GROWTH OF DEHN TWIST AND PSEUDO-ANOSOV CONJUGACY CLASSES IN TEICHMÜLLER SPACE

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CHAPTER I

Introduction

Athreya, Bufetov, Eskin and Mirzakhani [2] have shown the number of mapping class group lattice points intersecting a closed ball of radius R in Teichmüller space is asymptotic to e^{hR}, where h is the dimension of the Teichmüller space. In this thesis, we first show the number of Dehn twist lattice points intersecting a closed ball of radius R is coarsely asymptotic to $e^{\frac{h}{2}R}$. Moreover, we show the number of all multi-twists lattice points intersecting a closed ball of radius R grows coarsely at least at the rate of $R \cdot e^{\frac{h}{2}R}$. Furthermore, we show for any pseudo-Anosov mapping class f, there exists a power n, such that the number of lattice points of the f^n conjugacy class intersecting a closed ball of radius R is coarsely asymptotic to $e^{\frac{h}{2}R}$. At the last section, we discuss a few open questions and a conjecture. The main results in this thesis are from author's papers [14] and [13].

I.1 Motivations

One can study a group by understanding its "growth" in various ways. Consider G acting on a metric space S by isometries, one can measure the number of orbit or lattice points of G in a ball of radius R as R goes to infinity. For example, consider $\mathbb{Z}^3$ acting on $\mathbb{R}^3$ in the standard way, the number of lattice points of $\mathbb{Z}^3$ in a ball of radius R is roughly the volume of this ball, see [22] for example.

Let M be a compact, negatively curved Riemannnian manifold and denote $\tilde{M}$ its universal cover. Then its fundamental group $\pi_1(M)$ acts on $\tilde{M}$ by isometries. Given any $x \in \tilde{M}, R > 0$, let $B_R(x)$ denote the ball of radius R in $\tilde{M}$ centered at x. G.A. Margulis studied the growth rate of any orbit $\pi_1(M) \cdot y$ by intersecting with any metric balls $B_r(x)$. It's a classical result from Margulis that

Theorem I.1.1 (Margulis [27])**.** *There is a function $c\colon M \times M \to \mathbb{R}^+$ so that for every $x, y \in \tilde{M}$,*

$$|\pi_1(M) \cdot y \cap B_R(x)| \sim c(p(x), p(y))e^{hR}$$

where h equals to the dimension of the manifold, which is the topological entropy of the geodesic flow on the unit tangent bundle of M.

Here and throughout, the notation $f(R) \sim g(R)$ means $\lim_{R\to\infty} \frac{f(R)}{g(R)} = 1$.

Inspired by this result, Athreya, Bufetov, Eskin and Mirzakhani studied lattice point asymptotics in Teichmüller space. Let $S_{g,n}$ denote a closed surface of genus g with n punctures such that $3g - 3 + n > 0$, and we let $\mathrm{Mod}_{g,n}$ and $(\mathcal{T}_{g,n}, d_{\mathcal{T}})$ denote the corresponding mapping class group and Teichmüller space with

Teichmüller metric. Then $\mathrm{Mod}_{g,n}$ acts on $\mathcal{T}_{g,n}$ by isometries. We use $\mathrm{Mod}_g, \mathcal{T}_g$ to denote $\mathrm{Mod}_{g,0}, \mathcal{T}_{g,0}$ for simplicity. They showed the orbits of mapping class group have analogous asymptotics.

Theorem I.1.2 (Athreya, Bufetov, Eskin and Mirzakhani [2])**.** *For any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_g$, we have*

$$|\mathrm{Mod}_g \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \sim e^{hR}$$

Note in their original paper, there is a factor of $\Lambda(\mathcal{X})\Lambda(\mathcal{Y})$ in front of $e^{\frac{h}{2}R}$, Λ is called the Hubbard-Masur function. Mirzakhani later showed that Λ is a constant function, see [8]. Moreover, we recall the following result from Parkkonen and Paulin about the lattice point asymptotics for conjugacy classes of $\pi_1(M)$.

Theorem I.1.3 (Parkkonen, Paulin [36])**.** *Let G be a nontrivial conjugacy class of $\pi_1(M)$, for any $x \in \tilde{M}$, we have*

$$\lim_{R\to\infty} \frac{1}{R} \ln |G \cdot x \cap B_R(x)| = \frac{h}{2}.$$

Inspired by this result, we wish to explore the lattice point asymptotics for conjugacy classes of $\mathrm{Mod}_{g,n}$. In analog to the Classification of Hyperbolic Isometries (Theorem II.4.1), the Nielsen-Thurston Classification (Theorem II.1.3, Theorem II.2.1) says every element in Mod_g is one of the three types: periodic, reducible, or pseudo-Anosov. Let $PA \subset \mathrm{Mod}_g$ denote the subset of pseudo-Anosov elements. Maher showed pseudo-Anosov elements are generic in the following sense.

Theorem I.1.4 (Maher [26])**.** *For any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_g$, we have*

$$\frac{|PA \cdot \mathcal{Y} \cap B_R(\mathcal{X})|}{|\mathrm{Mod}_g \cdot \mathcal{Y} \cap B_R(\mathcal{X})|} \sim 1.$$

The above Theorems I.1.3, I.1.4, motivate us to explore the lattice point asymptotics for pseudo-Anosov conjugacy class subgroups. Moreover, it is natural to consider the asymptotic growth rate of reducible and periodic elements. A typical reducible element can be decomposed as a product of Dehn twists about disjoint simple closed curves and a partial pseudo-Anosov element on subsurfaces [5]. Dehn twists are also in a sense the most fundamental elements of mapping class groups, being both relatively elementary, yet sufficient to generate the mapping class group [17]. This motivates us to understand the asymptotic growth behavior of Dehn twist conjugacy class subgroups and pseudo-Anosov conjugacy class subgroups.

I.2 Notations

Throughout this paper we let $S_{g,n}$ denote a closed surface of genus g with n punctures such that $3g-3+n>0$, and we let $\mathrm{Mod}_{g,n}$, $(\mathcal{T}_{g,n}, d_{\mathcal{T}})$ and $\mathcal{M}_{g,n}$ denote the corresponding mapping class group, Teichmüller space with Teichmüller metric, and moduli space respectively. We use $h = 6g+2n-6$ to denote the dimension of $\mathcal{T}_{g,n}$. For any $\epsilon > 0$, we denote $\mathcal{T}^{\epsilon}_{g,n}$ the ϵ-thick part of $\mathcal{T}_{g,n}$. By saying α is a simple closed curve on $S_{g,n}$, we mean it's a non-trivial isotopy class of essential simple closed curves on $S_{g,n}$. We say the simple closed curve α is separating if the surface has two connected components after removing α. Otherwise, the surface only has one connected component after removing α, and we say α is a non-separating curve.

A multicurve α is a formal sum $\alpha = \sum_{i=1}^{k} a_i\alpha_i$ where $1 \leq k \leq \frac{h}{2}$, $a_i \in \mathbb{Z}\setminus\{0\}$, and α_i are pairwise disjoint simple closed curves on $S_{g,n}$. By this definition, simple closed curves are also multicurves. Let $\mathcal{ML}(\mathbb{Z})$ denote the set of multicurves on $S_{g,n}$ and let $\mathcal{S} \subset \mathcal{ML}(\mathbb{Z})$ denote the set of all simple closed curves. A multicurve $\alpha = \sum_{i=1}^{k} a_i\alpha_i$ is said to be positive if all coefficients are positive and is said to be negative if all coefficients are negative. Otherwise, we say α is of mixed sign. Two multicurves are of the same topological type if up to isotopy, there is an orientation-preserving homeomorphism taking one multicurve to another. For any $\gamma \in \mathcal{ML}(\mathbb{Z})$, we denote the multicurves of topological type γ by $\mathcal{ML}(\gamma)$. Since there are only finitely many topological types of simple closed curves on $S_{g,n}$, $\mathcal{S}$ is a finite union of sets of the form $\mathcal{ML}(\gamma)$, where γ are simple closed curves. Meanwhile, there are infinitely many topological types of multicurves, as can be seen by looking at the coefficients.

For reasons we will see, let's denote $\mathcal{ML}^*(\mathbb{Z})$ the set of multicurves $\alpha = \sum_{i=1}^{k} a_i\alpha_i$ satisfying one of the following conditions.

1. α is a weighted or unweighted simple closed curve, i.e., $k=1$.
2. α is positive or negative, i.e., all coefficients have the same sign.
3. α is of mixed sign where each $|a_i| \geq 2$.

For any simple closed curve α we let T_α denote the Dehn twist around α. In general, for any multicurve $\alpha = \sum_{i=1}^{k} a_i\alpha_i$, we define $T_\alpha = \prod_{i=1}^{k} T_{\alpha_i}^{a_i}$ and we call this a multi-twist. By this definition, Dehn twists are also multi-twists, and let's call them as twists in general. We will consider the following types subsets of $\mathrm{Mod}_{g,n}$ consisting of twists:

1. $[T_\gamma] = \{T_\alpha \mid \alpha \in \mathcal{ML}(\gamma)\} = \{fT_\gamma f^{-1} \mid f \in \mathrm{Mod}_{g,n}\}$, the set of twists about curves in $\mathcal{ML}(\gamma)$ or, equivalently, the conjugacy class of T_γ.
2. $\mathbf{D} = \{T_\alpha \mid \alpha \in \mathcal{S}\}$, the set of all Dehn twists without powers. $\mathbf{D}$ is a finite union of sets of the form $[T_\gamma]$, where γ are simple closed curves.

3. $\mathbf{M} = \{T_\alpha^k \mid \alpha \in \mathcal{S}, k \in \mathbb{Z}\}$, the set of all Dehn twists with any powers. $\mathbf{M}$ is a finite union of $\mathbf{M}_\gamma$, where each $\mathbf{M}_\gamma = \bigsqcup_{k\in\mathbb{Z}}[T_\gamma^k]$ is the infinite union of conjugacy classes of $T_\gamma^k, k \in \mathbb{Z}$.

4. $\mathbf{T} = \{T_\alpha \mid \alpha \in \mathcal{ML}(\mathbb{Z})\}$, the set of all twists.

We now introduce some notations. Let $A > 0$.

1. We say $f(x) \overset{+A}{\asymp} g(x)$ if $g(x) - A \leq f(x) \leq g(x) + A$ for any x.

2. We say $f(x) \overset{*A}{\asymp} g(x)$ if $\frac{1}{A} \cdot g(x) \leq f(x) \leq A \cdot g(x)$ for any x.

3. We say $f(R) \overset{*A}{\preceq} g(R)$ if for any $\lambda > 1$, there exists a $M(\lambda)$ such that $\frac{1}{\lambda A} \cdot f(R) \leq g(R)$ for any $R \geq M(\lambda)$.

4. We say $f(R) \overset{*A}{\sim} g(R)$ if $f(R) \overset{*A}{\preceq} g(R)$ and $g(R) \overset{*A}{\preceq} f(R)$.

Moreover, we say f, g are coarsely asymptotic if $f(R) \overset{*A}{\sim} g(R)$ for some coefficient A. Notice the notation $f(R) \sim g(R)$ is the same as $f(R) \overset{*1}{\sim} g(R)$, i.e. f, g are asymptotic when they are coarsely asymptotic with coefficient 1. Accordingly, we simply write $\preceq, \sim$ when $A = 1$.

I.3 Main Results

For any mapping class $\phi \in \mathrm{Mod}_{g,n}$, we use $[\phi] = \{f\phi f^{-1} \mid f \in \mathrm{Mod}_{g,n}\}$ to denote its conjugacy class. For simplicity of notation, we denote $\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi) = |[\phi] \cdot \mathcal{Y} \cap B_R(\mathcal{X})|$. The main results about the pseudo-Anosov conjugacy class subgroups are the followings.

Theorem A. Fix $S_{g,n}$ and $\epsilon > 0$, there exists a constant $A > 0$ such that given any ϵ-thick pseudo-Anosov element ϕ with translation distance $\lambda \geq A$ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$, there exists a corresponding $G(\mathcal{X}, \mathcal{Y}, \phi)$ such that

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi) \overset{G(\mathcal{X},\mathcal{Y},\phi)}{\sim} e^{\frac{h}{2}R}.$$

Corollary B. Fix $S_{g,n}$, given any pseudo-Anosov element ϕ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$. There exists a power N depending on ϕ such that for any $k \geq N$, there is a corresponding $G(\mathcal{X}, \mathcal{Y}, \phi, k)$ so that the following holds:

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) \overset{G(\mathcal{X},\mathcal{Y},\phi,k)}{\sim} e^{\frac{h}{2}R}.$$

In parallel with the Theorem I.1.3 above, we note the above Theorem A and Corollary B imply the following.

Corollary C. Fix $S_{g,n}$, given any pseudo-Anosov element ϕ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$, for all sufficiently large k we have

$$\lim_{R\to\infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) = \frac{h}{2}.$$

To state the main results about Dehn twist conjugacy class subgroups, recall $\mathcal{T}^{\epsilon}_{g,n}$ denotes the ϵ-thick part of $\mathcal{T}_{g,n}$ and $h = 6g - 6 + 2n$ denotes the dimension of $\mathcal{T}_{g,n}$. For any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}$, we define $F(\mathcal{X}, \mathcal{Y}) = e^{\frac{h}{2} d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})}$. For any multicurve $\alpha = \sum_{i=1}^{k} a_i\alpha_i$, we denote the sum of absolute coefficients as $c_\alpha = \sum_{i=1}^{k} |a_i|$ and we define $F_\alpha(\mathcal{X}, \mathcal{Y}) = (c_\alpha)^{\frac{h}{2}} e^{\frac{h}{2} d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})}$. The theorem below gives coarse asymptotics for $[T_\gamma]$ when $\gamma \in \mathcal{ML}^*(\mathbb{Z})$.

Theorem D. Given any $S_{g,n}$ and given any $\epsilon > 0$, there exists a $J > 0$ such that for any multicurve $\gamma \in \mathcal{ML}^*(\mathbb{Z})$ and for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}^{\epsilon}_{g,n}$, we have

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, T_\gamma) \overset{*JF_\gamma(\mathcal{X},\mathcal{Y})}{\sim} n_X(\gamma) \cdot e^{\frac{h}{2}R}$$

where $n_X(\gamma)$ is the corresponding Mirzakhani constant, see section II.7.

The above results says for example, the number of Dehn twist lattice points intersecting a closed ball of radius R in the Teichmüller space is coarsely asymptotic to $e^{\frac{h}{2}R}$. Note that any $\mathcal{X} \in \mathcal{T}_{g,n}$ lies in $\mathcal{T}^{\epsilon}_{g,n}$ for some ϵ, thus another way to phrase the theorem is by picking $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}$ first and then by picking any $\epsilon > 0$ such that $\mathcal{X}, \mathcal{Y} \in \mathcal{T}^{\epsilon}_{g,n}$. The constant J and the above results follow.

For simplicity of notation, we denote $\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{D}) = |\mathbf{D} \cdot \mathcal{Y} \cap B_R(\mathcal{X})|$, and similarly for the sets $\mathbf{M}, \mathbf{T}, \mathbf{T}([\underline{\gamma}])$ which we will see later.

Corollary E. Given $S_{g,n}$ and given any $\epsilon > 0$, for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}^{\epsilon}_{g,n}$, we have

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{D}) \overset{*JF(\mathcal{X},\mathcal{Y})}{\sim} n_X(\mathcal{S}) \cdot e^{\frac{h}{2}R}, \text{ if } h > 0,$$
$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{M}) \overset{*8JF(\mathcal{X},\mathcal{Y})}{\sim} n_X(\mathcal{S}) \cdot e^{\frac{h}{2}R}, \text{ if } \frac{h}{2} > 1$$

where $n_X(\mathcal{S})$ is the corresponding Mirzakhani constant, see section II.7.

We remark that when $\frac{h}{2} = 1$, $\mathcal{ML}(\mathbb{Z})$ is one dimensional and $\mathbf{M} = \mathbf{T}$. The coarse asymptotic for $\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{M})$ when $\frac{h}{2} = 1$ is separated out as a special case and treated in Corollary I.

In parallel with the Theorem I.1.3 and Corollary C above, Theorem D and Corollary E imply the following results.

Corollary F. Given any $S_{g,n}$ and given any $\epsilon > 0$, for any multicurve $\gamma \in \mathcal{ML}^*(\mathbb{Z})$ and for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}^{\epsilon}_{g,n}$, we have

$$\lim_{R\to\infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, T_\gamma) = \frac{h}{2}.$$
$$\lim_{R\to\infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{D}) = \frac{h}{2}, \text{ if } h > 0.$$
$$\lim_{R\to\infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{M}) = \frac{h}{2}, \text{ if } \frac{h}{2} > 1.$$

Our argument hinges on studying how the length of any simple closed geodesic τ on a hyperbolic structure $\mathcal{X}$ changes after applying a twist T_α. To this end, in Theorem IV.1.2 we obtain an explicit bound on the length of $\ell_{T_\alpha \mathcal{X}}(\tau)$ in terms of $\ell_{\mathcal{X}}(\tau)$, $\ell_{\mathcal{X}}(\alpha)$ and the intersection patterns between τ and α, up to additive error. We then use this Theorem IV.1.2, together with results of Choi, Rafi [6] and Lenzhen, Rafi, Tao [23], to realize a precise relationship between $\ell_{\mathcal{X}}(\alpha)$ and $d_{\mathcal{T}}(\mathcal{X}, T_\alpha \mathcal{X})$. This relation is stated in the following theorem.

Theorem G (Coarse Distance Formula)**.** Fix some $S_{g,n}$ and given any $\epsilon > 0$, there exists a constant $H > 0$ such that given any $\mathcal{X} \in \mathcal{T}^{\epsilon}_{g,n}$, we have

$$d_{\mathcal{T}}(\mathcal{X}, T_\alpha \mathcal{X}) \overset{+H}{\asymp} \log\left(\sum_{i=1}^{k} |a_i| \ell^2_{\mathcal{X}}(\alpha_i)\right)$$

for any $\alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}^*(\mathbb{Z})$.

Notice the above result only holds for $\alpha \in \mathcal{ML}^*(\mathbb{Z})$. Indeed, we have constructed a sequence of multicurves in $\mathcal{ML}(\mathbb{Z}) \setminus \mathcal{ML}^*(\mathbb{Z})$ for which Theorem G does not hold, see Remark IV.2.2. There exists a $H' > 0$ depends on $S_{g,n}$ and ϵ, so that for these multicurves the distances behave like

$$d_{\mathcal{T}}(\mathcal{X}, T_\alpha \mathcal{X}) \overset{+H'}{\asymp} \log\left(\sum_{i=1}^{k} |a_i| \ell_{\mathcal{X}}(\alpha_i)\right).$$

This leads to some interesting questions and motivates our Conjecture J in the last section.

Let $\underline{\gamma} = \sum_{i=1}^{k} \gamma_i$ denote a multicurve with all coefficients equal to one and of maximal dimension $k = \frac{h}{2}$. We say $\gamma = \sum_{i=1}^{k} a_i \gamma_i \in [\underline{\gamma}]$ if γ and $\underline{\gamma}$ are the same when without coefficients. Let's denote

$$\mathcal{ML}\left([\underline{\gamma}]\right) = \bigsqcup_{\gamma \in [\underline{\gamma}]} \mathcal{ML}(\gamma).$$

Notice $\mathcal{ML}([\underline{\gamma}])$ consists of infinity many conjugacy classes of multicurves. Let $\mathbf{T}([\underline{\gamma}])$ denote the set of all twists around curves in $\mathcal{ML}([\underline{\gamma}])$.

Theorem H. Given any $S_{g,n}$ such that $h > 0$, $\epsilon > 0$, and $\underline{\gamma} = \sum_{i=1}^{k} \gamma_i$ a multicurve with all coefficients equal to one and of maximal dimension $k = \frac{h}{2}$. There exists a number $f(\underline{\gamma})$ such that, for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}^{\epsilon}$,

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T}) \geq \Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T}([\underline{\gamma}])) \overset{*JF(\mathcal{X},\mathcal{Y})}{\succeq} f(\underline{\gamma}) \cdot R \cdot e^{\frac{h}{2}R}.$$

In particular, we can consider the case $\frac{h}{2} = 3g - 3 + n = 1$, where $S_{g,n}$ is either $S_{1,1}$ or $S_{0,4}$, and $\mathrm{Mod}_{g,n}, \mathcal{T}_{g,n}$ are $\mathrm{SL}_2(\mathbb{Z}), \mathbb{H}^2$ respectively. In this case, $\mathcal{ML}(\mathbb{Z})$ is one dimensional and we have $\mathcal{ML}(\mathbb{Z}) = \mathcal{ML}([\underline{\gamma}])$ for any simple closed curve $\underline{\gamma}$. In correspondence, $\mathbf{T} = \mathbf{M} = \mathbf{T}([\underline{\gamma}])$ is the set of all parabolic elements of $\mathrm{SL}_2(\mathbb{Z})$. There are many results about the asymptotic growth of lattice points in $\mathbb{H}^2$, see [16], [36] for example. The corollary below can also be interpreted as a coarse asymptotic for the number of parabolic lattice points of $\mathrm{SL}_2(\mathbb{Z})$ intersecting a closed ball of radius R in $\mathbb{H}^2$.

Corollary I. Given $S_{g,n}$ equal to $S_{1,1}$ or $S_{0,4}$ and given any $\epsilon > 0$. For any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}^{\epsilon}$, we have

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T}) \overset{*4JF(\mathcal{X},\mathcal{Y})}{\sim} n_X(\mathcal{S}) \cdot R \cdot e^{R}.$$

The upper bound in this Corollary follows from an alternation of the proof of Corollary E, see section IV.3, and the lower bound follows from previous Theorem H, see section IV.4.

Recall that in the ϵ-thick part of Teichmüller space, there is a uniformly bounded difference, depending on ϵ, between the Thurston metric and Teichmüller metric [23]. Thus many above results also hold for the Thurston metric after a slight variation.

In conclusion, these results again indicate the similarity of Teichmüller spaces and hyperbolic spaces in terms of lattice point asymptotics. In the meantime, we propose the Conjecture J below. If this holds true, in comparison with Theorem I.1.3, it shows a difference of Teichmüller spaces and hyperbolic spaces in terms of lattice point asymptotics. See section V for some open questions and an outlined idea about Conjecture J.

Conjecture J. Given any $S_{g,n}$ and given any $\epsilon > 0$, there exists a multicurve $\gamma \in \mathcal{ML}(\mathbb{Z})$ and $l > \frac{h}{2}$ such that for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}^{\epsilon}$, we have

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, T_\gamma) \overset{*JF_\gamma(\mathcal{X},\mathcal{Y})}{\sim} n_X(\gamma) \cdot e^{lR}.$$

As a result, we have

$$\lim_{R \to \infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, T_\gamma) = l > \frac{h}{2}.$$

CHAPTER II

Background

By saying a surface we mean a 2-dimensional topological manifold that is compact, connected and oriented. We say a surface is closed if it is compact and have no boundary. Let $S_{g,n}^b$ denote the surface of genus g with b boundary components and n punctures, the fundamental result from Mobius [42] states that every surface S is homeomorphic to $S_{g,n}^b$ for some $b, g, n \geq 0$. The Euler characteristic of such surface is $\chi(S) = 2 - 2g - (b + n)$. We say a surface is hyperbolic if $\chi(S) < 0$. In which case, there exists a complete, finite-area Riemannnian metric on S of constant curvature -1 where its boundaries, if non-empty, are totally geodesic. If it's a surface with any boundary components, then its universal cover $\tilde{S}$ is a simply connected Riemannnian 2-dimensional manifold of constant curvature -1, thus is isometric to the hyperbolic plane $\mathbb{H}^2$ by the Killing-Hopf Theorem [15] [21]. It follows that S is isometric to the quotient of $\mathbb{H}^2$ by a freely, properly discontinuous isometric action of $\pi_1(S)$.

