ACTIONS OF CUSP FORMS ON HOLOMORPHIC DISCRETE SERIES AND VON NEUMANN ALGEBRAS

By

Jun Yang

Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August 13, 2021

Nashville, Tennessee

Approved:

Professor Dietmar Bisch

Professor Jesse Peterson

Professor Bruce Hughes

Professor Larry Rolen

Professor Robert Scherrer

ACKNOWLEDGMENTS

I would like to thank my advisors Professor Vaughan F.R. Jones and Professor Dietmar Bisch. I am grateful to V. Jones for his valuable guidance, patient explanations, and insightful ideas. It is a tragedy that he suddenly passed away from us. I am grateful to D. Bisch for his great help and detailed guidance in mathematics and my career path.

I wish to thank Larry Rolen for many helpful suggestions and careful review. I would like to thank Florin Radulescu for his conversations and suggestions regarding this work.

I would like to thank my colleagues in mathematics, particularly Zhengwei Liu, Yunxiang Ren, Bin Gui, and Bin Sun, for the help with my research and career.

Finally, I would like to thank my wife Jia for accompanying me on my study and my life.

TABLE OF CONTENTS

Chapter 1

Introduction

A holomorphic discrete series representation is an infinite dimensional unitary representation of a semi-simple Lie group G, which is usually non-compact. There is a large family of discrete series representations of real reductive Lie groups and also an interesting family of admissible representations of algebraic groups over $\mathbb{R}$. Harish-Chandra proved such representations exist if and only if $\operatorname{rank} G = \operatorname{rank} K$, where K is a maximal compact subgroup with a non-finite center [17, 26]. Indeed, such a representation is realized as certain holomorphic functions on the bounded symmetric domain $\mathscr{D} = G/K$ with values in a highest weight representation (π, V_π) of K. In particular, they reduce to these highest weight representations when $G = K$, i.e., G is compact. As is the case with these finite dimensional highest weight representations, the holomorphic discrete series can also be described by the dominant weights of K.

The first non-trivial example is the discrete series of $SL(2,\mathbb{R})$, whose maximal compact subgroup is $SO(2)$ (up to conjugation). In this case, the symmetric domain $\mathscr{D} = SL(2,\mathbb{R})/SO(2)$ is just the open unit disk which is holomorphically isomorphic to the Poincaré upper half-plane $\mathbb{H}$. As the irreducible representations of $SO(2)$ are characterized by integers, the holomorphic discrete series of $SL(2,\mathbb{R})$ can be denoted by (L_m, H_m) where $m \in \mathbb{Z}$ and H_m is a certain subspace of the holomorphic functions on $\mathbb{H}$ [32]. Furthermore, for the modular group $\Gamma = SL(2,\mathbb{Z})$, the cusp forms are also holomorphic functions with some Γ-invariant properties. V. Jones found that the multiplication by a cusp form of weight p is in $B(H_m, H_{m+p})$ that intertwines the actions of $SL(2,\mathbb{Z})$ on H_m and H_{m+p}. More precisely, the multiplication operator

$$M_f \colon H_m \to H_{m+p},\ \phi(z) \mapsto f(z)\phi(z)$$

intertwines the actions of Γ on H_m and H_{m+p}, i.e., $M_f L_m(\gamma) = L_{m+p}(\gamma) M_f$ for all $\gamma \in SL(2,\mathbb{Z})$ [16].

Observe $PSL(2,\mathbb{Z})$ is an ICC group. (Recall that a group G is an infinite conjugacy classes group, or an ICC group for short if every non-trivial conjugacy class $C_h = \{g^{-1}hg | g \in G\}, h \neq 1$ is infinite.) Its group von Neumann algebra and the commutant are both factors of type II_1 (provided the formal dimension is finite). A natural question arises whether these operators composed with their adjoints, i.e., operators of the form $M_g^* M_f$, generate the commutant factor. (Note the adjoint M_f^* is more complicated than a single multiplication, see Section 4.3 or [16].) In 1994, F. Radulescu gave an affirmative answer by applying the Berezin quantization [33, 34].

Theorem 1.1 *[F. Radulescu, 1994] The von Neumann algebras generated by the forms $M_g^* M_f$ is the commutant of the II_1 factor $= L_m(PSL(2,\mathbb{Z}))''$, i.e.,*

$$\overline{\{span_{f,g} M_g^* M_f\}}^{w.o.} = L_m(PSL(2,\mathbb{Z}))',$$

where f, g run through the cusp forms of same weights.

But no result is known for other Fuchsian subgroups of $SL(2,\mathbb{R})$ or, more generally, lattices of a general real Lie group.

In the first part of this thesis, we generalize the result for $SL(2,\mathbb{Z})$ to the holomorphic discrete series of non-compact semi-simple real Lie groups. These representations can be denoted by (L_π, H_π), where $H_\pi = L^2_{\text{holo}}(\mathscr{D}, V_\pi)$ and (π, V_π) is an irreducible representation of K. We first use a generalized Berezin quantization to transfer each operator A in $B(H_\pi)$ to an $\mathrm{End}(V_\pi)$-valued function $S(A)(z)$ on $\mathscr{D}$ with some holomorphic properties (see Section 3.1). Once a discrete subgroup Γ of the Lie group G is given, we also give an explicit formula of a faithful normal tracial state on the commutant $L_\pi(\Gamma)' = B(H_\pi)^\Gamma = \{A \in B(H_\pi) | A L_\pi(\gamma) = L_\pi(\gamma) A, \forall \gamma \in \Gamma\}$ of the group von Neumann algebra $\overline{L_\pi(\Gamma)}^{\text{s.o.}}$.

Proposition 1.2 (The trace formula for $L_\pi(\Gamma)'$) *Assume π is an irreducible representation of K. Let $\tau \colon B(H_\pi) \to \mathbb{C}$ be the linear functional defined by*

$$\tau(A) = \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}} \mathrm{tr}(S(A)(z))d\mu(z), \qquad A \in B(H_\pi).$$

Then τ is a positive, faithful, normal, normalized trace on $L_\pi(\Gamma)'$. In particular, if Γ is an ICC group, τ is the unique normalized trace on the II_1 factor $L_\pi(\Gamma)'$.

We then generalize the classical Toeplitz operator $T_f \in B(H_\pi)$ associated with $f \in L^\infty(\mathscr{D})$ to an essentially bounded matrix Toeplitz operator T_f associated with an essentially bounded matrix-valued function f on $\mathscr{D}$. Then the Γ-invariant functions can be identified with the ones defined on $(\Gamma\backslash\mathscr{D}$. Using several formulas of the tracial state of these operators, we prove

Theorem 1.3 (Main Theorem I) *The commutant $L_\pi(\Gamma)'$ is generated by the Toeplitz operators of matrix-valued functions, i.e.,*

$$\overline{\langle T_f | f \in L^\infty(\Gamma\backslash\mathscr{D}, \mathrm{End}(V_\pi))\rangle}^{w.o.} = L_\pi(\Gamma)'.$$

In the second part, we consider cusp forms defined on real Lie groups, which were first studied by Harish-Chandra [18]. By definition, given a semi-simple Lie group G, an automorphic form on G is a complex (or complex vector-valued) function $f\colon G \to \mathbb{C}$ (or taking values in a finite dimensional representation V_ρ of K) which is K-right-finite (or right-equivariant), Γ-left-invariant and satisfies some analytic properties. Indeed, we focus on another type of automorphic form defined on the domain $\mathscr{D} = G/K$, which can be easily obtained from the ones defined on the group G. As for intertwining properties of the classical cusp forms of the modular group $SL(2,\mathbb{Z})$, we also show the existence of Γ-invariant bounded operators between these holomorphic function spaces from the cuspidal automorphic forms, or simply cusp forms, on general real Lie groups. Let $f\colon G(\text{or } \mathscr{D}) \to V_\rho$ be a cusp form of Γ of type (ρ, V_ρ) (here (ρ, V_ρ) is a representation of K), which is not always holomorphic as in the case of $SL(2,\mathbb{R})$. The multiplication operator M_f is no longer closed. We construct a Toeplitz-type operator

$$T_f\colon H_\pi \to H_{\rho\otimes\pi} \text{ given by } \phi(z) \mapsto P_{\rho\otimes\pi}(f \otimes \phi)(z),$$

where $P_{\rho\otimes\pi}$ is the projection from $L^2(\mathscr{D}, V_{\rho\otimes\pi})$ to the closed subspace $H_{\rho\otimes\pi} = L^2_{\text{hol}}(\mathscr{D}, V_{\rho\otimes\pi})$ which is square-integrable and not always irreducible. Then T_f also commutes with the actions of Γ on H_π and $H_{\rho\otimes\pi}$ respectively. This implies $T_f \in B(H_\pi, H_{\rho\otimes\pi})^\Gamma$ and $T_g^* T_f \in L_\pi(\Gamma)'$ if f, g are cusp forms of the same type. Our construction includes the earlier result on $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ as a special case [16, 34, 30, 23].

