Von Neumann Equivalence

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To my
parents and brother,
for never ending support

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Chapter 1

Introduction

Two infinite countable discrete groups Γ and Λ are *measure equivalent* if there is a σ-finite measure space (Ω, m) with a measurable, measure-preserving action of $\Gamma \times \Lambda$, so that both the actions $\Gamma \curvearrowright (\Omega, m)$ and $\Lambda \curvearrowright (\Omega, m)$ admit finite-measure fundamental domains $Y, X \subset \Omega$:

$$\Omega = \bigsqcup_{\gamma \in \Gamma} \gamma Y = \bigsqcup_{\lambda \in \Lambda} \lambda X.$$

This notion was introduced by Gromov in [Gro93, 0.5.E] in analogy with the topological notion of quasi-isometry for finitely generated groups and is fundamental in modern ergodic theory, especially to the study of measured group theory and orbit equivalence. The basic example of measure equivalent groups is when Γ and Λ are lattices in the same locally compact group G. In this case, Γ and Λ act on the left and right of G respectively, and these actions preserve the Haar measure on G.

For certain classes of groups, measure equivalence can be quite a coarse equivalence relation. For instance, the class of countable amenable groups splits into two measure equivalence classes, those that are finite, and those that are countably infinite [Dye59, Dye63, OW80]. Amenability is preserved under measure equivalence, as are other (non)-approximation type properties such as the Haagerup property or property (T). Outside the realm of amenable groups there are a number of powerful invariants to distinguish measure equivalence classes (for example, Gaboriau's celebrated result that states that measure equivalent groups have proportional ℓ^2-Betti numbers [Gab00]) and there are a number of striking rigidity results, such as Furman's work in [Fur99a, Fur99b] where he builds on the superrigidity results of Margulis [Mar75] and Zimmer [Zim84], or Kida's work in [Kid10, Kid11] where he considers measure equivalence for mapping class groups, or for classes of amalgamated free product groups.

If $\Gamma \curvearrowright (X, \mu)$ is a free probability measure-preserving action on a standard measure space, then associated to the action is its orbit equivalence relation, where equivalence classes are defined to be the orbits of the action. If $\Lambda \curvearrowright (Y, \nu)$ is another free probability measure-preserving action, then the actions are orbit equivalent if there is an isomorphism $\theta : X \to Y$ of measure spaces that preserves the orbit equivalence relations, i.e., $\theta(\Gamma \cdot x) = \Lambda \cdot \theta(x)$, for each $x \in X$. If $E \subset X$ is a positive measure subset, then one can also consider the restriction of the orbit equivalence relation to E. The two actions are stably orbit equivalent if there exist positive measure subsets $E \subset X$ and $F \subset Y$ such that the restricted equivalence relations are measurably isomorphic. A fundamental result in the study of measure equivalence is that two groups are measure equivalent if and only if they admit free probability measure-preserving actions that are stably orbit equivalent [Fur99a, Section 3] [Gab05, $P_{\rm ME}5$]. Moreover, in this case one can take the actions to be ergodic.

Also associated to each probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is the Murray-von Neumann crossed product von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma$ [MVN36]. This is the von Neumann subalgebra of $\mathcal{B}(L^2(X, \mu) \overline{\otimes} \ell^2\Gamma)$ that is generated by a copy of $L^\infty(X, \mu)$ acting on $L^2(X, \mu)$ by pointwise multiplication, together with a copy of the group Γ acting diagonally by $\sigma_\gamma \otimes \lambda_\gamma$, where σ_γ is the Koopman representation $\sigma_\gamma(f) = f \circ \gamma^{-1}$ and λ_γ is the left regular representation. The crossed product $L^\infty(X, \mu) \rtimes \Gamma$ is a finite von Neumann algebra with a normal faithful trace given by the vector state corresponding to $1 \otimes \delta_e \in L^2(X, \mu) \overline{\otimes} \ell^2\Gamma$, and if the action is free then this will be a factor if and only if the action is also ergodic, in which case $L^\infty(X, \mu)$ is a Cartan subalgebra of the crossed product. Non-free actions are also of interest in this setting. In particular, in the case when (X, μ) is trivial, this gives the group von Neumann algebra $L\Gamma$, which is a factor if and only if Γ is ICC, i.e., every non-trivial conjugacy class in Γ is infinite [MvN43].

A celebrated result of Singer shows that two free ergodic probability measure-preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are stably orbit equivalent if and only if their von Neumann crossed products are stably isomorphic in a way that preserves the Cartan subalgebras [Sin55]. Specifically, Singer showed that if $E \subset X$ and $F \subset Y$ are positive measure subsets and $\theta : E \to F$ is a measure space isomorphism, then θ preserves the orbit structure almost everywhere if and only if there exists an isomorphism of von Neumann algebras

$$\tilde{\theta} : 1_F(L^\infty(Y,\nu) \rtimes \Lambda)1_F \to 1_E(L^\infty(X,\mu) \rtimes \Gamma)1_E$$

such that $\tilde{\theta}(f) = f \circ \theta$ for all $f \in L^\infty(F, \nu_{|F})$.

Singer's result shows that the study of measure equivalence is closely connected to the study of finite von Neumann algebras, and there have been a number of instances where techniques from one field have been used to settle long-standing problems in the other. This exchange of ideas has especially thrived since the development of Popa's deformation/rigidity theory; see for instance [Pop06a, Pop06b, Pop06c, Pop07a, Pop08], or the survey papers [Pop07b, Vae06, Vae10, Ioa13, Ioa18], and the references therein.

Two groups Γ and Λ are W^**-equivalent* if they have isomorphic group von Neumann algebras, i.e., $L\Gamma \cong L\Lambda$. This is somewhat analogous to measure equivalence (although a closer analogy is made between measure equivalence and virtual W^*-equivalence, which for ICC groups asks for $L\Gamma$ and $L\Lambda$ to be virtually isomorphic in the sense that each factor is stably isomorphic to a finite index subfactor in the other factor [Pop86, Section 1.4]) and both equivalence relations preserve many of the same "approximation type" properties. These similarities led Shlyakhtenko to ask whether measure equivalence implied W^*-equivalence in the setting of ICC groups. It was shown in [CI11] that this is not the case, although the converse implication of whether W^*-equivalence implies measure equivalence is still open.

As with measure equivalence, we have a single W^*-equivalence class of ICC countably infinite amenable groups [Con76], which shows that W^*-equivalence is quite coarse. Yet there do exist countable ICC groups that are not W^*-equivalent to any other non-isomorphic group [IPV13, BV14, Ber15, CI18].

1.1 Measure equivalence and non-commutativity

Returning to discuss measure equivalence, if Γ and Λ have commuting actions on (Ω, m) and if $F \subset \Omega$ is a Borel fundamental domain for the action of Γ, then on the level of function spaces, the characteristic function 1_F gives a projection in $L^\infty(\Omega, m)$ such that the collection $\{1_{\gamma F}\}_{\gamma \in \Gamma}$ forms a partition of unity, i.e., $\sum_{\gamma \in \Gamma} 1_{\gamma F} = 1$. This notion generalizes quite nicely to the non-commutative setting where we will say that a *fundamental domain* for an action on a von Neumann algebra $\Gamma \curvearrowright^\sigma \mathcal{M}$ is a projection $p \in \mathcal{M}$ such that $\sum_{\gamma \in \Gamma} \sigma_\gamma(p) = 1$, where the convergence is in the strong operator topology.

Using this perspective for a fundamental domain we may then generalize the notion of measure equivalence by simply considering actions on non-commutative spaces.

Definition 1.1. Two groups Γ and Λ are *von Neumann equivalent*, written $\Gamma \sim_{vNE} \Lambda$, if there exists a von Neumann algebra $\mathcal{M}$ with a semi-finite normal faithful trace Tr and commuting, trace-preserving, actions of Γ and Λ on $\mathcal{M}$ such that the Γ and Λ-actions individually admit a finite-trace fundamental domain.

The proof of transitivity for measure equivalence is adapted in Proposition 3.9 below to show that von Neumann equivalence is a transitive relation. It is also clearly reflexive and symmetric, so that von Neumann equivalence is indeed an equivalence relation.

Von Neumann equivalence is clearly implied by measure equivalence, and, in fact, von Neumann equivalence is also implied by W^*-equivalence. Indeed, if $\theta : L\Gamma \to L\Lambda$ is a von Neumann algebra isomorphism, then we may take $\mathcal{M} = \mathcal{B}(\ell^2\Lambda)$ with the trace-preserving action $\sigma : \Gamma \times \Lambda \to \mathrm{Aut}(\mathcal{M})$ given by $\sigma_{(s,t)}(T) = \theta(\lambda_s)\rho_t T \rho_t^* \theta(\lambda_s^*)$, where $\rho : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the right regular representation, which commutes with operators in $L\Lambda$. It is then not difficult to see that the rank one projection p onto the subspace $\mathbb{C}\delta_e$ is a common fundamental domain for the actions of both Γ and Λ. In fact, we'll show below that virtual W^*-equivalence also implies von Neumann equivalence.

We introduce below a general induction procedure for inducing representations via von Neumann equivalence from Λ to Γ, and using these induced representations we show that some of the properties that are preserved for measure equivalence and W^*-equivalence are also preserved for von Neumann equivalence.

Theorem 1.2. *Amenability, property (T), and the Haagerup property are all von Neumann equivalence invariants.*

Theorem 1.3. *Weak amenability, weak Haagerup property, and the approximation property (AP) are von Neumann equivalence invariants.*

A group Γ is *properly proximal* if there does not exist a left-invariant state on the C^*-algebra $(\ell^\infty\Gamma/c_0\Gamma)^{\Gamma_r}$ consisting of elements in $\ell^\infty\Gamma/c_0\Gamma$ that are invariant under the right action of the group. Properly proximal groups were introduced in [BIP18], where a number of classes of groups were shown to be properly proximal, including non-elementary hyperbolic groups, convergence groups, non-amenable bi-exact groups, groups admitting proper 1-cocycles into non-amenable representations, and lattices in non-compact semi-simple Lie groups of arbitrary rank. It is also shown that the class of properly proximal groups is stable under commensurability up to finite kernels, and it was then asked if this class was also stable under measure equivalence [BIP18, Question 1(b)].

Proper proximality also has a dynamical formulation [BIP18, Theorem 4.3], and using this, together with our induction technique applied to isometric representations on dual Banach spaces, we show that the class of properly proximal groups is not only closed under measure equivalence but also under von Neumann equivalence.

Theorem 1.4. *If $\Gamma \sim_{vNE} \Lambda$ then Γ is properly proximal if and only if Λ is properly proximal.*

An example of Caprace, which appears in Section 5.C of [DTDW20], shows that the class of inner amenable groups is not closed under measure equivalence. Specifically, if p is a prime and F_p denotes the finite field with p elements, then the group $SL_3(F_p[t^{-1}]) \ltimes F_p[t, t^{-1}]^3$ is not inner amenable, although is measure equivalent to the inner amenable group $(SL_3(F_p[t^{-1}]) \ltimes F_p[t^{-1}]^3) \times F_p[t]^3$. Using the previous theorem we then answer another question from [BIP18] by providing

$SL_3(F_p[t^{-1}]) \ltimes F_p[t,t^{-1}]^3$ as an example of a non-inner amenable group that is also not properly proximal.

The notion of von Neumann equivalence also admits a generalization in the setting of finite von Neumann algebras.

Definition 1.5. Two finite von Neumann algebras M and N are *von Neumann equivalent*, written $M \sim_{vNE} N$, if there exists a semi-finite von Neumann algebra $\mathcal{M}$ containing commuting copies of M and $N^{\rm op}$ such that we have intermediate standard representations $M \subset \mathcal{B}(L^2(M)) \subset \mathcal{M}$ and $N^{\rm op} \subset \mathcal{B}(L^2(N)) \subset \mathcal{M}$ satisfying the property that finite-rank projections in $\mathcal{B}(L^2(M))$ and $\mathcal{B}(L^2(N))$ are finite projections in $\mathcal{M}$.

We show in Chapter 5 that this does indeed give an equivalence relation, which is coarser than the equivalence relation given by virtual isomorphism. Moreover, if $\mathcal{M}$ is a factor then we can associate an index $[M : N]_{\mathcal{M}}$, which is given by

$$[M : N]_{\mathcal{M}} = \mathrm{Tr}(p)/\mathrm{Tr}(q),$$

where Tr is a trace on $\mathcal{M}$ and p and q are rank 1 projections in $\mathcal{B}(L^2(M))$ and $\mathcal{B}(L^2(N))$ respectively. The connection to von Neumann equivalence for groups is given by the following theorem:

Theorem 1.6. *If Γ and Λ are countable groups, then $\Gamma \sim_{vNE} \Lambda$ if and only if $L\Gamma \sim_{vNE} L\Lambda$.*

We show in Theorem 5.8 that the set of indices for factorial self von Neumann couplings forms a subgroup $\mathcal{I}_{vNE}(M) < \mathbb{R}^*_+$, which we call the index group of M. If M is a factor then we show that the index group contains the square of the fundamental group of M. The fact that we have the square of the fundamental group instead of the fundamental group itself agrees with phenomena predicted by Connes and Shlyakhtenko in [CS05, Theorem 2.4] and leaves open the possibility that Gaboriau's theorem implying proportional ℓ^2-Betti numbers could still hold in the setting of von Neumann equivalence. However, we make no attempt to achieve this result here.

We also show that for a countable ICC group Γ there is a connection between the index group of $L\Gamma$ and the class $\mathcal{S}_{eqrel}(\Gamma)$ studied by Popa and Vaes in [PV10], which consists of fundamental groups for equivalence relations associated to free, ergodic, probability measure-preserving actions of Γ. Specifically, we show in Corollary 5.12 that $\mathcal{I}_{vNE}(L\Gamma)$ contains the group generated by all the groups in $\mathcal{S}_{eqrel}(\Gamma)$.

For the reader who may be more familiar with techniques coming from measured group theory, we end this thesis with an appendix where we give a direct proof in the measure equivalence setting that proper proximality is a measure equivalence invariant.

Majority of the content of this thesis is based on the joint work with Dr. Peterson, and Dr. Ruth in [IPR19], and some on the work of the author in [Ish21].

Chapter 2

Preliminaries

The main techniques we use in this thesis involve von Neumann algebras endowed with semi-finite normal traces. We briefly discuss some of the facts regarding semi-finite von Neumann algebras that we will use in the sequel. We refer the reader to [Tak02] for proofs of these facts.

2.1 Von Neumann algebras: Definition and Examples

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Equipped with the involution $x \mapsto x^*$ (adjoint of x) and with the operator norm, $\mathcal{B}(\mathcal{H})$ is a Banach $*$-algebra. One can consider the following weaker topologies on $\mathcal{B}(\mathcal{H})$:

- the *strong operator topology* (SOT): $x_i \to x$ if and only if $\|x_i\xi - x\xi\| \to 0$, for all $\xi \in \mathcal{H}$.
- the *weak operator topology* (WOT): $x_i \to x$ if and only if $\langle x_i\xi, \eta\rangle \to \langle x\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$.

A *von Neumann algebra* M on a Hilbert space $\mathcal{H}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains identity and is closed in the strong operator (or, equivalently weak operator) topology. One immediate example of a von Neumann algebra is $M = \mathcal{B}(\mathcal{H})$. When $\mathcal{H} = \mathbb{C}^n$, one gets the algebra $M_n(\mathbb{C})$ of $n \times n$-matrices with complex entries, the simplest example of a von Neumann algebra.

Definition 2.1. A von Neumann algebra M is called *tracial* if it admits a linear functional $\tau : M \to \mathbb{C}$, called a *trace*, which is

1. *positive*: $\tau(x^*x) \geq 0$, for all $x \in M$.
2. *faithful*: $\tau(x^*x) = 0$, for some $x \in M$, implies that $x = 0$.

3. *normal*: $\tau(\sum_{i\in I} p_i) = \sum_{i\in I} \tau(p_i)$, for any family $\{p_i\}_{i\in I}$ of mutually orthogonal projections.

4. *tracial*: $\tau(xy) = \tau(yx)$, for all $x, y \in M$.

A von Neumann algebra with trivial center [1] is called a *factor*. An infinite dimensional tracial factor is called a II_1 *factor*. Any II_1 factor M admits a unique trace τ such that $\tau(1) = 1$.

2.1.1 The standard representation

Any tracial von Neumann algebra (M, τ) has a canonical (or *standard*) representation on a Hilbert space. This representation is a particular case of the GNS construction. Consider the inner product $\langle x, y\rangle_\tau := \tau(y^*x), x, y \in M$ on M. Denote by $L^2(M)$ the completion of M with respect to the norm $\|x\|_2 = \tau(x^*x)^{1/2}, x \in M$. If $x \mapsto \hat{x} : M \to L^2(M)$ is the canonical embedding, then $\pi : M \to \mathcal{B}(L^2(M))$ given by

$$\pi(x)(\hat{y}) := \widehat{xy}, \qquad x, y \in M$$

defines a $*$-homomorphism since

$$\|xy\|_2^2 = \tau(y^*x^*xy) \leq \|x^*x\|\tau(y^*y) = \|x\|^2\|y\|_2^2.$$

π is called the *standard representation* of M. Since τ is a trace, the operator $J : \hat{x} \mapsto \widehat{x^*}$, is an antilinear isometry from $\widehat{M}$ onto itself, where $\widehat{M}$ is the image of M under the embedding $x \mapsto \hat{x} : M \to L^2(M)$. J extends to an antilinear surjective isometry of $L^2(M)$ still denoted by J. We say that J is the *canonical conjugation operator* on $L^2(M)$. One of the main features of the standard representation of M is that it makes M isomorphic to its commutant. More precisely, one has $JMJ = M'$ (e.g., see [AP17, Theorem 7.1.1]).

[1] only operators commuting with all of M are the scalar multiples of identity, i.e., $\mathcal{Z}(M) = M \cap M' = \mathbb{C}$, where M' denotes the commutant of M.

Definition 2.2. Let M be a von Neumann algebra and $B \subset M$ be a von Neumann subalgebra. A linear contraction $E : M \to B$ is called a *conditional expectation* if it satisfies

1. $E(b) = b$, for all $b \in B$.

2. $E(x) \geq 0$, for every $x \in M$ with $x \geq 0$.

3. $E(b_1 x b_2) = b_1 E(x) b_2$, for all $b_1, b_2 \in B$ and for all $x \in M$.

Proposition 2.3. *Let (M, τ) be a tracial von Neumann algebra and let $B \subset M$ be a von Neumann subalgebra. Then there exist a unique conditional expectation $E_B : M \to B$ which is trace preserving, i.e., $\tau \circ E_B = \tau$.*

Proof. Let $e_B : L^2(M) \to L^2(B)$ be the orthogonal projection, where $L^2(B) = \overline{\{\hat{b} \mid b \in B\}}^{\|\cdot\|_2}$. If $x \in M$ and $b \in B$, then $b e_B(\hat{x}) = e_B(\widehat{bx})$ and hence

$$\|b e_B(\hat{x})\|_2 = \|e_B(\widehat{bx})\|_2 \leq \|\widehat{bx}\|_2 = \|bx\|_2 \leq \|x\| \|b\|_2 = \|x\| \|\hat{b}\|_2.$$

Thus, there exists $T \in \mathcal{B}(L^2(B))$ such that $T(\hat{b}) = b e_B(\hat{x})$. Since $T \in B'$, we get that $T \in JBJ$, which gives that $e_B(\hat{x}) \in \widehat{B}$. We therefore have a linear map $E_B : M \to B$ given by $\widehat{E_B(x)} = e_B(\hat{x})$. One the checks that E_B satisfies all the conditions of Definition 2.2 and that $\tau \circ E_B = \tau$. $\square$

2.1.2 Group von Neumann algebras

Let Γ be a countable discrete group. A *unitary representation* of Γ on a Hilbert space $\mathcal{H}$ is group homomorphism $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ denotes the group of unitaries in $\mathcal{B}(\mathcal{H})$. Every countable group Γ has a canonical unitary representation $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$, called the *left regular representation*, defined by

$$\lambda_g(f)(h) = f(g^{-1}h),$$

for all $g, h \in \Gamma$ and $f \in \ell^2\Gamma$. If $\delta_g \in \ell^2\Gamma$ denotes the Dirac function at $g \in \Gamma$, then $\{\delta_g\}_{g\in\Gamma}$ is an orthonormal basis for $\ell^2\Gamma$, and we have $\lambda_g(\delta_h) = \delta_{gh}$ for all $g, h \in \Gamma$. Let

$$\mathcal{A} = \left\{ \sum_{g\in F} a_g \lambda_g \mid a_g \in \mathbb{C} \text{ for all } g \in F \text{ and } F \subset \Gamma \text{ is a finite subset} \right\}.$$

Then, $\mathcal{A} \subset \mathcal{B}(\ell^2\Gamma)$ is a $*$-subalgebra which is isomorphic to the complex group algebra $\mathbb{C}\Gamma$. The *group von Neumann algebra of* Γ, denoted $L\Gamma$, is defined as the weak operator closure of $\mathcal{A}$, that is, $L\Gamma := \overline{\mathcal{A}}^{\mathrm{WOT}}$. It is a tracial von Neumann algebra with a trace $\tau : L\Gamma \to \mathbb{C}$ given by $\tau(x) = \langle x\delta_e, \delta_e \rangle$. Recall that a group Γ is ICC (infinite conjugacy classes) if every non-trivial conjugacy class $\{ghg^{-1} \mid g \in \Gamma\}, h \neq e$ is infinite. $L\Gamma$ is a finite factor if and only if Γ is ICC (see [AP17, Proposition 1.3.9]

Example 2.4. There are plenty of countable ICC groups. The following are some of the simplest examples.

1. S_∞, the group of permutations of $\mathbb{N}$ fixing all but finitely many integers.
2. The free product group $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1, Γ_2 are arbitrary groups with $|\Gamma_1| > 1$ and $|\Gamma_2| > 2$. In particular, the free group $\mathbb{F}_n, n \geq 2$, on n generators is ICC.
3. $\mathrm{SL}_n(\mathbb{Z}) := \{A \in M_n(\mathbb{Z}) \mid \det(A) = 1\}, n \geq 3$ and n odd.

2.1.3 Group Measure Space Construction

We describe a fundamental construction associated to an action of a countable group Γ on a probability space (X, μ). The group von Neumann algebra $L\Gamma$ is a special case of this construction when X is reduced to a single point.

Definition 2.5. A *probability measure preserving* (p.m.p.) action $\Gamma \curvearrowright (X, \mu)$ of a countable group Γ on a probability space (X, μ) is a group homomorphism $\sigma : \Gamma \to \mathrm{Aut}(X, \mu)$. The action of $g \in \Gamma$ on $x \in X$ will be denoted by gx.

Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group Γ on a standard probability space (X, μ). Define a unitary representation $\sigma : \Gamma \to \mathcal{U}(L^2(X))$ by $\sigma_g(f)(x) = f(g^{-1}x)$ for all $f \in L^2(X)$. Note that $\sigma_g(L^\infty(X)) = L^\infty(X)$ for all $g \in \Gamma$. Denote $\mathcal{H} = L^2(X) \otimes \ell^2\Gamma$ and define a unitary representation $u : \Gamma \to \mathcal{U}(\mathcal{H})$ by $u_g = \sigma_g \otimes \lambda_g, g \in \Gamma$, where $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ is the left regular representation. We also consider the $*$-homomorphism $\pi : L^\infty(X) \to \mathcal{B}(\mathcal{H})$ given by $\pi(f)(\xi \otimes \delta_g) = f\xi \otimes \delta_g$, and view $L^\infty(X) \subset \mathcal{B}(\mathcal{H})$ via π. Then, for every $g \in \Gamma$ and for every $f \in L^\infty(X)$, it straightforward to check that

$$u_g f u_g^* = \sigma_g(f).$$

Definition 2.6. The *group measure space von Neumann algebra* $L^\infty(X) \rtimes \Gamma \subset \mathcal{B}(\mathcal{H})$ is defined as the WOT-closure of the linear span of $\{f u_g \mid f \in L^\infty(X), g \in \Gamma\}$. $L^\infty(X) \rtimes \Gamma$ comes with a faithful, normal tracial state $\tau : L^\infty(X) \rtimes \Gamma \to \mathbb{C}$, given by

$$\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e \rangle.$$

2.2 Semi-finite traces

A *trace* on a von Neumann algebra $\mathcal{M}$ is a function Tr on the positive cone $\mathcal{M}_+$ with values in the extended reals $[0, \infty]$ satisfying the following conditions:

(i) $\mathrm{Tr}(x + y) = \mathrm{Tr}(x) + \mathrm{Tr}(y), \quad x, y \in \mathcal{M}_+,$

(ii) $\mathrm{Tr}(\alpha x) = \alpha \,\mathrm{Tr}(x), \alpha \geq 0, \quad x \in \mathcal{M}_+,$

(iii) $\mathrm{Tr}(x^* x) = \mathrm{Tr}(x x^*), \quad x \in \mathcal{M}.$

A trace Tr is said to be *faithful* if $\mathrm{Tr}(x) > 0$ for any non-zero $x \in \mathcal{M}_+$, *semi-finite* if for every non-zero $x \in \mathcal{M}_+$ there exists a non-zero $y \in \mathcal{M}_+, y \leq x$ with $\mathrm{Tr}(y) < \infty$, *finite* if $\mathrm{Tr}(1) < \infty$, and *normal* if $\mathrm{Tr}(\sup_i x_i) = \sup_i \mathrm{Tr}(x_i)$ for every bounded increasing net $\{x_i\}$ in $\mathcal{M}_+$. A separable von Neumann algebra $\mathcal{M}$ is *semi-finite* if and only if it admits a faithful normal semi-finite trace.

If $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr, we set $\mathfrak{n}_{\mathrm{Tr}} = \{x \in \mathcal{M} \mid \mathrm{Tr}(x^*x) < \infty\}$, and $\mathfrak{m}_{\mathrm{Tr}} = \{\sum_{j=1}^n x_j^* y_j \mid x_j, y_j \in \mathfrak{n}_{\mathrm{Tr}}, 1 \leq j \leq n\}$. Both $\mathfrak{n}_{\mathrm{Tr}}$ and $\mathfrak{m}_{\mathrm{Tr}}$ are ideals in $\mathcal{M}$, and the trace Tr extents to a $\mathbb{C}$-valued linear functional on $\mathfrak{m}_{\mathrm{Tr}}$, which is called the *definition ideal* of Tr.

We let $L^1(\mathcal{M}, \mathrm{Tr})$ denote the completion of $\mathfrak{m}_{\mathrm{Tr}}$ under the norm $\|a\|_1 = \mathrm{Tr}(|a|)$, and then the bilinear form $\mathcal{M} \times \mathfrak{m}_{\mathrm{Tr}} \ni (x, a) \mapsto \mathrm{Tr}(xa)$ extends to the duality between $\mathcal{M}$ and $L^1(\mathcal{M}, \mathrm{Tr})$ so that we may identify $L^1(\mathcal{M}, \mathrm{Tr})$ with $\mathcal{M}_*$.

We let $L^2(\mathcal{M}, \mathrm{Tr})$ denote the Hilbert space completion of $\mathfrak{n}_{\mathrm{Tr}}$ under the inner product $\langle a, b\rangle_2 = \mathrm{Tr}(b^*a)$. Left multiplication of $\mathcal{M}$ on $\mathfrak{n}_{\mathrm{Tr}}$ then induces a normal faithful representation of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$, which is called the *standard representation*.

Restricting the conjugation operator from $\mathcal{M}$ to $\mathfrak{n}_{\mathrm{Tr}}$ induces an anti-linear isometry $J : L^2(\mathcal{M}, \mathrm{Tr}) \to L^2(\mathcal{M}, \mathrm{Tr})$, and we have $J\mathcal{M}J = \mathcal{M}' \cap \mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$. The von Neumann algebra $J\mathcal{M}J$ is canonically isomorphic to the opposite von Neumann algebra $\mathcal{M}^{\mathrm{op}}$ via the map $\mathcal{M}^{\mathrm{op}} \ni x^{\mathrm{op}} \mapsto Jx^*J$. We also have the induced trace on $\mathcal{M}^{\mathrm{op}}$ given by $\mathrm{Tr}(x^{\mathrm{op}}) = \mathrm{Tr}(x^*)$.

If $\mathcal{M}$ is a semi-finite factor, then it has a unique (up to scalar multiples) normal semi-finite faithful trace. In general, if Tr_1 and Tr_2 are normal semi-finite traces, then there is an injective positive operator a affiliated to the center $\mathcal{Z}(\mathcal{M})$ such that $\mathrm{Tr}_2(x) = \mathrm{Tr}_1(ax)$ for all $x \in \mathcal{M}_+$. In particular, the map $\mathfrak{n}_{\mathrm{Tr}_2} \ni x \mapsto a^{1/2}x \in \mathfrak{n}_{\mathrm{Tr}_1}$ extends to a unitary operator from $L^2(\mathcal{M}, \mathrm{Tr}_2)$ onto $L^2(\mathcal{M}, \mathrm{Tr}_1)$ that intertwines the representations of $\mathcal{M}$, and also intertwines the representations of $\mathcal{M}^{\mathrm{op}}$. Thus, up to isomorphism, the representation $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$ is independent of the choice of semi-finite normal faithful trace Tr, and we may use the notation $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}))$ if we wish to emphasize this fact.

If $\mathcal{H}$ is a Hilbert space and we have an embedding $\mathcal{B}(\mathcal{H}) \subset \mathcal{M}$, then setting $P = \mathcal{B}(\mathcal{H})' \cap \mathcal{M}$ we have an isomorphism $\mathcal{B}(\mathcal{H}) \overline{\otimes} P \cong \mathcal{M}$ that maps $T \otimes x$ to Tx for $T \in \mathcal{B}(\mathcal{H})$ and $x \in P$. Indeed, this is easy to verify in the case when $\mathcal{M}$ is a type I factor, and in general if we represent $\mathcal{M} \subset \mathcal{B}(\mathcal{K})$, then we have $\mathcal{B}(\mathcal{H}) \subset \mathcal{M} \subset \mathcal{B}(\mathcal{K}) \cong \mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}_0)$ for some Hilbert space $\mathcal{K}_0$. Thus, $\mathcal{M}' = \mathcal{M}' \cap \mathcal{B}(\mathcal{K}_0)$, so that $\mathcal{M} = \mathcal{M}'' = \mathcal{B}(\mathcal{H}) \overline{\otimes} (\mathcal{M} \cap \mathcal{B}(\mathcal{K}_0)) = \mathcal{B}(\mathcal{H}) \overline{\otimes} P$. There also then exists a unique semi-finite normal faithful trace Tr_P on P so that $\mathrm{Tr}_{\mathcal{M}} = \mathrm{Tr} \otimes \mathrm{Tr}_P$.

If we have two embeddings $\theta_1, \theta_2 : \mathcal{B}(\mathcal{H}) \to \mathcal{M}$, then $\theta_1(\mathcal{B}(\mathcal{H}))$ and $\theta_2(\mathcal{B}(\mathcal{H}))$ are conjugate by a unitary in $\mathcal{M}$ if and only if for some rank one projection $p \in \mathcal{B}(\mathcal{H})$ we have that $\theta_1(p)$ and $\theta_2(p)$ are Murray-von Neumann equivalent [2].

An element $x \in \mathcal{M}$ is *compact*, if for every $\varepsilon > 0$, there exists a projection $p \in \mathcal{M}$ such that $\|xp\| < \varepsilon$ and $1 - p$ is finite. If $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr, and $p \in \mathcal{M}$ is a finite-trace projection, then the map $x \mapsto \mathrm{Tr}(xp)$ is weak operator topology continuous.

Lemma 2.7. *Suppose $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr, $x \in \mathcal{M}$ is compact, and $\{p_i\}$ is a net of finite-trace projections such that $p_i \to 0$ in the weak operator topology. If $\{\mathrm{Tr}(p_i)\}$ is uniformly bounded, then, $\mathrm{Tr}(xp_i) \to 0$.*

Proof. Given $\varepsilon > 0$, there exists a projection $q \in \mathcal{M}$ such that $\|xq\| < \varepsilon$ and $\mathrm{Tr}(1 - q) < \infty$. Since $p_i \to 0$ in the weak operator topology, we get that $xp_i \to 0$ in the weak operator topology and hence $\mathrm{Tr}(p_i x(1 - q)) \to 0$. Moreover, we have that $\{\mathrm{Tr}(p_i)\}$ is uniformly bounded, say by C, whence it follows that

$$\limsup_i |\mathrm{Tr}(xp_i)| \leq \limsup_i (\|xq\| |\mathrm{Tr}(p_i)| + |\mathrm{Tr}(p_i x(1 - q))|) \leq \varepsilon C.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

2.3 Actions on semi-finite von Neumann algebras

If Γ is a discrete group and $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ is an action that preserves the trace Tr, then Γ preserves the $\|\cdot\|_1$-norm on $\mathfrak{m}_{\mathrm{Tr}}$ and hence the action extends to an action by isometries on $L^1(\mathcal{M}, \mathrm{Tr})$, and the dual of the action on $L^1(\mathcal{M}, \mathrm{Tr})$ agrees with the action on $\mathcal{M}$.

Restricted to $\mathfrak{n}_{\mathrm{Tr}}$ the action is also isometric with respect to $\|\cdot\|_2$ and hence gives a unitary

[2] two projections p and q in a von Neumann algebra $\mathcal{M}$ are *Murray-von Neumann equivalent* if there exits a partial isometry $u \in \mathcal{M}$ with $u^*u = p$ and $uu^* = q$.

representation in $\mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$, which is called the *Koopman representation* and denoted by $\sigma^0 : \Gamma \to \mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$. Note that considering $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$ via the standard representation, we have that the action $\sigma : \Gamma \to \mathrm{Aut}(\mathcal{M}, \mathrm{Tr})$ becomes unitarily implemented via the Koopman representation, i.e., for $x \in \mathcal{M}$ and $\gamma \in \Gamma$ we have $\sigma_\gamma(x) = \sigma^0_\gamma x \sigma^0_{\gamma^{-1}}$.

The *crossed product von Neumann algebra* $\mathcal{M} \rtimes \Gamma$ is defined to be the von Neumann subalgebra of $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}) \overline{\otimes} \ell^2\Gamma)$ generated by $\mathcal{M} \overline{\otimes} \mathbb{C}$ and $\{\sigma^0_\gamma \otimes \lambda_\gamma \mid \gamma \in \Gamma\}$. We use the notation $u_\gamma = \sigma^0_\gamma \otimes \lambda_\gamma$. Note that by Fell's absorption principle [3], the representation $\Gamma \ni \gamma \mapsto u_\gamma \in \mathcal{M} \rtimes \Gamma$ is conjugate to a multiple of the left regular representation and hence generates a copy of the group von Neumann algebra $L\Gamma$.

If P_e denotes the rank one projection onto $\mathbb{C}\delta_e \subset \ell^2\Gamma$, then we have a canonical conditional expectation from $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}) \overline{\otimes} \ell^2\Gamma)$ onto $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$ given by $T \mapsto (1 \otimes P_e)T(1 \otimes P_e)$ and then identifying $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$ with $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr})) \otimes \mathbb{C}P_e$. Restricting this to $\mathcal{M} \rtimes \Gamma$ gives a conditional expectation $E_{\mathcal{M}} : \mathcal{M} \rtimes \Gamma \to \mathcal{M}$. The trace on $\mathcal{M}$ then extends to a faithful normal semi-finite trace on $\mathcal{M} \rtimes \Gamma$ given by $\mathrm{Tr}(x) = \mathrm{Tr} \circ E_{\mathcal{M}}(x)$.

If we have a subgroup $\Gamma_0 < \Gamma$ and a Γ_0-invariant von Neumann subalgebra $\mathcal{M}_0$ such that $\mathcal{M}_0 \cap \mathfrak{m}_{\mathrm{Tr}}$ is weakly dense in $\mathcal{M}_0$, then the von Neumann algebra generated by $\mathcal{M}_0$ and Γ_0 is canonically isomorphic to the crossed product $\mathcal{M}_0 \rtimes \Gamma_0$, and so we have a canonical embedding of crossed products $\mathcal{M}_0 \rtimes \Gamma_0 \subset \mathcal{M} \rtimes \Gamma$.

A specific example of the crossed product construction that we will use below is when we consider $\ell^\infty\Gamma$ with its trace coming from counting measure, and the action of $\Gamma \curvearrowright^L \ell^\infty\Gamma$ is given by right multiplication $L_\gamma(f)(x) = f(x\gamma)$. In this case, by considering a Fell unitary, we obtain an isomorphism $\theta : \ell^\infty\Gamma \rtimes \Gamma \to \mathcal{B}(\ell^2\Gamma)$ such that $\theta(f)$ is the multiplication operator by f for $f \in \ell^\infty\Gamma$, while for $\gamma \in \Gamma$ we have $\theta(u_\gamma) = \lambda_\gamma$ gives the left-regular representation.

[3] Fell's absorption principle states that if π is a unitary representation of a group Γ on $\mathcal{H}$, then $\lambda \otimes \pi$ is unitarily equivalent to $\lambda \otimes 1_{\mathcal{H}}$. Here, λ is the left-regular representation and $1_{\mathcal{H}}$ is the trivial representation.

2.4 The basic construction

If $(\mathcal{M}, \mathrm{Tr})$ is a von Neumann algebra with a semi-finite normal faithful trace Tr, then conjugation on $\mathfrak{n}_{\mathrm{Tr}}$ induces an anti-linear isometry $J : L^2(\mathcal{M}, \mathrm{Tr}) \to L^2(\mathcal{M}, \mathrm{Tr})$, and we have $\mathcal{M}' = J\mathcal{M}J$. If $\mathcal{N} \subset \mathcal{M}$ is a von Neumann subalgebra then the *basic construction* is the von Neumann algebra

$$\langle \mathcal{M}, \mathcal{N} \rangle := (J\mathcal{N}J)' \subset \mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr})).$$

If $\mathcal{N}$ is semi-finite, the so is $\langle \mathcal{M}, \mathcal{N} \rangle$.

2.5 Tensor products of operator spaces

For the basic results we'll need from the theory of operator spaces and their tensor products, we refer the reader to [BLM04] or [Pis03]. A (concrete) *operator space* is a closed subspace $E \subset \mathcal{B}(\mathcal{H})$. Given operator spaces E and F, and a linear map $u : E \to F$, we define linear maps $u_n : \mathbb{M}_n(E) \to \mathbb{M}_n(F)$ by setting $u_n((x_{ij})) = (u(x_{ij}))$. The map u is *completely bounded* if the completely bounded norm $\|u\|_{\mathrm{cb}} = \sup_n \|u_n\|$ is finite.

We denote by $CB(E, F)$ the space of all completely bounded maps from E to F, which is a Banach space when given the completely bounded norm. We also endow $\mathbb{M}_n(CB(E, F))$ with the Banach space norms coming from the canonical isomorphism $\mathbb{M}_n(CB(E, F)) \cong CB(E, \mathbb{M}_n(F))$. Ruan's abstract matrix norm characterization for operator spaces shows that the norms on $\mathbb{M}_n(CB(E, F))$ give an operator space structure to $CB(E, F)$, i.e., $CB(E, F)$ is completely isometrically isomorphic to a concrete operator space. In particular, when $F = \mathbb{C}$ we obtain the dual operator space structure on E^*.

Any Banach space X can be endowed with an operator space structure by embedding X into the C^*-algebra $C((X^*)_1)$ of weak*-continuous functions on the unit ball of X^*, and where X is realized via the evaluation map. We denote this operator space structure by $\min(X)$. We may also consider the supremum of all operator space norms on X, and we denote this operator space structure by $\max(X)$. We then have completely isometrically $\min(X)^* = \max(X^*)$ and

$\max(X)^* = \min(X^*)$.

For a Hilbert space $\mathcal{H}$ there are two canonical operator space structures. The first is the *Hilbert column space* $\mathcal{H}^c$, which endows $\mathcal{H}$ with the operator space structure coming from the canonical isomorphism $\mathcal{H} \cong CB(\mathbb{C}, \mathcal{H})$. The second is the *Hilbert row space* $\mathcal{H}^r$, which endows $\mathcal{H}$ with the operator space structure coming from the canonical isomorphism $\mathcal{H} \cong CB(\mathcal{H}, \mathbb{C})$. As operator spaces we then have natural identifications $(\mathcal{H}^c)^* \cong \overline{\mathcal{H}}^r$ and $(\mathcal{H}^r)^* \cong \overline{\mathcal{H}}^c$. Unless otherwise stated, in the sequel we will endow any Hilbert space with its operator space structure as a Hilbert column space.

If $E \subset \mathcal{B}(\mathcal{H})$ and $F \subset \mathcal{B}(\mathcal{K})$ are operator spaces, the *minimal tensor product* $E \otimes_{\min} F$ is given by the completion of the algebraic tensor product $E \otimes F \subset \mathcal{B}(\mathcal{H} \overline{\otimes} \mathcal{K})$. The operator space structure on $E \otimes_{\min} F$ is independent of the concrete representations, and we have a completely isometric embedding

$$E \otimes_{\min} F \hookrightarrow CB(F^*, E)$$

where $u = \sum_{k=1}^n x_k \otimes y_k \in E \otimes F$ is associated to the map $\tilde{u} : F^* \to E$ given by $\tilde{u}(\psi) = \sum_{k=1}^n \psi(y_k) x_k$, for $\psi \in F^*$.

If E and F are operator spaces, then perhaps the simplest way to describe the *projective tensor product* is to define it as the completion of $E \otimes F$ when we embed $E \otimes F$ into the operator space $CB(E, F^*)^*$ via the map that assigns to $x \otimes y$ the functional $CB(E, F^*) \ni T \mapsto T(x)(y)$. We denote the operator space projective tensor product of E and F by $E \widehat{\otimes} F$. From [BP91, Proposition 5.4] we then have completely isometric isomorphisms

$$(E \widehat{\otimes} F)^* \cong CB(E, F^*) \cong CB(F, E^*). \tag{2.1}$$

We note that under the identification $(E \widehat{\otimes} F)^* \cong CB(E, F^*)$, the weak*-topology on bounded sets is given by pointwise weak*-convergence of operators.

In this article we will be mainly interested in dual operator spaces. We therefore find it convenient to use the notation E and F for operator spaces that are dual to operator spaces E_* and

F_* respectively. Every ultraweakly closed subspace E of $\mathcal{B}(\mathcal{H})$ is a dual operator space with a canonical predual $\mathcal{B}(\mathcal{H})_*/E_\perp$, where $E_\perp$ is the preannihilator of E. Conversely, if E is a dual operator space, then E is weak*-homeomorphically completely isometric to an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$.

If $E \subset \mathcal{B}(\mathcal{H})$ and $F \subset \mathcal{B}(\mathcal{K})$ are ultraweakly closed subspaces, then the *normal minimal tensor product* $E \overline{\otimes} F$ is the ultraweak completion of the algebraic tensor product $E \otimes F \subset \mathcal{B}(\mathcal{H} \overline{\otimes} \mathcal{K})$. This is independent of the concrete representations, and we have a weak*-homeomorphic completely isometric embedding

$$E \overline{\otimes} F \hookrightarrow (E_* \widehat{\otimes} F_*)^* \cong CB(E_*, F) \cong CB(F_*, E).$$

We will therefore identify $E \overline{\otimes} F$ as a subspace of $CB(E_*, F)$. We note that even in the case when $F = \mathcal{M}$ is a von Neumann algebra, this embedding will not be surjective in general. However, it follows from [Ble91, Theorem 2.5], [Rua92, Proposition 3.3] and [Kra91] that this embedding will be surjective whenever $F = \mathcal{M}$ is a von Neumann algebra with the σ-weak approximation property.

2.6 Hilbert C^*-modules

We refer the reader to [Lan95] for the basic properties of Hilbert C^*-modules. If A is a C^*-algebra and I is a set, then we let $\bigoplus_{i\in I} A$ denote the space of functions $(a_i)_{i\in I}$ such that $\sum_{i\in I} a_i^* a_i$ converges in A. This gives a Hilbert A-module where we have an A-valued inner product (linear in the second variable) given by

$$\langle (a_i)_{i\in I}, (b_i)_{i\in I} \rangle_A = \sum_{i\in I} a_i^* b_i.$$

If $\mathcal{H}$ is a Hilbert space then on the algebraic tensor product $A \otimes \mathcal{H}$ we have an A-valued inner product given by $\langle a \otimes \xi, b \otimes \eta \rangle_A = \langle \eta, \xi \rangle a^* b$. This inner product extends continuously to give a Hilbert A-module structure to $A \otimes_{\min} \mathcal{H}$ [BLM04, Theorem 8.2.17], where $\mathcal{H}$ is endowed with its operator space structure as a column Hilbert space. Choosing a basis $\{e_i\}_{i\in I}$ gives an identification between the Hilbert A-modules $A \otimes_{\min} \mathcal{H}$ and $\bigoplus_{i\in I} A$.

If $\mathcal{M}$ is a von Neumann algebra and I is a set then we let $\overline{\bigoplus}_{i\in I}\mathcal{M}$ denote the space of functions $(a_i)_{i\in I}$ such that $\sum_{i\in I} a_i^* a_i$ is bounded. If $(a_i)_{i\in I}$, $(b_i)_{i\in I} \in \overline{\bigoplus}_{i\in I}\mathcal{M}$ then we have ultraweak convergence of the sum

$$\langle (a_i)_{i\in I}, (b_i)_{i\in I}\rangle_{\mathcal{M}} = \sum_{i\in I} a_i^* b_i.$$

If $\mathcal{H}$ is a Hilbert space, then the Hilbert $\mathcal{M}$-module structure on $\mathcal{M} \otimes_{\min} \mathcal{H}$ has a unique extension to $\mathcal{M} \overline{\otimes} \mathcal{H}$ such that the inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ is separately ultraweakly continuous. In particular, $\mathcal{M} \overline{\otimes} \mathcal{H}$ will be self-dual in the sense of Paschke [Pas73], [Sch02, Proposition 2.9]. Choosing a basis $\{e_i\}_{i\in I}$ gives an identification between the Hilbert $\mathcal{M}$-modules $\mathcal{M} \overline{\otimes} \mathcal{H}$ and $\overline{\bigoplus}_{i\in I}\mathcal{M}$.

