ROUGH SOLUTIONS OF THE RELATIVISTIC EULER EQUATIONS

By

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CHAPTER 1

Introduction

This paper is concerned with the special relativistic Euler equations on the Minkowski background $(\mathbb{R}^{1+3}, \eta)$, where η is the Minkowski metric. For use throughout the article, we fix a coordinate system $\{x^\alpha\}_{\alpha=0,1,2,3}$, relative[1] to which $\eta_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$, where the speed of light is set to be 1. For these equations, there is considerable freedom in the choice of state-space variables, that is, the fundamental unknowns of the PDEs. In this work, we choose the logarithmic enthalpy h, the entropy s, and the four-velocity v, which is a future directed η-timelike vectorfield normalized as $\eta(v,v) = -1$. We allow for non-trivial vorticity. All other unknowns in the system can be considered as functions of the state-space variables. We denote the pressure as $\mathfrak{p} = \mathfrak{p}(h,s)$, the fluid density as $\rho = \rho(h,s)$ and speed of sound as $c := \sqrt{\frac{\partial \mathfrak{p}}{\partial \rho}}$. In this coordinate system, the relativistic Euler Equations can be expressed as[2]:

$$v^\kappa \partial_\kappa h + c^2 \partial_\kappa v^\kappa = 0, \tag{1.1a}$$

$$v^\kappa \partial_\kappa (v_\flat)_\alpha + \partial_\alpha h + (v_\flat)_\alpha v^\kappa \partial_\kappa h - q \partial_\alpha s = 0, \tag{1.1b}$$

$$v^\kappa \partial_\kappa s = 0, \tag{1.1c}$$

where $q := \frac{\theta}{H}$ is temperature over enthalpy, which can be expressed as $q = q(h,s)$. Also see Section 2.2-2.3 for the details.

Our work intimately depends on a new formulation of the equations derived by Disconzi-Speck [10], where the authors found that the flow splits into a "sound-wave-part" ("wave-part" for short) for (h,s,v) and a "transport-div-curl-part" ("transport-part" for short) for the vorticity ω and the entropy gradient S. Schematically, the geometric formulation takes the following form[3]:

[1] Throughout this article, we use the notation that Greek "space-time" indices take on the values $0,1,2,3$, while Latin "spatial indices" take on the values $1,2,3$. We use Einstein summation convention throughout the paper.

[2] For Greek and Latin indices, for any vectorfield or one-form V, we lower and raise indices with the Minkowski metric $\eta_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$ and its inverse by using the notation $(V_\flat)_\beta := \eta_{\alpha\beta} V^\alpha$ and $(V^\sharp)^\beta := (\eta^{-1})^{\alpha\beta} V_\alpha$.

[3] We denote schematic spatial partial derivatives and space-time partial derivatives by ∂ and $\boldsymbol{\partial}$ respectively. Also, we use the following schematic notations throughout the paper where A, B, C are arrays of variables:

- $\mathscr{L}[A](B)$ denotes any scalar-valued function that is linear in the components of B with coefficients that are a function of the components of A.
- $\mathscr{Q}[A](B,C)$ denotes any scalar-valued function that is quadratic in the components of B and C with coefficients that are a function of the components of A.

Wave equations

$$\square_{\mathbf{g}}\Psi = \mathscr{L}(\vec{\Psi})[\vec{\mathscr{C}},\mathscr{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}]. \tag{1.2}$$

Transport equations

$$\mathbf{B}\omega^{\alpha} = \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{1.3a}$$

$$\mathbf{B}(S^{\sharp})^{\alpha} = \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{1.3b}$$

Transport-Div-Curl system

$$\mathbf{B}\mathscr{C}^{\alpha} = \mathfrak{F}\mathscr{C}^{\alpha} := \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},(\boldsymbol{\partial}\omega,\boldsymbol{\partial}\vec{S},\boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{S}], \tag{1.4a}$$

$$\mathbf{B}\mathscr{D} = \mathfrak{F}\mathscr{D} := \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\omega] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},(\boldsymbol{\partial}\vec{S},\boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi},\vec{\omega},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{1.4b}$$

$$\mathrm{vort}^{\alpha}(S) = 0, \tag{1.4c}$$

$$\partial_{\alpha}\omega^{\alpha} = \mathscr{L}(\omega)[\boldsymbol{\partial}\vec{\Psi}], \tag{1.4d}$$

where $\vec{\Psi} := (v^0, v^1, v^2, v^3, h, s)$, $\mathbf{g} = \mathbf{g}(\vec{\Psi})$ is a solution-dependent Lorentzian metric which governs the geometry of sound waves, $\mathscr{C} \simeq \mathrm{vort}(\omega)$ and $\mathscr{D} \simeq \mathrm{div}S$ are special modified fluid variables, and $\mathbf{B}$ is the material derivative, which is parallel to v^{α}. See Definition 2.3 for the definition of ω, Definition 2.6 for the definition of S, and Section 2.4 for the precise definitions of $\mathbf{g},\mathscr{C},\mathscr{D},\mathbf{B}$ and more details of the geometric formulation of the equations. This formulation reveals miraculous regularity and geometric properties of the flow, which is used in a fundamental way in the present work. These geometric properties are not visible in first order equations (1.1).

In this paper, we show that under low regularity assumptions on the "wave-part" (see Section 3.5 for more details of "wave-part") of the initial data, the regularity of solutions of the relativistic Euler equations can be preserved for a short time. Specifically, we assume that the "wave-part" of the data belongs to H^{2+}, and that the "transport-part" $\mathscr{C}$ and $\mathscr{D}$ are in Hölder space $C^{0,0+}$. Our proof shows, in particular, that it is possible to avoid instantaneous shock formation, which in [21] was shown to occur in the irrotational case (i.e., for quasilinear wave equations) for initial data in H^2. See also [1, 2] for ill-posedness results in compressible MHD. **In particular, our regularity assumptions are optimal with respect to the "sound-wave-part" of the data.** One cannot hope to avoid singularities globally in time: it is known that, even in the irrotational and isentropic case, the compression of sound waves can cause shocks to develop from regular initial data in finite time. Moreover, in more than one space dimension and away from symmetry, these singularities are known to be stable as in [5].

For the irrotational and isentropic case in [5], the relativistic Euler equations reduce to a system of covariant quasilinear wave equations for the first derivatives $\psi_\alpha := \partial_\alpha \phi$ of a potential function ϕ of the following form:

$$\square_{\mathbf{g}(\psi)}(\psi_\alpha) = \mathscr{Q}(\psi)[\boldsymbol{\partial}\psi, \boldsymbol{\partial}\psi]. \tag{1.5}$$

Classical local well-posedness in $H^{(5/2)+}$ for the quasilinear wave system (1.5) can be obtained by applying energy estimates and Sobolev embedding, see Kato [13]. Starting in the late 90s, the regularity needed for local well-posedness for quasilinear wave equations was improved in a series of works by Bahouri-Chemin, Smith-Tataru and Klainerman-Rodnianski, see [3, 4, 31, 30, 16, 14]. The optimal result for low regularity H^{2+} of quasilinear wave equations was first achieved by Smith-Tataru in [26]. In [36], Wang reached the same result as in [26] by using a geometric approach. With the presence of vorticity, Disconzi-Luo-Mazzone-Speck, Wang, and Zhang-Andersson proved low-regularity local well-posedness result for the 3D compressible Euler equations in [9], [37] and [38] respectively. In all three works, the regularity of "wave-part" is in the optimal level H^{2+}. We will discuss the details of the assumptions for the data in [9] and [37] in Section 1.1.

Compared to the non-relativistic case, the first fundamental form of $\Sigma_t := \{t\} \times \mathbb{R}^3$ is no longer conformally flat in the relativistic case, leading to more complicated geometry. One of the main challenges of this paper is that, (1.2)-(1.4) seemingly suffers from a loss of derivative. This is because at the level of regularity, $\mathscr{C}, \mathscr{D} \simeq \boldsymbol{\partial}^2 \vec{\Psi}$, which is an issue since $\mathscr{C}$ and $\mathscr{D}$ show up as the source terms in the right-hand side of the wave equation (1.2). In [9], this was solved by using Hodge theory on the spacelike hypersurfaces Σ_t. In our case, the transport-div-curl system (1.4) is a space-time (non-elliptic Hodge) system, from which we have to extract a quasilinear elliptic Hodge system on the spacelike hypersurfaces, that is, we rewrite (see Proposition 5.9) the space-time div-curl system (1.4) into a spatial elliptic div-curl system with source terms that can be controlled only due to the special structure of the equations:

$$G^{ab} \partial_a ((\omega_\flat), S)_b = F, \tag{1.6}$$

$$\partial_a ((\omega_\flat), S)_b - \partial_b ((\omega_\flat), S)_a = H_{ab}. \tag{1.7}$$

In (1.6), $G^{-1} := G^{-1}(v)$ is the inverse of a Riemannian metric on constant-time hypersurfaces (see equation (5.18c) for the definition of G^{-1}). By using these structures, we are able to adequately control the vorticity and entropy gradient by using energy estimates for transport equations, elliptic estimates, Schauder estimates, and Littlewood-Paley theory.

We now state the main results of this paper.

Theorem 1.1 (Main theorem)**.** *Consider a smooth*[4] *solution to the relativistic Euler equations whose initial data on the initial Cauchy hypersurface* $\Sigma_0 := \{0\} \times \mathbb{R}^3$ *satisfies following assumptions for some real number* $2 < N < 5/2$, $0 < \alpha < 1$, $c_1 > 0$ *and* D*:*

1. ***"Wave part"****:* $\|h, v\|_{H^N(\Sigma_0)} \leq D$*,*

2. ***"Transport part"****:* $\|\omega\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)} \leq D$*, In addition, Modified fluid variables* $\mathscr{C}$ *and* $\mathscr{D}$ *(*$\mathscr{C} \simeq \mathrm{vort}(\omega)$ *and* $\mathscr{D} \simeq \mathrm{div} S$*, see Subsection 2.2.2 for the definition of operator* vort *and* ω*, Definition 2.6 for the definition of* S*, and Section 2.8 for the definition of* $\mathscr{C}$ *and* $\mathscr{D}$*) satisfy the Hölder-norm bound* $\|\mathscr{C}, \mathscr{D}\|_{C^{0,\alpha}(\Sigma_0)} \leq D$*,*

3. The image of data functions are contained in an interior of a compact subset $\mathscr{R}$ *(defined in Section 3.5) and the enthalpy* H *is positive, i.e.* $H \geq c_1 > 0$*.*

Then the solution's time of classical existence $T := T(D, \mathscr{R}) > 0$ *can be controlled in terms of only* D *and* $\mathscr{R}$*. Moreover, the Sobolev and some Hölder regularities of the data are propagated by the solution.*

See Section 1.5 for the main ideas behind proving Theorem 1.1.

1.1 Overview of Previous Low-Regularity Results

There have been many developments on low regularity problems for quasilinear wave equations and the non-relativistic 3D compressible Euler equations in past two decades. For quasilinear wave equations of the form (1.5), Bahouri-Chemin [4] and Tataru [30] independently showed local well-posedness with $H^{(2+\frac{1}{4})+}$ data. The improvements rely on Strichartz estimates based on Fourier integral parametrix representations. Bahouri-Chemin improved their earlier result to $H^{(2+\frac{1}{5})+}$ in [3]. Tataru pushed the results down to $H^{(2+\frac{1}{6})+}$ in [31] and Klainerman reached the same level in [16]. Klainerman-Rodnianski achieved $H^{(2+\frac{2-\sqrt{3}}{2})+}$ in [14]. Klainerman-Rodnianski proved $H^{2+\varepsilon}$ result for Eistein equations in wave coordinate in [18]. We also note that the L^2 curvature conjecture, which suggests that the time of existence of a classical solution to the Einstein-vacuum equations depends only on the L^2-norm of the curvature and a lower bound on the volume radius of the corresponding initial data set, was proved by Klainerman-Rodnianski-Szeftel in [20]. The optimal low regularity result H^{2+} for generic quasilinear wave equations was first achieved by Smith-Tataru in [26] by using wave-packets and properties of the geometry of characteristic light cones that were introduced in [14]. Besides the improvements over Sobolev exponents, a commuting vectorfield approach for Strichartz estimates was introduced by Klainerman in [16], and a fundamental decomposition of a Ricci component of **g** was used for improving the regularity in the causal geometry by Klainerman-Rodnianski in [14]. Recently,

[4]By smooth we mean as smooth as necessary for the analysis arguments to go through. We note that all of our quantitative estimates depend only on the Sobolev and Hölder norms.

Wang gave a second proof of the Smith-Tataru [26] result by using this geometric approach. The proof in Wang [36] relied on an upgraded version of Klainerman-Rodnianski's vectorfield method with the help of conformal energy estimates. We again emphasize that, for the general quasilinear wave equation of the form (1.5), it is impossible to prove any well-posedness result with data in H^2. Specifically, Lindblad provided an example of ill-posedness for a quasilinear wave equation with H^2 initial data in [21]. For the non-relativistic compressible Euler flow with vorticity and entropy, under the H^{2+} assumptions on "wave-part" and "transport part" of the data, Disconzi-Luo-Mazzone-Speck [9] and Wang [37] proved local well-posedness result for 3D compressible Euler equations. By assuming the Hölder regularity $C^{0,\alpha}$ for the data of the modified fluid variables $\mathscr{C}$ and $\mathscr{D}$ (our analogue of $\mathscr{C}$ and $\mathscr{D}$ are defined in Definition 2.8), the authors are able to prove a Schauder estimate in [9] for a transport-div-curl system in order to propagate the vorticity and entropy gradient along the waves. A non-local method of decomposing the velocity is given in [37] for the isentropic compressible Euler equations, which allowed Wang to remove the Hölder assumption on vorticity. In [38], Zhang-Andersson combined the methods in [26] and [37] to give an alternate proof of a similar result as in Wang [37].

1.2 A Brief Overview of the Strategy of the Proof

Klainerman [16], Klainerman-Rodnianski [14], and Wang [36] developed a geometric approach for proving the low regularity well-posedness for the quasilinear wave equations. The new formulation (1.2)-(1.4) provided by Disconzi-Speck [10] makes it possible to import the geometric techniques from [16, 14, 36] to the "sound-wave-part" of compressible Euler flow. The main difference with the wave problem is the addition of another characteristic speed into the problem, namely, the "transport-part". These two parts of the equations and solutions interact with each other, which creates substantial difficulties for understanding the Euler flow. See Section 1.3 for further discussions of the geometric formulation. See also Luk-Speck [23] for similar formulations for 3D isentropic compressible Euler equation and Speck [28] for 3D compressible Euler equations with any equation of state. The main tool for controlling the solution in the low-regularity setting, by using energy estimate (see Christodoulou [5, Chapter 1] for the energy current and its properties) and Littlewood-Paley theory, is the following estimates[5]:

$$\|(h,s,v)\|_{H^{2+\varepsilon}(\Sigma_t)} \lesssim \|(h,s,v)\|_{H^{2+\varepsilon}(\Sigma_0)} + \int_0^t \left(\|\boldsymbol{\partial}(h,s,v)\|_{L^\infty_x(\Sigma_\tau)} + 1\right) \|(h,s,v)\|_{H^{2+\varepsilon}(\Sigma_\tau)} \,\mathrm{d}\tau. \tag{1.8}$$

[5] We denote the constant-time hypersurface at time t by Σ_t. Moreover, $A \lesssim B$ means $A \leq C \cdot B$ for some universal constant C depending on region $\mathscr{R}$ and data norm D.

In order to make (1.8) useful, one needs to control $\|\boldsymbol{\partial}(h,s,v)\|_{L^1_t L^\infty_x}$. Since one is not able to apply Sobolev embedding to recover the bound below $H^{5/2+}$, we instead use a geometric approach to show the following **Strichartz estimates**: $\|\boldsymbol{\partial}(h,s,v)\|_{L^2_t L^\infty_x} \lesssim T_*^{2\delta}$. This is done by a bootstrap argument with bootstrap assumptions $\|\boldsymbol{\partial}(h,s,v)\|_{L^2_t L^\infty_x} \leq 1$, where T_* is the bootstrap time. In order to prove the Strichartz estimates, we apply a series of reductions. We reduce the Strichartz estimates to a decay estimates by using a $\mathscr{T}\mathscr{T}^*$ argument, then to a conformal energy estimate (see Definition 10.1 for the definition of conformal energy and Theorem 10.2 for boundness theorem for conformal energy) by Littlewood-Paley theory. See Section 1.5.4 for overview of the reduction, Section 4 for an extended overview of a global structure and Section 6-7 for details.

A crucial step in our geometric approach is the introduction of an acoustical function u satisfying the acoustical eikonal equation $\left(\mathbf{g}^{-1}\right)^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$, where the acoustical metric $\mathbf{g} = \mathbf{g}(h,v,s)$ is a Lorentzian metric (see Definition 2.10 for the definition of $\mathbf{g}$) distinct from the Minkowski metric. With the help of u, we construct a null frame and control the acoustic geometry along acoustic null cones, which are the level sets of u (see the figure on page 71). This allows us to derive suitable estimates for conformal energy. Disconzi-Luo-Mazzone-Speck [9] and Wang [37] showed that given good control over the "transport-part", we can run the machinery of Strichartz estimates for the "wave-part" where we treat the "transport-part" as a favorable source term. Thus, good control over the "transport-part" is a crucial component of our analysis. Inspired by the analysis of the 3D non-relativistic compressible Euler case in [9], we derive elliptic and Schauder estimates for the transport-div-curl systems to bound the $H^{1+\varepsilon}$ and $C^{0,\alpha}$ norms of $\mathscr{C}, \mathscr{D}$ when the wave part is rough.

The main new difficulty that is not found in 3D non-relativistic compressible Euler is: due to the space-time structure of the relativistic Euler flow, we encounter space-time velocity, vorticity, and the div-curl system where the ellipticity is not immediately apparent. To overcome this difficulty, we exploit two crucial aspects. We first notice that the v-directional derivative of the vorticity and entropy gradient is favorable due to the transport phenomena. To obtain control of v-orthogonal directional derivatives, we reduce the space-time div-curl system of vorticity and entropy gradient to a dynamic div-curl system on the constant-time hypersurfaces. By combining these special structures of relativistic Euler equations with Littlewood-Paley decomposition and properties of pseudodifferential operators, we derive estimates for vorticity and entropy.

We present the logical graph of this paper in Section 1.5.

1.3 Geometric Formulation of the Relativistic Euler Equations

Due to the coupling of sound waves with vorticity and entropy in the equations (1.2)-(1.4), when considering the relativistic Euler equations with an arbitrary equation of state, one needs to precisely and carefully split the dynamics into a "wave-part", which describes the propagation of sound waves, and a "transport-part", which describes the evolution of vorticity and entropy. For the 3D non-relativistic compressible Euler equations with any equation of state, Speck [28] derived a system consisting of geometric wave equations and transport-div-curl equations. This geometric formulation is used for the low-regularity problem in [9, 37]. See also Luk-Speck [22] and [23] for the geometric formulation of the compressible Euler in the barotropic case and its application to the shock formation problem. Disconzi-Speck [10] derived the geometric formulation of the relativistic Euler equations with vorticity and dynamic entropy that we used in this paper. It allows us to describe the influence of transport phenomena on the wave part of the system and the acoustic geometry with rough sound wave data given in the relativistic Euler flow. These geometric formulations have origins in Christodoulou and Christodoulou-Miao's proof of stable shock formation for the relativistic Euler equations and non-relativistic 3D compressible Euler equations in the irrotational and isentropic case [5][7].

The new geometric formulation from [10] (see (1.2)-(1.4) and Proposition 2.17 for the formulation) splits the dynamics into a "wave-part", which consists of geometric wave equations for the fluid variables h, v, s, and a "transport-div-curl-part", which governs the transport equations of special vorticity, entropy gradient and modified fluid variables $\mathscr{C}, \mathscr{D}$ ($\mathscr{C}, \mathscr{D}$ are special combinations of variables whose essential terms are the vorticity of vorticity and divergence of entropy gradient that are defined in Definition 2.8), and div-curl systems for the special vorticity and entropy gradient. The advantage of the geometric formulation is that one can do analysis on both "wave-part" and "transport-part", which are highly coupled. Here we briefly summarize the new formulation and its connection to establishing the Strichartz estimate:

- The "wave-part" of the formulation involves wave equations with principal part $\square_{\mathbf{g}}$. Properties of this operator are intimately related to the acoustic geometry, which is constructed via an acoustical function u. Here, u is a solution to the acoustical eikonal equation $\left(\mathbf{g}^{-1}\right)^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$, where the acoustical metric $\mathbf{g} = \mathbf{g}(h, v, s)$ is the Lorentzian metric defined in Definition 2.10. With the help of u, we construct a null frame and derive some transport equations as well as div-curl systems for some particular connection coefficients along acoustic null cones, which are the level sets of u (see the figure on page 71). We note that these equations for the connection coefficients are derived from basic geometry considerations and are independent of the relativistic Euler equations. By using a delicate decomposition of certain curvature components, which are highly tied to the geometric wave equations (2.29), we can control a large group of geometric quantities that are fundamental for deriving the conformal energy estimates. We emphasize already that achieving control of these geometric quantities is essential for controlling a certain conformal energy for solutions to the linear wave equation corresponding to the acoustical metric $\mathbf{g}$, i.e., solutions φ to the PDE $\square_{\mathbf{g}(\vec{\Psi})} \varphi = 0$. It is crucial to control the conformal energy in order to derive the decay estimates, which we again emphasize is the main ingredient needed to obtain the desired Strichartz estimate. We will describe the conformal energy and decay estimates in more detail in Section 1.5.

- The "transport-div-curl-part" of the formulation allows one to control the vorticity and entropy at one derivative level above standard estimates. The analysis uses transport estimates as well as Hodge estimates at constant-time hypersurfaces. This is highly non-trivial and more complicated compared to non-relativistic 3D compressible Euler because the Hodge system that we encounter is a space-time div-curl system. In total, we are able to show that the transport terms are "good" source terms in the wave equation estimates. We point out the vorticity and entropy gradient also appear in PDEs that we use to control the acoustic geometry because of the geometric wave equations (1.2). This shows that there are interactions between the vorticity, entropy, sound waves, and acoustic geometry.

1.4 Comparision with Low-regularity Results for 3D Non-relativistic Compressible Euler Equations

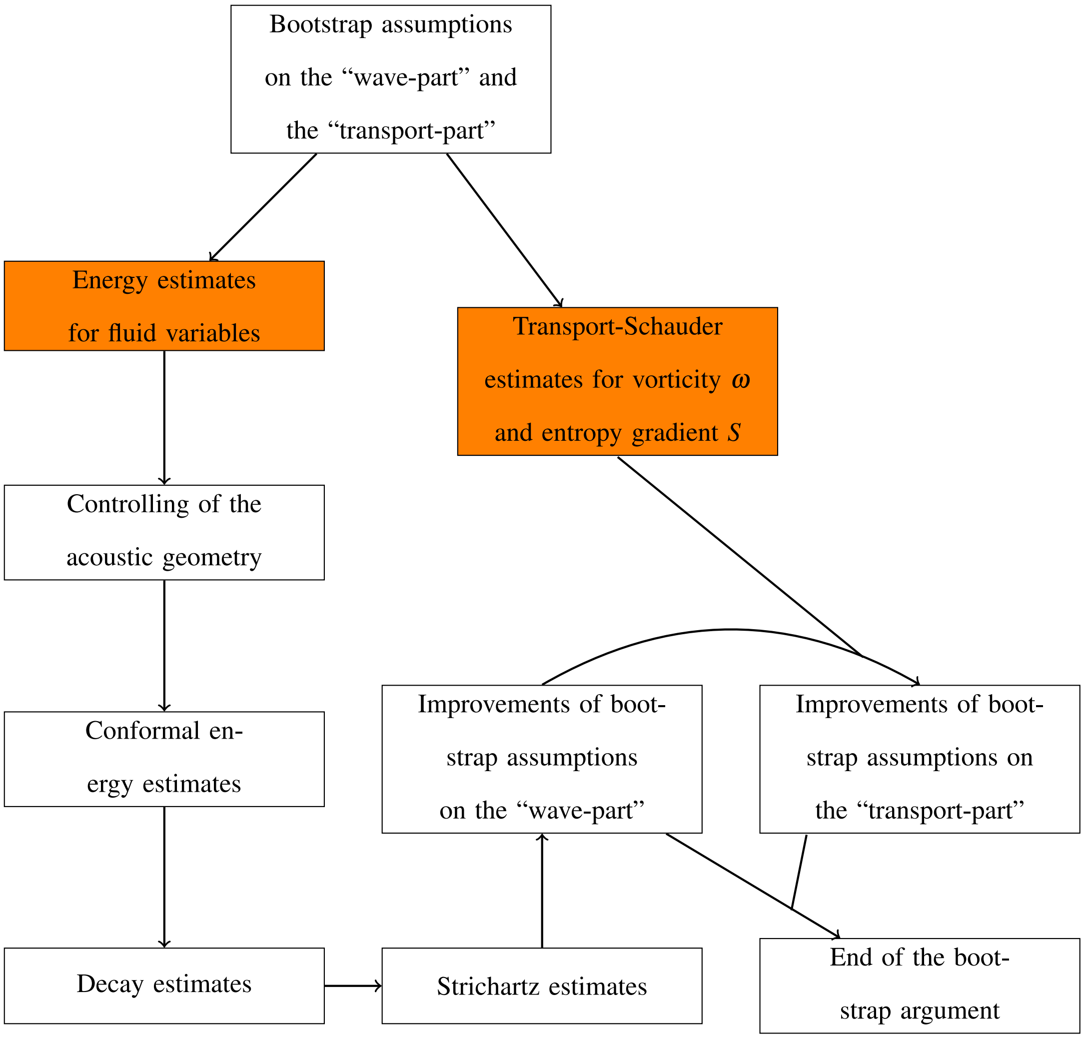
Recently, [9] and [37] proved low-regularity results for the 3D non-relativistic compressible Euler equations with the help of the geometric formulation in [28]. In [9], Disconzi-Luo-Mazzone-Speck showed that, if the "wave-part" of the data is initially in H^{2+}, and the "transport-part" $\mathscr{C}, \mathscr{D}$ are in $C^{0,\alpha}$, then the regularity of the data is preserved by solutions for short times. For barotropic flow, Wang proved a similar result by removing the Hölder assumption for $\mathscr{C}, \mathscr{D}$ and assuming the H^{2+} assumptions on the "transport-part" in [37]. For the relativistic Euler equations, we prove a similar result as in [9] for the 3D compressible Euler equations. That is, we allow any equation of state and we have the same level of regularity assumptions on the initial data. Due to the geometric nature of the relativistic Euler flow, the vorticity ω^α in this article is a space-time v-orthogonal vectorfield (see Definition 2.3) which solves a space-time transport-div-curl system. In the 3D non-relativistic compressible Euler case, the geometry of vorticity is much simpler: it is a Σ_t-tangent vector field and solves a div-curl system with constant coefficients on constant-time slices.

In the relativistic Euler equations, vorticity and entropy gradient satisfy transport equations in the v-direction. To control generic v-orthogonal (with respect to Minkowski metric) derivatives of vorticity and entropy gradient, we rely on a space-time div-curl system. Moreover, using the transport equations satisfied by the modified fluid variables $\mathscr{C}, \mathscr{D}$, we can control these quantities not only along constant-time slices, but also along null cones, which is fundamental in our work. To derive sufficient regularity for vorticity, entropy gradient, and modified fluid variables along constant-time slices, we rewrite the space-time div-curl system into a spatial div-curl system with source terms that can be controlled only due to the special structure of the equations. A crucial ingredient in our analysis is that the spatial divergence equation has the form $(G^{-1})^{ij}\partial_i(\omega_\flat)_j = ...$ where $G^{-1} := G^{-1}(v)$ is the inverse of a Riemannian metric on constant-time hypersurfaces (see equation (5.18c) for the definition of G^{-1}). Because the coefficient metric G of the divergence equation is Riemannian, by using the technique of freezing the spatial points, we are able to derive a localized div-curl system with constant coefficient principle terms, such that the Fourier transform of vorticity and entropy gradient is bounded in the frequency space by the source terms of the div-curl system. This allows us to control appropriate Hölder norms of $\partial \omega, \partial S$ in terms of the same Hölder norms of $\mathscr{C}, \mathscr{D}, \boldsymbol{\partial} \vec{\Psi}$. The analysis relies on the Littlewood-Paley theory as well as the standard theorem in pseudodifferential operators. We take a similar approach when deriving the elliptic div-curl estimates in L^2 space, where we need to control derivatives of vorticity and the entropy gradient by $\boldsymbol{\partial} \vec{\Psi}$, the modified fluid variables $\mathscr{C}$ and $\mathscr{D}$. Finally, we use the transport equations (1.4a)-(1.4b) and initial assumptions of $\mathscr{C}, \mathscr{D}$ to bound the Hölder norm of $\mathscr{C}, \mathscr{D}$ by $\partial \omega, \partial S$ to close the estimates for $\partial \omega, \partial S$.

1.5 Main Idea of the Proof of Theorem 1.1

Theorem 1.1 provides a priori estimates for smooth solutions, which is needed for a full proof of local well-posedness. The remaining aspects of a full proof of local well-posedness could be shown by deriving uniform estimates for sequences of smooth solutions and their differences. We refer readers to [26, Sections 2-3] for the proof of local well-posedness based on a priori estimates.

In this subsection, we present the logic of proofs in this paper, that is, the bootstrap argument. The colored steps involve new ingredients, where we need to do analysis based on the special structure of the relativistic Euler equations (see Subsection 1.5.1 and 1.5.3 for a discussion of these steps). The uncolored steps are introduced in the previous low-regularity problem works (see Subsections 1.5.4-1.5.7 for a discussion of these steps). We emphasize that, with the estimates we derive in the colored steps, the proofs of the uncolored steps are essentially the same as in [9, 36, 14, 4, 3, 26, 31, 16]. Hence, in this work, we provide all of the details for the colored steps, and give terse sketches for the uncolored steps with the appropriate citations.



1.5.1 Overview of elliptic and energy estimates

In this subsection, we provide an overview of how the energy estimates work and are related to the bootstrap assumptions (1.15a)-(1.15b). We provide representative energy estimates for wave variables, vorticity, and entropy gradient by using the basic energy estimates (see Section 5.1.1) and L^2 elliptic estimates (see Section 5.1.2). Then we leave the discussion of the key assumptions to future subsections and detailed estimates in Section 5.1.

We first consider the energy estimates for the "wave-part". Given any $2 < N < 5/2$, $0 < t \leq T_*$ where $0 < T_* \ll 1$ denotes the bootstrap time. By the vectorfield multiplier method and Littlewood-Paley calculus applied to the equations (1.2), we derive the following energy estimates for the "wave-part":

$$\left\|\partial\vec{\Psi}\right\|^2_{H^{N-1}(\Sigma_t)} \lesssim \left\|\partial\vec{\Psi}\right\|^2_{H^{N-1}(\Sigma_0)} + \int_0^t \left(\left\|\partial\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)} + 1\right)\left\|\partial\vec{\Psi},\mathscr{C},\mathscr{D}\right\|^2_{H^{N-1}(\Sigma_\tau)} \mathrm{d}\tau, \tag{1.9}$$

which is an analogue of (1.8). We will provide a detailed expression and its proof in Section 5.1.3.

To control $\vec{\mathscr{C}}, \mathscr{D}$ on the right-hand side of (1.9), we then consider the energy estimates for the "transport-part". We first need an important L^2 elliptic div-curl estimate

$$\left\|(\partial\vec{\omega},\partial\vec{S})\right\|_{L^2_x(\Sigma_t)} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|_{L^2_x(\Sigma_t)}. \tag{1.10}$$

We note that the proof of (1.10) requires a rewriting div-curl system (5.17a)-(5.17b) for vorticity and entropy gradient, where one has to exploit the structure of the relativistic Euler equations. By splitting the space-time div-curl systems into time and spatial directions of derivatives, taking the advantage of the transport equations (1.3a)-(1.3b) for ω and S, we write time derivative of vorticity and entropy gradient components as a combination of spatial derivatives of ω and S. We obtain a new **spatial** div-curl system of the form:

$$(G^{-1})^{ab}\partial_a((\omega_\flat),S)_b = F, \tag{1.11a}$$

$$\partial_a((\omega_\flat),S)_b - \partial_b((\omega_\flat),S)_a = H_{ab}, \tag{1.11b}$$

where $G^{-1} = G^{-1}(v)$ is Riemannian, $F = \mathscr{D} + l.o.t.$ and $H = \mathscr{C} + l.o.t.$. (1.11) is a PDE system on constant-time slices. Notice that equation (1.11a) is a quasilinear divergence equation while the analogue in [9] is a constant-coefficient equation. Then, by Littlewood-Paley estimates and a partition of unity argument, we prove (1.10) in Proposition 5.8.

As in Proposition 5.8, by (1.10) and Littlewood-Paley calculus, we also have, for $2 < N < 5/2$,

$$\left\|(\partial\vec{\omega},\partial\vec{S})\right\|_{H^{N-1}(\Sigma_t)} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|_{H^{N-1}(\Sigma_t)}. \tag{1.12}$$

By applying energy estimates and Littlewood-Paley calculus on evolution equations (1.4a)-(1.4b) for $\vec{\mathscr{C}}, \mathscr{D}$, we have the following energy estimate for $\vec{\mathscr{C}}, \mathscr{D}$:

$$\begin{aligned}\left\|\vec{\mathscr{C}},\mathscr{D}\right\|^2_{H^{N-1}(\Sigma_t)} \lesssim{}& \left\|\vec{\mathscr{C}},\mathscr{D}\right\|^2_{H^{N-1}(\Sigma_0)} \\ &+ \int_0^t \left(\left\|\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S}\right\|_{L^\infty_x(\Sigma_\tau)} + 1\right)\left\|\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S},\vec{\mathscr{C}},\mathscr{D}\right\|^2_{H^{N-1}(\Sigma_\tau)}\mathrm{d}\tau.\end{aligned} \tag{1.13}$$

By elliptic estimates (1.12), (1.13) and (1.9), we have

$$\begin{aligned}\left\|\partial\vec{\omega},\partial\vec{S}\right\|^2_{H^{N-1}(\Sigma_t)} &\lesssim \left\|\partial\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|^2_{H^{N-1}(\Sigma_t)} \\ &\lesssim \left\|\partial\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|^2_{H^{N-1}(\Sigma_0)} \\ &\quad + \int_0^t \left(\left\|\partial\vec{\Psi},\partial\vec{\omega},\partial\vec{S}\right\|_{L^\infty_x(\Sigma_\tau)} + 1 \right) \left\|\partial\vec{\Psi},\partial\vec{\omega},\partial\vec{S},\vec{\mathscr{C}},\mathscr{D}\right\|^2_{H^{N-1}(\Sigma_\tau)} \mathrm{d}\tau. \end{aligned} \tag{1.14}$$

The results of energy estimates are obtained in Subsection 5.1.3.

From (1.9), (1.13), (1.14) and Grönwall's inequality, we see that if $\left\|\partial\vec{\Psi},\partial\vec{\omega},\partial\vec{S}\right\|_{L^1_t L^\infty_x([0,t]\times\Sigma_t)}$ is bounded (note that $\mathscr{C} \simeq \mathrm{vort}(\omega) + l.o.t$, $\mathscr{D} \simeq \mathrm{div} S + l.o.t$), the Sobolev regularity of the data can be propogated by the solution. This drives us to set up a bootstrap argument with the bootstrap assumptions as introduced in the next subsection.

1.5.2 Bootstrap assumptions

As we made it clear in the previous subsection, our argument crucially relies on the boundness of the term $\left\|\partial\vec{\Psi},\partial\vec{\omega},\partial\vec{S}\right\|_{L^1_t L^\infty_x(\mathscr{M})}$. We prove the boundness of this via a bootstrap argument that we now describe:

Throughout the paper, $0 < T_* \ll 1$ denotes the bootstrap time. We assume that $\vec{\Psi},\vec{\omega},\vec{S}$ is a smooth solution to the relativistic Euler equations. For $\delta_0 > 0$ defined as in Section 3.4, we assume the following estimates hold:

$$\left\|\partial\vec{\Psi}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \left\|P_\nu \partial\vec{\Psi}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \leq 1, \tag{1.15a}$$

$$\left\|\partial\vec{\omega},\partial\vec{S}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \left\|P_\nu \partial\vec{\omega}, P_\nu\partial\vec{S}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \leq 1, \tag{1.15b}$$

where P_ν is the Littlewood-Paley projection (see Section 3.2 for definition). The Littlewood-Paley terms in the assumptions are needed for establishing the dyadic Strichartz estimate in order to improve the bootstrap assumptions.

In the classical local well-posedness problem of the relativistic Euler equations, the regularity assumptions are $(h,v,s) \in H^{(5/2)+}(\Sigma_0)$. This gives the $\partial\vec{\Psi} \in H^{(3/2)+}(\Sigma_0)$. One can recover the boundness assumption $\left\|\partial\vec{\Psi}\right\|_{L^\infty_x(\Sigma_t)}$ by standard energy estimates and Sobolev embedding $H^{3/2+} \hookrightarrow L^\infty$ at any constant-time hypersurface Σ_t. The lack of Sobolev embedding in the low-regularity level forces one to find a new machinery to improve the reasonable bootstrap assumptions (see Section 1.1 for introduction of previous results). Recovering the bootstrap assumptions occupies a large part of this article.

1.5.3 Transport-Schauder estimates for the transport-div-curl system

In this subsection, we explain how to improve the bootstrap assumption (1.15b). In particular, by applying Hölder's inequality in time, this will show $\left\|\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S}\right\|_{L^1_tL^\infty_x([0,T_*]\times\mathbb{R}^3)}$ is small. This improvement is conditional on (1.19), which we explain how to derive in Subsection 1.5.4.

Because of the lack of tools in Hodge estimates in L^∞ space, we have assumed a slight bit of extra regularity for the "transport-part". That is, we propagate the Hölder boundness of "transport-part" with given initial data $\left\|\vec{\mathscr{C}},\mathscr{D}\right\|_{C^{0,\alpha}_x(\Sigma_0)} \leq D$ where $0<\alpha<1$ and $D\in\mathbb{R}$. Besides using the transport equations (1.3a)-(1.3b) which exhibit source terms with surprisingly good structures, we also rely on the following Schauder-type estimate:

$$\left\|\partial\vec{\omega},\partial\vec{S}\right\|_{C^{0,\alpha}_x(\Sigma_t)} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|_{C^{0,\alpha}_x(\Sigma_t)}. \tag{1.16}$$

In order to control $\mathscr{C},\mathscr{D}$ on the right-hand side of (1.16), we use the transport equations (1.4a)-(1.4b) of $\mathscr{C}$ and $\mathscr{D}$, which are coupled to $\partial\omega,\partial S$ (see equation (1.4a)-(1.4b)). By combining the two, under bootstrap assumptions (1.15a), we can apply Grönwall's inequality to bound the vorticity and entropy gradient of the relativistic Euler flow. We will discuss this approach in more details below.

To derive Schauder estimates, we split the derivative of the vorticity ω and entropy gradient S (see Definition 2.3 and 2.6 for the definition of ω and S) into v-tangent direction and the Σ_t-tangent directions (v is transversal to Σ_t with respect to Minkowski metric). Now we highlight the following two features in our analysis:

- The v-tangent direction derivatives of vorticity and entropy gradient are favorable because of the transport phenomena. That is, by using transport equations (1.3a)-(1.3b), we are able to obtain the Hölder bound for v-tangent direction of vorticity and entropy gradient by using bootstrap assumptions.

- To control the Σ_t-tangent directional derivatives of vorticity and entropy gradient, we rely on a spacetime transport-div-curl system for ω and S. We note that it is qualitatively distinct from the case in [9] for 3D non-relativistic compressible Euler equations where the div-curl equations are spatial with constant coefficients.

We now explain how we derive Schauder estimates (1.16). We use the same div-curl system (1.11a)-(1.11b) as in the L^2 elliptic estimates. By partition of unity, Fourier transform, Littlewood-Paley theory and properties of pseudodifferential operators, we are able to bound $\left\|\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S}\right\|_{C^{0,\alpha}_x(\Sigma_t)}$ by $\left\|\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|_{C^{0,\alpha}_x(\Sigma_t)}$ as in (1.16), see Lemma 5.20 for the detailed proof. Then we bound $\left\|\vec{\mathscr{C}},\mathscr{D}\right\|_{C^{0,\alpha}_x(\Sigma_t)}$ by estimating solutions to the transport equations (1.4a)-(1.4b) as follows:

$$\left\|\vec{\mathscr{C}},\mathscr{D}\right\|_{C^{0,\delta_1}_x(\Sigma_t)} \lesssim 1+\int_0^t \left(\left\|\partial\vec{\Psi}\right\|_{C^{0,\delta_1}_x(\Sigma_\tau)}+1\right)\left\|\partial\vec{\Psi},\partial\vec{\omega},\partial\vec{S}\right\|_{C^{0,\delta_1}_x(\Sigma_\tau)} \mathrm{d}\tau. \tag{1.17}$$

Finally, combining (1.16) and (1.17), we use Grönwall's inequality and bootstrap assumptions to close the transport-Schauder type estimates

$$\left\|\partial\vec{\omega},\partial\vec{S}\right\|_{C^{0,\alpha}_x(\Sigma_t)} \lesssim \left\|\partial\vec{\Psi}\right\|_{C^{0,\alpha}_x(\Sigma_t)}. \tag{1.18}$$

We emphasize that later in the argument, we will integrate (1.18) in time and combine it with the improved Strichartz estimate (1.19) (which is obtained independently of (1.18)). These lead to a strict improvement of the bootstrap assumptions (1.15b) for T_* small. We provide full details of the Schauder estimates in Section 5.

1.5.4 Reductions of the Strichartz type estimates

Our argument above crucially relies on bounding $\left\|\partial\vec{\Psi}\right\|_{L^1_tL^\infty_x(\mathscr{M})}$. In this subsection, we explain how we derive strict improvements of bootstrap assumptions (1.15a), that is, we describe how to derive Strichartz estimates (1.19). By taking the advantage of the smallness of bootstrap time interval $[0,T_*]$, we will improve our bootstrap assumptions to the following Strichartz estimates:

$$\left\|\partial\vec{\Psi}\right\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)}+\sum_{\nu\geq 2}\nu^{2\delta_1}\left\|P_\nu\partial\vec{\Psi}\right\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}, \tag{1.19}$$

where $\delta>0$ is sufficiently small as in Subsection 3.4 and $8\delta_0<\delta_1<N-2$, where δ_0 is from the bootstrap assumptions (1.15a) and (1.15b). Notice that if T_* is small, then (1.19) is a strict improvement of (1.15a). We reduce the proof of (1.19) to the proof of estimates on the acoustic geometry by adopting the geometric approach of [36]. This reduction is done through the following steps: Improvement of bootstrap assumptions⟵Strichartz estimates⟵Decay estimates⟵Conformal energy estimates⟵Controlling of the acoustic geometry, where the left arrow indicates that the latter estimate implies the former. We remind readers of the logic diagram at the beginning of Section 1.5.

- *Reduction to dyadic Strichartz estimates.* The first step in the proof of (1.19) is to reduce Strichartz estimates to a dyadic Strichartz estimate. Specifically, for a fixed large dyadic frequency λ, we partition $[0,T_*]$ into disjoint union of sub-intervals $I_k:=[t_{k-1},t_k]$ of total number $\lesssim\lambda^{8\varepsilon_0}$ with $|I_k|\lesssim\lambda^{-8\varepsilon_0}$ (see Section 3.4 for the definition of ε_0). By Littlewood-Paley decomposition and Duhamel principle, the proof of (1.19) can be reduced to a dyadic Strichartz estimate

$$\|P_\lambda \partial \varphi\|_{L_t^q L_x^\infty([\tau, t_{k+1}]\times\mathbb{R}^3)} \lesssim \lambda^{\frac{3}{2}-\frac{1}{q}} \|\partial \varphi\|_{L_x^2(\Sigma_\tau)}, \tag{1.20}$$

where φ is a solution of the covariant linear wave equation

$$\square_{\mathbf{g}} \varphi = 0, \tag{1.21}$$

on the time interval I_k. In (1.20), $q > 2$ is any real number which is sufficiently close to 2 and $\tau \in [t_k, t_{k+1}]$. Here, we focus on large frequencies since control of small frequency is much easier due to Berstein inequalities.

- *Reduction to decay estimates.* For a large frequency λ, by rescaling the coordinates (see Section 6.2 for the rescaling) and using an abstract $\mathscr{T}\mathscr{T}^*$ argument, we can reduce the dyadic Strichartz estimates (1.20) to $L^2 - L^\infty$ decay estimates at any $t \in [0, T_{*;(\lambda)}]$ where $T_{*;(\lambda)}$ is the rescaled bootstrap time (see Section 6.2 for the definition of $T_{*;(\lambda)}$):

$$\|P_1 \mathbf{T}\varphi\|_{L_x^\infty(\Sigma_t)} \lesssim \left(\frac{1}{(1+|t-1|)^{\frac{2}{q}}} + d(t)\right)\left(\|\partial \varphi\|_{L_x^2(\Sigma_1)} + \|\varphi\|_{L_x^2(\Sigma_1)}\right), \tag{1.22}$$

where the timelike vectorfied $\mathbf{T}$ is $\mathbf{g}$-unit normal to Σ_t (that is defined in Definition 2.11) and φ is an arbitrary solution to the equation $\square_{\mathbf{g}}\varphi = 0$ on the time interval $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ with $\varphi(1,x)$ supported in the Euclidean ball B_R (see Theorem 6.9 for detailed definition of R). Moreover, the function $d(t)$ satisfies

$$\|d\|_{L_t^{\frac{q}{2}}([0,T_{*;(\lambda)}])} \lesssim 1, \tag{1.23}$$

for $q > 2$ sufficiently close to 2.

- *Reduction to conformal energy estimates.* By product estimates and Littlewood-Paley theory, we reduce the proof of (1.22) to a proof of following estimates for the conformal energy $\mathfrak{C}[\varphi](t)$ (see Section 10.1 for the definition of the conformal energy $\mathfrak{C}[\varphi](t)$) at time $t \in [1, T_{*;(\lambda)}]$:

$$\mathfrak{C}[\varphi](t) \lesssim (1+t)^{2\varepsilon}\left(\|\partial \varphi\|^2_{L_x^2(\Sigma_1)} + \|\varphi\|^2_{L_x^2(\Sigma_1)}\right), \tag{1.24}$$

where $\varepsilon > 0$ is an arbitrary small number and φ is an arbitrary solution to the equation $\square_{\mathbf{g}}\varphi = 0$ on

$[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ with $\varphi(1)$ supported in $B_R \subset \mathscr{M}^{(\mathrm{Int})} \bigcap \Sigma_1$ (see Section 7.1 for the definition of $\mathscr{M}^{(\mathrm{Int})}$) where $\varepsilon > 0$ is an arbitrary small number.

We emphasize that both the reduction of (1.22) to (1.24), as well as the very definition of $\mathfrak{C}[\varphi]$ require the acoustic geometry, where its sharp control is needed for deriving (1.24). We describe how to obtain such control in Subsection 1.5.5-1.5.7. We provide an overview of the structure over the reductions in Section 4 and more detailed discussions in Section 6.

1.5.5 Structures for the causal geometry of the acoustic space-time

In order to reduce the decay estimates to conformal energy estimates (see Subsection 1.5.6 and 4.4 for introduction and Section 10 for details), one needs sharp information about the acoustic geometry. In this subsection, we discuss the geometric framework that is crucial for our analysis. This part of the result is well-known and standard (see Section 1.1 for the introduction of the previous results). The central object of our geometric framework is the acoustical function u, which is defined as a solution of the acoustical eikonal equation $\left(\mathbf{g}^{-1}\right)^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$, where $\mathbf{g}^{-1}$ is the inverse of the normalized acoustical metric. We denote the level sets of u by $\mathscr{C}_u$, which are forward truncated null cones (defined in Section 7.1).

We construct a null frame, which consists of a null pair $L, \underline{L}$ and two spherical vectorfields $\{e_A\}_{A=1,2}$ (see Section 7.2 for detailed definitions). We derive transport and Hodge type equations for the Ricci coefficients. An important example is the Raychaudhuri equation (see Section 8 for definitions of connection coefficients and PDEs verified by geometric quantities):

$$L\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2 = -|\hat{\chi}|^2_{\not g} - k_{NN}\mathrm{tr}_{\not g}\chi - \mathbf{Ric}_{LL}. \tag{1.25}$$

With the help of a remarkable decomposition of the Ricci curvature tensor (see Definition 8.6 for the definition of $\mathbf{\Gamma}_\alpha$):

$$\mathbf{Ric}_{\alpha\beta} = -\frac{1}{2}\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi}) + \frac{1}{2}\left(\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha\right) + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}], \tag{1.26}$$

and the Bianchi identities, where we can substitute $\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi})$ in (1.26) by using relativistic Euler equations (1.2), we are able to obtain some estimates for the Ricci coefficients, which we will utilize in the following subsections. We emphasize that it is important to have exactly $\mathscr{C}, \mathscr{D}$ on the right-hand side of (1.2). $\mathscr{C}, \mathscr{D}$ satisfies the transport equations (1.4a)-(1.4b), which allows us to derive estimates along constant-time slices and null hypersurfaces (in Subsection 1.5.1) and control the acoustic geometry.

The advantage of using the acoustic geometry in this low regularity setting is that it reveals the dispersive properties of solutions to the wave equations. That is, for a solution ϕ of wave equation $\square_{\mathbf{g}}\phi = 0$, the derivatives which are tangent to the characteristic null cones $\mathscr{C}_u$ have better decay than the transversal derivatives parallel to the $\underline{L}$ direction. We have to control some geometric quantities for several reasons:

- The acoustic geometry that we set up must be well-defined. In particular, we have to rule out short-time shock formation due to the intersection of distinct null cones.

- To bound a suitably constructed weighted energy in order to derive decay estimates, a multiplier vectorfield method needs to be introduced. The multipliers we use are related to L and Σ_t-tangent sphere normal vector N. Since L and N depend on the wave variables (h,s,v), the acoustical eikonal function u, and their first derivatives, to control the weighted energy, one needs to control relative derivatives of the above quantities.

1.5.6 Control of the conformal energy

A crucial part in the reduction of Strichartz estimates is to derive the decay estimates. As we discussed in Section 1.5.4, we use the conformal energy method that was introduced by Wang in [36]. We need to consider both of equation $\square_{\mathbf{g}}\varphi = 0$ and the conformal wave equation $\square_{\widetilde{\mathbf{g}}}(e^{-\sigma}\varphi) = \cdots$ ($\widetilde{\mathbf{g}} := e^{2\sigma}\mathbf{g}$ where σ is the conformal factor defined in Definition 8.6) to control various terms via the energy method. We are interested in such equations because we have reduced the Strichartz estimates for solution φ of geometric equation $\square_{\mathbf{g}}\varphi = 0$ (see Subsection 1.5.4 for the reduction).

Since the metric $\widetilde{\mathbf{g}}$ is only smoother along null hypersurfaces, we have to first use the original wave equation $\square_{\mathbf{g}}\varphi = 0$ and choose $X = fN$ (see Section 10.2 for the definition of f and Section 7.2 for sphere normal vector N) as the multiplier. By using the divergence theorem for a modified current on an appropriate region, we get a Morawetz-type energy estimate where we obtain a uniform bound for the standard energy of φ along a union of a portion of the constant-time hypersurfaces and null cones.

Then we consider the conformal wave equation

$$e^{2\sigma}\square_{\widetilde{\mathbf{g}}}\varphi = \square_{\mathbf{g}}\varphi + 2\mathbf{D}^{\alpha}\sigma\mathbf{D}_{\alpha}\varphi. \tag{1.27}$$

We use the multiplier approach with $\widetilde{r}^{p}L$ type vectorfields in the region $\{\tau_1 \leq u \leq \tau_2\}\bigcap\{\widetilde{r} \geq R\}$ where $1 \leq \tau_1 < \tau_2 < T_{*;(\lambda)}$ to control the conformal energy in the exterior region and to provide energy decay for each null slice. Finally, we control the conformal energy in the interior region with the help of the argument in [8] by obtaining energy decay in each spatial-null slice.

The very definition of the conformal energy, as well as its analysis, requires delicate and precise control of the acoustic geometry. We state the boundness theorem of conformal energy in Theorem 10.2. By using our estimates on the Ricci coefficients, one could follow the steps listed in [9, Section 11] to prove Theorem 10.2. One could go through the details of the argument in [36, Section 7]. Also, readers could look into [16, Section 3] for initial ideas.

1.5.7 Control of the acoustic geometry

To control the conformal energy, we need to control the acoustic geometry. Klainerman-Rodnianski [14] and Wang [36] developed an approach of controlling the geometric quantities in the low regularity setting. In this paper, we control the acoustic geometry by following the approach in Wang [36].

First, we provide the PDEs verified by the geometric quantities in [14, Section 2]. We write down the geometric transport equations and div-curl system for the connection coefficients. These equations depend on our geometric formalism and are independent of the relativistic Euler equations. Secondly, we use the estimates for certain Ricci and Riemann curvature tensor components by using the decomposition of the Ricci curvature (1.26) in [14, Lemma 2.1] and the Bianchi identities. It is at this step that the structure of the relativistic Euler equations is used. Specifically, we can substitute $\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi})$ in (1.26) by using relativistic Euler equations (1.2). $\mathscr{C},\mathscr{D}$ on the right-hand side of (1.2) satisfies the transport-div-curl system, which allows us to derive elliptic estimates and control the acoustic geometry.

Then by combining the geometric transport equations and the aforementioned Ricci and Riemann curvature tensor components estimates, one can derive and analyze the equations for many acoustic variables. These include the important mass aspect function μ and the conformal factor σ, which carries the rough part of geometry. Finally, we derive mixed space-time norm estimates for all the quantities, which are needed in the conformal energy estimates. We omit the details of the proof of controlling the geometry since the argument follows the same as in [9, Section 10].

In the following few paragraphs, we show why the standard Morawetz energy estimates are insufficient. The deformation tensor ${}^{(K)}\boldsymbol{\pi} := \mathscr{L}_K\mathbf{g}$ is present in the standard Morawetz-type energy estimates, where $K := \frac{1}{2}\left(u^2\underline{L} + (2t-u)^2 L\right)$ and $\mathscr{L}$ is the Lie derivative. ${}^{(K)}\boldsymbol{\pi}$ can be expressed by the connection coefficients of the null frame. We need to control them with the help of the transport and Hodge type equations for geometric quantities.

After integrating by parts to obtain the Morawetz-type energy identity, one needs to control various derivatives of ${}^{(K)}\boldsymbol{\pi}$. In particular, $\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/}\chi$, which is the angular derivative of the expansion scalar, as well as the mass aspect function $\mu := \underline{L}\mathrm{tr}_{g\!\!\!/}\chi + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\mathrm{tr}_{g\!\!\!/}\underline{\chi}$ (see Definition 8.5 for the definition of these geometric quantities). In order to control $\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/}\chi$, we rely on the Raychaudhuri equation (1.25) commuted with $\nabla\!\!\!/$. To control the Ricci term in (1.25), we use (1.26) contracted with $L^\alpha L^\beta$. After commuting with $\nabla\!\!\!/$, we have to control the error term $L(\nabla\!\!\!/ \boldsymbol{\Gamma}_L)$. It turns out that it is difficult to control $\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/}\chi$ and $\nabla\!\!\!/ \boldsymbol{\Gamma}_L$ separately. Therefore, standard Morawetz energy estimates are insufficient in our case.

Therefore, to control all terms at a consistent level of regularity, we use the approach of Wang [36], which relies on renormalized quantities and a metric that is conformal to the acoustical metric, where the conformal factor is carefully constructed so that the null expansion scalar associated to the conformal metric $\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}$ is precisely $\mathrm{tr}_{g\!\!\!/}\chi + \mathbf{\Gamma}_L$. We are able to obtain the regularity theory of $\mathrm{tr}_{g\!\!\!/}\chi + \mathbf{\Gamma}_L$, while it seems impossible to treat them independently.

The conformal wave equation attempts to resolve the issues described above, but introduces the difficult conformal factor σ. Thus, in order to obtain sufficient regularity for the conformal factor σ, we must in fact control the modified mass aspect function $\check{\mu}$:

$$\check{\mu} := 2\Delta\!\!\!/\sigma + \mu - \mathrm{tr}_{g\!\!\!/}\chi k_{NN} + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\mathbf{\Gamma}_{\underline{L}}. \tag{1.28}$$

as well as the modified torsion $\nabla\!\!\!/\sigma + \zeta$. These quantities satisfy favorable transport and div-curl systems, i.e., the source terms have sufficient regularity, moreover, they have good decay properties. We stress that this analysis relies on obtaining careful control over the top derivatives of the specific vorticity and entropy gradient, since the modified fluid variables $\mathscr{C}, \mathscr{D}$ enter as source terms in various geometric equations, such as the Raychaudhuri equation.

1.6 Paper Outline

The structure of this article will follow the non-relativistic 3D compressible Euler in [9]. The logical graph of this paper is in Section 1.5.

- In Chapter 2, we first state the notations that we are going to use throughout the paper. Then we define the fluid variables and tensor fields, including the acoustical metric. We also introduce the geometric formulation of the relativistic Euler equations.

- In Chapter 3, we define the Littlewood-Paley projections, which are frequently used in our analysis. We provide the frequency-projected versions of the evolution equations (i.e., the relativistic Euler equations) in Lemma 3.3. We state the main theorem and the bootstrap assumptions in Theorem 3.4 and Section 3.6.

- In Chapter 4, we discuss the structure of the proofs which we will follow in the rest of the paper.

- In Chapter 5, we use the bootstrap assumptions to derive the energy, L^2 elliptic, and Schauder estimates for the fluid variables along constant-time hypersurfaces. Note that Schauder estimates will improve the bootstrap assumption (3.24c) in Section 3.6 after the bootstrap assumption (3.24b) is improved by Strichartz estimates.

- In Chapter 6, following the approach of Tataru [31] and Wang [36], we rescale the fluid solution and reduce the proof of the Strichartz estimates to the proof of a spatially localized decay estimate. The reduction is essentially the same as one used by Wang [36], which we refer to for various details.

- In Chapter 7, we implement nonlinear geometric optics by constructing an acoustical function u and setting up its geometry, including constructing an appropriate null frame. Finally, we define the conformal energy and state the boundness theorem of the conformal energy in Theorem 10.2, which plays a crucial role in deriving the decay estimates that were stated in Theorem 6.9.

- In Chapter 8, we prove the energy estimates for the fluid variables along the acoustical null hypersurfaces in Section 8.1. We define additional geometric quantities, including the connection coefficients of the null frame, conformal factors, mass aspect functions, and curvature tensor components.

- In Chapter 9. We restate the estimates from [36, 9] that yield control over geometry along the initial data hypersurface. We state the bootstrap assumptions satisfied by the rescaled fluid variables as well as the bootstrap assumptions for geometric quantities in Subsection 9.1.2. Then we state the main estimates for the geometric quantities in Proposition 9.1, followed by a discussion of its proof in Subsection 9.2. Then we give a full detailed proof of Proposition 9.1 in Section 9.4.1.

- In Chapter 10. We follow the method in [36]. First, We define the conformal energy in Definition 10.1 followed by the boundness theorem in Theorem 10.2. We give the bootstrap assumptions of conformal energy in Section 10.1.2. Then we spend rest of the Chapter proving Theorem 10.2 by using the multiplier approach.

CHAPTER 2

The Relativistic Euler Equations and Its Geometric Formulation

In this section, we provide the standard first-order relativistic Euler equations and a geometric reformulation of it, derived in [10]. The latter will be used throughout our analysis.

2.1 Notations

Greek "space-time" indices take on the values $0,1,2,3$, while Latin "spatial" indices take on the values $1,2,3$. In this article, for Greek and Latin indices, for any vectorfield or one-form V, we lower and raise indices with the Minkowski metric $\eta_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$ and its inverse by using the notation $(V_\flat)_\beta := \eta_{\alpha\beta}V^\alpha$ and $(V^\sharp)^\beta := (\eta^{-1})^{\alpha\beta}V_\alpha$. Similar notations apply to all tensorfields. Moreover, $\varepsilon_{\alpha\beta\gamma\delta}$ denotes the fully antisymmetric symbol normalized by $\varepsilon_{0123} = 1$. Note that $(\varepsilon^\sharp)^{0123} = -1$. We use Einstein summation throughout the paper.

We denote $\Sigma_t := \{(t',x^1,x^2,x^3) \in \mathbb{R}^{1+3} | t' \equiv t\}$ as the standard constant-time slice.

We denote the spatial partial derivatives by ∂ and the space-time partial derivatives by $\boldsymbol{\partial}$.

Remark 2.1. *Since $\eta_{\alpha\beta} := diag(-1,1,1,1)$, for any vectorfield or one-form V, we have the identities $\boldsymbol{\partial}(V_\flat)_a = \boldsymbol{\partial}(V^\sharp)^a$ and $\boldsymbol{\partial}(V_\flat)_0 = -\boldsymbol{\partial}(V^\sharp)^0$.*

2.2 Definitions of the Fluid Variables and Related Quantities

2.2.1 The basic fluid variables

The fluid velocity v^α is a future-directed four vector and normalized by $(v_\flat)_\alpha v^\alpha = -1$. $\mathfrak{p}$ denotes the pressure, ρ denotes the proper energy density, n denotes the proper number density, s denotes the entropy per particle, θ denotes the temperature, and

$$H = (\rho + \mathfrak{p})/n \tag{2.1}$$

is the enthalpy per particle. Thermodynamics supplies the following laws:

$$H = \left.\frac{\partial \rho}{\partial n}\right|_s, \qquad \theta = \frac{1}{n}\left.\frac{\partial \rho}{\partial s}\right|_n, \qquad \mathrm{d}H = \frac{\mathrm{d}\mathfrak{p}}{n} + \theta \mathrm{d}s, \tag{2.2}$$

where $\left.\frac{\partial}{\partial n}\right|_s$ denotes partial differentiation with respect to n at fixed s and $\left.\frac{\partial}{\partial s}\right|_n$ denotes partial differentiation with respect to s at fixed n.

2.2.2 v-orthogonal vorticity

Definition 2.2 (The v-orthogonal vorticity of a one form)**.** *Given a space-time one-form V, we define the corresponding v-orthogonal (with respect to Minkowski metric) vorticity vectorfield as follows:*

$$vort^{\alpha}(V) := -(\varepsilon^{\sharp})^{\alpha\beta\gamma\delta}(v_{\flat})_{\beta}\partial_{\gamma}V_{\delta}. \tag{2.3}$$

Definition 2.3 (The v-orthogonal vorticity vector field)**.** *We define the vorticity vector field ω^{α} as follows:*

$$\omega^{\alpha} := vort^{\alpha}(Hv) = -(\varepsilon^{\sharp})^{\alpha\beta\gamma\delta}(v_{\flat})_{\beta}\partial_{\gamma}(H(v_{\flat})_{\delta}). \tag{2.4}$$

2.2.3 Auxiliary fluid variables

Definition 2.4 (Logarithmic enthalpy)**.** *Let $\overline{H} > 0$ be a fixed constant value of the background enthalpy. We define the logarithmic enthalpy h as follows:*

$$h := \ln(H/\overline{H}). \tag{2.5}$$

Definition 2.5 (Temperature over enthalpy)**.** *We define the quantity q as follows:*

$$q := \frac{\theta}{H}. \tag{2.6}$$

Definition 2.6 (Entropy gradient one-form)**.** *We define the entropy gradient one-form S_{α} as follows:*

$$S_{\alpha} := \partial_{\alpha}s. \tag{2.7}$$

2.2.4 Equation of state and the speed of sound

Definition 2.7 (Partial derivatives with respect to h and s)**.** *If Q is a quantity that can be expressed as a function of (h,s), then*

$$Q_{;h} = Q_{;h}(h,s) := \left.\frac{\partial Q}{\partial h}\right|_{s}, \tag{2.8}$$

$$Q_{;s} = Q_{;s}(h,s) := \left.\frac{\partial Q}{\partial s}\right|_{h}. \tag{2.9}$$

We assume an equation of state of the form $\mathfrak{p} = \mathfrak{p}(\rho,s)$. The speed of sound c is defined as follows:

$$c := \sqrt{\left.\frac{\partial \mathfrak{p}}{\partial \rho}\right|_{s}}. \tag{2.10}$$

In the rest of the article, we view the speed of sound be a function of h and s: $c = c(h,s)$.

We restrict to the physically relevant regime where the speed of sound does not exceed the speed of light:

$$0 < c \leq 1. \tag{2.11}$$

2.3 Standard First-Order Equations

Considering s, h and $\{v^\alpha\}_{\alpha=0,1,2,3}$ to be the fundamental unknowns, as in [10, Section 3], the relativistic Euler equations take the form of a quasilinear hyperbolic system:

$$v^\kappa \partial_\kappa h + c^2 \partial_\kappa v^\kappa = 0, \tag{2.12a}$$

$$v^\kappa \partial_\kappa (v_\flat)_\alpha + \partial_\alpha h + (v_\flat)_\alpha v^\kappa \partial_\kappa h - q \partial_\alpha s = 0, \tag{2.12b}$$

$$v^\kappa \partial_\kappa s = 0. \tag{2.12c}$$

2.4 Modified Fluid Variables and the Geometric Wave-Transport Formulation

In this subsection we define several variables followed by the new formulation of relativistic Euler equations from [10].

Definition 2.8. *[10, Definition 2.8. Modified fluid variables] We define the modified fluid variables as follows:*

$$\begin{aligned}
\mathscr{C}^\alpha :=& vort^\alpha(\omega_\flat) + c^{-2} \varepsilon^{\alpha\beta\gamma\delta} (v_\flat)_\beta (\partial_\gamma h)(\omega_\flat)_\delta + (\theta - \theta_{;h})(S^\sharp)^\alpha (\partial_\kappa v^\kappa) \\
&+ (\theta - \theta_{;h}) v^\alpha \left\{ (S^\sharp)^\kappa \partial_\kappa h \right\} - (\theta - \theta_{;h})(S^\sharp)^\kappa \left\{ (\eta^{-1})^{\alpha\lambda} \partial_\lambda (v_\flat)_\kappa \right\},
\end{aligned} \tag{2.13}$$

$$\mathscr{D} := \frac{1}{n} \left\{ \partial_\kappa (S^\sharp)^\kappa \right\} + \frac{1}{n} \left\{ (S^\sharp)^\kappa \partial_\kappa h \right\} - \frac{1}{n} c^{-2} \left\{ (S^\sharp)^\kappa \partial_\kappa h \right\}. \tag{2.14}$$

Definition 2.9 (Acoustical metric and its inverse)**.** *Let η be the Minkowski as defined in Section 2.1. We define the acoustical metric $\mathbf{g}_{Acou_{\alpha\beta}}$ and its inverse $(\mathbf{g}^{-1}_{Acou})^{\alpha\beta}$ as follows*[1]*:*

$$\mathbf{g}_{Acou_{\alpha\beta}} := c^{-2} \eta_{\alpha\beta} + (c^{-2} - 1)(v_\flat)_\alpha (v_\flat)_\beta, \tag{2.15a}$$

$$(\mathbf{g}^{-1}_{Acou})^{\alpha\beta} := c^2 (\eta^{-1})^{\alpha\beta} + (c^2 - 1) v^\alpha v^\beta. \tag{2.15b}$$

Definition 2.10 (Adjusted acoustical metric and its inverse)**.** *We define the adjusted acoustical metric $\mathbf{g}_{\alpha\beta} =$*

[1] For convience, we write $\mathbf{g}^{\alpha\beta}_{Acou}$ instead of $(\mathbf{g}^{-1}_{Acou})^{\alpha\beta}$ in this paper.

$\mathbf{g}_{\alpha\beta}(\vec{\Psi})$ *and its inverse* $(\mathbf{g}^{-1})^{\alpha\beta} = (\mathbf{g}^{-1})^{\alpha\beta}(\vec{\Psi})$ *as follows*[2]*:*

$$\mathbf{g}_{\alpha\beta} = \mathbf{g}_{Acou_{\alpha\beta}}(-\mathbf{g}^{00}_{Acou}) = \left\{c^{-2}\eta_{\alpha\beta} + (c^{-2}-1)(v_\flat)_\alpha(v_\flat)_\beta\right\}\left\{c^2-(c^2-1)(v^0)^2\right\}, \tag{2.16a}$$

$$(\mathbf{g}^{-1})^{\alpha\beta} = \frac{\mathbf{g}^{\alpha\beta}_{Acou}}{-\mathbf{g}^{00}_{Acou}} = \frac{c^2\eta^{\alpha\beta} + (c^2-1)v^\alpha v^\beta}{c^2-(c^2-1)(v^0)^2}. \tag{2.16b}$$

We note that $\mathbf{g}^{\alpha\beta}_{Acou}\mathbf{g}_{Acou_{\alpha\gamma}} = \delta^\beta_\gamma$ *and* $\mathbf{g}^{\alpha\beta}(\mathbf{g}^{-1})_{\alpha\gamma} = \delta^\beta_\gamma$ *where* δ^β_γ *is the Kronecker delta. We emphasize that* $(\mathbf{g}^{-1})^{00} = -1$*. This helps us to simplify some of our formulas. Our bootstrap assumption will be so that* $0 < c \leq 1$*, so* $(\mathbf{g}^{-1}_{Acou})^{00} < 0$.

We lower and raise indices with the acoustic metric $\mathbf{g}$ and its inverse by using the notation $V_\beta = \mathbf{g}_{\alpha\beta}V^\alpha$ and $V^\beta = (\mathbf{g}^{-1})^{\alpha\beta}V_\alpha$. Note the difference between with raising and lowering indices with $\mathbf{g}$ versus η, see Section 2.1.

Definition 2.11. *We define the future-directed* $\mathbf{g}$*-timelike vectorfield*

$$\mathbf{T}^\alpha := -\mathbf{g}^{\alpha 0}. \tag{2.17}$$

We note that $\mathbf{g}(\mathbf{T},\mathbf{T}) = -1$*. We note that* $\mathbf{T}_\alpha = -\delta^0_\alpha$ *is the* $\mathbf{g}-$*unit normal to* Σ_t*, where* δ^0_α *is the Kronecker delta.*

Definition 2.12. *In Cartesian coordinates, the induced metric* g *and its inverse on constant-time hypersurface* Σ_t *from* $\mathbf{g}$ *are as follows:*

$$g_{ab} := \mathbf{g}_{ab} + \mathbf{T}_a\mathbf{T}_b, \tag{2.18a}$$

$$\left(g^{-1}\right)^{ab} := \mathbf{g}^{ab} + \mathbf{T}^a\mathbf{T}^b. \tag{2.18b}$$

By (2.16) and (2.17), one can compute that $g_{ab}\left(g^{-1}\right)^{bc} = \delta^c_a$*. We note that* g *can be also viewed as a spacetime tensor, that is,*

$$g_{\alpha\beta} := \mathbf{g}_{\alpha\beta} + \mathbf{T}_\alpha\mathbf{T}_\beta, \tag{2.19a}$$

$$\left(g^{-1}\right)^{\alpha\beta} := \mathbf{g}^{\alpha\beta} + \mathbf{T}^\alpha\mathbf{T}^\beta. \tag{2.19b}$$

Notice that by (2.17) and $(\mathbf{g}^{-1})^{00} = -1$, $\underline{\Pi}^\delta_\alpha := \mathbf{g}^{\beta\delta}g_{\alpha\beta}$ *can be viewed as* $\mathbf{g}-$*orthogonal projection operator onto* Σ_t.

[2]For convience, we write $\mathbf{g}^{\alpha\beta}$ instead of $(\mathbf{g}^{-1})^{\alpha\beta}$ in this paper.

Proposition 2.13. *The metric g and its inverse g^{-1} have the following spatial components relative to rectangular coordinates:*

$$(g^{-1})^{ab} = \left\{c^2 - (c^2-1)(v^0)^2\right\}^{-2}\left\{c^2\delta^{ab}\left[c^2-(c^2-1)(v^0)^2\right] + c^2(c^2-1)v^a v^b\right\}, \tag{2.20a}$$

$$g_{ab} = \left\{c^{-2}\delta_{ab} + (c^{-2}-1)(v_\flat)_a(v_\flat)_b\right\}\left\{c^2-(c^2-1)(v^0)^2\right\}. \tag{2.20b}$$

Using (2.16b), (2.17) and (2.18b), (2.20a) is obtained by direct computation. (2.20b) is obtained by the fact that $\mathbf{T}_a = 0$ for $a = 1,2,3$.

Definition 2.14 (Differential operators defined by $\mathbf{g}$)**.** $\mathbf{D}$ *denotes the Levi-Civita connection of* $\mathbf{g}$ *and* $\Box_{\mathbf{g}} := \mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta$ *denotes the corresponding covariant wave operator. In Cartesian coordinates, for scalar function* φ, $\Box_{\mathbf{g}}\varphi = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha\left(\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_\beta\varphi\right)$, *where* $|\mathbf{g}|$ *is the determinant of* $\mathbf{g}$.

Definition 2.15 (Arrays of variables)**.** *For convevience, in presenting the formulations and analysis, we define the following arrays of solution variables:*

$$\vec{v} := (v^0, v^1, v^2, v^3), \quad \vec{\omega} := (\omega^0, \omega^1, \omega^2, \omega^3), \quad \vec{S} := ((S^\sharp)^0, (S^\sharp)^1, (S^\sharp)^2, (S^\sharp)^3), \quad \vec{\mathscr{C}} := (\mathscr{C}^0, \mathscr{C}^1, \mathscr{C}^2, \mathscr{C}^3). \tag{2.21}$$

We also define the array $\vec{\Psi}$ *of wave variables, as follows:*

$$\vec{\Psi} := (v^0, v^1, v^2, v^3, h, s). \tag{2.22}$$

Definition 2.16 (Material derivative)**.** *We define the material derivative*[3] $\mathbf{B} = \mathbf{B}(\vec{\Psi})$ *as follows:*

$$\mathbf{B} := \frac{v^\alpha}{v^0}\partial_\alpha. \tag{2.23}$$

2.4.1 The geometric wave-transport formulation of the relativistic Euler equations

We use the following schematic notations throughout the paper where A, B, C are arrays of variables:

- $\mathscr{L}[A](B)$ denotes any scalar-valued function that is linear in the components of B with coefficients that are a function of the components of A.
- $\mathscr{Q}[A](B,C)$ denotes any scalar-valued function that is quadratic in the components of B and C with coefficients that are a function of the components of A.

[3] We note that in the non-relativistic case studied in [9], $\mathbf{B} = \partial_t + v^i\partial_i$.

Proposition 2.17. *[10, (3.1)-(3.12b). The geometric wave-transport formulation of the relativistic Euler equations] If $\Psi \in \{v^0, v^1, v^2, v^3, h, s\}$ solves the relativistic Euler equations (2.12), then $\Psi, \omega, S, \mathscr{C}, \mathscr{D}$ also satisfy the following:*

Wave equations

$$\square_{\mathbf{g}_{Acou}} \Psi = \mathscr{L}(\vec{\Psi})[\vec{\mathscr{C}}, \mathscr{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{2.24}$$

Transport equations

$$\mathbf{B}\omega^\alpha = \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{2.25a}$$

$$\mathbf{B}(S^\sharp)^\alpha = \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{2.25b}$$

Transport-Div-Curl system

$$\mathbf{B}\mathscr{C}^\alpha = \mathfrak{F}\mathscr{C}^\alpha := \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, (\boldsymbol{\partial}\omega, \boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S}], \tag{2.26a}$$

$$\mathbf{B}\mathscr{D} = \mathfrak{F}\mathscr{D} := \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\omega] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, (\boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi})] + \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{2.26b}$$

$$\mathrm{vort}^\alpha(S) = 0, \tag{2.26c}$$

$$\partial_\alpha \omega^\alpha = \mathscr{L}(\vec{\omega})[\boldsymbol{\partial}\vec{\Psi}]. \tag{2.26d}$$

Remark 2.18. *The div-curl system (2.26) in the geometric formulation of the relativistic Euler equations is a space-time div-curl system. This feature causes difficulties, as we want to derive estimates for vorticity and entropy gradient along the constant-time hypersurface Σ_t. To solve this issue, we rewrite the div-curl system into a dynamic spatial system along constant-time slices and apply theory in Littlewood-Paley decomposition as well as pseudodifferential operators in Section 5. These difficulties are not present in the non-relativistic 3D compressible Euler equations because the analogue of (2.26c)-(2.26d) is already a spatial div-curl system.*

We now provide some useful identities, which we are going to use throughout the rest of the paper.

Lemma 2.19 (Identities involving vorticity and entropy gradient)**.** *We list some useful identities in [10, Sec-*

tion 4] as follows:

$$\omega^\kappa (v_\flat)_\kappa = 0, \tag{2.27a}$$

$$v^\kappa \partial_\alpha (\omega_\flat)_\kappa = -\omega^\kappa \partial_\alpha (v_\flat)_\kappa, \tag{2.27b}$$

$$v^\kappa \partial_\alpha S_\kappa = -(S^\sharp)^\kappa \partial_\alpha (v_\flat)_\kappa, \tag{2.27c}$$

$$\begin{aligned}\partial_\gamma (\omega_\flat)_\delta - \partial_\delta (\omega_\flat)_\gamma =& \varepsilon_{\gamma\delta\kappa\lambda} v^\kappa \mathrm{vort}^\lambda(\omega) - (v^\kappa \partial_\kappa (\omega_\flat)_\delta)(v_\flat)_\gamma + v^\kappa (\partial_\delta (\omega_\flat)_\kappa)(v_\flat)_\gamma \\ &+ (v^\kappa \partial_\kappa (\omega_\flat)_\gamma)(v_\flat)_\delta - v^\kappa (\partial_\delta (\omega_\flat)_\kappa)(v_\flat)_\delta,\end{aligned} \tag{2.27d}$$

$$\begin{aligned}\partial_\gamma S_\delta - \partial_\delta S_\gamma =& \varepsilon_{\gamma\delta\kappa\lambda} v^\kappa \mathrm{vort}^\lambda(S) - (v^\kappa \partial_\kappa S_\delta)(v_\flat)_\gamma \\ &+ v^\kappa (\partial_\delta S_\kappa)(v_\flat)_\gamma + (v^\kappa \partial_\kappa S_\gamma)(v_\flat)_\delta - v^\kappa (\partial_\delta S_\kappa)(v_\flat)_\delta.\end{aligned} \tag{2.27e}$$

Discussion of the proof. (2.27a) follows from Definition 2.3. (2.27b) follows from taking ∂_α derivative of (2.27a). (2.27c) follows from taking ∂_α derivative of (2.12c).

To prove (2.27d) and (2.27e), we use Definition 2.3 to express $\mathrm{vort}^\lambda(V)$ for $V = \omega_\flat$ and $V = S$ respectively. Then, using the fact that

$$-\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\delta\theta\kappa\lambda} = \delta_\alpha^\theta \delta_\gamma^\kappa \delta_\beta^\lambda - \delta_\alpha^\theta \delta_\gamma^\lambda \delta_\beta^\kappa + \delta_\alpha^\lambda \delta_\gamma^\theta \delta_\beta^\kappa - \delta_\alpha^\lambda \delta_\gamma^\kappa \delta_\beta^\theta + \delta_\alpha^\kappa \delta_\gamma^\lambda \delta_\beta^\theta - \delta_\alpha^\kappa \delta_\gamma^\theta \delta_\beta^\lambda, \tag{2.28}$$

we rearrange the terms and obtain (2.27d) and (2.27e).

We refer readers to [10, Lemma 4.1] for detailed proofs. □

In the following proposition, we provide the geometric wave equation with respect to the rescaled acoustical metric $\mathbf{g}$, which is more convenient for us to derive energy estimates and construct geometry. We will use this equation in the rest of the paper.

Proposition 2.20 (Wave equations after rescaling the acoustical metric)**.** *Let* $\mathbf{g}$ *be as defined in Definition 2.10. If* $\Psi \in \{v^0, v^1, v^2, v^3, h, s\}$ *solves the relativistic Euler equations (2.12), then the following equation holds:*

$$\square_{\mathbf{g}}\Psi = \mathfrak{F}\Psi := \mathscr{L}(\vec{\Psi})[\vec{\mathscr{C}}, \mathscr{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{2.29}$$

Proof of Prop 2.20 using Prop 2.17 .

$$\begin{aligned}
\Box_{\mathbf{g}}\Psi &= \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha\left(\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_\beta\Psi\right) \\
&= \frac{1}{\sqrt{|\mathbf{g}_{Acou}|}}\left(-\mathbf{g}^{00}_{Acou}\right)^{-2}\partial_\alpha\left(\sqrt{|\mathbf{g}_{Acou}|}\left(-\mathbf{g}^{00}_{Acou}\right)\mathbf{g}^{\alpha\beta}_{Acou}\partial_\beta\Psi\right) \\
&= \frac{1}{\sqrt{|\mathbf{g}_{Acou}|}}\left(-\mathbf{g}^{00}_{Acou}\right)^{-1}\partial_\alpha\left(\sqrt{|\mathbf{g}_{Acou}|}\mathbf{g}^{\alpha\beta}_{Acou}\partial_\beta\Psi\right)+\left(-\mathbf{g}^{00}_{Acou}\right)^{-2}\mathbf{g}^{\alpha\beta}_{Acou}\partial_\alpha\left(\mathbf{g}^{00}_{Acou}\right)\partial_\beta\Psi \\
&= \left(-\mathbf{g}^{00}_{Acou}\right)^{-1}\Box_{\mathbf{g}_{Acou}}\Psi+\mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}].
\end{aligned} \tag{2.30}$$

Note that $\mathbf{g}^{\alpha\beta}_{Acou}$ is smooth function of $\vec{\Psi}$ and $\mathbf{g}^{00}_{Acou}\neq 0$. Therefore by combining (2.24) and (2.30), we obtain the desired equation. □

CHAPTER 3

Norms, Littlewood-Paley Projections, Statement of Main Results and Bootstrap Assumptions

In this chapter, we define the norms, define the standard Littlewood-Paley projections that we use in the analysis, and state our main results of the paper and bootstrap assumptions.

3.1 Norms

In this article, for functions f,g on a normed space $(X,\|\cdot\|_X)$, we use the notation $\|f,g\|_X := \|f\|_X + \|g\|_X$. Similarly, for an array of functions $\vec{U} = (U^1, U^2, \ldots, U^k)$, we define $\left\|\vec{U}\right\|_X := \sum_{a=1}^{k} \|U^a\|_X$. In particular, we use $\left|\vec{U}\right| := \sqrt{\sum_{i=1}^{k}(U^i)^2}$. For functions f and arrays $\vec{g}$, we also use $\|f,g\|_X := \|f\|_X + \|\vec{g}\|_X$.

Since the volume form on the constant-time hypersurface Σ_t induced by Minkowski metric η is $\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$, by identifying $(t,x^1,x^2,x^3) \in \Sigma_t$ with $(x^1,x^2,x^3) \in \mathbb{R}$, we define the standard Sobolev norm on Σ_t for $s \in \mathbb{R}$: $\|F\|_{H^s(\Sigma_t)} := \left\|\langle\xi\rangle^s \hat{F}(\xi)\right\|_{L^2_x(\Sigma_t)}$, where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ and $\hat{F}(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi} F(x)\mathrm{d}x$ is the Fourier transform of F..

We denote the standard Hölder semi-norm $\dot{C}^{0,\beta}_x$ and Hölder norm $C^{0,\beta}_x$, where $0<\beta<1$, of a function F with respect to flat metric on constant-time hypersurface Σ_t by

$$\|F\|_{\dot{C}^{0,\beta}_x(\Sigma_t)} := \sup_{x\neq y\in\Sigma_t} \frac{|F(x)-F(y)|}{|x-y|^\beta}, \tag{3.1}$$

$$\|F\|_{C^{0,\beta}_x(\Sigma_t)} := \sup_{x\in\Sigma_t}|F(x)| + \sup_{x\neq y\in\Sigma_t} \frac{|F(x)-F(y)|}{|x-y|^\beta}. \tag{3.2}$$

We also use the following mixed norms for function $F:\mathbb{R}^3\to\mathbb{R}$, where $1\le q_1<\infty$, $1\le q_2\le\infty$, and I is an interval of time:

$$\|F\|_{L^{q_1}_t L^{q_2}_x(I\times\Sigma_t)} := \left\{\int_I \|F\|^{q_1}_{L^{q_2}_x(\Sigma_\tau)}\mathrm{d}\tau\right\}^{1/q_1}, \qquad \|F\|_{L^\infty_t L^{q_2}_x(I\times\Sigma_t)} := \operatorname{ess\,sup}_{\tau\in I} \|F\|_{L^{q_2}_x(\Sigma_\tau)}, \tag{3.3}$$

$$\|F\|_{L^{q_1}_t C^{0,\beta}_x(I\times\Sigma_t)} := \left\{\int_I \|F\|^{q_1}_{C^{0,q_2}_x(\Sigma_\tau)}\mathrm{d}\tau\right\}^{1/q_1}, \qquad \|F\|_{L^\infty_t C^{0,q_2}_x(I\times\Sigma_t)} := \operatorname{ess\,sup}_{\tau\in I} \|F\|_{C^{0,q_2}_x(\Sigma_\tau)}. \tag{3.4}$$

If $\{F_\lambda\}_{\lambda\in 2^{\mathbb{N}}}$ is a dyadic-indexed sequence of functions on Σ_t, we define

$$\|F_\nu\|_{l^2_\nu L^2_x(\Sigma_t)} := \left(\sum_{\nu\ge1}\|F_\nu\|^2_{L^2_x(\Sigma_t)}\right)^{1/2}. \tag{3.5}$$

3.2 Littlewood-Paley Projections

We fix a smooth function $\psi = \psi(|\xi|) : \mathbb{R}^3 \to [0,1]$ supported on the frequency space annulus $\{\xi \in \mathbb{R}^3 | 1/2 \leq |\xi| \leq 2\}$ such that for $\xi \neq 0$, we have $\sum_{k\in\mathbb{Z}} \psi(2^k\xi) = 1$. For dyadic frequencies $\nu = 2^k$ with $k \in \mathbb{Z}$, we define the standard Littlewood-Paley projection P_ν, which acts on scalar functions $F : \mathbb{R} \to \mathbb{C}$, as follows:

$$P_\nu F(x) := \int_{\mathbb{R}^3} e^{2\pi i x\cdot\xi} \psi(\nu^{-1}\xi)\hat{F}(\xi)\mathrm{d}\xi, \tag{3.6}$$

where $\hat{F}(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi} F(x)\mathrm{d}x$ is the Fourier transform of F. If $I \subset 2^{\mathbb{Z}}$ is an interval of dyadic frequencies, then $P_I F := \sum_{\nu\in I} P_\nu F$, and $P_{\leq\nu} F := P_{[-\infty,\nu]} F$. For functions f, g, we use the schematic notation that $P_\nu(f,g)$ is a linear combination of $P_\nu f$ and $P_\nu g$, namely, $P_\nu f + P_\nu g$.

Proposition 3.1. *For a function F, standard results in Littlewood-Paley theory give the following:*

$$\|F\|_{H^s(\Sigma_t)} \approx \|F\|_{L^2_x(\Sigma_t)} + \left(\sum_{\nu>1} \nu^{2s} \|P_\nu F\|^2_{L^2_x(\Sigma_t)}\right)^{1/2}, \tag{3.7}$$

$$\|F\|_{C^{0,s}_x(\Sigma_t)} \approx \|F\|_{L^\infty_x(\Sigma_t)} + \sup_{\nu\geq 2} \nu^s \|P_\nu F\|_{L^\infty_x(\Sigma_t)}, \tag{3.8}$$

*where H^s is the standard Sobolev norm and $C^{0,s}$ is the standard Hölder norm. One can refer to [**?** , Section 1] and [32, A.1] for above results.*

The following two Lemmas consist of a commuted version of the equations. Lemma 3.2 commutes $\square_{\mathbf{g}}$ and $\mathbf{B}$ with $\boldsymbol{\partial}$ and is needed for below-top order estimates. Lemma 3.3 commutes $\square_{\mathbf{g}}$ and $\mathbf{B}$ with $P_\nu\boldsymbol{\partial}$ and is needed for the top order estimates.

Lemma 3.2 (Commuted equations satisfied by one derivative of the solution variables)**.** *We consider the solutions to the equations of Proposition 2.17, that is, if $\Psi \in \{v^0, v^1, v^2, v^3, h, s\}$ solves the relativistic Euler equations (2.12), the following equations hold:*

$$\begin{aligned}
\square_{\mathbf{g}}\boldsymbol{\partial}\Psi =& \mathscr{L}(\vec{\Psi})[\boldsymbol{\partial}\mathscr{C}, \boldsymbol{\partial}\mathscr{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi})[(\boldsymbol{\partial}\vec{\Psi})^3], && (3.9)\\
\mathbf{B}\boldsymbol{\partial}\mathscr{C}^\alpha =& \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, (\boldsymbol{\partial}^2\omega, \boldsymbol{\partial}^2\vec{S}, \boldsymbol{\partial}^2\vec{\Psi})] + \mathscr{Q}(\vec{\Psi})[(\boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S}), (\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\Psi})] && (3.10)\\
&+ \mathscr{L}(\vec{\Psi})[(\boldsymbol{\partial}\vec{\Psi})^2 \cdot (\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\omega})],\\
\mathbf{B}\boldsymbol{\partial}\mathscr{D} =& \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}^2\omega, \boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}^2\vec{S}] + \mathscr{Q}(\vec{\Psi})[(\boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S}), (\boldsymbol{\partial}\vec{S}, \boldsymbol{\partial}\vec{\Psi})] && (3.11)\\
&+ \mathscr{L}(\vec{\Psi})[(\boldsymbol{\partial}\vec{\Psi})^2 \cdot (\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{S})].
\end{aligned}$$

Sketch of the proof of Lemma 3.2. By commuting (2.29), (2.26a) and (2.26b) with $\boldsymbol{\partial}$, using relations (by Definition 2.8) $\mathscr{C} = \mathscr{L}(\vec{\Psi}, \vec{\omega}, \vec{S})[\boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{\Psi}]$ and $\mathscr{D} = \mathscr{L}[\boldsymbol{\partial}\vec{S}] + \mathscr{L}(\vec{S})[\boldsymbol{\partial}\vec{\Psi}]$, (3.9)-(3.11) are derived by straightforward computations. $\square$

The following Lemma provides the commuted equations with the Littlewood-Paley projections.

Lemma 3.3. *[9, Lemma 5.2, Lemma 5.4. Equations satisifed by the frequency-projected solution variables] For solutions to the equations of Proposition 2.17, the following equations hold:*

$$\square_{\mathbf{g}} P_\nu \boldsymbol{\partial}\Psi = \mathfrak{R}\boldsymbol{\partial}\vec{\Psi}, \tag{3.12}$$

$$\mathbf{B} P_\nu \boldsymbol{\partial}\mathscr{C}^\alpha = \mathfrak{R}\boldsymbol{\partial}\mathscr{C}^\alpha, \tag{3.13}$$

$$\mathbf{B} P_\nu \boldsymbol{\partial}\mathscr{D} = \mathfrak{R}\boldsymbol{\partial}\mathscr{D}, \tag{3.14}$$

where

$$\begin{aligned}
\mathfrak{R}\boldsymbol{\partial}\vec{\Psi} =& P_\nu \boldsymbol{\partial}\mathfrak{F}\Psi - \sum_{(\alpha,\beta)\neq(0,0)} P_\nu\left[\boldsymbol{\partial}\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\Psi\right] - \boldsymbol{\Gamma}^\alpha P_\nu \partial_\alpha \boldsymbol{\partial}\Psi \\
&+ \sum_{(\alpha,\beta)\neq(0,0)} \left[\mathbf{g}^{\alpha\beta} - P_{\leq\nu}\mathbf{g}^{\alpha\beta}\right] P_\nu \partial_\alpha\partial_\beta \boldsymbol{\partial}\Psi \\
&+ \sum_{(\alpha,\beta)\neq(0,0)} \left\{P_{\leq\nu}\mathbf{g}^{\alpha\beta} P_\nu \partial_\alpha\partial_\beta\boldsymbol{\partial}\Psi - P_\nu[\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\boldsymbol{\partial}\Psi]\right\},
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\mathfrak{R}\boldsymbol{\partial}\mathscr{C}^\alpha =& P_\nu \boldsymbol{\partial}\mathfrak{F}\mathscr{C}^\alpha - P_\nu\left[\boldsymbol{\partial}\left(\frac{v^a}{v^0}\right)\partial_a\mathscr{C}^\alpha\right] + \left[\frac{v^a}{v^0} - P_{\leq\nu}\left(\frac{v^a}{v^0}\right)\right] P_\nu \partial_a \boldsymbol{\partial}\mathscr{C}^\alpha \\
&+ P_{\leq\nu}\left(\frac{v^a}{v^0}\right) P_\nu \partial_a \boldsymbol{\partial}\mathscr{C}^\alpha - P_\nu\left[\frac{v^a}{v^0}\partial_a\boldsymbol{\partial}\mathscr{C}^\alpha\right],
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\mathfrak{R}\boldsymbol{\partial}\mathscr{D} =& P_\nu \boldsymbol{\partial}\mathfrak{F}\mathscr{D} - P_\nu\left[\boldsymbol{\partial}\left(\frac{v^a}{v^0}\right)\partial_a\mathscr{D}\right] + \left[\frac{v^a}{v^0} - P_{\leq\nu}\left(\frac{v^a}{v^0}\right)\right] P_\nu \partial_a \boldsymbol{\partial}\mathscr{D} \\
&+ P_{\leq\nu}\left(\frac{v^a}{v^0}\right) P_\nu \partial_a \boldsymbol{\partial}\mathscr{D} - P_\nu\left[\frac{v^a}{v^0}\partial_a\boldsymbol{\partial}\mathscr{D}\right].
\end{aligned} \tag{3.17}$$

where $\boldsymbol{\Gamma}_\alpha := \mathbf{g}^{\kappa\lambda}\boldsymbol{\Gamma}_{\alpha\kappa\lambda} = \mathbf{g}^{\kappa\lambda}\mathbf{g}_{\alpha\beta}\boldsymbol{\Gamma}^\beta_{\kappa\lambda}$ *and* $\boldsymbol{\Gamma}_{\alpha\kappa\lambda}$ *is the standard Christoffel symbols for* $\mathbf{g}$ *(defined in Definition 7.1).*

Moreover the following estimates hold for the remainders, where the l^2_ν*-seminorm is taken over dyadic frequencies with* $\nu > 1$*:*

$$\begin{aligned}
&\left\|\nu^{N-2}\left(\mathfrak{R}\boldsymbol{\partial}\vec{\Psi}, \mathfrak{R}\boldsymbol{\partial}\mathscr{C}^\alpha, \mathfrak{R}\boldsymbol{\partial}\mathscr{D}\right)\right\|_{l^2_\nu L^2_x(\Sigma_t)} \\
&\lesssim \left\|\partial(\vec{\mathscr{C}},\mathscr{D})\right\|_{H^{N-2}(\Sigma_t)} + \left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S})\right\|_{L^\infty_x(\Sigma_t)} + 1\right)\left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S})\right\|_{H^{N-1}(\Sigma_t)} + 1\right).
\end{aligned} \tag{3.18}$$

Discussion of the proof of Lemma 3.3. We omit the proof of equation (3.15),(3.16) and (3.17) since it follows from straightforward computations. We use bootstrap assumptions, product and commutator estimates for

Littlewood-Paley calculus to prove estimates (3.18). We refer readers to [9, Lemma 5.4] for the detailed proofs where the structure of the equations are the same as in this article. We note that we have $\mathbf{B}^a = \frac{v^a}{v^0}$, while $\mathbf{B}^a = v^a$ in [9], which doesn't change the proof or result in the estimates (3.18). □

3.3 Statement of Main Theorem

Using the notations introduced in the previous subsections, we precisely provide our assumptions on the data and the statement of the main theorem

Theorem 3.4 (Main theorem)**.** *Consider a solution to the relativistic Euler equations whose initial data satisfies following assumptions for some real number* $2 < N < 5/2$, $0 < \alpha < 1$, $c_1 > 0$ *and D:*

1. $\|h, v, \omega\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)} \leq D$,

2. $\|\mathscr{C}, \mathscr{D}\|_{C^{0,\alpha}(\Sigma_0)} \leq D$,

3. The data functions are contained in the interior of $\mathscr{R}$ *(See Definition 3.5 for definition of* $\mathscr{R}$*) and the enthalpy H is strictly positive, i.e.* $H \geq c_1 > 0$.

Then the solution's time of classical existence $T > 0$ *can be bounded from below in terms of D and* $\mathscr{R}$*. Moreover, the Sobolev and some Hölder regularity*[1] *of the data are propogated by the solution on the slab of classical existence.*

3.4 Choice of Parameters

In this subsection, we introduce several parameters that each of them either measures the regularity or plays a role in our analysis. We denote the assumed Sobolev regularity of the "wave-part" of the data and the Hölder regularity of the "transport-part" of the data by, respectively, $2 < N < 5/2$ and $0 < \alpha < 1$. For the purpose of analysis, we choose positive numbers $q, \varepsilon_0, \delta_0, \delta$ and δ_1 that satisfy the following conditions:

$$2 < q < \infty, \tag{3.19a}$$

$$0 < \varepsilon_0 := \frac{N-2}{10} < \frac{1}{10}, \tag{3.19b}$$

$$\delta_0 := \min\{\varepsilon_0^2, \frac{\alpha}{10}\}, \tag{3.19c}$$

$$0 < \delta := \frac{1}{2} - \frac{1}{q} < \varepsilon_0, \tag{3.19d}$$

$$\delta_1 := \min\{N - 2 - 4\varepsilon_0 - \delta(1 - 8\varepsilon_0), \alpha\} > 8\delta_0 > 0. \tag{3.19e}$$

More precisely, we consider N, α, ε_0 and δ_0 to be fixed throughout the paper, while q, δ and δ_1 will be treated as parameters.

[1] We note that the full Hölder regularity is not propogated, but rather we show only that $\left\|\vec{\Psi}, \mathscr{C}, \mathscr{D}\right\|_{C^{0,\delta_1}(\Sigma_t)}$ (see Section 3.4 for the definition of δ_1) is bounded by data.

3.5 Assumptions on the Initial Data

In this subsection, we provide the bootstrap assumptions that will be used in the proof of Theorem 3.4.