In this thesis, we generalize Theorem 1.1 to

1. Fuchsian subgroups of the first kind of $SL(2,\mathbb{R})$,

2. Lattices of real Lie groups (with holomorphic discrete series and $\dim_{\mathbb{C}} V_\pi = 1$).

The result on $SL(2,\mathbb{R})$ is obtained by proving certain existence theorems of meromorphic functions and holomorphic functions on the compact Riemann surface $\mathscr{F}^* = \Gamma/\mathbb{H}^*$. We apply Riemann-Roch theory for the proofs about meromorphic and holomorphic functions on $\mathscr{F}^*$. We prove there are enough cusp forms that can separate the points in the fundamental domain $\Gamma/\mathbb{H}$ of any Fuchsian group Γ of the first kind, i.e., Γ is a lattice.

For the most general case, we apply Baily-Borel compactification and show the Poincaré series are abundant to separate points in the fundamental domain $\mathscr{F} = \Gamma\backslash\mathscr{D}$. We always assume G has no normal $\mathbb{Q}$-subgroup of dimension 3 [3]. Finally, we prove the following theorem in Section 7 (see Theorem 7.1).

Theorem 1.4 (Main Theorem II) *The commutant $L_\pi(\Gamma)'$ can be generated by the cusp forms, i.e.,*

$$\overline{\langle \{span_{f,g} T_g^* T_f\} \otimes \operatorname{End}(V_\pi)\rangle}^{w.o.} = L_\pi(\Gamma)',$$

where f, g run through cusp forms for Γ of same types. Moreover, if $\dim_{\mathbb{C}} V_\pi = 1$, we have

$$\overline{\langle span_{f,g} T_g^* T_f\rangle}^{w.o.} = L_\pi(\Gamma)',$$

where $f, g \in \mathscr{A}^0_{\mathscr{D}}(\Gamma, \rho)$ for some $\rho \in \widehat{K}$ ($\widehat{K}$ is the set of equivalence classes of irreducible representations of K, and $\mathscr{A}^0_{\mathscr{D}}(\Gamma, \rho)$ is the space of cusp forms of type ρ, see Section 5.2).

Section 2 provides a brief discussion of the holomorphic discrete series representations and their realizations. Section 3 is devoted to the theory of the Berezin transform and construction of the matrix Toeplitz operators. We provide formulas for a trace τ on the finite von Neumann algebra $L_\pi(\Gamma)'$ In Section 4, we consider the extension of the Berezin transform of from $L_\pi(\Gamma)'$ to the standard module $L^2(L_\pi(\Gamma)', \tau)$of it. In Section 5, we construct Γ-intertwining operators from the cusp forms defined on the Lie group. Section 6 and 7 apply the results from previous sections to $SL(2,\mathbb{R})$ with its discrete subgroups and then to semi-simple real Lie groups with holomorphic discrete series. Then we prove our main result, Theorem 7.1.

Chapter 2

The Holomorphic Discrete Series

We review some basic facts about discrete series representations. Then we focus on the family of holomorphic discrete series representations and their construction. We refer to [17, 26, 29] for more details.

2.1 The discrete series representations

Let G be a locally compact unimodular group with Haar measure dg. Moreover, we are interested in the case that $G = G_{\mathbb{R}}$ is a connected semi-simple real Lie group. We assume K is a maximal compact subgroup of G and H is the Cartan subgroup of G. We will use the following notations.

- $\mathfrak{h}, \mathfrak{k}, \mathfrak{g}$: the Lie algebra of H, K, G respectively and $\mathfrak{h}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}$ are their complexifications;
- Δ, Δ_K: roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$;
- W_G, W_K: the Weyl groups of Δ, Δ_K;
- δ_G, δ_K: the respective half-sums of positive roots.

Furthermore, we have in mind G should be a non-compact group though we do not exclude the compact case. Let $\pi\colon G \to U(H)$ be a unitary representation of G where H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$. For vectors $u, v \in H$, one defines the coefficient

$$g \in G \mapsto c_{u,v}(g) = \langle \pi(g)u, v \rangle_H.$$

We obtain $c_{u,v}(h^{-1}g) = c_{u,\pi(h)v}(g)$ and $c_{u,v}(gh) = c_{\pi(h)u,v}(g)$ for all $g, h \in G$.

Definition 2.1 *Let π be a unitary representation of G. We say it is square-integrable if it has a non-zero square-integrable coefficient*

$$0 \neq c_{u,v} \in L^2(G, dg) \text{ for some } u, v \in H.$$

If π is irreducible, we call it a discrete series representation of G.

Theorem 2.2 *[32] Let π be a unitary irreducible representation of a locally compact group G. The following properties are equivalent:*

1. *There exist $u, v \in H$ such that $c_{u,v}$ is square-integrable.*

2. *For any $u, v \in H$, $c_{u,v}$ is square-integrable.*

3. *π is equivalent to a subrepresentation of the right regular representation $\rho: G \to U(L^2(G, dg))$.*

For each discrete series representation $\pi: G \to U(H)$, there is a parameter called *formal dimension* $d_\pi \in \mathbb{R}^+$ determined only by π, which is given by the following theorem.

Theorem 2.3 ([32]) *Let (π, H) be a discrete series representation of G. Then there is a constant $d_\pi \in \mathbb{R}_{\geq 0}$ such that*

$$\langle c_{u,v}, c_{x,y} \rangle_{L^2(G)} = d_\pi^{-1} \langle u, x \rangle_H \cdot \overline{\langle v, y \rangle_H}, \text{ for all } u, v, x, y \in H.$$

Moreover, if $(\pi, H), (\pi', H')$ are two discrete series representations that are not equivalent, then $\langle c_{u,v}, c_{u',v'} \rangle_{L^2(G)} = 0$, for all $u, v \in H$ and $u', v' \in H'$.

There is a criterion for the existence of discrete series representations proposed by Harish-Chandra and also proved by him. As in the case of the highest weight representations of compact Lie groups, these discrete series representations (up to unitary equivalence) can also be classified by their weights.

Theorem 2.4 (Harish-Chandra [17]) *The discrete series representations exist if and only if* $\operatorname{rank} G = \operatorname{rank} K$. *They are classified by* π_λ *with non-singular weight* $\lambda \in (i\mathfrak{h})'$ *such that* $\lambda + \delta_G$ *is analytically integral. Moreover,* $\pi_\lambda \cong \pi_{\lambda'}$ *if and only if* λ, λ' *are conjugate under* W_K.

Here $(i\mathfrak{h})'$ denotes the dual space of $i\mathfrak{h}$.

Note when G is compact, i.e., $G = K$, this theorem reduces to the theorem of highest weight representations. In particular, for a complex Lie group $G_{\mathbb{C}}$ with a compact real form $G_{\mathbb{R}}$, we have $\operatorname{rank} G_{\mathbb{C}} = 2 \operatorname{rank} G_{\mathbb{R}}$, and it never has discrete series representations. More details of the construction of these representations can be found in [26]. A geometric realization of these representations or the generalized Borel-Weil-Bott theorem using L^2-cohomology was conjectured by R. Langlands and then proved by W. Schmid [36].

Remark 2.5 *By Theorem 2.4, we can easily determine whether some classical groups have a discrete series or not. For instance:*

1. $SL(n,\mathbb{R})$ *has a discrete series only when* $n = 2$*, and* $SL(n,\mathbb{C})$ *has no discrete series.*

2. *Each* $SU(p,q)$ *has a discrete series for* $p, q \geq 1$.

3. $SO(p,q)$ *has a discrete series only when* $q = 2$.

4. *Each* $Sp(n,\mathbb{R})$ *has a discrete series, but* $Sp(n,\mathbb{C})$ *has none.*

2.2 Construction of the holomorphic discrete series

The holomorphic discrete series are the discrete series that can be represented in a natural way by Hilbert spaces of holomorphic functions. We refer to [17, 29, 13] for the relevant descriptions. From now on, we always assume G is a connected non-compact semi-simple real Lie group with $\operatorname{rank} G = \operatorname{rank} K$, and K has a non-finite center.

