}^{F(P)}.
\end{aligned}$$

It follows that $(F(\mathcal{C}), F(P)) \in \mathbf{FA}$.

Let $\iota^o \colon F(\mathcal{C}) \hookrightarrow \mathcal{D}$ be the inclusion functor and for each $d \in \mathrm{Ob}(F(\mathcal{C}))$, let $\iota^p_d \colon F(P)(d) \hookrightarrow Q(d)$ be the inclusion homomorphism. It is immediate that $\iota^p \colon F(P) \Rightarrow Q \circ \iota^o$ is natural. Let $\Omega \in \mathscr{L}_q$, $d_1, d_2 \in \mathrm{Ob}(F(\mathcal{C}))$ and $F^o(c_i) = d_i$. For $r \in F(P)(d_1 \hat{\times} d_2)$,

$$\begin{aligned}
\iota^p_{d_1} \circ \Omega^{F(P)}_{d_1,d_2}(r) &= \Omega_{d_1,d_2} \circ Q(a^{-1}_{F,c_1,c_2})(r) \\
&= \Omega_{\iota^o_{d_1},\iota^o_{d_2}} \circ Q(a^{-1}_{F,c_1,c_2}) \circ \iota^p_{d_1 \hat{\times} d_2}(r).
\end{aligned}$$

And,

$$\iota^p_{d \hat{\times} d}(Eq^{F(P)}) = Q(a_{F,c,c}) \circ Eq_{\iota^o d}.$$

It follows that $\iota = (\iota^o, \iota^p) \in \mathbf{FA}((F(\mathcal{C}), F(P)), (\mathcal{D}, Q))$.

Define $H = (H^o, H^p)$ by $H^o \colon \mathcal{C} \to F(\mathcal{C})$, where $H^o(c) := F^o(c)$, $H^o(f) := F^o(f)$ and H^p_c is the corestriction of F^p_c to $F(P)(F^o c)$. The naturality of $H^p \colon P \Rightarrow F(P) \circ H^o$ follows from the naturality of $F^p \colon P \Rightarrow Q \circ F^o$. By definition H^o strictly preserves finite products and $\iota \circ H = F$. Since ι^o is injective on objects and faithful, and ι^p_d is injective, for each $d \in \mathrm{Ob}(F(\mathcal{C}))$, H is uniquely defined so that $H \circ \iota = F$. Let $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$ and $r \in P(c_1 \times c_2)$. That H commutes with the quantifiers and equality follows from the fact that F does. Thus, $H \in \mathbf{FA}((\mathcal{C}, P), (F(\mathcal{C}), F(P)))$. □

Proposition 7. **FA** *has all products.*

Proof. The terminal object $1_{\mathbf{FA}}$ is $(*, 1_*)$ where $*$ is the terminal category and 1_* is its unique endofunctor. It is straightforward to verify that $(*, 1_*) \in \mathbf{FA}$ and that for each prop-category $(\mathcal{C}, P) \in \mathbf{FA}$, there is a unique morphism $!_{(\mathcal{C},P)} \colon (\mathcal{C}, P) \to (*, 1_*)$, where $!_{(\mathcal{C},P)} = (!_{\mathcal{C}}, !_P)$.

Let $\{(\mathcal{C}_i, P_i)\}_{i\in I} \subseteq \mathrm{Ob}(\mathbf{FA})$ be non-empty. Let $\prod_I \mathcal{C}_i$ denote the product of the categories $\{\mathcal{C}_i\}_{i\in I}$, that is, the category whose objects a specify for each $i \in I$, an object $a_i \in \mathrm{Ob}(\mathcal{C}_i)$, and whose morphisms $f\colon a \to b$ specify for each $i \in I$ a morphism $f_i \in \mathcal{C}_i(a_i, b_i)$. Composition $g \circ f$ of morphisms $f\colon a \to b$ and $g\colon b \to c$ is defined for each $i \in I$ as $(g \circ f)_i \coloneqq g_i \circ f_i\colon a_i \to c_i$. Since each $\mathcal{C}_i$ has finite products, we define 1 so that $1_i = 1_{\mathcal{C}_i}$ and for $a, b \in \mathrm{Ob}(\prod \mathcal{C}_i)$, we define $a \times b$ so that $(a \times b)_i \coloneqq a_i \times b_i$. For $k \in \{1, 2\}$, we define the projection maps $\pi_k^{a,b}$ by $\pi_{k,i}^{a,b} = \pi_k^{a_i,b_i}$. One verifies that this determines a product diagram for $a \times b$, where for every pair of maps $f\colon c \to a$ and $g\colon c \to b$, $\langle f, g\rangle_i \coloneqq \langle f_i, g_i\rangle$.

Now we define $\prod_I P_i\colon \prod_I \mathcal{C}_i \to \mathbf{Pos}$, on objects $a \in \mathrm{Ob}(\prod_I \mathcal{C}_i)$ by $(\prod_I P_i)(a) \coloneqq \prod_I P_i(a_i)$ and on morphisms $f \in \prod_I \mathcal{C}_i(a, b)$ by $(\prod_I P_i)(f) \coloneqq \prod_I P_i(f_i)$. Then for $f\colon a \to b$ and $g\colon b \to c$, $(\prod_I P_i)(g \circ f) = \prod_I P_i(g_i \circ f_i) \coloneqq \prod_I P_i(f_i)P_i(g_i) \coloneqq \prod_I P_i(f_i)\prod_I P_i(g_i) = (\prod_I P_i)(f) \circ (\prod_I P_i)(g)$. One also verifies that $\prod_I P_i(id_a) = id_{\prod_I P_i(a)}$, so that $\prod_I P_i$ is a contravariant functor and thus $(\prod_I \mathcal{C}_i, \prod_I P_i)$ is a prop-category.

For each $a \in \mathrm{Ob}(\prod_I \mathcal{C}_i)$, we define Eq_a and e_a so that $Eq_{a,i} \coloneqq Eq_{a_i}$ and $e_{a,i} \coloneqq e_{a_i}$. Let $b \in \mathrm{Ob}(\prod_I \mathcal{C}_i)$ and $c = a \times b$. Then $Eq_1 = e_{1\times 1}$ since for each $i \in I$,

$$Eq_{1,i} = Eq_{1_{\mathcal{C}_i}} = e_{1_{\mathcal{C}_i}\times 1_{\mathcal{C}_i}} = e_{1\times 1,i}.$$

Also,

$$Eq_c = (\prod_I P_i)\langle \pi_1^{a,b}\pi_1^{c,c}, \pi_1^{a,b}\pi_1^{c,c}\rangle Eq_a \otimes (\prod_I P_i)\langle \pi_2^{a,b}\pi_1^{c,c}, \pi_2^{a,b}\pi_1^{c,c}\rangle Eq_b,$$

since for each $j \in I$,

$$\begin{aligned}
&((\prod_I P_i)\langle \pi_1^{a,b}\pi_1^{c,c}, \pi_1^{a,b}\pi_1^{c,c}\rangle Eq_a \otimes (\prod_I P_i)\langle \pi_2^{a,b}\pi_1^{c,c}, \pi_2^{a,b}\pi_1^{c,c}\rangle Eq_b)_j \\
&= P_j\langle \pi_1^{a_j,b_j}\pi_1^{c_j,c_j}, \pi_1^{a_j,b_j}\pi_1^{c_j,c_j}\rangle Eq_{a_j} \otimes P_j\langle \pi_2^{a_j,b_j}\pi_1^{c_j,c_j}, \pi_2^{a_j,b_j}\pi_1^{c_j,c_j}\rangle Eq_{b_j} \\
&= Eq_{c,j}
\end{aligned}$$

For $\Omega \in \mathscr{L}_q$, $\Omega_{a,b}\colon \prod_I P_i(a \times b) \to \prod_I P_i(a)$ is defined so that $\Omega_{a,b,i} \coloneqq \Omega_{a_i,b_i}$. As with the conditions for equality, the remaining conditions for the quantifiers follow from the fact that they hold coordinate-wise. Therefore, $(\prod_I \mathcal{C}_i, \prod_I P_i) \in \mathrm{Ob}(\mathbf{FA})$.

