NEW EXAMPLES OF IRREDUCIBLE SUBFACTORS OF THE HYPERFINITE II_1 FACTOR WITH RATIONAL, NON-INTEGER INDEX

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INTRODUCTION

In [Jon83], Jones introduced the notion of index for an inclusion of II_1 factors and proved the striking theorem that the index of any subfactor is contained in the set

$$\{4\cos^2\frac{\pi}{n} \mid n \geq 3\} \cup [4,\infty].$$

Moreover, he showed that all these numbers are realized as indices of subfactors of the hyperfinite II_1 factor. In this paper, Jones proved that subfactors with indices less than 4 are automatically irreducible, but the examples of subfactors of the hyperfinite II_1 factor he gave with indices in $(4,\infty]$ are all reducible.

In [Pop93], Popa showed that any number above 4 is an index of an irreducible subfactor of some *non-hyperfinite* II_1 factor. It is an open problem to determine what all possible indices of *irreducible* subfactors of the hyperfinite II_1 factor are, but various examples and obstructions related to several different construction methods are known.

Commuting squares ([Pop83]) can be used to construct irreducible subfactors of the hyperfinite II_1 factor and many such examples were given by Haagerup and Schou in [HS89] (see [Sch90] for details). These examples typically arise from finite-dimensional commuting squares. In [Sch90], Schou showed that subfactors of the hyperfinite II_1 factor could be constructed from certain *infinite-dimensional* commuting squares as well and found examples of irreducible hyperfinite subfactors that cannot be constructed from finite-dimensional commuting squares.

In Chapter 1 we describe the method of constructing an irreducible subfactor of the hyperfinite II_1 factor from a (finite-dimensional or infinite-dimensional) symmetric commuting square as it appears in [Sch90].

In this construction, the resulting subfactor has index equal to the square of the norm of a certain connected, locally finite, bipartite graph that the commuting square is based on.

This somewhat limits what indices could potentially be obtained in this manner. Also, if this graph has a certain type of vertex with degree 1, then the subfactor is irreducible ([Wen88], [Sch90]).

In [Bis94b], Bisch used an infinite-dimensional symmetric commuting square to construct the first example of an irreducible subfactor of the hyperfinite II_1 factor whose index is a rational, non-integer number and thus not an algebraic integer. It cannot be constructed from a finite-dimensional commuting square. To construct this subfactor, Bisch used a commuting square based on graph called 4-star with A_∞-tail and obtained an irreducible hyperfinite subfactor with index 4.5. It is known that this subfactor has ”trivial” standard invariant consisting just of Temperley–Lieb algebras ([Haa94], see also [AMP15]).

In Chapter 2 we construct new examples of irreducible hyperfinite subfactors with rational, non-integer indices.

We first show that, for every $N \geq 4$, the square of the norm of a graph given by an N-star with A_∞-tail is $\frac{(N-1)^2}{N-2} = N + \frac{1}{N-2}$. This means that if there exists a commuting square based on any of these graphs, the resulting irreducible hyperfinite subfactor would have a rational, non-integer index. We show that this indeed is the case for $N \in \{5,\ 6,\ 7,\ 9\}$. There exist irreducible hyperfinite subfactors based on N-stars with A_∞-tail for these N and their indices are $\frac{16}{3} = 5.333\ldots$, $\frac{25}{4} = 6.25$, $\frac{36}{5} = 7.2$ and $\frac{64}{7} = 9.142\ldots$

For $N = 5$, we obtain the result by directly solving the equations arising from Ocneanu's bi-unitary condition ([Ocn88]) for the existence of a symmetric commuting square. For $N = 6$ and $N = 7$, we solve a subset of these equations numerically and then find the corresponding exact solutions. For $N = 7$ we obtain two distinct solutions, but we do not know whether the resulting subfactors are isomorphic.

Noticing certain symmetries in cases $N = 5$ and $N = 7$, we explore the possibility of generalizing the solutions from these cases to the case of arbitrary odd $N \geq 5$. This approach enables us to find a solution for $N = 9$.

CHAPTER 1

SYMMETRIC COMMUTING SQUARES

1.1 Preliminaries

Let $I = \{1,\ 2,\ \ldots,\ n\}$, for some $n \in \mathbb{N}$, or $I = \mathbb{N}$. We call a direct sum $A = \bigoplus_{i \in I} A_i$ of finite-dimensional von Neumann algebras $A_i \cong \mathrm{M}_{a_i}(\mathbb{C})$ for some $a_i \in \mathbb{N}$, $i \in I$, a (finite-dimensional if I is finite or infinite-dimensional if I is countably infinite) **multi-matrix algebra**. The **dimension vector** of A is $\vec{a} = (a_i)_{i \in I}$. A trace tr on A, which we will always assume to be not identically zero and positive, is determined by a **trace vector** $\vec{\alpha} = (\alpha_i)_{i \in I}$ where α_i is the trace of a non-zero minimal projection in A_i for $i \in I$. The trace is faithful if and only if all entries of its trace vector are positive. If $\mathrm{tr}(1) < \infty$, we call the trace **finite** and then $\|\vec{\alpha}\|_1 < \infty$ which implies $\|\vec{\alpha}\|_2 < \infty$. We say that the trace is **normalized** if $\mathrm{tr}(1) = 1$.

We will assume all inclusions of $*$-algebras to be unital, i.e. the notation $A \subset B$ will mean $1_A = 1_B$ whenever it makes sense. Let $A \subset B$ be an inclusion of two finite-dimensional or two infinite-dimensional multi-matrix algebras. Then $A = \bigoplus_{i \in I} A_i$ and $B = \bigoplus_{j \in J} B_j$ for simple summands A_i and B_j, $i \in I, j \in J$. Let $i \in I, j \in J$ and fix any non-zero minimal projection $p_{A_i} \leq z_{A_i}$ in A_i where z_{A_i} is the minimal central projection of A contained in A_i. Then $p_{A_i} z_{B_j}$ is a projection in B_j, where z_{B_j} is the minimal central projection of B contained in B_j. $p_{A_i} z_{B_j}$ can be decomposed into a sum of G_{ij} non-zero minimal projections in B_j. The number G_{ij} (called the **multiplicity** of A_i in B_j) does not depend on the inital choice of p_{A_i}. Let G be (finite or countably infinite) matrix such that $G = (G_{ij})_{i \in I, j \in J}$. We call G the **inclusion matrix** of $A \subset B$ and write $A \subset_G B$. All of its entries are non-negative integers. The (finite or countably infinite) bipartite graph whose bipartite adjacency matrix is G is called the **Bratteli diagram** ([Bra72]) or **inclusion graph** of $A \subset_G B$. We will denote this graph by Γ_G.

Let $A \subset B$ be an inclusion of finite von Neumann algebras and let tr be a normalized, faithful, normal trace on B. We call the orthogonal projection $e_A : \mathrm{L}^2(B, \mathrm{tr}) \to \mathrm{L}^2(A, \mathrm{tr}\big|_A)$ the **Jones projection**. B acts by left multiplication on $\mathrm{L}^2(B, \mathrm{tr})$ and we denote by $\langle B, e_A \rangle$ the von Neumann algebra generated by B and e_A as a subalgebra of $\mathrm{B}(\mathrm{L}^2(B, \mathrm{tr}))$. This is the **Jones basic construction** ([Jon83]).

If $A \subset_G B$ are (finite-dimensional or infinite-dimensional) multi-matrix algebras with dimension vectors $\vec{a}$ and $\vec{b}$, and trace vectors $\vec{\alpha}$ and $\vec{\beta}$ that define finite traces which agree on A, then $\vec{\alpha} = G\vec{\beta}$ and $\vec{b} = G^t \vec{a}$. Also $B' \subset_{G^t} A'$ and $B \subset_{G^t} \langle B, e_A \rangle$ ([Jon83] in finite-dimensional and [Sch90] in infinite-dimensional case).

We will only consider inclusions $A \subset_G B$ of multi-matrix algebras with inclusion graphs Γ_G that are:

(i) locally finite, hence all matrix polynomials in $\begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}$, $G^t G$ and GG^t have well-defined finite entries, and

(ii) connected, hence $\begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}$, $G^t G$ and GG^t are irreducible matrices ([GHJ89]).

Example 1.1. *Let $A = \mathbb{C} \oplus \mathrm{M}_2(\mathbb{C}) \oplus \mathbb{C}$ and $B = \mathrm{M}_5(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C})$. If $A \subset_G B$ and the inclusion is given by*

$$x \oplus \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \oplus z \mapsto \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & y_1 & y_2 & 0 \\ 0 & 0 & y_3 & y_4 & 0 \\ 0 & 0 & 0 & 0 & z \end{pmatrix} \oplus \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, \tag{1.1}$$

then its Bratteli diagram is

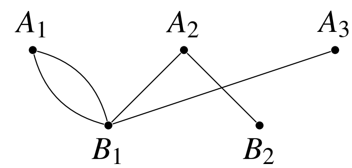

Figure 1: Bratteli diagram example

and its inclusion matrix is

$$G = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that any unital, injective $*$-homomorphism $\phi : A \to B$ between (both finite-dimensional or both infinite-dimensional) multi-matrix algebras A and B is, up to unitary conjugacy, of diagonal form (1.1). Also, the inclusion $A \subset_G B$ is determined by its inclusion matrix G (or Bratteli diagram Γ_G) only up to unitary conjugacy.

The notion of a *commuting square* was introduced by Popa in [Pop83]. Commuting squares arise naturally in classification and construction of subfactors ([Pop90], [Ocn88], [GHJ89], [JS97]). We recall the definition (see e.g. [GHJ89], [JS97]):

Definition 1.2. *Let A, B, C, D be finite von Neumann algebras such that*

$$\begin{array}{ccc} C & \subset & D \\ \cup & & \cup \\ A & \subset & B \end{array}$$

and let tr_D be a faithful, normal, finite trace on D. We call this a **commuting square** *if the unique trace-preserving (with respect to tr_D) conditional expectations $\mathrm{E}_A, \mathrm{E}_B, \mathrm{E}_C$ of D onto A, B, C satisfy $\mathrm{E}_A = \mathrm{E}_B \mathrm{E}_C = \mathrm{E}_C \mathrm{E}_B$. If A, B, C, D are multi-matrix algebras and the* *inclusions $A \subset B \subset D$ and $A \subset C \subset D$ are given by inclusion matrices $G,\ H,\ K,\ L$, we write*

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

Note that $GH = KL$. If moreover $G^tK = HL^t$, then we say that the commuting square is a **symmetric commuting square***. We call a commuting square* **finite-dimensional** *(resp.* **infinite-dimensional***) if the involved multi-matrix algebras are all finite-dimensional (resp. infinite-dimensional).*

The following result of Ocneanu ([Ocn88]) appears in this form in [Sch90] and gives a sufficient and necessary condition (the *bi-unitary condition*) for a symmetric commuting square to exist. A proof in the finite-dimensional case can also be found in [JS97] and [EK98].

Theorem 1.3. (Ocneanu's bi-unitary condition) *Let $G,\ H,\ K,\ L$ be bipartite adjacency matrices of connected, locally finite, bipartite graphs such that $GH = KL$ and $G^tK = HL^t$. (These matrices are all finite or all countably infinite.) Then the following are equivalent:*

(i) There exists a symmetric commuting square of multi-matrix algebras $A,\ B,\ C,\ D$

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

with respect to a faithful, finite trace tr_D on D.

(ii) There exist unitary matrices u, v of the form

$$u = \bigoplus_{(i,k)} u^{(i,k)}, \quad v = \bigoplus_{(j,l)} v^{(j,l)}$$

satisfying the **bi-unitary condition***, that is, their direct summands*

$$u^{(i,k)} = \left(u^{(i,k)}_{(j,\phi,\rho,l,\sigma,\psi)}\right)_{\substack{(i,\phi,j,\rho,k)\in S\\(i,\sigma,l,\psi,k)\in T}} \quad \text{for } (i,k) \text{ such that } (GH)_{ik} \neq 0, \text{ and}$$

$$v^{(j,l)} = \left(v^{(j,l)}_{(i,\phi,\rho,k,\sigma,\psi)}\right)_{\substack{(i,\phi,j,\rho,k)\in S\\(i,\sigma,l,\psi,k)\in T}} \quad \text{for } (j,l) \text{ such that } (G^tK)_{jl} \neq 0,$$

are unitary matrices where

$$v^{(j,l)}_{(i,\phi,\rho,k,\sigma,\psi)} = \sqrt{\frac{\alpha_i\gamma_k}{\beta_j\delta_l}}\, u^{(i,k)}_{(j,\phi,\rho,l,\sigma,\psi)} \quad \text{for all } (i,\phi,j,\rho,k) \in S \text{ and } (i,\sigma,l,\psi,k) \in T.$$

Here $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$, $\vec{\delta}$ denote the trace vectors of tr_D on A, B, C, D, and S and T are the sets of paths in the Bratteli diagrams of $A \subset_G B \subset_H D$ and $A \subset_K C \subset_L D$ (including multiplicities labeled by ϕ and ρ, resp. σ and ψ), i.e.

$$S = \left\{(i,\phi,j,\rho,k) \mid G_{ij}H_{jk} \neq 0,\ 1 \leq \phi \leq G_{ij},\ 1 \leq \rho \leq H_{jk}\right\} \quad \text{and}$$

$$T = \left\{(i,\sigma,l,\psi,k) \mid K_{il}L_{lk} \neq 0,\ 1 \leq \sigma \leq K_{il},\ 1 \leq \psi \leq L_{lk}\right\}.$$

If (ii) holds, we may asumme that A is commutative.

1.2 Construction of hyperfinite subfactors

The construction of a subfactor of the hyperfinite II_1 factor from a symmetric commuting square relies heavily on the Perron-Frobenius theorem ([Per07], [Fro12]) and its consequences.

Theorem 1.4. (Perron-Frobenius theorem [Fro12]) *Let T be an irreducible, finite square matrix with non-negative entries and λ its spectral radius (i.e. $\lambda = \|T\|$). Then:*

(i) λ is an eigenvalue of T with 1-dimensional eigenspace which contains a vector whose entries are all positive.

(ii) Any eigenvector of T whose entries are all positive is associated to the eigenvalue λ.

This eigenvalue and its positive eigenvectors (determined up to a positive scalar) are called the **Perron-Frobenius eigenvalue** *and* **Perron-Frobenius eigenvectors** *of* T.

The proof of Theorem 1.4 can also be found in [Gan59] or [Sen81]. To construct hyperfinite subfactors from infinite-dimensional symmetric commuting squares, a generalization of the Perron-Frobenius theorem is needed:

Theorem 1.5. (Perron-Frobenius type theorem) *Let* T *be the full adjacency matrix of a connected, locally finite, countably infinite graph. Then* T *is irreducible, symmetric (i.e.* $T^t = T$*) and* T^n *has well-defined finite entries for all* $n \in \mathbb{N}$*. If* $\vec{\xi}$ *is an eigenvector of* T *whose entries are all positive, then:*

(i) If $\|\vec{\xi}\|_2 < \infty$*, then* $\vec{\xi}$ *is proportional to the unique (up to a scalar) positive eigenvector of* T *associated with the largest eigenvalue* λ *of* T.

(ii) If T *has an* ℓ^2 *eigenvector whose entries are all positive, then* $\|\vec{\xi}\|_2 < \infty$.

If T *has an* ℓ^2 *eigenvector whose entries are all positive, then* $\lambda = \|T\|$*, for the operator norm in* $\mathrm{B}(\ell^2(\mathbb{N}))$*, and* λ *has a* 1*-dimensional eigenspace. This eigenvalue and its positive eigenvectors are called the* **Perron-Frobenius eigenvalue** *and* **Perron-Frobenius eigenvectors** *of* T *as in the finite case.*

This theorem was used in similar form by Schou in [Sch90]. The result follows from [Ken66], [Sen67] and [Moh82], see also [Sen81].

Corollary 1.6. *Let* T *be a finite or countably infinite, positive-semidefinite matrix and* $\vec{\xi}$ *its Perron-Frobenius eigenvector such that* $\|\vec{\xi}\|_2 = 1$*. Let* $\vec{\zeta}$ *be any* ℓ^2*-vector whose entries are indexed in the same way as those of* $\vec{\xi}$*. Then*

$$\left(\frac{T}{\|T\|}\right)^n \vec{\zeta} \xrightarrow[n\to\infty]{\|\cdot\|_2} \langle\vec{\zeta}|\vec{\xi}\rangle\vec{\xi}.$$

Proof. Define $f_n : [0,1] \to [0,1]$ by $f_n(x) = x^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Then $f_n \xrightarrow[n\to\infty]{\text{pointwise}} \chi_{\{1\}}$ and thus $\left(\frac{T}{\|T\|}\right)^n \xrightarrow[n\to\infty]{\text{s.o.}}$ projection onto $\mathbb{C}\vec{\xi}$. This implies the result. □

These theorems can be used to prove properties of traces in relation to the Jones basic construction applied to inclusions of multi-matrix algebras. The notion of a Markov trace first appears in [Jon83]:

Definition 1.7. *Let $A \subset_T B$ be multi-matrix algebras such that Γ_T is connected and locally finite, and let* tr *be a normalized, faithful trace on B. Then* tr *is* **λ-Markov** *for $\lambda > 0$ if there is an extension of* tr *to $\langle B, e_A \rangle$ such that $\mathrm{tr}(xe_A) = \lambda\, \mathrm{tr}(x)$ for all $x \in B$.*

Theorem 1.8. ([Jon83]) *Let $A \subset_T B$ be multi-matrix algebras such that Γ_T is connected and locally finite, and let* tr *be a normalized, faithful trace on B with trace vector $\vec{t}$. Let $\vec{s}$ be the trace vector of $\mathrm{tr}\big|_A$ on A. Then the following are equivalent:*

(i) tr *is a (λ^{-1})-Markov trace for $A \subset_T B$.*

(ii) $T^tT\vec{t} = \lambda\vec{t}$ and $TT^t\vec{s} = \lambda\vec{s}$, i.e. $\vec{t}$ and $\vec{s}$ are Perron-Frobenius eigenvectors of T^tT and TT^t, and λ is the Perron-Frobenius eigenvalue of T^tT and TT^t (hence $\lambda = \|T\|^2$).