Definition 3.5 (Regime of hyperbolicity)**.** *We define $\mathscr{R}$ as follows*

$$\mathscr{R} := \left\{(h, s, \vec{v}, \vec{\omega}, \vec{S}) \in \mathbb{R}^{14} |\ 0 < c \leq 1\right\}. \tag{3.20}$$

With N and α as in Section 3.4, we assume that

$$\textbf{“Wave-part”} \qquad \|h, \vec{v}\|_{H^N(\Sigma_0)} < \infty, \tag{3.21}$$

$$\textbf{“Transport-part”} \qquad \|s\|_{H^{N+1}(\Sigma_0)} + \|\vec{\omega}\|_{H^N(\Sigma_0)} + \left\|\vec{\mathscr{C}}, \mathscr{D}\right\|_{C^{0,\alpha}(\Sigma_0)} < \infty. \tag{3.22}$$

Assumptions (3.21) and (3.22) correspond to regularity assumptions on the “wave-part” and “transport-part” of the data respectively.

Let int$U$ denote the interior of the set U. We assume that there is a compact subset $\breve{\mathscr{R}}$ such that

$$\left(\vec{\Psi}, \vec{\omega}, \vec{S}\right)(\Sigma_0) \subset \text{int}\breve{\mathscr{R}} \subset \breve{\mathscr{R}} \subset \text{int}\mathscr{R}, \tag{3.23}$$

where $\mathscr{R}$ is defined in (3.20).

3.6 Bootstrap Assumptions

Throughout the article, $0 < T_* \ll 1$ denotes a bootstrap time that depends only on initial data. We assume that $\vec{\Psi}$ is a smooth[2] solution to the equation in Section 2.3 and the following estimates hold:

$$\left(\vec{\Psi}, \vec{\omega}, \vec{S}\right)([0, T_*] \times \mathbb{R}^3) \subset \mathscr{R}, \tag{3.24a}$$

$$\left\|\boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \left\|P_\nu \boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \leq 1, \tag{3.24b}$$

$$\left\|\partial\vec{\omega}, \partial\vec{S}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \left\|P_\nu \partial\vec{\omega}, P_\nu \partial\vec{S}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \leq 1. \tag{3.24c}$$

In Theorem 5.19, we derive an improvement of (3.24c), assuming T_* is sufficiently small. In Theorem 6.1, we derive an improvement of (3.24b). By fundamental theorem of calculus, equations (2.25a) and (2.25b), (3.24a) is a direct result of (3.24b) and (3.24c).

[2]By smooth we mean as smooth as neccessary for the analysis arguments to go through. Meanwhile, all of our quantitative estimates depend only on the Sobolev and Hölder norms.

CHAPTER 4

Structure of the Proofs in the Rest of the Article

In this section, we describe the structure of the proofs in the article. Our proofs rely on a bootstrap argument where the bootstrap assumptions are in Section 3.6. See Section 1.5 for the logic of the bootstrap argument. The main goal for us is to improve the bootstrap assumptions to the following Strichartz-type estimates:

$$\left\|\partial\vec{\Psi}\right\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)}+\sum_{\nu\geq 2}\nu^{2\delta_1}\left\|P_\nu\partial\vec{\Psi}\right\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)}\lesssim T_*^{2\delta}, \tag{4.1a}$$

$$\left\|\partial\vec{\omega},\partial\vec{S}\right\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)}+\sum_{\nu\geq 2}\nu^{2\delta_1}\left\|P_\nu\partial\vec{\omega},P_\nu\partial\vec{S}\right\|^2_{L^2_tL^\infty_x([0,T_*]\times\mathbb{R}^3)}\lesssim T_*^{2\delta}. \tag{4.1b}$$

We prove the (4.1a) through the following series of reductions, see Subsection 1.5.4 for an overview of the logic: Strichartz estimates⟵Decay estimates⟵Conformal energy estimates⟵Controlling of the acoustic null geometry.

To prove (4.1b), we prove a transport-Schauder type estimate in Section 5, which is **independent** of the proof of (4.1a). In Theorem 5.19, we obtain (4.1b) by combining the transport-Schauder estimate and (4.1a).

4.1 Similarities and Differences Compared to the 3D Compressible Euler Equations

Broadly speaking, we use the same machinery as in [9] to reduce the proof of the Strichartz estimates to geometric quantities that have to be controlled in order to derive a conformal energy estimate. This reduction was first introduced and developed in the context of low-regularity problems for quasilinear wave equations, as we discussed in Section 1.1. Disconzi-Luo-Mazzone-Speck [9] and Wang [37] have exploited the remarkable structure of the non-relativistic 3D compressible Euler equations to derive similar low-regularity well-posedness results in the presence of vorticity and entropy. The main purpose of this paper is to derive similar results for the relativistic Euler flow by using the remarkable structure of the equations derived by Disconzi-Speck in [10].

Two main differences in the present paper compared to the non-relativistic case are **1)** the first fundamental form of Σ_t is no longer conformally flat in the relativistic case, leading to more complicated geometry and **2)** the L^2 elliptic and Schauder estimates that we need to handle the vorticity and entropy are more complicated because unlike in the non-relativistic case, the Hodge systems that we study are quasilinear (instead of constant-coefficient).

4.2 Energy, L^2 Elliptic and Schauder Estimates in Section 5

In Section 5, first we prove the energy estimates for wave variables h, s, v and transport variables $\omega, S, \mathscr{C}, \mathscr{D}$ in Proposition 5.1. These estimates are essential to the local well-posedness theorem Theorem 3.4. We also need these estimates for controlling the acoustic geometry. We control the $H^{2+\varepsilon}$ norm of wave variables under the bootstrap assumptions by using the geometric energy method in Subsection 5.1.1 and commuted equations in Lemma 3.2 and Lemma 3.3. We refer readers to [35, Section 6] for the commutator estimates involving LP projections in fractional Sobolev spaces. We note that the L^2 elliptic estimates for transport variables $\omega, S, \mathscr{C}, \mathscr{D}$ in Proposition 5.8 is proven based on a rewritten dynamic div-curl system in Proposition 5.9.

We then prove the transport-Schauder estimates in the Hölder space C_x^{0,δ_1} for the transport variables $\omega, S, \mathscr{C}, \mathscr{D}$ in Theorem 5.19, which recovers the bootstrap assumptions (3.24c) assuming (4.1a). To prove these estimates, we use the div-curl system (5.19) and the transport equations (2.25a)-(2.26b). We prove Schauder estimates by some standard results in pseudodifferential operators as well as the Littlewood-Paley decomposition with the help of the equivalence between Hölder spaces and frequency spaces (3.8).

4.3 Reduction of Strichartz Estimates to Decay Estimates in Section 6

We state the Strichartz estimates for wave variables in Theorem 6.1, which improves the bootstrap assumptions (3.24b). Our reductions of the Strichartz estimates to the bounded conformal energy consists of several steps. In Section 6, we list several reductions from Strichartz estimates to a spatially localized decay estimate in Theorem 6.9. We first use Duhamel's principle to reduce Theorem 6.1 to a frequency localized version of Strichartz estimates in Theorem 6.2. Then after rescaling all the quantities with respect to the frequency in Subsection 6.2, we run a $\mathscr{T}\mathscr{T}^*$ argument to reduce the Theorem 6.2 to a decay estimate in Theorem 6.8. Finally, by Bernstein inequalities, partition of unity, and Sobolev embedding, we obtain the spatially localized version of decay estimates in Theorem 6.9. The reductions are by now standard, therefore we only state the reductions without proof. We refer to Wang [36, Section 3] for the details. It is crucial to derive the decay estimates (6.19). To further reduce the decay estimates to conformal energy estimates, we need a geometric setup, which is in Section 7, as we will discuss in the next subsection.

4.4 Geometric Setup in Section 7

To control the conformal energy, we use Wang's approach from [36], which relies on analysis on a conformal changed acoustic geometry. We reduce the decay estimates to the conformal energy estimates Theorem 10.2 in Section 10 via product estimates and Berstein inequality of Littlewood-Paley theory.

In order to define the conformal energy and do analysis based on the geometric structure of the relativistic

Euler equations, in Section 7.1-7.2, we construct the geometric null frame based on a solution u to the acoustical eikonal equation:

$$(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0. \tag{4.2}$$

Then we control the acoustic geometry based on the null frame that we just constructed. Specifically, we show that connect coefficients are bounded and has various decay properties. Note that given the control over the acoustic geometry, we have favorable estimates for the conformal energy as we discuss in Chpater 10. We will discuss the control of the acoustic geometry in the next two subsections.

4.5 Energy along Acoustic Null Hypersurfaces and Structures of the Acoustic Geomtry in Section 8

In Section 8, we prove the energy estimates for fluid variables along the acoustic null cones in Subsection 8.1. This is important since we need to control fluid variables along the null cones.

Then we list/compute transport equations and div-curl systems for geometric quantities. To start with, we list the connection coefficients in Definition 8.5. We define the conformal factor for the metric in Definition 8.6. We provide initial conditions for geometric cones in Proposition 8.12 and Proposition 8.13.

4.6 Control of the Acoustic Geometry in Section 9

We set up the bootstrap assumptions for geometric quantities in Section 9.1.2 and list the main estimates for the geometric quantities in Proposition 9.1. We give a discussion of the proof of Proposition 9.1 in Subsection 9.2, which improves the assumptions in Section 9.1.2. Proposition 9.1 is proven by applying strategies used in the transport equations and the div-curl systems of the geometric quantities. The proof follows the same in [9, Section 10]. We give a detailed proof in Section 9.4 with the help of various estimates of fluid, and lemmas for Sobolev and trace estimates derived in Section 9.3.

4.7 Control of the Conformal Energy in Section 10

Given the control of the acoustic geometry, we derive conformal energy estimates in Section 10. Note that this will finish our bootstrap argument since we have reduced the Strichartz estimates to decay estimates then to conformal energy estimates.

We set up the conformal energy and provide its basic properties in Section 10.1. Then, to control the conformal energy, we first prove a modified version of the standard energy estimate in Section 10.2. We use a multiplier approach for a conformally modified wave equation in Section 10.5 where the control of lower order terms, and differences between wave equation and its conformal modified version, are obtained in Section

10.3-10.4. By the estimates derived via the multiplier method described above, we prove decay estimates for the conformal energy in Section 10.6, thereby concluding the boundness theorem of the conformal energy.

The proof of the conformal energy estimates follows the same strategy used in [36, Section 7]. However, compared to [36], one needs to control additional quantities related to the fluid variables to obtain the same properties for geometric quantities.

CHAPTER 5

Energy, L^2 Elliptic and Schauder Estimates

In this section, we first derive the energy and L^2 elliptic estimates along constant-time hypersurfaces. Note that we obtain the same results as in [9, Section 4-5], where ρ in [9] plays the same role as h in this paper. Then we derive transport-Schauder type estimates for the vorticity and entropy gradient.

5.1 Energy and L^2 Elliptic Estimates

The following Proposition is the main result of the energy estimates.

Proposition 5.1 (Energy and elliptic estimates)**.** *Under the initial data and bootstrap assumptions of Section 3, smooth solutions to the relativistic Euler equations satisfy the following estimates for* $2 < N < 5/2$ *and* $t \in [0, T_*]$*:*

$$\sum_{k=0}^{2} \left\| \partial_t^k (h, \vec{v}, \vec{\omega}) \right\|_{H^{N-k}(\Sigma_t)} + \sum_{k=0}^{2} \left\| \partial_t^k s \right\|_{H^{N+1-k}(\Sigma_t)} + \sum_{k=0}^{1} \left\| \partial_t^k (\vec{\mathscr{C}}, \mathscr{D}) \right\|_{H^{N-1-k}(\Sigma_t)} \lesssim \|(h, \vec{v}, \vec{\omega})\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)} + 1. \tag{5.1}$$

Remark 5.2. *We note that* 1 *on the right-hand side of (5.1) is due to technical reasons. Specifically, as shown in Section 5.1.3,* 1 *can be replaced by* $\int_0^t \left\| \boldsymbol{\partial} (\vec{\Psi}, \vec{\omega}, \vec{S}) \right\|_{L_x^\infty(\Sigma_\tau)} \mathrm{d}\tau$. *By bootstrap assumptions (3.24b)-(3.24c) and Hölder's inequality, we could actually bound this term by* $T_*^{1/2}$.

We provide several key ingredients for proving Proposition 5.1 in the next two subsections. We provide the basic energy inequality for wave equations and transport equations in Subsection 5.1.1. We prove a crucial elliptic div-curl estimate in Subsection 5.1.2. We give the proof of Proposition 5.1 in Subsection 5.1.3. We refer readers to [9, Section 4-5] and [36, Section 2] for the energy estimates in the non-relativistic 3D compressible Euler equations case and the quasilinear wave equations case respectively.

5.1.1 The basic energy inequality for wave equations and transport equations

We provide the basic energy inequality for the wave equations in this subsection.

Definition 5.3 (Energy-momentum tensor, energy current, and deformation tensor)**.** *We define the energy-*

momentum tensor $Q_{\mu\nu}[\varphi]$ associated to a scalar function φ to be the following tensorfield:

$$Q_{\mu\nu}[\varphi] := \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha \varphi \partial_\beta \varphi. \tag{5.2}$$

Given φ and any multiplier vectorfield $\mathbf{X}$, we define the corresponding energy current ${}^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi]$ vectorfield as follows:

$$ {}^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi] := Q^{\alpha\beta}[\varphi]\mathbf{X}_\beta - \varphi^2\mathbf{X}^\alpha. \tag{5.3}$$

We define the deformation tensor of $\mathbf{X}$ as follows:

$$ {}^{(\mathbf{X})}\boldsymbol{\pi}_{\alpha\beta} := \mathbf{D}_\alpha\mathbf{X}_\beta + \mathbf{D}_\beta\mathbf{X}_\alpha, \tag{5.4}$$

where $\mathbf{D}$ is the Levi-Civita connection with respect to $\mathbf{g}$.

We have the following well-known divergence identity:

$$\mathbf{D}_\alpha {}^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi] = \square_{\mathbf{g}}\varphi(\mathbf{X}\varphi) + \frac{1}{2}Q^{\mu\nu}[\varphi]^{(\mathbf{X})}\boldsymbol{\pi}_{\mu\nu} - 2\varphi\mathbf{X}\varphi - \frac{1}{2}\varphi^2(\mathbf{g}^{-1})^{\mu\nu(\mathbf{X})}\boldsymbol{\pi}_{\mu\nu}. \tag{5.5}$$

We define the energy $\mathbb{E}[\varphi](t)$ as follows where $\mathbf{T}^\alpha := -\mathbf{g}^{\alpha 0}$ is the future-directed $\mathbf{g}$-timelike vectorfield defined in Definition 2.11:

$$\mathbb{E}[\varphi](t) := \int_{\Sigma_t} {}^{(\mathbf{T})}\mathbf{J}^\alpha[\varphi]\mathbf{T}_\alpha \mathrm{d}\varpi_g = \int_{\Sigma_t} \left(Q^{00}[\varphi] + \varphi^2\right)\mathrm{d}\varpi_g, \tag{5.6}$$

where $\mathrm{d}\varpi_g$ is the volume form on Σ_t with respect to g induced by $\mathbf{g}$.

Lemma 5.4 (Coerciveness of $\mathbb{E}$). *Under the bootstrap assumptions of Section 3.6, the following estimate holds for $t \in [0, T_*]$:*

$$\mathbb{E}[\varphi](t) \approx \|\varphi\|^2_{H^1(\Sigma_t)} + \|\partial_t\varphi\|^2_{L^2_x(\Sigma_t)}. \tag{5.7}$$

Proof of Lemma 5.4. First recall that $\mathbf{T}_a = 0$, so $g_{ab} = \mathbf{g}_{ab}$. Notice that since $0 < c(h,s) \leq 1$ and $(v^0)^2 \geq 1$, by direct computation and the bootstrap assumption (3.24a), we have

$$\mathrm{d}\varpi_g = \sqrt{\det g}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \left\{c^2 - (c^2-1)(v^0)^2\right\}^3\left\{c^{-6} + c^{-4}(c^{-2}-1)\left[(v^0)^2 - 1\right]\right\} \approx 1. \tag{5.8}$$

Then we compute $Q^{00}[\varphi]$. By (2.18b) and (2.20a), we have

$$\begin{aligned} Q^{00}[\varphi] =& \frac{1}{2}\left\{(\mathbf{T}\varphi)^2+(g^{-1})^{ab}\partial_a\varphi\partial_b\varphi\right\} \\ =& \frac{1}{2}\left\{(\mathbf{T}\varphi)^2+\frac{c^2\delta^{ab}\left[c^2-(c^2-1)(v^0)^2\right]+c^2(c^2-1)v^av^b}{\{c^2-(c^2-1)(v^0)^2\}^2}\partial_a\varphi\partial_b\varphi\right\}, \end{aligned} \tag{5.9}$$

where δ^{ab} is the Kronecker delta. Note that

$$\mathbf{T}=\partial_t+\frac{(c^2-1)v^av^0}{c^2-(c^2-1)(v^0)^2}\partial_a. \tag{5.10}$$

Then, since the speed of sound satisfies $0<c\leq 1$, it follows that (5.9) is coercive in $|\partial\varphi|$, since $(v^0)^2=1+\sum_{i=1,2,3}(v^i)^2$:

$$\begin{aligned} &\left\{c^2\delta^{ab}\left[c^2-(c^2-1)(v^0)^2\right]+c^2(c^2-1)v^av^b\right\}\partial_a\varphi\partial_b\varphi \\ &=c^4\left|\partial\varphi\right|^2-c^2(c^2-1)\left\{\delta^{ab}(v^0)^2-v^av^b\right\}\partial_a\varphi\partial_b\varphi \\ &\geq c^4\left|\partial\varphi\right|^2. \end{aligned} \tag{5.11}$$

By bootstrap assumptions that $|v^\alpha|$ are uniformly bounded and Young's inequality, we derive that $Q^{00}[\varphi]\lesssim|\boldsymbol{\partial}\varphi|^2$. Combined with (5.11), the desired estimates (5.7) follows. □

Lemma 5.5 (Basic energy inequality for the wave equations)**.** *Let φ be smooth on $[0,T_*]\times\mathbb{R}^3$. Under the bootstrap assumptions of Section 3.6, the following inequality holds for $t\in[0,T_*]$:*

$$\begin{aligned} \mathbb{E}[\varphi](t) \lesssim& \mathbb{E}[\varphi](0)+\int_0^t\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)}\mathbb{E}[\varphi](\tau)\mathrm{d}\tau \\ &+\int_0^t\|\square_{\mathbf{g}}\varphi\|_{L^2_x(\Sigma_\tau)}\|\boldsymbol{\partial}\varphi\|_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau. \end{aligned} \tag{5.12}$$

Proof of Lemma 5.5. We apply the divergence theorem on the space-time region $[0,t]\times\mathbb{R}^3$ relative to the volume form $\mathrm{d}\varpi_{\mathbf{g}}=\sqrt{\det g}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}\tau=\mathrm{d}\varpi_g\mathrm{d}\tau$. Note that $\mathbf{T}$ is the future-directed $\mathbf{g}$-unit normal to Σ_t. By (5.4), (5.5 and (5.6), with $\mathbf{X}:=\mathbf{T}$, we have:

$$\begin{aligned} \mathbb{E}[\varphi](t) =&\mathbb{E}[\varphi](0) \\ &-\int_0^t\int_{\Sigma_\tau}\left(\square_{\mathbf{g}}\varphi(\mathbf{T}\varphi)+\frac{1}{2}Q^{\mu\nu}[\varphi]^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}-2\varphi\mathbf{T}\varphi-\frac{1}{2}\varphi^2(\mathbf{g}^{-1})^{\mu\nu(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}\right)\mathrm{d}\varpi_g\mathrm{d}\tau. \end{aligned} \tag{5.13}$$

By bootstrap assumptions, we have $|\mathbf{T}\varphi|\lesssim|\boldsymbol{\partial}\varphi|$, $|Q^{\mu\nu}[\varphi]|\lesssim|\boldsymbol{\partial}\varphi|^2$ and $\left|{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}\right|\lesssim\left|\boldsymbol{\partial}\vec{\Psi}\right|$. Thus by

Cauchy-Schwarz inequality along Σ_τ and (5.7), we get the desired estimate. $\square$

Lemma 5.6 (Basic energy inequality for the transport equations)**.** *Let φ be smooth on $[0,T_*]\times\mathbb{R}^3$. Under the bootstrap assumptions of Section 3.6, the following inequality holds for $t\in[0,T_*]$:*

$$\|\varphi\|^2_{L^2_x(\Sigma_t)} \lesssim \|\varphi\|^2_{L^2_x(\Sigma_0)} + \int_0^t \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)} \|\varphi\|^2_{L^2_x(\Sigma_\tau)}\, \mathrm{d}\tau + \int_0^t \|\varphi\|_{L^2_x(\Sigma_\tau)} \|\mathbf{B}\varphi\|_{L^2_x(\Sigma_\tau)}\, \mathrm{d}\tau. \tag{5.14}$$

Proof of Lemma 5.6. Let $\mathbf{J}^\alpha := \varphi^2\mathbf{B}^\alpha$, then $\partial_\alpha\mathbf{J}^\alpha = 2\varphi\mathbf{B}\varphi + (\partial_\alpha\mathbf{B}^\alpha)\,\varphi^2$. We apply the divergence theorem on the space-time region $[0,t]\times\mathbb{R}^3$ relative to the Cartesian coordinates. Note that $\mathbf{J}^0 = \varphi^2$. By Cauchy-Schwarz inequality along Σ_τ, we obtain the desired estimates. $\square$

Remark 5.7. *We remark that for our implementation of the geometric energy method for wave equations, the timelike vectorfield* $\mathbf{T}$ *(defined in Definition 2.11) plays the same role as* $\mathbf{B}$ *(Note that* $\mathbf{B} = \partial_t + v^a\partial_a$ *in [9] is not the same as* $\mathbf{B} = \frac{v^\alpha}{v^0}\partial_\alpha$ *in this paper.) in [9, section 4.1]. All the arguments for geometric energy method for wave equations go through in the same fashion as in [9, section 4.1].*

5.1.2 Elliptic div-curl estiamtes in L^2 space

This subsection is dedicated to the proof of Proposition 5.8, which is a key ingredient in the proof of the energy estimates (5.1) for the $\vec{\omega},\vec{S},\vec{\mathscr{C}},\mathscr{D}$.

Proposition 5.8 (Elliptic div-curl estimates in L^2 space)**.** *Under the bootstrap assumptions in Section 3.6, the following estimates holds for ω and S:*

$$\left\|(\partial\vec{\omega},\partial\vec{S})\right\|_{L^2_x(\Sigma_t)} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|_{L^2_x(\Sigma_t)}. \tag{5.15}$$

Moreover, we also have the following H^{N-1} estimates:

$$\left\|(\partial\vec{\omega},\partial\vec{S})\right\|_{H^{N-1}(\Sigma_t)} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|_{H^{N-1}(\Sigma_t)} + \left\|\boldsymbol{\partial}\vec{\Psi}\right\|^4_{H^1(\Sigma_t)}. \tag{5.16}$$

Since we have to derive energy estimates on constant-time hypersurfaces, and the Hodge system (2.26) is a **space-time** div-curl system, we begin by deriving a **spatial** div-curl system for ω and S.

Proposition 5.9 (The div-curl system on constant-time hypersurfaces)**.** *Given the div-curl system (2.26), the* *following equations hold on Σ_t for vorticity ω and entropy gradient S:*

$$(G^{-1})^{ab}\partial_a(\omega_\flat)_b = F_\omega, \qquad (G^{-1})^{ab}\partial_a S_b = F_S, \tag{5.17a}$$

$$\partial_a(\omega_\flat)_b - \partial_b(\omega_\flat)_a = {}^{(\omega)}H_{ab}, \qquad \partial_a S_b - \partial_b S_a = {}^{(S)}H_{ab}, \tag{5.17b}$$

where

$$F_\omega = \mathscr{L}(\vec{\Psi},\vec{\varpi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \qquad F_S = \mathscr{L}(\vec{\Psi})\mathscr{D} + \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{5.18a}$$

$${}^{(\omega)}H_{ab} = \mathscr{L}(\vec{\Psi})\vec{\mathscr{C}} + \mathscr{L}(\vec{\Psi},\vec{\varpi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \qquad {}^{(S)}H_{ab} = \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}], \tag{5.18b}$$

$$(G^{-1})^{ab} = \delta^{ab} - \frac{v^a v^b}{(v^0)^2}. \tag{5.18c}$$

For convenience, we write the above 2 div-curl systems as follows, where $(\eta, F_\eta, {}^{(\eta)}H_{ab})$ which[1] *is either $(\omega_\flat, F_\omega, {}^{(\omega)}H_{ab})$ or $(S, F_S, {}^{(S)}H_{ab})$:*

$$(G^{-1})^{ab}\partial_a \eta_b = F_\eta, \tag{5.19a}$$

$$\partial_a \eta_b - \partial_b \eta_a = {}^{(\eta)}H_{ab}. \tag{5.19b}$$

Proof of Prop 5.9. For the div part (5.19a), by the equations (2.12c) and (2.27a), we write

$$\eta_0 = -\frac{\eta_b v^b}{v^0}. \tag{5.20}$$

Also by using transport equation (2.25a) and (2.25b), we have

$$\partial_0(\eta^\sharp)^b = -\frac{v^a \partial_a (\eta^\sharp)^b}{v^0} + \mathscr{L}(\vec{\Psi},\vec{\varpi},S)[\boldsymbol{\partial}\vec{\Psi}]. \tag{5.21}$$

Using (2.26d) for $\eta = \omega_\flat$, we write $\partial_0\omega^0 + \partial_a\omega^a = \mathscr{L}(\vec{\varpi})[\boldsymbol{\partial}\vec{\Psi}]$. By lowering the index $\partial_0\omega^0 = -\partial_0(\omega_\flat)_0$, equations (5.20) and (5.21), we prove (5.17a) for ω. Similarly, for $\eta = S$, by definition of $\mathscr{D}$ (2.14), we write $\partial_0(S^\sharp)^0 + \partial_a(S^\sharp)^a = \mathscr{L}(\vec{\Psi})\mathscr{D} + \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}]$. Using equations (5.20) and (5.21), we obtain the equation (5.17a) for S.

[1] We use the same notation throughout the remainder of the article.

Now we consider the curl part, note that we have the facts (2.27d) and (2.27e):

$$\begin{aligned}\partial_\gamma \eta_\delta - \partial_\delta \eta_\gamma =& \varepsilon_{\gamma\delta\kappa\lambda} v^\kappa \mathrm{vort}^\lambda(\eta) - (v^\kappa \partial_\kappa \eta_\delta)(v_\flat)_\gamma + v^\kappa(\partial_\delta \eta_\kappa)(v_\flat)_\gamma \\ &+ (v^\kappa \partial_\kappa \eta_\gamma)(v_\flat)_\delta - v^\kappa(\partial_\delta \eta_\kappa)(v_\flat)_\delta.\end{aligned} \tag{5.22}$$

Recall that

$$\mathscr{C}^\alpha = \mathrm{vort}^\alpha(\omega_\flat) + \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{5.23}$$

Hence for $\eta = \omega$, the first term on the right-hand side of (5.22) for ω is manifestly in ${}^{(\omega)}Hab$. Next, using $v^\kappa \partial_\kappa (\omega_\flat)_\delta = v^0 \mathbf{B}(\omega_\flat)_\delta$ and (2.25a), as well as (2.27b), we have that the right-hand side of (5.22) for ω is ${}^{(\omega)}Hab$. Similarly, by (2.26c), $v^\kappa \partial_\kappa S_\gamma = v^0 \mathbf{B} S_\gamma$, (2.25b) and (2.27c), we obtain the equation (5.17b) for S. □

Remark 5.10. *In terms of elliptic estimates, there is a major difference in Proposition 5.8 compared to the Hodge system of the non-relativistic 3D compressible Euler equations. In the non-relativistic case, the analogous elliptic equations are constant-coefficient div-curl equations along flat hypersurfaces of constant Cartesian time and, for example, the basic L^2 theory can be derived with the simple Hodge identity for Σ_t vectorfields $V \in H^1(\mathbb{R}^3; \mathbb{R}^3)$:*

$$\sum_{a,b=1}^{3} \left\| \partial_a V^b \right\|^2_{L^2_x(\mathbb{R}^3)} = \| divV \|^2_{L^2_x(\mathbb{R}^3)} + \| \mathrm{curl} V \|^2_{L^2_x(\mathbb{R}^3)}. \tag{5.24}$$

In contrast, the divergence equation (5.19a) has dynamic, solution-dependent coefficients.

Proposition 5.11 (The top-order div-curl system on constant-time hypersurfaces)**.** *Using the same notation as in Proposition 5.19, we have*

$$(G^{-1})^{ab} \partial_a (\partial \eta)_b = F_{\partial \eta}, \tag{5.25a}$$

$$\partial_a (\partial \eta)_b - \partial_b (\partial \eta)_a = {}^{(\partial\eta)}H_{ab}, \tag{5.25b}$$

where

$$F_{\partial \eta} = \partial F_\eta - \partial \left\{ (G^{-1})^{ab} \right\} \partial_a \eta, \tag{5.26a}$$

$${}^{(\partial\eta)}H_{ab} = \partial\, {}^{(\eta)}H_{ab}, \tag{5.26b}$$

Moreover, we have

$$(G^{-1})^{ab}\partial_a\partial P_\nu\eta_b = F_{\partial P_\nu\eta}, \tag{5.27a}$$

$$\partial_a\partial P_\nu\eta_b - \partial_b\partial P_\nu\eta_a = {}^{(\partial P_\nu\eta)}H_{ab}, \tag{5.27b}$$

where

$$F_{\partial P_\nu\eta} = P_\nu F_{\partial\eta} + [P_\nu, (G^{-1})^{ab}]\partial_a\partial\eta_b, \tag{5.28a}$$

$${}^{(\partial P_\nu\eta)}H_{ab} = P_\nu\left({}^{(\partial\eta)}H_{ab}\right). \tag{5.28b}$$

Proof of Proposition 5.11. (5.25) is the direct result of taking one spatial derivative of (5.17). By commuting Littlewood-Paley projection operator P_ν (defined in Section 3.2) with (5.25), we deduce (5.27). □

For convenience, we write the above 3 div-curl systems (5.19), (5.25) and (5.27) as follows, where (X, F, H_{ab}) is $(\eta, F_\eta, {}^{(\eta)}H_{ab})$ or $(\partial\eta, F_{\partial\eta}, {}^{(\partial\eta)}H_{ab})$ or $(\partial P_\nu\eta, F_{\partial P_\nu\eta}, {}^{(\partial P_\nu\eta)}H_{ab})$:

$$(G^{-1})^{ab}\partial_a X_b = F, \tag{5.29a}$$

$$\partial_a X_b - \partial_b X_a = H_{ab}. \tag{5.29b}$$

In order to derive elliptic estimates and Schauder estimates from the div-curl system (5.19), we need Lemma 5.12 provided below, which allows us to do estimates via Littlewood-Paley theory. We provide a partition of unity before the Lemma 5.12.

We want to apply the Fourier transform to a localized version of the div-curl system in Proposition 5.8. We consider the lattice $\mathscr{A} := \delta_2\mathbb{Z}^3$, where δ_2 is assumed to be small and will be determined in future analysis. Notice that $\{x_l\}_{l\in\mathbb{N}} := \mathscr{A} \subset \Sigma_t$ has points equally spread out, that is, for each x_l, there are 6 points in $\mathscr{A}$ such that the distance between x_l and any of them is δ_2. We define the family of functions $\{\psi_l\}_{l\in\mathbb{N}}$ as follows:

$$\psi_l(x) = \begin{cases} 1 & x \in B(x_l, \frac{1}{8}\delta_2), \\ \exp(\frac{4}{3\delta_2^2})\exp(\frac{1}{|x-x_l|^2-(\frac{7}{8}\delta_2)^2}) & x \in B(x_l, \frac{7}{8}\delta_2) - B(x_l, \frac{1}{8}\delta_2), \\ 0 & x \notin B(x_l, \frac{7}{8}\delta_2), \end{cases} \tag{5.30}$$

and set

$$\phi_l(x) := \frac{\psi_l(x)}{\sum\limits_k \psi_k(x)}. \tag{5.31}$$

We note that $\|\phi_l\|_{L^\infty(\Sigma_t)}, \|\partial\phi_l\|_{L^\infty(\Sigma_t)} \lesssim 1$.

We have constructed cut-off functions $\{\phi_l\}_{l\in\mathbb{N}} \subset C_0^\infty(\Sigma_t)$ such that $\phi_l = 1$ in $B(x_l, \frac{1}{8}\delta_2)$, $\mathrm{supp}(\phi_l) \subset B(x_l, \frac{7}{8}\delta_2)$, $\sum\limits_l \phi_l(x) = 1$ and for any $x_a, x_b \in \mathscr{A}$, $\phi_a(x) = \phi_b(x - x_a + x_b)$. We want to apply Fourier transform on a localized region, where $(G^{-1})^{ab}(x_l)$ is a constant and $(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)$ will be shown to be a controllable error term in the future analysis.

Lemma 5.12. *Given the Proposition 5.9 and Proposition 5.11 with x_l and ϕ_l, $l \in \mathbb{N}$, defined as above, let X be the solution of equations (5.29). Then the following identity holds in frequency space for $i = 1,2,3$:*

$$(G^{-1})^{ab}(x_l)\xi_a\xi_b\widehat{(\phi_l X_i)} = C\xi_i\underline{\hat{F}}^l + \sum_{k\neq i} C(G^{-1})^{ak}(x_l)\xi_a\underline{\hat{H}}^l_{ki}, \tag{5.32}$$

where

$$\begin{aligned}
\underline{F}^l =&(G^{-1})^{ab}(x_l)\partial_a(\phi_l X_b) = \phi_l F + (G^{-1})^{ab}(x_l)(\partial_a\phi_l)X_b \qquad (5.33\text{a})\\
&- \left[(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)\right]\partial_a(\phi_l X_b) - \left[(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)\right](\partial_a\phi_l)X_b,\\
\underline{H}^l_{ab} =&\partial_a(\phi_l X_b) - \partial_b(\phi_l X_a) = (\partial_a\phi_l)X_b - (\partial_b\phi_l)X_a + \phi_l H_{ab}. \qquad (5.33\text{b})
\end{aligned}$$

Proof of Lemma 5.12. By multiplying the div-curl system (5.19a) and (5.19b) by ϕ_l, we rewrite the system as follows:

$$\left\{(G^{-1})^{ab}(x_l) + (G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)\right\}\{\partial_a(\phi_l X_b) - (\partial_a\phi_l)X_b\} = \phi_l F, \tag{5.34a}$$

$$\partial_a(\phi_l X_b) - (\partial_a\phi_l)X_b - \partial_b(\phi_l X_a) + (\partial_b\phi_l)X_a = \phi_l H_{ab}. \tag{5.34b}$$

Taking the Fourier transform of (5.34a) and multiplying by ξ_1, we have

$$\xi_1\left\{(G^{-1})^{a1}(x_l)\xi_a\widehat{(\phi_l X_1)} + (G^{-1})^{a2}(x_l)\xi_a\widehat{(\phi_l X_2)} + (G^{-1})^{a3}(x_l)\xi_a\widehat{(\phi_l X_3)}\right\} = C\xi_1\underline{\hat{F}}^l, \tag{5.35}$$

where $C = \frac{1}{2\pi i}$ is a constant from Fourier transform, and

$$\begin{aligned}\underline{F}^l =&(G^{-1})^{ab}(x_l)\partial_a(\phi_l X_b) = \phi_l F + (G^{-1})^{ab}(x_l)(\partial_a\phi_l)X_b \\ &- \left[(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)\right]\partial_a(\phi_l X_b) - \left[(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)\right](\partial_a\phi_l)X_b.\end{aligned} \tag{5.36}$$

Similarly, taking the Fourier transform of (5.34b) and multiplying by $(G^{-1})^{a2}(x_l)\xi_a$ and $(G^{-1})^{a3}(x_l)\xi_a$, we have

$$(G^{-1})^{a2}(x_l)\xi_a\left\{\xi_2\widehat{(\phi_l X_1)} - \xi_1\widehat{(\phi_l X_2)}\right\} = C(G^{-1})^{a2}(x_l)\xi_a\hat{\underline{H}}^l_{21}, \tag{5.37a}$$

$$(G^{-1})^{a3}(x_l)\xi_a\left\{\xi_3\widehat{(\phi_l X_1)} - \xi_1\widehat{(\phi_l X_3)}\right\} = C(G^{-1})^{a3}(x_l)\xi_a\hat{\underline{H}}^l_{31}, \tag{5.37b}$$

where $C = \frac{1}{2\pi i}$ is a constant from Fourier transform, and

$$\underline{H}^l_{ab} = \partial_a(\phi_l X_b) - \partial_b(\phi_l X_a) = (\partial_a\phi_l)X_b - (\partial_b\phi_l)X_a + \phi_l H_{ab}. \tag{5.38}$$

Adding (5.35), (5.37a) and (5.37b), we obtain

$$(G^{-1})^{ab}(x_l)\xi_a\xi_b\widehat{(\phi_l X_1)} = C\xi_1\hat{\underline{F}}^l + C(G^{-1})^{a2}(x_l)\xi_a\hat{\underline{H}}^l_{21} + C(G^{-1})^{a3}(x_l)\xi_a\hat{\underline{H}}^l_{31}. \tag{5.39}$$

We use the same argument for X_2 and X_3. Hence for $i = 1,2,3$, we obtain

$$(G^{-1})^{ab}(x_l)\xi_a\xi_b\widehat{(\phi_l X_i)} = C\xi_i\hat{\underline{F}}^l + \sum_{k\neq i} C(G^{-1})^{ak}(x_l)\xi_a\hat{\underline{H}}^l_{ki}. \tag{5.40}$$

□

Lemma 5.13 (Positive definiteness of G^{-1})**.** *For any Σ_t-tangent one-form ξ, that is, $\xi_0 = 0$, we denote $|\xi|^2 := \sum_{i=1,2,3}\xi_i^2$. Then the following estimate holds for any $x_l \in \Sigma_t$, where G is defined in Proposition 5.9:*

$$C|\xi|^2 \leq (G^{-1})^{ab}(x_l)\xi_a\xi_b \leq |\xi|^2, \tag{5.41}$$

where $0 < C < 1$ is a constant depends only on $\|v\|_{L^\infty_x(\Sigma_t)}$, which is in turn controlled by the bootstrap assumption (3.24a).

Proof of Lemma 5.13. Using the definition of $(G^{-1})^{ab}$, we have

$$(G^{-1})^{ab}(x_l)\xi_a\xi_b = \left(\delta^{ab} - \frac{v^a v^b}{(v^0)^2}\right)\xi_a\xi_b = |\xi|^2 - \left(\frac{v^i\xi_i}{v^0}\right)^2. \tag{5.42}$$

Hence by the normalization $(v_\flat)_\alpha v^\alpha = -1$ in Section 2.2.1, we have

$$C|\xi|^2 \le (G^{-1})^{ab}(x_l)\xi_a\xi_b \le |\xi|^2, \tag{5.43}$$

where $0 < C < 1$ is a constant depends only on $\|v\|_{L^\infty_x(\Sigma_t)}$. □

Lemma 5.14. *For G defined in Proposition 5.9, the following inequality holds*

$$\left|(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)\right| \le C_3 \|v\|^4_{C_0^{0,\delta_0}(\Sigma_t)} |x-x_l|^{\delta_0} \le C_2^4 C_3 |x-x_l|^{\delta_0}, \tag{5.44}$$

where C_2, C_3 are constants (independent of l, x, a, b, δ_0).

Proof of Lemma 5.14. For Hölder continuous function $f, g \in C_0^{0,\delta_0}(\Sigma_t)$,

$$\begin{aligned} \frac{|f(x)g(x)-f(y)g(y)|}{|x-y|^{\delta_0}} =& \frac{|f(x)g(x)-f(x)g(y)|+|f(x)g(y)-f(y)g(y)|}{|x-y|^{\delta_0}} \\ \le& \|f\|_{L^\infty_x(\Sigma_t)}\|g\|_{\dot{C}_0^{0,\delta_0}(\Sigma_t)} + \|g\|_{L^\infty_x(\Sigma_t)}\|f\|_{\dot{C}_0^{0,\delta_0}(\Sigma_t)} \\ \le& 2\|f\|_{C_0^{0,\delta_0}(\Sigma_t)}\|g\|_{C_0^{0,\delta_0}(\Sigma_t)}. \end{aligned} \tag{5.45}$$

Therefore, by definition of G^{-1} in (5.18c) and the fact that $v^0 \ge 1$, substitute (f,g) in (5.45) by (v^a, v^b) and again by $(v^a v^b, \frac{1}{(v^0)^2})$, we have the estimate

$$\left|(G^{-1})^{ab}(x) - (G^{-1})^{ab}(x_l)\right| \le C_3 \|v\|^4_{C_0^{0,\delta_0}(\Sigma_t)} |x-x_l|^{\delta_0} \tag{5.46}$$

By Fundamental Theorem of Calculus, bootstrap assumptions (3.24) and (3.8), we have

$$\|v\|_{C_x^{0,\delta_0}(\Sigma_t)} \le \left\|\partial\vec{\Psi}\right\|_{L^1_t C_x^{0,\delta_0}(\Sigma_t)} + \|v\|_{C_x^{0,\delta_0}(\Sigma_0)} \lesssim \left\|\partial\vec{\Psi}\right\|_{L^1_t C_x^{0,\delta_0}(\Sigma_t)} + 1 \le C_2, \tag{5.47}$$

Combining (5.46) and (5.47), we have the desired result. □

Lemma 5.15 (Commutator estimates)**.** *For scalar function F, G, δ_1 defined as in Section 3.4, Littlewood-*

Paley projection operator P_ν defined in Section 3.2, we have the following estimates:

$$\|[P_\nu, G]F\|_{L^2(\Sigma_t)} \lesssim \nu^{-\delta_1} \|G\|_{C_x^{0,\delta_1}(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}. \tag{5.48}$$

Proof of Lemma 5.15. For ψ defined as in Section 3.2, we define $M(x)$ as follows:

$$M_\nu(x) := \mathscr{F}^{-1}(\psi(\nu^{-1}\xi)) = \int e^{ix\xi}\psi(\nu^{-1}\xi)\mathrm{d}\xi. \tag{5.49}$$

Then we have

$$\begin{aligned}[P_\nu, G]F &= \int M_\nu(x-y)\,(G(y)-G(x))\,F(y)\mathrm{d}y \\ &\leq \|G\|_{C_x^{0,\delta_1}(\Sigma_t)} \int M_\nu(x-y)\,|x-y|^{\delta_1}\,F(y)\mathrm{d}y.\end{aligned} \tag{5.50}$$

By Young's inequality, we have

$$\|[P_\nu, G]F\|_{L^2(\Sigma_t)} \lesssim \|G\|_{C_x^{0,\delta_1}(\Sigma_t)} \|F\|_{L^2(\Sigma_t)} \left\|M_\nu(x)\,|x|^{\delta_1}\right\|_{L^1(\Sigma_t)}. \tag{5.51}$$

By definition of Fourier transform and ψ in Section 3.2, we have

$$\int M_\nu(x)\,|x|^{\delta_1}\,\mathrm{d}x = \int \nu^3\hat{\psi}(-\nu x)\,|x|^{\delta_1}\,\mathrm{d}x \lesssim \nu^{-\delta_1}. \tag{5.52}$$

Combining above equations, we obtain the desired result. □

Lemma 5.16 (Control of the inhomogeneous terms)**.** *For $F_\eta, {}^{(\eta)}H, G^{-1}$ defined as in Proposition 5.9, and $F_{\partial\eta}, F_{\partial P_\nu\eta}, {}^{(\partial\eta)}H, {}^{(\partial P_\nu\eta)}H$ defined as in Proposition 5.11, we have the following estimates:*

$$\|F_\eta\|_{L^2(\Sigma_t)}, \left\|{}^{(\eta)}H\right\|_{L^2(\Sigma_t)} \leq C\left\|\boldsymbol{\partial}\vec{\Psi}, \vec{\mathscr{C}}, \mathscr{D}\right\|_{L^2(\Sigma_t)}, \tag{5.53}$$

$$\|F_{\partial\eta}\|_{L^2(\Sigma_t)}, \left\|{}^{(\partial\eta)}H\right\|_{L^2(\Sigma_t)} \leq C\left\|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathscr{C}}, \partial\mathscr{D}\right\|_{L^2(\Sigma_t)} + C\alpha^{-1}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|^2_{H^1(\Sigma_t)} \|\partial\eta\|_{L^2(\Sigma_t)} + C\alpha\left\|\partial^2\eta\right\|_{L^2(\Sigma_t)}, \tag{5.54}$$

$$\left\|\nu^{N-2}F_{\partial P_\nu\eta}\right\|_{l^2_\nu L^2(\Sigma_t)}, \left\|\nu^{N-2}\cdot{}^{(\partial P_\nu\eta)}H\right\|_{l^2_\nu L^2(\Sigma_t)} \lesssim \left(\left\|\partial^2\vec{\Psi}\right\|_{L^2(\Sigma_t)} + 1\right)\left\|\partial^2\eta\right\|_{L^2(\Sigma_t)} + \left\|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathscr{C}}, \partial\mathscr{D}\right\|_{H^{N-2}(\Sigma_t)}, \tag{5.55}$$

where C, α are constants, C is independent of α, and $\alpha > 0$ (small), which will be determined later.

Proof of Lemma 5.16. (5.53) is the direct result of taking $L^2(\Sigma_t)$ for (5.18).

Taking $L^2(\Sigma_t)$ for (5.26), we have

$$\|\partial F_\eta,\|_{L^2(\Sigma_t)}, \left\|\partial^{(\eta)}H\right\|_{L^2(\Sigma_t)} \le C\left\|\boldsymbol{\partial}^2\vec{\Psi},\partial\vec{\mathscr{C}},\partial\mathscr{D}\right\|_{L^2(\Sigma_t)}, \tag{5.56}$$

$$\begin{aligned}\left\|\partial\left\{(G^{-1})^{ab}\right\}\partial_a\eta\right\|_{L^2(\Sigma_t)} &\le C\left\|\partial\vec{\Psi}\right\|_{H^1(\Sigma_t)}\|\partial\eta\|^{\frac{1}{2}}_{L^2(\Sigma_t)}\|\partial^2\eta\|^{\frac{1}{2}}_{L^2(\Sigma_t)} \\ &\le C\alpha^{-1}\left\|\partial\vec{\Psi}\right\|^2_{H^1(\Sigma_t)}\|\partial\eta\|_{L^2(\Sigma_t)}+C\alpha\left\|\partial^2\eta\right\|_{L^2(\Sigma_t)},\end{aligned} \tag{5.57}$$

where for the first inequality in (5.57), we used the fact that $\|F\cdot G\|_{L^2(\Sigma_t)} \le C\|F\|^{\frac{1}{2}}_{L^2(\Sigma_t)}\|F\|^{\frac{1}{2}}_{H^1(\Sigma_t)}\|G\|_{H^1(\Sigma_t)}$ (see [9, (79b)]), for the second inequality in (5.57), we used Young's inequality.

Now we consider the proof of (5.55). Taking $l^2_\nu L^2(\Sigma_t)$ norm of (5.28), we have

$$\left\|\nu^{N-2}P_\nu F_{\partial\eta},\right\|_{l^2_\nu L^2(\Sigma_t)}, \left\|\nu^{N-2}P_\nu\left({}^{(\partial\eta)}H\right)\right\|_{l^2_\nu L^2(\Sigma_t)} \le C\left\|\boldsymbol{\partial}^2\vec{\Psi},\partial\vec{\mathscr{C}},\partial\mathscr{D}\right\|_{H^{N-2}(\Sigma_t)}+\left\|\partial^2\vec{\Psi}\right\|_{L^2(\Sigma_t)}\left\|\partial^2\eta\right\|_{L^2(\Sigma_t)}, \tag{5.58}$$

where the second term on the RHS of (5.58) is from the fact that (see [9, (81b)])

$$\left\|\nu^{N-2}P_\nu\left(\partial\left\{(G^{-1})^{ab}\right\}\partial_a\eta\right)\right\|_{l^2_\nu L^2(\Sigma_t)} \lesssim \left\|\partial\vec{\Psi}\right\|_{H^{N-\frac{3}{2}}(\Sigma_t)}\|\partial\eta\|_{H^1(\Sigma_t)}+\|\partial\eta\|_{H^{N-\frac{3}{2}}(\Sigma_t)}\left\|\partial\vec{\Psi}\right\|_{H^1(\Sigma_t)}. \tag{5.59}$$

What remains to be controlled is the commutator term $[P_\nu,(G^{-1})^{ab}]\partial_a\partial\eta_b$. By Lemma 5.15, where $G := (G^{-1})^{ab}$ and $F := \partial_a\partial\eta_b$, and (5.47), we have

$$\left\|\nu^{N-2}[P_\nu,(G^{-1})^{ab}]\partial_a\partial\eta_b\right\|_{l^2_\nu L^2(\Sigma_t)} \lesssim \left\|\partial^2\eta\right\|_{L^2(\Sigma_t)} \tag{5.60}$$

Combining above 3 estimates, we obtain (5.55). □

Lemma 5.17. *Let G^{-1} be defined as in (5.18c). F,H defined as in (5.29). For X the solution of the div-curl system (5.29), we have the following estimate:*

$$\|\partial X\|_{L^2_x(\Sigma_t)} \le C\|X,F,H\|_{L^2_x(\Sigma_t)}, \tag{5.61}$$

where C is a constant, which is independent of X,F,H.

Proof of Lemma 5.17. Throughout, X,F,G,H are the same as in (5.29), and $\underline{F},\underline{H}$ are defined in Lemma 5.12. Since $|\xi| \simeq 2\nu$ on support of $\widehat{P_\nu(\phi_l\eta_i)}$, by Littlewood-Paley estimate (3.7), (5.41) and (5.32), we remind ψ is defined in Section 3.2

$$\begin{aligned}\|\partial\{\phi_l X_i\}\|^2_{L^2_x(\Sigma_t)} &\le C_1\sum_{\nu>1}\|\nu P_\nu(\phi_l X_i)\|^2_{L^2_x(\Sigma_t)} \\ &\le C_1\sum_{\nu>1}\left\|\mathscr{F}^{-1}\left(\psi\left(\frac{|\xi|}{\nu}\right)|\nu|^{-1}\left\{\xi_i\underline{\hat{F}}^l+\sum_{k\neq i}(G^{-1})^{jk}(x_l)\xi_j\underline{\hat{H}}^l_{ki}\right\}\right)\right\|^2_{L^2_x(\Sigma_t)} \\ &\le C_1\left(\left\|\underline{F}^l\right\|^2_{L^2_x(\Sigma_t)}+\left\|\underline{H}^l\right\|^2_{L^2_x(\Sigma_t)}\right),\end{aligned} \tag{5.62}$$

where C_1 is a constant and $\mathscr{F}^{-1}$ is the Fourier inverse transform. By (5.33a), we have:

$$\begin{aligned}\left\|\underline{F}^l\right\|^2_{L^2_x(\Sigma_t)} \le \|\phi_l F\|^2_{L^2_x(\Sigma_t)} &+ \left\|(G^{-1})\right\|^2_{L^\infty_x(\Sigma_t)}\|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)} \\ &+\sup_{x\in B(x_l,\delta_2)}\left|(G^{-1})^{ab}(x)-(G^{-1})^{ab}(x_l)\right|^2\|\partial(\phi_l X)\|^2_{L^2_x(\Sigma_t)}.\end{aligned} \tag{5.63}$$

By (5.33b),

$$\left\|\underline{H}^l\right\|^2_{L^2_x(\Sigma_t)} \le \|\phi_l H\|^2_{L^2_x(\Sigma_t)} + 2\|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)}. \tag{5.64}$$

Let $C' := C_1C_2^4C_3$, where C_1, C_2, C_3 are constants from (5.62), (5.47), (5.44) respectively, and let δ_2 be small such that for all $x\in B(x_l,\delta_2)$,

$$C'|x-x_l|^{\delta_0} < \frac{1}{4}. \tag{5.65}$$

Hence by (5.62), (5.63), (5.64), (5.65), soaking the last term in the RHS of (5.63) to the left of (5.62) (by Lemma 5.14), we have:

$$\|\partial\{\phi_l X_i\}\|^2_{L^2_x(\Sigma_t)} \le C_4\left(\|\phi_l F\|^2_{L^2_x(\Sigma_t)}+\|\phi_l H\|^2_{L^2_x(\Sigma_t)}+\|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)}\right), \tag{5.66}$$

where C_4 is a constant.

Now we consider the $\|\partial X_i\|^2_{L^2_x(\Sigma_t)}$. By (5.66), we have:

$$\|\partial X_i\|^2_{L^2_x(\Sigma_t)} = \left\|\sum_l \partial\{\phi_l X_i\}\right\|^2_{L^2_x(\Sigma_t)} \le C_5\sum_l\|\partial\{\phi_l X_i\}\|^2_{L^2_x(\Sigma_t)} \le C_4C_5\left(\sum_l\|\phi_l(F,H)\|^2_{L^2_x(\Sigma_t)}+\|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)}\right). \tag{5.67}$$

For the first term on the RHS of (5.67), by (5.53), we have :

$$\sum_l \|\phi_l(F,H)\|^2_{L^2_x(\Sigma_t)} = \sum_l \int_{\Sigma_t} (\phi_l)^2(F^2+H^2)\mathrm{d}x \\ \leq \sum_l \|\phi_l\|^2_{L^\infty(\Sigma_t)} \int_{\Sigma_t \bigcap \{\partial\phi_l \neq 0\}} (F^2+H^2)\mathrm{d}x \\ \leq C_6 \|F,H\|^2_{L^2_x(\Sigma_t)}. \tag{5.68}$$

For the second term on the RHS of (5.67), we have:

$$\sum_l \|(\partial\phi_l)X\|^2_{L^2_x(\Sigma_t)} = \sum_l \int_{\Sigma_t} (\partial\phi_l)^2 X^2 \mathrm{d}x \leq C_7 \sum_l \|\partial\phi_l\|^2_{L^\infty(\Sigma_t)} \int_{\Sigma_t \bigcap \{\partial\phi_l \neq 0\}} X^2 \mathrm{d}x \leq C_8 \|X\|^2_{L^2_x(\Sigma_t)} \tag{5.69}$$

where the last inequality holds for both (5.68) and (5.69) since for each $x \in \Sigma_t$, there are finite many (at most 8) ϕ_l such that $\phi_l(x) \neq 0$.

Combining (5.67)-(5.69) and letting $C := C_4 C_5 (C_6 + C_8)$, we conclude the desired estimate. □

Proof of Proposition 5.8. (5.15) is a direct result by substituting (X,F,H) in (5.61) by $(\eta, F_\eta, {}^{(\eta)}H)$ and using the schematic definition $\eta = \partial\vec{\Psi}$, estimate (5.53).

Using (3.7), (5.16) can be proved in a similar fashion by using the following interpolation estimate:

$$\|\partial\eta_i\|^2_{H^{N-1}(\Sigma_t)} \leq \left\|\partial^2\eta_i\right\|^2_{L^2_x(\Sigma_t)} + C_1 \sum_\nu \left\|\nu^{N-2} P_\nu(\partial^2\eta_i)\right\|^2_{L^2_x(\Sigma_t)}, \tag{5.70}$$

where we bound the first term on the RHS of (5.70) as follows:

$$\left\|\partial^2\eta_i\right\|_{L^2_x(\Sigma_t)} \lesssim \left\|\partial\boldsymbol{\partial}\vec{\Psi}, \partial\mathscr{C}, \partial\mathscr{D}\right\|_{L^2_x(\Sigma_t)} + \left\|\boldsymbol{\partial}\vec{\Psi}\right\|^3_{H^1(\Sigma_t)}, \tag{5.71}$$

(5.71) is obtained by exactly the same method as in the proof of (5.15). That is, substituting $(X,F,H,\mathscr{C},\mathscr{D})$ in (5.61) by $(\partial\eta, F_{\partial_\eta}, {}^{(\partial_\eta)}H, \partial\mathscr{C}, \partial\mathscr{D})$ and using (5.54), we have

$$\left\|\partial^2\eta_i\right\|_{L^2_x(\Sigma_t)} \leq C\left\|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathscr{C}}, \partial\mathscr{D}\right\|_{L^2(\Sigma_t)} + C\alpha^{-1}\left\|\partial\vec{\Psi}\right\|^2_{H^1(\Sigma_t)} \|\partial\eta\|_{L^2(\Sigma_t)} + C\alpha\left\|\partial^2\eta\right\|_{L^2(\Sigma_t)}. \tag{5.72}$$

We pick α small ($C\alpha \leq \frac{1}{2}$) such that $C\alpha\left\|\partial^2\eta\right\|_{L^2(\Sigma_t)}$ can be soaked into the LHS of (5.72).

Now we consider the second term on the RHS of (5.70).

Substituting (X,F,H) in (5.61) by $(\partial P_\nu\eta, F_{\partial P_\nu \eta}, {}^{(\partial P_\nu \eta)}H)$, we have:

$$\left\|\partial^2 P_\nu \eta_i\right\|_{L^2_x(\Sigma_t)} \lesssim \left\|\partial P_\nu\eta, F_{\partial P_\nu\eta}, {}^{(\partial P_\nu\eta)}H\right\|_{L^2_x(\Sigma_t)} \tag{5.73}$$

Multiplying (5.73) by ν^{N-2} and taking the l^2_ν norm, we have:

$$\left\|\nu^{N-2}P_\nu(\partial^2\eta_i)\right\|_{l^2_\nu L^2_x(\Sigma_t)} \lesssim \left\|\nu^{N-2}\left(\partial P_\nu\eta, F_{\partial P_\nu\eta}, {}^{(\partial P_\nu\eta)}H\right)\right\|_{l^2_\nu L^2_x(\Sigma_t)}. \tag{5.74}$$

By (5.74) and (5.55), we have:

$$\left\|\nu^{N-2}P_\nu(\partial^2\eta_i)\right\|_{l^2_\nu L^2_x(\Sigma_t)} \lesssim \left(\left\|\partial^2\vec{\Psi}\right\|_{L^2(\Sigma_t)}+1\right)\left\|\partial^2\eta\right\|_{L^2(\Sigma_t)} + \left\|\boldsymbol{\partial}^2\vec{\Psi}, \partial\vec{\mathscr{C}}, \partial\mathscr{D}\right\|_{H^{N-2}(\Sigma_t)}. \tag{5.75}$$

By (5.70), (5.71) and (5.75), we have proved (5.16).

□

5.1.3 Proof of Proposition 5.1

Lemma 5.18 (Modified Grönwall's inequality)**.** *For non-negative scalar continuous function F, G and constant $\alpha > 0$, $\beta > 1$, if:*

$$F(t) \le \alpha + \int_0^t (F(\tau)+F(\tau)^\beta)G(\tau)\mathrm{d}\tau, \tag{5.76}$$

then:

$$F(t)^{\beta-1} \le \frac{1}{\left(1+\alpha^{-\beta+1}\right)\exp\left((-\beta+1)\int_0^t G(\tau)\mathrm{d}\tau\right)-1} \tag{5.77}$$

Proof of Lemma 5.18. Let $\tilde{F} := -\frac{1}{-\beta+1}\ln(1+F^{-\beta+1})$, then we have:

$$\tilde{F}'(t) \le G(t). \tag{5.78}$$

By the Fundamental Theorem of Calculus, we have:

$$1+F(t)^{-\beta+1} \ge (1+\alpha^{-\beta+1})\exp\left((-\beta+1)\int_0^t G(\tau)\mathrm{d}\tau\right). \tag{5.79}$$

Desired results follows by straightforward computation. □

Proof of Proposition 5.1. For N defined as in Section 3.4, we let

$$P_N(t) := \sum_{k=0}^{2}\left\|\boldsymbol{\partial}^k(\vec{\Psi}, \vec{\varpi}, \vec{S})\right\|^2_{H^{N-k}(\Sigma_t)} + \sum_{k=0}^{1}\left\|\boldsymbol{\partial}^k(\vec{\mathscr{C}}, \mathscr{D})\right\|^2_{H^{N-k-1}(\Sigma_t)}. \tag{5.80}$$

In this proof, we derive integral inequalities for $\sum_{k=0}^{2}\left\|\boldsymbol{\partial}^k\vec{\Psi}\right\|^2_{H^{N-k}(\Sigma_t)}$ and $\sum_{k=0}^{1}\left\|\boldsymbol{\partial}^k(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{H^{N-k-1}(\Sigma_t)}$ in $P_N(t)$, namely (5.82), (5.86), (5.88), (5.89). We then use elliptic estimates (5.15) and apply Grönwall's inequality to all the terms in $P_N(t)$ collectively.

The proof of Proposition 5.1 for $\vec{\Psi}$ combines the vectorfield multiplier method and Littlewood-Paley theory. That is, to derive the energy estimates at the top order, one integrate (5.5) and applies the divergence theorem using the energy current ${}^{(\mathbf{T})}\mathbf{J}^\alpha[\boldsymbol{\partial}\vec{\Psi}] := Q^{\alpha\beta}[\boldsymbol{\partial}\vec{\Psi}]\mathbf{T}_\beta - \mathbf{T}^\alpha(\boldsymbol{\partial}\vec{\Psi})^2$ and ${}^{(\mathbf{T})}\mathbf{J}^\alpha[P_\nu\boldsymbol{\partial}\vec{\Psi}] := Q^{\alpha\beta}[P_\nu\boldsymbol{\partial}\vec{\Psi}]\mathbf{T}_\beta - \mathbf{T}^\alpha(P_\nu\boldsymbol{\partial}\vec{\Psi})^2$ on the space-time region bounded by Σ_0 and Σ_t. Then by Lemma 5.5 with $\boldsymbol{\partial}\vec{\Psi}$ and $P_\nu\boldsymbol{\partial}\vec{\Psi}$ in a role of φ, we have, respectively:

$$\begin{aligned}\mathbb{E}[\boldsymbol{\partial}\vec{\Psi}](t) \lesssim& \mathbb{E}[\boldsymbol{\partial}\vec{\Psi}](0) + \int_0^t \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)}\mathbb{E}[\boldsymbol{\partial}\vec{\Psi}](\tau)\mathrm{d}\tau \\ &+ \int_0^t \left\|\square_{\mathbf{g}}\boldsymbol{\partial}\vec{\Psi}\right\|_{L^2_x(\Sigma_\tau)}\left[\mathbb{E}[\boldsymbol{\partial}\vec{\Psi}](\tau)\right]^{1/2}\mathrm{d}\tau,\end{aligned} \tag{5.81a}$$

$$\begin{aligned}\mathbb{E}[P_\nu\boldsymbol{\partial}\vec{\Psi}](t) \lesssim& \mathbb{E}[P_\nu\boldsymbol{\partial}\vec{\Psi}](0) + \int_0^t \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)}\mathbb{E}[P_\nu\boldsymbol{\partial}\vec{\Psi}](\tau)\mathrm{d}\tau \\ &+ \int_0^t \left\|\square_{\mathbf{g}}P_\nu\boldsymbol{\partial}\vec{\Psi}\right\|_{L^2_x(\Sigma_\tau)}\left[\mathbb{E}[P_\nu\boldsymbol{\partial}\vec{\Psi}](\tau)\right]^{1/2}\mathrm{d}\tau.\end{aligned} \tag{5.81b}$$

Then, we use equation (3.9) to substitute for $\square_{\mathbf{g}}\boldsymbol{\partial}\vec{\Psi}$, and we use equation (3.12) $\square_{\mathbf{g}}P_\nu\boldsymbol{\partial}\vec{\Psi}$ to substitute for the right-hand side of (5.81b). Multiplying (5.81b) by $\nu^{2(N-2)}$, summing over ν and using (3.7) , estimates (3.18) and Hölder's inequality, we have:

$$\begin{aligned}\left\|\boldsymbol{\partial}^2\vec{\Psi}\right\|^2_{H^{N-2}(\Sigma_t)} \lesssim& \left\|\boldsymbol{\partial}^2\vec{\Psi}\right\|^2_{H^{N-2}(\Sigma_0)} + \int_0^t \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)}\left\|\boldsymbol{\partial}^2\vec{\Psi}\right\|^2_{H^{N-2}(\Sigma_\tau)}\mathrm{d}\tau \\ &+ \int_0^t \Big\{ \left\|\partial(\vec{\mathscr{C}},\mathscr{D})\right\|_{H^{N-2}(\Sigma_\tau)}\left\|\boldsymbol{\partial}^2\vec{\Psi}\right\|_{H^{N-2}(\Sigma_\tau)} \\ &+ \left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}+1\right)\left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S})\right\|_{H^{N-1}(\Sigma_\tau)}+1\right)\left\|\boldsymbol{\partial}^2\vec{\Psi}\right\|_{H^{N-2}(\Sigma_\tau)}\Big\}\mathrm{d}\tau \\ \lesssim& P_N(0) + \int_0^t \left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}\mathrm{d}\tau + \int_0^t \left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}+1\right)P_N(\tau)\mathrm{d}\tau.\end{aligned} \tag{5.82}$$

For $\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{H^{N-1}(\Sigma_t)}$, we first have:

$$\left\|\boldsymbol{\partial}\vec{\Psi}\right\|^2_{H^{N-1}(\Sigma_t)} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_x(\Sigma_t)} + \left\|\boldsymbol{\partial}^2\vec{\Psi}\right\|^2_{H^{N-2}(\Sigma_t)}. \tag{5.83}$$

Then, by the fundamental theorem of Calculus in time, Minkowski integral inequality, and smallness of T_*, we have :

$$\left\|\boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_x(\Sigma_t)}=\int_{\Sigma_t}\left\{\boldsymbol{\partial}\vec{\Psi}(0,x)+\int_0^t\partial_t\boldsymbol{\partial}\vec{\Psi}(\tau,x)\mathrm{d}\tau\right\}^2\mathrm{d}x\lesssim P_N(0)+\left\|\boldsymbol{\partial}^2\vec{\Psi}\right\|^2_{H^{N-2}(\Sigma_t)}. \tag{5.84}$$

Similarly, we have:

$$\left\|\vec{\Psi}\right\|_{H^N(\Sigma_t)}\lesssim P_N(0)+\left\|\boldsymbol{\partial}^2\vec{\Psi}\right\|^2_{H^{N-2}(\Sigma_t)}. \tag{5.85}$$

Therefore, by adding (5.82)-(5.84), we have:

$$\sum_{k=0}^2\left\|\boldsymbol{\partial}^k\vec{\Psi}\right\|^2_{H^{N-k}(\Sigma_t)}\lesssim P_N(0)+\int_0^t\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}\mathrm{d}\tau+\int_0^t\left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}+1\right)P_N(\tau)\mathrm{d}\tau. \tag{5.86}$$

Now we derive top-order estimates for $\vec{\mathscr{C}}$ and $\mathscr{D}$. We apply the energy estimates (5.14) with $\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})$ and $P_\nu\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})$ in a role of φ respectively to obtain,

$$\begin{aligned}\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{L^2_x(\Sigma_t)}\lesssim&\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{L^2_x(\Sigma_0)}+\int_0^t\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)}\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau \\ &+\int_0^t\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L^2_x(\Sigma_\tau)}\left\|\mathbf{B}\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau,\end{aligned} \tag{5.87a}$$

$$\begin{aligned}\left\|P_\nu\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{L^2_x(\Sigma_t)}\lesssim&\left\|P_\nu\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{L^2_x(\Sigma_0)}+\int_0^t\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)}\left\|P_\nu\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau \\ &+\int_0^t\left\|P_\nu\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L^2_x(\Sigma_\tau)}\left\|\mathbf{B}P_\nu\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L^2_x(\Sigma_\tau)}\mathrm{d}\tau.\end{aligned} \tag{5.87b}$$

We use equations (3.10)-(3.11) to substitute for $\mathbf{B}\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})$, and we use equations (3.13) and (3.14) for $\mathbf{B}P_\nu\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})$ to substitute for the right-hand side of (5.87b). Multiplying (5.87b) by $\nu^{2(N-2)}$, summing over ν and using (3.7), substituting $\partial_t(S,\omega)$ with spatial derivatives $\partial(S,\omega)$ by (2.25a)-(2.25b), using estimates (3.18) and elliptic estimates (5.15)-(5.16), and we have:

$$\begin{aligned}&\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{H^{N-2}(\Sigma_t)}\lesssim\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{H^{N-2}(\Sigma_0)}+\int_0^t\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)}\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|^2_{H^{N-2}(\Sigma_\tau)}\mathrm{d}\tau \\ &+\int_0^t\left\{\left\|\partial(\vec{\mathscr{C}},\mathscr{D})\right\|_{H^{N-2}(\Sigma_\tau)}\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{H^{N-2}(\Sigma_\tau)}\right. \\ &\left.+\left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}+1\right)\left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{H^{N-1}(\Sigma_\tau)}+1\right)\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{H^{N-2}(\Sigma_\tau)}\mathrm{d}\tau\right\} \\ &\lesssim P_N(0)+\int_0^t\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}\mathrm{d}\tau+\int_0^t\left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}+1\right)(P_N(\tau)+P_N^3(\tau))\mathrm{d}\tau.\end{aligned} \tag{5.88}$$

For $\left\|\vec{\mathscr{C}},\mathscr{D}\right\|_{H^{N-1}(\Sigma_t)}$, using the same method as in (5.83)-(5.86), by (5.88), we have:

$$\begin{aligned}\left\|\vec{\mathscr{C}},\mathscr{D}\right\|^2_{H^{N-1}(\Sigma_t)} \lesssim & P_N(0)+\int_0^t \left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)} \mathrm{d}\tau \\ & +\int_0^t \left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}+1\right)(P_N(\tau)+P_N^3(\tau))\mathrm{d}\tau.\end{aligned} \tag{5.89}$$

Combining (5.82), (5.86), (5.88), (5.89) and elliptic estimates (5.15)-(5.16), we have:

$$P_N(t) \lesssim P_N(0)+\int_0^t \left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)} \mathrm{d}\tau+\int_0^t \left(\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S})\right\|_{L^\infty_x(\Sigma_\tau)}+1\right)\left(P_N(\tau)+P_N^3(\tau)\right)\mathrm{d}\tau. \tag{5.90}$$

By bootstrap assumptions (3.24b)-(3.24c), Hölder inequality in time and Grönwall's inequality (5.77), where $\beta=3$, we conclude the desired result. □

5.2 Schauder Estimates

In this section, we bound the Hölder norms of the modified fluid variables $\mathscr{C},\mathscr{D}$ and the derivatives of vorticity and entropy gradient. Moreover, in Proposition 5.19, we reduce the proof of the improvement of bootstrap assumption (3.24c) to the proof of the improvement of bootstrap assumption (3.24b).

Theorem 5.19 (Improvements of the bootstrap assumptions for the vorticity and entropy gradient)**.** *Let δ and δ_1 be as in Section 3.4. Under the initial data and bootstrap assumptions of Section 3, assume the improved estimates in Theorem 6.1 holds for $\boldsymbol{\partial}\vec{\Psi}$, that is,*

$$\left\|\boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)}+\sum_{\nu\geq 2}\nu^{2\delta_1}\left\|P_\nu\boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}, \tag{5.91}$$

then the following estimates hold:

$$\sum_{\nu\geq 2}\nu^{\delta_1}\left\|P_\nu(\partial\vec{\omega},\partial\vec{S})\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}. \tag{5.92}$$

We emphasize that the proof of the Strichartz estimates (6.1) is independent of (5.92).

We will prove Theorem 5.19 in Section 5.3. In this subsection, we derive a Schauder type estimate in the following lemma:

Lemma 5.20. *Let δ_1 be as in Section 3.4. Under the initial data and bootstrap assumptions of Section 3, the* *following estimates hold:*

$$\left\|\partial\vec{\omega},\partial\vec{S}\right\|_{C_x^{0,\delta_1}(\Sigma_t)} \lesssim \left\|\partial\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|_{C_x^{0,\delta_1}(\Sigma_t)}. \tag{5.93}$$

Proof of Lemma 5.20. We define the smooth function $\psi_1 = \psi_1(|\xi|) : \mathbb{R}^3 \to [0,1]$ such that $\psi_1(\xi) = 0$ for $0 \le |\xi| < \frac{1}{2}$ and $\psi_1(\xi) = 1$ for $|\xi| \ge 1$.

By Littlewood-Paley theory (3.8) and (5.32) where η, F, G, H are the same as in Proposition 5.9 and $\underline{F}, \underline{H}$ are defined in Lemma 5.12,

$$\begin{aligned}
&\|\partial_s(\phi_l\eta_i)\|_{\dot{C}_x^{0,\delta_1}(\Sigma_t)} \approx \sup_{\nu\ge 2}\left\|\nu^{\delta_1}P_\nu(\partial_s\phi_l\eta_i)\right\|_{L_x^\infty} \qquad (5.94)\\
&=\sup_{\nu\ge 2}\left\|C_1\nu^{\delta_1}\mathscr{F}^{-1}\left\{\psi(\nu^{-1}\xi)\frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a\xi_b}\left(\xi_i\hat{\underline{F}}^l+\sum_{k\ne i}(G^{-1})^{jk}(x_l)\xi_j\hat{\underline{H}}^l_{ki}\right)\right\}\right\|_{L_x^\infty}\\
&\le C_1\left\|\mathscr{F}^{-1}\left\{\psi_1(\xi)\frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a\xi_b}\xi_i\hat{\underline{F}}^l\right\}\right\|_{C_x^{0,\delta_1}(\Sigma_t)}\\
&\quad+C_1\sum_{k\ne i}\left\|\mathscr{F}^{-1}\left\{\psi_1(\xi)\frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a\xi_b}(G^{-1})^{jk}(x_l)\xi_j\hat{\underline{H}}^l_{ki}\right\}\right\|_{C_x^{0,\delta_1}(\Sigma_t)},
\end{aligned}$$

where C_1 is a constant, $\mathscr{F}^{-1}$ is the inverse Fourier transform operator and ψ is defined in Section 3.2. Now let's consider the first term in the right-hand side of last line of (5.94). For each fixed $s,i = 1,2,3$, we define function $p_{s,i}(\xi)$ as follows:

$$p_{s,i}(\xi) := \psi_1(\xi)\frac{\xi_s\xi_i}{(G^{-1})^{ab}(x_l)\xi_a\xi_b}. \tag{5.95}$$

The associated pseudodifferential operator $p(D_\xi; s,i)$ is defined by using Fourier integral representation as follows:

$$p_{s,i}(D_\xi)f(x) := \int p_{s,i}(\xi)\hat{f}(\xi)e^{ix\xi}\mathrm{d}\xi. \tag{5.96}$$

By direct computation and positive definiteness of G which is showed in Lemma 5.13, we have

$$\left|D_\xi^\alpha p_{s,i}(\xi)\right| \le C_{\alpha\beta}(1+|\xi|^2)^{\frac{-|\alpha|}{2}}. \tag{5.97}$$

So $p_{s,i}(\xi)$ is in the Hörmander class $S^0_{1,0}$ and $p_{s,i}(D)$ belongs to $OPS^0_{1,0}$. By the theory of pseudodifferential operators, we have $p_{s,i}(D_\xi) : C_x^{0,\delta_1} \to C_x^{0,\delta_1}$. We refer reader to [12, Chapter 18] for explicit definition of

Hörmander class and [32, Proposition 2.1.D] for the bounds of the operator $p_{s,i}(D_\xi)$. Therefore,

$$\left\| \mathscr{F}^{-1}\left\{ \psi_1(\xi)\frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a\xi_b}\xi_i \hat{\underline{F}}^l \right\} \right\|_{C_x^{0,\delta_1}(\Sigma_t)} \le C_4 \left\| \underline{F}^l \right\|_{C_x^{0,\delta_1}(\Sigma_t)}, \tag{5.98}$$

where C_4 is a constant (independent of s,i,j,k,l). Similarly, we have

$$\sum_{k\neq i} \left\| \mathscr{F}^{-1}\left\{ \psi_1(\xi)\frac{\xi_s}{(G^{-1})^{ab}(x_l)\xi_a\xi_b}(G^{-1})^{jk}(x_l)\xi_j \hat{\underline{H}}^l_{ki} \right\} \right\|_{C_x^{0,\delta_1}(\Sigma_t)} \le C_4 \sum_{k\neq i} \left\| \underline{H}^l_{ki} \right\|_{C_x^{0,\delta_1}(\Sigma_t)}. \tag{5.99}$$

Combining (5.94), (5.98) and (5.99), for any l,s,i, we have

$$\|\partial_s(\phi_l\eta_i)\|_{\dot{C}_x^{0,\delta_1}(\Sigma_t)} \le C_1C_4\left(\left\|\underline{F}^l\right\|_{C_x^{0,\delta_1}(\Sigma_t)} + \sum_{k\neq i}\left\|\underline{H}^l_{ki}\right\|_{C_x^{0,\delta_1}(\Sigma_t)} \right), \tag{5.100}$$

where the constant C_1,C_4 is independent of l,s,i. By (5.33a),

$$\begin{aligned} \left\|\underline{F}^l\right\|_{C_x^{0,\delta_1}(\Sigma_t)} \le{}& \|\phi_l F\|_{C_x^{0,\delta_1}(\Sigma_t)} + \left\|G^{-1}\right\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\partial\phi_l\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)} \\ &+ \sup_{x\in B(x_l,\delta_2)}\left|(G^{-1})^{ab}(x)-(G^{-1})^{ab}(x_l)\right| \|\partial(\phi_l\eta)\|_{C_x^{0,\delta_1}(\Sigma_t)}. \end{aligned} \tag{5.101}$$

By (5.33b),

$$\left\|\underline{H}^l\right\|_{C_x^{0,\delta_1}(\Sigma_t)} \le \|\phi_l H\|_{C_x^{0,\delta_1}(\Sigma_t)} + 2\|\partial\phi_l\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)}. \tag{5.102}$$

For C_2C_3 as in (5.44), C_1C_4 as in (5.100), by Lemma 5.14, we let δ_2 be small such that for all $x\in B(x_l,\delta_2)$,

$$C_1C_4\sup_{x\in B(x_l,\delta_2)}\left|(G^{-1})^{ab}(x)-(G^{-1})^{ab}(x_l)\right| < \frac{1}{4}. \tag{5.103}$$

Combining (5.100),(5.101), (5.102) and (5.103), absorbing last term in the RHS of (5.101) by the left, we have

$$\begin{aligned} \|\partial(\phi_l\eta_i)\|_{\dot{C}_x^{0,\delta_1}(\Sigma_t)} \lesssim{}& \|\phi_l F\|_{C_x^{0,\delta_1}(\Sigma_t)} + \left\|G^{-1}\right\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\partial\phi_l\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)} \\ &+ \|\phi_l H\|_{C_x^{0,\delta_1}(\Sigma_t)} + \|\partial\phi_l\|_{C_x^{0,\delta_1}(\Sigma_t)} \|\eta\|_{C_x^{0,\delta_1}(\Sigma_t)}, \end{aligned} \tag{5.104}$$

Now notice that

$$\|\partial\eta\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)} = \sup_{x,y\in\Sigma_t} \frac{|\partial\eta(x)-\partial\eta(y)|}{|x-y|^{\delta_1}} = \sup_{x,y\in\Sigma_t} \frac{\left|\partial\left(\sum_l \phi_l\eta\right)(x) - \partial\left(\sum_l \phi_l\eta\right)(y)\right|}{|x-y|^{\delta_1}} \tag{5.105}$$
$$\leq \sup_{x,y\in\Sigma_t} \sum_{\phi_l(x)\neq 0 \text{ or } \phi_l(y)\neq 0} \|\partial(\phi_l\eta)\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)}$$

Notice that for each $x\in\Sigma_t$, there are at most finitely many l's (at most 16) such that either $\phi_l(x)$ or $\phi_l(y)$ is non-zero. Hence, by definition of G^{-1}, F, H in (5.18), (5.104) and (5.105)

$$\|\partial\eta\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)} \lesssim \|F\|_{C^{0,\delta_1}_x(\Sigma_t)} + \left\|G^{-1}\right\|_{C^{0,\delta_1}_x(\Sigma_t)} \|\eta\|_{C^{0,\delta_1}_x(\Sigma_t)} + \|H\|_{C^{0,\delta_1}_x(\Sigma_t)} + \|\eta\|_{C^{0,\delta_1}_x(\Sigma_t)} \tag{5.106}$$
$$\lesssim \left\|\boldsymbol{\partial}\vec{\Psi}, \mathscr{C}, \mathscr{D}\right\|_{C^{0,\delta_1}_x(\Sigma_t)}.$$

We now bound $\|\partial\eta\|_{L^\infty_x(\Sigma_t)}$. For any point $z\in\Sigma_t$, there is a $y\in B(z,1)$ such that $|\partial\eta(y)|\leq\|\partial\eta\|_{L^2_x(\Sigma_t)}$, thus

$$|\partial\eta(z)| \leq |\partial\eta(y)| + 2\|\partial\eta\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)} \lesssim \|\partial\eta\|_{L^2_x(\Sigma_t)} + \|\partial\eta\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)}, \tag{5.107}$$

that is,

$$\|\partial\eta\|_{L^\infty_x(\Sigma_t)} \lesssim \|\partial\eta\|_{L^2_x(\Sigma_t)} + \|\partial\eta\|_{\dot{C}^{0,\delta_1}_x(\Sigma_t)}. \tag{5.108}$$

Combining with Proposition 5.1, we have

$$\|\partial\eta\|_{C^{0,\delta_1}_x(\Sigma_t)} \lesssim 1 + \left\|\boldsymbol{\partial}\vec{\Psi}, \mathscr{C}, \mathscr{D}\right\|_{C^{0,\delta_1}_x(\Sigma_t)}. \tag{5.109}$$

□

5.3 Estimates for $\|\mathscr{C},\mathscr{D}\|_{C^{0,\delta_1}_x(\Sigma_t)}$ via Transport Equations and the Proof of Theorem 5.19

In this subsection, we first estimate the Hölder norm of the modified fluid variables. We then prove Theorem 5.19. We will use the following lemma, which is a standard estimate for transport equations with Hölder data and Hölder inhomogeneities.

Lemma 5.21. *Let ϕ be a scalar function. If $F\in C^{0,\alpha}_x(\Sigma_\tau)$ with $\tau\in[0,t]$ and*

$$\mathbf{B}\phi = F, \tag{5.110}$$

then

$$\|\phi\|_{C^{0,\alpha}_x(\Sigma_t)} \lesssim \|\phi\|_{C^{0,\alpha}_x(\Sigma_0)} + \int_0^t \|F\|_{C^{0,\alpha}_x(\Sigma_\tau)} \mathrm{d}\tau. \tag{5.111}$$

Proof. Note that $\mathbf{B}^0 = 1$. Let γ be the integral curve of $\mathbf{B}$ such that

$$\gamma^i(0,x) = x^i, \tag{5.112}$$

$$\gamma^0(t,x) = t, \tag{5.113}$$

and

$$\partial_t \gamma^i(t,x) = \mathbf{B}^i(t,\gamma). \tag{5.114}$$

Then

$$\begin{aligned} |\gamma^i(t,x) - \gamma^i(t,y)| &\le |x^i - y^i| + \int_0^t \mathbf{B}^i(\tau,\gamma(\tau,x)) - \mathbf{B}^i(\tau,\gamma(\tau,y)) \mathrm{d}\tau \\ &\le |x^i - y^i| + \int_0^t \|\partial \mathbf{B}^i\|_{L^\infty_x(\Sigma_\tau)} |\gamma(\tau,x) - \gamma(\tau,y)| \mathrm{d}\tau. \end{aligned} \tag{5.115}$$

By Grönwall's inequality, we have

$$\frac{|\gamma(t,x) - \gamma(t,y)|}{|x-y|} \lesssim \exp\left(\int_0^t \|\partial \mathbf{B}\|_{L^\infty_x(\Sigma_\tau)} \mathrm{d}\tau \right). \tag{5.116}$$

Note that $\|\partial \mathbf{B}\|_{L^\infty_x(\Sigma_\tau)} \approx \left\|\partial \vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)}$. Similarly, considering $\bar{\gamma}(\tau) := \gamma(t-\tau)$ from Σ_t to Σ_0, we have

$$\frac{|x-y|}{|\gamma(t,x) - \gamma(t,y)|} \lesssim \exp\left(\int_0^t \|\partial \mathbf{B}\|_{L^\infty_x(\Sigma_\tau)} \mathrm{d}\tau \right). \tag{5.117}$$

Now we consider ϕ:

$$\partial_t(\phi \circ \gamma) = F \circ \gamma, \tag{5.118}$$

then

$$\phi \circ \gamma(t,x) - \phi \circ \gamma(t,y) = \phi \circ \gamma(0,x) - \phi \circ \gamma(0,y) + \int_0^t F(\tau,\gamma(\tau,x)) - F(\tau,\gamma(\tau,y)) \mathrm{d}\tau. \tag{5.119}$$

By (5.116), (5.117) and bootstrap assumption in Section 3.6, we have

$$\begin{aligned} \frac{|F(\tau,\gamma(\tau,x)) - F(\tau,\gamma(\tau,y))|}{|\gamma(t,x) - \gamma(t,y)|^\alpha} &= \frac{|F(\tau,\gamma(\tau,x)) - F(\tau,\gamma(\tau,y))|}{|\gamma(\tau,x) - \gamma(\tau,y)|^\alpha} \frac{|\gamma(\tau,x) - \gamma(\tau,y)|^\alpha}{|\gamma(t,x) - \gamma(t,y)|^\alpha} \\ &\lesssim \|F\|_{C^{0,\alpha}_x(\Sigma_\tau)}. \end{aligned} \tag{5.120}$$

□

Proof of Theorem 5.19. Now we consider equations (2.26a) and (2.26b):

$$\mathbf{B}\mathscr{C}^{\alpha} = \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},(\boldsymbol{\partial}\omega,\boldsymbol{\partial}\vec{S},\boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi},\vec{\varpi},\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{S}], \tag{5.121a}$$

$$\mathbf{B}\mathscr{D} = \mathscr{L}(\vec{\Psi},\vec{S})[\boldsymbol{\partial}\omega] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},(\boldsymbol{\partial}\vec{S},\boldsymbol{\partial}\vec{\Psi})] + \mathscr{Q}(\vec{S})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi},\vec{\varpi},\vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{5.121b}$$

By the Lemma 5.21 and bootstrap assumptions (3.24c), we have

$$\left\|\vec{\mathscr{C}},\mathscr{D}\right\|_{C_x^{0,\delta_1}(\Sigma_t)} \lesssim 1 + \int_0^t \left(\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{C_x^{0,\delta_1}(\Sigma_\tau)} + 1\right) \left\|\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\varpi},\boldsymbol{\partial}\vec{S}\right\|_{C_x^{0,\delta_1}(\Sigma_\tau)} \mathrm{d}\tau. \tag{5.122}$$

By (5.93), bootstrap assumptions and Grönwall's inequality, we have

$$\left\|\vec{\mathscr{C}},\mathscr{D}\right\|_{C_x^{0,\delta_1}(\Sigma_t)} \lesssim 1. \tag{5.123}$$

Integrating (5.93) in time, we have

$$\int_0^t \left\|\boldsymbol{\partial}\vec{\varpi},\boldsymbol{\partial}\vec{S}\right\|^2_{C_x^{0,\delta_1}(\Sigma_t)} \mathrm{d}t \lesssim \int_0^t 1 + \left\|\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|^2_{C_x^{0,\delta_1}(\Sigma_t)} \mathrm{d}t. \tag{5.124}$$

If (5.91) holds, using (5.123), the bootstrap assumption (3.24b) and the standard results in Littlewood-Paley (3.8), (5.92) is obtained by following estimate:

$$\int_0^t \left\|\boldsymbol{\partial}\vec{\varpi},\boldsymbol{\partial}\vec{S}\right\|^2_{C_x^{0,\delta_1}(\Sigma_t)} \mathrm{d}t \lesssim \int_0^t 1 + \left\|\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right\|^2_{C_x^{0,\delta_1}(\Sigma_t)} \mathrm{d}t \le T_*^{2\delta} + T_* \lesssim T_*^{2\delta}. \tag{5.125}$$

□

CHAPTER 6

Reduction of Strichartz Estimates and the Rescaled Solution

In this section, we state our main estimates as Theorem 6.1, which are the improvement of the bootstrap assumption (3.24b). We then provide a series of analytic reductions from the Strichartz estimates of Theorem 6.1 to the decay estimates of Theorem 6.9. We are quite terse in this section since the full proofs of these reductions are lengthy and difficult, yet standard. We refer readers to [9, 36, 14] for the detailed proofs.

Theorem 6.1 (Improvement of the Strichartz-type bootstrap assumptions for the wave variables)**.** *If $\delta > 0$ is sufficiently small as in Subseciton 3.4, then under the initial data and bootstrap assumptions of Section 3, the following estimates holds with a number $8\delta_0 < \delta_1 < N-2$:*

$$\left\|\boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_1} \left\|P_\nu \boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_t L^\infty_x([0,T_*]\times\mathbb{R}^3)} \lesssim T_*^{2\delta}. \tag{6.1}$$

We first reduce the proof of Theorem 6.1 to the proof of Strichartz estimates on small time intervals.

6.1 Partitioning of the Bootstrap Time Interval

Let λ be a fixed large number and let $0 < \varepsilon_0 < \frac{N-2}{5}$ be a fixed number as mentioned in Subsection 3.4. By the bootstrap assumptions, we can[1] partition $[0,T_*]$ into disjoint union of sub-intervals $I_k := [t_{k-1}, t_k]$ of total number $\lesssim \lambda^{8\varepsilon_0}$ with the properties that $|I_k| \leq \lambda^{-8\varepsilon_0} T_*$ and

$$\left\|\boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_t L^\infty_x(I_k\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \left\|P_\nu \boldsymbol{\partial}\vec{\Psi}\right\|^2_{L^2_t L^\infty_x(I_k\times\mathbb{R}^3)} \lesssim \lambda^{-8\varepsilon_0}, \tag{6.2a}$$

$$\left\|\partial\vec{\omega}, \partial\vec{S}\right\|^2_{L^2_t L^\infty_x(I_k\times\mathbb{R}^3)} + \sum_{\nu\geq 2} \nu^{2\delta_0} \left\|P_\nu (\partial\vec{\omega}, \partial\vec{S})\right\|^2_{L^2_t L^\infty_x(I_k\times\mathbb{R}^3)} \lesssim \lambda^{-8\varepsilon_0}. \tag{6.2b}$$

Now we reduce Theorem 6.1 to a frequency localized estimate.

Theorem 6.2 (Frequency localized Strichartz estimate)**.** *Let φ be a solution of*

$$\square_{\mathbf{g}} \varphi = 0 \tag{6.3}$$

on the time interval I_k. Then for any $q > 2$ sufficiently close to 2 and any $\tau \in [t_k, t_{k+1}]$, under the bootstrap

[1] The existence of such partition easily follows from the bootstrap assumptions, see [14, Remark 1.3].

assumptions, we have the following estimate:

$$\|P_\lambda \boldsymbol{\partial}\varphi\|_{L^q_t L^\infty_x([\tau,t_{k+1}]\times\mathbb{R}^3)} \lesssim \lambda^{\frac{3}{2}-\frac{1}{q}} \|\boldsymbol{\partial}\varphi\|_{L^2_x(\Sigma_\tau)}. \tag{6.4}$$

6.1.1 Dicussion of the reduction to Theorem 6.2 from Theorem 6.1

The proof that Theorem 6.1 follows from Theorem 6.2 is exactly the same as the proof of [9, Theorem 7.2]. The reason is that the proof relies only on: **1)** Duhamel's principle; **2)** our top order energy estimates (5.1), which is the same as the corresponding estimates in [9], and **3)** Littlewood-Paley estimates for the inhomogeneous terms in a frequency-projected version of the wave equations, when the wave equations in this paper have an identical schematic form to the ones in [9].

6.2 Rescaled Quantities and Rescaled Relativistic Euler Equations

In this subsection, in order to do further reductions, we consider the following coordinate change $(t,x) \mapsto (\lambda(t-t_k), \lambda x)$. Let

$$T_{*;(\lambda)} := \lambda(t_{k+1} - t_k). \tag{6.5}$$

Note that by construction,

$$0 \le T_{*;(\lambda)} \le \lambda|I_k| \le \lambda^{1-8\varepsilon_0} T_*. \tag{6.6}$$

Definition 6.3 (Rescaled quantities)**.** *First we define the following variables:*

$$\vec{\Psi}_{(\lambda)}(t,x) := \vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7a}$$

$$\vec{\omega}_{(\lambda)}(t,x) := \vec{\omega}(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7b}$$

$$\vec{S}^{\kappa}_{(\lambda)}(t,x) := \vec{S}^{\kappa}(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7c}$$

$$\theta_{(\lambda)} := \theta(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7d}$$

$$c_{(\lambda)} := c(t_k + \lambda^{-1}t, \lambda^{-1}x), \tag{6.7e}$$

$$\begin{aligned}\mathscr{C}^{\alpha}_{(\lambda)} := & vort^{\alpha}((\omega_{\flat})_{(\lambda)}) + c^{-2}_{(\lambda)} \varepsilon^{\alpha\beta\gamma\delta}((v_{\flat})_{(\lambda)})_{\beta}(\partial_{\gamma} h_{(\lambda)})(\omega_{\flat})_{(\lambda)\delta} + (\theta_{(\lambda)} \\ & - (\theta_{;h})_{(\lambda)})(S^{\sharp})^{\alpha}_{(\lambda)}(\partial_{\kappa}(v_{\flat})^{\kappa}_{(\lambda)}) \\ & + (\theta_{(\lambda)} - (\theta_{;h})_{(\lambda)})(v_{\flat})^{\alpha}_{(\lambda)}((S^{\sharp})^{\kappa}_{(\lambda)}\partial_{\kappa} h_{(\lambda)}) \\ & - (\theta_{(\lambda)} - (\theta_{;h})_{(\lambda)})(S^{\sharp})^{\kappa}_{(\lambda)} \left((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}(v_{\flat})_{(\lambda)\kappa}\right),\end{aligned} \tag{6.7f}$$

$$\mathscr{D}_{(\lambda)} := \frac{1}{n}(\partial_{\kappa}(S^{\sharp})^{\kappa}_{(\lambda)}) + \frac{1}{n}((S^{\sharp})^{\kappa}_{(\lambda)}\partial_{\kappa} h_{(\lambda)}) - \frac{1}{n}c^{-2}((S^{\sharp})^{\kappa}_{(\lambda)}\partial_{\kappa} h_{(\lambda)}). \tag{6.7g}$$

Then we define the following rescaled tensor fields:

$$(\mathbf{g}_{(\lambda)})_{\alpha\beta}(t,x) := \mathbf{g}_{\alpha\beta}(\vec{\Psi}_{(\lambda)}(t,x)), \tag{6.8a}$$

$$(g_{(\lambda)})_{\alpha\beta}(t,x) := g_{\alpha\beta}(\vec{\Psi}_{(\lambda)}(t,x)), \tag{6.8b}$$

$$\mathbf{B}^{\alpha}_{(\lambda)}(t,x) := \mathbf{B}^{\alpha}(\vec{\Psi}_{(\lambda)}(t,x)). \tag{6.8c}$$

Remark 6.4. *We note that after rescaling, the new initial constant-time hypersurface Σ_0 corresponds to the constant-time hypersurface that was denoted by Σ_{t_k} for some k in Section 1-5.*

The following proposition provides the equations satisfied by the rescaled quantities. We omit the straightforward proof.

Proposition 6.5 (The rescaled geometric wave-transport formulation of the relativistic Euler equations)**.** *Consider a solution to Proposition 2.17 . Let $\Psi \in \{v^0, v^1, v^2, v^3, h, s\}$, be the rescaled quantities from Definition 6.3. Then the rescaled quantities satisfy the following equations:*

Wave equations

$$\square_{\mathbf{g}_{(\lambda)}} \Psi_{(\lambda)} = \lambda^{-1} \mathscr{L}(\vec{\Psi}_{(\lambda)})[\vec{\mathscr{C}}_{(\lambda)}, \mathscr{D}_{(\lambda)}] + \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)}], \tag{6.9}$$

$$\square_{\mathbf{g}_{(\lambda)}} \boldsymbol{\partial}\Psi_{(\lambda)} = \lambda^{-1} \mathscr{L}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\mathscr{C}}_{(\lambda)}, \boldsymbol{\partial}\mathscr{D}_{(\lambda)}] + \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}^2\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)}] + \mathscr{L}(\vec{\Psi}_{(\lambda)})[(\boldsymbol{\partial}\vec{\Psi}_{(\lambda)})^3] \tag{6.10}$$

Transport equations

$$\mathbf{B}_{(\lambda)}\omega^{\alpha}_{(\lambda)} = \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}], \tag{6.11a}$$

$$\mathbf{B}_{(\lambda)}(S^{\sharp})^{\alpha}_{(\lambda)} = \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}]. \tag{6.11b}$$

Transport-Div-Curl system

$$\mathbf{B}_{(\lambda)}\mathscr{C}^{\alpha}_{(\lambda)} = \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, (\boldsymbol{\partial}\omega_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)})] + \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}], \tag{6.12a}$$

$$\begin{aligned}\mathbf{B}_{(\lambda)}\boldsymbol{\partial}\mathscr{C}^{\alpha}_{(\lambda)} =& \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, (\boldsymbol{\partial}^2\omega_{(\lambda)}, \boldsymbol{\partial}^2\vec{S}_{(\lambda)}, \boldsymbol{\partial}^2\vec{\Psi}_{(\lambda)})] \\ &+ \mathscr{Q}(\vec{\Psi}_{(\lambda)})[(\boldsymbol{\partial}^2\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{\omega}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}), (\boldsymbol{\partial}\vec{\omega}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)})] \\ &+ \mathscr{L}(\vec{\Psi}_{(\lambda)})[(\boldsymbol{\partial}\vec{\Psi}_{(\lambda)})^2 \cdot (\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}, \boldsymbol{\partial}\vec{\omega}_{(\lambda)})]\end{aligned} \tag{6.12b}$$

$$\begin{aligned}\mathbf{B}_{(\lambda)}\mathscr{D}_{(\lambda)} =& \mathscr{L}(\vec{S}_{(\lambda)}, \vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\omega_{(\lambda)}] \\ &+ \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, (\boldsymbol{\partial}\vec{S}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)})] + \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}],\end{aligned} \tag{6.12c}$$

$$\begin{aligned}\mathbf{B}_{(\lambda)}\boldsymbol{\partial}\mathscr{D}_{(\lambda)} =& \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}^2\omega_{(\lambda)}, \boldsymbol{\partial}^2\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}^2\vec{S}_{(\lambda)}] \\ &+ \mathscr{Q}(\vec{\Psi}_{(\lambda)})[(\boldsymbol{\partial}^2\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{\omega}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)}), (\boldsymbol{\partial}\vec{S}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)})] \\ &+ \mathscr{L}(\vec{\Psi}_{(\lambda)})[(\boldsymbol{\partial}\vec{\Psi}_{(\lambda)})^2 \cdot (\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{S}_{(\lambda)})]\end{aligned} \tag{6.12d}$$

$$\text{vort}^{\alpha}(S^{\sharp}_{(\lambda)}) = 0, \tag{6.12e}$$

$$\partial_{\alpha}\omega^{\alpha}_{(\lambda)} = \mathscr{L}(\omega_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}]. \tag{6.12f}$$

Remark 6.6. *For notation convenience, in the remainder of the article, we drop the sub and super scripts* (λ) *except for the rescaled time* $T_{*;(\lambda)}$.