Dual Hilbert $\mathcal{M}$-modules are naturally related to normal representations of $\mathcal{M}$ obtained via an internal tensor product $\mathcal{K} \overline{\otimes}_{\mathcal{M}} L^2(\mathcal{M})$. In the case when $\mathcal{M}$ has a finite trace τ, this is quite explicit, and as we will use this in the sequel, we describe this here. Given a Hilbert $\mathcal{M}$-module $\mathcal{K}$, we obtain a scalar-valued inner product $\langle \cdot, \cdot \rangle_\tau$ on $\mathcal{K}$ by $\langle \xi, \eta \rangle_\tau = \tau(\langle \eta, \xi \rangle_{\mathcal{M}})$. The completion gives a Hilbert space $\mathcal{K}_\tau$, and the right $\mathcal{M}$-module structure on $\mathcal{K}$ then extends to a normal representation of $\mathcal{M}^{\rm op}$ on $\mathcal{K}_\tau$.

Each vector $\xi \in \mathcal{K}$ then gives rise to a bounded right $\mathcal{M}$-modular map $L_\xi : L^2(\mathcal{M}, \tau) \to \mathcal{K}_\tau$ such that $L_\xi(x) = \xi x$ for all $x \in \mathcal{M} \subset L^2(\mathcal{M}, \tau)$. To see that L_ξ is bounded, just note that for $x \in \mathcal{M} \subset L^2(\mathcal{M}, \tau)$ we have

$$\|L_\xi(x)\|_\tau^2 = \tau(\langle \xi x, \xi x \rangle_{\mathcal{M}}) = \tau(x^* \langle \xi, \xi \rangle_{\mathcal{M}} x) \leq \|\xi\|^2 \|x\|_2^2.$$

Every bounded right $\mathcal{M}$-modular map arises in this way, and if $\xi, \eta \in \mathcal{K}$, then we can recover our inner product as $\langle \xi, \eta \rangle = L_\xi^* L_\eta \in (J\mathcal{M}J)' \cap \mathcal{B}(L^2(\mathcal{M}, \tau)) = \mathcal{M}$. The mapping $L : \mathcal{K} \to \mathcal{B}(L^2(\mathcal{M}, \tau), \mathcal{K}_\tau)$ is then isometric and gives a homeomorphism between the weak*-topology on $\mathcal{K}$ and the ultraweak topology on $\mathcal{B}(L^2(\mathcal{M}, \tau), \mathcal{K}_\tau)$.

As a consequence, if $X \subset \mathcal{K}$ is an $\mathcal{M}$-invariant subset, then X is weak*-dense in $\mathcal{K}$ if and only if X is dense in $\mathcal{K}_\tau$. Indeed, if X were not dense in $\mathcal{K}_\tau$, and if we let P denote the projection onto $X^\perp$ in $\mathcal{K}_\tau$, then P is right $\mathcal{M}$-modular, and $PL_\xi = 0$ for all $\xi \in X$. If we then took any non-zero right M-modular map $L \in \mathcal{B}(L^2(\mathcal{M}, \tau), P\mathcal{K}_\tau)$, then $L = L_\eta$ for some $\eta \in \mathcal{K}$, and $PL_\eta \neq 0$, showing that η is not in the weak*-closure of X.

Another consequence we shall use is that if $\mathcal{K}$ and $\mathcal{H}$ are two dual Hilbert $\mathcal{M}$-modules, $X \subset \mathcal{K}$ is a weak*-dense $\mathcal{M}$-invariant subset, and $V : X \to \mathcal{H}$ is a right $\mathcal{M}$-modular map that satisfies

$$\langle V\xi, V\eta\rangle = \langle \xi, \eta\rangle \tag{2.2}$$

for all $\xi, \eta \in X$, then V has an extension to $\mathcal{K}$ that satisfies (2.2) for $\xi, \eta \in \mathcal{K}$ and such that V is continuous with respect to the weak*-topologies. Indeed, if V_τ denotes the map V when viewed as a map between $\mathcal{K}_\tau$ and $\mathcal{H}_\tau$, then V_τ extends to an isometry, and we may define V by $L_{V\xi} = V_\tau L_\xi$.

2.7 Measurable functions into separable Banach spaces

If (X, μ) is a standard measure space and E is separable Banach space, then we let $L^1(X, \mu; E)$ denote the space of measurable functions $f : X \to E$ such that $\int \|f(x)\| \, d\mu(x) < \infty$, where we identify two functions if they agree almost everywhere. We have an isometric isomorphism $L^1(X, \mu) \hat{\otimes} E \to L^1(X, \mu; E)$, which takes an elementary tensor $f \otimes a$ to the function $x \mapsto f(x)a$; here $\hat{\otimes}$ represents the Banach space projective tensor product. If $E = (E_*)^*$ is dual to a separable Banach space then we let $L^\infty_{w^*}(X, \mu; E)$ denote the space of essentially bounded functions that are Borel with respect to the weak*-topology restricted to some ball in E that contains almost every point in the range of f, where we identify two functions if they agree almost everywhere. Note that since E_* is separable, the weak*-topology in E is compact and metrizable when restricted to any closed ball.

If $K \subset E$ is a weak*-compact subset, then we denote by $L^\infty_{w^*}(X, \mu; K)$ the subset of $L^\infty_{w^*}(X, \mu; E)$ consisting of those functions whose essential range is contained in K. We have an isometric isomorphism $L^\infty_{w^*}(X,\mu;E) \to (L^1(X,\mu;E_*))^*$ [Mon01, Section 2.2] given by the pairing

$$\langle f, g\rangle = \int \langle f(x), g(x)\rangle \, d\mu(x).$$

Thus we have isometric isomorphisms

$$L^\infty_{w^*}(X,\mu;E) \cong (L^1(X,\mu;E_*))^* \cong \mathcal{B}(L^1(X,\mu),E).$$

Proposition 2.8. *If $E = (E_*)^*$ is a dual Banach space and $K \subset E$ is a weak*-closed convex subset, then under the above isomorphism we have*

$$L^\infty_{w^*}(X,\mu;K) \cong \{\Xi \in \mathcal{B}(L^1(X,\mu),E) \mid \Xi(f) \in K \text{ for all } f \in L^1(X,\mu)_+, \|f\|_1 = 1\}.$$

Proof. We let $\Psi : L^\infty_{w^*}(X,\mu;E) \to \mathcal{B}(L^1(X,\mu),E)$ be the isomorphism from above, so that for $f \in L^\infty_{w^*}(X,\mu;E)$ and $g \in L^1(X,\mu)$ we have $\Psi(f)(g) = \int g(x)f(x)\,d\mu(x)$.

If $f \in L^\infty_{w^*}(X,\mu;K)$ and $g \in L^1(X,\mu)_+$ with $\|g\|_1 = 1$, then as K is convex and weak*-closed we have $\Psi(f)(g) = \int g(x)f(x)\,d\mu(x) \in K$. On the other hand, if $f \in L^\infty_{w^*}(X,\mu;E)$ is not in $L^\infty_{w^*}(X,\mu;K)$, then choose a point $m \in E \setminus K$ that is contained in the essential range of f. By the Hahn-Banach separation theorem, there exists a convex weak*-open neighborhood G of m such that $\overline{G} \cap K = \emptyset$. If we set $B = f^{-1}(G)$, then we have that $\mu(B) > 0$, and taking $g = \frac{1}{\mu(B)}1_B$, we have $\Psi(f)(g) = \frac{1}{\mu(B)}\int_B f(x)\,d\mu(x) \in \overline{G} \subset E \setminus K$. $\square$

2.8 Properly proximal groups

Suppose Γ is an infinite discrete group and we have an action by homeomorphisms on a non-empty Hausdorff topological space X. Recall that a pair of points $x, y \in X$ are called *proximal* if the orbit $\Gamma \cdot (x,y)$ has non-trivial intersection with every neighborhood of the diagonal $\Delta = \{(x,x) \mid x \in X\} \subset X^2$. We say a pair of points $x, y \in X$ are *properly proximal* if for every neighborhood O of Δ there is a finite set $F \subset \Gamma$ such that $(\Gamma \setminus F)\cdot(x,y) \subset O$. We say a point

$x \in X$ is *properly proximal* if any pair of points in the orbit $\Gamma \cdot x$ are properly proximal, and we say the action $\Gamma \curvearrowright X$ is *properly proximal* if the set of properly proximal points in X is dense.

Lemma 2.9. *Let E be a compact convex subset of a locally convex topological vector space X. Let $K \subset E$ be a compact convex subset and U be relatively open in E with $K \subset U$. Then there exists a convex set V such that $K \subset V \subset U$ and V is relatively open in E.*

Proof. For each point $x \in K$ choose an open convex neighborhood U_x of zero such that $(x + U_x + U_x) \cap E \subset U$. The family $\{x + U_x : x \in K\}$ is an open cover of K. Therefore there exists a finite subset $F \subset K$ such that $K \subset \bigcup\{x + U_x : x \in F\}$. Put $U_0 = \bigcap\{U_x : x \in F\}$. Then U_0 is both open and convex. The set $K + U_0$ is convex as a sum of convex sets, and it is open as a union of a family $\{x + U_0 : x \in K\}$ of open sets. The set $V = (K + U_0) \cap E$ is relatively open in E, and it is convex as an intersection of two convex sets. It is clear that $K \subset V$. Now, let $x \in V \subset K + U_0$ be an arbitrary point. Then there exists a point $z \in K$ such that $x \in z + U_0$. Also there exists a point $y \in F$ such that $z \in y + U_y$. Then $x \in y + U_y + U_0 \subset y + U_y + U_y$. Since $x \in E$, we see that $x \in U$. □

Lemma 2.10. *Suppose Γ acts on a compact Hausdorff space X. If $\Gamma \curvearrowright X$ is properly proximal, then so is the action $\Gamma \curvearrowright \mathrm{Prob}(X)$.*

Proof. First note that the embedding of X into $\mathrm{Prob}(X)$ as Dirac masses is a homeomorphism from X into $\mathrm{Prob}(X)$ with the weak*-topology. Thus, if $x \in X$ is a properly proximal point, then so is $\delta_{\{x\}} \in \mathrm{Prob}(X)$.

We now claim that the set of properly proximal points in $\mathrm{Prob}(X)$ is closed under taking convex combinations. Indeed, suppose $\eta_1, \ldots, \eta_n \in \mathrm{Prob}(X)$ are properly proximal, and $\eta = \sum_{i=1}^{k} t_i \eta_i$ with $\sum_{i=1}^{k} t_i = 1$. Let $\mathcal{O}^*$ be an open neighborhood of Δ^*, the diagonal of $\mathrm{Prob}(X) \times \mathrm{Prob}(X)$. In light of Lemma 2.9, we may assume without loss of generality that $\mathcal{O}^*$ is convex. For each $i = 1, \ldots, k$, there exists a finite subset $F_i \subset \Gamma$ such that $(\gamma\eta_i, \gamma g\eta_i) \in \mathcal{O}^*$ for all $g \in \Gamma, \gamma \notin F_i$. Set $F = \cup_{i=1}^{k} F_i$. Then, $(\gamma\eta_i, \gamma g\eta_i) \in \mathcal{O}^*$ for all $g \in \Gamma, \gamma \notin F$ and for all $i = 1, \ldots, k$. Since $\mathcal{O}^*$ is convex, it follows readily that $(\gamma\eta, \gamma g\eta) \in \mathcal{O}^*$ for all $g \in \Gamma, \gamma \notin F$.

The proof is then immediate, as the convex combination of Dirac masses is dense in $\mathrm{Prob}(X)$.

□

A *Banach* Γ*-module* consists of a pair (π, E), where E is a Banach space and $\pi : \Gamma \to \mathrm{Isom}(E)$ is an isometric representation of Γ on E. We will often drop the notation π and by abuse of notation refer to E as a Banach Γ-module. A dual Banach Γ-module consists of a dual Banach space of a Banach Γ-module, together with the natural dual representation of Γ. Note that for a dual Banach Γ-module (π, E), the Banach Γ-module to which it is dual is part of the data, and we denote this predual of E by E_* so that $E = (E_*)^*$. The weak*-topology on E will always refer to the weak*-topology with respect to this duality.

A group Γ is defined in [BIP18] to be *properly proximal* if there exists an action of Γ on a compact Hausdorff space X such that there is no Γ-invariant measure on X and such that $\mathrm{Prob}(X)$ has a properly proximal point. It will be easier for us here to consider actions on convex subsets of locally convex topological vector spaces, and so we reformulate proper proximality in this setting.

Proposition 2.11. *Let Γ be an infinite discrete group. The following are equivalent:*

(i) Γ *is properly proximal.*

(ii) Γ *has a properly proximal action on a compact Hausdorff space that does not have an invariant measure.*

(iii) *There is a dual Banach Γ-module E and a non-empty Γ-invariant weak*-compact convex subset $K \subset E$ such that K has a properly proximal point (with respect to the weak*-topology), but has no fixed point.*

(iv) *There is a dual Banach Γ-module E and a non-empty Γ-invariant weak*-compact convex subset $K \subset E$ such that the action $\Gamma \curvearrowright K$ is properly proximal (with respect to the weak*-topology) but has no fixed point.*

(v) Γ *has an action by affine homeomorphisms on a non-empty compact convex subset K of a locally convex topological vector space such that the action $\Gamma \curvearrowright K$ is properly proximal but*

has no fixed point.

Proof. (i) $\implies$ (iii) is trivial by considering the weak*-compact convex set of probability measures on a compact Hausdorff space. (i) $\implies$ (ii) is also trivial as we can restrict to the closure of an orbit of a properly proximal point. (ii) $\implies$ (iv) follows from Lemma 2.10. (iv) $\implies$ (v) is trivial.

Finally, both (iii) $\implies$ (i) and (v) $\implies$ (i) follow from the simple observations that if k is a properly proximal point in a compact Hausdorff space K, then $\delta_{\{k\}}$ is a properly proximal point in $\mathrm{Prob}(K)$, and if a compact convex set has an invariant measure, then the barycenter of such a measure gives a fixed point. $\square$

Given a dual Banach Γ-module (π, E), we let E_{mix} denote the set of all points $x \in E$ such that we have weak*-convergence $\lim_{\gamma \to \infty} \pi(\gamma)x = 0$. Note that E_{mix} is a norm-closed Γ-invariant subspace of E, so that (π, E_{mix}) is also a Banach Γ-module.

We also have a characterization of proper proximality in terms of bounded cohomology, which is of independent interest.

Proposition 2.12. *Let Γ be a discrete group. Then Γ is properly proximal if and only if there is a dual Banach Γ-module E such that the induced map $H^1_b(\Gamma, E_{\mathrm{mix}}) \to H^1_b(\Gamma, E)$ has non-trivial range.*

Proof. Suppose Γ is properly proximal and let $\Gamma \curvearrowright X$ be an action on a compact Hausdorff space such that $\mathrm{Prob}(X)$ has a properly proximal point η but X has no invariant measure. We let $E = \{\zeta \in \mathrm{Meas}(X) \mid \zeta(X) = 0\} \subset C(X)^*$ and define a bounded cocycle $c : \Gamma \to E$ by $c(\gamma) = \eta - \gamma\eta$. Since η is properly proximal we have that the cocycle c ranges in E_{mix}. If we had $c(\gamma) = \zeta - \gamma\zeta$ for some $\zeta \in E$ then it would follow that $\eta - \zeta \in C(X)^*$ is Γ-invariant, and as X has no Γ-invariant probability measure we must then have $\eta = \zeta \in E$. However, $\eta \notin E$ since $\eta(X) = 1$, and hence c represents a non-trivial cohomology class in $H^1_b(\Gamma, E)$.

Conversely, suppose E is a dual Banach Γ-module and $c : \Gamma \to E_{\mathrm{mix}}$ is a bounded cocycle that represents a non-trivial cohomology class in $H^1_b(\Gamma, E)$. Consider the associated isometric affine action on E given by $\alpha(\gamma)x = \gamma x + c(\gamma)$. Note that we have weak*-topology convergence

$\lim_{\gamma\to\infty} \alpha(\gamma)\alpha(g)\cdot 0 - \alpha(\gamma)\cdot 0 = \lim_{\gamma\to\infty} \gamma c(g) = 0$. Thus, 0 is a properly proximal point with respect to the action α.

If we let K be the weak*-closure of $c(\Gamma)$, then K is weak*-compact by the Banach-Alaoglu Theorem, and we have that $\Gamma \curvearrowright^{\alpha} K$ is properly proximal. If we had an invariant measure on K, then taking the barycenter would give a Γ-fixed point in E, which would contradict the fact that the cocycle c represents a non-trivial cohomology class in $H^1_b(\Gamma, E)$. Thus Γ is properly proximal by condition (2) in Proposition 2.11. □

2.9 Multipliers on discrete groups and associated multiplier algebras

Let Γ be an infinite countable discrete group. We will denote by $c_0(\Gamma)$, the space of all complex-valued functions on Γ vanishing at infinity, i.e., $f \in c_0(\Gamma)$ if for every $\varepsilon > 0$, there exists a finite set $F \subset \Gamma$ such that $|f(s)| < \varepsilon$ for all $s \in \Gamma \setminus F$. The space of all bounded complex-valued functions on Γ will be denoted by $\ell^\infty\Gamma$, and $c_{00}(\Gamma)$ will denote the space of all finitely supported functions on Γ. For a subset $E \subset \Gamma$, we denote the characteristic function of E by $\mathbf{1}_E$.

The *Fourier-Stieltjes algebra* of Γ, denoted by $B(\Gamma)$, is the set of all coefficient functions of unitary representations of Γ, that is, for every $\varphi \in B(\Gamma)$ there exists a unitary representation $(\pi, \mathcal{H})$ of Γ and vectors $\xi, \eta \in \mathcal{H}$ such that $\varphi(s) = \langle \pi(s)\xi, \eta\rangle$ for every $s \in \Gamma$. It is a Banach algebra with respect to the norm

$$\|\varphi\|_B = \inf \|\xi\|\|\eta\|,$$

where the infimum is taken over all representations of φ as above.

The *Fourier algebra* of Γ, denoted by $A[\Gamma]$, is the set of all coefficient functions associated to the left regular representation of Γ. It is the norm closure of the algebra of finitely supported functions in the algebra $B(\Gamma)$.

A *Herz-Schur multiplier* on Γ is a function $\varphi : \Gamma \to \mathbb{C}$ for which there exists a Hilbert space $\mathcal{H}$ and bounded functions $\xi, \eta : \Gamma \to \mathcal{H}$ such that

$$\varphi(t^{-1}s) = \langle \xi(s), \eta(t)\rangle \qquad s, t \in \Gamma.$$

The set $B_2(\Gamma)$ of all Herz-Schur multipliers on Γ is a Banach algebra with respect to the pointwise product and to the norm

$$\|\varphi\|_{B_2} = \inf \|\xi\|_\infty \|\eta\|_\infty,$$

where the infimum is taken over all representations of φ as above. It turns out that $B_2(\Gamma)$ is a dual space, and the predual $Q(\Gamma)$ of $B_2(\Gamma)$ is obtained by completing $\ell^1\Gamma$ in the norm

$$\|\varphi\|_Q = \left\{ \left| \sum_{s\in\Gamma} \varphi(s)u(s) \right| \mid u \in B_2(\Gamma), \|u\|_{B_2} \leq 1 \right\}$$

(see, [Her74, DCH85]).

Definition 2.13. Let Γ be a countable discrete group.

1. ([CCJ+01]) We say that Γ has the *Haagerup property* if there exists a net $\{\varphi_i\}$ of normalized (i.e., $\varphi_i(e) = 1$ for every i), positive definite functions on Γ such that $\varphi_i \in c_0(\Gamma)$ for every i, and $\varphi_i \to 1$ pointwise.

2. ([CH89]) We say that Γ is *weakly amenable* if there exists a net $\{\varphi_i\}$ of finitely supported functions on Γ converging pointwise to the constant function 1, and such that $\sup_i \|\varphi_i\|_{B_2} \leq C$. The *Cowling-Haagerup constant* $\Lambda_{\mathrm{cb}}(\Gamma)$ is the infimum of all constants C for which such a net $\{\varphi_i\}$ exists.

3. ([Knu16]) We say that Γ has the *weak Haagerup property* if there there exists a net $\{\varphi_i\}$ in $B_2(\Gamma) \cap c_0(\Gamma)$ such that $\sup_i \|\varphi_i\|_{B_2} \leq C$ and $\varphi_i \to 1$ pointwise. The *weak Haagerup constant* $\Lambda_{\mathrm{wcb}}(\Gamma)$ is the infimum of all constants C for which such a net $\{\varphi_i\}$ exists.

4. ([HK94]) We say that Γ has the *approximation property* (AP) if there exists a net $\{\varphi_i\}$ of finitely supported functions on Γ such that $\varphi_i \to 1$ in the $\sigma(B_2(\Gamma), Q(\Gamma))$-topology.

Remark 2.14. It is easy to see that every finitely supported function on Γ can be realized as a coefficient of the left regular representation and hence $c_{00}(\Gamma) \subset A[\Gamma]$. Therefore, by [CH89, Proposition 1.1], Γ is weakly amenable if and only if there exists a net $\{\varphi_i\}$ in the Fourier algebra $A[\Gamma]$ such that $\varphi_i \to 1$ pointwise and $\sup_i \|\varphi_i\|_{B_2} < \infty$.

Remark 2.15. The inclusion map from $B(\Gamma)$ into $B_2(\Gamma)$ is a contraction (see [DCH85, Corollary 1.8]), and so the $\sigma(B_2(\Gamma), Q(\Gamma))$-closure of any subset E of $B(\Gamma)$ contains the closure of E in the $B(\Gamma)$-norm. Hence Γ has (AP) if and only if the constant function 1 is in the $\sigma(B_2(\Gamma), Q(\Gamma))$-closure of $A[\Gamma]$ in $B_2(\Gamma)$.

Since $A[\Gamma] \subset B_2(\Gamma) \cap c_0(\Gamma)$, one always has $\Lambda_{\mathrm{wcb}}(\Gamma) \leq \Lambda_{\mathrm{cb}}(\Gamma)$, and a weakly amenable group has the weak Haagerup property. Similarly, as normalized, positive definite functions are Herz-Schur multipliers of norm one, it follows that if Γ has the Haagerup property then it has the weak Haagerup property and $\Lambda_{\mathrm{wcb}}(\Gamma) = 1$. It is well known that all weakly amenable groups have (AP), and there are non-weakly amenable groups with the (AP) as well (see [HK94]).

Chapter 3

Von Neumann equivalence

In this chapter we define the notion of a fundamental domain for the action of a discrete group on a von Neumann algebra and that of von Neumann equivalence of groups. We study their basic properties and prove some fundamental results.

3.1 Fundamental domains for actions on von Neumann algebras

Definition 3.1. Let $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ be an action of a discrete group Γ on a von Neumann algebra $\mathcal{M}$. A *fundamental domain* for the action is a projection $p \in \mathcal{M}$ so that $\{\sigma_\gamma(p)\}_{\gamma\in\Gamma}$ gives a partition of unity.

Note that if $p \in \mathcal{M}$ is a fundamental domain, then we obtain an inclusion $\theta_p : \ell^\infty\Gamma \to \mathcal{M}$ by $\theta_p(f) = \sum_{\gamma\in\Gamma} f(\gamma)\sigma_{\gamma^{-1}}(p)$. Moreover, this embedding is equivariant with respect to the Γ-actions, where $\Gamma \curvearrowright \ell^\infty\Gamma$ is the canonical right action given by $R_\gamma(f)(x) = f(x\gamma)$. Conversely, if $\theta : \ell^\infty\Gamma \to \mathcal{M}$ is an equivariant embedding, then $\theta(\delta_e)$ gives a fundamental domain.

