ON THE MODULI SPACE OF CONSTANT SCALAR CURVATURE KÄHLER METRICS ON COMPLEX SURFACES

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Chapter I

Introduction

The program of finding canonical metrics on surfaces began with Riemann, Poincaré, and Hilbert more than a century ago. Since then, many developments have been made including expanding the search to higher dimensional manifolds and the addition of numerous special metrics to consider. In particular, constant scalar curvature Kähler (cscK) and extremal metrics are some of the most prominent types of special metrics. Extremal metrics were first introduced by Calabi [Calabi, 1979] and are metrics which are critical points of a certain functional. In essence, extremal metrics are the umbrella that cscK and Kähler-Einstein sit under.

This thesis will focus on ruled surfaces and the existence/non-existence of constant scalar curvature Kähler metrics and extremal metrics, and a large portion will be done with Asymptotically Locally Euclidean (ALE) metrics in mind. One might be interested in ALE manifolds for a number of reasons. First, ALE manifolds are noteworthy in the realm of physics in the study of gravitational instantons – 4-dimensional complete Riemannian manifolds satisfying the vacuum Einstein equations. An important invariant of Asymptotically Euclidean (AE) manifolds known as the ADM mass was developed by Arnowitt-Deser-Misner [Arnowitt et al., 1961] with physics in mind. Both Bartnik [Bartnik, 1986] and Chruściel [Chruściel, 1985] developed the notion of mass further and uncovered some of its geometric underpinnings including showing that mass is independent of choice of coordinates at infinity. LeBrun [LeBrun, 1988] then sparked further intrigue when he produced scalar-flat ALE surfaces with negative mass– thus disproving the generalized positive action conjecture of Hawking and Pope [Hawking and Pope, 1978].

The metric fall-off conditions allow ALE manifolds to be relatively tame at infinity. This makes them an attractive class of non-compact manifolds to study. One can see this tameness used to great effect in the work of Arezzo and Pacard [Arezzo and Pacard, 2006] where they provide a method for constructing cscK metrics on blow-ups of compact, cscK manifolds admitting no non-zero holomorphic vector fields that vanish somewhere. They accomplish this by replacing a neighborhood of the blow-up point with a scalar-flat ALE space. The ALE condition plays a critical role in this surgery as a particular decay rate is required to control the analysis on the gluing overlap.

ALE manifolds have continued to be a large topic of study. Shortly after LeBrun's counterexamples, Kronheimer [Kronheimer, 1989b, Kronheimer, 1989a] produced Ricci-flat ALE 4-manifolds resulting from hyper-Kähler quotients and then showed that any simply-connected ALE hyper-Kähler 4-manifold must be isometric to one of his hyper-Kähler quotients. Șuvaina [Şuvaina, 2012] completes the classification of Ricci-flat ALE 4-manifolds by extending Kronheimer's approach to non simply-connected manifolds. Șuvaina-Rasdeaconu-Hein [Hein et al., 2020] characterize the complex structures of ALE manifolds given the group at infinity.

I.1 Summary of Results

The first result we give deals with presenting a new construction of ALE scalar-flat metrics on the line bundles $\mathcal{O}(-k)$. Tønnesen-Friedman [Tønnesen-Friedman, 1998] devised an approach to construct extremal metrics on a ruled surface where the base has genus at least 2. Her work is later expanded upon by Apostolov-Calderbank-Gauduchon-Tønnesen-Friedman in [Apostolov et al., 2008]. Hwang-Singer [Hwang and Singer, 2002] tackled a similar problem by developing a general method for constructing circle-invariant Kähler metrics using what's known as the momentum profile.

The momentum construction may be used to construct extremal metrics on the Hirzebruch surfaces Σ_k and on many well-known ALE metrics. However, we have found a way to obtain the ALE metrics as limits of the extremal metrics on Σ_k. We first show our method for $k = 1$, and obtain the following result:

Theorem A (Rizzo)**.** *There exists a family of extremal metrics on the* 1*st Hirzebruch surface,* Σ_1*, which converges to a scalar-flat, asymptotically Euclidean metric on* $\mathcal{O}(-1)$*. Moreover, this limit metric is in fact the Burn's metric.*

This can be generalized with minor additional difficulties to $k \geq 2$.

Theorem B (Rizzo)**.** *For each* $k \geq 2$*, there exists a family of extremal metrics on the* k*th Hirzebruch surface,* Σ_k*, which converges to a scalar-flat, asymptotically locally Euclidean metric on* $\mathcal{O}(-k)$*. When* $k = 2$*, this limit metric is Ricci-flat and is in fact the Eguchi-Hanson metric. When* $k > 2$*, the limit metric recovers LeBrun's negative mass metrics on* $\mathcal{O}(-k)$*.*

These results are pleasing in a number of ways. First, it aligns with one's geometric intuition about the situation. The limiting process amounts to letting the area of the infinity divisor of Σ_k

tend to infinity. This is, in some ways, how one might envision the opposite of the compactification procedure described in [Hein and LeBrun, 2016]. Second, it relates the existence of extremal metrics in the compact case, a heavily studied setting, to scalar-flat ALE metrics, a somewhat more mysterious setting. Put more concretely, we have a family of extremal metrics on Σ_k, a space where the existence of cscK metrics is obstructed, and by taking a limit, we are able to obtain a scalar flat metric on $O(-k)$. One could hope that our approach (or one of a similar flavor) might provide a method for finding the "right" choices for structures on ALE manifolds – the original impetus for this work.

As we mentioned above, there are no cscK metrics in any class on Σ_k. Nevertheless, if one blows up the manifolds enough, then the obstructions disappear. Furthermore, one only needs to consider the cases of blow-ups of $\mathbb{CP}^1 \times \mathbb{CP}^1$, rather than Σ_k, as many of the complex structures become biholomorphic after blowing up. This opens up the possibility to analyze the moduli space of Kähler metrics on these blow-ups by exhibiting classes where existence/non-existence of cscK metrics is known.

Separate from our construction above, we also tackle a problem of non-existence of cscK metrics on ruled surfaces. Existence of cscK metrics on certain blow-ups may be shown using parabolic stability due to Rollin-Singer [Rollin and Singer, 2009b]. We provide a complementary viewpoint by using destabilizing curves to show:

Theorem C (Rizzo)**.** *Let X be the 6-fold blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ corresponding to the parabolic structure consisting of 3 generic marked points with weight $\frac{1}{2}$, and let E_i, F_i, G_i denote the exceptional divisors of the blow-ups. Let:*

$$\Omega = a[H] + a[K] - e[E_1] - \alpha e[E_2] - f_1[F_1] - f_2[F_2] - g_1[G_1] - g_2[G_2]$$

For any $a > 0$, there are positive constants f_1, f_2, g_1, g_2, e_0 satisfying:

$$\frac{15}{8a} < \frac{4a - f_2 - g_2}{2a^2 - 2f_1^2 + 2f_1 f_2 - f_2^2 - 2g_1^2 + 2g_1 g_2 - g_2^2}$$

such that for any $\alpha \in \left(1, \frac{5}{4}\right]$ and $e < e_0$, the class Ω does not admit a cscK metric.

One reason why this complementary viewpoint is so nice is that it's rare for one to have a clear picture of both existence and non-existence of cscK metrics on the same complex manifold. Although we do not provide an entire picture, Theorem C is an initial step in the process.

Chapter II

Existence of Scalar-flat, ALE Metrics

II.1 General background

In this section, we will provide background on the concepts which are critical for our construction of scalar-flat, ALE metrics. We will begin setting the stage by discussing extremal metrics and the role they fill in the program of finding canonical metrics. We will then provide a brief overview of what ALE metrics are and discuss ADM mass – an important invariant for ALE Kähler manifolds.

II.1.1 Extremal Metrics

A major program in the field of complex geometry is determining canonical metrics on a given complex manifold; that is – finding the "best" metric in any given Kähler class $\Omega \in H^2(M, \mathbb{R})$. One candidate for such a metric would be a Kähler-Einstein metric i.e. a metric ω such that the Ricci form $\mathrm{Ric}(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$. However, as it turns out, obtaining a Kähler-Einstein metric in an arbitrary Kähler class is an ill-fated pursuit.

For example, let $M \subset \mathbb{CP}^3$ be a hypersurface of degree $d \geq 5$, and let $\pi\colon \tilde{M} \to M$ be the blow-up of M at a point p. The Kähler-Einstein condition tells us that:

$$-c_1(K_{\tilde{M}}) = c_1(\tilde{M}) = \frac{1}{2\pi}\,[\lambda\omega]$$

Therefore, the canonical bundle of $\tilde{M}$, $K_{\tilde{M}}$, must be either positive, negative, or zero. However, the adjunction formula tells us that $\tilde{M}$ has canonical bundle:

$$K_{\tilde{M}} = \pi^* K_M \otimes \mathcal{O}(E)$$

where E denotes the exceptional divisor of the blow-up, and as M is simply connected, we identify a line bundle with its first Chern class. Note that we are also using the notation $\mathcal{O}(E)$ to denote the line bundle associated to the divisor E. The adjunction formula tells us once more that:

$$K_M = K_{\mathbb{CP}^3}|_M \otimes N_M = \mathcal{O}(-4)|_M \otimes \mathcal{O}(d)|_M = \mathcal{O}(d-4)|_M > 0$$

Since $K_M > 0$ and $[E] < 0$, $K_{\tilde{M}}$ can neither be positive, negative, nor zero. Hence, $\tilde{M}$ cannot admit a Kähler-Einstein metric.

Therefore, it is necessary to consider a more general class of metrics. Calabi [Calabi, 1979] introduced the notion of extremal metrics.

Definition II.1.1. Let M be a compact Kähler manifold of complex dimension n and $\Omega \in H^2(M, \mathbb{R})$ a Kähler class. An extremal metric on M in the class Ω is a critical point of the functional

$$\mathrm{Cal}(\omega) = \int_M S(\omega)^2 \omega^n$$

for $\omega \in \Omega$, where $S(\omega)$ denotes the scalar curvature of the metric ω.

One can readily see that constant scalar curvature Kähler (cscK) metrics are extremal by first rewriting the Calabi functional:

$$\begin{aligned}\mathrm{Cal}(\omega) = \int_M S(\omega)^2 \omega^n &= \int_M S^2(\omega)\omega^n - 2\int_M \hat{S}^2\omega^n + 2\int_M \hat{S}^2 \omega^n \\ &= \int_M (S^2(\omega) - 2S(\omega)\hat{S} + \hat{S}^2)\omega^n + \int_M \hat{S}^2\omega^n \\ &= \int_M (S(\omega) - \hat{S})^2\omega^n + \int_M \hat{S}^2\omega^n \end{aligned} \tag{II.1}$$

where $\hat{S}$ denotes the average scalar curvature. Then since the first term in (II.1) is non-negative and the second is constant within a Kähler class, cscK metrics minimize the Calabi function since $S(\omega) = \hat{S}$.

However, not all extremal metrics are cscK. Our constructions in II.2 will provide examples of this phenomenon. An important characterization of extremal metrics comes from the Euler-Lagrange equation. For $f \colon M \to \mathbb{R}$, let $\mathrm{grad}^{1,0} f = g^{j\bar{k}}\partial_{\bar{k}} f$. Then:

Theorem II.1.2 ([Calabi, 1979])**.** *A metric ω on M is extremal if and only if grad$^{1,0}S(\omega)$ is a holomorphic vector field.*

Therefore, the task of determining whether a metric is extremal becomes a problem of differential equations.