Note that [Jon83] contains only the finite-dimensional case of Theorem 1.8. In [Sch90], Schou showed that the theorem also holds in the infinite-dimensional case.

Corollary 1.9. *Let $A \subset_T B$ be an inclusion of multi-matrix algebras such that Γ_T is connected and locally finite, and let* tr *be a normalized, faithful trace on B. If* tr *is λ-Markov for $A \subset_T B$, then the Markov extension of* tr *to $\langle B, e_A \rangle$ is also λ-Markov for* B $\subset_{T^t} \langle B, e_A \rangle$.

This relation between Perron-Frobenius eigenvectors and Markov traces can be used to obtain the following result about symmetric commuting squares:

Proposition 1.10. *Let*

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

be a symmetric commuting square of multi-matrix algebras with respect to a normalized, *faithful trace* tr_D *on* D *such that* $\Gamma_G, \Gamma_H, \Gamma_K, \Gamma_L$ *are connected and locally finite. Let* $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$, $\vec{\delta}$ *be the trace vectors of* tr_D *on* A, B, C, D*. Then:*

(i) $\|K\| = \|H\|$, tr_D *is the* $(\|H\|^{-2})$*-Markov trace of the inclusion* $B \subset_H D$ *and* $tr_D\big|_C$ *is the* $(\|K\|^{-2})$*-Markov trace of the inclusion* $A \subset_K C$*.*

(ii) $\|G\| = \|L\|$, tr_D *is the* $(\|L\|^{-2})$*-Markov trace of the inclusion* $C \subset_L D$ *and* $tr_D\big|_B$ *is the* $(\|G\|^{-2})$*-Markov trace of the inclusion* $A \subset_G B$*.*

(iii) $\vec{\delta}$ *is a Perron-Frobenius eigenvector of* H^tH *and* L^tL*.* $\vec{\gamma}$ *is a Perron-Frobenius eigenvector of* LL^t *and* K^tK*.* $\vec{\beta}$ *is a Perron-Frobenius eigenvector of* HH^t *and* G^tG*.* $\vec{\alpha}$ *is a Perron-Frobenius eigenvector of* GG^t *and* KK^t*.*

For details, see [Sch90] and [GHJ89].

Remark 1.11. *The following outlines the construction of a subfactor of the hyperfinite* II_1 *factor of finite index from a symmetric commuting square of (finite-dimensional or infinite-dimensional) multi-matrix algebras.*

The infinite-dimensional case was first described by Schou in [Sch90] and this outline follows his construction. In addition to results mentioned in this section, the construction also uses properties of projections in the Jones basic construction similar to those appearing in [Jon83].

Let

$$\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$$

be a symmetric commuting square of multi-matrix algebras, such that Γ_G, Γ_H, Γ_K, Γ_L are connected and locally finite, with respect to a normalized, faithful trace tr_{B_1} on B_1 given by a Perron-Frobenius eigenvector $\vec{\xi}$ of L^tL. We may and will assume that A_0 is commutative.

Define

$$G_j = \begin{cases} G, & j \text{ even} \\ G^t, & j \text{ odd} \end{cases}, \qquad L_j = \begin{cases} L, & j \text{ even} \\ L^t, & j \text{ odd} \end{cases} \qquad \text{and} \qquad T_j = \begin{cases} K, & j \text{ even} \\ H, & j \text{ odd} \end{cases}.$$

Define inductively $B_j = \langle B_{j-1}, e_{B_{j-2}} \rangle$ and $A_j = \{A_{j-1}, e_{B_{j-2}}\}''$ for $j \geq 2$ where $e_{B_{j-2}}$: $\mathrm{L}^2(B_{j-1}) \to \mathrm{L}^2(B_{j-2})$ are the Jones projections. Then $A_j \cong \langle A_{j-1}, e_{A_{j-2}} \rangle$ and

$$\begin{array}{ccc} B_{j-1} & \subset_{L_{j-1}} & B_j \\ \cup_{T_{j-1}} & & \cup_{T_j} \\ A_{j-1} & \subset_{G_{j-1}} & A_j \end{array}$$

is a symmetric commuting square of multi-matrix algebras for each $j \in \mathbb{N}$ with respect to the trace tr_{B_j} on B_j given by the trace vector $\|L\|^{-2j}L\vec{\xi}$ if j is even and $\|L\|^{-2j}\vec{\xi}$ if j is odd. This way we obtain a sequence of symmetric commuting squares

$$\begin{array}{ccccccccc} B_0 & \subset_L & B_1 & \subset_{L^t} & B_2 & \subset_L & B_3 & \cdots \\ \cup_K & & \cup_H & & \cup_K & & \cup_H & \\ A_0 & \subset_G & A_1 & \subset_{G^t} & A_2 & \subset_G & A_3 & \cdots \end{array} \tag{1.2}$$

Define $A_\infty = \cup_{j=0}^{\infty} A_j$ and $B_\infty = \cup_{j=0}^{\infty} B_j$. Since the traces tr_{B_j} (resp. tr_{A_j}) for $j \in \mathbb{N}$ extend one another, we can define a trace tr_{B_∞} (resp. tr_{A_∞}) on B_∞ (resp. A_∞) that extends each tr_{B_j} (resp. tr_{A_j}) for $j \in \mathbb{N}$. Define B (resp. A) to be the w.o.-closure of B_∞ (resp. A_∞) in $\mathrm{B}(\mathrm{L}^2(B_\infty, \mathrm{tr}_{B_\infty}))$. Since tr_{B_∞} extends tr_{A_∞}, A is equal to the w.o.-closure of A_∞ in $\mathrm{B}(\mathrm{L}^2(A_\infty, \mathrm{tr}_{A_\infty}))$. A and B are factors since uniqueness (up to a scalar) of the Perron-Frobenius eigenvector implies uniqueness of the (normalized) faithful, normal trace A and B. Thus $A \subset B$ is a hyperfinite subfactor and

$$[B : A] = \|H\|^2 = \|K\|^2. \tag{1.3}$$

Note that the index of a subfactor of the hyperfinite II_1 factor constructed in this manner from a *finite-dimensional* symmetric commuting square is always an algebraic integer. In [Sch90], Schou constructed irreducible hyperfinite subfactors whose indices are algebraic integers that do not arise as norms of finite graphs.

Subfactors $A \subset B$ constructed in this manner have the relative McDuff property ([Bis90]) as the Jones projections are non-trivial non-commuting central sequences $(e_{B_j})_{j=0}^{\infty}$ and $(e_{B_{j+1}})_{j=0}^{\infty}$ of B contained in A. This argument was given in [Bis94a].

1.3 Irreducibility

A basic property of the **relative commutant** $A' \cap B$ of a subfactor $A \subset B$ is that $[B : A] < \infty$ implies $\dim(A' \cap B) < \infty$ ([Jon83]). We say that a subfactor $A \subset B$ is **irreducible** if $\dim(A' \cap B) = 1$. The next theorem is Schou's generalization of Wenzl's irreducibility criterion ([Wen88]) to the case of infinite-dimensional symmetric commuting squares ([Sch90]). We first give two lemmas. Wenzl gave a proof of the first lemma for the finite-dimensional case in and Schou showed that it also holds in the infinite-dimensional case. The second lemma is a technical result of Schou used in his proof of Theorem 1.14.

Lemma 1.12. ([Sch90]) *Let*

$$A_0 \subset_G A_1 \subset_{G^t} A_2 \subset_G A_3 \cdots$$

and let B be as in the construction following Remark 1.11. Let tr_B *be the unique normalized, faithful, normal trace on B. Let* $j \in \mathbb{N}_0$. *Note that each* A_{2j} *has the same number of simple summands. Let* z_i^{2j} *be the minimal central projection of* A_{2j} *contained in its i-th simple summand for some* $i \in \mathbb{N}$. *Then there exists* $c_i > 0$ *such that* $\mathrm{tr}_B(z_i^{2j}) \geq c_i$ *for all* $j \in \mathbb{N}_0$. *A similar result holds for odd labeled* A_k*'s.*

Proof. Let $\vec{a}^{A_j}$ be the dimension vectors and $\vec{\alpha}^{A_j}$ the trace vectors (with respect to trace tr_B) of A_j, for $j \in \mathbb{N}_0$. Note that $\vec{a}^{A_{2j}} = (GG^t)^j \vec{a}^{A_0}$ and $\vec{\alpha}^{A_{2j}} = \|G\|^{-2j} \vec{\alpha}^{A_0}$ for $j \in \mathbb{N}_0$. Let z_i^{2j} be the minimal central projection of A_{2j} contained in its i-th simple summand for some $j \in \mathbb{N}_0$ and $i \in \mathbb{N}$. Then

$$\mathrm{tr}_B(z_i^{2j}) = a_i^{A_{2j}} \alpha_i^{A_{2j}} = \left((GG^t)^j \vec{a}^{A_0}\right)_i \|G\|^{-2j} \vec{\alpha}_i^{A_0} = \left(\left(\frac{GG^t}{\|G\|^2}\right)^j \vec{a}^{A_0}\right)_i \vec{\alpha}_i^{A_0}.$$

Note that the trace vector $\vec{\alpha}^{A_0}$ of tr_B on A_0 is given by the normalized Perron-Frobenius eigenvector of GG^t due to Proposition 1.10. Now, for $i \in \mathbb{N}$,

$$\lim_{j\to\infty} \mathrm{tr}_B(z_i^{2j}) = \lim_{j\to\infty} \left(\left(\frac{GG^t}{\|G\|^2}\right)^j \vec{a}^{A_0}\right)_i \vec{\alpha}_i^{A_0} = \left(\langle \vec{a}^{A_0} | \vec{\alpha}^{A_0} \rangle \vec{\alpha}^{A_0}\right)_i \frac{\vec{\alpha}_i^{A_0}}{\|\vec{\alpha}^{A_0}\|_2^2} = \frac{\left(\vec{\alpha}_i^{A_0}\right)^2}{\|\vec{\alpha}^{A_0}\|_2^2} > 0$$

where we used Corollary 1.6 for the second equality and the fact that tr_B is normalized for the last equality. This implies that, for every $i \in \mathbb{N}$, the sequence $\left(\mathrm{tr}_B(z_i^{2j})\right)_{j\in\mathbb{N}_0}$ of positive numbers is bounded from below, i.e. there exists $c_i > 0$ such that $\mathrm{tr}_B(z_i^{2j}) \geq c_i$ for all $j \in \mathbb{N}_0$. The proof for the odd labeled A_k's is similar. $\square$

Lemma 1.13. ([Sch90]) *Let $m \in \mathbb{N} \setminus \{1\}$. Let $\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m\}$ be a set of m different real numbers and let $\{t_1,\ t_2,\ \ldots,\ t_m\}$ be a set of m positive numbers with sum 1. Then there exists $\varepsilon > 0$ such that:*

If A is a II_1 factor, tr *its normalized, faithful, normal trace, and $a, b \in A$ are self-adjoint such that:*

(i) $a = \sum_{i=1}^m \alpha_i p_i$ where $p_i \in A$ are mutually orthogonal projections such that $\mathrm{tr}(p_i) = t_i$, for $1 \leq i \leq m$, and

(ii) b has strictly less than m spectral values and $\|b\| \leq \|a\|$,

then $\|b - a\|_2^2 \geq \varepsilon$.

Proof. This proof is taken from [Sch90]. Define

$$S = \left\{ (\beta, T) \;\middle|\; \begin{array}{l} \beta = (\beta_1, \ldots, \beta_{m-1}),\ 0 \le \beta_j \le \max_{1\le i\le m} |a_i| \text{ for } 1 \le j \le m-1; \\ T = (T_{ij}) \in \mathrm{M}_{m\times(m-1)}([0,1]),\ \sum_{j=1}^{m-1} T_{ij} = t_i \text{ for } 1 \le i \le m. \end{array} \right\}$$

and $F : S \to \mathbb{R}$ by $F(\beta, T) = \sum_{i=1}^{m} \sum_{j=1}^{m-1} (\alpha_i - \beta_j)^2 T_{ij}$. S is compact so F has a minimum on S. Let (β', T') be the point where this minimum is attained. There exists $i \in \{1, \ldots, m\}$ such that $\alpha_i \notin \{\beta'_1, \ldots, \beta'_{m-1}\}$ and then $t_i > 0$ implies that there exists $j \in \{1, \ldots, m-1\}$ such that $T'_{ij} > 0$. Therefore $F(\beta', T') > 0$. Set $\varepsilon = F(\beta', T')$. Let A be a II_1 factor and tr its normalized, faithful, normal trace.

Let $a, b \in A$ be such that they satisfy the assumptions of the lemma. Then $b = \sum_{j=1}^{m-1} \beta_j q_j$ for some $\beta_j \in [0, \max_{1\le i\le m} |a_i|]$ and $q_j \in A$ mutually orthogonal projections, for $1 \le j \le m-1$, such that $\sum_{j=1}^{m-1} q_j = 1$. Note that $\sum_{i=1}^{m} \mathrm{tr}(p_i) = 1$ implies $\sum_{i=1}^{m} p_i = 1$ so

$$a = \sum_{i=1}^{m} \sum_{j=1}^{m-1} \alpha_i p_i q_j, \quad \text{and}$$
$$b = \sum_{i=1}^{m} \sum_{j=1}^{m-1} \beta_j p_i q_j.$$

Now

$$\begin{aligned} \mathrm{tr}((a-b)^*(a-b)) &= \mathrm{tr}\left(\sum_{i=1}^{m} \sum_{j=1}^{m-1} (\alpha_i - \beta_j) q_j p_i \sum_{i'=1}^{m} \sum_{j'=1}^{m-1} (\alpha_{i'} - \beta_{j'}) p_{i'} q_{j'} \right) = \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m-1} \sum_{i'=1}^{m} \sum_{j'=1}^{m-1} (\alpha_i - \beta_j)(\alpha_{i'} - \beta_{j'}) \,\mathrm{tr}(q_j p_i p_{i'} q_{j'}) = \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m-1} (\alpha_i - \beta_j)^2 \,\mathrm{tr}(p_i q_j). \end{aligned}$$

Note that for $1 \le i \le m$ and $1 \le j \le m-1$

$$0 \le \mathrm{tr}(p_i q_j) \le \sum_{j=1}^{m-1} \mathrm{tr}(p_i q_j) = \mathrm{tr}(p_i) = t_i \le 1$$

so for $\beta = (\beta_1, \ldots, \beta_{m-1})$ and $T = (T_{ij})$ such that $T_{ij} = \operatorname{tr}(p_i q_j)$, for $1 \le i \le m$ and $1 \le j \le m-1$, we have $(\beta, T) \in S$. Then

$$\|a-b\|_2^2 = \operatorname{tr}((a-b)^*(a-b)) = F(\beta,T) \ge F(\beta',T') = \varepsilon.$$

□

The following theorem will be used to construct *irreducible* hyperfinite subfactors from infinite-dimensional symmetric commuting squares.

Theorem 1.14. ([Sch90]) *Let*

$$\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$$

be a symmetric commuting square of multi-matrix algebras such that Γ_G, Γ_H, Γ_K, Γ_L are connected and locally finite, with respect to a normalized, faithful trace tr_{B_1} on B_1 given by a Perron-Frobenius eigenvector $\vec{\xi}$ of $L^t L$. Let $A \subset B$ be the hyperfinite subfactor constructed from this commuting square as described after Remark 1.11. Then

$$\dim(A' \cap B) \le (\min\{\textit{1-norm of rows of } H \textit{ and } K\})^2.$$

Proof. This proof is taken from [Sch90]. Let $n \in \mathbb{N}$ be even, i.e. $n = 2l$ for some $l \in \mathbb{N}$. Denote $A_n = \bigoplus_{i=1}^{\infty} A_i^n$ and $B_n = \bigoplus_{i=1}^{\infty} B_i^n$ and let $a_n = (a_i^n)_{i=1}^{\infty}$ and $b_n = (b_i^n)_{i=1}^{\infty}$ be the dimension vectors of A_n and B_n. Then $A_{2l} \subset_K B_{2l}$. Let $m_0 = \min\{\text{1-norm of rows of } K\}$ and pick $i_0 \in \mathbb{N}$ such that i_0-th row of K has 1-norm equal to m_0.
Let $z_{i_0}^{2l}$ be the minimal central projection of A_{2l} contained in its i_0-th simple summand, i.e. in $A_{i_0}^{2l}$. Then $z_{i_0}^{2l} \in \mathscr{Z}(A_{2l}) \subset A'_{2l} \cap B_{2l}$. For $j \in \mathbb{N}$, let q_j^{2l} be the minimal central projection of B_{2l} contained in its j-th simple summand, i.e. $q_j^{2l} \in B_j^{2l}$. Then one can choose a matrix

representation of the inclusion $A_{2l} \subset B_{2l}$ such that the elements of $A_{2l}q_j^{2l}$ are of the form

$$\begin{pmatrix} x_1 & & & & \\ & \ddots & & & \\ & & x_1 & & \\ & & & x_2 & \\ & & & & \ddots \end{pmatrix}$$

where each $x_i \in A_i^{2l}$ is repeated K_{ij} times and the matrix has b_j^{2l} columns.