Remark 3.2. If $p \in \mathcal{M}$ is a fundamental domain, one can equivalently work with the Γ-equivariant normal inclusion $\theta_p : \ell^\infty\Gamma \to \mathcal{M}$ given by $\theta_p(f) = \sum_{\gamma\in\Gamma} f(\gamma)\sigma_\gamma(p)$. Here $\Gamma \curvearrowright \ell^\infty\Gamma$ is the canonical left action given by $L_\gamma(f)(x) = f(\gamma^{-1}x)$. This will be used in section 4.3.

Proposition 3.3. *Suppose $\mathcal{M}$ is a semi-finite von Neumann algebra with a semi-finite normal faithful trace* Tr*, and $\Gamma \curvearrowright^{\sigma} (\mathcal{M}, \mathrm{Tr})$ is a trace-preserving action that has a fundamental domain p. The following are true:*

(*i*) *The map $\tau(x) = \mathrm{Tr}(pxp)$ is independent of the fundamental domain p and defines a faithful normal semi-finite trace on $\mathcal{M}^\Gamma$. Here, $\mathcal{M}^\Gamma$ denotes the space of all Γ-fixed points.*

(*ii*) *There is a unitary operator* $\mathcal{F}_p : \ell^2\Gamma \,\overline{\otimes}\, L^2(\mathcal{M}^\Gamma, \tau) \to L^2(\mathcal{M}, \mathrm{Tr})$ *that satisfies*

$$\mathcal{F}_p(\delta_\gamma \otimes x) = \sigma_{\gamma^{-1}}(p)x,$$

for $x \in \mathfrak{n}_\tau \subset \mathcal{M}^\Gamma$ *and* $\gamma \in \Gamma$.

(*iii*) *The operator* $\mathcal{F}_p$ *satisfies*

$$\mathcal{F}_p(1 \otimes JxJ) = JxJ\mathcal{F}_p, \quad \mathcal{F}_p(\rho_\gamma \otimes 1) = \sigma^0_\gamma \mathcal{F}_p, \quad (f \otimes 1)\mathcal{F}_p = \mathcal{F}_p\theta_p(f), \tag{3.1}$$

for $x \in \mathcal{M}^\Gamma$, $\gamma \in \Gamma$, *and* $f \in \ell^\infty\Gamma$, *where* $\theta_p : \ell^\infty\Gamma \to \mathcal{M}$ *is the* Γ*-equivariant embedding given by* $\theta_p(f) = \sum_{\gamma\in\Gamma} f(\gamma)\sigma_{\gamma^{-1}}(p)$.

(*iv*) $\mathcal{F}_p^* \langle \mathcal{M}, \mathcal{M}^\Gamma \rangle \mathcal{F}_p = \mathcal{B}(\ell^2\Gamma) \,\overline{\otimes}\, \mathcal{M}^\Gamma$.

(*v*) *We have* $\mathcal{M} = W^*(\theta_p(\ell^\infty\Gamma), \mathcal{M}^\Gamma)$, *and, in fact,*

$$\mathrm{span}\{\theta_p(f)x \mid f \in \ell^\infty\Gamma, x \in \mathcal{M}^\Gamma\}$$

is strong operator topology dense in $\mathcal{M}$.

(*vi*) *If* $\alpha \in \mathrm{Aut}(\mathcal{M})$ *is an automorphism that preserves* Tr *and is* Γ*-equivariant, then* $\alpha_{|\mathcal{M}^\Gamma}$ *preserves* τ.

Proof. If $x \in \mathcal{M}^\Gamma$ such that $\tau(x^*x) = 0$, then as Tr is faithful we have $xp = 0$. We then have $x\sigma_\gamma(p) = \sigma_\gamma(xp) = 0$ for all $\gamma \in \Gamma$, and since $\sum_{\gamma\in\Gamma} \sigma_\gamma(p) = 1$ we then have $x = 0$, so that τ is faithful.

As Tr is semi-finite, there exists an increasing net of finite-trace projections $\{q_i\}_{i\in I}$ so that $q_i \to p$ in the weak operator topology. If we set $\tilde{q}_i = \sum_{\gamma\in\Gamma} \sigma_\gamma(q_i)$, then as p is a fundamental domain for Γ, it follows that $\{\tilde{q}_i\}_{i\in I}$ gives an increasing net of projections in $\mathcal{M}^\Gamma$ that converges in the weak operator topology to $\sum_{\gamma\in\Gamma} \sigma_\gamma(p) = 1$, and satisfies $\tau(\tilde{q}_i) = \mathrm{Tr}(q_i) < \infty$ for each $i \in I$. Therefore τ is semi-finite.

If q is another Γ-fundamental domain then we also have

$$\tau(x^*x) = \mathrm{Tr}(px^*xp) = \sum_{\gamma\in\Gamma} \mathrm{Tr}(px^*\sigma_\gamma(q)xp)$$
$$= \sum_{\gamma\in\Gamma} \mathrm{Tr}(\sigma_\gamma(q)xpx^*\sigma_\gamma(q)) = \sum_{\gamma\in\Gamma} \mathrm{Tr}(qx\sigma_{\gamma^{-1}}(p)x^*q) = \mathrm{Tr}(qxx^*q).$$

Thus τ is independent of the fundamental domain and defines a trace, proving (a).

If $x \in \mathfrak{n}_\tau \subset \mathcal{M}^\Gamma$, then $px \in \mathfrak{n}_{\mathrm{Tr}}$ and we have $\|x\|_\tau^2 = \tau(xx^*) = \mathrm{Tr}(pxx^*p) = \|px\|_{\mathrm{Tr}}^2$, so the map $\mathfrak{n}_\tau \ni x \mapsto px \in pL^2(\mathcal{M}, \mathrm{Tr})$ is isometric with respect to the trace norms. If $T \in \mathfrak{n}_{\mathrm{Tr}}$, then for each $\gamma \in \Gamma$ we set $a_\gamma^T = \sum_{\lambda\in\Gamma} \sigma_\lambda(pT\sigma_\gamma(p))$. Note that since p is a fundamental domain, this sum converges in the strong operator topology, and we have $a_\gamma^T \in \mathcal{M}^\Gamma$. We then compute

$$pT = \sum_{\gamma\in\Gamma} pT\sigma_\gamma(p) = \sum_{\gamma\in\Gamma} pa_\gamma^T,$$

where the sums converge in $pL^2(\mathcal{M}, \mathrm{Tr})$. Since $T \in \mathfrak{n}_{\mathrm{Tr}}$ was arbitrary, this shows that $\mathfrak{n}_\tau \ni x \mapsto px$ has dense range in $pL^2(\mathcal{M}, \mathrm{Tr})$, showing that this map extends to a unitary from $L^2(\mathcal{M}^\Gamma, \tau)$ onto $pL^2(\mathcal{M}, \mathrm{Tr})$. Since p is a fundamental domain we have a direct sum decomposition $L^2(\mathcal{M}, \mathrm{Tr}) = \sum_{\gamma\in\Gamma} \sigma_{\gamma^{-1}}(p)L^2(\mathcal{M}, \mathrm{Tr})$, and (ii) then follows easily.

Let $\gamma, g \in \Gamma$ and $x, y \in \mathfrak{n}_\tau$. The following three computations verify (iii):

$$\mathcal{F}_p(1 \otimes JxJ)(\delta_g \otimes y) = \mathcal{F}_p(\delta_g \otimes JxJy) = \sigma_{g^{-1}}(p)JxJy = JxJ\sigma_{g^{-1}}(p)y = JxJ\mathcal{F}_p(\delta_g \otimes y),$$

$$\mathcal{F}_p(\rho_\gamma \otimes 1)(\delta_g \otimes y) = \mathcal{F}_p(\delta_{g\gamma^{-1}} \otimes y) = \sigma_{\gamma g^{-1}}(p)y = \sigma_\gamma^0(\sigma_{g^{-1}}(p)y) = \sigma_\gamma^0\mathcal{F}_p(\delta_g \otimes y),$$

$$\mathcal{F}_p(f \otimes 1)(\delta_g \otimes y) = \mathcal{F}_p(f\delta_g \otimes y) = \mathcal{F}_p(f(g)\delta_g \otimes y) = f(g)\mathcal{F}_p(\delta_g \otimes y)$$
$$= f(g)\sigma_{g^{-1}}(p)y = \left(\sum_{\gamma\in\Gamma} f(\gamma)\sigma_{\gamma^{-1}}(p)\right)\sigma_{g^{-1}}(p)y = \theta_p(f)\sigma_{g^{-1}}(p)y = \theta_p(f)\mathcal{F}_p(\delta_g \otimes y).$$

To verify (iv), let $x, y \in \mathcal{M}^\Gamma, T \in \mathcal{B}(\ell^2\Gamma)$, and $\xi, \eta \in \mathfrak{n}_{\mathrm{Tr}}$. Then,

$$\begin{aligned}
\langle JyJ\mathcal{F}_p(T\otimes x)\mathcal{F}_p^*\xi, \eta\rangle &= \langle (T\otimes x)\mathcal{F}_p^*\xi, \mathcal{F}_p^* Jy^*J\eta\rangle \\
&= \langle (T\otimes x)\mathcal{F}_p^*\xi, (1\otimes Jy^*J)\mathcal{F}_p^*\eta\rangle \\
&= \langle \xi, \mathcal{F}_p(T^*\otimes x^*Jy^*J)\mathcal{F}_p^*\eta\rangle \\
&= \langle \xi, \mathcal{F}_p(1\otimes Jy^*J)(T^*\otimes x^*)\mathcal{F}_p^*\eta\rangle \\
&= \langle \xi, Jy^*J\mathcal{F}_p(T^*\otimes x^*)\mathcal{F}_p^*\eta\rangle \\
&= \langle \mathcal{F}_p(T\otimes x)\mathcal{F}_p^* JyJ\xi, \eta\rangle,
\end{aligned}$$

whence it follows that $\mathcal{F}_p(T\otimes x)\mathcal{F}_p^* \in (J\mathcal{M}^\Gamma J)' = \langle \mathcal{M}, \mathcal{M}^\Gamma\rangle$, and this establishes (iv).

As in part (ii), if $T \in \mathcal{M}$ and $\gamma_1, \gamma_2 \in \Gamma$, then

$$\sigma_{\gamma_1}(p)T\sigma_{\gamma_2}(p) = \sigma_{\gamma_1}(p)\left(\sum_{\gamma\in\Gamma}\sigma_\gamma(\sigma_{\gamma_1}(p)T\sigma_{\gamma_2}(p))\right) \in \mathrm{span}\{\theta_p(f)x \mid x\in\mathcal{M}^\Gamma, f\in\ell^\infty\Gamma\},$$

where the sum converges in the strong operator topology. We also have strong operator topology convergence

$$T = \sum_{\gamma_1,\gamma_2\in\Gamma}\sigma_{\gamma_1}(p)T\sigma_{\gamma_2}(p),$$

and hence (b) follows.

If $\alpha \in \mathrm{Aut}(\mathcal{M})$ is a Γ-equivariant automorphism that preserves Tr, and if p is a Γ-fundamental domain, then $\alpha(p)$ is also a Γ-fundamental domain, and hence for $x \in \mathcal{M}^\Gamma$ we have

$$\tau(\alpha(x^*x)) = \mathrm{Tr}(p\alpha(x^*x)p) = \mathrm{Tr}(\alpha(p)x^*x\alpha(p)) = \tau(x^*x),$$

showing (vi). □

Proposition 3.4. *Suppose* $\Gamma \curvearrowright (\mathcal{M}, \mathrm{Tr})$ *is a trace-preserving action with fundamental domain* p.

Then there exists a trace-preserving isomorphism $\Delta_p : \mathcal{M} \rtimes \Gamma \to \mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma$ *such that*

$$\Delta_p(u_\gamma) = \rho_\gamma \otimes 1, \quad \Delta_p(x) = \mathcal{F}_p^* x \mathcal{F}_p$$

for $\gamma \in \Gamma$, $x \in \mathcal{M} \subset \mathcal{M} \rtimes \Gamma$. *In particular, we have* $\mathcal{M} \rtimes \Gamma \cong \mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma \cong \langle \mathcal{M}, \mathcal{M}^\Gamma \rangle$.

Proof. We let $\mathcal{F}_p : \ell^2\Gamma \overline{\otimes} L^2(\mathcal{M}^\Gamma, \tau) \to L^2(\mathcal{M}, \mathrm{Tr})$ be the unitary from Proposition 3.3. We define a unitary operator $W \in \mathcal{U}(\ell^2\Gamma \overline{\otimes} L^2(\mathcal{M}^\Gamma, \tau) \overline{\otimes} \ell^2\Gamma)$ by $W(\delta_\gamma \otimes \xi \otimes \eta) = \delta_\gamma \otimes \xi \otimes \lambda_\gamma \eta$. We then check that $W(\mathcal{F}_p^* \otimes 1)$ gives a unitary intertwiner between $\mathcal{M} \rtimes \Gamma$ and $\mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma \overline{\otimes} \mathbb{C} = (\mathcal{F}_p^* \otimes 1)(\langle \mathcal{M}, \mathcal{M}^\Gamma \rangle \otimes \mathbb{C})(\mathcal{F}_p \otimes 1)$, which takes u_γ to $\rho_\gamma \otimes 1 \otimes 1$ for each $\gamma \in \Gamma$, and takes x to $\mathcal{F}_p^* x \mathcal{F}_p$ for each $x \in \mathcal{M} \subset \mathcal{M} \rtimes \Gamma$. □

We note that if $x \in \mathcal{M}$, then we also have an explicit form for $\Delta_p(x)$. Indeed, if we view $\mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma$ as $\mathcal{M}^\Gamma$-valued $\Gamma \times \Gamma$ matrices, then it's simple to check that $\Delta_p(x) = [x_{s,t}]_{s,t}$, where

$$x_{s,t} = \sum_{\gamma \in \Gamma} \sigma_\gamma(\sigma_{t^{-1}}(p) x \sigma_{s^{-1}}(p)) \in \mathcal{M}^\Gamma.$$

Proposition 3.5. *Suppose* $\Gamma \curvearrowright (\mathcal{M}, \mathrm{Tr})$ *is a trace-preserving action with fundamental domains* p *and* q. *Then, using the notation above, we have* $\mathcal{F}_q^* \mathcal{F}_p \in \mathcal{U}(L\Gamma \overline{\otimes} \mathcal{M}^\Gamma)$ *and* $\Delta_p(p)(\mathcal{F}_q^* \mathcal{F}_p) = (\mathcal{F}_q^* \mathcal{F}_p) \Delta_p(q)$.

Proof. By (3.1) and Proposition 3.4 we have $\mathcal{F}_q^* \mathcal{F}_p \in (\rho(\Gamma) \otimes \mathbb{C})' \cap \mathcal{B}(\ell^2\Gamma) \overline{\otimes} \mathcal{M}^\Gamma = L\Gamma \overline{\otimes} \mathcal{M}^\Gamma$. Moreover, by Proposition 3.4 we have

$$\Delta_p(p) \mathcal{F}_q^* \mathcal{F}_p = (\delta_e \otimes 1) \mathcal{F}_q^* \mathcal{F}_p = \mathcal{F}_q^* q \mathcal{F}_p = \mathcal{F}_q^* \mathcal{F}_p \Delta_p(q).$$

□

Definition 3.6. Let Λ and Γ be countable groups. A *von Neumann coupling* between Λ and Γ consists of a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr and a trace-preserving action $\Lambda \times \Gamma \curvearrowright \mathcal{M}$ such that there exist finite-trace fundamental domains q and p for the Λ and Γ-actions, respectively. The *index* of the von Neumann coupling is the ratio $\mathrm{Tr}(p)/\mathrm{Tr}(q)$ and is denoted by $[\Gamma : \Lambda]_{\mathcal{M}}$. This is well-defined by Proposition 3.3.

By Definition 1.1, Λ and Γ are von Neumann equivalent if there exists a von Neumann coupling between them.

Note that the notion of von Neumann equivalence coincides with measure equivalence when restricting to the case when $\mathcal{M}$ is abelian. Also, if we have an isomorphism $\theta : L\Lambda \to L\Gamma$, then setting $\mathcal{M} = \mathcal{B}(L^2(L\Gamma))$ we have an action of Γ by conjugation by ρ_γ, an action of Λ by conjugation by $\theta(u_\lambda)$, and a common fundamental domain P_e, so that if Γ and Λ are W^*-equivalent, then they are also von Neumann equivalent. More generally, we have the following construction:

Example 3.7. Suppose Λ and Γ are countable groups, we have trace-preserving actions $\Gamma \curvearrowright (M_1, \tau)$ and $\Lambda \curvearrowright (M_2, \tau)$, and a trace-preserving isomorphism $\theta : M_2 \rtimes \Lambda \to M_1 \rtimes \Gamma$ such that $\theta(M_1) = M_2$. Then θ extends to an isomorphism of basic constructions $\tilde{\theta} : \langle M_2 \rtimes \Lambda, M_2 \rangle \to \langle M_1 \rtimes \Gamma, M_1 \rangle$ such that $\tilde{\theta}(e_{M_2}) = e_{M_1}$.

For $\gamma \in \Gamma$ we have $[u_\gamma(Ju_\gamma J), e_{M_1}] = 0$ and hence $\Gamma \ni \gamma \mapsto \mathrm{Ad}(Ju_\gamma J)$ describes a trace-preserving action of Γ on $\langle M_1 \rtimes \Gamma, M_1 \rangle$, which pointwise fixes $M_1 \rtimes \Gamma$. In particular, we have that $\Lambda \ni \lambda \mapsto \mathrm{Ad}(\tilde{\theta}(u_\lambda))$ gives an action that commutes with the action of Γ, and we have that e_{M_1} gives a fundamental domain for both the Γ and Λ-actions. Therefore, $\langle M_1 \rtimes \Gamma, M_1 \rangle \cong M_1 \overline{\otimes} \mathcal{B}(\ell^2\Gamma)$ gives an index-one von Neumann coupling.

Remark 3.8. We have a partial converse of the previous example, which is in the spirit of Theorem 3.3 from [Fur99a]. If $\mathcal{M}$ is a von Neumann coupling, then by Proposition 3.4 we have isomorphisms $\mathcal{B}(\ell^2\Gamma) \overline{\otimes} (\mathcal{M}^\Gamma \rtimes \Lambda) \cong \mathcal{M} \rtimes (\Gamma \times \Lambda) \cong \mathcal{B}(\ell^2\Lambda) \overline{\otimes} (\mathcal{M}^\Lambda \rtimes \Gamma)$. Therefore $\mathcal{M}^\Gamma \rtimes \Lambda$ is a factor if and only if $\mathcal{M}^{\Lambda} \rtimes \Gamma$ is a factor, and in this case we have that $\mathcal{M}^{\Gamma} \rtimes \Lambda$ and $\mathcal{M}^{\Lambda} \rtimes \Gamma$ are stably isomorphic.

Just as in the case of measure equivalence, von Neumann equivalence is an equivalence relation. Reflexivity follows by considering the trivial von Neumann Γ-coupling $\ell^{\infty}\Gamma$. Symmetry is obvious, and transitivity follows from the following proposition.

Proposition 3.9. *Let* $(\mathcal{N}, \mathrm{Tr}_{\mathcal{N}})$ *and* $(\mathcal{M}, \mathrm{Tr}_{\mathcal{M}})$ *be* (Σ, Λ) *and* (Λ, Γ) *von Neumann couplings, respectively. We consider the natural action of* Σ, Λ*, and* Γ *on* $\mathcal{N} \overline{\otimes} \mathcal{M}$*, where* Λ *acts diagonally. Then* $\mathcal{N} \overline{\otimes} \mathcal{M}$ *has a* Λ*-fundamental domain, and the induced semi-finite trace on* $(\mathcal{N} \overline{\otimes} \mathcal{M})^{\Lambda}$ *gives a* (Σ, Γ) *von Neumann coupling with index*

$$[\Sigma : \Gamma]_{(\mathcal{N} \overline{\otimes} \mathcal{M})^{\Lambda}} = [\Sigma : \Lambda]_{\mathcal{N}} [\Lambda : \Gamma]_{\mathcal{M}}.$$

Proof. If q is a Λ-fundamental domain for $\mathcal{N}$, then $q \otimes 1$ gives a fundamental domain for the action on $\mathcal{N} \overline{\otimes} \mathcal{M}$. We therefore obtain an induced $\Sigma \times \Gamma$-invariant semi-finite normal faithful trace on $(\mathcal{N} \overline{\otimes} \mathcal{M})^{\Lambda}$ by Proposition 3.3.

If $p \in \mathcal{M}$ is a fundamental domain for Γ, then we see that $\sum_{\lambda \in \Lambda} \sigma_{\lambda}(q) \otimes \sigma_{\Lambda}(p) \in (\mathcal{N} \overline{\otimes} \mathcal{M})^{\Lambda}$ is a fundamental domain for Γ with trace $\mathrm{Tr}(q)\mathrm{Tr}(p) < \infty$. Similarly, if $r \in \mathcal{N}$ is a fundamental domain for Σ, and if $\tilde{q} \in \mathcal{M}$ is a fundamental domain for Λ, then $\sum_{\lambda \in \Lambda} \sigma_{\lambda}(r) \otimes \sigma_{\lambda}(\tilde{q}) \in (\mathcal{N} \overline{\otimes} \mathcal{M})^{\Lambda}$ is a fundamental domain for Σ with trace $\mathrm{Tr}(r)\mathrm{Tr}(\tilde{q}) < \infty$.

Hence, $(\mathcal{N} \overline{\otimes} \mathcal{M})^{\Lambda}$ is a (Σ, Γ) von Neumann coupling with index

$$[\Sigma : \Gamma]_{(\mathcal{N} \overline{\otimes} \mathcal{M})^{\Lambda}} = \mathrm{Tr}(q)\mathrm{Tr}(p)/\mathrm{Tr}(r)\mathrm{Tr}(\tilde{q}) = [\Sigma : \Lambda]_{\mathcal{N}} [\Lambda : \Gamma]_{\mathcal{M}}.$$

□

Chapter 4

Von Neumann equivalence and group approximation properties

We introduce induction procedures in this chapter. More precisely, we develop procedures for inducing group actions, unitary representations, and Herz-Schur multipliers via von Neumann equivalence from Λ to Γ, and using these we prove, respectively, Theorems 1.4, 1.2, and 1.3.

4.1 Inducing actions via semi-finite von Neumann algebras

If Γ is a group, then an *operator* Γ*-module* consists of a pair (π, E), where E is an operator space and $\pi : \Gamma \to CI(E)$ a homomorphism from Γ to the group of surjective complete isometries of E. A dual operator Γ-module consists of a dual operator space $E = (E_*)^*$ that is an operator Γ-module such that the action of Γ is dual to an action on E_*. Note that if $X = (X_*)^*$ is a dual Banach Γ-module, then we can regard X also as an operator Γ-module by endowing X_* with the operator space structure $\min(X_*)$, so that $\max(X) = (\min(X_*))^*$ becomes a dual operator Γ-module.

Definition 4.1. Let Γ and Λ be discrete groups and suppose that $\Gamma \times \Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ is a trace-preserving action on a semi-finite von Neumann algebra $\mathcal{M}$. Let E be a dual operator Λ-module.