In this thesis, we restrict ourselves to $S_{g,n}$ such that $3g - 3 + n > 0$, i.e., hyperbolic surfaces without any boundary components. We refer the reader to [9] for more background materials.

II.1 Mapping Class Group

Let $\text{Homeo}_{g,n}^+$ denote the group of all the orientation-preserving homeomorphisms of $S_{g,n}$ preserving the set of punctures, and let $\text{Homeo}_{g,n}^0$ denote the connected component of the identity. The mapping class group of $S_{g,n}$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms:

$$\mathrm{Mod}_{g,n} = \text{Homeo}_{g,n}^+/\text{Homeo}_{g,n}^0 = \text{Homeo}_{g,n}^+/\text{ isotopy}$$

Let $A = S^1 \times [0, 1]$ be an oriented annulus, the twist map $T\colon A \to A$ is defined to be $(\theta, t) \mapsto (\theta + 2\pi t, t)$, so T is a homeomorphism of A relative to its boundary. Let a be a representative of a simple closed curve α on $S_{g,n}$ and let N be a regular neighborhood of a. Pick some orientation-preserving homeomorphism $\phi\colon A \to N$, the Dehn twist about a is defined by

$$T_a(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) \text{ if } x \in N \\ \\ x \text{ if } x \in S_{g,n} \setminus N \end{cases} .$$

The isotopy class of T_a does not depend on choice of a in α. Thus T_α is an well-defined mapping class. If α is homotopically trivial, then T_α is trivial as well. Now given any multicurve $\alpha = \sum_{i=1}^k a_i\alpha_i$, the composition

$T_\alpha = \prod_{i=1}^k T_{\alpha_i}^{a_i}$ is called a multi-twist.

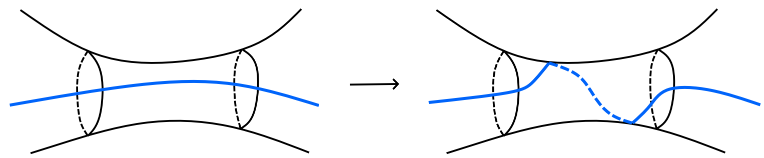

Figure II.1: Before and after a Dehn twist.

As we mentioned in the introduction, Dehn twists are in a sense the most fundamental elements of mapping class groups, see the well-known result below.

Theorem II.1.1 (Dehn-Lickorish Theorem [24])**.** *For $g \geq 0$, the mapping class group $\mathrm{Mod}(S_g)$ is generated by finitely many Dehn twists about non-separating simple closed curves.*

Moreover, Humphries [18] showed we need at least $2g+1$ such curves to generated the mapping class groups. Since Dehn twists are supported on simple closed curves, it's natural to study the action of Dehn twists on simple closed curves. One way to achieve that is via studying intersection numbers.

Given two simple closed curves α, β, the intersection number $i(\alpha, \beta)$ is defined to be $i(\alpha, \beta) = \min |a \cap b|$ where a, b are in the isotopy classes α, β respectively and $|a \cap b|$ denotes how many times a and b intersect. The following proposition of Ivanov shows how twists effect intersection numbers.

Proposition II.1.2 (Intersection Formula [19])**.** *Let $\alpha = \sum_{i=1}^k a_i \alpha_i$ be a multicurve on $S_{g,n}$, and $T_\alpha = \prod_{i=1}^k T_{\alpha_i}^{a_i}$ the corresponding twist. Given β, γ arbitrary simple closed curves on $S_{g,n}$. If α is positive or negative, we have*

$$\left| i(T_\alpha(\beta), \gamma) - \sum_{i=1}^n |k_i| i(\alpha_i, \beta) i(\alpha_i, \gamma) \right| \leq i(\beta, \gamma). \tag{II.1}$$

If α is of mixed sign, we have

$$\begin{aligned} &\sum_{i=1}^n (|k_i| - 2) i(\alpha_i, \beta) i(\alpha_i, \gamma) - i(\beta, \gamma) \\ &\leq i(T_\alpha(\beta), \gamma) \\ &\leq \sum_{i=1}^n |k_i| i(\alpha_i, \beta) i(\alpha_i, \gamma) + i(\beta, \gamma). \end{aligned} \tag{II.2}$$

Via the study of intersection numbers, one can show Dehn twists are infinite order elements. Dehn twists are classified as reducible elements according to Nielsen and Thurston. See their complete classification of mapping class group elements below.

Theorem II.1.3 (Nielsen-Thurston Classification [43]). *Given $f \in \mathrm{Mod}_{g,n}$, at least one of the following holds:*

1. *f is periodic, and some power of f is identity.*
2. *f is reducible, and f preserves some multicurve.*
3. *f is pseudo-Anosov, if it's not periodic nor reducible.*

Note one usually defines the concept of pseudo-Anosov relying on the notion of measured foliation introduced by Thurston [43]. As we are interested in the "hyperbolicty" of mapping class group, one can compare the Nielsen-Thurston Classification with the Classification of Hyperbolic Isometries, Theorem II.4.1. We will restate the Nielsen-Thurston Classification again in the next section, after introducing Teichmüller Space. One should observe the concepts of perodic, reducible, and pseudo-Anosov are reminiscent of the concepts of elliptic, parabolic, and hyperbolic respectively.

II.2 Teichmüller Space and Moduli Space

A hyperbolic structure $\mathcal{X}$ on $S_{g,n}$ is a pair (X, ϕ) where $\phi\colon S_{g,n} \to X$ is a homeomorphism and X is a hyperbolic surface. We say two hyperbolic structures $\mathcal{X} = (X, \phi), \mathcal{Y} = (Y, \psi)$ are isotopic if there is an isometry $I\colon X \to Y$ isotopic to $\psi \circ \phi^{-1}$. The Teichmüller space $\mathcal{T}_{g,n}$ is the set of hyperbolic structures on $S_{g,n}$ modulo isotopy. We let $\mathcal{X} = (X, \phi), \mathcal{Y} = (Y, \psi)$ denote elements in $\mathcal{T}_{g,n}$. One can also define the Teichmüller space via complex structures (marked Riemann surfaces) or view it as a representation space, see [9] for more interpretations.

Given any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}$, the Teichmüller distance between them is defined to be

$$d_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \inf_{f \sim \phi \circ \psi^{-1}} \log(K_f)$$

where the infimum is over all quasi-conformal homeomorphisms f isotopic to $\phi \circ \psi^{-1}$ and K_f is the quasi-conformal dilatation of f. Equipped with the Teichmüller metric, the Teichmüller space is a complete, unique geodesic metric space.

The mapping class group acts isometrically on $\mathcal{T}_{g,n}$ by changing the marking $(f, (X, \phi)) \mapsto (X, \phi \circ f^{-1})$. This action is properly discontinuous but not cocompact. The quotient $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\mathrm{Mod}_{g,n}$ is called the moduli space, and it is a non-compact orbifold parameterizing hyperbolic surfaces homeomorphic to $S_{g,n}$.

Given any $\mathcal{X} = (X, \phi) \in \mathcal{T}_{g,n}$ and given any isotopy class γ of nontrivial simple closed curves on X, there exists a unique geodesic in this free homotopy class. We define the length function on X by setting

$\ell_X(\gamma)$ equal to the length of this unique geodesic. We also let $\ell_{\mathcal{X}}(\alpha)$ denote $\ell_X(\phi(\alpha))$ for any simple closed curve α on $S_{g,n}$. For any multicurve $\alpha = \sum_{i=1}^{k} a_i\alpha_i$, we define $\ell_{\mathcal{X}}(\alpha) = \sum_{i=1}^{k} |a_i|\ell_{\mathcal{X}}(\alpha_i)$ to be its length.

A pair of pants is a closed surface of zero genus with three boundary components or punctures. A pants decomposition Γ of the surface $S_{g,n}$ is a collection of pairwise disjoint non-trivial simple closed curves $\gamma_1, \cdots, \gamma_{3g-3+n}$ on $S_{g,n}$, together they decompose the surface $S_{g,n}$ into $2g+n-2$ pairs of pants. Using pants decomposition and by introducing Fenchel-Nielsen coordinates, Fricke [11] showed that $\mathcal{T}_{g,n}$ is homeomorphic to $\mathbb{R}^{6g+2n-6}$ and its dimension is $6g+2n-6$.

By studying projective measured foliations and projective space of functionals on simple closed curves on the surfaces, one can define the Thurston boundary [10] of $\mathcal{T}_{g,n}$ and realize this boundary is homeomorphic to a sphere of dimension $6g+2n-7$. The action of $\mathrm{Mod}_{g,n}$ on the $\mathcal{T}_{g,n}$ extends continuously over the union with the boundary, in analogy with $\mathrm{Isom}^+(\mathbb{H}^2)$ acting on $\mathbb{H}^2$. We restate the Nielsen-Thurston Classification again, in comparison with the Classification of Hyperbolic Isometries, Theorem II.4.1.

Theorem II.2.1 (Nielsen-Thurston Classification [43]). *Given $f \in \mathrm{Mod}_{g,n}$, at least one of the following holds:*

1. *f is periodic. There is a single fixed point $\mathcal{X} \in \mathcal{T}_{g,n}$.*
2. *f is reducible. There could be a fixed point, corresponding to the fixed multicurve under f, in the Thurston boundary of $\mathcal{T}_{g,n}$.*
3. *f is pseudo-Anosov. It fixed a pair of points in the Thurston boundary, and these points correspond to stable and unstable foliations of $S_{g,n}$ preserved by f. Moreover, there is a stretch factor $\lambda_f > 1$ so f acts by stretching stable foliation by $\frac{1}{\lambda_f}$ and by stretching unstable foliation by λ_f.*

A theorem of Bers [4] says there exists a constant depending only on $S_{g,n}$ such that for every $\mathcal{X} \in \mathcal{T}_{g,n}$, there is a pants decomposition $\Gamma_{\mathcal{X}}$ of $\mathcal{X}$ in which each simple closed curve has length bounded above by this Bers' constant.

Given any $\epsilon > 0$, the ϵ-thick part of Teichmüller space is defined to be

$$\mathcal{T}_{g,n}^{\epsilon} = \{\mathcal{X} \in \mathcal{T}_{g,n} \mid \ell_{\mathcal{X}}(\alpha) \geq \epsilon \text{ for any simple closed curve } \alpha \text{ on } S_{g,n}\}$$

and consequently the ϵ-thick part of moduli space is $\mathcal{M}_{g,n}^{\epsilon} = \mathcal{T}_{g,n}^{\epsilon}/\mathrm{Mod}_{g,n}$. The Mumford compactness criterion [33] says $\mathcal{M}_{g,n}^{\epsilon}$ is compact for any $\epsilon > 0$.

II.3 Short Marking

For any $\mathcal{X} \in \mathcal{T}_{g,n}$, a short marking [6] $\mu_{\mathcal{X}}$ is a collection of simple closed curves $\{\eta_i\}_{i=1}^{3g-3+n} \cup \{\delta_i\}_{i=1}^{3g-3+n}$ on $S_{g,n}$ picked in the following way: First, choose a pant decomposition $\{\eta_i\}_{i=1}^{3g-3+n}$ by taking a curve η_1 on $S_{g,n}$ that is a shortest curve with respect to $\mathcal{X}$, and then a next shortest disjoint curve from the first, and so on until we complete a pants decomposition. Next, for each η_i, pick a shortest curve δ_i that intersects η_i and is disjoint from all other pants curves. For each i, we say η_i, δ_i is a pair. The collection of curves obtained in this way has the property that any two curves have intersection number bounded by 2. Note there could be a finite number of possible short markings corresponding to each $\mathcal{X} \in \mathcal{T}_{g,n}$, we fix one such short marking and call it the short marking $\mu_{\mathcal{X}}$. Moreover, given any $\epsilon > 0$, by Bers' Theorem and trigonometry, there exists $N > 0$ depending on ϵ and $S_{g,n}$ such that for any $\mathcal{X} \in \mathcal{T}_{g,n}^{\epsilon}$, all curves in the short marking $\mu_{\mathcal{X}}$ have length bounded above by N and bounded below by ϵ.

We recall a result from Choi and Rafi [6] stating that for any $\epsilon > 0$, the Teichmüller distance in the ϵ-thick part can be approximated by the maximum ratio of change of lengths of the short marking.

Theorem II.3.1 (Distance Formula [6])**.** *For any $\epsilon > 0$, there exists $c > 0$ depending on $S_{g,n}$ and ϵ such that for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}^{\epsilon}$*

$$d_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) \stackrel{+c}{\asymp} \log \max_{\gamma \in \mu_{\mathcal{X}}} \frac{\ell_{\mathcal{Y}}(\gamma)}{\ell_{\mathcal{X}}(\gamma)}.$$

We also recall that Lenzhen, Rafi, Tao [23] showed that for any simple closed curve on $S_{g,n}$, its length with respect to $\mathcal{X}$ can be estimated via its intersection pattern with the short marking $\mu_{\mathcal{X}}$.

Proposition II.3.2 (Length Formula [23])**.** *There exists $C \geq 1$ depending on $S_{g,n}$ such that for any simple closed curve β on $S_{g,n}$ and for any $\mathcal{X} \in \mathcal{T}_{g,n}$, we have*

$$\ell_{\mathcal{X}}(\beta) \stackrel{*C}{\asymp} \sum_{\gamma \in \mu_{\mathcal{X}}} i(\beta, \gamma) \ell_{\mathcal{X}}(\bar{\gamma})$$

where $\bar{\gamma}$ denotes the curve in the short marking paired with γ.

For a fixed ϵ, any curve γ in $\mu_{\mathcal{X}}, \mathcal{X} \in \mathcal{T}_{g,n}^{\epsilon}$ satisfies $\epsilon \leq \ell_{\mathcal{X}}(\gamma) \leq N$. We can therefore rewrite the above theorem and proposition for $\mathcal{T}_{g,n}^{\epsilon}$.

Lemma II.3.3. *For any $S_{g,n}$ and $\epsilon > 0$, there exists C depends on $S_{g,n}$ and c, N depends on $S_{g,n}$ and ϵ such that*

$$\log\left(\frac{1}{Ne^c} \max_{\gamma \in \mu_{\mathcal{X}}} \ell_{\mathcal{Y}}(\gamma)\right) \leq d_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) \leq \log\left(\frac{e^c}{\epsilon} \max_{\gamma \in \mu_{\mathcal{X}}} \ell_{\mathcal{Y}}(\gamma)\right) \tag{II.3}$$

$$\frac{\epsilon}{C} \sum_{\gamma \in \mu_{\mathcal{X}}} i(\beta, \gamma) \leq \ell_{\mathcal{X}}(\beta) \leq CN \sum_{\gamma \in \mu_{\mathcal{X}}} i(\beta, \gamma) \tag{II.4}$$

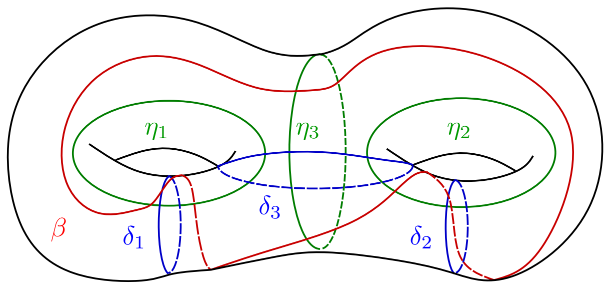

Figure II.2: A short marking $\mu_{\mathcal{X}} = \{\eta_1, \eta_2, \eta_3, \delta_1, \delta_2, \delta_3\}$ and a simple closed curve β on a hyperbolic surface X homeomorphic to S_2.

II.4 Upper Half Plane

Let $\mathbb{H}^2$ denote the upper half plane and let d denote the metric on $\mathbb{H}^2$ in the sense of $\mathbb{H}^2$ being a metric space. One common model of $\mathbb{H}^2$ is $\{x+iy \in \mathbb{C} \mid y > 0\}$ equipped with the Riemannnian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. In this model, geodesic are semicircles and half-lines perpendicular to the real axis intersect with the upper half plane. Another common model is the Poincaré disk $\{x+iy \in \mathbb{C} \mid r = x^2+y^2 < 1\}$ with the Riemannnian metric $ds^2 = 4\frac{dx^2+dy^2}{(1-r^2)^2}$. In this model, geodesic are semicircles and lines perpendicular to the circle intersect with the open disk. Recall by the Killing-Hopf Theorem [21] [15], upper half plane is the universal cover of hyperbolic surfaces without boundaries.

The group of orientation-preserving isometries of $\mathbb{H}^2$ is the group of Mobius transformation taking $\mathbb{H}^2$ to itself. This group is denoted as $\mathrm{Isom}^+(\mathbb{H}^2)$ and is isomorphic to

$$\mathrm{PSL}_2(\mathbb{R}) = \{\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{R}, ad - bc = 1\}.$$

Any such element acts on the upper half-plane by $z \mapsto \frac{az+b}{cz+d}$. Moreover, each $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ extends to a homeomorphism $\bar{f}$ on the closed Poincaré disk. It's well-known that we can apply the Browser Fixed Point Theorem and classify the non-trivial isometries of $\mathbb{H}^2$.

Theorem II.4.1 (Classification of Hyperbolic Isometries)**.** *Each $\bar{f}$ is of one of the following cases.*

1. *Elliptic. $\bar{f}$ has exactly one fixed point on $\mathbb{H}^2$, and f is a rotation around this point.*

   An example of such element would be $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, which is a rotation by π around i in the upper half-plane.

2. *Parabolic. $\bar{f}$ has exactly one fixed point on $\partial\mathbb{H}^2$.*

An example of such an element would be $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$*, a horizontal translation in the upper half-plane by* t*, fixing the boundary point* ∞*.*

3. *Hyperbolic.* $\bar{f}$ *has exactly two fixed point* a, b *on* $\partial\mathbb{H}^2$*, and* f *is a translation along the unique geodesic line* γ *from* a *to* b *with these two fixed points on the boundary acting like a source and a sink.*

   An example of such element would be $\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$ *, which is translation by length* $2\log(t)$ *along the vertical geodesic line* $(0,0)$ *to* ∞ *in the upper half-plane model.*

We observe that any two elements commuting means they must share the same fixed points, and vice versa. Moreover, since all the elements are isometries, the translation length is conjugate invariant.

We recall the following useful lemma in hyperbolic geometry.

Lemma II.4.2 (Collar Lemma [9])**.** *For any simple closed geodesic* γ *of length* ℓ *on a hyperbolic surface, it is contained in an embedded cylinder of diameter of order* ℓ^{-1}*, and the diameter is*

$$W(\gamma) = \sinh^{-1}\left(\frac{1}{\sinh(\frac{1}{2}\ell)}\right)$$

For any two closed sets $A, B \subset \mathbb{H}^2$ we let $d(A, B)$ denote the minimal distance between them. For any geodesic η in $\mathbb{H}^2$, we let π_η denote the closest point projection map, namely

$$\pi_\eta(x) = \{y \in \eta \mid d(x, y) = d(x, \eta)\}.$$

For any two points $x, y \in \mathbb{H}^2$, we let $[x, y]$ denote the unique geodesic connecting them. Given two points $x, y \in \mathbb{H}^2$ separated by a bi-infinite geodesic η and far away from η, we let $x_\eta \in \mathbb{H}^2$ denote the first point that the geodesic $[x, y]$ enters the L-neighborhood of η coming from the x side. Similarly, we can define $y_\eta \in \mathbb{H}^2$. If x is in the L-neighborhood of η to begin with, we just let $x_\eta = x$, and similarly for y.

Being a hyperbolic space, geodesics are strongly contracting in $\mathbb{H}^2$, see [1] for example. That is, there exists a constant L such that for any geodesic η and for any geodesic α that never enters the L-neighborhood of η, the diameter of $\pi_\eta(\alpha)$ is bounded by L. As a consequence, we have

Corollary II.4.3. *There exists a constant* L *such that for any bi-infinite geodesic* η *in* $\mathbb{H}^2$ *and for any two points* x, y *separated by* η*, we have*

$$d(x_\eta, \pi_\eta(x)) \leq 2L, d(y_\eta, \pi_\eta(y)) \leq 2L.$$

This is because we have $d(x_\eta, \pi_\eta(x)) \leq d(x_\eta, \pi_\eta(x_\eta)) + d(\pi_\eta(x_\eta), \pi_\eta(x)) \leq 2L$. Similarly for y_η.

Another important property of the projection map in $\mathbb{H}^2$ is that it's 1-Lipschitz. Viewing in the upper half plane model and up to isometry, we may assume η is the vertical line $x = 0$. For each point $(0, r) \in \eta$, the points projecting to $(0, r)$ are exactly the Euclidean semicircles of radius r centered at $(0, 0)$. Given two Euclidean semicircle centered at $(0, 0)$, the minimal distance between them are realized by the points intersecting the vertical line $x = 0$. This means

Lemma II.4.4. *π_η is 1-Lipschitz for any bi-infinite geodesic η in $\mathbb{H}^2$.*

II.5 Lifts of Twists

Given an oriented bi-infinite geodesic β in $\mathbb{H}^2$ and a number $l_\beta \in \mathbb{R}$, we can decompose $\mathbb{H}^2$ into two open pieces, one to the left of β and one to the right of β, and then regule the two pieces along β after translation according to l_β. When l_β is positive, we regule the pieces along β after translating distance $|l_\beta|$ to the left. When l_β is negative, we regule the pieces along β after translating distance $|l_\beta|$ to the right. This process is called shearing along β according to l_β, see [20] for more detail. We are mainly interested in what happens to geodesics after shearing. Let τ be a bi-infinite geodesic in $\mathbb{H}^2$ transverse to β and let τ' be the image of τ after shearing along β according to l_β, then τ' is a concatenation of two geodesic rays with a sub-segment of β of length l_β connecting these two rays' starting points, see Figure II.3 for an illustration in the Poincaré disk model.

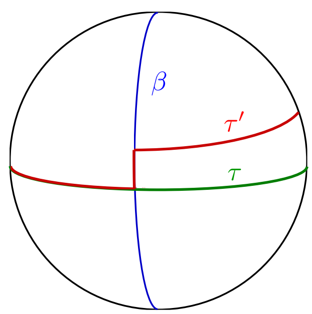

Figure II.3: After shearing along β according to l_β, τ becomes τ'.

Given $\mathcal{X} = (X, \phi) \in \mathcal{T}_{g,n}$ and let $p\colon \mathbb{H}^2 \to X$ be the universal cover. For any multicurve $\alpha = \sum_{i=1}^k a_i \alpha_i$, we let $\mathcal{A} = \{\alpha_i\}_{i=1}^k$ and let $\tilde{\mathcal{A}}$ denote the set of lifts of curves in $\mathcal{A}$. For each curve $\tilde{\alpha} \in \tilde{\mathcal{A}}$, we let $\alpha_{s(\tilde{\alpha})}$ denote the curve such that $\tilde{\alpha}$ is a lift of $\alpha_{s(\tilde{\alpha})}$. Note the complements of $\cup_{\tilde{\alpha} \in \tilde{\mathcal{A}}} \tilde{\alpha}$ are infinitely many open regions. Fixing one of these regions, we can shear along all these bi-infinite geodesics in $\tilde{\mathcal{A}}$ according to $a_{s(\tilde{a})} \ell_{\mathcal{X}}(\alpha_{s(\tilde{\alpha})})$, and this is called shearing according to α.

Now, given any simple closed geodesic β on $\mathcal{X}$, we let τ be a lift of β with a base point $q_0 \in \tau$. Fixing the region containing q_0, we can shear according to α. Let τ' denote the image of τ after shearing, then the projection of τ' is isotopic to the simple closed geodesic $T_\alpha(\beta)$. Let $q'_L, q'_R \in \partial\mathbb{H}^2$ denote the endpoints of τ'. The two end points $q'_L, q'_R \in \partial\mathbb{H}^2$ define a unique bi-infinite geodesic σ in $\mathbb{H}^2$ and σ is in the same isotopy class of τ', see Figure II.4. This means σ is a lift of the simple closed geodesic $p(\sigma) = T_\alpha(\beta)$. Similarly, one can obtain the simple closed geodesic $T_\alpha^{-1}(\beta)$ by shearing in the opposite direction.