For $z_{i_0}^{2l}q_j^{2l} \in A_{2l}q_j^{2l}$ then in the above form we have that

$$x_i = \begin{cases} 1_{a_{i_0}^{2l} \times a_{i_0}^{2l}}, & \text{if } i = i_0; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $A'_{2l} \cap B_{2l} \cong \bigoplus_{i,j=1}^{\infty} \mathrm{M}_{K_{ij}}(\mathbb{C})$ and $z_{i_0}^{2l}$ there corresponds to the identity in $\bigoplus_{j=1}^{\infty} \mathrm{M}_{K_{i_0 j}}(\mathbb{C})$ so $z_{i_0}^{2l}$ can be written as a sum of m_0 mutually orthogonal non-zero projections in $A'_{2l} \cap B_{2l}$ since m_0 is the 1-norm of the i_0-th row of K.

Let $\{p_j \mid j = 1,\ 2,\ \ldots,\ m\}$ be a maximal family of non-zero minimal projections contained in $A' \cap B$ whose sum is 1. This family is finite since $A' \cap B$ is finite-dimensional ($[B : A] < \infty$ and [Jon83]). Set $x = \sum_{j=1}^{m} \frac{j}{m} p_j$. Then $x = x^*$ and $\|x\| = 1$. By Kaplansky's density theorem there exists $(x_n)_{n=1}^{\infty} \subset (\cup_{n=1}^{\infty} B_n)_{\text{s.a.}}$ such that $\|x_n\| \leq 1$ and $x_n \xrightarrow[n\to\infty]{\text{s.o.}} x$, i.e. $\mathrm{d}_2(\cup_{n=1}^{\infty} B_n, x) = 0$.

Since $B_0 \subset B_1 \subset \ldots$ we can pick $x_n \in B_n$ with $x_{2l} \xrightarrow[l\to\infty]{\text{s.o.}} x$.

Set $y_{2l} = \mathrm{E}_{A'_{2l} \cap B_{2l}}(x_{2l}) = \mathrm{E}_{A'_{2l} \cap B}(x_{2l})$. Then

$$\|y_{2l} - x\|_2 \leq \left\|y_{2l} - \mathrm{E}_{A'_{2l} \cap B}(x)\right\|_2 + \left\|\mathrm{E}_{A'_{2l} \cap B}(x) - x\right\|_2.$$

Since $A'_{2l} \cap B \searrow A' \cap B$ we get

$$\|\mathrm{E}_{A'_{2l}\cap B}(x) - x\|_2 \xrightarrow[l\to\infty]{} \|\mathrm{E}_{A'\cap B}(x) - x\|_2 = 0$$

and

$$\left\|y_{2l} - \mathrm{E}_{A'_{2l}\cap B}(x)\right\|_2 = \left\|E_{A'_{2l}\cap B}(x_{2l} - x)\right\|_2 \leq \|x_{2l} - x\|_2 \xrightarrow[l\to\infty]{} 0$$

so $y_{2l} \xrightarrow[l\to\infty]{\text{s.o.}} x$.
Since $y_{2l} \in A'_{2l} \cap B_{2l}$ and $z_{i_0}^{2l} \in A_{2l}$, we have $[y_{2l}, z_{i_0}^{2l}] = 0$.
Now $y_{2l} z_{i_0}^{2l} \in z_{i_0}^{2l} B_{2l} z_{i_0}^{2l} \cong \bigoplus_j \mathrm{M}_{K_{i_0 j}}(\mathbb{C})$ which contains only m_0 non-zero minimal projections so $y_{2l} z_{i_0}^{2l}$ has at most m_0 spectral projections.
Since $z_{i_0} \in A$ and $p_j \in A' \cap B$ we have $\mathrm{tr}_B(p_j z_{i_0}^{2l}) = \mathrm{tr}_B(p_j)\,\mathrm{tr}_B(z_{i_0}^{2l}) \neq 0$, so $p_j z_{i_0}^{2l}$ is a non-zero projection and then $x z_{i_0}^{2l} = \sum_{j=1}^{m} \frac{j}{m} p_j z_{i_0}^{2l}$ has exactly m spectral projections.
Assume $m > m_0$. Now we use Lemma 1.13 with $\alpha_j = \frac{j}{m}$ and $t_j = \mathrm{tr}_B(p_j)$ for $j = 1, 2, \ldots, m$, and the II_1 factor $z_{i_0}^{2l} B z_{i_0}^{2l}$. By the lemma, there is $\varepsilon > 0$ such that for all $y \in z_{i_0}^{2l} B z_{i_0}^{2l}$ self-adjoint with $\|y\| \leq \|x z_{i_0}^{2l}\|$ and such that y has less than m spectral values, we have

$$\|y - x z_{i_0}^{2l}\|^2_{2, z_{i_0}^{2l} B z_{i_0}^{2l}} \geq \varepsilon.$$

The trace on $z_{i_0}^{2l} B z_{i_0}^{2l}$ is given by

$$\mathrm{tr}_{z_{i_0}^{2l} B z_{i_0}^{2l}}(\cdot) = \frac{\mathrm{tr}_B(\cdot)}{\mathrm{tr}_B(z_{i_0}^{2l})}$$

so

$$\|y - x z_{i_0}^{2l}\|_2^2 \geq \varepsilon\, \mathrm{tr}_B(z_{i_0}^{2l}).$$

We have that $\|xz_{i_0}^{2l}\| = 1$ and $\|y_{2l}z_{i_0}^{2l}\| \leq \|x_{2l}\| \leq 1$ so for $y = y_{2l}z_{i_0}^{2l}$ we get

$$\|y_{2l}z_{i_0}^{2l} - xz_{i_0}^{2l}\|_2^2 \geq \varepsilon\, \mathrm{tr}_B(z_{i_0}^{2l}) \geq \varepsilon c_{i_0} > 0$$

where $c_{i_0} > 0$ is the constant from Lemma 1.12 such that $\mathrm{tr}_B(z_{i_0}^{2l}) \geq c_{i_0}$ for all $l \in \mathbb{N}$. But we also have

$$\|y_{2l}z_{i_0}^{2l} - xz_{i_0}^{2l}\|_2 \leq \|z_{i_0}^{2l}\|\|y_{2l} - x\|_2 \xrightarrow[l\to\infty]{} 0$$

which is a contradiction, and hence we must have $m \leq m_0$.

We know that $A' \cap B \cong \bigoplus_{i=1}^r \mathrm{M}_{k_i}(\mathbb{C})$ for some $r, k_1, k_2, \dots, k_r \in \mathbb{N}$ such that $\sum_{i=1}^r k_i = m$. Now

$$\dim(A' \cap B) = \sum_{i=1}^r k_i^2 \leq \left(\sum_{i=1}^r k_i\right)^2 = m^2 \leq m_0^2$$

so $\dim(A' \cap B) \leq (\min\{\text{1-norm of rows of } K\})^2$. Proceeding similarly for $n \in \mathbb{N}$ odd and H instead of K finishes the proof. □

Corollary 1.15. (Wenzl's irreducibility criterion) *Let*

$$\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$$

be a symmetric commuting square of multi-matrix algebras such that Γ_G, Γ_H, Γ_K, Γ_L are connected and locally finite. Let $A \subset B$ be the hyperfinite II_1 factors constructed from this commuting square as described after Remark 1.11. If H or K contains a row with exactly one non-zero entry that is equal to 1*, then $A \subset B$ is irreducible.*

1.4 Ocneanu compactness

Ocneanu compactness provides a method for computing the relative commutant of the subfactor constructed from a symmetric commuting square in the *finite-dimensional* case.

Theorem 1.16. (Ocneanu compactness [Ocn90]) *Let*

$$\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$$

be a symmetric commuting square of finite-dimensional multi-matrix algebras such that $\Gamma_G, \Gamma_H, \Gamma_K, \Gamma_L$ *are connected. Let* $A \subset B$ *be the subfactor of the hyperfinite* II_1 *factor constructed from this commuting square as described after Remark 1.11. Then* $A' \cap B = A_1' \cap B_0$.

A proof of the theorem can be found in [JS97] or [EK98]. It allows one to compute (in principle) the standard invariant of such subfactors ([JS97]).

CHAPTER 2

NEW HYPERFINITE SUBFACTORS FROM N-STARS WITH A_∞-TAIL

In [Bis94b], Bisch constructed the first example of an irreducible subfactor of the hyperfinite II_1 factor whose index is a rational, non-integer number and thus not an algebraic integer. It was accomplished by giving an explicit construction of a symmetric commuting square of infinite-dimensional multi-matrix algebras based on an inclusion graph given by a 4-star with A_∞-tail.

Definition 2.1. *For N a positive integer, we call N-**star with** A_∞**-tail** the infinite bipartite graph*

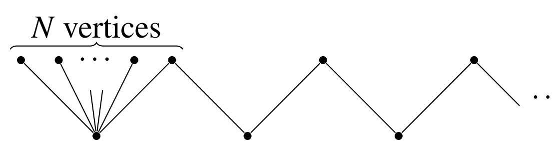

Figure 2: N-star with A_∞-tail

We will denote this graph by Γ_N.

Remark 2.2. *Graphs A_∞ and D_∞ (or more precisely, the $\mathrm{B}(\ell^2(\mathbb{N}))$ operators represented by their full adjacency matrices) do not have ℓ^2 Perron-Frobenius eigenvectors as they are infinite graphs that appear as principal graphs of amenable subfactors (*[Pop94]*).*

Remark 2.3. *Let Γ be a graph that is a concatenation of a finite graph and A_∞. Then the norm of Γ is greater than or equal to 2.*

Note that Γ_1 and Γ_2 are A_∞ and that Γ_3 is D_∞. Our goal is to find symmetric commuting squares based on inclusion graphs given by a N-stars with A_∞-tail so, due to Proposition 1.10 and Remark 2.2, in the following we consider a fixed $N \geq 4$.

2.1 Perron-Frobenius eigenvalue and eigenvector

We will enumerate vertices of Γ_N as follows:

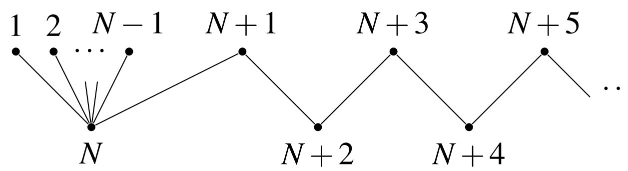

Figure 3: Labeled N-star with A_∞-tail

and we call the elements of its partition

$$\textbf{upper vertices} = \{1,\ 2,\ \ldots,\ N-1\} \cup \{N+2k+1 \mid k \in \mathbb{N}_0\}, \quad \text{and}$$
$$\textbf{lower vertices} = \{N+2k \mid k \in \mathbb{N}_0\}.$$

Let $\mathscr{G}$ be the full adjacency matrix of Γ_N. Its rows and columns are both enumerated by all vertices of Γ_N, i.e. by $\mathbb{N}$. $\mathscr{G}$ is symmetric since Γ_N is undirected and $\mathscr{G}$ is irreducible since Γ_N is connected. For $k \in \mathbb{N}$, $\mathscr{G}^k$ is again an infinite matrix with well-defined finite entries since Γ_N is locally finite. Then

$$\mathscr{G} \quad = \quad \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \\ N+1 \\ N+2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & N-1 & N & N+1 & N+2 & \cdots \end{array} \\ \left(\begin{array}{cccccccc} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 1 & 1 & \cdots & 1 & 0 & 1 & 0 & \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & & \vdots & & \ddots & \ddots & \ddots \end{array}\right) \end{array}.$$

We will call this the ω **form** of $\mathscr{G}$ (after the ordinal ω). This infinite matrix is a representation of a linear operator on $\ell^2(\mathbb{N})$ (by acting on sequences of complex numbers by matrix multiplication) which we will also call $\mathscr{G}$. Since the degrees of vertices of Γ_N are bounded by a constant, it follows that $\mathscr{G} \in \mathrm{B}(\ell^2(\mathbb{N}))$.

Let G be the bipartite adjacency matrix of Γ_N such that its rows are enumerated by upper vertices of Γ_N and its columns are enumerated by lower vertices of Γ_N. Then we can order rows and columns of $\mathscr{G}$ so that

$$\mathscr{G} \quad = \quad \begin{array}{c} \\ \text{upper} \\ \text{lower} \end{array} \begin{array}{c} \begin{array}{cc} \text{upper} & \text{lower} \end{array} \\ \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix} \end{array}.$$

We will call this the $\omega \cdot 2$ **form** of $\mathscr{G}$ (after the ordinal $\omega \cdot 2$). We have

$$G \quad = \quad \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N+1 \\ N+3 \\ N+5 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} N & N+2 & N+4 & N+6 & \cdots \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \ddots \\ 0 & 0 & 1 & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

and G is also in $\mathrm{B}(\ell^2(\mathbb{N}))$.

If $\mathscr{G}$ has an ℓ^2 eigenvector whose entries are all positive then due to Theorem 1.5 this eigenvector is a Perron-Frobenius eigenvector and the corresponding eigenvalue is its Perron-Frobenius eigenvalue. Since we want to use it to define a finite trace, we need this Perron-Frobenius eigenvector to also be ℓ^1.

$\mathscr{G}\vec{\xi} = \lambda\vec{\xi}$ is equivalent to:

$$\begin{aligned}
\xi_N &= \lambda\xi_1, \\
\xi_N &= \lambda\xi_2, \\
&\vdots \\
\xi_N &= \lambda\xi_{N-1}, \\
\xi_1 + \xi_2 + \ldots + \xi_{N-1} + \xi_{N+1} &= \lambda\xi_N, \\
\xi_{N+k-1} + \xi_{N+k+1} &= \lambda\xi_{N+k} \quad \text{for } k \in \mathbb{N}.
\end{aligned} \tag{2.1}$$

The last row of (2.1) is a recurrence relation with characteristic polynomial $x^2 - \lambda x + 1$ whose roots are $x_1 = \frac{\lambda+\sqrt{\lambda^2-4}}{2}$ and $x_2 = \frac{\lambda-\sqrt{\lambda^2-4}}{2}$. Therefore there exist $c_1, c_2 \in \mathbb{C}$ so that, for every $k \in \mathbb{N}$, $\xi_{N+k-1} = c_1 x_1^{k-1} + c_2 x_2^{k-1}$. Due to Remark 2.3, we are only interested in solutions such that $\lambda \geq 2$ and that implies $x_1 \geq 1 \geq x_2 > 0$. Due to Theorem 1.5, we are only interested in solutions such that $\|\vec{\xi}\|_2 < \infty$ so we must have $c_1 = 0$. Since we are looking for an eigenvector $\vec{\xi}$, we can set $c_2 = \lambda$ (or any other arbitrary non-zero scalar). Then we have $\xi_N = \lambda$ and $\xi_{N+1} = \lambda x_2$. Now it follows from (2.1) that $\xi_1 = \xi_2 = \ldots = \xi_{N-1} = 1$ and $N - 1 + \lambda x_2 = \lambda^2$. Solving the latter for $\lambda \geq 2$ (with the condition $N \geq 4$) gives a solution to (2.1):

$$\begin{aligned}
\lambda &= \frac{N-1}{\sqrt{N-2}}, \\
\xi_1 = \xi_2 = \ldots &= \xi_{N-1} = 1, \\
\xi_{N+k-1} &= \frac{N-1}{\left(\sqrt{N-2}\right)^k} \quad \text{for } k \in \mathbb{N}.
\end{aligned} \tag{2.2}$$

Since $0 < \frac{1}{\sqrt{N-2}} < 1$ for $N \geq 4$, it follows that $\|\vec{\xi}\|_1 = N - 1 + \frac{N-1}{\sqrt{N-2}-1} < \infty$, so $\vec{\xi}$ is ℓ^1 and ℓ^2. Since all entries of $\vec{\xi}$ are positive, it follows that λ is the Perron-Frobenius eigenvalue and $\vec{\xi}$ is a Perron-Frobenius eigenvector of $\mathscr{G}$ due to Theorem 1.5. This implies:

$$\|\Gamma_N\|^2 = \|\mathscr{G}\|^2 = \lambda^2 = \frac{(N-1)^2}{N-2}.$$

2.2 Bi-unitary condition

We want to find a symmetric commuting square where G is the inclusion matrix on the index defining side. A priori, the remaining inclusion matrices could be anything provided that certain Perron-Frobenius eigenvalues and eigenvectors agree. Polynomials in G and G^t are good candidates so we will attempt to find a symmetric commuting square of the form:

$$\begin{array}{ccc} C & \subset_{nG} & D \\ \cup_{G^t} & & \cup_G \\ A & \subset_{nG^t} & B \end{array} \tag{2.3}$$

Clearly $(nG^t)^t G^t = G(nG)^t$.