For $j \in I$, the projection morphism $\pi_j\colon (\prod_I \mathcal{C}_i, \prod_I P_i) \to (\mathcal{C}_j, P_j)$ consists of the projection functor $\pi_j^o\colon \prod_I \mathcal{C}_i \to \mathcal{C}_j$ and the natural transformation $\pi_j^p\colon \prod_I P_i \Rightarrow P_j \circ \pi_j^o$ defined for each object $a \in \mathrm{Ob}(\prod_I \mathcal{C}_i)$, by $\pi_{j,a}^p\colon \prod_I P_i(a) \to P_j(a_j)$, i.e. $\pi_{j,a}^p$ is the (monotone) projection homomorphism. For all

$f\colon a \to b \in \mathrm{Mor}(\prod_I \mathcal{C}_i)$,

$$
\begin{array}{ccccc}
b & & \prod_I P_i(b_i) & \xrightarrow{\pi^p_{j,b}} & P_j(b_j) \\
{\scriptstyle f^{op}}\downarrow & & {\scriptstyle \prod_I P_i(f_i)}\downarrow & & \downarrow{\scriptstyle P_j(f_j)} \\
a & & \prod_I P_i(a_i) & \xrightarrow{\pi^p_{j,a}} & P_j(a_j)
\end{array}
$$

commutes and so $\pi^p_j\colon \prod_i P_i \to P_j \circ \pi^o_j$ is a natural transformation. For all $a \in \mathrm{Ob}(\prod_I \mathcal{C}_i)$,

$$\pi^p_{j,a\times a}(Eq_a) = Eq_{a,j} = Eq_{a_j} = Eq_{\pi^o_j(a)}.$$

For $\Omega \in \mathscr{L}_q$, $a, b \in \mathrm{Ob}(\prod_I \mathcal{C}_i)$ and $A \in (\prod_I P_i)(a \times b)$,

$$\pi^p_{j,a} \circ \Omega_{a,b}(A) = \Omega_{a_j,b_j}(A_j) = \Omega_{\pi_j(a),\pi_j(b)} \circ \pi^p_{j,a\times b}(A).$$

It follows that $\pi_j\colon (\prod_I \mathcal{C}_i, \prod_I P_{\mathcal{C}_i}) \to (\mathcal{C}_j, P_{\mathcal{C}_j})$ is a prop-category morphism.

Let $(\mathcal{D}, Q) \in \mathbf{FA}$ and for each $j \in I$, let $F_j\colon (\mathcal{D}, Q) \to (\mathcal{C}_j, P_j)$ be a morphism in **FA**. Define $\langle F_i\rangle_I\colon (\mathcal{D}, Q) \to (\prod_I \mathcal{C}_i, \prod_I P_i)$, so that $\langle F_i\rangle^o_I \coloneqq \langle F^o_i\rangle_I$ and define $\langle F_i\rangle^p_I\colon Q \Rightarrow \prod_I P_i \circ \langle F_i\rangle^o_I$ for $d \in \mathrm{Ob}(\mathcal{D})$, by $\langle F_i\rangle^p_{I,d} \coloneqq \langle F^p_{i,d}\rangle_I$. Then for all $g\colon d_1 \to d_2 \in \mathrm{Mor}(\mathcal{D})$,

$$
\begin{array}{ccccc}
d_2 & & Q(d_2) & \xrightarrow{\langle F^p_{i,d_2}\rangle_I} & \prod_I(P_i \circ F^o_i(d_2)) \\
{\scriptstyle g^{op}}\downarrow & & {\scriptstyle Q(g)}\downarrow & & {\scriptstyle \prod_I(P_i \circ F^o_i(g))}\downarrow \\
d_1 & & P_{\mathcal{D}}(d_1) & \xrightarrow{\langle F^p_{i,d_1}\rangle_I} & \prod_I(P_i \circ F^o_i(d_1))
\end{array}
$$

commutes since

$$
\begin{aligned}
\prod_I (P_i \circ F^o_i(g)) \circ \langle F^p_{i,d_2}\rangle_I &= \langle P_i \circ F^o_i(g) \circ F^p_{i,d_2}\rangle_I \\
&= \langle F^p_{i,d_1} \circ Q(g)\rangle_I \\
&= \langle F^p_{i,d_1}\rangle_I \circ Q(g).
\end{aligned}
$$

It follows that $\langle F_i\rangle^p_I\colon Q \Rightarrow \prod_I P_i \circ \langle F_i\rangle^o_I$ is a natural transformation.

Let $d \in \mathrm{Ob}(\mathcal{D})$ and $a \in \mathrm{Mor}(\prod_I C_i)$ such that for each $j \in I$, $a_j\colon F^o_j(d \times d) \to F^o_j d \times F^o_j d$ is the change-of-product isomorphism. Then for each $j \in I$,

$$\langle F^p_{i,d\times d}\rangle_I (Eq_d)_j = F^p_{j,d\times d} Eq_d = P_j(a_j) Eq_{F^o_j d},$$

and so

$$\langle F_i\rangle^p_{I,d\times d} Eq_d = \prod_I P_i(a) Eq_{\langle F_i\rangle^o_I d}$$

Let $d_1, d_2 \in \mathrm{Ob}(\mathcal{D})$ and $a \in \mathrm{Mor}(\prod_I \mathcal{C}_i)$ such that for each $j \in I$, $a_j \colon F^o_j(d_1 \times d_2) \to F^o_j d_1 \times F^o_j d_2$ is the change-of-product isomorphism. Then, for $\Omega \in \mathscr{L}_q$,

$$\begin{aligned} \langle F_i\rangle^p_{I,d_1} \circ \Omega_{d_1,d_2} &= \langle F^p_{i,d_1} \circ \Omega_{d_1,d_2}\rangle_I \\ &= \langle \Omega_{F^o_i d_1, F^o_i d_2} \circ P(a_i^{-1}) \circ F^p_{i,d_1\times d_2}\rangle_I \\ &= \Omega_{\langle F_i\rangle^o_I d_1, \langle F_i\rangle^o_I d_2} \circ \prod_I P_i(a^{-1}) \circ \langle F_i\rangle^p_{I,d_1\times d_2}. \end{aligned}$$

Therefore, $\langle F_i\rangle_I$ is a morphism. Also, for each $j \in I$ and $d \in \mathrm{Ob}(\mathcal{D})$,

$$(\pi^p_{j,\langle F_i\rangle^o_I} \cdot \langle F_i\rangle^p_I)_d = \pi^p_{j,\langle F_i\rangle^o_I d} \circ \langle F^p_{i,d}\rangle_I = F^p_{j,d},$$

and

$$\pi^o_j \circ \langle F_i\rangle^o_I = F^o_j.$$

Thus for each $j \in I$, $\pi_j \circ \langle F_i\rangle_I = F_j$. If $\phi\colon (\mathcal{D}, Q) \to (\prod_I \mathcal{C}_I, \prod_I P_i)$ is another morphism such that for each $j \in I$, $\pi_j \circ \phi = F_j$, then $\phi^o = F^o_i$ and for each $d \in \mathrm{Ob}(\mathcal{D})$,

$$(\pi^p_{j,\phi^o} \cdot \phi^p)_d = \pi^p_{j,\langle F^o_i\rangle_I d} \circ \phi^p_d = F^p_{j,d}.$$