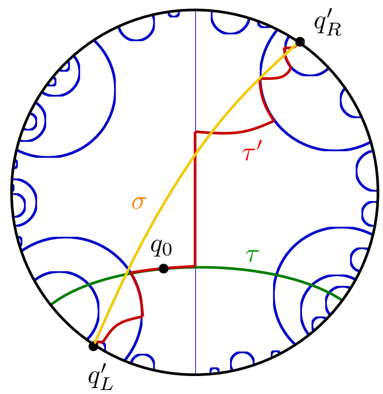

Figure II.4: After shearing according to α (blue curves are in $\tilde{A}$), the geodesic τ becomes τ', and the geodesic σ is uniquely defined by the endpoints of τ'.

For interested reader, what we are using here is a concrete case of an earthquake. There is a rich and developing theory about earthquake and earthquake flow, and proven to be a useful tool. For example, Kerckhoff [20] answered the Nielsen realization problem [34] using the notion of mapping class groups acting on surfaces using left earthquakes. To name a few more interesting results, see [44], [28], [35], [25], and [30].

II.6 Bass-Serre Tree

We briefly explain how to construct a Bass-Serre tree dual to an infinite collection of bi-infinite geodesics in $\mathbb{H}^2$ that arise from a covering map. In particular, one may imagine how to construct a Bass-Serre dual to the Figure II.4. See [41] for more detail about Bass-Serre trees in general, and see [7] for an example using Bass-Serre tree in $\mathbb{H}^2$.

Let $p\colon \mathbb{H}^2 \to S_{g,n}$ be a universal cover. Given $\mathcal{A} = \{\alpha_i\}_{i=1}^n$ a collection of disjoint simple closed curves on $S_{g,n}$, we let $\tilde{\mathcal{A}}$ denote the set of all liftings of curves in $\mathcal{A}$ to $\mathbb{H}^2$, and we let $\cup\tilde{\mathcal{A}}$ denote the union of all elements in $\tilde{\mathcal{A}}$. Define $\mathcal{Z}_{\mathcal{A}}$ to be the tree dual to $\tilde{\mathcal{A}}$ in $\mathbb{H}^2$. That is, $\mathcal{Z}_{\mathcal{A}} = (V_{\mathcal{A}}, E_{\mathcal{A}})$ is a graph such that each vertex in $V_{\mathcal{A}}$ corresponds to a connected component in $\mathbb{H}^2 \setminus \cup\tilde{\mathcal{A}}$ and each edge is dual to an element in $\tilde{\mathcal{A}}$. We label each edge by the element in $\tilde{\mathcal{A}}$ that it is dual to.

Denote the connected component corresponding to a vertex v as $C(v)$. Given two vertices $v, w \in V_{\mathcal{A}}$, $(v, w) \in E_{\mathcal{A}}$ if and only if $C(v), C(w)$ represent bordered connected components. Denote $d_{\mathcal{Z}}$ the metric on the tree $\mathcal{Z}_{\mathcal{A}}$ where the length of each edge has length 1, $(\mathcal{Z}_{\mathcal{A}}, d_{\mathcal{Z}})$ is a unique geodesic metric space.

By the Collar Lemma II.4.2, there exists a $r = \min\{W(\alpha_i)\}_{i=1}^n$ sufficiently small such that for any curve $\alpha \in \mathcal{A}$, $\mathcal{N}_r(\alpha)$ is an open annulus. We can define a π_1-equivariant, continuous and surjective map $\phi_{\mathcal{A}} \colon \mathbb{H}^2 \to \mathcal{Z}_{\mathcal{A}}$ such that each $\mathcal{N}_r(\tilde{\alpha})$ maps to an edge and each connected component in $\mathbb{H}^2 \setminus \cup_{\tilde{\alpha} \in \tilde{\mathcal{A}}} \mathcal{N}_r(\tilde{\alpha})$ gets mapped to a vertex.

Now, given any simple closed curve τ on $S_{g,n}$ and let $\tilde{\tau}$ be a lift of τ in $\mathbb{H}^2$. If τ does not intersect any curve in $\mathcal{A}$, then $\phi_{\mathcal{A}}(\tilde{\tau})$ is a vertex. Otherwise, denote

$$i(\tau, \mathcal{A}) = \sum_{i=1}^{n} i(\tau, \alpha_i)$$

the intersection number of τ with curves in $\mathcal{A}$, $\phi_{\mathcal{A}}(\tilde{\tau})$ is a bi-infinite geodesic in $(\mathcal{Z}_{\mathcal{A}}, d_{\mathcal{Z}})$. The hyperbolic isometry of $\mathbb{H}^2$ along $\tilde{\tau}$, with translation distance equals to the length of τ, is equivariant with respect to $\phi_{\mathcal{A}}$ and gives rise to an isometry $\rho_{\tilde{\tau}}$ of $(\mathcal{Z}_{\mathcal{A}}, d_{\mathcal{Z}})$ with translation length $i(\mathcal{A}, \tau)$ and translation axis $\phi_{\mathcal{A}}(\tilde{\tau})$. This means for any vertex s on the axis $\phi_{\mathcal{A}}(\tilde{\tau})$, we have $d_{\mathcal{Z}}(s, \rho_{\tilde{\tau}}(s)) = i(\tau, \mathcal{A})$.

II.7 Counting Simple Closed Geodesics

There are many interesting results about counting closed geodesics in a hyperbolic setting. Let $X \in \mathcal{M}_{g,n}$, we denote $c_X(L)$ the number of closed geodesics on X has length $\leq L$. one of the classical result from Delsarte, Huber, Selberg, and Margulis [27] showed that

$$c_X(L) \sim \frac{e^L}{L} \text{ as } L \to \infty.$$

Interestingly, $c_X(L)$ is independent of the topology and geometry of X. And there are many interesting results after, see [32], [39], [12], [40], [3] for more examples.

Since simple closed geodesics are generally non-generic among all closed geodesics [27], counting simple closed geodesics is expected to be a difficult question. Given γ a simple closed curve or multicurve on any $X \in \mathcal{M}_{g,n}$, we denote

$$s_X(L, \gamma) = |\{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma \mid \ell_X(\alpha) \leq L\}|$$

the number of simple closed geodesics on X of topological type γ and of hyperbolic length at most L. The following beautiful result is due to Mirzakhani.

Theorem II.7.1 (Counting Formula [31])**.** *Fix some $S_{g,n}$, given γ a simple closed curve or a multicurve on* *any $X \in \mathcal{M}_{g,n}$, we have*

$$s_X(L, \gamma) \sim n_X(\gamma) \cdot L^{6g+2n-6} \tag{II.5}$$

where $n_X(\gamma)$ depends on the hyperbolic structure X and the topological type of γ.

Later in the paper, we will count the sum of several topological types of multicurves. Thus we phrase the above Theorem II.7.1 in the following equivalent way.

Remark II.7.2. For any γ, X and $\lambda > 1$, there exist constants $n_X(\gamma)$ and $r_X(\gamma, \lambda)$ such that

$$\frac{1}{\lambda} \cdot n_X(\gamma) \cdot L^{6g+2n-6} \leq s_X(L, \gamma) \leq \lambda \cdot n_X(\gamma) \cdot L^{6g+2n-6} \tag{II.6}$$

for any $L \geq r_X(\gamma, \lambda)$.

It's also necessary for us to know how $n_X(\gamma)$ and $r_X(\gamma, \lambda)$ behave with respect to scaling the curve γ for later purposes.

Corollary II.7.3. *For any $\gamma, X, \lambda > 1$ and $c \in \mathbb{N}$, we have*

$$\begin{aligned} r_X(c \cdot \gamma, \lambda) &= c \cdot r_X(\gamma, \lambda) \\ n_X(c \cdot \gamma) &= \frac{n_X(\gamma)}{c^{6g+2n-6}} \end{aligned}$$

Proof. Indeed, we have

$$s_X(L, c \cdot \gamma) = s_X\left(\frac{L}{c}, \gamma\right)$$

and that

$$\frac{1}{\lambda} \cdot \frac{n_X(\gamma)}{c^{6g+2n-6}} \cdot L^{6g+2n-6} \leq s_X\left(\frac{L}{c}, \gamma\right) \leq \lambda \cdot \frac{n_X(\gamma)}{c^{6g+2n-6}} \cdot L^{6g+2n-6}$$

for any $\frac{L}{c} \geq r_X(\gamma, \lambda)$. This gives us the desired result. $\square$

Since there are only finitely many topological types of simple closed curves, we denote $n_X(\mathcal{S})$ the finite sum of $n_X(\gamma)$ where γ ranges over all topological types of simple closed curves on $S_{g,n}$. We will use the notation $s_X(L, \mathcal{S})$ to denote the number of all simple closed geodesics that have length bounded by L, and we will denote $r_X(\mathcal{S}, \lambda) = \max_{\gamma \in \mathcal{S}} r_X(\gamma, \lambda)$ for any $\lambda > 1$.

II.8 Projection Maps

Similar to hyperbolic isometrics acting on hyperbolic space, each pseudo-Anosov element $\phi \in \mathrm{Mod}_{g,n}$ acts on $\mathcal{T}_{g,n}$ by translating along its corresponding bi-infinite geodesic axis, denoted as $\mathrm{axis}(\phi)$ with translation distance denoted as $\lambda(\phi)$. Moreover, we say a pseudo-Anosov element $\phi \in \mathrm{Mod}_{g,n}$ is called ϵ-thick if its axis $\mathrm{axis}(\phi) \subset \mathcal{T}^{\epsilon}_{g,n}$.

For any $r > 0$ and for every closed set $W \subset \mathcal{T}_{g,n}$, denote $\mathcal{N}_r(W)$ the r-neighborhood of W. For every closed set $C \subset \mathcal{T}_{g,n}$, the closest point projection map is defined as follows

$$\pi_C(x) = \{y \in C \mid d(x,y) = d(x,C) = \inf_{z \in C} d(x,z)\}.$$

As one of the early works exploring negative curvature in Teichmüller space, the result below from Minsky [29] says that ϵ-thick geodesics in Teichmüller space satisfy the strongly contracting property.

Theorem II.8.1 (Minsky [29])**.** *There exists a constant $A > 0$ depending on $\epsilon, \chi(S)$ such that if $\mathcal{L}$ is an ϵ-thick geodesic in $\mathcal{T}_{g,n}$ and $d(\mathcal{X},\mathcal{L}) > A$, then we have*

$$\mathrm{diam}\left(\pi_{\mathcal{L}}\left(\mathcal{N}_{d(\mathcal{X},L)-A}\left(\mathcal{X}\right)\right)\right) \leq A$$

for any $\mathcal{X} \in \mathcal{T}_{g,n}$.

For $\mathcal{L}$ a geodesic in $\mathcal{T}_{g,n}$, we let $d^{\mathcal{L}}_{\pi}(C,W) = \mathrm{diam}\left(\pi_{\mathcal{L}}\left(C\right) \cup \pi_{\mathcal{L}}\left(W\right)\right)$. We can pick the constant A in Theorem II.8.1 in a way so that the following holds.

Corollary II.8.2 (Arzhantseva, Cashen, and Tao [1])**.** *Let $\mathcal{L}$ be an ϵ-thick geodesic in $\mathcal{T}_{g,n}$ and let $\mathcal{X},\mathcal{Y} \in \mathcal{T}_{g,n}$ be such that $d^{\mathcal{L}}_{\pi}(\mathcal{X},\mathcal{Y}) > A$, then*

$$d\left(\mathcal{X},\mathcal{Y}\right) \geq d\left(\mathcal{X},\pi_{\mathcal{L}}\left(\mathcal{X}\right)\right) + d^{\mathcal{L}}_{\pi}(\mathcal{X},\mathcal{Y}) + d\left(\pi_{\mathcal{L}}\left(\mathcal{Y}\right),\mathcal{Y}\right) - A.$$

Moreover, if $\mathcal{Y}$ happens to be on the geodesic $\mathcal{L}$, then $\pi_{\mathcal{L}}(\mathcal{Y}) = \{\mathcal{Y}\}$ and

$$d(\mathcal{X},\mathcal{Y}) \geq d\left(\mathcal{X},\pi_{\mathcal{L}}\left(\mathcal{X}\right)\right) + d\left(\pi_{\mathcal{L}}\left(\mathcal{X}\right),\mathcal{Y}\right) - A.$$

For any pseudo-Anosov element $\phi \in \mathrm{Mod}_{g,n}$, we denote $\pi_{\mathrm{axis}(\phi)}$ as π_ϕ. Since ϕ acts by translation along its axis, it commutes with the projection map π_ϕ. That is, for any $\mathcal{X} \in \mathcal{T}_{g,n}$, we have $\pi_\phi(\phi(\mathcal{X})) = \phi(\pi_\phi(\mathcal{X}))$.

By using Theorem II.8.1 and Corollary II.8.2, one can show if an ϵ-thick pseudo-Anosov element ψ has sufficiently large translation length, then the distance it translates a point is roughly twice the distance from the point to the axis. See Figure II.5 for an illustration.

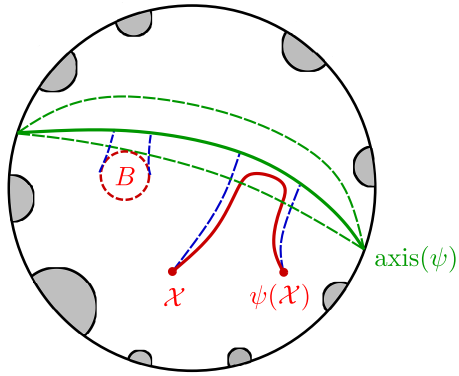

Figure II.5: Shaded area are ϵ-thin parts. Given a ϵ-thick pseudo-Anosov element ψ with $\lambda(\psi) > A$, the diameter of projection of any balls like B to $\operatorname{axis}(\psi)$ is bounded by A, see Theorem II.8.1. The geodesic from $\mathcal{X}$ to $\psi(\mathcal{X})$ fellow travels $\operatorname{axis}(\psi)$, see Corollary II.8.3.

Corollary II.8.3. *Let ϕ be a ϵ-thick pseudo-Anosov element with translation distance $\lambda(\phi) > A$. Then for any $\mathcal{X} \in \mathcal{T}_{g,n}$ and for any $\psi \in [\phi]$, we have*

$$2d(\mathcal{X}, \pi_\psi(\mathcal{X})) + \lambda(\phi) - A \leq d(\mathcal{X}, \psi(\mathcal{X})) \leq 2d(\mathcal{X}, \pi_\psi(\mathcal{X})) + \lambda(\phi) + 2A.$$

Proof. Since translation distance is invariant under conjugation, $\lambda(\psi) = \lambda(\phi) > A$ for any $\psi \in [\phi]$. Thus we have

$$d_\pi^\psi(\mathcal{X}, \psi(\mathcal{X})) = \operatorname{diam}(\pi_\psi(\mathcal{X}) \cup \pi_\psi(\psi(\mathcal{X}))) = \operatorname{diam}(\pi_\psi(\mathcal{X}) \cup \psi(\pi_\psi(\mathcal{X})))$$

where $\lambda(\phi) \leq \operatorname{diam}(\pi_\psi(\mathcal{X}) \cup \psi(\pi_\psi(\mathcal{X}))) \leq \lambda(\phi) + 2A$. Take any $\mathcal{X} \in \mathcal{T}_{g,n}$, by the triangle inequality, we have

$$\begin{aligned} d(\mathcal{X}, \psi(\mathcal{X})) \leq & d(\mathcal{X}, \pi_\psi(\mathcal{X})) + d_\pi^\psi(\mathcal{X}, \psi(\mathcal{X})) + d(\psi(\mathcal{X}), \pi_\psi(\psi(\mathcal{X}))) \\ \leq & 2d(\mathcal{X}, \pi_\psi(\mathcal{X})) + \lambda(\phi) + 2A. \end{aligned}$$

Meanwhile we can apply the previous Corollary II.8.2 and get

$$\begin{aligned} d(\mathcal{X}, \psi(\mathcal{X})) \geq & d(\mathcal{X}, \pi_\psi(\mathcal{X})) + d_\pi^\psi(\mathcal{X}, \psi(\mathcal{X})) + d(\psi(\mathcal{X}), \pi_\psi(\psi(\mathcal{X}))) - A \\ \geq & 2d(\mathcal{X}, \pi_\psi(\mathcal{X})) + \lambda(\phi) - A. \end{aligned}$$

The result follows. ∎

CHAPTER III

Growth of Pseudo-Anosov Conjugacy Classes in Teichmüller Space

III.1 Proof of Theorem A

By Theorem I.1.2, for any $\mathcal{X} \in \mathcal{T}_{g,n}$, we have

$$|\mathrm{Mod}_{g,n} \cdot \mathcal{X} \cap B_r(\mathcal{X})| \sim e^{hr}.$$

For any $r > 0$, define the set

$$\Omega_r(\mathcal{X}) = \{f \in \mathrm{Mod}_{g,n} \mid d(\mathcal{X}, f\mathcal{X}) \leq r\}$$

and denote N the maximal order of point stabilizer subgroups in $\mathrm{Mod}_{g,n}$ [20]. It follows that

$$|\mathrm{Mod}_{g,n} \cdot \mathcal{X} \cap B_r(\mathcal{X})| \leq |\Omega_r(\mathcal{X})| \leq N \cdot |\mathrm{Mod}_{g,n} \cdot \mathcal{X} \cap B_r(\mathcal{X})|,$$
$$e^{hr} \preceq |\Omega_r(\mathcal{X})| \preceq N \cdot e^{hr}.$$

Moreover, given any $\phi \in \mathrm{Mod}_{g,n}$, we have

$$\Gamma_r(\mathcal{X}, \mathcal{Y}, \phi) \leq |[\phi] \cap \Omega_r(\mathcal{X})| \leq N \cdot \Gamma_r(\mathcal{X}, \mathcal{Y}, \phi).$$

Combining things together, we have

$$\frac{1}{N} \cdot |[\phi] \cap \Omega_r(\mathcal{X})| \leq \Gamma_r(\mathcal{X}, \mathcal{Y}, \phi) \leq |[\phi] \cap \Omega_r(\mathcal{X})|. \tag{III.1}$$

We first prove a simplified version of the main theorem.

Theorem III.1.1. *For any $S_{g,n}$ and $\epsilon > 0$, there exists a constant $A > 0$ such that given any ϵ-thick pseudo-Anosov element ϕ with translation distance $\lambda \geq A$ and given any $\mathcal{X} \in \mathrm{axis}(\phi)$, there exists a corresponding constant $G(\mathcal{X}, \phi) > 0$ such that*

$$\Gamma_R(\mathcal{X}, \mathcal{X}, \phi) \overset{G(\mathcal{X},\phi)}{\sim} e^{\frac{h}{2}R}$$

Proof. Given $\phi, \mathcal{X}$ satisfying the assumptions. For any R, define

$$P_R^+ = \left\{\psi \in [\phi] \mid d\left(\mathcal{X}, \pi_\psi(\mathcal{X})\right) \leq \frac{R+A-\lambda}{2}\right\},$$
$$P_R^- = \left\{\psi \in [\phi] \mid d\left(\mathcal{X}, \pi_\psi(\mathcal{X})\right) \leq \frac{R-2A-\lambda}{2}\right\}.$$

Denote $\Omega_r(\mathcal{X})$ as $\Omega(r)$ for simplicity, by Corollary II.8.3 we have

$$P_R^- \subset [\phi] \cap \Omega(R) \subset P_R^+. \tag{III.2}$$

We now work towards obtaining an upper bound for $|P_R^+|$. Take any $\psi \in P_R^+$, there exists a $f \in \mathrm{Mod}_{g,n}$ such that $\psi = f\phi f^{-1}$. Since $\mathcal{X} \in \mathrm{axis}(\phi)$, $f(\mathcal{X})$ therefore lies on the $\mathrm{axis}(\psi)$. In particular, this means there exists a $k \in \mathbb{Z}$ such that

$$d\left(\psi^k \circ f(\mathcal{X}), \pi_\psi(\mathcal{X})\right) \leq \frac{\lambda}{2},$$
$$d\left(\psi^k \circ f(\mathcal{X}), \mathcal{X}\right) \leq d\left(\psi^k \circ f(\mathcal{X}), \pi_\psi(\mathcal{X})\right) + d\left(\mathcal{X}, \pi_\psi(\mathcal{X})\right) \leq \frac{R+A}{2}.$$

See Figure III.1 for an example.

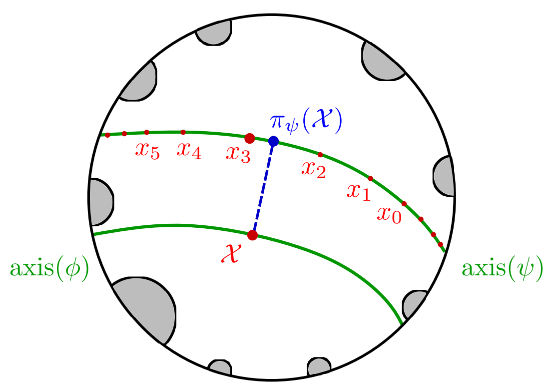

Figure III.1: Each x_i denotes $\psi^i \circ f(x)$ and distance between any two adjacent x_i is λ. The injective map maps $\mathcal{X}$ to x_3 since x_3 is the closest point to $\pi_\psi(\mathcal{X})$ in $\{x_i\}_{i\in\mathbb{Z}}$.

We claim one can define an injective map from $P_R^+ \to \Omega(\frac{R+A}{2})$ by sending ψ to $\psi^k f$. Indeed, if there is any another $\eta \in P_R^+, \eta \neq \psi$, $\eta = h\phi h^{-1}$ for some $h \in \mathrm{Mod}_{g,n}$, then $h(\mathcal{X}) \in \mathrm{axis}(\eta)$ and there exists a

$m \in \mathbb{Z}$ such that

$$d(\eta^m \circ h(\mathcal{X}), \pi_\eta(\mathcal{X})) \le \frac{\lambda}{2},$$
$$d(\eta^m \circ h(\mathcal{X}), \mathcal{X}) \le \frac{R+A}{2}.$$

We claim in this case $\psi^k f \neq \eta^m h$. Indeed, suppose they are equal, then

$$\psi = \psi^k \psi \psi^{-k} = \psi^k f \phi f^{-1} \psi^{-k} = \eta^m h \phi h^{-1} \eta^{-m} = \eta^m \eta \eta^{-m} = \eta.$$

However, this contradicts $\psi \neq \eta$. This means for R large, we can inject P_R^+ into $\Omega(\frac{R+A}{2})$, so that

$$\left|P_R^+\right| \le \left|\Omega\left(\frac{R+A}{2}\right)\right| \preceq e^{\frac{hA}{2}} \cdot e^{\frac{hR}{2}}. \tag{III.3}$$

To obtain the lower bound for $|P_R^-|$, we define

$$\mathcal{A}_R = \left\{\text{axis}(\psi) \mid \psi \in P_R^-\right\}.$$

This gives us a surjective map $F : P_R^- \to \mathcal{A}_R, \psi \mapsto \text{axis}(\psi)$, and each $\Theta \in \mathcal{A}_R$ has the form $\Theta = \text{axis}(f\phi f^{-1})$ for some $f \in \Omega(\frac{R-2A}{2})$. For any $L < \frac{R-2A-\lambda}{2}$, we define

$$A_R^L = \left\{\Theta \in \mathcal{A}_R \mid d\left(\mathcal{X}, \pi_\Theta(\mathcal{X})\right) > \frac{R-2A-\lambda}{2} - L\right\}$$

so that $\mathcal{A}_R^L \subset \mathcal{A}_R$. For each $\Theta \in \mathcal{A}_R$, we denote

$$H(\Theta) = \{f \in \Omega(\frac{R-2A}{2}) \mid \text{axis}(f\phi f^{-1}) = \Theta\},$$

which is a subset of $\Omega(\frac{R-2A}{2})$.