In $\omega \cdot 2$ form $\mathscr{G}\vec{\xi} = \lambda\vec{\xi}$ means

$$\begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}\begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} = \lambda \begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} \quad \text{where} \quad \begin{matrix} \text{upper} \\ \text{lower} \end{matrix}\begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} \quad \text{is the } \omega \cdot 2 \text{ form of } \vec{\xi}.$$

Note that $\vec{s}$ and $\vec{t}$ are both ℓ^1 and ℓ^2, and all of their entries are positive since this is true for $\vec{\xi}$.

$$\begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}\begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} = \begin{pmatrix} G\vec{t} \\ G^t\vec{s} \end{pmatrix} \quad \text{so} \quad \begin{matrix} G\vec{t} = \lambda\vec{s}, \\ G^t\vec{s} = \lambda\vec{t}. \end{matrix}$$

In $\omega \cdot 2$ form $\lambda^2\vec{\xi} = \mathscr{G}^2\vec{\xi}$ means

$$\lambda^2\begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}^2\begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} = \begin{pmatrix} GG^t & 0 \\ 0 & G^tG \end{pmatrix}\begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix} = \begin{pmatrix} GG^t\vec{s} \\ G^tG\vec{t} \end{pmatrix} \quad \text{so} \quad \begin{matrix} GG^t\vec{s} = \lambda^2\vec{s}, \\ G^tG\vec{t} = \lambda^2\vec{t}. \end{matrix}$$

GG^t and G^tG are symmetric, irreducible and have well-defined powers with finite entries. Thus $\vec{s}$ is a Perron-Frobenius eigenvector of GG^t, $\vec{t}$ is a Perron-Frobenius eigenvector of G^tG and $\lambda^2 = \|G\|^2$ is their Perron-Frobenius eigenvalue. If there is a symmetric commuting square of this form then due to (1.3) the index of the resulting subfactor of the hyperfinite II_1 factor would be

$$\|G\|^2 = \lambda^2 = \frac{(N-1)^2}{N-2} = N + \frac{1}{N-2}. \tag{2.4}$$

Since $N \geq 4$, all these potential indices are rational, non-integer numbers and thus not algebraic integers. Hence these indices cannot be obtained by using finite graphs. The values of the first few of these numbers are $4.5,\ 5.333\ldots,\ 6.25,\ 7.2,\ 8.166\ldots,\ 9.142\ldots,\ 10.125,\ \ldots$

Let $\vec{\alpha},\ \vec{\beta},\ \vec{\gamma},\ \vec{\delta}$ be the trace vectors of $A,\ B,\ C,\ D$ in (2.3) and let A be commutative, i.e. the algebra $\bigoplus_{i=1}^{\infty}\mathbb{C} = \ell^{\infty}(\mathbb{N})$. If we are to have a symmetric commuting square of this form, then due to Proposition 1.10 we need to have $\vec{\delta} = \vec{t}$ (up to a scalar) since it is the Markov trace (after normalization) for the inclusion $B \subset_G D$. Also, note that $\vec{t}$ is ℓ^1 so it defines a finite trace on D since the entries of the dimension vector of D are bounded by a constant. Then

$$\begin{aligned}
\vec{\gamma} &= nG\vec{\delta} = nG\vec{t} = n\lambda\vec{s}, \\
\vec{\beta} &= G\vec{\delta} = G\vec{t} = \lambda\vec{s}, \\
\vec{\alpha} &= nG^t\vec{\beta} = n\lambda G^t\vec{s} = n\lambda^2\vec{t},
\end{aligned}$$

so the traces are given as follows:

$$\begin{array}{ccccccc}
\vec{\gamma} = n\lambda\vec{s} & & & & & & \vec{\delta} = \vec{t} \\
& \nwarrow & & & & \nearrow & \\
& & C & \subset_{nG} & D & & \\
& & \cup_{G^t} & & \cup_{G} & & \\
& & A & \subset_{nG^t} & B & & \\
& \swarrow & & & & \searrow & \\
\vec{\alpha} = n\lambda^2\vec{t} & & & & & & \vec{\beta} = \lambda\vec{s}
\end{array}$$

These traces are not normalized, but we can apply results from Chapter 1 to their normalizations when necessary. Now

$$\sqrt{\frac{\alpha_i\delta_k}{\beta_j\gamma_l}} = \sqrt{\frac{n\lambda^2 t_i \cdot t_k}{\lambda s_j \cdot n\lambda s_l}} = \sqrt{\frac{t_i t_k}{s_j s_l}} \quad \text{for } i,j,k,l \in \mathbb{N}.$$

Let S and T be the sets of paths in $A \subset_{nG^t} B \subset_G D$ and $A \subset_{G^t} C \subset_{nG} D$, i.e.

$$S = \left\{(i,\phi,j,k) \mid (G^t)_{ij}G_{jk} \neq 0,\ 1 \leq \phi \leq n(G^t)_{ij}\right\} \text{ and}$$
$$T = \left\{(i,l,\psi,k) \mid (G^t)_{il}G_{lk} \neq 0,\ 1 \leq \psi \leq nG_{lk}\right\}.$$

Then by Theorem 1.3 the existence of a symmetric commuting square of this form is equivalent to the existence of a pair of unitaries u and v such that:

$$u = \bigoplus_{(i,k)} u^{(i,k)}, \quad u^{(i,k)} = \left(u^{(i,k)}_{(j,\phi,l,\psi)}\right)_{\substack{(i,\phi,j,k)\in S \\ (i,l,\psi,k)\in T}} \quad \text{for } (i,k) \text{ such that } (G^tG)_{ik} \neq 0,$$

$$v = \bigoplus_{(j,l)} v^{(j,l)}, \quad v^{(j,l)} = \left(v^{(j,l)}_{(i,\phi,k,\psi)}\right)_{\substack{(i,\phi,j,k)\in S \\ (i,l,\psi,k)\in T}} \quad \text{for } (j,l) \text{ such that } (GG^t)_{jl} \neq 0, \text{ and}$$

$$v^{(j,l)}_{(i,\phi,k,\psi)} = \sqrt{\frac{t_i t_k}{s_j s_l}} u^{(i,k)}_{(j,\phi,l,\psi)} \quad \text{for all } (i,\phi,j,k) \in S \text{ and } (i,l,\psi,k) \in T.$$

We want to find such unitaries u and v for (2.3).

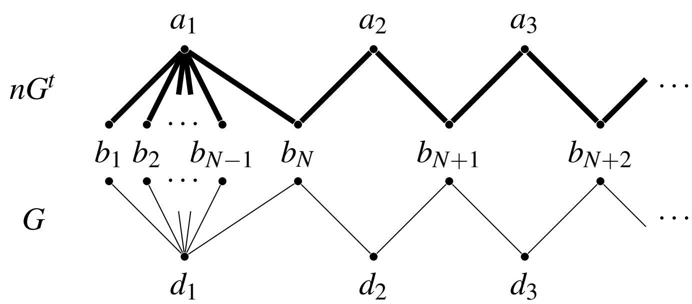

Figure 4: Paths in S

We will use labels of the form (a_i, ϕ, b_j, d_k) for paths in S

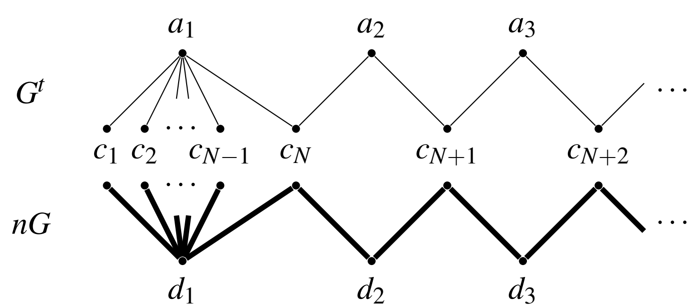

Figure 5: Paths in T

and labels of the form (a_i, c_l, ψ, d_k) for paths in T. Thick lines represent n edges.

Recall that in $\omega \cdot 2$ form we have

$$\vec{\xi} = \begin{matrix} \text{upper} \\ \text{lower} \end{matrix} \begin{pmatrix} \vec{s} \\ \vec{t} \end{pmatrix}$$

and upper $= \{1,\ 2,\ \ldots,\ N-1\} \cup \{N+2k+1 \mid k \in \mathbb{N}_0\}$ and lower $= \{N+2k \mid k \in \mathbb{N}_0\}$.

Using (2.2) it follows that

$$
\begin{aligned}
s_k = \xi_k &= 1 \text{ for } k = 1,\ 2,\ \dots,\ N-1, \\
s_{N+k} = \xi_{N+2k+1} &= \frac{N-1}{(N-2)^{k+1}} \quad \text{for} \quad k \in \mathbb{N}_0, \\
t_k = \xi_{N+2k-2} &= \frac{(N-1)\sqrt{N-2}}{(N-2)^k} \quad \text{for} \quad k \in \mathbb{N}.
\end{aligned} \tag{2.5}
$$

We see that u has block structure of the form

$$
u \quad = \quad
\begin{array}{c|c|c|c|c|c}
 & d_1 & d_2 & d_3 & d_4 & \cdots \\
\hline
a_1 & \times & \times & & & \cdots \\
\hline
a_2 & \times & \times & \times & & \cdots \\
\hline
a_3 & & \times & \times & \times & \\
\hline
a_4 & & & \times & \times & \ddots \\
\hline
\vdots & \vdots & \vdots & & \ddots & \ddots
\end{array}
$$

where $\times$ at (a_i, d_k) denotes a non-empty block $u^{(a_i,d_k)}$, i.e. there exists at least one S-path and at least one T-path from a_i to d_k.

Note that if we pick $a_i,\ b_j,\ c_l,\ d_k$ such that there exist $\phi,\ \psi$ for which $(a_i, \phi, b_j, d_k) \in S$ and $(a_i, c_l, \psi, d_k) \in T$ then actually $(a_i, \phi', b_j, d_k) \in S$ and $(a_i, c_l, \psi', d_k) \in T$ for every $\phi',\ \psi' \in \{1,\ 2,\ \dots,\ n\}$. So every $u^{(a_i,d_k)}$ (and every $v^{(b_j,c_l)}$) is built from some number of **elementary** $n \times n$ **blocks** (i.e. elements of $\mathrm{M}_n(\mathbb{C})$). In the case of $u^{(a_i,d_k)}$ we label these blocks by (b_j, c_l) and use the notation $u^{(a_i,d_k)}_{(b_j,c_l)} = \left(u^{(a_i,d_k)}_{(b_j,\phi,c_l,\psi)}\right)_{1\le\phi,\psi\le n}$. In the case of $v^{(b_j,c_l)}$ we label the blocks by (a_i, d_k) and write $v^{(b_j,c_l)}_{(a_i,d_k)} = \left(v^{(b_j,c_l)}_{(a_i,\phi,d_k,\psi)}\right)_{1\le\phi,\psi\le n}$. (For $a_i,\ b_j,\ c_l,\ d_k$ as above).

From Figure 4 and Figure 5 we deduce:

- For (a_1, d_1) all S-paths go through one of $b_1,\ b_2,\ \dots,\ b_N$ and all T-paths go through one of $c_1,\ c_2,\ \dots,\ c_N$ so $u^{(a_1,d_1)}$ consists of $N \cdot N = N^2$ elementary $n \times n$ blocks.

- For (a_i, d_i) s.t. $i > 1$ all S-paths go through b_{N+i-2} or b_{N+i-1} and all T-paths go through c_{N+i-2} or c_{N+i-1} so $u^{(a_i,d_i)}$ consists of $2 \cdot 2 = 4$ elementary $n \times n$ blocks.
- For (a_i, d_{i+1}) all S-paths go through b_{N+i-1} and all T-paths go through c_{N+i-1} so $u^{(a_i,d_{i+1})}$ consists of a single elementary $n \times n$ block.
- For (a_{i+1}, d_i) all S-paths go through b_{N+i-2} and all T-paths go through c_{N+i-2} so $u^{(a_{i+1},d_i)}$ consists of a single elementary $n \times n$ block.

Now we know that u is of the form

$$u \quad = \quad \begin{array}{c|c|c|c|c|c}
 & d_1 & d_2 & d_3 & d_4 & \cdots \\ \hline
a_1 & \begin{array}{c|ccccc} & c_1 & c_2 & \cdots & c_{N-1} & c_N \\ \hline b_1 & \square & \square & \cdots & \square & \square \\ b_2 & \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{N-1} & \square & \square & \cdots & \square & \square \\ b_N & \square & \square & \cdots & \square & \square \end{array} & \begin{array}{c|c} & c_N \\ \hline b_N & \square \end{array} & & & \cdots \\ \hline
a_2 & \begin{array}{c|c} & c_N \\ \hline b_N & \square \end{array} & \begin{array}{c|cc} & c_N & c_{N+1} \\ \hline b_N & \square & \square \\ b_{N+1} & \square & \square \end{array} & \begin{array}{c|c} & c_{N+1} \\ \hline b_{N+1} & \square \end{array} & & \cdots \\ \hline
a_3 & & \begin{array}{c|c} & c_{N+1} \\ \hline b_{N+1} & \square \end{array} & \begin{array}{c|cc} & c_{N+1} & c_{N+2} \\ \hline b_{N+1} & \square & \square \\ b_{N+2} & \square & \square \end{array} & \begin{array}{c|c} & c_{N+2} \\ \hline b_{N+2} & \square \end{array} & \\ \hline
a_4 & & & \begin{array}{c|c} & c_{N+2} \\ \hline b_{N+2} & \square \end{array} & \begin{array}{c|cc} & c_{N+2} & c_{N+3} \\ \hline b_{N+2} & \square & \square \\ b_{N+3} & \square & \square \end{array} & \ddots \\ \hline
\vdots & \vdots & \vdots & & \ddots & \ddots
\end{array} \qquad (2.6)$$

where at each (a_i, d_k) are blocks which constitute $u^{(a_i,d_k)}$ and each $\square$ represents some elementary $n \times n$ block $u^{(a_i,d_k)}_{(b_j,c_l)}$.

Since the elementary $n \times n$ block $u^{(a_i,d_k)}_{(b_j,c_l)}$ exists if and only if there exists an elementary $n \times n$

block $v^{(b_j,c_l)}_{(a_i,d_k)}$, we can read the block structure of v from the block structure of u:

$$
v \quad = \quad
\begin{array}{c|c|c|c|c|c|c|c|c}
 & c_1 & c_2 & \cdots & c_{N-1} & c_N & c_{N+1} & c_{N+2} & \cdots \\
\hline
b_1 & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \cdots & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & & & \cdots \\
\hline
b_2 & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \cdots & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & & & \cdots \\
\hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & \cdots \\
\hline
b_{N-1} & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \cdots & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & & & \cdots \\
\hline
b_N & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \cdots & \begin{smallmatrix} & d_1 \\ a_1 & \square \end{smallmatrix} & \begin{smallmatrix} & d_1 & d_2 \\ a_1 & \square & \square \\ a_2 & \square & \square \end{smallmatrix} & \begin{smallmatrix} & d_2 \\ a_2 & \square \end{smallmatrix} & & \cdots \\
\hline
b_{N+1} & & & & & \begin{smallmatrix} & d_2 \\ a_2 & \square \end{smallmatrix} & \begin{smallmatrix} & d_2 & d_3 \\ a_2 & \square & \square \\ a_3 & \square & \square \end{smallmatrix} & \begin{smallmatrix} & d_3 \\ a_3 & \square \end{smallmatrix} & \\
\hline
b_{N+2} & & & & & & \begin{smallmatrix} & d_3 \\ a_3 & \square \end{smallmatrix} & \begin{smallmatrix} & d_3 & d_4 \\ a_3 & \square & \square \\ a_4 & \square & \square \end{smallmatrix} & \ddots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots
\end{array}
\tag{2.7}
$$

Here at each (b_j,c_l) are blocks which constitute $v^{(b_j,c_l)}$ and each $\square$ represents some elementary $n\times n$ block $v^{(b_j,c_l)}_{(a_i,d_k)}$.

Note that

$$
v^{(b_j,c_l)}_{(a_i,\phi,d_k,\psi)} = \sqrt{\frac{t_i t_k}{s_j s_l}}\, u^{(a_i,d_k)}_{(b_j,\phi,c_l,\psi)} \quad \text{for all } (a_i,\phi,b_j,d_k)\in S \text{ and } (a_i,c_l,\psi,d_k)\in T
$$

is equivalent to

$$
v^{(b_j,c_l)}_{(a_i,d_k)} = \sqrt{\frac{t_i t_k}{s_j s_l}}\, u^{(a_i,d_k)}_{(b_j,c_l)} \quad \text{for all } (a_i,1,b_j,d_k)\in S \text{ and } (a_i,c_l,1,d_k)\in T.
$$

For each elementary $n\times n$ block in v (or u) we now use (2.5) to compute the corresponding value $\sqrt{\frac{t_i t_k}{s_j s_l}}$:

- If $1\le j,\ l\le N-1$ then $i=k=1$ and $\sqrt{\frac{t_i t_k}{s_j s_l}} = t_1 = \frac{N-1}{\sqrt{N-2}}$.