II.1.2 ALE Manifolds

The compact case is relatively well understood, so we will focus our attention on non-compact manifolds. Namely, we will be focusing on ALE manifolds. ALE manifolds are Riemmanian manifolds which asymptotically locally approximate the standard Euclidean metric on $\mathbb{R}^n$ with a sufficiently nice decay:

Definition II.1.3. A complete Riemannian manifold (M^n, g) is asymptotically locally Euclidean (ALE) if there exists a compact subset $K \subset M$ such that $M \setminus K$ has finitely many connected components $(M \setminus K)_i$, $i \in I$, and for each $i \in I$ there exists a finite subgroup $\Gamma_i \subseteq SO(n)$ acting freely on $\mathbb{R}^n \setminus B_R(0)$, and a diffeomorphism:

$$\psi_i\colon (M \setminus K)_i \to (\mathbb{R}^n \setminus B_R(0))/\Gamma_i$$

such that for some $\mu > 0$,

$$\nabla^k \left(\psi_*(g) - g_{\text{Euc}}\right) = O(r^{-\mu-k})$$

However, we will be focusing on ALE Kähler surfaces, so the above definition can be rephrased:

Definition II.1.4. A Kähler surface (M, g, J) is asymptotically locally Euclidean (ALE) if there exists a compact subset $K \subset M$ such that $M \setminus K$ has finitely many connected components $(M \setminus K)_i$, $i \in I$, and for each $i \in I$ there exists a finite subgroup $\Gamma_i \subseteq U(2)$ acting freely on $\mathbb{C}^2 \setminus B_R(0)$, and a diffeomorphism:

$$\psi_i\colon (M \setminus K)_i \to (\mathbb{C}^2 \setminus B_R(0))/\Gamma_i$$

such that for some $\mu > 0$,

$$\nabla^k \left(\psi_{i*}(g) - g_{\text{Euc}}\right) = O(r^{-\mu-k})$$

Remark. *The best rate of decay that one can hope for in the Kähler surface setting is $O(r^{-4})$. This occurs if and only if the surface is Ricci-flat.*

II.1.3 ADM Mass

The mass of an end of an ALE Riemannian manifold is an invariant coming from apparent mass in general relativity. It has the following somewhat involved definition:

Definition II.1.5. The mass of an end of an ALE Riemannian manifold (M^n, g) is the quantity:

$$\mathfrak{m}(M, g) = \lim_{\rho\to\infty} \frac{\mathbf{\Gamma}\left(\frac{n}{2}\right)}{4(n-1)\pi^{\frac{n}{2}}} \int_{S_\rho/\Gamma_j} (g_{ij,i} - g_{ii,j})\nu^j d\alpha_{\text{Euc}}$$

where the subscripts after the comma represent derivatives in the given asymptotic coordinate system, S_ρ is the Euclidean coordinate sphere of radius ρ, $d\alpha_{\text{Euc}}$ is the $(n-1)$-dimensional volume form induced on the sphere by the Euclidean metric, Γ_j is as in II.1.3, ν is the outward-pointing Euclidean unit normal vector, and $\mathbf{\Gamma}$ is the gamma function.

Since we are concerned with ALE Kähler surfaces, we have that $n = 4$ and the formula simplifies:

$$\mathfrak{m}(M, g) = \lim_{\rho\to\infty} \frac{1}{12\pi^2} \int_{S_\rho/\Gamma_j} (g_{ij,i} - g_{ii,j})\nu^j d\alpha_{\text{Euc}}$$

In [Hein and LeBrun, 2016], Hein and LeBrun prove many remarkable results about ALE Kähler manifolds and their mass. In order to do this, they impose the following fall-off conditions:

- the scalar curvature s of the C^2 metric g belongs to L^1; and
- in some asymptotic chart at each end of M^n, the components of the metric satisfy $g_{jk} - \delta_{jk} \in C^{1,\alpha}_{-\tau}$ for some $\tau > (n-2)/2$ and some $\alpha \in (0, 1)$

where the weighted Hölder spaces $C^{k,\alpha}_{-\tau}$ consist of $C^{k,\alpha}$ functions such that:

$$\left(\sum_{j=0}^{k} |x|^j |\nabla^j f|\right) + |x|^{k+\alpha} |\nabla^k f|_{C^{0,\alpha}(B_{|x|/10}(x))} = O(|x|^{-\tau})$$

Note that in the complex surface case $(n = 4)$, the second fall of condition must be amended to:

- in some asymptotic chart at each end of M^4, the components of the metric satisfy $g_{jk} - \delta_{jk} \in C^{2,\alpha}_{-\tau}$ for some $\tau > 1$ and some $\alpha \in (0, 1)$

Throughout this thesis, we will adopt the same fall-off conditions as Hein-LeBrun. We will now highlight two of their nicest results:

Theorem II.1.6 ([Hein and LeBrun, 2016])**.** *Any ALE Kähler manifold of complex dimension $m \geq 2$ has only one end.*

This allows them to identify a compactly supported representative of the first Chern class which plays a key role in their theorem below. Furthermore, since there is only one end, the mass becomes an invariant of the manifold itself rather than simply an invariant of an end. What's more, they prove that the mass takes the form:

Theorem II.1.7 ([Hein and LeBrun, 2016])**.** *The mass of an ALE Kähler manifold (M, g, J) of complex dimension $m \geq 2$ is given by:*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1(M)), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s d\mu$$

where $\clubsuit$ denotes the inverse of the inclusion of compactly support cohomology $H^2_c(M) \hookrightarrow H^2_{dR}(M)$ and $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^2_c(M)$ and $H^{2m-2}(M)$.

Not only is this formula noticeably more tractable but also reveals that the mass of a scalar-flat ALE Kähler manifold is determined by its Kähler class and first Chern class:

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1(M)), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}}$$

In §II.2, we will be able to see the mass of our constructed metrics as the coefficient of the $\log$ term of our potential.

II.2 Construction of Scalar-flat, ALE Metrics

We will be constructing one-parameter families of extremal metrics on the Hirzebruch surfaces $\Sigma_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k))$, and then, we will use these families to obtain scalar ALE metrics on the line bundle $p \colon \mathcal{O}(-k) \to \mathbb{CP}^1$. We begin by performing the construction on Σ_1 and then generalizing the approach to $k > 1$.

Our approach will be to look at metrics of the form $\omega = i\partial\overline{\partial} f(s)$ for a strictly convex function f and find the necessary conditions for which ω completes over $\mathbb{P}(0 \oplus \mathcal{O}(-k))$ and $\mathbb{P}(\mathcal{O} \oplus 0)$. These completion conditions provide us with boundary information to solve the extremal metric differential equation. This approach was originally developed by Tønnesen-Friedman [Tønnesen-Friedman, 1998], and then later, it was generalized by Hwang-Singer [Hwang and Singer, 2002]. In their paper, Hwang-Singer develop the momentum profile construction which is a very convenient way to frame Calabi's ansatz [Calabi, 1979] that makes use of symplectic geometry to frame f in terms of its Legendre transform. Gauduchon [Gauduchon, 2019] has provided a thorough summary of the technique from the perspective of toric geometry with Bach-Flat Hirzebruch surfaces in mind. The following computation takes after the presentation of the material given in [Székelyhidi, 2014]. It's also worth noting that this construction is largely the same as that of Gauduchon except we don't use machinery from toric geometry (at least explicitly). We deviate by giving more concrete coefficients for the constructed extremal metrics and by adding a new viewpoint to the construction of ALE scalar flat metrics on $\mathcal{O}(-k)$.

Our computations will be done in local coordinates on the copy of $\mathbb{C} \times \mathbb{C}$ sitting within $\mathcal{O}(-k)$.

$$\begin{tikzcd}
 & \Sigma_k \setminus D_\infty \arrow[d, "\cong"] \\
\mathbb{C} \times \mathbb{C} \arrow[r, hook] & \mathcal{O}(-k) \arrow[d, hook] \\
 & \Sigma_k
\end{tikzcd}$$

We let our coordinate be $s = \log|(z, w)|^2_{h^{(k)}}$ where z is the coordinate on $\mathbb{CP}^1$, w is the coordinate on the fiber of $\mathcal{O}(-k)$, and $|\cdot|^2_{h^{(k)}}$ is a fiberwise norm on $\mathcal{O}(-k)$. When $k = 1$, we choose the fiberwise norm to come from identifying $\mathbb{C}^2 \setminus 0$ with $\mathcal{O}(-1) \setminus D_0$ where D_0 is the divisor corresponding to the 0-section i.e. $\mathbb{P}(\mathcal{O} \oplus 0)$. In the chart $U_0 = (Z_0 \neq 0)$ of $\mathbb{CP}^1$, our cooordinates z and w are:

$$z = \frac{Z_1}{Z_0}, \quad w = Z_0$$

Furthermore, on U_0, we may write:

$$\mathcal{O}(-1)|_{U_0} = \{([1 : z], (w, wz)) \mid z \in \mathbb{C}, w \in \mathbb{C}\} \subset \mathbb{CP}^1 \times \mathbb{C}^2$$

From this perspective, it's straightforward to see that $\mathcal{O}(-1)$ inherits a hermitian inner product (and hence a norm) from the $\mathbb{C}^2$ component:

$$|(z, w)|^2_h = |(w, wz)|^2 = |w|^2 + |wz|^2 = |Z_0|^2 + |Z_1|^2 \tag{II.2}$$

For $k > 1$, the hermitian inner product h on $\mathcal{O}(-1)$ determines a hermitian inner product $h^{(k)}$ on $\mathcal{O}(-k)$ via the tensor product map $u \mapsto u^{\otimes k}$. Importantly, writing $w = u^{\otimes k}$:

$$|(z,w)|^2_{h^{(k)}} = |(z,u^{\otimes k})|^2_{h^{(k)}} = |(z,u)|^{2k}_h = \left(|Z_0|^2 + |Z_1|^2\right)^k \tag{II.3}$$

For clarity's sake, we will be normalizing the Fubini-Study metric such that $\int_{\mathbb{CP}^1} \omega_{FS} = 2\pi$. This is consistent with the normalization used in [Arezzo and Pacard, 2006] for the Burns-Simanca metric. With the choice of normalization, $\mathbb{CP}^1$ has constant scalar curvature 2 and $\frac{1}{2\pi}[\omega_{FS}]$ is a generator of $H^2(\mathbb{CP}^1, \mathbb{Z})$.

II.2.1 Asymptotically Euclidean Case

II.2.1.1 Construction

As mentioned above, we will be looking at metrics of the form $\omega = i\partial\overline{\partial} f(s)$ where f is a strictly convex function. We let our coordinate $s = \log|(z,w)|^2_h$ where z is the coordinate on $\mathbb{CP}^1$, w is the coordinate on the fiber obtained from a holomorphic trivialization, and $|\cdot|_h$ is the fiberwise norm on $\mathcal{O}(-1)$ described above:

$$|(z,w)|^2_h = |w|^2\left(|z|^2+1\right)$$

and hence s is of the form:

$$s = \log|w|^2 + \log\left(|z|^2+1\right)$$

We perform our computations at a point (z_0, w_0) such that $d\log(|z|^2+1) = 0$. Then:

$$\omega = i\partial\overline{\partial} f(s) = i\partial(f'(s)\overline{\partial} s) = if'(s)\partial\overline{\partial} s + if''(s)\partial s \wedge \overline{\partial} s$$

We may compute:

$$\begin{aligned}
\partial s &= \frac{1}{w} dw + \partial \log(|z|^2+1) \\
\overline{\partial} s &= \frac{1}{\overline{w}} d\overline{w} + \overline{\partial} \log(|z|^2+1) \\
\partial\overline{\partial} s &= \partial\overline{\partial} \log(|z|^2+1) = p^*\omega_{FS}
\end{aligned}$$

However, since the computations are performed at a point where $d\log(|z_0|^2+1)=0$, we have that:

$$\partial s = \frac{1}{w}dw, \quad \overline{\partial} s = \frac{1}{\overline{w}}d\overline{w}$$

Therefore:

$$\omega = f'(s)p^*\omega_{FS} + if''(s)\frac{dw\wedge d\overline{w}}{|w|^2}$$

This expression for ω also gives us that:

$$\omega^2 = \frac{1}{|w|^2}f'(s)f''(s)p^*\omega_{FS}\wedge(idw\wedge d\overline{w})$$

In order to determine whether or not ω is extremal, we will want to look at the second derivative of the scalar curvature of ω. To simplify this, we will look at the Legendre transform of f. Let $\tau = f'(s)$. Then the Legendre transform $F(\tau)$ is defined by the relation:

$$f(s) + F(\tau) = s\tau$$

We will use the Legendre transform to write our metric in terms of the momentum profile of Hwang-Singer [Hwang and Singer, 2002]. The momentum profile $\varphi\colon I \to \mathbb{R}$ is defined by:

$$\varphi(\tau) = \frac{1}{F''(\tau)}$$

Then differentiating the defining equation twice with respect to τ yields the relation:

$$\frac{ds}{d\tau} = F''(\tau) = \frac{1}{\varphi}$$

which then gives us the following two relations which will be of use later:

$$f''(s) = \frac{d\tau}{ds} = \frac{1}{F''(\tau)} = \varphi \tag{II.4}$$

$$\tau = \frac{d}{ds}f(s) = \varphi\frac{d}{d\tau}f(s) \tag{II.5}$$

We may then write:

$$\omega = \tau p^*\omega_{FS} + i\varphi(\tau)\frac{dw\wedge d\overline{w}}{|w|^2} \tag{II.6}$$

The following relations are enjoyed by the Legendre transform:

$$s = F'(\tau), \quad \frac{ds}{d\tau} = F''(\tau), \quad \varphi(\tau) = f''(s) = \frac{d\tau}{ds}$$