It follows that $\phi^p_d = \langle F_i\rangle^p_{I,d}$, and so $\phi = \langle F_i\rangle_I$. □

Proposition 8. *Let $\mathscr{L}$ be a first-order language and $\{(\mathcal{C}_i, P_i)\}_{i\in I} \subseteq \mathrm{Ob}(\mathbf{FA})$. Then*

$$\vDash_{(\prod_I \mathcal{C}_i, \prod_I P_i)} \supseteq \vDash_{\{(\mathcal{C}_i,P_i)\}_{i\in I}} .$$

Proof. If $I = \emptyset$, then $(\prod_I \mathcal{C}_i, \prod_I P_i) = 1_{\mathbf{FA}} = (*, 1_*)$ which satisfies every sequent-in-context and equation-in-context. Thus, $\vDash_{1_{\mathbf{FA}}} = \vDash_\emptyset$. Otherwise, suppose $T \vDash_{\{(\mathcal{C}_i,P_i)\}_{i\in I}} T'$ and let $S \in (\prod_I \mathcal{C}_i, \prod_I P_i)$ be a T-model. Then for each $i \in I$, $\pi_i(S)$ is a T-model in $(\mathcal{C}_i, P_i)$. By assumption $\pi_i(S)$ satisfies T' for all

$i \in I$ and it follows that S satisfies T'. □

As a consequence, for each logic $\mathcal{L}$, $\mathbf{FA}_{\mathcal{L}}$ has all products.

4.2 Fibered Homomorphism Theorem

In this section we present a fibered homomorphism theorem which closely mirrors the corresponding result from universal algebra. These results give an orthogonal factorization system for $\mathbf{FA}_{\mathcal{L}}$. For a general reference on factorization systems see Riehl (2008).

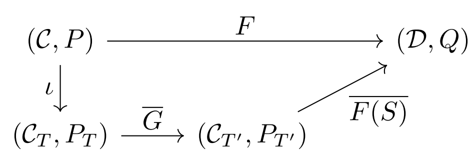
Fix a logic $\mathcal{L}$ and let $\mathscr{E}$ be the class of morphisms ϵ in $\mathbf{FA}_{\mathcal{L}}$ such that ϵ^o is bijective on objects, full and for each $c \in \mathrm{Ob}(\mathcal{C})$, ϵ^p_c is surjective. Let $\mathscr{M}$ be the collection of subprop-morphisms in $\mathbf{FA}_{\mathcal{L}}$.

Theorem 7 (Fibered Homomorphism Theorem)**.** *Let $F\colon (\mathcal{C}, P) \to (\mathcal{D}, Q)$ be a morphism in $\mathbf{FA}_{\mathcal{L}}$.*

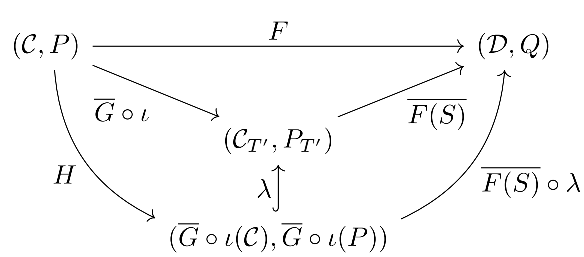
1. *F factors as $\psi \circ \epsilon$, for some $\epsilon \in \mathscr{E}$ and $\psi \in \mathscr{M}$.*

2. *If $K\colon (\mathcal{C}, P) \to (\mathcal{E}, R) \in \mathrm{Mor}(\mathbf{FA}_{\mathcal{L}})$ where K is a full, surjective on objects and for each $e \in \mathrm{Ob}(e)$, K^p_e is surjective, then there exists a unique morphism $H\colon (\mathcal{E}, R) \to (\mathcal{D}, Q)$ such that $HK = F$ if and only if $\ker K \leq \ker F$.*

Proof of Part 1 of Theorem 7. Let T be the theory of $(\mathcal{C}, P)$ and S the internal structure of $(\mathcal{C}, P)$. Let T' be the theory of $F(S)$ (which is an $\mathcal{L}^m$-theory) and G the generic T'-model in $(\mathcal{C}_{T'}, P_{T'})$. Let $\iota\colon (\mathcal{C}, P) \to (\mathcal{C}_T, P_T)$ be the morphism which with $\overline{S}$ witnesses the equivalence of $(\mathcal{C}, P)$ and $(\mathcal{C}_T, P_T)$. We have the following diagram:



It is straightforward to verify that the above diagram commutes. Since $\overline{G} \circ \iota$ is injective on objects, from Proposition 2, the following diagram commutes



where λ is the inclusion morphism and H the corestriction. Let $\epsilon = H$, $\psi = \overline{F(S)} \circ \lambda$ and $(\overline{G} \circ \iota(\mathcal{C}), \overline{G} \circ \iota(P)) = (\mathcal{E}, R)$. By construction, ϵ and ψ meet the conditions of the theorem. Since ι is a subprop-morphism, from Proposition 6, $(\mathcal{E}, R) \in \mathbf{FA}_{\mathcal{L}}$. □

Proof of part 2 of Theorem 7. ($\Longleftarrow$) Let $e_1 \in \mathrm{Ob}(\mathcal{E})$. Since K^o is surjective on objects, there exists $c_1 \in \mathrm{Ob}(\mathcal{C})$, such that $K^o(c_1) = e_1$. Define $H^o e_1 \coloneqq F^o c_1$. Let $e_2 \in \mathrm{Ob}(\mathcal{E})$, and $h_1 \colon e_1 \to e_2$. Then there exists $c_2 \in \mathrm{Ob}(\mathcal{C})$ such that $K^o c_2 = e_2$ and since K^o is full, there exists $f_1 \colon c_1 \to c_2$ such that $K^o f_1 = h_1$. Define $H^o h_1 \coloneqq F^o f_1$. Since $\ker K \leq \ker F$, H^o is well defined. If $h_2 \colon e_2 \to e_3$, let $f_2 \colon c_2 \to c_3$ such that $K^o f_2 = h_2$. Then

$$H^o(h_2 h_1) = H^o(K^o f_2 K^o f_1) = H^o(K^o(f_2 f_1)) = F^o(f_2 f_1) = H^o h_2 H^o h_1.$$

Also, $H^o(id_{e_1}) = H^o(K^o id_{c_1}) = F^o id_{c_1} = id_{F^o c_1} = id_{H^o e_1}$ and so H^o is a functor.