- If $j = N$ and $1 \le l \le N-1$ then $i = k = 1$ and $\sqrt{\frac{t_i t_k}{s_j s_l}} = \frac{t_1}{\sqrt{s_N}} = \sqrt{N-1}$.
- If $l = N$ and $1 \le j \le N-1$ then $i = k = 1$ and $\sqrt{\frac{t_i t_k}{s_j s_l}} = \frac{t_1}{\sqrt{s_N}} = \sqrt{N-1}$.
- If $j,\ l \ge N$ and $l = j+1$ then $i = k = j-N+2$ and $\sqrt{\frac{t_i t_k}{s_j s_l}} = \frac{t_{j-N+2}}{\sqrt{s_j s_{j+1}}} = 1$.
- If $j,\ l \ge N$ and $j = l+1$ then $i = k = l-N+2$ and $\sqrt{\frac{t_i t_k}{s_j s_l}} = \frac{t_{l-N+2}}{\sqrt{s_{l+1} s_l}} = 1$.
- If $j = l \ge N$ then $i,\ k \in \{j-N+1,\ j-N+2\}$ so:
  - If $i = k = j-N+1$ then $\sqrt{\frac{t_i t_k}{s_j s_l}} = \frac{t_{j-N+1}}{s_j} = \sqrt{N-2}$.
  - If $i = k = j-N+2$ then $\sqrt{\frac{t_i t_k}{s_j s_l}} = \frac{t_{j-N+2}}{s_j} = \frac{1}{\sqrt{N-2}}$.
  - If $i \neq j$ then $\sqrt{\frac{t_i t_k}{s_j s_l}} = \frac{\sqrt{t_{j-N+1} t_{j-N+2}}}{s_j} = 1$.

If we put each $\left(\sqrt{\frac{t_i t_k}{s_j s_l}}\right)^{-1}$ (i.e. the multiplicative inverse of the computed value) in the corresponding spot inside the shape of the block structure of v (or u) we get a matrix that tells us with what we need to multiply each elementary $n \times n$ block of v to get the corresponding elementary $n \times n$ block of u.

This matrix, in the shape of the block structure of u, is

$$
\begin{array}{c|ccccc|cc|cc|c}
 & & & d_1 & & & d_2 & & d_3 & & \cdots \\
\hline
 & \frac{\sqrt{N-2}}{N-1} & \frac{\sqrt{N-2}}{N-1} & \cdots & \frac{\sqrt{N-2}}{N-1} & \frac{1}{\sqrt{N-1}} & & & & & \\
 & \frac{\sqrt{N-2}}{N-1} & \frac{\sqrt{N-2}}{N-1} & \cdots & \frac{\sqrt{N-2}}{N-1} & \frac{1}{\sqrt{N-1}} & & & & & \\
a_1 & \vdots & \vdots & \ddots & \vdots & \vdots & & & & & \cdots \\
 & \frac{\sqrt{N-2}}{N-1} & \frac{\sqrt{N-2}}{N-1} & \cdots & \frac{\sqrt{N-2}}{N-1} & \frac{1}{\sqrt{N-1}} & & & & & \\
 & \frac{1}{\sqrt{N-1}} & \frac{1}{\sqrt{N-1}} & \cdots & \frac{1}{\sqrt{N-1}} & \frac{1}{\sqrt{N-2}} & 1 & & & & \\
\hline
a_2 & & & & & 1 & \sqrt{N-2} & 1 & & & \\
 & & & & & & 1 & \frac{1}{\sqrt{N-2}} & 1 & & \\
\hline
a_3 & & & & & & & 1 & \sqrt{N-2} & 1 & \ddots \\
 & & & & & & & & 1 & \frac{1}{\sqrt{N-2}} & \\
\hline
\vdots & & & \vdots & & & & & & \ddots & \ddots
\end{array}
\tag{2.8}
$$

We see in (2.6) that each $u^{(a_i,d_{i+1})}$ has only one elementary $n \times n$ block $u^{(a_i,d_{i+1})}_{(b_{N+i-1},c_{N+i-1})}$

which thus must be unitary. The same is true for $u^{(a_{i+1},d_i)}$ and $u^{(a_{i+1},d_i)}_{(b_{N+i-1},c_{N+i-1})}$. Then (2.7) shows that, for $i \geq 1$, each $v^{(b_{N+i-1},c_{N+i-1})}$ contains $v^{(b_{N+i-1},c_{N+i-1})}_{(a_i,d_{i+1})} = 1 \cdot u^{(a_i,d_{i+1})}_{(b_{N+i-1},c_{N+i-1})}$ and $v^{(b_{N+i-1},c_{N+i-1})}_{(a_{i+1},d_i)} = 1 \cdot u^{(a_{i+1},d_i)}_{(b_{N+i-1},c_{N+i-1})}$ which are both unitary.

Lemma 2.4. *Let*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

be a $2n \times 2n$ *block matrix where* $M_1,\ M_2,\ M_3,\ M_4$ *are* $n \times n$ *matrices such that* M_2 *and* M_3 *are unitary. Then the following are equivalent:*

(i) M *is unitary.*

(ii) $M_1 = M_4 = 0$.

Proof. Trivial. □

Applying Lemma 2.4 to $v^{(b_{N+i-1},c_{N+i-1})}$ we get that

$$\begin{aligned} v^{(b_{N+i-1},c_{N+i-1})}_{(a_i,d_i)} = v^{(b_{N+i-1},c_{N+i-1})}_{(a_{i+1},d_{i+1})} &= 0 \quad \text{for all } i \in \mathbb{N}, \text{ which implies} \\ u^{(a_i,d_i)}_{(b_{N+i-1},c_{N+i-1})} = u^{(a_{i+1},d_{i+1})}_{(b_{N+i-1},c_{N+i-1})} &= 0 \quad \text{for all } i \in \mathbb{N}. \end{aligned}$$

Hence we deduce that u has to be of the form

$$u = \begin{array}{c|ccccc|cc|cc|c}
 & & & d_1 & & & d_2 & & d_3 & & \cdots \\ \hline
 & \square & \square & \cdots & \square & \square & & & & & \\
 & \square & \square & \cdots & \square & \square & & & & & \\
a_1 & \vdots & \vdots & \vdots & \ddots & \vdots & & & & & \cdots \\
 & \square & \square & \cdots & \square & \square & & & & & \\
 & \square & \square & \cdots & \square & 0 & * & & & & \\ \hline
a_2 & & & & & * & 0 & * & & & \\
 & & & & & & * & 0 & * & & \\ \hline
a_3 & & & & & & & * & 0 & * & \ddots \\
 & & & & & & & & * & 0 & \\ \hline
\vdots & & & \vdots & & & & & & \ddots & \ddots
\end{array} \tag{2.9}$$

and v has to be of the form

$$v = \begin{array}{c|c|c|c|c|cc|cc|c}
 & c_1 & c_2 & \cdots & c_{N-1} & c_N & & c_{N+1} & & \cdots \\ \hline
b_1 & \square & \square & \cdots & \square & \square & & & & \cdots \\ \hline
b_2 & \square & \square & \cdots & \square & \square & & & & \cdots \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & & \cdots \\ \hline
b_{N-1} & \square & \square & \cdots & \square & \square & & & & \cdots \\ \hline
b_N & \square & \square & \cdots & \square & 0 & * & & & \\
 & & & & & * & 0 & * & & \\ \hline
b_{N+1} & & & & & & * & 0 & * & \ddots \\
 & & & & & & & * & 0 & \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & & \ddots
\end{array} \tag{2.10}$$

where each $\square$ and $*$ is an elementary $n \times n$ block.

Note that for every remaining non-zero elementary $n\times n$ block in all $v^{(b_j,c_l)}$ of v for $j,\ l\geq N$ (denoted by $*$) we can pick an arbitrary $n\times n$ unitary matrix. This is because each such $*$ in v is equal to some non-zero elementary $n\times n$ block in some $u^{(a_i,d_k)}$ of u for $i+k>2$ (again denoted by $*$) because the corresponding $\sqrt{\frac{t_it_k}{s_js_l}}=1$. Thus each such choice is consistent and all resulting $v^{(b_j,c_l)}$ for $j,\ l\geq N$ and $u^{(a_i,d_k)}$ for $i+k>2$ are unitary due to Lemma 2.4.

Observe that if u and v are bi-unitary matrices of this form and we fix their $\square$ entries, then for every choice of $n\times n$ unitaries for $*$ entries of v the resulting matrices u and v are bi-unitarily equivalent (see [JS97]), i.e. all such commuting squares are $*$-isomorphic and thus give rise to isomorphic subfactors.

All this proves the following proposition:

Proposition 2.5. *A symmetric commuting square of the form* (2.3) *exists if and only if there exist N^2-1 $n\times n$ unitary matrices $v^{(b_j,c_l)}=v^{(b_j,c_l)}_{(a_1,d_1)}$ such that the $(N\cdot n)\times(N\cdot n)$ matrix*

$$u^{(a_1,d_1)} = \begin{pmatrix} u^{(a_1,d_1)}_{(b_1,c_1)} & u^{(a_1,d_1)}_{(b_1,c_2)} & \cdots & u^{(a_1,d_1)}_{(b_1,c_{N-1})} & u^{(a_1,d_1)}_{(b_1,c_N)} \\ u^{(a_1,d_1)}_{(b_2,c_1)} & u^{(a_1,d_1)}_{(b_2,c_2)} & \cdots & u^{(a_1,d_1)}_{(b_2,c_{N-1})} & u^{(a_1,d_1)}_{(b_2,c_N)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{(a_1,d_1)}_{(b_{N-1},c_1)} & u^{(a_1,d_1)}_{(b_{N-1},c_2)} & \cdots & u^{(a_1,d_1)}_{(b_{N-1},c_{N-1})} & u^{(a_1,d_1)}_{(b_{N-1},c_N)} \\ u^{(a_1,d_1)}_{(b_N,c_1)} & u^{(a_1,d_1)}_{(b_N,c_2)} & \cdots & u^{(a_1,d_1)}_{(b_N,c_{N-1})} & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{N-2}}{N-1}v^{(b_1,c_1)} & \frac{\sqrt{N-2}}{N-1}v^{(b_1,c_2)} & \cdots & \frac{\sqrt{N-2}}{N-1}v^{(b_1,c_{N-1})} & \frac{1}{\sqrt{N-1}}v^{(b_1,c_N)} \\ \frac{\sqrt{N-2}}{N-1}v^{(b_2,c_1)} & \frac{\sqrt{N-2}}{N-1}v^{(b_2,c_2)} & \cdots & \frac{\sqrt{N-2}}{N-1}v^{(b_2,c_{N-1})} & \frac{1}{\sqrt{N-1}}v^{(b_2,c_N)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sqrt{N-2}}{N-1}v^{(b_{N-1},c_1)} & \frac{\sqrt{N-2}}{N-1}v^{(b_{N-1},c_2)} & \cdots & \frac{\sqrt{N-2}}{N-1}v^{(b_{N-1},c_{N-1})} & \frac{1}{\sqrt{N-1}}v^{(b_{N-1},c_N)} \\ \frac{1}{\sqrt{N-1}}v^{(b_N,c_1)} & \frac{1}{\sqrt{N-1}}v^{(b_N,c_2)} & \cdots & \frac{1}{\sqrt{N-1}}v^{(b_N,c_{N-1})} & 0 \end{pmatrix}$$

is also unitary.

2.3 Case $n = 1$, general N

Our goal is to find symmetric commuting squares based on N-stars with A_∞-tail of the form (2.3) for $n = 1$. According to Proposition 2.5 it is enough to find $N^2 - 1$ 1×1 unitary matrices (i.e. complex numbers of absolute value 1) $V_{j,k}$, for $1 \leq j,\ k \leq N$ and $j \neq N$ or $k \neq N$ such that

$$U = \begin{pmatrix} \frac{\sqrt{N-2}}{N-1}V_{1,1} & \frac{\sqrt{N-2}}{N-1}V_{1,2} & \cdots & \frac{\sqrt{N-2}}{N-1}V_{1,N-1} & \frac{1}{\sqrt{N-1}}V_{1,N} \\ \frac{\sqrt{N-2}}{N-1}V_{2,1} & \frac{\sqrt{N-2}}{N-1}V_{2,2} & \cdots & \frac{\sqrt{N-2}}{N-1}V_{2,N-1} & \frac{1}{\sqrt{N-1}}V_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sqrt{N-2}}{N-1}V_{N-1,1} & \frac{\sqrt{N-2}}{N-1}V_{N-1,2} & \cdots & \frac{\sqrt{N-2}}{N-1}V_{N-1,N-1} & \frac{1}{\sqrt{N-1}}V_{N-1,N} \\ \frac{1}{\sqrt{N-1}}V_{N,1} & \frac{1}{\sqrt{N-1}}V_{N,2} & \cdots & \frac{1}{\sqrt{N-1}}V_{N,N-1} & 0 \end{pmatrix}$$

is unitary.

Remark 2.6. *If U is such that its corresponding u and v satisfy the conditions of Proposition 2.5, then pre-multiplying or post-multiplying U by diagonal unitary matrices results with U' whose corresponding u' and v' again satisfy Proposition 2.5 and are bi-unitarily equivalent (see* [JS97]*) to u and v, i.e. their commuting squares are $*$-isomorphic and give rise to isomorphic subfactors. Additionaly, if we permute the first $N-1$ rows and/or columns of such U in any way, then the u' and v' of the resulting matrix U' will again satisfy Proposition 2.5 and, while strictly speaking they will not in general be bi-unitarily equivalent to u and v, due to* (2.8)*,* (2.9) *and* (2.10) *their commuting squares will be $*$-isomorphic and give rise to isomorphic subfactors. We will say that matrices U and U' obtained from one another by some finite sequence of these two types of transformations are* **equivalent***.*

Due to Remark 2.6 we are only interested in U up to equivalence, so we can take $V_{1,1} = V_{j,N} = V_{N,k} = 1$ for all $1 \leq j,\ k \leq N-1$ by multiplying rows and columns of U by appropriate scalars and then we have

$$U = \begin{pmatrix} \frac{\sqrt{N-2}}{N-1}V_{1,1} & \frac{\sqrt{N-2}}{N-1}V_{1,2} & \cdots & \frac{\sqrt{N-2}}{N-1}V_{1,N-1} & \frac{1}{\sqrt{N-1}} \\ \frac{\sqrt{N-2}}{N-1}V_{2,1} & \frac{\sqrt{N-2}}{N-1}V_{2,2} & \cdots & \frac{\sqrt{N-2}}{N-1}V_{2,N-1} & \frac{1}{\sqrt{N-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sqrt{N-2}}{N-1}V_{N-1,1} & \frac{\sqrt{N-2}}{N-1}V_{N-1,2} & \cdots & \frac{\sqrt{N-2}}{N-1}V_{N-1,N-1} & \frac{1}{\sqrt{N-1}} \\ \frac{1}{\sqrt{N-1}} & \frac{1}{\sqrt{N-1}} & \cdots & \frac{1}{\sqrt{N-1}} & 0 \end{pmatrix} \tag{2.11}$$

where also $V_{1,1} = 1$.

Remark 2.7. *Unless otherwise stated, U will be assumed to be of the form as in* (2.11) *(along with $V_{1,1} = 1$).*

U is unitary if and only if its rows form an orthonormal basis of $\mathbb{C}^N$ with respect to the standard inner product. Thus to find U, we are looking to solve the following system of equations:

$$\begin{aligned} \sum_{k=1}^{N-1} V_{j,k} = 0, &\quad \text{for } 1 \le j \le N-1, \\ \sum_{l=1}^{N-1} V_{j,l}\overline{V_{k,l}} = -\frac{N-1}{N-2}, &\quad \text{for } 1 \le j < k \le N-1, \\ |V_{j,k}| = 1 &\quad \text{for } 1 \le j,\ k \le N-1; \end{aligned} \tag{2.12}$$

or

$$\begin{aligned} \sum_{k=1}^{N-1} \cos\Phi_{j,k} = 0, &\quad \text{for } 1 \le j \le N-1, \\ \sum_{k=1}^{N-1} \sin\Phi_{j,k} = 0, &\quad \text{for } 1 \le j \le N-1, \\ \sum_{l=1}^{N-1} \cos(\Phi_{j,l} - \Phi_{k,l}) = -\frac{N-1}{N-2}, &\quad \text{for } 1 \le j < k \le N-1, \\ \sum_{l=1}^{N-1} \sin(\Phi_{j,l} - \Phi_{k,l}) = 0, &\quad \text{for } 1 \le j < k \le N-1; \end{aligned} \tag{2.13}$$

when those equations are separated into their real and imaginary parts, where $\Phi_{j,k} \in [0, 2\pi)$ is defined by $V_{j,k} = e^{i\cdot\Phi_{j,k}}$ (i.e. $\Phi_{j,k} = \arg(V_{j,k}) = \arg(U_{j,k})$) for all $1 \le j,\ k \le N-1$.