Let $\theta \in \operatorname{Aut}(\mathfrak{g})$ be a Cartan involution and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Then we have $\mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$. We also write $Z^* = (X + iY)^* = -\theta(X +$

$iY) = -X + iY$ for $Z = X + iY \in \mathfrak{g}_{\mathbb{C}}$ with $X, Y \in \mathfrak{g}$. For $g = \exp(X + iY) \in G_{\mathbb{C}}$, we write $g^* = \exp((X + iY)^*) = \exp(-X + iY) = \overline{g^{-1}}$. (The notation Z^* and g^* has this meaning only when we consider Lie algebras and Lie groups)

Consider the root space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$. We further obtain $\mathfrak{k}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha \in \Delta_K} \mathfrak{g}_{\alpha}$ and $\mathfrak{p}_{\mathbb{C}} = \sum_{\alpha \in \Delta_n} \mathfrak{g}_{\alpha}$ where $\mathfrak{p}_{\mathbb{C}}$ is the complexification of $\mathfrak{p}$ and $\Delta_n = \Delta - \Delta_K$ is the set of noncompact roots. Let $\Delta^+, \Delta_K^+, \Delta_n^+$ be selected sets of positive roots, positive compact roots, and positive non-compact roots, respectively. We set $\mathfrak{p}^+ = \sum_{\alpha \in \Delta_n^+} \mathfrak{g}_{\alpha}$ and $\mathfrak{p}^- = \sum_{\alpha \in \Delta_n^-} \mathfrak{g}_{\alpha}$

Let P^+ and P^- be the analytic subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$ respectively. There is a diffeomorphism $(z^+, k, z^-) \mapsto z^+ \cdot k \cdot z^-$ from $P^+ \times K_{\mathbb{C}} \times K^-$ to an open submanifold of $G_{\mathbb{C}}$ containing G [20]. Following [29], we also introduce the projections $\zeta \colon P^+ K_{\mathbb{C}} P^- \to P^+$, $\kappa \colon P^+ K_{\mathbb{C}} P^- \to K_{\mathbb{C}}$ and $\xi \colon P^+ K_{\mathbb{C}} P^- \to P^-$. Then the map $\phi \colon G/K \to \mathfrak{p}^+$ given by $\phi(gK) = \log(\zeta(g))$ induces a diffeomorphism from G/K to a bounded domain $\mathscr{D} \subset \mathfrak{p}^+$. The domain $\mathscr{D}$ is an irreducible Hermitian symmetric space of non-compact type [20, 13] and is known as the *Harish-Chandra realization* of G/K. We will identify $\mathscr{D}$ with G/K in the following sections.

Let $B(\cdot, \cdot)$ be the Killing form. We have

Theorem 2.6 ([35]) *The bounded symmetric domain can be given by*

$$\mathscr{D} = \{z \in \mathfrak{p}^+ | \|\operatorname{ad} z\| < \sqrt{2}\}$$

where the norm is the operator norm on $\mathfrak{g}_{\mathbb{C}}$ equipped with the positive definite Hermitian form $-B(X, \theta \overline{Y})$.

Recall the map $\kappa \colon P^+ K_{\mathbb{C}} P^- \to K_{\mathbb{C}}$ defined above. We define a map from $\mathscr{D} \times \mathscr{D}$ to $K_{\mathbb{C}}$, which is also denoted by κ, by

$$\kappa(z, w) = \kappa(\exp(w^*) \exp(z))^{-1}, \, z, w \in \mathscr{D}.$$

We define a map $J \colon G \times \mathscr{D} \to K_{\mathbb{C}}$ by

$$J\colon G\times\mathscr{D}\to K_{\mathbb{C}} \text{ by } J(g,z)=\kappa(g\exp z),\ g\in G, z\in\mathscr{D}.$$

The map J is usually called the *canonical automorphy factor* of G, which satisfies the following properties:

(i) $J(g,z)$ is C^∞ in the first variable and holomorphic in the second one,

(ii) J is a 1-cocycle, i.e., $J(gh,w)=J(g,hw)J(h,w)$ for $g,h\in G$, $w\in\mathscr{D}$,

(iii) $J(k,0)=k$ if $k\in K$.

For κ, we have an alternative definition by the following properties: (i)$\kappa(0,0)=e$, (ii) $\kappa(z,w)$ is holomorphic in z, (iii) $\kappa(z,w)=\kappa(w,z)^*$ and (iv)$\kappa(gz,gw)=J(g,z)\kappa(z,w)J(g,w)^*$.

Remark 2.7 *Let ρ be a unitary representation of K (so it can be extended to $K_{\mathbb{C}}$). For $g\in K_{\mathbb{C}}$, we have $\rho(g^*)=\rho(g)^*$, the adjoint operator associated with $\rho(g)$.*

Indeed, we assume $\rho(g)=\exp(X+iY)$ with $X,Y\in\rho(\mathfrak{k}_{\mathbb{R}})$. Then $\rho(g^*)=\exp(-X+iY)$. As ρ is unitary on K, $X+X^*=0$. So we obtain $\exp(-X+iY)=\exp(X^*-i\cdot Y^*)=\exp((X+iY)^*)=\rho(g)^*$.

Now we can construct the holomorphic discrete series of G. Let (π,V_π) be a finite dimensional unitary representation of the compact subgroup K and $\langle,\rangle_\pi\colon V_\pi\times V_\pi\to\mathbb{C}$ is a K-invariant inner product. We also let (π,V_π) denote the representation extended to $K_{\mathbb{C}}$. Here we do not assume the irreducibility of π.

Consider the space

$$\mathrm{Map}(\mathscr{D},V_\pi)=\{f\colon\mathscr{D}\to V_\pi | f \text{ is measurable}\}$$

and the following inner product on it:

$$\langle f,h\rangle=\int_{\mathscr{D}}\langle\pi(\kappa(z,z)^{-1})f(z),h(z)\rangle_\pi d\mu(z),\ f,h\in\mathrm{Map}(\mathscr{D},V_\pi)$$

where μ is the G-invariant measure on $\mathscr{D}$ given by

$$d\mu(z)=\det\mathrm{ad}_{\mathfrak{p}^+}\kappa^{-1}(z,z)dz,$$

and dz is the Euclidean measure on $\mathscr{D} \subset \mathfrak{p}^+$ [35]. Following Remark 2.7 above, one can check it is positive-definite. We define

$$L^2(\mathscr{D}, V_\pi) = \{f \in \mathrm{Map}(\mathscr{D}, V_\pi) | \langle f, f \rangle < \infty\},$$

and also the subspace spanned by holomorphic functions

$$H_\pi = L^2_{\mathrm{hol}}(\mathscr{D}, V_\pi) = \{f \in L^2(\mathscr{D}, V_\pi) | f \text{ is holomorphic}\}$$

where the inner product restricted to H_π will be written as $\langle , \rangle_{H_\pi}$. It can be shown that H_π is closed subspace of $L^2(\mathscr{D}, V_\pi)$. The action L_π of G on H_π is given by

$$L_\pi(g) f(z) = \pi(J(g^{-1}, z)^{-1}) f(g^{-1} z),\ f \in H_\pi, g \in G, z \in \mathscr{D}.$$

It can be proved that this representation (L_π, H_π) is a unitary representation of G [13].

Please note the unitary representations (L_π, H_π) are square-integrable, i.e., $H_\pi \subset L^2(G)$. The *holomorphic discrete series* of G are the (L_π, H_π)'s, where π is an irreducible representation of K.

Theorem 2.8 ([17, 26]) *Assume π is an irreducible representation of K with highest weight Λ, then H_π is nonzero if and only if $(\Lambda + \delta_G)(H_\beta) < 0$ for all $\beta \in \Delta_n^+$. In this case, it is irreducible.*

The formal dimensions of these representations can be given by explicit formulas [29, 13]. Now we let P_π be the orthogonal projection from $L^2(\mathscr{D}, V_\pi)$ to H_π. Let L_π be the action of G on $L^2(\mathscr{D}, V_\pi)$ defined by

$$L_\pi(g) f(z) = \pi(J(g^{-1}, z)^{-1}) f(g^{-1} z),\ f \in H_\pi, g \in G, z \in \mathscr{D},\ f \in L^2(\mathscr{D}, V_\pi).$$

This is to say H_π is a G-invariant subspace and $(L_\pi, L^2(\mathscr{D}, V_\pi))$ also gives us a well-defined unitary representation.

Proposition 2.9 *For $g \in G$, $L_\pi(g)$ and P_π commute on $L^2(\mathscr{D}, V_\pi)$, i.e., $L_\pi(g) P_\pi f = P_\pi L_\pi(g)(f)$ for $f \in L^2(\mathscr{D}, V_\pi)$.*

Proof: Take $h \in L^2(\mathscr{D}, V_\pi)$. Assume $h = h_0 \oplus h_1$ with $h_0 = P_\pi(h), h_1 = (1 - P_\pi)(h)$. We can show L_π leaves $H_\pi^\perp$ invariant: $\langle L_\pi(g) f_1, h_0 \rangle_{L^2} = \langle f_1, L_\pi(g^{-1}) h_0 \rangle = 0$. Hence $L_\pi(g) P_\pi f = P_\pi L_\pi(g) f$. □

2.3 Discrete subgroups and von Neumann dimensions

Recall that semi-simple real Lie groups are unimodular. Let G be a connected semi-simple real Lie group with the Haar measure dg. Let Γ be a discrete subgroup of G. Let $D \subset G$ be the *fundamental domain* for Γ, that is D satisfies:

1. $\gamma_1 D \cap \gamma_2 D$ has null measure with respect to dg if $\gamma_1 \neq \gamma_2 \in \Gamma$,
2. $G \backslash \cup_{\gamma \in \Gamma} \gamma D$ has null measure with respect to dg.