(i) Letting Γ act trivially on E, we obtain an isometric action $\Gamma \curvearrowright \mathcal{M}_* \widehat{\otimes} E_*$, and hence a dual action $\Gamma \curvearrowright CB(\mathcal{M}_*, E) = (\mathcal{M}_* \widehat{\otimes} E_*)^*$, which we may then restrict to $(\mathcal{M} \overline{\otimes} E)^\Lambda$. We call $(\mathcal{M} \overline{\otimes} E)^\Lambda$ the *dual operator* Γ*-module induced from* E.

(ii) If $K \subset E$ is a non-empty convex weak*-closed subset that is Λ-invariant, then, considering the embedding $\mathcal{M} \overline{\otimes} E \subset CB(\mathcal{M}_*, E)$, we let $\mathcal{M} \overline{\otimes} K$ denote those maps $\Xi \in CB(\mathcal{M}_*, E)$ such that $\Xi(\varphi) \in K$ for each normal state φ. We then have that $\mathcal{M} \overline{\otimes} K \subset \mathcal{M} \overline{\otimes} E$ is a convex subset that is invariant under the actions of Γ and Λ. Hence we have an action $\Gamma \curvearrowright (\mathcal{M} \overline{\otimes} K)^\Lambda$, which we refer to as the Γ*-action induced from the* Λ*-action* $\Lambda \curvearrowright K$.

As motivation for Definition 4.1, note that if (X, μ) is a standard measure space and $\mathcal{M} = L^\infty(X, \mu)$, then Proposition 2.8 gives an identification between $L^\infty(X, \mu) \overline{\otimes} K$ and $L^\infty_{w^*}(X, \mu; K)$, so that $(L^\infty(X, \mu) \overline{\otimes} K)^\Lambda$ can be identified as the space of Λ-equivariant measurable functions from X to K.

Lemma 4.2. *Using the notation above, if K is weak*-compact, then $\mathcal{M} \overline{\otimes} K$ is a weak*-compact subset of $\mathcal{M} \overline{\otimes} E$.*

Proof. Since K is weak*-compact, it is bounded, and hence $\mathcal{M} \overline{\otimes} K$ is a norm bounded subset of $\mathcal{M} \overline{\otimes} E$. Viewing elements in $\mathcal{M} \overline{\otimes} K$ as maps from $\mathcal{M}_*$ to E, we then have that the weak*-topology coincides with the topology of pointwise weak*-convergence. Since K is weak*-closed, it follows that $\mathcal{M} \overline{\otimes} K$ is also weak*-closed, hence weak*-compact by the Banach-Alaoglu Theorem. □

Proposition 4.3. *Using the notation above, suppose that $\mathcal{M}^\Gamma$ has a normal Λ-invariant finite trace τ. Then there exists a Γ-fixed point in $(\mathcal{M} \overline{\otimes} K)^\Lambda$ if and only if there exists a Λ-fixed point in K.*

Proof. If $k_0 \in K$ is fixed by Λ, then we have that $1 \otimes k_0 \in (\mathcal{M} \overline{\otimes} K)^\Lambda$ is clearly Γ-invariant.

Conversely, suppose $\Xi \in (\mathcal{M} \overline{\otimes} K)^\Lambda \subset CB(\mathcal{M}_*, K)^\Lambda$ is Γ-invariant. Under the Banach space isomorphism $CB(\mathcal{M}_*, E) \cong CB(E_*, \mathcal{M})$, we see that we may make the identification $CB(\mathcal{M}_*, K)^\Gamma \cong CB((\mathcal{M}^\Gamma)_*, K)$, so that we may view Ξ as a completely bounded Λ-equivariant map from $(\mathcal{M}^\Gamma)_*$ into E taking states into K. Since Λ preserves the trace τ on $\mathcal{M}^\Gamma$, we have that $\Xi(\tau) \in K$ is Λ-invariant. □

Lemma 4.4. *Let $E = (E_*)^*$ be a dual operator space, and let $\mathcal{M}$ be a von Neumann algebra. Suppose $x, y \in \mathcal{M}$, and $\{p_i\}_{i \in I}$ is a family of pairwise orthogonal projections in $\mathcal{M}$. If $\{a_i\}_{i \in I} \subset E$ is any uniformly bounded family in E, then the sum $\sum_{i \in I} x p_i y \otimes a_i$ converges weak* in $\mathcal{M} \overline{\otimes} E$. Moreover, we have*

$$\| \sum_{i \in I} x p_i y \otimes a_i \| \leq \|x\| \|y\| (\sup_{i \in I} \|a_i\|).$$

Proof. By representing E as an ultraweakly closed subspace of a Hilbert space $\mathcal{H}$, it suffices to show this when $E = \mathcal{B}(\mathcal{H})$. Since $\{p_i\}_i$ are pairwise orthogonal, we then have that $\sum_{i\in I} p_i \otimes a_i$ converges ultraweakly and hence so does

$$\sum_{i\in I} xp_i y \otimes a_i = (x \otimes 1)\left(\sum_{i\in I} p_i \otimes a_i\right)(y \otimes 1).$$

Moreover,

$$\|\sum_{i\in I} xp_i y \otimes a_i\| \leq \|x \otimes 1\|\|\sum_{i\in I} p_i \otimes a_i\|\|y \otimes 1\| = \|x\|\|y\| \sup_{i\in I} \|a_i\|.$$

□

Lemma 4.5. *Let E be a dual operator Λ-module, suppose $K \subset E$ is a non-empty Λ-invariant convex weak*-closed subset, and let $\mathcal{M}$ be a von Neumann algebra on which Λ acts. If the action of Λ on $\mathcal{M}$ has a fundamental domain p then the map $\chi_p : K \to (\mathcal{M} \overline{\otimes} K)^\Lambda$ defined by*

$$\chi_p^k = \sum_{\lambda\in\Lambda} \sigma_\lambda(p) \otimes \lambda k$$

gives a well-defined, weak*-continuous affine isometric map. In particular, $(\mathcal{M} \overline{\otimes} K)^\Lambda$ is non-empty in this case.

Proof. We first note that the sum defining χ_p^k converges weak* by Lemma 4.4. It is also easy to see by a change of variables that we have $\chi_p^k \in (\mathcal{M} \overline{\otimes} E)^\Lambda$.

If φ is a normal state on $\mathcal{M}$, then we obtain a probability measure μ on Λ given by $\mu(\lambda) = \varphi(\sigma_\lambda(p))$. Viewing χ_p^k as a map from $\mathcal{M}_*$ to E, we see that it takes φ to the element $\int \lambda k \, d\mu(\lambda) \in K$, so that χ_p maps into $(\mathcal{M} \overline{\otimes} K)^\Lambda$.

We clearly have that χ_p is affine, and by Lemma 4.4 we have $\|\chi_p\| \leq \|k\|$. Also, $\|k\| = \|p \otimes k\| \leq \|\chi_p^k\|$ so that χ_p is isometric. Finally, note that if $k_i \to k$ weak*, then for each $\eta \in \mathcal{M}_*$ we have $\sum_{\lambda\in\Lambda} |\eta(\sigma_\lambda(p))| \leq \|\eta\|$. It then follows that $\sum_{\lambda\in\Lambda} \eta(\sigma_\lambda(p))\lambda k_i \to \sum_{\lambda\in\Lambda} \eta(\sigma_\lambda(p))\lambda k$ weak*, and since $\eta \in \mathcal{M}_*$ was arbitrary, this shows that $\chi_p^{k_i} \to \chi_p^k$ weak*. □

4.1.1 Properly proximal actions

In this section we show that if Λ has a fundamental domain and the Γ-action on $\mathcal{M}$ is mixing, then points that are properly proximal for a Λ-action can be induced to points that are Γ-properly proximal. At the heart of the argument is Lemma 4.7, which allows us to compare the induction maps χ_p and χ_q from Lemma 4.5 corresponding to different fundamental domains p and q.

Lemma 4.6. *Let $\mathcal{M}$ be von Neumann algebra, and fix $\varphi \in \mathcal{M}_*$. If a sequence $x_n \in \mathcal{M}$ converges in the ultrastrong topology to 0, then $\lim_{n\to\infty} \|x_n\varphi\|_* = 0$.*

Proof. By considering the polar decomposition of φ, it is enough to consider the case when φ is a state. Since $x_n \to 0$ in the ultrastrong topology, we have $x_n^* x_n \to 0$ in the ultraweak topology. Hence by Cauchy-Schwarz we have

$$\|x_n\varphi\|_* = \sup_{a\in\mathcal{M}, \|a\|\le 1} |\varphi(ax_n)| \le \varphi(aa^*)^{1/2}\varphi(x_n^* x_n)^{1/2} \to 0.$$

□

Lemma 4.7. *Suppose $\Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ is a trace-preserving action, E is a dual operator Λ-module, and a point $k \in E$ is properly proximal. Fix $A \in \mathfrak{m}_{\mathrm{Tr}}$ and suppose $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathrm{Aut}(\mathcal{M}, \mathrm{Tr})$ is a sequence of trace-preserving automorphisms commuting with the action of Λ and such that $\alpha_n(A) \to 0$ in the weak operator topology. Then for any finite-trace fundamental domains $p, q \in \mathcal{P}(\mathcal{M})$, we have weak*-convergence*

$$\lim_{n\to\infty} \chi_p^k(\alpha_n(A)) - \chi_q^k(\alpha_n(A)) = 0,$$

where $\chi_p^k, \chi_q^k \in (\mathcal{M} \overline{\otimes} E)^\Lambda$ are defined as in Lemma 4.5.

Proof. Consider the trace-preserving embedding $\Delta_p : \mathcal{M} \to \mathcal{B}(\ell^2\Lambda) \overline{\otimes} \mathcal{M}^\Lambda$ as given in Proposition 3.4. This then gives a corresponding restriction map from $(\mathcal{B}(\ell^2\Lambda) \overline{\otimes} \mathcal{M}^\Lambda)_*$ to $\mathcal{M}_*$, and by composing this with χ_p^k (where we view $\chi_p^k \in CB(\mathcal{M}_*, E)$), we obtain a map (which we still denote by χ_p^k) from $(\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda)_*$ into E. This map is Λ-equivariant, where Λ acts on $(\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda)_*$ by conjugation with $\rho_\gamma \otimes 1$.

Note also that the isomorphism $\mathcal{M} \rtimes \Gamma \cong \mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda$ shows that the automorphisms α_n extend to trace-preserving automorphisms of $\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda$, which we also denote by α_n, and which fix $R\Gamma \otimes \mathbb{C}$. Part (vi) of Proposition 3.3 applied to Λ acting by conjugation on $\mathcal{M} \rtimes \Lambda$ then shows that α_n also preserves the finite trace on $L\Gamma\,\overline{\otimes}\,\mathcal{M}^\Lambda = R\Gamma' \cap (\mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,\mathcal{M}^\Lambda)$.

Fix $A \in \mathfrak{m}_{\mathrm{Tr}\otimes\tau} \subset \mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda$ and suppose that $\alpha_n(A) \to 0$ weakly. Fix $t \in \Lambda$, $u \in L\Gamma\,\overline{\otimes}\,\mathcal{M}^\Lambda$ and $v \in \mathcal{M}^\Lambda$. We have

$$
\begin{aligned}
&\chi_p^k((\lambda_t \otimes v)u\alpha_n(A) - u\alpha_n(A)(\lambda_t \otimes v)) \\
&\qquad = \sum_{s\in\Lambda} (\mathrm{Tr}\otimes\tau)(((\lambda_t \otimes v)u\alpha_n(A) - u\alpha_n(A)(\lambda_t \otimes v))(\rho_s P_e \rho_s^* \otimes 1))sk \\
&\qquad = \sum_{s\in\Lambda} (\mathrm{Tr}\otimes\tau)((\lambda_t \otimes v)u\alpha_n(A)(\rho_s P_e \rho_s^* \otimes 1))(sk - st^{-1}k).
\end{aligned}
$$

Since we have weak operator topology convergence $u\alpha_n(A) \to 0$, and since τ is a finite trace on $\mathcal{M}^\Lambda$, it follows that for any finite set $F \subset \Lambda$ we have

$$
\sum_{s\in F} (\mathrm{Tr}\otimes\tau)((\lambda_t \otimes v)u\alpha_n(A)(\rho_s P_e \rho_s^* \otimes 1)) \to 0.
$$

Since k is properly proximal, and since $\{\alpha_n(A)\}_n$ is uniformly bounded in trace norm, it follows that we have weak*-convergence

$$
\chi_p^k((\lambda_t \otimes x)u\alpha_n(A) - u\alpha_n(A)(\lambda_t \otimes x)) \to 0.
$$

Taking linear combinations of vectors of the form $\lambda_t \otimes v$, it follows that for all $z \in \mathbb{C}\Lambda \otimes_{\mathrm{alg}} \mathcal{M}^\Lambda$, we have weak*-convergence

$$
\chi_p^k(zu\alpha_n(A) - u\alpha_n(A)z) \to 0. \tag{4.1}
$$

If $\{z_m\}_m \subset L\Lambda \overline{\otimes} \mathcal{M}^\Lambda$ is uniformly bounded, then z_n converge to 0 in the ultrastrong* topology if and only if z_n converge in $\|\cdot\|_2$ with respect to the trace. If this is the case, then as α_n are trace-preserving, we have from Lemma 4.6 that

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\|\alpha_n^{-1}(z_m u)A\|_{\mathrm{Tr}\otimes\tau} = \lim_{m\to\infty}\sup_{n\in\mathbb{N}}\|A\alpha_n^{-1}(z_m u)\|_{\mathrm{Tr}\otimes\tau} = 0.$$

Kaplansky's Density Theorem then shows that we have (4.1) for all $z \in L\Gamma \overline{\otimes} \mathcal{M}^\Lambda$ and all $u \in L\Gamma \overline{\otimes} \mathcal{M}^\Lambda$.

By Proposition 3.5 there exists a unitary $u \in \mathcal{U}(L\Lambda \overline{\otimes} \mathcal{M}^\Lambda)$ so that $\chi_q^k(A) = \chi_p^k(uAu^*)$ for all $A \in \mathcal{M}_*$. We then have weak*-convergence

$$\chi_p^k(\alpha_n(A)) - \chi_q^k(\alpha_n(A)) = \chi_p^k(u^*(u\alpha_n(A)) - (u\alpha_n(A))u^*) \to 0.$$

□

Proposition 4.8. *Suppose* $\Gamma \times \Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ *is a trace-preserving action such that the action of* Λ *has a finite-trace fundamental domain and the Koopman representation* $\Gamma \curvearrowright L^2(\mathcal{M}, \mathrm{Tr})$ *is mixing. Suppose* E *is a dual operator* Λ*-module and* $K \subset E$ *is a non-empty convex weak***-compact* Λ*-invariant subset. If the action* $\Lambda \curvearrowright K$ *has a point that is properly proximal, then so does the induced action* $\Gamma \curvearrowright (\mathcal{M} \overline{\otimes} K)^\Lambda$.

Proof. We fix a point $k \in K$ that is properly proximal for the action $\Lambda \curvearrowright K$. Given a finite-trace Λ-fundamental domain $p \in \mathcal{M}$, we let $\chi_p : K \to (\mathcal{M} \overline{\otimes} K)^\Lambda$ be defined by $\chi_p^k = \sum_{s\in\Lambda} \sigma_s(p) \otimes sk$ as in Lemma 4.5, and we view χ_p^k as a Λ-equivariant map from $\mathcal{M}_*$ to E^*.

Fix $g \in \Gamma$, and suppose $\{\gamma_n\}_n \subset \Gamma$ is such that $\gamma_n \to \infty$. If $A \in \mathfrak{m}_{\mathrm{Tr}}$, then as the action of Γ is mixing, we have that $\sigma_{\gamma_n}(A)$ converges to 0 in the weak operator topology. Therefore, if we consider the Λ-fundamental domain $q = \sigma_g(p)$, then by Lemma 4.7 we have weak*-convergence

$$\chi_p^k(\sigma_{\gamma_n^{-1}}(A)) - \chi_p^k(\sigma_{g^{-1}\gamma_n^{-1}}(A)) = \chi_p^k(\sigma_{\gamma_n^{-1}}(A)) - \chi_q^k(\sigma_{\gamma_n^{-1}}(A)) \to 0.$$

As the set of such A is dense in $\mathcal{M}_*$, the result follows. □

Theorem 4.9. *Suppose* $\Gamma \times \Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ *is a trace-preserving action such that* $\mathcal{M}^\Gamma$ *has a normal* Λ*-invariant finite trace, the action of* Λ *on* $\mathcal{M}$ *has a finite-trace fundamental domain, and the Koopman representation* $\Gamma \curvearrowright L^2(\mathcal{M}, \mathrm{Tr})$ *is mixing. If* Λ *is properly proximal, then so is* Γ.

Proof. This follows from Propositions 2.11, 4.3 and 4.8. □

Proof of Theorem 1.4. From Proposition 3.3, the existence of a fundamental domain for Γ implies that the Koopman representation is a multiple of the left-regular representation, and hence is mixing for any infinite group. The result then follows from Theorem 4.9. □

4.2 Inducing unitary representations

Suppose $\Lambda \curvearrowright^\sigma (\mathcal{M}, \mathrm{Tr})$ is a trace-preserving action on a semi-finite von Neumann algebra and $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a unitary representation. In Section 2.6 we gave a dual Hilbert $\mathcal{M}$-module structure to $\mathcal{M} \,\overline{\otimes}\, \mathcal{H}$ that satisfies

$$\langle a \otimes \xi, b \otimes \eta \rangle_{\mathcal{M}} = \langle \eta, \xi \rangle a^* b$$

for all $a, b \in \mathcal{M}$ and $\xi, \eta \in \mathcal{H}$. Thus for $s \in \Lambda$ and $x, y \in \mathcal{M} \,\overline{\otimes}\, \mathcal{H}$, we have

$$\langle (\sigma_s \otimes \pi(s))x, (\sigma_s \otimes \pi(s))y \rangle_{\mathcal{M}} = \sigma_s(\langle x, y \rangle_{\mathcal{M}}). \tag{4.2}$$

The space of fixed points $(\mathcal{M} \,\overline{\otimes}\, \mathcal{H})^\Lambda$ then becomes a dual Hilbert $\mathcal{M}^\Lambda$-module.

Left multiplication of $\mathcal{M}$ on itself gives a normal representation of $\mathcal{M}$ on $\mathcal{M} \,\overline{\otimes}\, \mathcal{H}$. Thus the space of fixed points $(\mathcal{M} \,\overline{\otimes}\, \mathcal{H})^\Lambda$ is endowed with a normal $\mathcal{M}^\Lambda$-representation.

If τ is a faithful normal trace on $\mathcal{M}^\Lambda$, we obtain a positive definite scalar-valued inner product on $(\mathcal{M} \,\overline{\otimes}\, \mathcal{H})^\Lambda$ by $\langle y, x \rangle = \tau(\langle x, y \rangle_{\mathcal{M}})$. We denote the corresponding Hilbert space completion as $(\mathcal{M} \,\overline{\otimes}\, \mathcal{H})^\Lambda_\tau$, which we then see is an $\mathcal{M}^\Lambda$-correspondence in the sense of Connes [Con95, Chapter 5, Appendix B].

If we are also given a trace-preserving action $\Gamma \curvearrowright^{\sigma} (\mathcal{M}, \mathrm{Tr})$ that commutes with the Λ-action, then we see that (4.2) also holds for the Γ-action. Hence if Γ preserves the trace τ on $\mathcal{M}^{\Lambda}$, we obtain a unitary representation $\Gamma \curvearrowright (\mathcal{M} \overline{\otimes} \mathcal{H})^{\Lambda}_{\tau}$.

Definition 4.10. Suppose $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a unitary representation and $\Gamma \times \Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ is a trace-preserving action on a semi-finite von Neumann algebra such that the action of Λ admits a finite-trace fundamental domain. We let τ denote the Γ-invariant trace on $\mathcal{M}^{\Lambda}$ given by Proposition 3.3. We say that the representation $\Gamma \curvearrowright (\mathcal{M} \overline{\otimes} \mathcal{H})^{\Lambda}_{\tau}$ is *induced* from π, and we denote this representation by $\pi_{\mathcal{M}}$.

As an $\mathcal{M}^{\Lambda}$-correspondence, we say that $(\mathcal{M} \overline{\otimes} \mathcal{H})^{\Lambda}_{\tau}$ is the correspondence induced from π.

Proposition 4.11. *Suppose $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a unitary representation and $\Lambda \curvearrowright^{\sigma} (\mathcal{M}, \mathrm{Tr})$ is a trace-preserving action on a semi-finite von Neumann algebra that has a finite-trace fundamental domain p. There exists an isomorphism of dual Hilbert $\mathcal{M}^{\Lambda}$-modules $V_p : \mathcal{M}^{\Lambda} \overline{\otimes} \mathcal{H} \to (\mathcal{M} \overline{\otimes} \mathcal{H})^{\Lambda}$ such that*

$$V_p(x \otimes \xi) = \sum_{t \in \Lambda} \sigma_t(p) x \otimes \pi(t)\xi$$

for all $x \in \mathcal{M}^{\Lambda}$ and $\xi \in \mathcal{H}$.

Proof. Note first that by Lemma 4.4, when restricted to the algebraic tensor product, the map $V_p : \mathcal{M}^{\Lambda} \otimes \mathcal{H} \to (\mathcal{M} \overline{\otimes} \mathcal{H})^{\Lambda}$ is well-defined. Moreover, for $x, y \in \mathcal{M}$ and $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned}
\langle V_p(x \otimes \xi), V_p(y \otimes \eta) \rangle_{\mathcal{M}} &= \sum_{s,t \in \Lambda} \langle \pi(s)\eta, \pi(t)\xi \rangle x^* \sigma_t(p) \sigma_s(p) y \\
&= \langle \eta, \xi \rangle x^* y \\
&= \langle x \otimes \xi, y \otimes \eta \rangle_{\mathcal{M}^{\Lambda}}.
\end{aligned}$$

As described in Section 2.6, it follows that V_p has a weak*-continuous extension $V_p : \mathcal{M}^{\Lambda} \overline{\otimes} \mathcal{H} \to (\mathcal{M} \overline{\otimes} \mathcal{H})^{\Lambda}$ that preserves the inner product; and to see that V is surjective, it suffices to show that the range of V_p is dense when viewed as a map into $(\mathcal{M} \overline{\otimes} \mathcal{H})^{\Lambda}_{\tau}$, where τ is the trace given by Proposition 3.3, i.e., $\tau(x) = \mathrm{Tr}(pxp)$ for $x \in \mathcal{M}^{\Lambda}$.

Suppose therefore that we have $\zeta_0 \in (\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda_\tau$, orthogonal to the range of V. Note that since

$$\langle \xi, \eta \rangle_\tau = \tau(\langle \xi, \eta \rangle) = \mathrm{Tr}(p\langle \xi, \eta \rangle p) = \mathrm{Tr}(\langle \xi p, \eta p \rangle)$$

for all $\xi, \eta \in (\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda$, we may view ζ_0 as an element in $(\mathcal{M}p \overline{\otimes} \mathcal{H})_{\mathrm{Tr}}$, which is the completion of $\mathcal{M}p \overline{\otimes} \mathcal{H}$ with respect to the inner product given by $\langle \xi, \eta \rangle_{\mathrm{Tr}} = \mathrm{Tr}(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathcal{M}p \overline{\otimes} \mathcal{H}$.