The first and the third row of equations in (2.12) imply that for each $j \in \{1,\ 2,\ \ldots,\ N-1\}$ the numbers $V_{j,1},\ V_{j,2},\ \ldots,\ V_{j,N-1}$ are vectors in the complex plane that correspond to the sides of some equilateral $(N-1)$-gon, i.e. the rows of the upper left $(N-1)\times(N-1)$ submatrix of U represent sides of some equilateral $(N-1)$-gons in the complex plane. Note that these $(N-1)$-gons can be non-convex, degenerate or self-intersecting.

Analogous statements are also true for the columns of U, so this proves the following:

Lemma 2.8. *Let U be as in Remark 2.7. Then the rows and the columns of its corresponding matrix $\Phi = (\Phi_{j,k})_{1\le j,k\le N-1}$ consist of directions of sides of some equilateral $(N-1)$-gons in the complex plane.*

Note that Lemma 2.8 does not hold for all matrices equivalent to U, because changing the last row (resp. column) of U means that the other rows (resp. columns) do not need to have this property anymore.

We will denote the rows of U by R_j and columns of U by C_k, i.e. we define

$$R_j = (U_{j,k})_{1\le k\le N}, \quad \text{for } 1 \le j \le N,$$
$$C_k = (U_{j,k})_{1\le j\le N}, \quad \text{for } 1 \le k \le N.$$

2.3.1 Case $n = 1, N = 4$

For $N = 4$, equilateral 3-gons are equilateral triangles.

Remark 2.9. *The directions of sides of an equilateral triangle in the complex plane form the set $\{\alpha,\ \alpha+\frac{2\pi}{3},\ \alpha+\frac{4\pi}{3}\}$ for some $\alpha \in [0, \frac{2\pi}{3})$. The following is an example of an equilateral triangle in the complex plane whose sides are labeled by their directions in this manner:*

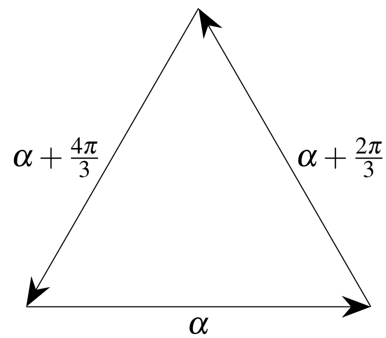

Figure 6: Equilateral triangle in the complex plane

Assume that, for $N = 4$, there exists a unitary matrix U as in Remark 2.7. Then we can apply Lemma 2.8 to its corresponding matrix Φ. Note that $V_{1,1} = 1$ implies $\Phi_{1,1} = 0$. Now, without loss of generality (by reordering rows and columns if necessary), Remark 2.9 implies that

$$\Phi = \begin{pmatrix} 0 & \frac{2\pi}{3} & \frac{4\pi}{3} \\ \frac{2\pi}{3} & * & * \\ \frac{4\pi}{3} & * & * \end{pmatrix}.$$

Using Remark 2.9 again, we get

$$\Phi = \begin{pmatrix} 0 & \frac{2\pi}{3} & \frac{4\pi}{3} \\ \frac{2\pi}{3} & \frac{4\pi}{3} & 0 \\ \frac{4\pi}{3} & 0 & \frac{2\pi}{3} \end{pmatrix}.$$

But now $R_1 = \left(\frac{\sqrt{2}}{3},\, -\frac{\sqrt{2}}{6}+\frac{\sqrt{6}}{6}i,\, -\frac{\sqrt{2}}{6}-\frac{\sqrt{6}}{6}i,\, \frac{1}{2}\right)$ and $R_2 = \left(-\frac{\sqrt{2}}{6}+\frac{\sqrt{6}}{6}i,\, -\frac{\sqrt{2}}{6}-\frac{\sqrt{6}}{6}i,\, \frac{\sqrt{2}}{3},\, \frac{1}{2}\right)$ so $\langle R_1|R_2\rangle = -\frac{1}{12}-\frac{\sqrt{3}}{3}i \neq 0$. Thus U is not unitary which is a contradiction.

Thus according to Proposition 2.5 there are no symmetric commuting squares of the form (2.3) for $N = 4$ and $n = 1$. This is the reason why Bisch's construction of an irreducible hyperfinite subfactor with index 4.5 is more involved and is based on a symmetric commuting square of the form (2.3) for $N = 4$ and $n = 2$ ([Bis94b]).

2.3.2 Case $n = 1, N = 5$

For $N = 5$, equilateral 4-gons are equilateral quadrilaterals, i.e. rhombuses.

Remark 2.10. *The directions of sides of a rhombus in the complex plane form the multiset* $\{\alpha,\ \beta,\ \alpha+\pi,\ \beta+\pi\}$ *for some* $\alpha,\ \beta \in [0,\pi)$. *The following is an example of a rhombus in the complex plane whose sides are labeled by their directions in this manner:*

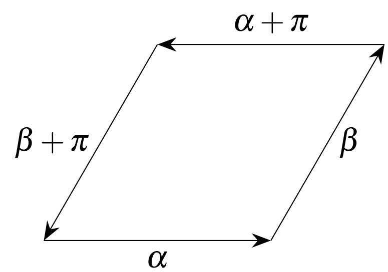

Figure 7: Rhombus in the complex plane

We want to find unitary matrices U of the form from Remark 2.7. We can apply Lemma 2.8 to its corresponding matrix Φ. Note that $V_{1,1} = 1$ implies $\Phi_{1,1} = 0$, so, without loss of generality (by reordering rows and columns if necessary), Remark 2.10 implies that we need to have

$$\Phi \quad = \quad \begin{pmatrix} 0 & \pi & \alpha & \alpha+\pi \\ \pi & * & * & * \\ \beta & * & * & * \\ \beta+\pi & * & * & * \end{pmatrix} \quad \text{for some } \alpha,\ \beta \in [0,2\pi).$$

Note that additions of directions and equalities of directions are considered $\mod 2\pi$.

If $\Phi_{2,2} = 0$, then Remark 2.10 implies that $\Phi_{2,3} = \gamma$ and $\Phi_{2,4} = \gamma + \pi$ for some $\gamma \in [0,2\pi)$. But then $\langle R_1 | R_2 \rangle \neq 0$ so we must have $\Phi_{2,2} \neq 0$. Thus, due to Remark 2.10, without loss of generality (by reordering rows and columns if necessary), we must have

$$\Phi = \begin{pmatrix} 0 & \pi & \alpha & \alpha+\pi \\ \pi & \gamma+\pi & 0 & \gamma \\ \beta & 0 & * & * \\ \beta+\pi & \gamma & * & * \end{pmatrix} \quad \text{for some } \alpha,\ \beta,\ \gamma \in [0,2\pi).$$

If any of $\alpha,\ \beta,\ \gamma$ is 0 or π, then $\langle R_1|R_2\rangle \neq 0$ or $\langle C_1|C_2\rangle \neq 0$ so we must have $\alpha,\ \beta,\ \gamma \notin \{0,\ \pi\}$.

If $\Phi_{3,3} = \pi$, then Remark 2.10 implies that

$$\Phi = \begin{pmatrix} 0 & \pi & \alpha & \alpha+\pi \\ \pi & \gamma+\pi & 0 & \gamma \\ \beta & 0 & \pi & \beta+\pi \\ \beta+\pi & \gamma & \alpha+\pi & * \end{pmatrix} \quad \text{for some } \alpha,\ \beta,\ \gamma \in [0,2\pi).$$

Thus we have

$$\begin{aligned} & \langle R_2|R_3\rangle = 0 \\ \implies \quad & \frac{3}{16}(-e^{-i\beta} - e^{i\gamma} - 1 - e^{i(\gamma-\beta)}) + \frac{1}{4} = 0 \\ \implies \quad & (1+e^{-i\beta})(1+e^{i\gamma}) = \frac{4}{3} \end{aligned}$$

Similarly, $\langle C_2|C_3\rangle = 0 \implies \left(1+e^{-i\alpha}\right)\left(1+e^{i\gamma}\right) = \frac{4}{3}$ so $\alpha = \beta$ and then Remark 2.10 applied to the 4th row (or column) gives $\alpha = \beta = \gamma$. But now $\langle R_1|R_3\rangle \neq 0$, so we must have $\Phi_{3,3} \neq \pi$. Then, due to Remark 2.10, we must have $\Phi_{3,3} = \alpha+\pi = \beta+\pi$ which implies

$\Phi_{3,4} = \Phi_{4,3} = \pi$ and $\alpha = \beta = \gamma$ so

$$\Phi = \begin{pmatrix} 0 & \pi & \alpha & \alpha+\pi \\ \pi & \alpha+\pi & 0 & \alpha \\ \alpha & 0 & \alpha+\pi & \pi \\ \alpha+\pi & \alpha & \pi & 0 \end{pmatrix} \quad \text{for some } \alpha \in [0, 2\pi).$$

Solving $\langle R_1 | R_2 \rangle = 0$ for $e^{i\alpha}$ gives two solutions: $e^{i\alpha} = \frac{1}{3} \pm \frac{2\sqrt{2}}{3} i$. Set $z = \frac{1}{3} + \frac{2\sqrt{2}}{3} i$. Two matrices U

$$\begin{pmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4}z & \frac{1}{2} \\ -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}z & \frac{1}{2} \\ \frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}\bar{z} & -\frac{\sqrt{3}}{4}\bar{z} & \frac{1}{2} \\ -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4}\bar{z} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}\bar{z} & \frac{1}{2} \\ \frac{\sqrt{3}}{4}\bar{z} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4}\bar{z} & -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{\sqrt{3}}{4}\bar{z} & \frac{\sqrt{3}}{4}\bar{z} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

that correspond to these solutions are both unitary. Note that the second matrix is equivalent to the first one via

$$\rightarrow \begin{pmatrix} \frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4}z & \frac{1}{2} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4}z & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4}z & \frac{1}{2} \\ -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4}z & \frac{1}{2} \\ \frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{\sqrt{3}}{4}z & \frac{\sqrt{3}}{4}z & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

where in the first step we multiply the first four rows by z and the last column by $\bar{z}$, and in the second step we permute the rows by (23) and the columns by $(13)(24)$. So there is a unique (up to equivalence) unitary matrix U:

$$U = \begin{pmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{12}+\frac{\sqrt{6}}{6}i & -\frac{\sqrt{3}}{12}-\frac{\sqrt{6}}{6}i & \frac{1}{2} \\ -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{12}-\frac{\sqrt{6}}{6}i & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{12}+\frac{\sqrt{6}}{6}i & \frac{1}{2} \\ \frac{\sqrt{3}}{12}+\frac{\sqrt{6}}{6}i & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{12}-\frac{\sqrt{6}}{6}i & -\frac{\sqrt{3}}{4} & \frac{1}{2} \\ -\frac{\sqrt{3}}{12}-\frac{\sqrt{6}}{6}i & \frac{\sqrt{3}}{12}+\frac{\sqrt{6}}{6}i & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \qquad (2.14)$$

Since $\alpha = \arg(\frac{1}{3}+\frac{2\sqrt{2}}{3}i) = \arccos\frac{1}{3}$, its corresponding matrix Φ is

$$\Phi = \begin{pmatrix} 0 & \pi & \arccos\frac{1}{3} & \arccos\frac{1}{3}+\pi \\ \pi & \arccos\frac{1}{3}+\pi & 0 & \arccos\frac{1}{3} \\ \arccos\frac{1}{3} & 0 & \arccos\frac{1}{3}+\pi & \pi \\ \arccos\frac{1}{3}+\pi & \arccos\frac{1}{3} & \pi & 0 \end{pmatrix}.$$

All rhombuses of Φ are the same (up to permutation of sides):

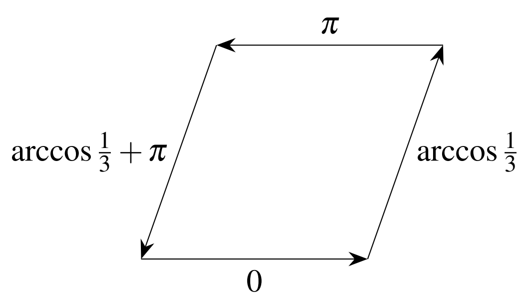

Figure 8: Rhombus for $N = 5$ and $n = 1$

In the context of Remark 2.10, we have $\alpha = 0$ and $\beta = \arccos\frac{1}{3}$.

It follows from Proposition 2.5 that there exists a symmetric commuting square of the form (2.3) for $N = 5$ and $n = 1$. Thus, by (1.3), (2.4) and Corollary 1.15, there is an irreducible subfactor of the hyperfinite II_1 factor constructed from this commuting square whose index is $\frac{(N-1)^2}{N-2} = \frac{16}{3} = 5.333\ldots$

2.3.3 Case $n = 1, N = 6$ and $N = 7$

We studied the question of existence of a unitary matrix U from (2.11) for $N = 4$ (in Subsection 2.3.1) and for $N = 5$ (in Subsection 2.3.2) by using a parametrization of equilateral triangles (stated in Remark 2.9) and a parametrization of rhombuses (stated in Remark 2.10). While such parametrizations are possible for equilateral $(N-1)$-gons for $N \geq 6$, in these cases the number of parameters ($N-2$, or $N-3$ when considered up to rotation) and additional conditions on these parameters make them less useful for solving the problem directly.

The examples for $N = 6$ and $N = 7$ were discovered by solving the equations in (2.13) numerically and then finding the corresponding exact solutions.

For $N = 6$, there exists at least one unitary matrix U of the form as in Remark 2.7:

$$U = \begin{pmatrix} \frac{2}{5} & -\frac{1}{10}+\frac{\sqrt{15}}{10}i & -\frac{1}{10}+\frac{\sqrt{15}}{10}i & -\frac{1}{10}-\frac{\sqrt{15}}{10}i & -\frac{1}{10}-\frac{\sqrt{15}}{10}i & \frac{1}{\sqrt{5}} \\ -\frac{1}{10}+\frac{\sqrt{15}}{10}i & \frac{2}{5} & -\frac{1}{10}-\frac{\sqrt{15}}{10}i & -\frac{1}{10}-\frac{\sqrt{15}}{10}i & -\frac{1}{10}+\frac{\sqrt{15}}{10}i & \frac{1}{\sqrt{5}} \\ -\frac{1}{10}+\frac{\sqrt{15}}{10}i & -\frac{1}{10}-\frac{\sqrt{15}}{10}i & \frac{2}{5} & -\frac{1}{10}+\frac{\sqrt{15}}{10}i & -\frac{1}{10}-\frac{\sqrt{15}}{10}i & \frac{1}{\sqrt{5}} \\ -\frac{1}{10}-\frac{\sqrt{15}}{10}i & -\frac{1}{10}-\frac{\sqrt{15}}{10}i & -\frac{1}{10}+\frac{\sqrt{15}}{10}i & \frac{2}{5} & -\frac{1}{10}+\frac{\sqrt{15}}{10}i & \frac{1}{\sqrt{5}} \\ -\frac{1}{10}-\frac{\sqrt{15}}{10}i & -\frac{1}{10}+\frac{\sqrt{15}}{10}i & -\frac{1}{10}-\frac{\sqrt{15}}{10}i & -\frac{1}{10}+\frac{\sqrt{15}}{10}i & \frac{2}{5} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{pmatrix}.$$

Its corresponding matrix Φ is

$$\Phi = \begin{pmatrix} 0 & \arccos\frac{-1}{4} & \arccos\frac{-1}{4} & -\arccos\frac{-1}{4} & -\arccos\frac{-1}{4} \\ \arccos\frac{-1}{4} & 0 & -\arccos\frac{-1}{4} & -\arccos\frac{-1}{4} & \arccos\frac{-1}{4} \\ \arccos\frac{-1}{4} & -\arccos\frac{-1}{4} & 0 & \arccos\frac{-1}{4} & -\arccos\frac{-1}{4} \\ -\arccos\frac{-1}{4} & -\arccos\frac{-1}{4} & \arccos\frac{-1}{4} & 0 & \arccos\frac{-1}{4} \\ -\arccos\frac{-1}{4} & \arccos\frac{-1}{4} & -\arccos\frac{-1}{4} & \arccos\frac{-1}{4} & 0 \end{pmatrix}.$$

Note that $-\arccos\frac{-1}{4} \notin [0,2\pi)$ and that it is equal $\mod 2\pi$ to $\arccos\frac{1}{4}+\pi \in [0,2\pi)$.

All equilateral pentagons (5-gons) of Φ (from Lemma 2.8) are the same (up to permutation of sides). When its sides are ordered so that it is convex, it is degenerate, i.e. it is similar to a triangle with sides $1,\ 2,\ 2$:

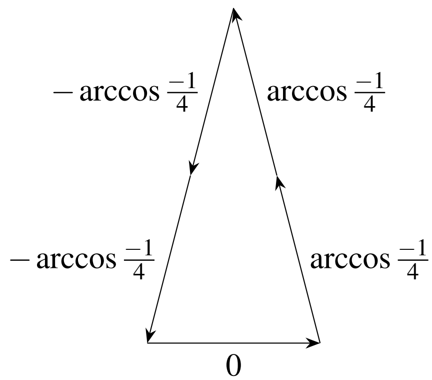

Figure 9: Equilateral pentagon for $N = 6$ and $n = 1$

It follows from Proposition 2.5 that there exists a symmetric commuting square of the form (2.3) for $N = 6$ and $n = 1$ and, by (1.3), (2.4) and Corollary 1.15, there is an irreducible subfactor of the hyperfinite II_1 factor constructed from this commuting square whose index is $\frac{(N-1)^2}{N-2} = \frac{25}{4} = 6.25$.