For the fixed Haar measure dg on G, we call the measure of D the *covolume* of Γ and denote it by $\mathrm{covol}(\Gamma)$. Note that if $d'g = \lambda dg$, then $\mathrm{covol}'(\Gamma) = \lambda \cdot \mathrm{covol}(\Gamma)$.

By definition, a *von Neumann algebra M* is a C^*-subalgebra of $B(H)$, the algebra of bounded linear operators on a Hilbert space H, such that it coincides with its double commutant $M = M''$, or equivalently, it contains the identity operator on H and is closed in the strong operator topology $M = \overline{M}^{\text{s.o.}}$. M is a *factor* if it has trivial center, that is $M' \cap M = \mathbb{C} \cdot \mathrm{id}$. Further, we call a factor a *II_1 factor*, if it is infinite dimensional and admits a positive, faithful, normal, and normalized trace [22].

Let $\mathscr{L}\Gamma \subset B(l^2(\Gamma))$ be the group von Neumann algebra associated to Γ, i.e., the strong operator closure of the left regular representation of its group algebra on the Hilbert space $l^2(\Gamma)$.

Definition 2.10 *A group G is called an infinite conjugacy classes group, or an ICC group for short, if every non-trivial conjugacy class $C_h = \{g^{-1}hg | g \in G\}, h \neq 1$ is infinite.*

We have the following well-known result [16].

Theorem 2.11 *The group von Neumann algebra $M = \mathscr{L}\Gamma$ is a factor of type II_1 if and only if Γ is nontrivial and ICC.*

Note that for any discrete series representation $L\colon G \to U(H)$, $L|_\Gamma$ gives a representation of $\mathscr{L}\Gamma$ [16].

For a II_1 factor M represented on a separable Hilbert space H, there is a von Neumann dimension $\dim_M(H) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ with the following properties [21]:

(i) $\dim_M(H) = \dim_M(K)$ if and only if H, K are isomorphic as left M-modules,

(ii) $0 < \dim_M(H) < \infty$ if and only if $M' \cap B(H)$ is also a II_1 factor,

(iii) $\dim_M(H \oplus K) = \dim_M(H) + \dim_M(K)$,

(iv) $\dim_M(L^2(M)) = 1$.

The von Neumann dimension can be related to the formal dimension by the following theorem [16].

Theorem 2.12 *Let G be a connected semi-simple real Lie group with Haar measure dg and Γ a discrete subgroup of G. Suppose $L\colon G \to U(H)$ is an irreducible discrete series representation with formal dimension d_L. Assume Γ is an ICC group and $M = L(\Gamma)''$ which is a II_1 factor. Then*

$$\dim_M(H) = \mathrm{covol}(\Gamma) \cdot d_L.$$

Let (L_π, H_π) be the holomorphic discrete series representation of G associated with an irreducible representation (π, V_π), which is constructed in Section 2.2. We define the commutant by

$$A_\pi = \{A \in B(H_\pi) | AL_\pi(\gamma) = L_\pi(\gamma)A,\ \forall \gamma \in \Gamma\},$$

which is equivalent to $A_\pi = L_\pi(\Gamma)' \cap B(H_\pi)$. Assume Γ is a discrete subgroup with finite covolume, $\mathrm{covol}(\Gamma) < \infty$. By Theorem 2.12, we have $\dim_M(H_\pi) < \infty$.

Corollary 2.13 *Assume Γ is a discrete subgroup that is ICC and has finite covolume. If π is irreducible, then A_π is a type II_1 factor.*

In general, A_π is not a factor but always a finite von Neumann algebra. This will be proved in the next section by exhibiting a positive, faithful, normal, tracial state on it.

Chapter 3

Berezin Transform, Toeplitz Operators and Trace Formulas

In this section, we will study the holomorphic discrete series representations (L_π, H_π) of a semi-simple real Lie group G restricted to a lattice $\Gamma \subset G$ and the related von Neumann algebras. We construct some Berezin quantizations and generalized Toeplitz operators, emphasizing the Γ-invariant properties. We obtain several explicit formulas of a trace on the commutant.

3.1 Berezin symbols and a trace

The Berezin quantization is defined for the upper half-plane $\mathbb{H} = \{z = x+iy \in \mathbb{C} | y > 0\}$. It may also be defined for the open unit disk $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ by the Cayley transform, which sends $z \in \mathbb{H}$ to $\frac{z-i}{z+i} \in \mathbb{D}$ [4, 5]. This is a special case of the bounded symmetric domain $\mathscr{D} = G/K$ when $G = SL(2,\mathbb{R})$ and $K = SO(2)$. We will focus on its generalization to the bounded symmetric domain $\mathscr{D} = G/K$ for a Lie group G with holomorphic discrete series as introduced in the previous sections.

Let (L_π, H_π) be a nontrivial square-integrable representation of G which is associated to the finite dimensional unitary representation (π, V_π) of a maximal compact subgroup K. Recall that $H_\pi = L^2_{\text{holo}}(\mathscr{D}, V_\pi)$. Here we do not assume it is irreducible (and (π, V_π) neither by Theorem 2.4). For any $z \in D$, the evaluation function $K_z \colon H_\pi \to V_\pi$ given by $f \mapsto f(z)$ is continuous and bounded [29]. Hence its adjoint operator $E_z = K_z^* \colon V_\pi \to H_\pi$ is defined by

$$\langle K_z f, v\rangle_\pi = \langle f(z), v\rangle_\pi = \langle f, E_z(v)\rangle_{H_\pi}, \forall f \in H_\pi, v \in V_\pi,$$

where $\langle\cdot,\cdot\rangle_\pi$ is the K-invariant inner product on V_π.

Let $G = NAK$ be the Iwasawa decomposition. There is a smooth embedding

$$i\colon \mathscr{D} \cong G/K \hookrightarrow NA \subset G,\, z \mapsto g_z.$$

Please note we have $z = \dot{g}_z = g_z \cdot K$ as a coset in G/K. We denote $h_z = \kappa(g_z) \in K_{\mathbb{C}}$ for $z \in \mathscr{D}$ and also let $H_z = \pi(h_z^{-1}) \in GL(V_\pi)$ (see Section 2.2 for the map $\kappa\colon P^+K_{\mathbb{C}}P^- \to K_{\mathbb{C}}$).

The following result and its proof can be found in [29, 11].

Lemma 3.1 *(i) There exists a constant $c_\pi \in \mathbb{R}_{>0}$ such that $E_z^*E_w = c_\pi \pi(\kappa(z,w))$ for $z,w \in \mathscr{D}$.*

(ii) $E_{gz} = L_\pi(g)E_z\pi(J(g,z)^*)$ for $g \in G, z \in \mathscr{D}$.

Now we define four *Berezin symbols* for operators in $B(H_\pi)$. Recall $c_\pi \in \mathbb{R}$ is the contant given in Lemma 3.1.

Definition 3.2 *For an operator $A \in B(H_\pi)$, the Berezin symbols of A are defined as the following* $\mathrm{End}(V_\pi)$*-valued function:*

1. $K_A(z,w) = E_z^*AE_w$,
2. $R(A)(z,w) = \frac{1}{c_\pi}H_wK_A(w,z)H_z^* = \frac{1}{c_\pi}H_wE_w^*AE_zH_z^*$,
3. $S(A)(z) = R(A)(z,z) = \frac{1}{c_\pi}H_zK_A(z,z)H_z^* = \frac{1}{c_\pi}H_zE_z^*AE_zH_z^*$,
4. $Q(A)(z) = \frac{1}{c_\pi}K_A(z,z)H_z^*H_z = \frac{1}{c_\pi}E_z^*AE_zH_z^*H_z$

where $z,w \in \mathscr{D}$.

Note $K_A, R(A)$ are maps from $\mathscr{D} \times \mathscr{D}$ to $\mathrm{End}(V_\pi)$ and $S(A), Q(A)$ are maps from $\mathscr{D}$ to $\mathrm{End}(V_\pi)$.

Let U be a non-empty open subset of $\mathbb{C}^N \times \mathbb{C}^N$. Given a function $f(z,w)\colon U \to \mathbb{C}$, we call it *sesqui-holomorphic* if f is holomorphic in both z and $\bar{w}$.