6.2.1 Consequences of the bootstrap assumptions

After rescaling in Subsection 6.2, assuming bootstrap assumptions (3.24b)-(3.24c), as in [9, Section 10.2.1], by standard computation based on Littlewood-Paley calculus, we have the following consequences of the bootstrap assumptions:

Estimates by using bootstrap assumptions of variables

$$\begin{aligned}&\left\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathscr{C}}, \mathscr{D}\right\|_{L^2_t L^\infty_x(\mathscr{M})} \\ &+ \lambda^{\delta_0}\sqrt{\sum_{\nu>2}\nu^{2\delta_0}\left\|P_\nu\left(\mathrm{f}(\vec{\Psi}, \vec{\omega}, \vec{S})(\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathscr{C}}, \mathscr{D})\right)\right\|^2_{L^2_t L^\infty_x(\mathscr{M})}} \lesssim \lambda^{-1/2-4\varepsilon_0}.\end{aligned} \tag{6.13}$$

6.3 Further Reduction of the Strichartz Estimates

By the rescaling in the Section 6.2 and direct computation, to prove Theorem 6.2, it is equivalent to show the following Strichartz estimate on $[0,T_{*;(\lambda)}]$ with respect to LP projection on the frequency domain $\{1/2 \leq |\xi| \leq 2\}$.

Theorem 6.7. *Assume the bootstrap assumptions. For any solution φ of $\square_{\mathbf{g}}\varphi = 0$ on the slab $[0,T_{*;(\lambda)}]\times\mathbb{R}^3$, the following estimate holds:*

$$\|P_1\boldsymbol{\partial}\varphi\|_{L^q_t L^\infty_x([0,T_{*;(\lambda)}]\times\mathbb{R}^3)} \lesssim \|\boldsymbol{\partial}\varphi\|_{L^2_x(\Sigma_0)}, \tag{6.14}$$

where $q > 2$ is sufficiently close to 2 and $\mathbf{g}$ is the rescaled metric $\mathbf{g}_{(\lambda)}$ defined in (6.8a).

The proof of Theorem 6.7 crucially relies on the following decay estimate.

Theorem 6.8 (Decay estimate)**.** *There exists a large number Λ such that for any $\lambda \geq \Lambda$ and any solution φ of the equation $\square_{\mathbf{g}}\varphi = 0$ on $[0,T_{*;(\lambda)}]\times\mathbb{R}^3$, there is a function $d(t)$ satisfying*

$$\|d\|_{L_t^{\frac{q}{2}}([0,T_{*;(\lambda)}])} \lesssim 1 \tag{6.15}$$

such that for $q > 2$ sufficiently close to 2 and any $t \in [0,T_{*;(\lambda)}]$, the following decay estimate holds[2]*:*

$$\|P_1\mathbf{T}\varphi\|_{L^\infty_x(\Sigma_t)} \lesssim \left(\frac{1}{(1+t)^{\frac{2}{q}}} + d(t)\right)\left(\sum_{m=0}^{3}\|\partial^m\varphi\|_{L^1_x(\Sigma_0)} + \sum_{m=0}^{2}\|\partial^m\partial_t\varphi\|_{L^1_x(\Sigma_0)}\right). \tag{6.16}$$

The proof of Theorem 6.7 using Theorem 6.8 is based on a $\mathscr{T}\mathscr{T}^*$ argument[3] (see [14, Section 8.6] and [18, Section 8.30]).

Theorem 6.8 can be further reduced to the following spatially localized version.

Theorem 6.9 (Spatially localized version of decay estimate)**.** *Let R be a fixed radius*[4] *such that*

$$B_R(p) \subset B_{\frac{1}{2}}(p, g_{(\lambda)}), \qquad \forall p \in \Sigma_t, 0 \leq t \leq T_{*;(\lambda)}, \tag{6.17}$$

where $B_\rho(p, g_{(\lambda)})$ is the geodesic ball[5] *centered at p with radius ρ and $g_{(\lambda)}$ is the rescaled induced metric of*

[2] $\mathbf{T}_\flat = -\mathrm{d}t$, see Definition 2.11 for the definition of $\mathbf{T}$.

[3] This argument comes from functional analysis, which does not require the structure of the relativistic Euler equations.

[4] The radius R will be used in the following sections as well with the same definition. The existence of such an R is guaranteed by properties of g, namely, (2.20b), that ensures g is comparable to the Euclidean metric on Σ_t under the bootstrap assumptions.

[5] The notation $B_\rho(p, g_{(\lambda)})$ will be used in the remainder of the article. This is consistent with the notation that is used in [9] and [36]

$\mathbf{g}$ *on* Σ_t *(defined in (6.8b)). Then there exists a large number* Λ *and a function* $d(t)$ *satisfying*

$$\|d\|_{L_t^{\frac{q}{2}}([0,T_{*;(\lambda)}])} \lesssim 1, \tag{6.18}$$

such that for any $\lambda \geq \Lambda$ *and any solution* φ *of the equation* $\square_{\mathbf{g}}\varphi = 0$ *on* $[0,T_{*;(\lambda)}] \times \mathbb{R}^3$ *with* $\varphi(1,x)$ *supported in the Euclidean ball* B_R*, there is* $q > 2$ *sufficiently close to 2 and any* $t \in [0,T_{*;(\lambda)}]$*, the following estimate holds:*

$$\|P_1\mathbf{T}\varphi\|_{L_x^\infty(\Sigma_t)} \lesssim \left(\frac{1}{(1+|t-1|)^{\frac{2}{q}}} + d(t)\right)\left(\|\boldsymbol{\partial}\varphi\|_{L_x^2(\Sigma_1)} + \|\varphi\|_{L_x^2(\Sigma_1)}\right). \tag{6.19}$$

The proof of Theorem 6.8 using Theorem 6.9 can be done via an approach involving the Bernstein inequalities of LP projection, partition of unity[6] of φ and Sobolev embedding $W^{2,1} \hookrightarrow L^2$. We refer readers to [14, Section 8.5] for detailed proof.

To summarize, in this section we have reduced Theorem 6.1 to Theorem 6.9. Before we can proceed, we need to introduce the geometric setup. We will discuss the proof of Theorem 6.9 in Section 10.1.1.

[6]We take a sequence of Euclidean balls $\{B_I\}$ of radius R such that their union covers $\mathbb{R}^3$. For each ball B_R, it is centered at $\gamma_{\mathbf{z}}(1)$ (defined in Section 7.1.2) for some $\mathbf{z}$.

CHAPTER 7

Geometric Setup and Conformal Energy

In this section, we first construct acoustic geometry. It is deeply coupled to the relativistic Euler equations (via the acoustic metric $\mathbf{g}$) and is crucial in our analysis. In Section 10, we provide the conformal energy, which is based on acoustic geometry, and derive estimates for it. This is a fundamental ingredientin the vectorfield method we use to derive Strichartz estimates.

Definition 7.1 (Christoffel symbols)**.** *We define the Christoffel symbols* $\mathbf{\Gamma}_{\alpha\kappa\lambda}$ *and* $\mathbf{\Gamma}_{\kappa\lambda}^{\beta}$ *with the rescaled metric* $\mathbf{g}$ *to be as follows:*

$$\mathbf{\Gamma}_{\alpha\kappa\lambda} := \frac{1}{2}\left(\partial_\kappa \mathbf{g}_{\alpha\lambda} + \partial_\lambda \mathbf{g}_{\alpha\kappa} - \partial_\alpha \mathbf{g}_{\kappa\lambda}\right), \tag{7.1}$$

$$\mathbf{\Gamma}_{\kappa\lambda}^{\beta} := \mathbf{g}^{\alpha\beta}\mathbf{\Gamma}_{\alpha\kappa\lambda}. \tag{7.2}$$

7.1 Construction of the Acoustical Function

The goal of this subsection is to construct the geometry based on a solution u to the acoustical eikonal equation[1]:

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0. \tag{7.3}$$

7.1.1 Point z and integral curve $\gamma_{\mathbf{z}}(t)$

Let $\mathbf{z} \in \Sigma_0$ be an arbitrarily fixed point[2] in the rescaled space-time $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$, where $T_{*;(\lambda)}$ is defined in (6.5). We let $\gamma_{\mathbf{z}}(t)$ denote the integral curve of the future-directed vectorfield $\mathbf{T}$ emanating from the point $\mathbf{z}$. We say that $\gamma_{\mathbf{z}}(t)$ is the cone-tip axis. Specifically, the point $\mathbf{z}$ depends on the partition of unity of Σ_1 used in the proof of Theorem 6.8 using Theorem 6.9. More specifically, $\mathbf{z}$ is going to be the tip of the cone (constructed in Subsection 7.1.2) such that $\gamma_{\mathbf{z}}(1)$ is the center of the Euclidean ball B_R (as in Theorem 6.9). We note that the estimates, constants and parameters in Section 7-8 are independent of $\mathbf{z}$.

7.1.2 The interior and exterior solution u

For convenience, in the rest of the article, for vectorfields X, Y, we denote $\langle X, Y\rangle := \mathbf{g}(X, Y)$. First we show that $\mathbf{T}$ is geodesic.

[1]For more details of the geometric construction of u, we refer reader to [6, Chapter 9 and Chapter 14].

[2]Note that in the original spacetime $[0, T_*] \times \mathbb{R}^3$, $\mathbf{z}$ is a point in Σ_{t_k} for some k.

Proof of $\mathbf{D_T T}=0$. We consider the $t-x$ coordinates with respect to $\mathbf{g}$.

$$\begin{aligned}\langle \mathbf{D_T T}, \partial_\mu\rangle &= \partial^\gamma t \langle \mathbf{D}_\gamma \partial t, \partial_\mu\rangle \\ &= \partial^\gamma t \partial_\gamma \langle \partial t, \partial_\mu\rangle - \partial^\gamma t \langle \partial t, \mathbf{D}_\gamma \partial_\mu\rangle \\ &= \partial^\gamma t \partial_\gamma (\mathbf{g}_{\mu\alpha}\partial^\alpha t) - \partial^\gamma t \langle \partial t, \mathbf{\Gamma}^\beta_{\mu\gamma}\partial_\beta\rangle \\ &= \partial^\gamma t \partial_\mu \partial_\gamma t - \partial^\gamma t \langle \partial t, \mathbf{D}_\mu \partial\gamma\rangle \\ &= \partial^\gamma t \partial_\mu \langle \partial t, \partial_\gamma\rangle - \partial^\gamma t \langle \partial t, \mathbf{D}_\mu \partial_\gamma\rangle \\ &= \langle \mathbf{D}_\mu \partial t, \partial t\rangle = \frac{1}{2}\partial_\mu \langle \partial_t, \partial_t\rangle \\ &= \frac{1}{2}\partial_\mu(-1) = 0.\end{aligned} \tag{7.4}$$

□

7.1.2.0.1 The interior solution u.

We let $\{L_\omega|_{\mathbf{z}}\}_{\omega\in\mathbb{S}^2}$ be the family of future-directed null vectors (parameterized by $\mathbb{S}^2$, where the parameterization will be uniquely determined in the paragraph of the exterior solution u) in the tangent space $T_{\mathbf{z}}\mathscr{M}$ such that $\langle L_\omega|_{\mathbf{z}}, \mathbf{T}\rangle = -1$. To propagate L_ω along the cone-tip axis $\gamma_{\mathbf{z}}(t)$, for any $\mathbf{p}\in\gamma_{\mathbf{z}}(t)$ (as defined in Subsection 7.1.1) and $\omega\in\mathbb{S}^2$, we define $L_\omega|_{\mathbf{p}}$ by solving the parallel transport equation $\mathbf{D_T}L_\omega = 0$ with initial conditions $L_\omega|_{\mathbf{z}}$. We note that for any $\mathbf{p}\in\gamma_{\mathbf{z}}(t)$, $\langle L_\omega|_{\mathbf{p}}, \mathbf{T}\rangle = -1$ since $\mathbf{T}$ is geodesic. Then, for each $u\in[0, T_{*;(\lambda)}]$ and $\omega\in\mathbb{S}^2$, there exists a unique null geodesic $\Upsilon_{u,\omega}(t)$, where $t\in[u, T_{*;(\lambda)}]$, emanating from $\mathbf{p}=\gamma_{\mathbf{z}}(u)$ with $\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_{u,\omega}\big|_{t=u} = L_\omega$ and $\Upsilon^0_{u,\omega}(t) = t$. Specifically, $\Upsilon_{u,\omega}(t)$ is constructed by solving the following "geodesic" ODE system[3]:

$$\begin{aligned}\ddot{\Upsilon}^\alpha_{u,\omega}(t) =& -\mathbf{\Gamma}^\alpha_{\kappa\lambda}\big|_{\Upsilon_{u,\omega}(t)}\dot{\Upsilon}^\kappa_{u,\omega}(t)\dot{\Upsilon}^\lambda_{u,\omega}(t) \\ &+\frac{1}{2}\,[\mathscr{L}_{\mathbf{T}}\mathbf{g}]_{\kappa\lambda}\big|_{\Upsilon_{u,\omega}(t)}(\dot{\Upsilon}^\kappa_{u,\omega}(t) - \mathbf{T}^\kappa|_{\Upsilon_{u,\omega}(t)})(\dot{\Upsilon}^\lambda_{u,\omega}(t) - \mathbf{T}^\lambda\Big|_{\Upsilon_{u,\omega}(t)})\dot{\Upsilon}^\alpha_{u,\omega}(t),\end{aligned} \tag{7.5a}$$

$$\Upsilon^\alpha_{u,\omega}(u) = \gamma^\alpha_{\mathbf{z}}(u), \quad \dot{\Upsilon}^\alpha_{u,\omega}(u) = L^\alpha_\omega, \tag{7.5b}$$

where $\ddot{\Upsilon}^\alpha_{u,\omega} := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Upsilon^\alpha_{u,\omega}$, $\dot{\Upsilon}^\alpha_{u,\omega} := \frac{\mathrm{d}}{\mathrm{d}t}\Upsilon^\alpha_{u,\omega}$, $\mathbf{\Gamma}$ is the Christoffel symbol of $\mathbf{g}$ and $\mathscr{L}_{\mathbf{T}}\mathbf{g}$ is the Lie derivative of $\mathbf{g}$ with respect to $\mathbf{T}$. The curve $t\to\Upsilon_{u,\omega}(t)$ is a non-affinely parameterized null geodesic such that the vectorfield $L^\alpha_{u,\omega} := \frac{\mathrm{d}}{\mathrm{d}t}\Upsilon^\alpha_{u,\omega}$ is null and normalized by $L^0_{u,\omega} = 1$. In fact, this vectorfield coincides (in the interior region) with the vectorfield L defined in (7.13) below. We define the truncated null cone $\mathscr{C}_u$ to be $\mathscr{C}_u := \bigcup_{\omega\in\mathbb{S}^2, t\in[u,T_{*;(\lambda)}]}\Upsilon_{u,\omega}(t)$. We define the acoustical function u by asserting that its level sets $\{u=u'\}$

[3]See [9, Section 9.4.1] for detailed explanation of this ODE system, where the correct analog of $\mathbf{T}$ is denoted by $\mathbf{B}$ there.

are $\mathscr{C}_{u'}$. For $u \in [0, T_{*;(\lambda)}]$ and $t \in [u, T_{*;(\lambda)}]$, we let $S_{t,u} := \mathscr{C}_u \bigcap \Sigma_t$. For $u \neq t$, $S_{t,u}$ is a smooth surface diffeomorphic[4] to $\mathbb{S}^2$. We define $\mathscr{M}^{(\mathrm{Int})} := \bigcup_{t \in [0, T_{*;(\lambda)}], 0 \le u \le t} S_{t,u}$. For each $\omega \in \mathbb{S}^2$ and $u \in [0, T_{*;(\lambda)}]$, we define the angular coordinate functions $\{\omega^A\}_{A=1,2}$ to be constant along each fixed null geodesics $\Upsilon_{u,\omega}(t)$ and to coincide with standard angular coordinates on $\mathbb{S}^2$ at the tip $\mathbf{p}$, which corresponds to $t = u$.

7.1.2.0.2 The exterior solution u.

Now we extend the foliation of space-time by null hypersurfaces to a neighborhood of $\bigcup_{t \in [0, T_{*;(\lambda)}], 0 \le u \le t} S_{t,u}$ in $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$. Let $w_* = \frac{4}{5} T_{*;(\lambda)}$. Using the arguments in [29], we can guarantee[5] that there is a neighborhood $\mathfrak{O} \in \Sigma_0$ contained in the geodesic ball $B_{T_{*;(\lambda)}}(\mathbf{z}, g_{(\lambda)})$, where $g_{(\lambda)}$ is the frequency rescaled induced metric of $\mathbf{g}$ on Σ_0 defined in (6.8b), such that $\mathfrak{O}$ can be foliated by the level sets $S_{0,-w}$ of a positive function w taking all values in $[0, w_*]$ with $w(\mathbf{z}) = 0$ and each $S_{0,-w}$ for positive w is diffeomorphic to $\mathbb{S}^2$. Fix a diffeomorphism $\omega \to \Phi_\omega(w_*)$ from $\mathbb{S}^2$ to $S_{0,-w_*}$. Then, for each point $\mathbf{p} = \Phi_\omega(w_*)$, denoting the lapse $a := \left((g^{-1})^{cd} \partial_c w \partial_d w\right)^{-\frac{1}{2}}$ with $a(\mathbf{z}) = 1$ and $a \approx 1$; see Proposition 8.12 (we note that the proof of Proposition 8.12 is independent of the construction of u), there is a unique integral curve[6] $w \to \Phi_\omega(w)$ of the vectorfield $a^2 (g^{-1})^{ic} \partial_c w$ in Σ_0 with $\Phi_\omega(w_*) = \mathbf{p}$ and such that this integral curve can be extended to $\mathbf{z}$, i.e. $\Phi_\omega(0) = \mathbf{z}$ (the extendibility of Φ_ω to $\mathbf{z}$ follows by estimates (8.63h) and the fundamental theorem of Calculus). Denoting $\dot{\Phi}_\omega := \frac{\mathrm{d}}{\mathrm{d}w} \Phi_\omega$, we then define $N_\omega|_{\mathbf{z}} := \dot{\Phi}_\omega(0)$ and $L_\omega|_{\mathbf{z}} := N_\omega|_{\mathbf{z}} + \mathbf{T}|_{\mathbf{z}}$. Note that the diffeomorphism $\omega \to L_\omega|_{\mathbf{z}}$ is uniquely determined by the vector field $a^2 (g^{-1})^{ic} \partial_c w$, and it is precisely this diffeomorphism that appears in our construction of the interior solution described above, since $\langle N_\omega|_{\mathbf{z}} + \mathbf{T}|_{\mathbf{z}}, \mathbf{T}|_{\mathbf{z}} \rangle = -1$ and $\langle N_\omega|_{\mathbf{z}} + \mathbf{T}|_{\mathbf{z}}, N_\omega|_{\mathbf{z}} + \mathbf{T}|_{\mathbf{z}} \rangle = 0$ (because of $a(\mathbf{z}) = 1$). By such construction, for each $w \in (0, w_*]$ and any $\mathbf{p} \in S_{0,-w}$, there exists a $\omega \in \mathbb{S}^2$ such that $\mathbf{p} = \Phi_\omega(w)$ and such that the outward unit normal (in Σ_0) to $S_{0,-w}$ at $\mathbf{p}$ is $N_{w,\omega} := \dot{\Phi}_\omega(w)$. We set $L_{w,\omega} := N_{w,\omega} + \mathbf{T}|_{\Phi_\omega(w)}$, which is a null vector in $T_{\mathbf{p}}\mathscr{M}$. Then, with $u = -w$, there is a unique (non-affinely parameterized) null geodesic $\Upsilon_{u,\omega}$ emanating from $\mathbf{p}$ and solving (7.5a) with the initial condition $\frac{\mathrm{d}}{\mathrm{d}t} \Upsilon_{u,\omega}\big|_{t=0} = L_{-u,\omega}$ and $\Upsilon_{u,\omega}(0) = \mathbf{p}$. We define the null cone $\mathscr{C}_u$ to be $\mathscr{C}_u := \bigcup_{\omega \in \mathbb{S}^2, t \in [0, T_{*;(\lambda)}]} \Upsilon_{u,\omega}(t)$. We define the acoustical function u by asserting that its level sets $\{u = u'\}$ are $\mathscr{C}_{u'}$. Let $S_{t,u} := \mathscr{C}_u \bigcap \Sigma_t$. We note that $\mathscr{C}_u$'s are the outgoing truncated null cones, that is, $\mathscr{C}_u := \bigcup_{t \in [0, T_{*;(\lambda)}]} S_{t,u}$. We define $\mathscr{M}^{(\mathrm{Ext})} := \bigcup_{t \in [0, T_{*;(\lambda)}], u \in [-w_*, 0)} S_{t,u}$. For each $\omega \in \mathbb{S}^2$ and $u \in [-w_*, 0)$, we define the angular coordinate functions $\{\omega^A\}_{A=1,2}$ to be constant along null geodesics $\Upsilon_{u,\omega}(t)$ and to coincide with the angular coordinates $\{\omega^A\}_{A=1,2}$ on Σ_0 provided by the above construction; note that on $\Sigma_0 \backslash \{\mathbf{z}\}$, by construction, the

[4] $S_{t,u}$ is a smooth surface diffeomorphic to $\mathbb{S}^2$ is a direct result of Proposition 9.9

[5] The existence of w-foliation for $w \in [0, \varepsilon]$ with a small $\varepsilon > 0$ can be proven by Nash-Moser implicit function theorem and such foliation can be extended to w_* by an argument of continuity (see [29]).

[6] The existence and uniqueness of such integral curve are ensured by the estimates on Σ_0 in Proposition 8.12. We refer readers to [9, Subsection 9.4.2] for details.

angular coordinate functions $\{\omega^A\}_{A=1,2}$ are constant along the integral curves of the vectorfield $a^2(g^{-1})^{ic}\partial_c w$.

We define the space-time region $\mathscr{M}$ as follows:

$$\mathscr{M} = \mathscr{M}^{(\mathrm{Int})} \bigcup \mathscr{M}^{(\mathrm{Ext})}. \tag{7.6}$$

By the constructions above, we have constructed the geometric coordinates (t, u, ω^A) in $\mathscr{M}$.

See Figure 7.2 for the figure of the geometry.

7.2 Geometric Quantities

Definition 7.2 (The radial variable)**.** *Recall that*

$$0 \leq T_{*;(\lambda)} \leq \lambda^{1-8\varepsilon_0} T_*. \tag{7.7}$$

We define the geometric radial variable $\widetilde{r}$ as follows:

$$\widetilde{r} := t - u. \tag{7.8}$$

Since we have that $t \in [0, T_{*;(\lambda)}]$ and $u \in [-w_*, t]$ in $\mathscr{M}$, where $w_* := \frac{4}{5} T_{*;(\lambda)}$, we have

$$0 \leq \widetilde{r} < 2T_{*;(\lambda)} = 2\lambda^{1-8\varepsilon_0} T_*, \qquad -\frac{4}{5}\lambda^{1-8\varepsilon_0} T_* \leq u \leq \lambda^{1-8\varepsilon_0} T_*. \tag{7.9}$$

We will silently use estimates (7.9) throughout the paper.

Definition 7.3 (Acoustic vectorfields and scalar functions)**.** *We define the null vector field*

$$L_{(Geo)} := -\mathbf{g}^{\alpha\beta} \partial_\beta u \partial_\alpha. \tag{7.10}$$

Note that by (7.3), we have $\mathbf{D}_{L_{(Geo)}} L_{(Geo)} = 0$.

We define the null lapse b as follows:

$$b := (\sqrt{g^{ij}\partial_i u \partial_j u})^{-1}. \tag{7.11}$$

We define the vector field N as follows:

$$N := -b g^{ij} \partial_i u \partial_j. \tag{7.12}$$

Note that $N|_{\Sigma_0} = N_\omega$, where N_ω is defined in Section 7.1.

We define the principal null vector fields L and $\underline{L}$ as follows:

$$L := \mathbf{T} + N, \qquad \underline{L} := \mathbf{T} - N. \tag{7.13}$$

Notice that by Definition 2.12, (7.3) and (7.12), we have the identity $\mathbf{T}u = |\nabla u|_g = \frac{1}{b}$. Then we have

$$L = bL_{(Geo)}. \tag{7.14}$$

We have following basic properties:

$$\langle \mathbf{T}, \mathbf{T} \rangle = -1, \qquad \langle N, N \rangle = 1, \tag{7.15}$$

$$\langle \mathbf{T}, N \rangle = 0, \qquad \langle L, L \rangle = 0, \tag{7.16}$$

$$\langle L, \underline{L} \rangle = -2, \qquad \langle \underline{L}, \underline{L} \rangle = 0. \tag{7.17}$$

We define $g\!\!\!/$ to be the inverse of $g\!\!\!/^{-1}$, and $g\!\!\!/^{-1}$ as follows:

$$(g\!\!\!/^{-1})^{\alpha\beta} := \mathbf{g}^{\alpha\beta} + \frac{1}{2}L^\alpha \underline{L}^\beta + \frac{1}{2}\underline{L}^\alpha L^\beta. \tag{7.18}$$

It is easy to check that $g\!\!\!/$ is an induced metric of $\mathbf{g}$ on $S_{t,u}$. We fix a pair of locally defined unit orthogonal spherical vectorfields[7] on $S_{t,u}$ by $\{e_A\}_{A=1,2}$ such that $(g\!\!\!/^{-1})^{\alpha\beta} = \sum_{A=1,2} e_A^\alpha e_A^\beta$. We call $L, \underline{L}, e_1, e_2$ a null frame for the geometry.

We denote the Levi-Civita connection on $S_{t,u}$ with respect to $g\!\!\!/$ by $\nabla\!\!\!/$.

As we discussed in Section 7.1, the angular coordinate functions $\{\omega^A\}_{A=1,2}$ satisfy the equation $L(\omega^A) = 0$ along null cones. By this construction of angular coordinates, with respect to geometric coordinates (t, u, ω^A), we have

$$L = \frac{\partial}{\partial t}, \qquad N = -b^{-1}\frac{\partial}{\partial u} + Y^A \frac{\partial}{\partial \omega^A}. \tag{7.19}$$

By construction in Section 7.1.2, $Y^A = 0$ on Σ_0.

Definition 7.4 ($S_{t,u}$-tangent tensorfields)**.** *We define the* $\mathbf{g}$*-orthogonal projection $\Pi\!\!\!/$ onto $S_{t,u}$, where δ^α_β is the*

[7]In the rest of the paper, we automatically sum over A if there are two A's in the expression.

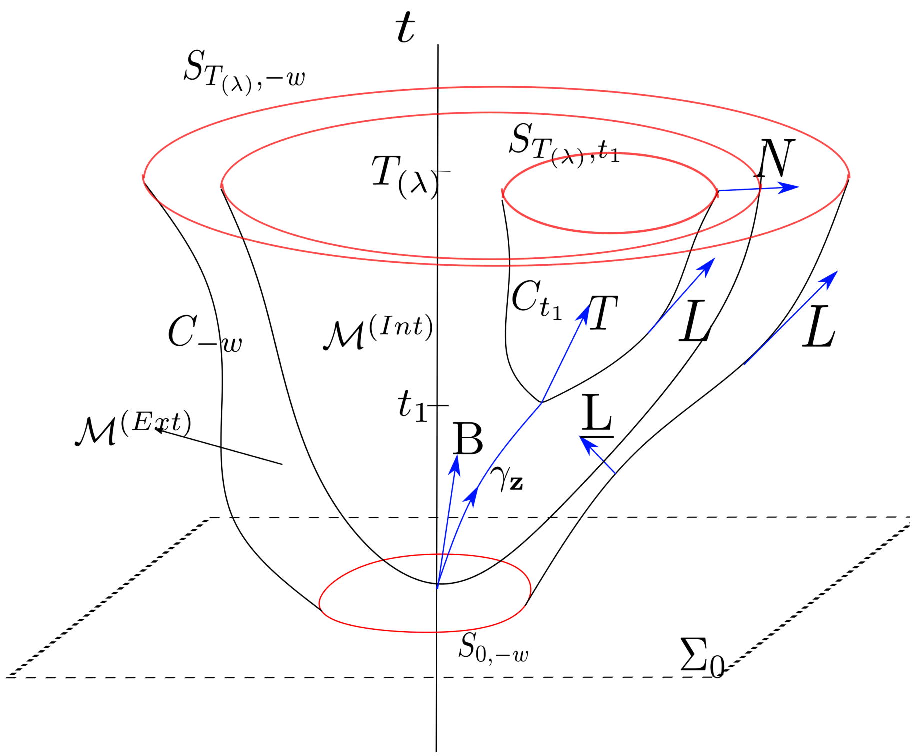


Figure 7.1: The geometric construction out of acoustical function

Kronecker delta, as follows:

$$\Pi\!\!\!/^{\alpha}_{\beta} := \delta^{\alpha}_{\beta} + \frac{1}{2} L^{\alpha} \underline{L}_{\beta} + \frac{1}{2} \underline{L}^{\alpha} L_{\beta}, \tag{7.20}$$

We use the notation $|\xi|_{g\!\!\!/}$ *to denote the norm of the* $S_{t,u}$*-tangent tensorfield* $\xi = \Pi\!\!\!/\xi$ *with respect to* $g\!\!\!/$*, that is,*

$$|\xi|^2_{g\!\!\!/} := g\!\!\!/(\xi, \xi). \tag{7.21}$$

We use $\mathrm{tr}_{g\!\!\!/}\xi$ *to denote the trace of a* $\binom{0}{2}$ $S_{t,u}$*-tangent tensorfield* ξ *with respect to* $g\!\!\!/$*:*

$$\mathrm{tr}_{g\!\!\!/}\xi := g\!\!\!/^{AB} \xi_{AB}. \tag{7.22}$$

We define $\hat{\xi}$ *to be the trace-free part of the* $\binom{0}{2}$ $S_{t,u}$*-tangent tensorfield* ξ*:*

$$\hat{\xi} := \xi - \frac{1}{2}(\mathrm{tr}_{g\!\!\!/}\xi) g\!\!\!/. \tag{7.23}$$

CHAPTER 8

Energy along Acoustic Null Hypersurfaces and the Acoustic Geometry

In this section, we derive energy estimates along acoustic null hypersurfaces, which is necessary for obtaining the mixed-norm estimates in Proposition 9.1. Then we introduce the notation for many geometric quantities, followed by their bootstrap assumptions. Their improved estimates are in Subsection 9.1. The proof of estimates for geometric quantities is obtained by transport equation and div-curl estimates for the acoustic quantities, decomposition of Ricci curvature components, trace and Sobolev inequalities. We omit the proof of these estimates since they are the same as in [9, Section 10].

8.1 Energy Estimates along Acoustic Null Hypersurfaces

In this subsection, we define acoustic null fluxes and derive energy estimates along acoustic null hypersurfaces. These estimates are necessary for deriving the mixed-norm estimates for the acoustical function quantities in Proposition 9.1.

Definition 8.1 (Acoustic null fluxes)**.** *For functions φ defined on $\mathscr{C}_u$, we define the acoustic null fluxes $\mathscr{F}_{(wave)}[\varphi;\mathscr{C}_u]$ and $\mathscr{F}_{(transport)}[\varphi;\mathscr{C}_u]$ as follows:*

$$\mathscr{F}_{(wave)}[\varphi;\mathscr{C}_u] := \int_{\mathscr{C}_u} \left((L\varphi)^2 + |\nabla\!\!\!/\varphi|^2_{g\!\!\!/}\right) \mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t, \tag{8.1}$$

$$\mathscr{F}_{(transport)}[\varphi;\mathscr{C}_u] := \int_{\mathscr{C}_u} \varphi^2 \mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t, \tag{8.2}$$

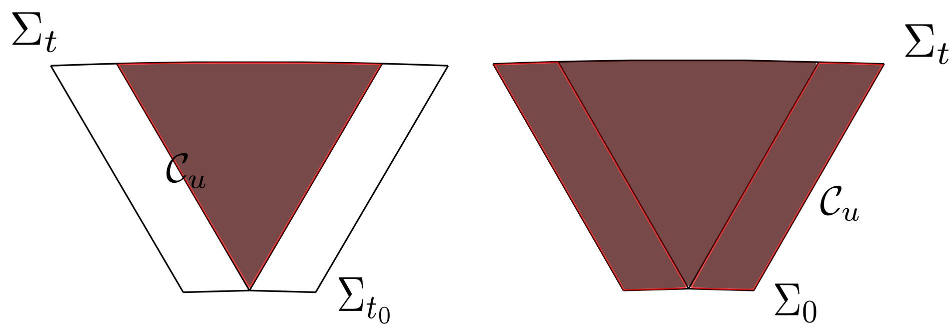
where $\mathrm{d}\varpi_{g\!\!\!/}$ is the volume form of the induced metric $g\!\!\!/$ on the $S_{t,u}$-sphere from $\mathbf{g}$.

Proposition 8.2 (Energy estimates along acoustic null hypersurfaces)**.** *Under the initial data, bootstrap assumptions and the standard energy estimates Proposition 5.1, we have the following estimates along null hypersurfaces $\mathscr{C}_u$ for $u \in [-w_*, T_{*;(\lambda)}]$:*

$$\mathscr{F}_{(wave)}[\boldsymbol{\partial}\vec{\Psi};\mathscr{C}_u] + \sum_{\nu>1} \nu^{2(N-2)} \mathscr{F}_{(wave)}[P_\nu \boldsymbol{\partial}\vec{\Psi};\mathscr{C}_u] \lesssim \lambda^{-1}, \tag{8.3}$$

$$\mathscr{F}_{(transport)}[\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D});\mathscr{C}_u] + \sum_{\nu>1} \nu^{2(N-2)} \mathscr{F}_{(transport)}[P_\nu \boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D});\mathscr{C}_u] \lesssim \lambda^{-1}. \tag{8.4}$$

Remark 8.3. *We note that Proposition 5.1 does not concern rescaled quantities while (8.3)-(8.4) concern rescaled quantities, as defined in Definition 6.3.*



(a) When $t_0 > 0$.

(b) When $t_0 = 0$.

Figure 8.1: The regions that the divergence theorem is applied on.

Proof of Proposition 8.2. We first prove (10.8). We apply the divergence theorem to the energy current ${}^{(\mathbf{T})}\mathbf{J}^{\alpha}[\varphi] := Q^{\alpha\beta}[\varphi]\mathbf{T}_{\beta}$, where $Q_{\alpha\beta}$ is the energy momentum tensor defined in (5.3), over the region bounded by $\mathscr{C}_u$, Σ_{t_0} and Σ_t where $t_0 := \max\{0,u\}$ (see figure 8.1 for the region on which the divergence theorem is applied). Notice that $\mathbf{D}_{\alpha}{}^{(\mathbf{T})}\mathbf{J}^{\alpha}[\varphi] = \square_{\mathbf{g}}\varphi(\mathbf{T}\varphi) + \frac{1}{2}Q^{\mu\nu}[\varphi]{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}$ where ${}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}$ is define in (5.4). The same proof of Lemma 5.4 reveals the coercivity ${}^{(\mathbf{T})}\mathbf{J}^{\alpha}[\varphi] = Q^{00} \approx |\boldsymbol{\partial}\varphi|^2$. It is straightforward to check that

$$ {}^{(\mathbf{T})}\mathbf{J}^{\alpha}[\varphi]L_{\alpha} = \left((L\varphi)^2 + |\nabla\!\!\!/\varphi|^2_{g\!\!\!/}\right). \tag{8.5}$$

Thus we have

$$\begin{aligned}\mathscr{F}_{(wave)}[\varphi;\mathscr{C}_u] = &\int_{\Sigma_t} |\boldsymbol{\partial}\varphi|^2 \,\mathrm{d}\varpi_g - \int_{\Sigma_{t_0}} |\boldsymbol{\partial}\varphi|^2 \,\mathrm{d}\varpi_g \\ &+ \int_{\bigcup\limits_{u'\geq u}\mathscr{C}_{u'}} \left(\square_{\mathbf{g}}\varphi(\mathbf{T}\varphi) + \frac{1}{2}Q^{\mu\nu}[\varphi]{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}\right)\mathrm{d}\varpi_{\mathbf{g}}.\end{aligned} \tag{8.6}$$

Note that $\vec{\Psi}$ are rescaled quantities defined in Definition 6.3. First of all, we have $|\mathbf{T}\varphi| \lesssim |\boldsymbol{\partial}\varphi|$, $|Q^{\mu\nu}[\varphi]| \lesssim |\boldsymbol{\partial}\varphi|^2$ and $\left|{}^{(\mathbf{T})}\boldsymbol{\pi}_{\mu\nu}\right| \lesssim \left|\boldsymbol{\partial}\vec{\Psi}\right| + 1$. Next we substitute $\boldsymbol{\partial}\vec{\Psi}$ for φ, and use equation (6.10) to substitute $\square_{\mathbf{g}}\boldsymbol{\partial}\vec{\Psi}$. By Cauchy-Schwarz inequality, estimate $\int_{\bigcup\limits_{u'\geq u}\mathscr{C}_{u'}} \square_{\mathbf{g}}\varphi(\mathbf{T}\varphi)\mathrm{d}\varpi_{\mathbf{g}} \leq \int_0^{T_{*;(\lambda)}}\int_{\Sigma_\tau} |\square_{\mathbf{g}}\varphi|\,|\mathbf{T}\varphi|\,\mathrm{d}\varpi_g \mathrm{d}\tau$, bootstrap assumptions (6.13), and energy estimates in Proposition 5.1, we obtain the desired estimates.

The proofs for $P_\nu \boldsymbol{\partial}\vec{\Psi}$ are of the same fashion.

To prove (8.4), for each $\beta = 0,1,2,3$, we apply divergence theorem for energy fluxes ${}^{(\mathbf{B})}\mathbf{J}^\alpha := \left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 \mathbf{B}^\alpha$ over the region bounded by $\mathscr{C}_u$, Σ_{t_0} and Σ_t as follows:

$$
\begin{aligned}
-\int_{\mathscr{C}_u} \left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 \mathbf{B}^\alpha L_\alpha \mathrm{d}\varpi_g \mathrm{d}t = \int_{\Sigma_{t_0}} \left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 \mathbf{B}^\alpha \mathbf{T}_\alpha \mathrm{d}\varpi_g - \int_{\Sigma_t} \left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 \mathbf{B}^\alpha \mathbf{T}_\alpha \mathrm{d}\varpi_g \\
- \int_{\bigcup_{u' \geq u} \mathscr{C}_{u'}} \mathbf{D}_\alpha(\left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 \mathbf{B}^\alpha) \mathrm{d}\varpi_{\mathbf{g}}.
\end{aligned}
\tag{8.7}
$$

By the fact that $\mathbf{B}^\alpha = \mathbf{T}^\alpha + \mathrm{f}(\vec{\Psi})$, $\mathbf{B}$ is timelike and (9.119) (The proof of (9.119) is independent of (8.4)), we have $\mathbf{B}^\alpha L_\alpha \approx -1$ and $\mathbf{B}^\alpha \mathbf{T}_\alpha = \mathbf{B}^0 = -1$. By commuted equation (6.12b), we have

$$
\begin{aligned}
\mathbf{D}_\alpha(\left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 \mathbf{B}^\alpha) =& 2\left|\boldsymbol{\partial}\mathscr{C}^\beta\right| \mathbf{B}(\boldsymbol{\partial}\mathscr{C}^\beta) + \left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 \left(\partial_\alpha \mathbf{B}^\alpha + \mathbf{\Gamma}_\beta \mathbf{B}^\beta\right) \\
\lesssim& \left|\boldsymbol{\partial}\mathscr{C}^\beta\right| \mathbf{B}(\boldsymbol{\partial}\mathscr{C}^\beta) + \left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)} \\
\lesssim& \left|\boldsymbol{\partial}\mathscr{C}^\beta\right| \boldsymbol{\partial}^2\vec{\Psi}(\boldsymbol{\partial}\omega, \boldsymbol{\partial}S) + \left|\boldsymbol{\partial}\mathscr{C}^\beta\right| \boldsymbol{\partial}\vec{\Psi} \cdot (\boldsymbol{\partial}^2\vec{\Psi}, \boldsymbol{\partial}^2\omega, \boldsymbol{\partial}^2 S) \\
&+ (\left|\boldsymbol{\partial}\mathscr{C}^\beta\right|^2 + \left|\boldsymbol{\partial}\vec{\Psi}\right|^2) \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_x(\Sigma_\tau)},
\end{aligned}
\tag{8.8}
$$

where $\mathbf{\Gamma}$ is the contracted Christoffel symbols of $\mathbf{g}$ defined as $\mathbf{\Gamma}_\alpha := \mathbf{g}^{\kappa\lambda}\Gamma_{\alpha\kappa\lambda} = \mathbf{g}^{\kappa\lambda}\mathbf{g}_{\alpha\beta}\Gamma^\beta_{\kappa\lambda}$. Combining (8.7) and (8.8), and using the rescaled bootstrap assumptions (6.13), energy estimates in Proposition 5.1, we have

$$
\mathscr{F}_{(transport)}[\boldsymbol{\partial}\vec{\mathscr{C}}; \mathscr{C}_u] \lesssim \lambda^{-1}. \tag{8.9}
$$

The proofs for $\boldsymbol{\partial}\mathscr{D}$ and $P_\nu \boldsymbol{\partial}(\vec{\mathscr{C}}, \mathscr{D})$ are of the same fashion. □

Remark 8.4. *We refer readers to [9, Proposition 6.1] for the energy estimate along null cones for 3D non-relativistic compressible Euler case where* **B** *coincides with* **T***. We note that in [9, 36], authors derive energy estimate along null cones before rescaling (with respect to* λ*). However, the results coincide with ours when energy estimates along null cones are rescaled. That is, our estimate is sufficient to obtain the mix-norm estimates for fluid variables, which serves the same purpose as the estimates in [9, 36].*

8.2 Connection Coefficients

8.2.1 Levi-Civita connections, angular operators and curvatures

If ξ is a space-time tensor, then $\not\Pi\xi$ denotes its $\mathbf{g}$-orthogonal projection onto $S_{t,u}$. If both V and ξ are $S_{t,u}$-tangent, we denote the covariant derivative on $S_{t,u}$ with respect to the induced metric $\not g$ as $\not\nabla_V \xi$. Then we define $\not\mathbf{D}_V \xi := \not\Pi \mathbf{D}_V \xi$, where V is a vector and $\mathbf{D}_V \xi$ is the covariant derivative of ξ in the V direction. Note that $\not\mathbf{D}_V \xi := \not\nabla_V \xi$ when both V and ξ are $S_{t,u}$-tangent.

We let $\mathbf{Riem}_{\alpha\beta\gamma\delta}$ denote the Riemann curvature tensor of $\mathbf{g}$ and $\mathbf{Ric}_{\alpha\beta} := \mathbf{g}^{\gamma\delta}\mathbf{Riem}_{\gamma\alpha\delta\beta}$. We use the notation that

$$\langle \mathbf{D}_X \mathbf{D}_Y W - \mathbf{D}_Y \mathbf{D}_X W, Z\rangle = \mathbf{Riem}_{ZWXY} + \langle \mathbf{D}_{[X,Y]} W, Z\rangle, \tag{8.10}$$

where X,Y,W,Z are vectorfields, $[\cdot,\cdot]$ is the Lie bracket and $\langle\cdot,\cdot\rangle := \mathbf{g}(\cdot,\cdot)$.

Definition 8.5 (Connection coefficients)**.** *We define the second fundamental form k of Σ_t to be the $\binom{0}{2}$ Σ_t-tangent tensor such that the following relation holds for all Σ_t-tangent vectorfields X and Y:*

$$k(X,Y) := -\mathbf{g}(\mathbf{D}_X \mathbf{T}, Y). \tag{8.11}$$

We denote a pair of unit orthogonal spherical vectorfields on $S_{t,u}$ by $\{e_A\}_{A=1,2}$. We define the second fundamental form θ of $S_{t,u}$, the null second fundamental form χ of $S_{t,u}$, and $\underline{\chi}$ to be the following type $\binom{0}{2}$ $S_{t,u}$-tangent tensors:

$$\theta_{AB} := \mathbf{g}(\mathbf{D}_A N, e_B), \tag{8.12a}$$

$$\chi_{AB} := \mathbf{g}(\mathbf{D}_A L, e_B), \qquad \underline{\chi}_{AB} := \mathbf{g}(\mathbf{D}_A \underline{L}, e_B). \tag{8.12b}$$

We define the torsion ζ and $\underline{\zeta}$ to be the following $S_{t,u}$-tangent one-forms:

$$\zeta_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, e_A), \qquad \underline{\zeta}_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, e_A). \tag{8.13}$$

k, θ χ, and $\underline{\chi}$ are symmetric and the following relations hold:

$$k = \frac{1}{2}\mathscr{L}_{\mathbf{T}} g = \frac{1}{2}\mathscr{L}_{\mathbf{T}}\mathbf{g}, \tag{8.14a}$$

$$\chi = \frac{1}{2}\not\mathscr{L}_L \not g = \frac{1}{2}\not\mathscr{L}_L \mathbf{g}, \qquad \underline{\chi} = \frac{1}{2}\not\mathscr{L}_{\underline{L}} \not g = \frac{1}{2}\not\mathscr{L}_{\underline{L}}\mathbf{g}, \tag{8.14b}$$

where $\mathscr{L}$ is the standard Lie derivative and $\not\mathscr{L} = \not\Pi\mathscr{L}$. Also, we have following identities:

$$\mathbf{D}_N N = -\nabla\!\!\!/ \ln b - \frac{1}{2}k_{NN}\underline{L} - \frac{1}{2}k_{NN}L, \qquad \mathbf{D}_A N = \theta_{AB}e_B - \frac{1}{2}k_{AN}L - \frac{1}{2}k_{AN}\underline{L}, \tag{8.15}$$

$$\mathbf{D}_A L = \chi_{AB}e_B - k_{AN}L, \qquad \mathbf{D}_A\underline{L} = \underline{\chi}_{AB}e_B + k_{AN}\underline{L}, \tag{8.16a}$$

$$\mathbf{D}_L L = -k_{NN}L, \qquad \mathbf{D}_L\underline{L} = 2\underline{\zeta}_A e_A + k_{NN}\underline{L}, \tag{8.16b}$$

$$\mathbf{D}_{\underline{L}} L = 2\zeta_A e_A + k_{NN}L, \qquad \mathbf{D}_L e_A = \mathbf{D}\!\!\!\!/_L e_A + \underline{\zeta}_A L, \tag{8.16c}$$

$$\mathbf{D}_B e_A = \nabla\!\!\!/_B e_A + \frac{1}{2}\chi_{AB}\underline{L} + \frac{1}{2}\underline{\chi}_{AB}L, \qquad \mathbf{D}_{\underline{L}}\underline{L} = -2(\nabla\!\!\!/_A \ln b)e_A - k_{NN}\underline{L}, \tag{8.16d}$$

$$\mathbf{D}_{\underline{L}} e_A = \mathbf{D}\!\!\!\!/_{\underline{L}} e_A + \zeta_A\underline{L} - \nabla\!\!\!/_A \ln b L. \tag{8.16e}$$

Morevoer, we have the following relations between some connection coefficients:

$$\chi_{AB} = \theta_{AB} - k_{AB}, \qquad \underline{\chi}_{AB} = -\theta_{AB} - k_{AB}, \qquad \underline{\zeta}_A = -k_{AN}, \qquad \zeta_A = \nabla\!\!\!/_A \ln b + k_{AN}. \tag{8.17}$$

The computation of the above identities relies on the following fact: Let X be a vectorfield, then

$$X = \mathbf{g}(X, e_A)e_A - \frac{1}{2}\mathbf{g}(X, L)\underline{L} - \frac{1}{2}\mathbf{g}(X, \underline{L})L. \tag{8.18}$$

8.2.2 Proof of (8.14)-(8.17)

Proof of $\chi = \frac{1}{2}\mathscr{L}\!\!\!/_L g\!\!\!/ = \frac{1}{2}\mathscr{L}\!\!\!/_L \mathbf{g}$.

$$\mathscr{L}_L \mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\gamma}\mathbf{D}_\beta L^\gamma + \mathbf{g}_{\beta\gamma}\mathbf{D}_\alpha L^\gamma = \langle \mathbf{D}_\alpha L, \partial_\beta\rangle + \langle \mathbf{D}_\beta L, \partial_\alpha\rangle. \tag{8.19}$$

By definition of χ,

$$\chi_{AB} = \frac{1}{2}(\langle \mathbf{D}_A L, e_B\rangle + \langle \mathbf{D}_B L, e_A\rangle) = \frac{1}{2}(\mathscr{L}\!\!\!/_L \mathbf{g})_{AB}. \tag{8.20}$$

$$\begin{aligned}\mathscr{L}_L \mathbf{g}_{\alpha\beta} &= \mathbf{g}_{\alpha\gamma}\mathbf{D}_\beta L^\gamma + \mathbf{g}_{\beta\gamma}\mathbf{D}_\alpha L^\gamma \\ &= \left(g\!\!\!/_{\alpha\gamma} - \frac{1}{2}L_\alpha\underline{L}_\gamma - \frac{1}{2}\underline{L}_\alpha L_\gamma\right)\mathbf{D}_\beta L^\gamma + \left(g\!\!\!/_{\beta\gamma} - \frac{1}{2}L_\beta\underline{L}_\gamma - \frac{1}{2}\underline{L}_\beta L_\gamma\right)\mathbf{D}_\alpha L^\gamma.\end{aligned} \tag{8.21}$$

Since $\langle L, e_A \rangle = \langle \underline{L}, e_A \rangle = 0$,

$$\mathscr{L}_L \not{g} = \mathscr{L}_L \mathbf{g}. \tag{8.22}$$

□

Proof of $\mathbf{D}_A L = \chi_{AB} e_B - k_{AN} L$.

$$\langle \mathbf{D}_A L, e_B \rangle = \chi_{AB}, \tag{8.23a}$$

$$\langle \mathbf{D}_A L, L \rangle = e_A \langle L, L \rangle = 0, \tag{8.23b}$$

$$\langle \mathbf{D}_A L, \underline{L} \rangle = \langle \mathbf{D}_A L, L - 2N \rangle = -2 \langle \mathbf{D}_A (\mathbf{T} + N), N \rangle = 2k_{AN}. \tag{8.23c}$$

□

Proof of $\mathbf{D}_A \underline{L} = \underline{\chi}_{AB} e_B + k_{AN} \underline{L}$.

$$\langle \mathbf{D}_A \underline{L}, e_B \rangle = \underline{\chi}_{AB}, \tag{8.24a}$$

$$\langle \mathbf{D}_A \underline{L}, \underline{L} \rangle = 0, \tag{8.24b}$$

$$\langle \mathbf{D}_A \underline{L}, L \rangle = -\langle \mathbf{D}_A L, \underline{L} \rangle = -2k_{AN}. \tag{8.24c}$$

□

To prove $\mathbf{D}_L L = -k_{NN} L$, we shall first prove that $L_{(Geo)}$ is geodesic.

Proof of $\mathbf{D}_{L_{(Geo)}} L_{(Geo)} = 0$. We note that by (7.10), $L^{\alpha}_{(Geo)} = \partial^{\alpha} u$ and $(L_{(Geo)})_{\beta} = \partial_{\beta} u$. Then we have

$$\begin{aligned}
\langle \mathbf{D}_{L_{(Geo)}} L_{(Geo)}, \partial_{\mu} \rangle &= \partial^{\gamma} u \langle \mathbf{D}_{\gamma} \partial u, \partial_{\mu} \rangle \\
&= \partial^{\gamma} u \partial_{\gamma} \langle \partial u, \partial_{\mu} \rangle - \partial^{\gamma} u \langle \partial u, \mathbf{D}_{\gamma} \partial_{\mu} \rangle \\
&= \partial^{\gamma} u \partial_{\gamma} (\mathbf{g}_{\mu\alpha} \partial^{\alpha} u) - \partial^{\gamma} u \langle \partial u, \mathbf{\Gamma}^{\beta}_{\mu\gamma} \partial_{\beta} \rangle \\
&= \partial^{\gamma} u \partial_{\mu} \partial_{\gamma} u - \partial^{\gamma} u \langle \partial u, \mathbf{D}_{\mu} \partial \gamma \rangle \\
&= \partial^{\gamma} u \partial_{\mu} \langle \partial u, \partial_{\gamma} \rangle - \partial^{\gamma} u \langle \partial u, \mathbf{D}_{\mu} \partial_{\gamma} \rangle \\
&= \langle \mathbf{D}_{\mu} \partial u, \partial u \rangle = 0.
\end{aligned} \tag{8.25}$$

□

Proof of $\mathbf{D}_L L = -k_{NN} L$.

$$\mathbf{D}_L L = \mathbf{D}_{bL_{(Geo)}}(bL_{(Geo)}) = b\mathbf{D}_{L_{(Geo)}} bL_{(Geo)} = \frac{1}{b}\mathbf{D}_L(bL). \tag{8.26a}$$

$$\begin{aligned}\langle \mathbf{D}_L L, \underline{L}\rangle &= \langle \mathbf{D}_L L, L-2N\rangle = -2\langle \mathbf{D}_{\mathbf{T}+N}(\mathbf{T}+N), N\rangle \\ &= -2\langle \mathbf{D}_{\mathbf{T}}\mathbf{T}, N\rangle - 2\langle \mathbf{D}_N \mathbf{T}, N\rangle \\ &= 2k_{NN}.\end{aligned} \tag{8.26b}$$

□

So we have $\frac{1}{b}\mathbf{D}_L b = -k_{NN}$ as well.

Proof of $\mathbf{D}_L \underline{L} = 2\underline{\zeta}_A e_A + k_{NN}\underline{L}$.

$$\langle \mathbf{D}_L \underline{L}, e_A\rangle = 2\underline{\zeta}_A, \tag{8.27a}$$

$$\langle \mathbf{D}_L \underline{L}, L\rangle = -\langle \underline{L}, \mathbf{D}_L L\rangle = 2\langle \mathbf{D}_{\mathbf{T}}\mathbf{T}, L\rangle - 2k_{NN}, \tag{8.27b}$$

$$\langle \mathbf{D}_L \underline{L}, \underline{L}\rangle = 0. \tag{8.27c}$$

□

Proof of $\mathbf{D}_{\underline{L}} L = 2\zeta_A e_A + k_{NN} L$.

$$\langle \mathbf{D}_{\underline{L}} L, e_A\rangle = 2\zeta_A, \tag{8.28a}$$

$$\langle \mathbf{D}_{\underline{L}} L, L\rangle = 0, \tag{8.28b}$$

$$\langle \mathbf{D}_{\underline{L}} L, \underline{L}\rangle = \langle \mathbf{D}_{\underline{L}} L, L-2N\rangle = -2\langle \mathbf{D}_{\mathbf{T}-N}(\mathbf{T}+N), N\rangle = 2\langle \mathbf{D}_N \mathbf{T}, N\rangle = -2k_{NN}. \tag{8.28c}$$

□

Proof of $\mathbf{D}_B e_A = \nabla\!\!\!/_B e_A + \frac{1}{2}\chi_{AB}\underline{L} + \frac{1}{2}\underline{\chi}_{AB} L$.

$$\langle \mathbf{D}_B e_A, L\rangle = -\langle \mathbf{D}_B L, e_A\rangle = -\chi_{AB}, \tag{8.29a}$$

$$\langle \mathbf{D}_B e_A, \underline{L}\rangle = -\underline{\chi}_{AB}. \tag{8.29b}$$

□

Proof of $\mathbf{D}_N N = -\nabla\!\!\!/ \ln b - \frac{1}{2}k_{NN}\underline{L} - \frac{1}{2}k_{NN}L$.

$$\langle \mathbf{D}_N N, L\rangle = \langle \mathbf{D}_N N, \underline{L}\rangle = \langle \mathbf{D}_N N, \mathbf{T}\rangle = -\langle \mathbf{D}_N \mathbf{T}, N\rangle = k_{NN}. \tag{8.30}$$

Since the computation is local, we choose a local coordinate system such that $\mathbf{\Gamma}_{ijk} = 0$. In this coordinate we have

$$N = -b\partial^j u \partial_j. \tag{8.31}$$

We note that $e_A(u) = 0$,

$$\begin{aligned} \langle \mathbf{D}_N N, e_A\rangle =& e_A^a N^i \{\mathbf{g}(\partial_i N, \partial_a) + \mathbf{\Gamma}_{iaN}\} \\ =& e_A^a b\partial^i u \{\partial_i(b\partial_a u)\} \\ =& \frac{1}{2}b^2 e_A^a \partial_a(\partial^i u \partial_i u) \\ =& \frac{1}{2}b^2 e_A(\frac{1}{b^2}) = -\frac{1}{b}e_A(b) = -e_A(\ln b). \end{aligned} \tag{8.32}$$

□

Proof of $\zeta_A = \nabla\!\!\!/_A \ln b + k_{AN}$.

$$2\zeta_A = \langle \mathbf{D}_{\underline{L}} L, e_A\rangle = \langle \mathbf{D}_{L-2N} L, e_A\rangle = -2\langle \mathbf{D}_N(\mathbf{T}+N), e_A\rangle = 2k_{AN} + 2\nabla\!\!\!/_A \ln b. \tag{8.33}$$

□

Proof of $\underline{\zeta}_A = -k_{AN}$.

$$\begin{aligned} 2\underline{\zeta}_A &= \langle \mathbf{D}_L \underline{L}, e_A\rangle = \langle \mathbf{D}_L \underline{L} + \mathbf{D}_{\underline{L}} L, e_A\rangle - \langle \mathbf{D}_{\underline{L}} L, e_A\rangle \\ &= \langle \mathbf{D}_{\mathbf{T}+N}(\mathbf{T}-N) + \mathbf{D}_{\mathbf{T}-N}(\mathbf{T}+N), e_A\rangle - 2\zeta_A \\ &= -2\langle \mathbf{D}_N N, e_A\rangle - 2\zeta_A \\ &= 2\nabla\!\!\!/_A \ln b - 2\zeta_A = -2k_{AN}. \end{aligned} \tag{8.34}$$

□

Proof of $\mathbf{D}_{\underline{L}}\underline{L} = -2\nabla\!\!\!/ \ln b - k_{NN}\underline{L}$.

$$\langle \mathbf{D}_{\underline{L}}\underline{L}, e_A\rangle = \langle \mathbf{D}_N N - \mathbf{D}_N \mathbf{T} - \mathbf{D}_{\mathbf{T}} N, e_A\rangle, \tag{8.35a}$$

$$= -\langle \mathbf{D}_{\underline{L}} L + 2\mathbf{D}_N \mathbf{T}, e_A\rangle = -2\zeta_A + 2k_{AN} = -2\nabla\!\!\!/_A \ln b$$

$$\langle \mathbf{D}_{\underline{L}}\underline{L}, \underline{L}\rangle = 0, \tag{8.35b}$$

$$\langle \mathbf{D}_{\underline{L}}\underline{L}, L\rangle = \langle \mathbf{D}_{\mathbf{T}-N}(\mathbf{T}-N), \mathbf{T}+N\rangle \tag{8.35c}$$

$$= -\langle \mathbf{D}_{\mathbf{T}} N, \mathbf{T}\rangle - \langle \mathbf{D}_N \mathbf{T}, N\rangle + \langle \mathbf{D}_N N, \mathbf{T}\rangle$$

$$= 2k_{NN}.$$

□

Proof of $\mathbf{D}_{\underline{L}} e_A = \mathbf{D}\!\!\!\!/_{\underline{L}} e_A + \zeta_A \underline{L} - (\nabla\!\!\!/_A \ln b)L$.

$$\langle \mathbf{D}_{\underline{L}} e_A, L\rangle = -\langle e_A, \mathbf{D}_{\underline{L}} L\rangle = -2\zeta_A, \tag{8.36a}$$

$$\langle \mathbf{D}_{\underline{L}} e_A, \underline{L}\rangle = -\langle e_A, \mathbf{D}_{\underline{L}}\underline{L}\rangle = 2\nabla\!\!\!/_A \ln b. \tag{8.36b}$$

□

8.2.3 Conformal metric, initial conditions on Σ_0 and on the cone-tip axis for the acoustical function u

Definition 8.6 (Conformal factor and the modified null second fundamental form and acoustical quantities)**.** *In interior region* $\mathscr{M}^{(Int)}$ *(defined in Section 7.1.2), we define* σ *to be the solution to the following transport initial value problem:*

$$L\sigma(t,u,\omega) = \frac{1}{2}[\mathbf{\Gamma}_L](t,u,\omega) \qquad u \in [0, T_{*;(\lambda)}], t \in [u, T_{*;(\lambda)}], \omega \in \mathbb{S}^2, \tag{8.37a}$$

$$\sigma(u,u,\omega) = 0, \qquad u \in [0, T_{*;(\lambda)}], \omega \in \mathbb{S}^2, \tag{8.37b}$$

where $\mathbf{\Gamma}_L := \mathbf{\Gamma}_\alpha L^\alpha$ *and* $\mathbf{\Gamma}_\alpha := \mathbf{g}^{\kappa\lambda}\mathbf{\Gamma}_{\alpha\kappa\lambda} = \mathbf{g}^{\kappa\lambda}\mathbf{g}_{\alpha\beta}\mathbf{\Gamma}^{\beta}_{\kappa\lambda}$.

We define the conformal space-time metric and the Riemannian metric that induces on $S_{t,u}$ *as follows:*

$$\widetilde{\mathbf{g}} := e^{2\sigma}\mathbf{g}, \qquad \widetilde{g\!\!\!/} := e^{2\sigma} g\!\!\!/. \tag{8.38}$$

We define the null second fundamental forms of the metric to be the following symmetric $S_{t,u}$*-tangent* *tensors:*

$$\tilde{\chi} := \frac{1}{2}\mathscr{L}_L \tilde{\not{g}}, \qquad \underline{\tilde{\chi}} := \frac{1}{2}\mathscr{L}_{\underline{L}} \tilde{\not{g}}. \tag{8.39}$$

Using (8.38)-(8.39), it follows that $\chi, \underline{\chi}$ *and* $\tilde{\chi}, \underline{\tilde{\chi}}$ *are related by:*

$$\tilde{\chi} = e^{2\sigma}\left(\chi + (L\sigma)\not{g}\right), \qquad \underline{\tilde{\chi}} = e^{2\sigma}\left(\underline{\chi} + (\underline{L}\sigma)\not{g}\right), \tag{8.40a}$$

$$\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi} = \mathrm{tr}_{\not{g}}\chi + 2L\sigma = \mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L, \qquad \mathrm{tr}_{\tilde{\not{g}}}\underline{\tilde{\chi}} = \mathrm{tr}_{\not{g}}\underline{\chi} + 2\underline{L}\sigma, \tag{8.40b}$$

$$\chi = \frac{1}{2}\left(\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi} - \mathbf{\Gamma}_L\right)\not{g} + \hat{\chi}, \qquad \underline{\chi} = \frac{1}{2}\left(\mathrm{tr}_{\tilde{\not{g}}}\underline{\tilde{\chi}} - 2\underline{L}\sigma\right)\not{g} + \underline{\hat{\chi}}. \tag{8.40c}$$

We define the following:

$$\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(Small)} := \mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L - \frac{2}{\tilde{r}} = \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi} - \frac{2}{\tilde{r}}. \tag{8.41}$$

We note that the first equality in (8.41) holds in the whole region $\mathscr{M}$*, while the second equality holds only in the interior region* $\mathscr{M}^{(Int)}$.

Definition 8.7 (Average values on $S_{t,u}$ and spherevolume ratio)**.** *For scalar functions* f*, we define the average value of* f*, denoted by* $\overline{f}$*, as follows:*

$$\overline{f} := \frac{1}{|S_{t,u}|_{\not{g}}}\int_{S_{t,u}} f d\varpi_{\not{g}}, \qquad |S_{t,u}|_{\not{g}} := \int_{S_{t,u}} 1 d\varpi_{\not{g}}. \tag{8.42}$$

We define $\mathrm{d}\varpi_{\not{e}}$ *to be the standard volume form on the Euclidean sphere. We define the volume form* $\mathrm{d}\varpi_{\not{g}}$, $\mathrm{d}\varpi_{\tilde{\not{g}}}$ *and* $\mathrm{d}\varpi_{\mathbf{g}}$ *as follows:*

$$\mathrm{d}\varpi_{\not{g}} := \sqrt{\det\not{g}}\,\mathrm{d}\varpi_{\not{e}}, \tag{8.43}$$

$$\mathrm{d}\varpi_{\tilde{\not{g}}} := \sqrt{\det\tilde{\not{g}}}\,\mathrm{d}\varpi_{\not{e}}, \tag{8.44}$$

$$\mathrm{d}\varpi_{\mathbf{g}} := b\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}\varpi_{\not{g}}. \tag{8.45}$$

We define the spherevolume ratio as follows:

$$\upsilon := \frac{\sqrt{\det\not{g}}}{\sqrt{\det\not{e}}}, \tag{8.46}$$

$$\tilde{\upsilon} := \frac{\sqrt{\det\tilde{\not{g}}}}{\sqrt{\det\not{e}}} \tag{8.47}$$

Proposition 8.8 (Evolution equations for the lapse and volume)**.**

$$L\upsilon = \upsilon \mathrm{tr}_{\not g}\chi, \tag{8.48a}$$

$$\underline{L}\upsilon = \upsilon \mathrm{tr}_{\not g}\underline{\chi}, \tag{8.48b}$$

$$Lb = -bk_{NN}. \tag{8.48c}$$

Proof of $L\upsilon = \upsilon \mathrm{tr}_{\not g}\chi$. By (8.14b)

$$L\sqrt{|\not g|} = \frac{1}{2\sqrt{|\not g|}}|\not g|\mathrm{tr}(\not g^{-1}\mathscr{L}_L\not g) = \sqrt{|\not g|}\mathrm{tr}((\not g^{-1})^{\alpha\beta}\chi_{\beta\gamma}) = \sqrt{|\not g|}\mathrm{tr}_{\not g}\chi. \tag{8.49}$$

□

Proof of $Lb = -bk_{NN}$. By (8.16) and (7.14), we have

$$\frac{1}{b}\mathbf{D}_L b \cdot L_{(Geo)} = \mathbf{D}_L L = -k_{NN}L. \tag{8.50}$$

□

Lemma 8.9 (Evolution equation for the average value on $S_{t,u}$)**.** *For scalar functions f, we have*

$$L\overline{f} + \mathrm{tr}_{\not g}\chi\overline{f} = \left(\mathrm{tr}_{\not g}\chi - \overline{\mathrm{tr}_{\not g}\chi}\right)\overline{f} + \overline{Lf + \mathrm{tr}_{\not g}\chi f}. \tag{8.51}$$

Proof of (8.51). With the help of (8.48a), we have

$$\begin{aligned} L\overline{f} &= L\left(\frac{1}{|S_{t,u}|_{\not g}}\int_{S_{t,u}} f d\varpi_{\not g}\right) \qquad (8.52)\\ &= -|S_{t,u}|_{\not g}^{-2}L(|S_{t,u}|_{\not g})\int_{S_{t,u}} f d\varpi_{\not g} + \frac{1}{|S_{t,u}|_{\not g}}\int_{S_{t,u}}\left(Lf d\varpi_{\not g} + fL(\varpi_{\not g})\right)\\ &= -\overline{\mathrm{tr}_{\not g}\chi}\cdot\overline{f} + \overline{Lf + \mathrm{tr}_{\not g}\chi f}. \end{aligned}$$

□

In the following definition, we define mass aspect function μ and its modified version $\check{\mu}$, as well as modified torsion $\tilde{\zeta}$. These objects are defined to avoid loss of regularity of the acoustical eikonal function.

Definition 8.10 (Modified mass aspect function)**.** *We define the mass aspect function μ to be the following* *scalar function*

$$\mu := \underline{L}\mathrm{tr}_{\not g}\chi + \frac{1}{2}\mathrm{tr}_{\not g}\chi\mathrm{tr}_{\not g}\underline{\chi}. \tag{8.53}$$

We now define the modified mass aspect function $\check{\mu}$ *to be the following scalar function:*

$$\check{\mu} := 2\not\Delta\sigma + \underline{L}\mathrm{tr}_{\not g}\chi + \frac{1}{2}\mathrm{tr}_{\not g}\chi\mathrm{tr}_{\not g}\underline{\chi} - \mathrm{tr}_{\not g}\chi k_{NN} + \frac{1}{2}\mathrm{tr}_{\not g}\chi\mathbf{\Gamma}_{\underline{L}}. \tag{8.54}$$

In $\mathscr{M}^{(Int)}$*, we define* $\not\mu$ *to be the* $S_{t,u}$*-tangent one-form that satisfies the following Hodge system on* $S_{t,u}$*:*

$$\not{\mathrm{div}}\not\mu = \frac{1}{2}(\check{\mu} - \bar{\check{\mu}}) \qquad\qquad \not{\mathrm{curl}}\not\mu = 0. \tag{8.55}$$

In $\mathscr{M}^{(Int)}$*, we define the modified torsion* $\tilde{\zeta}$ *to be the following* $S_{t,u}$*-tangent one-form:*

$$\tilde{\zeta} := \zeta + \not\nabla\sigma. \tag{8.56}$$

Definition 8.11 (Norms of $S_{t,u}$-tangent tensorfields)**.** *Let* $\not e = \not e(\omega)$ *be the canonical metric on* $\mathbb{S}^2$ *and* $\{\omega^A\}_{A=1,2}$ *are local angular coordinates on* $\mathbb{S}^2$*. For* $p \in [1,\infty)$*, We define the Lebesgue norms* L^p_ω *and* $L^p_{\not g}$ *for* $S_{t,u}$*-tangent tensorfields* ξ *as follows:*

$$\|\xi\|_{L^p_\omega(S_{t,u})} := \left(\int_{\omega\in\mathbb{S}^2} |\xi|^p_{\not g}\, \mathrm{d}\varpi_{\not e}\right)^{1/p}, \qquad \|\xi\|_{L^p_{\not g}(S_{t,u})} := \left(\int_{\omega\in\mathbb{S}^2} |\xi|^p_{\not g}\, \mathrm{d}\varpi_{\not g}\right)^{1/p}. \tag{8.57}$$

Since we can have a parallel transport along $\mathbb{S}^2$ *that preserves the tensor products and contractions, that is,* $\Phi^m_n(\omega_{(2)};\omega_{(1)})$ *is the parallel transport with respect to* $\not e$ *from the vector space of type* $\binom{m}{n}$ *tensors at* $\omega_{(2)} \in \mathbb{S}^2$ *to the vector space of type* $\binom{m}{n}$ *tensors at* $\omega_{(1)} \in \mathbb{S}^2$*, for* $\xi = \xi(\omega)$ *a type* $\binom{m}{n}$ *tensorfield on* $\mathbb{S}^2$*, we define the* L^∞ *norm* L^∞_ω *and Hölder norm* $C^{0,\alpha}_\omega$ *of* ξ *as follows:*

$$\|\xi\|_{L^\infty_\omega(S_{t,u})} := \operatorname*{ess\,sup}_{\omega\in\mathbb{S}^2} |\xi|_{\not g}, \tag{8.58}$$

$$\begin{aligned}\|\xi\|_{C^{0,\alpha}_\omega(S_{t,u})} := {}& \|\xi\|_{L^\infty_\omega(S_{t,u})} \\ &+ \sup_{0<d_{\not e}(\omega_{(1)},\omega_{(2)})<\frac{\pi}{2}} \frac{|\xi(t,u,\omega_{(1)}) - \Phi^m_n(\omega_{(2)};\omega_{(1)})\circ\xi(t,u,\omega_{(2)})|_{\not g(t,u,\omega_{(1)})}}{d^\alpha_{\not e}(\omega_{(1)},\omega_{(2)})},\end{aligned} \tag{8.59}$$

where $d^\alpha_{\not e}(\omega_{(1)},\omega_{(2)})$ is the distance between $\omega_{(1)},\omega_{(2)} \in \mathbb{S}^2$ with respect to $\not e$. In particular, $d^\alpha_{\not e}(\omega_{(1)},\omega_{(2)}) \leq \pi$.

In the following two propositions, we list the estimates of the initial foliation. These estimates are crucial for the well-defined geometric setup in Section 7.1 and controlling the acoustic geometry.

Proposition 8.12. *[9, Proposition 9.8], [36, Proposition 4.3. Existence and properties of the initial foliation].*

On Σ_0, there exists a function $w = w(x)$ on the domain $0 \le w \le w_* := \frac{4}{5}T_{*;(\lambda)}$, such that $w(\mathbf{z}) = 0$ and each level set $S_{0,w}$ is diffeomorphic to $\mathbb{S}^2$ and

$$\mathrm{tr}_{\not g}\theta + k_{NN} = \frac{2}{aw} + \mathrm{tr}_g k - \mathbf{\Gamma}_L, \qquad a(\mathbf{z}) = 1. \tag{8.60}$$

By (8.41), $\widetilde{r}(0,-u) = w$, and the fact that $\chi_{AB} = \theta_{AB} - k_{AB}$, (8.60) is equivalent to

$$\mathrm{tr}_{\widetilde{\not g}}\tilde{\chi}^{(Small)}|_{\Sigma_0} = \frac{2(1-a)}{aw}, \qquad \text{for } 0 \le w \le w_*, \tag{8.61}$$

where we define the lapse a as follows:

$$a = \left(\sqrt{g^{cd}\partial_c w \partial_d w}\right)^{-1}. \tag{8.62}$$

Note that $\frac{\partial}{\partial w} = aN|_{\Sigma_0}$. Then on $\Sigma_0^{w_*} := \bigcup_{0\le w\le w_*} S_{0,w}$, for $0 < 1 - \frac{2}{q_*} < N-2$, there hold

$$|a-1| \lesssim \lambda^{-4\varepsilon_0} \le \frac{1}{4}, \qquad \left\| w^{-1/2}(a-1)\right\|_{L^\infty_w C^{0,1-\frac{2}{q_*}}_\omega(\Sigma_0^{w_*})} \lesssim \lambda^{-1/2}, \qquad \upsilon := \frac{\sqrt{\det \not g}}{\sqrt{\det \not e}} \approx w^2. \tag{8.63a}$$

$$\left\| w^{\frac{1}{2}-\frac{2}{q_*}}(\hat{\theta}, \not\nabla \ln a)\right\|_{L^\infty_w L^{q_*}_{\not g}(\Sigma_0^{w_*})}, \|\not\nabla \ln a\|_{L^2_w L^\infty_\omega(\Sigma_0^{w_*})}, \|\hat{\chi}\|_{L^2_w L^\infty_\omega(\Sigma_0^{w_*})} \lesssim \lambda^{-1/2}. \tag{8.63b}$$

$$\max_{A,B=1,2}\left\| w^{-2}\not g\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - \not e\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right)\right\|_{L^\infty_x(\Sigma_0^{w_*})} \lesssim \lambda^{-4\varepsilon_0}, \tag{8.63c}$$

$$\max_{A,B,C=1,2}\left\| \frac{\partial}{\partial\omega^C}\left(w^{-2}\not g\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - \not e\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right)\right)\right\|_{L^{q_*}_\omega(S_{0,w})} \lesssim \lambda^{-4\varepsilon_0}, \tag{8.63d}$$

$$\left\| w^{\frac{1}{2}-\frac{2}{q_*}}\not\nabla \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L^{q_*}_{\not g}(S_{0,w})} \lesssim \lambda^{-1/2}. \tag{8.63e}$$

In addition,

$$\left\| w \mathrm{tr}_{\tilde{g}} \tilde{\chi}^{(Small)} \right\|_{L^\infty_x(\Sigma_0^{w_*})} \lesssim \lambda^{-4\varepsilon_0}, \tag{8.63f}$$

$$\left\| w^{3/2} \nabla\!\!\!/ \mathrm{tr}_{\tilde{g}} \tilde{\chi}^{(Small)} \right\|_{L^\infty_w L^p_\omega(\Sigma_0^{w_*})} + \left\| w^{1/2} \mathrm{tr}_{\tilde{g}} \tilde{\chi}^{(Small)} \right\|_{L^\infty_w C^{0,1-\frac{2}{p}}_\omega(\Sigma_0^{w_*})} \lesssim \lambda^{1/2}. \tag{8.63g}$$

Finally,

$$\sum_{i,j=1,2,3} \left| w \Pi\!\!\!/_j^a \partial_a N^i - \Pi\!\!\!/_j^i \right| = \mathscr{O}(w) \text{ as } w \downarrow 0. \tag{8.63h}$$

Proposition 8.13. *[9, Lemma 9.9. Initial conditions on the cone-tip axis tied to the acoustical function] On any null cone $\mathscr{C}_u$ initiating from a point on the time axis $0 \le t = u \le T_{*;(\lambda)}$ there hold*

$$\begin{aligned}
&\mathrm{tr}_{g\!\!\!/} \chi - \frac{2}{\widetilde{r}}, \widetilde{r}\, \mathrm{tr}_{\tilde{g}} \tilde{\chi}^{(Small)}, |\hat{\chi}|_{g\!\!\!/}, \left| \widetilde{r} \Pi\!\!\!/_j^a \partial_a L^i - \Pi\!\!\!/_j^i \right|, b-1, |\zeta|_{g\!\!\!/}, \sigma, \\
&\widetilde{r} \left| \nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/} \chi \right|_{g\!\!\!/}, \widetilde{r}^2 \left| \nabla\!\!\!/ \mathrm{tr}_{\tilde{g}} \tilde{\chi}^{(Small)} \right|_{g\!\!\!/}, \widetilde{r} |\nabla\!\!\!/ \hat{\chi}|_{g\!\!\!/}, \widetilde{r} |\nabla\!\!\!/ b|_{g\!\!\!/}, \widetilde{r} |\nabla\!\!\!/ \zeta|_{g\!\!\!/}, \widetilde{r} |\nabla\!\!\!/ \sigma|_{g\!\!\!/}, \\
&\widetilde{r}^2 \Delta\!\!\!/ b, \widetilde{r}^2 \Delta\!\!\!/ \sigma, \widetilde{r}^2 \mu, \widetilde{r}^2 \check{\mu} \\
&= \mathscr{O}(\widetilde{r}) \text{ as } t \downarrow u,
\end{aligned} \tag{8.64a}$$

$$\lim_{t \downarrow u} \left\| \underline{\zeta}, k \right\|_{L^\infty(S_{t,u})} < \infty. \tag{8.64b}$$

Moreover,

$$\lim_{t \downarrow u} \widetilde{r}^{-2} g\!\!\!/ \left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B} \right) = e\!\!\!/ \left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B} \right), \tag{8.64c}$$

$$\lim_{t \downarrow u} \widetilde{r}^{-2} \frac{\partial}{\partial \omega^C} g\!\!\!/ \left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B} \right) = \frac{\partial}{\partial \omega^C} e\!\!\!/ \left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B} \right). \tag{8.64d}$$

Discussion of the proof of Proposition 8.12 and Proposition 8.13. The existence of such initial foliation can be proved by Nash-Moser implicit function theorem (see [29]). The proof of the estimates in Prop.8.12 relies on the energy estimates (5.1). The reason is that Prop.8.12 yields a foliation and estimates on the hypersurface Σ_0 with respect to the rescaled coordinates, and this hypersurface corresponds to the hyperfaces Σ_{t_k} for each k with respect to original space-time (see Remark 6.4 and Subsection 7.1.2 for the description of Σ_0 in rescaled space-time). The point is that we need the energy estimates of (5.1) to control the fluid along the "old" Σ_{t_k}. We refer the reader to [36, Appendix C] for the proof of the estimates in Prop.8.12 and in Prop.8.13. □

8.3 PDEs Verified by Geometric Quantities

In order to control the acoustic geometry, we need to derive the estimates for various geometric quantities. We provide the transport-Hodge type of PDEs in Lemma 8.14. Then we use relativistic Euler equations to express the curvature terms in the right hand side of these PDEs in the following subsection.

Lemma 8.14. *[14, Section 5.4, PDEs verified by connection coefficients] The following evolution equations hold relative to a null frame*

$$L\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2 = -|\hat{\chi}|^2_{\not g} - k_{NN}\mathrm{tr}_{\not g}\chi - \mathbf{Ric}_{LL}, \tag{8.65a}$$

$$\not{\mathbf{D}}_L\hat{\chi}_{AB} + (\mathrm{tr}_{\not g}\chi)\hat{\chi}_{AB} = -k_{NN}\hat{\chi}_{AB} - \left(\mathbf{Riem}_{LALB} - \frac{1}{2}\mathbf{Ric}_{LL}\delta_{AB}\right), \tag{8.65b}$$

$$\not{\mathbf{D}}_L\zeta_A + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\zeta_A = -(k_{BN}+\zeta_B)\hat{\chi}_{AB} - \frac{1}{2}\mathrm{tr}_{\not g}\chi k_{AN} + \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}. \tag{8.65c}$$

$$L\mathrm{tr}_{\not g}\underline{\chi} + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\mathrm{tr}_{\not g}\underline{\chi} = 2\not{\mathrm{div}}\underline{\zeta} + k_{NN}\mathrm{tr}_{\not g}\underline{\chi} - \hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + 2\left|\underline{\zeta}\right|^2_{\not g} + \mathbf{Riem}_{A\underline{L}LA}, \tag{8.65d}$$

$$\begin{aligned}\not{\mathbf{D}}_L\hat{\underline{\chi}}_{AB} + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\hat{\underline{\chi}}_{AB} = &-\frac{1}{2}(\mathrm{tr}_{\not g}\underline{\chi})\hat{\chi}_{AB} + 2\not\nabla_A\underline{\zeta}_B \\ &- \not{\mathrm{div}}\underline{\zeta}\delta_{AB} + k_{NN}\hat{\underline{\chi}}_{AB} + \left(2\underline{\zeta}_A\underline{\zeta}_B - \left|\underline{\zeta}\right|^2_{\not g}\delta_{AB}\right) \\ &- \left(\hat{\chi}_{AC}\hat{\underline{\chi}}_{BC} - \frac{1}{2}\hat{\underline{\chi}}_{CD}\hat{\chi}_{CD}\delta_{AB}\right) + \mathbf{Riem}_{B\underline{L}LA} - \frac{1}{2}\mathbf{Riem}_{C\underline{L}LC}\delta_{AB},\end{aligned} \tag{8.65e}$$

$$\underline{L}\mathrm{tr}_{\not g}\chi + \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}\mathrm{tr}_{\not g}\chi = 2\not{\mathrm{div}}\zeta + k_{NN}\mathrm{tr}_{\not g}\chi - \hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + 2|\zeta|^2_{\not g} + \mathbf{Riem}_{AL\underline{L}A}, \tag{8.65f}$$

$$\begin{aligned}\not{\mathbf{D}}_{\underline{L}}\hat{\chi}_{AB} + \frac{1}{2}(\mathrm{tr}_{\not g}\underline{\chi})\hat{\chi}_{AB} = &-\frac{1}{2}(\mathrm{tr}_{\not g}\chi)\hat{\underline{\chi}}_{AB} + 2\not\nabla_A\zeta_B \\ &- \not{\mathrm{div}}\zeta\delta_{AB} + k_{NN}\hat{\chi}_{AB} + \left(2\zeta_A\zeta_B - |\zeta|^2_{\not g}\delta_{AB}\right) \\ &- \left(\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC} - \frac{1}{2}\hat{\underline{\chi}}_{CD}\hat{\chi}_{CD}\delta_{AB}\right) + \mathbf{Riem}_{BL\underline{L}A} - \frac{1}{2}\mathbf{Riem}_{CL\underline{L}C}\delta_{AB}.\end{aligned} \tag{8.65g}$$

$$\not{\mathrm{div}}\hat{\chi}_A + \hat{\chi}_{AB}k_{BN} = \frac{1}{2}(\not\nabla_A\mathrm{tr}_{\not g}\chi + k_{AN}\mathrm{tr}_{\not g}\chi) + \mathbf{Riem}_{BLBA}, \tag{8.65h}$$

$$\not{\mathrm{div}}\zeta = \frac{1}{2}(\mu - k_{NN}\mathrm{tr}_{\not g}\chi - 2|\zeta|^2_{\not g} - |\hat{\chi}|^2_{\not g} - 2k_{AB}\hat{\chi}_{AB}) - \frac{1}{2}\mathbf{Riem}_{A\underline{L}LA}, \tag{8.65i}$$

$$\not{\mathrm{curl}}\zeta = \frac{1}{2}\varepsilon^{AB}\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC} - \frac{1}{2}\varepsilon^{AB}\mathbf{Riem}_{BL\underline{L}A}. \tag{8.65j}$$

8.3.1 Proof of Lemma 8.14

In the proof, we (secretly) use (8.16) and (8.10) whenever they are needed. Also, we note that for $S_{t,u}$-tangent tensor X, Y, $\mathrm{tr}_{\not g}(X_A, Y_B) = \delta_{AB}X_AY_B$. We give proof for all the equations in Lemma 8.14. Also, we refer readers to [14] for derivations of many equations.