For $N = 7$, there exist at least two non-equivalent unitary matrices U of the form as in Remark 2.7.

The first solution U is:

$$U = \begin{pmatrix}
\frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{10}+\frac{2\sqrt{5}}{15}i & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & -\frac{\sqrt{5}}{6}i & -\frac{\sqrt{5}}{6}i & \frac{1}{\sqrt{6}} \\
\frac{\sqrt{5}}{10}+\frac{2\sqrt{5}}{15}i & \frac{\sqrt{5}}{6} & -\frac{\sqrt{5}}{6}i & -\frac{\sqrt{5}}{6}i & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & \frac{1}{\sqrt{6}} \\
-\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & -\frac{\sqrt{5}}{6}i & \frac{\sqrt{5}}{6} & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & -\frac{\sqrt{5}}{6}i & \frac{\sqrt{5}}{10}+\frac{2\sqrt{5}}{15}i & \frac{1}{\sqrt{6}} \\
-\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & -\frac{\sqrt{5}}{6}i & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & \frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{10}+\frac{2\sqrt{5}}{15}i & -\frac{\sqrt{5}}{6}i & \frac{1}{\sqrt{6}} \\
-\frac{\sqrt{5}}{6}i & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & -\frac{\sqrt{5}}{6}i & \frac{\sqrt{5}}{10}+\frac{2\sqrt{5}}{15}i & \frac{\sqrt{5}}{6} & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & \frac{1}{\sqrt{6}} \\
-\frac{\sqrt{5}}{6}i & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & \frac{\sqrt{5}}{10}+\frac{2\sqrt{5}}{15}i & -\frac{\sqrt{5}}{6}i & -\frac{2\sqrt{5}}{15}+\frac{\sqrt{5}}{10}i & \frac{\sqrt{5}}{6} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0
\end{pmatrix}. \tag{2.15}$$

Its corresponding matrix Φ is

$$\Phi = \begin{pmatrix} 0 & \arccos\frac{3}{5} & \arccos\frac{-4}{5} & \arccos\frac{-4}{5} & \frac{3\pi}{2} & \frac{3\pi}{2} \\ \arccos\frac{3}{5} & 0 & \frac{3\pi}{2} & \frac{3\pi}{2} & \arccos\frac{-4}{5} & \arccos\frac{-4}{5} \\ \arccos\frac{-4}{5} & \frac{3\pi}{2} & 0 & \arccos\frac{-4}{5} & \frac{3\pi}{2} & \arccos\frac{3}{5} \\ \arccos\frac{-4}{5} & \frac{3\pi}{2} & \arccos\frac{-4}{5} & 0 & \arccos\frac{3}{5} & \frac{3\pi}{2} \\ \frac{3\pi}{2} & \arccos\frac{-4}{5} & \frac{3\pi}{2} & \arccos\frac{3}{5} & 0 & \arccos\frac{-4}{5} \\ \frac{3\pi}{2} & \arccos\frac{-4}{5} & \arccos\frac{3}{5} & \frac{3\pi}{2} & \arccos\frac{-4}{5} & 0 \end{pmatrix}.$$

All equilateral hexagons (6-gons) of Φ are the same (up to permutation of sides). When its sides are ordered so that it is convex, it is degenerate, i.e. it is similar to a kite with sides 1, 1, 2, 2:

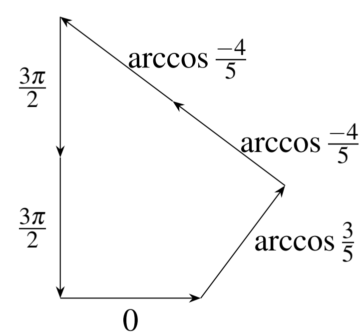

Figure 10: First equilateral hexagon for $N = 7$ and $n = 1$

The second solution U is

$$U = \begin{pmatrix} \frac{\sqrt{5}}{6} & z & \frac{\sqrt{5}}{30}+\frac{\sqrt{30}}{15}i & -\frac{\sqrt{5}}{6} & -z & -\frac{\sqrt{5}}{30}-\frac{\sqrt{30}}{15}i & \frac{1}{\sqrt{6}} \\ z & -\frac{\sqrt{5}}{30}-\frac{\sqrt{30}}{15}i & -z & \frac{\sqrt{5}}{30}+\frac{\sqrt{30}}{15}i & -\frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{6} & \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{30}+\frac{\sqrt{30}}{15}i & -z & \frac{\sqrt{5}}{6} & -\frac{\sqrt{5}}{30}-\frac{\sqrt{30}}{15}i & z & -\frac{\sqrt{5}}{6} & \frac{1}{\sqrt{6}} \\ -\frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{30}+\frac{\sqrt{30}}{15}i & -\frac{\sqrt{5}}{30}-\frac{\sqrt{30}}{15}i & z & \frac{\sqrt{5}}{6} & -z & \frac{1}{\sqrt{6}} \\ -z & -\frac{\sqrt{5}}{6} & z & \frac{\sqrt{5}}{6} & -\frac{\sqrt{5}}{30}-\frac{\sqrt{30}}{15}i & \frac{\sqrt{5}}{30}+\frac{\sqrt{30}}{15}i & \frac{1}{\sqrt{6}} \\ -\frac{\sqrt{5}}{30}-\frac{\sqrt{30}}{15}i & \frac{\sqrt{5}}{6} & -\frac{\sqrt{5}}{6} & -z & \frac{\sqrt{5}}{30}+\frac{\sqrt{30}}{15}i & z & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \end{pmatrix} \quad (2.16)$$

where $z = \frac{3\sqrt{30}-2\sqrt{5}}{60} + \frac{6\sqrt{5}+\sqrt{30}}{60}i$. Its corresponding matrix Φ is

$$\Phi = \begin{pmatrix} 0 & \alpha & \arccos\frac{1}{5} & \pi & \alpha+\pi & \arccos\frac{1}{5}+\pi \\ \alpha & \arccos\frac{1}{5}+\pi & \alpha+\pi & \arccos\frac{1}{5} & \pi & 0 \\ \arccos\frac{1}{5} & \alpha+\pi & 0 & \arccos\frac{1}{5}+\pi & \alpha & \pi \\ \pi & \arccos\frac{1}{5} & \arccos\frac{1}{5}+\pi & \alpha & 0 & \alpha+\pi \\ \alpha+\pi & \pi & \alpha & 0 & \arccos\frac{1}{5}+\pi & \arccos\frac{1}{5} \\ \arccos\frac{1}{5}+\pi & 0 & \pi & \alpha+\pi & \arccos\frac{1}{5} & \alpha \end{pmatrix}$$

where $\alpha = \arccos\frac{3\sqrt{6}-2}{10}$.

All equilateral hexagons of Φ are again the same (up to permutation of sides):

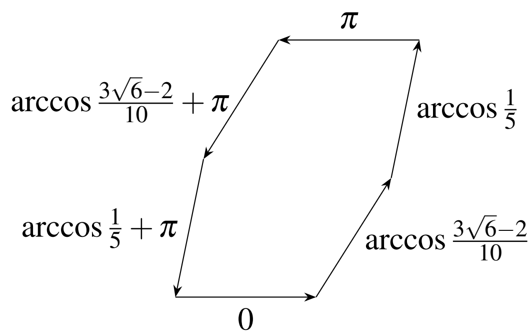

Figure 11: Second equilateral hexagon for $N = 7$ and $n = 1$

It follows from Proposition 2.5 that there exists a symmetric commuting square of the form (2.3) for $N = 7$ and $n = 1$ and, by (1.3), (2.4) and Corollary 1.15, there is an irreducible subfactor of the hyperfinite II_1 factor constructed from this commuting square whose index is $\frac{(N-1)^2}{N-2} = \frac{36}{5} = 7.2$.

We do not know if the subfactors obtained from the two non-equivalent symmetric commuting squares based on the above connections are isomorphic. This appears to be a hard problem.

2.4 Case $n = 1$, odd N

Note that if N is odd, then $N-1$ is even. It is easier to find somewhat symmetric non-trivial equilateral $(N-1)$-gons when $N-1$ is even because one can simply pick any $\frac{N-1}{2}$ directions and their additive inverses (i.e. add π). We can then order the sides of this $(N-1)$-gon so that it has parallel opposite sides. For $N = 5$, due to geometric constraints, all rhombuses are of this form, but for $N = 7$ there exist equilateral hexagons which are not of this form. Indeed, for $N = 7$, the first solution (2.15) is not of this form, but the second solution (2.16) is of this form. We will be looking for solutions of this form for general odd N.

Let $N \geq 4$ be an odd number and let $M = \frac{N-3}{2}$. We will follow the notational convention of Section 2.3. Let U be of the form as in Remark 2.7 such that its corresponding matrix Φ is

$$\Phi = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{M-1} & \alpha_M & \alpha_0+\pi & \alpha_1+\pi & \cdots & \alpha_{M-1}+\pi & \alpha_M+\pi \\ \alpha_1 & \alpha_2 & \cdots & \alpha_M & \alpha_0 & \alpha_M+\pi & \alpha_0+\pi & \cdots & \alpha_{M-2}+\pi & \alpha_{M-1}+\pi \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{M-1} & \alpha_M & \cdots & \alpha_{M-3} & \alpha_{M-2} & \alpha_2+\pi & \alpha_3+\pi & \cdots & \alpha_0+\pi & \alpha_1+\pi \\ \alpha_M & \alpha_0 & \cdots & \alpha_{M-2} & \alpha_{M-1} & \alpha_1+\pi & \alpha_2+\pi & \cdots & \alpha_M+\pi & \alpha_0+\pi \\ \alpha_0+\pi & \alpha_M+\pi & \cdots & \alpha_2+\pi & \alpha_1+\pi & \alpha_M & \alpha_{M-1} & \cdots & \alpha_1 & \alpha_0 \\ \alpha_1+\pi & \alpha_0+\pi & \cdots & \alpha_3+\pi & \alpha_2+\pi & \alpha_{M-1} & \alpha_{M-2} & \cdots & \alpha_0 & \alpha_M \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{M-1}+\pi & \alpha_{M-2}+\pi & \cdots & \alpha_0+\pi & \alpha_M+\pi & \alpha_1 & \alpha_0 & \cdots & \alpha_3 & \alpha_2 \\ \alpha_M+\pi & \alpha_{M-1}+\pi & \cdots & \alpha_1+\pi & \alpha_0+\pi & \alpha_0 & \alpha_M & \cdots & \alpha_2 & \alpha_1 \end{pmatrix} \tag{2.17}$$

for $\alpha_0 = 0$ and $\alpha_j \in [0, 2\pi)$ for $1 \leq j \leq M$. More precisely:

$$\Phi_{j,k} = \begin{cases} \alpha_{((j+k-2) \bmod (M+1))}, & 1 \leq j,k \leq M+1; \\ \alpha_{((-j+k) \bmod (M+1))} + \pi, & 1 \leq j \leq M+1 < k \leq 2M+2; \\ \alpha_{((j-k) \bmod (M+1))} + \pi, & 1 \leq k \leq M+1 < j \leq 2M+2; \\ \alpha_{((-j-k+1) \bmod (M+1))}, & M+1 < j,k \leq 2M+2. \end{cases}$$

Here mod denotes the least non-negative remainder.

Note that Φ satisfies equations from the first and the second row in (2.13) automatically since $\cos x + \cos(x+\pi) = \sin x + \sin(x+\pi) = 0$ for all x. We will now show that this is also true for the equations from the fourth row in (2.13).

Lemma 2.11. *Matrix Φ of the form as in* (2.17) *satisfies equations from the fourth row in*

(2.13) *for any choice of* $\alpha_j \in [0, 2\pi)$ *for* $1 \leq j \leq M$. *The equations in question are*

$$\sum_{l=1}^{N-1} \sin(\Phi_{j,l} - \Phi_{k,l}) = 0, \quad \textit{for } 1 \leq j < k \leq N-1.$$

Proof. Note that $N-1 = 2M+2$. For $1 \leq j < k \leq 2M+2$ and $1 \leq l \leq 2M+2$, let

$$\begin{aligned} \mathrm{S}(j,k,l) &= \sin(\Phi_{j,l} - \Phi_{k,l}) \quad \text{and} \\ \mathrm{E}(j,k) &= \sum_{l=1}^{2M+2} \mathrm{S}(j,k,l). \end{aligned}$$

Denote $S_1 = \{1,\ 2,\ \ldots,\ M+1\}$ and $S_2 = \{M+2,\ M+3,\ \ldots,\ 2M+2\}$. We want to show that $\mathrm{E}(j,k) = 0$ for all $1 \leq j < k \leq 2M+2$.

Case 1: $1 \leq j < k \leq M+1$

Let $1 \leq l \leq M+1$ and $l' = M+2+((j+k+l-3) \bmod (M+1))$. Note that $l \mapsto l'$ is a bijection between S_1 and S_2. We have

$$\begin{aligned} &(-j+l') \bmod (M+1) = \\ =&(-j+M+2+((j+k+l-3) \bmod (M+1))) \bmod (M+1) = \\ =&(-j+M+2+j+k+l-3) \bmod (M+1) = \\ =&(k+l-2) \bmod (M+1) \end{aligned}$$

and, similarly, $(-k+l') \bmod (M+1) = (j+l-2) \bmod (M+1)$. This implies

$$\begin{aligned} \Phi_{j,l'} - \Phi_{k,l'} &= \alpha_{((-j+l') \bmod (M+1))} + \pi - (\alpha_{((-k+l') \bmod (M+1))} + \pi) = \\ &= \alpha_{((k+l-2) \bmod (M+1))} - \alpha_{((j+l-2) \bmod (M+1))} = \\ &= \Phi_{k,l} - \Phi_{j,l} \end{aligned}$$

so

$$
\begin{aligned}
\mathrm{S}(j,k,l)+\mathrm{S}(j,k,l') &= \sin(\Phi_{j,l}-\Phi_{k,l})+\sin(\Phi_{j,l'}-\Phi_{k,l'}) = \\
&= \sin(\Phi_{j,l}-\Phi_{k,l})+\sin(\Phi_{k,l}-\Phi_{j,l}) = \\
&= 0
\end{aligned}
$$

which combined with the fact that $l \mapsto l'$ is a bijection between S_1 and S_2 gives

$$
\mathrm{E}(j,k) = \sum_{l=1}^{2M+2} \mathrm{S}(j,k,l) = 0, \quad \text{for } 1 \le j < k \le M+1.
$$

Case 2: $M+1 < j < k \le 2M+2$

Let $1 \le l \le M+1$ and $l' = M+2+((-j-k+l) \bmod (M+1))$. Note that $l \mapsto l'$ is a bijection between S_1 and S_2. We have

$$
\begin{aligned}
&(-j-l'+1) \bmod (M+1) = \\
=&(-j-(M+2+((-j-k+l) \bmod (M+1)))+1) \bmod (M+1) = \\
=&(-j-M-2+j+k-l+1) \bmod (M+1) = \\
=&(k-l) \bmod (M+1)
\end{aligned}
$$

and, similarly, $(-k-l'+1) \bmod (M+1) = (j-l) \bmod (M+1)$. This implies

$$
\begin{aligned}
\Phi_{j,l'}-\Phi_{k,l'} &= \alpha_{((-j-l'+1) \bmod (M+1))} - \alpha_{((-k-l'+1) \bmod (M+1))} = \\
&= \alpha_{((k-l) \bmod (M+1))} + \pi - (\alpha_{((j-l) \bmod (M+1))} + \pi) = \\
&= \Phi_{k,l} - \Phi_{j,l}
\end{aligned}
$$

so

$$\begin{aligned}
\mathrm{S}(j,k,l)+\mathrm{S}(j,k,l') &= \sin(\Phi_{j,l}-\Phi_{k,l})+\sin(\Phi_{j,l'}-\Phi_{k,l'}) = \\
&= \sin(\Phi_{j,l}-\Phi_{k,l})+\sin(\Phi_{k,l}-\Phi_{j,l}) = \\
&= 0
\end{aligned}$$

which combined with the fact that $l \mapsto l'$ is a bijection between S_1 and S_2 gives

$$\mathrm{E}(j,k) = \sum_{l=1}^{2M+2} \mathrm{S}(j,k,l) = 0, \quad \text{for } M+1 < j < k \leq 2M+2.$$

Case 3: $1 \leq j \leq M+1 < k \leq 2M+2$

Let $1 \leq l \leq M+1$ and $l' = 1 + ((-j+k-l+1) \bmod (M+1))$. Note that since

$$\begin{aligned}
(l')' &= 1 + ((-j+k-l'+1) \bmod (M+1)) = \\
&= 1 + (((-j+k-(1+((-j+k-l+1) \bmod (M+1))))+1) \bmod (M+1)) = \\
&= 1 + ((-j+k-1+j-k+l-1+1) \bmod (M+1)) = \\
&= 1 + ((l-1) \bmod (M+1)) = \\
&= l
\end{aligned}$$