Theorem 3.3 ([6] II.4.) *Assume a complex function $f(z,w)$ of $2N$ complex variables $z = z_1,\ldots,z_N$ and $w = w_1,\ldots,w_N$ is given in a neighborhood of the origin $(0,0)$. If f is sesqui-holomorphic function and $f(z,\bar{z}) = 0$ for all z, then we have $f = 0$.*

Now we fix the bounded domain $\mathscr{D}$ with the measure μ given in Section 2.2. We give some properties of the Berezin symbols. The result concerning only K_A can be found in [11].

Proposition 3.4 *(i)* $K_{A^*}(z,w) = K_A(w,z)^*$.

(ii) $K_A(z,w)$ *is holomorphic in* z *and anti-holomorphic in* w.

(iii) The correspondences $A \mapsto K_A(z,w)$, $A \mapsto K_A(z,z)$ *and* $K_A(z,w) \mapsto K_A(z,z)$ *are all injective.*

(iv) $K_A(gz,gw) = \pi(J(g,z))K_{L_\pi(g)^{-1}AL_\pi(g)}(z,w)\pi(J(g,w))^*$.

Proof: For (i), we have $K_A(w,z)^* = (E_w^* A E_z)^* = E_z^* A^* E_w = K_{A^*}(z,w)$.

For (ii), observe $K_A(z,w)(v) = (AE_w(v))(z)$ is a holomorphic function of z as it belongs to H_π. Furthermore, by (i), it is anti-holomorphic in w.

For (iii), we first show $K_A(z,w)$ determines A. Take $\phi \in H_\pi$ and consider the inner product $\langle (A\phi)(z), v\rangle_\pi$.

$$\begin{aligned}
\langle (A\phi)(z), v\rangle_\pi &= \langle \phi, A^* E_z(v)\rangle_{H_\pi} \\
&= \int_{\mathscr{D}} \langle \pi(\kappa(w,w)^{-1}) E_w^* \phi, E_w^* A^* E_z(v)\rangle_\pi d\mu(w) \\
&= \int_{\mathscr{D}} \langle \pi(\kappa(w,w)^{-1}) \phi(w), K_A(z,w)^*(v)\rangle_\pi d\mu(w)
\end{aligned}$$

Hence $K_A(z,w)$ determines A. For the injectivity of the map $K_A(z,w) \mapsto K_A(z,z)$, we consider the $K_A(z,w) = [k_{i,j}(z,w)]_{n\times n}$ (here $n = \dim V_\pi$) such that each $k_{i,j}(z,w)\colon \mathscr{D}\times\mathscr{D} \to \mathbb{C}$ is a sesqui-holomorphic function. Then it immediately follows from Theorem 3.3.

Moreover, (iv) is a direct consequence of (ii) of Lemma 3.1. $\square$

The Berezin symbol $S(A)$ also has some similar properties.

Proposition 3.5 *Given* $A \in B(H_\pi)$, *we have:*

(i) The maps $A \mapsto K_A, R(A), S(A), Q(A)$ *are all injective.*

(ii) $S(A)(z)^* = S(A^*)(z)$ *and* $R(A)(z,w)^* = R(A^*)(w,z)$ *for* $z, w \in \mathscr{D}$*.*

(iii) $S(I)(z) = I_{V_\pi}$ *for each* $z \in \mathscr{D}$*.*

(iv) The spectral radius of $S(A)(z)$ *on* V_π *is bounded by* $\|A\|$ *for each* $z \in \mathscr{D}$*.*

(v) For $g \in G, z \in \mathscr{D}$*, we have*

$$S(A)(gz) = \pi(k(g,z)^{-1})S(L_\pi(g)^{-1}AL_\pi(g))(z)\pi(k(g,z))$$

where $k(g,z) = h_z^{-1}\kappa(g\exp z)^{-1}h_{gz}$*.*

Proof: (i) and (ii) follow Proposition 3.4.

For (iii), note $K_I(z,z) = E_z^* E_z = c_\pi \pi(\kappa(z,z)^{-1})$ by Lemma 3.1(i), it suffices to prove $\pi(h_z h_z^*) = \pi(\kappa(z,z)^{-1})$. We write $g_z = \exp z \cdot h_z \cdot p$ with $p \in P^-$ (see the definition of κ in Section 2.2). As $g_z \in G = G_{\mathbb{R}}$, we have $g_z^* = \overline{g_z^{-1}} = g_z^{-1}$ and $\pi(g_z^*)\pi(g_z) = I_v$. Hence $\pi(p)^*\pi(h_z)^*\pi(\exp z)^*\pi(\exp z)\pi(h_z)\pi(p) = I_V$. We obtain $\pi(h_z)^*\pi(\exp z)^*\pi(\exp z)\pi(h_z) = I_V$ and also $\pi(h_z)^*\pi(\kappa(\exp z^* \exp z))\pi(h_z) = I_V$ where the middle term is just $\pi(\kappa(z,z)^{-1})$.

For (iv), we assume λ is an eigenvalue of A of the maximal modulus and $v \neq 0$ is the corresponding eigenvector. Note $S(I)(z) = I_{V_\pi}$, we have

$$\begin{aligned} |\lambda| &= \left|\frac{\langle S(A)(z)v, v\rangle_\pi}{\langle v, v\rangle_\pi}\right| = \left|\frac{\langle S(A)(z)v, v\rangle_\pi}{\langle S(I)(z)v, v\rangle_\pi}\right| = \left|\frac{c_\pi^{-1} \cdot \langle AE_zH_z^*v, E_zH_z^*v\rangle_\pi}{c_\pi^{-1} \cdot \langle E_zH_z^*v, E_zH_z^*v\rangle_\pi}\right| \\ &= \left|\frac{\langle AE_zH_z^*v, E_zH_z^*v\rangle_\pi}{\langle E_zH_z^*v, E_zH_z^*v\rangle_\pi}\right| \leq \frac{\|AE_zH_z^*v\|_\pi \cdot \|E_zH_z^*v\|_\pi}{\|E_zH_z^*v\|_\pi^2} \leq \|A\|. \end{aligned}$$

So $\lambda \leq \|A\|$.

For (v), we also assume $g_z = \exp z \cdot h_z \cdot p$. Following [20], the action $G \curvearrowright \mathscr{D}$ induced from $G \curvearrowright G/K$ is given as $g \cdot z = \log \zeta(g \exp z)$. Hence we have

$$gg_z = g\exp z h_z y = \exp(g \cdot z)\kappa(g\exp z)\xi(g\exp z)h_z y.$$

We have $g_{g\cdot z} = \exp(g \cdot z)h_{g\cdot z}y'$ for some $y' \in P^-$. Note $g\dot{g}_z = \dot{g_g z}$ in G/K, there is some $k = k(g,z) \in K$ such that $g_{g\cdot z} = gg_z k$. So we obtain $\kappa(g\exp z)\xi(g\exp z)h_z yk = h_{g\cdot z}y'$ and further $\kappa(g\exp z)h_z k = h_{g\cdot z}$ by applying κ. Now we can apply Proposition 3.4(iv) and obtain

$$\begin{aligned} S(A)(g\cdot z) &= \frac{1}{c_\pi}\pi(h_{g\cdot z}^{-1})\pi(\kappa(g\exp z))E_z^* L_\pi(g)^{-1}AL_\pi(g)\pi(\kappa(g\exp z))^*\pi(h_{g\cdot z}^{-1})^* \\ &= \frac{1}{c_\pi}\pi(k)^{-1}\pi(h_z^{-1})E_z^* L_\pi(g)^{-1}AL_\pi(g)\pi(h_z^{-1})^*\pi(k) \\ &= \pi(k)^{-1}S(L_\pi(g)^{-1}AL_\pi(g))(z)\pi(k) \end{aligned}$$

□

Parts (ii), (iii) and (v) of Proposition 3.5 are first proved by B. Cahen [11].

Now we are able give an explicit formula for the projection P_π.