By Corollary II.8.2, for any $\Theta \in \mathcal{A}_R^L$, there are at most $\frac{2(L+A)}{\lambda} + 2$ many $f \in H(\Theta)$ satisfying $\text{axis}(f\phi f^{-1}) = \Theta$ since $d(\mathcal{X}, \pi_\Theta(\mathcal{X})) \in \left(\frac{R-2A-\lambda}{2} - L, \frac{R-2A-\lambda}{2}\right]$. In the example of Figure III.2, there are six such f for this Θ. This means

$$\left|\mathcal{A}_R^L\right| \ge \frac{\lambda}{2(L+A+\lambda)} \cdot \sum_{\Theta \in \mathcal{A}_R^L} \left|H\left(\Theta\right)\right|. \tag{III.4}$$

For any element $f \in \Omega(\frac{R-2A-\lambda}{2})$, let's denote $\Theta_f = \text{axis}(f\phi f^{-1})$, then each f is exactly one of the following types.

(a) Θ_f never enters $B_{\frac{R-2A-\lambda}{2}-L}(\mathcal{X})$.

(b) Θ_f enters $B_{\frac{R-2A-\lambda}{2}-L}(\mathcal{X})$ and $d(\mathcal{X}, f(\mathcal{X})) \leq \frac{R-2A-\lambda}{2} - L$.

(c) Θ_f enters $B_{\frac{R-2A-\lambda}{2}-L}(\mathcal{X})$ and $d(\mathcal{X}, f(\mathcal{X})) > \frac{R-2A-\lambda}{2} - L$.

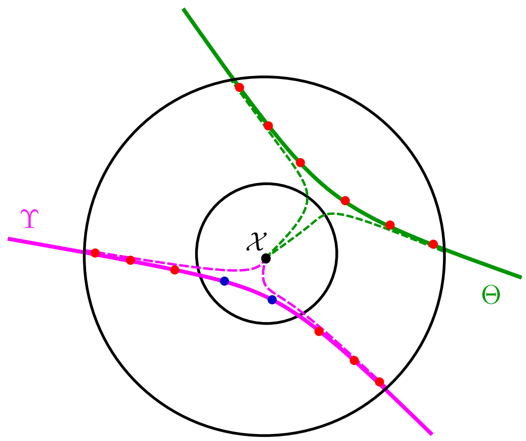

Figure III.2: Θ is of type (a) and Υ is of type (c). The lengths of Θ and Υ intersecting $B_{\frac{R-2A-\lambda}{2}}$ can be approximated by Corollary II.8.2, which showed as the dotted geodesic segments.

The union of type (a) elements is $\bigsqcup_{\Theta\in\mathcal{A}_R^L} H(\Theta)$, and the union of type (b) elements are $\Omega\left(\frac{R-2A-\lambda}{2} - L\right) \subset \Omega\left(\frac{R-2A}{2} - L\right)$. By Corollary II.8.2, we notice there are at most $\frac{2(L+A)}{\lambda}$ many type (c) elements can share the same axis, and the numbers of axes going through $B_{\frac{R-2A-\lambda}{2}-L}(\mathcal{X})$ is bounded by $\left|\Omega\left(\frac{R-2A}{2} - L\right)\right|$. In the example of Figure III.2, there are six f satisfying type (c) conditions sharing the axis Υ. Notice there are two f realize $\Upsilon = \Theta_f$ but not satisfy the type (c) assumption. Since type (a), (b), (c) elements compose $\Omega(\frac{R-2A-\lambda}{2})$, we have

$$\sum_{\Theta\in\mathcal{A}_R^L} |H(\Theta)| \geq \left|\Omega\left(\frac{R-2A-\lambda}{2}\right)\right| - \left(1 + \frac{2(L+A)}{\lambda}\right)\cdot\left|\Omega\left(\frac{R-2A}{2} - L\right)\right|.$$

Moreover, we let L be a constant satisfy $e^{hL} > 2\cdot e^{h\frac{\lambda}{2}}\cdot N\left(1 + \frac{2(L+A)}{\lambda}\right)$, then

$$\begin{aligned}\sum_{\Theta\in\mathcal{A}_R^L} |H(\Theta)| &\succeq e^{\frac{h(R-2A-\lambda)}{2}} - \left(1 + \frac{2(L+A)}{\lambda}\right)\cdot N\cdot e^{\frac{h(R-2A)}{2} - hL} \\ &\succeq e^{\frac{h}{2}R}\cdot e^{-hA}\cdot\left(\frac{1}{e^{h\frac{\lambda}{2}}} - \frac{N\cdot\left(1 + \frac{2(L+A)}{\lambda}\right)}{e^{hL}}\right) \\ &\succeq e^{\frac{h}{2}R}\cdot\frac{1}{2e^{h(\frac{\lambda}{2}+A)}},\end{aligned} \tag{III.5}$$

and this lower bound is nontrivial.

Thus, to construct the lower bound for $|P_R^-|$, we let L be a constant satisfy $e^{hL} > 2 \cdot e^{h\frac{\lambda}{2}} \cdot N\left(1 + \frac{2(L+A)}{\lambda}\right)$. Apply formulas (III.4) (III.5) from above, for R large we have

$$\begin{aligned} \left|P_R^-\right| &\geq |\mathcal{A}_R| \geq \left|\mathcal{A}_R^L\right| \\ &\geq \frac{\lambda}{2(L+A+\lambda)} \cdot \sum_{\Theta \in \mathcal{A}_R^L} |H(\Theta)| \\ &\succeq e^{\frac{h}{2}R} \cdot \frac{\lambda}{2(L+A+\lambda)e^{hA}} \cdot \frac{1}{2e^{h(\frac{\lambda}{2}+A)}}. \end{aligned} \tag{III.6}$$

Finally, combining formulas (III.1), (III.2), (III.6) we have

$$|[\phi] \cdot \mathcal{X} \cap B_R(\mathcal{X})| \geq \frac{1}{N} \cdot |[\phi] \cap \Omega(R)| \geq \frac{1}{N} \cdot \left|P_R^-\right| \succeq G_L(\mathcal{X}, \phi) \cdot e^{\frac{h}{2}R}$$

where

$$G_L(\mathcal{X}, \phi) = \frac{\lambda}{2N(L+A+\lambda)e^{hA}} \cdot \frac{1}{2e^{h(\frac{\lambda}{2}+A)}}.$$

And combining formulas (III.1), (III.2), (III.3) we have

$$|[\phi] \cdot \mathcal{X} \cap B_R(\mathcal{X})| \leq |[\phi] \cap \Omega(R)| \leq P_R^+ \preceq G_U(\mathcal{X}, \phi) \cdot e^{\frac{h}{2}R}$$

where

$$G_U(\mathcal{X}, \phi) = N e^{\frac{hA}{2}}.$$

Recall $f(R) \overset{A}{\preceq} g(R)$ is the same as $f(R) \overset{1}{\preceq} Ag(R)$. Thus we have

$$e^{\frac{h}{2}R} \overset{G_L^{-1}(\mathcal{X},\phi)}{\preceq} |[\phi] \cdot \mathcal{X} \cap B_R(\mathcal{X})| \overset{G_U(\mathcal{X},\phi)}{\preceq} e^{\frac{h}{2}R}$$

This means by setting

$$G(\mathcal{X}, \phi) = \max\{G_L^{-1}(\mathcal{X}, \phi), G_U(\mathcal{X}, \phi)\}$$

we obtain the desired result. $\square$

Now we are ready to prove the general case.

Theorem A. Fix $S_{g,n}$ and $\epsilon > 0$, there exists a constant $A > 0$ such that given any ϵ-thick pseudo-Anosov element ϕ with translation distance $\lambda \geq A$ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$, there exists a corresponding $G(\mathcal{X}, \mathcal{Y}, \phi)$ such that

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi) \overset{G(\mathcal{X},\mathcal{Y},\phi)}{\sim} e^{\frac{h}{2}R}.$$

Proof of Theorem A. Take any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}$, and let D be the maximum between $d(\mathcal{X}, \pi_\phi(\mathcal{X}))$ and $d(\pi_\phi(\mathcal{X}), \mathcal{Y})\}$. We then have

$$|[\phi] \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \geq |[\phi] \cdot \pi_\phi(\mathcal{X}) \cap B_{R-D}(\mathcal{X})| \geq |[\phi] \cdot \pi_\phi(\mathcal{X}) \cap B_{R-2D}(\pi_\phi(\mathcal{X}))| \,,$$
$$|[\phi] \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \leq |[\phi] \cdot \pi_\phi(\mathcal{X}) \cap B_{R+D}(\mathcal{X})| \leq |[\phi] \cdot \pi_\phi(\mathcal{X}) \cap B_{R+2D}(\pi_\phi(\mathcal{X}))| \,.$$

By applying these inequalities and by applying Theorem III.1.1 to ϕ and $\pi_\phi(\mathcal{X})$, without loss of generality, we get the desired result by setting $G(\mathcal{X}, \mathcal{Y}, \phi) = G(\pi_\phi(\mathcal{X}), \phi) \cdot e^{hD}$. □

III.2 Proof of Corollary B and Corollary C

Corollary B. Fix $S_{g,n}$, given any pseudo-Anosov element ϕ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$. There exists a power N depending on ϕ such that for any $k \geq N$, there is a corresponding $G(\mathcal{X}, \mathcal{Y}, \phi, k)$ so that the following holds:

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) \overset{G(\mathcal{X},\mathcal{Y},\phi,k)}{\sim} e^{\frac{h}{2}R}.$$

Proof of Corollary B. Given ϕ, we pick ϵ so that $\mathrm{axis}(\phi)$ is in $\mathcal{T}^{\epsilon}_{g,k}$. Since $\lambda(\phi^k) = k \cdot \lambda(\phi)$ for any pseudo-Anosov element ϕ, there exists a $N(\phi)$ such that $\lambda(\phi^k) \geq A$ for any $k \geq N(\phi)$. We now can apply Theorem A, and the corresponding error constant G depends on $\mathcal{X}, \mathcal{Y}, \phi, k$. □

Corollary C. Fix $S_{g,n}$, given any pseudo-Anosov element ϕ and given any $\mathcal{X}, \mathcal{Y}$ in $\mathcal{T}_{g,n}$, for all sufficiently large k we have

$$\lim_{R\to\infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) = \frac{h}{2}.$$

Proof of Corollary C. Assuming the conditions, we can apply the Corollary B. This means for any $k \geq N$

and for any $\delta > 1$, there exists a $M(\delta)$ such that

$$\frac{1}{\delta G(\mathcal{X}, \mathcal{Y}, \phi, k)} \cdot e^{\frac{h}{2}R} \leq \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) \leq \delta G(\mathcal{X}, \mathcal{Y}, \phi, k) \cdot e^{\frac{h}{2}R}$$

for any $R \geq M(\delta)$. Let $\epsilon > 0$, one can pick $\delta > 0$ and pick $M(\epsilon) \geq M(\delta)$ so that

$$\begin{aligned}
&\delta G(\mathcal{X}, \mathcal{Y}, \phi, k) \leq e^{\epsilon \frac{h}{2}R}, \\
&e^{-\epsilon \frac{h}{2}R} \leq \frac{1}{\delta G(\mathcal{X}, \mathcal{Y}, \phi, k)},
\end{aligned}$$

for any $R \geq M(\epsilon)$. This implies for any $\epsilon > 0$, we have

$$\begin{aligned}
&e^{(1-\epsilon)\frac{h}{2}R} \leq \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) \leq e^{(1+\epsilon)\frac{h}{2}R}, \\
&(1-\epsilon)\frac{h}{2}R \leq \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) \leq (1+\epsilon)\frac{h}{2}R, \\
&(1-\epsilon)\frac{h}{2} \leq \frac{1}{R}\ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) \leq (1+\epsilon)\frac{h}{2},
\end{aligned}$$

whenever $R \geq M(\epsilon)$. That is,

$$\lim_{R\to\infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) = \frac{h}{2}.$$

This finishes the proof. $\square$

CHAPTER IV

Growth Rate of Dehn Twist Lattice Points in Teichmüller Space

IV.1 The Effect of Twisting on Hyperbolic Length

In this section, we study how the length of simple closed geodesics on a hyperbolic surface change after applying a twist. In the next section, we use the results below to estimate how far a point in Teichmüller space moves after applying a twist.

As our first result, we may obtain the following estimate from the length formula (II.4) and intersection formula (II.1).

Proposition IV.1.1. *Fix some $\epsilon > 0$. Given a multicurve $\alpha = \sum_{i=1}^{k} a_i \alpha_i$ and a simple closed curve τ on a hyperbolic surface $\mathcal{X} \in \mathcal{T}^{\epsilon}_{g,n}$, there exists a constant A depends only on $S_{g,n}$ and ϵ such that*

$$A\left(\sum_{i=1}^{k} |a_i| i(\alpha_i, \tau)\ell_{\mathcal{X}}(\alpha_i) + \ell_{\mathcal{X}}(\tau)\right) \geq \ell_{T_\alpha \mathcal{X}}(\tau) \geq \frac{1}{A}\left(\sum_{i=1}^{k}(|a_i| - 2) i(\alpha_i, \tau)\ell_{\mathcal{X}}(\alpha_i) - \ell_{\mathcal{X}}(\tau)\right).$$

Furthermore, if α is positive or negative, the lower bound can be sharpened to

$$\ell_{T_\alpha \mathcal{X}}(\tau) \geq \frac{1}{A}\left(\sum_{i=1}^{k} |a_i| i(\alpha_i, \tau)\ell_{\mathcal{X}}(\alpha_i) - \ell_{\mathcal{X}}(\tau)\right).$$

Proof. By the length formula (II.4), we know

$$CN \sum_{\gamma \in \mu_X} i(T_\alpha^{-1}(\tau), \gamma) \geq \ell_{T_\alpha \mathcal{X}}(\tau) \geq \frac{\epsilon}{C} \sum_{\gamma \in \mu_X} i(T_\alpha^{-1}(\tau), \gamma).$$

We can apply the signed intersection formula (II.2) to approximate $i(T_\alpha^{-1}(\tau), \gamma)$. This allows us to expand the above inequality into the following:

$$\begin{aligned} &CN \sum_{\gamma \in \mu_{\mathcal{X}}} \left(\sum_{i=1}^{k} |a_i| i(\alpha_i, \gamma) i(\alpha_i, \tau) + i(\tau, \gamma)\right) \\ &\geq \ell_{T_\alpha \mathcal{X}}(\tau) \\ &\geq \frac{\epsilon}{C} \sum_{\gamma \in \mu_{\mathcal{X}}} \left(\sum_{i=1}^{k}(|a_i| - 2) i(\alpha_i, \gamma) i(\alpha_i, \tau) - i(\tau, \gamma)\right). \end{aligned}$$

By switching the order of summations $\sum_{\gamma \in \mu_{\mathcal{X}}}$ and $\sum_{i=1}^{k}$ and by applying the length formula (II.4) again, we obtain the result in the proposition.

If α is positive or negative, we can use the intersection formula (II.1), and going through the same proof give us the sharpened lower bound. $\square$

Note the above proposition provides a good estimate for the length of multicurves up to a multiplicative error. This error arises from our repeated use of length formula (II.4). Below we propose a more generalized result that leads to removing this multiplicative error. Let $\lfloor \cdot \rfloor_0$ denote the 0 threshold function.

Theorem IV.1.2. *Given a multicurve $\alpha = \sum_{i=1}^{k} a_i \alpha_i$ and a simple closed curve τ on any hyperbolic structure $\mathcal{X}$, we have*

$$\begin{aligned} &\ell_{\mathcal{X}}(\tau) + \sum_{i=1}^{k} i(\tau, \alpha_i)|a_i|\ell_{\mathcal{X}}(\alpha_i) \\ &\geq \ell_{T_\alpha \mathcal{X}}(\tau) \\ &\geq \sum_{i=1}^{k} i(\tau, \alpha_i) \cdot \Big\lfloor (|a_i| - 2) \cdot \ell_{\mathcal{X}}(\alpha_i) - 2\ell_{\mathcal{X}}(\tau) - L \Big\rfloor_0 \end{aligned} \tag{IV.1}$$

where L is a constant that depends on $\mathbb{H}^2$.

Proof. Fix the hyperbolic structure we may assume curves are geodesics. Given a multicurve $\alpha = \sum_{i=1}^{k} a_i \alpha_i$, we denote $\mathcal{A} = \{\alpha_i\}_{i=1}^{k}$ and denote $\tilde{\mathcal{A}}$ the set of all liftings of curves in $\mathcal{A}$ to $\mathbb{H}^2$. Let $\mathcal{Z}_{\mathcal{A}}$ denote the corresponding Bass-Serre tree, see section II.6.

For each $\beta \in \tilde{\mathcal{A}}$, we denote $\psi_\beta \in \pi_1(S_{g,n})$ the corresponding hyperbolic isometry in $\mathbb{H}^2$. If $\beta, \gamma \in \tilde{\mathcal{A}}$ are lifts of the same $\alpha \in \mathcal{A}$, then ψ_β, ψ_γ are conjugate to each other and have the same translation distance equal to $\ell_{\mathcal{X}}(\alpha)$. This also means there exists an isometry ψ in $\mathbb{H}^2$ that sends γ to β. We can choose this isometry up to composing with any power of ψ_β or pre-composing with any power of ψ_γ. In particular, suppose there are geodesic segments $\beta' \subset \beta, \gamma' \subset \gamma$ such that their length are less than $\ell_{\mathcal{X}}(\alpha)$, we can choose the isometry ψ in a way such that β' and $\psi(\gamma')$ both lie on β and intersect the same fundamental domain of the action of ψ_β.

Given a simple closed curve τ, we denote

$$i(\tau, \alpha) = m = \sum_{i=1}^{k} i(\tau, \alpha_i) = \sum_{i=1}^{k} m_i.$$

In the case of $i(\tau, \alpha) = 0$, T_α has no effect on τ and the theorem holds true. We may assume $i(\tau, \alpha) \geq 1$. Let $\tilde{\tau}$ be a lifting of τ and say it has end points $q_L, q_R \in \partial \mathbb{H}^2$. $\tilde{\tau}$ is therefore a bi-infinite geodesic in $\mathbb{H}^2$

transverse to $\tilde{\mathcal{A}}$, and $\phi_{\mathcal{A}}(\tilde{\tau})$ is a bi-infinite geodesic in $\mathcal{Z}_{\mathcal{A}}$, say the edges are labeled by

$$\mathcal{B} = \{\cdots, \beta_{-2}, \beta_{-1}, \beta_0, \beta_1, \beta_2, \cdots\}.$$

For each β_i, let's denote $\pi_{\beta_i}(\beta_{i-1}), \pi_{\beta_i}(\beta_{i+1})$ as $\beta_{i,L}, \beta_{i,R}$ respectively. Define a index function $s\colon \mathbb{N} \to \{1, \cdots, k\}$ so that $\alpha_{s(i)} \in \mathcal{A}$ is the simple closed curve such that β_i is a lift of $\alpha_{s(i)}$. For each i, we claim $d(\beta_{i,L}, \beta_{i,R}) \leq 2\ell_{\mathcal{X}}(\tau)$. In the case of $i(\tau, \mathcal{A}) = 1$, we pick $\kappa \subset \tilde{\tau}$ to be the geodesic segment between the points $\tilde{\tau} \cap \beta_{i-1}$ and $\tilde{\tau} \cap \beta_{i+1}$. Then κ is a concatenation of two consecutive path liftings of τ and $\ell_{\mathcal{X}}(\kappa) = 2\ell_{\mathcal{X}}(\tau)$. For $i(\tau, \mathcal{A}) \geq 2$, we pick κ to be the path lifting of τ starting from $\beta_{i-1} \cap \tilde{\tau}$. In any case, $\kappa \subset \tilde{\tau}$ goes through $\beta_{i-1}, \beta_i, \beta_{i+1}$ and has length bounded by $2\ell_{\mathcal{X}}(\tau)$. By Lemma II.4.4 we know the projection maps π_{β_i} are 1-Lipschitz, thus the distance between projections of the two endpoints of κ on β_i is smaller than the length of κ, which is less than $2\ell_{\mathcal{X}}(\tau)$. Since the projections of the two endpoints lie in $\beta_{i,L}, \beta_{i,R}$ respectively, we have $d(\beta_{i,L}, \beta_{i,R}) \leq 2\ell_{\mathcal{X}}(\tau)$.

Fix some point $q_0 \in \tilde{\tau}$ and let $\tilde{\tau}'$ be $\tilde{\tau}$ after shearing according to $-\alpha$ fixing the component of q_0, see section II.5. The projection of $\tilde{\tau}'$ to the surface $\mathcal{X}$ has length equal to

$$\ell_{\mathcal{X}}(\tau) + \sum_{i=1}^{k} i(\tau, \alpha_i)|a_i|\ell_{\mathcal{X}}(\alpha_i).$$

Denote the end points of $\tilde{\tau}'$ as $q'_L, q'_R \in \partial\mathbb{H}^2$. Let σ be the geodesic with end points $q'_L, q'_R \in \partial\mathbb{H}^2$, then σ is a lift of the geodesic $T_\alpha^{-1}(\tau)$ and its image $\phi_{\mathcal{A}}(\sigma)$ is a geodesic in $\mathcal{Z}_{\mathcal{A}}$. Since the projection of $\tilde{\tau}'$ is in the isotopy class $T_\alpha^{-1}(\tau)$, the upper bound in (IV.1) follows.

Once $\tilde{\tau}'$ leaves a connected component of $\mathbb{H}^2 \setminus \cup\tilde{\mathcal{A}}$, it never comes back. This means $\phi_{\mathcal{A}}(\tilde{\tau}')$ does not back track in $\mathcal{Z}_{\mathcal{A}}$ so $\phi_{\mathcal{A}}(\tilde{\tau}')$ is a geodesic path in $\mathcal{Z}_{\mathcal{A}}$. Since σ shares the same endpoints with $\tilde{\tau}'$ and since $\mathcal{Z}_{\mathcal{A}}$ is a unique geodesic space, we have $\phi_{\mathcal{A}}(\tilde{\tau}') = \phi_{\mathcal{A}}(\sigma)$. Denote the edge labels of $\phi_{\mathcal{A}}(\sigma)$ as

$$\mathcal{F} = \{\cdots, \eta_{-2}, \eta_{-1}, \eta_0, \eta_1, \eta_2, \cdots\}.$$

For each η_i, let's denote $\pi_{\eta_i}(\eta_{i-1}), \pi_{\eta_i}(\eta_{i+1})$ as $\eta_{i,L}, \eta_{i,R}$ respectively. Since each β_i and η_i are lifts of the same curve, we can use the same index function s denoting $\alpha_{s(i)} \in \mathcal{A}$ the simple closed curve such that η_i is a lift of $\alpha_{s(i)}$. Since for each i, the triples $(\tilde{\tau}, \beta_i, \beta_{i+1})$ and $(\tilde{\tau}', \eta_i, \eta_{i+1})$ realize the same intersection pattern on the surface, $\eta_{i,L}, \eta_{i,R}$ are translations of $\beta_{i,L}, \beta_{i,R}$ respectively and have the same diameters respectively.