Proof of $L\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2 = -|\hat{\chi}|^2_{\not g} - k_{NN}\mathrm{tr}_{\not g}\chi - \mathbf{Ric}_{LL}$ *in (8.65a).*

$$\not{\mathbf{D}}_L\chi_{AB} = L(\chi_{AB}) - \chi(\not{\mathbf{D}}_L e_A, B) - \chi(A, \not{\mathbf{D}}_L e_B), \tag{8.66a}$$

$$L(\chi_{AB}) = \langle \mathbf{D}_L\mathbf{D}_A L, e_B\rangle + \langle \mathbf{D}_A L, \mathbf{D}_L e_B\rangle, \tag{8.66b}$$

$$\langle \mathbf{D}_L\mathbf{D}_A L, e_B\rangle = \mathbf{Riem}_{BLLA} + \langle \mathbf{D}_A\mathbf{D}_L L, e_B\rangle + \langle \mathbf{D}_{[L,A]}L, e_B\rangle, \tag{8.66c}$$

$$\langle \mathbf{D}_A\mathbf{D}_L L, e_B\rangle = -\langle \mathbf{D}_L L, \mathbf{D}_A e_B\rangle = -k_{NN}\chi_{AB}, \tag{8.66d}$$

$$\langle \mathbf{D}_{[L,A]}L, e_B\rangle = \langle \mathbf{D}_{\mathbf{D}_L e_A - \mathbf{D}_A L}L, e_B\rangle = \langle \mathbf{D}_{\not{\mathbf{D}}_L e_A}L, e_B\rangle - \chi_{AC}\chi_{BC} = \chi(B, \not{\mathbf{D}}_L e_A) - \chi_{AC}\chi_{BC}, \tag{8.66e}$$

$$\langle \mathbf{D}_A L, \mathbf{D}_L e_B\rangle = \langle \mathbf{D}_A L, \not{\mathbf{D}}_L e_B - k_{BN}L\rangle = \chi(A, \not{\mathbf{D}}_L e_B). \tag{8.66f}$$

Combining the above equations, we have

$$\not{\mathbf{D}}_L\chi_{AB} = -k_{NN}\chi_{AB} - \chi_{AC}\chi_{BC} - \mathbf{Riem}_{BLAL}. \tag{8.67}$$

Notice that $\mathbf{g}(e_A, L) = \mathbf{g}(e_A\underline{L}) = \mathbf{g}(\underline{L}, \underline{L}) = 0$ and $\mathbf{Riem}_{\underline{L}LLL} = \mathbf{Riem}_{LL\underline{L}L} = \mathbf{Riem}_{LLLL} = 0$, so

$$\mathbf{Ric}_{LL} = \mathbf{g}(e_A, e_B)\mathbf{Riem}_{ALBL} = \not g(e_A, e_B)\mathbf{Riem}_{ALBL}. \tag{8.68}$$

Take the $\not g$-trace of (8.67). For the second term on the right,

$$\begin{aligned} \mathrm{tr}_{\not g}(\chi_{AC}\chi_{BC}) &= \mathrm{tr}_{\not g}\left(\left(\hat{\chi}_{AC} + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not g_{AC}\right)\left(\hat{\chi}_{BC} + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not g_{BC}\right)\right) \\ &= \left(\hat{\chi}_{AC} + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not g_{AC}\right)\left(\hat{\chi}_{AC} + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not g_{AC}\right) \\ &= |\hat{\chi}|^2_{\not g} + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2, \end{aligned} \tag{8.69}$$

where we used that $\mathrm{tr}_{\not g}\chi\not g_{AC}\hat{\chi}_{AC} = \mathrm{tr}_{\not g}\chi\hat{\chi}_{AA} = 0$ and $\not g_{AC}\not g_{AC} = \not g_{AA} = 2$.
Therefore

$$L\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2 = -|\hat{\chi}|^2_{\not g} - k_{NN}\mathrm{tr}_{\not g}\chi - \mathbf{Ric}_{LL}. \tag{8.70}$$

□

Proof of $\not{\mathbf{D}}_L\hat{\chi}_{AB} + (\mathrm{tr}_{\not g}\chi)\hat{\chi}_{AB} = -k_{NN}\hat{\chi}_{AB} - \left(\mathbf{Riem}_{LALB} - \frac{1}{2}\mathbf{Ric}_{LL}\delta_{AB}\right)$ *in (8.65b).* By (8.67) and (8.65a), we have

$$\not{D}_L \hat{\chi}_{AB} = \not{D}_L \left(\chi_{AB} - \frac{1}{2} \mathrm{tr}_{\not{g}} \chi \not{g}_{AB} \right) \tag{8.71a}$$
$$= - k_{NN} \chi_{AB} - \chi_{AC} \chi_{BC} - \mathbf{Riem}_{BLAL} + \frac{1}{2} \not{g}_{AB} \left(|\hat{\chi}|^2_{\not{g}} + k_{NN} \mathrm{tr}_{\not{g}} \chi + \mathbf{Ric}_{LL} + \frac{1}{2} (\mathrm{tr}_{\not{g}} \chi)^2 \right).$$

$$-k_{NN} \chi_{AB} + \frac{1}{2} \not{g}_{AB} k_{NN} \mathrm{tr}_{\not{g}} \chi = -k_{NN} \hat{\chi}_{AB}. \tag{8.71b}$$

$$\chi_{AC} \chi_{BC} = \left(\hat{\chi}_{AC} + \frac{1}{2} \mathrm{tr}_{\not{g}} \chi \not{g}_{AC} \right) \left(\hat{\chi}_{BC} + \frac{1}{2} \mathrm{tr}_{\not{g}} \chi \not{g}_{BC} \right) \tag{8.71c}$$
$$= \hat{\chi}_{AC} \hat{\chi}_{BC} + \frac{1}{2} \hat{\chi}_{AC} \mathrm{tr}_{\not{g}} \chi \not{g}_{BC} + \frac{1}{2} \hat{\chi}_{BC} \mathrm{tr}_{\not{g}} \chi \not{g}_{AC} + \frac{1}{4} (\mathrm{tr}_{\not{g}} \chi)^2 \not{g}_{AC} \not{g}_{BC}.$$

Notice that $\hat{\chi}_{11} = -\hat{\chi}_{22}$,

$$\hat{\chi}_{AC} \hat{\chi}_{BC} = \begin{pmatrix} \hat{\chi}^2_{11} + \hat{\chi}^2_{12} & 0 \\ 0 & \hat{\chi}^2_{22} + \hat{\chi}^2_{12} \end{pmatrix} = \frac{1}{2} \mathrm{tr}_{\not{g}} (\hat{\chi}_{AC} \hat{\chi}_{BC}) \not{g}_{AB} = \frac{1}{2} |\hat{\chi}|^2_{\not{g}} \not{g}_{AB}. \tag{8.71d}$$

$$\frac{1}{2} \hat{\chi}_{AC} \mathrm{tr}_{\not{g}} \chi \not{g}_{BC} = \frac{1}{2} \hat{\chi}_{BC} \mathrm{tr}_{\not{g}} \chi \not{g}_{AC} = \frac{1}{2} \hat{\chi}_{AB} \mathrm{tr}_{\not{g}} \chi. \tag{8.71e}$$

Combine above equations and by the fact $\not{g}_{AB} = \delta_{AB}$, we get the desired equation. □

Proof of $\not{D}_L \zeta_A + \frac{1}{2} (\mathrm{tr}_{\not{g}} \chi) \zeta_A = -(k_{BN} + \zeta_B) \hat{\chi}_{AB} - \frac{1}{2} \mathrm{tr}_{\not{g}} \chi k_{AN} + \frac{1}{2} \mathbf{Riem}_{ALL\underline{L}}$ *in (8.65c).*

$$\not{D}_L \zeta_A = L(\zeta_A) - \frac{1}{2} \langle \mathbf{D}_{\underline{L}} L, \not{D}_L e_A \rangle \tag{8.72a}$$
$$L(\zeta_A) = \frac{1}{2} \langle \mathbf{D}_L \mathbf{D}_{\underline{L}} L, e_A \rangle + \frac{1}{2} \langle \mathbf{D}_{\underline{L}} L, \mathbf{D}_L e_A \rangle, \tag{8.72b}$$
$$\langle \mathbf{D}_L \mathbf{D}_{\underline{L}} L, e_A \rangle = \langle \mathbf{D}_{\underline{L}} \mathbf{D}_L L, e_A \rangle + \langle \mathbf{D}_{[L,\underline{L}]} L, e_A \rangle + \mathbf{Riem}_{ALL\underline{L}}, \tag{8.72c}$$
$$\langle \mathbf{D}_{\underline{L}} \mathbf{D}_L L, e_A \rangle = -k_{NN} \langle \mathbf{D}_{\underline{L}} L, e_A \rangle, \tag{8.72d}$$
$$\langle \mathbf{D}_{[L,\underline{L}]} L, e_A \rangle = \langle \mathbf{D}_{\mathbf{D}_L \underline{L} - \mathbf{D}_{\underline{L}} L} L, e_A \rangle = 2(\underline{\zeta}_B - \zeta_B) \chi_{AB} + k_{NN} \langle \mathbf{D}_{\underline{L}} L, e_A \rangle - k_{NN} \langle \mathbf{D}_L L, e_A \rangle, \tag{8.72e}$$
$$\langle \mathbf{D}_{\underline{L}} L, \mathbf{D}_L e_A \rangle = \langle \mathbf{D}_{\underline{L}} L, \not{D}_L e_A + \underline{\zeta}_A L \rangle = \langle \mathbf{D}_{\underline{L}} L, \not{D}_L e_A \rangle. \tag{8.72f}$$

Combining above equations, we have

$$\not{D}_L \zeta_A = (\underline{\zeta}_B - \zeta_B)\chi_{AB} + \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}} \tag{8.73}$$
$$= -(k_{BN} + \zeta_B)\left(\hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\not{g}_{AB}\right) + \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}$$
$$= -(k_{BN} + \zeta_B)\hat{\chi}_{AB} - \frac{1}{2}\mathrm{tr}_{\not{g}}\chi(k_{AN} + \zeta_A) + \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}.$$

□

Proof of $L\mathrm{tr}_{\not{g}}\underline{\chi} + \frac{1}{2}(\mathrm{tr}_{\not{g}}\chi)\mathrm{tr}_{\not{g}}\underline{\chi} = 2\not{\mathrm{div}}\,\underline{\zeta} + k_{NN}\mathrm{tr}_{\not{g}}\underline{\chi} - \hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + 2\left|\underline{\zeta}\right|^2_{\not{g}} + \mathbf{Riem}_{A\underline{L}LA}$ *in (8.65d).*

$$\not{D}_L\underline{\chi}_{AB} = L\underline{\chi}_{AB} - \underline{\chi}(\not{D}_L e_A, e_B) - \underline{\chi}(e_A, \not{D}_L e_B), \tag{8.74a}$$

$$L\underline{\chi}_{AB} = \langle \mathbf{D}_L\mathbf{D}_A\underline{L}, e_B\rangle + \langle \mathbf{D}_A\underline{L}, \mathbf{D}_L e_B\rangle, \tag{8.74b}$$

$$\langle \mathbf{D}_L\mathbf{D}_A\underline{L}, e_B\rangle = \langle \mathbf{D}_A\mathbf{D}_L\underline{L}, e_B\rangle + \langle \mathbf{D}_{[L,A]}\underline{L}, e_B\rangle + \mathbf{Riem}_{B\underline{L}LA}. \tag{8.74c}$$

$$\langle \mathbf{D}_A\mathbf{D}_L\underline{L}, e_B\rangle = 2\langle \mathbf{D}_A(\underline{\zeta}_C e_C), e_B\rangle + k_{NN}\langle \mathbf{D}_A\underline{L}, e_B\rangle \tag{8.74d}$$
$$= 2(e_A\langle \underline{\zeta}_C e_C, e_B\rangle - \langle \underline{\zeta}_C e_C, \mathbf{D}_A e_B\rangle) + k_{NN}\langle \underline{\chi}_{AD} e_D + k_{AN}\underline{L}, e_B\rangle$$
$$= 2\left(e_A(\underline{\zeta}_B) - \langle \underline{\zeta}_C e_C, \not{\nabla}_A e_B + \frac{1}{2}\chi_{AB}\underline{L} + \frac{1}{2}\underline{\chi}_{AB}L\rangle\right) + k_{NN}\underline{\chi}_{AB}$$
$$= 2\not{\nabla}_A\underline{\zeta}_B + k_{NN}\underline{\chi}_{AB}.$$

$$\langle \mathbf{D}_{[L,A]}\underline{L}, e_B\rangle = \langle \mathbf{D}_{\mathbf{D}_L A - \mathbf{D}_A L}\underline{L}, e_B\rangle \tag{8.74e}$$
$$= \langle \mathbf{D}_{\not{D}_L e_A}\underline{L}, e_B\rangle + (\underline{\zeta}_A + k_{AN})\langle 2\underline{\zeta}_C e_C + k_{NN}L, e_B\rangle - \chi_{AC}\underline{\chi}_{BC}$$
$$= \underline{\chi}(e_B, \not{D}_L e_A) - \chi_{AC}\underline{\chi}_{BC}.$$

$$\langle \mathbf{D}_A\underline{L}, \mathbf{D}_L e_B\rangle = \langle \mathbf{D}_A\underline{L}, \not{D}_L e_B + \underline{\zeta}_B L\rangle = \underline{\chi}(e_A, \not{D}_L e_B) - 2k_{AN}\underline{\zeta}_B. \tag{8.74f}$$

Combine above equations, we have

$$\not{D}_L\underline{\chi}_{AB} = 2\not{\nabla}_A\underline{\zeta}_B + k_{NN}\underline{\chi}_{AB} - \chi_{AC}\underline{\chi}_{BC} + 2\underline{\zeta}_A\underline{\zeta}_B + \mathbf{Riem}_{B\underline{L}LA}. \tag{8.75}$$

Take $\not{g}$-trace,

$$L\mathrm{tr}_{\not{g}}\underline{\chi} = 2\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\underline{\zeta} + k_{NN}\mathrm{tr}_{\not{g}}\underline{\chi} - \chi_{AC}\underline{\chi}_{AC} + 2\left|\underline{\zeta}\right|^2_{\not{g}} + \mathbf{Riem}_{A\underline{L}LA} \tag{8.76}$$

$$\begin{aligned}\chi_{AC}\underline{\chi}_{AC} &= \left(\hat{\chi}_{AC} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\not{g}_{AC}\right)\left(\underline{\hat{\chi}}_{AC} + \frac{1}{2}\mathrm{tr}_{\not{g}}\underline{\chi}\not{g}_{AC}\right) \\ &= \hat{\chi}_{AC}\underline{\hat{\chi}}_{AC} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\mathrm{tr}_{\not{g}}\underline{\chi}.\end{aligned} \tag{8.77}$$

Here we used the fact that $\hat{\chi}$ and $\underline{\hat{\chi}}$ are traceless. □

Proof of $\not{D}_L\underline{\hat{\chi}}_{AB} + \frac{1}{2}(\mathrm{tr}_{\not{g}}\chi)\underline{\hat{\chi}}_{AB}$ *in (8.65e).* By (8.75),

$$\begin{aligned}\not{D}_L\underline{\hat{\chi}}_{AB} + \frac{1}{2}L\mathrm{tr}_{\not{g}}\underline{\chi}\not{g}_{AB} =& 2\not{\nabla}_A\underline{\zeta}_B + k_{NN}\left(\underline{\hat{\chi}}_{AB} + \frac{1}{2}\mathrm{tr}_{\not{g}}\underline{\chi}\not{g}_{AB}\right) \\ & - \left(\hat{\chi}_{AC} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\not{g}_{AC}\right)\left(\underline{\hat{\chi}}_{BC} + \frac{1}{2}\mathrm{tr}_{\not{g}}\underline{\chi}\not{g}_{BC}\right) + 2\underline{\zeta}_A\underline{\zeta}_B + \mathbf{Riem}_{B\underline{L}LA}.\end{aligned} \tag{8.78a}$$

$$\begin{aligned}\left(\hat{\chi}_{AC} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\not{g}_{AC}\right)\left(\underline{\hat{\chi}}_{BC} + \frac{1}{2}\mathrm{tr}_{\not{g}}\underline{\chi}\not{g}_{BC}\right) =& \hat{\chi}_{AC}\underline{\hat{\chi}}_{BC} + \frac{1}{2}\underline{\hat{\chi}}_{AB}\mathrm{tr}_{\not{g}}\chi \\ & + \frac{1}{2}\hat{\chi}_{AB}\mathrm{tr}_{\not{g}}\underline{\chi} + \frac{1}{4}\not{g}_{AB}\mathrm{tr}_{\not{g}}\chi\mathrm{tr}_{\not{g}}\underline{\chi}.\end{aligned} \tag{8.78b}$$

By using (8.65d), we have

$$\begin{aligned}&\not{D}_L\underline{\hat{\chi}}_{AB} + \frac{1}{2}\not{g}_{AB}\left(2\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\underline{\zeta} + k_{NN}\mathrm{tr}_{\not{g}}\underline{\chi} - \hat{\chi}_{CD}\underline{\hat{\chi}}_{CD} + 2\left|\underline{\zeta}\right|^2_{\not{g}} + \mathbf{Riem}_{C\underline{L}LC}\right) \\ =& 2\not{\nabla}_A\underline{\zeta}_B + k_{NN}\left(\underline{\hat{\chi}}_{AB} + \frac{1}{2}\mathrm{tr}_{\not{g}}\underline{\chi}\not{g}_{AB}\right) - \hat{\chi}_{AC}\underline{\hat{\chi}}_{BC} - \frac{1}{2}\underline{\hat{\chi}}_{AB}\mathrm{tr}_{\not{g}}\chi \\ & - \frac{1}{2}\hat{\chi}_{AB}\mathrm{tr}_{\not{g}}\underline{\chi} + 2\underline{\zeta}_A\underline{\zeta}_B + \mathbf{Riem}_{B\underline{L}LA}.\end{aligned} \tag{8.79}$$

We get the desired equation after absorbing cancellations. □

Proof of $\underline{L}\mathrm{tr}_{\not{g}}\chi + \frac{1}{2}\mathrm{tr}_{\not{g}}\underline{\chi}\mathrm{tr}_{\not{g}}\chi = 2\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\zeta + k_{NN}\mathrm{tr}_{\not{g}}\chi - \hat{\chi}_{AB}\underline{\hat{\chi}}_{AB} + 2\left|\zeta\right|^2_{\not{g}} + \mathbf{Riem}_{AL\underline{L}A}$ *in (8.65f).*

$$\not{D}_{\underline{L}}\chi_{AB} = \underline{L}(\chi_{AB}) - \hat{\chi}(\not{D}_{\underline{L}}e_A, B) - \chi(A,,\not{D}_{\underline{L}}e_B) \tag{8.80a}$$

$$\underline{L}(\chi_{AB}) = \langle \mathbf{D}_{\underline{L}}\mathbf{D}_A L, e_B\rangle + \langle \mathbf{D}_A L, \mathbf{D}_{\underline{L}}e_B\rangle \tag{8.80b}$$

$$\langle \mathbf{D}_{\underline{L}}\mathbf{D}_A L, e_B\rangle = \mathbf{Riem}_{BL\underline{L}A} + \langle \mathbf{D}_A\mathbf{D}_{\underline{L}}L, e_B\rangle + \langle \mathbf{D}_{[\underline{L},A]}L, e_B\rangle. \tag{8.80c}$$

$$\begin{aligned}\langle \mathbf{D}_A \mathbf{D}_{\underline{L}} L, e_B\rangle &= 2\langle \mathbf{D}_A(\zeta_C e_C), e_B\rangle + k_{NN}\chi_{AB} \\ &= 2e_A(\zeta_B) - 2\langle \zeta_C e_C, \mathbf{D}_A e_B\rangle + k_{NN}\chi_{AB} \\ &= 2e_A(\zeta_B) - 2\langle \zeta_C e_C, \nabla\!\!\!/_A e_B\rangle + k_{NN}\chi_{AB} \\ &= 2\nabla\!\!\!/_A \zeta_B + + k_{NN}\chi_{AB}.\end{aligned} \tag{8.80d}$$

$$\begin{aligned}\langle \mathbf{D}_{[\underline{L},A]} L, e_B\rangle &= \langle \mathbf{D}_{\mathbf{D}_{\underline{L}} e_A - \mathbf{D}_A \underline{L}} L, e_B\rangle \\ &= \langle \mathbf{D}_{\mathbf{D}\!\!\!/_{\underline{L}} e_A + \zeta_A \underline{L} - \underline{\chi}_{AC} e_C - k_{AN}\underline{L}} L, e_B\rangle \\ &= \chi(\mathbf{D}\!\!\!/_{\underline{L}} e_A, B) + 2\zeta_A\zeta_B - \underline{\chi}_{AC}\chi_{BC} - 2k_{AN}\zeta_B.\end{aligned} \tag{8.80e}$$

$$\langle \mathbf{D}_A L, \mathbf{D}_{\underline{L}} e_B\rangle = \langle \mathbf{D}_A L, \mathbf{D}\!\!\!/_{\underline{L}} e_B + \zeta_B \underline{L}\rangle = \chi(A, \mathbf{D}\!\!\!/_{\underline{L}} e_B) + 2k_{AN}\zeta_B. \tag{8.80f}$$

Combining above equations, we have

$$\mathbf{D}\!\!\!/_{\underline{L}} \chi_{AB} = 2\nabla\!\!\!/_A \zeta_B + k_{NN}\chi_{AB} - \underline{\chi}_{AC}\chi_{BC} + 2\zeta_A\zeta_B + \mathbf{Riem}_{BL\underline{L}A}. \tag{8.81}$$

Taking $g\!\!\!/$-trace, we have

$$\underline{L}\mathrm{tr}_{g\!\!\!/}\chi = 2\mathrm{di}\!\!\!/\mathrm{v}\,\zeta + k_{NN}\mathrm{tr}_{g\!\!\!/}\chi - \chi_{AB}\underline{\chi}_{AB} + 2|\zeta|^2_{g\!\!\!/} + \mathbf{Riem}_{AL\underline{L}A}. \tag{8.82}$$

We obtain the desired equation by using (8.77). □

Proof of $\mathbf{D}\!\!\!/_{\underline{L}}\hat{\chi}_{AB} + \frac{1}{2}(\mathrm{tr}_{g\!\!\!/}\underline{\chi})\hat{\chi}_{AB}$ *in (8.65g).* By (8.81), we have

$$\begin{aligned}\mathbf{D}\!\!\!/_{\underline{L}}\hat{\chi}_{AB} + \frac{1}{2}\underline{L}\mathrm{tr}_{g\!\!\!/}\chi g\!\!\!/_{AB} =& 2\nabla\!\!\!/_A\zeta_B + k_{NN}\left(\hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi g\!\!\!/_{AB}\right) \\ &- \left(\hat{\underline{\chi}}_{AC} + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi} g\!\!\!/_{AC}\right)\left(\hat{\chi}_{BC} + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi g\!\!\!/_{BC}\right) + 2\zeta_A\zeta_B + \mathbf{Riem}_{BL\underline{L}A}.\end{aligned} \tag{8.83}$$

Use (8.82) and (8.78b),

$$\begin{aligned}&\mathbf{D}\!\!\!/_{\underline{L}}\hat{\chi}_{AB} + \frac{1}{2}g\!\!\!/_{AB}\left(2\mathrm{di}\!\!\!/\mathrm{v}\,\zeta + k_{NN}\mathrm{tr}_{g\!\!\!/}\chi - \hat{\chi}_{CD}\hat{\underline{\chi}}_{CD} + 2|\zeta|^2_{g\!\!\!/} + \mathbf{Riem}_{CL\underline{L}C}\right) \\ =& 2\nabla\!\!\!/_A\zeta_B + k_{NN}\left(\hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi g\!\!\!/_{AB}\right) - \hat{\underline{\chi}}_{AC}\hat{\chi}_{BC} \\ &- \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi g\!\!\!/_{BC}\hat{\underline{\chi}}_{AC} - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi} g\!\!\!/_{AC}\hat{\chi}_{BC} + 2\zeta_A\zeta_B + \mathbf{Riem}_{BL\underline{L}A}.\end{aligned} \tag{8.84}$$

□

Proof of $\operatorname{d\!i\!/\!v}\hat\chi_A + \hat\chi_{AB}k_{BN} = \frac{1}{2}(\nabla\!\!\!/_A \mathrm{tr}_{g\!\!\!/}\chi + k_{AN}\mathrm{tr}_{g\!\!\!/}\chi) + \mathbf{Riem}_{BLBA}$ *in (8.65h).*

$$\nabla\!\!\!/_C\chi_{AB} = e_C(\chi_{AB}) - \chi(\nabla\!\!\!/_C e_A, B) - \chi(A, \nabla\!\!\!/_C e_B), \tag{8.85a}$$

$$e_C(\chi_{AB}) = \langle \mathbf{D}_C\mathbf{D}_A L, e_B\rangle + \langle \mathbf{D}_A L, \mathbf{D}_C e_B\rangle, \tag{8.85b}$$

$$\langle \mathbf{D}_C\mathbf{D}_A L, e_B\rangle = \langle \mathbf{D}_A\mathbf{D}_C L, e_B\rangle + \langle \mathbf{D}_{[e_C,e_A]} L, e_B\rangle + \mathbf{Riem}_{BLCA}, \tag{8.85c}$$

$$\langle \mathbf{D}_A\mathbf{D}_C L, e_B\rangle = e_A(\chi_{BC}) - \langle \mathbf{D}_C L, \mathbf{D}_A e_B\rangle, \tag{8.85d}$$

$$\langle \mathbf{D}_C L, \mathbf{D}_A e_B\rangle = \langle \mathbf{D}_C L, \nabla\!\!\!/_A e_B + \frac{1}{2}\chi_{AB}\underline{L} + \frac{1}{2}\underline{\chi}_{AB} L\rangle = \chi(C, \nabla\!\!\!/_A e_B) + k_{CN}\chi_{AB}, \tag{8.85e}$$

$$\langle \mathbf{D}_{[e_C,e_A]} L, e_B\rangle = \langle \mathbf{D}_{\nabla\!\!\!/_C e_A - \nabla\!\!\!/_A e_C} L, e_B\rangle = \chi(B, \nabla\!\!\!/_C e_A) - \chi(B, \nabla\!\!\!/_A e_C). \tag{8.85f}$$

Combining above equations,

$$\begin{aligned}\nabla\!\!\!/_C\chi_{AB} =& e_A(\chi_{BC}) - \chi(B, \nabla\!\!\!/_A e_C) + k_{AN}\chi_{BC}\\ & - \chi(C, \nabla\!\!\!/_A e_B) - k_{CN}\chi_{AB} - \chi(A, \nabla\!\!\!/_C e_B) + \mathbf{Riem}_{BLCA}\\ =& \nabla\!\!\!/_A\chi_{BC} + k_{AN}\chi_{BC} - k_{CN}\chi_{AB} - \chi(A, \nabla\!\!\!/_C e_B) + \mathbf{Riem}_{BLCA}.\end{aligned} \tag{8.86}$$

Writing χ as the sum of trace part and trace-free part, we have

$$\begin{aligned}\nabla\!\!\!/_C\left(\hat\chi_{AB} + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi g\!\!\!/_{AB}\right) =& \nabla\!\!\!/_A\chi_{BC} + k_{AN}\chi_{BC}\\ & - k_{CN}\left(\hat\chi_{AB} + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi g\!\!\!/_{AB}\right) - \chi(A, \nabla\!\!\!/_C e_B) + \mathbf{Riem}_{BLCA}.\end{aligned} \tag{8.87}$$

Taking trace, we have

$$\operatorname{d\!i\!/\!v}\hat\chi_A + \frac{1}{2}\nabla\!\!\!/_A\mathrm{tr}_{g\!\!\!/}\chi = \nabla\!\!\!/_A\mathrm{tr}_{g\!\!\!/}\chi + k_{AN}\mathrm{tr}_{g\!\!\!/}\chi - k_{BN}\hat\chi_{AB} - \frac{1}{2}k_{AN}\mathrm{tr}_{g\!\!\!/}\chi + \mathbf{Riem}_{BLBA}. \tag{8.88}$$

□

Proof of $\operatorname{d\!i\!/\!v}\zeta = \frac{1}{2}(\mu - k_{NN}\mathrm{tr}_{g\!\!\!/}\chi - 2|\zeta|^2_{g\!\!\!/} - |\hat\chi|^2_{g\!\!\!/} - 2k_{AB}\hat\chi_{AB}) - \frac{1}{2}\mathbf{Riem}_{A\underline{L}LA}$ *in (8.65i).* Recall definition of μ (8.53) and equation (8.82):

$$\underline{L}\mathrm{tr}_{g\!\!\!/}\chi = 2\operatorname{d\!i\!/\!v}\zeta + k_{NN}\mathrm{tr}_{g\!\!\!/}\chi - \chi_{AB}\underline{\chi}_{AB} + 2|\zeta|^2_{g\!\!\!/} + \mathbf{Riem}_{AL\underline{L}A}. \tag{8.89}$$

Notice that

$$-\chi_{AB}\underline{\chi}_{AB} = -\chi_{AB}(-2k_{AB} - \chi_{AB}) \tag{8.90}$$
$$= \left(\hat{\chi}_{AB} + \frac{1}{2}\not{g}_{AB}\mathrm{tr}_{\not{g}}\chi\right)\left(2k_{AB} + \left(\hat{\chi}_{AB} + \frac{1}{2}\not{g}_{AB}\mathrm{tr}_{\not{g}}\chi\right)\right)$$
$$= 2k_{AB}\hat{\chi}_{AB} + \mathrm{tr}_{\not{g}}k\mathrm{tr}_{\not{g}}\chi + |\hat{\chi}|^2_{\not{g}} + \frac{1}{2}(\mathrm{tr}_{\not{g}}\chi)^2$$
$$= 2k_{AB}\hat{\chi}_{AB} - \frac{1}{2}(\mathrm{tr}_{\not{g}}\chi + \mathrm{tr}_{\not{g}}\underline{\chi})\mathrm{tr}_{\not{g}}\chi + |\hat{\chi}|^2_{\not{g}} + \frac{1}{2}(\mathrm{tr}_{\not{g}}\chi)^2$$
$$= 2k_{AB}\hat{\chi}_{AB} - \frac{1}{2}\mathrm{tr}_{\not{g}}\underline{\chi}\mathrm{tr}_{\not{g}}\chi + |\hat{\chi}|^2_{\not{g}}.$$

Using above fact and (8.82), we obtain the desired equation. □

Proof of $\mathrm{cu\not{r}l}\zeta = \frac{1}{2}\varepsilon^{AB}\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC} - \frac{1}{2}\varepsilon^{AB}\mathbf{Riem}_{BL\underline{L}A}$ *in (8.65j).* By (8.80a), we know $\not{\mathbf{D}}_{\underline{L}}\chi_{AB}$ is $S_{t,u}$ symmetric tensor. Contract ε^{AB} against equation (8.81), we have

$$2\mathrm{cu\not{r}l}\zeta = \varepsilon^{AB}\left(\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC} + \frac{1}{2}\hat{\chi}_{AB}\mathrm{tr}_{\not{g}}\underline{\chi} + \frac{1}{2}\hat{\underline{\chi}}_{AB}\mathrm{tr}_{\not{g}}\chi + \frac{1}{4}\not{g}_{AB}\mathrm{tr}_{\not{g}}\chi\mathrm{tr}_{\not{g}}\underline{\chi}\right) - \varepsilon^{AB}\mathbf{Riem}_{BL\underline{L}A} \tag{8.91}$$
$$= \varepsilon^{AB}\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC} - \varepsilon^{AB}\mathbf{Riem}_{BL\underline{L}A}. \tag{8.92}$$

□

8.4 Curvature Decompositions

In this subsection, we decompose Riemann and Ricci curvature tensor components with the help of geometric structure equations and the relativistic Euler equations. Then, in Prop. 8.20, we rewrite the PDEs from Lemma 8.14 into the version that will be used in the analysis.

Lemma 8.15. *[9, Lemma 9.5. Identities for the derivatives of some scalar functions] With* $\mathrm{d}f$ *denoting the space gradient of the scalar function* f*, we have the following identities*

$$\not{\Pi}\cdot\mathrm{d}(\vec{L},\underline{\vec{L}},N) = \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right), \tag{8.93a}$$
$$\not{\Pi}\cdot\mathrm{d}\mathrm{f}_{(\vec{L})} = \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right), \tag{8.93b}$$
$$\mathrm{d}(\vec{L},\underline{\vec{L}},N) = \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(Small)}, \hat{\chi}, \zeta, \widetilde{r}^{-1}\right), \tag{8.93c}$$
$$\mathrm{d}\mathrm{f}_{(\vec{L})} = \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(Small)}, \hat{\chi}, \zeta, \widetilde{r}^{-1}\right). \tag{8.93d}$$

Where $\mathrm{f}_{(\vec{L})}$ *are polynomials of Cartesian components of* L *with smooth functions of* $\vec{\Psi}$ *as coefficients.*

In the proof of Lemma 8.15, we (secretly) use (8.16) whenever they are needed.

Proof of $d\vec{L} = \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\not{g}}\tilde{\chi}^{(Small)}, \hat{\chi}, \zeta, \widetilde{r}^{-1}\right)$. Notice that $\mathbf{T} = \mathrm{f}(\vec{\Psi})$ and $(k_{ij}, \boldsymbol{\Gamma}) = \mathrm{f} \cdot \boldsymbol{\partial}\vec{\Psi}$. Also, we have $(N, \underline{\vec{L}}) = \mathrm{f}_{(\vec{L})}$ and $\boldsymbol{\Gamma}_L = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, hence

$$\mathbf{D}_\beta L^\alpha = \partial_\beta L^\alpha + \Gamma^\alpha_{\beta\gamma} L^\gamma, \tag{8.94}$$

$$\begin{aligned} \mathbf{D} L^\alpha &= \langle \mathbf{D} L^\alpha, e_A \rangle e_A - \frac{1}{2}\langle \mathbf{D} L^\alpha, L \rangle \underline{L} - \frac{1}{2}\langle \mathbf{D} L^\alpha, \underline{L} \rangle L \\ &= (\mathbf{D}_A L^\alpha)\, e_A - \left(\frac{1}{2}\mathbf{D}_L L^\alpha\right) \underline{L} - \left(\frac{1}{2}\mathbf{D}_{\underline{L}} L^\alpha\right) L \\ &= (\chi_{AB} e^\alpha_B - k_{AN} L^\alpha)\, e_A + \left(\frac{1}{2}k_{NN} L^\alpha\right) \underline{L} - \left(\zeta_A e^\alpha_A + \frac{1}{2}k_{NN} L^\alpha\right) L \\ &= \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta, \widetilde{r}^{-1}\right). \end{aligned} \tag{8.95}$$

Here we use the fact that

$$\langle \mathbf{D} L^\alpha, X \rangle = \mathbf{g}^{\beta\gamma} \mathbf{D}_\beta L^\alpha X_\gamma = \mathbf{D}_\beta L^\alpha X^\beta = \mathbf{D}_X L^\alpha. \tag{8.96}$$

□

It's obvious from above that

$$\not{\Pi} \cdot d\vec{L} = \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right). \tag{8.97}$$

We obtain the identities that involve N and $\underline{\vec{L}}$ by chain rule and $(N, \underline{\vec{L}}) = \mathrm{f}_{(\vec{L})}$.

Lemma 8.16. *[14, Lemma 2.1. Ricci curvature component decompostions] We decompose the Ricci curvature tensor components as follows. Here we view* $\mathbf{g}_{\alpha\beta}$ *as a scalar function.*

$$\mathbf{Ric}_{\alpha\beta} = -\frac{1}{2}\square_{\mathbf{g}} \mathbf{g}_{\alpha\beta}(\vec{\Psi}) + \frac{1}{2}\left(\mathbf{D}_\alpha \boldsymbol{\Gamma}_\beta + \mathbf{D}_\beta \boldsymbol{\Gamma}_\alpha\right) + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{8.98}$$

Proof of $\mathbf{Ric}_{\alpha\beta}$.

$$\begin{aligned} \mathbf{Riem}_{\rho\alpha\mu\beta} &= \langle \left(\mathbf{D}_\mu \mathbf{D}_\beta - \mathbf{D}_\beta \mathbf{D}_\mu\right) \partial_\alpha, \partial_\rho \rangle \\ &= \langle \partial_\rho, \left(\mathbf{D}_\mu \boldsymbol{\Gamma}^\gamma_{\alpha\beta}\right) \partial_\gamma + \boldsymbol{\Gamma}^\gamma_{\alpha\beta} \boldsymbol{\Gamma}^\sigma_{\mu\gamma} \partial_\sigma - \left(\mathbf{D}_\beta \boldsymbol{\Gamma}^\delta_{\mu\alpha}\right) \partial_\delta - \boldsymbol{\Gamma}^\kappa_{\beta\delta} \partial_\kappa \rangle \\ &= \mathbf{g}_{\rho\gamma} \left(\mathbf{D}_\mu \boldsymbol{\Gamma}^\gamma_{\alpha\beta}\right) + \mathbf{g}_{\rho\sigma} \boldsymbol{\Gamma}^\gamma_{\alpha\beta} \boldsymbol{\Gamma}^\sigma_{\mu\gamma} - \mathbf{g}_{\rho\delta} \left(\mathbf{D}_\beta \boldsymbol{\Gamma}^\delta_{\mu\alpha}\right) - \mathbf{g}_{\kappa\rho} \boldsymbol{\Gamma}^\delta_{\mu\alpha} \boldsymbol{\Gamma}^\kappa_{\beta\delta} \end{aligned} \tag{8.99}$$

$$= \partial_\mu \partial_\alpha \mathbf{g}_{\rho\beta} + \partial_\beta \partial_\rho \mathbf{g}_{\mu\alpha} - \partial_\mu \partial_\rho \mathbf{g}_{\alpha\beta} - \partial_\alpha \partial_\beta \mathbf{g}_{\rho\mu} + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{8.100}$$

$$\mathbf{Ric}_{\alpha\beta} = \mathbf{g}^{\rho\mu}\mathbf{Riem}_{\rho\alpha\mu\beta} \tag{8.101}$$
$$= \mathbf{D}_\mu \mathbf{\Gamma}^\mu_{\alpha\beta} + \mathbf{\Gamma}^\gamma_{\alpha\beta}\mathbf{\Gamma}^\mu_{\mu\gamma} - \mathbf{D}_\beta\mathbf{\Gamma}^\mu_{\mu\alpha} - \mathbf{\Gamma}^\delta_{\mu\alpha}\mathbf{\Gamma}^\mu_{\beta\delta}.$$

$$\begin{aligned}\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta} =& \mathbf{g}^{\mu\rho}\mathbf{D}_\mu\mathbf{D}_\rho\left(\mathbf{g}_{\alpha\beta}\right) \\ =& \mathbf{g}^{\mu\rho}\mathbf{D}_\mu\left(\langle \mathbf{\Gamma}^\sigma_{\rho\alpha}\partial_\sigma, \partial_\beta\rangle + \langle\partial_\alpha, \mathbf{\Gamma}^\kappa_{\rho\beta}\partial_\kappa\rangle\right) \\ =& \mathbf{g}^{\mu\rho}\left(\mathbf{D}_\mu\mathbf{\Gamma}^\sigma_{\rho\alpha}\right)\mathbf{g}_{\sigma\beta} + \mathbf{\Gamma}^\sigma_{\rho\alpha}\mathbf{\Gamma}^\gamma_{\mu\sigma}\mathbf{g}_{\gamma\beta} \\ & + \mathbf{\Gamma}^\sigma_{\rho\alpha}\mathbf{\Gamma}^\delta_{\mu\beta}\mathbf{g}_{\sigma\delta} + \mathbf{g}^{\mu\rho}\left(\mathbf{D}_\mu\mathbf{\Gamma}^\kappa_{\rho\beta}\right)\mathbf{g}_{\alpha\kappa} + \mathbf{\Gamma}^\kappa_{\rho\beta}\mathbf{\Gamma}^\gamma_{\alpha\mu}\mathbf{g}_{\gamma\kappa} + \mathbf{\Gamma}^\kappa_{\rho\beta}\mathbf{\Gamma}^\delta_{\mu\kappa}\mathbf{g}_{\alpha\delta}.\end{aligned} \tag{8.102}$$

$$\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha = \mathbf{g}^{\rho\mu}\mathbf{g}_{\kappa\beta}\mathbf{D}_\alpha\mathbf{\Gamma}^\kappa_{\rho\mu} + \mathbf{g}^{\rho\mu}\mathbf{g}_{\kappa\alpha}\mathbf{D}_\beta\mathbf{\Gamma}^\kappa_{\rho\mu}. \tag{8.103}$$

We only need to compare the highest order terms since the terms of the form $\mathbf{\Gamma}\cdot\mathbf{\Gamma}$ are absorbed by $\mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]$.

$$\begin{aligned}& \mathbf{g}_{\mu\gamma}\left(\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha - \square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}\right) \\ =& \partial_\alpha\left(\partial_\gamma\mathbf{g}_{\beta\mu} + \partial_\mu\mathbf{g}_{\gamma\beta} - \partial_\beta\mathbf{g}_{\gamma\mu}\right) + \partial_\beta\left(\partial_\mu\mathbf{g}_{\alpha\gamma} + \partial_\alpha\mathbf{g}_{\mu\gamma} - \partial_\gamma\mathbf{g}_{\alpha\mu}\right) + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] \\ & - \partial_\mu\left(\partial_\gamma\mathbf{g}_{\alpha\beta} + \partial_\alpha\mathbf{g}_{\gamma\beta} - \partial_\gamma\mathbf{g}_{\alpha\beta} + \partial_\beta\mathbf{g}_{\alpha\gamma} - \partial_\alpha\mathbf{g}_{\gamma\beta}\right) + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] \\ =& 2\mathbf{D}_\mu\mathbf{\Gamma}_{\gamma\alpha\beta} - \mathbf{D}_\beta\mathbf{\Gamma}_{\gamma\alpha\mu} - \mathbf{D}_\alpha\mathbf{\Gamma}_{\gamma\mu\beta} + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] \\ =& 2\mathbf{g}_{\mu\gamma}\mathbf{Ric}_{\alpha\beta} + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}].\end{aligned} \tag{8.104}$$

Here we used the fact that Ricci curvature tensor is symmetric. □

Corollary 8.17. *[14, 36, 9] Let ξ denote schematically the $S_{t,u}$-tangent one-forms and symmetric $\binom{0}{2}$ $S_{t,u}$-tangent tensors with the property $\xi = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$[1]. We note that $\vec{\Psi}$ is the rescaled solution variable defined in Definition 6.3. Then with the help of (8.98), we can decompose following Ricci curvature tensor components:*

$$\mathbf{Ric}_{LL} = L\mathbf{\Gamma}_L + k_{NN}\mathbf{\Gamma}_L + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{8.105a}$$
$$\mathbf{Ric}_{L\underline{L}} = \frac{1}{2}(L\mathbf{\Gamma}_{\underline{L}} + \underline{L}\mathbf{\Gamma}_L) + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, \zeta)\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{8.105b}$$
$$\mathbf{Ric}_{LA} = \nabla\!\!\!/_A\xi^{(1)} + \mathcal{D}\!\!\!/_L\xi^{(2)}_A + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{8.105c}$$
$$\mathbf{Ric}_{AB} = \nabla\!\!\!/_A\xi_B + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.105d}$$

[1]We use the symbol ξ in the same way in the rest of the article.

In the proof of all identities in Corollary 8.17, we use (8.98).

Proof of $\mathbf{Ric}_{LL} = L\mathbf{\Gamma}_L + k_{NN}\mathbf{\Gamma}_L + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}$. By chain rule and equation (6.9), we have

$$-\frac{1}{2}\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi})L^\alpha L^\beta = \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.106}$$

$$\frac{1}{2}\left(\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha\right)L^\alpha L^\beta = L\mathbf{\Gamma}_L - \mathbf{D}_L(L^\beta)\mathbf{\Gamma}_\beta = L\mathbf{\Gamma}_L + \left(k_{NN}L^\beta + \mathbf{\Gamma}^\beta_{LL}\right)\mathbf{\Gamma}_\beta. \tag{8.107}$$

□

Proof of $\mathbf{Ric}_{L\underline{L}} = \frac{1}{2}(L\mathbf{\Gamma}_{\underline{L}} + \underline{L}\mathbf{\Gamma}_L) + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta)\cdot\boldsymbol{\partial}\vec{\Psi}$.

$$\begin{aligned}\left(\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha\right)L^\alpha\underline{L}^\beta =&(L\mathbf{\Gamma}_{\underline{L}} + \underline{L}\mathbf{\Gamma}_L) - \mathbf{D}_L(\underline{L}^\beta)\mathbf{\Gamma}_\beta - \mathbf{D}_{\underline{L}}(L^\alpha)\mathbf{\Gamma}_\alpha \\ =&(L\mathbf{\Gamma}_{\underline{L}} + \underline{L}\mathbf{\Gamma}_L) - \left(2\underline{\zeta}_A e^\beta_A - k_{NN}\underline{L}^\beta + \mathbf{\Gamma}^\beta_{L\underline{L}}\right)\mathbf{\Gamma}_\beta \\ &- \left(2\zeta_A e^\alpha_A - k_{NN}L^\alpha + \mathbf{\Gamma}^\alpha_{\underline{L}L}\right)\mathbf{\Gamma}_\alpha.\end{aligned} \tag{8.108}$$

□

Proof of $\mathbf{Ric}_{LA} = (\nabla\!\!\!/,\mathbf{D}\!\!\!/_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}$.

$$\mathbf{D}_\alpha\mathbf{\Gamma}_\beta L^\alpha e^\beta_A = \mathbf{D}\!\!\!/_L\xi_A \tag{8.109}$$

$$\mathbf{D}_\beta\mathbf{\Gamma}_\alpha L^\alpha e^\beta_A = e_A(\mathbf{\Gamma}_L) - \mathbf{D}_A(L^\alpha)\mathbf{\Gamma}_\alpha = \nabla\!\!\!/_A\xi - (\chi_{AB}e^\alpha_B - k_{AN}L^\alpha - \mathbf{\Gamma}^\alpha_{AL})\mathbf{\Gamma}_\alpha. \tag{8.110}$$

□

Proof of $\mathbf{Ric}_{AB} = \nabla\!\!\!/_A\xi_B + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}$.

$$\mathbf{D}_A\mathbf{\Gamma}_B = \nabla\!\!\!/_A\xi_B. \tag{8.111}$$

□

Lemma 8.18 (Curvature component decomposition under frames)**.** *Let e_a, e_b, e_c, e_d be an arbitrary bounded frame. We can decompose Riemann Curvature as follows.*

$$\mathbf{Riem}_{abcd} = \mathbf{D}_a\pi_{bcd} + \mathbf{D}_b\pi_{dac} - \mathbf{D}_a\pi_{bdc} - \mathbf{D}_b\pi_{acd} + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}]. \tag{8.112}$$

where

$$\pi_{\alpha\beta\gamma} := \partial_\gamma \mathbf{g}_{\alpha\beta}. \tag{8.113}$$

Proof of Lemma 8.18.

$$\begin{aligned}
\mathbf{D}_a \pi_{bcd} =& b^\alpha c^\mu d^\beta \mathbf{D}_a(\partial_\beta \mathbf{g}_{\alpha\gamma}) \\
=& a^\rho \partial_\rho \left(b^\alpha c^\mu d^\beta \partial_\beta \mathbf{g}_{\alpha\mu} \right) - \pi_{\mathbf{D}_a bcd} - \pi_{b\mathbf{D}_a cd} - \pi_{bc\mathbf{D}_a d} \\
=& a^\rho b^\alpha c^\mu d^\beta \partial_\rho \partial_\beta \mathbf{g}_{\alpha\mu} + c^\mu d^\beta a^\rho \partial_\rho (b^\alpha) \partial_\beta \mathbf{g}_{\alpha\mu} + b^\alpha c^\mu a^\rho \partial_\rho (d^\beta) \partial_\beta \mathbf{g}_{\alpha\mu} + d^\beta b^\alpha a^\rho \partial_\rho (c^\mu) \partial_\beta \mathbf{g}_{\alpha\mu} \\
& - \pi_{\mathbf{D}_a bcd} - \pi_{b\mathbf{D}_a cd} - \pi_{bc\mathbf{D}_a d},
\end{aligned} \tag{8.114}$$

where, in1 particular,

$$\begin{aligned}
b^\alpha c^\mu a^\rho \partial_\rho (d^\beta) \partial_\beta \mathbf{g}_{\alpha\mu} - \pi_{bc\mathbf{D}_a d} &= b^\alpha c^\mu a^\rho \partial_\rho (d^\beta) \partial_\beta \mathbf{g}_{\alpha\mu} - b^\alpha c^\mu (\mathbf{D}_a d)^\beta \partial_\beta \mathbf{g}_{\alpha\mu} \\
&= b^\alpha c^\mu \partial_\beta \mathbf{g}_{\alpha\mu} \left(a^\rho \partial_\rho (d^\beta) - (\mathbf{D}_a d)^\beta \right) \\
&= b^\alpha c^\mu \partial_\beta \mathbf{g}_{\alpha\mu} \left(\mathbf{\Gamma}^\beta_{ad} \right).
\end{aligned} \tag{8.115}$$

Use similar argument to other π's, we have

$$\mathbf{D}_a \pi_{bcd} = a^\rho b^\alpha c^\mu d^\beta \partial_\rho \partial_\beta \mathbf{g}_{\alpha\mu} + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{8.116}$$

Using same method to all of the $\mathbf{D}\pi$ and comparing to (8.100), we obtain the desired equation. We also refer readers to [17, Proposition 4.1] for more details of the proof. □

Corollary 8.19. *With the help of (8.112) and Corollary 8.17, we can decompose following Riemann curvature* *tensor components:*

$$\mathbf{Riem}_{LALB} = (\nabla\!\!\!/, \mathbf{D}\!\!\!/_L)\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{8.117a}$$

$$\mathbf{Riem}_{CALB} = \nabla\!\!\!/\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{8.117b}$$

$$\mathbf{Riem}_{ABAB} = \mathrm{di}\!\!\!/\mathrm{v}\,\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{8.117c}$$

$$\mathbf{Riem}_{ABBL} = \nabla\!\!\!/_A\xi + \mathrm{di}\!\!\!/\mathrm{v}\,\xi_A + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{8.117d}$$

$$\mathbf{Riem}_{ALL\underline{L}} = (\nabla\!\!\!/, \mathbf{D}\!\!\!/_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{8.117e}$$

$$\mathbf{Riem}_{AL\underline{L}A} = \mathrm{di}\!\!\!/\mathrm{v}\,\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{8.117f}$$

$$\varepsilon^{AB}\mathbf{Riem}_{AL\underline{L}B} = \mathrm{cu}\!\!\!/\mathrm{rl}\,\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}. \tag{8.117g}$$

In the proof of Corollary 8.19, we use the same notation as Lemma 8.18. In particular, recall (8.113) for definition of π. Also, we (secretly) use (8.16) whenever they are needed. We also refer readers to [9, Lemma 9.6] for more details of the proof.

Proof of $\mathbf{Riem}_{LALB} = (\nabla\!\!\!/, \mathbf{D}\!\!\!/_L)\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}$.

$$\begin{aligned}\mathbf{Riem}_{LALB} =& \mathbf{D}_L\pi_{ALB} + \mathbf{D}_A\pi_{BLL} - \mathbf{D}_L\pi_{ABL} - \mathbf{D}_A\pi_{LLB} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ =& \mathbf{D}\!\!\!/_L\xi^{(1)}_{AB} + \nabla\!\!\!/_A\xi^{(2)}_B - \pi_{A\mathbf{D}_L LB} - \pi_{B\mathbf{D}_A LL} - \pi_{BL\mathbf{D}_A L} + \pi_{AB\mathbf{D}_L L} \\ & + 2\pi_{\mathbf{D}_A LLB} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ =& (\nabla\!\!\!/, \mathbf{D}\!\!\!/_L)\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}.\end{aligned} \tag{8.118}$$

Where

$$\xi^{(1)}_{\alpha\beta} = \pi_{\alpha L\beta} - \pi_{\alpha\beta L}, \qquad \xi^{(2)}_{\alpha} = \pi_{\alpha LL} - \pi_{LL\alpha}. \tag{8.119}$$

□

Proof of $\mathbf{Riem}_{CALB} = \nabla\!\!\!/\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}$.

$$\begin{aligned}\mathbf{Riem}_{CALB} &= \mathbf{D}_C\pi_{ALB} + \mathbf{D}_A\pi_{BCL} - \mathbf{D}_C\pi_{ABL} - \mathbf{D}_A\pi_{CLB} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ &= \nabla\!\!\!/_C\xi_{AB} - \nabla\!\!\!/_A\xi_{CB} - \pi_{A\mathbf{D}_C LB} - \pi_{BC\mathbf{D}_A L} + \pi_{AB\mathbf{D}_C L} + \pi_{C\mathbf{D}_A LB} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ &= \nabla\!\!\!/\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}.\end{aligned} \tag{8.120}$$

Where $\xi_{\alpha\beta} = \pi_{\alpha L\beta} - \pi_{\alpha\beta L}$. □

Proof of $\mathbf{Riem}_{ABAB} = \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}$.

$$\begin{aligned} \mathbf{Riem}_{ABAB} =& \mathbf{D}_A\pi_{BAB} + \mathbf{D}_B\pi_{BAA} - \mathbf{D}_A\pi_{BBA} - \mathbf{D}_B\pi_{AAB} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ =& \not{\nabla}_A\xi_A^{(1)} + \not{\nabla}_B\xi_B^{(2)} - \pi_{\not{\nabla}_A BAB} - \pi_{BA\not{\nabla}_A B} - \pi_{B\not{\nabla}_B AA} - \pi_{BA\not{\nabla}_B A} \\ &+ \pi_{\not{\nabla}_A BBA} + \pi_{B\not{\nabla}_A BA} + \pi_{\not{\nabla}_B AAB} + \pi_{A\not{\nabla}_B AB} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ =& \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\xi + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}. \end{aligned} \tag{8.121}$$

Where

$$\xi_\alpha^{(1)} = \pi_{B\alpha B} - \pi_{BB\alpha}, \qquad \xi_\alpha^{(2)} = \pi_{\alpha AA} - \pi_{AA\alpha}. \tag{8.122}$$

□

Proof of $\mathbf{Riem}_{ABBL} = \not{\nabla}_A\xi + \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\xi_A + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}$.

$$\begin{aligned} \mathbf{Riem}_{ABBL} =& \mathbf{D}_A\pi_{BBL} + \mathbf{D}_B\pi_{LAB} - \mathbf{D}_A\pi_{BLB} - \mathbf{D}_B\pi_{ABL} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ =& \not{\nabla}_A\xi^{(1)} - 2\pi_{\not{\nabla}_A BBL} - \pi_{BB\mathbf{D}_A L} + \pi_{\not{\nabla}_A BLB} + \pi_{BL\not{\nabla}_A B} \\ &+ \mathbf{D}_B(\pi_{LAB} - \pi_{ABL}) + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ =& \not{\nabla}_A\xi^{(1)} + \not{\nabla}_B\xi_{AB}^{(2)} - \pi_{BB\mathbf{D}_A L} - 2\pi_{\not{\nabla}_A BBL} \\ &+ \pi_{\not{\nabla}_A BLB} + \pi_{BL\not{\nabla}_A B} - \pi_{\mathbf{D}_B LAB} + \pi_{AB\mathbf{D}_B L} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi} \\ =& \not{\nabla}_A\xi^{(1)} + \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\xi_A^{(2)} + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}. \end{aligned} \tag{8.123}$$

Here we define $\binom{0}{2}$ $S_{t,u}$-tangent tensor $\xi_{\alpha\beta}^{(2)} := \pi_{L\alpha\beta} - \pi_{\alpha\beta L}$. □

Proof of $\mathbf{Riem}_{ALL\underline{L}} = (\not{\nabla}, \not{\mathbf{D}}_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}$.

$$\mathbf{Riem}_{ALL\underline{L}} = \mathbf{Ric}_{LA} + \mathbf{Riem}_{ABBL}. \tag{8.124}$$

Use (8.105c) and (8.117d),

$$\mathbf{Riem}_{ALL\underline{L}} = (\not{\nabla}, \not{\mathbf{D}}_L)\xi + \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\xi_A + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}. \tag{8.125}$$

□

Proof of $\mathbf{Riem}_{AL\underline{L}A} = \mathrm{d}\!\!/\mathrm{iv}\,\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}$.

$$\mathbf{Ric}_{AB} = -\frac{1}{2}\mathbf{Riem}_{ALB\underline{L}} - \frac{1}{2}\mathbf{Riem}_{A\underline{L}BL} + \mathbf{Riem}_{ACBC}. \tag{8.126}$$

$$\mathbf{Riem}_{AL\underline{L}A} = -\frac{1}{2}\delta^{AB}(\mathbf{Riem}_{ALB\underline{L}} + \mathbf{Riem}_{A\underline{L}BL}) = \delta^{AB}\mathbf{Ric}_{AB} - \delta^{AB}\mathbf{Riem}_{ACBC}. \tag{8.127}$$

By (8.105d) and (8.117c), we obtain the desired equation. □

Proof of $\varepsilon^{AB}\mathbf{Riem}_{AL\underline{L}B} = \mathrm{cu}\!\!/\mathrm{rl}\,\xi + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}$. By Bianchi identity,

$$\mathbf{Riem}_{A\underline{L}BL} + \mathbf{Riem}_{ABL\underline{L}} + \mathbf{Riem}_{AL\underline{L}B} = 0. \tag{8.128}$$

we have

$$\mathbf{Riem}_{A\underline{L}BL} - \mathbf{Riem}_{ALB\underline{L}} = -\mathbf{Riem}_{AB\underline{L}L}, \tag{8.129}$$

$$2\varepsilon^{AB}\mathbf{Riem}_{AL\underline{L}B} = \varepsilon^{AB}\mathbf{Riem}_{AB\underline{L}L}. \tag{8.130}$$

$$\begin{aligned}\mathbf{Riem}_{AB\underline{L}L} =& \mathbf{D}_A\pi_{B\underline{L}L} + \mathbf{D}_B\pi_{LA\underline{L}} - \mathbf{D}_A\pi_{BL\underline{L}} - \mathbf{D}_B\pi_{A\underline{L}L} + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi} \\ =& \nabla\!\!\!/_A\xi_B - \nabla\!\!\!/_B\xi_A - \pi_{B\mathbf{D}_A\underline{L}L} - \pi_{B\underline{L}\mathbf{D}_A L} - \pi_{\mathbf{D}_B LA\underline{L}} - \pi_{LA\mathbf{D}_B\underline{L}} \\ & + \pi_{B\mathbf{D}_A L\underline{L}} + \pi_{BL\mathbf{D}_A\underline{L}} + \pi_{A\mathbf{D}_B\underline{L}L} + \pi_{A\underline{L}\mathbf{D}_B L} + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi} \\ =& \nabla\!\!\!/_A\xi_B - \nabla\!\!\!/_B\xi_A + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}.\end{aligned} \tag{8.131}$$

Where $\xi_\alpha = \pi_{\alpha\underline{L}L} - \pi_{\alpha L\underline{L}}$

$$\varepsilon^{AB}\mathbf{Riem}_{AB\underline{L}L} = 2\mathrm{cu}\!\!/\mathrm{rl}\,\xi + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.132}$$

□

8.5 Main version of the PDEs Verified by the Acoustical Quantities

Proposition 8.20. *[9, Section 9.9.3. PDEs verified by the modified acoustical quantities] Recall that the vectorfields $L,\underline{L},N,e_A$ and nullapse b in Section 7.3. ω^A is the local coordinates in $\mathbb{S}^2$ (defined in Definition 8.11). Spherevolume ratio υ is defined in (8.46). Connection coefficients $\xi,\underline{\chi},\zeta$ are defined in Definition 8.5. Conformal factor σ and conformally modified connection coefficients $\tilde{\chi},\tilde{\underline{\chi}},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)}$ are define in Definition*

8.6. Mass aspect function μ *and its modified version* $\check{\mu}, \not{\mu}$ *and modified torsion* $\tilde{\zeta}$ *are defined in Definition 8.10. The following identities hold where the terms on the right hand sides are displayed schematically:*

Transport equations involving the Cartisian components L^i ***and*** N^i

$$LL^i = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}, \qquad LN^i = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}. \tag{8.133a}$$

Moreover, along Σ_0*(where* $w = \widetilde{r} = -u$ *and* $a = b$*, see Proposition 8.12), we have*

$$\frac{\partial}{\partial w} L^i = a \cdot \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} - \not{\nabla} a, \qquad \frac{\partial}{\partial w} N^i = a \cdot \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} - \not{\nabla} a. \tag{8.133b}$$

Transport equations involving the Cartesian components $\Theta^i_{(A)}$

$$\Theta^i_{(A)} := \frac{1}{\widetilde{r}} \left(\frac{\partial}{\partial \omega^A} \right)^i. \tag{8.134}$$

Then the following evolution equation hold in $\mathscr{M}$*:*

$$L\Theta^i_{(A)} = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}} \tilde{\chi}^{(Small)}, \hat{\chi}) \cdot \vec{\Theta}_{(A)}, \tag{8.135}$$

$$N \frac{\partial}{\partial \omega^A}^i = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \frac{\partial}{\partial \omega^A} + \not{\nabla}_A \ln b \cdot N. \tag{8.136}$$

Moreover, along Σ_0*,*

$$\frac{\partial}{\partial w} \Theta^i_{(A)} = a \cdot \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \hat{\chi}) \cdot \vec{\Theta}_{(A)} + \mathrm{f}_{(\vec{L})} \cdot \not{\nabla} a \cdot \vec{\Theta}_{(A)}. \tag{8.137}$$

Transport equations connected to the trace of χ

$$L\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)} + \frac{2}{\tilde{r}}\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)} = \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi} - |\hat{\chi}|_{\not g}^2 + \frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)}\cdot\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)}, \tag{8.138a}$$

$$\begin{aligned}\not{D}_L\not\nabla\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)} + \frac{3}{\tilde{r}}\not\nabla\mathrm{tr}_{\not g}\tilde{\chi}^{(Small)} =& \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\not\nabla(\vec{\mathscr{C}},\mathscr{D}) \\ &+ \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right) \\ &+ \mathrm{f}_{(\vec{L})}\cdot\not\nabla\boldsymbol{\partial}\vec{\Psi}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\tilde{r}^{-1}\right) \\ &+ \mathrm{f}_{(\vec{L})}\cdot\not\nabla\hat{\chi}\cdot\hat{\chi} + \mathrm{f}_{(\vec{L})}\cdot\not\nabla\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi}\right) \\ &+ \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi},\end{aligned} \tag{8.138b}$$

$$\begin{aligned}L\left(\frac{1}{2}\mathrm{tr}_{\not g}\tilde{\chi}\upsilon\right) - \frac{1}{4}(\mathrm{tr}_{\not g}\chi)^2\nu - \frac{1}{2}(L\ln b)\,\mathrm{tr}_{\not g}\tilde{\chi}\upsilon - |\not\nabla\sigma|_{\not g}^2\,\upsilon& \\ = \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\cdot\upsilon + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\upsilon& \\ - \frac{1}{2}|\hat{\chi}|_{\not g}^2\cdot\upsilon - |\not\nabla\sigma|_{\not g}^2\,\upsilon.&\end{aligned} \tag{8.138c}$$

PDEs involving $\hat{\chi}$

$$\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\hat{\chi} = \not\nabla\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)} + \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\xi^{(1)} + \not\nabla\xi^{(2)} + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{8.139a}$$

$$\not{D}_L\hat{\chi} + (\mathrm{tr}_{\not g}\chi)\hat{\chi} = (\not\nabla,\not{D}_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.139b}$$

The transport equation for ζ

$$\not{D}_L\zeta_A + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\zeta = (\not\nabla,\not{D}_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi} + \mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\hat{\chi}. \tag{8.140}$$

The transport equation for b

$$Lb = b\cdot\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.141}$$

Transport equation for $\not g$ *Along the integral curves of L, parameterized by t, we have, with* $\not e$ *the standard* *round metric on the Euclidean unit sphere $\mathbb{S}^2$, the following identity:*

$$\begin{aligned}
&\frac{d}{dt}\left\{\widetilde{r}^{-2}\mathfrak{g}\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-\mathfrak{e}\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right\} \\
&=\left(\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)}-\mathbf{\Gamma}_L\right)\left\{\widetilde{r}^{-2}\mathfrak{g}\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-\mathfrak{e}\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right\} \\
&\quad+\left(\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)}-\mathbf{\Gamma}_L\right)\mathfrak{e}\!\!/\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)+\frac{2}{\widetilde{r}^2}\hat{\chi}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right).
\end{aligned} \tag{8.142}$$

Transport equations for υ ***and*** $\nabla\!\!\!/\upsilon$

$$L\ln\left(\widetilde{r}^{-2}v\right)=\mathrm{tr}_{\mathfrak{g}\!\!/}\chi-\frac{2}{\widetilde{r}}=\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)}-\mathbf{\Gamma}_L, \tag{8.143a}$$

$$L\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)+\frac{1}{2}(\mathrm{tr}_{\mathfrak{g}\!\!/}\chi)\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)=\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)+\nabla\!\!\!/\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)}-\nabla\!\!\!/(\mathbf{\Gamma}_L). \tag{8.143b}$$

An algebraic identity for μ *The mass aspect function* μ *verifies the following identity:*

$$\begin{aligned}
\mu=&\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})+\mathrm{di}\!\!/\mathrm{v}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta) \\
&+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}.
\end{aligned} \tag{8.144}$$

The transport equation for $\nabla\!\!\!/\sigma$ *In the interior region* $\mathscr{M}^{(Int)}$*,* $\nabla\!\!\!/\sigma$ *verifies the following transport equation:*

$$L\nabla\!\!\!/\sigma+\frac{1}{2}\mathrm{tr}_{\mathfrak{g}\!\!/}\chi\nabla\!\!\!/\sigma=\frac{1}{2}\nabla\!\!\!/(\mathbf{\Gamma}_L)-\hat{\chi}\cdot\nabla\!\!\!/\sigma. \tag{8.145}$$

The transport equation for $\check{\mu}$ *The modified mass aspect function* $\check{\mu}$ *verifies the following transport equation:*

$$L\check{\mu}+(\mathrm{tr}_{\mathfrak{g}\!\!/}\chi)\check{\mu}=\mathfrak{I}_{(1)}+\mathfrak{I}_{(2)}. \tag{8.146}$$

Where

$$\mathfrak{I}_{(1)}=\widetilde{r}^{-1}\mathrm{di}\!\!/\mathrm{v}\,\xi+\widetilde{r}^{-2}\xi, \tag{8.147a}$$

$$\begin{aligned}
\mathfrak{I}_{(2)}=&\lambda^{-1}\cdot\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\left(\vec{\mathscr{C}},\mathscr{D}\right)+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)},\widetilde{r}^{-1}\right) \\
&+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\nabla\!\!\!/\widetilde{\zeta}+\mathrm{f}_{(\vec{L})}\cdot\left(\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi},\nabla\!\!\!/\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)}\right)\cdot\nabla\!\!\!/\sigma \\
&+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\nabla\!\!\!/\sigma+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta\right)\cdot\boldsymbol{\partial}^2\vec{\Psi} \\
&+\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\zeta\right) \\
&+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\mathfrak{g}\!\!/}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right).
\end{aligned} \tag{8.147b}$$

The Hodge system for ζ *The torsion* ζ *satisfies the following Hodge system on* $S_{t,u}$*:*

$$\begin{aligned} \text{d}\!\!/\text{iv}\,\zeta =& \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)+\text{d}\!\!/\text{iv}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi} \quad (8.148a)\\ &+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}+\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\zeta\right),\\ \text{cu}\!\!/\text{rl}\,\zeta =& \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)+\text{cu}\!\!/\text{rl}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}. \quad (8.148b) \end{aligned}$$

The Hodge system for $\tilde{\zeta}$ *The modified torsion* $\tilde{\zeta}$ *satisfies the following Hodge system on* $S_{t,u}$*:*

$$\begin{aligned} \text{d}\!\!/\text{iv}\,\tilde{\zeta}-\frac{1}{2}\check{\mu} =& \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)+\text{d}\!\!/\text{iv}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta \quad (8.149a)\\ &+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi},\\ \text{cu}\!\!/\text{rl}\,\tilde{\zeta} =& \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)+\text{cu}\!\!/\text{rl}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}. \quad (8.149b) \end{aligned}$$

The Hodge system for $\tilde{\zeta}-\mu\!\!\!/$ *The difference* $\tilde{\zeta}-\mu\!\!\!/$ *satisfies the following Hodge system on* $S_{t,u}$ [2]*:*

$$\begin{aligned} \text{d}\!\!/\text{iv}\left(\tilde{\zeta}-\mu\!\!\!/\right) =& \text{d}\!\!/\text{iv}\,\xi+\left(\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)-\lambda^{-1}\overline{\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)}\right) \quad (8.150a)\\ &+\left(\mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta-\overline{\mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta}\right)+\left(\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}-\overline{\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}}\right)\\ &+\left(\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}-\overline{\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}}\right),\\ \text{cu}\!\!/\text{rl}\left(\tilde{\zeta}-\mu\!\!\!/\right) =& \text{cu}\!\!/\text{rl}\,\xi+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}. \quad (8.150b) \end{aligned}$$

A decomposition of $\mu\!\!\!/$ ***and a Hodge-transport system for the constituent parts*** *In* $\mathscr{M}^{(Int)}$*, we can decompose the* $\mu\!\!\!/$ *as follows:*

$$\mu\!\!\!/ = \mu\!\!\!/_{(1)}+\mu\!\!\!/_{(2)}. \quad (8.151)$$

where $\mu\!\!\!/_{(1)}$ *and* $\mu\!\!\!/_{(2)}$ *verify the following Hodge-transport PDE systems:*

$$\text{d}\!\!/\text{iv}\left(\mathbf{D}\!\!\!/_L\,\mu\!\!\!/_{(1)}+\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\,\mu\!\!\!/_{(1)}\right)=\mathfrak{I}_{(1)}-\overline{\mathfrak{I}_{(1)}}, \quad (8.152a)$$

$$\text{cu}\!\!/\text{rl}\left(\mathbf{D}\!\!\!/_L\,\mu\!\!\!/_{(1)}+\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\,\mu\!\!\!/_{(1)}\right)=0. \quad (8.152b)$$

[2] The barred terms appear in the equation (8.149a) because $\text{d}\!\!/\text{iv}(\tilde{\zeta}-\mu\!\!\!/)$ must have vanishing average

$$\text{d}\mathrm{i}\!\!/\!\text{v}\left(\not{\mathbf{D}}_L \not{\mu}_{(2)} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi \not{\mu}_{(2)}\right) = \mathfrak{J}_{(2)} - \overline{\mathfrak{J}_{(2)}} + \hat{\chi}\cdot\not{\nabla}\not{\mu} + \left(\not{\nabla}\boldsymbol{\partial}\vec{\Psi}, \not{\nabla}\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(Small)}\right)\cdot\not{\mu} \tag{8.153a}$$

$$+ \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\not{\mu} + \left(\mathrm{tr}_{\not{g}}\chi - \overline{\mathrm{tr}_{\not{g}}\chi}\right)\overline{\check{\mu}},$$

$$\mathrm{cu}\!\!/\!\mathrm{rl}\left(\not{\mathbf{D}}_L \not{\mu}_{(2)} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi \not{\mu}_{(2)}\right) = \hat{\chi}\cdot\not{\nabla}\not{\mu} + \left(\not{\nabla}\boldsymbol{\partial}\vec{\Psi}, \not{\nabla}\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(Small)}\right)\cdot\not{\mu} \tag{8.153b}$$

$$+ \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\not{\mu}.$$

subject to the following initial conditions along the cone-tip axis for $u \in [0, T_{*;(\lambda)}]$*:*

$$\lim_{t\downarrow u}\widetilde{r}\left|\not{\mu}_{(1)} - \not{\mu}\right|_{\not{g}} = \mathscr{O}(\widetilde{r}), \qquad \lim_{t\downarrow u}\widetilde{r}\left|\not{\mu}_{(2)}\right|_{\not{g}} = \mathscr{O}(\widetilde{r}). \tag{8.154}$$

Proof of $LL^i = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$ *in (8.133a).*

$$LL^i = \mathbf{D}_L L^i - \boldsymbol{\Gamma}^i_{LL} = -k_{NN}L^i - \boldsymbol{\Gamma}^i_{LL} = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.155}$$

□

Proof of $LN^i = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$ *in (8.133a).*

$$L\underline{L}^i = \mathbf{D}_L\underline{L}^i - \boldsymbol{\Gamma}^i_{L\underline{L}} = -2k_{AN}e^i_A + k_{NN}L^i - \boldsymbol{\Gamma}^i_{L\underline{L}} = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{8.156}$$

$$LN^i = \frac{1}{2}LL^i - \frac{1}{2}L\underline{L}^i = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.157}$$

□

Proof of $\frac{\partial}{\partial w}N^i = a\cdot\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi} - \not{\nabla}a$ *in (8.133b).* Using (8.15), we find that

$$\frac{\partial}{\partial w}N^i = aNN^i = -a\not{\nabla}\ln a - \frac{a}{2}k_{NN}L^i - \frac{a}{2}k_{NN}\underline{L}^i - a\boldsymbol{\Gamma}^i_{NN}, \tag{8.158}$$

$$= -\not{\nabla}a + a\cdot\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.159}$$

□

Proof of $\frac{\partial}{\partial w}L^i = a\cdot \mathrm{f}_{(\vec{L})}\cdot \boldsymbol{\partial}\vec{\Psi} - \nabla\!\!\!/ a$ *in (8.133b).*

$$\frac{\partial}{\partial w}L^i = \frac{\partial}{\partial w}\mathbf{T}^i + \frac{\partial}{\partial w}N^i, \tag{8.160}$$
$$\frac{\partial}{\partial w}\mathbf{T}^i = aN\cdot d\mathbf{T}^i = a\cdot \mathrm{f}_{(\vec{L})}\cdot \boldsymbol{\partial}\vec{\Psi}. \tag{8.161}$$

□

Proof of $L\Theta^i_{(A)} = \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}^{(Small)}, \hat{\chi})\cdot \vec{\Theta}_{(A)}$ *in (8.135).*

$$L\Theta^i_{(A)} = L\left(\frac{1}{\tilde{r}}\frac{\partial}{\partial\omega^A}\right) = -\frac{1}{\tilde{r}^2}L(\tilde{r})\left(\frac{\partial}{\partial\omega^A}\right)^i + \frac{1}{\tilde{r}}L\left(\frac{\partial}{\partial\omega^A}\right)^i. \tag{8.162}$$

Note that $\frac{\partial}{\partial t} = L$ relative to geometric coordinates. So L commutes with $\frac{\partial}{\partial\omega^A}$.

$$\begin{aligned} L\left(\frac{\partial}{\partial\omega^A}\right)^i &= \frac{\partial}{\partial\omega^A}L^i = \chi_{\frac{\partial}{\partial\omega^A}B}e^i_B - k_{\frac{\partial}{\partial\omega^A}N}L^i \\ &= \left(\hat{\chi}_{\frac{\partial}{\partial\omega^A}B} + \frac{1}{2}\left(\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}^{(\mathrm{Small})} + \frac{2}{\tilde{r}} - 2\boldsymbol{\Gamma}_L\right)g\!\!\!/_{\frac{\partial}{\partial\omega^A}B}\right)e^i_B + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\frac{\partial}{\partial\omega^A} \\ &= \frac{1}{\tilde{r}}\frac{\partial}{\partial\omega^A} + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}\right)\cdot\frac{\partial}{\partial\omega^A}. \end{aligned} \tag{8.163}$$

Combining above equations and using the fact $L(\tilde{r}) = 1$, we obtain the desired equation. □

Proof of $\frac{\partial}{\partial w}\Theta^i_{(A)} = a\cdot \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\hat{\chi})\cdot\vec{\Theta}_{(A)} + \mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/ a\cdot\vec{\Theta}_{(A)}$ *in (8.137).* We are computing on Σ_0, where $w = \tilde{r} = -u, a = b$.

$$\frac{\partial}{\partial w}\Theta^i_{(A)} = \frac{\partial}{\partial w}\left(\frac{1}{\tilde{r}}\frac{\partial}{\partial\omega^A}\right) = -\frac{1}{\tilde{r}^2}\frac{\partial}{\partial w}(\tilde{r})\frac{\partial}{\partial\omega^A} + \frac{1}{\tilde{r}}\frac{\partial}{\partial w}\left(\frac{\partial}{\partial\omega^A}\right)^i, \tag{8.164}$$
$$\frac{\partial}{\partial w}\left(\frac{\partial}{\partial\omega^A}\right)^i = \frac{\partial}{\partial\omega^A}\left(\frac{\partial}{\partial w}\right)^i = \frac{\partial}{\partial\omega^A}(aN^i) = \left(\frac{\partial}{\partial\omega^A}a\right)N^i + a\frac{\partial}{\partial\omega^A}N^i, \tag{8.165}$$
$$\begin{aligned} \frac{\partial}{\partial\omega^A}N^i &= \theta_{\frac{\partial}{\partial\omega^A}B}e^i_B - k_{\frac{\partial}{\partial\omega^A}B}N^i = \chi_{\frac{\partial}{\partial\omega^A}B}e^i_B + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\frac{\partial}{\partial\omega^A} \\ &= \left(\hat{\chi}_{\frac{\partial}{\partial\omega^A}B} + \frac{1}{2}\left(\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}^{(\mathrm{Small})} + \frac{2}{\tilde{r}} - \boldsymbol{\Gamma}_L\right)g\!\!\!/_{\frac{\partial}{\partial\omega^A}B}\right)e^i_B + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\frac{\partial}{\partial\omega^A}. \end{aligned} \tag{8.166}$$

We then use the properties on the initial foliation:

$$\mathrm{tr}_{g\!\!\!/}\theta + k_{NN} = \frac{2}{aw} + \mathrm{tr}_g k - \boldsymbol{\Gamma}_L, \tag{8.167}$$
$$\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}^{(\mathrm{Small})}|_{\Sigma_0} = \frac{2(1-a)}{aw}, \qquad \text{for } 0\le w\le w_*. \tag{8.168}$$

Combining (8.168) and (8.166), we have:

$$\frac{\partial}{\partial \omega^A} N^i = \frac{1}{aw}\frac{\partial}{\partial \omega^A} + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \hat{\chi}\right) \cdot \frac{\partial}{\partial \omega^A}. \tag{8.169}$$

Combining (8.169), (8.165),(8.164) and noticing that $\frac{\partial}{\partial w}\widetilde{r} = 1$, we conclude the desired equation. □

Proof of $N\frac{\partial}{\partial \omega^A}^i = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \frac{\partial}{\partial \omega^A} + \nabla\!\!\!/_A \ln b \cdot N$ *in (8.136).* Combining (8.166) and (8.196) (the proof of (8.196) is independent of the proof of Proposition 8.20), we have

$$N\frac{\partial}{\partial \omega^A}^i = k_{\frac{\partial}{\partial \omega^A}B} e_B^i + \nabla\!\!\!/_A \ln b \cdot N^i \tag{8.170}$$

□

Proof of $L\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(Small)} + \frac{2}{\widetilde{r}}\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(Small)}$ *in (8.138a).* Using (8.65a), we have

$$\begin{aligned} & L\left(\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})} + \frac{2}{\widetilde{r}} - \boldsymbol{\Gamma}_L\right) + \frac{1}{2}\left(\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})} + \frac{2}{\widetilde{r}} - \boldsymbol{\Gamma}_L\right)^2 \\ & = -|\hat{\chi}|^2_{g\!\!/} - k_{NN}\left(\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})} + \frac{2}{\widetilde{r}} - \boldsymbol{\Gamma}_L\right) - \mathbf{Ric}_{LL}. \end{aligned} \tag{8.171}$$

Hence,

$$\begin{aligned} L\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})} + \frac{2}{\widetilde{r}}\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})} &= \frac{1}{2}\left(\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})}\right)^2 - |\hat{\chi}|^2_{g\!\!/} - \frac{2}{\widetilde{r}^2} + \frac{2}{\widetilde{r}^2}L(\widetilde{r}) + L\boldsymbol{\Gamma}_L - \frac{1}{2}\boldsymbol{\Gamma}_L^2 + \frac{2}{\widetilde{r}}\boldsymbol{\Gamma}_L \\ &+ \mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})}\boldsymbol{\Gamma}_L - k_{NN}\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})} - k_{NN}\frac{2}{\widetilde{r}} + k_{NN}\boldsymbol{\Gamma}_L \\ &- \left(L\boldsymbol{\Gamma}_L + k_{NN}\boldsymbol{\Gamma}_L + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi}\right) \\ =& \frac{1}{2}\left(\mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})}\right)^2 - |\hat{\chi}|^2_{g\!\!/} + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}. \end{aligned} \tag{8.172}$$

□

Proof of $D\!\!\!/_L \nabla\!\!\!/ \mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(Small)} + \frac{2}{\widetilde{r}}\nabla\!\!\!/ \mathrm{tr}_{g\!\!/}\widetilde{\chi}^{(Small)}$ *in (8.138b).* In the computation below we should view $\nabla\!\!\!/_A \mathrm{tr}_{\widetilde{g}\!\!/}\widetilde{\chi}^{(\mathrm{Small})}$ as a one-form.

$$\not{\mathbf{D}}_L \nabla\!\!\!/_A \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} - \nabla\!\!\!/_A \not{\mathbf{D}}_L \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} = L e_A (\mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}) - \nabla\!\!\!/_{\not{\mathbf{D}}_L e_A} \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} - e_A L \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} \tag{8.173}$$

$$= ([L,A] - \not{\mathbf{D}}_L e_A)\, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}$$

$$= (\mathbf{D}_L e_A - \not{\mathbf{D}}_L e_A - \mathbf{D}_A L)\, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}$$

$$= (-k_{AN} L - \chi_{AB} e_B + k_{AN} L)\, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}$$

$$= \left(-\hat{\chi}_{AB} - \frac{1}{2} \left(\mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} + \frac{2}{\tilde{r}} - \mathbf{\Gamma}_L \right) g\!\!\!/_{AB} \right) e_B \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}$$

$$= \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi} \right) \nabla\!\!\!/_A \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} - \frac{1}{\tilde{r}} \nabla\!\!\!/_A \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}.$$

Now we apply (8.65a) to $\nabla\!\!\!/_A \not{\mathbf{D}}_L \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}$ and use the fact $\nabla\!\!\!/_A \tilde{r} = 0$ to obtain:

$$\nabla\!\!\!/_A \left(L \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} \right) = \nabla\!\!\!/_A \left(-\frac{2}{\tilde{r}} \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} \right) + \lambda^{-1} \nabla\!\!\!/_A \left(\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) \right) \tag{8.174}$$

$$+ \nabla\!\!\!/_A \left(\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \tilde{r}^{-1} \right) \cdot \boldsymbol{\partial} \vec{\Psi} \right)$$

$$- \nabla\!\!\!/_A \left(|\hat{\chi}|^2_{g\!\!\!/} \right) + \nabla\!\!\!/_A \left(\frac{1}{2} \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} \right)$$

$$= -\frac{2}{\tilde{r}} \nabla\!\!\!/_A \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} + \lambda^{-1} \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \tilde{r}^{-1} \right) \cdot (\vec{\mathscr{C}}, \mathscr{D})$$

$$+ \lambda^{-1} \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/_A (\vec{\mathscr{C}}, \mathscr{D})$$

$$+ \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \tilde{r}^{-1} \right) \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \tilde{r}^{-1} \right) \cdot \boldsymbol{\partial} \vec{\Psi}$$

$$+ \mathrm{f}_{(\vec{L})} \cdot \left(\nabla\!\!\!/_A \boldsymbol{\partial} \vec{\Psi}, \nabla\!\!\!/_A \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} \right) \cdot \boldsymbol{\partial} \vec{\Psi} + \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \tilde{r}^{-1} \right) \cdot \nabla\!\!\!/_A \boldsymbol{\partial} \vec{\Psi}$$

$$- 2 |\hat{\chi}|_{g\!\!\!/} \nabla\!\!\!/_A \hat{\chi} + \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} \nabla\!\!\!/_A \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}.$$

Combining (8.173)-(8.174), we have:

$$\not{\mathbf{D}}_L \nabla\!\!\!/ \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} + \frac{3}{\tilde{r}} \nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/} \tilde{\chi}^{(\mathrm{Small})} = \lambda^{-1} \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ (\vec{\mathscr{C}}, \mathscr{D}) \tag{8.175}$$

$$+ \lambda^{-1} \mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \tilde{r}^{-1} \right)$$

$$+ \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ \boldsymbol{\partial} \vec{\Psi} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \tilde{r}^{-1} \right)$$

$$+ \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ \hat{\chi} \cdot \hat{\chi} + \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi} \right)$$

$$+ \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \tilde{r}^{-1} \right) \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}, \tilde{r}^{-1} \right) \cdot \boldsymbol{\partial} \vec{\Psi}.$$

□

Proof of $L\left(\frac{1}{2}\mathrm{tr}_{\not g}\tilde{\chi}\upsilon\right)$ *in (8.138c).* We need to use (8.48a), (8.65a) and (8.105a) for the computation below.

$$L\left(\frac{1}{2}\mathrm{tr}_{\not g}\tilde{\chi}\upsilon\right)=\frac{1}{2}L\left\{(\mathrm{tr}_{\not g}\chi+\mathbf{\Gamma}_L)\upsilon\right\}=\frac{1}{2}L(\mathrm{tr}_{\not g}\chi\upsilon)+\frac{1}{2}L(\mathbf{\Gamma}_L\upsilon),\tag{8.176}$$

$$L(\mathbf{\Gamma}_L\upsilon)=L\mathbf{\Gamma}_L\upsilon+\mathbf{\Gamma}_L\upsilon\mathrm{tr}_{\not g}\chi,\tag{8.177}$$

$$\begin{aligned}L(\mathrm{tr}_{\not g}\chi\upsilon)=&\upsilon\left(\mathrm{tr}_{\not g}\chi\right)^2+\upsilon(L\mathrm{tr}_{\not g}\chi)\\=&\upsilon\left(\mathrm{tr}_{\not g}\chi\right)^2+\upsilon\left(-\frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2-|\hat{\chi}|^2_{\not g}-k_{NN}\mathrm{tr}_{\not g}\chi\right)\\&-\upsilon\left(L\mathbf{\Gamma}_L+k_{NN}\mathbf{\Gamma}_L+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})+\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right).\end{aligned}\tag{8.178}$$

Combining (8.175)-(8.176) and noticing that

$$\upsilon k_{NN}\mathrm{tr}_{\not g}\chi+\upsilon k_{NN}\mathbf{\Gamma}_L=\upsilon k_{NN}\mathrm{tr}_{\not g}\tilde{\chi}=\upsilon\mathrm{tr}_{\not g}\tilde{\chi}L(\ln b),\tag{8.179}$$

we get the desired equation. □

Proof of $\mathrm{d\not{i}v}\hat{\chi}=\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(Small)}+\mathrm{d\not{i}v}\xi+\not\nabla\xi+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}$ *in (8.139a).* Using equations (8.65h) and (8.117d), we deduce:

$$\mathrm{d\not{i}v}\hat{\chi}_A=-\hat{\chi}_{AB}k_{BN}+\frac{1}{2}\not\nabla_A\mathrm{tr}_{\not g}\chi+\frac{1}{2}k_{AN}\mathrm{tr}_{\not g}\chi+\mathbf{Riem}_{BLBA},\tag{8.180a}$$

$$\not\nabla_A\mathrm{tr}_{\not g}\chi=\not\nabla_A\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}-\mathbf{\Gamma}_L+\frac{2}{\tilde{r}}\right)=\not\nabla_A\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}-\not\nabla_A\mathbf{\Gamma}_L,\tag{8.180b}$$

$$\begin{aligned}\not\nabla_A\mathbf{\Gamma}_L&=e_A(\mathbf{\Gamma}_L)-(\chi_{AB}e^\alpha_B-k_{AN}L^\alpha-\mathbf{\Gamma}^\alpha_{AL})\mathbf{\Gamma}_\alpha\\&=\not\nabla\xi+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi},\end{aligned}\tag{8.180c}$$

$$\mathbf{Riem}_{BLBA}=\not\nabla_A\xi+\mathrm{d\not{i}v}\xi_A+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}.\tag{8.180d}$$

Combining (8.180), we conclude the desired equation. □

Proof of $\not{\mathbf{D}}_L\hat{\chi}+(\mathrm{tr}_{\not g}\chi)\hat{\chi}=(\not\nabla,\not{\mathbf{D}}_L)\xi+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}$ *in (8.139b).* Using equations (8.65b), (8.105a) and (8.117a), we deduce

$$\not{\mathbf{D}}_L\hat{\chi}_{AB}+(\mathrm{tr}_{\not g}\chi)\hat{\chi}_{AB}=-k_{NN}\hat{\chi}_{AB}-\left(\mathbf{Riem}_{LALB}-\frac{1}{2}\mathbf{Ric}_{LL}\delta_{AB}\right),\tag{8.181a}$$

$$\mathbf{Riem}_{LALB}=(\not\nabla,\not{\mathbf{D}}_L)\xi+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi},\tag{8.181b}$$

$$\mathbf{Ric}_{LL}=L\mathbf{\Gamma}_L+k_{NN}\mathbf{\Gamma}_L+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})+\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi},\tag{8.181c}$$

$$L\mathbf{\Gamma}_L=L(\mathbf{\Gamma}_L)-(-k_{NN}L^\alpha\mathbf{\Gamma}^\alpha_{LL})\mathbf{\Gamma}_\alpha=\not{\mathbf{D}}_L\xi-\mathrm{f}_{(\vec{L})}\boldsymbol{\partial}\vec{\Psi}\boldsymbol{\partial}\vec{\Psi}.\tag{8.181d}$$

Combining (8.181) finishes the proof. □

Proof of $\not{\mathbf{D}}_L \zeta_A + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\zeta$ *in (8.140).* We obtain the result by combining equations (8.65c) and (8.117e):

$$\not{\mathbf{D}}_L \zeta_A + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\zeta_A = -(k_{BN}+\zeta_B)\hat{\chi}_{AB} - \frac{1}{2}\mathrm{tr}_{\not g}\chi k_{AN} + \frac{1}{2}\mathbf{Riem}_{AL L\underline{L}}, \tag{8.182}$$

$$\mathbf{Riem}_{AL L\underline{L}} = (\not\nabla, \not{\mathbf{D}}_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.183}$$

□

Proof of (8.142).

$$\chi\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right) = \hat{\chi}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right) + \frac{1}{2}\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})} + \frac{2}{\widetilde{r}} - \boldsymbol{\Gamma}_L\right)\not g\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right). \tag{8.184}$$

Using the fact $L(\widetilde{r}) = 1$ and $\mathbf{D}_L \frac{\partial}{\partial\omega^A} = 0$, we obtain

$$\begin{aligned} L\left\{\widetilde{r}^{-2}\not g\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right\} &= -2\widetilde{r}^{-3}\not g\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right) + 2\widetilde{r}^{-2}\chi\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right) \\ &= 2\widetilde{r}^{-2}\hat{\chi}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right) + \widetilde{r}^{-2}\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L\right)\not g\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right). \end{aligned} \tag{8.185}$$

This finishes the proof. □

Proof of $L\ln\left(\widetilde{r}^{-2}v\right) = \mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}} = \mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(Small)} - \boldsymbol{\Gamma}_L$ *in (8.143a).* We use equation (8.48a) and the fact that $L(\widetilde{r}) = 1$ to deduce:

$$\begin{aligned} L\ln\left(\widetilde{r}^{-2}v\right) &= \widetilde{r}^{2}\upsilon^{-1}\left(-2\widetilde{r}^{-3}L(\widetilde{r})\upsilon + \widetilde{r}^{-2}L\upsilon\right) \\ &= \widetilde{r}^{2}\upsilon^{-1}\left\{-2\widetilde{r}^{-3}L(\widetilde{r})\upsilon + \widetilde{r}^{-2}\upsilon\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L + \frac{2}{\widetilde{r}}\right)\right\} \\ &= \mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L. \end{aligned} \tag{8.186}$$

□

Proof of $L\not\nabla\ln\left(\widetilde{r}^{-2}v\right) + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)\not\nabla\ln\left(\widetilde{r}^{-2}v\right) = \mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\not\nabla\ln\left(\widetilde{r}^{-2}v\right) + \not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(Small)} - \not\nabla(\boldsymbol{\Gamma}_L)$ *in (8.143b).*

$$\begin{aligned} (L\not\nabla_A - \not\nabla_A L)\ln\left(\widetilde{r}^{-2}v\right) &= ([L,A] - \not{\mathbf{D}}_L e_A)\ln\left(\widetilde{r}^{-2}v\right) \\ &= -\chi_{AB}e_B\ln\left(\widetilde{r}^{-2}v\right), \end{aligned} \tag{8.187a}$$

$$\not\nabla_A L\ln\left(\widetilde{r}^{-2}v\right) = \not\nabla_A\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L\right), \tag{8.187b}$$

$$\chi_{AB}e_B\ln\left(\widetilde{r}^{-2}v\right) = \hat{\chi}_{AB}e_B\ln\left(\widetilde{r}^{-2}v\right) + \frac{1}{2}\mathrm{tr}_{\not g}\chi e_A\ln\left(\widetilde{r}^{-2}v\right). \tag{8.187c}$$

Combining (8.187), we conclude the desired equation. □

Proof of the algebraic identity of μ *in (8.144).* We will use (8.48b) and the fact that $\underline{L}(\widetilde{r}) = L(\widetilde{r}) - 2N(\widetilde{r}) = 1 - \frac{2}{b}$ to get the following equation:

$$\begin{aligned}
\underline{L}\ln\left(\widetilde{r}^{-2}v\right) &= \widetilde{r}^2\upsilon^{-1}\left(-2\widetilde{r}^{-3}\underline{L}(\widetilde{r})\upsilon + \widetilde{r}^{-2}\underline{L}\upsilon\right) \\
&= \widetilde{r}^2\upsilon^{-1}\left(-2\widetilde{r}^{-3}\left(1-\frac{2}{b}\right)\upsilon + \widetilde{r}^{-2}\mathrm{tr}_{\not g}\underline{\chi}\upsilon\right) \\
&= \mathrm{tr}_{\not g}\underline{\chi} - \frac{2}{\widetilde{r}} + \frac{4}{b\widetilde{r}}.
\end{aligned} \tag{8.188}$$

Combining (8.188) with (8.143a), we also have:

$$N\ln\left(\widetilde{r}^{-2}v\right) = \mathrm{tr}_{\not g}\theta - \frac{2}{b\widetilde{r}}. \tag{8.189}$$

Therefore,

$$\begin{aligned}
(L\underline{L} - \underline{L}L)\ln\left(\widetilde{r}^{-2}v\right) &= L\left(\mathrm{tr}_{\not g}\underline{\chi} - \frac{2}{\widetilde{r}} + \frac{4}{b\widetilde{r}}\right) - \underline{L}\left(\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}\right) \\
&= L\mathrm{tr}_{\not g}\underline{\chi} - \underline{L}\mathrm{tr}_{\not g}\chi + \frac{4}{\widetilde{r}}L\left(b^{-1}\right) \\
&= L\mathrm{tr}_{\not g}\underline{\chi} - \underline{L}\mathrm{tr}_{\not g}\chi + \frac{4}{b\widetilde{r}}k_{NN}.
\end{aligned} \tag{8.190}$$

Meanwhile we have $[L,\underline{L}] = 2\left(\underline{\zeta}_A - \zeta_A\right)e_A + k_{NN}\underline{L} - k_{NN}L$. Thus we have

$$\begin{aligned}
L\mathrm{tr}_{\not g}\underline{\chi} - \underline{L}\mathrm{tr}_{\not g}\chi + \frac{4}{b\widetilde{r}}k_{NN} =& [L,\underline{L}]\ln\left(\widetilde{r}^{-2}v\right) = 2\left(\underline{\zeta}_A - \zeta_A\right)e_A\ln\left(\widetilde{r}^{-2}v\right) \\
&+ k_{NN}\underline{L}\ln\left(\widetilde{r}^{-2}v\right) - k_{NN}L\ln\left(\widetilde{r}^{-2}v\right) \\
=& 2\left(\underline{\zeta}_A - \zeta_A\right)e_A\ln\left(\widetilde{r}^{-2}v\right) - 2k_{NN}\left(\mathrm{tr}_{\not g}\theta - \frac{2}{b\widetilde{r}}\right).
\end{aligned} \tag{8.191}$$

which can be rewritten as:

$$L\mathrm{tr}_{\not g}\underline{\chi} - \underline{L}\mathrm{tr}_{\not g}\chi = 2\left(\underline{\zeta}_A - \zeta_A\right)e_A\ln\left(\widetilde{r}^{-2}v\right) - 2k_{NN}\mathrm{tr}_{\not g}\theta. \tag{8.192}$$

Combining (8.192) with (8.65d), we have:

$$\begin{aligned}\mu =& 2\mathrm{d}\!\!/\mathrm{iv}\,\underline{\zeta} + k_{NN}\mathrm{tr}_{\not g}\underline{\chi} - \hat{\chi}_{AB}\underline{\hat{\chi}}_{AB} + 2\left|\underline{\zeta}\right|^2_{\not g} + \mathbf{Riem}_{A\underline{L}LA} \\ &- 2\left(\underline{\zeta}_A - \zeta_A\right) e_A \ln\left(\widetilde{r}^{-2}v\right) - 2k_{NN}\mathrm{tr}_{\not g}\theta \\ =& 2\mathrm{d}\!\!/\mathrm{iv}\,\underline{\zeta} + k_{NN}\mathrm{tr}_{\not g}\chi - \hat{\chi}_{AB}\left(-2k_{AB} - \hat{\chi}_{AB}\right) + 2\left|\underline{\zeta}\right|^2_{\not g} \\ &+ \mathbf{Riem}_{A\underline{L}LA} - 2\left(\underline{\zeta}_A - \zeta_A\right) e_A \ln\left(\widetilde{r}^{-2}v\right).\end{aligned} \tag{8.193}$$

Using (8.117f) and noticing that $\underline{\zeta}_A = -k_{AN}$, we conclude the desired equation. □

Proof of $L\nabla\!\!\!/\sigma + \frac{1}{2}\mathrm{tr}_{\not g}\chi\nabla\!\!\!/\sigma = \frac{1}{2}\nabla\!\!\!/(\mathbf{\Gamma}_L) - \hat{\chi}\cdot\nabla\!\!\!/\sigma$ *in (8.145).*

$$L\nabla\!\!\!/_A\sigma = \nabla\!\!\!/_A L\sigma - \chi_{AB}\nabla\!\!\!/_B\sigma = \frac{1}{2}\nabla\!\!\!/_A(\mathbf{\Gamma}_L) - \hat{\chi}_{AB}\nabla\!\!\!/_B\sigma - \frac{1}{2}\mathrm{tr}_{\not g}\chi\nabla\!\!\!/_A\sigma. \tag{8.194}$$

□

We need several preliminary ingredients before deriving the transport equation for $\check{\mu}$.