Lemma 3.6 *Given any $\phi \in L^2(\mathscr{D}, V_\pi)$, its image under P_π is given by*

$$(P_\pi\phi)(z) = \int_{\mathscr{D}} E_z^* E_w H_w^* H_w \phi(w) d\mu(w).$$

Proof: As $S(I)(w) = I_{V_\pi}$, we have $\frac{1}{c_\pi}H_w E_w^* E_w z H_w^* = I_{V_\pi}$. Then, by Lemma 3.1(i), we get $\pi(\kappa(w,w)^{-1}) = H_w^* H_w$. Now let $v \in V_\pi, z \in \mathscr{D}$ and consider the inner product $\langle f, E_z(v)\rangle_{H_\pi}$. We have

$$\begin{aligned} \langle (P_\pi\phi)(z), v\rangle_\pi = \langle E_Z^*(P_\pi\phi), v\rangle_\pi = \langle P_\pi\phi, E_z v\rangle_{H_\pi} &= \langle \phi, E_z v\rangle_{L^2} \\ &= \int_{\mathscr{D}} \langle \pi(\kappa(w,w)^{-1})\phi(w), E_w^* E_z v\rangle_\pi d\mu(w) \\ &= \int_{\mathscr{D}} \langle H_w\phi(w), H_w E_w^* E_z v\rangle_\pi d\mu(w) \\ &= \langle \int_{\mathscr{D}} E_z^* E_w H_w^* H_w \phi(w) d\mu(w), v\rangle_\pi \end{aligned}$$

which completes the proof. □

We denote by $\mathrm{tr} = \mathrm{tr}_\pi$ the normalized trace on $\mathrm{End}(V_\pi)$.

Corollary 3.7 *Let $A \in B(H_\pi)$ such that it commutes with the action of Γ, i.e., $AL_\pi(\gamma) = L_\pi(\gamma)A$ for any $\gamma \in \Gamma$. Then we have*

1. $\mathrm{tr}(S(A)(z))$ *is Γ-invariant,*

2. $\mathrm{tr}(R(A)(z,w)R(A)(z,w)^*)$ *is Γ-invariant.*

Proof: The first statment follows from the fact $\mathrm{tr}_\pi(S(A)(gz)) = \mathrm{tr}_\pi(S(L_\pi(g)^{-1}AL_\pi(g))(z))$ in Proposition 3.5 (v).

For the second statement, let $\gamma \in \Gamma$ and $\tilde{z} \in G$ be the inverse image of $z \in \mathscr{D}$. Note that

$$\begin{aligned}\pi(J(\gamma,z))^* H_{\gamma z}^* H_{\gamma z}\pi(J(\gamma,z)) &= \pi(J(\gamma,z))^*\pi(J(\tilde{\gamma z},0)^{-1})^*\pi(J(\tilde{\gamma z},0)^{-1})\pi(J(\gamma,z)) \\ &= (\pi(J(\tilde{\gamma z},0)^{-1})\pi(J(\gamma,z)))^* \cdot (\pi(J(\tilde{\gamma z},0)^{-1})\pi(J(\gamma,z))) \\ &= \pi(J(\tilde{z},0)^{-1})^*\pi(J(\tilde{z},0)^{-1}) = H_z^* H_z\end{aligned}$$

Then we have

$$\begin{aligned}&\mathrm{tr}(R(A)(\gamma z,\gamma w)R(A)(\gamma z,\gamma w)^*) \\ =&\frac{1}{c_\pi^2}\,\mathrm{tr}(H_{\gamma w}K_A(\gamma w,\gamma z)H_{\gamma z}^* H_{\gamma z}K_{A^*}(\gamma z,\gamma w)H_{\gamma w}^*) \\ =&\frac{1}{c_\pi^2}\,\mathrm{tr}(H_{\gamma w}\pi(J(\gamma,w))K_A(w,z)\pi(J(\gamma,z))^*H_{\gamma z}^*H_{\gamma z}\pi(J(\gamma,z))K_{A^*}(z,w)\pi(J(\gamma,w))^*H_{\gamma w}^*) \\ =&\frac{1}{c_\pi^2}\,\mathrm{tr}(K_A(w,z)\pi(J(\gamma,z))^*H_{\gamma z}^*H_{\gamma z}\pi(J(\gamma,z))K_{A^*}(z,w)\pi(J(\gamma,w))^*H_{\gamma w}^*H_{\gamma w}\pi(J(\gamma,w))) \\ =&\frac{1}{c_\pi^2}\,\mathrm{tr}(H_wK_A(w,z)H_z^*H_zK_{A^*}(z,w)H_w^*) = \mathrm{tr}(R(A)(z,w)R(A)(z,w)^*).\end{aligned}$$

□

Let $\mathscr{F} = \Gamma\backslash\mathscr{D}$ be the fundamental domain of the left action of Γ on $\mathscr{D} = G/K$. Suppose the discrete group Γ is a lattice, we have $\mu(\mathscr{F})$ is finite. Recall $A_\pi = B(H_\pi)^\Gamma = \{A \in B(H_\pi) | AL_\pi(\gamma) = L_\pi(\gamma)A, \forall\gamma\in\Gamma\}$ is the commutant.

Proposition 3.8 *Assume π is an irreducible representation of K. Let $\tau\colon B(H_\pi)\to\mathbb{C}$ be the linear functional defined by*

$$\tau(A) = \tfrac{1}{\mu(\mathscr{F})}\textstyle\int_{\mathscr{F}} \mathrm{tr}(S(A)(z))d\mu(z),\ A\in B(H_\pi).$$

Then τ is a positive, faithful, normal, normalized trace on A_π.

In particular, if Γ is an ICC group, τ is the unique normalized trace on the II_1 factor A_π.

Proof: We first show $|\tau(A)| < \infty$ for all $A \in B(H_\pi)$. By Proposition 3.5 (iv), we know $|\operatorname{tr}(S(A)(z))| \leq \|A\|$ and the integral is finite as $\mu(\mathscr{F})$ is finite.

Note $\tau(I) = 1$ and $\tau(A^*A) \geq 0$ since $S(A^*A)(z) = S(A^*A)(z)^*$ by Proposition 3.5 (v). Suppose $\dim_{\mathbb{C}} V_\pi = n$ and take an orthonormal basis $\{v_i, 1 \leq i \leq n\}$ of V_π. One has

$$\begin{aligned}
\tau(A^*A) =& \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}} \operatorname{tr}(S(A^*A)(z))d\mu(z) = \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}} \operatorname{tr}(\frac{1}{c_\pi} H_z E_z^* A^* A E_z H_z^*)d\mu(z) \\
=& \frac{1}{n \cdot c_\pi \cdot \mu(\mathscr{F})}\int_{\mathscr{F}} \sum_{i=1}^{n} \langle H_z E_z^* A^* A E_z H_z^* v_i, v_i\rangle_\pi d\mu(z) \\
=& \frac{1}{n \cdot c_\pi \cdot \mu(\mathscr{F})}\int_{\mathscr{F}} \sum_{i=1}^{n} \langle A E_z H_z^* v_i, A E_z H_z^* v_i\rangle_{H_\pi} d\mu(z) \\
=& \frac{1}{n \cdot c_\pi \cdot \mu(\mathscr{F})}\int_{\mathscr{F}} \sum_{i=1}^{n} \left(\int_{\mathscr{D}} \langle \pi(\kappa(w,w)^{-1})(A E_z H_z^* v_i)(w), (A E_z H_z^* v_i)(w)\rangle_\pi d\mu(w)\right) d\mu(z) \\
=& \frac{1}{n \cdot c_\pi \cdot \mu(\mathscr{F})}\int_{\mathscr{F}} \sum_{i=1}^{n} \left(\int_{\mathscr{D}} \langle H_w E_w^* A E_z H_z^* v_i, H_w E_w^* A E_z H_z^* v_i\rangle_\pi d\mu(w)\right) d\mu(z) \\
=& \frac{1}{c_\pi \cdot \mu(\mathscr{F})}\int_{\mathscr{F}} \left(\int_{\mathscr{D}} \operatorname{tr}((H_z E_z^* A^* E_w H_w^*)(H_w E_w^* A E_z H_z))d\mu(w)\right) d\mu(z) \\
=& \frac{1}{c_\pi \cdot \mu(\mathscr{F})}\int_{\mathscr{F}\times\mathscr{D}} \operatorname{tr}((H_z E_z^* A^* E_w H_w^*)(H_w E_w^* A E_z H_z))d\mu^2(z,w).
\end{aligned}$$

Here we are able to take the integral over the product space $\mathscr{F} \times \mathscr{D}$ since the integral is finite. Similarly, we obtain

$$\tau(AA^*) = \frac{1}{c_\pi \cdot \mu(\mathscr{F})}\int_{\mathscr{F}\times\mathscr{D}} \operatorname{tr}((H_z E_z^* A E_w H_w^*)(H_w E_w^* A^* E_z H_z))d\mu^2(z,w).$$

Consider the diagonal action of Γ on $\mathscr{D} \times \mathscr{D}$. Both of the two integrations are over a Γ-fundamental domain of $\mathscr{D} \times \mathscr{D}$. As the measure μ^2 is Γ-invariant and the integrand is Γ-invariant by Lemma 3.7, we replace it with the integration over another fundamental domain $(z,w) \in \mathscr{D} \times \mathscr{F}$, which is equivalent to swapping z,w. Note the integrand is Γ-invariant under the same action of Γ on $\mathscr{D} \times \mathscr{D}$ by Lemma 3.7. Hence the integration above is invariant if we swap z,w. This is to say $\tau(A^*A) = \tau(AA^*)$ and τ is a trace.