We may then write:

$$\omega = \tau p^*\omega_{FS} + \varphi(\tau)\frac{idw\wedge d\overline{w}}{|w|^2}$$

and:

$$\omega^2 = \frac{1}{|w|^2}\tau\varphi(\tau)p^*\omega_{FS}\wedge idw\wedge d\overline{w}$$

Therefore, we may compute the Ricci form:

$$\rho = -i\partial\overline{\partial}\log(\tau\varphi(\tau)) + p^*\rho_{\mathbb{CP}^1} = -i\partial\overline{\partial}\log(\tau\varphi(\tau)) + 2p^*\omega_{FS}$$

where we made use of the fact that $\rho_{\mathbb{CP}^1} = 2\omega_{FS}$. Let $\psi = \log(\tau\varphi(\tau))$. Then:

$$\begin{aligned}\frac{d\psi}{ds} &= \frac{\varphi^2 + \tau\varphi\varphi'}{\tau\varphi} = \frac{\varphi}{\tau} + \varphi' \\ \frac{d^2\psi}{ds^2} &= \frac{\tau\varphi\varphi' - \varphi^2}{\tau^2} + \varphi''\varphi\end{aligned}$$

where above we begin writing $\varphi = \varphi(\tau)$ for simplicity. Therefore:

$$\partial\overline{\partial}\psi = \frac{d\psi}{ds}\partial\overline{\partial}s + \frac{d^2\psi}{ds^2}\partial s\wedge\overline{\partial}s$$

Hence:

$$\rho = \left(\frac{\varphi^2 - \tau\varphi\varphi'}{\tau^2} - \varphi''\varphi\right)\frac{idw\wedge d\overline{w}}{|w|^2} + \left(2 - \frac{\varphi}{\tau} - \varphi'\right)p^*\omega_{FS}$$

Now, we may finally compute the scalar curvature $S(\tau)$:

$$
\begin{aligned}
S(\tau) = \frac{\rho \wedge \omega}{\omega^2} &= \frac{\tau\left(\frac{\varphi^2 - \tau\varphi\varphi'}{\tau^2} - \varphi''\varphi\right) + \varphi\left(2 - \frac{\varphi}{\tau} - \varphi'\right)}{\varphi\tau} \\
&= \frac{1}{\tau^2}\left(\varphi - \tau\varphi' - \varphi''\tau^2\right) + \frac{1}{\tau}(2 - \frac{\varphi}{\tau} - \varphi') \\
&= \frac{2}{\tau} + \frac{1}{\tau^2}(\varphi - \tau\varphi' - \varphi''\tau^2 - \varphi - \varphi'\tau) \\
&= \frac{2}{\tau} + \frac{1}{\tau}(-2\varphi' - \tau\varphi'') \\
&= \frac{2}{\tau} - \frac{(\tau\varphi)''}{\tau}
\end{aligned}
$$

Lemma II.2.1. *The metric ω is extremal if and only if $S''(\tau) = 0$*

Proof. The metric ω is extremal if and only if $\mathrm{grad}^{1,0} S(\tau)$ is a holomorphic vector field. We have that:

$$\mathrm{grad}^{1,0} S(\tau) = g^{j\overline{k}} \partial_{\overline{k}} S(\tau) = \frac{|w|^2}{f''(s)} \cdot \frac{S'(\tau) f''(s)}{\overline{w}} \frac{\partial}{\partial w} = w S'(\tau) \frac{\partial}{\partial w}$$

This is holomorphic if:

$$w\varphi S''(\tau) \frac{\partial}{\partial w} = \overline{\partial}\left(w S'(\tau)\frac{\partial}{\partial w}\right) = \overline{\partial}\mathrm{grad}^{1,0} S(\tau) = 0$$

which is the case if and only if $S''(\tau) = 0$. $\square$

In other words, to obtain an extremal metric, we need to solve the differential equation $S''(\tau) = 0$. However, there is no guarantee that a solution to this differential equation will complete over the 0 and ∞-divisors of Σ_1. To do this, let $(1, m)$ be the range of τ for some $m > 1$. Completing over the 0 and ∞-divisors means imposing conditions on φ and φ' at the values $\tau = 1, m$:

Lemma II.2.2. *The metric ω completes over the 0 and ∞-divisors if and only if*

$$\lim_{\tau \to 1} \varphi(\tau) = \lim_{\tau \to m} \varphi(\tau) = 0$$

$$\lim_{\tau \to 1} \varphi'(\tau) = 1, \quad \lim_{\tau \to m} \varphi'(\tau) = -1$$

Proof. The S^1 symmetry of the z component of s tells us that we need only look in the fiber direction. This corresponds to the second term of our metric:

$$\varphi\frac{idw\wedge d\overline{w}}{|w|^2}$$

Let $r=|w|$. Then we have that $s=2\log r$. To complete the metric over $w=0$, φ must be of the form:

$$\varphi(\tau)=c_2r^2+c_3r^3+c_4r^4+\cdots \tag{II.7}$$

And using the fact that $\varphi\frac{d}{d\tau}=\frac{d}{ds}=\frac{r}{2}\frac{d}{dr}$:

$$\varphi(\tau)\varphi'(\tau)=\frac{d}{ds}\varphi(\tau)=c_2r^2+\frac{3c_3}{2}r^3+2c_4r^4+\cdots \tag{II.8}$$

(II.7) and (II.8) then tell us:

$$\varphi'(\tau)=1+O(r) \tag{II.9}$$

We may similarly look at the conditions to complete across the infinity divisor. We perform a change of coordinates letting $l=\frac{1}{w}$, so:

$$f''(s)\frac{idw\wedge d\overline{w}}{|w|^2}=f''(s)|l|^2i\left(-\frac{1}{l^2}dl\right)\wedge\left(-\frac{1}{\bar{l}^2}d\bar{l}\right)=f''(s)\frac{idl\wedge d\bar{l}}{|l|^2}$$

Now, we let $r=|l|$. Then $s=-2\log r$ and $\varphi\frac{d}{d\tau}=\frac{d}{ds}=-\frac{r}{2}\frac{d}{dr}$. Then proceeding as before, φ must be of the form:

$$\varphi(\tau)=c_2r^2+c_3r^3+c_4r^4+\cdots$$

And using the facts that $\varphi\frac{d}{d\tau}=\frac{d}{ds}=\frac{r}{2}\frac{d}{dr}$:

$$\varphi(\tau)\varphi'(\tau)=\frac{d}{ds}\varphi(\tau)=-c_2r^2-\frac{3c_3}{2}r^3-2c_4r^4-\cdots,$$

telling us that:

$$\varphi'(\tau) = -1 + O(r) \tag{II.10}$$

The expressions for φ in both coordinate systems above, tell us that:

$$\lim_{\tau\to 1} \varphi(\tau) = \lim_{\tau\to m} \varphi(\tau) = 0$$

Furthermore, (II.9) and (II.10) give:

$$\lim_{\tau\to 1} \varphi'(\tau) = 1, \quad \lim_{\tau\to m} \varphi'(\tau) = -1$$

$\square$

It is important to note that the choice of m is of critical importance. As mentioned previously, in order for φ to define a bona fide metric ω, ω must be positive definite. One can see from (II.6) that this will be the case as long as φ is positive on the interval $(1, m)$. Second, we will later show that the Kähler class of the resulting metric is determined by m. Due to this dependence, we will begin indexing φ_m to keep track of m.

Therefore to obtain an extremal metric, we must solve the ODE:

$$\left(\frac{2}{\tau} - \frac{(\tau\varphi_m)''}{\tau}\right)'' = 0 \tag{II.11}$$

$$\varphi_m(1) = \varphi_m(m) = 0, \quad \varphi_m'(1) = 1, \quad \varphi_m'(m) = -1$$

or equivalently:

$$2 - (\tau\varphi_m)'' = A_m\tau + B_m\tau^2$$

$$\varphi_m(1) = \varphi_m(m) = 0, \quad \varphi_m'(1) = 1, \quad \varphi_m'(m) = -1$$

Integrating with respect to τ twice yields:

$$2\tau - (\tau\varphi_m)' = \frac{A_m}{2}\tau^2 + \frac{B_m}{3}\tau^3 + C_m$$

$$\tau^2 - \tau\varphi_m = \frac{A_m}{6}\tau^3 + \frac{B_m}{12}\tau^4 + C_m\tau + D_m$$
$$\varphi_m = \tau - \frac{A_m}{6}\tau^2 - \frac{B_m}{12}\tau^3 - C_m - \frac{D_m}{\tau}$$

The boundary conditions give us the system:

$$\begin{cases} \frac{1}{6}A_m + \frac{1}{12}B_m + C_m + D_m = 1 \\ \frac{m^2}{6}A_m + \frac{m^3}{12}B_m + C_m + \frac{1}{m}D_m = m \\ \frac{1}{3}A_m + \frac{1}{4}B_m - D_m = 0 \\ \frac{m}{3}A_m + \frac{m^2}{4}B_m - \frac{1}{m^2}D_m = 2 \end{cases}$$

Solving this system yields the following values:

$$\begin{aligned} A_m &= \frac{6(m^2-3)}{(m-1)(m^2+4m+1)} \\ B_m &= \frac{24}{(m-1)(m^2+4m+1)} \\ C_m &= \frac{m(m^2-3)}{(m-1)(m^2+4m+1)} \\ D_m &= \frac{2m^2}{(m-1)(m^2+4m+1)} \end{aligned} \tag{II.12}$$

Lemma II.2.3. *The* $(1,1)$*-form* ω_m *corresponding to* φ_m *with coefficients given by* (II.12) *is a well-defined Kähler metric for all* $m > 1$.

Proof. As mentioned previously, in order for ω_m to be a bonafide metric, φ_m must be positive on $(1,m)$. Given the boundary conditions, we know that φ_m is positive on $(1,m)$ near the endpoints. It then suffices to show that φ_m is concave down on $(1,m)$. This is straightforward as:

$$\varphi_m'' = -\frac{A_m}{3} - \frac{B}{2}\tau - \frac{2D}{\tau^3} = -\frac{2}{\tau^3(m-1)(m^2+4m+1)}\left(6\tau^4 + (m^2-3)\tau^3 + 4m^2\right)$$

Since $m > 1$, we know that the first factor is negative on $(1,m)$, so we need only check that $q(\tau) = 6\tau^4 + (m^2-3)\tau^3 + 4m^2 > 0$ on $(1,m)$. We can see that:

$$q'(\tau) = 24\tau^3 + 3(m^2-3)\tau^2 = 3\tau^2(8\tau + m^2 - 3)$$

Then $q' > 0$ when $\tau > \frac{3-m^2}{8}$. Given that $\tau \geq 1 > \frac{1}{4} \geq \frac{3-m^2}{8}$, we have that $q' > 0$ on $(1, m)$. This combined with the fact that $q(1) = 5m^2 + 3 > 0$, tells us that $q > 0$ on $(1, m)$, and hence $\varphi_m'' < 0$ on $(1, m)$. □

Hence, we have obtained a one-parameter family of Kähler metrics on Σ_1.

II.2.1.2 Kähler Class of ω_m

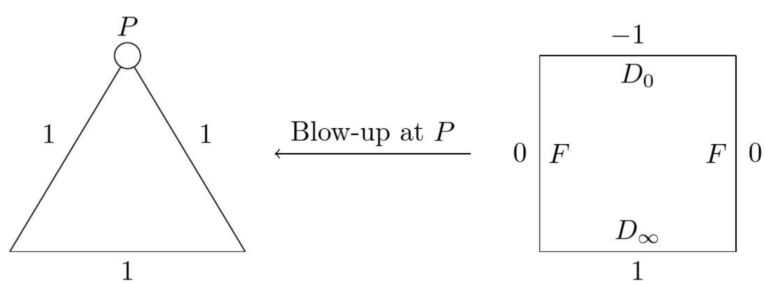
Now that we've constructed our family of metrics (and verified that they are in fact metrics), one might be curious about some of their characteristics. We will see that we have a metric for each Kähler class of Σ_1.

Proposition II.2.4. *The Kähler class of the constructed metrics are:*

$$[\omega_m] = 2\pi \left(m \left[D_\infty \right] - \left[D_0 \right] \right)$$

where $[D_\infty]$ and $[D_0]$ represent the Poincaré duals of the ∞ and 0 divisors respectively.