Lemma 8.21 (Commutator formulas)**.** *If* ξ_A *be a* $S_{t,u}$*-tangent covector and* f *is a scalar, then:*

$$[L,\nabla\!\!\!/_A]f = -\chi_{AB}\nabla\!\!\!/_B f, \tag{8.195}$$

$$[N,\nabla\!\!\!/_A]f = -\theta_{AB}e_B f + \nabla\!\!\!/_A \ln b\cdot N, \tag{8.196}$$

$$\mathbf{D}\!\!\!\!/_L\nabla\!\!\!/_B\xi_A - \nabla\!\!\!/_B\mathbf{D}\!\!\!\!/_L\xi_A = -\chi_{AB}k_{CN}\xi_C + \chi_{BC}k_{AN}\xi_C - \chi_{BC}\nabla\!\!\!/_C\xi_A + \mathbf{Riem}_{ACLB}\xi_C. \tag{8.197}$$

Proof. We only prove (8.197) since the (8.195)-(8.196) can be proved using similar but simpler ideas. First, we have the following two equations:

$$\nabla\!\!\!/_B(\mathbf{D}\!\!\!\!/_L e_A) = \mathbf{D}\!\!\!\!/_B(\mathbf{D}_L e_A + k_{AN}L) = \mathbf{D}\!\!\!\!/_B(\mathbf{D}_L e_A) + \chi_{BC}k_{AN}e_C, \tag{8.198a}$$

$$\mathbf{D}\!\!\!\!/_L(\nabla\!\!\!/_B e_A) = \mathbf{D}\!\!\!\!/_L\left(\mathbf{D}_B e_A - \frac{1}{2}\chi_{AB}\underline{L} - \frac{1}{2}\underline{\chi}_{AB}L\right) = \mathbf{D}\!\!\!\!/_L(\mathbf{D}_B e_A) + \chi_{AB}k_{CN}e_C. \tag{8.198b}$$

Now we compute $\mathbf{D}\!\!\!\!/_L\nabla\!\!\!/_B\xi_A - \mathbf{D}_L\mathbf{D}_B\xi_A$:

$$\begin{aligned}\mathbf{D}\!\!\!\!/_L\nabla\!\!\!/_B\xi_A &= L(\nabla\!\!\!/_B\xi_A) - \nabla\!\!\!/_{\mathbf{D}\!\!\!\!/_L e_B}\xi_A - (\nabla\!\!\!/_B\xi)(\mathbf{D}\!\!\!\!/_L e_A) \\ &= L(\nabla\!\!\!/_B\xi_A) - \nabla\!\!\!/_{\mathbf{D}\!\!\!\!/_L e_B}\xi_A - e_B\xi(\mathbf{D}\!\!\!\!/_L\xi_A) + \xi\left\{\nabla\!\!\!/_B(\mathbf{D}\!\!\!\!/_L e_A)\right\} \\ &= L(\nabla\!\!\!/_B\xi_A) - \nabla\!\!\!/_{\mathbf{D}\!\!\!\!/_L e_B}\xi_A - e_B\xi(\mathbf{D}\!\!\!\!/_L\xi_A) + \xi\left\{\mathbf{D}\!\!\!\!/_B(\mathbf{D}_L e_A)\right\} + \xi(\chi_{BC}k_{AN}e_C).\end{aligned} \tag{8.199}$$

$$\mathbf{D}_L\mathbf{D}_B\xi_A = L(\mathbf{D}_B\xi_A) - \mathbf{D}_{\mathbf{D}_L e_B}\xi_A - (\mathbf{D}_B\xi)(\mathbf{D}_L e_A) \tag{8.200}$$
$$= L(\mathbf{D}_B\xi_A) - \slashed{\mathbf{D}}_{\slashed{\mathbf{D}}_L e_B - k_{BN}L}\xi_A - e_B\xi(\slashed{\mathbf{D}}_L e_A) - \xi\left\{\slashed{\mathbf{D}}_B(\mathbf{D}_L e_A)\right\}.$$

So

$$\slashed{\mathbf{D}}_L\slashed{\nabla}_B\xi_A - \mathbf{D}_L\mathbf{D}_B\xi_A = \chi_{BC}k_{AN}\xi_C - k_{BN}\slashed{\mathbf{D}}_L\xi_A. \tag{8.201}$$

Now we compute $\slashed{\nabla}_B\slashed{\mathbf{D}}_L\xi_A - \mathbf{D}_B\mathbf{D}_L\xi_A$. Here we use the fact that $\slashed{\nabla}_B L = \slashed{\Pi}\mathbf{D}_B(\slashed{\Pi}L) = 0$.

$$\slashed{\nabla}_B\slashed{\mathbf{D}}_L\xi_A = e_B(\slashed{\mathbf{D}}_L\xi_A) - (\slashed{\mathbf{D}}_L\xi)(\slashed{\nabla}_B e_A) \tag{8.202}$$
$$= e_B(\slashed{\mathbf{D}}_L\xi_A) - L\xi(\slashed{\nabla}_B e_A) + \xi\left\{\slashed{\mathbf{D}}_L(\slashed{\nabla}_B e_A)\right\}$$
$$= e_B(\slashed{\mathbf{D}}_L\xi_A) - L\xi(\slashed{\nabla}_B e_A) + \xi(\slashed{\mathbf{D}}_L(\mathbf{D}_B e_A)) + \xi(\chi_{AB}k_{CN}e_C).$$

$$\mathbf{D}_B\mathbf{D}_L\xi_A = e_B(\mathbf{D}_L\xi_A) - (\mathbf{D}_L\xi)(\mathbf{D}_B e_A) - \mathbf{D}_{\mathbf{D}_B L}\xi_A \tag{8.203}$$
$$= e_B(\mathbf{D}_L\xi_A) - L\xi(\mathbf{D}_B e_A) + \xi(\slashed{\mathbf{D}}_L(\mathbf{D}_B e_A)) - \slashed{\mathbf{D}}_{\chi_{BC}e_C - k_{BN}L}\xi_A$$
$$= e_B(\mathbf{D}_L\xi_A) - L\xi(\mathbf{D}_B e_A) + \xi\left\{\slashed{\mathbf{D}}_L(\mathbf{D}_B e_A)\right\} - \chi_{BC}\slashed{\nabla}_C\xi_A + k_{BN}\slashed{\mathbf{D}}_L\xi_A.$$

So

$$\slashed{\nabla}_B\slashed{\mathbf{D}}_L\xi_A - \mathbf{D}_B\mathbf{D}_L\xi_A = \chi_{AB}k_{CN}\xi_C + \chi_{BC}\slashed{\nabla}_C\xi_A - k_{BN}\slashed{\mathbf{D}}_L\xi_A. \tag{8.204}$$

We use (8.201), (8.204) and the fact that $\mathbf{D}_L\mathbf{D}_B\xi_A - \mathbf{D}_B\mathbf{D}_L\xi_A = \mathbf{Riem}_{ACLB}\xi_C$ to obtain the following:

$$\slashed{\mathbf{D}}_L\slashed{\nabla}_B\xi_A - \slashed{\nabla}_B\slashed{\mathbf{D}}_L\xi_A = -\chi_{AB}k_{CN}\xi_C + \chi_{BC}k_{AN}\xi_C + \chi_{BC}\slashed{\nabla}_C\xi_A + \mathbf{Riem}_{ACLB}\xi_C. \tag{8.205}$$

□

Proposition 8.22 (Transport equation for $\slashed{\Delta}\sigma$)**.** *The angular Laplacian of conformal factor verifies the fol-*

lowing transport equation:

$$L\not\Delta\sigma + \mathrm{tr}_{\not g}\chi\not\Delta\sigma = \frac{1}{2}\not\Delta(\mathbf{\Gamma}_L) - 2\hat{\chi}\cdot\not\nabla^2\sigma - \not\nabla_B \mathrm{tr}_{\not g}\chi\not\nabla_B\sigma - k_{BN}\mathrm{tr}_{\not g}\chi\not\nabla_B\sigma \tag{8.206}$$
$$+ 2\hat{\chi}_{AB}k_{AN}\not\nabla_B\sigma + 2\mathbf{Riem}_{ABLA}\not\nabla_B\sigma$$
$$= \frac{1}{2}\not\Delta(\mathbf{\Gamma}_L) + \mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\not\nabla^2\sigma + \mathrm{f}_{(\vec{L})}\cdot\left(\not\nabla\boldsymbol{\partial}\vec{\Psi}, \not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(Small)}\right)\cdot\not\nabla\sigma$$
$$+ \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\not\nabla\sigma.$$

Proof. Here we will use both commutator formulas in Lemma 8.21 to conclude $[L,\not\Delta]\sigma$:

$$L\not\Delta\sigma - \not\Delta L\sigma = \delta^{AB}\left(\not\mathbf{D}_L\not\nabla_B\not\nabla_A\sigma - \not\nabla_B\not\mathbf{D}_L\not\nabla_A\sigma - \not\nabla_B\chi_{AC}\not\nabla_C\sigma\right) \tag{8.207}$$
$$= \left(-\mathrm{tr}_{\not g}\chi k_{BN} + \chi_{AB}k_{AN} + \mathbf{Riem}_{ABLA}\right)\not\nabla_B\sigma - \chi_{AB}\not\nabla_B\not\nabla_A\sigma - \not\nabla_A\chi_{AB}\not\nabla_B\sigma$$
$$= \left(-\mathrm{tr}_{\not g}\chi k_{BN} + \chi_{AB}k_{AN} + \mathbf{Riem}_{ABLA}\right)\not\nabla_B\sigma$$
$$- \left(\hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not g_{AB}\right)\not\nabla_B\not\nabla_A\sigma - \not\nabla_A\chi_{AB}\not\nabla_B\sigma$$
$$= \left(-\mathrm{tr}_{\not g}\chi k_{BN} + \chi_{AB}k_{AN} + \mathbf{Riem}_{ABLA}\right)\not\nabla_B\sigma$$
$$- \hat{\chi}\cdot\not\nabla^2\sigma - \frac{1}{2}\mathrm{tr}_{\not g}\chi\not\Delta\sigma - \not\nabla_A\chi_{AB}\not\nabla_B\sigma.$$

Now we compute $\not\nabla_A\chi_{AB}\not\nabla_B\sigma$ with the help of (8.65h)

$$\not\nabla_A\chi_{AB}\not\nabla_B\sigma = \not\nabla_A\left(\hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not g_{AB}\right)\not\nabla_B\sigma + \hat{\chi}\cdot\not\nabla^2\sigma + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not\Delta\sigma \tag{8.208}$$
$$= \not{\mathrm{div}}\,\hat{\chi}_B\not\nabla_B\sigma + \frac{1}{2}\not\nabla_A\mathrm{tr}_{\not g}\chi\not\nabla_A\sigma + \hat{\chi}\cdot\not\nabla^2\sigma + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not\Delta\sigma$$
$$= (\not\nabla_B\mathrm{tr}_{\not g}\chi)\not\nabla_B\sigma + \frac{1}{2}k_{BN}\mathrm{tr}_{\not g}\chi\not\nabla_B\sigma + \mathbf{Riem}_{ALAB}\not\nabla_B\sigma$$
$$- \hat{\chi}_{AB}k_{AN}\not\nabla_B\sigma + \hat{\chi}\cdot\not\nabla^2\sigma + \frac{1}{2}\mathrm{tr}_{\not g}\chi\not\Delta\sigma.$$

Combining (8.207)-(8.208) and writing $\chi_{AB}k_{AN}\not\nabla_B\sigma = \hat{\chi}_{AB}k_{AN}\not\nabla_B\sigma + \frac{1}{2}\mathrm{tr}_{\not g}\chi k_{BN}\not\nabla_B\sigma$, we conclude the desired equation. □

Proposition 8.23 (Identity for $\underline{L}(k_{NN})$)**.** *The following identity holds:*

$$\underline{L}(k_{NN}) + L(k_{NN}) = \mathbf{Ric}_{L\underline{L}} + 4k_{AN}\zeta_A + \mathbf{Riem}_{AL\underline{L}A}. \tag{8.209}$$

Proof. By the definition of k, we have:

$$\begin{aligned}\underline{L}(k_{NN})+L(k_{NN}) =& -\langle \mathbf{D}_{\underline{L}}\mathbf{D}_N\mathbf{T},N\rangle - \langle \mathbf{D}_L\mathbf{D}_N\mathbf{T},N\rangle - 2\langle \mathbf{D}_N\mathbf{T},\mathbf{D}_{\mathbf{T}}N\rangle \\ =& -\frac{1}{8}\left(\mathbf{Riem}_{\underline{L}LL\underline{L}} - \mathbf{Riem}_{\underline{L}L\underline{L}L} - \mathbf{Riem}_{L\underline{L}L\underline{L}} + \mathbf{Riem}_{L\underline{L}\underline{L}L}\right) \\ & -2\langle \mathbf{D}_N\mathbf{T},\mathbf{D}_{\mathbf{T}}N\rangle \\ =& -\frac{1}{2}\mathbf{Riem}_{\underline{L}LL\underline{L}} - \frac{1}{8}\langle \mathbf{D}_{L-\underline{L}}(L+\underline{L}),\mathbf{D}_{L+\underline{L}}(L-\underline{L})\rangle \\ =& \mathbf{Ric}_{L\underline{L}} + \mathbf{Riem}_{AL\underline{L}A} + 2\zeta_A k_{AN}.\end{aligned} \tag{8.210}$$

□

Proposition 8.24 (Decomposition of the wave operator under null frame)**.** *We have following decompositions for* $\square_{\mathbf{g}} f$*,* $\square_{\mathbf{g}} L^\alpha$ *and* $\square_{\mathbf{g}}(\mathbf{\Gamma}_L)$*, where* f *is a scalar function:*

$$\square_{\mathbf{g}} f = -L\underline{L}f + \Delta\!\!\!/ f - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\underline{L}f - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi}Lf - 2k_{AN}\nabla\!\!\!\!/_A f + k_{NN}\underline{L}f, \tag{8.211a}$$

$$\begin{aligned}\square_{\mathbf{g}} L^\alpha =& \left(-\hat{\chi}_{AB}k_{BN} + \frac{1}{2}(\nabla\!\!\!\!/_A\mathrm{tr}_{g\!\!\!/}\chi + k_{AN}\mathrm{tr}_{g\!\!\!/}\chi) + \mathbf{Riem}_{BLBA}\right)e_A^\alpha \\ & + \chi_{AB}\left(\frac{1}{2}\chi_{AB}\underline{L}^\alpha + \frac{1}{2}\underline{\chi}_{AB}L^\alpha\right) - \left(\mathbf{D}_A(k_{AN})L^\alpha + k_{AB}\theta_{AB}L^\alpha\right) \\ & - k_{AN}\left(\chi_{AB}e_B^\alpha - k_{AN}L^\alpha\right) \\ & - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi(2\zeta_A e_A^\alpha + k_{NN}L^\alpha) + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi}k_{NN}L^\alpha \\ & + \underline{L}(k_{NN})L + 2\zeta_A\left(k_{NN}e_A^\alpha + \chi_{AB}e_B^\alpha - k_{AN}L^\alpha\right) - \frac{1}{2}\mathbf{Riem}^\alpha_{LL\underline{L}},\end{aligned} \tag{8.211b}$$

$$\begin{aligned}\square_{\mathbf{g}}(\mathbf{\Gamma}_L) =& \underline{L}(k_{NN})\mathbf{\Gamma}_L + \widetilde{r}^{-2}\xi \\ & + \mathrm{f}_{(\vec{L})}\cdot\left(\hat{\chi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(Small)},\boldsymbol{\partial}\vec{\Psi},\zeta\right)\cdot\boldsymbol{\partial}^2\vec{\Psi} \\ & + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(Small)},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(Small)},\boldsymbol{\partial}\vec{\Psi}\right) \\ & + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\nabla\!\!\!\!/\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(Small)} + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi} \\ & + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\cdot\boldsymbol{\partial}\vec{\Psi} + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(Small)},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi},\zeta\right)\cdot\boldsymbol{\partial}\vec{\Psi}.\end{aligned} \tag{8.211c}$$

Proof of $\square_{\mathbf{g}} f$ *in (8.211a).* We will use the fact that $\Delta\!\!\!/ f = \nabla\!\!\!\!/_A\nabla\!\!\!\!/_A f = \nabla\!\!\!\!/_A(\nabla\!\!\!\!/_A f) - \nabla\!\!\!\!/_{\nabla\!\!\!\!/_A e_A} f$ and the decomposition of the metric $\mathbf{g}$ in (7.18).

$$\Box_{\mathbf{g}} f = \left(e_A^\alpha e_A^\alpha - \frac{1}{2} L^\alpha \underline{L}^\beta - \frac{1}{2} \underline{L}^\alpha L^\beta \right) \mathbf{D}_\alpha \mathbf{D}_\beta f, \tag{8.212}$$

$$e_A^\alpha e_A^\alpha \mathbf{D}_\alpha \mathbf{D}_\beta f = \mathbf{D}_A \left(\mathbf{D}_A f \right) - \mathbf{D}_{\mathbf{D}_A e_A} f = \not{\Delta} f - \frac{1}{2} \mathrm{tr}_{\not{g}} \chi \underline{L} f - \frac{1}{2} \mathrm{tr}_{\not{g}} \underline{\chi} L f, \tag{8.213}$$

$$-\frac{1}{2} L^\alpha \underline{L}^\beta \mathbf{D}_\alpha \mathbf{D}_\beta f = -\frac{1}{2} \mathbf{D}_L (\mathbf{D}_{\underline{L}} f) + \frac{1}{2} (\mathbf{D}_L \underline{L}) f = -\frac{1}{2} L \underline{L} f - k_{AN} e_A f + \frac{1}{2} k_{NN} \underline{L} f, \tag{8.214}$$

$$\begin{aligned} -\frac{1}{2} L^\alpha \underline{L}^\beta \mathbf{D}_\alpha \mathbf{D}_\beta f &= -\frac{1}{2} \mathbf{D}_{\underline{L}} (\mathbf{D}_L f) + \frac{1}{2} (\mathbf{D}_{\underline{L}} L) f = -\frac{1}{2} \underline{L} L f + \zeta_A e_A f + \frac{1}{2} k_{NN} L f \\ &= -\frac{1}{2} L \underline{L} f + \frac{1}{2} [L, \underline{L}] f + \zeta_A e_A f + \frac{1}{2} k_{NN} L f \\ &= -\frac{1}{2} L \underline{L} f - k_{AN} e_A f + \frac{1}{2} k_{NN} \underline{L} f. \end{aligned} \tag{8.215}$$

Combining (8.212)-(8.215), we obtain (8.211a). $\Box$

Proof of $\Box_{\mathbf{g}} L^\alpha$ *in (8.211b).* Since we have $L^\beta \underline{L}^\gamma \mathbf{D}_\beta \mathbf{D}_\gamma L^\alpha - \underline{L}^\beta L^\gamma \mathbf{D}_\beta \mathbf{D}_\gamma L^\alpha = \mathbf{Riem}^\alpha_{LL\underline{L}}$, the decomposition of the wave operator can be written as follows:

$$\Box_{\mathbf{g}} L^\alpha = \left(e_A^\beta e_A^\gamma - \underline{L}^\beta L^\gamma \right) \mathbf{D}_\beta \mathbf{D}_\gamma L^\alpha - \frac{1}{2} \mathbf{Riem}^\alpha_{LL\underline{L}}. \tag{8.216}$$

By the fact that

$$\begin{aligned} \chi(e_A, \not{\nabla}_A e_B) e_B^\alpha &= \langle \mathbf{D}_A L, \not{\nabla}_A e_B \rangle e_B \\ &= \langle \chi_{AC} e_C - k_{AN} L, \mathbf{D}_A e_B - \frac{1}{2} \chi_{AB} \underline{L} - \frac{1}{2} \underline{\chi}_{AB} L \rangle e_B \\ &= \left(\chi_{AC} \langle e_C, \mathbf{D}_A e_B \rangle - k_{AN} \langle L, \mathbf{D}_A e_B \rangle - \chi_{AB} k_{AN} \right) e_B \\ &= -\chi_{AC} \langle \mathbf{D}_A e_C, e_B \rangle e_B \\ &= -\chi_{AB} \not{\nabla}_A e_B. \end{aligned} \tag{8.217}$$

We compute $e_A^\beta e_A^\gamma \mathbf{D}_\beta \mathbf{D}_\gamma L^\alpha$ as follows:

$$\begin{aligned}
e_A^\beta e_A^\gamma \mathbf{D}_\beta \mathbf{D}_\gamma L^\alpha =& \mathbf{D}_A \mathbf{D}_A L^\alpha = \mathbf{D}_A (\mathbf{D}_A L^\alpha) - \mathbf{D}_{\mathbf{D}_A e_A} L^\alpha \\
=& \mathbf{D}_A (\chi_{AB} e_B^\alpha - k_{AN} L^\alpha) - \mathbf{D}_{\nabla\!\!\!/_A e_A} L^\alpha - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi \mathbf{D}_{\underline{L}} L^\alpha - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi} \mathbf{D}_L L^\alpha \\
=& \mathrm{div\!\!\!/}\,\chi_B e_B^\alpha + \chi(\nabla\!\!\!/_A e_A, B) e_B^\alpha + \chi(A, \nabla\!\!\!/_A e_B) e_B^\alpha \\
& + \chi_{AB} (\mathbf{D}_A e_B) - e_A(k_{AN}) L^\alpha - k_{AN} (\mathbf{D}_A L^\alpha) \\
& - \mathbf{D}_{\nabla\!\!\!/_A e_A} L^\alpha - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi (2\zeta_A e_A^\alpha + k_{NN} L^\alpha) + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi} k_{NN} L^\alpha \\
=& \mathrm{div\!\!\!/}\,\chi_A e_A^\alpha + \chi(A, \nabla\!\!\!/_A e_B) e_B^\alpha + \chi_{AB} \left(\nabla\!\!\!/_A e_B^\alpha + \frac{1}{2}\chi_{AB} \underline{L}^\alpha + \frac{1}{2}\underline{\chi}_{AB} L^\alpha \right) \\
& - (\mathbf{D}_A(k_{AN}) L^\alpha + k_{AB}\theta_{AB} L^\alpha) - k_{AN} (\chi_{AB} e_B^\alpha - k_{AN} L^\alpha) \\
& - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi (2\zeta_A e_A^\alpha + k_{NN} L^\alpha) + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi} k_{NN} L^\alpha \\
=& \mathrm{div\!\!\!/}\,\chi_A e_A^\alpha + \chi_{AB} \left(\frac{1}{2}\chi_{AB} \underline{L}^\alpha + \frac{1}{2}\underline{\chi}_{AB} L^\alpha \right) \\
& - (\mathbf{D}_A(k_{AN}) L^\alpha + k_{AB}\theta_{AB} L^\alpha) - k_{AN} (\chi_{AB} e_B^\alpha - k_{AN} L^\alpha) \\
& - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi (2\zeta_A e_A^\alpha + k_{NN} L^\alpha) + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi} k_{NN} L^\alpha
\end{aligned} \tag{8.218}$$

Using the equation (8.65h), we have

$$\begin{aligned}
e_A^\beta e_A^\gamma \mathbf{D}_\beta \mathbf{D}_\gamma L^\alpha = & \left(\nabla\!\!\!/_A \mathrm{tr}_{g\!\!\!/}\chi + k_{AN} \mathrm{tr}_{g\!\!\!/}\chi - k_{BN}\chi_{AB} + \mathbf{Riem}_{BLBA} \right) e_A^\alpha \\
& + \chi_{AB} \left(\frac{1}{2}\chi_{AB} \underline{L}^\alpha + \frac{1}{2}\underline{\chi}_{AB} L^\alpha \right) - (\mathbf{D}_A(k_{AN}) L^\alpha + k_{AB}\theta_{AB} L^\alpha) \\
& - k_{AN} (\chi_{AB} e_B^\alpha - k_{AN} L^\alpha) \\
& - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi (2\zeta_A e_A^\alpha + k_{NN} L^\alpha) + \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi} k_{NN} L^\alpha .
\end{aligned} \tag{8.219}$$

Now we compute $-\underline{L}^\beta L^\gamma \mathbf{D}_\beta \mathbf{D}_\gamma L^\alpha$.

$$\begin{aligned}
-\underline{L}^\beta L^\gamma \mathbf{D}_\beta \mathbf{D}_\gamma L^\alpha &= -\mathbf{D}_{\underline{L}}(\mathbf{D}_L) L^\alpha + \mathbf{D}_{\mathbf{D}_{\underline{L}} L} L \\
&= \mathbf{D}_{\underline{L}}(k_{NN} L) + \mathbf{D}_{2\zeta_A e_A + k_{NN} L} L^\alpha \\
&= \underline{L}(k_{NN}) L + k_{NN}(2\zeta_A e_A^\alpha + k_{NN} L^\alpha) + 2\zeta_A \mathbf{D}_A L^\alpha + k_{NN} \mathbf{D}_L L^\alpha \\
&= \underline{L}(k_{NN}) L + 2\zeta_A (k_{NN} e_A^\alpha + \chi_{AB} e_B^\alpha - k_{AN} L^\alpha) .
\end{aligned} \tag{8.220}$$

Combining (8.216), (8.219) and (8.220), we conclude (8.211b). □

Proof of $\square_{\mathbf{g}}(\mathbf{\Gamma}_L)$ *in (8.211c).*

$$\square_{\mathbf{g}}(\mathbf{\Gamma}_L) = \square_{\mathbf{g}}\mathbf{\Gamma}\cdot L + \square_{\mathbf{g}}L\cdot\mathbf{\Gamma} + 2\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{\Gamma}\cdot\mathbf{D}_\beta L \tag{8.221}$$

$$\begin{aligned} 2\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{\Gamma}\cdot\mathbf{D}_\beta L &= \left(e_A^\alpha e_A^\beta - L^\alpha \underline{L}^\beta - \underline{L}^\alpha L^\beta\right)\mathbf{D}_\alpha\mathbf{\Gamma}\cdot\mathbf{D}_\beta L \\ &= 2\mathbf{D}_A\mathbf{\Gamma}\cdot\mathbf{D}_A L - \mathbf{D}_L\mathbf{\Gamma}\cdot\mathbf{D}_{\underline{L}}L - \mathbf{D}_{\underline{L}}\mathbf{\Gamma}\cdot\mathbf{D}_L L \\ &= 2\mathbf{D}_A\mathbf{\Gamma}\left(\chi_{AB}e_B - k_{AN}L\right) - \mathbf{D}_L\mathbf{\Gamma}\left(2\zeta_A e_A + k_{NN}L\right) + \mathbf{D}_{\underline{L}}\mathbf{\Gamma}\left(k_{NN}L\right) \\ &= \mathbf{D}\mathbf{\Gamma}\cdot \mathrm{f}_{(\vec{L})}\cdot\left(\hat{\chi}, \mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}, \zeta\right) \\ &= \mathrm{f}_{(\vec{L})}\cdot\left(\hat{\chi}, \mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}, \zeta\right)\cdot\boldsymbol{\partial}^2\vec{\Psi} \end{aligned} \tag{8.222}$$

$$\square_{\mathbf{g}}\mathbf{\Gamma}\cdot L = \square_{\mathbf{g}}\boldsymbol{\partial}\mathbf{g}\cdot L = \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}^2\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{8.223}$$

By (8.211b), we have:

$$\begin{aligned} \square_{\mathbf{g}}L\cdot\mathbf{\Gamma} = &\left(\nabla\!\!\!/_A \mathrm{tr}_{\not g}\chi + k_{AN}\mathrm{tr}_{\not g}\chi - k_{BN}\chi_{AB} + \mathbf{Riem}_{BLBA}\right)\mathbf{\Gamma}_A \\ &+ \chi_{AB}\left(\frac{1}{2}\chi_{AB}\mathbf{\Gamma}_{\underline{L}} + \frac{1}{2}\underline{\chi}_{AB}\mathbf{\Gamma}_L\right) - \left(\mathbf{D}_A(k_{AN}) + k_{AB}\theta_{AB}\right)\mathbf{\Gamma}_L \\ &- k_{AN}\left(\chi_{AB}\mathbf{\Gamma}_B - k_{AN}\mathbf{\Gamma}_L\right) \\ &- \frac{1}{2}\mathrm{tr}_{\not g}\chi(2\zeta_A\mathbf{\Gamma}_A + k_{NN}\mathbf{\Gamma}_L) + \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}k_{NN}\mathbf{\Gamma}_L \\ &+ \underline{L}(k_{NN})\mathbf{\Gamma}_L + 2\zeta_A\left(k_{NN}\mathbf{\Gamma}_A + \chi_{AB}\mathbf{\Gamma}_B - k_{AN}\mathbf{\Gamma}_L\right) - \frac{1}{2}\mathbf{Riem}^\alpha_{\ L\underline{L}L}\mathbf{\Gamma}_\alpha \\ =&\frac{1}{2}|\chi|^2_{\not g}\mathbf{\Gamma}_{\underline{L}} + \frac{1}{2}\chi_{AB}\underline{\chi}_{AB}\mathbf{\Gamma}_L + \underline{L}(k_{NN})\mathbf{\Gamma}_L \\ &- L\mathbf{\Gamma}_{\underline{L}}\mathbf{\Gamma}_L - \underline{L}\mathbf{\Gamma}_L\mathbf{\Gamma}_L - \mathbf{Riem}_{A\underline{L}AL}\mathbf{\Gamma}_L - \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}\mathbf{\Gamma}_A \\ &+ \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\zeta\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi},\widetilde{r}^{-1},\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}\right) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}^2\vec{\Psi} \\ &+ \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\left(\nabla\!\!\!/\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right) \\ =&\underline{L}(k_{NN})\mathbf{\Gamma}_L \\ &+ \widetilde{r}^{-2}\xi + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\left(\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\left(\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi}\right) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}^2\vec{\Psi} \\ &+ \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\nabla\!\!\!/\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})} \\ &+ \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\cdot\boldsymbol{\partial}\vec{\Psi} + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi},\zeta\right)\cdot\boldsymbol{\partial}\vec{\Psi}. \end{aligned} \tag{8.224}$$

The term $\widetilde{r}^{-2}\xi$ in (8.224) comes from terms $\frac{1}{2}|\chi|^2_{\not g}\mathbf{\Gamma}_{\underline{L}}$ and $\frac{1}{2}\chi_{AB}\underline{\chi}_{AB}\mathbf{\Gamma}_L$. And above we used:

$$
\begin{aligned}
-\frac{1}{2}\mathbf{Riem}^{\alpha}_{LL\underline{L}}\mathbf{\Gamma}_\alpha &= \mathbf{Riem}_{\underline{L}LL\underline{L}}\mathbf{\Gamma}_L - \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}\mathbf{\Gamma}_A \\
&= (-2\mathbf{Ric}_{L\underline{L}} - \mathbf{Riem}_{A\underline{L}AL})\mathbf{\Gamma}_L - \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}\mathbf{\Gamma}_A + \mathbf{Riem}_{BLBA}\mathbf{\Gamma}_A \\
&= -L\mathbf{\Gamma}_{\underline{L}}\mathbf{\Gamma}_L - \underline{L}\mathbf{\Gamma}_L\mathbf{\Gamma}_L - \mathbf{Riem}_{A\underline{L}AL}\mathbf{\Gamma}_L - \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}\mathbf{\Gamma}_A.
\end{aligned}
\tag{8.225}
$$

Combining (8.221), (8.222), (8.223) and (8.224), we obtain (8.211c.) □

Now we compute $L\mu + \mathrm{tr}_{\not g}\chi\mu$.

Transport equation for μ. We will use (8.65a), (8.65f) and (8.65g). Notice that $\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC}\hat{\chi}_{AB} = 0$.

$$
\begin{aligned}
L\mu + \mathrm{tr}_{\not g}\chi\mu =& L\underline{L}\mathrm{tr}_{\not g}\chi + \frac{1}{2}L(\mathrm{tr}_{\not g}\chi\mathrm{tr}_{\not g}\underline{\chi}) + \mathrm{tr}_{\not g}\chi\left(\underline{L}\mathrm{tr}_{\not g}\chi + \frac{1}{2}\mathrm{tr}_{\not g}\chi\mathrm{tr}_{\not g}\underline{\chi}\right) \\
=& [L,\underline{L}]\mathrm{tr}_{\not g}\chi + \underline{L}L\mathrm{tr}_{\not g}\chi + \frac{1}{2}L(\mathrm{tr}_{\not g}\chi\mathrm{tr}_{\not g}\underline{\chi}) + \mathrm{tr}_{\not g}\chi\underline{L}\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2\mathrm{tr}_{\not g}\underline{\chi} \\
=& 2(\underline{\zeta}_A - \zeta_A)\not\nabla_A\mathrm{tr}_{\not g}\chi + k_{NN}(\underline{L} - L)\mathrm{tr}_{\not g}\chi \\
&+ \underline{L}L\mathrm{tr}_{\not g}\chi + \frac{1}{2}L(\mathrm{tr}_{\not g}\chi\mathrm{tr}_{\not g}\underline{\chi}) + \mathrm{tr}_{\not g}\chi\underline{L}\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2\mathrm{tr}_{\not g}\underline{\chi}.
\end{aligned}
\tag{8.226}
$$

By (8.65a), we have

$$
\begin{aligned}
L\mu + \mathrm{tr}_{\not g}\chi\mu =& 2(\underline{\zeta}_A - \zeta_A)\not\nabla_A\mathrm{tr}_{\not g}\chi + k_{NN}\underline{L}\mathrm{tr}_{\not g}\chi - k_{NN}L\mathrm{tr}_{\not g}\chi \\
&+ \underline{L}\left(-\frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2 - |\hat{\chi}|^2_{\not g} - k_{NN}\mathrm{tr}_{\not g}\chi - \mathbf{Ric}_{LL}\right) + \frac{1}{2}L(\mathrm{tr}_{\not g}\chi\mathrm{tr}_{\not g}\underline{\chi}) + \mathrm{tr}_{\not g}\chi\underline{L}\mathrm{tr}_{\not g}\chi \\
&+ \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2\mathrm{tr}_{\not g}\underline{\chi} \\
=& 2(\underline{\zeta}_A - \zeta_A)\not\nabla_A\mathrm{tr}_{\not g}\chi + k_{NN}\underline{L}\mathrm{tr}_{\not g}\chi - k_{NN}L\mathrm{tr}_{\not g}\chi \\
&+ \left(-\mathrm{tr}_{\not g}\chi\underline{L}\mathrm{tr}_{\not g}\chi - 2\hat{\chi}\cdot\not\mathbf{D}_{\underline{L}}\hat{\chi} - k_{NN}\underline{L}\mathrm{tr}_{\not g}\chi - \mathrm{tr}_{\not g}\chi\underline{L}k_{NN} - \underline{L}\mathbf{Ric}_{LL}\right) \\
&+ \frac{1}{2}L(\mathrm{tr}_{\not g}\chi\mathrm{tr}_{\not g}\underline{\chi}) + \mathrm{tr}_{\not g}\chi\underline{L}\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2\mathrm{tr}_{\not g}\underline{\chi}.
\end{aligned}
\tag{8.227}
$$

By (8.65a), (8.65f) and (8.65g), we have:

$$
\begin{aligned}
L\mu + \mathrm{tr}_{\not g}\chi\mu =& 2(\underline{\zeta}_A - \zeta_A)\not\nabla_A \mathrm{tr}_{\not g}\chi - k_{NN} L\mathrm{tr}_{\not g}\chi + \mathrm{tr}_{\not g}\underline{\chi}\,|\hat{\chi}|^2_{\not g} + \mathrm{tr}_{\not g}\chi\hat{\underline{\chi}}_{AB}\hat{\chi}_{AB} - 4\not\nabla_A\zeta_B\hat{\chi}_{AB} \\
& - 2k_{NN}|\hat{\chi}|^2_{\not g} - 4\zeta_A\zeta_B\hat{\chi}_{AB} + 2\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC}\hat{\chi}_{AB} - 2\hat{\chi}_{AB}\mathbf{Riem}_{BL\underline{L}A} \\
& - \mathrm{tr}_{\not g}\chi \underline{L} k_{NN} - \underline{L}\mathbf{Ric}_{LL} + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2\mathrm{tr}_{\not g}\underline{\chi} \\
& + \frac{1}{2}\Big(-|\hat{\chi}|^2_{\not g}\mathrm{tr}_{\not g}\underline{\chi} - \mathrm{tr}_{\not g}\underline{\chi}\mathbf{Ric}_{LL} \\
& + \mathrm{tr}_{\not g}\chi\Big(2\not{\mathrm{div}}\,\underline{\zeta} - \hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + 2|\underline{\zeta}|^2_{\not g} + \mathbf{Riem}_{A\underline{L}LA}\Big) - (\mathrm{tr}_{\not g}\chi)^2\mathrm{tr}_{\not g}\underline{\chi}\Big) \\
=& 2(\underline{\zeta}_A - \zeta_A)\not\nabla_A \mathrm{tr}_{\not g}\chi - k_{NN} L\mathrm{tr}_{\not g}\chi + \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}\,|\hat{\chi}|^2_{\not g} \\
& - 4\not\nabla_A\zeta_B\hat{\chi}_{AB} - 4\zeta_A\zeta_B\hat{\chi}_{AB} - 2\hat{\chi}_{AB}\mathbf{Riem}_{BL\underline{L}A} \\
& - \mathrm{tr}_{\not g}\chi\left(\mathbf{Ric}_{L\underline{L}} - L(k_{NN}) + \mathbf{Riem}_{AL\underline{L}A} + 2k_{AN}\zeta_A\right) \\
& - \underline{L}\mathbf{Ric}_{LL} - \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}\mathbf{Ric}_{LL} + \frac{1}{2}\mathrm{tr}_{\not g}\chi\Big(2\not{\mathrm{div}}\,\underline{\zeta} + \hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + 2|\underline{\zeta}|^2_{\not g} + \mathbf{Riem}_{A\underline{L}LA}\Big) \\
=& - \mathrm{tr}_{\not g}\chi\mathbf{Ric}_{L\underline{L}} - \underline{L}\mathbf{Ric}_{LL} - \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}\mathbf{Ric}_{LL} \\
& - k_{NN}L\mathrm{tr}_{\not g}\chi + \mathrm{tr}_{\not g}\chi L k_{NN} + \widetilde{r}^{-1}\not{\mathrm{div}}\,\xi + \widetilde{r}^{-2}\xi \\
& + \mathrm{f}_{(\vec{L})}\cdot\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\zeta\right) + \mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\zeta\right)\cdot\boldsymbol{\partial}^2\vec{\Psi} \\
& + \mathrm{f}_{(\vec{L})}\cdot\not\nabla\zeta\cdot\hat{\chi} + \mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta\cdot\hat{\chi} + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\zeta\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right) \\
& + \lambda^{-1}\cdot\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\left(\vec{\mathscr{C}},\mathscr{D}\right) + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right) \\
& + \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right)\cdot\boldsymbol{\partial}\vec{\Psi} \\
& + \mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\left(\hat{\chi},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\hat{\chi}.
\end{aligned}
\tag{8.228}
$$

The term $\widetilde{r}^{-2}\xi$ in (8.228) comes from term $\mathrm{tr}_{\not g}\chi\mathbf{Riem}_{A\underline{L}LA}$ while $\widetilde{r}^{-1}\not{\mathrm{div}}\,\xi$ comes from the term $\mathrm{tr}_{\not g}\chi\not{\mathrm{div}}\,\zeta$

Let

$$
R(\mu) = -\mathrm{tr}_{\not g}\chi\mathbf{Ric}_{L\underline{L}} - \underline{L}\mathbf{Ric}_{LL} - \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}\mathbf{Ric}_{LL}. \tag{8.229}
$$

We have:

$$R(\mu) = -L\underline{L}\mathbf{\Gamma}_L - \frac{1}{2}\mathrm{tr}_{\not g}\chi\underline{L}\mathbf{\Gamma}_L - \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}L\mathbf{\Gamma}_L + k_{NN}\underline{L}\mathbf{\Gamma}_L \tag{8.230}$$
$$- \frac{1}{2}\mathrm{tr}_{\not g}\chi L\mathbf{\Gamma}_{\underline{L}} + 2\left(\underline{\zeta}_A - \zeta_A\right)e_A\mathbf{\Gamma}_L - k_{NN}L\mathbf{\Gamma}_L - \underline{L}k_{NN}\mathbf{\Gamma}_L - \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}k_{NN}\mathbf{\Gamma}_L$$
$$= \Box_{\mathbf{g}}\mathbf{\Gamma}_L - \not\Delta\mathbf{\Gamma}_L + 2k_{AN}\not\nabla_A\mathbf{\Gamma}_L$$
$$- \frac{1}{2}\mathrm{tr}_{\not g}\chi L\mathbf{\Gamma}_{\underline{L}} + 2\left(\underline{\zeta}_A - \zeta_A\right)e_A\mathbf{\Gamma}_L - k_{NN}L\mathbf{\Gamma}_L - \underline{L}k_{NN}\mathbf{\Gamma}_L - \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}k_{NN}\mathbf{\Gamma}_L.$$

Inserting (8.230) into (8.228), we have:

$$L\mu + \mathrm{tr}_{\not g}\chi\mu = \Box_{\mathbf{g}}\mathbf{\Gamma}_L - \not\Delta\mathbf{\Gamma}_L \tag{8.231}$$
$$- \frac{1}{2}\mathrm{tr}_{\not g}\chi L\mathbf{\Gamma}_{\underline{L}} - \underline{L}k_{NN}\mathbf{\Gamma}_L - k_{NN}L\mathrm{tr}_{\not g}\chi + \mathrm{tr}_{\not g}\chi Lk_{NN} + \widetilde{r}^{-1}\not{\mathrm{div}}\,\xi + \widetilde{r}^{-2}\xi$$
$$+ \mathrm{f}_{(\vec{L})}\cdot\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\zeta\right) + \mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\zeta\right)\cdot\boldsymbol{\partial}^2\vec{\Psi}$$
$$+ \mathrm{f}_{(\vec{L})}\cdot\not\nabla\zeta\cdot\hat{\chi} + \mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta\cdot\hat{\chi}$$
$$+ \lambda^{-1}\cdot\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\left(\vec{\mathscr{C}},\mathscr{D}\right) + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)$$
$$+ \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right)\cdot\boldsymbol{\partial}\vec{\Psi}$$
$$+ \mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\left(\hat{\chi},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\hat{\chi}$$
$$+ \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\zeta\right)\cdot\boldsymbol{\partial}\vec{\Psi}.$$

□

With the help of (8.231), (8.211c) and (8.206), we now can derive the transport equation for $\check{\mu}$.

Proof of $L\check{\mu}+\mathrm{tr}_{\not g}\chi\check{\mu}$ *in (8.146).*

$$\begin{aligned}
L\check{\mu}+\mathrm{tr}_{\not g}\chi\check{\mu} =& L\mu+\mathrm{tr}_{\not g}\chi\mu+2\left(L+\mathrm{tr}_{\not g}\chi\right)\not\Delta\sigma \\
&-\left(L+\mathrm{tr}_{\not g}\chi\right)\left(k_{NN}\mathrm{tr}_{\not g}\chi\right)+\frac{1}{2}\left(L+\mathrm{tr}_{\not g}\chi\right)\left(\mathrm{tr}_{\not g}\chi\mathbf{\Gamma}_L\right) \\
=&-\frac{1}{2}\mathrm{tr}_{\not g}\chi L\mathbf{\Gamma}_{\underline{L}}-\underline{L}k_{NN}\mathbf{\Gamma}_L-k_{NN}L\mathrm{tr}_{\not g}\chi+\mathrm{tr}_{\not g}\chi Lk_{NN} \\
&-Lk_{NN}\mathrm{tr}_{\not g}\chi-k_{NN}L\mathrm{tr}_{\not g}\chi-k_{NN}(\mathrm{tr}_{\not g}\chi)^2 \\
&+\frac{1}{2}L\mathrm{tr}_{\not g}\chi\mathbf{\Gamma}_{\underline{L}}+\frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2\mathbf{\Gamma}_{\underline{L}}+\frac{1}{2}\mathrm{tr}_{\not g}\chi L\mathbf{\Gamma}_{\underline{L}}+\underline{L}(k_{NN})\mathbf{\Gamma}_L \\
&-2\hat{\chi}_{AB}\mathbf{Riem}_{BL\underline{L}A}+\widetilde{r}^{-2}\xi+\widetilde{r}^{-1}\mathrm{d}\not{\mathrm{iv}}\xi \\
&+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\not\nabla^2\sigma+\mathrm{f}_{(\vec{L})}\cdot\left(\not\nabla\boldsymbol{\partial}\vec{\Psi},\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}\right)\cdot\not\nabla\sigma \\
&+\mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\boldsymbol{\partial}\vec{\Psi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\not\nabla\sigma+\mathrm{f}_{(\vec{L})}\cdot\left(\hat{\chi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\zeta\right)\cdot\boldsymbol{\partial}^2\vec{\Psi} \\
&+\mathrm{f}_{(\vec{L})}\cdot\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\zeta\right)+\mathrm{f}_{(\vec{L})}\cdot\not\nabla\zeta\cdot\hat{\chi} \\
&+\lambda^{-1}\cdot\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\left(\vec{\mathscr{C}},\mathscr{D}\right)+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right) \\
&+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\zeta,\hat{\chi}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right).
\end{aligned}\tag{8.232}$$

Let M be the first 11 terms of the RHS of (8.232), after rearranging the terms and cancellations, we have:

$$\begin{aligned}
M =&-k_{NN}L\mathrm{tr}_{\not g}\chi-k_{NN}L\mathrm{tr}_{\not g}\chi-k_{NN}(\mathrm{tr}_{\not g}\chi)^2+\frac{1}{2}L\mathrm{tr}_{\not g}\chi\mathbf{\Gamma}_{\underline{L}}+\frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2\mathbf{\Gamma}_{\underline{L}} \\
=&\left(2k_{NN}-\frac{1}{2}\mathbf{\Gamma}_{\underline{L}}\right)\left(|\hat{\chi}|^2_{\not g}+k_{NN}\mathrm{tr}_{\not g}\chi+\mathbf{Ric}_{LL}\right)+\frac{1}{4}(\mathrm{tr}_{\not g}\chi)^2\mathbf{\Gamma}_L \\
=&\widetilde{r}^{-2}\xi+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}\cdot\boldsymbol{\partial}\vec{\Psi}+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi} \\
&+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)\cdot\boldsymbol{\partial}\vec{\Psi} \\
&+\mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\widetilde{r}^{-1}\right)\cdot\left(\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\boldsymbol{\partial}\vec{\Psi}.
\end{aligned}\tag{8.233}$$

The term $\widetilde{r}^{-2}\xi$ in (8.233) comes from the term $\frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2\mathbf{\Gamma}_{\underline{L}}$. Combine (8.233) and (8.232), we obtain:

$$\begin{aligned} L\check{\mu}+\mathrm{tr}_{\not g}\chi\check{\mu} =& \widetilde{r}^{-2}\xi+\widetilde{r}^{-1}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\xi \\ &+\lambda^{-1}\cdot \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\left(\vec{\mathscr{C}},\mathscr{D}\right)+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\left(\vec{\mathscr{C}},\mathscr{D}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right) \\ &+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\not\nabla\widetilde{\zeta}+\mathrm{f}_{(\vec{L})}\cdot\left(\not\nabla\boldsymbol{\partial}\vec{\Psi},\not\nabla\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}\right)\cdot\not\nabla\sigma \\ &+\mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\boldsymbol{\partial}\vec{\Psi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\not\nabla\sigma+\mathrm{f}_{(\vec{L})}\cdot\left(\hat{\chi},\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\zeta\right)\cdot\boldsymbol{\partial}^2\vec{\Psi} \\ &+\mathrm{f}_{(\vec{L})}\cdot\not\nabla\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\zeta\right) \\ &+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\zeta,\hat{\chi}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right). \end{aligned} \tag{8.234}$$

□

Proof of a decomposition of $\not\mu$ and Hodge-transport systems related to this decompostion in (8.151)-(8.153b). We will use (8.51) and commutator formula (8.197) in the equations below.

$$\begin{aligned} \not\nabla_A\left(\not{\mathbf{D}}_L\not\mu+\frac{1}{2}\mathrm{tr}_{\not g}\chi\not\mu\right)_B =& \not\nabla_A\not{\mathbf{D}}_L\not\mu_B-\not{\mathbf{D}}_L\not\nabla_A\not\mu_B+\not{\mathbf{D}}_L\not\nabla_A\not\mu_B+\frac{1}{2}\not\nabla_A\mathrm{tr}_{\not g}\chi\not\mu_B+\frac{1}{2}\mathrm{tr}_{\not g}\chi\not\nabla_A\not\mu_B \\ =& \left(\chi_{AB}k_{CN}-\chi_{AC}k_{BN}-\mathbf{Riem}_{BCLA}\right)\not\mu_C \\ &+\hat{\chi}_{AC}\not\nabla_C\not\mu_B+\frac{1}{2}\not\nabla_A\mathrm{tr}_{\not g}\chi\not\mu_B \\ &+\mathrm{tr}_{\not g}\chi\not\nabla_A\not\mu_B+\not{\mathbf{D}}_L\not\nabla_A\not\mu_B. \end{aligned} \tag{8.235}$$

$$\begin{aligned} \mathrm{d}\not{\mathrm{i}}\mathrm{v}\left(\not{\mathbf{D}}_L\not\mu+\frac{1}{2}\mathrm{tr}_{\not g}\chi\not\mu\right) =& \left(\mathrm{tr}_{\not g}\chi k_{BN}-\chi_{AB}k_{AN}-\mathbf{Riem}_{ABLA}\right)\not\mu_B \\ &+\hat{\chi}_{AB}\not\nabla_B\not\mu_A+\frac{1}{2}\not\nabla_A\mathrm{tr}_{\not g}\chi\not\mu_A \\ &+\frac{1}{2}\left(L\left(\check{\mu}-\overline{\check{\mu}}\right)+\mathrm{tr}_{\not g}\chi\left(\check{\mu}-\overline{\check{\mu}}\right)\right) \\ =& \left(\mathrm{tr}_{\not g}\chi k_{BN}-\chi_{AB}k_{AN}-\mathbf{Riem}_{ABLA}\right)\not\mu_B \\ &+\hat{\chi}_{AB}\not\nabla_B\not\mu_A+\frac{1}{2}\not\nabla_A\mathrm{tr}_{\not g}\chi\not\mu_A \\ &+\frac{1}{2}\left\{\left(L\check{\mu}+\mathrm{tr}_{\not g}\chi\check{\mu}\right)-\overline{L\check{\mu}+\mathrm{tr}_{\not g}\chi\check{\mu}}-\left(\mathrm{tr}_{\not g}\chi-\overline{\mathrm{tr}_{\not g}\chi}\right)\overline{\check{\mu}}\right\} \\ =& \left(\mathrm{tr}_{\not g}\chi k_{BN}-\chi_{AB}k_{AN}-\mathbf{Riem}_{ABLA}\right)\not\mu_B \\ &+\hat{\chi}_{AB}\not\nabla_B\not\mu_A+\frac{1}{2}\not\nabla_A\mathrm{tr}_{\not g}\chi\not\mu_A \\ &+\mathfrak{J}^{(1)}-\overline{\mathfrak{J}^{(1)}}+\mathfrak{J}^{(2)}-\overline{\mathfrak{J}^{(2)}}-\frac{1}{2}\left(\mathrm{tr}_{\not g}\chi-\overline{\mathrm{tr}_{\not g}\chi}\right)\overline{\check{\mu}}. \end{aligned} \tag{8.236}$$

$$\mathrm{c\slashed{u}rl}\left(\slashed{\mathbf{D}}_L\slashed{\mu}+\frac{1}{2}\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}\right)=\varepsilon^{AB}\left(-\chi_{AC}k_{BN}-\mathbf{Riem}_{BCLA}\right)\slashed{\mu}_C + \varepsilon^{AB}\hat{\chi}_{AC}\slashed{\nabla}_C\slashed{\mu}_B+\frac{1}{2}\varepsilon^{AB}\slashed{\nabla}_A\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}_B. \tag{8.237}$$

We decompose $\slashed{\mu}=\slashed{\mu}_{(1)}+\slashed{\mu}_{(2)}$ where $\slashed{\mu}_{(1)}$ and $\slashed{\mu}_{(2)}$ verify the following:

$$\mathrm{d\slashed{i}v}\left(\slashed{\mathbf{D}}_L\slashed{\mu}_{(1)}+\frac{1}{2}\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}_{(1)}\right)=\mathfrak{I}_{(1)}-\overline{\mathfrak{I}_{(1)}}, \tag{8.238a}$$

$$\begin{aligned}\mathrm{d\slashed{i}v}\left(\slashed{\mathbf{D}}_L\slashed{\mu}_{(2)}+\frac{1}{2}\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}_{(2)}\right)=&\mathfrak{I}_{(2)}-\overline{\mathfrak{I}_{(2)}}-\frac{1}{2}\left(\mathrm{tr}_{\slashed{g}}\chi-\overline{\mathrm{tr}_{\slashed{g}}\chi}\right)\overline{\breve{\mu}}\\&+\left(\mathrm{tr}_{\slashed{g}}\chi k_{BN}-\chi_{AB}k_{AN}-\mathbf{Riem}_{ABLA}\right)\slashed{\mu}_B\\&+\hat{\chi}_{AB}\slashed{\nabla}_B\slashed{\mu}_A+\frac{1}{2}\slashed{\nabla}_A\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}_A\\=&\mathfrak{I}^{(2)}-\overline{\mathfrak{I}^{(2)}}+\left(\mathrm{tr}_{\slashed{g}}\chi-\overline{\mathrm{tr}_{\slashed{g}}\chi}\right)\overline{\breve{\mu}}\\&+\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi},\mathrm{tr}_{\tilde{\slashed{g}}}\tilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\slashed{\mu}\\&+\mathrm{f}_{(\vec{L})}\cdot\left(\slashed{\nabla}\boldsymbol{\partial}\vec{\Psi},\slashed{\nabla}\mathrm{tr}_{\tilde{\slashed{g}}}\tilde{\chi}^{(\mathrm{Small})}\right)\cdot\slashed{\mu}+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\slashed{\nabla}\slashed{\mu}.\end{aligned} \tag{8.238b}$$

By definition of $\slashed{\mu}$ (8.55), it's obvious to see that

$$\mathrm{c\slashed{u}rl}\left(\slashed{\mathbf{D}}_L\slashed{\mu}_{(1)}+\frac{1}{2}\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}_{(1)}\right)=0. \tag{8.239}$$

Since $\slashed{\mu}_{(1)}$ correspond to the terms $\mathrm{tr}_{\slashed{g}}\chi\slashed{\nabla}_A\slashed{\mu}_B$ and $\slashed{\mathbf{D}}_L\slashed{\nabla}_A\slashed{\mu}_B$ in (8.235), we have:

$$\begin{aligned}\mathrm{c\slashed{u}rl}\left(\slashed{\mathbf{D}}_L\slashed{\mu}_{(2)}+\frac{1}{2}\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}_{(2)}\right)=&\mathrm{c\slashed{u}rl}\left(\slashed{\mathbf{D}}_L\slashed{\mu}+\frac{1}{2}\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}\right)\\=&\varepsilon^{AB}\left(-\chi_{AC}k_{BN}-\mathbf{Riem}_{BCLA}\right)\slashed{\mu}_C\\&+\varepsilon^{AB}\hat{\chi}_{AC}\slashed{\nabla}_C\slashed{\mu}_B+\frac{1}{2}\varepsilon^{AB}\slashed{\nabla}_A\mathrm{tr}_{\slashed{g}}\chi\slashed{\mu}_B\\=&\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\slashed{g}}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\slashed{\mu}\\&+\mathrm{f}_{(\vec{L})}\cdot\left(\slashed{\nabla}\boldsymbol{\partial}\vec{\Psi},\slashed{\nabla}\mathrm{tr}_{\tilde{\slashed{g}}}\tilde{\chi}^{(\mathrm{Small})}\right)\cdot\slashed{\mu}+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\slashed{\nabla}\slashed{\mu}.\end{aligned} \tag{8.240}$$

□

CHAPTER 9

Control of the Acoustic Geometry

In this section, we prove the estimates for geometric quantities that are listed in Proposition 9.1.

9.1 Restatement of Bootstrap Assumptions and Estimates for Quantities Constructed out of the Acoustical Eikonal Equation

In this section, we restate the consequence of bootstrap assumptions of fluid variables, vorticity, and entropy gradient that we obtained in (6.13), followed by the bootstrap assumptions for acoustic geometry. Then we state the main estimates for the acoustical function quantities in Prop.9.1. The estimates in Prop.9.1 are needed for our conformal energy estimates and hence to close the whole bootstrap argument. We provide a discussion of the proof of Prop.9.1 in Section 9.2 via a bootstrap argument, where the bootstrap assumptions are listed in Section 9.1.2. For the complete details of the proof, we refer readers to [9, Section 10] and [36, Section 5-6].

9.1.1 The fixed number p

In the rest of the article, p denotes a fixed number with

$$0 < \delta_0 < 1 - \frac{2}{p} < N - 2, \tag{9.1}$$

where δ_0 is defined in Section 3.4.

9.1.2 Bootstrap assumptions for geometric quantities

After rescaling in Subsection 6.2, we make several bootstrap assumptions for the quantities in acoustic geometry. These assumptions will be recovered and improved by estimates in Proposition 9.1

For the rescaled solution variables defined in Definition 6.3, we recall the bootstrap assumptions (6.13) :

Estimates by using bootstrap assumptions of variables

$$\begin{aligned} &\left\|\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\omega}, \boldsymbol{\partial}\vec{S}, \vec{\mathscr{C}}, \mathscr{D}\right\|_{L_t^2 L_x^\infty(\mathscr{M})} \\ &+ \lambda^{\delta_0}\sqrt{\sum_{\nu>2}\nu^{2\delta_0}\left\|P_\nu\left(\mathrm{f}(\vec{\Psi},\vec{\omega},\vec{S})(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S},\vec{\mathscr{C}},\mathscr{D})\right)\right\|^2_{L_t^2 L_x^\infty(\mathscr{M})}} \lesssim \lambda^{-1/2-4\varepsilon_0}. \end{aligned} \tag{9.2}$$

We now make a few more bootstrap assumptions for the acoustical geometry:

Bootstrap assumptions for the acoustical geometry

$$\max_{A,B=1,2} \left\| \widetilde{r}^{-2} \not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) - \not{e}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) \right\|_{L^\infty_x(\mathscr{M})} \lesssim \lambda^{-\varepsilon_0}, \tag{9.3a}$$

$$\max_{A,B,C=1,2} \left\| \frac{\partial}{\partial \omega^C} \left(\widetilde{r}^{-2} \not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) - \not{e}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) \right) \right\|_{L^\infty_t L^p_\omega(\mathscr{C}_u)} \lesssim \lambda^{-\varepsilon_0}. \tag{9.3b}$$

Also,

$$\left\| \mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta \right\|_{L^2_t C^{0,\delta_0}_\omega(\mathscr{C}_u)} \lesssim \lambda^{-1/2+2\varepsilon_0}. \tag{9.4}$$

Moreover,

$$\left\| \widetilde{r}\left(\mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta \right) \right\|_{L^p_\omega(S_{t,u})} \le 1, \tag{9.5a}$$

$$\| b - 1 \|_{L^\infty_\omega(S_{t,u})} \le \frac{1}{2}, \tag{9.5b}$$

$$\left\| \mathrm{tr}_{\not{g}} \theta - \frac{2}{\widetilde{r}} \right\|_{L^3(\Sigma_t)} \le 1. \tag{9.5c}$$

Finally, we assume that the following estimates hold in the interior region $\mathscr{M}^{(\mathrm{Int})}$ (defined in Section 7.1):

$$\left\| \mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi} \right\|_{L^2_t L^\infty_u C^{0,\delta_0}_\omega(\mathscr{M}^{(\mathrm{Int})})} \le \lambda^{-1/2}, \quad \|\zeta\|_{L^2_t L^\infty_x(\mathscr{M}^{(\mathrm{Int})})} \le \lambda^{-1/2}, \quad \|\not{\nabla}\sigma\|_{L^2_u L^2_t L^\infty_\omega(\mathscr{M}^{(\mathrm{Int})})} \le 1. \tag{9.6}$$

Proposition 9.1 (The main estimates for the eikonal function quantities)**.** *Under the bootstrap assumptions, we have the following estimates where* $2 < q \le 4$*:*

Estimates for connection coefficients

$$\left\| \mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta \right\|_{L^2_t L^p_\omega(\mathscr{C}_u)}, \left\| \widetilde{r} \not{\mathbf{D}}_L \left(\mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta \right) \right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \lesssim \lambda^{-1/2}, \tag{9.7a}$$

$$\left\| \widetilde{r}^{1/2} \left(\mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta \right) \right\|_{L^\infty_t L^p_\omega(\mathscr{C}_u)} \lesssim \lambda^{-1/2}, \tag{9.7b}$$

$$\left\| \widetilde{r} \left(\mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta \right) \right\|_{L^\infty_t L^p_\omega(\mathscr{C}_u)} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.7c}$$

$$\widetilde{r}\mathrm{tr}_{\widetilde{g}}\widetilde{\chi} \approx 1, \tag{9.8a}$$

$$\left\| \widetilde{r}^{1/2}\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)} \right\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-1/2}, \tag{9.8b}$$

$$\left\| \widetilde{r}^{3/2}\nabla\!\!\!/\,\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)} \right\|_{L_t^\infty L_u^\infty L_\omega^p(\mathscr{M})} \lesssim \lambda^{-1/2}, \tag{9.8c}$$

$$\left\| \widetilde{r}\left(\nabla\!\!\!/\,\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \nabla\!\!\!/\,\hat{\chi}\right) \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}, \tag{9.8d}$$

$$\left\| \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta \right\|_{L_t^2 C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.8e}$$

In addition, the null lapse b verifies the following:

$$\left\| \frac{b^{-1}-1}{\widetilde{r}} \right\|_{L_t^2 L_x^\infty(\mathscr{M})}, \left\| \frac{b^{-1}-1}{\widetilde{r}^{1/2}} \right\|_{L_t^\infty L_u^\infty L_\omega^{2p}(\mathscr{M})}, \left\| \widetilde{r}(\mathbf{D}_L, \nabla\!\!\!/)\left(\frac{b^{-1}-1}{\widetilde{r}}\right) \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.9}$$

Moreover, we have:

$$\left\| \mathrm{f}_{(\vec{L})} \right\|_{L_t^\infty L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \lesssim 1. \tag{9.10}$$

Furthermore,

$$\left\| \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \mathrm{tr}_{g\!\!\!/}\chi - \frac{2}{\widetilde{r}} \right\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}, \tag{9.11}$$

$$\|\zeta\|_{L_t^2 L_x^\infty(\mathscr{M})} \lesssim \lambda^{-1/2-3\varepsilon_0}, \qquad \|\zeta\|_{L_t^{\frac{q}{2}} L_x^\infty(\mathscr{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}. \tag{9.12}$$

Improved estimates in the interior region

$$\left\| \frac{b^{-1}-1}{\widetilde{r}} \right\|_{L_t^2 L_x^\infty(\mathscr{M}^{(Int)})} \lesssim \lambda^{-1/2-4\varepsilon_0}, \tag{9.13}$$

$$\left\| \widetilde{r}^{1/2}\left(\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta\right) \right\|_{L_\omega^{2p} L_t^\infty(\mathscr{C}_u)} \lesssim \lambda^{-1/2}, \qquad \text{if } \mathscr{C}_u \subset \mathscr{M}^{(Int)}, \tag{9.14}$$

$$\left\| \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \mathrm{tr}_{g\!\!\!/}\chi - \frac{2}{\widetilde{r}} \right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(Int)})} \lesssim \lambda^{-1/2-3\varepsilon_0}. \tag{9.15}$$

Estimates for the geometric angular coordinate components of $g\!\!\!/$

$$\max_{A,B=1,2} \left\| \widetilde{r}^{-2} g\!\!\!/\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - e\!\!\!/\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) \right\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-4\varepsilon_0} \tag{9.16a}$$

$$\max_{A,B,C=1,2} \left\| \frac{\partial}{\partial\omega^C}\left(\widetilde{r}^{-2} g\!\!\!/\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - e\!\!\!/\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right)\right) \right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.16b}$$

***Estimates for* υ *and* b**

$$\upsilon := \frac{\sqrt{\det \not g}}{\sqrt{\det \not e}} \approx \widetilde{r}^2, \tag{9.17a}$$

$$\|b-1\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-4\varepsilon_0} < \frac{1}{4}. \tag{9.17b}$$

Furthermore,

$$\left\|\widetilde{r}^{1/2}\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L_t^\infty L_u^\infty L_\omega^p(\mathscr{M})}, \left\|\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)}, \left\|\widetilde{r}L\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.18}$$

***Estimates for* μ *and* $\nabla\!\!\!/ \zeta$**

$$\|\widetilde{r}\mu, \widetilde{r}\nabla\!\!\!/ \zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.19}$$

***Interior region estimates for* σ**

$$\left\|\widetilde{r}^{1/2}L\sigma\right\|_{L_t^\infty L_\omega^{2p}(\mathscr{C}_u)} \lesssim \lambda^{-1/2-2\varepsilon_0}, \left\|\widetilde{r}^{1/2}\nabla\!\!\!/ \sigma\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}, \|\nabla\!\!\!/ \sigma\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}, \qquad \textit{if } \mathscr{C}_u \subset \mathscr{M}^{(Int)}, \tag{9.20a}$$

$$\|\sigma\|_{L^\infty\left(\mathscr{M}^{(Int)}\right)} \lesssim \lambda^{-8\varepsilon_0}, \tag{9.20b}$$

$$\left\|\widetilde{r}^{-1/2}\sigma\right\|_{L^\infty\left(\mathscr{M}^{(Int)}\right)} \lesssim \lambda^{-1/2-4\varepsilon_0}. \tag{9.20c}$$

***Interior region estimates for* σ*,* μ*,* $\tilde{\zeta}$ *and* $\not\mu$**

$$\|\nabla\!\!\!/ \sigma\|_{L_u^2 L_t^2 C_\omega^{0,\delta_0}(\mathscr{M}^{(Int)})}, \|\widetilde{r}\check{\mu}, \widetilde{r}\nabla\!\!\!/ \tilde{\zeta}\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(Int)})} \lesssim \lambda^{-4\varepsilon_0}, \tag{9.21a}$$

$$\left\|\widetilde{r}^{\frac{3}{2}}\check{\mu}\right\|_{L_u^2 L_t^\infty L_\omega^p(\mathscr{M}^{(Int)})} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.21b}$$

In addition,

$$\|\widetilde{r}\nabla\!\!\!/ \not\mu, \not\mu\|_{L_t^2 L_u^2 L_\omega^p(\mathscr{M}^{(Int)})}, \|\not\mu\|_{L_t^2 L_u^2 C_\omega^{0,\delta_0}(\mathscr{M}^{(Int)})} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.22}$$

Decomposition of* $\nabla\!\!\!/ \sigma$ *and corresponding estimates in the interior region*: In* $\mathscr{M}^{(Int)}$*, we can decompose* $\nabla\!\!\!/ \sigma$ *as follows:*

$$\nabla\!\!\!/ \sigma = -\zeta + (\tilde{\zeta} - \not\mu) + \not\mu_{(1)} + \not\mu_{(2)}, \tag{9.23}$$

where the following asymptotic conditions near the cone-tip axis are satisfied:

$$\widetilde{r}\not{\mu}_{(1)}(t,u,\omega), \widetilde{r}\not{\mu}_{(2)}(t,u,\omega) = \mathscr{O}(\widetilde{r}) \text{ as } t \downarrow u. \tag{9.24}$$

Moreover,

$$\left\|\tilde{\zeta} - \not{\mu}\right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M}^{(Int)})}, \left\|\not{\mu}_{(1)}\right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M}^{(Int)})} \lesssim \lambda^{-1/2-4\varepsilon_0}, \tag{9.25a}$$

$$\left\|\not{\mu}_{(2)}\right\|_{L_u^2 L_t^\infty L_\omega^\infty(\mathscr{M}^{(Int)})} \lesssim \lambda^{-1/2-3\varepsilon_0}. \tag{9.25b}$$

9.2 Discussion of the proof of Proposition 9.1

In this subsection, we discuss the proof of Proposition 9.1. For the details of the proof, we refer readers to [9, Section 10] and [36, Section 5-6].

The proof consists of 3 main steps:

1. First of all, in Lemma 8.14 and Proposition 8.20, we derived the transport equations along null hypersurfaces and div-curl systems[1] on $S_{t,u}$ verified by the geometric quantities. These equations are derived using basic differential geometry and, at the appropriate spots, using the relativistic Euler equations for substitution for reasons further described in Step 2 below. The key point is that *all of the equations we obtain have the exact same schematic structure as the equations in [9].* We refer readers to [14, Section 2] for the PDEs satisfied by connection coefficients and mass aspect function μ. We refer readers to [36, Section 6] for the PDEs of conformal factor σ, modified torsion $\tilde{\zeta}$ and modified mass aspect function $\check{\mu}, \not{\mu}$.

2. Secondly, as showed in Corollary 8.17 and Corollary 8.19, certain Ricci and Riemann curvature tensor components, which appear as source terms in the PDEs that we just obtained in the previous step, are rewritten by using Bianchi identities and the decomposition of the following Ricci curvature (see Lemma 8.16):

$$\mathbf{Ric}_{\alpha\beta} = -\frac{1}{2}\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi}) + \frac{1}{2}\left(\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha\right) + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{9.26}$$

It is at this step that the wave equation (2.29) of the geometric formulation of the relativistic Euler equations is used to substitute for the term $\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta}(\vec{\Psi})$, as we alluded to in the previous step. We again

[1] These div-curl systems depend on the acoustic geometry and are independent of the structure of the relativistic Euler equations. Therefore, these div-curl systems are completely unrelated to the ones that we derived for the vorticity and entropy gradient in Section 5.

emphasize that following this substitution, one obtains equations of the exact same schematic form as in [9].

After the substitution, one is faced with controlling the source terms in the geometric equations in mixed space-time norms. The source terms depend on $\boldsymbol{\partial}\vec{\Psi}, \vec{\omega}, \vec{S}, \vec{\mathscr{C}}, \mathscr{D}$ and have the same schematic structure as in [9]. This step requires trace inequalities and Sobolev inequalities, as we will discuss in Proposition 9.3, which are provided by [9, Proposition 10.2] and [36, (5.34)-(5.39)]; the proofs of these inequalities are the same as in [9], and rely on bootstrap assumptions (3.24b)-(3.24c), energy estimates on constant-time hypersurfaces (5.1), and energy estimates on null hypersurfaces (8.3)-(8.4).

3. After one has controlled the source terms in the geometric PDEs for the acoustic geometry from Step 1, one uses a transport lemma and div-curl estimates to obtain various mixed space-time norms estimates for $\widetilde{r}^p$ weighted acoustic quantities in Proposition 9.1. We provide the proof of Proposition 9.1 in Section 9.4. We also refer readers to [9, Section 10.9] and [36, Section 5-6] for the detailed proof. We emphasize that the proof is the same as in [9] because it relies only on the bootstrap assumptions (3.24b)-(3.24c) and source term bounds from Step 2, ingredients which are already available to us at this step in the proof.

We now point out some differences between the non-relativistic 3D compressible Euler equations and the relativistic Euler equations in terms of the control of acoustic geometry. Besides the different acoustic metric $\mathbf{g}$ in this paper compared to [9], the $\mathbf{g}$-Σ_t-normal vectorfield is $\mathbf{T}$ (see (2.17)) in this paper, while in the non-relativistic case [9], it is $\mathbf{B} = \partial_t + v^a\partial_a$. Although these differences have necessitated changes to some of the *proofs* earlier in the paper (such as the proof of the energy estimates on constant-time hypersurfaces and the acoustic null hypersurfaces), these changes do not have any effect on the proofs of the estimate for the acoustic geometry; it is for this reason that we refer to [9, Section 10] for the details behind the proof of Proposition 9.1.

9.3 Preparations for the Proofs of Main Estimates for the Eikonal Function Quatities

Proposition 9.2 (Hardy-Littlewood maximal function)**.** *If $f = f(t)$ is a scalar function that is defined on the interval I, we define the Hardy-Littlewood maximal function $\mathscr{M}(f)(t)$ as follows:*

$$\mathscr{M}(f)(t) := \sup_{t' \in I \bigcup (\infty, t)} \frac{1}{|t-t'|} \int_{t'}^{t} f(\tau) \mathrm{d}\tau. \tag{9.27}$$

We will use the following well-known estimate, valid for $1 < Q \leq \infty$*:*

$$\|\mathscr{M}(f)\|_{L^Q(I)} \lesssim \|f\|_{L^Q(I)}. \tag{9.28}$$

Proposition 9.3 (Calculus and elliptic estimates)**.** *Under the bootstrap assumptions, we have the following estimates for* $S_{t,u}$*-tangent tensorfields* ξ*:*

Comparison of $S_{t,u}$***-norms with different volume forms****: If* $1 \leq Q < \infty$*, we have*

$$\|\xi\|_{L^Q_{g}(S_{t,u})} \approx \left\|\widetilde{r}^{\frac{2}{Q}}\xi\right\|_{L^Q_{\omega}(S_{t,u})}. \tag{9.29}$$

Trace inequalities

$$\left\|\widetilde{r}^{-\frac{1}{2}}\xi\right\|_{L^2_{g}(S_{t,u})} + \|\xi\|_{L^4_{g}(S_{t,u})} \lesssim \|\xi\|_{H^1(\Sigma_t)}. \tag{9.30}$$

Sobolev and Morrey-type inequalities

$$\|\xi\|_{L^2_u L^2_{\omega}(\Sigma_t)} \lesssim \|\xi\|_{H^1(\Sigma_t)}, \tag{9.31}$$

$$\left\|\widetilde{r}^{\frac{1}{2}}\xi\right\|^2_{L^{2p}_{\omega}L^{\infty}_t(\mathscr{C}_u)} \lesssim \left(\|\widetilde{r}\,\mathbf{D}\!\!\!/_L\xi\|_{L^p_{\omega}L^2_t(\mathscr{C}_u)} + \|\xi\|_{L^p_{\omega}L^2_t(\mathscr{C}_u)}\right)\|\xi\|_{L^{\infty}_{\omega}L^2_t(\mathscr{C}_u)}. \tag{9.32}$$

Furthemore, if $2 < Q < \infty$*, we have*

$$\|\xi\|_{L^Q_{\omega}(S_{t,u})} \lesssim \|\widetilde{r}\nabla\!\!\!/\xi\|^{1-\frac{2}{Q}}_{L^2_{\omega}(S_{t,u})}\|\xi\|^{\frac{2}{Q}}_{L^2_{\omega}(S_{t,u})} + \|\xi\|_{L^2_{\omega}(S_{t,u})}, \tag{9.33a}$$

$$\|\xi\|_{C^{0,1-\frac{2}{Q}}_{\omega}(S_{t,u})} \lesssim \|\widetilde{r}\nabla\!\!\!/\xi\|_{L^Q_{\omega}(S_{t,u})} + \|\xi\|_{L^2_{\omega}(S_{t,u})}. \tag{9.33b}$$

In addition, if $2 \leq Q$*, we have*

$$\left\|\widetilde{r}^{\frac{1}{2}-\frac{1}{Q}}\xi\right\|^2_{L^{2Q}_{g}L^{\infty}_u(\Sigma_t)} \lesssim \left(\|\widetilde{r}(\mathbf{D}\!\!\!/_N,\nabla\!\!\!/)\xi\|_{L^Q_{\omega}L^2_u(\Sigma_t)} + \|\xi\|_{L^Q_{\omega}L^2_u(\Sigma_t)}\right)\|\xi\|_{L^{\infty}_{\omega}L^2_u(\Sigma_t)}. \tag{9.34}$$

Finally, if $0 < 1 - \frac{2}{Q} < N - 2$*, then for any scalar function* f*, we have*

$$\|\widetilde{r}f\|_{L^2_u L^Q_{\omega}(\Sigma_t)} \lesssim \|f\|_{H^{N-2}(\Sigma_t)}. \tag{9.35a}$$

Lemma 9.4 (Lemmas for trace inequalities and Sobolev inequalities)**.** *We also refer readers to [6, 33] for the original ideas of the proof.*

Under the bootstrap assumptions, the following estimate holds with ξ as a $S_{t,u}$-tangent tensorfield.

$$\left\|\widetilde{r}^{-\frac{1}{2}}\xi\right\|_{L^2_{\not{g}}(S_{t,u})} \lesssim \|\partial\xi\|_{L^2(\Sigma_t)} + \|\xi\|_{L^2(\Sigma_t)}. \tag{9.36}$$

Proof of Lemma 9.4. Recall that $\widetilde{r} \approx t-u$. By the bootstrap assumption (9.3a), we have

$$\|\xi\|_{L^2_{\not{g}}(S_{t,u})} \approx \int_{\mathbb{S}^2} |\xi|^2 (t-u)^2 \mathrm{d}\varpi_{\not{e}}. \tag{9.37}$$

For a fixed $\omega \in \mathbb{S}^2$, let

$$\xi_{(1)} := \frac{2}{t-u}\int_0^{\frac{t-u}{2}} \xi(-u'+u, -\omega'+\omega)\mathrm{d}u', \tag{9.38}$$

$$\xi_{(2)} := \xi_{(1)} - \xi, \tag{9.39}$$

where $-u'+u$ is viewed as a space radial variable and ω' will be determined later.

It is suffice to prove that:

$$\left\|\widetilde{r}^{-\frac{1}{2}}\xi_{(2)}\right\|_{L^2_{\not{g}}(S_{t,u})} \lesssim \|\partial\xi\|_{L^2(\Sigma_t)}, \tag{9.40}$$

$$\left\|\widetilde{r}^{-\frac{1}{2}}\xi_{(1)}\right\|_{L^2_{\not{g}}(S_{t,u})} \lesssim \|\xi\|_{H^1(\Sigma_t)}. \tag{9.41}$$

We first prove (9.40). By definition and the Fundamental Theorem of Calculus, we have:

$$\xi_{(2)} = \frac{2}{t-u}\int_0^{\frac{t-u}{2}}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}l}\xi(u-lu', \omega-l\omega')\mathrm{d}l\mathrm{d}u'. \tag{9.42}$$

By definition of N (as in (7.12)), and bootstrap assumption (9.5b), let ω' be such that $(-u'+u, -\omega'+\omega)$ and (u,ω) are on the same integral curve of some N. Then, we have:

$$\left|\frac{\mathrm{d}}{\mathrm{d}l}\xi(u-lu', \omega-l\omega')\right|^2 \lesssim \left|u'\not{\mathbf{D}}_N\xi\right|^2. \tag{9.43}$$

For the last inequality, we used equation (8.136), initial condition (8.63b), (8.64a) to show that $u' \lesssim \widetilde{r}$, combining above equations, we have:

$$\left\|\widetilde{r}^{-\frac{1}{2}}\xi_{(2)}\right\|_{L^2_{\not g}(S_{t,u})} \lesssim \widetilde{r}^{-\frac{1}{2}}\int_0^1 \mathrm{d}l \int_0^{\frac{t-u}{2}} \widetilde{r}\left\|\not{D}_N\xi(-lu'+u,\omega)\right\|_{L^2_\omega(S_{t,-lu'+u})}\mathrm{d}u' \tag{9.44}$$

$$\lesssim \widetilde{r}^{-\frac{1}{2}}\int_0^1 l^{-1}\int_0^{\frac{l(t-u)}{2}} \widetilde{r}\|\not{D}_N\xi(-y+u,\omega)\|_{L^2_\omega(S_{t,-y+u})}\mathrm{d}y\mathrm{d}l$$

$$\lesssim \widetilde{r}^{-\frac{1}{2}}\int_0^{\frac{t-u}{2}}\int_{\frac{2y}{t-u}}^1 l^{-1}\mathrm{d}l\,\|\not{D}_N\xi(-y+u,\omega)\|_{L^2_{\not g}(S_{t,-y+u})}\,\mathrm{d}y$$

$$\lesssim (\widetilde{r})^{-\frac{1}{2}}\int_0^{\frac{t-u}{2}} \ln(\frac{t-u}{2y})\,\|\not{D}_N\xi(-y+u,\omega)\|_{L^2_{\not g}(S_{t,-y+u})}\,\mathrm{d}y,$$

where in the first inequality, we used (9.42), (9.43) and the Fubini's theorem; in the second inequality, we let $y = lu'$; and in the third inequality, we swapped the order of integration and used (9.29) (the proof of (9.29) does not require Lemma 9.4).

Then, using the Hölder's inequality, writing $\ln(\frac{t-u}{2y}) = \ln(\widetilde{r}/2) - \ln(y)$ and substituting y by $\frac{\widetilde{r}}{2}z$, we have:

$$\left\|\widetilde{r}^{-\frac{1}{2}}\xi_{(2)}\right\|_{L^2_{\not g}(S_{t,u})} \lesssim \widetilde{r}^{-\frac{1}{2}}\left\{\left(\int_0^{\widetilde{r}/2}\ln^2(\widetilde{r}/2)\mathrm{d}y\right)^{1/2} + \left(\frac{\widetilde{r}}{2}\int_0^1 \ln^2(\frac{\widetilde{r}}{2}z)\mathrm{d}z\right)^{1/2}\right\}\|\not{D}_N\xi\|_{L^2(\Sigma_t)} \lesssim \|\partial\xi\|_{L^2(\Sigma_t)} \tag{9.45}$$

To show (9.41), by $\widetilde{r}' := t-u+u'$, the Minkowski's inequality, the Hölder's inequality and the Sobolev embedding $L^6 \hookrightarrow H^1$, we have:

$$\|\xi_{(1)}\|_{L^2_{\not g}(S_{t,u})} \lesssim \int_0^{\frac{t-u}{2}} \left\|(\widetilde{r}')^{1/3}\xi\right\|_{L^6_\omega}(\widetilde{r}')^{-1/3}\mathrm{d}u' \lesssim \widetilde{r}^{1/2}\|\xi\|_{L^6(\Sigma_t)} \lesssim \widetilde{r}^{1/2}\|\xi\|_{H^1(\Sigma_t)}. \tag{9.46}$$

□

9.3.1 Proof of Proposition 9.3

Proof of (9.29).

$$\|\xi\|_{L^Q_{\not g}(S_{t,u})} = \left(\int_{\mathbb{S}^2}|\xi|^Q_{\not g}\,d\varpi_{\not g}\right)^{\frac{1}{Q}} \approx \left(\int_{\mathbb{S}^2}\widetilde{r}^2|\xi|^Q_{\not g}\,d\varpi_{\not e}\right)^{\frac{1}{Q}} = \left\|\widetilde{r}^{\frac{2}{Q}}\xi\right\|_{L^Q_\omega(S_{t,u})}. \tag{9.47}$$

□

Proof of (9.30).

$$\left\|\widetilde{r}^{-\frac{1}{2}}\xi\right\|^2_{L^2_{\not g}(S_{t,u})} = \left(\int_{\mathbb{S}^2}\widetilde{r}^{-1}|\xi|^2_{\not g}\,d\varpi_{\not g}\right) \le \|\widetilde{r}^{-1}\|_{L^2_{\not g}(S_{t,u})}\left(\int_{\mathbb{S}^2}|\xi|^4_{\not g}\,d\varpi_{\not g}\right)^{\frac{1}{2}} \approx \|\xi\|^2_{L^4_{\not g}(S_{t,u})}. \tag{9.48}$$

We now prove $\|\xi\|_{L^4_{\not g}(S_{t,u})} \lesssim \|\xi\|_{H^1(\Sigma_t)}$. For fixed t and u, let φ be a cut-off function of u verifying $0 \leq \varphi \leq 1$, $\varphi(u) = 1$ and $\mathrm{supp}(\varphi) \subset [u - \frac{t-u}{2}, u + \frac{t-u}{2}]$. Integrating on the region $\bigcup\limits_{u \leq u' \leq \frac{t+u}{2}} S_{t,u'}$, by the Fundamental Theorem of Calcules, we have:

$$\|\xi\|^4_{L^4_{\not g}(S_{t,u})} = -\int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\not{D}_N |\varphi \xi|^4 + \mathrm{tr}_{\not g}\theta \, |\varphi \xi|^4 \right) b \mathrm{d}\varpi_{\not g} \mathrm{d}u' \tag{9.49}$$

$$= -\int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\varphi^4 |\xi|^3 \cdot \not{D}_N \xi + \varphi^3 |\xi|^4 \cdot \not{D}_N \varphi + \mathrm{tr}_{\not g}\theta \, |\varphi \xi|^4 \right) b \mathrm{d}\varpi_{\not g} \mathrm{d}u'.$$

For the first term on the RHS of (9.49), by bootstrap assumption (9.5b), the Hölder's inequality and Sobolev embedding $L^6 \hookrightarrow H^1$, we have:

$$\left| \int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\varphi^4 |\xi|^3 \cdot \not{D}_N \xi \right) b \mathrm{d}\varpi_{\not g} \mathrm{d}u' \right| \lesssim \|\not{D}_N \xi\|_{L^2(\Sigma_t)} \left\| |\xi|^3 \right\|_{L^2(\Sigma_t)} \lesssim \|\xi\|^4_{H^1(\Sigma_t)}, \tag{9.50}$$

For the second term on the RHS of (9.49), by bootstrap assumption (9.5b), and noticing that $|\not{D}_N \varphi| \lesssim \frac{1}{t-u}$, we have

$$\left| \int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\varphi^3 |\xi|^4 \cdot \not{D}_N \varphi \right) b \mathrm{d}\varpi_{\not g} \mathrm{d}u' \right| \lesssim \left| \frac{1}{t-u} \int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} |\xi|^4 \mathrm{d}\varpi_{\not g} \mathrm{d}u' \right|. \tag{9.51}$$

By Lemma 9.4,

$$\int_{S_{t,u'}} |\xi|^4 \mathrm{d}\varpi_{\not g} \lesssim \left\| \widetilde{r}^{-\frac{1}{2}} \xi \right\|_{L^2_{\not g}(S_{t,u'})} \left\| |\xi|^3 \right\|_{L^2_{\not g}(S_{t,u'})} \widetilde{r}^{\frac{1}{2}} \lesssim \|\xi\|_{H^1(\Sigma_t)} \left\| |\xi|^3 \right\|_{L^2_{\not g}(S_{t,u'})} \widetilde{r}^{\frac{1}{2}}. \tag{9.52}$$

Then by (9.29), (9.51), (9.52), the Hölder's inequality, the Sobolev embedding $L^6 \hookrightarrow H^1$, we have:

$$\left| \frac{1}{t-u} \int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} |\xi|^4 \mathrm{d}\varpi_{\not g} \mathrm{d}u' \right| \lesssim \frac{1}{t-u} \|\xi\|_{H^1(\Sigma_t)} \left\| |\xi|^3 \right\|_{L^2(\Sigma_t)} \left(\int_{u-\frac{t-u}{2}}^{u} \widetilde{r}(t,u') \mathrm{d}u' \right)^{1/2} \tag{9.53}$$

$$\lesssim \|\xi\|^4_{H^1(\Sigma_t)}.$$

Now we consider the third term on the RHS of (9.49), that is, $\int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\mathrm{tr}_{\not g}\theta \, |\varphi \xi|^4 \right) b \mathrm{d}\varpi_{\not g} \mathrm{d}u'$. By bootstrap assumptions (6.13), (9.4) and (9.5b) and the Hölder's inequality, we have:

$$\int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\mathrm{tr}_{\not g}\theta \, |\varphi \xi|^4 \right) b \mathrm{d}\varpi_{\not g} \mathrm{d}u' \lesssim \int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\widetilde{r}^{-1} |\varphi \xi|^4 \right) \mathrm{d}\varpi_{\not g} \mathrm{d}u' \tag{9.54}$$

$$+ \int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\left| \mathrm{tr}_{\not g}\theta - \frac{2}{\widetilde{r}} \right| |\varphi \xi|^4 \right) \mathrm{d}\varpi_{\not g} \mathrm{d}u'$$

$$\lesssim \int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\widetilde{r}^{-1} |\xi|^4 \right) \mathrm{d}\varpi_{\not g} \mathrm{d}u' + \left\| \mathrm{tr}_{\not g}\theta - \frac{2}{\widetilde{r}} \right\|_{L^3(\Sigma_t)} \|\xi\|^4_{L^6(\Sigma_t)}.$$

By bootstrap assumptions (9.5c) where we have $\left\|\mathrm{tr}_{\not g}\theta - \frac{2}{\widetilde{r}}\right\|_{L^3(\Sigma_t)} \leq 1$, and Sobolev embedding $L^6 \hookrightarrow H^1$ for $\|\xi\|^4_{L^6(\Sigma_t)}$, we have:

$$\int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\mathrm{tr}_{\not g}\theta\, |\varphi\xi|^4\right) b \mathrm{d}\varpi_{\not g} \mathrm{d}u' \lesssim \int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\widetilde{r}^{-1} |\xi|^4\right) \mathrm{d}\varpi_{\not g} \mathrm{d}u' + \|\xi\|^4_{H^1(\Sigma_t)}. \tag{9.55}$$

Using Lemma 9.4, the Hólder inequality and the Sobolev embedding $L^6 \hookrightarrow H^1$, we have:

$$\int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \left(\widetilde{r}^{-1} |\xi|^4\right) \mathrm{d}\varpi_{\not g} \mathrm{d}u' \lesssim \|\xi\|_{H^1(\Sigma_t)} \left\||\xi|^3\right\|_{L^2(\Sigma_t)} \left(\int_{u-\frac{t-u}{2}}^{u} \int_{S_{t,u'}} \widetilde{r}^{-1}(t,u') \mathrm{d}u'\right)^{1/2} \lesssim \|\xi\|^4_{H^1(\Sigma_t)}. \tag{9.56}$$

□

Proof of (9.31). Replace ξ by $\xi^{1/2}$ in (9.49). By the same approach, we derive the following:

$$\|\xi\|^2_{L^2_{\not g}(S_{t,u})} \lesssim \|\xi\|_{H^1(\Sigma_t)} \|\xi\|_{L^2(\Sigma_t)}. \tag{9.57}$$

Integrating (9.57) with respect to u on Σ_t, we obtain (9.31). □

Proof of (9.32). We define $\mathscr{C}_{u,\omega} := \mathscr{C}_u \bigcap \{\omega^A = \omega\}$. For any $t \in [0, T_{*;(\lambda)}]$, let ϕ be a smooth function verifying $0 \leq \phi \leq 1$, $\phi(t) = 1$ and $supp\phi \subset [\frac{1}{2}t, \frac{3}{2}t]$. Since $|L(\phi)| \lesssim \widetilde{r}^{-1}$, by the Fundamental Theorem of Calcules, we have:

$$\begin{aligned}\left(\widetilde{r}^p |\xi|^{2p}_{\not g} \phi^{2p}\right)(t) &= \int_0^t L\left(\widetilde{r}^p |\xi|^{2p}_{\not g}\right) \phi^{2p} + \widetilde{r}^p |\xi|^{2p}_{\not g} L\left(\phi^{2p}\right) \mathrm{d}\tau \\ &\lesssim \int_0^t \widetilde{r}^{p-1} |\xi|^{2p}_{\not g} + \widetilde{r}^p |\xi|^{2p-2}_{\not g} \not{\mathbf{D}}_L \xi \cdot \xi \mathrm{d}\tau.\end{aligned} \tag{9.58}$$

Hence,

$$\begin{aligned}&\left\|\widetilde{r}^{\frac{1}{2}} \xi\right\|^{2p}_{L^{2p}_\omega L^\infty_t(\mathscr{C}_u)} \lesssim \int_{\mathbb{S}^2} \int_0^t \widetilde{r}^{p-1} |\xi|^{2p}_{\not g} + \widetilde{r}^p |\xi|^{2p-2}_{\not g} \not{\mathbf{D}}_L \xi \cdot \xi \mathrm{d}\tau d\varpi_{\not e} \\ &\lesssim \int_{\mathbb{S}^2} \left(\|\xi\|_{L^2_t(\mathscr{C}_{u,\omega})} \|\xi\|_{L^2_t(\mathscr{C}_{u,\omega})} + \|\xi\|_{L^2_t(\mathscr{C}_{u,\omega})} \|\widetilde{r}\not{\mathbf{D}}_L \xi\|_{L^2_t(\mathscr{C}_{u,\omega})}\right) \left\|\widetilde{r}^{p-1} \xi^{2p-2}\right\|_{L^\infty_t(\mathscr{C}_{u,\omega})} d\varpi_{\not e} \\ &\lesssim \|\xi\|_{L^\infty_\omega L^2_t(\mathscr{C}_u)} \left\|r^{\frac{1}{2}} \xi\right\|^{2p-2}_{L^{2p}_\omega L^\infty_t(\mathscr{C}_u)} \left(\|\widetilde{r}\not{\mathbf{D}}_L \xi\|_{L^p_\omega L^2_t(\mathscr{C}_u)} + \|\xi\|_{L^p_\omega L^2_t(\mathscr{C}_u)}\right).\end{aligned} \tag{9.59}$$

This yields the desired result. □

Proof of (9.33b). We first consider the Euclidean sphere case, applying the Morrey's inequality and the

Gagliardo–Nirenberg interpolation inequality with dimension $n = 2$, ans using Young's inequality, we have:

$$\begin{aligned}\|\xi\|_{C^{0,1-\frac{2}{Q}}_{\omega}(S_{t,u})} &\lesssim \left\|\nabla_{\not e}\xi\right\|_{L^Q_{\omega}(S_{t,u})} + \|\xi\|_{L^Q_{\omega}(S_{t,u})} \\ &\lesssim \left\|\nabla_{\not e}\xi\right\|_{L^Q_{\omega}(S_{t,u})} + \left\|\nabla_{\not e}\xi\right\|^{1-\frac{2}{Q}}_{L^2_{\omega}(S_{t,u})} \|\xi\|^{\frac{2}{Q}}_{L^2_{\omega}(S_{t,u})} + \|\xi\|_{L^2_{\omega}(S_{t,u})} \\ &\lesssim \left\|\nabla_{\not e}\xi\right\|_{L^Q_{\omega}(S_{t,u})} + \left\|\nabla_{\not e}\xi\right\|^{1-\frac{2}{Q}}_{L^Q_{\omega}(S_{t,u})} \|\xi\|^{\frac{2}{Q}}_{L^2_{\omega}(S_{t,u})} + \|\xi\|_{L^2_{\omega}(S_{t,u})} \\ &\lesssim \left\|\nabla_{\not e}\xi\right\|_{L^Q_{\omega}(S_{t,u})} + \|\xi\|_{L^2_{\omega}(S_{t,u})}.\end{aligned} \tag{9.60}$$

Noticing that for $\binom{m}{n}$ tensorfield ξ, we have:

$$|\xi|_{\not e} \approx \widetilde{r}^{m-n}\,|\xi|_{\not g}. \tag{9.61}$$

Thus, for any $0 < p < \infty$, there holds:

$$\left\|\left|\nabla_{\not e}\xi\right|_{\not e}\right\|_{L^p_{\omega}(S_{t,u})} \approx \left\|\widetilde{r}\nabla_{\not e}\xi\right\|_{L^p_{\omega}(S_{t,u})}. \tag{9.62}$$

Now we consider the difference between round metric $\not e$ and the geometric sphere metric $\not g$. We have:

$$\nabla\!\!\!/\,\xi = \nabla_{\not e}\xi + (\mathbf{\Gamma} - \mathbf{\Gamma}_{(\not e)})\xi, \tag{9.63}$$

where $\mathbf{\Gamma}_{(\not e)}$ are the Christoffel symbols of $\not e$ relative to Euclidean sphere coordinates ω^A. By the bootstrap assumptions (9.3a)-(9.3b), we have:

$$\left\|\widetilde{r}^{m-n}\left|(\mathbf{\Gamma} - \mathbf{\Gamma}_{(\not e)})\xi\right|_{\not e}\right\|_{L^Q_{\omega}(S_{t,u})} \lesssim \lambda^{-\varepsilon_0}\,\|\xi\|_{L^{\infty}_{\omega}(S_{t,u})}. \tag{9.64}$$

The RHS of (9.64) can be absorbed in the left hand side of (9.60). □

Proof of (9.33a). We use the similar approach as in the proof of (9.33b), for the Euclidean sphere case, we have:

$$\left\||\xi|_{\not e}\right\|_{L^Q_{\omega}(S_{t,u})} \lesssim \left\|\left|\nabla_{\not e}\xi\right|_{\not e}\right\|^{1-\frac{2}{Q}}_{L^2_{\omega}(S_{t,u})} \left\||\xi|_{\not e}\right\|^{\frac{2}{Q}}_{L^2_{\omega}(S_{t,u})} + \left\||\xi|_{\not e}\right\|_{L^2_{\omega}(S_{t,u})}. \tag{9.65}$$

The proof of (9.65) can be reduced, by a partition of unity, to the case where ξ has compact support in a local chart $U \subset S_{t,u}$. Then we can apply the Gagliardo–Nirenberg interpolation inequality with dimension $n = 2$. Now we consider the difference between Euclidean round metric $\not e$ and the geometric sphere metric $\not g$. We have:

$$\nabla\!\!\!/ \xi = \nabla_{\not e}\xi + (\mathbf{\Gamma} - \mathbf{\Gamma}_{(\not e)})\xi. \tag{9.66}$$

where $\mathbf{\Gamma}_{(\not e)}$ are the Christoffel symbols of $\not e$ relative to Euclidean sphere coordinates ω^A. By the bootstrap assumptions (9.3a)-(9.3b), by Hölder's inequality, we have

$$\left\| \widetilde{r}^{m-n} \left|(\mathbf{\Gamma} - \mathbf{\Gamma}_{(\not e)})\xi\right|_{\not e} \right\|_{L^2_\omega(S_{t,u})} \lesssim \left\| \mathbf{\Gamma} - \mathbf{\Gamma}_{(\not e)} \right\|_{L^p_\omega(S_{t,u})} \|\xi\|_{L_\omega^{\frac{2p}{p-2}}(S_{t,u})} \lesssim \lambda^{-4\varepsilon_0} \|\xi\|_{L^\infty_\omega(S_{t,u})} \widetilde{r}^{\frac{p-2}{p}}. \tag{9.67}$$

Note that $\widetilde{r} \lesssim \lambda^{1-8\varepsilon_0}$. Hence, for p sufficiently close to 2, we have:

$$\left\| \widetilde{r}^{m-n} \left|(\mathbf{\Gamma} - \mathbf{\Gamma}_{(\not e)})\xi\right|_{\not e} \right\|_{L^2_\omega(S_{t,u})} \lesssim \|\xi\|_{L^\infty_\omega(S_{t,u})}. \tag{9.68}$$

(9.33a) is then obtained by applying (9.33b). □

Proof of (9.34). The proof is similar to the proof of (9.32). We use a cut-off function of u with support in $u \in [-\frac{4}{5}T_{*;(\lambda)}, t]$ instead of $t \in [0, T_{*;(\lambda)}]$, and we consider that the derivative vectorfield $\frac{\partial}{\partial u}$ along Σ_t instead of L along $\mathscr{C}_u$. Also noticing that $\left|\mathbf{D}\!\!\!/_{\frac{\partial}{\partial u}}\xi\right|_{\not g} \lesssim |(\mathbf{D}\!\!\!/_N, \nabla\!\!\!/)\xi|_{\not g}$ and:

$$\left\| \widetilde{r}^{\frac{1}{2}-\frac{1}{Q}}\xi \right\|^2_{L^{2Q}_{\not g} L^\infty_u(\Sigma_t)} = \left\| \widetilde{r}^{\frac{1}{2}}\xi \right\|^2_{L^{2Q}_\omega L^\infty_u(\Sigma_t)}, \tag{9.69}$$

we obtain the desired estimate. □

Proof of (9.35a). We decompose f as follows:

$$f = \sum_{\nu>1} P_\nu f + P_{\le 1} f. \tag{9.70}$$

Using (9.33a), we have

$$\widetilde{r}\|P_\nu f\|_{L^Q_\omega(S_{t,u})} \lesssim \left\|\nabla_{\not e} P_\nu f\right\|^{1-\frac{2}{Q}}_{L^2_{\not g}(S_{t,u})} \|P_\nu f\|^{\frac{2}{Q}}_{L^2_{\not g}(S_{t,u})} + \|P_\nu f\|_{L^2_{\not g}(S_{t,u})}. \tag{9.71}$$

Integrating (9.71) with respect to u along Σ_t, using the finite band property and bootstrap assumption (9.3a)-(9.3b), and summing over $\nu > 1$, when $0 < 1 - \frac{2}{Q} < N - 2$, we have:

$$\sum_{\nu>1} \|\widetilde{r}P_\nu f\|_{L^2_u L^Q_\omega(\Sigma_t)} \lesssim \sum_{\nu>1} \left(\nu^{1-\frac{2}{Q}} + 1\right) \|P_\nu f\|_{L^2_u L^2_{\not g}(\Sigma_t)} \lesssim \|f\|_{H^{N-2}(\Sigma_t)}. \tag{9.72}$$

For the low frequency terms, we use Bernstein's inequality:

$$\|\widetilde{r}P_{\leq 1}f\|_{L^2_u L^Q_\omega(\Sigma_t)} \approx \|P_{\leq 1}f\|_{L^2_u L^Q_{g\!\!/}(\Sigma_t)} \lesssim \|f\|_{L^2(\Sigma_t)}. \tag{9.73}$$

□

Lemma 9.5 (Transport lemma)**.** *Let m be a constant, and let ξ and $\mathfrak{F}$ be $S_{t,u}$-tangent tensorfields such that the following transport equation holds along the null cone portion $\mathscr{C}_u \subset \mathscr{M}$:*

$$\not{D}_L \xi + m\mathrm{tr}_{g\!\!/}\chi\xi = \mathfrak{F}. \tag{9.74}$$

Then we have the following identities, where $[u]_+ := max\{u, 0\}$:

$$(\upsilon^m \xi)(t,u,\omega) = \lim_{\tau \downarrow [u]_+} (\upsilon^m \xi)(\tau,u,\omega) + \int_{[u]_+}^t (\upsilon^m \mathfrak{F})(\tau,u,\omega)\mathrm{d}\tau, \tag{9.75}$$

$$(\widetilde{r}^{2m}\xi)(t,u,\omega) = \lim_{\tau \downarrow [u]_+} (\widetilde{r}^{2m}\xi)(\tau,u,\omega) + \int_{[u]_+}^t (\widetilde{r}^{2m}\mathfrak{F})(\tau,u,\omega) + m\left(\widetilde{r}^{2m}\left(\frac{2}{\widetilde{r}} - \mathrm{tr}_{g\!\!/}\chi\right)\xi\right)(\tau,u,\omega)\mathrm{d}\tau. \tag{9.76}$$

Similarly, if ξ, $\mathfrak{F}$ and $\mathfrak{G}$ are $S_{t,u}$-tangent tensorfields such that the following transport equation holds:

$$\not{D}_L \xi + \frac{2m}{\widetilde{r}}\xi = \mathfrak{G}\cdot\xi + \mathfrak{F}, \tag{9.77}$$

and if

$$\|\mathfrak{G}\|_{L^\infty_\omega L^1_t(\mathscr{C}_u)} \leq C, \tag{9.78}$$

then under the bootstrap assumptions, the following estimates holds (where the implicit constants in the estimates below depend on the constant C on (9.78)):

$$\left|\widetilde{r}^{2m}\xi\right|_{g\!\!/}(t,u,\omega) \lesssim \lim_{\tau \downarrow [u]_+} \left|\widetilde{r}^{2m}\xi\right|_{g\!\!/}(\tau,u,\omega) + \int_{[u]_+}^t \left|\widetilde{r}^{2m}\mathfrak{F}\right|_{g\!\!/}(\tau,u,\omega)\mathrm{d}\tau. \tag{9.79}$$

Proof of (9.75). By (9.74), we have:

$$L(\upsilon^m\xi) = \upsilon^m \not{D}_L \xi + m\upsilon^{m-1}(L\upsilon)\xi = \upsilon^m\left(\not{D}_L\xi + m\mathrm{tr}_{g\!\!/}\chi\right) = \upsilon^m \mathfrak{F}. \tag{9.80}$$

Integrating υ^m along null geodesics, we get the desired equation. Proof of (9.76) is by the same process. □

Proof of (9.79). By (9.77), we have:

$$L\left(\left|\widetilde{r}^{2m}\xi\right|_{\not g}\right)=\widetilde{r}^{2m}\not{\mathbf{D}}_L\xi\cdot\frac{\xi}{|\xi|_{\not g}}+2m\widetilde{r}^{2m-1}|\xi|_{\not g} \tag{9.81}$$
$$=\widetilde{r}^{2m}\left(-\frac{2m}{\widetilde{r}}\xi+\mathfrak{G}\cdot\xi+\mathfrak{F}\right)\cdot\frac{\xi}{|\xi|_{\not g}}+2m\widetilde{r}^{2m-1}|\xi|_{\not g}$$
$$=\widetilde{r}^{2m}\left(\mathfrak{G}|\xi|_{\not g}+\mathfrak{F}\cdot\frac{\xi}{|\xi|_{\not g}}\right).$$

Integrating $\left|\widetilde{r}^{2m}\xi\right|_{\not g}$ along null geodesics and then using the Grownwall's inequality, we get the desired estimate. □

Proposition 9.6 (Estimates for the fluid variables)**.** *$\vec{\Psi},\vec{\varpi},\vec{S},\vec{\mathscr{C}},\mathscr{D}$ are rescaled variables defined in Definition 6.3. Under the bootstrap assumptions, for any $2\leq Q\leq p$, where $0<\delta_0<1-\frac{2}{p}<N-2$, the following estimates hold on $\mathscr{M}$:*

$$\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right\|_{L_u^2L_\omega^p(\Sigma_t)},\left\|\widetilde{r}^{1/2}\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_u^\infty L_\omega^{2p}(\Sigma_t)}\lesssim\lambda^{-1/2}, \tag{9.82a}$$
$$\left\|\widetilde{r}^{1-\frac{2}{Q}}\left(\boldsymbol{\partial}^2(\vec{\Psi},\vec{\varpi},\vec{S}),\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right)\right\|_{L_u^2L_{\not g}^Q(\Sigma_t)}\lesssim\lambda^{-1/2}, \tag{9.82b}$$
$$\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\lesssim\lambda^{-1/2-4\varepsilon_0}, \tag{9.82c}$$
$$\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2-4\varepsilon_0}, \tag{9.82d}$$
$$\left\|\widetilde{r}\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\lesssim\lambda^{1/2-12\varepsilon_0}, \tag{9.82e}$$
$$\left\|(\not\nabla,\not{\mathbf{D}}_L)\boldsymbol{\partial}\vec{\Psi}\right\|_{L^2(\mathscr{C}_u)},\left\|\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L^2(\mathscr{C}_u)},\left\|\widetilde{r}^{1-\frac{2}{p}}(\not\nabla,\not{\mathbf{D}}_L)\left(\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_t^2L_{\not g}^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}, \tag{9.82f}$$
$$\left\|\widetilde{r}^{1/2}\boldsymbol{\partial}\left(\vec{\Psi},\vec{\varpi},\vec{S}\right)\right\|_{L_u^2L_t^\infty L_\omega^{2p}(\mathscr{M})}\lesssim\lambda^{-4\varepsilon_0}. \tag{9.82g}$$

Moreover, for any smooth function f*, we have:*

$$\left\|\widetilde{r}(\not\nabla,\not{\mathbf{D}}_L)\left(\mathrm{f}\left(\vec{\Psi},\vec{\varpi},\vec{S},\vec{L}\right)\left(\boldsymbol{\partial}\vec{\Psi},\vec{\mathscr{C}},\mathscr{D}\right)\right)\right\|_{L_t^2L_\omega^Q(\mathscr{C}_u)}\lesssim\lambda^{-1/2}, \tag{9.83a}$$
$$\left\|\widetilde{r}\boldsymbol{\partial}\left(\mathrm{f}\left(\vec{\Psi},\vec{\varpi},\vec{S},\vec{L}\right)\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\varpi},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right)\right\|_{L_u^2L_\omega^Q(\Sigma_t)}\lesssim\lambda^{-1/2}, \tag{9.83b}$$
$$\left\|\widetilde{r}^{1/2}\mathrm{f}\left(\vec{\Psi},\vec{L}\right)\boldsymbol{\partial}\left(\vec{\Psi},\vec{\varpi},\vec{S}\right)\right\|_{L_u^2L_t^\infty L_\omega^{2p}(\mathscr{M})}\lesssim\lambda^{-4\varepsilon_0}. \tag{9.83c}$$

Proof of (9.82b). By using (9.35a), and rescaling the top order energy estimates (5.1), we have:

$$\left\|\widetilde{r}^{1-\frac{2}{Q}}\left(\boldsymbol{\partial}^2(\vec{\Psi},\vec{\omega},\vec{S}),\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right)\right\|_{L_u^2L_{\not g}^Q(\Sigma_t)} \approx \left\|\widetilde{r}\left(\boldsymbol{\partial}^2(\vec{\Psi},\vec{\omega},\vec{S}),\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right)\right\|_{L_u^2L_\omega^Q(\Sigma_t)} \lesssim \left\|\boldsymbol{\partial}^2(\vec{\Psi},\vec{\omega},\vec{S}),\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{H^{N-2}(\Sigma_t)} \lesssim \lambda^{-1/2}. \tag{9.84}$$

□

Proof of (9.82a). By (9.31), (9.33a), (9.35a), (9.82b) and the energy estimates (5.1), we have:

$$\begin{aligned}\left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right\|_{L_u^2L_\omega^p(\Sigma_t)} &\lesssim \left\|\widetilde{r}\not\nabla\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_u^2L_\omega^2(\Sigma_t)}^{1-\frac{2}{p}} \left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right\|_{L_u^2L_\omega^2(\Sigma_t)}^{\frac{2}{p}} \\ &\quad + \left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right\|_{L_u^2L_\omega^2(\Sigma_t)} \\ &\lesssim \left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right\|_{H^1(\Sigma_t)} \lesssim \lambda^{-1/2}.\end{aligned} \tag{9.85}$$

For estimate of the second norm, by (9.34), (9.33b), (9.35a), (9.31), (9.82b) and the energy estimates (5.1), we have:

$$\begin{aligned}&\left\|\widetilde{r}^{1/2}\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_\omega^{2p}L_u^\infty(\Sigma_t)}^2 \lesssim \left\|\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_u^2L_\omega^\infty(\Sigma_t)} \cdot \\ &\left(\left\|\widetilde{r}(\mathbf{D}_N,\not\nabla)\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_\omega^QL_u^2(\Sigma_t)} + \left\|\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_\omega^QL_u^2(\Sigma_t)}\right) \\ &\lesssim \left(\left\|\widetilde{r}\not\nabla\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_u^2L_\omega^Q(\Sigma_t)} + \left\|\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_u^2L_\omega^2(\Sigma_t)}\right) \cdot \\ &\left(\left\|\widetilde{r}(\mathbf{D}_N,\not\nabla)\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_\omega^QL_u^2(\Sigma_t)} + \left\|\left(\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right)\right\|_{L_\omega^QL_u^2(\Sigma_t)}\right) \\ &\lesssim \left\|\boldsymbol{\partial}(\vec{\Psi},\vec{\omega},\vec{S}),\vec{\mathscr{C}},\mathscr{D}\right\|_{H^{N-1}(\Sigma_t)} \lesssim \lambda^{-1}.\end{aligned} \tag{9.86}$$

□

Proof of (9.82g). Using the fact that $u \lesssim \lambda^{1-8\varepsilon_0}$ in $\mathscr{M}$ and (9.82a), we deduce:

$$\begin{aligned}\left\|\widetilde{r}^{1/2}\mathrm{f}\left(\vec{\Psi},\vec{L}\right)\boldsymbol{\partial}\left(\vec{\Psi},\vec{\omega},\vec{S}\right)\right\|_{L_u^2L_t^\infty L_\omega^{2p}(\mathscr{M})} &\lesssim \lambda^{1/2-4\varepsilon_0}\left\|\widetilde{r}^{1/2}\mathrm{f}\left(\vec{\Psi},\vec{L}\right)\boldsymbol{\partial}\left(\vec{\Psi},\vec{\omega},\vec{S}\right)\right\|_{L_u^\infty L_t^\infty L_\omega^{2p}(\mathscr{M})} \\ &\lesssim \lambda^{-4\varepsilon_0}.\end{aligned} \tag{9.87}$$

□

Proof of other estimates in Proposition 9.6. (9.82c)-(9.82e) are direct results of rescaled bootstrap assumptions (9.2).