Let $1_{\mathcal{E}}$ be the terminal element in $\mathcal{E}$. Then there exists $c \in \mathrm{Ob}(\mathcal{C})$ such that $K^o c = 1_{\mathcal{E}}$. Since K^o preserves finite products $K^o 1_{\mathcal{C}}$ is a terminal element in $\mathcal{E}$ and so $K^o 1_{\mathcal{C}} \cong 1_{\mathcal{E}}$. Since H^o is a functor, it preserves isomorphisms and so $H^o 1_{\mathcal{E}} \cong H^o K^o 1_{\mathcal{C}} = F^o 1_{\mathcal{C}}$, and so $H^o 1_{\mathcal{E}}$ is terminal in $\mathcal{D}$ since F^o preserves finite products. Let $e_1, e_2 \in \mathrm{Ob}(\mathcal{E})$ and $c_1, c_2, c \in \mathrm{Ob}(\mathcal{C})$, such that $K^o c_i = e_i$ and $K^o c = e_1 \times e_2$. For $i \in \{1, 2\}$, let $f_i \colon c \to c_i$ such that $K^o f_i = \pi_i^{e_1, e_2}$ and let $\alpha = \langle K^o \pi_1^{c_1, c_2}, K^o \pi_2^{c_1, c_2} \rangle$. Since K^o preserves finite products, α is an isomorphism. For $i \in \{1, 2\}$, $K^o \pi_i^{c_1, c_2} K^o \langle f_1, f_2 \rangle = K^o f_i = \pi_i^{e_1, e_2}$ and so $K^o \langle f_1, f_2 \rangle = \alpha^{-1}$. Then $H^o(\alpha^{-1}) = F^o \langle f_1, f_2 \rangle$ is an isomorphism and for $i \in \{1, 2\}$, $F^o \pi_i^{c_1, c_2} \circ F^o \langle f_1, f_2 \rangle = F^o f_i$. Thus since F^o preserves finite products, $F^o \langle f_1, f_2 \rangle$ is a change-of-product isomorphism from $(F^o c, F^o f_1, F^o f_2)$ to $(F^o(c_1 \times c_2), F^o \pi_1^{c_1, c_2}, F^o \pi_2^{c_1, c_2})$. It follows that $(H^o(e_1 \times e_2), H^o \pi_1^{e_1, e_2}, H^o \pi_2^{e_1, e_2}) = (F^o c, F^o f_1, F^o f_2)$ is a product diagram for $H^o e_1, H^o e_2$ and so H^o preserves finite products.

Let $e \in \mathrm{Ob}(\mathcal{E})$, and $r \in R(e)$. Then there exists $c_1 \in \mathrm{Ob}(\mathcal{C})$ and $r_1 \in P(c_1)$ such that $K^p_{c_1}(r_1) = r$. Define $H^p_e(r) \coloneqq F^p_{c_1}(r_1)$. If $r_2 \in P(c_2)$, such that $K^p_{c_2}(r_2) = r$, then $(r_1, r_2), (r_2, r_1) \in \ker K \subseteq \ker F$ and so $F^p_{c_1}(r_1) = F^p_{c_2}(r_2)$. It follows that H^p_e is well-defined and by definition, $H^p_e \circ K^p_{c_1} = F^p_{c_1}$. Let $r_1, r_2 \in R(e)$ such that $r_1 \leq r_2$. Then there exists $c \in \mathrm{Ob}(\mathcal{C})$ and $r'_1, r'_2 \in P(c)$ such that $K^p_c(r'_i) = r_i$ for $i \in \{1, 2\}$. Then $(r'_1, r'_2) \in \ker K \subseteq \ker F$, and so $H^p_e(r_1) = F^p_c(r'_1) \leq F^p_c(r'_2) = H^p_e(r_2)$. Thus H^p_e is monotone.

Let $h \colon e_1 \to e_2$ and $f \colon c_1 \to c_2$ such that $K^o f = h$. Then

$$\begin{aligned} Q(H^o h) \circ H^p_{e_2} \circ K^p_{c_2} &= Q(F^o f) \circ F^p_{c_2} = F^p_{c_1} \circ P(f) \\ &= H^p_{e_1} \circ K^p_{c_1} \circ P(f) = H^p_{e_1} \circ R(h) \circ K^p_{c_2}, \end{aligned}$$

and since $K^p_{c_2}$ is surjective, $Q(H^o h) \circ H^p_{e_2} = H^p_{e_1} \circ R(h)$. Thus, $H^p \colon R \Rightarrow Q H^o$ is a natural transformation.

Let $e \in \mathrm{Ob}(\mathcal{E})$. Then there exists $c, c' \in \mathrm{Ob}(\mathcal{C})$, such that $K^o c' = e \times e$ and $K^o c = e$. Then for

$i \in \{1, 2\}$, there exists $f_i : c' \to c$ such that $K^o f_i = \pi_i^{e,e}$. Then for $i \in \{1, 2\}$,

$$K^o \pi_i^{c,c} K^o \langle f_1, f_2 \rangle = K^o f_i = \pi_i^{e,e}.$$

It follow that $K^o \langle f_1, f_2 \rangle = \alpha^{-1}$, where $\alpha = \langle K^o \pi_1^{c,c}, K^o \pi_2^{c,c} \rangle$. Define:

$$a \coloneqq \langle F^o \pi_1^{c,c}, F^o \pi_2^{c,c} \rangle \quad b \coloneqq \langle H^o \pi_1^{e,e}, H^o \pi_2^{e,e} \rangle.$$

Then

$$aF^o \langle f_1, f_2 \rangle = \langle F^o f_1, F^o f_2 \rangle = \langle H^o \pi_1^{e,e}, H^o \pi_2^{e,e} \rangle = b,$$

and so $F^o \langle f_1, f_2 \rangle = a^{-1} b$. Also note that

$$F^p_{c \times c}(Eq_c) = Q(a)(Eq_{F^o c}) \quad \text{and} \quad K^p_{c \times c}(Eq_c) = R(\alpha)(Eq_{K^o c}).$$

Then

$$\begin{aligned} Q(b)(Eq_{H^o e}) &= Q(b)(Eq_{F^o c}) = Q(b)(Q(a^{-1}) F^p_{c \times c} Eq_c) \\ &= Q(a^{-1} b) F^p_{c \times c}(Eq_c) = Q(F^o \langle f_1, f_2 \rangle) F^p_{c \times c}(Eq_c) \\ &= F^p_{c'} P(\langle f_1, f_2 \rangle)(Eq_c) = H^p_{e \times e} K^p_{c'} P(\langle f_1, f_2 \rangle)(Eq_c) \\ &= H^p_{e \times e} R(K^o \langle f_1, f_2 \rangle) K^p_{c \times c}(Eq_c) = H^p_{e \times e}(R(\alpha^{-1}) K^p_{c \times c}(Eq_c)) \\ &= H^p_{e \times e}(Eq_{K^o c}) = H^p_{e \times e}(Eq_e). \end{aligned}$$

Let $e_1, e_2 \in \mathrm{Ob}(\mathcal{E})$, and $c_1, c_2, c \in \mathrm{Ob}(\mathcal{C})$ such that $K^o c_i = e_i$ and $K^o c = e_1 \times e_2$. Let $f_i : c \to c_i$ such that $K^o f_i = \pi_i^{e_1, e_2}$ and define

$$a \coloneqq \langle F^o \pi_1^{c_1, c_2}, F^o \pi_2^{c_1, c_2} \rangle \quad b \coloneqq \langle H^o \pi_1^{e_1, e_2}, H^o \pi_2^{e_1, e_2} \rangle.$$