Proof. We may identify Σ_1 with the blow-up of $\mathbb{CP}^2$:



Through this identification, we have that $H^2(\Sigma_1, \mathbb{R}) = H^2(\mathrm{Bl}_P\mathbb{CP}^2, \mathbb{R}) = \langle [D_\infty], [D_0] \rangle$. Therefore, $[\omega_m]$ may be written:

$$[\omega_m] = a_m \left[D_\infty \right] + b_m \left[D_0 \right]$$

In order to solve for a_m and b_m, one needs the area of D_∞ and the area of an arbitrary fiber. The area of D_∞ computed as follows:

$$\text{Area } D_\infty = \int_{D_\infty} \omega_m = m \int_{\mathbb{CP}^1} \omega_{FS} = 2\pi m$$

Now, we compute the area of a fiber. Fix $z \in \mathbb{CP}^1$, and let F be the fiber of $\mathcal{O}(-1)$ over z. We perform the substitution $u = w\sqrt{|z|^2+1}$ and then express the integrand in terms of polar coordinates:

$$\begin{aligned}
\text{Area } F = \int_F \omega_m = \int_{\mathbb{C}\setminus 0} f''(s)\frac{idw \wedge d\overline{w}}{|w|^2} &= \int_{\mathbb{C}\setminus 0} f''(2\log|u|^2)\frac{idu \wedge d\overline{u}}{|u|^2} \\
&= \int_0^{2\pi}\int_0^{\infty} \frac{2f''(2\log r)}{r} dr d\theta \\
&= 2\pi\left(\lim_{r\to\infty} f'(2\log r) - \lim_{s\to 0} f'(2\log r)\right) \\
&= 2\pi\left(\lim_{s\to\infty} f'(s) - \lim_{s\to -\infty} f'(s)\right) \\
&= 2\pi(m-1)
\end{aligned}$$

We will then make use of the following intersection numbers:

$$D_\infty \cdot D_\infty = 1, \quad D_\infty \cdot D_0 = 0, \quad D_\infty \cdot F = 1$$
$$D_0 \cdot D_0 = -1, \quad D_0 \cdot F = 1, \quad F \cdot F = 0$$

By De Rham's theorem, we then have that:

$$2\pi m = \text{Area } D_\infty = (a_m D_\infty + b_m D_0)\cdot D_\infty = a_m$$
$$2\pi(m-1) = \text{Area } F = (a_m D_\infty + b_m D_0)\cdot F = a_m + b_m$$

Then solving for b_m, we get that:

$$[\omega_m] = 2\pi\left(m\,[D_\infty] - [D_0]\right)$$

□

Remark. *One can see that we have in fact an extremal metric for every Kähler class of Σ_1. The Kähler cone of Σ_1 is:*

$$\mathcal{K} = \left\{ a\,[D_\infty] - b\,[D_0] \;\middle|\; a > 0, b > 0, a > b \right\}$$

which upto rescaling is:

$$\mathcal{K}|_{\mathbb{R}>0} = \left\{2\pi\left(m\left[D_\infty\right] - \left[D_0\right]\right) \,\middle|\, m > 1\right\}$$

which is precisely the classes of our constructed extremal metrics.

II.2.1.3 The Burns Metric

The Burns metric is a metric of great importance. It was first introduced by Dan Burns [Burns, 1986] by defining a Kähler form on $\mathbb{C}^2 \setminus 0$:

$$\omega = i\partial\overline{\partial}\left(|z|^2 + a\log|z|^2\right) \tag{II.13}$$

for some $a > 0$. Burns then completed this metric by attaching a $\mathbb{CP}^1$ at the origin to obtain the Burns metric. It is clear to see that the completed metric is then a metric on the blow-up of $\mathbb{C}^2$ at the origin or, in other words, on the total space of $\mathcal{O}(-1)$. Note that Burns in fact defined a family of metrics parametrized by a. The coefficient a controls the area of the $\mathbb{CP}^1$ that was added when completing the metric. For our purposes, we will take $a = 1$ in order to stay consistent with our normalization of the area of the $\mathbb{CP}^1$ to be 2π.

Proposition II.2.5. *The Burns metric is ALE.*

Proof. In order to see that the Burns metric is ALE, one must first convert to the standard Euclidean coordinates and look at the asymptotics as $|z|$ gets large. Equation (II.2) gives that the metric in terms of the standard Euclidean coordinates is:

$$\omega_{\text{Burns}}(Z_0, Z_1) = i\partial\overline{\partial}\left(|Z_0|^2 + |Z_1|^2 + \log\left(|Z_0|^2 + |Z_1|^2\right)\right)$$

Comparing to the Euclidean metric $\omega_{euc} = i\partial\overline{\partial}(|Z_0|^2 + |Z_1|^2)$ yields:

$$\begin{aligned}\omega_{\text{Burns}} - \omega_{euc} =& i\partial\overline{\partial}\left(\log\left(|Z_0|^2 + |Z_1|^2\right)\right)\\ =& i\partial\left(\frac{Z_0}{|Z_0|^2 + |Z_1|^2}d\overline{Z}_0 + \frac{Z_1}{|Z_0|^2 + |Z_1|^2}d\overline{Z}_1\right)\\ =& i\left(\frac{|Z_1|^2}{\left(|Z_0|^2 + |Z_1|^2\right)^2}dZ_0 \wedge d\overline{Z}_0 + \frac{|Z_0|^2}{\left(|Z_0|^2 + |Z_1|^2\right)^2}dZ_1 \wedge d\overline{Z}_1\right.\end{aligned}$$

$$+\frac{Z_0\overline{Z}_1}{(|Z_0|^2+|Z_1|^2)^2}dZ_1\wedge d\overline{Z}_0+\frac{\overline{Z}_0Z_1}{(|Z_0|^2+|Z_1|^2)^2}dZ_0\wedge d\overline{Z}_1\Bigg)$$

Letting $r^2 = |Z_0|^2 + |Z_1|^2$, we can see that $\omega_\infty - \omega_{euc} = O(r^{-2})$. Hence, the metric that we have constructed is ALE of order 2. $\square$

One can also arrive at the Burns metric in a way similar to our construction above in that it is the solution to the differential equation:

$$2-(\tau\varphi)''=0 \tag{II.14}$$
$$\varphi(1)=0, \quad \varphi'(1)=1$$

One arrives at this ODE following the same procedure as above with two differences. The first being that we are solving the differential equation $S(\tau)=0$ instead of $S''(\tau)=0$. The second being that we omit the boundary conditions on $\varphi(m)$ and $\varphi'(m)$. This is because we no longer are concerned with completing over D_∞, so we don't need control over φ and φ' as s gets large. In this case, the range of τ is $(1,\infty)$.

Integrating (II.14) twice shows that φ is of the form:

$$\varphi=\tau+C+\frac{D}{\tau}$$

Substituting in for the boundary conditions gives the following system:

$$\begin{cases}1+C+D=0\\ 1=1-D\end{cases}\implies\begin{cases}C=-1\\ D=0\end{cases}$$

Hence, $\varphi_\infty(\tau)=\tau-1$, and upto a constant f_∞ may be written upto a constant as:

$$f_\infty(s)=\int\frac{\tau}{\tau-1}d\tau=\tau-1+\log(\tau-1)$$

Furthermore:

$$\log|(z,w)|_h^2=s=\int\frac{1}{\tau-1}d\tau=\log(\tau-1)+C$$

Note that the choice of constant C amounts to rescaling the hermitian metric h, so we shall take C to be 0 for simplicity. Our potential is then:

$$f_\infty = |(z,w)|_h^2 + \log|(z,w)|_h^2 = |Z_0|^2 + |Z_1|^2 + \log\left(|Z_0|^2 + |Z_1|^2\right)$$

which is precisely the potential for the Burns metric mentioned in (II.13) with our normalization of $a = 1$.

This construction of the Burns metric has been known for some time. However, there is a quite appealing connection between the family that we constructed in II.2.1.1 and the Burns metric that has gone unnoticed up until now.

Theorem II.2.6 (Rizzo)**.** *Let $\{\omega_m\}$ denote the metrics whose momentum profile satisfies:*

$$\left(\frac{2}{\tau} - \frac{(\tau\varphi_m)''}{\tau}\right)'' = 0$$
$$\varphi_m(1) = \varphi_m(m) = 0, \quad \varphi_m'(1) = 1, \quad \varphi_m'(m) = -1$$

Then $\omega_m|_{\mathcal{O}(-1)} \to \omega_{\textit{Burns}}$ as m goes to ∞. Here, convergence is meant in the sense that their momentum profiles converge (i.e. $\varphi_m \to \varphi_{\textit{Burns}}$).

Proof. Letting m tend to infinity, we can see that coefficients of φ_m approach:

$$\lim_{m\to\infty} A_m = \lim_{m\to\infty} \frac{6(m^2-3)}{(m-1)(m^2+4m+1)} = 0$$
$$\lim_{m\to\infty} B_m = \frac{24}{(m-1)(m^2+4m+1)} = 0$$
$$\lim_{m\to\infty} C_m = \lim_{m\to\infty} \frac{m(m^2-3)}{(m-1)(m^2+4m+1)} = -1$$
$$\lim_{m\to\infty} D_m = \lim_{m\to\infty} \frac{2m^2}{(m-1)(m^2+4m+1)} = 0$$

Therefore:

$$\lim_{m\to\infty} \varphi_m = \lim_{m\to\infty} \left(\tau - \frac{A_m}{6}\tau^2 - \frac{B_m}{12}\tau^3 - C_m - \frac{D_m}{\tau}\right)$$
$$= \tau - 1$$

$$= \varphi_{\text{Burns}}$$

$\square$

One way that we can intuitively make sense of this limiting procedure is by looking at how it affects the areas of D_0, D_∞, and F. Recall that we computed:

$$\text{Area } D_\infty = 2\pi m$$
$$\text{Area } F = 2\pi(m-1)$$

Furthermore, we can compute:

$$\text{Area } D_0 = \int_{D_0} \omega_m = \int_{\mathbb{CP}^1} \omega_{FS} = 2\pi \tag{II.15}$$

As we can see, the areas of D_∞ and an arbitrary fiber F grow linearly with respect to m while the area of D_0 is unaffected. We can therefore understand this limiting process as simply letting the divisor at infinity grow to be infinitely large. In [Hein and LeBrun, 2016], Hein and LeBrun carefully devise a method for compactifying n-dimensional ALE manifolds that amounts to effectively adding a $\mathbb{CP}^{n-1}$ at infinity. From this perspective, one can loosely view our limiting process as a sort of opposite of their compactifying procedure where we are increasing the size of and then removing the $\mathbb{CP}^1$ at infinity.

II.2.2 Asymptotically Locally Euclidean Case

II.2.2.1 Construction

The construction on Σ_k is largely the same as it is for Σ_1. The main difference is that k terms begin to appear throughout the computation, but the general architecture of the construction is the exact same. We present the computation below, making special note where the introduction of k matters.

As before, we let our coordinate $s = \log|(z,w)|^2_{h^{(k)}}$ where z is the coordinate on $\mathbb{CP}^1$ and w is the coordinate on the fiber of $\mathcal{O}(-k)$. Now, however, our fiberwise norm is given by:

$$|(z,w)|^2_{h^{(k)}} = |w|^2(|z|^2+1)^k$$

for some function $h\colon \mathbb{CP}^1 \to \mathbb{R}$, and hence s is of the form:

$$s = \log|w|^2 + k\log(|z|^2 + 1)$$

The computations are performed at a point (z_0, w_0) such that $d\log(|z|^2 + 1) = 0$. Then:

$$\omega_k = i\partial\overline{\partial}f(s) = i\partial(f'(s)\overline{\partial}s) = if'(s)\partial\overline{\partial}s + if''(s)\partial s \wedge \overline{\partial}s$$

We may compute:

$$\begin{aligned}
\partial s &= \frac{1}{w}dw + k\partial\log(|z|^2 + 1)\\
\overline{\partial}s &= \frac{1}{\overline{w}}d\overline{w} + k\overline{\partial}\log(|z|^2 + 1)\\
\partial\overline{\partial}s &= k\partial\overline{\partial}\log(|z|^2 + 1) = kp^*\omega_{FS}
\end{aligned}$$

However, since the computations are performed at a point where $d\log(|z_0|^2 + 1) = 0$, we have that:

$$\partial s = \frac{1}{w}dw, \quad \overline{\partial}s = \frac{1}{\overline{w}}d\overline{w}$$

Therefore:

$$\omega_k = kf'(s)p^*\omega_{FS} + if''(s)\frac{dw \wedge d\overline{w}}{|w|^2}$$