The first and second estimates of (9.82f) are direct results of energy estimates along acoustic null hypersurfaces (8.3)-(8.4). The last estimates of (9.82f) is [36, Lemma 5.5], we use the Sobolev inequality (9.33a) and the following inequality in [34, Proposition 2.7]:

$$\sum_{l>1} \left\| l^{N-2} \not{P}_l (\not{D}_L, \not{\nabla}) f \right\|^2_{L^2_t L^2_{\not{g}}(\mathscr{C}_u)} \lesssim \sum_{\nu>1} \nu^{2(N-2)} \left(\mathscr{F}_{(wave)}[P_\nu f; \mathscr{C}_u] + \|P_\nu f\|_{H^1(\Sigma_t)} \right) + \mathscr{F}_{(wave)}[P_\nu f; \mathscr{C}_u] + \|f\|^2_{H^1(\Sigma_t)}, \tag{9.88}$$

where $\not{P}_l$ is the Littlewood-Paley projection operator on $S_{t,u}$. We obtain the results by the energy estimates along constant-time hypersurfaces (5.1) and along acoustic null hypersurfaces (8.3)-(8.4).
(9.83a)-(9.83c) are the results of (9.82f), (9.82b) and (9.82g) respectively. □

Proposition 9.7 (Needed estimates to recover and improve the bootstrap assumption)**.** *Under the bootstrap assumptions, the following estimates hold whenever $q > 2$ is sufficiently close to 2, where p is the same as in the previous proposition.*

Estimates for time-integrated terms

$$\lambda^{-1} \left\| \widetilde{r}^{-1} \int_{[u]_+}^t \left| \widetilde{r}(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^2_t L^\infty_x(\mathscr{M})} \lesssim \lambda^{-1/2-12\varepsilon_0}, \tag{9.89a}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-2} \int_{[u]_+}^t \left| \widetilde{r}^2(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^2_t L^\infty_x(\mathscr{M})} \lesssim \lambda^{-1/2-12\varepsilon_0}, \tag{9.89b}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-2} \int_{[u]_+}^t \left| \widetilde{r}^2(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L_t^{\frac{q}{2}} L^\infty_x(\mathscr{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}+2\right)}, \tag{9.89c}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-1/2} \int_{[u]_+}^t \left| \widetilde{r}(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^\infty_t L^\infty_u L^p_\omega(\mathscr{M})} \lesssim \lambda^{-1/2-12\varepsilon_0}, \tag{9.89d}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-3/2} \int_{[u]_+}^t \left| \widetilde{r}^2(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^\infty_t L^\infty_u L^p_\omega(\mathscr{M})} \lesssim \lambda^{-1/2-12\varepsilon_0}, \tag{9.89e}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-1} \int_{[u]_+}^t \left| \widetilde{r}^2(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-16\varepsilon_0}, \tag{9.89f}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-\frac{3}{2}} \int_{[u]_+}^t \left| \widetilde{r}^2(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-1/2-12\varepsilon_0}, \tag{9.89g}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-2} \int_{[u]_+}^t \left| \widetilde{r}^2(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-1-8\varepsilon_0}. \tag{9.89h}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-3/2} \int_{[u]_+}^t \left| \widetilde{r}^3 \not{\nabla}(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^\infty_t L^\infty_u L^p_\omega(\mathscr{M})} \lesssim \lambda^{-1/2-8\varepsilon_0}, \tag{9.90a}$$

$$\lambda^{-1} \left\| \widetilde{r}^{-2} \int_{[u]_+}^t \left| \widetilde{r}^3 \not{\nabla}(\vec{\mathscr{C}}, \mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^2_t L^\infty_u L^p_\omega(\mathscr{M})} \lesssim \lambda^{-1/2-8\varepsilon_0}. \tag{9.90b}$$

$$\lambda^{-1}\left\|\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{3}(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_t^\infty L_u^\infty L_\omega^p(\mathcal{M})}\lesssim\lambda^{-1/2-12\varepsilon_0}, \quad (9.91a)$$

$$\lambda^{-1}\left\|\widetilde{r}^{-2}\int_{[u]_+}^{t}\left|\widetilde{r}^{3}(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{M})}\lesssim\lambda^{-1/2-12\varepsilon_0}, \quad (9.91b)$$

$$\lambda^{-1}\left\|\widetilde{r}^{-1}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{M})}\lesssim\lambda^{-16\varepsilon_0}, \quad (9.91c)$$

$$\lambda^{-1}\left\|\widetilde{r}^{-1/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{M})}\lesssim\lambda^{-16\varepsilon_0}. \quad (9.91d)$$

$$\lambda^{-1}\left\|\widetilde{r}^{-1}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\boldsymbol{\partial}\vec{\mathscr{C}},\boldsymbol{\partial}\mathscr{D})\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{M})}\lesssim\lambda^{-12\varepsilon_0}, \quad (9.92a)$$

$$\lambda^{-1}\left\|\widetilde{r}^{-1/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\boldsymbol{\partial}\vec{\mathscr{C}},\boldsymbol{\partial}\mathscr{D})\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{M})}\lesssim\lambda^{-12\varepsilon_0}. \quad (9.92b)$$

space-time norm estimates

$$\lambda^{-1}\left\|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{M})}\lesssim^{-1/2-8\varepsilon_0}, \quad (9.93a)$$

$$\lambda^{-1}\left\|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L_t^{\frac{q}{2}} L_u^\infty L_\omega^p(\mathcal{M})}\lesssim^{-\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}+1\right)}, \quad (9.93b)$$

$$\lambda^{-1}\left\|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{M})}\lesssim^{-12\varepsilon_0}, \quad (9.93c)$$

$$\lambda^{-1}\left\|\widetilde{r}\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L_u^2 L_t^1 L_\omega^p(\mathcal{M})}\lesssim^{-\frac{1}{2}-8\varepsilon_0}, \quad (9.93d)$$

$$\lambda^{-1}\left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\right\|_{L_u^2 L_t^1 L_\omega^p(\mathcal{M})}\lesssim^{-\frac{1}{2}-10\varepsilon_0}. \quad (9.93e)$$

9.3.2 Discussion of the proof of Prop.9.7

The proof of Prop.9.7 relies on Prop.9.6 and facts that $\widetilde{r}(\tau,u)\lesssim\widetilde{r}(t,u)$ when $\tau\le t$ and $T_{*;(\lambda)}\lesssim\lambda^{1-8\varepsilon_0}$. We refer reader to [9, Lemma 10.5] for the detailed proof.

We provide Lemma 9.8, Proposition 9.9 and Corollary 9.10 as follows. These propositions are crucial for controlling the geometry. In particular, we will use (9.104)-(9.105) to prove Proposition 9.1. We give a discussion of the proof of Lemma 9.8, Prop.9.9 and Corollary 9.10 in Section 9.3.3.

Lemma 9.8 (Preliminary results for controlling the integral curves of L)**.** *Let $d_{e\!\!\!/}(\omega_{(1)},\omega_{(2)})$ denote the distance between two points $\omega_{(1)}$ and $\omega_{(2)}$ in $\mathbb{S}^2$ with respect to the standard Euclidean metric $e\!\!\!/$. Under the bootstrap assumptions, we have the following results hold:*

Results along Σ_0: *For each $\omega\in\mathbb{S}^2$, $\lim_{w\downarrow 0}N^i(0,w,\omega)$, $\lim_{w\downarrow 0}L^i(0,w,\omega)$, $\lim_{w\downarrow 0}\Theta^i_{(A)}(0,w,\omega)$ exist. Furthermore, for* *each $\omega \in \mathbb{S}^2$, we have*

$$g_{cd}\Theta_{(A)}^c(0,0,\omega)\Theta_{(B)}^d(0,0,\omega) = \not{g}(\omega)\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) + \mathscr{O}\left(\lambda^{-4\varepsilon_0}\right). \tag{9.94}$$

In addition, for $\mathbf{z}$ *defined as in Section 7.1, we have:*

$$x^i(0,w,\omega) = \mathbf{z}^i + w\left(N^i(0,0,\omega) + \mathscr{O}\left(\lambda^{-4\varepsilon_0}\right)\right), \tag{9.95a}$$

$$N^i(0,w,\omega) = N^i(0,0,\omega) + \mathscr{O}\left(\lambda^{-4\varepsilon_0}\right), \tag{9.95b}$$

$$L^i(0,w,\omega) = L^i(0,0,\omega) + \mathscr{O}\left(\lambda^{-4\varepsilon_0}\right), \tag{9.95c}$$

$$\Theta_{(A)}^i(0,w,\omega) = \Theta_{(A)}^i(0,0,\omega) + \mathscr{O}\left(\lambda^{-4\varepsilon_0}\right). \tag{9.95d}$$

Moreover,

$$\frac{\partial}{\partial \omega^A} N^i(0,0,\omega) = \frac{\partial}{\partial \omega^A} L^i(0,0,\omega) = \Theta_{(A)}^i(0,0,\omega). \tag{9.96}$$

In addition,

$$\sum_{i=1}^{3}\left|N^i(0,0,\omega_{(1)}) - N^i(0,0,\omega_{(2)})\right| = \sum_{i=1}^{3}\left|L^i(0,0,\omega_{(1)}) - L^i(0,0,\omega_{(2)})\right| \approx d_{\not{g}}(\omega_{(1)},\omega_{(2)}). \tag{9.97}$$

Finally,

$$\|L^\alpha\|_{L_u^\infty C_\omega^{0,\delta_0}(\Sigma_0)} \lesssim 1. \tag{9.98}$$

Results along the cone-tip axis*: For each* $(u,\omega) \in [0, T_{*;(\lambda)}] \times \mathbb{S}^2$, $\lim_{t \downarrow u} \Theta_{(A)}^i(t,u,\omega)$ *exists. Furthermore,*

$$g_{cd}\Theta_{(A)}^c(t,t,\omega)\Theta_{(B)}^d(t,t,\omega) = \not{g}(\omega)\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) + \mathscr{O}(\lambda^{-\varepsilon_0}). \tag{9.99}$$

Moreover,

$$N^i(t,t,\omega) = N^i(0,0,\omega) + \mathscr{O}\left(\lambda^{-8\varepsilon_0}\right), \tag{9.100a}$$

$$L^i(t,t,\omega) = L^i(0,0,\omega) + \mathscr{O}\left(\lambda^{-8\varepsilon_0}\right), \tag{9.100b}$$

$$\Theta_{(A)}^i(t,t,\omega) = \Theta_{(A)}^i(0,0,\omega) + \mathscr{O}\left(\lambda^{-4\varepsilon_0}\right). \tag{9.100c}$$

In addition,

$$\frac{\partial}{\partial \omega^A} N^i(t,t,\omega) = \frac{\partial}{\partial \omega^A} L^i(t,t,\omega) = \Theta^i_{(A)}(t,t,\omega). \tag{9.101}$$

Results in $\mathscr{M}$***:*** *For* $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}], t \in [[u]_+, T_{*;(\lambda)}]$ *and* $\omega \in \mathbb{S}^2$*, we have*

$$L^i(t,u,\omega) = L^i(0,0,\omega) + \mathscr{O}\left(\lambda^{-4\varepsilon_0}\right), \tag{9.102a}$$

$$\Theta^i_{(A)}(t,u,\omega) = \Theta^i_{(A)}(0,0,\omega) + \mathscr{O}\left(\lambda^{-4\varepsilon_0}\right). \tag{9.102b}$$

Proposition 9.9 (Control of the integral curves of L in $\mathscr{M}$)**.** *Let* $\Upsilon_{u;\omega}(t)$ *be the family of null geodesic curves normalized by* $\Upsilon^0_{u;\omega}(t) = t$*. Then the following estimate holds for* $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$ *and* $t \in [[u]_+, T_{*;(\lambda)}]$*:*

$$\sum_{\alpha=0}^{3} \left| \Upsilon^\alpha_{u;\omega_{(1)}}(t) - \Upsilon^\alpha_{u;\omega_{(2)}}(t) \right| \approx \widetilde{r} d_{\not{g}}(\omega_{(1)}, \omega_{(2)}). \tag{9.103}$$

Corollary 9.10. *We have the following estimates:*

$$\left\| \partial \vec{\Psi}, \partial \vec{\omega}, \partial \vec{S}, \vec{\mathscr{C}}, \mathscr{D} \right\|_{L^2_t L^\infty_u C^{0,\delta_0}_\omega(\mathscr{M})} \lesssim \lambda^{-1/2-3\varepsilon_0}. \tag{9.104}$$

Moreover,

$$\left\| \vec{\Psi}, \vec{\omega}, \vec{S} \right\|_{L^\infty_t L^\infty_u C^{0,\delta_0}_\omega(\mathscr{M})} \lesssim 1. \tag{9.105}$$

9.3.3 Discussion of the proof of Lemma 9.8, Prop.9.9 and Corollary 9.10

Lemma 9.8, Proposition 9.9 can be proved by using transport equations (8.133a)-(8.137), estimates (9.33b), (9.83b), (9.83c), and initial conditions (8.63c), (8.63h), (8.61), (8.63a). We refer reader to [9, Section 10.6] for the detailed proofs of Lemma 9.8, Prop.9.9 and Corollary 9.10 where the exact same argument can go through in this article.

Proposition 9.11. *[9, Lemma 10.9. Estimates for transport equations along the integral curve of L in Holder spaces] Let the scalar function* ϕ *be a smooth solution to the following inhomogeneous transport equation with the data given on* $S_{[u]_+,u}$ *:*

$$L\phi(t,u,\omega) = \mathfrak{F}(t,u,\omega). \tag{9.106}$$

(9.107)

Since $\{\omega^A\}_{A=1,2}$ are constant along integral curves of L, we have

$$\|\phi\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim \|\phi\|_{C^{0,\delta_0}_\omega(S_{[u]_+,u})} + \int_{[u]_+}^t \|\mathfrak{F}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})}. \tag{9.108}$$

Moreover,

$$\left\|\int_{[u]_+}^t \mathfrak{F}(\tau,u,\omega)\mathrm{d}\tau\right\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim \int_{[u]_+}^t \|\mathfrak{F}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})}. \tag{9.109}$$

The proof of Proposition 9.11 relies on the Fundamental Theorem of Calculus and the fact that $L(\omega^A) = 0$.

Lemma 9.12 (Calderon-Zygmund and Schauder-type Hodge estimates on $S_{t,u}$)**.** *If ξ is an $S_{t,u}$-tangent one-form and $2 \leq Q \leq p$, then*

$$\|\nabla\!\!\!/\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} + \left\|\widetilde{r}^{-1}\xi\right\|_{L^Q_{g\!\!\!/}(S_{t,u})} \lesssim \|\mathrm{di\!\!\!/v}\,\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} + \|\mathrm{cur\!\!\!/l}\,\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})}. \tag{9.110}$$

Similarly, if ξ is an $S_{t,u}$-tangent type $\binom{0}{2}$ symmetric trace-free tensorfield, then

$$\|\nabla\!\!\!/\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} + \left\|\widetilde{r}^{-1}\xi\right\|_{L^Q_{g\!\!\!/}(S_{t,u})} \lesssim \|\mathrm{di\!\!\!/v}\,\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})}. \tag{9.111}$$

Moreover, let ξ be an $S_{t,u}$-tangent type $\binom{0}{2}$ symmetric trace-free tensorfield, let $\mathfrak{F}_{(1)}$ be a scalar function, let $\mathfrak{F}_{(2)}$ be a type $\binom{0}{2}$ symmetric trace-free tensorfield, and let $\mathfrak{G}$ be an $S_{t,u}$-tangent one-form. Assume that

$$\mathrm{di\!\!\!/v}\,\xi = \nabla\!\!\!/\mathfrak{F}_{(1)} + \mathrm{di\!\!\!/v}\,\mathfrak{F}_{(2)} + \mathfrak{G}. \tag{9.112}$$

Let $2 < Q < \infty$, and let Q' be defined by $\frac{1}{2} + \frac{1}{Q} = \frac{1}{Q'}$. Then the following estimate holds:

$$\|\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} \lesssim \sum_{i=1,2} \left\|\mathfrak{F}_{(i)}\right\|_{L^Q_{g\!\!\!/}(S_{t,u})} + \|\mathfrak{G}\|_{L^{Q'}_{g\!\!\!/}(S_{t,u})}. \tag{9.113}$$

In addition, if $2 < Q < \infty$, then

$$\|\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \lesssim \sum_{i=1,2} \left\|\mathfrak{F}_{(i)}\right\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} + \|\widetilde{r}\mathfrak{G}\|_{L^Q_\omega(S_{t,u})}. \tag{9.114}$$

Similarly, assume that ξ, $\mathfrak{F}_{(1)}$, and $\mathfrak{F}_{(2)}$ are $S_{t,u}$-tangent one-forms and $\mathfrak{G}_{(1)}$ and $\mathfrak{G}_{(2)}$ are scalar functions s.t *the following Hodge system is satisfied:*

$$\not{\mathrm{div}}\,\xi = \not{\mathrm{div}}\,\mathfrak{F}_{(1)} + \mathfrak{G}_{(1)}, \tag{9.115a}$$

$$\not{\mathrm{curl}}\,\xi = \not{\mathrm{curl}}\,\mathfrak{F}_{(2)} + \mathfrak{G}_{(2)}. \tag{9.115b}$$

Then ξ *satisfies (9.113) and (9.114) with* $\mathfrak{G} := (\mathfrak{G}_{(1)}, \mathfrak{G}_{(2)})$.

Finally, assume that ξ, $\mathfrak{F} = \left(\mathfrak{F}_{(1)}, \mathfrak{F}_{(2)}\right)$, *and* $\mathfrak{G}$ *are* $S_{t,u}$ *tensorfields of the type from the previous paragraphs. Assume that* $\mathfrak{F}$ *is the* $S_{t,u}$*-projection of a space-time tensorfield* $\tilde{\mathfrak{F}}$ *or is a contraction of a space-time tensorfield* $\tilde{\mathfrak{F}}$ *against L,* $\underline{L}$*, or N. If* $Q > 2$, $1 \le c < \infty$, *and* $\delta' > 0$ *is sufficiently small, then the following estimates hold, where* $\vec{\tilde{\mathfrak{F}}}$ *denotes the array of (scalar) Cartesian component function of* $\tilde{\mathfrak{F}}$*:*

$$\|\xi\|_{L^\infty_\omega(S_{t,u})} \lesssim \left\| \nu^{\delta'} P_\nu \vec{\tilde{\mathfrak{F}}} \right\|_{l^c_\nu L^\infty_\omega(S_{t,u})} + \left\| \vec{\tilde{\mathfrak{F}}} \right\|_{L^\infty_\omega(S_{t,u})} + \left\| \tilde{r}^{1-\frac{2}{Q}} \mathfrak{G} \right\|_{L^Q_{\not g}(S_{t,u})}. \tag{9.116}$$

Discussion of the proof of Lemma 9.12. Aside from (9.114), these estimates are a restatement of [36, Lemma 5.8, Proposition 5.9, Proposition 5.10]. (9.114) is a perturbation result of the standard Schauder-type estimate for Hodge system on $\mathbb{S}^2$ where the error terms are controlled by (9.113) and the bootstrap assumptions. □

9.4 Proofs of the Main Estimates for the Eikonal Function Quatities

With the help of the previous subsection, we now prove all the estimates for geometric quantities that are listed in Prop. 9.1

Proposition 9.13 (Frequently used estimates)**.** *We have the following estimates, which we use silently in this section. Recall that we have the bound for t and* $\tilde{r}$*:*

$$0 < t \lesssim \lambda^{1-8\varepsilon_0}, \tag{9.117}$$

$$0 < \tilde{r} \lesssim \lambda^{1-8\varepsilon_0}. \tag{9.118}$$

Also for $\mathrm{f}_{(\vec{L})}$, *we have*

$$\left\| \mathrm{f}_{(\vec{L})} \right\|_{L^\infty(\mathscr{M})} \lesssim 1. \tag{9.119}$$

In addition, for scalar functions ϕ and ψ on $S_{t,u}$ and smooth function f*, we have*

$$\begin{aligned}\|\phi\cdot\psi\|_{C^{0,\delta_0}_{\omega}(S_{t,u})} &\lesssim \|\phi\|_{L^\infty(S_{t,u})}\|\psi\|_{\dot{C}^{0,\delta_0}_{\omega}(S_{t,u})} + \|\psi\|_{L^\infty(S_{t,u})}\|\phi\|_{\dot{C}^{0,\delta_0}_{\omega}(S_{t,u})} \\ &\lesssim \|\psi\|_{C^{0,\delta_0}_{\omega}(S_{t,u})}\|\phi\|_{C^{0,\delta_0}_{\omega}(S_{t,u})}.\end{aligned} \tag{9.120}$$

$$\|\mathrm{f}\circ\phi\|_{C^{0,\delta_0}_{\omega}(S_{t,u})} \lesssim 1+\|\phi\|_{C^{0,\delta_0}_{\omega}(S_{t,u})}. \tag{9.121}$$

Moreover, for scalar functions f, and $1\le Q<p\le\infty$, we have

$$\|f\|_{L^Q_{\omega}(S_{t,u})} \le \|f\|_{L^p_{\omega}(S_{t,u})}|S_{t,u}|_{\not{g}}^{\frac{1}{Q}-\frac{1}{p}} \lesssim \|f\|_{L^p_{\omega}(S_{t,u})}. \tag{9.122}$$

Finally, we have the following estimates for average of scalar functions among $S_{t,u}$ (defined in Definition 8.7):

$$|\overline{f}| \lesssim \|f\|_{L^Q_{\omega}(S_{t,u})}, \qquad \|\overline{f}\|_{L^Q_{\omega}(S_{t,u})} \lesssim \|f\|_{L^Q_{\omega}(S_{t,u})}. \tag{9.123}$$

Proof of Proposition 9.13. (9.117) and (9.118) are from (7.7), (7.9) respectively. (9.119) is a direct result of (9.98). (9.120)-(9.121) are standard results in Hölder space. (9.122) and (9.123) are direct results of the Hölder's inequality and the Minkowski's inequality. □

9.4.1 Proof of Proposition 9.1

With the help of the previous results, we are now ready to control the geometric quantities. Recall $[u]_+ := max\{u,0\}$.

Proof of $\upsilon\approx\widetilde{r}^2$ in (9.17a). Recall equation (8.143a), we have:

$$L\left(\widetilde{r}^{-2}\upsilon\right) = \left(\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})} - \mathbf{\Gamma}_L\right)\left(\widetilde{r}^{-2}\upsilon\right). \tag{9.124}$$

Integrating (9.124) along the integral curve of L, we have:

$$\left(\widetilde{r}^{-2}\upsilon\right)(t,u,\omega) = \lim_{\tau\downarrow[u]_+}\left(\widetilde{r}^{-2}\upsilon\right)(\tau,u,\omega) + \int_{[u]_+}^{t}\left(\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})} - \mathbf{\Gamma}_L\right)\left(\widetilde{r}^{-2}\upsilon\right)\mathrm{d}\tau. \tag{9.125}$$

Using Grownwall's inequality, bootstrap assumptions for $\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}$ and $\boldsymbol{\partial}\vec{\Psi}$, and initial conditions for $\widetilde{r}^{-2}\upsilon$, we have:

$$\left(\widetilde{r}^{-2}\upsilon\right)(t,u,\omega) \lesssim \lim_{\tau\downarrow[u]_+}\left(\widetilde{r}^{-2}\upsilon\right)\exp\left(\left\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}-\boldsymbol{\Gamma}_L\right\|_{L_t^1L_\omega^\infty(\mathscr{C}_u)}\right) \lesssim 1. \tag{9.126}$$

We get $\left(\widetilde{r}^{-2}\upsilon\right)(t) \gtrsim 1$ by applying similar argument to:

$$\left(-\widetilde{r}^{-2}\upsilon\right)(t,u,\omega) = \lim_{\tau\downarrow[u]_+}\left(-\widetilde{r}^{-2}\upsilon\right) + \int_{[u]_+}^t \left(\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}-\boldsymbol{\Gamma}_L\right)\left(-\widetilde{r}^{-2}\upsilon\right)\mathrm{d}\tau. \tag{9.127}$$

□

Proof of $\|b-1\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-4\varepsilon_0} < \frac{1}{4}$ in (9.17b). Recall equation (8.141). Taking the initial condition for the two separate cases $u<0$ and $u\geq 0$ into consideration, we have

$$b = 1 + \int_{[u]_+}^t b\cdot \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\mathrm{d}\tau, \qquad u\geq 0, \tag{9.128a}$$

$$b = a + \int_{[u]_+}^t b\cdot \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\mathrm{d}\tau, \qquad u< 0, \tag{9.128b}$$

where a is defined in (8.62). Using Grownwall's inequality, bootstrap assumption for $\boldsymbol{\partial}\vec{\Psi}$ and initial condition (8.63a), we conclude the desired estimate. □

Proof of $\left\|\mathrm{f}_{(\vec{L})}\right\|_{L_t^\infty L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \lesssim 1$ in (9.10).

$$\left\|\mathrm{f}_{(\vec{L})}\right\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim 1 + \left\|\vec{\Psi}\right\|_{C_\omega^{0,\delta_0}(S_{t,u})} + \left\|\vec{L}\right\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim 1 + \left\|\vec{L}\right\|_{C_\omega^{0,\delta_0}(S_{t,u})}. \tag{9.129}$$

Recalling equation (8.133a) and using (9.108), we have:

$$\begin{aligned}\left\|\vec{L}\right\|_{C_\omega^{0,\delta_0}(S_{t,u})} &\lesssim \left\|\vec{L}\right\|_{C_\omega^{0,\delta_0}(S_{[u]_+,u})} + \int_{[u]_+}^t \left\|\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{C_\omega^{0,\delta_0}(S_{\tau,u})}\mathrm{d}\tau \\ &\lesssim \left\|\vec{L}\right\|_{C_\omega^{0,\delta_0}(S_{[u]_+,u})} + \int_{[u]_+}^t \left\|\mathrm{f}_{(\vec{L})}\right\|_{C_\omega^{0,\delta_0}(S_{\tau,u})}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{C_\omega^{0,\delta_0}(S_{\tau,u})}\mathrm{d}\tau.\end{aligned} \tag{9.130}$$

By Grownwall's inequality, we have:

$$\left\|\mathrm{f}_{(\vec{L})}\right\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim \left(1+\left\|\vec{L}\right\|_{C_\omega^{0,\delta_0}(S_{[u]_+,u})}\right)\exp\left(\int_{[u]_+}^t\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{C_\omega^{0,\delta_0}(S_{\tau,u})}\mathrm{d}\tau\right). \tag{9.131}$$

By (7.7) and (9.104),

$$\int_{[u]_+}^{t} \left\| \boldsymbol{\partial}\vec{\Psi} \right\|_{C^{0,\delta_0}_{\omega}(S_{\tau,u})} \mathrm{d}\tau \lesssim \lambda^{-7\varepsilon_0}. \tag{9.132}$$

Combining (9.101), (9.100c), (9.96) and (9.98), we have:

$$\left\| \vec{L} \right\|_{C^{0,\delta_0}_{\omega}(S_{[u]_+,u})} \lesssim 1 + \sum_{i=1}^{3} \max_{A=1,2} \left\| \Theta^i_{(A)}(t,t,\omega) \right\|_{L^\infty_\omega(S_{[u]_+,u})} \lesssim 1 + \sum_{\alpha} \| L^\alpha \|_{L^\infty_u C^{0,\delta_0}_\omega(\Sigma_0)} \lesssim 1. \tag{9.133}$$

Combining (9.131), (9.132) and (9.133) we conclude the desired estimate. □

Proof of $\|\hat{\chi}\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.7a).* Recalling equation (8.139b) and using the transport identity (9.76), we have:

$$\begin{aligned} \left| \widetilde{r}^2 \hat{\chi} \right|_{\not{g}} (t,u,\omega) &\leq \lim_{\tau \downarrow [u]_+} \left| \widetilde{r}^2 \hat{\chi} \right|_{\not{g}} (\tau,u,\omega) \\ &+ \int_{[u]_+}^{t} \lambda^{-1} \left| \widetilde{r}^2 (\vec{\mathscr{C}},\mathscr{D}) \right|_{\not{g}} + \left| \widetilde{r}^2 (\not{\nabla}, \not{\mathbf{D}}_L) \xi \right|_{\not{g}} + \left| \widetilde{r}^2 \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1} \right) \cdot \boldsymbol{\partial}\vec{\Psi} \right|_{\not{g}} \mathrm{d}\tau. \end{aligned} \tag{9.134}$$

Dividing (9.134) by $\widetilde{r}^2(t,u)$ and taking the norm, we have:

$$\begin{aligned} \|\hat{\chi}\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \leq & \left\| \widetilde{r}^{-2} \left(\lim_{\tau \downarrow [u]_+} \widetilde{r}^2 \hat{\chi} \right) \right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \\ &+ \left\| \lambda^{-1} \widetilde{r}^{-2} \int_{[u]_+}^{t} \left| \widetilde{r}^2 (\vec{\mathscr{C}},\mathscr{D}) \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} + \left\| \widetilde{r}^{-2} \int_{[u]_+}^{t} \left| \widetilde{r}^2 (\not{\nabla}, \not{\mathbf{D}}_L) \xi \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \\ &+ \left\| \widetilde{r}^{-2} \int_{[u]_+}^{t} \left| \widetilde{r}^2 \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not{g}}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1} \right) \cdot \boldsymbol{\partial}\vec{\Psi} \right|_{\not{g}} \mathrm{d}\tau \right\|_{L^2_t L^p_\omega(\mathscr{C}_u)}. \end{aligned} \tag{9.135}$$

We now consider the initial conditions. When $u \geq 0$, by the initial condition on the cone-tip (8.64a), $\lim_{\tau \downarrow [u]_+} \left| \widetilde{r}^2 \hat{\chi} \right|_{\not{g}} = 0$. When $u < 0$, $\lim_{\tau \downarrow [u]_+} \left| \widetilde{r}^2 \hat{\chi} \right|_{\not{g}} = w^2 \hat{\chi}(0,u,\omega)$. Then, by (8.63b) and (9.82a), we have:

$$\left\| w^{1/2} \hat{\chi}(0,u,\omega) \right\|_{L^p_\omega(S_w)} \lesssim \left\| w^{1/2} \theta(0,u,\omega) \right\|_{L^p_\omega(S_w)} + \left\| w^{1/2} \boldsymbol{\partial}\vec{\Psi}(0,u,\omega) \right\|_{L^p_\omega(S_w)} \lesssim \lambda^{-1/2}. \tag{9.136}$$

Then,

$$\begin{aligned} \left\| \widetilde{r}^{-2} w^2 \hat{\chi}(0,u,\omega) \right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} &\lesssim \left(\int_0^{T_{*;(\lambda)}} \left(\widetilde{r}^{-2} w^{3/2} \lambda^{-1/2} \right)^2 \mathrm{d}\tau \right)^{1/2} \\ &= \left\{ \lambda^{-1} w^3 \left(- \frac{1}{(\tau+w)^3} \Big|_0^{T_{*;(\lambda)}} \right) \right\}^{1/2} \lesssim \lambda^{-1/2}. \end{aligned} \tag{9.137}$$

Now we estimate other terms in (9.135). By (9.89b),

$$\left\|\lambda^{-1}\widetilde{r}^{-2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2-12\varepsilon_0}. \tag{9.138}$$

Using the estimate for Hardy-Littlewood maximal function (9.28) and estimate (9.82f), we have

$$\left\|\widetilde{r}^{-2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\not{\nabla},\not{\mathbf{D}}_L)\xi\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\left\|\frac{1}{t-[u]_+}\int_{[u]_+}^{t}\left|\widetilde{r}(\not{\nabla},\not{\mathbf{D}}_L)\xi\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \tag{9.139}$$

$$\lesssim\|\widetilde{r}(\not{\nabla},\not{\mathbf{D}}_L)\xi\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}. \tag{9.140}$$

Similarly, by the estimate for Hardy-Littlewood maximal function (9.28), bootstrap assumptions (6.13), (9.4), and (9.5a), we have:

$$\begin{aligned}&\left\|\widetilde{r}^{-2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \qquad (9.141)\\ &\lesssim\left\|\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\\ &\lesssim\left\|\widetilde{r}\left(\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}+\lambda^{1/2-4\varepsilon_0}\left\|\widetilde{r}^{1/2}\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\\ &\lesssim\lambda^{-1/2-4\varepsilon_0}.\end{aligned}$$

Combining (9.134)-(9.141) and we conclude the desired estimate. □

Proof of $\|\widetilde{r}^{1/2}\hat{\chi}\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}$ *in (9.7b).* Dividing the equation (9.134) by $\widetilde{r}^{3/2}(t,u)$ and taking the norm, we have:

$$\begin{aligned}\left\|\widetilde{r}^{1/2}\hat{\chi}\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\leq&\left\|\widetilde{r}^{-3/2}\left(\lim_{\tau\downarrow[u]_+}\widetilde{r}^{2}\hat{\chi}\right)\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)} \qquad (9.142)\\ &+\left\|\lambda^{-1}\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\\ &+\left\|\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\not{\nabla},\not{\mathbf{D}}_L)\xi\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\\ &+\left\|\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}.\end{aligned}$$

We control the initial condition as in the previous proof. In particular, using (9.136) for $u<0$, we have:

$$\left\|\widetilde{r}^{-3/2}\left(\lim_{\tau\downarrow[u]_+}\widetilde{r}^{2}\hat{\chi}\right)\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}. \tag{9.143}$$

By (9.89e),

$$\left\|\lambda^{-1}\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{-1/2-12\varepsilon_0}. \tag{9.144}$$

By Minkowski's integral inequality and Hölder's inequality, also using (9.82f), we have:

$$\begin{aligned}\left\|\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}(\not{\nabla},\not{\mathbf{D}}_L)\xi\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathcal{C}_u)} &\leq \sup_t\left\|\widetilde{r}^{-1/2}(t,u)\int_{[u]_+}^{t}|\widetilde{r}(\not{\nabla},\not{\mathbf{D}}_L)\xi|_{\not{g}}\,\mathrm{d}\tau\right\|_{L_\omega^p(S_{t,u})} \\ &\lesssim \sup_t\frac{(t-[u]_+)^{1/2}}{\widetilde{r}^{1/2}(t,u)}\|\widetilde{r}(\not{\nabla},\not{\mathbf{D}}_L)\xi\|_{L_t^2L_\omega^p(\mathcal{C}_u)} \\ &\lesssim\lambda^{-1/2}.\end{aligned} \tag{9.145}$$

By the same argument as above and also using (9.141), we have

$$\begin{aligned}&\left\|\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{2}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathcal{C}_u)} \\ &\lesssim\sup_t\frac{(t-[u]_+)^{1/2}}{\widetilde{r}^{1/2}(t,u)}\left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)} \\ &\lesssim\lambda^{-1/2-4\varepsilon_0}.\end{aligned} \tag{9.146}$$

Combining (9.142)-(9.146) and we conclude the desired estimate. □

Proof of $\|\widetilde{r}\not{\mathbf{D}}_L\hat{\chi}\|_{L_t^2L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{-1/2}$ *in (9.7a).* Consider equation (8.139b):

$$\begin{aligned}\widetilde{r}\not{\mathbf{D}}_L\hat{\chi}=&\left(\widetilde{r}\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}+1\right)\hat{\chi}+\widetilde{r}(\not{\nabla},\not{\mathbf{D}}_L)\xi+\lambda^{-1}\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) \\ &+\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}.\end{aligned} \tag{9.147}$$

By (9.4) and (9.7b), we have:

$$\left\|\widetilde{r}\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}\cdot\hat{\chi}\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{1/2-4\varepsilon_0}\left\|\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^\infty(\mathcal{C}_u)}\left\|\widetilde{r}^{1/2}\hat{\chi}\right\|_{L_t^\infty L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{-1/2-2\varepsilon_0}. \tag{9.148}$$

By (9.82e), (9.82f) and (9.141), we have:

$$\begin{aligned}&\|\widetilde{r}(\not{\nabla},\not{\mathbf{D}}_L)\xi\|_{L_t^2L_\omega^p(\mathcal{C}_u)},\left\|\lambda^{-1}\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)}, \\ &\left\|\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)}\lesssim\lambda^{-1/2}.\end{aligned} \tag{9.149}$$

Combining (9.147)-(9.149) with (9.7a) for $\|\hat{\chi}\|_{L_t^2L_\omega^p(\mathcal{C}_u)}$, we conclude the desired estimates. □

Proof of $\|\zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.7a).* Considering equation (8.140) and using (9.76), we have:

$$
\begin{aligned}
|\widetilde{r}\zeta|_{\not g} \leq & \lim_{\tau\downarrow[u]_+} |\widetilde{r}\zeta|_{\not g}(\tau,u,\omega) \\
& + \int_{[u]_+}^t \lambda^{-1} \left|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not g} + |\widetilde{r}(\not\nabla,\not\mathbf{D}_L)\xi|_{\not g} \\
& + \left|\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}\right|_{\not g} + \left|\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \zeta \cdot \hat{\chi}\right|_{\not g} \mathrm{d}\tau.
\end{aligned}
\tag{9.150}
$$

Dividing (9.150) by $\widetilde{r}(t,u)$ and taking the $L_t^2 L_\omega^p$ norm, we have:

$$
\begin{aligned}
\|\zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \leq & \left\|\widetilde{r}^{-1}\left(\lim_{\tau\downarrow[u]_+} \widetilde{r}\zeta\right)\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \\
& + \left\|\lambda^{-1}\widetilde{r}^{-1}\int_{[u]_+}^t \left|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not g} \mathrm{d}\tau\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\|\widetilde{r}^{-1}\int_{[u]_+}^t |\widetilde{r}(\not\nabla,\not\mathbf{D}_L)\xi|_{\not g}\,\mathrm{d}\tau\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \\
& + \left\|\widetilde{r}^{-1}\int_{[u]_+}^t \left|\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}, \zeta, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}\right|_{\not g} \mathrm{d}\tau\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \\
& + \left\|\widetilde{r}^{-1}\int_{[u]_+}^t |\widetilde{r}\zeta\cdot\hat{\chi}|_{\not g}\,\mathrm{d}\tau\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)}.
\end{aligned}
\tag{9.151}
$$

We now consider the initial condition. When $u \geq 0$, by the initial condition on the cone-tip (8.64a), $\widetilde{r}^{-1}\left(\lim_{\tau\downarrow[u]_+} \widetilde{r}\zeta\right) = 0$. When $u < 0$, we use (8.17), (8.63b), and bootstrap assumption for $\boldsymbol{\partial}\vec{\Psi}$ to deduce:

$$
\begin{aligned}
\left\|\widetilde{r}^{-1}\left(\lim_{\tau\downarrow[u]_+} \widetilde{r}\zeta\right)\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} & \lesssim \left\|\widetilde{r}^{-1} w \not\nabla \ln a\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\|\widetilde{r}^{-1} w \boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \\
& \lesssim \lambda^{-1/2}\int_0^{T_{*;(\lambda)}} \frac{w}{(\tau+w)^2}\mathrm{d}\tau + \lambda^{-1/2-4\varepsilon_0} \\
& \lesssim \lambda^{-1/2}.
\end{aligned}
\tag{9.152}
$$

By (9.89a),

$$
\left\|\lambda^{-1}\widetilde{r}^{-1}\int_{[u]_+}^t \left|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not g} \mathrm{d}\tau\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2-12\varepsilon_0}.
\tag{9.153}
$$

Using the same method as in (9.139) and (9.141), we have :

$$
\left\|\widetilde{r}^{-1}\int_{[u]_+}^t |\widetilde{r}(\not\nabla,\not\mathbf{D}_L)\xi|_{\not g}\,\mathrm{d}\tau\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2},
\tag{9.154}
$$

$$
\left\|\widetilde{r}^{-1}\int_{[u]_+}^t \left|\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}, \zeta, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}\right|_{\not g} \mathrm{d}\tau\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2-4\varepsilon_0}.
\tag{9.155}
$$

Using the estimate for Hardy-Littlewood maximal function (9.28), bootstrap assumption (9.6), and the estimate previously proven for $\widetilde{r}^{1/2}\hat{\chi}$ in (9.7b), we have:

$$\left\|\widetilde{r}^{-1}\int_{[u]_+}^{t}|\widetilde{r}\zeta\cdot\hat{\chi}|_{\not g}\,\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\|\widetilde{r}\zeta\cdot\hat{\chi}\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{1/2-4\varepsilon_0}\left\|\widetilde{r}^{1/2}\hat{\chi}\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\|\zeta\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\lesssim\lambda^{-1/2-4\varepsilon_0}. \tag{9.156}$$

Combining (9.150)-(9.156), we conclude the desired estimate. □

Proof of $\left\|\widetilde{r}^{1/2}\zeta\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}$ *in (9.7b).* Dividing the equation (9.150) by $\widetilde{r}^{1/2}(t,u)$ and taking the $L_t^\infty L_\omega^p$ norm, we have:

$$\begin{aligned}\left\|\widetilde{r}^{1/2}\zeta\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\leq&\left\|\widetilde{r}^{-1/2}\left(\lim_{\tau\downarrow[u]_+}\widetilde{r}\zeta\right)\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\\&+\left\|\lambda^{-1}\widetilde{r}^{-1/2}\int_{[u]_+}^{t}\left|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not g}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}+\left\|\widetilde{r}^{-1/2}\int_{[u]_+}^{t}|\widetilde{r}(\nabla\!\!\!/,\mathbf{D}\!\!\!\!/_L)\xi|_{\not g}\,\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\\&+\left\|\widetilde{r}^{-1/2}\int_{[u]_+}^{t}\left|\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{\not g}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\\&+\left\|\widetilde{r}^{-1/2}\int_{[u]_+}^{t}|\widetilde{r}\zeta\cdot\hat{\chi}|_{\not g}\,\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}.\end{aligned} \tag{9.157}$$

Let's consider the initial condition. When $u\geq 0$, by the initial condition on the cone-tip (8.64a), $\widetilde{r}^{-1/2}\left(\lim_{\tau\downarrow[u]_+}\widetilde{r}\zeta\right)=0$. When $u<0$, we use (8.17), (8.63b), and the estimate (9.82a) to deduce:

$$\left\|\widetilde{r}^{-1/2}\left(\lim_{\tau\downarrow[u]_+}\widetilde{r}\zeta\right)\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\lesssim\left\|w^{1/2}\nabla\!\!\!/\ln a\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}+\left\|\widetilde{r}^{1/2}\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}. \tag{9.158}$$

By (9.89d), we have:

$$\left\|\lambda^{-1}\widetilde{r}^{-1/2}\int_{[u]_+}^{t}\left|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not g}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2-12\varepsilon_0}. \tag{9.159}$$

Using the same method as in (9.145) and (9.146), we have :

$$\left\|\widetilde{r}^{-1/2}\int_{[u]_+}^{t}|\widetilde{r}(\nabla\!\!\!/,\mathbf{D}\!\!\!\!/_L)\xi|_{\not g}\,\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}, \tag{9.160}$$

$$\left\|\widetilde{r}^{-1/2}\int_{[u]_+}^{t}\left|\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})},\zeta,\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{\not g}\mathrm{d}\tau\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2-4\varepsilon_0}. \tag{9.161}$$

With the help of (9.156), we have:

$$\left\| \widetilde{r}^{-1/2} \int_{[u]_+}^{t} |\widetilde{r}\zeta \cdot \hat{\chi}|_{\not{g}} \, \mathrm{d}\tau \right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)} \lesssim \sup_t \widetilde{r}^{-1/2} \int_{[u]_+}^{t} \|\widetilde{r}\zeta \cdot \hat{\chi}\|_{L_\omega^p(S_{\tau,u})} \, \mathrm{d}\tau \tag{9.162}$$
$$\lesssim \sup_t \frac{(t-[u]_+)^{1/2}}{(t-u)^{1/2}} \|\widetilde{r}\zeta \cdot \hat{\chi}\|_{L_t^2 L_\omega^p(\mathscr{C}_u)}$$
$$\lesssim \lambda^{-1/2-4\varepsilon_0}.$$

Combining (9.157)-(9.162) and we conclude the desired estimate. □

Proof of $\|\widetilde{r}\not{\mathbf{D}}_L \zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.7a).* Consider equation (8.140):

$$\widetilde{r}\not{\mathbf{D}}_L \zeta = \zeta + \widetilde{r}(\not{\nabla}, \not{\mathbf{D}}_L)\xi + \lambda^{-1}\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) + \widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi} \tag{9.163}$$
$$+ \widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \zeta \cdot \left(\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \boldsymbol{\partial}\vec{\Psi}\right).$$

By (6.13), (9.4) and (9.7b), we have:

$$\left\| \widetilde{r}\left(\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \boldsymbol{\partial}\vec{\Psi}\right) \cdot \zeta \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{1/2-4\varepsilon_0} \left\| \mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \boldsymbol{\partial}\vec{\Psi} \right\|_{L_t^2 L_\omega^\infty(\mathscr{C}_u)} \left\| \widetilde{r}^{1/2}\zeta \right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)} \tag{9.164}$$
$$\lesssim \lambda^{-1/2-2\varepsilon_0}.$$

By (9.82e), (9.82f) and (9.141), we have:

$$\|\widetilde{r}(\not{\nabla}, \not{\mathbf{D}}_L)\xi\|_{L_t^2 L_\omega^p(\mathscr{C}_u)}, \left\| \lambda^{-1}\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)}, \tag{9.165}$$
$$\left\| \widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi} \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}.$$

Combining (9.163)-(9.165) with (9.7a) for $\|\zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)}$, we conclude the desired estimates. □

Proof of $\left\|\widetilde{r}^{1/2}(\hat{\chi}, \zeta)\right\|_{L_\omega^{2p} L_t^\infty(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ *when* $\mathscr{C}_u \subset \mathscr{M}^{(Int)}$ *in (9.14).* By the Sobolev inequality (9.32), the previously proven estimate (9.7a), and the bootstrap assumptions (9.6), we have:

$$\left\| \widetilde{r}^{\frac{1}{2}}(\hat{\chi}, \zeta) \right\|_{L_\omega^{2p} L_t^\infty(\mathscr{C}_u)}^2 \lesssim \left(\|\widetilde{r}\not{\mathbf{D}}_L(\hat{\chi}, \zeta)\|_{L_\omega^p L_t^2(\mathscr{C}_u)} + \|(\hat{\chi}, \zeta)\|_{L_\omega^p L_t^2(\mathscr{C}_u)} \right) \|(\hat{\chi}, \zeta)\|_{L_\omega^\infty L_t^2(\mathscr{C}_u)} \lesssim \lambda^{-1}. \tag{9.166}$$

In deriving (9.166), we also used Minkowski's inequality for integrals to switch the order of L_t, L_ω norms. □

Proof of $\widetilde{r}\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi} \approx 1$ *in (9.8a).* It is sufficient to show $\left|\widetilde{r}\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}\right|_{\not{g}} \lesssim \lambda^{-4\varepsilon_0}$. We plug (8.138a) into (9.77), where $\mathfrak{G} = \mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$. Then dividing the estimate (9.79) by $\widetilde{r}(t,u)$, we have:

$$\begin{aligned} \left|\widetilde{r}\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} \lesssim & \widetilde{r}^{-1} \lim_{\tau\downarrow[u]_+} \left|\widetilde{r}^2\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} \\ & + \widetilde{r}^{-1}\int_{[u]_+}^t \left|\widetilde{r}^2\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \widetilde{r}^2\mathrm{f}_{(\vec{L})}\cdot\left(\hat{\chi},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\boldsymbol{\partial}\vec{\Psi} + \widetilde{r}^2\left|\hat{\chi}\right|_{g}^2\right|_{g} \mathrm{d}\tau. \end{aligned} \tag{9.167}$$

We now consider the initial conditions. When $u \geq 0$, by the initial condition on the cone-tip (8.64a), $\widetilde{r}^{-1}\lim_{\tau\downarrow[u]_+}\left|\widetilde{r}^2\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} \leq \lim_{\tau\downarrow[u]_+}\left|\widetilde{r}\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} = 0$. When $u < 0$, by (8.61) and (8.63a), we have:

$$\widetilde{r}^{-1}\lim_{\tau\downarrow[u]_+}\left|\widetilde{r}^2\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} = \frac{1}{\widetilde{r}}\frac{2w(1-a)}{a} \leq \frac{2(1-a)}{a} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.168}$$

By (9.89f), we have:

$$\widetilde{r}^{-1}\int_{[u]_+}^t \widetilde{r}^2\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\mathrm{d}\tau \lesssim \lambda^{-16\varepsilon_0}. \tag{9.169}$$

By bootstrap assumptions (6.13) and (9.4), we have:

$$\widetilde{r}^{-1}\int_{[u]_+}^t \left|\widetilde{r}^2\mathrm{f}_{(\vec{L})}\cdot(\hat{\chi},\boldsymbol{\partial}\vec{\Psi})\cdot(\hat{\chi},\boldsymbol{\partial}\vec{\Psi})\right|_{g}\mathrm{d}\tau \lesssim \lambda^{1-8\varepsilon_0}\left\|\hat{\chi},\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\left\|\hat{\chi},\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.170}$$

$$\widetilde{r}^{-1}\int_{[u]_+}^t \left|\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{g}\mathrm{d}\tau \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\lambda^{1/2-4\varepsilon_0} \lesssim \lambda^{-8\varepsilon_0}. \tag{9.171}$$

Combining (9.167)-(9.171) and we conclude the desired estimate. □

$\left\|\widetilde{r}\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-4\varepsilon_0}$ follows from the proof of $\widetilde{r}\mathrm{tr}_{\tilde{g}}\tilde{\chi} \approx 1$.

Proof of $\left\|\widetilde{r}^{1/2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathit{Small})}\right\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-1/2}$ *in (9.8b).* Dividing (9.167) by $\widetilde{r}^{1/2}$, we have

$$\begin{aligned} \left|\widetilde{r}^{1/2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} \lesssim & \widetilde{r}^{-3/2} \lim_{\tau\downarrow[u]_+} \left|\widetilde{r}^2\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} \\ & + \widetilde{r}^{-3/2}\int_{[u]_+}^t \left|\widetilde{r}^2\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D}) + \widetilde{r}^2\mathrm{f}_{(\vec{L})}\cdot\left(\hat{\chi},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\boldsymbol{\partial}\vec{\Psi} + \widetilde{r}^2\left|\hat{\chi}\right|_{g}^2\right|_{g} \mathrm{d}\tau. \end{aligned} \tag{9.172}$$

We consider the initial conditions. When $u \geq 0$, by the initial condition on the cone-tip (8.64a), we have:

$$\widetilde{r}^{-3/2}\lim_{\tau\downarrow[u]_+}\left|\widetilde{r}^2\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} \leq \widetilde{r}^{-1/2}\lim_{\tau\downarrow[u]_+}\left|\widetilde{r}\mathrm{tr}_{\tilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} \lesssim \lim_{\tau\downarrow[u]_+}\widetilde{r}^{1/2} = 0. \tag{9.173}$$

When $u < 0$, by (8.61) and (8.63b), we have:

$$\widetilde{r}^{-3/2} \lim_{\tau \downarrow [u]_+} \left| \widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right|_{g\!\!/} = \frac{1}{\widetilde{r}^{3/2}} \frac{2w(1-a)}{a} \leq w^{-1/2}(1-a) \lesssim \lambda^{-1/2}. \tag{9.174}$$

By (9.89g), we have:

$$\lambda^{-1} \left\| \widetilde{r}^{-\frac{3}{2}} \int_{[u]_+}^{t} \left| \widetilde{r}^2 (\vec{\mathscr{C}}, \mathscr{D}) \right|_{g\!\!/} \mathrm{d}\tau \right\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-1/2 - 12\varepsilon_0}. \tag{9.175}$$

By bootstrap assumptions (6.13) and (9.4), we have:

$$\begin{aligned} \widetilde{r}^{-3/2} \int_{[u]_+}^{t} \left| \widetilde{r}^2 \mathrm{f}_{(\vec{L})} \cdot (\hat{\chi}, \boldsymbol{\partial} \vec{\Psi}) \cdot (\hat{\chi}, \boldsymbol{\partial} \vec{\Psi}) \right|_{g\!\!/} \mathrm{d}\tau \lesssim & \lambda^{1/2 - 4\varepsilon_0} \left\| \hat{\chi}, \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_\omega^\infty(\mathscr{C}_u)} \left\| \hat{\chi}, \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_\omega^\infty(\mathscr{C}_u)} \\ \lesssim & \lambda^{-1/2 - 4\varepsilon_0}. \end{aligned} \tag{9.176}$$

$$\begin{aligned} \widetilde{r}^{-3/2} \int_{[u]_+}^{t} \left| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial} \vec{\Psi} \right|_{g\!\!/} \mathrm{d}\tau \lesssim & \widetilde{r}^{-1/2} \int_{[u]_+}^{t} \left\| \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial} \vec{\Psi} \right\|_{L_x^\infty(\Sigma_t)} \mathrm{d}\tau \lesssim \frac{(t - [u]_+)^{1/2}}{(t-u)^{1/2}} \left\| \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_x^\infty(\mathscr{M})} \\ \lesssim & \lambda^{-1/2 - 4\varepsilon_0}. \end{aligned} \tag{9.177}$$

Combining (9.172)-(9.177) and we conclude the desired estimate. □

$\left\| \widetilde{r}^{1/2} \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ and $\left\| \widetilde{r}^{1/2} \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right\|_{L_\omega^{2p} L_t^\infty(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ follows from the previously proven estimate $\left\| \widetilde{r}^{1/2} \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right\|_{L^\infty(\mathscr{M})} \lesssim \lambda^{-1/2}$.

Proof of $\left\| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathit{Small})} \right\|_{L_t^2 C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.8e).* Using equation (8.138a) and dividing $L(\widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})})$ by $\widetilde{r}^2(t,u)$, we have:

$$\begin{aligned} \left| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right|_{g\!\!/} \leq & \widetilde{r}^{-2} \lim_{\tau \downarrow [u]_+} \left| \widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right|_{g\!\!/} + \widetilde{r}^{-2} \int_{[u]_+}^{t} \left| \widetilde{r}^2 \lambda^{-1} \mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) \mathrm{d}\tau \right|_{g\!\!/} \\ & + \widetilde{r}^{-2} \int_{[u]_+}^{t} \left| \widetilde{r}^2 \mathrm{f}_{(\vec{L})} \cdot \left(\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}, \boldsymbol{\partial} \vec{\Psi} \right) \cdot \boldsymbol{\partial} \vec{\Psi} \right. \\ & \left. + \widetilde{r}^2 \left| \hat{\chi} \right|_{g\!\!/}^2 + \widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \cdot \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right|_{g\!\!/} \mathrm{d}\tau. \end{aligned} \tag{9.178}$$

Using (9.108), we have:

$$\begin{aligned} \left\| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim & \left\| \widetilde{r}^{-2} \lim_{\tau \downarrow [u]_+} \left(\widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right) \right\|_{C_\omega^{0,\delta_0}(\mathscr{C}_u)} + \lambda^{-1} \left\| \vec{\mathscr{C}}, \mathscr{D} \right\|_{L_t^1 C_\omega^{0,\delta_0}(\mathscr{C}_u)} \\ & + \left\| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial} \vec{\Psi}, \hat{\chi} \right\|_{L_t^2 C_\omega^{0,\delta_0}(\mathscr{C}_u)}^2 + \mathscr{M} \left(\left\| \boldsymbol{\partial} \vec{\Psi} \right\|_{C_\omega^{0,\delta_0}(S_{t,u})} \right). \end{aligned} \tag{9.179}$$

We now consider the initial condition. When $u \geq 0$, by (8.64a), $\lim_{\tau\downarrow[u]_+}\left(\widetilde{r}^2\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right)=0$, we have:

$$\begin{aligned}\widetilde{r}^2\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})} \leq & \int_{[u]_+}^{t}\left|\widetilde{r}^2\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\right|_{\not g}\mathrm{d}\tau \\ & + \int_{[u]_+}^{t}\left|\widetilde{r}^2\mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right. \\ & \left.+\widetilde{r}^2\left|\hat{\chi}\right|_{\not g}^2+\widetilde{r}^2\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\cdot\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right|_{\not g}\mathrm{d}\tau.\end{aligned} \tag{9.180}$$

When $u<0$, by (8.61), we have:

$$\left\|\widetilde{r}^{-2}\lim_{\tau\downarrow[u]_+}\left(\widetilde{r}^2\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right)\right\|^2_{L_t^2C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \int_0^{T_{*;(\lambda)}}\left(\frac{w^{3/2}}{\widetilde{r}^2}\lambda^{-1/2}\right)^2\mathrm{d}\tau \leq \lambda^{-1}. \tag{9.181}$$

Using the estimate for Hardy-Littlewood maximal function (9.28) and estimate (9.104), we have:

$$\left\|\mathscr{M}\left(\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{C_\omega^{0,\delta_0}(S_{t,u})}\right)\right\|_{L_t^2} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \lambda^{-1/2-3\varepsilon_0}. \tag{9.182}$$

By bootstrap assumptions (6.13), (9.4), and estimate (9.104), we have:

$$\lambda^{-1}\left\|\vec{\mathscr{C}},\mathscr{D}\right\|_{L_t^1C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \lambda^{-1-7\varepsilon_0}, \tag{9.183}$$

$$\left\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right\|^2_{L_t^2C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \lambda^{-1+4\varepsilon_0}. \tag{9.184}$$

Taking the L_t^2 norm of (9.179), combining (9.180)-(9.184), we conclude the desired result. □

$\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ follows directly from $\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\|_{L_t^2C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$.

Proof of $\|\widetilde{r}\not{\mathbf{D}}_L\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathit{Small})}\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.7a).* Using equation (8.138a), we have:

$$\begin{aligned}\widetilde{r}\not{\mathbf{D}}_L\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})} =& \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}+\widetilde{r}\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})+\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi} \\ & +\widetilde{r}\left|\hat{\chi}\right|_{\not g}^2+\widetilde{r}\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\cdot\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}.\end{aligned} \tag{9.185}$$

By (9.93a), we have:

$$\lambda^{-1}\left\|\widetilde{r}(\vec{\mathscr{C}},\mathscr{D})\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \lesssim^{-1/2-8\varepsilon_0}. \tag{9.186}$$

Using the same method as in (9.141), we have:

$$\left\|\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2-4\varepsilon_0}. \tag{9.187}$$

By bootstrap assumtions (9.4) and the previously proven results (9.7b), we have:

$$\begin{aligned}&\left\|\widetilde{r}\left(\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta\right)\cdot\left(\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta\right)\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \\ &\lesssim\lambda^{1/2-4\varepsilon_0}\left\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta\right\|_{L_t^2C_\omega^{0,\delta_0}(\mathscr{C}_u)}\left\|\widetilde{r}^{1/2}\left(\hat{\chi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\zeta\right)\right\|_{L_t^\infty L_\omega^p(\mathscr{C}_u)} \\ &\lesssim\lambda^{-1/2-2\varepsilon_0}.\end{aligned} \tag{9.188}$$

Combining (9.185)-(9.188) and we conclude the desired estimate. □

Remark 9.14. *From (9.7a), (9.141), (9.161), (9.188) and (9.164), it follows:*

$$\left\|\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta\right)\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}. \tag{9.189}$$

Proof of $\left\|\widetilde{r}\left(\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\nabla\!\!\!/\hat{\chi}\right)\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}$ *in (9.8d).* First, we bound $\left\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}$. Plugging equation (8.138b) into (9.77), where $\mathfrak{G}=\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi}\right)$, then dividing by $\widetilde{r}^2(t,u)$, we have:

$$\begin{aligned}\left|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right|_{g\!\!\!/}\lesssim&\,\widetilde{r}^{-2}\lim_{\tau\downarrow[u]_+}\left|\widetilde{r}^3\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right|_{g\!\!\!/}+\lambda^{-1}\widetilde{r}^{-2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/(\vec{\mathscr{C}},\mathscr{D})\right|_{g\!\!\!/}\mathrm{d}\tau \\ &+\lambda^{-1}\widetilde{r}^{-2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\right|_{g\!\!\!/}\mathrm{d}\tau \\ &+\widetilde{r}^{-2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)+\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\hat{\chi}\cdot\hat{\chi}\right|_{g\!\!\!/}\mathrm{d}\tau \\ &+\widetilde{r}^{-2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{g\!\!\!/}\mathrm{d}\tau.\end{aligned} \tag{9.190}$$

Let's consider the initial conditions. When $u\geq 0$, we use the estimate for Hardy-Littlewood maximal function (9.28) and (8.64a) to deduce $\left\|\widetilde{r}^{-2}\lim_{\tau\downarrow[u]_+}(\widetilde{r}^3\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})})\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}=0$. When $u<0$, we use the initial condition (8.63g) to deduce:

$$\left\|\widetilde{r}^{-2}\lim_{\tau\downarrow[u]_+}(\widetilde{r}^3\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})})\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}\left\|\widetilde{r}^{-2}w^{3/2}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}. \tag{9.191}$$

By (9.90b), we have:

$$\lambda^{-1}\left\|\widetilde{r}^{-2}\int_{[u]_+}^{t}\left|\widetilde{r}^3 \mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/(\vec{\mathscr{C}},\mathscr{D})\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2-8\varepsilon_0}. \tag{9.192}$$

By (9.91b), we have:

$$\lambda^{-1}\left\|\widetilde{r}^{-2}\int_{[u]_+}^{t}\left|\widetilde{r}^3 \mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2-12\varepsilon_0}. \tag{9.193}$$

By the estimate for Hardy-Littlewood maximal function (9.28), bootstrap assumptions (6.13), (9.4), and estimate (9.82f), we have:

$$\begin{aligned}
&\left\|\widetilde{r}^{-2}\int_{[u]_+}^{t}\widetilde{r}^3 \mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \qquad (9.194)\\
&\lesssim\left\|\int_{[u]_+}^{t}\left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^p(S_{\tau,u})}\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_\omega^\infty(S_{\tau,u})}\mathrm{d}\tau\right\|_{L_t^2}+\left\|\widetilde{r}^{-1}\int_{[u]_+}^{t}\left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^p(S_{\tau,u})}\mathrm{d}\tau\right\|_{L_t^2}\\
&\lesssim\lambda^{1/2-4\varepsilon_0}\left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}+\left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\\
&\lesssim\lambda^{-1/2}.
\end{aligned}$$

By the estimate for Hardy-Littlewood maximal function (9.28), bootstrap assumptions (6.13) , (9.4), (9.5a), and estimate (9.82e), we have:

$$\begin{aligned}
&\left\|\widetilde{r}^{-2}\int_{[u]_+}^{t}\widetilde{r}^3 \mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \qquad (9.195)\\
\lesssim&\left\|\int_{[u]_+}^{t}\left(\left\|\widetilde{r}\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\right\|_{L_\omega^p(S_{\tau,u})},\left\|\widetilde{r}\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^p(S_{\tau,u})}\right)\right.\\
&\left.\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_\omega^\infty(S_{\tau,u})}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^\infty(S_{\tau,u})}\mathrm{d}\tau\right\|_{L_t^2}\\
&+\left\|\widetilde{r}^{-1}\int_{[u]_+}^{t}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^\infty(S_{\tau,u})}\mathrm{d}\tau\right\|_{L_t^2}\\
\lesssim&\lambda^{1/2-4\varepsilon_0}\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}+\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\\
\lesssim&\lambda^{-1/2-4\varepsilon_0}.
\end{aligned}$$

By bootstrap assumption (9.4), we have:

$$\begin{aligned}
\left\|\widetilde{r}^{-2}\int_{[u]_+}^{t}\widetilde{r}^3 \mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\hat{\chi}\cdot\hat{\chi}\mathrm{d}\tau\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}&\lesssim\left\|\|\widetilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\|\hat{\chi}\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\right\|_{L_t^2} \qquad (9.196)\\
&\lesssim\lambda^{-2\varepsilon_0}\|\widetilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2L_\omega^p(\mathscr{C}_u)}.
\end{aligned}$$

Combining (9.190)-(9.196) and we have:

$$\left\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}+\lambda^{-2\varepsilon_0}\left\|\widetilde{r}\nabla\!\!\!/\hat{\chi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}. \tag{9.197}$$

Now we consider $\|\widetilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2L_\omega^p(\mathscr{C}_u)}$. Plugging equation (8.139a) into the Hodge estimate (9.111), we have:

$$\begin{aligned}\|\widetilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \lesssim& \left\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \\ &+\left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}+\left\|\widetilde{r}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}.\end{aligned} \tag{9.198}$$

By (9.82f) and (9.189), we have:

$$\|\widetilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}+\left\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}. \tag{9.199}$$

Combining (9.197) and (9.199), we have the desired estimates $\left\|\widetilde{r}\left(\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\nabla\!\!\!/\hat{\chi}\right)\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\lesssim\lambda^{-1/2}$. □

Proof of $\left\|\widetilde{r}^{3/2}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\right\|_{L_t^\infty L_u^\infty L_\omega^p(\mathscr{M})}\lesssim\lambda^{-1/2}$ *in (9.8c).* Plugging equation (8.138b) into (9.77), where $\mathfrak{G}=\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi}\right)$, then dividing by $\widetilde{r}^{3/2}(t,u)$, we have:

$$\begin{aligned}&\left|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right|_{g\!\!\!/} \\ \lesssim&\,\widetilde{r}^{-3/2}\lim_{\tau\downarrow[u]_+}\left|\widetilde{r}^3\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right|_{g\!\!\!/}+\lambda^{-1}\widetilde{r}^{-3/2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/(\vec{\mathscr{C}},\mathscr{D})\right|_{g\!\!\!/}\mathrm{d}\tau \\ &+\lambda^{-1}\widetilde{r}^{-3/2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\right|_{g\!\!\!/}\mathrm{d}\tau \\ &+\widetilde{r}^{-3/2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)+\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\hat{\chi}\cdot\hat{\chi}\right|_{g\!\!\!/}\mathrm{d}\tau \\ &+\widetilde{r}^{-3/2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{g\!\!\!/}\mathrm{d}\tau.\end{aligned} \tag{9.200}$$

We now consider the initial conditions. When $u\geq 0$, we use (8.64a) to deduce $\widetilde{r}^{-3/2}\lim_{\tau\downarrow[u]_+}\left(\widetilde{r}^3\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right)=0$. When $u<0$, we use the initial condition (8.63g) to deduce $\left\|\widetilde{r}^{-3/2}\lim_{\tau\downarrow[u]_+}\left|\widetilde{r}^3\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right|_{g\!\!\!/}\right\|_{L_u^\infty L_\omega^p(\Sigma_0)}\lesssim\lambda^{-1/2}$. By (9.90a), we have:

$$\left\|\lambda^{-1}\widetilde{r}^{-3/2}\int_{[u]_+}^t\left|\widetilde{r}^3\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/(\vec{\mathscr{C}},\mathscr{D})\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_t^\infty L_u^\infty L_\omega^p(\mathscr{M})}\lesssim\lambda^{-1/2-8\varepsilon_0}. \tag{9.201}$$

By (9.91a), we have:

$$\left\|\lambda^{-1}\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{3}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_t^\infty L_u^\infty L_\omega^p(\mathscr{M})}\lesssim\lambda^{-1/2-12\varepsilon_0}. \tag{9.202}$$

By bootstrap assumptions (6.13) and (9.4), estimate (9.82f), we have:

$$\begin{aligned}
&\left\|\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{3}\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_\omega^p(S_{t,u})} \\
&\lesssim\lambda^{1/2-4\varepsilon_0}\left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(S_{t,u})}\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^\infty(S_{t,u})}+\frac{(t-[u]_+)^{1/2}}{(t-u)^{1/2}}\left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)} \\
&\lesssim\lambda^{-1/2}.
\end{aligned} \tag{9.203}$$

By bootstrap assumptions (6.13), (9.4) and (9.82a), we have:

$$\begin{aligned}
&\left\|\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{3}\mathrm{f}_{(\vec{L})}\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_\omega^p(S_{t,u})} \\
\lesssim&\lambda^{1/2-4\varepsilon_0}\int_{[u]_+}^{t}\left\|\widetilde{r}\left(\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\right\|_{L_\omega^p(S_{\tau,u})}\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_\omega^\infty(S_{\tau,u})}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^\infty(S_{\tau,u})}\mathrm{d}\tau \\
&+\lambda^{1-8\varepsilon_0}\int_{[u]_+}^{t}\left\|\widetilde{r}^{1/2}\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^p(S_{\tau,u})}\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_\omega^\infty(S_{\tau,u})}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^\infty(S_{\tau,u})}\mathrm{d}\tau \\
&+\widetilde{r}^{-1/2}\int_{[u]_+}^{t}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^\infty(S_{\tau,u})}\mathrm{d}\tau \\
\lesssim&\lambda^{1/2-4\varepsilon_0}\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}+\frac{(t-[u]_+)^{1/2}}{(t-u)^{1/2}}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)} \\
\lesssim&\lambda^{-1/2-4\varepsilon_0}.
\end{aligned} \tag{9.204}$$

By bootstrap assumptions (6.13) and the previously proven result (9.8d), we have:

$$\left\|\widetilde{r}^{-3/2}\int_{[u]_+}^{t}\left|\widetilde{r}^{3}\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\hat{\chi}\cdot\hat{\chi}\right|_{\not{g}}\mathrm{d}\tau\right\|_{L_\omega^p(S_{\tau,u})}\lesssim\lambda^{1/2-4\varepsilon_0}\|\widetilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\|\hat{\chi}\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\lesssim\lambda^{-1/2-2\varepsilon_0}. \tag{9.205}$$

□

Proof of $\left\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\mathrm{tr}_{\not{g}}\chi-\frac{2}{\widetilde{r}},\hat{\chi}\right\|_{L_t^2L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(Int)})}\lesssim\lambda^{-1/2-3\varepsilon_0}$ *in (9.15).* We first bound $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}$. Using equation (8.138a) and initial condition (8.64a), and dividing $L(\widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})})$ by $\widetilde{r}^2(t,u)$, we have

$$\left|\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}\right|_{g} \lesssim \widetilde{r}^{-2} \int_{[u]_+}^{t} \left|\widetilde{r}^2 \lambda^{-1} \mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D})\right|_{g} \mathrm{d}\tau + \widetilde{r}^{-2} \int_{[u]_+}^{t} \left|\widetilde{r}^2 \mathrm{f}_{(\vec{L})} \cdot \left(\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}, \boldsymbol{\partial} \vec{\Psi}\right) \cdot \boldsymbol{\partial} \vec{\Psi} + \widetilde{r}^2 |\hat{\chi}|_{g}^2 + \widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \cdot \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}\right|_{g} \mathrm{d}\tau. \tag{9.206}$$

Using (9.108), we have:

$$\left\|\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_u^\infty C_\omega^{0,\delta_0}(\Sigma_t^{(\mathrm{Int})})} \lesssim \lambda^{-1} \left\|\vec{\mathscr{C}}, \mathscr{D}\right\|_{L_t^1 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} + \left\|\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial} \vec{\Psi}, \hat{\chi}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})}^2 + \mathscr{M}\left(\left\|\boldsymbol{\partial} \vec{\Psi}\right\|_{L_u^\infty C_\omega^{0,\delta_0}(\Sigma_t^{(\mathrm{Int})})}\right). \tag{9.207}$$

Using the estimate for Hardy-Littlewood maximal function (9.28) and estimate (9.104), we have:

$$\left\|\mathscr{M}\left(\left\|\boldsymbol{\partial} \vec{\Psi}\right\|_{L_u^\infty C_\omega^{0,\delta_0}(\Sigma_t^{(\mathrm{Int})})}\right)\right\|_{L_t^2} \lesssim \left\|\boldsymbol{\partial} \vec{\Psi}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \lambda^{-1/2-3\varepsilon_0}. \tag{9.208}$$

By bootstrap assumption (6.13), (9.6) and (9.104), we have:

$$\lambda^{-1} \left\|\vec{\mathscr{C}}, \mathscr{D}\right\|_{L_t^1 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \lambda^{-1-7\varepsilon_0}, \tag{9.209}$$

$$\left\|\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial} \vec{\Psi}, \hat{\chi}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{C}_u)}^2 \lesssim \lambda^{-1}. \tag{9.210}$$

Taking the L_t^2 norm of (9.206), combining (9.206)-(9.210), we conclude the desired result for $\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}$.

Now, since $\mathrm{tr}_{g}\chi - \frac{2}{\widetilde{r}} = \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L$, we have:

$$\left\|\mathrm{tr}_{g}\chi - \frac{2}{\widetilde{r}}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} = \left\|\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} + \left\|\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial} \vec{\Psi}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})}. \tag{9.211}$$

By the previously proven result (9.10) and estimate (9.104), we have:

$$\left\|\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial} \vec{\Psi}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} \lesssim \left\|\mathrm{f}_{(\vec{L})}\right\|_{L_t^\infty L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} \left\|\boldsymbol{\partial} \vec{\Psi}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-3\varepsilon_0}. \tag{9.212}$$

Combining (9.211)-(9.212) with the result for $\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}$, we conclude the proof for $\mathrm{tr}_{g}\chi - \frac{2}{\widetilde{r}}$.

We now prove $\|\hat{\chi}\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-3\varepsilon_0}$. Plugging equation (8.139a) into the Hodge estimate

(9.114) with $Q := \frac{2}{1-\delta_0}$, where we recall $\delta_0 < 1 - \frac{2}{p}$, $Q < p$, we have:

$$\begin{aligned}\|\hat{\chi}\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} \lesssim & \left\| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\| \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1} \right) \cdot \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}.\end{aligned} \tag{9.213}$$

By bootstrap assumptions (6.13), (9.5a) and (9.82a), we have:

$$\begin{aligned}& \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1} \right) \cdot \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ \lesssim & \left\| \left(\lambda^{1/2-4\varepsilon_0} \left\| \widetilde{r}^{1/2} \boldsymbol{\partial} \vec{\Psi} \right\|_{L_\omega^p(S_{t,u})}, \left\| \widetilde{r} \left(\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1} \right) \right\|_{L_\omega^p(S_{t,u})} \right) \left\| \boldsymbol{\partial} \vec{\Psi} \right\|_{L_\omega^\infty(S_{t,u})} \right\|_{L_t^2 L_u^\infty} \\ \lesssim & \left\| \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M})} \lesssim \lambda^{-1/2-4\varepsilon_0}.\end{aligned} \tag{9.214}$$

Combining (9.213)-(9.214) with previously proven result for $\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}$ and estimate (9.104), we conclude the desired estimate. ☐

Proof of $\left\| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(Small)}, \hat{\chi}, \mathrm{tr}_{g} \chi - \frac{2}{\widetilde{r}} \right\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}$ *in (9.11).* We first bound $\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}$. Using equation (8.138a) and dividing by $\widetilde{r}^2(t,u)$, we have:

$$\begin{aligned}\left| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right|_{g} \lesssim & \widetilde{r}^{-2} \lim_{\tau \downarrow [u]_+} \left| \widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right|_{g} + \widetilde{r}^{-2} \int_{[u]_+}^t \left| \widetilde{r}^2 \lambda^{-1} \mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) \right|_{g} \mathrm{d}\tau \\ & + \widetilde{r}^{-2} \int_{[u]_+}^t \left| \widetilde{r}^2 \mathrm{f}_{(\vec{L})} \cdot \left(\mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}, \boldsymbol{\partial} \vec{\Psi} \right) \cdot \boldsymbol{\partial} \vec{\Psi} \right. \\ & \left. + \widetilde{r}^2 |\hat{\chi}|_{g}^2 + \widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \cdot \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right|_{g} \mathrm{d}\tau.\end{aligned} \tag{9.215}$$

Using (9.108), by bootstrap assumptions (6.13), (9.4) and (9.104), we have:

$$\begin{aligned}\left\| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right\|_{L_u^\infty C_\omega^{0,\delta_0}(\Sigma_t)} \lesssim & \left\| \widetilde{r}^{-2} \lim_{\tau \downarrow [u]_+} \left(\widetilde{r}^2 \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})} \right) \right\|_{L_u^\infty C_\omega^{0,\delta_0}(\Sigma_t)} + \lambda^{-1} \left\| \vec{\mathscr{C}}, \mathscr{D} \right\|_{L_t^1 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \\ & + \left\| \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial} \vec{\Psi}, \hat{\chi} \right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})}^2 + \mathscr{M} \left(\left\| \boldsymbol{\partial} \vec{\Psi} \right\|_{L_u^\infty C_\omega^{0,\delta_0}(\Sigma_t)} \right).\end{aligned} \tag{9.216}$$

We now consider the initial condition. When $u \geq 0$, by (8.64a), $\lim_{\tau\downarrow[u]_+}\left(\widetilde{r}^2\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right) = 0$, we have:

$$\left|\widetilde{r}^2\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g} \lesssim \int_{[u]_+}^{t}\left|\widetilde{r}^2\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathscr{C}},\mathscr{D})\right|_{g}\mathrm{d}\tau + \int_{[u]_+}^{t}\widetilde{r}^2\left|\mathrm{f}_{(\vec{L})}\cdot\left(\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1},\boldsymbol{\partial}\vec{\Psi}\right)\cdot\boldsymbol{\partial}\vec{\Psi} + \widetilde{r}^2\left|\hat{\chi}\right|_{g}^2 + \widetilde{r}^2\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})}\cdot\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right|_{g}\mathrm{d}\tau. \tag{9.217}$$

When $u < 0$, by (8.61), we have:

$$\left\|\widetilde{r}^{-2}\lim_{\tau\downarrow[u]_+}\left(\widetilde{r}^2\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right)\right\|_{L_t^{\frac{q}{2}}L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \lesssim \left(\int_0^{T_{*;(\lambda)}}\left(\widetilde{r}^{-1/2}\lambda^{-1/2}\right)^{\frac{q}{2}}\mathrm{d}\tau\right)^{\frac{2}{q}} \lesssim \left(\lambda^{(1-8\varepsilon_0)(1-\frac{q}{4})}\lambda^{-\frac{q}{4}}\right)^{\frac{2}{q}} = \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}. \tag{9.218}$$

Using the estimate for Hardy-Littlewood maximal function (9.28), (9.104), and Hölder's inequality in t, we have:

$$\left\|\mathscr{M}\left(\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_u^\infty C_\omega^{0,\delta_0}(\Sigma_t)}\right)\right\|_{L_t^{\frac{q}{2}}} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^{\frac{q}{2}}L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})}\lambda^{(1-8\varepsilon_0)\left(1-\frac{q}{4}\right)\frac{2}{q}} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)-3\varepsilon_0}. \tag{9.219}$$

By bootstrap assumptions (6.13) and (9.4), and (9.104), we have:

$$\left\|\lambda^{-1}\left\|\vec{\mathscr{C}},\mathscr{D}\right\|_{L_t^1 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})}\right\|_{L_t^{\frac{q}{2}}} \lesssim \lambda^{(1-8\varepsilon_0)\frac{2}{q}-1-7\varepsilon_0} = \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)-11\varepsilon_0}, \tag{9.220}$$

$$\left\|\left\|\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})}^2\right\|_{L_t^{\frac{q}{2}}} \lesssim \lambda^{(1-8\varepsilon_0)\frac{2}{q}-1+4\varepsilon_0} = \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}. \tag{9.221}$$

Combining (9.215)-(9.221) and we conclude the desired estimate for $\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})}$.

Now since $\mathrm{tr}_{g}\chi - \frac{2}{\widetilde{r}} = \mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L$, we have:

$$\left\|\mathrm{tr}_{g}\chi - \frac{2}{\widetilde{r}}\right\|_{L_t^{\frac{q}{2}}L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} = \left\|\mathrm{tr}_{\widetilde{g}}\tilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^{\frac{q}{2}}L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} + \left\|\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^{\frac{q}{2}}L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})}. \tag{9.222}$$

By previously proven result (9.10) and previous estimate (9.219), we have:

$$\left\|\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^{\frac{q}{2}}L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \lesssim \left\|\mathrm{f}_{(\vec{L})}\right\|_{L_t^\infty L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^{\frac{q}{2}}L_u^\infty C_\omega^{0,\delta_0}(\mathscr{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)-3\varepsilon_0}. \tag{9.223}$$

Combining the above with the result for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}$, we conclude the proof for $\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}$.

Now we prove $\|\hat{\chi}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\mathcal{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}$. Plugging equation (8.139a) into the Hodge estimate (9.114) with $Q := \frac{2}{1-\delta_0}$, and recalling $\delta_0 < 1 - \frac{2}{p}$, $Q < p$, we have:

$$\|\hat{\chi}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\mathcal{M})} \lesssim \left\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\mathcal{M})} + \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\mathcal{M})} + \left\|\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^{\frac{q}{2}} L_u^\infty L_\omega^p(\mathcal{M})}. \tag{9.224}$$

Using previous estimate (9.214), we have:

$$\begin{aligned}
&\left\|\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^{\frac{q}{2}} L_u^\infty L_\omega^p(\mathcal{M})} \\
&\lesssim \lambda^{(1-8\varepsilon_0)\left(1-\frac{q}{4}\right)\frac{2}{q}} \left\|\widetilde{r}\mathrm{f}_{(\vec{L})} \cdot \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right) \cdot \boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{M})} \\
&\lesssim \lambda^{(1-8\varepsilon_0)\left(1-\frac{q}{4}\right)\frac{2}{q}} \lambda^{-1/2-4\varepsilon_0} \\
&= \lambda^{\frac{2}{q}-1-4\varepsilon_0(\frac{4}{q}-1)-4\varepsilon_0}.
\end{aligned} \tag{9.225}$$

Combining (9.224)-(9.225) with previously proven result for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}$ and (9.223), we conclude the desired estimates. □

Proof of $\|\hat{\chi}\|_{L_t^2 C_\omega^{0,\delta_0}(\mathcal{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.8e).* Using the Sobolev inequality (9.33b) with $Q := p$, and the previously proven estimates (9.8d) and (9.7a), we have:

$$\|\hat{\chi}\|_{L_t^2 C_\omega^{0,\delta_0}(\mathcal{C}_u)} \lesssim \|\widetilde{r}\nabla\!\!\!\!/\hat{\chi}\|_{L_t^2 L_\omega^p(\mathcal{C}_u)} + \|\hat{\chi}\|_{L_t^2 L_\omega^2(\mathcal{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.226}$$

□

Proof of (9.16a) and (9.16b). We first prove (9.16a). Integrating equation (8.142) along the integral curves of L, we find that:

$$\begin{aligned}
&\left\{\widetilde{r}^{-2}\not g\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - \not e\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right)\right\} \\
&= \lim_{t\downarrow[u]_+}\left\{\widetilde{r}^{-2}\not g\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - \not e\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right)\right\} \\
&\quad + \int_{[u]_+}^t \left(\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L\right)\left\{\widetilde{r}^{-2}\not g\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - \not e\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right)\right\} d\tau \\
&\quad + \int_{[u]_+}^t \left(\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L\right)\not e\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) + \frac{2}{\widetilde{r}^2}\hat{\chi}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) d\tau.
\end{aligned} \tag{9.227}$$

Using the initial conditions (8.63c) and (8.64c), noticing that $\hat{\chi}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)=\widetilde{r}^2\hat{\chi}_{AB}$, and using Grownwall's inequality, we have:

$$\left|\widetilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-\not{e}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right|_{\not{g}} \lesssim\left(\lambda^{-4\varepsilon_0}+\left\|\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right\|_{L_t^1L_\omega^\infty(\mathscr{C}_u)}\right)\exp\left(\left\|\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})},\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^1L_\omega^\infty(\mathscr{C}_u)}\right). \tag{9.228}$$

By bootstrap assumptions (6.13), and the previously proven result (9.11) with $q:=4$, we have:

$$\left\|\widetilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-\not{e}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right\|_{L^\infty(\mathscr{M})}\lesssim\lambda^{-4\varepsilon_0}. \tag{9.229}$$

We now prove (9.16b), we first apply $\frac{\partial}{\partial\omega^C}$ to equation (8.142). Note that $\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})}-\boldsymbol{\Gamma}_L=\mathrm{tr}_{\not{g}}\chi-\frac{2}{\widetilde{r}}$. Since L and $\frac{\partial}{\partial\omega^C}$ commute, we have:

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{\partial}{\partial\omega^C}\left[\widetilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-\not{e}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right]\right\}=\frac{\partial}{\partial\omega^C}\mathrm{tr}_{\not{g}}\chi\widetilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\\&+\left(\mathrm{tr}_{\not{g}}\chi-\frac{2}{\widetilde{r}}\right)\frac{\partial}{\partial\omega^C}\left\{\widetilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-\not{e}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right\}\\&+\left(\mathrm{tr}_{\not{g}}\chi-\frac{2}{\widetilde{r}}\right)\frac{\partial}{\partial\omega^C}\not{e}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)+2\widetilde{r}^{-2}\frac{\partial}{\partial\omega^C}\hat{\chi}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right).\end{aligned} \tag{9.230}$$

Integrating equation along the integral curves of L, taking the L^p_ω norm, and using the initial condition (8.63d), (8.64d), and then applying Grownwall's inequality, we have:

$$\begin{aligned}&\left\|\frac{\partial}{\partial\omega^C}\left\{\widetilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-\not{e}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right\}\right\|_{L^p_\omega(S_{t,u})}\\&\lesssim\left(\lambda^{-4\varepsilon_0}+\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\not{g}}\chi\|_{L_t^1L_\omega^p(\mathscr{C}_u)}+\left\|\left(\mathrm{tr}_{\not{g}}\chi-\frac{2}{\widetilde{r}}\right)\frac{\partial}{\partial\omega^C}\not{e}\right\|_{L_t^1L_\omega^p(\mathscr{C}_u)}\right.\\&\left.\quad+\|\widetilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^1L_\omega^p(\mathscr{C}_u)}+\|\boldsymbol{\Gamma}\cdot\hat{\chi}\|_{L_t^1L_\omega^p(\mathscr{C}_u)}\right)\\&\quad\cdot\exp\left(\left\|\mathrm{tr}_{\not{g}}\chi-\frac{2}{\widetilde{r}}\right\|_{L_t^1L_\omega^\infty(\mathscr{C}_u)}\right).\end{aligned} \tag{9.231}$$

By bootstrap assumptions (6.13), and the previously proven results (9.7a), (9.8d) and (9.11) with $q:=4$, we have:

$$\left\|\frac{\partial}{\partial\omega^C}\left\{\widetilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)-\not{e}\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right)\right\}\right\|_{L^p_\omega(S_{t,u})}\lesssim\lambda^{-4\varepsilon_0}. \tag{9.232}$$

□

Proof of $\left\|\frac{b^{-1}-1}{\widetilde{r}}\right\|_{L_t^2L_x^\infty(\mathcal{M})} \lesssim \lambda^{-1/2}$ *in (9.9).* We first bound $\left\|\frac{b^{-1}-1}{\widetilde{r}}\right\|_{L_t^2L_x^\infty(\mathcal{M}^{(\text{Int})})}$. Integrating equation (8.141) along along the integral curves of L emanating from the cone-tip, and using the initial condition (8.64a), we have:

$$b^{-1}-1 = -\int_u^t \left(b^{-1}-1\right) \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \mathrm{d}\tau - \int_u^t \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \mathrm{d}\tau. \tag{9.233}$$

Using Grownwall's inequality and the bootstrap assumption (6.13), we have:

$$\left|\frac{b^{-1}-1}{\widetilde{r}}\right|_{g} \lesssim \mathcal{M}\left(\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_\omega(S_{t,u})}\right). \tag{9.234}$$

Hence,

$$\left\|\frac{b^{-1}-1}{\widetilde{r}}\right\|_{L_t^2L_x^\infty(\mathcal{M}^{(\text{Int})})} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_x^\infty(\mathcal{M})} \lesssim \lambda^{-1/2-4\varepsilon_0}. \tag{9.235}$$

Now we consider the case when $u<0$. Integrating equation (8.141) along along the integral curves of L emanating from Σ_0, we have:

$$\left|\frac{b^{-1}-1}{\widetilde{r}}\right|_{g} \leq \left|\frac{b^{-1}-a^{-1}}{\widetilde{r}}\right|_{g} + \left|\frac{a^{-1}-1}{\widetilde{r}}\right|_{g} \lesssim \mathcal{M}\left(\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^\infty_\omega(S_{t,u})}\right) + \left|\frac{a^{-1}-1}{\widetilde{r}}\right|_{g}. \tag{9.236}$$

By initial condition (8.63a), we have:

$$\left\|\frac{a^{-1}-1}{\widetilde{r}}\right\|_{L_t^2L_x^\infty(\mathcal{M})} \lesssim \left\|\frac{a^{-1}-1}{w^{1/2}}\right\|_{L_x^\infty(\mathcal{M})} \left(\int_0^{T_{*;(\lambda)}} \frac{w}{(\tau+w)^2}\mathrm{d}\tau\right)^{1/2} \lesssim \lambda^{-1/2}. \tag{9.237}$$

Combining (9.236)-(9.237), we have:

$$\left\|\frac{b^{-1}-1}{\widetilde{r}}\right\|_{L_t^2L_x^\infty(\mathcal{M}^{(\text{Ext})})} \lesssim \lambda^{-1/2}. \tag{9.238}$$

Combining (9.235) and (9.238), we conclude the desired estimate. □

Proof of $\left\|\widetilde{r}(\mathbf{D}_L, \nabla\!\!\!/)\left(\frac{b^{-1}-1}{\widetilde{r}}\right)\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.9).* We first prove $\left\|\widetilde{r}\mathbf{D}\!\!\!\!/_L\left(\frac{b^{-1}-1}{\widetilde{r}}\right)\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}$. By (8.141) and the fact that $L(\widetilde{r})=1$, we have:

$$\widetilde{r}\mathbf{D}\!\!\!\!/_L\left(\frac{b^{-1}-1}{\widetilde{r}}\right) = -b^{-1}\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} - \frac{b^{-1}-1}{\widetilde{r}}. \tag{9.239}$$

By the bootstrap assumption (6.13), the previously proven result (9.17b), and the result (9.234) for $\frac{b^{-1}-1}{\widetilde{r}}$ in the last proof, we have:

$$\left\|\widetilde{r}\mathbf{D}_L\left(\frac{b^{-1}-1}{\widetilde{r}}\right)\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)} + \left\|\frac{b^{-1}-1}{\widetilde{r}}\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.240}$$

Now we bound $\widetilde{r}\nabla\!\!\!/\left(\frac{b^{-1}-1}{\widetilde{r}}\right)$. Recall (8.17), $\zeta = \nabla\!\!\!/\ln b + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$. Therefore,

$$\widetilde{r}\nabla\!\!\!/\left(\frac{b^{-1}-1}{\widetilde{r}}\right) = -b^{-1}\nabla\!\!\!/\ln b = b^{-1}\left(-\zeta + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\right). \tag{9.241}$$

By the bootstrap assumption (6.13), and the proven results (9.7a) and (9.17b), we have:

$$\left\|\widetilde{r}\nabla\!\!\!/\left(\frac{b^{-1}-1}{\widetilde{r}}\right)\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.242}$$

□

Proof of $\left\|\frac{b^{-1}-1}{\widetilde{r}^{1/2}}\right\|_{L_t^\infty L_u^\infty L_\omega^{2p}(\mathcal{M})} \lesssim \lambda^{-1/2}$ *in (9.9).* By the Sobolev inequality (9.32), we have:

$$\begin{aligned}&\left\|\frac{b^{-1}-1}{\widetilde{r}^{1/2}}\right\|^2_{L_t^\infty L_u^\infty L_\omega^{2p}(\mathcal{M})} \\ &\lesssim \left(\left\|\widetilde{r}\mathbf{D}_L\left(\frac{b^{-1}-1}{\widetilde{r}}\right)\right\|_{L_t^2L_u^\infty L_\omega^p(\mathcal{M})} + \left\|\frac{b^{-1}-1}{\widetilde{r}}\right\|_{L_t^2L_u^\infty L_\omega^p(\mathcal{M})}\right)\left\|\frac{b^{-1}-1}{\widetilde{r}}\right\|_{L_t^2L_u^\infty L_\omega^\infty(\mathcal{M})}.\end{aligned} \tag{9.243}$$

Using the proven first and third estimates of (9.9), we conclude the desired result. □

Proof of (9.18). We first prove $\left\|\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)} \lesssim \lambda^{-1/2}$. Plugging equation (8.143b) into estimate (9.76), we have:

$$\begin{aligned}\left|\widetilde{r}\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/} =& \lim_{t\downarrow\tau}\left|\widetilde{r}\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/} + \int_{[u]_+}^t \left|\widetilde{r}\left(\frac{1}{2}\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})} - \mathbf{\Gamma}_L\right) + \mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\right)\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/} \mathrm{d}\tau \\ &+ \int_{[u]_+}^t \left|\widetilde{r}\nabla\!\!\!/\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})} - \mathbf{\Gamma}_L\right)\right|_{g\!\!\!/}\mathrm{d}\tau.\end{aligned} \tag{9.244}$$

Applying the Grownwall's inequality, and using the proven results (9.11) with $q := 4$, we have:

$$\begin{aligned}\left|\widetilde{r}\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/} \lesssim& \left(\lim_{t\downarrow\tau}\left|\widetilde{r}\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/} + \int_{[u]_+}^t \left|\widetilde{r}\nabla\!\!\!/\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})} - \mathbf{\Gamma}_L\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right) \\ &\cdot\exp\left(\left\|\mathrm{tr}_{g\!\!\!/}\chi - \frac{2}{\widetilde{r}}, \hat{\chi}\right\|_{L_t^1L_\omega^\infty(\mathcal{C}_u)}\right) \\ \lesssim& \lim_{t\downarrow\tau}\left|\widetilde{r}\nabla\!\!\!/\ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/} + \int_{[u]_+}^t \left|\widetilde{r}\nabla\!\!\!/\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})} - \mathbf{\Gamma}_L\right)\right|_{g\!\!\!/}\mathrm{d}\tau.\end{aligned} \tag{9.245}$$

We now consider the initial conditions. When $u \geq 0$, by initial condition (8.64c) and (8.64d), we have $\lim_{t\downarrow\tau} \left|\widetilde{r}\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/} = 0$. When $u < 0$, by (8.63e), $\left\|\lim_{t\downarrow\tau} \widetilde{r}^{1/2}\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L^p_\omega(S_w)} \lesssim \lambda^{-1/2}$. Now dividing both sides of (9.245) by $\widetilde{r}(t,u)$, then take the norms and use the estimate for Hardy-Littlewood maximal function (9.28) to deduce:

$$\left\|\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \lesssim \lambda^{-1/2}\left(\int_0^{T_{*;(\lambda)}} \frac{w}{(\tau+w)^2}\mathrm{d}\tau\right)^{1/2} + \left\|\widetilde{r}\left(\nabla\!\!\!/ \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \nabla\!\!\!/ \boldsymbol{\partial}\vec{\Psi}\right)\right\|_{L^2_t L^p_\omega(\mathscr{C}_u)}. \tag{9.246}$$

By estimate (9.82f) and proven result (9.8d), we conclude the desired estimate.