Fixing $s \in \Lambda$, $x \in \mathcal{M}^\Lambda$, $\zeta \in (\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda$ and $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \langle \sigma_s(p)xp \otimes \xi, \zeta p \rangle_{\mathrm{Tr}} &= \mathrm{Tr}(\langle \sigma_s(p)x \otimes \xi, \zeta p \rangle) \\ &= \sum_{t \in \Lambda} \mathrm{Tr}(p\sigma_t(\langle \sigma_s(p)x \otimes \xi, \zeta p \rangle)) \\ &= \tau(\langle \sum_{t \in \Lambda} \sigma_t(p)x \otimes \pi(ts^{-1})\xi, \zeta \rangle) \\ &= \langle V_p(x \otimes \pi(s^{-1})\xi), \zeta \rangle_\tau. \end{aligned}$$

Approximating ζ_0 by elements in $(\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda$ and viewing ζ_0 as an element in $(\mathcal{M}p \overline{\otimes} \mathcal{H})_{\mathrm{Tr}}$, it follows that

$$\langle \sigma_s(p)xp \otimes \xi, \zeta_0 \rangle_{\mathrm{Tr}} = 0.$$

By part (b) of Proposition 3.3 we have that $\mathrm{span}\{\sigma_s(p)x \otimes \xi \mid s \in \Lambda, x \in \mathcal{M}^\Lambda, \xi \in \mathcal{H}\}$ is weak*-dense in $\mathcal{M} \overline{\otimes} \mathcal{H}$, and hence it follows that $\zeta_0 = 0$. $\square$

A motivating example is when $\mathcal{M} = \mathcal{B}(\ell^2\Lambda)$ and the action $\sigma : \Lambda \to \mathrm{Aut}(\mathcal{B}(\ell^2\Lambda))$ is given by $\sigma_t(T) = \rho_t T \rho_t^*$, where $\rho : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the right-regular representation. Then $\mathcal{M}^\Lambda = L\Lambda$ and the above process describes a method of inducing representations of Λ to normal Hilbert $L\Lambda$-bimodules.

There is another, extensively used, method of inducing representations to normal Hilbert bimodules, which was originally discovered by Connes (see [Con82, Cho83, CJ85, Pop86]). Given a unitary representation $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$, set $\mathcal{K} = \ell^2\Lambda \overline{\otimes} \mathcal{H}$, and consider the representations $\lambda \otimes \pi$, and $1 \otimes \rho$ of Λ in $\mathcal{U}(\mathcal{K})$. The Fell unitary $U : \mathcal{K} \to \mathcal{K}$ given by $U(\delta_t \otimes \xi) = \delta_t \otimes \pi(t)\xi$ satisfies

$U(\lambda \otimes \pi)U^* = \lambda \otimes 1$, and thus both representations $\lambda \otimes \pi$ and $\rho \otimes 1$ extend to give commuting normal representations of $L\Lambda$ and $L\Lambda^{\rm op}$ in $\mathcal{B}(\mathcal{K})$.

The following proposition shows that, for this example, the induced bimodule described in Definition 4.10 is isomorphic to Connes' induced bimodule.

Proposition 4.12. *Let Λ be a discrete group, and $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ a unitary representation. Then there exists a unitary $V : \ell^2\Lambda \overline{\otimes} \mathcal{H} \to (\mathcal{B}(\ell^2\Lambda) \overline{\otimes} \mathcal{H})^\Lambda_\tau$ that induces an isomorphism of $L\Lambda$-bimodules.*

Proof. For $r \in \Lambda$ we let p_r be the rank-one projection onto $\mathbb{C}\delta_r \subset \ell^2\Lambda$. We let $V_{p_e} : L\Lambda \overline{\otimes} \mathcal{H} \to (\mathcal{B}(\ell^2\Lambda) \overline{\otimes} \mathcal{H})^\Lambda$ be as in Proposition 4.11. If $s, t \in \Lambda$ and $\xi \in \mathcal{H}$, then

$$\begin{aligned}(\lambda_s \otimes 1)V_{p_e}(\delta_t \otimes \xi) &= (\lambda_s \otimes 1)\sum_{r\in\Lambda} p_r\lambda_t \otimes \pi(r^{-1})\xi \\ &= \sum_{r\in\Lambda} p_{sr}\lambda_{st} \otimes \pi(r^{-1})\xi \\ &= V_{p_e}(\lambda_s \otimes \pi(s))(\delta_t \otimes \xi).\end{aligned}$$

Viewing $L\Gamma$ as a dense subspace of $\ell^2\Gamma$ and taking completions shows that V_{p_e} extends to a unitary $V_{p_e} : \ell^2\Gamma \overline{\otimes} \mathcal{H} \to (\mathcal{B}(\ell^2\Lambda) \overline{\otimes} \mathcal{H})^\Lambda_\tau$ that intertwines the $L\Gamma$-module structures defined above. As V_{p_e} is also right $L\Gamma$-modular, the result follows easily. □

Lemma 4.13. *Suppose $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ and $\rho : \Lambda \to \mathcal{U}(\mathcal{K})$ are unitary representations and $\Lambda \curvearrowright^\sigma (\mathcal{M}, \mathrm{Tr})$ is a trace-preserving action on a semi-finite von Neumann algebra. For finite-trace fundamental domains $p, q \in \mathcal{M}$, let V_p and V_q respectively be the maps defined in Proposition 4.11. Suppose G is a finite set and we have functions $\xi : G \to \mathcal{K}$ and $\xi_i : G \to \mathcal{H}$ such that $\sup_{i,k\in G} \|\xi_i^k\| < \infty$, and for all $t \in \Lambda$ and $k, \ell \in G$ we have*

$$\langle \pi(t)\xi_i^k, \xi_i^\ell\rangle \to \langle \rho(t)\xi^k, \xi^\ell\rangle.$$

Then for all $x, y \in \mathcal{M}^\Lambda$ and for all $k, \ell \in G$, we have

$$\langle V_p(x \otimes \xi_i^k), V_q(y \otimes \xi_i^\ell)\rangle_\tau \to \langle V_p(x \otimes \xi^k), V_q(y \otimes \xi^\ell)\rangle_\tau.$$

Proof. We compute

$$\begin{aligned}\langle V_p(x \otimes \xi_i^k), V_q(y \otimes \xi_i^\ell)\rangle &= \sum_{s,t\in\Lambda} \langle \sigma_s(p)x \otimes \pi(s)\xi_i^k, \sigma_t(q)y \otimes \pi(t)\xi_i^\ell\rangle \\ &= \sum_{s,t\in\Lambda} x^*\sigma_s(p)\sigma_t(q)y\langle \pi(t)\xi_i^\ell, \pi(s)\xi_i^k\rangle \\ &= \sum_{s\in\Lambda} x^* \left(\sum_{t\in\Lambda} \sigma_t(\sigma_s(p)q)\right) y\langle \xi_i^\ell, \pi(s)\xi_i^k\rangle. \end{aligned} \tag{4.3}$$

We have

$$\sum_{s\in\Lambda} \tau\left(\sum_{t\in\Lambda} \sigma_t(\sigma_s(p)q)\right) = \sum_{s\in\Lambda} \mathrm{Tr}(p\sigma_{s^{-1}}(q)) = \mathrm{Tr}(p) < \infty,$$

and hence given $\varepsilon > 0$ there exists a finite set $F \subset \Lambda$ such that setting

$$y_F = \sum_{s\notin F}\sum_{t\in\Lambda} \sigma_t(\sigma_s(p)q),$$

for all $k, \ell \in G$ we have

$$|\tau(x^*y_Fy)| \leq \tau(x^*y_Fx)^{1/2}\tau((y)^*y_Fy)^{1/2} \leq \|x\|\|y\|\tau(y_F) < \varepsilon.$$

Thus,

$$\begin{aligned}&\limsup_{i\to\infty} |\langle V_p(x \otimes \xi_i^k), V_q(y \otimes \xi_i^\ell)\rangle_\tau - \langle V_p(x \otimes \xi^k), V_q(y \otimes \xi^\ell)\rangle_\tau| \\ &\quad\leq \varepsilon \sup_i \|\xi_i^k\|^{1/2}\|\xi_i^\ell\|^{1/2} + \limsup_{i\to\infty} \sum_{s\in F} \tau\left(x^*\left(\sum_{t\in\Lambda}\sigma_t(\sigma_s(p)q)\right)y\right) \\ &\qquad\cdot |\langle \xi_i^\ell, \pi(s)\xi_i^k\rangle - \langle \xi^\ell, \rho(s)\xi^k\rangle|\end{aligned}$$

$$= \varepsilon \sup_i \|\xi_i^k\|^{1/2} \|\xi_i^\ell\|^{1/2}.$$

As $\varepsilon > 0$ was arbitrary, the result follows. □

Lemma 4.14. *Suppose $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a mixing representation and $\Lambda \curvearrowright^\sigma (\mathcal{M}, \mathrm{Tr})$ is a trace-preserving action on a semi-finite von Neumann algebra. Suppose we have finite-trace Λ-fundamental domains $p_i \in \mathcal{M}$ such that $p_i \to 0$ in the weak operator topology. Then for any Λ-fundamental domain p and $\xi, \eta \in \mathcal{H}$, we have*

$$\lim_{i\to\infty} \sup_{x,y \in (\mathcal{M}^\Lambda)_1} |\langle V_p(x \otimes \xi), V_{p_i}(y \otimes \eta)\rangle_\tau| = 0.$$

Proof. Fix $p \in \mathcal{M}$ a finite-trace fundamental domain and $\xi, \eta \in \mathcal{H}$. Then for $x, y \in \mathcal{M}^\Lambda$ we may compute as in (4.3)

$$\langle V_p(x \otimes \xi), V_{p_i}(y \otimes \eta)\rangle_\tau = \sum_{s \in \Lambda} x^* \left(\sum_{t \in \Lambda} \sigma_t(\sigma_s(p) p_i) \right) y \langle \eta, \pi(s)\xi\rangle.$$

Fix $\varepsilon > 0$. Since π is a mixing representation, there exists $F \subset \Lambda$ finite so that $|\langle \eta, \pi(s)\xi\rangle| < \varepsilon$ for all $s \notin F$. As $p_i \to 0$ weakly, we have

$$\lim_{i\to\infty} \sum_{s \in F} \left| \tau\left(\sum_{t \in \Lambda} \sigma_t(\sigma_s(p) p_i) \right) \right| = 0.$$

Hence

$$\begin{aligned} &\limsup_{i\to\infty} \sup_{x,y \in (\mathcal{M}^\Lambda)_1} |\langle V_p(x \otimes \xi), V_{p_i}(y \otimes \eta)\rangle_\tau| \\ &\qquad \leq \limsup_{i\to\infty} \sup_{x,y \in (\mathcal{M}^\Lambda)_1} \sum_{s \notin F} \left| \tau\left(x^* \left(\sum_{t \in \Lambda} \sigma_t(\sigma_s(p) p_i) \right) y \right) \langle \eta, \pi(s)\xi\rangle \right| < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, and since vectors of the form $x \otimes \xi$ span a weak*-dense subset of $\mathcal{M}^\Lambda \mathbin{\overline{\otimes}} \mathcal{H}$, the result follows. □

The following proposition generalizes results in Section 8 from [Fur99b]. In the case when $\mathcal{M}$ is associated to a W^*-equivalence as in Proposition 4.12, this follows from results in [Cho83, CJ85].

Proposition 4.15. *Suppose $\pi : \Lambda \to U(\mathcal{H})$ and $\rho : \Lambda \to U(\mathcal{K})$ are two unitary representations of Λ, and $\Gamma \times \Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ is a von Neumann coupling. The following hold:*

(i) If $\pi \prec \rho$, then $\pi_{\mathcal{M}} \prec \rho_{\mathcal{M}}$.

(ii) If π is mixing, then $\pi_{\mathcal{M}}$ is mixing.

(iii) $\lambda_{\mathcal{M}}$ is a multiple of the left-regular representation of Γ.

(iv) If π is weak mixing, then $\pi_{\mathcal{M}}$ has no non-zero invariant vectors.

Proof. Suppose first that $\pi \prec \rho$. Replacing ρ with $\rho^{\oplus\infty}$, we may assume that ρ has infinite multiplicity. Fix G a finite set, and suppose $\xi : G \to \mathcal{K}$ is a map. Since $\pi \prec \rho$, there exists a net $\xi_i : G \to \mathcal{H}$ such that for all $t \in \Lambda$, we have $\langle \pi(t)\xi_i^k, \xi_i^\ell \rangle \to \langle \rho(t)\xi^k, \xi^\ell \rangle$. By Lemma 4.13, for all $x, y \in \mathcal{M}^\Lambda$ and $\gamma \in \Gamma$, we then have

$$\begin{aligned}
\langle (\sigma_\gamma \otimes 1)V_p(x \otimes \xi_i^k), V_p(y \otimes \xi_i^\ell) \rangle_\tau &= \langle V_{\sigma_\gamma(p)}(\sigma_\gamma(x) \otimes \xi_i^k), V_p(y \otimes \xi_i^\ell) \rangle_\tau \\
&\to \langle V_{\sigma_\gamma(p)}(\sigma_\gamma(x) \otimes \xi^k), V_p(y \otimes \xi^\ell) \rangle_\tau \\
&= \langle (\sigma_\gamma \otimes 1)V_p(x \otimes \xi^k), V_p(y \otimes \xi^\ell) \rangle_\tau.
\end{aligned}$$

As elements of the form $x \otimes \xi$ span a dense subset of $(\mathcal{M}^\Lambda \overline{\otimes} \mathcal{H})_\tau$ this then shows (i).

If π is mixing and $\gamma \to \infty$, then for a fixed Λ-fundamental domain $p \in \mathcal{M}$, we have that $\sigma_\gamma(p) \to 0$ weakly. Hence Lemma 4.14 shows that for all $\xi, \eta \in \mathcal{H}$ and $x, y \in \mathcal{M}^\Lambda$, we have

$$\lim_{\gamma \to \infty} \langle (\sigma_\gamma \otimes 1)V_p(x \otimes \xi), V_p(y \otimes \eta) \rangle_\tau = \lim_{\gamma \to \infty} \langle V_{\sigma_\gamma(p)}\sigma_\gamma(x) \otimes \xi, V_p(y \otimes \eta) \rangle_\tau = 0.$$

Thus $\pi_{\mathcal{M}}$ is also mixing, which then shows (ii).

We define the map $\mathcal{F} : (\mathcal{M} \overline{\otimes} \ell^2\Lambda)^\Lambda \to \mathcal{M}$ by $\mathcal{F}(\xi) = \langle 1 \otimes \delta_e, \xi \rangle_{\mathcal{M}}$. For $x \in \mathcal{M}^\Lambda$ and $t \in \Lambda$, we then have $\mathcal{F}(V_p(x \otimes \delta_t)) = \langle 1 \otimes \delta_e, V_p(x \otimes \delta_t) \rangle = \sigma_{t^{-1}}(p)x$. Hence if we also have $y \in \mathcal{M}^\Lambda$ and $s \in \Lambda$, then

$$\begin{aligned}
\langle \mathcal{F}(V_p(x \otimes \delta_t)), \mathcal{F}(V_p(y \otimes \delta_s)) \rangle_{\mathrm{Tr}} &= \delta_{s,t} \mathrm{Tr}(x^* \sigma_{t^{-1}}(p) y) \\
&= \delta_{s,t} \tau(\sigma_t(yx^*)) \\
&= \delta_{s,t} \tau(x^* y) \\
&= \langle x \otimes \delta_t, y \otimes \delta_s \rangle_\tau \\
&= \langle V_p(x \otimes \delta_t), V_p(y \otimes \delta_s) \rangle_\tau .
\end{aligned}$$

Thus, $\mathcal{F}$ extends to an isometry $\mathcal{F} : (\mathcal{M} \overline{\otimes} \ell^2\Lambda)^\Lambda_\tau \to L^2(\mathcal{M}, \mathrm{Tr})$. Moreover, by part (b) of Proposition 3.3 we see that $\mathrm{span}\{\mathcal{F}(V_p(x \otimes \delta_t)) \mid x \in \mathcal{M}^\Lambda, t \in \Lambda\}$ is dense in $L^2(M, \mathrm{Tr})$, hence $\mathcal{F}$ is unitary.

As $\mathcal{F}$ commutes with the action of Γ, we then see that $\mathcal{F}$ implements an intertwiner between the representation $\lambda_{\mathcal{M}}$ and the Koopman representation on $L^2(\mathcal{M}, \mathrm{Tr})$. Since Γ has a finite-measure fundamental domain, the latter representation is isomorphic to an amplification of the left regular representation by part (ii) of Proposition 3.3. This then establishes (iii).

We now suppose that $\pi_{\mathcal{M}}$ has a non-zero invariant vector in $(\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda_\tau$. First, note that this then implies that there is a non-zero Γ-invariant vector in $(\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda$. Indeed, if $\xi \in (\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda_\tau$ is a Γ-invariant vector, then we may approximate ξ by some $\eta \in (\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda$ so that $\|\xi - \eta\|_\tau < \frac{1}{2}\|\xi\|$. If we let ξ_0 be the unique element of minimal $\| \cdot \|_\tau$ in the $\| \cdot \|_\tau$-closed convex closure hull of $\{\pi_{\mathcal{M}}(\gamma)\eta \mid \gamma \in \Gamma\}$, then ξ_0 is also Γ-invariant, and we have $\|\xi_0 - \xi\| \leq \|\eta - \xi\| < \frac{1}{2}\|\xi\|$, so that ξ_0 is non-zero. Closed balls in $\mathcal{M} \overline{\otimes} \mathcal{H}$ are weak*-compact by the Banach-Alaoglu theorem and hence we see that $\xi_0 \in (\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda \subset (\mathcal{M} \overline{\otimes} \mathcal{H})^\Lambda_\tau$.

We therefore have a non-zero vector in $(\mathcal{M} \overline{\otimes} \mathcal{H})^{\Gamma \times \Lambda} \cong (\mathcal{M}^\Gamma \overline{\otimes} \mathcal{H})^\Lambda$. Recall that we endow $\mathcal{H}$ with its column operator space structure coming from the isomorphism $\mathcal{H} \cong \mathcal{B}(\mathbb{C}, \mathcal{H})$. We therefore consider $\xi_0 \in (\mathcal{M}^\Gamma \overline{\otimes} \mathcal{B}(\mathbb{C}, \mathcal{H}))^\Lambda$, and we then obtain a non-zero positive operator $|\xi_0| \in$

$(\mathcal{M}^\Gamma \overline{\otimes} HS(\mathcal{H}))^\Lambda$, where $HS(\mathcal{H})$ denotes the space of Hilbert-Schmidt operators on $\mathcal{H}$. As $\tau_\Gamma \otimes \mathrm{Tr}$ gives a faithful trace on $\mathcal{M}^\Gamma \overline{\otimes} \mathcal{B}(\mathcal{H})$, we then obtain a non-zero Λ-invariant vector $(\tau_\Gamma \otimes \mathrm{id})(|\xi_0|) \in HS(\mathcal{H})$. This then shows that π is not weak mixing, establishing (iv). □

Proof of Theorem 1.2. Amenability is characterized by having the left regular representation weakly contain the trivial representation, thus (i) and (iii) in Proposition 4.15 show that amenability is preserved under von Neumann equivalence.

Similarly, the Haagerup property is characterized by having a mixing representation that weakly contains the trivial representation. Thus, (i) and (ii) in Proposition 4.15 show that the Haagerup property is preserved under von Neumann equivalence.

Finally, if Γ has property (T) and π is a representation of Λ that weakly contains the trivial representation, then since $1_\mathcal{M}$ contains the trivial representation for Γ, it follows that $\pi_\mathcal{M}$ also weakly contains the trivial representation. Property (T) then implies that $\pi_\mathcal{M}$ contains non-zero Γ-invariant vectors, and by (iv) in Proposition 4.15 it follows that π is not weak mixing. It then follows from [BV93, Theorem 1] that Λ also has property (T). □

4.3 Inducing Herz-Schur Multipliers

We present the proof of Theorem 1.3 in this section. The proof relies on the following analogue of Lemma 2.1 in [Jol14].

Lemma 4.16. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite-trace Tr and let $\Lambda \curvearrowright^\sigma (\mathcal{M}, \mathrm{Tr})$ be a trace-preserving action with a finite-trace fundamental domain p. Suppose $\Gamma \curvearrowright (\mathcal{M}, \mathrm{Tr})$ is another trace-preserving action that commutes with the Λ-action. For $\varphi \in \ell^\infty\Lambda$, define $\hat{\varphi} : \Gamma \to \mathbb{C}$ by*

$$\hat{\varphi}(\gamma) := \frac{1}{\mathrm{Tr}(p)} \mathrm{Tr}(\sigma_\gamma(\theta_p(\varphi))p) = \frac{1}{\mathrm{Tr}(p)} \mathrm{Tr}(\theta_p(\varphi)\sigma_{\gamma^{-1}}(p)), \qquad \gamma \in \Gamma,$$

where $\theta_p : \ell^\infty\Lambda \hookrightarrow \mathcal{M}$ is the Λ-equivariant embedding.

(a) *If $\varphi \in B_2(\Lambda)$ is a Herz-Schur multiplier on Λ, then $\hat{\varphi}$ is a Herz-Schur multiplier on Γ and $\|\hat{\varphi}\|_{B_2} \leq \|\varphi\|_{B_2}$. Moreover, if φ is positive definite, then so is $\hat{\varphi}$.*

(b) *If $\Gamma \curvearrowright (\mathcal{M}, \mathrm{Tr})$ is mixing, i.e., the Koopman representation $\Gamma \curvearrowright L^2(\mathcal{M}, \mathrm{Tr})$ is mixing, and if $\varphi \in c_0(\Lambda)$, then $\hat{\varphi} \in c_0(\Gamma)$. In particular, if $\varphi \in B_2(\Lambda) \cap c_0(\Lambda)$, then $\hat{\varphi} \in B_2(\Gamma) \cap c_0(\Gamma)$.*

Proof. Since p is a finite-trace fundamental domain, it follows that $\hat{\varphi}$ is well-defined and $\|\hat{\varphi}\|_\infty \leq \|\varphi\|_\infty$. Let $\xi, \eta : \Lambda \to \mathcal{H}_0$ be bounded functions from Λ into a Hilbert space $\mathcal{H}_0$ such that $\varphi(t^{-1}s) = \langle \xi(s), \eta(t)\rangle, \;\; s, t \in \Lambda$.