Also, let $\alpha \coloneqq \langle K^o \pi_1^{c_1,c_2}, K^o \pi_2^{c_1,c_2} \rangle$. Then for $\Omega \in \mathscr{L}_q$

$$
\begin{aligned}
&= \Omega_{H^o e_1, H^o e_2} \circ Q(b^{-1}) \circ H^p_{e_1 \times e_2} \circ R(\alpha^{-1}) \circ K^p_{c_1 \times c_2} \\
&= \Omega_{F^o c_1, F^o c_2} \circ Q(a^{-1}) \circ Q(b^{-1} a) \circ H^p_{e_1 \times e_2} \circ R(\alpha^{-1}) \circ K^p_{c_1 \times c_2} \\
&= \Omega_{F^o c_1, F^o c_2} \circ Q(a^{-1}) \circ Q(H^o \alpha) \circ H^p_{e_1 \times e_2} \circ R(\alpha^{-1}) \circ K^p_{c_1 \times c_2} \\
&= \Omega_{F^o c_1, F^o c_2} \circ Q(a^{-1}) \circ H^p_{K^o(c_1 \times c_2)} \circ R(\alpha) \circ R(\alpha^{-1}) \circ K^p_{c_1 \times c_2} \\
&= \Omega_{F^o c_1, F^o c_2} \circ Q(a^{-1}) \circ H^p_{K^o(c_1 \times c_2)} \circ K^p_{c_1 \times c_2} \\
&= \Omega_{F^o c_1, F^o c_2} \circ Q(a^{-1}) \circ F^p_{c_1 \times c_2} \\
&= F^p_{c_1} \circ \Omega_{c_1, c_2} \\
&= H^p_{e_1} \circ K^p_{c_1} \circ \Omega_{c_1, c_2} \\
&= H^p_{e_1} \circ \Omega_{K^o c_1, K^o c_2} \circ R(\alpha^{-1}) \circ K^p_{c_1 \times c_2} \\
&= H^p_{e_1} \circ \Omega_{e_1, e_2} \circ R(\alpha^{-1}) \circ K^p_{c_1 \times c_2}.
\end{aligned}
$$

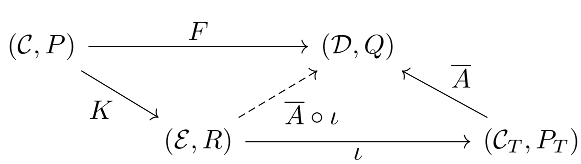
Since $R(\alpha^{-1}) \circ K^p_{c_1 \times c_2}$ is surjective it follows that

$$\Omega_{H^o e_1, H^o e_2} \circ Q(b^{-1}) \circ H^p_{e_1 \times e_2} = H^p_{e_1} \circ \Omega_{e_1, e_2}.$$

Thus H is a morphism in **FA** and $H \circ K = F$.

($\implies$). Since $\mathcal{K}$ is full, surjective on objects and for each $e \in \mathrm{Ob}(\mathcal{E})$, K^p_e is surjective, there is only one possible definition for H. One may verify that if $\ker K \nleq \ker F$, then either H^o is ill-defined or for some $e \in \mathrm{Ob}(\mathcal{E})$, H^p_e is ill-defined or not monotone. □

Remark 2. Like part 1 of Theorem 7, the internal logic can also be used to prove part 2. In what follows we sketch the proof of the ($\impliedby$) direction. Let S be the internal structure in $(\mathcal{E}, R)$ and T its theory. We construct a T-model $A \in (\mathcal{D}, Q)$, so that



commutes.

For each $e \in \mathrm{Ob}(\mathcal{E})$, let $c_e \in \mathrm{Ob}(\mathcal{C})$ such that $K^o c_e = e$. For each $h \in \mathrm{Mor}(\mathcal{E})$, let $f_h \in \mathrm{Mor}(\mathcal{C})$, such that $K^o f_h = h$ and for each $e \in \mathrm{Ob}(\mathcal{E})$ and $r \in R(e)$, let $q_r \in P(c)$ for some $c \in \mathrm{Ob}(\mathcal{C})$ such that

$K_c^p(q_r) = r$. For each each sort e, $A[\![e]\!] \coloneqq F^o c_e$, for each unary function symbol $h\colon e_1 \times \cdots \times e_n \to e$, $A[\![h]\!] \coloneqq F^o f_h$ and each unary relation symbol $r \in R(e_1 \times \cdots \times e_n)$, $A[\![r]\!] \coloneqq F^p_{c_{\bar{e}}} q_r$. For each n-ary function symbol $h\colon e_1, \ldots, e_n \to e$, we define $A[\![h]\!] \coloneqq f_h \circ a_{\bar{e}}^{-1}$ where $a_{\bar{e}}\colon F^o(c_{\bar{e}}) \to A[\![e_1]\!] \times \cdots \times A[\![e_n]\!]$ is the change-in-product isomorphism. Similarly, for each n-ary relation symbol $r \subseteq e_1, \ldots, e_n$, we define $A[\![r]\!] \coloneqq Q(a_{\bar{e}})^{-1} \circ F^p_{c_{\bar{e}}}(q_r)$.

By induction, one may prove that for each term $M : e\,[\Gamma]$, where $\Gamma = x_1 : e_1, \ldots, x_n : e_n$, that $A[\![M : e\,[\Gamma]]\!] = F^o f_{S[\![M:e\,[\Gamma]]\!]} \circ a_{\bar{e}}^{-1}$, and for each formula-in-context $\phi\,[\Gamma]$, that $A[\![\phi[\Gamma]]\!] = Q(a_{\bar{e}}^{-1}) \circ F^p_{c_{\bar{e}}}(q_{S[\![\phi[\Gamma]]\!]})$. It follows that A satisfies each equation-in-context in T, and if $\phi_1, \ldots, \phi_n \vdash \phi_{n+1}\,[\Gamma]$ is in T, then

$$\begin{aligned}
& S[\![\phi_1 \otimes \ldots \otimes \phi_n\,[\Gamma]]\!] \leq S[\![\phi_{n+1}[\Gamma]]\!] \\
\implies\ & K_c^p(q_{S[\![\phi_1 \otimes \ldots \otimes \phi_n[\Gamma]]\!]}) \leq K_c^p(q_{S[\![\phi_{n+1}[\Gamma]]\!]}) \\
\implies\ & F_c^p(q_{S[\![\phi_1 \otimes \ldots \otimes \phi_n[\Gamma]]\!]}) \leq F_c^p(q_{S[\![\phi_{n+1}[\Gamma]]\!]}) \\
\implies\ & F_c^p(q_{S[\![\phi_1[\Gamma]]\!]}) \otimes \ldots \otimes F_c^p(q_{S[\![\phi_n[\Gamma]]\!]}) \leq F_c^p(q_{S[\![\phi_{n+1}[\Gamma]]\!]}) \\
\implies\ & A[\![\phi_1[\Gamma]]\!] \otimes \ldots \otimes A[\![\phi_n[\Gamma]]\!] \leq A[\![\phi_{n+1}[\Gamma]]\!].
\end{aligned}$$

It follows that A is a T-model. By construction, we have $K \circ \overline{A} \circ \iota = F$.

Proposition 9. *The classes $\mathscr{E}$ and $\mathscr{M}$ form a factorization system in* $\mathbf{FA}_{\mathcal{L}}$.