Now we bound $\widetilde{r}^{1/2}\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)$. Dividing both sides of (9.245) by $\widetilde{r}^{1/2}$, taking the L^p_ω norm, by estimate (9.82f) and proven result (9.8d), we have:

$$\left\|\widetilde{r}^{1/2}\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L^p_\omega(S_{t,u})} \lesssim \lambda^{-1/2} + \frac{(t-[u]_+)^{1/2}}{(t-u)^{1/2}}\left\|\widetilde{r}\left(\nabla\!\!\!/ \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \nabla\!\!\!/ \boldsymbol{\partial}\vec{\Psi}\right)\right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.247}$$

Now we prove for $\widetilde{r}L\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)$. Using equation (8.143b), we have:

$$\begin{aligned} &\left|\widetilde{r}L\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/} \\ &\lesssim \left|\widetilde{r}\nabla\!\!\!/\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L\right) + \widetilde{r}\left(\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}, \hat{\chi}\right)\cdot \nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right) + \nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right|_{g\!\!\!/}. \end{aligned} \tag{9.248}$$

By the bootstrap assumptions (6.13), the estimate (9.82f), and the proven results (9.11) with $q := 4$, (9.8d), we have:

$$\begin{aligned} \left\|\widetilde{r}L\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \lesssim & \left\|\widetilde{r}\left(\nabla\!\!\!/ \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \nabla\!\!\!/ \boldsymbol{\partial}\vec{\Psi}\right)\right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \\ & + \lambda^{1/2-4\varepsilon_0}\left\|\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}, \hat{\chi}\right\|_{L^2_t L^\infty_\omega(\mathscr{C}_u)}\left\|\widetilde{r}^{1/2}\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L^\infty_t L^p_\omega(\mathscr{C}_u)} \\ & + \left\|\nabla\!\!\!/ \ln\left(\widetilde{r}^{-2}v\right)\right\|_{L^2_t L^p_\omega(\mathscr{C}_u)} \\ \lesssim & \lambda^{-1/2}. \end{aligned} \tag{9.249}$$

□

Proof of $\|\widetilde{r}\nabla\!\!\!/\zeta\|_{L^2_t L^p_\omega(\mathscr{C}_u)}, \|\zeta\|_{L^2_t C^{0,\delta_0}_\omega(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.19) and (9.8e).* Plugging equation (8.148a) and (8.148b) into the Hodge estimate (9.110) with $Q := p$, we have:

$$\|\widetilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \left\| \lambda^{-1} \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\| \widetilde{r} \nabla\!\!\!/ \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot \hat{\chi} \cdot \hat{\chi} \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot \zeta \cdot \zeta \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ \ln\left(\widetilde{r}^{-2} v\right) \cdot (\boldsymbol{\partial} \vec{\Psi}, \zeta) \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\widetilde{g}\!\!\!/} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}) \cdot \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)}. \tag{9.250}$$

By estimates (9.82d), (9.82f), the already proven results (9.7a), (9.7c), (9.11), (9.18), (9.189), we have:

$$\|\widetilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2} + \lambda^{-4\varepsilon_0} \|\zeta\|_{L_t^2 L_\omega^\infty(\mathscr{C}_u)}. \tag{9.251}$$

Using the Sobolev inequality (9.33b) with $Q := p$, and the already proven result (9.7a), we have:

$$\|\zeta\|_{L_t^2 C_\omega^{0,\delta_0}(\mathscr{C}_u)} \lesssim \|\widetilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \|\zeta\|_{L_t^2 L_\omega^2(\mathscr{C}_u)} \lesssim \lambda^{-1/2} + \|\widetilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\mathscr{C}_u)}. \tag{9.252}$$

Combining (9.251) and (9.252), we conclude the desired estimates. □

Proof of $\|\widetilde{r}\mu\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$ *in (9.19).* Using equation (8.144), by the estimates (9.82d), (9.82f), and already proven results (9.7a), (9.7c), (9.8e), (9.18), (9.189), we have:

$$\begin{aligned}\|\widetilde{r}\mu\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim & \left\| \lambda^{-1} \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot (\vec{\mathscr{C}}, \mathscr{D}) \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\| \widetilde{r} \nabla\!\!\!/ \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} + \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot \hat{\chi} \cdot \hat{\chi} \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \\ & + \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ \ln\left(\widetilde{r}^{-2} v\right) \cdot (\boldsymbol{\partial} \vec{\Psi}, \zeta) \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \\ & + \left\| \widetilde{r} \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\widetilde{g}\!\!\!/} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}) \cdot \boldsymbol{\partial} \vec{\Psi} \right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \\ \lesssim & \lambda^{-1/2}. \end{aligned} \tag{9.253}$$

□

Proof of $\|\zeta\|_{L_t^2 L_x^\infty(\mathscr{M}^{(Int)})} \lesssim \lambda^{-1/2-3\varepsilon_0}$ *and* $\|\zeta\|_{L_t^{\frac{q}{2}} L_x^\infty(\mathscr{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-1\right)}$ *in (9.12).* Plugging equations (8.148a) and (8.148b) into the Hodge estimate (9.116) with $Q := p$, $c = 2$, $\mathfrak{F} = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, we have:

$$\begin{aligned}\|\zeta\|_{L_t^2 L_x^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim & \left\| \mathrm{v}^{\delta'} P_\mathrm{v} \left(\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \right) \right\|_{L_t^2 L_u^\infty l_\mathrm{v}^2 L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\| \boldsymbol{\partial}\vec{\Psi} \right\|_{L_t^2 L_x^\infty(\mathscr{M}^{(\mathrm{Int})})} + \left\| \lambda^{-1} \widetilde{r}(\vec{\mathscr{C}}, \mathscr{D}) \right\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ & + \|\widetilde{r}(\hat{\chi} \cdot \hat{\chi}, \zeta \cdot \zeta)\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\| \widetilde{r}(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}} \widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi} \right\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\| \widetilde{r} \nabla\!\!\!/ \ln\left(\widetilde{r}^{-2} v\right) \cdot (\boldsymbol{\partial}\vec{\Psi}, \zeta) \right\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}. \end{aligned} \tag{9.254}$$

Using the estimates (9.93a), (9.104), the already proven results (9.7c), (9.15) for $\hat{\chi}$, (9.18), and (9.141), we have:

$$\|\zeta\|_{L_t^2 L_x^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-3\varepsilon_0} + \lambda^{-4\varepsilon_0} \|\zeta\|_{L_t^2 L_x^\infty(\mathscr{M}^{(\mathrm{Int})})}. \tag{9.255}$$

By absorbing the second term on RHS of (9.255) into the left, we conclude the desired result for $L_t^2 L_x^\infty$ norm. To bound the $L_t^{\frac{q}{2}} L_x^\infty$ norm, we apply the Hölder's inequality to (9.254) and use the same argument given above to deduce:

$$\|\zeta\|_{L_t^{\frac{q}{2}} L_x^\infty(\mathscr{M})} \lesssim \lambda^{\frac{2}{q}-1-4\varepsilon_0\left(\frac{4}{q}-\frac{1}{4}\right)} + \lambda^{-4\varepsilon_0} \|\zeta\|_{L_t^{\frac{q}{2}} L_x^\infty(\mathscr{M})}. \tag{9.256}$$

By absorbing the second term on RHS of (9.256) into the left, we conclude the desired result. □

Proof of (9.20a)-(9.20c). Using equation (8.37a) and initial condition for σ (8.37b), we have:

$$\sigma = \frac{1}{2} \int_u^t \boldsymbol{\Gamma}_L \mathrm{d}\tau. \tag{9.257}$$

Using the bootstrap assumptions (6.13), we find that:

$$\|\sigma\|_{L^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-8\varepsilon_0}, \tag{9.258}$$

$$\left\| \widetilde{r}^{-1/2} \sigma \right\|_{L^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim \sup_{t,u} \frac{(t-u)^{1/2}}{\widetilde{r}^{1/2}} \lambda^{-1/2-4\varepsilon_0} \lesssim \lambda^{-1/2-4\varepsilon_0}. \tag{9.259}$$

To prove $\|\nabla\!\!\!/ \sigma\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$, we plug equation (8.145) into (9.76), and use the initial condition (8.64a) to deduce:

$$|\widetilde{r}\nabla\!\!\!/\sigma|_{\not g} \lesssim \int_u^t \widetilde{r}\left(\left|\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right|_{\not g} + \left|\left(\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}, \hat{\chi}\right)\cdot\nabla\!\!\!/\sigma\right|_{\not g}\right)\mathrm{d}\tau. \tag{9.260}$$

Using the Grownwall's inequality, the bootstrap assumption (6.13) and the already proven results (9.8e), we have:

$$|\widetilde{r}\nabla\!\!\!/\sigma|_{\not g} \lesssim \int_u^t \widetilde{r}\left|\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right|_{\not g}\mathrm{d}\tau \exp\left(\left\|\mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}, \hat{\chi}\right\|_{L_t^1 L_\omega^\infty(\mathscr{C}_u)}\right) \lesssim \int_u^t \widetilde{r}\left|\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right|_{\not g}\mathrm{d}\tau. \tag{9.261}$$

Dividing both sides of (9.261) by $\widetilde{r}(t,u)$, taking the $L_t^2 L_\omega^p$ norms, using the estimate for Hardy-Littlewood maximal function (9.28) and the estimate (9.83a), we have:

$$\|\nabla\!\!\!/\sigma\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}, \tag{9.262}$$

as desired. To show $\left\|\widetilde{r}^{1/2}\nabla\!\!\!/\sigma\right\|_{L_\omega^p L_t^\infty(\mathscr{C}_u)} \lesssim \lambda^{-1/2}$, dividing estimate (9.261) by $\widetilde{r}^{1/2}(t,u)$, then taking the L_ω^p norms, we have:

$$\left\|\widetilde{r}^{1/2}\nabla\!\!\!/\sigma\right\|_{L_\omega^p(S_{t,u})} \lesssim \frac{(t-u)^{1/2}}{\widetilde{r}^{1/2}}\left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2 L_\omega^p(\mathscr{C}_u)} \lesssim \lambda^{-1/2}. \tag{9.263}$$

We now prove $\left\|\widetilde{r}^{1/2}L\sigma\right\|_{L_t^\infty L_\omega^{2p}(\mathscr{C}_u)} \lesssim \lambda^{-1/2-2\varepsilon_0}$. By the Sobolev equality (9.32), the estimate (9.82d) and (9.83a), we have:

$$\left\|\widetilde{r}^{1/2}L\sigma\right\|^2_{L_t^\infty L_\omega^{2p}(\mathscr{C}_u)} \lesssim \left(\left\|\widetilde{r}\not{\mathbf{D}}_L\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^p L_t^2(\mathscr{C}_u)} + \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^p L_t^2(\mathscr{C}_u)}\right)\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^\infty L_t^2(\mathscr{C}_u)} \lesssim \lambda^{-1-4\varepsilon_0}. \tag{9.264}$$

□

Proof of $\left\|\widetilde{r}\left(\nabla\!\!\!/\check{\mu}, \nabla\!\!\!/\widetilde{\zeta}\right)\right\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(Int)})} \lesssim \lambda^{-4\varepsilon_0}$ *in (9.21a).* In this paragraph, we are assuming $\mathscr{C}_u \subset \mathscr{M}^{(\mathrm{Int})}$. We will silently use the fact that $0 \le u \le t \le T_{*;(\lambda)} \lesssim \lambda^{1-8\varepsilon_0}$ silently. We start by deriving a preliminary estimate for $\left\|\widetilde{r}\nabla\!\!\!/\widetilde{\zeta}\right\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}$. Plugging equations (8.149a) and (8.149b) into the Hodge estimate (9.110), we have:

$$\begin{aligned}\|\widetilde{r}\nabla\!\!\!/\widetilde{\zeta}\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim{}& \left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} + \lambda^{-1}\left\|\widetilde{r}(\vec{\mathscr{C}}, \mathscr{D})\right\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ &+ \|\widetilde{r}(\zeta\cdot\zeta, \hat{\chi}\cdot\hat{\chi})\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ &+ \left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} + \|\widetilde{r}\check{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}.\end{aligned} \tag{9.265}$$

By estimates (9.82f), (9.93c), (9.189), we have:

$$\left\|\widetilde{r}\nabla\!\!\!/\tilde{\zeta}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-4\varepsilon_0} + \left\|\widetilde{r}\check{\mu}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}. \tag{9.266}$$

We now bound $\left\|\widetilde{r}\check{\mu}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}$. Plugging (8.146) into (9.76), and using the initial condition on cone-tip (8.64a), we have:

$$\left|\widetilde{r}^2\check{\mu}\right|_{g\!\!\!/} \lesssim \int_u^t \widetilde{r}^2\left(\left|\mathfrak{I}_{(1)}\right|_{g\!\!\!/} + \left|\mathfrak{I}_{(2)}\right|_{g\!\!\!/}\right)\mathrm{d}\tau + \int_u^t \widetilde{r}^2\left|\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}\right|_{g\!\!\!/}\check{\mu}\mathrm{d}\tau. \tag{9.267}$$

Using the Grownwall's inequality, the bootstrap assumption (6.13) and the already proven estimate (9.15), we have:

$$\begin{aligned}\left|\widetilde{r}^2\check{\mu}\right|_{g\!\!\!/} &\lesssim \int_u^t \widetilde{r}^2\left(\left|\mathfrak{I}_{(1)}\right|_{g\!\!\!/} + \left|\mathfrak{I}_{(2)}\right|_{g\!\!\!/}\right)\mathrm{d}\tau \exp\left(\left\|\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^1L_\omega^\infty(\mathscr{C}_u)}\right) \\ &\lesssim \int_u^t \widetilde{r}^2\left(\left|\mathfrak{I}_{(1)}\right|_{g\!\!\!/} + \left|\mathfrak{I}_{(2)}\right|_{g\!\!\!/}\right)\mathrm{d}\tau.\end{aligned} \tag{9.268}$$

We divide estimate (9.268) by $\widetilde{r}(t,u)$ and then take the $L_u^2L_t^2L_\omega^p$ norm. We now estimate terms in $\mathfrak{I}_{(1)}$ and $\mathfrak{I}_{(2)}$ defined in (8.147). First, by the estimate for Hardy-Littlewood maximal function (9.28), estimates (9.82f), (9.82a), we have:

$$\left\|\widetilde{r}^{-1}\int_u^t \widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-4\varepsilon_0}, \tag{9.269}$$

$$\left\|\widetilde{r}^{-1}\int_u^t \boldsymbol{\partial}\vec{\Psi}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.270}$$

Next, by the estimates (9.92a), (9.91c), we have:

$$\lambda^{-1}\left\|\widetilde{r}^{-1}\int_u^t \left|\widetilde{r}^2\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-12\varepsilon_0}, \tag{9.271}$$

$$\lambda^{-1}\left\|\widetilde{r}^{-1}\int_{[u]_+}^t \left|\widetilde{r}^2(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\varpi},\boldsymbol{\partial}\vec{S})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-16\varepsilon_0}. \tag{9.272}$$

By the already proven estimate (9.15), we have:

$$\begin{aligned}\left\|\widetilde{r}^{-1}\int_u^t \widetilde{r}^2\left|\nabla\!\!\!/\tilde{\zeta}\cdot\hat{\chi}\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} &\lesssim \left\|\left\|\widetilde{r}\nabla\!\!\!/\tilde{\zeta}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\left\|\hat{\chi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\right\|_{L_u^2L_t^2} \\ &\lesssim \lambda^{-7\varepsilon_0}\left\|\widetilde{r}\nabla\!\!\!/\tilde{\zeta}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}.\end{aligned} \tag{9.273}$$

By the bootstrap assumptions (9.6), and the estimates (9.82f), (9.8d) and (9.189), we have:

$$\begin{aligned}&\left\|\widetilde{r}^{-1}\int_u^t\left|\widetilde{r}^2\nabla\!\!\!/\sigma\cdot\left(\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi},\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\\&\lesssim\left\|\|\nabla\!\!\!/\sigma\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\left\|\widetilde{r}\left(\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi},\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right)\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\right\|_{L_u^2L_t^2}\\&\lesssim\lambda^{-4\varepsilon_0}.\end{aligned}\tag{9.274}$$

By the bootstrap assumptions (6.13), the already proven estimate (9.8d) and (9.12), we have:

$$\begin{aligned}&\left\|\widetilde{r}^{-1}\int_u^t\widetilde{r}^2\left|\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})}\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\\1&\lesssim\left\|\left\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi},\zeta\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\right\|_{L_u^2L_t^2}\\&\lesssim\lambda^{-11\varepsilon_0}.\end{aligned}\tag{9.275}$$

By the bootstrap assumptions (6.13), already proven result (9.15), and the estimate (9.189), we have:

$$\begin{aligned}&\left\|\widetilde{r}^{-1}\int_u^t\widetilde{r}^2\left|\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\\&\lesssim\left\|\left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta\right)\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\right\|_{L_u^2L_t^2}\\&\lesssim\lambda^{-11\varepsilon_0}.\end{aligned}\tag{9.276}$$

By the bootstrap assumptions (6.13), the estimates (9.82b), (9.12), (9.15), and the Fubini's theorem, we have:

$$\begin{aligned}&\left\|\widetilde{r}^{-1}\int_u^t\widetilde{r}^2\left|(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta)\cdot\boldsymbol{\partial}^2\vec{\Psi}\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\\&\lesssim\left\|\left\|\widetilde{r}\boldsymbol{\partial}^2\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}\!\!\!/}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta\right\|_{L_t^2L_\omega^\infty(\mathscr{C}_u)}\right\|_{L_u^2L_t^2}\\&\lesssim\lambda^{-7\varepsilon_0}\left\|\widetilde{r}\boldsymbol{\partial}^2\vec{\Psi}\right\|_{L_t^2L_u^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\\&\lesssim\lambda^{-11\varepsilon_0}.\end{aligned}\tag{9.277}$$

$\mathfrak{I}_{(1)}$ is bounded by (9.269)-(9.270), and $\mathfrak{I}_{(2)}$ is bounded by (9.271)-(9.277). Therefore, we conclude

$$\|\widetilde{r}\check{\mu}\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\lesssim\lambda^{-4\varepsilon_0}+\lambda^{-7\varepsilon_0}\left\|\widetilde{r}\nabla\!\!\!/\tilde{\zeta}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}.\tag{9.278}$$

Combining this estimate with the estimate (9.266) for $\left\|\widetilde{r}\nabla\!\!\!/\tilde{\zeta}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}$, and soaking the second term on the RHS of (9.266) to the LHS of (9.278), we conclude the desired results. □

Proof of $\|\widetilde{r}^{3/2}\check{\mu}\|_{L^2_u L^\infty_t L^p_\omega(\mathscr{M}^{(Int)})} \lesssim \lambda^{-4\varepsilon_0}$ *in (9.21b).* We divide (9.268) by $\widetilde{r}^{1/2}(t,u)$ and then take the $L^2_u L^\infty_t L^p_\omega$ norms. We first estimate the terms in $\mathfrak{J}_{(1)}$ and $\mathfrak{J}_{(2)}$ defined in (8.147).

By the estimates (9.82f), (9.82a), we have:

$$\left\|\widetilde{r}^{-1/2}\int_u^t \widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\mathrm{d}\tau\right\|_{L^2_u L^\infty_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} \lesssim \left\|\widetilde{r}\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi}\right\|_{L^2_u L^2_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-4\varepsilon_0}, \tag{9.279}$$

$$\left\|\widetilde{r}^{-1/2}\int_u^t \boldsymbol{\partial}\vec{\Psi}\mathrm{d}\tau\right\|_{L^2_u L^\infty_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} \lesssim \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L^2_u L^2_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.280}$$

By the estimates (9.92b), (9.91d), we have:

$$\lambda^{-1}\left\|\widetilde{r}^{-1/2}\int_u^t \left|\widetilde{r}^2\boldsymbol{\partial}(\vec{\mathscr{C}},\mathscr{D})\right|_{\not{g}}\mathrm{d}\tau\right\|_{L^2_u L^\infty_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-12\varepsilon_0}, \tag{9.281}$$

$$\lambda^{-1}\left\|\widetilde{r}^{-1/2}\int_{[u]_+}^t \left|\widetilde{r}^2(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S})\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\right|_{\not{g}}\mathrm{d}\tau\right\|_{L^2_u L^\infty_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-16\varepsilon_0}. \tag{9.282}$$

By the proven estimates (9.15) and (9.21a), we have:

$$\begin{aligned}\left\|\widetilde{r}^{-1/2}\int_u^t \widetilde{r}^2\left|\nabla\!\!\!/\tilde{\zeta}\cdot\hat{\chi}\right|_{\not{g}}\mathrm{d}\tau\right\|_{L^2_u L^\infty_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} &\lesssim \lambda^{1/2-4\varepsilon_0}\left\|\|\widetilde{r}\nabla\!\!\!/\tilde{\zeta}\|_{L^2_t L^p_\omega(\mathscr{C}_u)}\|\hat{\chi}\|_{L^2_t L^\infty_\omega(\mathscr{C}_u)}\right\|_{L^2_u L^\infty_t} \\ &\lesssim \lambda^{-11\varepsilon_0}.\end{aligned} \tag{9.283}$$

By the bootstrap assumption (9.6), the estimates (9.82f), (9.8d) and (9.189), we have:

$$\begin{aligned}&\left\|\widetilde{r}^{-1/2}\int_u^t \left|\widetilde{r}^2\nabla\!\!\!/\sigma\cdot\left(\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi},\nabla\!\!\!/\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right)\right|_{\not{g}}\mathrm{d}\tau\right\|_{L^2_u L^\infty_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} \\ &\lesssim \lambda^{1/2-4\varepsilon_0}\Bigg\|\|\nabla\!\!\!/\sigma\|_{L^2_t L^\infty_\omega(\mathscr{C}_u)} \\ &\cdot\left\|\widetilde{r}\left(\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi},\nabla\!\!\!/\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right)\right\|_{L^2_t L^p_\omega(\mathscr{C}_u)}\Bigg\|_{L^2_u L^\infty_t} \\ &\lesssim \lambda^{-4\varepsilon_0}.\end{aligned} \tag{9.284}$$

By the bootstrap assumptions (6.13), proven estimates (9.8d) and (9.12), we have:

$$\begin{aligned}&\left\|\widetilde{r}^{-1/2}\int_u^t \widetilde{r}^2\left|\nabla\!\!\!/\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta)\right|_{\not{g}}\mathrm{d}\tau\right\|_{L^2_u L^\infty_t L^p_\omega(\mathscr{M}^{(\mathrm{Int})})} \\ &\lesssim \lambda^{1/2-4\varepsilon_0}\left\|\left\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L^2_t L^p_\omega(\mathscr{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi},\zeta\right\|_{L^2_t L^\infty_\omega(\mathscr{C}_u)}\right\|_{L^2_u L^\infty_t} \lesssim \lambda^{-11\varepsilon_0}.\end{aligned} \tag{9.285}$$

By bootstrap assumptions (6.13), proven result (9.15), estimate (9.189), we have:

$$
\begin{aligned}
&\left\|\widetilde{r}^{-1/2}\int_u^t \widetilde{r}^2 \left|\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta, \widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \hat{\chi}\right)\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^\infty L_\omega^p(\mathcal{M}^{(\mathrm{Int})})} && (9.286)\\
&\lesssim \lambda^{1/2-4\varepsilon_0}\left\|\left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta, \widetilde{r}^{-1}\right)\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta\right)\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)}\right.\\
&\left.\cdot\left\|\boldsymbol{\partial}\vec{\Psi}, \hat{\chi}\right\|_{L_t^2L_\omega^\infty(\mathcal{C}_u)}\right\|_{L_u^2L_t^\infty}\\
&\lesssim \lambda^{-11\varepsilon_0}.
\end{aligned}
$$

By bootstrap assumptions (6.13), estimates (9.82b), (9.12), (9.15), using Fubini's theorem, we have:

$$
\begin{aligned}
&\left\|\widetilde{r}^{-1/2}\int_u^t \widetilde{r}^2 \left|(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta)\cdot\boldsymbol{\partial}^2\vec{\Psi}\right|_{g\!\!\!/}\mathrm{d}\tau\right\|_{L_u^2L_t^\infty L_\omega^p(\mathcal{M}^{(\mathrm{Int})})} && (9.287)\\
&\lesssim \lambda^{1/2-4\varepsilon_0}\left\|\left\|\widetilde{r}\boldsymbol{\partial}^2\vec{\Psi}\right\|_{L_t^2L_\omega^p(\mathcal{C}_u)}\left\|\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \zeta\right\|_{L_t^2L_\omega^\infty(\mathcal{C}_u)}\right\|_{L_u^2L_t^\infty}\\
&\lesssim \lambda^{-7\varepsilon_0}\left\|\widetilde{r}\boldsymbol{\partial}^2\vec{\Psi}\right\|_{L_t^2L_u^2L_\omega^p(\mathcal{M}^{(\mathrm{Int})})}\\
&\lesssim \lambda^{-11\varepsilon_0}.
\end{aligned}
$$

$\mathfrak{J}_{(1)}$ is bounded by (9.279)-(9.280), and $\mathfrak{J}_{(2)}$ is bounded by (9.281)-(9.287). Combining all the estimates for terms in $\mathfrak{J}_{(1)}$ and $\mathfrak{J}_{(2)}$, we conclude the desired result. □

Proof of $\|\nabla\!\!\!/\sigma\|_{L_u^2L_t^2C_\omega^{0,\delta_0}(\mathcal{M}^{(Int)})} \lesssim \lambda^{-4\varepsilon_0}$ *in (9.21a).* Plugging (8.56) into the Sobolev equality (9.33b) with $Q := p$, we have:

$$
\|\nabla\!\!\!/\sigma\|_{L_u^2L_t^2C_\omega^{0,\delta_0}(\mathcal{M}^{(\mathrm{Int})})} \lesssim \left\|\widetilde{r}(\nabla\!\!\!/\tilde{\zeta}, \nabla\!\!\!/\zeta)\right\|_{L_u^2L_t^2L_\omega^p(\mathcal{M}^{(\mathrm{Int})})} + \|\nabla\!\!\!/\sigma\|_{L_u^2L_t^2L_\omega^2(\mathcal{M}^{(\mathrm{Int})})}. \tag{9.288}
$$

By the proven estimates (9.19), (9.20a), (9.21a) for $\left\|\widetilde{r}\nabla\!\!\!/\tilde{\zeta}\right\|_{L_u^2L_t^2L_\omega^p(\mathcal{M}^{(\mathrm{Int})})}$, we conclude the desired estimate. □

Proof of (9.22). Plugging (8.55) into the Hodge estimate (9.110), using (9.123) and the proven estimate (9.21a), we have:

$$
\|\widetilde{r}\nabla\!\!\!/\mu\!\!\!/, \mu\!\!\!/\|_{L_t^2L_u^2L_\omega^p(\mathcal{M}^{(\mathrm{Int})})} \lesssim \|\widetilde{r}\check{\mu}\|_{L_t^2L_u^2L_\omega^p(\mathcal{M}^{(\mathrm{Int})})} + \left\|\widetilde{r}\bar{\mu}\right\|_{L_t^2L_u^2L_\omega^p(\mathcal{M}^{(\mathrm{Int})})} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.289}
$$

By the Sobolev inequality (9.33b) with $Q := p$, we have:

$$\|\not{\mu}\|_{L_t^2 L_u^2 C_\omega^{0,\delta_0}(\mathscr{M}^{(\mathrm{Int})})} \lesssim \|\widetilde{r}\not{\nabla}\not{\mu}\|_{L_t^2 L_u^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} + \|\not{\mu}\|_{L_t^2 L_u^2 L_\omega^2(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-4\varepsilon_0}. \tag{9.290}$$

□

Proof of $\left\|\tilde{\zeta} - \not{\mu}\right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M}^{(Int)})} \lesssim \lambda^{-1/2-4\varepsilon_0}$ *in (9.25a).* Plugging equations (8.150a) and (8.150b) into the Hodge estimate (9.116) with $Q := p$, $c := 2$ and $\delta' \leq \delta_0$, we have:

$$\begin{aligned}\left\|\tilde{\zeta} - \not{\mu}\right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim & \left\|\nu^{\delta'} P_\nu \boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2 L_u^\infty l_\nu^2 L_\omega^\infty\left(\mathscr{M}^{(\mathrm{Int})}\right)} + \left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \\ & + \lambda^{-1}\left\|\widetilde{r}(\vec{\mathscr{C}}, \mathscr{D})\right\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ & + \|\widetilde{r}(\zeta\cdot\zeta, \hat{\chi}\cdot\hat{\chi})\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^2 L_u^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}.\end{aligned} \tag{9.291}$$

By the bootstrap assumptions (6.13), the estimates (9.93a), (9.141), and the proven results (9.7c), (9.12) and (9.15), we conclude the desired bound:

$$\left\|\tilde{\zeta} - \not{\mu}\right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-4\varepsilon_0}. \tag{9.292}$$

□

Proof of (9.24). By (8.154), it suffices to show $\lim_{t\downarrow u}\widetilde{r}\not{\mu} = \mathscr{O}(\widetilde{r})$. Plugging definition (8.55) into the Hodge estimate (9.116) with $\mathfrak{F} := 0$ and $Q := p$, and using the initial condition (8.64a), we have:

$$\left\|\lim_{t\downarrow u}\widetilde{r}\not{\mu}\right\|_{L_\omega^\infty(S_{t,u})} \lesssim \left\|\lim_{t\downarrow u}\widetilde{r}^2\check{\mu}\right\|_{L_\omega^p(S_{t,u})} = \mathscr{O}(\widetilde{r}). \tag{9.293}$$

□

Proof of $\left\|\not{\mu}_{(1)}\right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M}^{(Int)})} \lesssim \lambda^{-1/2-4\varepsilon_0}$ *in (9.25a).* Let $\mathfrak{H} := \not{\mathbf{D}}_L\not{\mu}_{(1)} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\not{\mu}_{(1)}$. By (9.76) and the initial condition for $\widetilde{r}\not{\mu}_{(1)}$, we have:

$$\left|\widetilde{r}\not{\mu}_{(1)}\right|_{\not{g}} = \int_u^t \left|\widetilde{r}\mathfrak{H} + \widetilde{r}\left(\mathrm{tr}_{\tilde{\not{g}}}\tilde{\chi}^{(\mathrm{Small})} + \boldsymbol{\partial}\vec{\Psi}\right)\not{\mu}_{(1)}\right|_{\not{g}} \mathrm{d}\tau. \tag{9.294}$$

Using the Grownwall's inequality, by the bootstrap assumption (6.13) and the proven estimate (9.15), we have:

$$\left|\widetilde{r}\not{\mu}_{(1)}\right|_{\not{g}} \lesssim \int_u^t |\widetilde{r}\mathfrak{H}|_{\not{g}}\,\mathrm{d}\tau \exp\left(\left\|\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}, \boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^1 L_\omega^\infty(\mathscr{C}_u)}\right) \lesssim \int_u^t |\widetilde{r}\mathfrak{H}|_{\not{g}}\,\mathrm{d}\tau. \tag{9.295}$$

Dividing both sides of (9.295) by $\widetilde{r}$, taking the norm, and then substituting equations (8.152a) and (8.152b) into the Hodge estimate (9.116) with $\mathfrak{F} := \widetilde{r}^{-1}\mathrm{d}\not{i}\mathrm{v}\,\xi$, $Q := p$, $c := 2$ and $\delta' \leq \delta_0$, we have:

$$\begin{aligned}\left\|\not{\mu}_{(1)}\right\|_{L_t^2 L_u^\infty L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim \Bigg\|\widetilde{r}^{-1}\int_u^t \widetilde{r}\Bigg(\widetilde{r}^{-1}\left\|\nu^{\delta'}P_\nu \boldsymbol{\partial}\vec{\Psi}\right\|_{l_\nu^2 L_\omega^\infty(S_{\tau,u})} + \left\|\widetilde{r}^{-1}\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^\infty(S_{\tau,u})} \\ + \left\|\widetilde{r}^{-1}\boldsymbol{\partial}\vec{\Psi}\right\|_{L_\omega^p(S_{\tau,u})}\Bigg)\mathrm{d}\tau\Bigg\|_{L_t^2 L_u^\infty}.\end{aligned} \tag{9.296}$$

By the estimate for Hardy-Littlewood maximal function (9.28) and bootstrap assumptions (6.13), we conclude the desired estimate. □

Proof of $\left\|\not{\mu}_{(2)}\right\|_{L_u^2 L_t^\infty L_\omega^\infty(\mathscr{M}^{(Int)})} \lesssim \lambda^{-1/2-3\varepsilon_0}$ *in (9.25b).* Using the same argument as in the proof of (9.25a), we have

$$\left|\widetilde{r}\not{\mu}_{(2)}\right|_{\not{g}} \lesssim \int_u^t |\widetilde{r}\mathfrak{H}|_{\not{g}}\,\mathrm{d}\tau, \tag{9.297}$$

where $\mathfrak{H} := \not{\mathbf{D}}_L\not{\mu}_{(2)} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\not{\mu}_{(2)}$. Now divide both sides of (9.297) by $\widetilde{r}$ and take the norms. Notice that:

$$\left\|\not{\mu}_{(2)}\right\|_{L_u^2 L_t^\infty L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim \left\|\widetilde{r}^{-1}\int_u^t |\widetilde{r}\mathfrak{H}|_{\not{g}}\,\mathrm{d}\tau\right\|_{L_u^2 L_t^\infty L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim \|\mathfrak{H}\|_{L_u^2 L_t^1 L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})}. \tag{9.298}$$

Applying equations (8.153a) and (8.153b) into Hodge estimate (9.116) with $\mathfrak{F} := 0$ and $Q := p$, we have:

$$\begin{aligned}\|\mathfrak{H}\|_{L_u^2 L_t^1 L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim & \left\|\widetilde{r}\mathfrak{I}_{(2)}\right\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} + \|\widetilde{r}\hat{\chi}\cdot\not{\nabla}\not{\mu}\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\|\widetilde{r}\left(\not{\nabla}\boldsymbol{\partial}\vec{\Psi}, \not{\nabla}\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}\right)\cdot\not{\mu}\right\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\not{\mu}\right\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\ & + \left\|\widetilde{r}\left(\mathrm{tr}_{\not{g}}\chi - \overline{\mathrm{tr}_{\not{g}}\chi}\right)\check{\mu}\right\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}.\end{aligned} \tag{9.299}$$

By (9.93d), we have:

$$\lambda^{-1}\left\|\widetilde{r}\boldsymbol{\partial}(\vec{\mathscr{C}}, \mathscr{D})\right\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-8\varepsilon_0}. \tag{9.300}$$

By (9.93e), we have:

$$\lambda^{-1}\left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\omega},\boldsymbol{\partial}\vec{S}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}^{-1}\right)\right\|_{L_u^2L_t^1L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\lesssim^{-\frac{1}{2}-10\varepsilon_0}. \tag{9.301}$$

By the proven estimates (9.21a) and (9.15), we have:

$$\left\|\widetilde{r}\nabla\!\!\!/\widetilde{\zeta}\cdot\hat{\chi}\right\|_{L_u^2L_t^1L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\lesssim\left\|\widetilde{r}\nabla\!\!\!/\widetilde{\zeta}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\left\|\widetilde{r}\nabla\!\!\!/\widetilde{\zeta}\cdot\hat{\chi}\right\|_{L_u^\infty L_t^2L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})}\lesssim\lambda^{-1/2-7\varepsilon_0}. \tag{9.302}$$

By (9.82f), (9.8d), (9.189) and (9.21a), we have:

$$\begin{aligned}
&\left\|\widetilde{r}\nabla\!\!\!/\sigma\cdot\left(\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi},\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right)\right\|_{L_u^2L_t^1L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim\|\nabla\!\!\!/\sigma\|_{L_u^2L_t^2L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})}\left\|\widetilde{r}\left(\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi},\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\widetilde{r}^{-1}\right)\cdot\boldsymbol{\partial}\vec{\Psi}\right)\right\|_{L_u^\infty L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim\lambda^{-1/2-4\varepsilon_0}.
\end{aligned} \tag{9.303}$$

By (9.8d), (6.13) and (9.12), we have:

$$\begin{aligned}
\left\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta)\right\|_{L_u^2L_t^1L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}&\lesssim\left\|\widetilde{r}\nabla\!\!\!/\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\left\|\boldsymbol{\partial}\vec{\Psi},\zeta\right\|_{L_u^\infty L_t^2L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim\lambda^{-1/2-3\varepsilon_0}.
\end{aligned} \tag{9.304}$$

By (9.189), (6.13) and (9.15), we have:

$$\begin{aligned}
&\left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta\right)\cdot(\boldsymbol{\partial}\vec{\Psi},\hat{\chi})\right\|_{L_u^2L_t^1L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim\left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta,\widetilde{r}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\hat{\chi},\zeta\right)\right\|_{L_u^2L_t^2L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\left\|\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right\|_{L_u^\infty L_t^2L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim\lambda^{-1/2-7\varepsilon_0}.
\end{aligned} \tag{9.305}$$

By (9.82b), (6.13), (9.15) and (9.12), we have:

$$\begin{aligned}
\left\|\widetilde{r}\left(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\zeta\right)\cdot\boldsymbol{\partial}^2\vec{\Psi}\right\|_{L_u^2L_t^1L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}&\lesssim\left\|\widetilde{r}\boldsymbol{\partial}^2\vec{\Psi}\right\|_{L_u^2L_t^\infty L_\omega^p(\mathscr{M}^{(\mathrm{Int})})}\cdot \\
&\qquad\left\|\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\zeta\right\|_{L_u^\infty L_t^1L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim\lambda^{-1/2-7\varepsilon_0}.
\end{aligned} \tag{9.306}$$

By (9.22) and (9.15), we have:

$$\|\widetilde{r}\hat{\chi}\cdot \nabla\!\!\!/ \mu\!\!\!/\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \lesssim \|\widetilde{r}\nabla\!\!\!/ \mu\!\!\!/\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \|\hat{\chi}\|_{L_u^\infty L_t^2 L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \lesssim \lambda^{-1/2-7\varepsilon_0}. \tag{9.307}$$

By (9.82f), (9.8d), (9.189) and (9.22), we have:

$$\begin{aligned}
&\left\|\widetilde{r}\mu\!\!\!/ \cdot \left(\nabla\!\!\!/ \boldsymbol{\partial}\vec{\Psi}, \nabla\!\!\!/ \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}\right)\right)\right\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim \|\mu\!\!\!/\|_{L_u^2 L_t^2 L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \cdot \\
&\left\|\widetilde{r}\left(\nabla\!\!\!/ \boldsymbol{\partial}\vec{\Psi}, \nabla\!\!\!/ \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}, \widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}, \hat{\chi}\right)\right)\right\|_{L_u^\infty L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim \lambda^{-1/2-4\varepsilon_0}.
\end{aligned} \tag{9.308}$$

Note that $\overline{\frac{1}{\widetilde{r}}} = \frac{1}{\widetilde{r}}$ (see Definition 8.7 for the definition of $\overline{\frac{1}{\widetilde{r}}}$). By (6.13), (9.15), and (9.21a), we have:

$$\begin{aligned}
\left\|\widetilde{r}\left(\mathrm{tr}_{g\!\!\!/}\chi - \overline{\mathrm{tr}_{g\!\!\!/}\chi}\right)\check{\mu}\right\|_{L_u^2 L_t^1 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} &\lesssim \left\|\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(\mathrm{Small})}\right\|_{L_u^\infty L_t^2 L_\omega^\infty(\mathscr{M}^{(\mathrm{Int})})} \|\widetilde{r}\check{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathscr{M}^{(\mathrm{Int})})} \\
&\lesssim \lambda^{-1/2-7\varepsilon_0}.
\end{aligned} \tag{9.309}$$

Combining (9.298)-(9.309), we conclude the desired estimate. □

CHAPTER 10

Conformal Energy Estimates

In this section, with the control of the acoustic geometry that we derived in Proposition 9.1, we prove the boundness theorem for the conformal energy in Theorem 10.2.

10.1 Setup of the Conformal Energy

In order to give the definition of our conformal energy, we fix two smooth non-negative cut-off functions $\underline{\varpi}$ and ϖ, depending only on t and u, such that the following hold:

$$\underline{\varpi} = \begin{cases} 1 & \text{on } 0 \le u \le t \\ 0 & \text{on } u \le -\frac{t}{4} \end{cases}, \qquad \varpi = \begin{cases} 1 & \text{on } 0 \le u \le \frac{1}{2}t \\ 0 & \text{if } u \ge \frac{3}{4}t \text{ or } u \le -\frac{t}{4} \end{cases}. \tag{10.1}$$

Also, see Figure 10.1 for an illustration of the regions.

Definition 10.1 (Conformal energy)**.** *For any scalar φ vanishing outside $\mathscr{M}^{(Int)}$ (defined in Section 7.1), we define the conformal energy $\mathfrak{C}[\varphi]$as follows:*

$$\mathfrak{C}[\varphi](t) = \mathfrak{C}^{(i)}[\varphi](t) + \mathfrak{C}^{(e)}[\varphi](t), \tag{10.2}$$

where

$$\mathfrak{C}^{(i)}[\varphi](t) = \int_{\Sigma_t} (\underline{\varpi} - \varpi) t^2 \left(|\mathbf{D}\varphi|^2 + \left|\widetilde{r}^{-1}\varphi\right|^2 \right) \mathrm{d}\varpi_g, \tag{10.3a}$$

$$\mathfrak{C}^{(e)}[\varphi](t) = \int_{\Sigma_t} \varpi \left(\widetilde{r}^2 \left|\mathbf{D}_L\varphi\right|^2 + \widetilde{r}^2 \left|\nabla\!\!\!\!/ \varphi\right|^2 + |\varphi|^2 \right) \mathrm{d}\varpi_g. \tag{10.3b}$$

In this section, we prove the following theorem.

Theorem 10.2 (Boundness theorem)**.** *Let φ be any solution of*

$$\square_{\mathbf{g}} \varphi = 0 \tag{10.4}$$

on $[0, T_{*;(\lambda)}] \times \Sigma_t$ with $\varphi(1)$ supported in $B_R \subset \mathscr{M}^{(Int)} \bigcap \Sigma_1$, where B_R is defined in Theorem 6.9. Then under

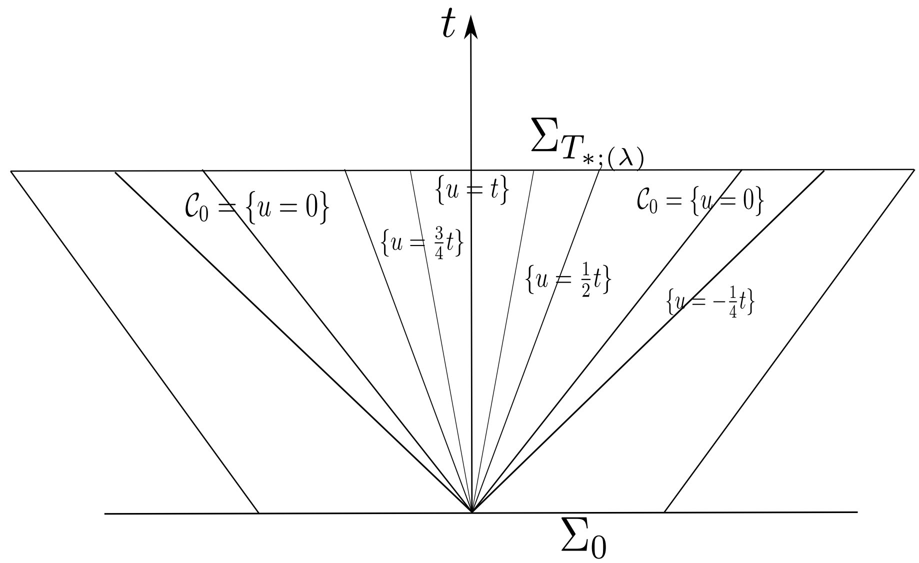


Figure 10.1: The illustration of the regions in (10.1)

bootstrap assumptions, for $t \in [1, T_{*;(\lambda)}]$*, the conformal energy of* φ *satisfies the estimate:*

$$\mathfrak{C}[\varphi](t) \lesssim (1+t)^{2\varepsilon} \left(\|\partial\varphi\|^2_{L^2(\Sigma_1)} + \|\varphi\|^2_{L^2(\Sigma_1)} \right), \tag{10.5}$$

where $\varepsilon > 0$ *is an arbitrary small number and the constants in "* $\lesssim$ *" can depend on* ε*.*

10.1.1 Reduction of Theorem 6.9

The proof of Theorem 6.9 by using Theorem 10.2 is done via product estimates and the Berstein inequality of Littlewood-Paley theory. We refer reader to [14, Section 8] and [36, Section 4] for the detailed proof.

10.1.2 Bootstrap assumptions for the conformal energy

Let $C_0 := \|\partial\varphi\|_{L^2(\Sigma_1)} + \|\varphi\|_{L^2(\Sigma_1)}$, for ε_0 defined as in Section 3.4, we make the following assumptions:

$$\mathfrak{C}[\varphi](t) \leq \lambda^{2\varepsilon_0} C_0^2, \tag{10.6a}$$

$$\|\varphi\|_{L^2_u L^2_\omega(\Sigma_t)} \leq (t+1)^{-1} \lambda^{\varepsilon_0} C_0. \tag{10.6b}$$

Theorem 10.3 (Boundness of the conformal energy)**.** *The bootstrap assumptions can be improved as follows*

$$\mathfrak{C}[\varphi](t) \lesssim (t+1)^{2\varepsilon} C_0^2, \tag{10.7}$$

where $\varepsilon > 0$ can be arbitrarily close to 0 and the constants in "$\lesssim$" can depend on ε. Since $t \leq T_{*;(\lambda)} \leq \lambda^{1-8\varepsilon_0} T_*$ as in (7.7) and $\varepsilon > 0$ can be arbitrarily close to 0 (In particular, $\varepsilon < \varepsilon_0$), (10.7) is an improvement of (10.6a).

The improvement consists of proving three estimates for solutions to equation (10.4):

$$\int_{\Sigma_t} |(t+1)\nabla\!\!\!/\varphi|^2 + |(t+1)L\varphi|^2 \, \mathrm{d}\varpi_g \lesssim (t+1)^{2\varepsilon} C_0^2, \tag{10.8a}$$

$$\int_{\Sigma_t \bigcap \{u \geq \frac{t}{2}\}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \, \mathrm{d}\varpi_g \lesssim (1+t)^{-2} C_0^2, \tag{10.8b}$$

$$\|\varphi\|_{L_u^2 L_\omega^2(\Sigma_t)} \leq (t+1)^{-1+\varepsilon} C_0. \tag{10.8c}$$

We notice that (10.8c) improves (10.6b).

Recall that the energy momentum tensor $Q_{\mu\nu}[\varphi]$:

$$Q_{\mu\nu}[\varphi] := \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi. \tag{10.9}$$

Definition 10.4 (Energy current)**.** *We recall the energy current ${}^{(X)}\mathbf{J}[\varphi]$ as follows:*

$${}^{(X)}\mathbf{J}_\alpha[\varphi] := Q_{\alpha\beta}[\varphi]X^\beta. \tag{10.10}$$

For $\square_{\mathbf{g}}\varphi = 0$, we have:

$$\mathbf{D}^{\alpha(X)}\mathbf{J}_\alpha[\varphi] = \frac{1}{2}Q^{\alpha\beta}[\varphi]^{(X)}\boldsymbol{\pi}_{\alpha\beta}, \tag{10.11}$$

where ${}^{(X)}\boldsymbol{\pi}_{\alpha\beta} := \mathbf{D}_\alpha X_\beta + \mathbf{D}_\beta X_\alpha$ is the deformation tensor of X.

Moreover,

$${}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{T}^\alpha = \frac{1}{2}\left((\mathbf{T}\varphi)^2 + |\partial\varphi|_g^2\right), \tag{10.12}$$

$${}^{(\mathbf{T})}\mathbf{J}^\alpha[\varphi]L_\alpha = \left((L\varphi)^2 + |\nabla\!\!\!/\varphi|_{g\!\!\!/}^2\right). \tag{10.13}$$

Definition 10.5 (Modified energy current)**.** *We define f,Θ to be as follows:*

$$f := \beta - \frac{\beta}{(1+\widetilde{r})^{\alpha}}, \tag{10.14}$$

$$\Theta := \widetilde{r}^{-1}\left(\beta - \frac{\beta}{(1+\widetilde{r})^{\alpha}}\right), \tag{10.15}$$

where α,β are positive constants, which will be determined later, such that $\beta\alpha = 2$.

We introduce the modified weighted energy:

$$\tilde{Q}[\varphi](t) = \int_{\Sigma_t} {}^{(X)}\widetilde{\mathbf{J}}_{\mu}[\varphi]\mathbf{T}^{\mu}\mathrm{d}x, \tag{10.16}$$

where ${}^{(X)}\widetilde{\mathbf{J}}_{\mu}[\varphi]$ is the modified energy current:

$${}^{(X)}\widetilde{\mathbf{J}}_{\mu}[\varphi] = Q_{\mu\nu}[\varphi]X^{\mu} + \frac{1}{2}\Theta\partial_{\mu}(\varphi^2) - \frac{1}{2}\varphi^2\partial_{\mu}\Theta. \tag{10.17}$$

By direct computation, we have:

$$\mathbf{D}^{\alpha(X)}\widetilde{\mathbf{J}}_{\alpha}[\varphi] = \square_{\mathbf{g}}\varphi(X\varphi + \Theta\varphi) + \Theta\mathbf{D}^{\alpha}\varphi\mathbf{D}_{\alpha}\varphi - \frac{1}{2}\square_{\mathbf{g}}\Theta\varphi^2 + \frac{1}{2}Q^{\alpha\beta}[\varphi]^{(X)}\boldsymbol{\pi}_{\alpha\beta}. \tag{10.18}$$

Also, by (8.46), (9.17a), (9.17b), we have:

$$\mathrm{d}\varpi_{\mathbf{g}} \approx \widetilde{r}^2\mathrm{d}t\mathrm{d}u\mathrm{d}\varpi_{\not{e}}. \tag{10.19}$$

10.1.3 Discussion of the proof of Theorem 10.2

By choosing $X = fN$ and using divergence theorem for modified current on appropriate region, we get a Morawetz-type energy estimate to obtain the uniform bound for a standard energy of φ along a union of a portion of the constant-time hypersurfaces and null cones.

Then we consider the conformal changed wave equation $\square_{\widetilde{\mathbf{g}}}(e^{-\sigma}\varphi) = \cdots$. We use the multiplier approach with $\widetilde{r}^pL$ type vectorfields in the region $\{\tau_1 \leq u \leq \tau_2\}\bigcap\{\widetilde{r} \geq R\}$ where $1 \leq \tau_1 < \tau_2 < T_{*;(\lambda)}$ to control the conformal energy in the exterior region and to provide energy decay along null slices. Finally we control the conformal energy in the interior region with the help of the argument in [8] regarding obtaining energy decay in each spatial-null slice.

The proof of Theorem 10.2 will close the reduction of Strichartz estimate. One could follow the steps listed in [9, Section 11] to obtain the estimates of conformal energy with the control of Ricci coefficients given. One could go through the detail of the argument in [36, Section 7]. Also reader could look into [16, Section 3] for initial ideas.

Lemma 10.6 (Identities involving deformation tensors)**.** *With $X = fN$ with f as defined in Definition 10.5, the following identities hold:*

$$ {}^{(X)}\boldsymbol{\pi}_{NN} = 2Nf, \qquad {}^{(X)}\boldsymbol{\pi}_{AN} = f(k_{AN} - \zeta_A), \tag{10.20a} $$

$$ {}^{(X)}\boldsymbol{\pi}_{AB} = 2f\theta_{AB}, \qquad {}^{(X)}\boldsymbol{\pi}_{\mathbf{T}\mathbf{T}} = 0, \tag{10.20b} $$

$$ {}^{(X)}\boldsymbol{\pi}_{\mathbf{T}N} = fk_{NN}. \tag{10.20c} $$

10.1.3.1 Proof of Lemma 10.6

We will silently use the fact that $\nabla\!\!\!/ f = 0$.

Proof of ${}^{(X)}\boldsymbol{\pi}_{NN} = 2Nf$. By (8.15),

$$ {}^{(X)}\boldsymbol{\pi}_{NN} = 2\langle \mathbf{D}_N(fN), N\rangle = 2Nf. \tag{10.21} $$

□

Proof of ${}^{(X)}\boldsymbol{\pi}_{AN} = f(k_{AN} - \zeta_A)$. By (8.15) and (8.17),

$$ {}^{(X)}\boldsymbol{\pi}_{AN} = \langle \mathbf{D}_A(fN), N\rangle + \langle \mathbf{D}_N(fN), A\rangle = f\nabla\!\!\!/_A \ln b = f(k_{AN} - \zeta_A). \tag{10.22} $$

□

Proof of ${}^{(X)}\boldsymbol{\pi}_{AB} = 2f\theta_{AB}$. By (8.12a),

$$ {}^{(X)}\boldsymbol{\pi}_{AB} = \langle \mathbf{D}_A(fN), B\rangle + \langle \mathbf{D}_B(fN), A\rangle = 2f\theta_{AB}. \tag{10.23} $$

□

Proof of ${}^{(X)}\boldsymbol{\pi}_{\mathbf{T}\mathbf{T}} = 0$. By (8.16) and (8.17),

$$ {}^{(X)}\boldsymbol{\pi}_{\mathbf{T}\mathbf{T}} = 2f\langle \mathbf{D}_{\mathbf{T}}(fN), \mathbf{T}\rangle = \frac{1}{4}\langle \mathbf{D}_{L+\underline{L}}(L-\underline{L}), L+\underline{L}\rangle = f\langle \zeta_A e_A, L+\underline{L}\rangle = 0. \tag{10.24} $$

□

Proof of ${}^{(X)}\boldsymbol{\pi}_{\mathbf{T}N} = fk_{NN}$. Note that $\mathbf{T}f = 0$. By (8.16) and (8.17),

$$\begin{aligned} {}^{(X)}\boldsymbol{\pi}_{\mathbf{T}N} =& \langle \mathbf{D}_{\mathbf{T}}(fN), N\rangle + \langle \mathbf{D}_N(fN), \mathbf{T}\rangle = f\langle \mathbf{D}_N N, \mathbf{T}\rangle \\ =& \frac{1}{8} f \langle \mathbf{D}_{L-\underline{L}}(L-\underline{L}), L+\underline{L}\rangle = fk_{NN}. \end{aligned} \tag{10.25}$$

□

Definition 10.7 (Truncated region)**.** *We define the truncated constant-time hypersurface* $\Sigma_{\tau,u'}$ *as follows:*

$$\Sigma_{\tau,u'} := \bigcup_{\tau=t,u'\leq u\leq t} S_{t,u}. \tag{10.26}$$

For $R \leq 1 \leq \tau_1 < \tau \leq T_{*;(\lambda)}$ *and* $R < R' < \tau - \tau_1 + R$*, we define the null-spatial slice as follows:*

$$\mathscr{C}_u[\tau_1,\tau] := \mathscr{C}_u \bigcap \{\tau_1 \leq t \leq \tau\}, \tag{10.27}$$

$$\Sigma^{\tau}_{\tau_1,R'} := \{\widetilde{r} \leq R', t = \tau_1\} \bigcup \{u = \tau_1 - R', \tau_1 \leq t \leq \tau\}. \tag{10.28}$$

We denote $\mathscr{M}^{\tau}_{\tau_1,R'}$ *the region enclosed by* $\Sigma^{\tau}_{\tau_1,R'}$ *and* $\Sigma_{\tau,\tau_1-R'}$*. We will use* $\mathbf{n}$ *to denote its normal vectorfields in* $(\mathbb{R}^{1+3},\mathbf{g})$.

We define the region $\mathscr{T}^{\tau}_{\tau_1,R}$*,* $K^{\tau}_{\tau_1,R}$ *and* $D^{\tau_2,\tau}_{\tau_1,R}$ *as follows:*

$$\mathscr{T}^{\tau}_{\tau_1,R} := \mathscr{M}^{\tau}_{\tau_1,R} \bigcap \{\widetilde{r} \geq R\}, \tag{10.29}$$

$$K^{\tau}_{\tau_1,R} := \mathscr{M}^{\tau}_{\tau_1,R} \bigcap \{\widetilde{r} \leq R\}, \tag{10.30}$$

$$D^{\tau_2,\tau}_{\tau_1,R} := \left(\mathscr{M}^{\tau}_{\tau_1,R} - \mathscr{M}^{\tau}_{\tau_2,R}\right) \bigcap \{\widetilde{r} \geq R\}. \tag{10.31}$$

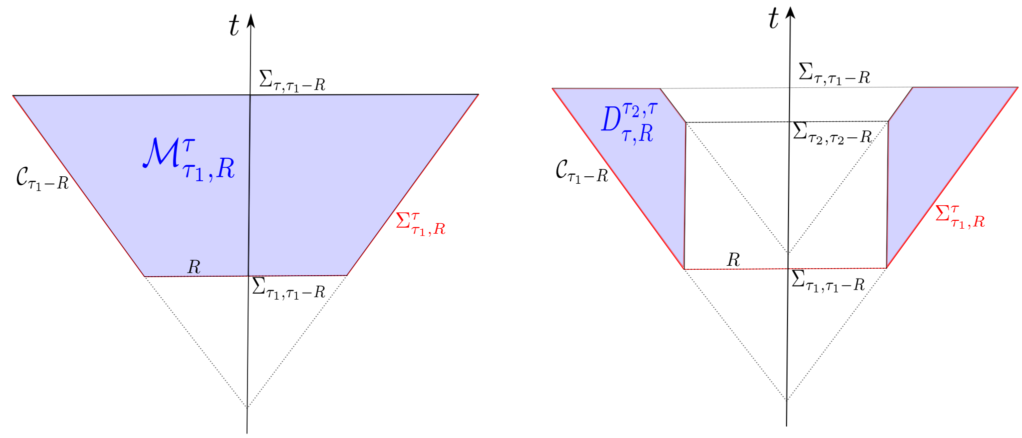
We first give the following energy estimates.

Lemma 10.8. *Let* φ *be any solution of* $\Box_{\mathbf{g}}\varphi = 0$ *on* $[1, T_{*;(\lambda)}] \times \Sigma_t$*. Given* $1 \leq \tau_1 < \tau_2 < \tau \leq T_{*;(\lambda)}$ *and* $0 \leq R' \leq \tau_2 - \tau_1 + R$*. For a fixed large number* λ*, there hold*

$$\int_{\Sigma_{\tau,\tau_1-R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g \lesssim \int_{\Sigma^{\tau}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}}, \tag{10.32}$$

and

$$\int_{\{t=\tau,\tau_1-R\leq u\leq \tau_2-R'\}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g + \int_{\Sigma^{\tau}_{\tau_2,R'}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} \leq 2\int_{\Sigma^{\tau}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}}. \tag{10.33}$$



(a) $\Sigma^{\tau}_{\tau_1,R}$ and $\mathscr{M}^{\tau}_{\tau_1,R}$.

(b) $D^{\tau_2,\tau}_{\tau_1,R}$.

Figure 10.2: The regions defined in Definition 10.7.

Proof of Lemma 10.8. By applying divergence theorem over the region $\mathscr{M}^{\tau}_{\tau_1,R}$ with ${}^{(\mathbf{T})}\mathbf{J}_{\alpha}[\varphi]$ as a multiplier, we have:

$$\int_{\Sigma_{\tau,\tau_1-R}} {}^{(\mathbf{T})}\mathbf{J}_{\alpha}[\varphi]\mathbf{n}^{\alpha} \mathrm{d}\varpi_g = \int_{\Sigma^{\tau}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_{\alpha}[\varphi]\mathbf{n}^{\alpha} \mathrm{d}\varpi_{\mathbf{g}} - \frac{1}{2}\int_{\mathscr{M}^{\tau}_{\tau_1,R}} Q^{\alpha\beta\,(\mathbf{T})}\boldsymbol{\pi}_{\alpha\beta} \mathrm{d}\varpi_{\mathbf{g}}. \tag{10.34}$$

By the bootstrap assumptions (6.13), $\left\|{}^{(\mathbf{T})}\boldsymbol{\pi}\right\|_{L^1_t L^\infty_x(\mathscr{M})} \lesssim \lambda^{-8\varepsilon_0}$. Also using the coerciveness (10.12), we obtain

$$\int_{\mathscr{M}^{\tau}_{\tau_1,R}} \left|Q^{\alpha\beta\,(\mathbf{T})}\boldsymbol{\pi}_{\alpha\beta}\right| \mathrm{d}\varpi_{\mathbf{g}} \lesssim \lambda^{-8\varepsilon_0} \sup_{\tau_1\leq t\leq\tau} \int_{\Sigma_{\tau,\tau_1-R}} {}^{(\mathbf{T})}\mathbf{J}_{\alpha}[\varphi]\mathbf{n}^{\alpha} \mathrm{d}\varpi_g. \tag{10.35}$$

Therefore, by (10.34) and (10.35), we have

$$\sup_{\tau_1\leq t\leq\tau} \int_{\Sigma_{\tau,\tau_1-R}} {}^{(\mathbf{T})}\mathbf{J}_{\alpha}[\varphi]\mathbf{n}^{\alpha} \mathrm{d}\varpi_g \lesssim \int_{\Sigma^{\tau}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_{\alpha}[\varphi]\mathbf{n}^{\alpha} \mathrm{d}\varpi_{\mathbf{g}}, \tag{10.36}$$

which yields (10.32). Moreover,

$$\int_{\mathscr{M}^{\tau}_{\tau_1,R}} \left|Q^{\alpha\beta\,(\mathbf{T})}\boldsymbol{\pi}_{\alpha\beta}\right| \mathrm{d}\varpi_{\mathbf{g}} \lesssim \lambda^{-8\varepsilon_0} \int_{\Sigma^{\tau}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_{\alpha}[\varphi]\mathbf{n}^{\alpha} \mathrm{d}\varpi_{\mathbf{g}}. \tag{10.37}$$

We now prove (10.33). By applying divergence theorem over the region $\mathscr{M}^{\tau}_{\tau_1,R} - \mathscr{M}^{\tau}_{\tau_2,R}$ with ${}^{(\mathbf{T})}\mathbf{J}_{\alpha}[\varphi]$ as a multiplier, we have:

$$\int_{\{t=\tau,\tau_1-R\le u\le\tau_2-R'\}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g + \int_{\Sigma^\tau_{\tau_2,R'}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} \tag{10.38}$$
$$= \int_{\Sigma^\tau_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} - \frac{1}{2}\int_{\mathscr{M}^\tau_{\tau_1,R}-\mathscr{M}^\tau_{\tau_2,R}} Q^{\alpha\beta\,(\mathbf{T})}\boldsymbol{\pi}_{\alpha\beta}\mathrm{d}\varpi_{\mathbf{g}}.$$

Since $\mathscr{M}^\tau_{\tau_1,R}-\mathscr{M}^\tau_{\tau_2,R}\subset\mathscr{M}^\tau_{\tau_1,R}$, we conclude the desired estimate (10.33) from (10.37)-(10.38) whenever λ is large. □

Lemma 10.9. *For $u'\ge 0$ and any scalar function φ, we have the following estimate:*

$$\|\varphi\|_{L^2_uL^2_\omega(\Sigma_{t,u'})} \lesssim \|\widetilde{r}N\varphi\|_{L^2_uL^2_\omega(\Sigma_{t,u'})} + \left\|\widetilde{r}^{-\frac{1}{2}}\varphi\right\|_{L^2_{\not{g}}(S_{t,u'})}. \tag{10.39}$$

Proof of Lemma 10.9. By $Nu=-b^{-1}$, we have:

$$bN(\widetilde{r}^m\varphi^2) = 2b\widetilde{r}^m\varphi N\varphi + m\widetilde{r}^{m-1}\varphi^2, \tag{10.40}$$

where m is a constant. Integrating (10.40) over $\Sigma_{t,u'}$, we have:

$$\int_{\Sigma_{t,u'}} m\widetilde{r}^{m-1}\varphi^2\mathrm{d}\varpi_g + \int_{\Sigma_{t,u'}} 2b\widetilde{r}^m\varphi N\varphi\mathrm{d}\varpi_g = \int_{S_{t,u'}} \widetilde{r}^m\varphi^2\mathrm{d}\varpi_{\not{g}}. \tag{10.41}$$

Setting $m=1$ in (10.41), by Hölder's inequality and the norm comparison (9.29), we conclude (10.39). □

We now give estimates for modified energy current. We will use the following lemma in Section 10.2.

Lemma 10.10. *For $1\le\tau_1<\tau_2$, with $X=fN$, where f is defined in Definition 10.17, we have:*

$$\int_{\Sigma_{\tau_2}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_g \lesssim \int_{\Sigma^{\tau_2}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L^2_\omega(S_{\tau_2,\tau_1-R})}, \tag{10.42}$$

$$\int_{\Sigma^{\tau_2}_{\tau_1,R}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}} \lesssim \int_{\Sigma^{\tau_2}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L^2_\omega(S_{\tau_2,\tau_1-R})} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L^2_\omega(S_{\tau_1,\tau_1-R})}. \tag{10.43}$$

Proof of Lemma 10.10. We first prove (10.42). By (10.17), (10.12) and Cauchy-Schwarz inequality, we have:

$$\int_{\Sigma_{\tau_2}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_g \lesssim \int_{\Sigma_{\tau_2}} (\mathbf{T}\varphi)^2+(N\varphi)^2\mathrm{d}\varpi_g + \|\varphi\|^2_{L^2_uL^2_\omega(\Sigma_{\tau_2})} \tag{10.44}$$
$$\lesssim \int_{\Sigma_{\tau_2}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_g + \|\varphi\|^2_{L^2_uL^2_\omega(\Sigma_{\tau_2})}.$$

By (9.29), (10.44), Lemma 10.9 and (10.32), we conclude the desired estimate (10.42).

We now prove (10.43). By the same method we used to prove (10.42), we have:

$$\int_{\Sigma_{\tau_1,\tau_1-R}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g \lesssim \int_{\Sigma_{\tau_1,R}^{\tau_2}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L^2_\omega(S_{\tau_1,\tau_1-R})}. \tag{10.45}$$

On the null part of $\Sigma_{\tau_1,R}^{\tau_2}$ where $\mathbf{n}=L$, we have:

$$ {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha = \frac{1}{2}f\left(|L\varphi|^2 - |\nabla\!\!\!/\varphi|_{g\!\!\!/}\right) - \frac{1}{2}\varphi^2 L\Theta + \Theta\varphi L\varphi \lesssim {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha + \left|\widetilde{r}^{-1}\varphi\right|^2. \tag{10.46}$$

Combining (10.45) and (10.46), using (8.46) and (9.17a), we have:

$$\int_{\Sigma_{\tau_1,R}^{\tau_2}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} \lesssim \int_{\Sigma_{\tau_1,R}^{\tau_2}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L^2_\omega(S_{\tau_1,\tau_1-R})} + \int_{\mathbb{S}^2}\int_{\tau_1}^{\tau_2}\varphi^2 \mathrm{d}t\mathrm{d}\varpi_{e\!\!\!/}. \tag{10.47}$$

By integrating the identity $\varphi^2 = L(\widetilde{r}\varphi^2) - 2\widetilde{r}\varphi L\varphi$ along $\mathscr{C}_{\tau_1-R}$, and using Young's inequality, we have:

$$\begin{aligned}\int_{\mathbb{S}^2}\int_{\tau_1}^{\tau_2}\varphi^2 \mathrm{d}t\mathrm{d}\varpi_{e\!\!\!/} &= \int_{S_{\tau_2,\tau_1-R}} \widetilde{r}\varphi^2 \mathrm{d}\varpi_{e\!\!\!/} - \int_{S_{\tau_1,\tau_1-R}} \widetilde{r}\varphi^2 \mathrm{d}\varpi_{e\!\!\!/} + \int_{\mathbb{S}^2}\int_{\tau_1}^{\tau_2} 2\widetilde{r}\varphi L\varphi \mathrm{d}t\mathrm{d}\varpi_{e\!\!\!/} \\ &\lesssim \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L^2_\omega(S_{\tau_2,\tau_1-R})} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L^2_\omega(S_{\tau_1,\tau_1-R})} + \int_{\Sigma_{\tau_1,R}^{\tau_2}} |L\varphi|^2 \mathrm{d}\varpi_{\mathbf{g}}.\end{aligned} \tag{10.48}$$

Combining (10.47) and (10.48), we conclude the desired estimate (10.43). □

10.2 Integrated Energy Estimates

In this Subsection, we prove the integrated energy estimates on a interior region of the $\mathscr{M}^{(\mathrm{Int})}\bigcap\{t\geq 1\}$ in Proposition 10.11.

Proposition 10.11. *Let φ be any solution of $\Box_{\mathbf{g}}\varphi = 0$ on $[1, T_{*;(\lambda)}]\times\Sigma_t$ with $\varphi[1]$ supported with B_R (defined in Theorem 6.9). Given $1 \leq \tau_1 < \tau_2 \leq T_{*;(\lambda)}$ and $R \leq R' < 2R$, then we have:*

$$\int_{\tau_1}^{\tau_2}\int_{\widetilde{r}\leq R'} (\mathbf{T}\varphi)^2 + (N\varphi)^2 + |\nabla\!\!\!/\varphi|^2 + \widetilde{r}^{-1}\varphi^2 \mathrm{d}\varpi_{\mathbf{g}} \lesssim \int_{\Sigma_1} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g. \tag{10.49}$$

Proof of Proposition 10.11. Proposition 10.11 follows from Lemma 10.12 by letting $\tau_1 = 1$ and the fact that that $\varphi = 0$ in $\{1\leq t\leq T_{*;(\lambda)}\} - \mathscr{M}_{1,1}^{T_{*;(\lambda)}}$, which is a direct result of $\varphi[1]$ supported with B_R and finite speed propagation property of wave. □

Lemma 10.12. *Let φ be any solution of $\Box_{\mathbf{g}}\varphi = 0$ on $[1, T_{*;(\lambda)}]\times\Sigma_t$ with $\varphi[1]$ supported with B_R. Given*

$1 \leq \tau_1 < \tau_2 \leq T_{*;(\lambda)}$, $R \leq R' < 2R$ *and* $\alpha = 2\varepsilon_0$, *we have:*

$$\int_{\mathscr{M}_{\tau_1,R}^{\tau_2}} \left(\frac{(\mathbf{T}\varphi)^2 + (N\varphi)^2 + |\nabla\!\!\!/\varphi|^2}{(1+\widetilde{r})^{\alpha+1}} + \frac{\varphi^2}{\widetilde{r}(1+\widetilde{r})^{\alpha+2}} \right) \mathrm{d}\varpi_{\mathbf{g}} \lesssim \int_{\Sigma_{\tau_1,R}^{\tau_2}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} + \left\| \widetilde{r}^{\frac{1}{2}}\varphi \right\|^2_{L^2_\omega(S_{\tau_2,\tau_1-R})} + \left\| \widetilde{r}^{\frac{1}{2}}\varphi \right\|^2_{L^2_\omega(S_{\tau_1,\tau_1-R})}. \tag{10.50}$$

Proof of Lemma 10.12. We denote the left hand side of (10.50) by

$$\mathscr{J} := \int_{\mathscr{M}_{\tau_1,R}^{\tau_2}} \left(\frac{(\mathbf{T}\varphi)^2 + (N\varphi)^2 + |\nabla\!\!\!/\varphi|^2}{(1+\widetilde{r})^{\alpha+1}} + \frac{\varphi^2}{\widetilde{r}(1+\widetilde{r})^{\alpha+2}} \right) \mathrm{d}\varpi_{\mathbf{g}}. \tag{10.51}$$

We apply the divergence theorem over the region $\mathscr{M}_{\tau_1,R}^{\tau_2}$ with modified current ${}^{(X)}\widetilde{\mathbf{J}}$ with $X = fN$ where f is defined in (10.15). By (10.18) and $\Box_{\mathbf{g}}\varphi = 0$, we have:

$$\int_{\Sigma_{\tau_1,R}^{\tau_2}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} - \int_{\Sigma_{\tau_2,\tau_1-R}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{T}^\alpha \mathrm{d}\varpi_g = \int_{\mathscr{M}_{\tau_1,R}^{\tau_2}} \left(\Theta \mathbf{D}^\alpha\varphi \mathbf{D}_\alpha\varphi - \frac{1}{2}(\Box_{\mathbf{g}}\Theta)\varphi^2 + \frac{1}{2}Q^{\alpha\beta}[\varphi]^{(X)}\boldsymbol{\pi}_{\alpha\beta} \right) \mathrm{d}\varpi_{\mathbf{g}}, \tag{10.52}$$

where

$$\frac{1}{2}Q^{\alpha\beta}[\varphi]^{(X)}\boldsymbol{\pi}_{\alpha\beta} = \frac{1}{2}{}^{(X)}\boldsymbol{\pi}_{NN}Q^{NN} + {}^{(X)}\boldsymbol{\pi}_{AN}Q^{AN} + \frac{1}{2}{}^{(X)}\boldsymbol{\pi}_{AB}Q^{AB} + {}^{(X)}\boldsymbol{\pi}_{\mathbf{T}N}Q^{\mathbf{T}N}. \tag{10.53}$$

By Lemma 10.6 and the decomposition $\mathbf{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi = -(\mathbf{T}\varphi)^2 + (N\varphi)^2 + |\nabla\!\!\!/\varphi|_{g\!\!\!/}$, we have:

$$\frac{1}{2}{}^{(X)}\boldsymbol{\pi}_{NN}Q^{NN} = \frac{1}{2}Nf\left[(\mathbf{T}\varphi)^2 + (N\varphi)^2 - |\nabla\!\!\!/\varphi|_{g\!\!\!/}\right], \tag{10.54a}$$

$$\frac{1}{2}{}^{(X)}\boldsymbol{\pi}_{AB}Q^{AB} = \frac{1}{2}f\mathrm{tr}_{g\!\!\!/}\theta\left[(\mathbf{T}\varphi)^2 - (N\varphi)^2\right] + f\hat{\theta}_{AB}Q^{AB}, \tag{10.54b}$$

$${}^{(X)}\boldsymbol{\pi}_{\mathbf{T}N}Q^{\mathbf{T}N} = fk_{NN}(\mathbf{T}\varphi)(N\varphi). \tag{10.54c}$$

Therefore, (10.52) can be expressed as:

$$\begin{aligned} &\int_{\Sigma_{\tau_1,R}^{\tau_2}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} - \int_{\Sigma_{\tau_2,\tau_1-R}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{T}^\alpha \mathrm{d}\varpi_g \\ &= \int_{\mathscr{M}_{\tau_1,R}^{\tau_2}} \Big\{ \frac{1}{2}(N\varphi)^2\left(Nf - \mathrm{tr}_{g\!\!\!/}\theta f + 2\Theta\right) + \frac{1}{2}(\mathbf{T}\varphi)^2\left(Nf + \mathrm{tr}_{g\!\!\!/}\theta f - 2\Theta\right) + \frac{1}{2}|\nabla\!\!\!/\varphi|^2_{g\!\!\!/}(-Nf + 2\Theta) \\ &\quad + fk_{NN}(\mathbf{T}\varphi)(N\varphi) + f\left(\hat{\theta}_{AB}Q^{AB} + {}^{(X)}\boldsymbol{\pi}_{AN}Q^{AN}\right) - \frac{1}{2}(\Box_{\mathbf{g}}\Theta)\varphi^2 \Big\} \mathrm{d}\varpi_{\mathbf{g}}. \end{aligned} \tag{10.55}$$

We first consider the first three terms on the right hand side of (10.55). Recall Definition 10.5 for

α, β, Θ, f. We have:

$$\Theta f = \widetilde{r}^{-1} f = \frac{\beta\left[(1+\widetilde{r})^{\alpha}-1\right]}{\widetilde{r}(1+\widetilde{r})^{\alpha}} \geq \frac{2}{(1+\widetilde{r})^{\alpha+1}}, \tag{10.56}$$

$$N f = \frac{2b^{-1}}{(1+\widetilde{r})^{\alpha+1}}, \tag{10.57}$$

$$\mathrm{tr}_{\not g}\theta f = \frac{2}{\widetilde{r}} f + \left(\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}\right) f + \mathrm{tr}_{\not g} k f. \tag{10.58}$$

By (9.17b) and (10.56)-(10.58), we have:

$$\begin{aligned} \frac{(N\varphi)^2 + (\mathbf{T}\varphi)^2 + |\nabla\!\!\!/ \varphi|^2_{\not g}}{(1+\widetilde{r})^{\alpha+1}} \lesssim & \frac{1}{2}(N\varphi)^2\left(Nf - \mathrm{tr}_{\not g}\theta f + 2\Theta\right) + \frac{1}{2}(\mathbf{T}\varphi)^2\left(Nf + \mathrm{tr}_{\not g}\theta f - 2\Theta\right) \\ & + \frac{1}{2}|\nabla\!\!\!/ \varphi|^2_{\not g}(-Nf + 2\Theta) \\ & + \frac{1}{2}\left(\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}} + \mathrm{tr}_{\not g} k\right) f\left[(\mathbf{T}\varphi)^2 - (N\varphi)^2\right]. \end{aligned} \tag{10.59}$$

Now we consider the term $f k_{NN}(\mathbf{T}\varphi)(N\varphi) + f\left(\hat{\theta}_{AB} Q^{AB} + {}^{(X)}\boldsymbol{\pi}_{AN} Q^{AN}\right)$. By Lemma 10.6, Lemma 10.8, (10.12), estimates (9.8e) and bootstrap assumptions (6.13), note that $\theta = \chi + k$, we have:

$$\left|\int_{\mathscr{M}^{\tau_2}_{\tau_1,R}} f k_{NN}(\mathbf{T}\varphi)(N\varphi) + f\left(\hat{\theta}_{AB} Q^{AB} + {}^{(X)}\boldsymbol{\pi}_{AN} Q^{AN}\right) \mathrm{d}\boldsymbol{\varpi}_{\mathbf{g}}\right| \tag{10.60}$$

$$\begin{aligned} & \lesssim T^{\frac{1}{2}}_{*;(\lambda)} \left\|\hat{\theta}, \zeta, k\right\|_{L^2_t L^\infty_x(\mathscr{M})} \|\mathbf{D}\varphi\|^2_{L^\infty_t L^2_x(\mathscr{M})} \\ & \lesssim \lambda^{-4\varepsilon_0} \int_{\Sigma^{\tau_2}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\boldsymbol{\varpi}_{\mathbf{g}}. \end{aligned} \tag{10.61}$$

Now we consider the term $-\square_{\mathbf{g}}\Theta\varphi^2$. By decomposition formula (8.211a) $L\widetilde{r} = 1$, $\underline{L}\widetilde{r} = 1 - 2b^{-1}$, $\nabla\!\!\!/ \widetilde{r} = 0$ and (9.17b) we have:

$$\begin{aligned} \square_{\mathbf{g}}\Theta = & -L\underline{L}\Theta + \Delta\!\!\!/ \Theta - \frac{1}{2}\mathrm{tr}_{\not g}\chi\underline{L}\Theta - \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi} L\Theta - 2k_{AN}\nabla\!\!\!/_A\Theta + k_{NN}\underline{L}\Theta \\ = & \left(\Theta'' + \frac{2}{\widetilde{r}}\Theta'\right)\left(2b^{-1}-1\right) + \frac{1}{2}\left(\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}\right)\left(2b^{-1}-1\right)\Theta' \\ & - \frac{1}{2}\left(\mathrm{tr}_{\not g}\underline{\chi} + \frac{2}{\widetilde{r}}\right)\Theta' + \frac{\Theta'}{\widetilde{r}}\left(2-2b^{-1}\right) - k_{NN}\Theta', \end{aligned} \tag{10.62}$$

where $\Theta' := \frac{\partial}{\partial\widetilde{r}}\Theta$. By direct computation and (9.17b), we have $-\left(\Theta'' + \frac{2}{\widetilde{r}}\Theta'\right)\left(2b^{-1}-1\right) \lesssim \frac{1}{\widetilde{r}(1+\widetilde{r})^{\alpha+2}}$ and

$0 \le -\Theta' \le \frac{\beta}{(1+\widetilde{r})^2}$. Thus,

$$\left(\frac{1}{\widetilde{r}(1+\widetilde{r})^{\alpha+2}}\right) \lesssim -\Box_{\mathbf{g}}\Theta \lesssim \left(\frac{1}{\widetilde{r}(1+\widetilde{r})^{\alpha+2}}\right) + \left(\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}, \mathrm{tr}_{\not g}\underline{\chi} + \frac{2}{\widetilde{r}}, \frac{b^{-1}-1}{\widetilde{r}}, k_{NN}\right)(1+\widetilde{r})^{-2}. \tag{10.63}$$

Now we consider the term $\left(\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}, \mathrm{tr}_{\not g}\underline{\chi} + \frac{2}{\widetilde{r}}, \frac{b^{-1}-1}{\widetilde{r}}, k_{NN}\right)(1+\widetilde{r})^{-2}\varphi^2$ in the term $\int_{\mathscr{M}^{\tau_2}_{\tau_1,R}} -(\Box_{\mathbf{g}}\Theta)\varphi^2 \mathrm{d}\varpi_{\mathbf{g}}$ in (10.55). First, using Hölder's inequality, we have:

$$\begin{aligned}&\int_{\mathscr{M}^{\tau_2}_{\tau_1,R}} \left|\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}, \mathrm{tr}_{\not g}\underline{\chi} + \frac{2}{\widetilde{r}}, \frac{b^{-1}-1}{\widetilde{r}}, k_{NN}\right|(1+\widetilde{r})^{-2}\varphi^2 \mathrm{d}\varpi_{\mathbf{g}} \\ &\lesssim \widetilde{r}^{\frac{1}{2}}(1+\widetilde{r})^{\alpha-\frac{1}{2}} \\ &\cdot\left\|\widetilde{r}^{\frac{1}{2}}\left(\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}, \mathrm{tr}_{\not g}\underline{\chi} + \frac{2}{\widetilde{r}}, \frac{b^{-1}-1}{\widetilde{r}}, k_{NN}\right)\right\|_{L^\infty_t L^\infty_u L^2_\omega(\mathscr{M})}\left\|\frac{\varphi^2\widetilde{r}}{(1+\widetilde{r})^{\alpha+\frac{3}{2}}}\right\|_{L^1_t L^1_u L^2_\omega(\Sigma_t)}\end{aligned} \tag{10.64}$$

We apply (9.33a) with $p=4$ on $\frac{\varphi^2\widetilde{r}}{(1+\widetilde{r})^{\alpha+\frac{3}{2}}}$ and use $T_{*;(\lambda)} \lesssim \lambda^{1-8\varepsilon_0}$. By Hölder's inequality, we have:

$$\left\|\frac{\varphi^2\widetilde{r}}{(1+\widetilde{r})^{\alpha+\frac{3}{2}}}\right\|_{L^1_u L^2_\omega(\Sigma_t)} \lesssim (1+t)^{\frac{1}{2}}\left(\left\|\frac{\not\nabla\varphi}{(1+\widetilde{r})^{\frac{\alpha+1}{2}}}\right\|_{L^2_u L^2_\omega(\Sigma_t)}\left\|\frac{\varphi}{(1+\widetilde{r})^{\frac{\alpha+1}{2}}\widetilde{r}^{\frac{1}{2}}}\right\|_{L^2_u L^2_\omega(\Sigma_t)} + \left\|\frac{\varphi}{(1+\widetilde{r})^{\frac{\alpha+1}{2}}\widetilde{r}^{\frac{1}{2}}}\right\|_{L^2_u L^2_\omega(\Sigma_t)}\right). \tag{10.65}$$

Therefore,

$$\left\|\frac{\varphi^2\widetilde{r}}{(1+\widetilde{r})^{\alpha+\frac{3}{2}}}\right\|_{L^1_t L^1_u L^2_\omega(\Sigma_t)} \lesssim \lambda^{\frac{1}{2}-4\varepsilon_0}\mathscr{J}. \tag{10.66}$$

Combining (10.64) and (10.66), choosing $\alpha = 2\varepsilon_0$, and using (9.7b),(9.9) and (9.82a), we have:

$$\int_{\mathscr{M}^{\tau_2}_{\tau_1,R}}\left(\mathrm{tr}_{\not g}\chi - \frac{2}{\widetilde{r}}, \mathrm{tr}_{\not g}\underline{\chi} + \frac{2}{\widetilde{r}}, \frac{b^{-1}-1}{\widetilde{r}}, k_{NN}\right)(1+\widetilde{r})^{-2}\varphi^2 \mathrm{d}\varpi_{\mathbf{g}} \lesssim \lambda^{-2\varepsilon_0}\mathscr{J}. \tag{10.67}$$

Combining (10.55), (10.59), (10.60), (10.63) and (10.67), we have:

$$\begin{aligned}&\int_{\mathscr{M}^{\tau_2}_{\tau_1,R}}\left(\frac{(\mathbf{T}\varphi)^2 + (N\varphi)^2 + |\not\nabla\varphi|^2}{(1+\widetilde{r})^{\alpha+1}} + \frac{\varphi^2}{\widetilde{r}(1+\widetilde{r})^{\alpha+2}}\right)\mathrm{d}\varpi_{\mathbf{g}} \\ &\lesssim \lambda^{-4\varepsilon_0}\int_{\Sigma^{\tau_2}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} + \left|\int_{\Sigma^{\tau_2}_{\tau_1,R}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g - \int_{\Sigma_{\tau_2,\tau_1-R}} {}^{(X)}\widetilde{\mathbf{J}}_\alpha[\varphi]\mathbf{T}^\alpha \mathrm{d}\varpi_g\right|.\end{aligned} \tag{10.68}$$

By Lemma 10.10, we conclude the desired estimate (10.50). □

10.3 Control of Lower Order Terms

In this subsection, we control various terms that will appear in the multiplier method in the next subsections. We first introduce several energies that are useful to our future analysis.

Definition 10.13 (Byproduct energies)**.** *Let φ be any solution of $\Box_{\mathbf{g}}\varphi = 0$ on $[1, T_{*;(\lambda)}] \times \Sigma_t$ with $\varphi[1]$ supported with B_R. Let $R \le R' < 2R$. The conformal flux $\mathfrak{CF}_{m,R'}[\varphi]$ is defined as follows:*

$$\mathfrak{CF}_{m,R'}[\varphi](u,\tau) := \int_{u+R'}^{\tau} \int_{S_{t,u}} \widetilde{r}^m \left| L(\upsilon^{\frac{1}{2}}\varphi)^2 \right| \mathrm{d}\varpi_{\not g} \mathrm{d}t. \tag{10.69}$$

We define

$$\tilde{\varphi} := e^{-\sigma}\varphi. \tag{10.70}$$

We use the following notation U_t for any region $U \subset \{1 \le t \le T_{*;(\lambda)}, u \ge 0\}$ with the property that there exist two positive functions $u_0(t)$ and $u_1(t)$ on $[1, T_{*;(\lambda)}]$ such that

$$U_t := U \bigcap \Sigma_t = \bigcup_{u_0(t) \le u \le u_1} S_{t,u}. \tag{10.71}$$

Then the energy $\mathscr{E}_{m,U,R'}[\varphi]$ and its modified version $\widetilde{\mathscr{E}}_{m,U,R'}[\varphi]$ are defined as follows:

$$\mathscr{E}_{m,U,R'}[\varphi](t) := \int_{U_t \bigcap \{\widetilde{r} \ge R'\}} \widetilde{r}^m \left(\left| L(\upsilon^{\frac{1}{2}}\varphi) \right|^2 + |\nabla\!\!\!/ \tilde{\varphi}|^2_{\not g} e^{2\sigma}\upsilon \right) \mathrm{d}\varpi_{\not g} \mathrm{d}u, \tag{10.72}$$

$$\widetilde{\mathscr{E}}_{m,U,R'}[\varphi] := \left\| \widetilde{r}^{\frac{m}{2}} \varphi \right\|_{L^2_u L^2_\omega (U_t \bigcap \{\widetilde{r} \ge R'\})} + \mathscr{E}_{m,U,R'}[\varphi](t). \tag{10.73}$$

We define the following error term energy:

$$\begin{aligned} \mathscr{B}^{\tau}_{\tau_1} &:= (\tau - \tau_1 + R)^{-1} \left(\left\| \widetilde{r}^{\frac{1}{2}} L(\upsilon^{\frac{1}{2}}\varphi) \right\|^2_{L^2_t L^2_\omega (\mathscr{C}_{\tau_1 - R}[\tau_1,\tau])} + \left\| \widetilde{r}^{\frac{1}{2}} \varphi \right\|^2_{L^2_t L^2_\omega (\mathscr{C}_{\tau_1 - R}[\tau_1,\tau])} \right) \\ &\quad + \int_{\Sigma^{\tau}_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi] \mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} + \left\| \widetilde{r}^{\frac{1}{2}} \varphi \right\|^2_{L^2_\omega (S_{\tau_1,\tau_1 - R})}, \end{aligned} \tag{10.74}$$

where $\mathscr{C}_{\tau_1 - R}[\tau_1, \tau]$ is defined in (10.27).

We now derive estimates for some low order terms.

Lemma 10.14. *Recall (10.29), that is, $\mathscr{T}^{\tau}_{\tau_1,R} := \mathscr{M}^{\tau}_{\tau_1,R} \bigcap \{\widetilde{r} \ge R\}$. Then, for $m = 1,2$ with $\Phi_1(\tau) = 1$ and*

$\Phi_2(\tau) = (\log(2+\tau))^{\frac{1}{2}}$, *we have:*

$$\sup_{\tau_1 \le t \le \tau} \left\| \widetilde{r}^{\frac{m}{2}} \varphi \right\|_{L^2_u L^2_\omega(\mathscr{T}^\tau_{\tau_1,R} \bigcap \Sigma_t)} \lesssim \Phi_m(\tau) \left\| \widetilde{r}^{\frac{m-1}{2}} L(\upsilon^{\frac{1}{2}} \varphi) \right\|_{L^2_u L^2_t L^2_\omega(\mathscr{T}^\tau_{\tau_1,R})} + R^{\frac{m}{2}} \left(\int_{\tau_1}^{\tau} \int_{S_{t,t-R}} \varphi^2 \mathrm{d}\varpi_{\not g} \mathrm{d}t \right)^{\frac{1}{2}}. \tag{10.75}$$

Moreover,

$$\sup_{\tau_1 \le t \le \tau} \|\varphi\|^2_{L^2_{\not g}(S_{t,u})} \lesssim \|\varphi\|^2_{L^2_{\not g}(S_{\tau_1,u})} + \left\| \widetilde{r}^{\frac{1}{2}} L(\upsilon^{\frac{1}{2}} \varphi) \right\|^2_{L^2_t L^2_\omega(\mathscr{C}_u[\tau_1,\tau])} + \left\| \widetilde{r}^{\frac{1}{2}} \varphi \right\|^2_{L^2_t L^2_\omega(\mathscr{C}_u[\tau_1,\tau])}, \tag{10.76}$$

$$\sup_{\tau_1 \le t \le \tau} \left\| \widetilde{r}^{-\frac{1}{2}} \varphi \right\|^2_{L^2_{\not g}(S_{t,u})} \lesssim \left\| \widetilde{r}^{-\frac{1}{2}} \varphi \right\|^2_{L^2_{\not g}(S_{\tau_1,u})} + \left\| L(\upsilon^{\frac{1}{2}} \varphi) \right\|^2_{L^2_t L^2_\omega(\mathscr{C}_u[\tau_1,\tau])}. \tag{10.77}$$

In addition,

$$\left\| \widetilde{r}^{\frac{1}{2}} \varphi \right\|_{L^2_t L^2_\omega(\mathscr{C}_{\tau_1-R}[\tau_1,\tau])} \lesssim \log\left(\frac{\tau - \tau_1 + R}{R} \right) \cdot \left[R \|\varphi\|_{L^2_\omega(S_{\tau_1,\tau_1-R})} + \left\| \widetilde{r}^{\frac{1}{2}} L(\upsilon^{\frac{1}{2}} \varphi) \right\|_{L^2_t L^2_\omega(\mathscr{C}_{\tau_1-R}[\tau_1,\tau])} \right]. \tag{10.78}$$

10.3.1 Proof of Lemma 10.14

In this subsection, we prove Lemma 10.14, which consists of the proof of (10.75)-(10.78).

Proof of (10.75). By (8.48a), we have:

$$L(\upsilon^{\frac{1}{2}} \varphi) = \upsilon^{\frac{1}{2}} \left[L\varphi + \frac{\varphi}{\widetilde{r}} + \frac{\varphi}{2} \left(\mathrm{tr}_{\not g} \chi - \frac{2}{\widetilde{r}} \right) \right]. \tag{10.79}$$

Then by (9.17a),

$$\widetilde{r}^{\frac{m+1}{2}} \left| L\varphi + \frac{\varphi}{\widetilde{r}} \right| \lesssim \widetilde{r}^{\frac{m-1}{2}} \left| L(\upsilon^{\frac{1}{2}} \varphi) \right| + \widetilde{r}^{\frac{m+1}{2}} \left| \left(\mathrm{tr}_{\not g} \chi - \frac{2}{\widetilde{r}} \right) \varphi \right|. \tag{10.80}$$

Taking the $L_t^2 L_u^2 L_\omega^2$ norm on $\mathscr{T}_{\tau_1,R}^{\tau}$, and using (9.15), we have:

$$\begin{aligned}\left\|\widetilde{r}^{\frac{m+1}{2}}\left(L\varphi+\frac{\varphi}{\widetilde{r}}\right)\right\|_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})} &\lesssim \left\|\widetilde{r}^{\frac{m+1}{2}}\left(\mathrm{tr}_{\not g}\chi-\frac{2}{\widetilde{r}}\right)\varphi\right\|_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})} \\ &\quad+\left\|\widetilde{r}^{\frac{m-1}{2}}L(\upsilon^{\frac{1}{2}}\varphi)\right\|_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})} \\ &\lesssim \lambda^{-7\varepsilon_0}\sup_{\tau_1\le t\le\tau}\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|_{L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau}\bigcap\Sigma_t)} \\ &\quad+\left\|\widetilde{r}^{\frac{m-1}{2}}L(\upsilon^{\frac{1}{2}}\varphi)\right\|_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})}.\end{aligned} \tag{10.81}$$

Now we prove (10.75) with $m=1$. By the identity $\widetilde{r}^2\left(L\varphi+\frac{\varphi}{\widetilde{r}}\right)^2=\widetilde{r}^2(L\varphi)^2+L(\widetilde{r}\varphi^2)$, we have:

$$\begin{aligned}\left\|\widetilde{r}\left(L\varphi+\frac{\varphi}{\widetilde{r}}\right)\right\|^2_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})} &= \|\widetilde{r}L\varphi\|^2_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})} \\ &\quad+\int_{\tau_1-R}^{\tau-R}\int_{\mathbb{S}^2}\widetilde{r}\varphi(\tau,u,\omega)\mathrm{d}\varpi_{\not g}\mathrm{d}u-R\int_{\tau_1}^{\tau}\int_{\mathbb{S}^2}\varphi(t,t-R,\omega)\mathrm{d}\varpi_{\not g}\mathrm{d}t.\end{aligned} \tag{10.82}$$

Noticing that by (8.48a) and (9.17a), we have:

$$\|\widetilde{r}L\varphi\|_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})}\lesssim\left\|L(\upsilon^{\frac{1}{2}}\varphi)\right\|_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})}. \tag{10.83}$$

Combining (10.81)-(10.83), letting $m=1$, we have:

$$\begin{aligned}\sup_{\tau_1\le t\le\tau}\left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|_{L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau}\bigcap\Sigma_t)} &\lesssim\left(R\int_{\tau_1}^{\tau}\int_{\mathbb{S}^2}\varphi(t,t-R,\omega)\mathrm{d}\varpi_{\not g}\mathrm{d}t\right)^{\frac{1}{2}} \\ &\quad+\lambda^{-7\varepsilon_0}\sup_{\tau_1\le t\le\tau}\left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|_{L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau}\bigcap\Sigma_t)}+\left\|L(\upsilon^{\frac{1}{2}}\varphi)\right\|_{L_t^2L_u^2L_\omega^2(\mathscr{T}_{\tau_1,R}^{\tau})},\end{aligned} \tag{10.84}$$

which gives (10.75) with $m=1$.

To prove (10.75) with $m=2$, we integrate $L(\upsilon^{\frac{1}{2}}\varphi)$ along null geodesics over $\mathscr{C}_u$ and use Hölder's inequality to obtain:

$$\begin{aligned}\left\|\upsilon^{\frac{1}{2}}\varphi\right\|_{L_\omega^2(S_{\tau,u})} &\le\left\|\int_{u+R}^{\tau}L(\upsilon^{\frac{1}{2}}\varphi)\mathrm{d}t\right\|_{L_\omega^2}+\left\|\upsilon^{\frac{1}{2}}\varphi\right\|_{L_\omega^2(S_{u+R,u})} \\ &\lesssim(\log(2+\tau))^{\frac{1}{2}}\left\|\widetilde{r}^{\frac{1}{2}}L(\upsilon^{\frac{1}{2}}\varphi)\right\|_{L_t^2L_\omega^2(\mathscr{C}_u)}+\left\|\upsilon^{\frac{1}{2}}\varphi\right\|_{L_\omega^2(S_{u+R,u})}.\end{aligned} \tag{10.85}$$

Taking the L_u^2 norm on $\mathscr{T}_{\tau_1,R}^{\tau}\bigcap\Sigma_t$, and using (9.17a), we conclude the desired estimate (10.75). □

Proof of (10.78). Using (9.17a), dividing (10.85) by $\widetilde{r}$, taking the L_t^2 norm on $[\tau_1,\tau]$ and using Hölder's inequality, we conclude the desired estimates. □

Proof of (10.76). Taking the $L_t^2 L_\omega^2$ norm of the identity $\widetilde{r}(\widetilde{r}L\varphi+\varphi)^2 = \widetilde{r}(\widetilde{r}L\varphi)^2 - \widetilde{r}\varphi^2 + L(\widetilde{r}^2\varphi^2)$ over $\mathscr{C}_u[\tau_1,\tau]$, using the Fundamental Theorem of Calculus and (10.83), we have:

$$\begin{aligned} \|\varphi\|^2_{L^2_{\not g}(S_{\tau,u})} \lesssim & \left\|\widetilde{r}^{\frac{1}{2}}(\widetilde{r}L\varphi+\varphi)\right\|^2_{L_t^2L_\omega^2(\mathscr{C}_u[\tau_1,\tau])} + \|\varphi\|^2_{L^2_{\not g}(S_{\tau_1,u})} \\ & + \left\|\widetilde{r}^{\frac{1}{2}}L(\upsilon^{\frac{1}{2}}\varphi)\right\|^2_{L_t^2L_\omega^2(\mathscr{C}_u[\tau_1,\tau])} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L_t^2L_\omega^2(\mathscr{C}_u[\tau_1,\tau])}. \end{aligned} \tag{10.86}$$

Combining (10.86) and (10.80) with $m=2$, we conclude the desired result. □

Proof of (10.77). Integrating the identity $\widetilde{r}^2\left(L\varphi+\frac{\varphi}{\widetilde{r}}\right)^2 = \widetilde{r}^2(L\varphi)^2 + L(\widetilde{r}\varphi^2)$ over $\mathscr{C}_u[\tau_1,\tau]$, using the Fundamental Theorem of Calculus and (10.83), we have:

$$\begin{aligned} \left\|\widetilde{r}^{-\frac{1}{2}}\varphi\right\|^2_{L^2_{\not g}(S_{\tau,u})} \lesssim & \left\|\widetilde{r}(L\varphi+\frac{\varphi}{\widetilde{r}})\right\|^2_{L_t^2L_\omega^2(\mathscr{C}_u[\tau_1,\tau])} + \left\|\widetilde{r}^{-\frac{1}{2}}\varphi\right\|^2_{L^2_{\not g}(S_{\tau_1,u})} \\ & + \left\|L(\upsilon^{\frac{1}{2}}\varphi)\right\|^2_{L_t^2L_\omega^2(\mathscr{C}_u[\tau_1,\tau])} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L_t^2L_\omega^2(\mathscr{C}_u[\tau_1,\tau])}. \end{aligned} \tag{10.87}$$

Combining (10.87) and (10.80) with $m=1$, we obtain the desired result. □

We now derive estimates in terms of $\mathscr{B}^\tau_{\tau_1}$. $\mathscr{B}^\tau_{\tau_1}$ serves as an important bridge between various terms, since it controls most of the energies together with the initial energy C_0. We will finally control $\mathscr{B}^\tau_{\tau_1}$ in Proposition 10.28 and Corollary 10.29.