In order to determine whether or not the metric is extremal, we want to look at the second derivative of the scalar curvature. We may once again express ω in terms of the Legendre transform variable $\tau = f'(s)$ and the momentum profile φ:

$$\omega_k = k\tau p^*\omega_{FS} + i\varphi\frac{dw \wedge d\overline{w}}{|w|^2} \tag{II.16}$$

and

$$\omega_k^2 = \frac{1}{|w|^2}k\tau\varphi p^*\omega_{FS} \wedge (idw \wedge d\overline{w})$$

The Ricci form of ω is then given by:

$$\rho = -i\partial\overline{\partial}\log(k\tau\varphi) + p^*\rho_{\mathbb{CP}^1} = -i\partial\overline{\partial}\log(\tau\varphi) + 2p^*\omega_{FS}$$

Let $\psi = \log(\tau\varphi(\tau))$. Then using (II.4):

$$\frac{d\psi}{ds} = \frac{\varphi^2 + \tau\varphi\varphi'}{\tau\varphi} = \frac{\varphi}{\tau} + \varphi'$$
$$\frac{d^2\psi}{ds^2} = \frac{\tau\varphi\varphi' - \varphi^2}{\tau^2} + \varphi''\varphi$$

where above we begin writing $\varphi = \varphi(\tau)$ for simplicity. Therefore:

$$\partial\overline{\partial}\psi = \frac{d\psi}{ds}\partial\overline{\partial}s + \frac{d^2\psi}{ds^2}\partial s \wedge \overline{\partial}s = \left(-\frac{\varphi}{\tau} - \varphi'\right)kp^*\omega_{FS} + \left(\frac{\varphi^2 - \tau\varphi\varphi'}{\tau^2} - \varphi''\varphi\right)\frac{idw \wedge d\overline{w}}{|w|^2}$$

Hence:

$$\rho = \left(2 - \frac{k\varphi}{\tau} - k\varphi'\right)p^*\omega_{FS} + \left(\frac{\varphi^2 - \tau\varphi\varphi'}{\tau^2} - \varphi''\varphi\right)\frac{idw \wedge d\overline{w}}{|w|^2}$$

Now we may finally compute $S(\tau)$: Then we have that:

$$\begin{aligned}
S(\tau) = \frac{\rho \wedge \omega_k}{\omega_k^2} &= \frac{k\tau\left(\frac{\varphi^2 - \tau\varphi\varphi'}{\tau^2} - \varphi''\varphi\right) + \varphi\left(2 - \frac{k\varphi}{\tau} - k\varphi'\right)}{k\varphi\tau} \\
&= \frac{1}{\tau^2}\left(\varphi - \tau\varphi' - \tau^2\varphi''\right) + \frac{1}{k\tau}\left(2 - \frac{k\varphi}{\tau} - k\varphi'\right) \\
&= \frac{2}{k\tau} + \frac{1}{\tau^2}(\varphi - \tau\varphi' - \tau^2\varphi'' - \varphi - \tau\varphi') \\
&= \frac{2}{k\tau} - \frac{1}{\tau}(2\varphi' + \tau\varphi'') \\
&= \frac{2}{k\tau} - \frac{(\tau\varphi)''}{\tau}
\end{aligned}$$

Both lemma II.2.1 and lemma II.2.2 immediately apply. Therefore, we must solve the ODE:

$$\left(\frac{2}{k\tau} - \frac{(\tau\varphi_{k,m})''}{\tau}\right)'' = 0 \tag{II.17}$$

$$\varphi_{k,m}(1) = \varphi_{k,m}(m) = 0, \quad \varphi'_{k,m}(1) = 1, \quad \varphi'_{k,m}(m) = -1$$

which is equivalent to solving:

$$\frac{2}{k} - (\tau\varphi_{k,m})'' = A_{k,m}\tau + B_{k,n}\tau^2$$
$$\varphi_{k,m}(1) = \varphi_{k,m}(m) = 0, \quad \varphi'_{k,m}(1) = 1, \quad \varphi'_{k,m}(m) = -1$$

We may integrate both sides twice with respect to τ to obtain:

$$\frac{\tau^2}{k} - \tau\varphi_{k,m} = \frac{A_{k,m}}{6}\tau^3 + \frac{B_{k,m}}{12}\tau^4 + C_{k,m}\tau + D_{k,m} \tag{II.18}$$

Then by rearranging terms and dividing by τ, we get:

$$\varphi_{k,m} = \frac{\tau}{k} - \frac{A_{k,m}}{6}\tau^2 - \frac{B_{k,m}}{12}\tau^3 - C_{k,m} - \frac{D_{k,m}}{\tau}$$

Inputting the boundary conditions gives the following system of equations:

$$\begin{cases} \frac{1}{6}A_{k,m} + \frac{1}{12}B_{k,m} + C_{k,m} + D_{k,m} = \frac{1}{k} \\ \frac{m^2}{6}A_{k,m} + \frac{m^3}{12}B_{k,m} + C_{k,m} + \frac{1}{m}D_{k,m} = \frac{m}{k} \\ \frac{1}{3}A_{k,m} + \frac{1}{4}B_{k,m} - D_{k,m} = \frac{1-k}{k} \\ \frac{m}{3}A_{k,m} + \frac{m^2}{4}B_{k,m} - \frac{1}{m^2}D_{k,m} = \frac{k+1}{k} \end{cases}$$

Solving the system gives that:

$$\begin{aligned} A_{k,m} &= \frac{-6\left((k(m^2+1) - 2(m-1)(m+1)\right)}{k(m-1)(m^2+4m+1)} \\ B_{k,m} &= \frac{12\left(k(m+1) - m + 1\right)}{k(m-1)(m^2+4m+1)} \\ C_{k,m} &= \frac{-m\left(k(m^2+1) - 2(m-1)(m+1)\right)}{k(m-1)(m^2+4m+1)} \\ D_{k,m} &= \frac{m^2\left(k(m+1) - m + 1\right)}{k(m-1)(m^2+4m+1)} \end{aligned} \tag{II.19}$$

Lemma II.2.7. *The $(1,1)$-form corresponding to $\varphi_{k,m}$ with coefficients given by* (II.19) *is a well-defined Kähler metric for all $m > 1$ and all $k \geq 2$.*

Proof. As before, we need $\varphi_{k,m}$ to be positive on $(1, m)$. Let

$$q(\tau) = -\frac{B_{k,m}}{12}\tau^4 - \frac{A_{k,m}}{6}\tau^3 + \frac{1}{k}\tau^2 - C_{k,m}\tau - D_{k,m}$$

Then $\varphi_{k,m} = \frac{1}{\tau}q(\tau)$. Since $\tau > 0$ on $(1, m)$, we need only check that $q(\tau) > 0$ on $(1, m)$. We have the following inequalities for all $k \geq 2$ and $m > 1$:

$$k(m^2+1) - 2(m-1)(m+1) = (k-2)m^2 + k + 2 > 0$$

$$k(m+1) - m + 1 = (k-1)m + k + 1 > 0$$

$$k(m-1)(m^2+4m+1) > 0$$

By comparing the above inequalities to (II.19), one has that $A_{k,m}, C_{k,m} < 0$ and $B_{k,m}, D_{k,m} > 0$. Hence, the coefficients of $q(\tau)$ have the following sign pattern: $-+++-$. Descartes' rule of signs implies that $\varphi_{k,m}$ can have at most 2 positive roots – which we already know to be at $\tau = 1, m$ from the ODE's boundary conditions. This combined with the fact that $\varphi'_{k,m}(1) = 1$ implies that $\varphi_{k,m} > 0$ on $(1, m)$. Therefore, $\varphi_{k,m}$ defines a bonafide metric for all $m > 1$. □

Hence, we have obtained a one-parameter family of Kähler metrics on Σ_k.

II.2.2.2 Kähler Class of $\omega_{k,m}$

We compute the Kähler classes of these metrics.

Proposition II.2.8. *The Kähler class of the constructed metrics are:*

$$[\omega_{k,m}] = 2\pi\left(m\,[D_\infty] - [D_0]\right)$$

where $[D_\infty]$ and $[D_0]$ represent the Poincaré duals of the ∞ and 0 divisors respectively.

Proof. Since $H^2(\Sigma_k, \mathbb{R})$ is generated by $[D_\infty]$ and $[D_0]$, then $[\omega_{k,m}]$ may be written:

$$[\omega_{k,m}] = a_{k,m}\,[D_\infty] + b_{k,m}\,[D_0]$$

In order to solve for $a_{k,m}$ and $b_{k,m}$, one can find them using the area of D_0 and the area of an arbitrary fiber F. The area of D_0 computed as follows:

$$\text{Area } D_0 = k \int_{\mathbb{CP}^1} \omega_{FS} = 2\pi k$$

and that of F:

$$\begin{aligned}
\text{Area } F = \int_{\mathbb{C}\backslash 0} f''(s) \frac{i dw \wedge d\overline{w}}{|w|^2} &= \int_0^{2\pi} \int_0^{\infty} \frac{2f''(2\log r)}{r} dr d\theta \\
&= 2\pi \left(\lim_{s\to\infty} f'(s) - \lim_{s\to-\infty} f'(s) \right) \\
&= 2\pi(m-1)
\end{aligned}$$

We will then make use of the following intersection numbers:

$$D_\infty \cdot D_\infty = k, \quad D_\infty \cdot D_0 = 0, \quad D_\infty \cdot F = 1$$
$$D_0 \cdot D_0 = -k, \quad D_0 \cdot F = 1, \quad F \cdot F = 0$$

We then have that:

$$2\pi k = \text{Area } D_0 = (a_{k,m} D_\infty + b_{k,m} D_0) \cdot D_0 = -k b_{k,m}$$
$$2\pi(m-1) = \text{Area } F = (a_{k,m} D_\infty + b_{k,m} D_0) \cdot F = a_{k,m} + b_{k,m}$$

Hence, by solving for $a_{k,m}$ and $b_{k,m}$, we get that:

$$[\omega_{k,m}] = 2\pi \left(m \left[D_\infty \right] - [D_0] \right)$$

□

Remark. *The same argument as in Remark II.2.1.2 holds here for all k. The Kähler cone of Σ_k is:*

$$\mathcal{K} = \left\{ a \left[D_\infty \right] - b \left[D_0 \right] \,\middle|\, ka > 0, kb > 0, a > b \right\}$$

which up to rescaling is:

$$\mathcal{K}|_{\mathbb{R}>0} = \left\{2\pi\left(m\left[D_\infty\right] - \left[D_0\right]\right) \;\middle|\; m > 1\right\}$$

Hence, we have an extremal metric in every Kähler class of Σ_k.

II.2.2.3 Limit Metrics

Now, we will consider what happens when m tends to infinity. More specifically, we will be shifting our attention from complete metrics on Σ_k to non-complete metrics on $\Sigma_k \setminus D_\infty$. Letting $m \to \infty$, the coefficients approach:

$$\begin{aligned} A_{k,\infty} &= B_{k,\infty} = 0 \\ C_{k,\infty} &= \frac{2-k}{k} \\ D_{k,\infty} &= \frac{k-1}{k} \end{aligned}$$

In the case that $k = 2$, we have that $C_{2,\infty} = 0$ as well. The vanishing of $C_{2,\infty}$ impacts the process of solving for the potential, so we will consider the cases $k = 2$ and $k > 2$ separately.

II.2.2.4 Ricci-flat case ($k = 2$)

Much like the $k = 1$ case, we can convert back in terms of the variable s:

$$\begin{aligned} \varphi_{2,m} &= \frac{1}{2}(\tau - \tau^{-1}) \\ f_{2,\infty} = 2\int \frac{\tau}{\tau - \tau^{-1}} d\tau &= 2\int \frac{\tau^2}{\tau^2 - 1} d\tau = 2\tau + \log(\tau - 1) - \log(\tau + 1) \\ \log|(z,w)|_h^2 = s &= 2\int \frac{1}{\tau - \tau^{-1}} d\tau = \log(\tau - 1) + \log(\tau + 1) + C \end{aligned}$$

where $f_{2,\infty}$ is upto a constant. Therefore, we have: $\tau = \sqrt{\frac{|(z,w)|^2_{h^{(2)}}}{C} + 1}$ and hence:

$$f_{2,\infty} = 2\sqrt{\frac{|(z,w)|^2_{h^{(2)}}}{C} + 1} + \log\left(\sqrt{\frac{|(z,w)|^2_{h^{(2)}}}{C} + 1} - 1\right) - \log\left(\sqrt{\frac{|(z,w)|^2_{h^{(2)}}}{C} + 1} + 1\right) + C$$

$$
\begin{aligned}
=&2\sqrt{\frac{\left(|Z_0|^2+|Z_1|^2\right)^2}{C}+1}+\log\left(\sqrt{\frac{\left(|Z_0|^2+|Z_1|^2\right)^2}{C}+1}-1\right) \\
&-\log\left(\sqrt{\frac{\left(|Z_0|^2+|Z_1|^2\right)^2}{C}+1}+1\right)+C
\end{aligned}
$$

which is the potential for an Eguchi-Hanson metric. Note that when checking the ALE condition, we have the freedom to work in a diffeomorphism at infinity of our choosing. Therefore, we change into the coordinate r given by $r^2 = 2\sqrt{\frac{(|Z_0|^2+|Z_1|^2)^2}{C}+1}$, and we have:

$$
f_{2,\infty} = r^2 + \log(r^2-1) - \log(r^2+1) + C
$$

Taking two derivatives we can see:

$$
\begin{aligned}
\frac{d^2}{dr^2}f_{2,\infty} = \frac{d}{dr}\left(2r + \frac{2r}{r^2-1} - \frac{2r}{r^2+1}\right) &= 2 - \frac{2+2r^2}{(r^2-1)^2} - \frac{2-2r^2}{(r^2+1)^2} \\
&= 2 - \frac{12r^4+4}{(r^2-1)^2(r^2+1)^2} \\
&= 2 + O(r^{-4})
\end{aligned}
$$

so the ALE condition is satisfied. Moreover, we can see that this metric in fact obtains the Ricci flat decay rate of $\mathcal{O}(r^{-4})$.