$l \mapsto l'$ is an involution on S_1. We have

$$\begin{aligned}
&(j+l'-2) \bmod (M+1) = \\
=&(j+(1+(-j+k-l+1) \bmod (M+1)))-2) \bmod (M+1) = \\
=&(j+1-j+k-l+1-2) \bmod (M+1) = \\
=&(k-l) \bmod (M+1)
\end{aligned}$$

and then

$$(k-l') \text{ mod } (M+1) = (j+l-2) \text{ mod } (M+1)$$

follows from the fact that $l \mapsto l'$ is an involution on S_1. This implies

$$\begin{aligned}\Phi_{j,l'} - \Phi_{k,l'} &= \alpha_{((j+l'-2) \text{ mod } (M+1))} - (\alpha_{((k-l') \text{ mod } (M+1))} + \pi) = \\ &= \alpha_{((k-l) \text{ mod } (M+1))} + \pi - \alpha_{((j+l-2) \text{ mod } (M+1))} - 2\pi = \\ &= \Phi_{k,l} - \Phi_{j,l} - 2\pi.\end{aligned}$$

If $l \neq l'$, then

$$\begin{aligned}\mathrm{S}(j,k,l) + \mathrm{S}(j,k,l') &= \sin(\Phi_{j,l} - \Phi_{k,l}) + \sin(\Phi_{j,l'} - \Phi_{k,l'}) = \\ &= \sin(\Phi_{j,l} - \Phi_{k,l}) + \sin(\Phi_{k,l} - \Phi_{j,l} - 2\pi) = \\ &= 0\end{aligned}$$

and if $l = l'$, then

$$\begin{aligned}0 =& \mathrm{S}(j,k,l) - \mathrm{S}(j,k,l') = \sin(\Phi_{j,l} - \Phi_{k,l}) - \sin(\Phi_{j,l'} - \Phi_{k,l'}) = \\ &= \sin(\Phi_{j,l} - \Phi_{k,l}) - \sin(\Phi_{k,l} - \Phi_{j,l} - 2\pi) = \\ &= 2\sin(\Phi_{j,l} - \Phi_{k,l}) = 2\,\mathrm{S}(j,k,l), \quad \text{so} \\ 0 =& \mathrm{S}(j,k,l)\end{aligned}$$

which combined with the fact that $l \mapsto l'$ is an involution on S_1 gives

$$\sum_{l=1}^{M+1} \mathrm{S}(j,k,l) = 0. \tag{2.18}$$

Let $M+1 < l \leq 2M+2$ and $l' = M+2+((j-k-l) \bmod (M+1))$. Note that since

$$\begin{aligned}
(l')' &= M+2+((j-k-l') \bmod (M+1)) = \\
&= M+2+((j-k-(M+2+((j-k-l) \bmod (M+1)))) \bmod (M+1)) = \\
&= M+2+((j-k-M-2-j+k+l) \bmod (M+1)) = \\
&= M+2+((l-M-2) \bmod (M+1)) = \\
&= l
\end{aligned}$$

$l \mapsto l'$ is an involution on S_2. We have

$$\begin{aligned}
&(-j+l') \bmod (M+1) = \\
=&(-j+(M+2+((j-k-l) \bmod (M+1)))) \bmod (M+1) = \\
=&(-j+M+2+j-k-l) \bmod (M+1) = \\
=&(-k-l+1) \bmod (M+1)
\end{aligned}$$

and then

$$(-k-l'+1) \bmod (M+1) = (-j+l) \bmod (M+1)$$

follows from the fact that $l \mapsto l'$ is an involution on S_2. This implies

$$\begin{aligned}
\Phi_{j,l'} - \Phi_{k,l'} &= \alpha_{((-j+l') \bmod (M+1))} + \pi - \alpha_{((-k-l'+1) \bmod (M+1))} = \\
&= \alpha_{((-k-l+1) \bmod (M+1))} - (\alpha_{((-j+l) \bmod (M+1))} + \pi) + 2\pi = \\
&= \Phi_{k,l} - \Phi_{j,l} + 2\pi.
\end{aligned}$$

If $l \neq l'$, then

$$\begin{aligned}
\mathrm{S}(j,k,l)+\mathrm{S}(j,k,l') &= \sin(\Phi_{j,l}-\Phi_{k,l})+\sin(\Phi_{j,l'}-\Phi_{k,l'}) = \\
&= \sin(\Phi_{j,l}-\Phi_{k,l})+\sin(\Phi_{k,l}-\Phi_{j,l}+2\pi) = \\
&= 0
\end{aligned}$$

and if $l = l'$, then

$$\begin{aligned}
0 =& \mathrm{S}(j,k,l)-\mathrm{S}(j,k,l') = \sin(\Phi_{j,l}-\Phi_{k,l})-\sin(\Phi_{j,l'}-\Phi_{k,l'}) = \\
=& \sin(\Phi_{j,l}-\Phi_{k,l})-\sin(\Phi_{k,l}-\Phi_{j,l}+2\pi) = \\
=& 2\sin(\Phi_{j,l}-\Phi_{k,l}) = 2\,\mathrm{S}(j,k,l), \quad \text{so} \\
0 =& \mathrm{S}(j,k,l)
\end{aligned}$$

which combined with the fact that $l \mapsto l'$ is an involution on S_2 gives

$$\sum_{l=M+2}^{2M+2} \mathrm{S}(j,k,l) = 0. \tag{2.19}$$

Now from (2.18) and (2.19) it follows that

$$\mathrm{E}(j,k) = \sum_{l=1}^{M+1} \mathrm{S}(j,k,l) + \sum_{l=M+2}^{2M+2} \mathrm{S}(j,k,l) = 0, \quad \text{for } 1 \leq j \leq M+1 < k \leq 2M+2.$$

□

This means that that it is enough to find matrix Φ of the form as in (2.17) that satisfies equations from the third row in (2.13). These equations are

$$\sum_{l=1}^{N-1} \cos(\Phi_{j,l}-\Phi_{k,l}) = -\frac{N-1}{N-2}, \quad \text{for } 1 \leq j < k \leq N-1. \tag{2.20}$$

2.4.1 Case $n = 1, N = 9$

We will now find a matrix Φ of the form as in (2.17) that satisfies (2.20) for $N = 9$. Taking $\alpha_0 = 0$ into account, the system of equations (2.20) contains these distinct equations:

$$\begin{aligned} 2\cos(\alpha_1) + 2\cos(\alpha_3) + 2\cos(\alpha_1 - \alpha_2) + 2\cos(\alpha_2 - \alpha_3) &= -\frac{8}{7}, \\ 4\cos(\alpha_2) + 4\cos(\alpha_1 - \alpha_3) &= -\frac{8}{7}, \\ -2\cos(\alpha_3) - 2\cos(\alpha_1 - \alpha_2) - 2\cos(\alpha_1 - \alpha_3) &= \frac{6}{7}, \\ -2\cos(\alpha_1) - 2\cos(\alpha_2) - 2\cos(\alpha_2 - \alpha_3) &= \frac{6}{7}. \end{aligned} \tag{2.21}$$

For $1 \leq j \leq 3$ let x_j be such that

$$\cos(\alpha_j) = -\frac{1}{7} + x_j$$

and for $1 \leq j < k \leq 3$ let $x_{j,k}$ be such that

$$\cos(\alpha_j - \alpha_k) = -\frac{1}{7} + x_{j,k}.$$

Substituting these into (2.21) gives the following system of equations:

$$\begin{aligned} x_1 + x_3 + x_{1,2} + x_{2,3} &= 0, \\ x_2 + x_{1,3} &= 0, \\ x_3 + x_{1,2} + x_{1,3} &= 0, \\ x_1 + x_2 + x_{2,3} &= 0. \end{aligned}$$

Considered as a system of linear equations its solution is

$$\begin{aligned} x_{1,2} &= x_2 - x_3, \\ x_{1,3} &= -x_2, \\ x_{2,3} &= -x_1 - x_2, \end{aligned} \tag{2.22}$$
$$x_1,\ x_2,\ x_3, \quad \text{are free parameters.}$$

For $1 \leq j < k \leq 3$ we have

$$\begin{aligned} &-\frac{1}{7} + x_{j,k} = \cos(\alpha_j - \alpha_k) = \cos(\alpha_j)\cos(\alpha_k) + \sin(\alpha_j)\sin(\alpha_k) \\ \Longrightarrow\ &\sin^2(\alpha_j)\sin^2(\alpha_k) = \left(-\frac{1}{7} + x_{j,k} - \cos(\alpha_j)\cos(\alpha_k)\right)^2 \\ \Longrightarrow\ &(1-\cos^2(\alpha_j))(1-\cos^2(\alpha_k)) = \left(-\frac{1}{7} + x_{j,k} - \cos(\alpha_j)\cos(\alpha_k)\right)^2 \\ \Longrightarrow\ &\left(1 - \left(-\frac{1}{7} - x_j\right)^2\right)\left(1 - \left(-\frac{1}{7} - x_k\right)^2\right) = \left(-\frac{1}{7} + x_{j,k} - \left(-\frac{1}{7} - x_j\right)\left(-\frac{1}{7} - x_k\right)\right)^2 \end{aligned}$$

which after using (2.22) gives a system of polynomial equations:

$$\begin{aligned} \left(1 - \left(-\frac{1}{7} - x_1\right)^2\right)\left(1 - \left(-\frac{1}{7} - x_2\right)^2\right) &= \left(-\frac{1}{7} + x_2 - x_3 - \left(-\frac{1}{7} - x_1\right)\left(-\frac{1}{7} - x_2\right)\right)^2, \\ \left(1 - \left(-\frac{1}{7} - x_1\right)^2\right)\left(1 - \left(-\frac{1}{7} - x_3\right)^2\right) &= \left(-\frac{1}{7} - x_2 - \left(-\frac{1}{7} - x_1\right)\left(-\frac{1}{7} - x_3\right)\right)^2, \\ \left(1 - \left(-\frac{1}{7} - x_2\right)^2\right)\left(1 - \left(-\frac{1}{7} - x_3\right)^2\right) &= \left(-\frac{1}{7} - x_1 - x_2 - \left(-\frac{1}{7} - x_2\right)\left(-\frac{1}{7} - x_3\right)\right)^2. \end{aligned}$$

This system has a solution

$$\begin{aligned} x_1 &= \frac{2\sqrt{2}}{7}, \\ x_2 &= -\frac{4\sqrt{2}}{7}, \\ x_3 &= -\frac{2\sqrt{2}}{7}. \end{aligned}$$

One possible matrix Φ of the form as in (2.17) that comes from this solution is characterized by values

$$\alpha_1 = \arccos \frac{-1+2\sqrt{2}}{7},$$
$$\alpha_2 = -\arccos \frac{-1-4\sqrt{2}}{7},$$
$$\alpha_3 = \arccos \frac{-1-2\sqrt{2}}{7}.$$

Note that matrix Φ with these entries satisfies (2.21). Matrix U that corresponds to this Φ is

$$U = \begin{pmatrix} \frac{\sqrt{7}}{8} & z_1 & z_2 & z_3 & -\frac{\sqrt{7}}{8} & -z_1 & -z_2 & -z_3 & \frac{1}{2\sqrt{2}} \\ z_1 & z_2 & z_3 & \frac{\sqrt{7}}{8} & -z_3 & -\frac{\sqrt{7}}{8} & -z_1 & -z_2 & \frac{1}{2\sqrt{2}} \\ z_2 & z_3 & \frac{\sqrt{7}}{8} & z_1 & -z_2 & -z_3 & -\frac{\sqrt{7}}{8} & -z_1 & \frac{1}{2\sqrt{2}} \\ z_3 & \frac{\sqrt{7}}{8} & z_1 & z_2 & -z_1 & -z_2 & -z_3 & -\frac{\sqrt{7}}{8} & \frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{7}}{8} & -z_3 & -z_2 & -z_1 & z_3 & z_2 & z_1 & \frac{\sqrt{7}}{8} & \frac{1}{2\sqrt{2}} \\ -z_1 & -\frac{\sqrt{7}}{8} & -z_3 & -z_2 & z_2 & z_1 & \frac{\sqrt{7}}{8} & z_3 & \frac{1}{2\sqrt{2}} \\ -z_2 & -z_1 & -\frac{\sqrt{7}}{8} & -z_3 & z_1 & \frac{\sqrt{7}}{8} & z_3 & z_2 & \frac{1}{2\sqrt{2}} \\ -z_3 & -z_2 & -z_1 & -\frac{\sqrt{7}}{8} & \frac{\sqrt{7}}{8} & z_3 & z_2 & z_1 & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 \end{pmatrix}$$

where

$$z_1 = \frac{-\sqrt{7}+2\sqrt{14}}{56} + \frac{\sqrt{70+7\sqrt{2}}}{28} i,$$
$$z_2 = \frac{-\sqrt{7}-4\sqrt{14}}{56} - \frac{\sqrt{28-14\sqrt{2}}}{28} i,$$
$$z_3 = \frac{-\sqrt{7}-2\sqrt{14}}{56} + \frac{\sqrt{70-7\sqrt{2}}}{28} i.$$

This U is unitary and is of the form as in Remark 2.7. It follows already from the form of Φ in (2.17) that all its equilateral octagons (8-gons) (from Lemma 2.8) are the same (up to permutation of sides) and they are of the following form:

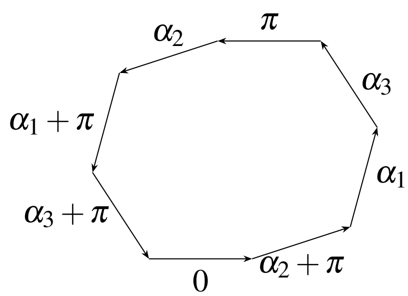

Figure 12: Equilateral octagon for $N = 9$ and $n = 1$

It follows from Proposition 2.5 that there exists a symmetric commuting square of the form (2.3) for $N = 9$ and $n = 1$ and, by (1.3), (2.4) and Corollary 1.15, there is an irreducible subfactor of the hyperfinite II_1 factor constructed from this commuting square whose index is $\frac{(N-1)^2}{N-2} = \frac{64}{7} = 9.142\ldots$

This method can similarly be used to obtain the solution (2.14) for $N = 5$ and the second solution (2.16) for $N = 7$. For $N \geq 11$, obtaining a solution in this manner that is simple enough so that it can be written out explicitly does not seem feasible.

2.5 Summary and remarks

Regarding the existence of symmetric commuting squares based on the inclusion graph N-star with A_∞-tail of the form (2.3) for $n = 1$, we have shown that:

(i) they do not exist for $N = 4$,

(ii) there exists exactly one (up to equivalence as in Remark 2.6) for $N = 5$,

(iii) there exists at least one for $N = 6$,

(iv) there exist at least two non-equivalent (as in Remark 2.6) ones for $N = 7$, and

(v) there exists at least one for $N = 9$.

We summarize the newly constructed hyperfinite subfactors in the following theorem:

Theorem 2.12. *There exists an irreducible subfactor of the hyperfinite II_1 factor constructed from a symmetric commuting square based on N-star with A_∞-tail for:*

(i) $N = 5$ *with index* $\frac{16}{3} = 5 + \frac{1}{3} = 5.333\ldots,$

(ii) $N = 6$ *with index* $\frac{25}{4} = 6 + \frac{1}{4} = 6.25,$

(iii) $N = 7$ *with index* $\frac{36}{5} = 7 + \frac{1}{5} = 7.2,$

(iv) $N = 9$ *with index* $\frac{64}{7} = 9 + \frac{1}{7} = 9.142\ldots$

All these indices are rational, non-integer numbers and thus not algebraic integers implying that these subfactors cannot be obtained from finite-dimensional symmetric commuting squares.

Numerical computations suggest that there are no commuting squares of the form (2.3) for $N = 8$ and $n = 1$, but that there exist such commuting squares for $n = 1$ and several consecutive numbers $N \geq 10$. They also suggest that there exists such a commuting square for $N = 8$ and $n = 2$.

Regarding the existence of such commuting squares for $n = 1$ and odd N whose corresponding matrix Φ is of the form as in (2.17), numerical computations suggest that they exist for many consecutive odd numbers $N \geq 11$.

There exists a symmetric commuting square based on the inclusion graph 4-star with A_∞-tail of the form (2.3) for $n = 2$, this is the commuting square that gives rise to the subfactor of index 4.5 of Bisch from [Bis94b]. Due to Haagerup ([Haa94]), its principal graph is A_∞. Subfactors constructed in this dissertation that arise from commuting squares based on inclusion graphs N-star with A_∞-tail of the form (2.3) for $N \geq 5$ all have indices greater than 5.25 and are thus beyond the scope of the current classification of small index subfactors ([AMP15]). It is an open problem to determine their principal graphs, or more generally, their standard invariant.

There has been a lot of recent interest in quantum Fourier analysis and the bi-unitary connections we have computed are explicit examples of what is called quantum Fourier transform in [JJL$^+$20]. It is not clear what the quantum symmetries are that they transform, so there is more interesting work that can be done here.

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