Let $\mathcal{H} = L^2(\mathcal{M}, \mathrm{Tr})\overline{\otimes}\mathcal{H}_0$. Note that, for $\gamma \in \Gamma$, we have

$$\sum_{s\in\Lambda} \|\sigma_\gamma(\sigma_s(p))p\|_2^2 \|\xi(s)\|^2 \leq \|\xi\|_\infty^2 \sum_{s\in\Lambda} \mathrm{Tr}(p\sigma_\gamma(\sigma_s(p))) = \|\xi\|_\infty^2 \,\mathrm{Tr}(p) < \infty.$$

Therefore, $\hat{\xi}, \hat{\eta} : \Gamma \to L^2(\mathcal{M}, \mathrm{Tr})\overline{\otimes}\mathcal{H}_0$ given below are well-defined.

$$\hat{\xi}(\gamma) := \frac{1}{\sqrt{\mathrm{Tr}(p)}} \sum_{s\in\Lambda} \sigma_\gamma(\sigma_s(p))p \otimes \xi(s), \qquad \hat{\eta}(\gamma) := \frac{1}{\sqrt{\mathrm{Tr}(p)}} \sum_{t\in\Lambda} \sigma_\gamma(\sigma_t(p))p \otimes \eta(t).$$

One has, for every $\gamma \in \Gamma$,

$$\|\hat{\xi}(\gamma)\|^2 = \frac{1}{\mathrm{Tr}(p)} \sum_{s\in\Lambda} \langle \xi(s), \xi(s)\rangle \,\mathrm{Tr}(\sigma_s(p)\sigma_{\gamma^{-1}}(p)) \leq \frac{1}{\mathrm{Tr}(p)} \|\xi\|_\infty^2 \,\mathrm{Tr}\left(\sum_{s\in\Lambda} \sigma_s(p)\sigma_{\gamma^{-1}}(p)\right) = \|\xi\|_\infty^2.$$

Thus, $\|\hat{\xi}\|_\infty \leq \|\xi\|_\infty$. Similarly, $\|\hat{\eta}\|_\infty \leq \|\eta\|_\infty$. Finally, for $\gamma_1, \gamma_2 \in \Gamma$, we have

$$\begin{aligned}
\langle \hat{\xi}(\gamma_1), \hat{\eta}(\gamma_2)\rangle &= \frac{1}{\mathrm{Tr}(p)} \left\langle \sum_{s\in\Lambda} \sigma_{\gamma_1}(\sigma_s(p))p \otimes \xi(s), \sum_{t\in\Lambda} \sigma_{\gamma_2}(\sigma_t(p))p \otimes \eta(t) \right\rangle \\
&= \frac{1}{\mathrm{Tr}(p)} \sum_{s,t\in\Lambda} \langle \xi(s), \eta(t)\rangle \,\mathrm{Tr}(p\sigma_{\gamma_2}(\sigma_t(p))\sigma_{\gamma_1}(\sigma_s(p))) \\
&= \frac{1}{\mathrm{Tr}(p)} \sum_{t\in\Lambda} \mathrm{Tr}\left(p\sigma_{\gamma_2}(\sigma_t(p))\sigma_{\gamma_1}\left(\sum_{s\in\Lambda} \varphi(t^{-1}s)\sigma_s(p)\right)\right) \\
&= \frac{1}{\mathrm{Tr}(p)} \sum_{t\in\Lambda} \mathrm{Tr}(p\sigma_{\gamma_2}(\sigma_t(p))\sigma_{\gamma_1}(\sigma_t(\theta_p(\varphi))))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\operatorname{Tr}(p)} \sum_{t \in \Lambda} \operatorname{Tr}(\sigma_{\gamma_2^{-1}}(p)\sigma_t(p\sigma_{\gamma_2^{-1}\gamma_1}(\theta_p(\varphi)))) \\
&= \frac{1}{\operatorname{Tr}(p)} \operatorname{Tr}\left(\sum_{t \in \Lambda} \sigma_{t^{-1}}(\sigma_{\gamma_2^{-1}}(p)) p \sigma_{\gamma_2^{-1}\gamma_1}(\theta_p(\varphi)) \right) \\
&= \frac{1}{\operatorname{Tr}(p)} \operatorname{Tr}(\sigma_{\gamma_2^{-1}\gamma_1}(\theta_p(\varphi))p) \\
&= \hat{\varphi}(\gamma_2^{-1}\gamma_1).
\end{aligned}
$$

Therefore, $\hat{\varphi}$ is a Herz-Schur multiplier with $\|\hat{\varphi}\|_{B_2} \leq \|\varphi\|_{B_2}$. Furthermore, if φ is positive definite, then one can take $\eta = \xi$ and it is straightforward to see that $\hat{\varphi}$ is positive definite on Γ as well. (In fact, let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the GNS-triple associated to φ. Then, as $\varphi(s) = \langle \pi_\varphi(s)\xi_\varphi, \xi_\varphi \rangle$ for every $s \in \Lambda$, we see that the function $s \mapsto \xi(s) = \pi_\varphi(s)\xi_\varphi$ works.)

Note that, if $\varphi \in c_0(\Lambda)$, then $\theta_p(\varphi)$ is compact. Since the action of Γ is mixing, $\sigma_\gamma(p) \to 0$ in the weak operator topology as $\gamma \to \infty$, and (b) now follows from Lemma 2.7. ☐

Proposition 4.17. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr and $\Lambda \curvearrowright^{\sigma} (\mathcal{M}, \operatorname{Tr})$ be a trace-preserving action with a finite-trace fundamental domain p. Suppose $\Gamma \curvearrowright (\mathcal{M}, \operatorname{Tr})$ is another trace-preserving action that commutes with the Λ-action. Consider the map $\Phi : \ell^\infty\Lambda \to \ell^\infty\Gamma$ defined by $\Phi(\varphi) = \hat{\varphi}$, where $\hat{\varphi}$ is defined as in Lemma 4.16. Then Φ is a contractive linear mapping from $B_2(\Lambda)$ into $B_2(\Gamma)$, and is*

(a) *continuous on norm bounded sets with respect to the topology of poitwise convergence.*

(b) $\sigma(B_2(\Lambda), Q(\Lambda))$-$\sigma(B_2(\Gamma), Q(\Gamma))$ *continuous.*

Proof. It is clear that $\Phi : B_2(\Lambda) \to B_2(\Gamma)$ is linear, and that it is contractive follows from Lemma 4.16(a). Moreover, $\Phi : \ell^\infty\Lambda \to \ell^\infty\Gamma$ is also a linear contraction.

(a) Let $\varphi_i \to 0$ in $B_2(\Lambda)$ pointwise and let $\|\varphi_i\|_{B_2} < C$ for every i. Since $\|\cdot\|_\infty \leq \|\cdot\|_{B_2}$, after passing to a subnet if necessary, we may assume that $\varphi_i \to 0$ weak*. By Lemma **??** it follows that $\theta_p(\varphi_i) \to 0$ in the weak operator topology. Therefore, $\operatorname{Tr}(\theta_p(\varphi_i)\sigma_{\gamma^{-1}}(p)) \to 0$ for every $\gamma \in \Gamma$, and hence $\hat{\varphi}_i \to 0$ pointwise.

(b) Since $\Phi : B_2(\Lambda) \to B_2(\Gamma)$ is a linear contraction, the dual $\Phi^* : B_2(\Gamma)^* \to B_2(\Lambda)^*$ is continuous. Therefore, to prove that Φ is $\sigma(B_2(\Lambda), Q(\Lambda))$-$\sigma(B_2(\Gamma), Q(\Gamma))$ continuous, it suffices to show that Φ maps $Q(\Gamma)$ into $Q(\Lambda)$. To this end, notice that a similar argument as in the proof of previous part shows that $\Phi : \ell^\infty\Lambda \to \ell^\infty\Gamma$ is normal, whence it follows that the dual map Φ^* maps $\ell^1\Gamma$ into $\ell^1\Lambda$. Since $\ell^1\Gamma$ and $\ell^1\Lambda$ are dense, respectively, in $Q(\Gamma)$ and $Q(\Lambda)$, it follows that $\Phi^*(Q(\Gamma)) \subset Q(\Lambda)$.

□

Proof of Theorem 1.3. Suppose $\Lambda_{\text{cb}}(\Lambda) \leq C$, and choose a net $\{\varphi_i\}$ of finitely supported functions on Λ such that $\sup_i \|\varphi_i\|_{B_2} \leq C$, and $\varphi_i \to 1$ pointwise. It follows from Lemma 4.16 that $\hat{\varphi}_i \in B_2(\Gamma)$ and $\|\hat{\varphi}_i\|_{B_2} \leq C$ for every i. Since each φ_i is finitely supported, we have that $\hat{\varphi}_i$ is a coefficient of the Koopman representation $\sigma^0 : \Gamma \to \mathcal{U}(L^2(\mathcal{M}, \text{Tr}))$. Moreover, the existence of a fundamental domain for Γ implies that σ^0 is a multiple of the left-regular representation [IPR19, Proposition 4.2] and hence $\hat{\varphi}_i \in A[\Gamma]$ for all i. From Proposition 4.17, we also have that $\hat{\varphi}_i \to 1$ pointwise. This shows that $\Lambda_{\text{cb}}(\Gamma) \leq C$, and Γ is weakly amenable.

If the net $\{\varphi_i\}$ is in $c_0(\Lambda) \cap B_2(\Lambda)$, then by Lemma 4.16(b), $\hat{\varphi}_i \in c_0(\Gamma) \cap B_2(\Gamma)$ for every i. Now the same argument as in the previous paragraph shows that if Λ has the weak Haagerup property, then Γ has the weak Haagerup property and $\Lambda_{\text{wcb}}(\Gamma) \leq \Lambda_{\text{wcb}}(\Lambda)$.

If Λ has (AP), then 1 is in the $\sigma(B_2(\Lambda), Q(\Lambda))$-closure of finitely supported functions on Λ. From the first paragraph we have that if φ is a finitely supported function on Λ, then $\hat{\varphi} \in A[\Gamma]$. Therefore, Proposition 4.17 gives that 1 is in the $\sigma(B_2(\Gamma), Q(\Gamma))$-closure of $A[\Gamma]$ inside $B_2(\Gamma)$, whence it follows that Γ has (AP). □

Remark 4.18. The above methods provide an alternate proof for the stability of the Haagerup property under vNE as follows. Note that $\theta_p(\varphi)p = \varphi(e)p$. Therefore, if $\varphi(e) = 1$, then $\hat{\varphi}(e) = 1$. Moreover, if φ is a normalized positive definite function, then there exists a Hilbert space $\mathcal{H}_\varphi$, a unitary representation $\pi_\varphi : \Lambda \to \mathcal{U}(\mathcal{H}_\varphi)$, and a unit vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that $\varphi(s) = \langle \pi_\varphi(s)\xi_\varphi, \xi_\varphi \rangle, s \in \Lambda$ [BO08, Theorem 2.5.11]. In particular, $\|\varphi\|_\infty \leq 1$. Therefore, the arguments in the previous paragraphs show that the Haagerup property is invariant under von Neumann equivalence.

Chapter 5

Von Neumann equivalence for finite von Neumann algebras

The notion of von Neumann equivalence also admits a generalization in the setting of finite von Neumann algebras which we study in this chapter.

Definition 5.1. Let $M \subset \mathcal{M}$ be an inclusion of semi-finite von Neumann algebras. A *fundamental domain* for M inside of $\mathcal{M}$ consists of a realization of the standard representation $M \subset \mathcal{B}(L^2(M))$ as an intermediate von Neumann subalgebra $M \subset \mathcal{B}(L^2(M)) \subset \mathcal{M}$. The fundamental domain is *finite* if finite-rank projections in $\mathcal{B}(L^2(M))$ are mapped to finite projections in $\mathcal{M}$.

Note that if $P = \mathcal{B}(L^2(M))' \cap \mathcal{M}$, then we have an isomorphism

$$\mathcal{M} \cong \mathcal{B}(L^2(M)) \mathbin{\overline{\otimes}} P$$

where M acts standardly as $M \otimes \mathbb{C}$.

Lemma 5.2. *Let $M \subset \mathcal{M}$ be an inclusion of von Neumann algebras with $\mathcal{M}$ being semi-finite and M being finite and σ-finite. Then any two fundamental domains for M are conjugate by a unitary in $M' \cap \mathcal{M}$.*

Proof. Let $\mathcal{C}_{\mathrm{Tr}}$ be a faithful normal semi-finite center-valued trace on $\mathcal{M}$, and let τ be a faithful normal trace on M. Let $\mathcal{F}$ denote the collection of finite dimensional subspaces of $M \subset L^2(M, \tau)$, which we order by inclusion. Then the net $\{P_V\}_{V \in \mathcal{F}} \subset \mathcal{B}(L^2(M, \tau))$ converges to the identity in the strong operator topology. If $\theta_1, \theta_2 : \mathcal{B}(L^2(M, \tau)) \to \mathcal{M}$ are embeddings that restrict to the identity on M, and if $V \in \mathcal{F}$ has an orthonormal basis $\{x_1, \ldots, x_k\} \subset M$, then we have

$$\mathcal{C}_{\mathrm{Tr}}(\theta_1(P_{\mathbb{C}\hat{1}})\theta_2(P_V)) = \sum_{i=1}^{k} \mathcal{C}_{\mathrm{Tr}}(\theta_1(P_{\mathbb{C}\hat{1}})\theta_2(x_j P_{\mathbb{C}\hat{1}} x_j^*))$$

$$= \sum_{i=1}^{k} \mathcal{C}_{\mathrm{Tr}}(\theta_1(x_j^* P_{\mathbb{C}\hat{1}} x_j)\theta_2(P_{\mathbb{C}\hat{1}}))$$
$$= \mathcal{C}_{\mathrm{Tr}}(\theta_1(P_{V^*})\theta_2(P_{\mathbb{C}\hat{1}})).$$

Taking the limit over $\mathcal{F}$, we see that $\mathcal{C}_{\mathrm{Tr}}(\theta_1(P_{\mathbb{C}\hat{1}})) = \mathcal{C}_{\mathrm{Tr}}(\theta_2(P_{\mathbb{C}\hat{1}}))$.

Thus, $\theta_1(P_{\mathbb{C}\hat{1}})$ and $\theta_2(P_{\mathbb{C}\hat{1}})$ are Murray-von Neumann equivalent projections in $\mathcal{M}$. It follows that $\theta_1(\mathcal{B}(L^2(M,\tau)))$ and $\theta_2(\mathcal{B}(L^2(M,\tau)))$ are conjugate by a unitary in $\mathcal{M}$, and as M is standardly represented on $L^2(M,\tau)$, we may then find such a unitary $u \in \mathcal{M}$ so that

$$uxu^* = u\theta_1(x)u^* = \theta_2(x) = x$$

for all $x \in M$, and hence $u \in M' \cap \mathcal{M}$. □

Definition 5.3. A *von Neumann coupling* between two finite, σ-finite von Neumann algebras M and N consists of a semi-finite von Neumann algebra $\mathcal{M}$, together with embeddings of M and N^{op} into $\mathcal{M}$ such that $N^{\mathrm{op}} \subset M' \cap \mathcal{M}$ and such that each inclusion $M \subset \mathcal{M}$ and $N^{\mathrm{op}} \subset \mathcal{M}$ has a finite fundamental domain.

We use the notation $\mathcal{M} = {}_M\mathcal{M}_N$ to indicate that $\mathcal{M}$ is a von Neumann coupling between M and N. Two von Neumann couplings ${}_M\mathcal{M}_N$ and ${}_M\mathcal{N}_N$ are isomorphic if there exists an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ that restricts to the identity on M and N, respectively.

Proposition 5.4. *Let $\mathcal{M} = {}_M\mathcal{M}_N$ be a von Neumann coupling between M and N. Fix a normal faithful semi-finite center-valued trace $\mathcal{C}_{\mathrm{Tr}}$ on $\mathcal{M}$, and consider the quantity*

$$[M : N]_{\mathcal{M}} := \mathcal{C}_{\mathrm{Tr}}(p)/\mathcal{C}_{\mathrm{Tr}}(q)$$

where p and q are rank-one projections in $\mathcal{B}(L^2(M))$ and $\mathcal{B}(L^2(N))$ respectively. Then this gives an invertible operator affiliated to $\mathcal{Z}(\mathcal{M})$, which is independent of the choice of $\mathcal{C}_{\mathrm{Tr}}$ as well as the fundamental domains for $M \subset \mathcal{B}(L^2(M)) \subset \mathcal{M}$ and $N \subset \mathcal{B}(L^2(N)) \subset \mathcal{M}$.

Proof. Considering the decomposition $\mathcal{M} \cong \mathcal{B}(L^2(M)) \overline{\otimes} P$ we see that non-zero projections in $\mathcal{B}(L^2(M))$ have central support equal to 1. Therefore $\mathcal{C}_{\mathrm{Tr}}(p)/\mathcal{C}_{\mathrm{Tr}}(q)$ gives an invertible operator affiliated to $\mathcal{Z}(\mathcal{M})$.

As two faithful semi-finite center-valued traces are related by a positive injective operator affiliated to $\mathcal{Z}(\mathcal{M})$, we see that the quantity $\mathcal{C}_{\mathrm{Tr}}(p)/\mathcal{C}_{\mathrm{Tr}}(q)$ is independent of $\mathcal{C}_{\mathrm{Tr}}$. Also, by Lemma 5.2 fundamental domains must be conjugate in $\mathcal{M}$ and hence the quantities $\mathrm{Tr}_{\mathcal{M}}(p)$ and $\mathrm{Tr}_{\mathcal{M}}(q)$ are each independent of the choice of fundamental domain. □

Definition 5.5. The quantity $[M : N]_{\mathcal{M}} \in \mathcal{Z}(\mathcal{M})$ is the *index* of the coupling $\mathcal{M}$. The *index group* of M is the subset of $\mathbb{R}_+^*$ consisting of all indices for factorial self-couplings of M and is denoted by $\mathcal{I}_{vNE}(M)$.

Note that in Theorem 5.8 below, we justify the terminology by showing that the index group is indeed a subgroup of $\mathbb{R}_+^*$.

Suppose $\mathcal{M} = {}_M\mathcal{M}_N$ and $\mathcal{N} = {}_N\mathcal{M}_Q$ are M-N and N-Q von Neumann couplings, respectively. Choose fundamental domains $\theta_{\mathcal{M}} : \mathcal{B}(L^2(N)) \to \mathcal{M}$ for $N^{\mathrm{op}} \subset \mathcal{M}$ and $\theta_{\mathcal{N}} : \mathcal{B}(L^2(N)) \to \mathcal{N}$ for $N \subset \mathcal{N}$. Set $P_1 = \theta_{\mathcal{M}}(\mathcal{B}(L^2(N)))' \cap \mathcal{M}$ and $P_2 = \theta_{\mathcal{N}}(\mathcal{B}(L^2(N)))' \cap \mathcal{N}$. Then we have isomorphisms

$$\tilde{\theta}_{\mathcal{M}} : P_1 \overline{\otimes} \mathcal{B}(L^2(N)) \to \mathcal{M}, \qquad \tilde{\theta}_{\mathcal{N}} : \mathcal{B}(L^2(N)) \overline{\otimes} P_2 \to \mathcal{N}$$

such that $\tilde{\theta}_{\mathcal{M}}(a \otimes x) = a\theta_{\mathcal{M}}(x)$ and $\tilde{\theta}_{\mathcal{N}}(x \otimes b) = \theta_{\mathcal{N}}(x)b$ for $a \in P_1$, $b \in P_2$ and $x \in \mathcal{B}(L^2(N))$.

We then define the fusion (or composition) of the couplings ${}_M\mathcal{M}_N$ and ${}_N\mathcal{M}_Q$ to consist of the von Neumann algebra

$$\mathcal{M} \overline{\otimes}_N \mathcal{N} := P_1 \overline{\otimes} \mathcal{B}(L^2(N)) \overline{\otimes} P_2,$$

endowed with the embeddings of M and Q via the inclusions $\tilde{\theta}_{\mathcal{M}}^{-1} \times 1$ of $\mathcal{M}$ and $1 \times \tilde{\theta}_{\mathcal{N}}^{-1}$ of $\mathcal{N}$ given respectively by

$$\mathcal{M} \ni x \mapsto \tilde{\theta}_{\mathcal{M}}^{-1}(x) \otimes 1 \in \mathcal{M} \overline{\otimes}_N \mathcal{N}, \quad \mathcal{N} \ni x \mapsto 1 \otimes \tilde{\theta}_{\mathcal{N}}^{-1}(x) \in \mathcal{M} \overline{\otimes}_N \mathcal{N}.$$

Note that $\mathcal{Z}(\mathcal{M}) \subset P_1$ and $\mathcal{Z}(\mathcal{N}) \subset P_2$, so we have an inclusion

$$\mathcal{Z}(\mathcal{M}) \overline{\otimes} \mathcal{Z}(\mathcal{N}) \subset \mathcal{M} \overline{\otimes}_N \mathcal{N}.$$

Proposition 5.6. *Using the notation above, the von Neumann algebra $\mathcal{M} \overline{\otimes}_N \mathcal{N}$ gives a von Neumann coupling between M and Q with index*

$$[M : Q]_{\mathcal{M} \overline{\otimes}_N \mathcal{N}} = [M : N]_{\mathcal{M}} \otimes [N : Q]_{\mathcal{N}}.$$

Moreover, up to isomorphism, this coupling is independent of the choice of fundamental domains for the inclusions $N \subset \mathcal{M}$ and $N \subset \mathcal{N}$.

Proof. We have

$$(\tilde{\theta}_{\mathcal{M}}^{-1} \times 1)(M) \subset (\tilde{\theta}_{\mathcal{M}}^{-1} \times 1)(N^{\mathrm{op}\prime} \cap \mathcal{M}) = P_1 \overline{\otimes} N \overline{\otimes} \mathbb{C},$$

while

$$(1 \times \tilde{\theta}_{\mathcal{N}}^{-1})(Q) \subset (1 \times \tilde{\theta}_{\mathcal{N}}^{-1})(N' \cap \mathcal{N}) = \mathbb{C} \overline{\otimes} N^{\mathrm{op}} \overline{\otimes} P_2,$$

so that the copies of M and Q in $\mathcal{M} \overline{\otimes}_N \mathcal{N}$ commute. Since we have isomorphisms

$$\mathcal{M} \overline{\otimes} P_2 \cong \mathcal{M} \overline{\otimes}_N \mathcal{N} \cong P_1 \overline{\otimes} \mathcal{N},$$

and since P_1 and P_2 are finite, we then have finite fundamental domains for M and Q. We let $p \in \mathcal{M}$ and $q \in \mathcal{N}$ be minimal projections in fundamental domains for M and Q respectively. We also let $\mathcal{C}_i$ denote the center-valued trace on P_i for $i = 1, 2$, and we define

$$\mathcal{C}_{\mathcal{M}} = \mathcal{C}_1 \otimes \mathrm{Tr}_{\mathcal{B}(L^2(N))} \text{ and } \mathcal{C}_{\mathcal{N}} = \mathrm{Tr}_{\mathcal{B}(L^2(N))} \otimes \mathcal{C}_2.$$

Then we have

$$[M : Q]_{\mathcal{M} \overline{\otimes}_N \mathcal{N}} = (\mathcal{C}_1 \otimes \mathrm{Tr}_{\mathcal{B}(L^2(N))} \otimes \mathcal{C}_2)(\tilde{\theta}_{\mathcal{M}}^{-1}(p) \otimes 1) / (\mathcal{C}_1 \otimes \mathrm{Tr}_{\mathcal{B}(L^2(N))} \otimes \mathcal{C}_2)(1 \otimes \tilde{\theta}_{\mathcal{N}}^{-1}(q))$$

$$= \mathcal{C}_{\mathcal{M}}(p) \otimes \mathcal{C}_2(1)/\mathcal{C}_1(1) \otimes \mathcal{C}_{\mathcal{N}}(q)$$
$$= [M : N]_{\mathcal{M}} \otimes [N : Q]_{\mathcal{N}}.$$

Suppose now that we have fundamental domains for $N^{\mathrm{op}} \subset \mathcal{M}$ and $N \subset \mathcal{N}$ given respectively by $\phi_{\mathcal{M}} : \mathcal{B}(L^2(N)) \to \mathcal{M}$ and $\phi_{\mathcal{N}} : \mathcal{B}(L^2(N)) \to \mathcal{N}$. We set

$$R_1 = \phi_{\mathcal{M}}(\mathcal{B}(L^2(N)))' \cap \mathcal{M} \text{ and } R_2 = \phi_{\mathcal{N}}(\mathcal{B}(L^2(N)))' \cap \mathcal{N},$$

and we define the isomorphisms $\tilde{\phi}_{\mathcal{M}}$ and $\tilde{\phi}_{\mathcal{N}}$ as above.