II.2.2.5 Non Ricci-flat case ($k > 2$)

In the $k > 2$ case, the relevant equations are:

$$
\begin{aligned}
\varphi_{k,m} &= \frac{\tau}{k} - \frac{2-k}{k} - \frac{k-1}{k\tau} \\
f_{k,\infty} &= \int \frac{\tau}{\frac{\tau}{k} - \frac{2-k}{k} - \frac{k-1}{k}\tau^{-1}}d\tau = \int \frac{k\tau^2}{\tau^2-(2-k)\tau-(k-1)}d\tau \\
\log|(z,w)|^2_{h^{(k)}} &= s = \int \frac{k\tau}{\tau^2-(2-k)\tau-(k-1)}d\tau
\end{aligned}
$$

On can use partial fractions to see:

$$
\frac{k\tau}{\tau^2-(2-k)\tau-(k-1)} = \frac{1}{\tau-1} + \frac{k-1}{\tau+k-1}
$$

$$\frac{k\tau^2}{\tau^2-(2-k)\tau-(k-1)} = \frac{\tau}{\tau-1} + \frac{(k-1)\tau}{\tau+k-1}$$

Therefore:

$$f_{k,\infty} = \int \frac{\tau}{\tau-1} + \frac{(k-1)\tau}{\tau+k-1} d\tau = k\tau + \log(\tau-1) - (k-1)^2\log(\tau+k-1) + C$$

Similarly:

$$\begin{aligned}
\log|(z,w)|^2_{h^{(k)}} = s &= \int \frac{1}{\tau-1} + \frac{k-1}{\tau+k-1} d\tau \\
&= \log(\tau-1) + (k-1)\log(\tau+k-1) + C \\
&= \log\tau + \log\left(1-\tau^{-1}\right) + (k-1)\log\tau + (k-1)\log\left(1-(1-k)\tau^{-1}\right) + C \\
&= \log\tau^k + \log\left(1-\tau^{-1}\right) + (k-1)\log\left(1-(1-k)\tau^{-1}\right) + C
\end{aligned}$$

As before, we choose to work in a different coordinate at infinity r given by $r^{2k} = |(z,w)|^2_{h^{(k)}}$ (i.e. $r^2 = |Z_0|^2 + |Z_1|^2$). Then for large enough τ:

$$\begin{aligned}
\log r^{2k} &= \log\tau^k + O(\tau^{-1}) \\
r^{2k} &\sim \tau^k
\end{aligned}$$

Or, in other words, $r^2 \sim \tau$. Therefore, we may look into the asymptotics of $f_{k,\infty}$:

$$\begin{aligned}
f_{k,\infty} &= k\tau + \log(\tau-1) - (k-1)^2\log(\tau+k-1) + C \\
&\sim kr^2 + \log r^2 - (k-1)^2\log(r^2) + C \\
&= kr^2 + k(2-k)\log r^2 + C
\end{aligned}$$

Note that since our metrics are constrained to the same toric symmetries as the metrics in [Calderbank and Singer, 2004], they are in fact the same metrics. This is because the toric symmetries define the differential equation that we solve, and therefore by uniqueness of solutions, our metrics and Calderbank-Singer's are one and the same. Calderbank-Singer identify their metrics with LeBrun's metrics in [LeBrun, 1988], so our method recovers the LeBrun metrics.

Corollary II.2.9. *The mass of our constructed metrics is positive when* $k = 1$*, zero when* $k = 2$*, and negative when* $k > 2$*.*

Proof. In LeBrun's computation [LeBrun, 1988], he shows that the sign of the $\log$ term aligns with the sign of the mass. Therefore, the mass is positive when $k = 1$, zero when $k = 2$, and negative when $k > 2$ which agrees with LeBrun's result. $\square$

Chapter III

Regions of cscK metrics on the Kähler Cone

In this chapter, we will look at existence and non-existence of cscK metrics on rational surfaces. While this approach could be applied to various ruled surfaces, we will demonstrate the approach on a particular 6-fold blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$. This example is fascinating for a number of reasons. Namely, existence and non-existence are often treated separately in the literature – that is, it's rare for existence and non-existence of cscK metrics to both be described on a given manifold. Normally, one or the other is the focus. Although we will not provide a complete classification of the Kähler cone, we will begin painting a clearer picture of existence and non-existence.

We will describe the specifics of the blow-up in the next section, however, it's worth noting some particularly useful qualities of our choice of blow-up before we begin. First and foremost, it is a straightforward and topologically quite simple manifold to work with. Second, this blow-up falls in an interesting location in the theory of canonical metrics as it is has non-definite first Chern class, and hence, falls outside the purview of the Kähler-Einstein program. Finally, we note that blow-up has positive scalar curvature. The positive scalar curvature setting is often more rigid than the zero and negative scalar curvature cases which makes the results more surprising.

III.0.1 Parabolic Stability

In order to get a more complete picture of existence of cscK metrics on blow-ups of ruled surfaces, we will briefly discuss results about parabolic stability due to [Rollin and Singer, 2005]. It involves putting a parabolic structure on a ruled surface and determining the slope of holomorphic sections:

Definition III.0.1. Let $\pi : M \to \Sigma$ be a geometrically ruled surface. Then a parabolic structure on M consists of the following:

- A set of points $P_1, P_2, \ldots, P_n \in \Sigma$;
- A point $Q_j \in \pi^{-1}(P_j)$ for each j;
- A weight $\alpha_j \in (0, 1) \cap \mathbb{Q}$ for each j.

The slope of a holomorphic section $S\colon \Sigma \to M$ is then defined to be:

$$\mu(S) = S^2 + \sum_{Q_j \notin S} \alpha_j - \sum_{Q_j \in S} \alpha_j$$

where S^2 represents the self-intersection number of S.

The parabolic structure is (parabolically) stable if $\mu(S) > 0$ for all holomorphic sections S. Rollin-Singer explore other related notions of semi-/polystability [Rollin and Singer, 2009b], but we will only need stable for our purposes.

A parabolic structure determines a sequence of blow-ups on the ruled surface M. Let $P_j \in \Sigma$ be one of the selected points in (III.0.1) and let $\alpha_j = \frac{p}{q}$. We obtain the Hirzebruch-Jung continued fraction decomposition:

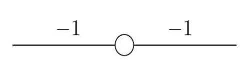
$$\frac{p}{q} = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cdots \cfrac{1}{e_l}}}$$

as well as one for $1 - \alpha_j$

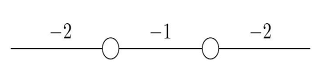
$$1 - \alpha_j = \frac{q-p}{q} = \cfrac{1}{e'_1 - \cfrac{1}{e'_2 - \cdots \cfrac{1}{e'_m}}}$$

These decompositions are unique if one assumes $e_j, e'_j \geq 2$ for all j.

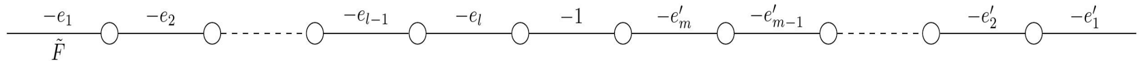
The iterated blow-up is then constructed by first blowing up at the point Q_j. This can be represented as follows:



Proceeding by further blowing-up the intersection point, one obtains the following:



Iterating this process, by choosing one of the intersection points and blowing-up, one arrives at:



Rollin-Singer show in [Rollin and Singer, 2005] that there is a unique choice of blow-up points to obtain the above diagram. Note that the curve labeled $\tilde{F}$ is the proper transform of the original fiber.

If the parabolic structure has multiple points, this process is then repeated for any remaining parabolic points. Rollin-Singer then show [Rollin and Singer, 2009b, Theorem D] that such an iterated blow-up of a parabolically stable ruled surface admits a cscK metric.

III.0.2 Example of Parabolically Stable Ruled Surfaces

We will now provide one of the simplest examples of a parabolically stable ruled surface. We will now describe a parabolic structure on the ruled surface $\pi\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$. Take distinct $P_1, P_2, P_3 \in \mathbb{CP}^1$ and $Q_j \in \pi^{-1}(P_j)$ with weights $\alpha_j = \frac{1}{2}$. Let X be the iterated blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ obtained from the procedure described above.

Rollin-Singer provide an argument that this is a stable parabolic structure in [Rollin and Singer, 2009a] which we will reproduce here.

Proposition III.0.2. *The parabolic structure on $\pi\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ described in the prior paragraph is parabolically stable for generic Q_1, Q_2, Q_3.*

Proof. The formula for the slope of this parabolic structure is:

$$\mu(S) = S^2 + \sum_{Q_j \notin S} \frac{1}{2} - \sum_{Q_j \in S} \frac{1}{2}$$

Since any holomorphic section $S \subseteq \mathbb{CP}^1 \times \mathbb{CP}^1$ is the graph of a meromorphic function f, one has that the self-intersection of S is $S^2 = 2\deg(f)$. Therefore, the only way that $\mu(S) \leq 0$ is if $\deg(f) = 0$ and at least two Q_j lie on S. In other words, this parabolic structure will be stable so long as no two Q_j fall on the same constant section which is true generically. $\square$

Rollin-Singer show that parabolic stability guarantees the existence of cscK metrics by using the parabolic structure to construct a cscK orbifold. They then construct a $\mathbb{CP}^1$-bundle over the orbifold and quotient the fiber bundle by the orbifold fundamental group. Locally, this quotient introduces two singularities at the 0 and ∞ sections of the fibers of the marked points P_i. These singularities can be resolved which results in the Hirzebruch-Jung string corresponding the weight $1/2$ as described above. The resolution of the singularities along the 3 distinguished fibers admits a cscK metric provided that the areas of the exceptional divisors of each resolution are small and the proper transform of the singular fibers is large (relative to the exceptional divisors).

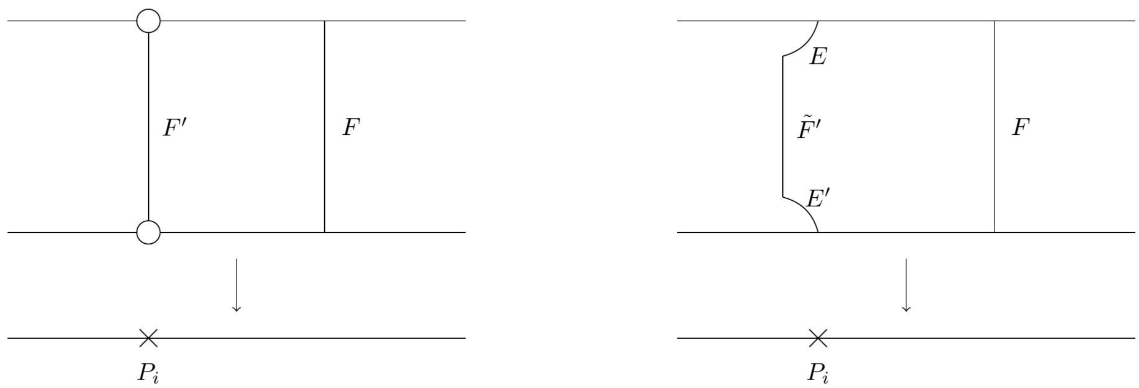


Figure III.1: The left diagram depicts a singular and smooth fiber in the quotient of the $\mathbb{CP}^1$-bundle prior to the resolution. Note that the fiber F' has half of the area of a generic fiber due to the quotienting procedure. The diagram on the right depicts the resolution where E and E' denote the exceptional divisors introduced by the resolution and $\tilde{F}'$ is the proper transform of F'.