Proof. Note that both $\mathscr{M}$ and $\mathcal{E}$ contain all the isomorphisms in $\mathbf{FA}_{\mathcal{L}}$ and are closed under composition. From Part 1 of Theorem 7, each $F \in \mathrm{Mor}(\mathbf{FA}_{\mathcal{L}})$ can be factored as $m \circ e$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$. Now consider the following commuting solid diagram of morphisms in $\mathbf{FA}_{\mathcal{L}}$:

$$\begin{array}{ccc}
\cdot & \xrightarrow{\ u\ } & \cdot \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\
\cdot & \overset{\exists! w}{\dashrightarrow} & \cdot \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\
\cdot & \xrightarrow{\ v\ } & \cdot
\end{array}$$

We want to to show there is a unique morphism w making the small squares commute. Note that $\mathrm{Th}(e) \subseteq \mathrm{Th}(vme) = \mathrm{Th}(m'e'u) = \mathrm{Th}(e'u)$, and since e is bijective on objects, $\ker(e) \leq \ker(e'u)$. From part 2 of Theorem 7, there exists a unique morphism w such that $we = e'u$. Moreover it is straightforward to verify that the morphisms in $\mathscr{E}$ are epimorphisms in $\mathbf{FA}_{\mathcal{L}}$ and so e is an epimorphism. From this fact and the fact that the large rectangle and small top square commute, the small bottom square must commute as well. $\square$

4.3 Algebraic Characterizations of Logical Closure Operators

Let Sg be a single-sorted algebraic signature, Alg_{Sg} the category of set-valued Sg-algebras and Eq_{Sg} the class of Sg-equations. Then there is an adjunction

$$\mathscr{P}(Eq_{Sg}) \underset{Eq(\cdot)}{\overset{\mathrm{Alg}(\cdot)}{\rightleftarrows}} \mathscr{P}(\mathrm{Alg}_{Sg})^{op} \quad \bot \tag{4.1}$$

Where $\mathscr{P}$ is the operation of taking the powerset and both $\mathscr{P}(Eq_{Sg})$ and $\mathscr{P}(\mathrm{Alg}_{Sg})$ are ordered by inclusion. $\mathrm{Alg}(\cdot)$ takes a class of algebras to the collection of all Sg-equations they satisfy and $Eq(\cdot)$ takes a collection of Sg-equations to the class of all Sg-algebras that satisfy them. This adjunction determines a closure operator $Eq \circ \mathrm{Alg}$ on $\mathscr{P}(Eq_{Sg})$, and a closure operator $\mathrm{Alg} \circ Eq$ on $\mathscr{P}(\mathrm{Alg}_{Sg})$. From Birkhoff's Completeness Theorem for equational logic, $Eq \circ \mathrm{Alg}(\Theta)$ is the closure of Θ under the derivation rules of equational logic. Birkhoff's HSP Theorem asserts $\mathrm{Alg} \circ Eq(\cdot)$ is $\mathbb{HSP}(\cdot)$ where $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$ are the operations which close a class of algebras under the operations of taking homomorphic images, subalgebras and products respectively Birkhoff (1935).

In the categorical semantics for equational logic, Alg_{Sg} is equivalent to $\mathbf{FP}(\mathcal{C}_{Sg}, \mathbf{Set})$ where $\mathbf{FP}$ is the 2-category of categories with finite products and product preserving functors. Birkhoff's HSP Theorem interpreted in the categorical semantics says $\mathcal{Y} \subseteq \mathrm{Ob}(\mathbf{FP}(\mathcal{C}_{Sg}, \mathbf{Set}))$ is an equational class iff $\mathbb{HSP}(\mathcal{Y}) = \mathcal{Y}$, where

1. $\mathbb{P}(\mathcal{Y})$ is the smallest class containing $\mathcal{Y}$ stable under products.
2. $\mathbb{S}(\mathcal{Y})$ is the smallest class containing $\mathcal{Y}$ such that whenever $F \in \mathcal{Y}$ and $\eta\colon K \Rightarrow F$ is a monomorphism, then $K \in \mathbb{S}(\mathcal{Y})$.
3. $\mathbb{H}(\mathcal{Y})$ is the smallest class containing $\mathcal{Y}$ such that whenever $F \in \mathcal{Y}$, and $\eta\colon F \Rightarrow K$ is a regular epimorphism, then $K \in \mathbb{H}(\mathcal{Y})$.

For an analogous characterization for a multi-sorted algebraic signature see Adámek et al. (2011). Since we are considering first-order logics, we are interested in a similar result where $\mathscr{P}(Eq_{Sg})$ is replaced by the complete lattice of all Sg-theories Th_{Sg} and algebras by first-order structures. If Sg is single-sorted and $\mathscr{L}$ is a language for classical first-order logic, then the Tarskian Sg-structures of classical first-order logic can be identified with morphisms $F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathbf{Set}, \mathscr{P})$, where $\mathscr{P}$ is the preimage functor. It is well-known that closing a class of Tarskian structures under their common theory, is equivalent to closing the class under ultraproducts, isomorphic copies and ultraroots (Hodges, 1993, p. 454). In what follows, we develop an analogous result for an arbitrary first-order logic $\mathcal{L}$ and its general prop-categorical semantics.

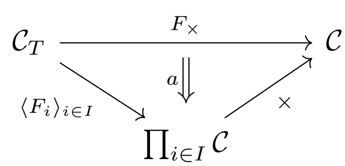
Let $\mathrm{Mod}_{Sg}^{\mathcal{L}}$ be the collection of all Sg-structures in $\mathbf{FA}_{\mathcal{L}}$. Then we are interested in characterizing $\mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}$, where

$$\mathrm{Th}_{Sg} \underset{\mathrm{Th}(\cdot)}{\overset{\mathrm{Mod}_{(\cdot)}}{\rightleftarrows}} \perp \mathscr{P}(\mathrm{Mod}_{Sg}^{\mathcal{L}})^{op}.$$

We take $\mathrm{Mod}_{Sg}^{\mathcal{L}}$ to be the collection of all morphisms $F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}, P)$ such that $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})$. Then for each $\mathcal{Y} \subseteq \mathrm{Mod}_{Sg}^{\mathcal{L}}$, $\mathrm{Th}(\mathcal{Y})$ is the Sg-theory whose assertions are $\{a \in A_{Sg} : \forall F \in \mathcal{Y}, G[\![a]\!] \in \ker F\}$ and for each $T \in \mathrm{Th}_{Sg}$, $\mathrm{Mod}_T = \{F \in \mathrm{Mod}_{Sg}^{\mathcal{L}} : G[\![T]\!] \subseteq \ker F\}$. If $\mathbf{FA}_{\mathcal{L}}$ forms a complete semantics for $\mathcal{L}$, $\mathrm{Th}(\mathrm{Mod}_T)$ is the Sg-theory whose assertions are $\{a \in \mathrm{A}_{Sg} : T \vdash_{\mathcal{L}} a\}$. For $\mathcal{Y} \subseteq \mathrm{Mod}_{Sg}^{\mathcal{L}}$, we wish to characterize $\mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$. Let $\{F_i\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}_i, P_i)\}_{i\in I} \subseteq \mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$. Then each $(\mathcal{C}_i, P_i) \in \mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})$, and from Proposition 8, $(\prod \mathcal{C}_i, \prod P_i) \in \mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})$, and so $\langle F_i \rangle_{i\in I} \in \mathrm{Mod}_{Sg}^{\mathcal{L}}$. Moreover, $\langle F_i \rangle_{i\in I}$ satisfies each assertion satisfied by all F_i and so $\langle F_i \rangle_{i\in I} \in \mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$. We define the **(external) product of** $\{F_i\}_{i\in I}$ to be $\langle F_i \rangle_{i\in I}$ and let $\mathbb{P}(\mathcal{Y})$ denote the closure of $\mathcal{Y}$ under external products. And so, $\mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$ is stable under taking external products.