The relative location of $\eta_{i,L}, \eta_{i,R}$ is the same as the relative location of $\beta_{i,L}$ and $\psi_{\beta_i}^{a_{s(i)}}(\beta_{i,R})$, see Figure IV.1 for an illustration. Recall for any point x on any β_i and for any $t \in \mathbb{Z}$, we have $d(x, \psi^t_{\beta_i}(x)) = |t|\ell_{\mathcal{X}}(\alpha_{s(i)})$. Since both $\mathrm{diam}(\beta_{i,L}), \mathrm{diam}(\beta_{i,R})$ are bounded by $\ell_{\mathcal{X}}(\alpha_{s(i)})$, and since $d(\beta_{i,L}, \beta_{i,R}) \leq 2\ell_{\mathcal{X}}(\tau)$, we have $\mathrm{diam}(\beta_{i,L} \cup \beta_{i,R}) \leq 2\ell_{\mathcal{X}}(\alpha_{s(i)}) + 2\ell_{\mathcal{X}}(\tau)$. It follows that

$$\begin{aligned} d(\eta_{i,L}, \eta_{i,R}) &\geq |a_{s(i)}|\ell_{\mathcal{X}}(\alpha_{s(i)}) - \mathrm{diam}(\beta_{i,L} \cup \beta_{i,R}) \\ &\geq (|a_{s(i)}| - 2)\ell_{\mathcal{X}}(\alpha_{s(i)}) - 2\ell_{\mathcal{X}}(\tau). \end{aligned}$$

Denote

$$D_i = (|a_{s(i)}| - 2)\ell_{\mathcal{X}}(\alpha_{s(i)}) - 2\ell_{\mathcal{X}}(\tau)$$

so that $d(\eta_{i,L}, \eta_{i,R}) \geq D_i$ for any i.

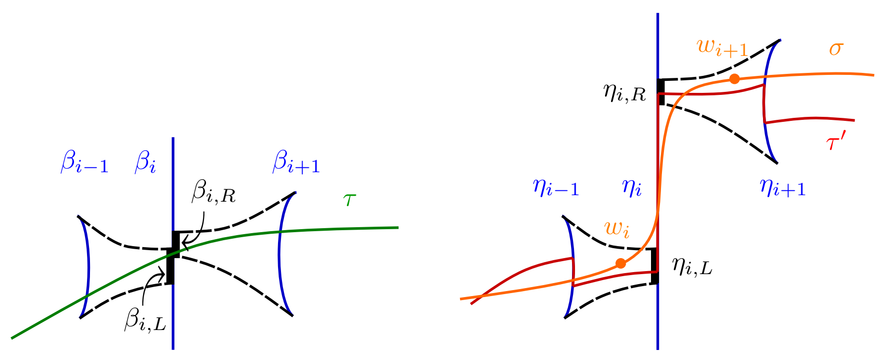

Figure IV.1: Before and after shearing.

Denote the sequence of points $\{w_i\}_{i\in\mathbb{N}}$ on σ such that each w_i is the first point on σ entering the L-neighborhood of η_i from left. Since $\pi_{\eta_i}(q'_L) \in \eta_{i,L}$, by Corollary II.4.3, for each i we have

$$d(\pi_{\eta_i}(w_i), \eta_{i,L}) \leq d(\pi_{\eta_i}(w_i), \pi_{\eta_i}(q'_L)) \leq 2L.$$

Moreover, all these points are equivalent under translation of σ, i.e, for any i we have

$$d(w_i, w_{i+m}) = \ell_{\mathcal{X}}(T_\alpha^{-1}(\tau)) = \ell_{T_\alpha \mathcal{X}}(\tau).$$

By Lemma II.4.4, we know projection π_{η_i} is 1-Lipschitz for any i. Since $\pi_{\eta_i}(\eta_j) \subset \eta_{i,R}$, for any $i < j$ we have

$$d(w_i, w_j) \geq d(\eta_{i,L}, \eta_{i,R}) - 4L \geq \lfloor D_i - 4L \rfloor_0. \tag{IV.2}$$

See Figure IV.1 for an illustration.

We use $a \ll b$ to denote that σ goes through the point a first and then the point b from left to right. We claim for any i such that $D_i > 4L$ and for any $j > i$, we have $w_i \ll w_j$. Indeed, suppose $w_j \ll w_i$, then by definition the geodesic segment from w_j to w_i completely lies outside the L-neighborhood of η_i, and this means $d(\pi_{\eta_i}(w_i), \pi_{\eta_i}(w_j)) \leq L$ because geodesics are strongly contracting in $\mathbb{H}^2$, see section II.4. Meanwhile, we know

$$d(\eta_{i,R}, \pi_{\eta_i}(w_j)) \leq d(\pi_{\eta_i}(\eta_j), \pi_{\eta_i}(w_j)) \leq d(\eta_j, w_j) = L$$

because $\pi_{\eta_i}(\eta_j) \subset \eta_{i,R}$ and because the projection map π_{η_i} is 1-Lipschitz. Combining with the previous fact $d(\pi_{\eta_i}(w_i), \eta_{i,L}) \leq 2L$, we conclude

$$\begin{aligned} D_i &\leq d(\eta_{i,L}, \eta_{i,R}) \\ &\leq d(\pi_{\eta_i}(w_i), \eta_{i,L}) + d(\pi_{\eta_i}(w_i), \pi_{\eta_i}(w_j)) + d(\eta_{i,R}, \pi_{\eta_i}(w_j)) \\ &\leq 2L + L + L = 4L. \end{aligned}$$

And this contradicts $D_i > 4L$. Therefore, we have a pattern of ordering

$$\cdots \ll w_0 \ll w_1 \ll \cdots \ll w_{m-1} \ll w_m \ll \cdots$$

on σ provided that each $D_i > 4L$. In this case we have

$$\ell_{T_\alpha \mathcal{X}}(\tau) = d(w_0, w_m) \geq \sum_{i=1}^{m} d(w_{i-1}, w_i) \geq \sum_{i=1}^{m} \lfloor D_i - 4L \rfloor_0. \tag{IV.3}$$

If for some i, $D_i \leq 4L$, we can delete the point w_i from our sequence and we only need to measure $d(w_{i-1}, w_{i+1})$ instead. The same result (IV.3) holds. Replacing $4L$ by L gives us the lower bound in (IV.1). $\square$

While the above Theorem IV.1.2 no longer has multiplicative error, we are not yet able to provide an effective lower bound for multicurves with mixed sign and with each coefficient having absolute value ≤ 2. The Proposition IV.1.5 below takes one more step and will lead to an effective lower bound for "long" multicurves with mixed sign and with each coefficient having absolute value ≥ 2. Before that, we make the following two remarks that would help us establish Proposition IV.1.5.

Remark IV.1.3. We notice the following in the proof of Theorem IV.1.2.

Recall that we denote $i(\tau,\alpha) = m = \sum_{i=1}^{k} i(\tau,\alpha_i) = \sum_{i=1}^{k} m_i$. Let's fix m-many consecutive lifts in $\mathcal{B}$ and denote it as $\mathcal{B}_m \subset \mathcal{B}$. Take any α_i in the multicurve α and without loss of generality, say

$$\beta_{i(1)}, \cdots, \beta_{i(m_i)}$$

are all the lifts of this α_i in $\mathcal{B}_m$. As discussed in the proof of Theorem IV.1.2, for all $1 \leq j \leq m_i$, there exist isometries ϕ_j sending $\beta_{i(j)}$ to $\beta_{i(1)}$ such that all $\beta_{i(1),R}, \phi_j(\beta_{i(j),R})$ lie on $\beta_{i(1)}$. For any distinct pair $\beta_{i(j_1)}, \beta_{i(j_2)}$ where $1 \leq j_1, j_2 \leq m_i$, the orbits $\left\langle \psi_{\beta_{i(1)}} \right\rangle \cdot \phi_{j_1}(\beta_{i(j_1),R})$ and $\left\langle \psi_{\beta_{i(1)}} \right\rangle \cdot \phi_{j_2}(\beta_{i(j_2),R})$ are either the same or completely disjoint. That is, for any distinct pair $\beta_{i(j_1),R}, \beta_{i(j_2),R}$, either $\phi_{j_1}(\beta_{i(j_1),R}) = \phi_{j_2}(\beta_{i(j_2),R})$ or they are disjoint.

Thus except the repetitive ones, we can further assume all $\beta_{i(1),R}, \psi_j(\beta_{i(j),R})$ are disjoint, lie on $\beta_{i(1)}$, and lie in a same fundamental domain of the action of $\psi_{\beta_{i(1)}}$. Denote the intersection of this fundamental domain with $\beta_{i(1)}$ as $\overline{\beta^R_{i(1)}}$, and it follows $\overline{\beta^R_{i(1)}} \subset \beta_{i(1)}$ is a path lifting of α_i and $\operatorname{diam}(\overline{\beta^R_{i(1)}}) = \ell_{\mathcal{X}}(\alpha_i)$. This means we have

$$\operatorname{diam}\left(\beta_{i(1),R} \cup \phi_2(\beta_{i(2),R}) \cup \cdots \cup \phi_{m_i}(\beta_{i(m_i),R})\right) \leq \ell_{\mathcal{X}}(\alpha_i).$$

After removing the repetitive ones, the disjoint union of all these right neighbor projections $\phi_j(\beta_{j,R})$ can be arranged into $\overline{\beta^R_{i(1)}}$, a geodesic segment of diameter $\ell_{\mathcal{X}}(\alpha_i)$. One can do the same thing to all the left neighbors, and the union of all these left neighbor projections $\phi_j(\beta_{j,L})$ can be arranged into $\overline{\beta^L_{i(1)}}$, a geodesic segment of diameter $\ell_{\mathcal{X}}(\alpha_i)$

Remark IV.1.4. Continuing on Remark IV.1.3, we recall there are m_i many intersections points between α_i and τ, and let's denote the set of these points on the surface as $X_i = \{x_1, \cdots, x_{m_i}\}$. On one hand, we can lift X_i to $Y_i = \{y_1, \cdots, y_{m_i}\}$, where each $y_j = \tilde{\tau} \cap \beta_{i(j)}, \beta_{i(j)} \in \mathcal{B}_m$. On the other hand, we can lift X_i to $Z_i = \{z_1, \cdots, z_{m_i}\}$ where each $z_j = \phi_j(y_j)$, so all points in Z_i lie in a geodesic segment of diameter $\ell_{\mathcal{X}}(\alpha_i)$, namely, $\overline{\beta^R_{i(1)}}$.

For any of these intersection points z_j, we denote the corresponding lift of τ as $\tilde{\tau}_j$. The left neighbor of z_j is defined to be the previous $\tilde{\alpha} \in \mathcal{A}$ that $\tilde{\tau}_j$ intersects, and the right neighbor of z_j is the next $\tilde{\alpha} \in \mathcal{A}$ that $\tilde{\tau}_j$ intersects. Now, we notice the union of all these right neighbors projections of Z_i are exactly the union of right neighbor projections we considered in Remark IV.1.3 and lie in $\overline{\beta^R_{i(1)}}$. Similarly for the left neighbors.

Proposition IV.1.5. *Given a multicurve $\alpha = \sum_{i=1}^{k} a_i\alpha_i$ and a simple closed curve τ on any hyperbolic* *structure $\mathcal{X}$. Let $K \in (0,1)$ be a constant, we have*

$$\ell_{T_\alpha \mathcal{X}}(\tau) \geq \sum_{i=1}^{k} \min\{\mathcal{L}_1^i, \mathcal{L}_2^i\} \tag{IV.4}$$

where

$$\mathcal{L}_1^i = i(\tau, \alpha_i) \cdot \Big\lfloor (|a_i| - 2 + K) \cdot \ell_{\mathcal{X}}(\alpha_i) - 2\ell_{\mathcal{X}}(\tau) - L \Big\rfloor_0,$$
$$\mathcal{L}_2^i = \Big\lfloor i(\tau, \alpha_i) - \frac{K\ell_{\mathcal{X}}(\alpha_i) + 4\ell_{\mathcal{X}}(\tau)}{W(\tau)} \Big\rfloor_0 \cdot \Big\lfloor (|a_i| - 1 - K) \cdot \ell_{\mathcal{X}}(\alpha_i) - 2\ell_{\mathcal{X}}(\tau) - L \Big\rfloor_0.$$

Proof. We will use the similar notations and ideas from Theorem IV.1.2. Recall we denote $i(\tau, \alpha) = m = \sum_{i=1}^{k} i(\tau, \alpha_i) = \sum_{i=1}^{k} m_i$. And recall $\phi_{\mathcal{A}}(\tilde{\tau})$ is a bi-infinite geodesic in $\mathcal{Z}_{\mathcal{A}}$, and its edges are labeled by

$$\mathcal{B} = \{\cdots, \beta_{-2}, \beta_{-1}, \beta_0, \beta_1, \beta_2, \cdots\}.$$

Define the index function $s\colon \mathbb{N} \to \{1, \cdots, k\}$ so that $\alpha_{s(t)} \in \mathcal{A}$ is the simple closed curve such that β_t is a lift of $\alpha_{s(t)}$. Let $K \in (0,1)$, and we consider two different scenarios.

If $\operatorname{diam}(\beta_{t,L} \cup \beta_{t,R}) \leq (2-K)\ell_{\mathcal{X}}(\alpha_{s(t)}) + 2\ell_{\mathcal{X}}(\tau)$ for all t, following the argument from Theorem IV.1.2, we can set

$$D_t = (|a_{s(t)}| - 2 + K)\ell_{\mathcal{X}}(\alpha_{s(t)}) - 2\ell_{\mathcal{X}}(\tau),$$

and we have $d(\eta_{t,L}, \eta_{t,R}) \geq D_t$ for all t. Following the same equation (IV.3) and the same argument gives us the lower bound

$$\ell_{T_\alpha \mathcal{X}}(\tau) \geq \sum_{i=1}^{k} i(\tau, \alpha_i) \cdot \Big\lfloor (|a_i| - 2 + K) \cdot \ell_{\mathcal{X}}(\alpha_i) - 2\ell_{\mathcal{X}}(\tau) - L \Big\rfloor_0.$$

In the second scenario, we have $\operatorname{diam}(\beta_{t,L} \cup \beta_{t,R}) \geq (2-K)\ell_{\mathcal{X}}(\alpha_{s(t)}) + 2\ell_{\mathcal{X}}(\tau)$ for some t. Let's $i = s(t)$ for simplicity. For this t, since both $\operatorname{diam}(\beta_{t,L})$ and $\operatorname{diam}(\beta_{t,R})$ are bounded by $\ell_{\mathcal{X}}(\alpha_i)$ respectively, we have both

$$\operatorname{diam}(\beta_{t,L}), \operatorname{diam}(\beta_{t,R}) \geq (1-K)\ell_{\mathcal{X}}(\alpha_i).$$

Since the $\beta_{t,L}$ is exhausting an interval length of at least $(1-K)\ell_{\mathcal{X}}(\alpha_i)$, as we discussed above in Remark IV.1.3, the diameter of the union of all other right neighbor projections is bounded by $\ell_{\mathcal{X}}(\alpha_i) - (1-K)\ell_{\mathcal{X}}(\alpha_i)$, that is, $K\ell_{\mathcal{X}}(\alpha_{s(t)})$. Similarly, the diameter of the union of all left neighbor projections except $\beta_{t,L}$, is bounded by $K\ell_{\mathcal{X}}(\alpha_i)$.

Denote $i = s(t)$ and let $\beta_{i(1)}, \cdots, \beta_{i(m_i)}$ denote distinct lifts of α_i in $\mathcal{B}_m$ with $\beta_{i(1)} = \beta_t$, see Remark IV.1.3. Define X_i, Y_i, Z_i just as in Remark IV.1.4. We say a z_j is in vain if its left neighbor is β_{t-1} and its right neighbor is β_{t+1}, and we say z_j is effective otherwise. Notice any points in Z_i is within distance $2\ell_{\mathcal{X}}(\tau)$ of its left neighbor projection and its right neighbor projection, see the proof of Theorem IV.1.2.

If $\beta_{t,L} \cap \beta_{t,R}$ is empty, all points in vain lie in a geodesic segment of length $4\ell_{\mathcal{X}}(\tau)$. If $\beta_{t,L} \cap \beta_{t,R}$ is nontrivial, since

$$(2-K)\ell_{\mathcal{X}}(\alpha_i) + 2\ell_{\mathcal{X}}(\tau) \leq \operatorname{diam}(\beta_{t,L} \cup \beta_{t,R}) \leq 2\ell_{\mathcal{X}}(\alpha_i) + 2\ell_{\mathcal{X}}(\tau),$$

we have $\operatorname{diam}(\beta_{t,L} \cap \beta_{t,R}) \leq K\ell_{\mathcal{X}}(\alpha_i)$, and all points in vain is within $2\ell_{\mathcal{X}}(\tau)$-neighborhood of $\beta_{t,L} \cap \beta_{t,R}$. In any case, all points in vain can be arranged in a geodesic segment that has length bounded by $\operatorname{diam}(\beta_{i,L} \cap \beta_{i,R}) + 4\ell_{\mathcal{X}}(\tau)$. By Collar Lemma II.4.2, there are at most $\frac{K\ell_{\mathcal{X}}(\alpha_i)+4\ell_{\mathcal{X}}(\tau)}{W(\tau)}$ many intersections points in vain.

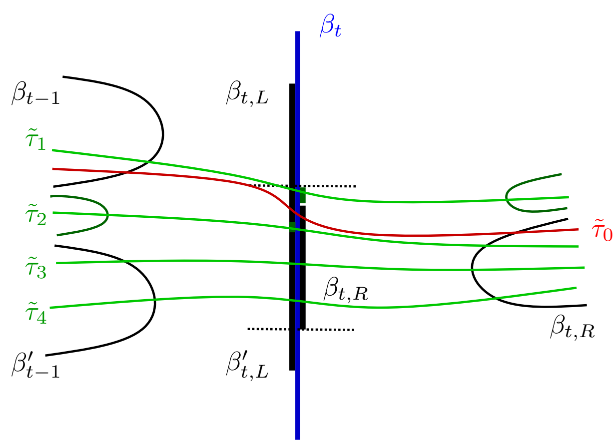

Figure IV.2: In between the two dotted lines is the geodesic segment $\overline{\beta_t^R}$. β'_{t-1} and β_{t-1} differ by ψ_{β_t}, thus their projections $\beta_{t,L}, \beta'_{t,L}$ differ by ψ_{β_t}. $\tilde{\tau}_0$ is in vain and thus is not "counted" in $T_i(K)$. Lifts like $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4$ are effective and hence will realize a translation distance no less than $D_i(K)$ after twisting. $\tilde{\tau}_1$ is of case (1), $\tilde{\tau}_2$, $\tilde{\tau}_3$, and $\tilde{\tau}_4$ is of case (2).

Thus, realized by α_i and β_t, there are at least $T_i(K)$ many effective intersection points, where

$$T_i(K) = \Big\lfloor i(\alpha_i, \tau) - \frac{K\ell_{\mathcal{X}}(\alpha_i) + 4\ell_{\mathcal{X}}(\tau)}{W(\tau)} \Big\rfloor_0.$$

For each effective intersection point z_j, it's of exactly one of the following cases.

1. Its left neighbor is β_{t-1} where its projection is $\beta_{t,L}$, and its right neighbor projection is being squeezed

into an interval of length bounded by $K\ell_{\mathcal{X}}(\alpha_i)$.

2. Its left neighbor projection is being squeezed into an interval of length bounded by $K\ell_{\mathcal{X}}(\alpha_i)$, and its right neighbor is β_{t+1} where its projection is $\beta_{t,L}$.

3. Both of its left and right neighbor projection are being squeezed into an interval of length bounded by $K\ell_{\mathcal{X}}(\alpha_i)$ respectively.

In any case, the diameter of the union of its left and right projection is bounded by

$$(1+K)\ell_{\mathcal{X}}(\alpha_i)+2\ell_{\mathcal{X}}(\tau)=\max\left\{\ell_{\mathcal{X}}(\alpha_i)+K\ell_{\mathcal{X}}(\alpha_i)+2\ell_{\mathcal{X}}(\tau),2K\ell_{\mathcal{X}}(\alpha_i)+2\ell_{\mathcal{X}}(\tau)\right\},$$

which is like the upper bound for $\operatorname{diam}(\beta_{i,L}\cup\beta_{i,R})$ in the proof of Theorem IV.1.2. Apply the same argument from Theorem IV.1.2 about "D_i", for any effective intersection point z_j, and for its corresponding y_j and $\beta_{i(j)}$, it realizes a distance no less than

$$D_i(K)=\lfloor(|a_i|-1-K)\cdot\ell_{\mathcal{X}}(\alpha_i)-2\ell_{\mathcal{X}}(\tau)-L\rfloor_0$$

after twisting.

For example, let's consider the situation in Figure IV.2. z_0 is the only one in vain and is not "included" in $T_i(K)$. z_1, z_2 realize translation distances no less than

$$\lfloor(|a_i|-1)\cdot\ell_{\mathcal{X}}(\alpha_i)-2\ell_{\mathcal{X}}(\tau)-L\rfloor_0$$

after twisting, because the union of projections of their left and right neighbors are bounded by $\ell_{\mathcal{X}}(\alpha_i)$. z_3, z_4 realize translation distances no less than

$$\lfloor(|a_i|-1-K)\cdot\ell_{\mathcal{X}}(\alpha_i)-2\ell_{\mathcal{X}}(\tau)-L\rfloor_0$$

after twisting since the union of projections of their left and right neighbors are bounded by $(1+K)\ell_{\mathcal{X}}(\alpha_i)$.

Finally, following the same procedure from Theorem IV.1.2 and only counting the sum of minimum distances realized by effective intersection points, we have

$$\ell_{T_\alpha\mathcal{X}}(\tau)\geq\sum_{i=1}^{k}T_i(K)\cdot D_i(K).$$

This gives us the desired result. □

Notice that the bounds in Proposition IV.1.1, Theorem IV.1.2, Proposition IV.1.5 involve both the lengths $\ell_{\mathcal{X}}(\alpha_i)$ and the intersection numbers $i(\tau, \alpha_i)$. Note also that the lower bounds we obtain are all vacuous in the case where the multicurve α is of mixed sign with all coefficients having absolute value 1. The following example shows that there cannot exist a general lower bound, on the order of $\sum_{i=1}^{k} i(\tau, \alpha_i)\ell_{\mathcal{X}}(\alpha_i)$ as in above results, that is effective in this case.

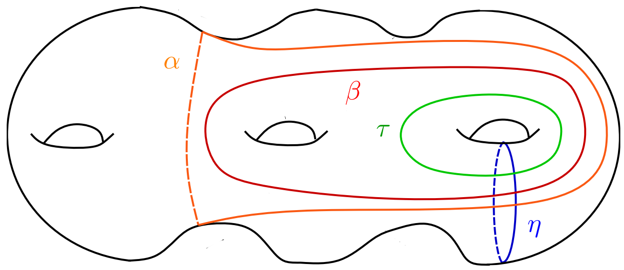

Figure IV.3: A special example.

Example IV.1.6. Consider simple closed curves $\alpha, \beta, \eta, \tau$ on a hyperbolic surface $\mathcal{X}$ as in the Figure IV.3 above. We can define two sequences of simple closed curves

$$\alpha_i = T_\eta^i(\alpha), \beta_j = T_\eta^j(\beta),$$

and note that the lengths $\ell_{\mathcal{X}}(\alpha_n), \ell_{\mathcal{X}}(\beta_n)$ and intersection numbers $i(\tau, \alpha_n), i(\tau, \beta_n)$ all tend to infinity with n. Denote the multicurve $\gamma_n = \alpha_n - \beta_n$ for each n, then $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of multicurves with mixed sign and with each coefficient having absolute value equal to 1. Hence neither Proposition IV.1.1, Theorem IV.1.2, nor Proposition IV.1.5 provides an effective lower bound on $\ell_{T_{\gamma_n}\mathcal{X}}(\tau)$. One can use train tracks [37] to study the images $T_{\gamma_n}(\tau)$ is, and then use length formula (II.4) to verify that

$$\ell_{T_{\gamma_n}\mathcal{X}}(\tau) \overset{*}{\asymp} \ell_{\mathcal{X}}(\gamma_n) = \ell_{\mathcal{X}}(\alpha_n) + \ell_{\mathcal{X}}(\beta_n)$$

up to an uniform multiplicative error for all $n \in \mathbb{N}$. In particular, we notice the intersection numbers $i(\tau, \alpha_n), i(\tau, \beta_n)$, which go to infinity as n goes to infinity, do not play any roles in the length of $\ell_{\mathcal{X}}(\gamma_n)$. Thus there does not exist a constant λ such that

$$\ell_{T_{\gamma_n}\mathcal{X}}(\tau) \geq \lambda i(\tau, \alpha_n)\ell_{\mathcal{X}}(\alpha_n) + \lambda i(\tau, \beta_n)\ell_{\mathcal{X}}(\beta_n).$$

IV.2 Proof of Theorem G, Coarse Distance Formula

In this section we will adopt the results from the previous section to establish a coarse distance formula, estimating how far a point in Teichmüller space moves after applying a twist.