This manifold can be achieved through the iterated blow-up procedure outlined above. It will be easier for our computation later to use a basis for homology that results from the iterated blow-up, so we introduce it here. We let E_i, F_i, and G_i denote the ith exceptional divisor of the iterated blow-ups at Q_1, Q_2, and Q_3 respectively. We may represent this pictorially as follows

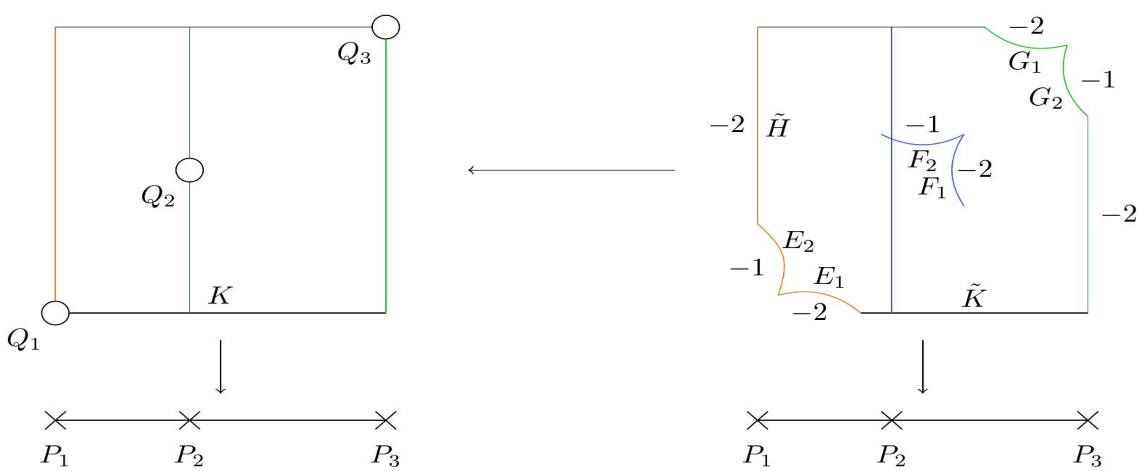


Figure III.2: Iterated blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$. The numbers denote the self-intersection number of the corresponding curve.

Note that in this basis, the exceptional divisors from the desingularizations correspond to the curves labeled with subscript 1 and the proper transforms of fibers (for example $\tilde{H}$ in the above picture) whereas the proper transform of the fibers in the desingularization corresponds to the curves labeled with subscript 2.

III.1 Stability

A fundamental concept in this thesis is the algebro-geometric notion of stability. There are numerous versions of stability each with their own particular flavor. Mumford [Mumford, 1977] introduced stability in the context of geometric invariant theory in order to characterize problematic points when taking quotients of schemes by a group action. However, in our case, we care more about the link between stability and existence of canonical metrics. A famous conjecture is the field of complex geometry is the following:

Conjecture (Yau-Tian-Donaldson)**.** *Let (M, L) be a polarized manifold and suppose that M has discrete holomorphic automorphism group. Then M admits a cscK metric in $c_1(L)$ if and only if (M, L) is K-stable.*

We will not delve into the specifics of K-stability as it will not feature in this thesis. However, it is useful to understand some of the results that live around it. The $\Longleftarrow$ direction of this conjecture (and its other formulations) is often viewed as the "harder" direction. This is due to the fact that it relies on using an algebraic condition to produce a solution to a difficult PDE whereas the other direction uses the PDE solution to prove the algebraic condition. While the $\Longleftarrow$ direction is still being tackled, Stoppa [Stoppa, 2009] proved the $\Longrightarrow$ direction:

Theorem III.1.1 (Stoppa)**.** *If $c_1(L)$ contains a cscK metric and $Aut(X, L)$ is discrete, then (X, L) is K-stable.*

We will instead focus on the notion of slope stability due to Ross-Thomas [Ross and Thomas, 2006] and parabolic stability due to Rollin-Singer [Rollin and Singer, 2009b].

III.1.1 Slope Stability

Let X be a smooth polarized complex manifold of dimension n with ample line bundle L. The pair (X, L) is referred to as a polarized manifold and L is the polarization. Let the Hilbert polynomial of (X, L) be:

$$\mathcal{H}(k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

Then one may define the slope of a polarized manifold:

Definition III.1.2. The slope of the polarized manifold (X, L) is:

$$\mu(X, L) = \frac{a_1}{a_0}$$

The asymptotic Riemann-Roch theorem gives that:

$$\mathcal{H}(k) = \chi(L^{\otimes k}) = \frac{\int_X c_1(L)^n}{n!} k^n - \frac{\int_X c_1(L)^{n-1} \cdot c_1(K_X)}{2(n-1)!} k^{n-1} + O(k^{n-2})$$

Hence, one has the following expressions:

$$a_0 = \frac{1}{n!} \int_X c_1(L)^n, \quad a_1 = -\frac{1}{2(n-1)!} \int_X c_1(L)^{n-1} \cdot c_1(K_X)$$

which implies that:

$$\mu(X, L) = -\frac{n \int_X c_1(L)^{n-1} \cdot c_1(K_X)}{\int_X c_1(L)^n}$$

The final piece needed to understand the obstruction given by slope stability is the slope of a submanifold $Z \subset X$. Let $\pi \colon \hat{X} \to X$ be the blow-up of X along Z with exceptional divisor E and let $\mathcal{J}_Z$ denote the ideal sheaf of Z. One then has the following definition

Definition III.1.3. The Seshadri constant of Z is

$$\begin{aligned} \epsilon(Z, X, L) &= \sup\{c : \pi^* L \otimes \mathcal{O}(-cE) \text{ is ample on } \hat{X}\} \\ &= \max\{c : \pi^* L \otimes \mathcal{O}(-cE) \text{ is nef on } \hat{X}\} \end{aligned}$$

For $x \in (0, \epsilon(Z, X, L))$, $\left(\hat{X}, \pi^* L \otimes \mathcal{O}(-xE)\right)$ is a polarized manifold, so one can once again look at its Hilbert polynomial except now we allow x to vary:

$$\chi((\pi^* L \otimes \mathcal{O}(-xE))^{\otimes k}) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2}), k \gg 0, xk \in \mathbb{N}$$

where by the asymptotic Riemann-Roch theorem again:

$$a_0(x) = \frac{1}{n!}\int_{\hat{X}} c_1(\pi^* L \otimes \mathcal{O}(-xE))^n, \quad a_1(x) = -\frac{1}{2(n-1)!}\int_{\hat{X}} c_1(\pi^* L \otimes \mathcal{O}(-xE))^{n-1} \cdot c_1(K_{\hat{X}})$$

Definition III.1.4. Let $\mathcal{J}_Z$ denote the ideal sheaf of the submanifold Z. The slope of Z with respect to c is

$$\mu_c(\mathcal{J}_Z, L) = \frac{\int_0^c a_1(x) + \frac{a_0'(x)}{2} dx}{\int_0^c a_0(x) dx}$$

Observe how this definition is very similar to definition III.1.2. Here, however, the values a_0 and a_1 are replaced with the averages $a_0(x)$ and $a_1(x)$ over the interval $(0, c)$. What's more, the numerator has an $\frac{a_0'(x)}{2}$ term in the integrand. Ross-Thomas refer to this as a correction term of sorts which is to account for the difference between the Hilbert polynomial of a 2-component normal crossing variety and the sum of its components' Hilbert polynomials. One can now define slope stability:

Definition III.1.5. (X, L) is slope stable with respect to Z if $\mu_c(\mathcal{J}_Z, L) < \mu(X, L)$ for every $c \in (0, \epsilon(Z, X, L))$, and for $c = \epsilon(Z, X, L)$ if $\epsilon(Z, X, L)$ is rational and the global sections $L^{\otimes k} \otimes \mathcal{J}_Z^{\epsilon(Z,X,L)k}$ saturate $\mathcal{J}_Z^{\epsilon(Z,X,L)k}$ for $k \gg 0$. (X, L) is slope stable if it is slope stable with respect to all subschemes Z.

In the above definition, saying that the global sections $L^{\otimes k} \otimes \mathcal{J}_Z^{\epsilon(Z,X,L)k}$ saturate $\mathcal{J}_Z^{\epsilon(Z,X,L)k}$ means that the global sections generate the line bundle $\pi^* L \otimes \mathcal{O}(-\epsilon(Z, X, L)kE)$. Ross and Thomas prove the following theorem relating slope stability and K-stability:

Theorem III.1.6 ([Ross and Thomas, 2006])**.** *Suppose (X, L) is K-stable. Then it is slope stable with respect to any smooth subscheme Z.*

Said another way, if (X, L) has a destabilizing subscheme (that is a subscheme which (X, L) is is not slope stable with respect to), then (X, L) is not K-stable, and hence, the class $c_1(L)$ does not contain a cscK metric. Therefore, slope stability gives an obstruction to the existence of cscK metrics in the compact setting.

It is often easier, however, to instead work with the related notion of quotient slope.

Definition III.1.7. Let $\tilde{a}_i(x)$ be defined by:

$$\chi(\pi^* L^{\otimes k}/(\pi^* L \otimes \mathcal{O}(-xE))^{\otimes k}) = \tilde{a}_0(x)k^n + \tilde{a}_1 k^{n-1} + O(k^{n-2})$$

The quotient slope of Z with respect to c is:

$$\mu_c(\mathcal{O}_Z, L) = \frac{\int_0^c \tilde{a}_1(x) + \frac{\tilde{a}_0'(x)}{2} dx}{\int_0^c \tilde{a}_0(x) dx} = \frac{\int_0^c a_1(x) + \frac{a_0'(x)}{2} dx - ca_1}{\int_0^c a_0(x) dx - ca_0}$$

which is finite for $0 < c \leq \epsilon(Z, X, L)$

Note that for $0 < B < D$:

$$\frac{A}{B} < \frac{C}{D} \iff \frac{C}{D} < \frac{C-A}{D-B} \iff \frac{A}{B} < \frac{C-A}{D-B}$$

Hence, the above formulas for slope, slope of Z, and quotient slope of Z enjoy the following implications:

$$\mu_c(\mathcal{J}_Z, L) < \mu(X, L) \iff \mu(X, L) < \mu_c(\mathcal{O}_Z, L) \iff \mu_c(\mathcal{J}_Z, L) < \mu_c(\mathcal{O}_Z, L)$$

Importantly, the stability inequality gets reversed when looking at quotient slope.

The above formulas for slope and quotient slope with respect to Z are fairly involved. Thankfully, there is a more tractable formula for quotient slope of a divisor Z:

Theorem III.1.8 ([Ross and Thomas, 2006]). *Suppose that Z is a divisor in (X, L). Then:*

$$\mu_c(\mathcal{O}_Z, L) = \frac{n\left(L^{n-1} \cdot Z - \sum_{j=1}^{n-1} \binom{n-1}{j} \frac{(-c)^j}{j+1} L^{n-1-j} \cdot Z^j \cdot (K_X \otimes \mathcal{O}(Z))\right)}{2\sum_{j=1}^{n} \binom{n}{j} \frac{(-c)^j}{j+1} L^{n-j} \cdot Z^j}$$

In the complex surface case, this simplifies down to:

Corollary III.1.9 ([Ross and Thomas, 2006]). *Let Z be a smooth curve in a smooth polarised* *surface* (X, L)*. Then:*

$$\mu(X, L) = -\frac{K_X \cdot L}{L^2}$$
$$\mu_c(\mathcal{O}_Z, L) = \frac{3\left(2L \cdot Z - c(K_X \cdot Z + Z^2)\right)}{2c(3L \cdot Z - cZ^2)}$$

and if Z *is a smooth rational curve:*

$$\mu_c(\mathcal{O}_Z, L) = \frac{3(L \cdot Z + c)}{c(3L \cdot Z - cZ^2)}$$

These formulas are notably easier to work with, and therefore, will be the main computational tool we use for slope stability. Importantly, a smooth rational curve Z destabilizes the polarization (X, L) if for some $c \in (0, \epsilon(Z, X, L)]$:

$$\frac{3(L \cdot Z + c)}{c(3L \cdot Z - cZ^2)} < -\frac{K_X \cdot L}{L^2}$$

III.2 The Destabilizing Curve

We will now examine nonexistence on the same iterated blow-up using slope stability. We will be looking at classes in $H^2(X)$ of the form:

$$\Omega = a[H] + a[K] - e[E_1] - \alpha e[E_2] - f_1[F_1] - f_2[F_2] - g_1[G_1] - g_2[G_2] \tag{III.1}$$