Let $F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}, P) \in \mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$. For each $(\mathcal{D}, Q) \in \mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})$ and $H \in \mathbf{FA}_{\mathcal{L}}((\mathcal{C}, P), (\mathcal{D}, Q))$, $H \circ F \in \mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$. We say $H \circ F$ is an **(external) homomorphic image of** F and let $\mathbb{H}(\mathcal{Y})$ denote the closure of $\mathcal{Y}$ under external homomorphic images. If $H\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{E}, R)$ and $\iota\colon (\mathcal{E}, R) \to (\mathcal{C}, P)$ are morphisms in $\mathbf{FA}_{\mathcal{L}}$ such that $F = \iota \circ H$ and ι is a subprop-morphism, then we call H an **(external) submodel of** F and let $\mathbb{S}(\mathcal{Y})$ denote the closure of $\mathcal{Y}$ under taking external submodels. Since $G[\![\mathrm{Th}(\mathcal{Y})]\!] \subseteq \ker F$, $\iota \circ H = F$ and ι is a subprop-morphism, $G[\![\mathrm{Th}(\mathcal{Y})]\!] \subseteq \ker H$ and so $H \in \mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$.

Remark 3. If we consider T-algebras as product preserving functors $\{F_i\colon \mathcal{C}_T \to \mathcal{C}\}_{i\in I}$, where $\mathcal{C}$ has arbitrary products then,



commutes up to a change-in-product natural isomorphism a, where $\times$ is the right adjoint to the diagonal functor $\mathrm{Diag}\colon \mathcal{C} \to \prod_{i\in I} \mathcal{C}$ and $F_\times$ is the usual "internal" product of the algebras $\{F_i\}_{i\in I}$. Since $\times$ is faithful, $\langle F_i \rangle_{i\in I}$ and $F_\times$ satisfy the same equations. Then the external product $\langle F_i \rangle_{i\in I}$ satisfies the same equational theory as the internal product $F_\times$. For first-order models, the internal product need not correspond to the external. The issue is that for $c \in \mathrm{Ob}(\prod_{i\in I} \mathcal{C})$, in general, $\prod_{i\in I} P(c_i) \ncong P(\times(c))$.

Theorem 8. *Let $\mathcal{L}$ be a logic and Sg a small signature (not a proper class). For each $\mathcal{Y} \subseteq \mathrm{Mod}_{Sg}^{\mathcal{L}}$,*

$\mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y}) = \mathbb{HSP}(\mathcal{Y})$.

Proof. If $\mathcal{Y}$ is the class of all Sg-$\mathcal{L}$ structures of a given theory T, then we showed $\mathcal{Y}$ is closed under taking (external) products, submodels and homomorphic images and so $\mathbb{HSP}(\mathcal{Y}) \subseteq \mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$.

In the other direction, let $T = \mathrm{Th}(\mathcal{Y})$, $F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}, P)$ be a T-model, where $(\mathcal{C}, P) \in \mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})$ and let $\{T_i\}_{i \in I}$ be the collection of all Sg-$\mathcal{L}$ theories such that there exists $F_i\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}_i, P_i) \in \mathcal{Y}$ such that $\mathrm{Th}(F_i) = T_i$. Then $T = \cap_{i \in I} T_i$, and define $\epsilon_T := \overline{G}\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}_T, P_T)$, where $G \in (\mathcal{C}_T, P_T)$ is the generic T-model. Then $\ker \epsilon_T \leq \ker \langle F_i \rangle_I$, and so from Theorem 7, there exists $\iota\colon (\mathcal{C}_T, P_T) \to (\prod_I \mathcal{C}_i, \prod_I P_i)$ such that

$$\begin{array}{ccc} (\mathcal{C}_{Sg}, P_{Sg}) & \xrightarrow{\quad \langle F_i \rangle_I \quad} & (\prod_I \mathcal{C}_i, \prod_I P_i) \\ & {\scriptstyle \epsilon_T} \searrow \qquad \nearrow {\scriptstyle \iota} & \\ & (\mathcal{C}_T, P_T) & \end{array}$$

commutes. Moreover, since $\mathrm{Th}(\langle F_i \rangle_I) = T$, ι is a subprop-morphism. From Proposition 6, $(\mathcal{C}_T, P_T) \in \mathrm{Ob}(\mathbf{FA}_{\mathcal{L}})$ and so $\epsilon_T \in \mathbb{SP}(\mathcal{Y}) = \mathcal{Y}$. Since F is a T-model, $\ker \epsilon_T \leq \ker F$. From Theorem 7, there exists a morphism $H\colon (\mathcal{C}_T, P_T) \to (\mathcal{C}, P)$, such that

$$\begin{array}{ccc} (\mathcal{C}_{Sg}, P_{Sg}) & \xrightarrow{\quad F \quad} & (\mathcal{C}, P) \\ & {\scriptstyle \epsilon_T} \searrow \qquad \nearrow {\scriptstyle H} & \\ & (\mathcal{C}_T, P_T) & \end{array}$$

commutes and so $F \in \mathbb{HSP}(\mathcal{Y})$. Therefore, $\mathbb{HSP}(\mathcal{Y}) \supseteq \mathrm{Mod}_{(\cdot)} \circ \mathrm{Th}(\mathcal{Y})$. $\square$

Let Log be the partial order of all logics. We now consider the following adjunction:

$$\mathrm{Log} \underset{\vDash_{(\cdot)}}{\overset{\mathrm{Ob}(\mathbf{FA}_{(\cdot)})}{\rightleftarrows}} \perp \mathscr{P}(\mathrm{Ob}(\mathbf{FA}))^{op} \tag{4.2}$$

In order to give a characterization of the closure operator $\mathrm{Ob}(\mathbf{FA}_{\vDash_{(\cdot)}})$, we first restrict the logics in Log to some fixed signature Sg. Let Log^{Sg} be the collection of all logics restricted to Sg-assertions and we denote the corresponding restrictions of $\mathrm{Ob}(\mathbf{FA}_{(\cdot)})$ and $\vDash_{(\cdot)}$, $\mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)})$ and $\vDash^{Sg}_{(\cdot)}$ respectively.

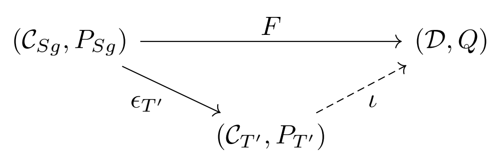
$$\mathrm{Log}^{Sg} \underset{\vDash^{Sg}_{(\cdot)}}{\overset{\mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)})}{\rightleftarrows}} \perp \mathscr{P}(\mathrm{Ob}(\mathbf{FA}))^{op} \tag{4.3}$$

In the context of (the untyped) equational logic Adjunction 4.3 corresponds to the following: Let $\mathscr{L}_\omega$ be an algebraic signature and V a set of variables of cardinality λ. Let Eq_V be the collection of $\mathscr{L}_\omega$-equations over V which we identify with Fm_V^2, the square of the formula-algebra. A collection of $\mathscr{L}_\omega$-algebras $\mathcal{A}$ determines a structural closure operator $\vDash_{\mathcal{A}}$, defined by $\Theta \vDash_{\mathcal{A}} \epsilon = \delta$ if and only if for all homomorphisms $h\colon Fm_V \to A$, where $A \in \mathcal{A}$, if $\Theta \subseteq \ker h$, then $\epsilon = \delta \in \ker h$. Observe that $\vDash_{\mathbb{SP}(\mathcal{A})} = \vDash_{\mathcal{A}}$, but $\mathbb{SP}(\mathcal{A})$ may not be the largest collection of algebras defining the same consequence. In Blok and Jónsson (2006) it is shown that the largest such class of $\mathscr{L}_\omega$-algebras is $\mathbb{U}_\lambda\mathbb{SP}(\mathcal{A})$, where $B \in \mathbb{U}_\lambda(\mathcal{A})$ if every λ-generated subalgebra of B is in $\mathcal{A}$.