By Lemma 5.2 there exist unitaries $u \in N^{\mathrm{op}\prime} \cap \mathcal{M}$ and $v \in N' \cap \mathcal{N}$ so that $\phi_{\mathcal{M}} = \mathrm{Ad}(u) \circ \theta_{\mathcal{M}}$ and $\phi_{\mathcal{N}} = \mathrm{Ad}(v) \circ \theta_{\mathcal{N}}$. We then have $R_1 = uP_1u^*$ and $R_2 = vP_2v^*$.

We consider the isomorphism $\alpha : R_1 \overline{\otimes} \mathcal{B}(L^2(N)) \overline{\otimes} R_2 \to P_1 \overline{\otimes} \mathcal{B}(L^2(N)) \overline{\otimes} P_2$ given by $\alpha = \mathrm{Ad}(u^*) \otimes \mathrm{id} \otimes \mathrm{Ad}(v^*)$. Under this isomorphism, the inclusion of M coming from the fundamental domains $\phi_{\mathcal{M}}$ and $\phi_{\mathcal{N}}$ is given by $\alpha \circ (\tilde{\phi}_{\mathcal{M}}^{-1} \times 1)$ and still maps M into $P_1 \overline{\otimes} N \otimes \mathbb{C}$. Similarly, the new inclusion of Q again maps into $\mathbb{C} \otimes N^{\mathrm{op}} \overline{\otimes} P_2$.

If we restrict α to $P_1 \overline{\otimes} \mathcal{B}(L^2(N))$ and consider the automorphism

$$\beta = \alpha \circ \tilde{\phi}_{\mathcal{M}}^{-1} \circ \tilde{\theta}_{\mathcal{M}} \in \mathrm{Aut}(P_1 \overline{\otimes} \mathcal{B}(L^2(N)),$$

then for $a \in P_1$ and $x \in \mathcal{B}(L^2(N))$ we have

$$\begin{aligned}
\beta(a \otimes x) &= \alpha \circ \tilde{\phi}_{\mathcal{M}}^{-1}(a\theta_{\mathcal{M}}(x)) \\
&= \alpha \circ \tilde{\phi}_{\mathcal{M}}^{-1}(u^*(uau^*)\phi_{\mathcal{M}}(x)u) \\
&= \alpha \circ \mathrm{Ad}(\tilde{\phi}_{\mathcal{M}}^{-1}(u^*))(uau^* \otimes x) \\
&= \mathrm{Ad}(\alpha(\tilde{\phi}_{\mathcal{M}}^{-1}(u^*)))(a \otimes x).
\end{aligned}$$

Hence, $\beta = \mathrm{Ad}(\alpha(\tilde{\phi}_{\mathcal{M}}^{-1}(u^*)))$, and if we set $U = \tilde{\phi}_{\mathcal{M}}^{-1}(u)$, then we see that $U \in R_1 \overline{\otimes} N \subset R_1 \overline{\otimes} \mathcal{B}(L^2(N))$ and the map $\mathrm{Ad}(U) \circ \alpha^{-1}$ intertwines the two inclusions of M coming from the choice of fundamental domains. Similarly, if we set $V = \tilde{\phi}_{\mathcal{N}}^{-1}(v)$, then $V \in N^{\rm op} \overline{\otimes} R_2 \subset \mathcal{B}(L^2(N)) \overline{\otimes} R_2$ and the map $\mathrm{Ad}(U) \circ \alpha^{-1}$ intertwines the two inclusions of Q coming from the fundamental domains. We then have that U and V commute and the isomorphism $\mathrm{Ad}(UV) \circ \alpha^{-1}$ intertwines both the inclusions of M and Q. □

Definition 5.7. Two finite, σ-finite von Neumann algebras M and N are *von Neumann equivalent*, denoted $M \sim_{vNE} N$, if there exists a von Neumann coupling between them.

Von Neumann equivalence is indeed an equivalence relation. Reflexivity follows by considering the trivial von Neumann coupling $\mathcal{B}(L^2(M, \tau))$ with the standard embeddings of M and $M^{\rm op}$. If ${}_M\mathcal{M}_N$ is a von Neumann coupling between M and N, then $\mathcal{M}^{\rm op}$ gives a von Neumann coupling between N and M (with index $[N : M]_{\mathcal{M}^{\rm op}} = [M : N]_{\mathcal{M}}^{-1}$), showing that this relation is symmetric, while transitivity of this relation follows from Proposition 5.6. We also see that by considering the index it follows that $\mathcal{I}_{vNE}(M)$ is a subgroup of $\mathbb{R}_+^*$. We record all these facts in the following theorem.

Theorem 5.8. *Von Neumann equivalence gives an equivalence relation on the collection of finite, σ-finite von Neumann algebras. Given a finite, σ-finite von Neumann algebra M, we have that $\mathcal{I}_{vNE}(M)$ is a subgroup of $\mathbb{R}_+^*$, which only depends on the von Neumann equivalence class of M.*

Two finite factors M and N are virtually isomorphic if there exists a normal Hilbert M-N-bimodule $\mathcal{H}$ that has finite von Neumann dimension as both an M module and an N module. Two ICC groups Γ and Λ are virtually W^*-equivalent if $L\Gamma$ and $L\Lambda$ are virtually isomorphic. The notion of virtual isomorphism of factors was first studied by Popa in [Pop86, Section 1.4], while the terminology was coined more recently in [PS20, Section 4.1].

Theorem 5.9. *Let M and N be finite factors, and suppose that $\mathcal{H}$ is a Hilbert M-N-bimodule that is finite as both an M-module and an N-module. Then M and N are von Neumann equivalent, and a von Neumann coupling $\mathcal{M}$ may be chosen so that*

$$[M : N]_{\mathcal{M}} = \dim_M(\mathcal{H}) \cdot \dim_N(\mathcal{H})^{-1}.$$

Proof. Let $\mathcal{H}$ be a Hilbert M-N-bimodule that is finite as both an M-module and an N-module. Let R denote the hyperfinite II$_1$ factor and set $\mathcal{M} = R \overline{\otimes} \mathcal{B}(\mathcal{H})$.

Suppose $t = \dim_M(\mathcal{H}) < \infty$, and take $k \in \mathbb{N}$ so that $k > t$. If we take a projection $p \in \mathbb{M}_n(\mathbb{C}) \overline{\otimes} M^{\rm op}$ such that $(\mathrm{Tr} \otimes \tau)(p) = t$, then we have an isomorphism of inclusions between $M \subset \mathcal{B}(\mathcal{H})$ and $pM \subset p(\mathbb{M}_n(\mathbb{C}) \overline{\otimes} \mathcal{B}(L^2(M, \tau)))p$.

If we now take a projection $q \in R$ so that $\tau(q) = t/n$, then we have that q and p are equivalent projections in $R \overline{\otimes} \mathbb{M}_n(\mathbb{C}) \overline{\otimes} M^{\rm op}$ and hence we see that we have an isomorphism of inclusions between $M \subset R \overline{\otimes} \mathcal{B}(\mathcal{H})$ and $qM \subset qRq \overline{\otimes} \mathbb{M}_n(\mathbb{C}) \overline{\otimes} \mathcal{B}(L^2(M, \tau))$. In particular, we then see that we have a fundamental domain for the inclusion $M \subset R \overline{\otimes} \mathcal{B}(\mathcal{H})$. Moreover, the trace of a rank-one projection in this fundamental domain will be $n\tau(q) = t$.

We similarly see that the inclusion $N \subset R \overline{\otimes} \mathcal{B}(\mathcal{H})$ has a fundamental domain, and the trace of a rank-one projection in its fundamental domain will be $\dim_N(\mathcal{H})$. Thus, $\mathcal{M}$ is a von Neumann coupling between M and N with index $\dim_M(\mathcal{H}) \dim_N(\mathcal{H})^{-1}$. □

Corollary 5.10. *If M is a II$_1$ factor and $s, t > 0$, then M^t and M^s have a von Neumann coupling $\mathcal{M}$ that satisfies*

$$[M^t : M^s]_{\mathcal{M}} = t^2/s^2.$$

Consequently, we have an inclusion $\mathcal{F}(M)^2 \subset \mathcal{I}_{vNE}(M)$.

We may now show the relationship between von Neumann equivalence for groups and for finite von Neumann algebras as stated in Theorem 1.6.

Proof of Theorem 1.6. We first suppose that $\mathcal{M}$ is an $L\Gamma$-$L\Lambda$ von Neumann coupling. If $p \in \mathcal{Z}(\mathcal{M})$ is a non-trivial central projection, then $p\mathcal{M}$ is also an $L\Gamma$-$L\Lambda$ von Neumann coupling, hence we may assume that $\mathcal{M}$ is σ-finite and fix a semi-finite normal faithful trace Tr on $\mathcal{M}$.

We identify Γ (resp. Λ) as a subgroup of $\mathcal{U}(L\Gamma)$ (resp. $\mathcal{U}(L\Lambda)$) and then consider the commuting trace-preserving actions of Γ and Λ on $\mathcal{M}$ given by conjugation. If we have a fundamental domain $L\Gamma \subset \mathcal{B}(\ell^2\Gamma) \subset \mathcal{M}$, then the rank-one projection onto the subspace spanned by $\delta_e \in \ell^2\Gamma$ gives a finite-trace fundamental domain for the conjugation action of Γ on $\mathcal{M}$. We similarly have a finite-trace fundamental domain for the action of Λ on $\mathcal{M}$, and hence we see that $\mathcal{M}$ is then a Γ-Λ von Neumann coupling.

Now suppose that $(\mathcal{M}, \mathrm{Tr})$ is a Γ-Λ von Neumann coupling. We set $\mathcal{N} = \mathcal{M} \rtimes (\Gamma \times \Lambda)$. We then have embeddings $L\Gamma, L\Lambda \subset \mathcal{N}$. A Γ-fundamental domain in $\mathcal{M}$ gives a Γ-equivariant embedding $\ell^\infty\Gamma \subset \mathcal{M}$ and hence we get an embedding of von Neumann algebras

$$\mathcal{B}(\ell^2\Gamma) \cong \ell^\infty\Gamma \rtimes \Gamma \subset \mathcal{M} \rtimes \Gamma \subset \mathcal{N}.$$

Thus $\mathcal{N}$ has an $L\Gamma$ fundamental domain. Moreover, if P_e is the rank-one projection onto the span of $\delta_e \in \ell^2\Gamma$, then we have $P_e \in \mathcal{M} \subset \mathcal{N}$, and therefore the fundamental domain for $L\Gamma$ has finite trace and so must be finite. We similarly have a finite-trace fundamental domain for $L\Lambda$ in $\mathcal{N}$, and hence $\mathcal{N}$ is an $L\Gamma$-$L\Lambda$ von Neumann coupling. □

The analogue of the index group has also been considered in the setting of measure equivalence. For instance, in [Gab02, Section 2.2] or [Gab05, Question 2.8] Gaboriau considered the set of indices of all ergodic self measure equivalence couplings of a group Γ. For minimally almost periodic groups [NW40] any non-trivial ergodic probability measure-preserving action is weak mixing, and a simple argument then shows that the composition of two ergodic measure equivalence self-couplings is again ergodic. This then shows that for minimally almost periodic groups, the set of indices of all ergodic self measure equivalence couplings is a subgroup of $\mathbb{R}^*_+$. It is not clear, however, if this set is a group in general, or that it is a measure equivalence invariant, as the composition of ergodic measure equivalence couplings need not be ergodic in general. For ICC groups, at least, we have the following relationship between indices for ergodic measure equivalence couplings and the index group for the group von Neumann algebra:

Proposition 5.11. *Suppose* (Ω, m) *is an ergodic ME-self-coupling of an ICC group* Γ*, then* $[\Gamma : \Gamma]_\Omega \in \mathcal{I}_{vNE}(L\Gamma)$.

Proof. We see from the proof of Theorem 1.6 that if (Ω, m) is an ergodic measure equivalence self-coupling of Γ, then $L^\infty(\Omega, m) \rtimes (\Gamma \times \Gamma)$ is a von Neumann self-coupling for $L\Gamma$, and if $L^\infty(\Omega, m) \rtimes (\Gamma \times \Gamma)$ is a factor, then the indices for these couplings agree. Thus it suffices to show that under these hypotheses, we have that $L^\infty(\Omega, m) \rtimes (\Gamma \times \Gamma)$ is a factor.

If we let Γ_i denote the ith copy of Γ in $\Gamma \times \Gamma$, then Proposition 3.4 shows that the fundamental domain for Γ_1 leads to an isomorphism

$$L^\infty(\Omega, m) \rtimes (\Gamma \times \Gamma) \cong (L^\infty(\Omega/\Gamma_1) \rtimes \Gamma_2) \overline{\otimes} \mathcal{B}(\ell^2\Gamma_1).$$

Since Γ_2 is ICC, and since $\Gamma_2 \curvearrowright \Omega/\Gamma_1$ is an ergodic and measure-preserving action on a finite measure space, Murray and von Neumann's proof of factoriality of $L\Gamma_2$ [**?**] shows that we have

$$L\Gamma_2' \cap (L^\infty(\Omega/\Gamma_1) \rtimes \Gamma_2) \subset L^\infty(\Omega/\Gamma_1)^{\Gamma_2} = \mathbb{C}.$$

Hence $L^\infty(\Omega/\Gamma_1) \rtimes \Gamma_2$ is a factor, and so is

$$(L^\infty(\Omega/\Gamma_1) \rtimes \Gamma_2) \overline{\otimes} \mathcal{B}(\ell^2\Gamma_1) \cong L^\infty(\Omega, m) \rtimes (\Gamma \times \Gamma).$$

□

In [PV10] Popa and Vaes study the collection $\mathcal{S}_{\text{eqrel}}(\Gamma)$ of fundamental groups for equivalence relations associated to free, ergodic, probability measure-preserving actions of Γ. Each element in such a fundamental group gives rise to an ergodic measure equivalence coupling with the same index [Fur99a, Theorem 3.3], and hence we obtain the following corollary.

Corollary 5.12. *For a countable ICC group Γ, we have $\mathcal{F} < \mathcal{I}_{vNE}(L\Gamma)$ for all $\mathcal{F} \in \mathcal{S}_{\text{eqrel}}(\Gamma)$.*

As an example, the previous corollary applied to Theorem 1.3 in [PV10] shows that for $n \geq 3$ we have $\mathbb{Q}_+^* < \mathcal{I}_{vNE}(L(\mathbb{Z}^n \rtimes SL(n, \mathbb{Z})))$.

If M is von Neumann equivalent to a factor with fundamental group $\mathbb{R}_+^*$, then we have $\mathcal{I}_{vNE}(M) = \mathbb{R}_+^*$. It would be interesting to have examples of von Neumann algebras, or even groups Γ, such that $\mathcal{I}_{vNE}(L\Gamma)$ is not $\mathbb{R}_+^*$. Or, in view of the previous proposition, examples when $\mathcal{I}_{vNE}(L\Gamma)$ is non-trivial and discrete.

Chifan and Ioana in [CI11] gave examples of groups that are orbit equivalent (and hence also von Neumann equivalent) but that are not W^*-equivalent. Popa and Shlyakhtenko showed in [PS20, Propostion 4.3] that these are not even virtually W^*-equivalent (and additional examples with this property are also given). Combining this result with Theorem 1.6 shows that von Neumann equivalence for groups (resp. for von Neumann algebras) is strictly coarser than virtual W^*-equivalence (resp. virtual isomorphism). We also note that Bannon, Marrakchi, and Ozawa showed recently in [BMO20] that property Gamma of Murray and von Neumann [MvN43] is a virtual isomorphism invariant, while Theorem 1.6 together with Effros's Theorem [Eff75] and Caprace's example [DTDW20, Section 5.C] show that property Gamma is not an invariant of von Neumann equivalence.

The related problem of finding groups that are W^*-equivalent but not measure equivalent remains open (see [CI11]). We also do not know examples of groups that are von Neumann equivalent but not measure equivalent.

Appendix

Measure equivalence and properly proximal groups

For the benefit of the reader who may be less familiar with von Neumann algebras, we include here a separate proof that proper proximality is a measure equivalence invariant. We refer the reader to [Zim84] or [Fur11] for preliminary results on measure equivalence and cocycles.

If E_* is a separable Banach space and (X, μ) is a standard Borel space, then we denote by $L^1(X; E_*)$ the set of norm-integrable Borel functions from X to E_*, where we identify two functions if they agree almost everywhere. This is naturally a Banach space with norm $\|f\| = \int \|f(x)\| \, d\mu$. We set $E = (E_*)^*$ and let $L^\infty(X; E)$ denote the space of measurable, essentially bounded functions from X to E, where E is given the Borel structure coming from the weak*-topology, and we identify functions that agree almost everywhere. We have a natural identification of $L^\infty(X; E)$ with $L^1(X; E_*)^*$ via the pairing $\langle \varphi, f \rangle = \int \varphi_x(f_x) d\mu(x)$. If $K \subset E$ is a weak*-compact convex subset, then $L^\infty(X; K)$ gives a weak*-compact convex subset of $L^\infty(X; E)$.

If E is a dual Banach Λ-module and $K \subset E$ is a non-empty weak*-compact convex Λ-invariant subset, $\Gamma \curvearrowright (X, \mu)$ is a probability measure-preserving action, and $\alpha : \Gamma \times X \to \Lambda$ is a cocycle, then we obtain an induced affine action of Γ on $L^\infty(X; K)$ by

$$(\gamma \cdot f)(x) = \alpha(\gamma, \gamma^{-1} x) f(\gamma^{-1} x).$$

Let (Ω, m) be an ME-coupling of two groups Γ and Λ, and let $X \subset \Omega$ be fundamental domains for the Λ-actions. Under the identification $\Omega/\Lambda \cong X$ given by $\Lambda\omega \mapsto \Lambda\omega \cap X$, the action $\Gamma \curvearrowright \Omega/\Lambda$ translates to

$$\gamma \cdot x = \alpha(\gamma, x)\gamma x,$$

where α is the Zimmer cocycle, which is defined by the property that $\alpha(\gamma, x)$ is the unique element in Λ such that $\alpha(\gamma, x)\gamma x \in X$.

The following result is well known.

Proposition 5.13. *There exists a Γ-fixed point in $L^\infty(\Omega/\Lambda; K)$ if and only if there exists a Λ-fixed point in K.*

Proof. Suppose $\xi : \Omega/\Lambda \to K$ is a Borel map that satisfies $\xi(\gamma x) = \alpha(\gamma, x)\xi(x)$. We then define the map $\tilde{\xi} : \Omega \to K$ by $\tilde{\xi}(\lambda, x) = \lambda^{-1}\xi(x)$, where we identify here Ω with $\Lambda \times (\Omega/\Lambda)$. Then as $\tilde{\xi}$ is invariant under the induced Γ-action and is equivariant with respect to the Λ-action, we therefore obtain a Λ-equivariant map from $\Omega/\Gamma \to K$. Integrating this map with respect to the Λ-invariant measure on Ω/Γ then gives a Λ-fixed point. □

We recall from Proposition 2.11 that a group Λ is properly proximal if there exists a dual Banach Λ-module E and a non-empty weak*-compact convex Λ-invariant subset $K \subset E$ such that K has a properly proximal point, but has no fixed point.

A cocycle $\alpha : \Gamma \times X \to \Lambda$ is *proper* if for all $\varepsilon > 0$ and $F \subset \Lambda$ finite, there exists $F' \subset \Gamma$ finite such that $\mu(\{x \mid \alpha(\gamma, \gamma^{-1}x) \in F\}) < \varepsilon$ for all $\gamma \in \Gamma \setminus F'$. It's easy to see that a cocycle coming from an ME-coupling is proper.

Proposition 5.14. *If the action $\Lambda \curvearrowright K$ is properly proximal, and if the cocycle α is proper, then the induced action $\Gamma \curvearrowright L^\infty(X; K)$ is properly proximal.*

Proof. We assume for simplicity that K is contained in the unit ball of E^*. Fix $k \in K$ such that for all $h \in \Lambda$ we have $\lim_{\lambda\to\infty} \lambda hk - \lambda k = 0$. We view $k \in L^\infty(X; K)$ as a constant function. Fix $g \in \Gamma$, $\varepsilon > 0$, and $\mathcal{F} \subset L^1(X; E)$ a finite collection of step functions with finite range F_0 contained in the unit ball of E. Fix a set $X_0 \subset X$ such that $\mu(X_0) > 1 - \varepsilon$ and such that $x \mapsto \alpha(g, x)$ ranges in a finite set $F_{00} \subset \Lambda$.

Since k is a convergence point for Λ, there exists a finite set $F'_{00} \subset \Lambda$ such that for all $\lambda \in \Lambda\setminus F'_{00}$ we have $|\langle \lambda hk - \lambda k, a\rangle| < \varepsilon$ for all $h \in F_{00}, a \in F_0$. As the cocycle α is proper, there exists a finite set $G_0 \subset \Gamma$, so that if $E_\gamma = \{x \in X \mid \alpha(\gamma, \gamma^{-1}x) \notin F'_{00}\}$, then $\mu(E_\gamma) > 1 - \varepsilon$ for all $\gamma \in \Gamma \setminus G_0$. For $\gamma \in \Gamma \setminus G_0$, and $f \in \mathcal{F}$ we then have

$$|\langle \gamma gk - \gamma k, f\rangle| = \left| \int \langle \alpha(\gamma g, g^{-1}\gamma^{-1}x)k - \alpha(\gamma, \gamma^{-1}x)k, f(x)\rangle d\mu(x) \right|$$

$$
\begin{aligned}
&\leq \int |\langle \alpha(\gamma, \gamma^{-1}x)\alpha(g, g^{-1}\gamma^{-1}x)k - \alpha(\gamma, \gamma^{-1}x)k, f(x)\rangle| d\mu(x) \\
&\leq \varepsilon + \int_{\gamma g X_0} \sup_{h \in F_{00}} \sup_{a \in F_0} |\langle \alpha(\gamma, \gamma^{-1}x)hk - \alpha(\gamma, \gamma^{-1}x)k, a\rangle| d\mu(x) \\
&\leq 2\varepsilon + \int_{\gamma g X_0 \cap E_\gamma} \sup_{h \in F_{00}} \sup_{a \in F_0} |\langle \alpha(\gamma, \gamma^{-1}x)hk - \alpha(\gamma, \gamma^{-1}x)k, a\rangle| d\mu(x) < 3\varepsilon.
\end{aligned}
$$

Since simple functions are dense in $L^1(X; E)$, it follows that k is a convergence point for the action $\Gamma \curvearrowright L^\infty(X; K)$. □

Corollary 5.15. *If two groups Γ and Λ are measure equivalent, then Γ is properly proximal if and only if Λ is properly proximal.*

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