The lemma below provides a lower bound of the length of $\ell_{T_\alpha \mathcal{X}}(\kappa)$ where α is a multicurve of mixed sign where each coefficient has absolute value ≥ 2, and κ is a particular curve in the short marking $\mu_{\mathcal{X}}$.

Lemma IV.2.1. *Fix some $\epsilon > 0$, there exists $E, Q > 0$ depends on ϵ so the following holds true. Given any $\mathcal{X} \in \mathcal{T}_{g,n}^{\epsilon}$, let $\alpha = \sum_{i=1}^{k} a_i \alpha_i$ be a multicurve where each coefficient has absolute value ≥ 2, and satisfying $\ell_{\mathcal{X}}(\alpha) \geq E$. Let j denote an index such that $|a_j| \ell_{\mathcal{X}}^2(\alpha_j) = \max_{1 \leq i \leq k} |a_i| \ell_{\mathcal{X}}^2(\alpha_i)$. Let $\kappa \in \mu_{\mathcal{X}}$ be a marking curve such that $i(\alpha_j, \kappa) = \max_{\eta \in \mu_{\mathcal{X}}} i(\alpha_j, \eta)$, we have*

$$\ell_{T_\alpha \mathcal{X}}(\kappa) \geq \frac{1}{Q} \cdot i(\alpha_j, \kappa) |a_j| \ell_{\mathcal{X}}(\alpha_j). \tag{IV.5}$$

Proof. Since $k \leq \frac{h}{2}$, $\ell_{\mathcal{X}}(\alpha) \geq E$ and since any simple closed curve has length $\geq \epsilon$, we have

$$|a_j| \ell_{\mathcal{X}}(\alpha_j) \geq \sqrt{|a_j|^2 \ell_{\mathcal{X}}^2(\alpha_j)} \geq \sqrt{|a_j| \ell_{\mathcal{X}}^2(\alpha_j)} \geq \sqrt{\frac{2\epsilon E}{h}}.$$

First, we consider the case $|a_j| \geq 3$. Since $|a_j| - 2 \geq \frac{1}{3}|a_j|$ for any $|a_j| \geq 3$, by assuming $E \geq \frac{18h(2N+L)^2}{\epsilon}$, we have

$$\begin{aligned}
(|a_j| - 2)\ell_{\mathcal{X}}(\alpha_j) &\geq \frac{1}{3}|a_j| \ell_{\mathcal{X}}(\alpha_j) \geq \frac{1}{3} \cdot \sqrt{\frac{\epsilon E}{k}} \geq 2(2N + L), \\
(|a_j| - 2)\ell_{\mathcal{X}}(\alpha_j) - 2N - L &\geq \frac{1}{6}|a_j| \ell_{\mathcal{X}}(\alpha_j).
\end{aligned}$$

Thus for this particular j and κ, since $\ell_{\mathcal{X}}(\kappa) \leq N$ we have

$$i(\alpha_j, \kappa) \left[(|a_j| - 2)\ell_{\mathcal{X}}(\alpha_j) - 2\ell_{\mathcal{X}}(\kappa) - L \right] \geq \frac{1}{6} \cdot i(\alpha_j, \kappa) |a_j| \ell_{\mathcal{X}}(\alpha_j).$$

Apply Theorem IV.1.2, we conclude

$$\ell_{T_\alpha \mathcal{X}}(\kappa) \geq \frac{1}{6} \cdot i(\alpha_j, \kappa) |a_j| \ell_{\mathcal{X}}(\alpha_j).$$

Now, we consider the case $|a_j| = 2$. Recall by fixing ϵ, any short marking curve corresponding to any $\mathcal{X} \in \mathcal{T}_{g,n}^{\epsilon}$ has length bounded on top by N. By Collar Lemma II.4.2, there exists a W depends on ϵ such that collar width of any short marking curve corresponding to any $\mathcal{X} \in \mathcal{T}_{g,n}^{\epsilon}$ has length bounded below by W.

This means we have

$$\frac{K\ell_{\mathcal{X}}(\alpha_j)+4\ell_{\mathcal{X}}(\kappa)}{W(\kappa)} \leq \frac{hKCNi(\alpha_j,\kappa)+4N}{W}.$$

Since we have $\ell_{\mathcal{X}}(\alpha_j) \geq \frac{1}{2}\sqrt{\frac{2\epsilon E}{h}}$, by length formula (II.4),

$$i(\alpha_j,\kappa) \geq \frac{\ell(\alpha_j)}{hCN} \geq \frac{1}{hCN}\sqrt{\frac{\epsilon E}{2h}}.$$

Let $E \geq \frac{32hN^2}{\epsilon K^2}$ so that

$$hKCNi(\alpha_j,\kappa) \geq 4N,$$

and take the constant $K = \min\{\frac{W}{4hCN}, \frac{1}{2}\}$, we have

$$\frac{hKCNi(\alpha_j,\kappa)+4N}{W} \leq \frac{2hCNK}{W} i(\alpha_j,\kappa) \leq \frac{1}{2} i(\alpha_j,\kappa). \tag{IV.6}$$

Moreover, by further assuming $E \geq \frac{12^2h(2N+L)^2}{2\epsilon}$, we have $\ell_{\mathcal{X}}(\alpha_j) \geq 6(2N+L)$. Since $|a_j| = 2, K \leq \frac{1}{2}$, we have

$$(|a_j|-1-K)\cdot \ell_{\mathcal{X}}(\alpha_i) - 2N - L \geq \frac{1}{2}\ell_{\mathcal{X}}(\alpha_j) - 2N - L \geq \frac{1}{3}\ell_{\mathcal{X}}(\alpha_j). \tag{IV.7}$$

Now we can apply Proposition IV.1.5. In the $\mathcal{L}_1$ case, $|a_j| = 2$ and the lower bound is

$$\ell_{T_\alpha\mathcal{X}}(\kappa) \geq i(\alpha_j,\kappa)\left(K\ell_{\mathcal{X}}(\alpha_j) - 2N - L\right).$$

By further assuming

$$E \geq \max\left\{\frac{12^2h(2N+L)^2}{2\epsilon}, \frac{h}{2\epsilon}\left(\frac{16hCN(2N+L)}{W}\right)^2\right\},$$

we have

$$\begin{aligned} &K\ell_{\mathcal{X}}(\alpha_j) - 2N - L \geq \frac{1}{2}K\ell_{\mathcal{X}}(\alpha_j), \\ &\ell_{T_\alpha\mathcal{X}}(\kappa) \geq \min\left\{\frac{W}{8hCN}, \frac{1}{4}\right\} \cdot i(\alpha_j,\kappa)\ell_{\mathcal{X}}(\alpha_j). \end{aligned}$$

In the $\mathcal{L}_2$ case, our previous formulas (IV.6), (IV.7) guarantee

$$\begin{aligned}\ell_{T_\alpha\mathcal{X}}(\tau) &\geq \left\lfloor i(\kappa,\alpha_j) - \frac{K\ell_{\mathcal{X}}(\alpha_j)+2\ell_{\mathcal{X}}(\kappa)}{W(\kappa)}\right\rfloor_0 \cdot \left\lfloor (|a_j|-1-K)\cdot \ell_{\mathcal{X}}(\alpha_j) - 2\ell_{\mathcal{X}}(\kappa) - L\right\rfloor_0 \\ &\geq \frac{1}{2} i(\kappa,\alpha_j)\cdot\frac{1}{3}\ell_{\mathcal{X}}(\alpha_j) \geq \frac{1}{6}\cdot i(\kappa,\alpha_j)\ell_{\mathcal{X}}(\alpha_j).\end{aligned}$$

Let

$$\begin{aligned}&E = \max\left\{\frac{12^2h(2N+L)^2}{\epsilon}, \frac{32hN^2}{\epsilon K^2}, \frac{h}{2\epsilon}\left(\frac{16hCN(2N+L)}{W}\right)^2\right\}, \\ &Q = \min\left\{\frac{1}{6}, \frac{W}{8hCN}\right\}.\end{aligned}$$

The result follows. □

Now we are ready to prove the main result of this section.

Theorem G (Coarse Distance Formula)**.** Fix some $S_{g,n}$ and given any $\epsilon > 0$, there exists a constant $H > 0$ such that given any $\mathcal{X} \in \mathcal{T}^{\epsilon}_{g,n}$, we have

$$d_{\mathcal{T}}(\mathcal{X}, T_\alpha\mathcal{X}) \overset{+H}{\asymp} \log\left(\sum_{i=1}^{k}|a_i|\ell^2_{\mathcal{X}}(\alpha_i)\right) \tag{IV.8}$$

for any $\alpha = \sum_{i=1}^{k} a_i\alpha_i \in \mathcal{ML}^*(\mathbb{Z})$.

Proof. By the distance formula (II.3) and our formula (IV.1), we have

$$\begin{aligned}d_{\mathcal{T}}(\mathcal{X}, T_\alpha\mathcal{X}) &\leq \log\left(\frac{e^c}{\epsilon}\max_{\tau\in\mu_{\mathcal{X}}}\ell_{T_\alpha\mathcal{X}}(\tau)\right) \\ &\leq \log\left(\frac{Ne^c}{\epsilon} + \frac{e^c}{\epsilon}\cdot\max_{\tau\in\mu_{\mathcal{X}}}\sum_{i=1}^{k} i(\tau,\alpha_i)|a_i|\ell_{\mathcal{X}}(\alpha_i)\right) \\ &\leq \log\left(\frac{Ne^c}{\epsilon} + \frac{e^c}{\epsilon}\cdot\sum_{\tau\in\mu_{\mathcal{X}}}\sum_{i=1}^{k} i(\tau,\alpha_i)|a_i|\ell_{\mathcal{X}}(\alpha_i)\right) \\ &= \log\left(\frac{Ne^c}{\epsilon} + \frac{e^c}{\epsilon}\cdot\sum_{i=1}^{k}\left(\sum_{\tau\in\mu_{\mathcal{X}}} i(\tau,\alpha_i)\right)|a_i|\ell_{\mathcal{X}}(\alpha_i)\right) \\ &\leq \log\left(\frac{Ne^c}{\epsilon} + \frac{e^cC}{\epsilon^2}\cdot\sum_{i=1}^{k}|a_i|\ell^2_{\mathcal{X}}(\alpha_i)\right)\end{aligned}$$

where last inequality holds by applying the length formula (II.4). Since we always have $|a_i|\ell^2_{\mathcal{X}}(\alpha_i) \geq \epsilon^2$, by using equality $\log(a+b) = \log(1+\frac{a}{b}) + \log(b)$ we have

$$d_{\mathcal{T}}(\mathcal{X}, T_\alpha \mathcal{X}) \leq \log\left(1 + \frac{\frac{Ne^c}{\epsilon}}{\frac{e^c C}{\epsilon^2}\cdot \sum_{i=1}^k |a_i|\ell^2_{\mathcal{X}}(\alpha_i)}\right) + \log\left(\frac{e^c C}{\epsilon^2}\cdot \sum_{i=1}^k |a_i|\ell^2_{\mathcal{X}}(\alpha_i)\right)$$
$$\leq \log\left(1 + \frac{N}{h\epsilon C}\right) + \log\left(\frac{e^c C}{\epsilon^2}\right) + \log\left(\sum_{i=1}^k |a_i|\ell^2_{\mathcal{X}}(\alpha_i)\right).$$

This gives us the upper bound in (IV.8) after setting appropriate H.

Now we work toward the lower bound in (IV.8). Let's assume that $\ell_{\mathcal{X}}(\alpha) \geq E$, E from Lemma IV.2.1. First, we consider the case that α is positive or negative, i.e., all coefficients have the same sign. In this case, by applying Proposition IV.1.1, similar to the argument obtaining the upper bound, we have

$$d_{\mathcal{T}}(\mathcal{X}, T_\alpha \mathcal{X}) \geq \log\left(\frac{1}{Ne^c}\max_{\tau\in\mu_{\mathcal{X}}} \ell_{T_\alpha \mathcal{X}}(\tau)\right)$$
$$\geq \log\left(\frac{1}{ANe^c}\right) + \log\left(\max_{\tau\in\mu_{\mathcal{X}}} \sum_{i=1}^k |a_i| i(\alpha_i,\tau)\ell_{\mathcal{X}}(\alpha_i) - N\right)$$
$$\geq \log\left(\sum_{i=1}^k |a_i|\ell^2_X(\alpha_i)\right) - \log\left(2hACN^2 e^c\right)$$

Now, we consider the case where α is of mixed sign where each coefficient has absolute value ≥ 2. Let j be the index of α such that $|a_j|\ell^2_{\mathcal{X}}(\alpha_j) = \max_{1\leq i\leq k}|a_i|\ell^2_{\mathcal{X}}(\alpha_i)$ and let $\kappa \in \mu_{\mathcal{X}}$ be the curve realizes $\max_{\eta\in\mu_{\mathcal{X}}} i(\alpha_j, \eta)$, then by previous Lemma IV.2.1 we have

$$\max_{\eta\in\mu_{\mathcal{X}}} \ell_{T_\alpha\mathcal{X}}(\eta) \geq \ell_{T_\alpha\mathcal{X}}(\kappa) \geq \frac{1}{Q}\cdot i(\alpha_j,\kappa)|a_j|\ell_{\mathcal{X}}(\alpha_j).$$

Apply the length formula (II.4), we have

$$\max_{\eta\in\mu_{\mathcal{X}}} \ell_{T_\alpha\mathcal{X}}(\eta) \geq \frac{1}{Q} i(\alpha_j,\kappa)|a_j|\ell_{\mathcal{X}}(\alpha_j) = \frac{1}{Q}\max_{\eta\in\mu_{\mathcal{X}}} i(\alpha_j,\eta)|a_j|\ell_{\mathcal{X}}(\alpha_j)$$
$$\geq \frac{1}{Q}\frac{\sum_{\eta\in\mu_{\mathcal{X}}} i(\alpha_j,\eta)}{h}|a_j|\ell_{\mathcal{X}}(\alpha_j) \geq \frac{1}{hQCN}|a_j|\ell^2_{\mathcal{X}}(\alpha_j)$$
$$\geq \frac{1}{hQCN}\max_{1\leq i\leq k}|a_i|\ell^2_{\mathcal{X}}(\alpha_i).$$

Apply the distance formula (II.3), we have

$$\begin{aligned} d_{\mathcal{T}}(\mathcal{X}, T_\alpha \mathcal{X}) &\geq \log\left(\frac{1}{hQCN^2e^c} \max_{1\leq i\leq k} |a_i|\ell^2_{\mathcal{X}}(\alpha_i)\right) \\ &\geq \log\left(\frac{1}{h^2QCN^2e^c} \sum_{i=1}^{k} |a_i|\ell^2_{\mathcal{X}}(\alpha_i)\right) \\ &= \log\left(\sum_{i=1}^{k} |a_i|\ell^2_{\mathcal{X}}(\alpha_i)\right) - \log\left(h^2QCN^2e^c\right). \end{aligned}$$

And this gives us the lower bound in (IV.8) after setting appropriate H. Notice the above holds true for any $\alpha \in \mathcal{ML}^*_{\geq E}(\mathbb{Z})$. For any $\alpha \in \mathcal{ML}(\mathbb{Z})$ that has length bounded by E, we have $\log\left(\sum_{i=1}^{k} |a_i|\ell^2_{\mathcal{X}}(\alpha_i)\right) \leq \log\left(\ell^2_{\mathcal{X}}(\alpha)\right) \leq 2\log(E)$.

Finally, we set

$$\begin{aligned} H = \max\{ &\log\left(1 + \frac{N}{h\epsilon C}\right) + \log\left(\frac{e^cC}{\epsilon^2}\right), \\ &\log\left(2hACN^2e^c\right), \log\left(h^2QCN^2e^c\right), 2\log(E)\}. \end{aligned}$$

The result follows. ∎

Remark IV.2.2. Consider Example IV.1.6, if τ is chosen to be a short marking curve, using similar idea in Theorem G, we would have

$$d_{\mathcal{T}}(\mathcal{X}, T_{\gamma_n}\mathcal{X}) \stackrel{+H}{\asymp} \log\left(\ell_{\mathcal{X}}(\alpha_n) + \ell_{\mathcal{X}}(\beta_n)\right)$$

for some H. This implies our coarse distance formula does not hold for this sequence of multicurves $\{\gamma_n\} \subset \mathcal{ML}(\mathbb{Z}) \setminus \mathcal{ML}^*(\mathbb{Z})$.

IV.3 Proof of Theorem D, Corollary E, and Corollary F

Assume the conditions in Theorem D, let D be one of the three sets $[T_\gamma]$, $\mathbf{D}$, $\mathbf{M}$. Since any mapping class is an isometry for $\mathcal{T}_{g,n}$, given any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}^\epsilon_{g,n}$, we notice

$$|D\cdot\mathcal{X} \cap B_{R-d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})}(\mathcal{X})| \leq |D\cdot\mathcal{Y}\cap B_R(\mathcal{X})| \leq |D\cdot\mathcal{X}\cap B_{R+d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})}(\mathcal{X})|. \qquad \text{(IV.9)}$$

Also, recall that

$$D\cdot\mathcal{X}\cap B_R(\mathcal{X}) = \{g\cdot\mathcal{X}\in\mathcal{T}_{g,n} \mid g\in D, d_{\mathcal{T}}(g\cdot\mathcal{X},\mathcal{X})\leq R\}.$$

Since twists never stabilize any point in $\mathcal{T}_{g,n}$, we have

$$|D \cdot \mathcal{X} \cap B_R(\mathcal{X})| = |\{g \in D \mid d_{\mathcal{T}}(g \cdot \mathcal{X}, \mathcal{X}) \leq R\}|.$$

For simplicity of notation, we use $t = \frac{h}{2}$ to denote half the dimension of $\mathcal{T}_{g,n}$. Recall we say $f(R) \overset{*A}{\sim} g(R)$ if for any $\lambda > 1$, there exists a $U(\lambda)$ such that $\frac{1}{\lambda A} \leq \frac{g(R)}{f(R)} \leq \lambda A$ for any $R \geq U(\lambda)$. We are now ready to prove Theorem D and Corollary E. For each case and for any $\lambda > 1$, we will compute the corresponding $U(\lambda)$.

Theorem D. Given any $S_{g,n}$ and given any $\epsilon > 0$, there exists a $J > 0$ such that for any multicurve $\gamma \in \mathcal{ML}^*(\mathbb{Z})$ and for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}^{\epsilon}_{g,n}$, we have

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, T_\gamma) \overset{*JF_\gamma(\mathcal{X},\mathcal{Y})}{\sim} n_X(\gamma) \cdot e^{\frac{h}{2}R}$$

where $n_X(\gamma)$ is the corresponding Mirzakhani constant, see section II.7.

Proof of Theorem D. Let $\gamma = \sum_{i=1}^k a_i \gamma_i$ be a multicurve, then $c_\alpha = c_\gamma$ for any $\alpha \in \mathcal{ML}(\gamma)$. We consider the corresponding set of twists around curves of topological type γ

$$[T_\gamma] = \{T_\alpha \mid \alpha \in \mathcal{ML}(\gamma)\}.$$

Define

$$\begin{aligned}
\mathcal{S}_R^{\pm} &= \left\{ \alpha = \sum_{i=1}^k a_i \alpha_i \in \mathcal{ML}(\gamma) \mid \sum_{i=1}^k |a_i| \ell^2_{\mathcal{X}}(\alpha_i) \leq e^{R \pm H} \right\}, \\
\mathcal{S}_R^{++} &= \left\{ \alpha = \sum_{i=1}^k a_i \alpha_i \in \mathcal{ML}(\gamma) \mid \ell_{\mathcal{X}}(\alpha) \leq \sqrt{c_\alpha} \cdot e^{(R+H)/2} \right\}, \\
\mathcal{S}_R^{--} &= \left\{ \alpha = \sum_{i=1}^k a_i \alpha_i \in \mathcal{ML}(\gamma) \mid \ell_{\mathcal{X}}(\alpha) \leq e^{(R-H)/2} \right\}.
\end{aligned}$$

By the coarse distance formula (IV.8), we have

$$|\mathcal{S}_R^-| \leq |[T_\gamma] \cdot \mathcal{X} \cap B_R(\mathcal{X})| \leq |\mathcal{S}_R^+|.$$

Since

$$\sum_{i=1}^k |a_i| \ell^2_{\mathcal{X}}(\alpha_i) \leq \sum_{i=1}^k |a_i|^2 \ell^2_{\mathcal{X}}(\alpha_i) \leq (\ell_{\mathcal{X}}(\alpha))^2$$

we have $\mathcal{S}_R^{--} \subset \mathcal{S}_R^{-}$. Moreover, by Schwartz inequality, we have

$$(\ell_{\mathcal{X}}(\alpha))^2 = (\sum_{i=1}^{k} |a_i|\ell_{\mathcal{X}}(\alpha_i))^2 \leq (\sum_{i=1}^{k} |a_i|) \cdot \sum_{i=1}^{k} |a_i|\ell_{\mathcal{X}}^2(\alpha_i) = c_\alpha \cdot \sum_{i=1}^{k} |a_i|\ell_{\mathcal{X}}^2(\alpha_i) \tag{IV.10}$$

so $\mathcal{S}_R^{+} \subset \mathcal{S}_R^{++}$. Together this means

$$|\mathcal{S}_R^{--}| \leq |[T_\gamma] \cdot \mathcal{X} \cap B_R(\mathcal{X})| \leq |\mathcal{S}_R^{++}|.$$

Mirzakhani's counting formula (II.6) tells us for any $\lambda > 1$, we have

$$|\mathcal{S}_R^{--}| = s_X(e^{(R-H)/2}, \gamma) \geq \frac{1}{\lambda} \cdot n_X(\gamma) \cdot e^{t(R-H)},$$
$$|\mathcal{S}_R^{++}| = s_X(\sqrt{c_\alpha} \cdot e^{(R+H)/2}, \gamma) \leq \lambda \cdot n_X(\gamma) \cdot c_\gamma^t \cdot e^{t(R+H)}$$

whenever $R \geq U(\lambda) = 2\log(r_X(\gamma, \lambda)) + H$. This means

$$\frac{1}{\lambda} \cdot n_X(\gamma) \cdot e^{t(R-H)} \leq |[T_\gamma] \cdot \mathcal{X} \cap B_R(\mathcal{X})| \leq \lambda \cdot c_\gamma^t \cdot n_X(\gamma) \cdot e^{t(R+H)}$$

whenever $R \geq U(\lambda)$. By equation (IV.9), we have

$$\frac{1}{\lambda \cdot e^{t(d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})+H)}} \leq \frac{|[T_\gamma] \cdot \mathcal{Y} \cap B_R(\mathcal{X})|}{n_X(\gamma)e^{tR}} \leq \lambda \cdot c_\gamma^t \cdot e^{t(d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})+H)}$$

whenever $R \geq U(\lambda)$. Recall that we denote $F_\gamma(\mathcal{X}, \mathcal{Y}) = (c_\gamma)^t e^{t d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})}$. By setting $J = e^{tH}$, we are done with the case $D = [T_\gamma]$ and Theorem D follows. $\square$

Corollary E. Given $S_{g,n}$ and given any $\epsilon > 0$, for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}^{\epsilon}$, we have

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{D}) \overset{*JF(\mathcal{X},\mathcal{Y})}{\sim} n_X(\mathcal{S}) \cdot e^{\frac{h}{2}R}, \text{ if } h > 0,$$
$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{M}) \overset{*8JF(\mathcal{X},\mathcal{Y})}{\sim} n_X(\mathcal{S}) \cdot e^{\frac{h}{2}R}, \text{ if } \frac{h}{2} > 1$$

where $n_X(\mathcal{S})$ is the corresponding Mirzakhani constant, see section II.7.