Note $\{E_z H_z^* v_i | z \in \mathscr{D}, 1 \leq i \leq n\}$ spans a dense subspace of H_π. If $A \neq 0$, we have $\|AE_{z_0} H_{z_0}^* v_i\|^2 > 0$ for some $z_0 \in \mathscr{D}$ and i. As $AE_z H_z^* v_i$ is continuous in z, we have $\|AE_z H_z^* v_i\|^2 >$

0 in a neighbourhood N_{z_0} of z_0 whose measure $\mu(N_{z_0})$ is strictly positive. From the equality above, we also have

$$\tau(A^*A) = \frac{1}{n\cdot c_\pi\cdot\mu(\mathscr{F})}\int_{\mathscr{F}}\sum_{i=1}^{n}\langle AE_zH_z^*v_i, AE_zH_z^*v_i\rangle_{H_\pi}d\mu(z)$$

Hence $\tau(A^*A) > 0$ and τ is faithful.

For the normality, it suffices to prove τ is completely additive [22]. Take a family of mutually orthogonal projections $\{p_j | j \in J\}$ in A_π and let $p = \sum_{j\in J} p_i$. We have

$$\tau(p) = \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}}\mathrm{tr}(S(p)(z))d\mu(z) = \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}}\sum_{j\in J}\mathrm{tr}(S(p_j)(z))d\mu(z),$$

which converges since $\tau(p) \leq \tau(1) = 1$. Moreover, as $0 \leq \mathrm{tr}(S(p_j)(z)) \leq \mathrm{tr}(S(p)(z))$, we have $\mathrm{tr}(S(p_j)(z)) \in L^1(\mathscr{F},\mu)$. By the Fubini Theorem, we obtain

$$\frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}}\sum_{j\in J}\mathrm{tr}(S(p_j)(z))d\mu(z) = \sum_{j\in J}\frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}}\mathrm{tr}(S(p_j)(z))d\mu(z) = \sum_{j\in J}\tau(p_j).$$

Hence τ is normal. □

3.2 Toeplitz operators of matrix-valued functions

In this section, we define the generalized Toeplitz operators associated with $\mathrm{End}(V_\pi)$-valued functions on the bounded symmetric domain $\mathscr{D}$. Then we focus on the Γ-invariant case and give another formula of the trace of the Toeplitz operator.

Let H_π be the holomorphic discrete series (or square-integrable) representations defined in Section 2. Recall that P_π is the orthogonal projection from $L^2(\mathscr{D},V_\pi)$ onto $H_\pi = L^2_{\mathrm{hol}}(\mathscr{D},V_\pi)$ and $H_z \in GL(V_\pi)$ defined in Section 3.1.

Now we consider a measurable $\mathrm{End}(V_\pi)$-valued function f on $\mathscr{D}$. For any $\phi \in L^2(\mathscr{D},V_\pi)$, one may wonder that when the multiplication operator $M_f \colon \phi \mapsto f\cdot\phi$ is bounded. Indeed, we have

$$\begin{aligned}\langle f\cdot\phi, f\cdot\phi\rangle_{L^2(\mathscr{D},V_\pi)} &= \int_{\mathscr{D}}\langle H_zf(z)\phi(z), H_zf(z)\phi(z)\rangle_\pi d\mu(z)\\ &= \int_{\mathscr{D}}\langle H_zf(z)H_z^{-1}H_z\phi(z), H_zf(z)H_z^{-1}H_z\phi(z)\rangle_\pi d\mu(z)\\ &\leq \int_{\mathscr{D}}\|H_zf(z)H_z^{-1}\|_{\mathrm{op}}^2\langle H_z\phi(z), H_z\phi(z)\rangle_\pi d\mu(z),\end{aligned}$$

where $\|\cdot\|_{\text{op}}$ is the operator norm on the finite dimensional Hilbert space V_π. Hence if $\|H_z f(z) H_z^{-1}\|_{\text{op}}$ is essentially bounded on $\mathscr{D}$, say $\|H_z f(z) H_z^{-1}\|_{\text{op}} \leq C$ for all $z \in \mathscr{D}$, we will certainly get $\|M_f\|_{L^2(\mathscr{D},V_\pi)} \leq C$.

We define the following two spaces

1. $L^\infty_H(\mathscr{D}, \text{End}(V_\pi)) = \{f \colon \mathscr{D} \to \text{End}(V_\pi) \text{ measurable} \mid \|H_z f(z) H_z^{-1}\|_{\text{op}} \in L^\infty(\mathscr{D})\}$.

2. $L^\infty(\mathscr{D}, \text{End}(V_\pi)) = \{f \colon \mathscr{D} \to \text{End}(V_\pi) \text{ measurable} \mid \|f\|_F \in L^\infty(\mathscr{D})\}$.

Please note for a measurable space (X,μ), we call a function $f \colon X \to \text{End}(V_\pi)$ measurable if $f = [f_{i,j}]_{1\leq i,j\leq n}$ and $f_{i,j} \colon X \to \mathbb{C}$ is measurable for all $1 \leq i, j \leq n$.

We also let $\|A\|_F = \text{Tr}(A^*A)^{1/2}$ be the *Frobenius norm* of a square matrix A where Tr is the trace that is not normalized (or the sum of the diagonal elements).

Lemma 3.9 *We have $f(z) \in L^\infty_H(\mathscr{D}, \text{End}(V_\pi))$ iff $\|H_z f(z) H_z^{-1}\|_F$ is essentially bounded i.e., $H_z f(z) H_z^{-1} \in L^\infty(\mathscr{D}, \text{End}(V_\pi))$.*

Proof: It follows by the fact that the operator norm is always bounded by the Frobenius norm (i.e., $\|A\|_{\text{op}} \leq \|A\|_F$ for square matrix A). □

Definition 3.10 *For any $f \in L^\infty_H(\mathscr{D}, \text{End}(V_\pi))$, we define the Toeplitz operator in $B(H_\pi)$ associated to f by*

$$T_f = P_\pi \cdot M_f \cdot P_\pi = P_\pi \cdot M_f.$$

where M_f is the multiplication operator by f on H_π. More precisely, for any $\phi \in H_\pi$, the operator acts on it by

$$(T_f \circ \phi)(z) = P_\pi(f \circ \phi)(z)$$

where $(f \circ \phi)(z) = f(z)\phi(z) \in V_\pi$.

Remark 3.11 *When f takes values in the center of* $\text{End}(V_\pi)$*, it can be identified with a scalar-valued function. In this case, T_f is just the classical Toeplitz operator associated* *with $f \in L^\infty(\mathscr{D})$. For more details on classical Toeplitz operators associated with functions on the open unit disk, we refer to [9, 19, 40].*

Proposition 3.12 *We have $T_f \in B(H_\pi)$ for each $f \in L^\infty_H(\mathscr{D}, \mathrm{End}(V_\pi))$ with the following properties:*

(i) $T: f \mapsto T_f$ is linear, i.e., $T_{\alpha f+\beta h} = \alpha T_f + \beta T_h$ for $\alpha, \beta \in \mathbb{C}$ and $f, h \in L^\infty(\mathscr{D}, \mathrm{End}(V_\pi))$,

(ii) $T^*_{f(z)} = T_{H_z^{-1}(H_z^*)^{-1}f(z)^*H_z^*H_z}$ where f^* is given pointwise by $f^*(z) = f(z)^*$ acting on V_π. In particular, if $f \in L^\infty(\mathscr{D})$, $T^*_f = T_{\overline{f}}$.

Proof: As M_f is bounded, it is clear that $T_f = P_\pi M_f P_\pi$ is bounded on H_π.

The linearity is straightforward. For the adjoint T^*_f, let $\phi, \eta \in H_\pi$ and consider the following inner product:

$$\begin{aligned}
\langle \phi, T^*_f \eta \rangle_{H_\pi} = \langle T_f \phi, \eta \rangle_{H_\pi} &= \langle M_f \phi, \eta \rangle_{L^2} \\
&= \int_{\mathscr{D}} \langle H_w f(z)\phi(z), H_z \eta(z) \rangle_\pi d\mu(z) \\
&= \int_{\mathscr{D}} \langle H_z f(z) H_w^{-1} H_z \phi(z), H_z \eta(z) \rangle_\pi d\mu(z) \\
&= \int_{\mathscr{D}} \langle H_z \phi(z), (H_z^{-1})^* f(z)^* H_z^* H_z \eta(z) \rangle_\pi d\mu(z) \\
&= \int_{\mathscr{D}} \langle H_z \phi(z), H_z (H_z^{-1}(H_z^{-1})^* f(z)^* H_z^* H_z) \eta(z) \rangle_\pi d\mu(z) \\
&= \langle \phi(z), M_{H_z^{-1}(H_z^{-1})^* f(z)^* H_z^* H_z} \eta(z) \rangle_{L^2} \\
&= \langle \phi(z), T_{H_z^{-1}(H_z^{-1})^* f(z)^* H_z^* H_z} \eta(z) \rangle_{H_\pi}.
\end{aligned}$$

This implies $T^*_{f(z)} = T_{H_z^{-1}(H_z^{-1})^* f(z)^* H_z^* H_z}$. □

Now we consider the left action of G on $\mathscr{D}$ and also on $L^\infty(\mathscr{D}, \mathrm{End}(V_\pi))$ given by $g \cdot f(z) = f(g^{-1}z)$.