Lemma III.2.1. *For any* $a > 0$*, there are positive constants* f_1, f_2, g_1, g_2, e_0 *such that for any* $\alpha \in (1, 2)$ *and* $e < e_0$*, the class* Ω *is Kähler.*

Proof. For the sake of clarity, we will walk through the iterated blow-up procedure and our notation. First, consider a single blow-up at a point $\pi : \hat{X} \to X_0$ with exceptional divisor E. Given a Kähler class $\Omega_0 \in H^2(X_0, \mathbb{R})$, we know that $\pi^*(\Omega_0) - x[E] > 0$ for any x small enough. Letting $X_0 = \mathbb{CP}^{\nVdash} \times \mathbb{CP}^1$, we will iterate this procedure twice at both Q_2 and Q_3 (a total of 4 blow-ups). Hence, for a Kähler class $a[H] + a[K] \in H^2(X_0, \mathbb{R})$, the class $\Omega_1 = a[H] + a[K] - f_1[F_1] - f_2[F_2] - g_1[G_1] - g_2[G_2] > 0$ for small enough f_i, g_i. Although it isn't necessary for our current argument, it's worth noting for the next theorem that we may stipulate the f_i's and g_i's satisfy the

the following inequality:

$$\frac{15}{8a} < \frac{4a - f_2 - g_2}{2a^2 - 2f_1^2 + 2f_1 f_2 - f_2^2 - 2g_1^2 + 2g_1 g_2 - g_2^2}$$

The procedure above is a quick and painless method for showing positivity, however, it comes with the price that the coefficients indexed with 2 depend on those indexed with 1. In our later argument, this dependence won't be a problem for the coefficients of $[F_i]$ and $[G_i]$. It will, however, be problematic for the coefficients of $[E_i]$.

We instead use the Nakai-Moishezon criterion to handle the coefficients of $[E_i]$. Let X_1 denote the 4-fold blow-up of X_0 described in the first paragraph. Let $\pi \colon X \to X_1$ be the iterated blow-up of X_1 at Q_1, and let Ω be as in (III.1). Recall the for surfaces, the Nakai-Moishezon criterion says that $\Omega > 0$ if and only if $\Omega^2 > 0$ and $\Omega \cdot C > 0$ for every curve $C \subseteq X$. We will see that the Nakai-Moishezon criterion is satisfied for small enough e.

We begin by obtaining the inequality $\Omega \cdot C > 0$ on some notable curves:

Curve	Inequality
E_1	$2e > \alpha e$
E_2	$\alpha e > e$
$\tilde{H} = H - E_1 - 2E_2$	$a > \alpha e$
$\tilde{K} = K - E_1 - E_2$	$a > e$

So for Ω to be positive, one needs $1 < \alpha < 2$. We must also consider a general curve C. Assume that C is a curve on X other than E_1, E_2, $\tilde{H}$, and $\tilde{K}$. If C does not intersect E_1 or E_2, then $C = \pi^*(C')$ for some holomorphic curve $C' \subseteq X_1$ in which case:

$$\Omega \cdot C = \Omega \cdot \pi^*(C') = \Omega_1 \cdot C' > 0$$

since $\Omega_1 > 0$ on X_1.

Now, assume that C intersects E_1 or E_2, and let $C' = \pi(C) \subseteq X_1$. Then $[C] = \pi^*[C'] -$

$x_1[E_1] - x_2[E_2]$. Since $C \neq E_1, E_2, \tilde{H}, \tilde{K}$ by assumption, we have the inequalities:

$$C \cdot E_1 = 0 - x_1 E_1^2 - x_2 E_1 \cdot E_2 = 2x_1 - x_2 \geq 0$$
$$C \cdot E_2 = 0 - x_1 E_1 \cdot E_2 - x_2 E_2^2 = -x_1 + x_2 \geq 0$$

Therefore, we have that $2x_1 \geq x_2 \geq x_1 \geq 0$. Since C intersects at least one of E_1 and E_2, at least one of $C \cdot E_1$ and $C \cdot E_2$ is, in fact, greater than or equal to 1.

We may also compute:

$$C \cdot \tilde{H} = (C' - x_1 E_1 - x_2 E_2) \cdot (H - E_1 - 2E_2) \tag{III.2}$$
$$= C' \cdot H + 0 + x_1 E_1^2 + 2x_1 E_1 \cdot E_2 + x_2 E_1 \cdot E_2 + 2x_2 E_2^2 \tag{III.3}$$
$$= C' \cdot H - 2x_1 + 2x_1 + x_2 - 2x_2 = C' \cdot H - x_2 \geq 0 \tag{III.4}$$

so $C' \cdot H \geq x_2$. Similarly:

$$\begin{aligned} C \cdot \tilde{K} &= (C' - x_1 E_1 - x_2 E_2) \cdot (K - E_1 - E_2) \\ &= C' \cdot K + 0 + x_1 E_1^2 + x_1 E_1 \cdot E_2 - 0 + x_2 E_1 \cdot E_2 + x_2 E_2^2 \\ &= C' \cdot K - 2x_1 + x_1 + x_2 - x_2 = C' \cdot K - x_1 \geq 0 \end{aligned}$$

so $C' \cdot K \geq x_1$. Then:

$$\begin{aligned} \Omega \cdot C &= (\pi^*(\Omega_1) - eE_1 - \alpha e E_2) \cdot (C' - x_1 E_1 - x_2 E_2) \\ &= \Omega_1 \cdot C' - 0 + e x_1 E_1^2 + e x_2 E_1 \cdot E_2 + \alpha e x_1 E_1 \cdot E_2 + \alpha e x_2 E_2^2 \\ &= \Omega_1 \cdot C' - 2e x_1 + e x_2 + \alpha e x_1 - \alpha e x_2 \\ &= \Omega_1 \cdot C' - e\left(x_1(2-\alpha) + x_2(\alpha - 1)\right) \end{aligned}$$

Since α ranges from 1 to 2, the quantity $x_1(2-\alpha) + x_2(\alpha-1)$ ranges from x_1 to x_2. Hence, we have that $\Omega \cdot C > \Omega_1 \cdot C' - e x_2$. We will now compute $\Omega_1 \cdot C'$. Recall that $\Omega_1 = a[H] + a[K] - \sum\limits_i f_i[F_i] - \sum\limits_i g_i[G_i]$. By shrinking if necessary, we may assume that $f_1, g_1 < \frac{a}{20}$ and $f_2, g_2 < \frac{a}{10}$.

Let $C' = m[H] + n[K] - \sum_i s_i[F_i] - \sum_i t_i[G_i]$. Then:

$$\begin{aligned}\Omega_1 \cdot C' &= \left(a[H] + a[K] - \sum_i f_i[F_i] - \sum_i g_i[G_i]\right) \cdot \left(m[H] + n[K] - \sum_i s_i[F_i] - \sum_i t_i[G_i]\right) \\ &= an + am + (f_1[F_1] + f_2[F_2]) \cdot (s_1[F_1] + s_2[F_2]) + (g_1[G_1] + g_2[G_2]) \cdot (t_1[G_1] + t_2[G_2])\end{aligned}$$

Since $f_1 < f_2 < \frac{a}{10}$, we have that:

$$(f_1[F_1] + f_2[F_2]) \cdot (s_1[F_1] + s_2[F_2]) = -s_1(2f_1 - f_2) - s_2(f_2 - f_1) > -(s_1 + s_2)\frac{a}{10}$$

Similarly, we have that $(g_1[G_1] + g_2[G_2]) \cdot (t_1[G_1] + t_2[G_2]) > -(t_1 + t_2)\frac{a}{10}$. Note that since C' is a holomorphic curve, we must have that $s_1 \leq m$, $s_2 \leq n$, $t_1 \leq m$, and $t_2 \leq n$. Therefore:

$$\begin{aligned}\Omega \cdot C > \Omega_1 \cdot C' - ex_2 &> a(n+m) - (s_1 + s_2 + t_1 + t_2)\frac{a}{10} - ex_2 \\ &\geq a(n+m) - (n+m)\frac{a}{5} - ex_2\end{aligned}$$

Finally, (III.2) tells us that $n = C' \cdot H \geq x_2$, so:

$$\Omega \cdot C > a(n+m) - (n+m)\frac{a}{5} - ex_2 > a(n+m) - (n+m)\frac{a}{5} - (n+m)e$$

which is greater than 0 provided that $e < \frac{4a}{5}$. $\square$

Note that the above lemma tells us information about the Seshadri constant $\epsilon(E_1, X, L)$; namely that $\epsilon(E_1, X, L) \geq \min\{\frac{4a}{5}, \frac{a}{\alpha}\}$.

Now that we know have a grasp on which classes are Kähler, we may begin finding classes where cscK metrics are obstructed. To accomplish this, we will show that the curve E_1 is destabilizes (M, L) where L is a line bundle such that $c_1(L) = \Omega$ as defined above (III.1) with rational coefficients. Note that from a technical standpoint, in order for Ω to represent $c_1(L)$, it must be an integral cohomology class which it is not. However, we selected our coefficients such that $\Omega \in H^2(X, \mathbb{Q})$. Then $k\Omega \in H^2(X, \mathbb{Z})$ for some k, and therefore we may find a line bundle L such that $c_1(L) = k\Omega$. Then, $k\Omega$ admits cscK metrics if and only if Ω does, so we can assume that our class Ω has integer coefficients.

Now we will prove our destabilizing results:

Theorem III.2.2 (Rizzo)**.** *For any $a > 0$, there are positive constants f_1, f_2, g_1, g_2, e_0 satisfying:*

$$\frac{15}{8a} < \frac{4a - f_2 - g_2}{2a^2 - 2f_1^2 + 2f_1 f_2 - f_2^2 - 2g_1^2 + 2g_1 g_2 - g_2^2}$$

such that for any $\alpha \in \left(1, \frac{5}{4}\right]$ and $e < e_0$, the class Ω (III.1) *does not admit a cscK metric.*

Proof. Let Ω be as in the previous lemma. Recall that E_1 destabilizes if in terms of the quotient slope:

$$\mu_c(\mathcal{O}_{E_1}, L) < \mu(X, L)$$

for some $c \in (0, \epsilon(E_1, X, L)]$ where $\mu_c(\mathcal{O}_{E_1}, L)$ represents the quotient slope of E_1 with respect to c and $\mu(X, L)$ is the slope of (X, L). We will take $c = \frac{4a}{5}$. Since $1 < \alpha \leq \frac{5}{4}$, we have that $\frac{4a}{5} \leq \frac{a}{\alpha}$. Therefore, we are guaranteed that $c \in (0, \epsilon(E_1, X, L)]$.

Using the fact that $-c_1(K_X) = c_1(X) = 2H + 2K - (E_1 + F_1 + G_1) - 2(E_2 + F_2 + G_3)$ and corollary 5.3 in [Ross and Thomas, 2006], we may compute:

$$\mu(X, L) = -\frac{K_X \cdot L}{L^2} = \frac{4a - \alpha e - f_2 - g_2}{2a^2 - 2f_1^2 + 2f_1 f_2 - f_2^2 - 2g_1^2 + 2g_1 g_2 - g_2^2 - e^2(\alpha^2 - 2\alpha + 2)}$$

and:

$$\mu_{\frac{4a}{5}}(\mathcal{O}_{E_1}, L) = \frac{3(L \cdot E_1 + c)}{c(3L \cdot E_1 - cE_1^2)} = \frac{3((2-\alpha)e + \frac{4a}{5})}{\frac{4a}{5}(3(2-\alpha)e + \frac{8a}{5})}$$

Therefore, E_1 is a destabilizing curve if

$$\frac{3((2-\alpha)e + \frac{4a}{5})}{\frac{4a}{5}(3(2-\alpha)e + \frac{8a}{5})} < \frac{4a - \alpha e - f_2 - g_2}{2a^2 - 2f_1^2 + 2f_1 f_2 - f_2^2 - 2g_1^2 + 2g_1 g_2 - g_2^2 - e^2(\alpha^2 - 2\alpha + 2)}$$

Taking the limit as $e \to 0$:

$$\frac{15}{8a} < \frac{4a - f_2 - g_2}{2a^2 - 2f_1^2 + 2f_1 f_2 - f_2^2 - 2g_1^2 + 2g_1 g_2 - g_2^2}$$

which is satisfied due to our choices of f_i's and g_i's. Continuity tells us that our destabilizing inequality holds for e small enough. $\square$

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