Proof of Corollary E. We observe in the above proof of Theorem D, when γ is a simple closed curve, $c_\gamma = 1$ and for $R \geq 2\log(r_X(\gamma, \lambda)) + H$ we have

$$\frac{1}{\lambda \cdot e^{t(d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})+H)}} \leq \frac{|\mathbf{D} \cdot \mathcal{Y} \cap B_R(\mathcal{X})|}{n_X(\gamma)e^{tR}} \leq \lambda \cdot e^{t(d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})+H)}.$$

Summing up all the topological types of simple closed curves, we have

$$\frac{1}{\lambda \cdot e^{t(d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})+H)}} \leq \frac{|\mathbf{D}\cdot\mathcal{Y}\cap B_R(\mathcal{X})|}{n_X(\mathcal{S})e^{tR}} \leq \lambda \cdot e^{t(d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})+H)}$$

whenever $R \geq U(\lambda) = 2\log(r_X(\mathcal{S},\lambda)) + H$. Recall we denote $F(\mathcal{X},\mathcal{Y}) = e^{td_{\mathcal{T}}(\mathcal{X},\mathcal{Y})}$ and $J = e^{tH}$. Thus we are done with the case $D = \mathbf{D}$ and the first result of Corollary E follows.

Now, we consider the set of Dehn twists with powers

$$\mathbf{M} = \{T_\alpha^n \mid \alpha \in \mathcal{S}, n \in \mathbb{Z}\setminus\{0\}\}.$$

Define

$$M_R^{\pm} = \left\{T_\alpha^n \in \mathbf{M} \mid |n|\ell_{\mathcal{X}}^2(\alpha) \leq e^{R\pm H}\right\},$$
$$\mathcal{S}_{R,n}^{\pm} = \left\{\alpha \in \mathcal{S} \mid \ell_{\mathcal{X}}(\alpha) \leq \frac{e^{(R\pm H)/2}}{\sqrt{|n|}}\right\}$$

so that

$$|M_R^{\pm}| = \sum_{n\in\mathbb{Z}\setminus\{0\}} |S_{R,n}^{\pm}| = 2\cdot\sum_{n\in\mathbb{N}} |S_{R,n}^{\pm}|.$$

Thus we only need to consider $n \in \mathbb{N}$. Since we are in $\mathcal{T}_{g,n}^{\epsilon}$, $S_{R,n}^{\pm}$ is empty when $n \geq \frac{e^{R\pm H}}{\epsilon^2}$, we have

$$|M_R^{\pm}| = 2\cdot\sum_{n=1}^{\frac{e^{R\pm H}}{\epsilon^2}} |S_{R,n}^{\pm}|.$$

Fix some $\lambda > 1$ and let's assume $r_X(\mathcal{S},\lambda) \geq \epsilon$. Let's also asssume that $R \geq 2\log(r_X(\mathcal{S},\lambda))$ so that $e^{R+H} \geq r_X^2(\mathcal{S},\lambda)$. For simplicity let's denote

$$a = \frac{e^{R+H}}{\epsilon^2}, b = \frac{e^{R+H}}{r_X^2(\mathcal{S},\lambda)}.$$

Then the above assumptions say $a \geq b \geq 1$. By Corollary II.7.3 and the Mirzakhani's counting formula (II.6), we have

$$\mathcal{S}_{R,n}^{+} = s_X(\frac{e^{(R+H)/2}}{\sqrt{n}},\mathcal{S}) \leq \lambda\cdot n_X(\mathcal{S})\cdot e^{t(R+H)}\cdot\frac{1}{n^t}$$

provided that $n \leq \min\{a, b\} = b$. Notice now we have

$$|M_R^+| \leq 2 \cdot \sum_{n=1}^{a} |\mathcal{S}_{R,n}^+| \leq 2 \cdot \sum_{n=1}^{b} |\mathcal{S}_{R,n}^+| + 2 \cdot \sum_{n=b}^{a} |\mathcal{S}_{R,n}^+|$$

where

$$\sum_{n=1}^{b} |\mathcal{S}_{R,n}^+| \leq \lambda \cdot n_X(\mathcal{S}) \cdot e^{t(R+H)} \cdot \sum_{n=1}^{b} \frac{1}{n^t},$$
$$\sum_{n=b+1}^{a} |\mathcal{S}_{R,n}^+| \leq \sum_{n=b+1}^{a} |\mathcal{S}_{R,b}^+| \leq \sum_{n=1}^{a} |\mathcal{S}_{R,b}^+| \leq \frac{s_X(r_X(\mathcal{S}, \lambda), \mathcal{S})}{\epsilon^2} \cdot e^{R+H}$$

whenever $R \geq c_\lambda = 2\log(r_X(\mathcal{S}, \lambda))$.

When $t > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^t}$ converges and is bounded by 2. By assuming R is even larger,

$$R \geq d_\lambda = \log\left(\frac{s_X(r_X(\mathcal{S}, \lambda), \mathcal{S})}{2\lambda\epsilon^2 n_X(\mathcal{S})}\right),$$

bigger exponential "wins" and we have

$$2\lambda \cdot n_X(\mathcal{S}) \cdot e^{t(R+H)} \geq \frac{s_X(r_X(\mathcal{S}, \lambda), \mathcal{S})}{\epsilon^2} \cdot e^{R+H}$$

Thus we have

$$|M_R^+| \leq 8\lambda \cdot n_X(\mathcal{S}) \cdot e^{t(R+H)}$$

whenever $R \geq \max\{c_\lambda, d_\lambda\}$. By the coarse distance formula (IV.8) we have

$$|\mathcal{S}_R^{--}| \leq |\mathbf{M} \cdot \mathcal{X} \cap B_R(\mathcal{X})| \leq |M_R^+|,$$

$|\mathcal{S}_R^{--}|$ is from the proof of Theorem D. Similar to the previous cases, we have

$$\frac{1}{\lambda e^{t(d_{\mathcal{T}}(\mathcal{X}, \mathcal{Y})+H)}} \leq \frac{|\mathbf{M} \cdot \mathcal{Y} \cap B_R(\mathcal{X})|}{n_X(\mathcal{S})e^{tR}} \leq \lambda \cdot 8e^{t(d_{\mathcal{T}}(\mathcal{X}, \mathcal{Y})+H)}$$

whenever $R \geq U(\lambda) = \max\{r_X(\mathcal{S}, \lambda), c_\lambda, d_\lambda\}$. The second result of Corollary E follows. □

Corollary F. Given any $S_{g,n}$ and given any $\epsilon > 0$, for any multicurve $\gamma \in \mathcal{ML}^*(\mathbb{Z})$ and for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}^{\epsilon}$, we have

$$\lim_{R\to\infty}\frac{1}{R}\ln\Gamma_R(\mathcal{X},\mathcal{Y},T_\gamma)=\frac{h}{2}.$$
$$\lim_{R\to\infty}\frac{1}{R}\ln\Gamma_R(\mathcal{X},\mathcal{Y},\mathbf{D})=\frac{h}{2},\text{ if } h>0.$$
$$\lim_{R\to\infty}\frac{1}{R}\ln\Gamma_R(\mathcal{X},\mathcal{Y},\mathbf{M})=\frac{h}{2},\text{ if } \frac{h}{2}>1.$$

Proof of Corollary F. This is similar to how Corollary B implies Corollary C. □

Proof of upper bound of Corollary I. We now consider $t=\frac{h}{2}=1$. Since in this case $\mathcal{ML}(\mathbb{Z})$ is 1 dimensional, we have $\mathbf{T}=\mathbf{M}$.

Notice from the above proof of $\mathbf{M}$, when $t=1$, $\sum_{n=1}^{b}\frac{1}{n}\leq\log(b+1)$ where

$$\log(b+1)=\log(\frac{e^{R+H}}{r_X^2(\mathcal{S},\lambda)}+1)\leq R$$

by assuming $r_X(\mathcal{S},\lambda)$ sufficiently large. Moreover, we have

$$\lambda\cdot n_X(\mathcal{S})\cdot R\cdot e^{R+H}\geq\frac{s_X(r_X(\mathcal{S},\lambda),\mathcal{S})}{\epsilon^2}\cdot e^{R+H}$$

when R is large $\left(R\geq l_\lambda=\frac{s_X(r_X(\mathcal{S},\lambda),\mathcal{S})}{\lambda\epsilon^2 n_X(\mathcal{S})}\right)$. Thus we have

$$|M_R^+|\leq 4\lambda\cdot n_X(\mathcal{S})\cdot R\cdot e^{R+H}$$

whenever $R\geq U(\lambda)=\max\{c_\lambda,l_\lambda\}$. Similar to previous arguments, this shows

$$|\mathbf{M}\cdot\mathcal{Y}\cap B_R(\mathcal{X})|\overset{*4JF(\mathcal{X},\mathcal{Y})}{\preceq}n_X(\mathcal{S})\cdot R\cdot e^{\frac{h}{2}R}$$

when $\frac{h}{2}=1$. □

IV.4 Proof of Theorem H and Corollary I

Let $\underline{\gamma}$ be a multicurve satisfying the conditions in Theorem H. Given $s\in\mathbb{N}$, we denote

$$[\underline{\gamma},s]=\{\gamma\in[\underline{\gamma}]\mid c_\gamma=s\},$$
$$\mathcal{ML}([\underline{\gamma},s])=\bigsqcup_{\gamma\in[\underline{\gamma},s]}\mathcal{ML}(\gamma),$$

and we denote $\#[\underline{\gamma}, s]$ the number of $\gamma \in [\underline{\gamma}], c_\gamma = s$. Indeed, this number equals, up to $\mathrm{Mod}_{g,n}$, the number of topological types of curves composing the set $\mathcal{ML}([\underline{\gamma}, s])$. For any $l < s \in \mathbb{N}$, we denote

$$[\underline{\gamma}, s, l] = \left\{ \gamma = \sum_{i=1}^{k} a_i \gamma_i \in [\underline{\gamma}, s] \mid |a_i| \geq l \text{ for any i} \right\},$$

and $\mathcal{ML}([\underline{\gamma}, s, l]), \#[\underline{\gamma}, s, l]$ are similarly defined as above.

Lemma IV.4.1. *Let $\underline{\gamma} = \sum_{i=1}^{k} \gamma_i$ be a multicurve with all coefficients equal to one and of maximal dimension $k = \frac{h}{2}$. For $s \geq h-2$ we have*

$$\#[\underline{\gamma}, s] \geq \frac{s^{k-1}}{2^{k-1}(k-1)!}. \tag{IV.11}$$

In particular, there exists a t such that for any $s \geq h-2$, we have

$$\#[\underline{\gamma}, s, \frac{s}{t}] \geq \frac{1}{2}\#[\underline{\gamma}, s] \geq \frac{s^{k-1}}{2^{k}(k-1)!}. \tag{IV.12}$$

Proof. The number $\#[\underline{\gamma}, s]$ equals to the number of ordered k-tuples $(x_1, \cdots, x_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^{k} x_i = s$. It's a standard combinatorics fact this number is $\binom{s-1}{k-1}$, which is greater than $\frac{s^{k-1}}{2^{k-1}(k-1)!}$ whenever $s \geq 2(k-1) = h-2$.

For any $x = (x_1, \cdots, x_k) \in \mathbb{R}_+^k$ we denote δ_x the corresponding dirac measure and denote $\|x\| = \sum_{i=1}^{k} x_i$. We define the following sets

$$\begin{aligned}
C &= \{x \in \mathbb{R}_+^k \mid \|x\| = 1\}, \\
C^t &= \{x \in C \mid x_i \geq \frac{1}{t} \text{ for any } i\}, \\
C_s &= \{x \in \mathbb{N}^k \mid \|x\| = s\}, \\
C_s^{\frac{s}{t}} &= \{x \in C_s \mid x_i \geq \frac{s}{t} \text{ for any i}\}
\end{aligned}$$

where $t, s \in \mathbb{N}$. Define the measures $\delta_s = \sum_{x \in C_s} \delta_{\frac{x}{s}}$ and $\delta_s^{\frac{s}{t}} = \sum_{x \in C_s^{\frac{s}{t}}} \delta_{\frac{x}{s}}$. Denote the standard probability measure on C as μ, a classic measure theory result says the following ratio converges, and we have

$$\lim_{s \to \infty} \frac{\#[\underline{\gamma}, s, \frac{t}{s}]}{\#[\underline{\gamma}, s]} = \lim_{s \to \infty} \frac{\delta_s^{\frac{s}{t}}(C)}{\delta_s(C)} = \frac{\mu(C^t)}{\mu(C)}.$$

Thus by picking a t large enough the second equation (IV.12) above holds true. $\square$

Corollary IV.4.2. *Let $\underline{\gamma} = \sum_{i=1}^{k} \gamma_i$ be a multicurve with all coefficients equal to one and of maximal* *dimension $k = \frac{h}{2}$. For any $\gamma \in [\underline{\gamma}, s, \frac{s}{t}]$, we have*

$$\begin{aligned}
&(\ell_{\mathcal{X}}(\gamma))^2 \geq \frac{s}{t} \cdot \sum_{i=1}^{k} |a_i| \ell^2_{\mathcal{X}}(\gamma_i), \\
&\frac{s}{t} \cdot \ell_{\mathcal{X}}(\underline{\gamma}) \leq \ell_{\mathcal{X}}(\gamma) \leq s \cdot \ell_{\mathcal{X}}(\underline{\gamma}).
\end{aligned} \tag{IV.13}$$

This means for any $\gamma \in [\underline{\gamma}, s, \frac{s}{t}]$ and any $\lambda > 1$, we have

$$s_X(L, \gamma) \geq s_X(L, s \cdot \underline{\gamma}) \geq \frac{1}{\lambda} \cdot \frac{L^h}{s^h} n_X(\underline{\gamma}) \tag{IV.14}$$

for $L \geq s \cdot r_X(\underline{\gamma}, \lambda)$.

Proof. The first two equations follow from the definition of $[\underline{\gamma}, s, \frac{s}{t}]$. The third equation follows from Corollary II.7.3. $\square$

Now we are ready to prove the Theorem H.

Theorem H. Given any $S_{g,n}$ such that $h > 0$, $\epsilon > 0$, and $\underline{\gamma} = \sum_{i=1}^{k} \gamma_i$ a multicurve with all coefficients equal to one and of maximal dimension $k = \frac{h}{2}$. There exists a number $f(\underline{\gamma})$ such that, for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}^{\epsilon}_{g,n}$,

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T}) \geq \Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T}([\underline{\gamma}])) \overset{*JF(\mathcal{X},\mathcal{Y})}{\succeq} f(\underline{\gamma}) \cdot R \cdot e^{\frac{h}{2}R}.$$

Proof of Theorem H. Define

$$\begin{aligned}
&\mathcal{S}_R = \mathbf{T}([\underline{\gamma}]) \cdot \mathcal{X} \cap B_R(\mathcal{X}), \\
&\mathcal{S}^-_R = \left\{ \alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}([\underline{\gamma}]) \mid \sum_{i=1}^{k} |a_i| \ell^2_{\mathcal{X}}(\alpha_i) \leq e^{R-H} \right\}, \\
&\mathcal{S}^-_R(s) = \left\{ \alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}([\underline{\gamma}, s]) \mid \sum_{i=1}^{k} |a_i| \ell^2_{\mathcal{X}}(\alpha_i) \leq e^{R-H} \right\}, \\
&\mathcal{S}^-_R(s, \frac{s}{t}) = \left\{ \alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}([\underline{\gamma}, s, \frac{s}{t}]) \mid \ell_{\mathcal{X}}(\alpha) \leq \sqrt{\frac{s}{t}} \cdot e^{\frac{R-H}{2}} \right\}.
\end{aligned}$$

Notice for any $\alpha \in \mathcal{S}^-_R(s, \frac{s}{t})$, by previous Corollary IV.4.2 we have

$$\frac{s}{t} \cdot \sum_{i=1}^{k} |a_i| \ell^2_{\mathcal{X}}(\alpha_i) \leq (\ell_{\mathcal{X}}(\alpha))^2 \leq \frac{s}{t} \cdot e^{R-H}$$

so that $S^-_R(s, \frac{s}{t}) \subset \mathcal{S}^-_R(s)$.

Fix some $\lambda > 1$, given that

$$\sqrt{\frac{s}{t}} \cdot e^{\frac{R-H}{2}} \geq s \cdot r_X(\underline{\gamma}, \lambda) \text{ and } s \geq h-2,$$

and by Theorem II.7.1 and formulas (IV.12), (IV.14), we have

$$\begin{aligned}
\left|\mathcal{S}_R^-(s, \frac{s}{t})\right| &\geq \sum_{\gamma \in [\underline{\gamma}, s, \frac{s}{t}]} s_X\left(\sqrt{\frac{s}{t}} \cdot e^{\frac{R-H}{2}}, \gamma\right) \\
&\geq \frac{1}{\lambda} \cdot \#[\underline{\gamma}, s, \frac{s}{t}] \cdot \frac{\left(\sqrt{\frac{s}{t}} \cdot e^{\frac{R-H}{2}}\right)^h}{s^h} \cdot n_X(\underline{\gamma}) \\
&\geq \frac{1}{\lambda} \cdot \frac{s^{k-1}}{2^k(k-1)!} \cdot \frac{e^{k(R-H)}}{s^k t^k} \cdot n_X(\underline{\gamma}) \\
&= \frac{1}{\lambda} \cdot \frac{n_X(\underline{\gamma}) \cdot e^{k(R-H)}}{2^k(k-1)! t^k} \cdot \frac{1}{s}
\end{aligned}$$

That is,

$$h-2 \leq s \leq \frac{e^{R-H}}{t \cdot r_X^2(\underline{\gamma}, \lambda)}.$$

Thus, we have

$$\begin{aligned}
|\mathcal{S}_R| \geq |\mathcal{S}_R^-| &= \sum_{s \in \mathbb{N}} |\mathcal{S}_R^-(s)| \\
&\geq \sum_{s=h-2}^{\frac{e^{R-H}}{t \cdot r_X^2(\underline{\gamma}, \lambda)}} \left|\mathcal{S}_R^-(s, \frac{s}{t})\right| \\
&\geq \frac{1}{\lambda} \cdot \frac{n_X(\underline{\gamma}) \cdot e^{k(R-H)}}{2^k(k-1)! t^k} \cdot \sum_{s=h-2}^{\frac{e^{R-H}}{t \cdot r_X^2(\underline{\gamma}, \lambda)}} \frac{1}{s}
\end{aligned}$$

By assuming R is large,

$$R \geq U(\lambda) = 2\left(H + \log(t \cdot r_X(\underline{\gamma}, \lambda)) + \log(h-2)\right),$$

we have

$$\sum_{s=h-2}^{\frac{e^{R-H}}{t \cdot r_X^2(\underline{\gamma}, \lambda)}} \frac{1}{s} \geq \frac{R}{2}.$$

We now have

$$|\mathcal{S}_R| \geq \frac{1}{\lambda} \cdot \frac{n_X(\underline{\gamma})}{2^{k+1}(k-1)!t^k} \cdot R \cdot e^{\frac{hR}{2}}.$$

Similar to the proof of Theorem D, we have

$$\begin{aligned} \left|\mathbf{T}([\underline{\gamma}]) \cdot \mathcal{Y} \cap B_R(\mathcal{X})\right| &\geq \left|S_{R-d_{\mathcal{T}}(\mathcal{X},\mathcal{Y})}\right| \\ &\geq \frac{1}{\lambda JF(\mathcal{X},\mathcal{Y})} \cdot f(\underline{\gamma}) \cdot R \cdot e^{\frac{hR}{2}} \end{aligned}$$

whenever $R \geq U(\lambda)$, and

$$f(\underline{\gamma}) = \frac{n_X(\underline{\gamma})}{2^{k+1}(k-1)!t^k}$$

This concludes the proof of Theorem H. $\square$

Finally, we discuss about how Corollary I follows from previous results.

Corollary I. Given $S_{g,n}$ equal to $S_{1,1}$ or $S_{0,4}$ and given any $\epsilon > 0$. For any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}^{\epsilon}$, we have

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T}) \overset{*4JF(\mathcal{X},\mathcal{Y})}{\sim} n_X(\mathcal{S}) \cdot R \cdot e^R.$$

Proof of Corollary I. When $\frac{h}{2} = 1$, $\mathcal{ML}(\mathbb{Z})$ is one dimensional. Take any simple closed curve $\underline{\gamma}$, then it's maximal dimension and $\mathcal{ML}(\mathbb{Z}) = \mathcal{ML}([\underline{\gamma}])$. As a special case of Theorem H, we have $f(\underline{\gamma}) = n_X(\underline{\gamma}) = n_X(\mathcal{S})$ and

$$n_X(\mathcal{S}) \cdot R \cdot e^R \overset{*JF(\mathcal{X},\mathcal{Y})}{\preceq} |\mathbf{T} \cdot \mathcal{Y} \cap B_R(\mathcal{X})|$$

This gives us the lower bound. The upper bound follows from an alternation of proof of Corollary E, see proof of upper bound of Corollary I in Section IV.3. This concludes the result. $\square$

Remark IV.4.3. We briefly discuss about the difficultly using Theorem G to obtain an upper bound estimate for the coarse asymptotic rate of $|\mathbf{T} \cdot \mathcal{Y} \cap B_R(\mathcal{X})|$. When we are dealing with a conjugacy class of multicurves, say the conjugacy class of $\gamma = \sum_{i=1}^k a_i\gamma_i$, we have a "bounding relation" (IV.10) between $\ell_{\mathcal{X}}^2(\gamma)$ and $\sum_{i=1}^k |a_i|\ell_{\mathcal{X}}^2(\gamma_i)$ depends only on the coefficients of γ, and later we use this relation to estimate the number of corresponding lattice points inside a ball of radius R. In the case of $\mathcal{ML}([\underline{\gamma}])$, $\underline{\gamma}$ being a maximal dimensional multicurve with all coefficients equal to one, we consider a subset of $\mathcal{ML}([\underline{\gamma}])$ with "balanced weights" so that a uniform “bounding relation” (IV.13) holds. This idea in fact works for multicurves with“balanced weights”. Namely, for any $m \geq 0$, we can define the following subset of multicurves

$$\mathcal{ML}(\mathbb{Z}, m) = \{\alpha = \sum_{i=1}^{k} a_i \alpha_i \in \mathcal{ML}(\mathbb{Z}) \mid |a_i| \geq \frac{c_\alpha}{m} \text{ for each } i\}$$

and denote $\mathbf{T}(m)$ the corresponds set of twists. By using similar ideas one can show

$$|\mathbf{T}(m) \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \overset{*JF(\mathcal{X},\mathcal{Y})}{\preceq} s^{\frac{h}{2}} \cdot n_X(\mathcal{S}) \cdot R \cdot e^{\frac{h}{2}R}$$

However, for a sequence of multicurves $\{\gamma_j\}_{j \in \mathbb{N}}$ such that γ_j is outside $\mathcal{ML}(\mathbb{Z}, j)$, the possible “bounding relations” get more and more coarse, and does not yield a uniform upper bound as above for $\mathcal{ML}(\mathbb{Z})$.