Proposition 10.15. *Recall (10.74):*

$$\begin{aligned} \mathscr{B}^\tau_{\tau_1} :=& (\tau-\tau_1+R)^{-1}\left(\left\|\widetilde{r}^{\frac{1}{2}}L(\upsilon^{\frac{1}{2}}\varphi)\right\|^2_{L_t^2L_\omega^2(\mathscr{C}_{\tau_1-R}[\tau_1,\tau])} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L_t^2L_\omega^2(\mathscr{C}_{\tau_1-R}[\tau_1,\tau])}\right) \\ & + \int_{\Sigma^\tau_{\tau_1,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} + \left\|\widetilde{r}^{\frac{1}{2}}\varphi\right\|^2_{L^2_\omega(S_{\tau_1,\tau_1-R})}. \end{aligned} \tag{10.88}$$

Then, for $R \le R' \le 2R$, we have the following estimates:

$$\int_{K^\tau_{\tau_1,R'}} \left({}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{T}^\alpha + \frac{\varphi^2}{\widetilde{r}}\right)\mathrm{d}\varpi_{\mathbf{g}} \lesssim \mathscr{B}^\tau_{\tau_1}. \tag{10.89}$$

Proof of Proposition 10.15. Proposition 10.15 follows directly from Lemma 10.12 and (10.76). □

Lemma 10.16. *We have the following estimate:*

$$R^2\int_{\tau_1}^\tau\int_{S_{t,t-R}} \varphi^2 \mathrm{d}\varpi_{\not g}\mathrm{d}t \lesssim \mathscr{B}^\tau_{\tau_1}. \tag{10.90}$$

Proof of Lemma 10.16. By integrating the identity $N(\widetilde{r}^2\varphi^2) = 2\widetilde{r}\varphi^2 + 2\widetilde{r}^2\varphi N\varphi$ in the integral curve of N over $0 \leq \widetilde{r} \leq R$, we have:

$$R^2\varphi^2 = 2\int_0^R \left(\widetilde{r}\varphi^2 + \widetilde{r}^2\varphi N\varphi\right) \mathrm{d}\widetilde{r} \lesssim \int_0^R \widetilde{r}\varphi^2 + \widetilde{r}^2\varphi^2 + \widetilde{r}^2(N\varphi)^2 \mathrm{d}\widetilde{r}. \tag{10.91}$$

Integrating (10.91) over $[\tau_1, \tau] \times \mathbb{S}^2$ and using (9.17a), (10.19), we have:

$$R^2 \int_{\tau_1}^{\tau} \int_{S_{t,t-R}} \varphi^2 \mathrm{d}\varpi_{\not{g}} \mathrm{d}t \lesssim \int_{K^{\tau}_{\tau_1,R'}} \varphi^2(1 + \frac{1}{\widetilde{r}}) + (N\varphi)^2 \, \mathrm{d}\varpi_{\mathbf{g}} \tag{10.92}$$

By Proposition 10.15, we conclude the desired estimate (10.90). □

Corollary 10.17. *We have the following estimate:*

$$\sup_{\tau_1 \leq t \leq \tau} \left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|^2_{L^2_u L^2_\omega(\mathscr{T}^{\tau}_{\tau_1,R} \bigcap \Sigma_t)} \lesssim \Phi_m(\tau)^2 \left\|\widetilde{r}^{\frac{m-1}{2}} L(\upsilon^{\frac{1}{2}}\varphi)\right\|^2_{L^2_u L^2_t L^2_\omega(\mathscr{T}^{\tau}_{\tau_1,R})} + \mathscr{B}^{\tau}_{\tau_1}. \tag{10.93}$$

In particular, in the region $\{R \leq \widetilde{r} \leq 2R\} \bigcap \{\tau_1 - R \leq u \leq \tau - R\}$, *we have:*

$$\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|^2_{L^2_u L^2_\omega(\{R \leq \widetilde{r} \leq 2R\} \bigcap \{\tau_1 - R \leq u \leq \tau - R\})} \lesssim \mathscr{B}^{\tau}_{\tau_1}. \tag{10.94}$$

Proof of Corollary 10.17. Corollary 10.17 follows from (10.75) and Lemma 10.16. □

10.4 Comparison Results

In this subsection, we compare the norms of φ related terms and of $\tilde{\varphi}$ related terms. Since we use $\tilde{\varphi}$ related terms in the multiplier method while we need estimates for φ related terms, we need to translate between norms associate with these two variables.

Proposition 10.18. *Recall (10.70) that* $\tilde{\varphi} := e^{-\sigma}\varphi$. *We define* $U := \{1 \leq t \leq T_{*;(\lambda)}, u \geq 0, R \leq \widetilde{r} \leq 2R\}$. *Then the following estimate holds:*

$$\int_U \left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2 + |\nabla\!\!\!/ \tilde{\varphi}|^2_{\not{g}}\,\upsilon\right) b\mathrm{d}u\mathrm{d}t\mathrm{d}\varpi_{\not{g}} \lesssim \int_U \left({}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{T}^\alpha + \frac{\varphi^2}{\widetilde{r}}\right) \mathrm{d}\varpi_{\mathbf{g}}. \tag{10.95}$$

Proof of Proposition 10.18. By definition of $\tilde{\varphi}$, we have:

$$\nabla\!\!\!/ \tilde{\varphi} = e^{-\sigma}(\nabla\!\!\!/ \varphi - \varphi \nabla\!\!\!/ \sigma). \tag{10.96}$$

Then by (10.79), (10.96), $\upsilon^{\frac{1}{2}} \approx \widetilde{r}$ and (9.20b), we have $e^{-\sigma} \approx 1$, and:

$$\begin{aligned}
&\int_U \left(\left| L(\upsilon^{\frac{1}{2}} \varphi) \right|^2 + |\nabla\!\!\!/ \tilde{\varphi}|^2_{g\!\!\!/}\, \upsilon \right) b \mathrm{d}u \mathrm{d}t \mathrm{d}\varpi_{e\!\!\!/} && (10.97)\\
&\lesssim \int_U \upsilon \left[|L\varphi|^2 + \frac{\varphi^2}{\widetilde{r}^2} + \varphi^2 \left(\mathrm{tr}_{g\!\!\!/} \chi - \frac{2}{\widetilde{r}} \right)^2 + |\nabla\!\!\!/ \varphi|^2_{g\!\!\!/} + \varphi^2 \, |\nabla\!\!\!/ \sigma|^2_{g\!\!\!/} \right] b \mathrm{d}u \mathrm{d}t \mathrm{d}\varpi_{e\!\!\!/}.
\end{aligned}$$

By definition of U, $\widetilde{r} \approx R$. Thus all the terms on the right hand side of (10.97) are bounded by the right hand side of (10.95) except

$$\int_U \varphi^2 \left(\nabla\!\!\!/ \sigma, \mathrm{tr}_{g\!\!\!/} \chi - \frac{2}{\widetilde{r}} \right)^2 \upsilon b \mathrm{d}\varpi_{e\!\!\!/} \mathrm{d}t \mathrm{d}u. \tag{10.98}$$

Therefore it suffices to show that:

$$\int_U \varphi^2 \left(\nabla\!\!\!/ \sigma, \mathrm{tr}_{g\!\!\!/} \chi - \frac{2}{\widetilde{r}} \right)^2 \upsilon b \mathrm{d}\varpi_{e\!\!\!/} \mathrm{d}t \mathrm{d}u \lesssim \int_U \left({}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi] \mathbf{T}^\alpha + \frac{\varphi^2}{\widetilde{r}} \right) \mathrm{d}\varpi_{\mathbf{g}}. \tag{10.99}$$

We take $q > 2$ with $0 < 1 - \frac{2}{q} < N - 2$ and q_* such that $\frac{2}{q} + \frac{1}{q_*} = 1$. Then by Hölder's inequality, we have

$$\int_U \varphi^2 \left(\nabla\!\!\!/ \sigma, \mathrm{tr}_{g\!\!\!/} \chi - \frac{2}{\widetilde{r}} \right)^2 \upsilon b \mathrm{d}\varpi_{e\!\!\!/} \mathrm{d}t \mathrm{d}u \leq \left\| \widetilde{r}^{\frac{1}{2}} \left(\nabla\!\!\!/ \sigma, \mathrm{tr}_{g\!\!\!/} \chi - \frac{2}{\widetilde{r}} \right) \right\|^2_{L^\infty L^q_\omega (U)} \left\| \widetilde{r}^{\frac{1}{2}} \varphi \right\|^2_{L^2_t L^2_u L^{2q_*}_\omega (U)}. \tag{10.100}$$

By (9.33a) and Young's inequality, we have:

$$\left\| \widetilde{r}^{\frac{1}{2}} \varphi \right\|^2_{L^2_t L^2_u L^{2q_*}_\omega (U)} \lesssim \int_U \left(|\nabla\!\!\!/ \varphi|^2_{g\!\!\!/} + \frac{\varphi^2}{\widetilde{r}} \right) \mathrm{d}\varpi_{\mathbf{g}} \lesssim \int_U \left({}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi] \mathbf{T}^\alpha + \frac{\varphi^2}{\widetilde{r}} \right) \mathrm{d}\varpi_{\mathbf{g}}. \tag{10.101}$$

By the fact that $\mathrm{tr}_{g\!\!\!/} \chi - \frac{2}{\widetilde{r}} = \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})} - \boldsymbol{\Gamma}_L$, (9.7b), (9.20a) and (9.82a), we have:

$$\left\| \widetilde{r}^{\frac{1}{2}} \left(\nabla\!\!\!/ \sigma, \mathrm{tr}_{g\!\!\!/} \chi - \frac{2}{\widetilde{r}} \right) \right\|^2_{L^\infty L^q_\omega (U)} \lesssim \lambda^{-1}. \tag{10.102}$$

Combining (10.97)-(10.102), we conclude the desired estimate (10.95). □

Lemma 10.19. *Given $R \leq R' \leq 2R$ and $m \in \mathbb{R}$. We define $U := \{1 \leq t \leq T_{*;(\lambda)}, u \geq 0, \widetilde{r} \geq R\}$ and $U_t := U \bigcap \Sigma_t$.*

Then the following comparison results hold:

$$\left\|\widetilde{r}^{\frac{m}{2}}\nabla\!\!\!/\tilde{\varphi}\right\|_{L_u^2L_{\not g}^2(U_t)}+\left\|\widetilde{r}^{\frac{m}{2}}\tilde{\varphi}\right\|_{L_u^2L_\omega^2(U_t)}\approx\left\|\widetilde{r}^{\frac{m}{2}}\nabla\!\!\!/\varphi\right\|_{L_u^2L_{\not g}^2(U_t)}+\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|_{L_u^2L_\omega^2(U_t)},\tag{10.103a}$$

$$\left\|\widetilde{r}^{\frac{m}{2}}L(\upsilon^{\frac{1}{2}}\varphi)\right\|_{L_u^2L_\omega^2(U_t)}+\left\|\widetilde{r}^{\frac{m}{2}}\nabla\!\!\!/\tilde{\varphi}\right\|_{L_u^2L_{\not g}^2(U_t)}+\left\|\widetilde{r}^{\frac{m}{2}}\tilde{\varphi}\right\|_{L_u^2L_\omega^2(U_t)}\tag{10.103b}$$

$$\approx\left\|\widetilde{r}^{\frac{m}{2}}L\varphi\right\|_{L_u^2L_{\not g}^2(U_t)}+\left\|\widetilde{r}^{\frac{m}{2}}\nabla\!\!\!/\varphi\right\|_{L_u^2L_{\not g}^2(U_t)}+\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|_{L_u^2L_\omega^2(U_t)}.$$

Proof of Lemma 10.19. By identities (10.79) and (10.96) and the estimates (9.17a), (9.29), it suffices to show that

$$\left\|\widetilde{r}^{\frac{m}{2}}\varphi\left(\mathrm{tr}_{\not g}\chi-\frac{2}{\widetilde{r}},\nabla\!\!\!/\sigma\right)\right\|_{L_u^2L_{\not g}^2(U_t)}\lesssim\left\|\widetilde{r}^{\frac{m}{2}}\nabla\!\!\!/\varphi\right\|_{L_u^2L_{\not g}^2(U_t)}+\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|_{L_u^2L_\omega^2(U_t)}.\tag{10.104}$$

We take $q>2$ with $0<1-\frac{2}{q}<N-2$ and q_* such that $\frac{2}{q}+\frac{1}{q_*}=1$. Then by (9.17a), (9.118) and Hölder's inequality, we have:

$$\left\|\widetilde{r}^{\frac{m}{2}}\varphi\left(\mathrm{tr}_{\not g}\chi-\frac{2}{\widetilde{r}},\nabla\!\!\!/\sigma\right)\right\|_{L_u^2L_{\not g}^2(U_t)}\lesssim\lambda^{\frac{1}{2}-4\varepsilon_0}\left\|\widetilde{r}^{\frac{1}{2}}\left(\nabla\!\!\!/\sigma,\mathrm{tr}_{\not g}\chi-\frac{2}{\widetilde{r}}\right)\right\|_{L_u^\infty L_\omega^q(U_t)}\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|_{L_u^2L_\omega^{2q_*}(U_t)}.\tag{10.105}$$

By (9.33a) and Young's inequality, we have

$$\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|_{L_u^2L_\omega^{2q_*}(U_t)}\lesssim\left\|\widetilde{r}^{\frac{m}{2}}\nabla\!\!\!/\varphi\right\|_{L_u^2L_{\not g}^2(U_t)}+\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|_{L_u^2L_\omega^2(U_t)}.\tag{10.106}$$

Combining (10.104)-(10.106) with (10.102), we obtain the desired estimates. □

Lemma 10.20. *For $1\le\tau_1<t<\tau$ and $\tau_1-R\le u\le\tau-R$, the region $\mathscr{T}_{\tau_1,R}^{\tau}$ defined in Definition 10.7, the following estimates hold:*

$$\mathscr{E}_{m,U,R}[\varphi](t)\lesssim\mathscr{E}_{m,U,2R}[\varphi](t)+\mathscr{B}_{\tau_1}^{\tau},\tag{10.107a}$$

$$\widetilde{\mathscr{E}}_{m,U,R}[\varphi](t)\lesssim\widetilde{\mathscr{E}}_{m,U,2R}[\varphi](t)+\mathscr{B}_{\tau_1}^{\tau},\tag{10.107b}$$

$$\mathfrak{CF}_{m,R}[\varphi](u,\tau)\lesssim\mathfrak{CF}_{m,2R}[\varphi](u,\tau)+\mathscr{B}_{\tau_1}^{\tau},\tag{10.107c}$$

where $\mathscr{E}_{m,U,R}[\varphi],\widetilde{\mathscr{E}}_{m,U,R}[\varphi],\mathfrak{CF}_{m,R}[\varphi]$ and $\mathscr{B}_{\tau_1}^{\tau}$ are defined in Definition 10.13.

Proof of Lemma 10.20. We let $D:=\mathscr{T}_{\tau_1,R}^{\tau}\bigcap\{R\le\widetilde{r}\le 2R\}$ and $D_t:=D\bigcap\Sigma_t$. By definitions of $\mathscr{E}_{m,U,R}[\varphi],\widetilde{\mathscr{E}}_{m,U,R}$

and $\mathfrak{C}\mathfrak{F}_{m,R}[\varphi]$, it suffices to show that :

$$\int_{D_t} \widetilde{r}^m \left(\left| L(\upsilon^{\frac{1}{2}}\varphi) \right|^2 + |\nabla\!\!\!/ \tilde{\varphi}|^2_{g\!\!\!/}\, \upsilon e^{2\sigma} + \varphi^2 \right) \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}u \lesssim \mathscr{B}^{\tau}_{\tau_1}, \tag{10.108a}$$

$$\int_{u+R}^{\hat{\tau}} \int_{S_{t,u}} \widetilde{r}^m \left| L(\upsilon^{\frac{1}{2}}\varphi) \right|^2 \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}t \lesssim \mathscr{B}^{\tau}_{\tau_1}, \qquad \hat{\tau} := \min(u+2R, \tau). \tag{10.108b}$$

By (10.79), (10.96), $\upsilon^{\frac{1}{2}} \approx \widetilde{r} \approx R$ and $e^{\sigma} \approx 1$ (by (9.20b)), we have:

$$\begin{aligned} \int_{D_t} \widetilde{r}^m \left(\left| L(\upsilon^{\frac{1}{2}}\varphi) \right|^2 + |\nabla\!\!\!/ \tilde{\varphi}|^2_{g\!\!\!/}\, \upsilon e^{2\sigma} + \varphi^2 \right) \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}u \lesssim{}& \left\| \widetilde{r}^{\frac{m}{2}+1} \varphi \left(\mathrm{tr}_{g\!\!\!/}\chi - \frac{2}{\widetilde{r}}, \nabla\!\!\!/ \sigma \right) \right\|_{L^2_u L^2_\omega(D_t)} \\ &+ \|(L, \nabla\!\!\!/)\varphi\|_{L^2_u L^2_\omega(D_t)} + \|\varphi\|_{L^2_u L^2_\omega(D_t)}, \end{aligned} \tag{10.109a}$$

$$\begin{aligned} \int_{u+R}^{\hat{\tau}} \int_{S_{t,u}} \widetilde{r}^m \left| L(\upsilon^{\frac{1}{2}}\varphi) \right|^2 \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}t \lesssim{}& \left\| \widetilde{r}^{\frac{m}{2}+1} \varphi \left(\mathrm{tr}_{g\!\!\!/}\chi - \frac{2}{\widetilde{r}} \right) \right\|_{L^2_t L^2_\omega(\mathscr{C}_u \bigcap D)} \\ &+ \|(L, \nabla\!\!\!/)\varphi\|_{L^2_t L^2_\omega(\mathscr{C}_u \bigcap D)} + \|\varphi\|_{L^2_t L^2_\omega(\mathscr{C}_u \bigcap D)}. \end{aligned} \tag{10.109b}$$

By (10.105), (10.106), (10.102) and $\widetilde{r} \approx R$, we have:

$$\int_{D_t} \widetilde{r}^m \left(\left| L(\upsilon^{\frac{1}{2}}\varphi) \right|^2 + |\nabla\!\!\!/ \tilde{\varphi}|^2_{g\!\!\!/}\, \upsilon e^{2\sigma} + \varphi^2 \right) \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}u \lesssim \|(L, \nabla\!\!\!/)\varphi\|_{L^2_u L^2_\omega(D_t)} + \|\varphi\|_{L^2_u L^2_\omega(D_t)}, \tag{10.110a}$$

$$\int_{u+R}^{\hat{\tau}} \int_{S_{t,u}} \widetilde{r}^m \left| L(\upsilon^{\frac{1}{2}}\varphi) \right|^2 \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}t \lesssim \|(L, \nabla\!\!\!/)\varphi\|_{L^2_t L^2_\omega(\mathscr{C}_u \bigcap D)} + \|\varphi\|_{L^2_t L^2_\omega(\mathscr{C}_u \bigcap D)}. \tag{10.110b}$$

By (10.33) and (10.89), we have:

$$\|(L, \nabla\!\!\!/)\varphi\|_{L^2_u L^2_\omega(D_t)} + \|(L, \nabla\!\!\!/)\varphi\|_{L^2_t L^2_\omega(\mathscr{C}_u \bigcap D)} + \|\varphi\|_{L^2_u L^2_\omega(D_t)} \lesssim \mathscr{B}^{\tau}_{\tau_1}. \tag{10.111}$$

For $\|\varphi\|_{L^2_t L^2_\omega(\mathscr{C}_u \bigcap D)}$, we take the L^2_t norm of (10.41) over $\mathscr{C}_u \bigcap D$ and use (10.89) to deduce:

$$\|\varphi\|^2_{L^2_t L^2_\omega(\mathscr{C}_u \bigcap D)} \lesssim \int_{K^{\tau}_{\tau_1, 2R}} \left({}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{T}^\alpha + \frac{\varphi^2}{\widetilde{r}} \right) \mathrm{d}\varpi_{\mathbf{g}} \lesssim \mathscr{B}^{\tau}_{\tau_1}. \tag{10.112}$$

Combining (10.108)-(10.112) and we conclude the desired estimates (10.107). □

10.5 Multiplier Approach

In this subsection, we apply divergence theorem for $L(\widetilde{r}^m \tilde{\varphi}) \cdot e^{3\sigma} \Box_{\widetilde{\mathbf{g}}} \tilde{\varphi}$ over region $D^{\tau_2,\tau}_{\tau_1,R}$ (see Definition 10.7 for the definition of $D^{\tau_2,\tau}_{\tau_1,R}$), which gives us a potential of deriving decay estimates for the energy $\int_{\Sigma_{\tau,\tau_1-R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g$. Then the rest of the work in Subsection 10.5.1 is to control the error terms, which requires sharp estimates of the acoustic geometry.

We consider φ as a solution of $\Box_{\mathbf{g}}\varphi = 0$. Recall Definition 8.6 that $\widetilde{\mathbf{g}} := e^{2\sigma}\mathbf{g}$, $\mathrm{tr}_{\tilde{\not g}}\tilde{\chi} = \mathrm{tr}_{\not g}\chi + 2L\sigma = \mathrm{tr}_{\not g}\chi + \mathbf{\Gamma}_L$ and $\mathrm{tr}_{\tilde{\not g}}\underline{\tilde{\chi}} = \mathrm{tr}_{\not g}\underline{\chi} + 2\underline{L}\sigma$. For any scalar function, by (8.211a),

$$\Box_{\mathbf{g}} f = -\underline{L}Lf + \not\Delta f - \frac{1}{2}\mathrm{tr}_{\not g}\chi\underline{L}f - \frac{1}{2}\mathrm{tr}_{\not g}\underline{\chi}Lf + 2\zeta_A e_A f + k_{NN}Lf. \tag{10.113}$$

By direct computation and using (8.212), we have:

$$\begin{aligned} e^{2\sigma}\Box_{\widetilde{\mathbf{g}}}\varphi =& \Box_{\mathbf{g}}\varphi + 2\mathbf{D}^{\alpha}\sigma\mathbf{D}_{\alpha}\varphi \\ =& \not\Delta\varphi - \underline{L}L\varphi - \left(\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\underline{\tilde{\chi}} - k_{NN}\right)L\varphi - \frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\underline{L}\varphi + 2\left(\zeta + \not\nabla\sigma\right)\cdot\not\nabla\varphi. \end{aligned} \tag{10.114}$$

Replacing φ by $\tilde{\varphi}$ in (10.114), we have:

$$\begin{aligned} e^{2\sigma}\Box_{\widetilde{\mathbf{g}}}\tilde{\varphi} =& \not\Delta\tilde{\varphi} - \underline{L}L\tilde{\varphi} - \left(\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\underline{\tilde{\chi}} - k_{NN}\right)L\tilde{\varphi} - \frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\underline{L}\tilde{\varphi} + 2\left(\zeta + \not\nabla\sigma\right)\cdot\not\nabla\tilde{\varphi} \\ =& \not\Delta\tilde{\varphi} - \underline{L}\left(L\tilde{\varphi} + \frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right) - \left(\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\underline{\tilde{\chi}} - k_{NN}\right)\left(L\tilde{\varphi} + \frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right) \\ & + 2\left(\zeta + \not\nabla\sigma\right)\cdot\not\nabla\tilde{\varphi} + \frac{1}{2}\left(\underline{L}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi} + \frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\mathrm{tr}_{\tilde{\not g}}\underline{\tilde{\chi}} - \mathrm{tr}_{\tilde{\not g}}\tilde{\chi}k_{NN}\right)\tilde{\varphi}. \end{aligned} \tag{10.115}$$

Meanwhile,

$$\begin{aligned} e^{3\sigma}\Box_{\widetilde{\mathbf{g}}}\tilde{\varphi} &= e^{-\sigma}\frac{1}{\sqrt{|\mathbf{g}|}}\partial_{\alpha}\left\{e^{2\sigma}\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_{\beta}\left(e^{-\sigma}\varphi\right)\right\} \\ &= e^{-\sigma}\frac{1}{\sqrt{|\mathbf{g}|}}\partial_{\alpha}\left\{e^{\sigma}\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_{\beta}\varphi - e^{\sigma}\varphi\sqrt{|\mathbf{g}|}\mathbf{g}^{\alpha\beta}\partial_{\beta}\sigma\right\} \\ &= \Box_{\mathbf{g}}\varphi - \left(\Box_{\mathbf{g}}\sigma + \mathbf{D}^{\alpha}\sigma\mathbf{D}_{\alpha}\sigma\right)\varphi. \end{aligned} \tag{10.116}$$

Combining (10.115) and (10.116), we have:

$$\begin{aligned} 0 =& \not\Delta\tilde{\varphi} - \underline{L}\left(L\tilde{\varphi} + \frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right) - \left(\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\underline{\tilde{\chi}} - k_{NN}\right)\left(L\tilde{\varphi} + \frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right) + 2\left(\zeta + \not\nabla\sigma\right)\cdot\not\nabla\tilde{\varphi} \\ & + \left(\frac{1}{2}\underline{L}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi} + \frac{1}{4}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\mathrm{tr}_{\tilde{\not g}}\underline{\tilde{\chi}} - \frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}k_{NN} + \Box_{\mathbf{g}}\sigma + \mathbf{D}^{\alpha}\sigma\mathbf{D}_{\alpha}\sigma\right)\tilde{\varphi}. \end{aligned} \tag{10.117}$$

Using (10.113) to substitute for $\Box_{\mathbf{g}}\sigma$, we have the following identity for the terms in parenthesis on the last line of (10.117):

$$\begin{aligned}&\frac{1}{2}\underline{L}\mathrm{tr}_{\tilde{g}}\tilde{\chi}+\frac{1}{4}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\mathrm{tr}_{\tilde{g}}\underline{\tilde{\chi}}-\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}k_{NN}+\Box_{\mathbf{g}}\sigma+\mathbf{D}^{\alpha}\sigma\mathbf{D}_{\alpha}\sigma\\&=\frac{1}{2}\underline{L}\mathrm{tr}_{g\!\!/}\chi+\frac{1}{4}\mathrm{tr}_{g\!\!/}\chi\mathrm{tr}_{g\!\!/}\underline{\chi}-\frac{1}{2}\mathrm{tr}_{g\!\!/}\chi k_{NN}+\Delta\!\!\!/\sigma+|\nabla\!\!\!/\sigma|^2_{g\!\!/}+2\zeta\cdot\nabla\!\!\!/\sigma.\end{aligned}\tag{10.118}$$

Recall that $\mu:=\underline{L}\mathrm{tr}_{g\!\!/}\chi+\frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\mathrm{tr}_{g\!\!/}\underline{\chi}$. Hence, combining (10.117) and (10.118), we have:

$$\begin{aligned}0=&\Delta\!\!\!/\tilde{\varphi}-\underline{L}\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)-\left(\frac{1}{2}\mathrm{tr}_{\tilde{g}}\underline{\tilde{\chi}}-k_{NN}\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)+2\left(\zeta+\nabla\!\!\!/\sigma\right)\cdot\nabla\!\!\!/\tilde{\varphi}\\&+\left(\frac{1}{2}\mu-\frac{1}{2}\mathrm{tr}_{g\!\!/}\chi k_{NN}+\Delta\!\!\!/\sigma+|\nabla\!\!\!/\sigma|^2_{g\!\!/}+2\zeta\cdot\nabla\!\!\!/\sigma\right)\tilde{\varphi}.\end{aligned}\tag{10.119}$$

We multiply (10.119) by $\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\tilde{r}^m$ and integrate over region $D^{\tau_2,\tau}_{\tau_1,R}$ (see Definition 10.7 for the definition of $D^{\tau_2,\tau}_{\tau_1,R}$) to obtain:

$$\begin{aligned}0=&\int_{D^{\tau_2,\tau}_{\tau_1,R}}\left(\Delta\!\!\!/\tilde{\varphi}+2\left(\zeta+\nabla\!\!\!/\sigma\right)\cdot\nabla\!\!\!/\tilde{\varphi}\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\tilde{r}^m b\,\mathrm{d}\varpi_{\tilde{g}}\mathrm{d}u\mathrm{d}t\\&-\int_{D^{\tau_2,\tau}_{\tau_1,R}}\left\{\frac{1}{2}\underline{L}\left[\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)^2\right]+\left(\frac{1}{2}\mathrm{tr}_{\tilde{g}}\underline{\tilde{\chi}}-k_{NN}\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)^2\right\}\tilde{r}^m b\,\mathrm{d}\varpi_{\tilde{g}}\mathrm{d}u\mathrm{d}t\\&+\int_{D^{\tau_2,\tau}_{\tau_1,R}}\left(\frac{1}{2}\mu-\frac{1}{2}\mathrm{tr}_{g\!\!/}\chi k_{NN}+\Delta\!\!\!/\sigma+|\nabla\!\!\!/\sigma|^2_{g\!\!/}+2\zeta\cdot\nabla\!\!\!/\sigma\right)\tilde{\varphi}\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\tilde{r}^m b\,\mathrm{d}\varpi_{\tilde{g}}\mathrm{d}u\mathrm{d}t,\end{aligned}\tag{10.120}$$

where $b\,\mathrm{d}\varpi_{\tilde{g}}\mathrm{d}u\mathrm{d}t=e^{2\sigma}b\,\mathrm{d}\varpi_{g\!\!/}\mathrm{d}u\mathrm{d}t$. We consider the first term in (10.120). By integrating by part on $S_{t,u}$, using Lemma 8.197 and $\chi=\hat{\chi}+\frac{1}{2}\mathrm{tr}_{g\!\!/}\chi g\!\!/$, we have:

$$\begin{aligned}&\int_{S_{t,u}}\left(\Delta\!\!\!/\tilde{\varphi}+2\left(\zeta+\nabla\!\!\!/\sigma\right)\cdot\nabla\!\!\!/\tilde{\varphi}\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\tilde{r}^m e^{2\sigma}b\,\mathrm{d}\varpi_{g\!\!/}\\=&\int_{S_{t,u}}-\tilde{r}^m\nabla\!\!\!/\tilde{\varphi}\nabla\!\!\!/\left[e^{2\sigma}b\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\right]+2\left(\zeta+\nabla\!\!\!/\sigma\right)\cdot\nabla\!\!\!/\tilde{\varphi}\cdot\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)e^{2\sigma}\tilde{r}^m b\,\mathrm{d}\varpi_{g\!\!/}\\=&\int_{S_{t,u}}\left\{-\nabla\!\!\!/\tilde{\varphi}\nabla\!\!\!/\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)+2\left(\zeta-\nabla\!\!\!/\log b\right)\cdot\nabla\!\!\!/\tilde{\varphi}\cdot\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\right\}\tilde{r}^m e^{2\sigma}b\,\mathrm{d}\varpi_{g\!\!/}\\=&\int_{S_{t,u}}\left\{-\nabla\!\!\!/\tilde{\varphi}\left[\mathbf{D}\!\!\!\!/_L\nabla\!\!\!/\tilde{\varphi}+[\nabla\!\!\!/,L]\tilde{\varphi}+\nabla\!\!\!/\left(\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\right]\right.\\&\left.+2\left(\zeta-\nabla\!\!\!/\log b\right)\cdot\nabla\!\!\!/\tilde{\varphi}\cdot\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\right\}\tilde{r}^m e^{2\sigma}b\,\mathrm{d}\varpi_{g\!\!/}\\=&\int_{S_{t,u}}\left\{-\frac{1}{2}L\left(|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!/}\right)-\frac{1}{2}\left(\mathrm{tr}_{\tilde{g}}\tilde{\chi}+\mathrm{tr}_{g\!\!/}\chi\right)|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!/}-\frac{1}{2}\tilde{\varphi}\nabla\!\!\!/\mathrm{tr}_{\tilde{g}}\tilde{\chi}\nabla\!\!\!/\tilde{\varphi}-\nabla\!\!\!/\tilde{\varphi}\cdot\nabla\!\!\!/\tilde{\varphi}\cdot\hat{\chi}\right\}\tilde{r}^m e^{2\sigma}b\\&+\left\{\left(2\zeta-\nabla\!\!\!/\log b\right)\cdot\nabla\!\!\!/\tilde{\varphi}\cdot\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\varphi}\right)\right\}\tilde{r}^m e^{2\sigma}b\,\mathrm{d}\varpi_{g\!\!/}.\end{aligned}\tag{10.121}$$

Integrating (10.121) over $D^{\tau_2,\tau}_{\tau_1,R}$, combining (10.120), and using the fact that $e^{\sigma}\approx 1$, we have:

$$
\begin{aligned}
&\frac{1}{2}\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m \Big\{L\left(|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\right)+(\mathrm{tr}_{g\!\!\!/}\chi+\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi})\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/} \\
&+\underline{L}\left[\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right)^2\right]+\mathrm{tr}_{\tilde{g}\!\!\!/}\underline{\tilde{\chi}}\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right)^2\Big\}\mathrm{d}\varpi_{\tilde{g}\!\!\!/}\mathrm{d}u\mathrm{d}t \\
=&\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m \Big\{(2\zeta-\nabla\!\!\!/\log b)\cdot\nabla\!\!\!/\tilde{\varphi}\cdot\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right) \\
&+k_{NN}\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right)^2-\nabla\!\!\!/\tilde{\varphi}\cdot\nabla\!\!\!/\tilde{\varphi}\cdot\hat{\chi}\Big\}\mathrm{d}\varpi_{\tilde{g}\!\!\!/}\mathrm{d}u\mathrm{d}t \\
&+\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\tilde{\varphi}\Big\{\left(\frac{1}{2}\mu-\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi k_{NN}+\Delta\!\!\!/\sigma+|\nabla\!\!\!/\sigma|^2_{g\!\!\!/}+2\zeta\cdot\nabla\!\!\!/\sigma\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right) \\
&-\frac{1}{2}\nabla\!\!\!/\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\nabla\!\!\!/\tilde{\varphi}\Big\}\mathrm{d}\varpi_{\tilde{g}\!\!\!/}\mathrm{d}u\mathrm{d}t.
\end{aligned}
\tag{10.122}
$$

Proposition 10.21. *Let $1\le\tau_1<\tau_2\le\tau$. For $m\in\mathbb{N}$, we have:*

$$
\begin{aligned}
&2\int_{\tau_2}^{\tau}\int_{S_{t,\tau_2-R}}\widetilde{r}^m\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t \\
&+\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^{m-1}\left(m\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2+(2-m)\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma}\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t \\
&+\int_{\tau_1-R}^{\tau_2-R}\int_{S_{\tau,u}} b\widetilde{r}^m\left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2+|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma}\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u \\
=&\int_{\tau_1}^{\tau_2}\int_{S_{t,t-R}}\widetilde{r}^m\left(b\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma}-(2-b)\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t \\
&+2\int_{\tau_1}^{\tau}\int_{S_{t,\tau_1-R}}\widetilde{r}^m\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t+\mathrm{error}(m,R),
\end{aligned}
\tag{10.123}
$$

where

$$
\begin{aligned}
&\mathrm{error}(m,R) \\
=&2\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\tilde{\varphi}\Big\{\left(\frac{1}{2}\mu-\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi k_{NN}+\Delta\!\!\!/\sigma+|\nabla\!\!\!/\sigma|^2_{g\!\!\!/}+2\zeta\cdot\nabla\!\!\!/\sigma\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right)-\frac{1}{2}\nabla\!\!\!/\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\nabla\!\!\!/\tilde{\varphi}\Big\}\mathrm{d}\varpi_{\tilde{g}\!\!\!/}\mathrm{d}u\mathrm{d}t \\
&+\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\,|\nabla\!\!\!/\tilde{\varphi}|_{g\!\!\!/}\cdot\left(|L\tilde{\varphi}|,\left|\frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right|\right)\left(k,\mathbf{\Gamma}_L,\zeta,\mathrm{tr}_{g\!\!\!/}\chi-\frac{2}{\widetilde{r}},\hat{\chi},\frac{b-1}{\widetilde{r}}\right)\mathrm{d}\varpi_{\tilde{g}\!\!\!/}\mathrm{d}u\mathrm{d}t.
\end{aligned}
\tag{10.124}
$$

Proof of Proposition 10.21. By (9.17a), (9.17b), we have $b\mathrm{d}\varpi_{\tilde{g}\!\!\!/}\mathrm{d}u\mathrm{d}t\approx b\widetilde{r}^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t$, we only need to consider the left hand side of (10.122) since the other terms are in $\mathrm{error}(m,R)$ already. First of all, by integrating by parts with respect to L and using $L\tilde{\upsilon} = \mathrm{tr}_{\tilde{g}}\tilde{\chi}\tilde{\upsilon}$ ($\tilde{\upsilon}$ is defined in (8.47)), we have:

$$\begin{aligned}\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m L\left(|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\right)\tilde{\upsilon}\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t =& \int_{\tau_1-R}^{\tau_2-R}\int_{S_{\tau,u}} b\widetilde{r}^m\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma}\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u \\ &-\int_{\tau_1}^{\tau_2}\int_{S_{t,t-R}} b\widetilde{r}^m\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma}\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t \\ &-\int_{D^{\tau_2,\tau}_{\tau_1,R}} L\,(b\widetilde{r}^m\tilde{\upsilon})\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t,\end{aligned} \tag{10.125}$$

where

$$\int_{D^{\tau_2,\tau}_{\tau_1,R}} L\,(b\widetilde{r}^m\tilde{\upsilon})\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t = \int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^{m-1}(m+2)\tilde{\upsilon}\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t + \mathrm{error}(m,R). \tag{10.126}$$

For the second term on the left hand side of (10.122), using (8.41), we have:

$$\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\tilde{\upsilon}(\mathrm{tr}_{g\!\!\!/}\chi + \mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi})\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t = 4\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^{m-1}\tilde{\upsilon}\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t + \mathrm{error}(m,R). \tag{10.127}$$

For the last two terms of the left hand side of (10.122), we use $\underline{L}\tilde{\upsilon} = \mathrm{tr}_{\tilde{g}\!\!\!/}\underline{\tilde{\chi}}\tilde{\upsilon}$ and the identity

$$\left(L\tilde{\varphi} + \frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right)^2\tilde{\upsilon} = \left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2, \tag{10.128}$$

and use $\underline{L} = L - 2N$, $bN\widetilde{r} = 1$ and integrate by parts with respect to L and N to obtain:

$$\begin{aligned}&\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\left\{\underline{L}\left[\left(L\tilde{\varphi} + \frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right)^2\right] + \mathrm{tr}_{\tilde{g}\!\!\!/}\underline{\tilde{\chi}}\left(L\tilde{\varphi} + \frac{1}{2}\mathrm{tr}_{\tilde{g}\!\!\!/}\tilde{\chi}\tilde{\varphi}\right)^2\right\}\mathrm{d}\varpi_{\tilde{g}\!\!\!/}\mathrm{d}u\mathrm{d}t \\ &= \int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\underline{L}\left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t \\ &= \int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m L\left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t - 2\int_{D^{\tau_2,\tau}_{\tau_1,R}} \widetilde{r}^m N\left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t,\end{aligned} \tag{10.129}$$

where

$$\begin{aligned}&\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m L\left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t \\ &= \int_{\tau_1-R}^{\tau_2-R}\int_{S_{\tau,u}} b\widetilde{r}^m\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u - \int_{\tau_1}^{\tau_2}\int_{S_{t,t-R}} b\widetilde{r}^m\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t \\ &\quad - \int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^{m-1}m\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t + \mathrm{error}(m,R),\end{aligned} \tag{10.130}$$

and

$$\int_{D^{\tau_2,\tau}_{\tau_1,R}} \widetilde{r}^m N\left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2\right) \mathrm{d}\varpi_{\not{g}}\mathrm{d}u\mathrm{d}t = -\int_{\tau_2}^{\tau}\int_{S_{t,\tau_2-R}} \widetilde{r}^m \left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2 \mathrm{d}\varpi_{\not{g}}\mathrm{d}t \tag{10.131}$$
$$-\int_{\tau_1}^{\tau_2}\int_{S_{t,t-R}} \widetilde{r}^m \left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2 \mathrm{d}\varpi_{\not{g}}\mathrm{d}t + \int_{\tau_1}^{\tau}\int_{S_{t,\tau_1-R}} \widetilde{r}^m \left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2 \mathrm{d}\varpi_{\not{g}}\mathrm{d}t$$
$$-\int_{D^{\tau_2,\tau}_{\tau_1,R}} \widetilde{r}^{m-1} m \left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2 \mathrm{d}\varpi_{\not{g}}\mathrm{d}u\mathrm{d}t.$$

Combining (10.125), (10.125), (10.126), (10.127), (10.129), (10.130), (10.131) and using $\widetilde{r}^{m-1}(b-2) = 2\widetilde{r}^m\left(\frac{b-1}{\widetilde{r}}\right) - \widetilde{r}^{m-1}b$, we conclude the desired identity (10.123). □

10.5.1 Bounding the weighted energy

In this subsection, we control the error term error(m,R), which appears in the Proposition 10.21.

Lemma 10.22. *For $2 < Q < \infty$, $m \in \mathbb{R}$, we have:*

$$\left\|\widetilde{r}^{\frac{m-1}{2}}\varphi\right\|_{L^2_u L^Q_\omega(\Sigma_t)} \lesssim \left\|\widetilde{r}^{\frac{m-1}{2}}\not{\nabla}\varphi\right\|_{L^2_u L^2_{\not{g}}(\Sigma_t)} + \left\|\widetilde{r}^{\frac{m-3}{2}}\varphi\right\|_{L^2_u L^2_{\not{g}}(\Sigma_t)}. \tag{10.132}$$

Proof. (10.132) follows from integrating (9.33a) on Σ_t. □

Lemma 10.23. *Let $1 \le \tau_1 < T_{*;(\lambda)}$. Under the bootstrap assumptions (10.6), we have the followings estimate:*

$$\|\not{\nabla}\varphi\|_{L^2_u L^2_{\not{g}}(\Sigma_t)} + \|\varphi\|_{L^2_u L^2_\omega(\Sigma_t)} + \|\not{\nabla}\tilde{\varphi}\|_{L^2_u L^2_{\not{g}}(\Sigma_t)} \lesssim (t+1)^{-1}\lambda^{\varepsilon_0}C_0, \tag{10.133}$$
$$\left\|\widetilde{r}^{\frac{m-1}{2}}\varphi\right\|_{L^2_t L^2_u L^Q_\omega(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_1,R})} \lesssim (\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}\lambda^{\varepsilon_0}C_0, \tag{10.134}$$

where $m = 1,2$, $2 < Q < \infty$ and C_0 is defined in Section 10.1.2.

Proof of Lemma 10.23. (10.133) is a direct result of (10.6) and (10.103a). By $\widetilde{r} < 1 \le \tau_1 < T_{*;(\lambda)} \le \lambda^{1-8\varepsilon_0}$, (10.132) and (10.133), we have:

$$\left\|\widetilde{r}^{\frac{m-1}{2}}\varphi\right\|_{L^2_t L^2_u L^Q_\omega(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_1,R})} \le \left(\int_{\tau_1}^{T_{*;(\lambda)}}(1+t)^{m-3}\mathrm{d}t\right)^{\frac{1}{2}}\lambda^{\varepsilon_0}C_0 \lesssim (\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}\lambda^{\varepsilon_0}C_0. \tag{10.135}$$

□

Now we control the error term in Proposition 10.21:

$$\begin{aligned}&\mathrm{error}(m,R) \qquad (10.136)\\ =&2\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\tilde{\varphi}\left\{\left(\frac{1}{2}\mu-\frac{1}{2}\mathrm{tr}_{\not g}\chi k_{NN}+\not\Delta\sigma+|\not\nabla\sigma|^2_{\not g}+2\zeta\cdot\not\nabla\sigma\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right)-\frac{1}{2}\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\not\nabla\tilde{\varphi}\right\}\mathrm{d}\varpi_{\tilde{\not g}}\mathrm{d}u\mathrm{d}t\\ &+\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m|\not\nabla\tilde{\varphi}|_{\not g}\cdot\left(|L\tilde{\varphi}|,\left|\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right|\right)\left(k,\mathbf{\Gamma}_L,\zeta,\mathrm{tr}_{\not g}\chi-\frac{2}{\widetilde{r}},\hat{\chi},\frac{b-1}{\widetilde{r}}\right)\mathrm{d}\varpi_{\tilde{\not g}}\mathrm{d}u\mathrm{d}t.\end{aligned}$$

Lemma 10.24. *For $1\leq\tau_1<\tau_2\leq\tau\leq T_{*;(\lambda)}$ and $m=1,2$, we have:*

$$\begin{aligned}\mathrm{error}(m,R)\lesssim\lambda^{-3\varepsilon_0}\Bigg\{&(\tau_1+1)^{m-2}C_0^2+\sup_{\tau_1\leq t\leq\tau}\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|^2_{L^2_uL^2_\omega(D^{\tau_2,\tau}_{\tau_1,R}\cap\Sigma_t)} \qquad (10.137)\\ &+\sup_{\tau_1\leq t\leq\tau}\mathscr{E}_{m,U,R}[\varphi](t)+\sup_{\tau_1-R\leq u\leq\tau_2-R}\mathfrak{CF}_{m,R}[\varphi](u,\tau)\Bigg\},\end{aligned}$$

where $\mathscr{E}_{m,U,R}[\varphi],\mathfrak{CF}_{m,R}[\varphi]$ are defined in Definition 10.13 with $U:=D^{\tau_2,\tau}_{\tau_1,R}$.

Proof of Lemma 10.24. By Proposition 9.1, we have:

$$\left\|k,\mathbf{\Gamma}_L,\zeta,\mathrm{tr}_{\not g}\chi-\frac{2}{\widetilde{r}},\hat{\chi},\frac{b-1}{\widetilde{r}}\right\|_{L^2_tL^\infty_x(\mathscr{M})}\lesssim\lambda^{-\frac{1}{2}-3\varepsilon_0}. \qquad (10.138)$$

Therefore, we have:

$$\begin{aligned}&\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\mathscr{Q}\left(|\not\nabla\tilde{\varphi}|_{\not g},\left|L\tilde{\varphi},\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right|\right)\left(k,\mathbf{\Gamma}_L,\zeta,\mathrm{tr}_{\not g}\chi-\frac{2}{\widetilde{r}},\hat{\chi},\frac{b-1}{\widetilde{r}}\right)\mathrm{d}\varpi_{\tilde{\not g}}\mathrm{d}u\mathrm{d}t \qquad (10.139)\\ &\lesssim\lambda^{-7\varepsilon_0}\sup_{\tau_1\leq t\leq\tau}\mathscr{E}_{m,U,R}[\varphi](t).\end{aligned}$$

Now we consider the first term on the right hand side of (10.136). First, let $\frac{1}{Q}+\frac{1}{Q'}=\frac{1}{2}$, by (9.20b) and (10.133), we have:

$$\begin{aligned}&\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\tilde{\varphi}\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\not\nabla\tilde{\varphi}\mathrm{d}\varpi_{\tilde{\not g}}\mathrm{d}u\mathrm{d}t \qquad (10.140)\\ &\lesssim\int_{\tau_1}^{\tau}\left\|\widetilde{r}^{\frac{3}{2}}\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\right\|_{L^\infty_uL^{Q'}_\omega(\Sigma_t)}\left\|\widetilde{r}^{m-\frac{1}{2}}\varphi\right\|_{L^2_uL^Q_\omega(D^{\tau_2,\tau}_{\tau_1,R})}\|\not\nabla\tilde{\varphi}\|_{L^2_uL^2_{\not g}(\Sigma_t)}\mathrm{d}t\\ &\lesssim\lambda^{\varepsilon_0}C_0\int_{\tau_1}^{\tau}(1+t)^{-1}\left\|\widetilde{r}^{\frac{3}{2}}\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\right\|_{L^\infty_uL^{Q'}_\omega(\Sigma_t)}\left\|\widetilde{r}^{m-\frac{1}{2}}\varphi\right\|_{L^2_uL^Q_\omega(D^{\tau_2,\tau}_{\tau_1,R}\cap\Sigma_t)}\mathrm{d}t,\end{aligned}$$

where by (10.132) and $\widetilde{r} \leq t$, we have:

$$\left\| \widetilde{r}^{m-\frac{1}{2}} \varphi \right\|_{L_u^2 L_\omega^Q (D_{\tau_1,R}^{\tau_2,\tau} \cap \Sigma_t)} \lesssim t^{\frac{m-1}{2}} \left(\left\| \widetilde{r}^{\frac{m}{2}} \nabla\!\!\!/ \varphi \right\|_{L_u^2 L_{g\!\!\!/}^2 (D_{\tau_1,R}^{\tau_2,\tau} \cap \Sigma_t)} + \left\| \widetilde{r}^{\frac{m}{2}} \varphi \right\|_{L_u^2 L_\omega^2 (D_{\tau_1,R}^{\tau_2,\tau} \cap \Sigma_t)} \right). \tag{10.141}$$

Combining (10.140) and (10.141), we have:

$$\begin{aligned} \int_{D_{\tau_1,R}^{\tau_2,\tau}} b\widetilde{r}^m \tilde{\varphi} \nabla\!\!\!/ \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi} \nabla\!\!\!/ \tilde{\varphi} \mathrm{d}\varpi_{\tilde{g\!\!\!/}} \mathrm{d}u \mathrm{d}t \lesssim & \lambda^{\varepsilon_0} C_0 \int_{\tau_1}^{\tau} (1+t)^{\frac{m-3}{2}} \left\| \widetilde{r}^{\frac{3}{2}} \nabla\!\!\!/ \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi} \right\|_{L_u^\infty L_\omega^{Q'} (\Sigma_t)} \\ & \cdot \left(\left\| \widetilde{r}^{\frac{m}{2}} \nabla\!\!\!/ \varphi \right\|_{L_u^2 L_{g\!\!\!/}^2 (D_{\tau_1,R}^{\tau_2,\tau} \cap \Sigma_t)} + \left\| \widetilde{r}^{\frac{m}{2}} \varphi \right\|_{L_u^2 L_\omega^2 (D_{\tau_1,R}^{\tau_2,\tau} \cap \Sigma_t)} \right) \mathrm{d}t. \end{aligned} \tag{10.142}$$

By (9.8c) and $\nabla\!\!\!/ \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi} = \nabla\!\!\!/ \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi}^{(\mathrm{Small})}$, we have, for $m = 1, 2$,

$$\begin{aligned} & \int_{D_{\tau_1,R}^{\tau_2,\tau}} b\widetilde{r}^m \tilde{\varphi} \nabla\!\!\!/ \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi} \nabla\!\!\!/ \tilde{\varphi} \mathrm{d}\varpi_{\tilde{g\!\!\!/}} \mathrm{d}u \mathrm{d}t \\ & \lesssim \lambda^{-3\varepsilon_0} (\tau_1+1)^{\frac{m-2}{2}} C_0 \sup_{\tau_1 \leq t \leq \tau} \left(\mathscr{E}_{m,U,R}(t)^{\frac{1}{2}} + \left\| \widetilde{r}^{\frac{m}{2}} \varphi \right\|_{L_u^2 L_\omega^2 (D_{\tau_1,R}^{\tau_2,\tau} \cap \Sigma_t)} \right). \end{aligned} \tag{10.143}$$

We have bounded the first term on the right hand side of (10.136) by the right hand side of (10.137).

Next, we recall (9.23) and (8.54):

$$\nabla\!\!\!/ \sigma = -\zeta + (\tilde{\zeta} - \not{\mu}) + \not{\mu}_{(1)} + \not{\mu}_{(2)}, \tag{10.144}$$

$$\check{\mu} := 2\Delta\!\!\!/ \sigma + \underline{L} \mathrm{tr}_{g\!\!\!/} \chi + \frac{1}{2} \mathrm{tr}_{g\!\!\!/} \chi \mathrm{tr}_{g\!\!\!/} \underline{\chi} - \mathrm{tr}_{g\!\!\!/} \chi k_{NN} + \frac{1}{2} \mathrm{tr}_{g\!\!\!/} \chi \mathbf{\Gamma}_{\underline{L}}. \tag{10.145}$$

Using (10.128), we rewrite the remaining terms in (10.136) as follows:

$$\begin{aligned} & \int_{D_{\tau_1,R}^{\tau_2,\tau}} b\widetilde{r}^m \tilde{\varphi} \left\{ \left(\frac{1}{2} \mu - \frac{1}{2} \mathrm{tr}_{g\!\!\!/} \chi k_{NN} + \Delta\!\!\!/ \sigma + |\nabla\!\!\!/ \sigma|_{g\!\!\!/}^2 + 2\zeta \cdot \nabla\!\!\!/ \sigma \right) \left(L\tilde{\varphi} + \frac{1}{2} \mathrm{tr}_{\tilde{g\!\!\!/}} \tilde{\chi} \tilde{\varphi} \right) \right\} \mathrm{d}\varpi_{\tilde{g\!\!\!/}} \mathrm{d}u \mathrm{d}t \\ = & \int_{D_{\tau_1,R}^{\tau_2,\tau}} b\widetilde{r}^m \not{\mu}_{(2)} \cdot \nabla\!\!\!/ \sigma \cdot \left(\upsilon^{\frac{1}{2}} \varphi \right) \cdot L \left(\upsilon^{\frac{1}{2}} \varphi \right) \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}u \mathrm{d}t \\ & + \int_{D_{\tau_1,R}^{\tau_2,\tau}} b\widetilde{r}^m \left\{ -\zeta + (\tilde{\zeta} - \not{\mu}) + \not{\mu}_{(1)} \right\} \cdot \nabla\!\!\!/ \sigma \cdot \left(\upsilon^{\frac{1}{2}} \varphi \right) \cdot L \left(\upsilon^{\frac{1}{2}} \varphi \right) \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}u \mathrm{d}t \\ & + \int_{D_{\tau_1,R}^{\tau_2,\tau}} b\widetilde{r}^m \left(\check{\mu}, \mathrm{tr}_{g\!\!\!/} \chi \mathbf{\Gamma}_{\underline{L}} \right) \cdot \left(\upsilon^{\frac{1}{2}} \varphi \right) \cdot L \left(\upsilon^{\frac{1}{2}} \varphi \right) \mathrm{d}\varpi_{g\!\!\!/} \mathrm{d}u \mathrm{d}t. \end{aligned} \tag{10.146}$$

Let $\frac{1}{Q_\star}-\frac{1}{Q}=\frac{1}{2}$ and $\frac{1}{p}+\frac{1}{Q_\star}=1$, where p is defined in Section 9.1.1. By (9.17a) and (10.134), we have:

$$\begin{aligned}
&\left\|\widetilde{r}^{m-\frac{1}{2}}\left(\upsilon^{\frac{1}{2}}\varphi\right)\cdot L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^2_uL^1_tL^{Q_\star}_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
\lesssim&\left\|\widetilde{r}^{\frac{m+1}{2}}\varphi\right\|_{L^2_uL^2_tL^Q_\omega(D^{\tau_2,\tau}_{\tau_1,R})}\left\|\widetilde{r}^{\frac{m}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^\infty_uL^2_tL^2_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
\lesssim&\lambda^{1-7\varepsilon_0}(\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}C_0\left\|\widetilde{r}^{\frac{m}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^\infty_uL^2_tL^2_\omega(D^{\tau_2,\tau}_{\tau_1,R})}.
\end{aligned} \tag{10.147}$$

Combining (10.147), (9.20a), (9.25b), we have:

$$\begin{aligned}
&\left|\int_{D^{\tau_2,\tau}_{\tau_1,R}}b\widetilde{r}^m\not\mu_{(2)}\cdot\not\nabla\sigma\cdot\left(\upsilon^{\frac{1}{2}}\varphi\right)\cdot L\left(\upsilon^{\frac{1}{2}}\varphi\right)\mathrm{d}\varpi_{\not g}\mathrm{d}u\mathrm{d}t\right| \\
\lesssim&\left\|\not\mu_{(2)}\right\|_{L^2_uL^\infty_tL^\infty_\omega(D^{\tau_2,\tau}_{\tau_1,R})}\left\|\widetilde{r}^{\frac{1}{2}}\not\nabla\sigma\right\|_{L^\infty_uL^\infty_tL^p_\omega(D^{\tau_2,\tau}_{\tau_1,R})}\left\|\widetilde{r}^{m-\frac{1}{2}}\left(\upsilon^{\frac{1}{2}}\varphi\right)\cdot L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^2_uL^1_tL^{Q_\star}_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
\lesssim&\lambda^{-10\varepsilon_0}(\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}C_0\left\|\widetilde{r}^{\frac{m}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^\infty_uL^2_tL^2_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
\lesssim&\lambda^{-10\varepsilon_0}(\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}C_0\sup_{\tau_1-R\le u\le\tau_2-R}\mathfrak{CF}_{m,R}[\varphi](u,\tau)^{\frac{1}{2}}.
\end{aligned} \tag{10.148}$$

Also, by (9.12), (9.20a), (9.25a), we have:

$$\begin{aligned}
&\left|\int_{D^{\tau_2,\tau}_{\tau_1,R}}b\widetilde{r}^m\left\{-\zeta+(\tilde{\zeta}-\not\mu)+\not\mu_{(1)}\right\}\cdot\not\nabla\sigma\cdot\left(\upsilon^{\frac{1}{2}}\varphi\right)\cdot L\left(\upsilon^{\frac{1}{2}}\varphi\right)\mathrm{d}\varpi_{\not g}\mathrm{d}u\mathrm{d}t\right| \\
\lesssim&\left\|\zeta,(\tilde{\zeta}-\not\mu),\not\mu_{(1)}\right\|_{L^2_tL^\infty_uL^\infty_\omega(D^{\tau_2,\tau}_{\tau_1,R})}\left\|\widetilde{r}^{\frac{1}{2}}\not\nabla\sigma\right\|_{L^\infty_uL^\infty_tL^p_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
&\cdot\left\|\widetilde{r}^{m-\frac{1}{2}}\left(\upsilon^{\frac{1}{2}}\varphi\right)\cdot L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^2_tL^1_uL^{Q_\star}_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
\lesssim&\lambda^{-10\varepsilon_0}(\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}C_0\left\|\widetilde{r}^{\frac{m}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^\infty_tL^2_uL^2_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
\lesssim&\lambda^{-10\varepsilon_0}(\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}C_0\sup_{\tau_1\le t\le\tau}\mathscr{E}_{m,U,R}[\varphi](t)^{\frac{1}{2}}.
\end{aligned} \tag{10.149}$$

Moreover, by (9.7b) and (9.21b), we have:

$$\begin{aligned}
&\left|\int_{D^{\tau_2,\tau}_{\tau_1,R}}b\widetilde{r}^m\left(\check{\mu},\mathrm{tr}_{\not g}\chi\mathbf{\Gamma}_{\underline{L}}\right)\cdot\left(\upsilon^{\frac{1}{2}}\varphi\right)\cdot L\left(\upsilon^{\frac{1}{2}}\varphi\right)\mathrm{d}\varpi_{\not g}\mathrm{d}u\mathrm{d}t\right| \\
\lesssim&\left\|\widetilde{r}^{\frac{3}{2}}\left(\check{\mu},\mathrm{tr}_{\not g}\chi\mathbf{\Gamma}_{\underline{L}}\right)\right\|_{L^2_tL^\infty_uL^p_\omega(D^{\tau_2,\tau}_{\tau_1,R})}\left\|\widetilde{r}^{m-\frac{3}{2}}\left(\upsilon^{\frac{1}{2}}\varphi\right)\cdot L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^2_uL^1_tL^{Q_\star}_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
\lesssim&\lambda^{-3\varepsilon_0}(\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}C_0\left\|\widetilde{r}^{\frac{m}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^\infty_uL^2_tL^2_\omega(D^{\tau_2,\tau}_{\tau_1,R})} \\
\lesssim&\lambda^{-3\varepsilon_0}(\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}C_0\sup_{\tau_1-R\le u\le\tau_2-R}\mathfrak{CF}_{m,R}[\varphi](u,\tau)^{\frac{1}{2}}.
\end{aligned} \tag{10.150}$$

Combining (10.146), (10.148), (10.149) and (10.150), we have:

$$\int_{D^{\tau_2,\tau}_{\tau_1,R}} b\widetilde{r}^m\tilde{\varphi}\left\{\left(\frac{1}{2}\mu-\frac{1}{2}\mathrm{tr}_{\not g}\chi k_{NN}+\not\Delta\sigma+|\not\nabla\sigma|^2_{\not g}+2\zeta\cdot\not\nabla\sigma\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right)\right\}\mathrm{d}\varpi_{\tilde{\not g}}\mathrm{d}u\mathrm{d}t \tag{10.151}$$
$$\lesssim\lambda^{-3\varepsilon_0}(\tau_1+1)^{\frac{m}{2}-1}(\ln\lambda)^{\frac{1}{2}}C_0\left(\sup_{\tau_1\le t\le\tau}\mathscr{E}_{m,U,R}[\varphi](t)^{\frac{1}{2}}+\sup_{\tau_1-R\le u\le\tau_2-R}\mathfrak{CF}_{m,R}[\varphi](u,\tau)^{\frac{1}{2}}\right).$$

Combining (10.139), (10.143) and (10.151), we conclude the desired result (10.137). □

Proposition 10.25. *Let $1\le\tau_1\le T_{*;(\lambda)}$, $U=\mathscr{T}^{T_{*;(\lambda)}}_{\tau_1,R}$ (defined in (10.29)), $m=1,2$ with $\Phi_1(\tau)=1$ and $\Phi_2(\tau)=(\log(2+\tau))^{\frac{1}{2}}$. The following holds:*

$$\sup_{\tau_1\le t\le T_{*;(\lambda)}}\left\|\Phi_m^{-1}\widetilde{r}^{\frac{m}{2}}\varphi\right\|_{L^2_uL^2_\omega(U\bigcap\Sigma_t)}+\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|_{L^2_uL^2_tL^2_\omega(U)} \tag{10.152}$$
$$+\sup_{\tau_1\le t\le T_{*;(\lambda)}}\mathscr{E}_{m,U,R}[\varphi](t)+\sup_{\tau_1-R\le u\le\tau_2-R}\mathfrak{CF}_{m,R}[\varphi](u,T_{*;(\lambda)})$$
$$\lesssim\mathfrak{CF}_{m,R}[\varphi](\tau_1-R,T_{*;(\lambda)})+(\tau_1+1)^{m-2}C_0^2+\sup_{\tau_1\le t\le T_{*;(\lambda)}}\mathscr{B}^t_{\tau_1},$$

where $\mathscr{B}^t_{\tau_1}$ is defined in (10.74).

Proof of Proposition 10.25. Let $U=D^{\tau_2,\tau}_{\tau_1,R}$. We apply (10.123) on $U\bigcap\{R\le R'\le 2R\}$:

$$\mathfrak{CF}_{m,R'}[\varphi](\tau_2-R,\tau)+\mathscr{E}_{m,U,R'}[\varphi](\tau)+\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_uL^2_tL^2_\omega(U\bigcap\{\widetilde{r}\ge R'\})} \tag{10.153}$$
$$\lesssim\int_{\tau_1+R'-R}^{\tau_2+R'-R}\int_{S_{t,t-R'}}\widetilde{r}^m\left(|\not\nabla\tilde{\varphi}|^2_{\not g}\upsilon e^{2\sigma}+\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2\right)\mathrm{d}\varpi_{\not g}\mathrm{d}t$$
$$+\mathfrak{CF}_{m,R'}[\varphi](\tau_1-R,\tau)+\mathrm{error}(m,\widetilde{r}\ge R'),$$

where

$$\mathrm{error}(m,\widetilde{r}\ge R') \tag{10.154}$$
$$=2\int_{U\bigcap\{R\le R'\le 2R\}} b\widetilde{r}^m\tilde{\varphi}\left\{\left(\frac{1}{2}\mu-\frac{1}{2}\mathrm{tr}_{\not g}\chi k_{NN}+\not\Delta\sigma+|\not\nabla\sigma|^2_{\not g}+2\zeta\cdot\not\nabla\sigma\right)\left(L\tilde{\varphi}+\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right)-\frac{1}{2}\not\nabla\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\not\nabla\tilde{\varphi}\right\}\mathrm{d}\varpi_{\tilde{\not g}}\mathrm{d}u\mathrm{d}t$$
$$+\int_{U\bigcap\{R\le R'\le 2R\}} b\widetilde{r}^m\mathscr{Q}\left(|\not\nabla\tilde{\varphi}|_{\not g},\left|L\tilde{\varphi},\frac{1}{2}\mathrm{tr}_{\tilde{\not g}}\tilde{\chi}\tilde{\varphi}\right|\right)\left|k,\mathbf{\Gamma}_L,\zeta,\mathrm{tr}_{\not g}\chi-\frac{2}{\widetilde{r}},\hat{\chi},\frac{b-1}{\widetilde{r}}\right|\mathrm{d}\varpi_{\tilde{\not g}}\mathrm{d}u\mathrm{d}t.$$

Integrating (10.153) with respect to R', dividing by R, and using (10.128), we have:

$$\begin{aligned}
&\mathfrak{CF}_{m,2R}[\varphi](\tau_2-R,\tau)+\mathscr{E}_{m,U,2R}[\varphi](\tau)+\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_uL^2_tL^2_\omega(U\bigcap\{\widetilde{r}\geq 2R\})} \\
\lesssim&\frac{1}{R}\int_R^{2R}\int_{\tau_1+R'-R}^{\tau_2+R'-R}\int_{S_{t,t-R'}}\widetilde{r}^m\left(|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma}+\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t\mathrm{d}R' \\
&+\mathfrak{CF}_{m,R}[\varphi](\tau_1-R,\tau)+\operatorname{error}(m,R).
\end{aligned}\tag{10.155}$$

By (10.95), (10.89), and $\widetilde{r}\leq 2R<2$, we have:

$$\begin{aligned}
&\frac{1}{R}\int_R^{2R}\int_{\tau_1+R'-R}^{\tau_2+R'-R}\int_{S_{t,t-R'}}\widetilde{r}^m\left(|\nabla\!\!\!/\tilde{\varphi}|^2\,\upsilon e^{2\sigma}+\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t\mathrm{d}R' \\
&\leq\int_{U\bigcap\{R\leq R'\leq 2R\}}\left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2+|\nabla\!\!\!/\tilde{\varphi}|^2\,\upsilon\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t\lesssim\mathscr{B}^t_{\tau_1}.
\end{aligned}\tag{10.156}$$

Combining (10.155) and (10.156), and using Lemma 10.20, we have:

$$\begin{aligned}
&\mathfrak{CF}_{m,R}[\varphi](\tau_2-R,\tau)+\mathscr{E}_{m,U,2R}[\varphi](\tau)+\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_uL^2_tL^2_\omega(U\bigcap\{\widetilde{r}\geq 2R\})} \\
&\lesssim\mathfrak{CF}_{m,2R}[\varphi](\tau_2-R,\tau)+\mathscr{E}_{m,U,2R}[\varphi](\tau)+\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_uL^2_tL^2_\omega(U\bigcap\{\widetilde{r}\geq 2R\})} \\
&\lesssim\sup_{\tau_1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_{\tau_1}+\mathfrak{CF}_{m,R}[\varphi](\tau_1-R,\tau)+\operatorname{error}(m,R).
\end{aligned}\tag{10.157}$$

Combining (10.157) and (10.137), we have:

$$\begin{aligned}
&\mathfrak{CF}_{m,R}[\varphi](\tau_2-R,\tau)+\mathscr{E}_{m,U,2R}[\varphi](\tau)+\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_uL^2_tL^2_\omega(U\bigcap\{\widetilde{r}\geq R'\})} \\
\lesssim&\lambda^{-3\varepsilon_0}\Bigg\{(\tau_1+1)^{m-2}C_0^2+\sup_{\tau_1\leq t\leq\tau}\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|^2_{L^2_uL^2_\omega(D^{\tau_2,\tau}_{\tau_1,R}\bigcap\Sigma_t)} \\
&+\sup_{\tau_1\leq t\leq\tau}\mathscr{E}_{m,U,R}[\varphi](t)+\sup_{\tau_1-R\leq u\leq\tau_2-R}\mathfrak{CF}_{m,R}[\varphi](u,\tau)\Bigg\} \\
&+\sup_{\tau_1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_{\tau_1}+\mathfrak{CF}_{m,R}[\varphi](\tau_1-R,T_{*;(\lambda)}).
\end{aligned}\tag{10.158}$$

By (10.93), (10.95), and (10.89), we have:

$$\left\|\widetilde{r}^{\frac{m}{2}}\varphi\right\|^2_{L^2_uL^2_\omega(D^{\tau_2,\tau}_{\tau_1,R}\bigcap\Sigma_t)}\lesssim\lambda^{-\varepsilon_0}\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_uL^2_tL^2_\omega(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_1,R})},\tag{10.159}$$

$$\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_uL^2_tL^2_\omega(U\bigcap\{R\leq\widetilde{r}\leq 2R\})}\lesssim\lambda^{-\varepsilon_0}\sup_{\tau_1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_{\tau_1}.\tag{10.160}$$

By Lemma 10.20, we have:

$$\begin{aligned}&\mathfrak{C}\mathfrak{F}_{m,R}[\varphi](\tau_2-R,\tau)+\mathscr{E}_{m,U,R}[\varphi](\tau)+\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_uL^2_tL^2_\omega(U\bigcap\{\widetilde{r}\geq R'\})} \\ \lesssim&\sup_{\tau_1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_{\tau_1}+(\tau_1+1)^{m-2}C_0^2+\mathfrak{C}\mathfrak{F}_{m,R}[\varphi](\tau_1-R,T_{*;(\lambda)}) \\ &+\lambda^{-2\varepsilon_0}\left\{\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_tL^2_uL^2_\omega(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_1,R})}\right. \\ &\left.+\sup_{\tau_1\leq t\leq\tau}\mathscr{E}_{m,U,R}[\varphi](t)+\sup_{\tau_1-R\leq u\leq\tau_2-R}\mathfrak{C}\mathfrak{F}_{m,R}[\varphi](u,\tau)\right\}.\end{aligned} \tag{10.161}$$

Taking the sup over t in (10.161) and absorbing the last 3 terms on the right hand side back into the left (for λ large), we deduce:

$$\begin{aligned}&\left\|\widetilde{r}^{\frac{m-1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_tL^2_uL^2_\omega(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_1,R})}+\sup_{\tau_1\leq t\leq\tau}\mathscr{E}_{m,U,R}[\varphi](t)+\sup_{\tau_1-R\leq u\leq\tau_2-R}\mathfrak{C}\mathfrak{F}_{m,R}[\varphi](u,\tau) \\ \lesssim&\sup_{\tau_1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_{\tau_1}+(\tau_1+1)^{m-2}C_0^2+\mathfrak{C}\mathfrak{F}_{m,R}[\varphi](\tau_1-R,T_{*;(\lambda)}).\end{aligned} \tag{10.162}$$

Combining (10.162) and (10.93), we conclude the desired result (10.152). □

Corollary 10.26. *Under the assumptions of Proposition 10.25, the following estimate holds:*

$$\mathrm{error}(m,R)\lesssim\lambda^{-2\varepsilon_0}\left\{(\tau_1+1)^{m-2}C_0^2+\mathfrak{C}\mathfrak{F}_{m,R}[\varphi](\tau_1-R,T_{*;(\lambda)})+\sup_{\tau_1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_{\tau_1}\right\}. \tag{10.163}$$

Proof. (10.163) follows directly from (10.137) and (10.152). □

Proposition 10.27. *The following estimate holds:*

$$\int_{\mathscr{T}^{T_{*;(\lambda)}}_{1,R}}\widetilde{r}\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2\mathrm{d}\varpi_{\not{g}}\mathrm{d}u\mathrm{d}t\lesssim C_0^2. \tag{10.164}$$

Proof of Proposition 10.27. We consider (10.157) with $m=2,\tau_1=1,\tau_2=\tau=T_{*;(\lambda)}$. By (10.163), we have:

$$\left\|\widetilde{r}^{\frac{1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_tL^2_uL^2_\omega(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_1,R}\bigcap\{\widetilde{r}\geq 2R\})}\lesssim C_0^2+\sup_{1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_1+\mathfrak{C}\mathfrak{F}_{2,R}[\varphi](\tau_1-R,T_{*;(\lambda)}). \tag{10.165}$$

The last term is 0 since $\varphi[1]$ supported with B_R and finite speed propagation property of wave. $\sup_{1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_1\lesssim C_0^2$ by its definition. Therefore, we have

$$\left\|\widetilde{r}^{\frac{1}{2}}L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right\|^2_{L^2_tL^2_uL^2_\omega(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_1,R}\bigcap\{\widetilde{r}\geq 2R\})}\lesssim C_0^2. \tag{10.166}$$

Combining (10.160) and (10.166), we conclude the desired result. □

10.6 Decay Estimates for the Conformal Energy

In this subsection, we prove a decay estimate for the energy $\int_{\Sigma_{\tau,\tau_1-R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha d\varpi_g$. We first derive estimates for $\mathscr{B}^t_\tau$.

Proposition 10.28. *There is a sequence $\{\tau_n\}_{n\geq 0}\subset[1,T_{*;(\lambda)}]$ satisfying $c_1\tau_n\leq\tau_{n+1}\leq c_2\tau_n$ for some $c_2>c_1>0$ such that*

$$\int_{\tau_n}^{T_{*;(\lambda)}}\int_{S_{t,\tau_n-R}}\widetilde{r}\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2 d\varpi_{\not g}dt+R^2\int_{S_{\tau_n,\tau_n-R}}\varphi^2 d\varpi_{\not g}\lesssim(1+\tau_n)^{-1}C_0^2. \tag{10.167}$$

Moreover, for $\tau_n\leq\tau\leq T_{*;(\lambda)}$, we have:

$$\int_{\tau_n}^{\tau}\int_{S_{t,\tau_n-R}}\widetilde{r}\varphi^2 d\varpi_{\not g}dt\lesssim[\ln(\tau-\tau_n+R)]^2(1+\tau_n)^{-1}C_0^2. \tag{10.168}$$

Proof of Proposition 10.28. By Proposition 10.27, Lemma 10.16 and $\sup_{1\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_1\lesssim C_0^2$, we have:

$$\int_{\mathscr{T}^{T_{*;(\lambda)}}_{1,R}}\widetilde{r}\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2 d\varpi_{\not g}dudt+R^2\int_1^{T_{*;(\lambda)}}\int_{S_{t,t-R}}\varphi^2 d\varpi_{\not g}dt\leq CC_0^2, \tag{10.169}$$

for some constant C. Suppose (10.167) does not hold, that is, for any pair of constant c_1,c_2 and any sequence sequence $\{\tau_n\}_{n\geq 0}\subset[1,T_{*;(\lambda)}]$ satisfying $c_1\tau_n\leq\tau_{n+1}\leq c_2\tau_n$, there is $\tau_i\in\{\tau_n\}_{n\geq 0}$ such that $[c_1\tau_i,c_2\tau_i]\subset[1,T_{*;(\lambda)}]$ and

$$\begin{aligned}&\int_{c_1\tau_i}^{c_2\tau_i}\left\{\int_{\tau}^{T_{*;(\lambda)}}\int_{S_{t,\tau_i-R}}\widetilde{r}\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2 d\varpi_{\not g}dt+R^2\int_{S_{\tau_i,\tau_i-R}}\varphi^2 d\varpi_{\not g}\right\}d\tau\\&\geq C'\int_{c_1\tau_i}^{c_2\tau_i}(1+\tau)^{-1}C_0^2 d\tau\\&\geq C'\ln\frac{c_2}{c_1}C_0^2.\end{aligned} \tag{10.170}$$

Pick c_1,c_2 such that $C'\ln\frac{c_2}{c_1}>C$, then (10.170) contradict to (10.169). We have proved (10.167). (10.168) is a direct result of (10.78) and (10.167). □

Corollary 10.29. *Let sequence $\{\tau_n\}_{n\geq 0}\subset[1,T_{*;(\lambda)}]$ be as in Proposition 10.28. We have:*

$$\sup_{\tau_n\leq t\leq T_{*;(\lambda)}}\mathscr{B}^t_{\tau_n}\lesssim(1+\tau_n)^{-1}C_0^2+\int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha d\varpi_{\mathbf{g}}. \tag{10.171}$$

Proof. (10.171) follows directly from the definition of $\mathscr{B}^t_\tau$ and (10.167). □

Now we prove the decay estimate for $\int_{\Sigma_{\tau,\tau_1-R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha d\varpi_g$.

Proposition 10.30 (Decay estimates for energy)**.** *Let φ be any solution of $\Box_{\mathbf{g}}\varphi = 0$ with $\varphi[1]$ supported on $B_R \subset \mathscr{M}^{(Int)} \bigcap \{t = 1\}$ (B_R is defined in Theorem 6.9). Then for $1 \le \tau \le T_{*;(\lambda)}$, we have the following:*

$$\int_{\Sigma^{T_{*;(\lambda)}}_{\tau,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha d\varpi_{\mathbf{g}} \lesssim (1+\tau)^{-2}C_0^2. \tag{10.172}$$

Proof of Proposition 10.30. Let $\{\tau_n\}_{n\ge 0} \subset [1, T_{*;(\lambda)}]$ be as in Proposition 10.28. For $\tau_{n+1} \in [1, T_{*;(\lambda)} - R]$, we apply (10.123) with $m = 1$ on $D^{\tau_{n+1},T_{*;(\lambda)}}_{\tau_n,R} \bigcap\{\widetilde{r} \ge R'\}$ with $R \le R' \le 2R$ to deduce:

$$\begin{aligned}
&\int_{D^{\tau_{n+1},T_{*;(\lambda)}}_{\tau_n,R} \bigcap\{\widetilde{r}\ge R'\}} b\left\{\left|L\left(\upsilon^{\frac12}\varphi\right)\right|^2 + |\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma}\right\} d\varpi_{g\!\!\!/} du dt \\
\le &\int_{\tau_n+R'-R}^{\tau_{n+1}+R'-R}\int_{S_{t,t-R'}} \widetilde{r}^m\left(b\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma} - (2-b)\left|L(\upsilon^{\frac12}\varphi)\right|^2\right) d\varpi_{g\!\!\!/} dt \\
&+ 2\int_{\tau_n+R'-R}^{T_{*;(\lambda)}}\int_{S_{t,\tau_n-R}} \widetilde{r}\left|L\left(\upsilon^{\frac12}\varphi\right)\right|^2 d\varpi_{g\!\!\!/} dt + \mathrm{error}(1,R).
\end{aligned} \tag{10.173}$$

By (10.163), (10.167) and (10.171), we have:

$$\mathrm{error}(1,R) \lesssim \lambda^{-2\varepsilon_0}\left\{(\tau_n+1)^{-1}C_0^2 + \sup_{\tau_1\le t\le T_{*;(\lambda)}} \mathscr{B}^t_{\tau_n}\right\} \lesssim (\tau_n+1)^{-1}C_0^2 + \int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha d\varpi_{\mathbf{g}}. \tag{10.174}$$

Also, by (10.95), (10.89) and (10.171), we have:

$$\begin{aligned}
&\int_R^{2R}\int_{\tau_n+R'-R}^{\tau_{n+1}+R'-R}\int_{S_{t,t-R'}} \widetilde{r}^m\left(b\,|\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma} - (2-b)\left|L(\upsilon^{\frac12}\varphi)\right|^2\right) d\varpi_{g\!\!\!/} dt dR' \\
\lesssim &(\tau_n+1)^{-1}C_0^2 + \int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha d\varpi_{\mathbf{g}}.
\end{aligned} \tag{10.175}$$

We now integrate (10.173) with respect to R' for $R \le R' \le 2R$. Notice that the left hand side is decreasing and the second term on the right hand side is bounded by energy. By (10.174) and (10.175) we have:

$$\begin{aligned}
&\int_{D^{\tau_{n+1},T_{*;(\lambda)}}_{\tau_n,R} \bigcap\{\widetilde{r}\ge 2R\}} b\left\{\left|L\left(\upsilon^{\frac12}\varphi\right)\right|^2 + |\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}\,\upsilon e^{2\sigma}\right\} d\varpi_{g\!\!\!/} du dt \\
\lesssim &(\tau_n+1)^{-1}C_0^2 + \int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha d\varpi_{\mathbf{g}}.
\end{aligned} \tag{10.176}$$

Combining (10.176), (10.95), (10.89) and (10.171), we have:

$$\begin{aligned}&\int_{D_{\tau_n,R}^{\tau_{n+1},T_{*;(\lambda)}}} b\left\{\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2+|\nabla\!\!\!/\tilde{\varphi}|_{g\!\!\!/}^2\,\upsilon e^{2\sigma}\right\}\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u\mathrm{d}t\\&\lesssim(\tau_n+1)^{-1}C_0^2+\int_{\Sigma_{\tau_n,R}^{T_{*;(\lambda)}}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}}.\end{aligned}\tag{10.177}$$

By (10.89), we have:

$$\begin{aligned}\int_{\tau_n}^{\tau_{n+1}}\int_{\Sigma_{\tau,R}^{T_{*;(\lambda)}}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}}\mathrm{d}\tau&=\int_{\tau_n}^{\tau_{n+1}}\int_{\Sigma_\tau\bigcap\{\tilde{r}\le R\}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{g}\mathrm{d}\tau\\&\quad+\int_{\tau_n}^{\tau_{n+1}}\int_{\mathscr{C}_{\tau-R}\bigcap\{\widetilde{r}\ge R\}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}}\mathrm{d}\tau\\&\lesssim\sup_{\tau_1\le t\le T_{*;(\lambda)}}\mathscr{B}^t_{\tau_n}\\&\quad+\int_{\tau_n}^{\tau_{n+1}}\int_{\mathscr{C}_{\tau-R}\bigcap\{\widetilde{r}\ge R\}}b\upsilon\left(|L\varphi|^2+|\nabla\!\!\!/\varphi|^2_{g\!\!\!/}\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t\mathrm{d}\tau.\end{aligned}\tag{10.178}$$

Now we consider the last term of (10.178). First we have:

$$\begin{aligned}&b\left\{\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2+|\nabla\!\!\!/\tilde{\varphi}|_{g\!\!\!/}^2\,\upsilon e^{2\sigma}\right\}=b\left(|L\varphi|^2+\upsilon\,|\nabla\!\!\!/\varphi|^2_{g\!\!\!/}\right)+\frac{1}{2}L\left(\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}b\upsilon\varphi^2\right)\\&+b\left\{-\frac{1}{2}L\left(\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}\upsilon\right)+\left(\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\right)^2\upsilon-\frac{1}{2}L(\ln b)\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}\upsilon+|\nabla\!\!\!/\sigma|^2_{g\!\!\!/}\,\upsilon\right\}\varphi^2\\&+b\upsilon\left\{\mathbf{\Gamma}_L\varphi L\varphi+2\varphi\nabla\!\!\!/\varphi\cdot\nabla\!\!\!/\sigma\right\},\end{aligned}\tag{10.179}$$

where we write the last two lines of (10.179) schematically as follows:

$$\begin{aligned}&b\left\{-\frac{1}{2}L\left(\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}\upsilon\right)+\left(\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\right)^2\upsilon-\frac{1}{2}L(\ln b)\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}\upsilon+|\nabla\!\!\!/\sigma|^2_{g\!\!\!/}\,\upsilon\right\}\varphi^2\\&+b\upsilon\left\{\mathbf{\Gamma}_L\varphi L\varphi+2\varphi\nabla\!\!\!/\varphi\cdot\nabla\!\!\!/\sigma\right\}\\&\lesssim\upsilon\varphi^2\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi},\mathrm{tr}_{\tilde{g\!\!\!/}}\tilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right)+\upsilon\varphi^2\,|\nabla\!\!\!/\sigma|^2_{g\!\!\!/}+\upsilon\left\{\mathbf{\Gamma}_L\varphi L\varphi+2\varphi\nabla\!\!\!/\varphi\cdot\nabla\!\!\!/\sigma\right\}.\end{aligned}\tag{10.180}$$

Let $\frac{1}{Q}+\frac{1}{Q'}=\frac{1}{2}$. To handle the first two terms on the right hand side of (10.180), by (9.7b), (9.20a), bootstrap assumptions (6.13) and (10.134), we have:

$$\begin{aligned}&\int_{\tau_n}^{\tau_{n+1}}\int_{\mathscr{C}_{\tau-R}\bigcap\{\widetilde{r}\geq R\}}\upsilon\varphi^2\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right)+\upsilon\varphi^2\left|\nabla\!\!\!/\sigma\right|_{g\!\!\!/}^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t\mathrm{d}\tau \qquad (10.181)\\&\lesssim\lambda^{\frac{1}{2}-4\varepsilon_0}\|\varphi\|^2_{L_t^2L_u^2L_\omega^Q(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_n,R})}\left\|\widetilde{r}^{\frac{3}{2}}\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(\mathrm{Small})},\widetilde{r}^{-1}\right)\cdot\left(\boldsymbol{\partial}\vec{\Psi},\hat{\chi}\right),\left|\nabla\!\!\!/\sigma\right|_{g\!\!\!/}^2\right\|_{L_t^\infty L_u^\infty L_\omega^{\frac{Q'}{2}}(\mathscr{M}^{(\mathrm{Int})})}\\&\lesssim\lambda^{-6\varepsilon_0}(\tau_n+1)^{-1}C_0^2.\end{aligned}$$

Also, by (9.20a), (10.133) and (10.134), we have:

$$\begin{aligned}&\int_{\tau_n}^{\tau_{n+1}}\int_{\mathscr{C}_{\tau-R}\bigcap\{\widetilde{r}\geq R\}}\upsilon\left|\varphi\nabla\!\!\!/\varphi\cdot\nabla\!\!\!/\sigma\right|\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t\mathrm{d}\tau \qquad (10.182)\\&\lesssim\|\varphi\|_{L_t^2L_u^2L_\omega^Q(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_n,R})}\|\nabla\!\!\!/\varphi\|_{L_t^2L_u^2L_{g\!\!\!/}^2(\mathscr{T}^{T_{*;(\lambda)}}_{\tau_n,R})}\|\widetilde{r}\nabla\!\!\!/\sigma\|_{L_t^\infty L_u^\infty L_\omega^{Q'}(\mathscr{M}^{(\mathrm{Int})})}\\&\lesssim\lambda^{-2\varepsilon_0}(\tau_n+1)^{-1}C_0^2.\end{aligned}$$

Moreover, by bootstrap assumptions (9.2), (10.6), we have

$$\begin{aligned}&\int_{\tau_n}^{\tau_{n+1}}\int_{\mathscr{C}_{\tau-R}\bigcap\{\widetilde{r}\geq R\}}\upsilon\left|\varphi L\varphi\boldsymbol{\Gamma}_L\right|\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t\mathrm{d}\tau \qquad (10.183)\\&\lesssim\sup_{\tau_n\leq t\leq\tau_{n+1}}\|\varphi\|_{L_u^2L_\omega^2(\Sigma_t)}\left\|\widetilde{r}^2L\varphi\right\|_{L_t^\infty L_u^2L_\omega^2(\mathscr{M}^{(\mathrm{Int})})}\left\|\boldsymbol{\partial}\vec{\Psi}\right\|_{L_t^1L_x^\infty(\mathscr{M})}\\&\lesssim\lambda^{-5\varepsilon_0}(\tau_n+1)^{-1}C_0^2.\end{aligned}$$

Integrating (10.179) over $\mathscr{C}_{\tau-R}\bigcap\{\widetilde{r}\geq R\}$ and combining it with (10.177), (10.178), (10.181), (10.182) and (10.183), we have:

$$\begin{aligned}\int_{\tau_n}^{\tau_{n+1}}\int_{\Sigma^{T_{*;(\lambda)}}_{\tau,R}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}}\mathrm{d}\tau&\lesssim\int_{\tau_n}^{\tau_{n+1}}\int_{S_{\tau,\tau-R}}b\upsilon\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}\varphi\mathrm{d}\varpi_g\mathrm{d}\tau \qquad (10.184)\\&\quad+(\tau_n+1)^{-1}C_0^2+\int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}}\\&\lesssim(\tau_n+1)^{-1}C_0^2+\int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}},\end{aligned}$$

where by (10.90) and (10.171),

$$\int_{\tau_n}^{\tau_{n+1}}\int_{S_{\tau,\tau-R}}b\upsilon\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}\varphi\mathrm{d}\varpi_g\mathrm{d}\tau\lesssim(\tau_n+1)^{-1}C_0^2+\int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}}{}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha\mathrm{d}\varpi_{\mathbf{g}}. \qquad (10.185)$$

By (10.33), we have:

$$(\tau_{n+1}-\tau_n)\int_{\Sigma^{T_{*;(\lambda)}}_{\tau_{n+1},R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} \lesssim \int_{\tau_n}^{\tau_{n+1}}\int_{\Sigma^{T_{*;(\lambda)}}_{\tau,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}}\mathrm{d}\tau$$
$$\lesssim (\tau_n+1)^{-1}C_0^2 + \int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}}. \tag{10.186}$$

For any $\tau \in [1, T_{*;(\lambda)}]$, there is τ_n such that $c_1\tau_n \leq \tau \leq c_2\tau_n$. Then by (10.33) again, we have:

$$\int_{\Sigma^{T_{*;(\lambda)}}_{\tau_n,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_{\mathbf{g}} \lesssim (\tau_n+1)^{-2}C_0^2 \lesssim (\tau+1)^{-2}C_0^2. \tag{10.187}$$

□

10.7 Proof of the Boundness Theorem of the Conformal Energy

In this subsection, we prove the boundness theorem of the conformal energy, that is, Theorem 10.3. First we show (10.8b). By (10.172) and (10.32), we have:

$$\int_{\Sigma_t\bigcap\{u\geq \frac{t}{2}\}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g \lesssim \int_{\Sigma^t_{t/2+R,R}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g \lesssim (1+t)^{-2}C_0^2. \tag{10.188}$$

(10.8a) follows from (10.8b) in $\mathscr{M}^{(\mathrm{Int})}\bigcap\{u\geq\frac{t}{2}\}$. For $\mathscr{M}^{(\mathrm{Int})}\bigcap\{u\leq\frac{3}{4}t\}$, we use (10.93) and (10.152) with $m=2$ to deduce:

$$\int_{\Sigma_t\bigcap\{u\leq\frac{3}{4}t\}} \widetilde{r}^2\left(\left|L(\upsilon^{\frac{1}{2}}\varphi)\right|^2 + |\nabla\!\!\!/\tilde{\varphi}|^2_{g\!\!\!/}e^{2\sigma}\upsilon\right)\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}u \lesssim C_0^2, \tag{10.189}$$

$$\|\varphi\|_{L^2_uL^2_\omega(\Sigma_t\bigcap\{u\leq\frac{3}{4}t\})} \lesssim (t+1)^{-1+\varepsilon_0}\|\widetilde{r}\varphi\|_{L^2_uL^2_\omega(\Sigma_t\bigcap\{u\leq\frac{3}{4}t\})} \lesssim (t+1)^{-1+\varepsilon_0}C_0. \tag{10.190}$$

Therefore (10.8b) holds. It remains for us to prove (10.8c) in $\Sigma_t\bigcap\{u\geq\frac{t}{2}\}$. Let the sequence $\{\tau_n\}_{n\geq 0}\subset [1, T_{*;(\lambda)}]$ be as in Proposition 10.28. There is τ_n such that $c_1\tau_n\leq\frac{t}{2}\leq c_2\tau_n$. Let $u_n=\tau_n-R$. By (10.39), we have:

$$\|\varphi\|^2_{L^2_uL^2_\omega(\Sigma_t\bigcap\{u\geq u_n\})} \lesssim \|\widetilde{r}N\varphi\|^2_{L^2_uL^2_\omega(\Sigma_t\bigcap\{u\geq u_n\})} + \left\|\widetilde{r}^{-\frac{1}{2}}\varphi\right\|^2_{L^2_{g\!\!\!/}(S_{t,u_n})}, \tag{10.191}$$

where by integrating $L\left(\upsilon^{\frac{1}{2}}\varphi\right)$ along $\mathscr{C}_{u_n}$, we have:

$$\left\|\widetilde{r}^{-\frac{1}{2}}\varphi\right\|^2_{L^2_{g\!\!\!/}(S_{t,u_n})} \lesssim R^2\int_{S_{\tau_n,u_n}}\varphi^2\mathrm{d}\varpi_{g\!\!\!/} + \ln(t-\tau_n+R)\int_{\tau_n}^t\int_{S_{t',u_n}}\widetilde{r}\left|L\left(\upsilon^{\frac{1}{2}}\varphi\right)\right|^2\mathrm{d}\varpi_{g\!\!\!/}\mathrm{d}t'. \tag{10.192}$$

Combining (10.167), (10.191) and (10.192), we have:

$$\|\varphi\|^2_{L^2_u L^2_\omega(\Sigma_t \bigcap \{u \geq u_n\})} \lesssim (1+t)^{-2+2\varepsilon_0} C_0^2 + \int_{\Sigma_{\tau_n}} {}^{(\mathbf{T})}\mathbf{J}_\alpha[\varphi]\mathbf{n}^\alpha \mathrm{d}\varpi_g \lesssim (1+t)^{-2+2\varepsilon_0} C_0^0. \tag{10.193}$$

The desired result follows since $\{u \geq \frac{t}{2}\} \subset \{u \geq u_n\}$.

Appendices

Notations

In this appendix, we gather some notation that we use throughout the article.

Symbol	Reference
$\eta, \eta^{-1}, \varepsilon_{\gamma\delta\kappa\lambda}, \Sigma_t, \partial, \boldsymbol{\partial}$	Section 2.1
$v, \mathfrak{p}, n, s, \theta, H$	Subsection 2.2.1
$\mathrm{vort}^{\alpha}(\cdot)$	Definition 2.2
ω	Definition 2.3
h	Definition 2.4
q	Definition 2.5
S	Definition 2.6
c	Subsection 2.2.4
$\mathscr{C}, \mathscr{D}$	Definition 2.8
$\mathbf{g}_{Acou}, \mathbf{g}_{Acou}^{-1}$	Definition 2.9
$\mathbf{g}, \mathbf{g}^{-1}$	Definition 2.10
$\mathbf{T}$	Definition 2.11
g, g^{-1}	Definition 2.12
$\mathbf{D}, \square_{\mathbf{g}}$	Definition 2.14
$\vec{v}, \vec{\omega}, \vec{S}, \vec{\mathscr{C}}, \vec{\Psi}, \mathbf{B}$	Definition 2.15
$\mathscr{L}, \mathscr{Q}$	Subsection 2.4.1
$\mathfrak{F}\mathscr{C}^{\alpha}, \mathfrak{F}\mathscr{D}$	Proposition 2.17
$\mathfrak{F}\Psi$	Proposition 2.20
$\nu, P_{\nu}, P_I, P_{\leq \nu}$	Section 3.2
$\mathfrak{R}\boldsymbol{\partial}\Psi, \mathfrak{R}\partial\mathscr{C}^{\alpha}, \mathfrak{R}\partial\mathscr{D}$	Lemma 3.3
$q, \varepsilon_0, \delta_0, \delta, \delta_1$	Section 3.4
$\mathscr{R}, \breve{\mathscr{R}}$	Section 3.5

Symbol	Reference
T_*	Section 3.6
$Q, {}^{(X)}\mathbf{J}, {}^{(X)}\boldsymbol{\pi}, \mathbb{E}, d\varpi_g$	Definition 5.3
λ	Section 6.1
$T_{*;(\lambda)}$	Section 6.2
$u, \mathscr{C}_u, S_{t,u}, \mathscr{M}^{(\mathrm{Int})}, \mathscr{M}^{(\mathrm{Ext})}, \mathscr{M}$	Section 7.1
$\widetilde{r}, w_*$	Definition 7.2
$L_{(Geo)}, b, N, L, \underline{L}, g\!\!\!/, e_A, \omega^A$	Definition 7.3
$\Pi\!\!\!/, \lvert\xi\rvert_{g\!\!\!/}, \mathrm{tr}_{g\!\!\!/}\xi, \hat{\xi}$	Definition 7.4
$\mathscr{F}_{(wave)}, \mathscr{F}_{(transport)}, d\varpi_{g\!\!\!/}$	Definition 8.1
$\mathfrak{C}, \mathfrak{C}^{(i)}, \mathfrak{C}^{(e)}$	Definition 10.1
$\mathbf{D}\!\!\!/, \nabla\!\!\!\!/, \Delta\!\!\!\!/, \mathbf{Ric}, \mathbf{Riem}$	Section 8.2.1
$k, \theta, \chi, \underline{\chi}, \underline{\zeta}, \underline{\zeta}, \mathscr{L}, \mathscr{L}\!\!\!/$	Definition 8.5
$\sigma, \boldsymbol{\Gamma}_L, \widetilde{\mathbf{g}}, \tilde{\chi}, \underline{\tilde{\chi}}, \mathrm{tr}_{\tilde{g\!\!\!/}}, \tilde{\chi}^{(\mathrm{Small})}$	Definition 8.6
μ	Proposition 8.10
$\phi\!\!\!/$	Definition 8.11
$a, w, \frac{\partial}{\partial w}, \frac{\partial}{\partial \omega^A}, \upsilon$	Proposition 8.12
$\mathrm{f}_{(\vec{L})}$	Lemma 8.15
$\lvert S_{t,u}\rvert_{g\!\!\!/}, \overline{f}$	Definition 8.7
$\breve{\mu}, \mu\!\!\!/, \tilde{\zeta}$	Definition 8.54
$\boldsymbol{\Theta}_{(A)}$	Proposition 8.20
p	Subsection 9.1.1

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