Taking the operation $\mathbb{U}_\lambda$ as inspiration, where λ up to renaming specifies the propositional signature, given a signature Sg, and $\mathcal{X}$ a collection of prop-categories, we define $\mathbb{U}_{Sg}(\mathcal{X})$ so that $(\mathcal{D}, Q) \in \mathbb{U}_{Sg}(\mathcal{X})$, if each classifying sub-prop-category $(\mathcal{C}_T, P_T)$ of $(\mathcal{D}, Q)$ is in $\mathcal{X}$, where $\mathrm{Sg}(T) = Sg$. For $\mathcal{X} \subseteq \mathrm{Ob}(\mathbf{FA})$, we take $\mathbb{P}(\mathcal{X})$ to be the closure of $\mathcal{X}$ under taking products and $\mathbb{S}(\mathcal{X})$ to be the closure of $\mathcal{X}$ under taking sub-prop-categories.

Theorem 9. *For $\mathcal{X} \subseteq \mathrm{Ob}(\mathbf{FA})$, and Sg a small signature (Not a proper class),* $\mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X}) = \mathbb{U}_{Sg}\mathbb{SP}(\mathcal{X})$.

Proof. From Proposition 6 and Proposition 8, $\mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X})$ is stable under $\mathbb{S}$ and $\mathbb{P}$. Let $(\mathcal{D}, Q) \in \mathbb{U}_{Sg}(\mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X}))$ and suppose $T \vDash_{\mathcal{X}} a$. Let $F\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{D}, Q)$ be a T-model in $(\mathcal{D}, Q)$ and $T' = \mathrm{Th}(F)$. From Theorem 7 there exists $\iota\colon (\mathcal{C}_{T'}, P_{T'}) \to (\mathcal{D}, Q)$ which makes the diagram



commute. Moreover, since $\mathrm{Th}(F) = \mathrm{Th}(\epsilon_{T'})$, ι is a subprop-morphism. Then $(\mathcal{C}_{T'}, P_{T'}) \in \mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X})$, because $(\mathcal{D}, Q) \in \mathbb{U}_{Sg}(\mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X}))$. Since $T \vDash_{\mathcal{X}} a$ and $G[\![T]\!] \subseteq \ker \epsilon_{T'}$, $G[\![a]\!] \in \ker \epsilon_{T'}$. Thus $G[\![a]\!] \in \ker F$ and so $T \vDash_{(\mathcal{D},Q)} a$. It follows that $(\mathcal{D}, Q) \in \mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X})$ and so $\mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X})$ is stable under $\mathbb{U}_{Sg}$. Therefore, $\mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X}) \supseteq \mathbb{U}_{Sg}\mathbb{SP}(\mathcal{X})$.

Now let $(\mathcal{D}, Q) \in \mathrm{Ob}(\mathbf{FA}^{Sg}_{(\cdot)}) \circ \vDash^{Sg}_{(\cdot)} (\mathcal{X})$ and consider a subprop-morphism $\iota\colon (\mathcal{C}_T, P_T) \to (\mathcal{D}, Q)$ such that $\mathrm{Sg}(T) = Sg$. If $a \in A_{Sg} \setminus \mathrm{A}(T)$, then the generic T-model G in $(\mathcal{C}_T, P_T)$ does not satisfy a and so $T \nvDash_{(\mathcal{C}_T, P_T)} a$. Since $\vDash_{(\mathcal{C}_T,P_T)} \supseteq \vDash_{(\mathcal{D},Q)} \supseteq \vDash_{\mathcal{X}}$ for each $a \notin T$, there exists a T-model $F_a\colon (\mathcal{C}_{Sg}, P_{Sg}) \to (\mathcal{C}_a, P_a)$ such that $(\mathcal{C}_a, P_a) \in \mathcal{X}$ and $G[\![a]\!] \notin \ker F_a$. Then $\ker(\epsilon_T) \leq \ker(\langle F_a \rangle_{a \notin T})$ and so there exists a morphism $\lambda\colon (\mathcal{C}_T, P_T) \to \prod_{a \notin T}(\mathcal{C}_a, P_a)$ such that

$$\begin{array}{ccc} (\mathcal{C}_{Sg}, P_{Sg}) & \xrightarrow{\langle F_a \rangle_{a \notin T}} & \prod_{a \notin T}(\mathcal{C}_a, P_a) \\ & \searrow^{\epsilon_T} \quad \nearrow^{\lambda} & \\ & (\mathcal{C}_T, P_T) & \end{array}$$

commutes. Moreover, $\mathrm{Th}(\epsilon_T) = \mathrm{Th}(\langle F_a \rangle_{a \notin T})$ and so λ is a subprop-morphism. Thus $(\mathcal{C}_T, P_T) \in \mathbb{SP}(\mathcal{X})$ and so $(\mathcal{D}, Q) \in \mathbb{U}_{Sg}\mathbb{SP}(\mathcal{X})$. □

Adjunction 4.3 is relevant when one wants to consider a logic over a fixed signature. However, in categorical logic, where internal logic is used, it makes sense to have a logic be independent of a particular signature. Thus we also provide a characterization of the closure $\mathrm{Ob}(\mathbf{FA}_{\vDash_{(\cdot)}})$ in Adjunction 4.3. For $\mathcal{X} \subseteq \mathrm{Ob}(\mathbf{FA})$, we define $\mathbb{U}(\mathcal{X})$ by $(\mathcal{D}, Q) \in \mathbb{U}(\mathcal{X})$, if each classifying sub-prop-category $(\mathcal{C}_T, P_T)$ of $(\mathcal{D}, Q)$ is in $\mathcal{X}$. That is, $(\mathcal{D}, Q) \in \mathbb{U}(\mathcal{X})$ if and only if for each signature Sg, $(\mathcal{D}, Q) \in \mathbb{U}_{Sg}(\mathcal{X})$. For the following result, we must assume all prop-categories and signatures are small.

Theorem 10. $\mathrm{Ob}(\mathbf{FA}_{\vDash_{(\cdot)}}) = \mathbb{USP}(\cdot)$.

Proof. Clearly $\mathrm{Ob}(\mathbf{FA}_{\vDash_{\mathcal{X}}})$ is stable under $\mathbb{S}$ and $\mathbb{P}$. Suppose $(\mathcal{D}, Q) \in \mathbb{U}(\mathrm{Ob}(\mathbf{FA}_{\vDash_{\mathcal{X}}}))$, and let T be the theory of $(\mathcal{D}, Q)$. Then $(\mathcal{C}_T, P_T) \in \mathbf{FA}_{\vDash_{\mathcal{X}}}$ and since $\vDash_{(\mathcal{D},W)} = \vDash_{(\mathcal{C}_T, P_T)}$, $(\mathcal{D}, Q) \in \mathrm{Ob}(\mathbf{FA}_{\vDash_{\mathcal{X}}})$.

Now suppose $(\mathcal{D}, Q) \in \mathrm{Ob}(\mathbf{FA}_{\vDash_{\mathcal{X}}})$. Then $\vDash_{(\mathcal{D},Q)} \supseteq \vDash_{\mathcal{X}}$. Following the proof of Theorem 9, one shows for each signature Sg, $(\mathcal{D}, Q) \in \mathbb{U}_{Sg}\mathbb{SP}(\mathcal{X})$ and so $(\mathcal{D}, Q) \in \mathbb{USP}(\mathcal{X})$. □

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