CHAPTER V

Discussions

In Example IV.1.6, we have constructed a sequence of multicurves in $\mathcal{ML}(\mathbb{Z})\setminus\mathcal{ML}^*(\mathbb{Z})$ for which Theorem G does not hold, see Remark IV.2.2. Moreover, There exists a $H' > 0$ depends on $S_{g,n}$ and ϵ, so that for these multicurves the distances behave like

$$d_{\mathcal{T}}(\mathcal{X}, T_\alpha \mathcal{X}) \overset{+H'}{\asymp} \log\left(\sum_{i=1}^{k} |a_i| \ell_{\mathcal{X}}(\alpha_i)\right)$$

for any $\mathcal{X} \in \mathcal{T}_{g,n}^{\epsilon}$. This leads to the following question.

Question V.0.1. *for $\alpha \in \mathcal{ML}(\mathbb{Z}) \setminus \mathcal{ML}^*(\mathbb{Z})$, how does the length of any simple closed geodesic τ on a hyperbolic structure $\mathcal{X}$ changes after applying a twist T_α? How far does a point move in Teichmüller space after applying the corresponding twist T_α?*

Recall that Mirzakhani's Theorem II.7.1 says

$$|\{\alpha \in \mathcal{ML}(\cdot) \mid \ell_X(\alpha) \le e^{\frac{R}{2}}\}| \sim n_X(\cdot) \cdot e^{\frac{h}{2}R},$$

which is at the same coarse asymptotic rate of $|D \cdot \mathcal{Y} \cap B_R(\mathcal{X})|$ for the three cases as in Theorem D and Corollary E. Moreover, Mirzakhani [31] also proves that for any $X \in \mathcal{M}_{g,n}$, there exists a constant n_X such that

$$|\{\alpha \in \mathcal{ML}(\mathbb{Z}) \mid \ell_X(\alpha) \le e^{\frac{R}{2}}\}| \sim n_X \cdot e^{\frac{h}{2}R}.$$

We may wonder whether $\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T})$ is coarsely asymptotic to $n_X \cdot e^{\frac{h}{2}R}$ as well? This turns out to be false. Namely, Theorem H showed there is a subset $\mathcal{ML}([\gamma]) \subset \mathcal{ML}(\mathbb{Z})$ such that $\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T}([\underline{\gamma}]))$ is at least coarsely asymptotic to $R \cdot e^{\frac{h}{2}R}$, forcing a lower bound for the coarse asymptotic rate of $\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T})$. In the Remark IV.4.3, we discussed the difficultly using Theorem G to obtain an upper bound estimate for the coarse asymptotic rate of $\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T})$. A natural question prompted by above is the following.

Question V.0.2. *What is the coarse asymptotics for $\Gamma_R(\mathcal{X}, \mathcal{Y}, \mathbf{T})$?*

Two more general questions we may ask:

Question V.0.3. *For a set of twists D with known coarse asymptotics, we may next ask for more precise asymptotics, i.e., what is the best coarse asymptotic coefficient J we can achieve?*

Question V.0.4. *What about the asymptotics growth behavior of thin pseudo-Anosov conjugacy classes?*

Inspired by Example IV.1.6 and Remark IV.2.2, we conjecture there exists a multicurve γ, such that the number of lattice points of the γ conjugacy class intersecting a closed ball of radius R is coarsely asymptotic to e^{lR}, l strictly greater than $\frac{h}{2}$. If this conjecture holds true, in comparison with Theorem I.1.3, it shows a difference of Teichmüller spaces and hyperbolic spaces in terms of lattice point asymptotics.

Conjecture J. Given any $S_{g,n}$ and given any $\epsilon > 0$, there exists a multicurve $\gamma \in \mathcal{ML}(\mathbb{Z})$ and $l > \frac{h}{2}$ such that for any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}^{\epsilon}$, we have

$$\Gamma_R(\mathcal{X}, \mathcal{Y}, T_\gamma) \overset{*JF_\gamma(\mathcal{X},\mathcal{Y})}{\sim} n_X(\gamma) \cdot e^{lR}.$$

As a result, we have

$$\lim_{R\to\infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, T_\gamma) = l > \frac{h}{2}.$$

In particular, let's consider the following example illustrated by Figure V.1. Let $\mathcal{X}$ be a hyperbolic surface that is homeomorphic to S_g for some large g. We can cut S_g open along the simple closed curve δ and decompose S_g as two subsurfaces $S_{g-2}^1 \cup S_2^1$ as the Figure V.1 below, so that $\delta = \partial S_{g-2}^1 = \partial S_2^1$. Let $\mathcal{T}_{g-2,1}, \mathrm{Mod}_{g-2,1}$ denote the Teichmüller space, and mapping class group of the S_{g-2}^1 that preserves the boundary δ pointwise, respectively. The dimension of $\mathcal{T}_{g-2,1}$ equals to $6g-16$, which is strictly greater than $3g-3$ when g is large.

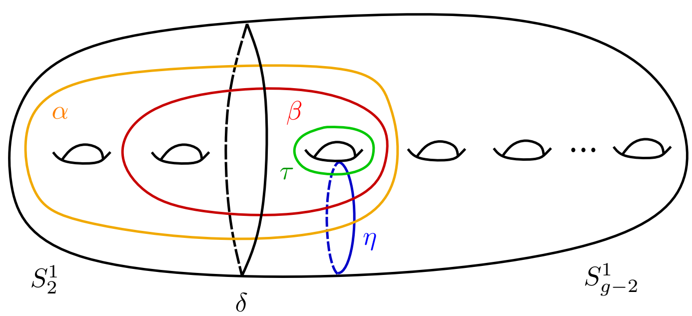

Figure V.1: S_g with large g can be decomposed as S_{g-2}^1 and S_2^1.

Note that $\mathrm{Mod}_{g-2,1} \cdot \gamma$ is a subset of $\mathcal{ML}(\gamma) = \mathrm{Mod}_g \cdot \gamma$ for any simple closed curve γ on S_g. Thus assume the conditions in Conjecture J, we have

$$|D\left(\mathrm{Mod}_{g-2,1} \cdot \gamma\right) \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \subset \Gamma_R(\mathcal{X}, \mathcal{Y}, T_\gamma)$$

where $D\left(\mathrm{Mod}_{g-2,1}\cdot\gamma\right)$ denote the set $\{T_\alpha \mid \alpha \in \mathrm{Mod}_{g-2,1}\cdot\gamma\}$.

We now explicitly conjecture the following, which would imply the above Conjecture J holds true.

Conjecture K. Assume the conditions in Conjecture J and in Figure V.1, we let $\gamma = \gamma_0 = \alpha - \beta$. We have

$$\left|D\left(\mathrm{Mod}_{g-2,1}\cdot\gamma\right)\cdot\mathcal{Y}\cap B_R(\mathcal{X})\right| \overset{*JF_\gamma(\mathcal{X},\mathcal{Y})}{\sim} n_X(\gamma)\cdot e^{(6g-16)R}.$$

As a result, we have

$$\lim_{R\to\infty}\frac{1}{R}\ln\Gamma_R(\mathcal{X},\mathcal{Y},T_\gamma) \geq 6g-16.$$

Remark V.0.5. Given simple closed curves α,β,τ,η as shown in the Figure V.1 and Conjecture K, denote $\gamma_n = T_\eta^n(\gamma_0)$. Let $\tilde{\tau}$ be a lift of τ to $\mathbb{H}^2$ and let $\tilde{T}(\tilde{\tau},\gamma_n)$ be $\tilde{\tau}$ after shearing according to γ_n, see Section II.5. Denote $T(\tau,\gamma_n)$ the projection of $\tilde{T}(\tilde{\tau},\gamma_n)$ back to the surface, so it's in the free homotopy class of the simple closed geodesic $T_{\gamma_n}(\tau)$ on $\mathcal{X}$.

One can check $T(\tau,\gamma_1)$ would be the curve as shown in the left Figure V.2. And $T(\tau,\gamma_1)$ is carried on the train track in the right Figure V.2. After tightening $T(\tau,\gamma_1)$ to the geodesic in its free homotopy class $T_{\gamma_1}(\tau)$, one can check $T_{\gamma_1}(\tau)$ would be the curve as shown in the left Figure V.3, and $T_{\gamma_1}(\tau)$ is carried on the "reduced" train track in the right Figure V.3. In general, one can check the $T_{\gamma_n}(\tau)$ can be carried by the train track in Figure V.4.

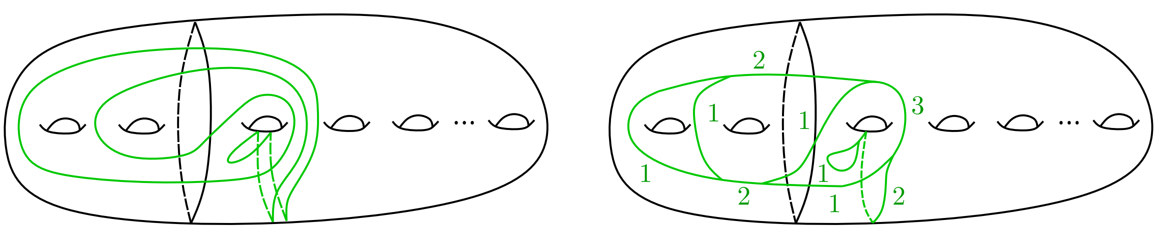

Figure V.2: The simple closed curve $T(\gamma,1)$ and a train track carrying it.

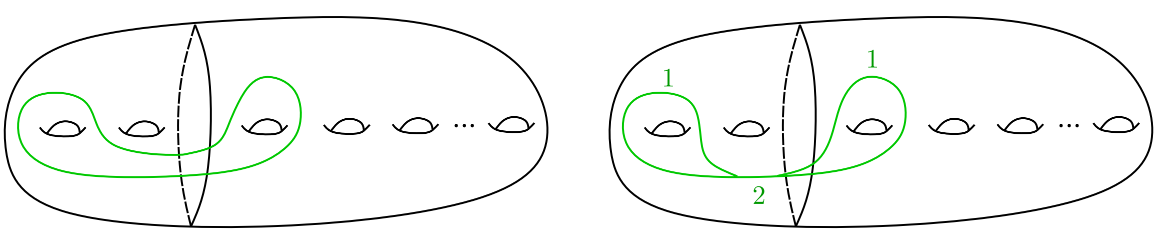

Figure V.3: The simple closed geodesic $T_{\gamma_1}(\tau)$ and a train track carrying it.

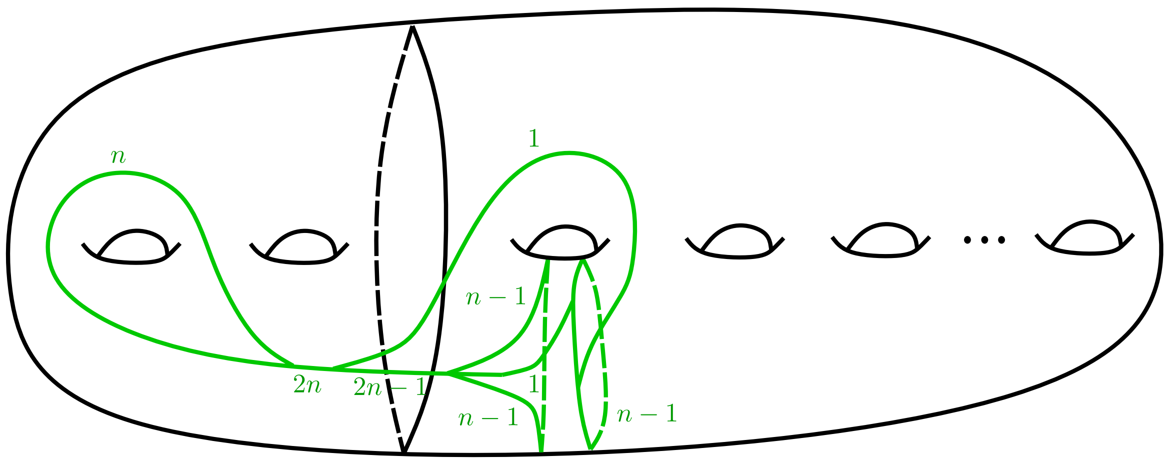

Figure V.4: A general train track that can carry all $T_{\gamma_n}(\tau)$.

Similar to Example IV.1.6 and Remark IV.2.2, one should expect

$$\ell_{\mathcal{X}}\left(T_{\gamma_n}(\tau)\right) \overset{*}{\asymp} \ell_{\mathcal{X}}(\gamma_n).$$

Now, we outline our idea about Conjecture K.

Similar to the proof of Theorem D, it's enough to show

$$\left|D\left(\mathrm{Mod}_{g-2,1} \cdot \gamma\right) \cdot \mathcal{X} \cap B_R(\mathcal{X})\right| \overset{*J}{\sim} n_X(\gamma) \cdot e^{(6g-16)R}.$$

By the Distance Formula II.3, we only need to count the following sets

$$\{f \cdot \gamma \mid f \in \mathrm{Mod}_{g-2,1}, \max_{\kappa \in \mu_{\mathcal{X}}} \log \ell_X(T_{f(\gamma)}\kappa) < R \pm H\}, \tag{V.1}$$

H is some constant error term.

Next, for any $f \in \mathrm{Mod}_{g-2,1}$, let κ_f denote the curve in $\mu_{\mathcal{X}}$ realizing the maximum

$$\ell_X(T_{f(\gamma)}\kappa_f) = \max_{\kappa \in \mu_{\mathcal{X}}} \ell_X(T_{f(\gamma)}\kappa).$$

Motivated by Remark V.0.5, we conjecture

$$\ell_{\mathcal{X}}(f(\gamma)) \overset{*}{\asymp} \ell_X(T_{f(\gamma)}\kappa_f) \tag{V.2}$$

so that

$$\log \ell_{\mathcal{X}}(f(\gamma)) \stackrel{+}{\asymp} \log \ell_X(T_{f(\gamma)}\kappa_f) = \max_{\kappa\in\mu_{\mathcal{X}}} \log \ell_X(T_{f(\gamma)}\kappa).$$

Notice Remark V.0.5 considers the case that f is in the cyclic subgroup $\langle T_\eta\rangle \leq \mathrm{Mod}_{g-2,1}$. In which case, we have

$$\ell_{\mathcal{X}}(f(\gamma)) \stackrel{*}{\asymp} \ell_{\mathcal{X}}(T_{f(\gamma)}\tau).$$

To prove or disprove Formula V.2, perhaps one could think "combinatorially" and start with constructing a finite train track Γ on S_g so that any curve, including $\eta, \alpha, \beta, \tau, f(\alpha), f(\beta), f(\tau)$, can be carried on Γ. Understand and generalizing the Figure V.4 might be a good first step to approach this question.

By Formula V.2, we can simplify the Formula V.1 and only need to count the following set

$$\{f\cdot\gamma \mid f \in \mathrm{Mod}_{g-2,1}, \ell_{\mathcal{X}}(f(\gamma)) < e^{R\pm H}\}. \tag{V.3}$$

Ideally, we would like to apply Mirzakhani's Counting Formula II.7.1 to the Formula V.3 with the surface being $S_{g-2,1}$. However, $f(\gamma)$ does not sit in $S_{g-2,1}$ necessarily. Thus, we purpose two possible routes "solving" this issue.

For the first route, let $S_{g',n'}$ denote a subsurface of $S_{g,n}$, we define

$$s_X^{g',n'}(L,\gamma) = \left|\{\alpha \in \mathrm{Mod}_{g',n'}\cdot\gamma \mid \ell_X(\alpha) \leq L\}\right|.$$

We conjecture the following variation of Mirzakhani's Counting Formula, Theorem II.7.1.

Conjecture L. Fix a subsurface $S_{g',n'}$ of $S_{g,n}$, given γ a simple closed curve or a multicurve on any $X \in \mathcal{M}_{g,n}$, we have

$$s_X^{g',n'}(L,\gamma) \sim n_X^{g',n'}(\gamma)\cdot L^{6g'+2n'-6}$$

where $n_X^{g',n'}(\gamma)$ depends on $S_{g',n'}$, $S_{g,n}$, the hyperbolic structure X, and the topological type of γ.

For the second route, we would like to find a curve γ' in $S_{g-2,1}$ that behaves similarly with γ under the action of $\mathrm{Mod}_{g-2,1}$. The idea is based on the observations from Rafi's combinatorial model for the Teichmüller metric and subsurface projections, see [38]. Although, we remark that, taking the honest subsurface projection of γ to $S_{g-2,1}$ does not work.

Conjecture M. There exists a A and a multicurve $\gamma' \subset S_{g-2,1}$ such that, for any $f \in \mathrm{Mod}_{g-2,1}$, we have

$$\ell_X(f(\gamma)) \overset{+A}{\asymp} \ell_X(f(\gamma')).$$

If the Formula V.2 and the Conjecture L holds true, by applying Conjecture L to Formula V.3, we obtain the desired result.

If the Formula V.2 and the Conjecture M holds true, by applying Conjecture M to Formula V.3, we have

$$\{f \cdot \gamma' \mid f \in \mathrm{Mod}_{g-2,1}, \ell_{\mathcal{X}}(f(\gamma)) < e^{R \pm H}\}. \tag{V.4}$$

By applying Mirzakhani's Counting Formula II.7.1 to the Formula V.4 with the surface being $S_{g-2,1}$, we obtain the desired result.

References

[1] Arzhantseva, G. N., Cashen, C. H., and Tao, J. (2015). Growth tight actions. *Pacific J. Math.*, 278(1):1–49.

[2] Athreya, J., Bufetov, A., Eskin, A., and Mirzakhani, M. (2012). Lattice point asymptotics and volume growth on Teichmüller space. *Duke Math. J.*, 161(6):1055–1111.

[3] Athreya, J. S., Lalley, S. P., Sapir, J., and Wroten, M. (2017). Local geometry of random geodesics on negatively curved surfaces. *arXiv preprint arXiv:1708.09830*.

[4] Bers, L. (1985). An inequality for Riemann surfaces. In *Differential geometry and complex analysis*, pages 87–93. Springer, Berlin.

[5] Birman, J. S., Lubotzky, A., and McCarthy, J. (1983). Abelian and solvable subgroups of the mapping class groups. *Duke Math. J.*, 50(4):1107–1120.

[6] Choi, Y.-E. and Rafi, K. (2007). Comparison between Teichmüller and Lipschitz metrics. *J. Lond. Math. Soc. (2)*, 76(3):739–756.

[7] Dowdall, S. (2011). Dilatation versus self-intersection number for point-pushing pseudo-anosov homeomorphisms. *Journal of Topology*, 4(4):942–984.

[8] Dumas, D. (2015). Skinning maps are finite-to-one. *Acta Math.*, 215(1):55–126.

[9] Farb, B. and Margalit, D. (2012). *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ.

[10] Fathi, A., Laudenbach, F., and Poénaru, V. (2021). *Thurston's Work on Surfaces (MN-48)*, volume 48. Princeton University Press.

[11] Fricke, R. and Klein, F. (1965). *Vorlesungen über die Theorie der automorphen Funktionen. Band 1: Die gruppentheoretischen Grundlagen. Band II: Die funktionentheoretischen Ausführungen und die Andwendungen*, volume 4 of *Bibliotheca Mathematica Teubneriana, Bände 3*. Johnson Reprint Corp., New York; B. G. Teubner Verlagsgesellschaft, Stuttg art.

[12] Gekhtman, I., Taylor, S. J., and Tiozzo, G. (2019). A central limit theorem for random closed geodesics: Proof of the chas–li–maskit conjecture. *Advances in Mathematics*, 358:106852.

[13] Han, J. (2021a). Growth of pseudo-Anosov conjugacy classes in Teichmüller space. *submitted*.

[14] Han, J. (2021b). Growth rate of Dehn twist lattice points in Teichmüller space. *submitted*.

[15] Hopf, H. (1926). Zum clifford-kleinschen raumproblem. *Mathematische Annalen*, 95(1):313–339.

[16] Huber, H. (1956). Über eine neue Klasse automorpher Funktionen und ein Gitterpunktproblem in der hyperbolischen Ebene. I. *Comment. Math. Helv.*, 30:20–62 (1955).

[17] Humphries, S. P. (1979a). Generators for the mapping class group. In *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*, volume 722 of *Lecture Notes in Math.*, pages 44–47. Springer, Berlin.

[18] Humphries, S. P. (1979b). Generators for the mapping class group. In *Topology of low-dimensional manifolds*, pages 44–47. Springer.

[19] Ivanov, N. V. (1992). *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI. Translated from the Russian by E. J. F. Primrose and revised by the author.

[20] Kerckhoff, S. P. (1983). The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265.

[21] Killing, W. (1891). Ueber die clifford-klein'schen raumformen. *Mathematische Annalen*, 39(2):257–278.

[22] Lax, P. D. and Phillips, R. S. (1982). The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. *J. Functional Analysis*, 46(3):280–350.

[23] Lenzhen, A., Rafi, K., and Tao, J. (2012). Bounded combinatorics and the Lipschitz metric on Teichmüller space. *Geom. Dedicata*, 159:353–371.

[24] Lickorish, W. B. (1964). A finite set of generators for the homeotopy group of a 2-manifold. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 60, pages 769–778. Cambridge University Press.

[25] Lindenstrauss, E. and Mirzakhani, M. (2008). Ergodic theory of the space of measured laminations. *International Mathematics Research Notices*, 2008(9):rnm126–rnm126.

[26] Maher, J. (2010). Asymptotics for pseudo-Anosov elements in Teichmüller lattices. *Geom. Funct. Anal.*, 20(2):527–544.

[27] Margulis, G. A. (2004). *On some aspects of the theory of Anosov systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Translated from the Russian by Valentina Vladimirovna Szulikowska.

[28] McMullen, C. (1998). Complex earthquakes and teichmüller theory. *Journal of the American Mathematical Society*, 11(2):283–320.

[29] Minsky, Y. N. (1996). Quasi-projections in Teichmüller space. *J. Reine Angew. Math.*, 473:121–136.

[30] Mirzakhani, M. (2008a). Ergodic theory of the earthquake flow. *International Mathematics Research Notices*, 2008(9):rnm116–rnm116.

[31] Mirzakhani, M. (2008b). Growth of the number of simple closed geodesics on hyperbolic surfaces. *Ann. of Math. (2)*, 168(1):97–125.

[32] Mirzakhani, M. and Petri, B. (2019). Lengths of closed geodesics on random surfaces of large genus. *Commentarii Mathematici Helvetici*, 94(4):869–889.

[33] Mumford, D. (1971). A remark on Mahler's compactness theorem. *Proc. Amer. Math. Soc.*, 28:289–294.

[34] Nielsen, J. (1932). Studies on the topology of the closed two-sided surfaces. iii. *Acta Mathematica*, 58:87–167.

[35] Papadopoulos, A. (1991). On thurston's boundary of teichmüller space and the extension of earthquakes. *Topology and its Applications*, 41(3):147–177.

[36] Parkkonen, J. and Paulin, F. (2015). On the hyperbolic orbital counting problem in conjugacy classes. *Math. Z.*, 279(3-4):1175–1196.

[37] Penner, R. C. and Harer, J. L. (1992). *Combinatorics of train tracks*, volume 125 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ.

[38] Rafi, K. (2007). A combinatorial model for the teichmüller metric. *GAFA Geometric And Functional Analysis*, 17(3):936–959.

[39] Rivin, I. (2001). Simple curves on surfaces. *Geometriae Dedicata*, 87(1):345–360.

[40] Sapir, J. (2016). Bounds on the number of non-simple closed geodesics on a surface. *International Mathematics Research Notices*, 2016(24):7499–7545.

[41] Scott, P. and Wall, T. (1979). Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge-New York.

[42] Thomassen, C. (1992). The jordan-schönflies theorem and the classification of surfaces. *The American Mathematical Monthly*, 99(2):116–130.

[43] Thurston, W. P. (1988). On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431.

[44] Thurston, W. P. (1998). Minimal stretch maps between hyperbolic surfaces. *arXiv preprint math/9801039*.