Proposition 3.13 *For any $f \in L^\infty_H(\mathscr{D}, \mathrm{End}(V_\pi))$, We have*

$$L_\pi(g) T_{f(z)} L_\pi(g)^* = T_{\pi(J(g^{-1},z))^{-1}(g \cdot f)(z)\pi(J(g^{-1},z))}.$$

Hence if $f(g^{-1}\cdot z)=\pi(J(g^{-1},z))f(z)\pi(J(g^{-1},z))^{-1}$ for all $\gamma\in\Gamma, z\in\mathscr{D}$, then T_f commutes with the action of Γ. In particular, for $f\in L^\infty(\mathscr{D})$, T_f commutes with the action of Γ if f is Γ-invariant.

Proof: Let $\phi\in H_\pi$. We have

$$\begin{aligned}
&(L_\pi(g)T_fL_\pi(g)^*\phi)(z)\\
=&L_\pi(g)P_\pi M_f\pi(J(g,z))^{-1}\phi(gz)=P_\pi L_\pi(g)f(z)\pi(J(g,z))^{-1}\phi(gz)\\
=&P_\pi\pi(J(g^{-1},z))^{-1}f(g^{-1}z)\pi(J(g^{-1},z))\pi(J(g,z))^{-1}\pi(J(g,g^{-1}z))^{-1}\phi(z)\\
=&P_\pi M_{\pi(J(g^{-1},z))^{-1}f(g^{-1}z)\pi(J(g^{-1},z))}\phi(z)\\
=&T_{\pi(J(g^{-1},z))^{-1}(g\cdot f)(z)\pi(J(g^{-1},z))}\phi(z),
\end{aligned}$$

where we use $\pi(J(g,z))^{-1}\pi(J(g,g^{-1}z))^{-1}=I$. Hence

$$L_\pi(g)T_{f(z)}L_\pi(g)^*=T_{\pi(J(g^{-1},z))^{-1}(g\cdot f)(z)\pi(J(g^{-1},z))}.$$

If $f\in L^\infty(\mathscr{D})$, i.e., $f(z)\in\mathbb{C}$, we have $f(z)=g\cdot f(z)=f(g^{-1}z)$. □

Now we define $R\colon\mathscr{D}\times\mathscr{D}\to\mathrm{End}(V_\pi)$ by

$$R(w,z)=H_zE_z^*E_wH_w^*.$$

Note $R(w,z)^*=R(z,w)$ and it is indeed the element $c_\pi\cdot R(I)(z,w)$ in $\mathrm{End}(V_\pi)$. Moreover, we let $\delta\colon\mathscr{D}\times\mathscr{D}\to\mathrm{End}(V_\pi)$ given by

$$\delta(z,w)=R(w,z)R(w,z)^*=H_zE_z^*E_wH_w^*H_wE_w^*E_zH_z^*,$$

which is a positive operator in $\mathrm{End}(V_\pi)$.

Lemma 3.14 *For $f\in L^\infty_H(\mathscr{D},\mathrm{End}(V_\pi))$, we have*

$$S(T_f)(z)=\tfrac{1}{c_\pi}\textstyle\int_{\mathscr{D}}R(w,z)(H_wf(w)H_w^{-1})R(w,z)^*d\mu(w).$$

If $f\in L^\infty(\mathscr{D})$, $S(T_f)(z)=\frac{1}{c_\pi}\int_{\mathscr{D}}f(w)\delta(z,w)d\mu(w)$.

Proof: Take any $u, v \in V_\pi$ and consider the following inner product.

$$\begin{aligned}\langle S(T_f)(z)u,v\rangle_\pi =&\frac{1}{c_\pi}\langle T_f E_z H_z^* u, E_z H_z^* v\rangle_{H_\pi} = \frac{1}{c_\pi}\langle M_f E_z H_z^* u, E_z H_z^* v\rangle_{L^2}\\ =&\frac{1}{c_\pi}\int_{\mathscr{D}}\langle H_w^* H_w E_w^* M_f E_z H_z^* u, E_w^* E_z H_z^* v\rangle_\pi d\mu(w)\\ =&\frac{1}{c_\pi}\int_{\mathscr{D}}\langle H_w^* H_w f(w) E_w^* E_z H_z^* u, E_w^* E_z H_z^* v\rangle_\pi d\mu(w)\\ =&\frac{1}{c_\pi}\int_{\mathscr{D}}\langle H_z E_z^* E_w H_w^* H_w f(w) E_w^* E_z H_z^* u, v\rangle_\pi d\mu(w)\\ =&\langle\frac{1}{c_\pi}\int_{\mathscr{D}} R(w,z)(H_w f(w) H_w^{-1}) R(w,z)^* d\mu(w) u, v\rangle_\pi\end{aligned}$$

If $f(z) \in \mathbb{C}\cdot I \in \mathrm{End}(V_\pi)$, we can further obtain $R(w,z)(H_w f(w) H_w^{-1})R(w,z)^* = f(w)\delta(z,w)$.

$\square$

Now we identify the Γ-invariant function in $L^\infty(\mathscr{D})$ with $L^\infty(\mathscr{F})$, i.e., $L^\infty(\mathscr{F}) = L^\infty(\mathscr{D})^\Gamma$. By Proposition 3.13, the Toeplitz operator gives a map $T\colon L^\infty(\mathscr{F}) \to B(H_\pi)^\Gamma = A_\pi$ by $f \mapsto T_f$.

For the $\mathrm{End}(V_\pi)$-valued Toeplitz operators, consider the extension form $L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$ to $L_H^\infty(\mathscr{D}, \mathrm{End}(V_\pi))$ given in Proposition 3.13 by

$$\gamma\cdot f(z) = f(\gamma^{-1}z) = \pi(J(\gamma^{-1},z))f(z)\pi(J(\gamma^{-1},z))^{-1}, \forall\gamma\in\Gamma, z\in\mathscr{F}$$

This establishes a map form $L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$ to A_π:

$$T\colon L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi)) \to B(H_\pi)^\Gamma = A_\pi \text{ by } f \mapsto T_f.$$

Proposition 3.15 *Given $A \in A_\pi$ and $f \in L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$, we have*

$$\tau(AT_f) = \tfrac{1}{\mu(\mathscr{F})}\textstyle\int_{\mathscr{F}} \mathrm{tr}(f(z)Q(A)(z))d\mu(z).$$

Proof: We let $v_1, \ldots, v_n$ be an orthonormal basis of V_π. By Lemma 3.6 and Proposition

3.8, we obtain:

$$
\begin{aligned}
\tau(AT_f) =& \frac{1}{\mu(\mathscr{F})} \operatorname{tr}(S(AT_f)(z)) d\mu(z) \\
=& \frac{1}{n\cdot c_\pi n\cdot \mu(\mathscr{F})} \int_{\mathscr{F}} \sum_{i=1}^{n} \langle H_z E_z^* A T_f E_z H_z^* v_i, v_i \rangle_\pi d\mu(z) \\
=& \frac{1}{nc_\pi \mu(\mathscr{F})} \sum_{i=1}^{n} \int_{\mathscr{F}} \langle T_f E_z H_z^* v_i, A^* E_z H_z^* v_i \rangle_{H_\pi} d\mu(z) \\
=& \frac{1}{nc_\pi \mu(\mathscr{F})} \sum_{i=1}^{n} \int_{\mathscr{F}} \langle M_f E_z H_z^* v_i, A^* E_z H_z^* v_i \rangle_{L^2} d\mu(z) \\
=& \frac{1}{nc_\pi \mu(\mathscr{F})} \sum_{i=1}^{n} \int_{\mathscr{F}} \big(\int_{\mathscr{D}} \langle H_w f(w) E_w^* E_z H_z^* v_i, H_w E_w^* A^* E_z H_z^* v_i \rangle_\pi d\mu(w) \big) d\mu(z) \\
=& \frac{1}{c_\pi \mu(\mathscr{F})} \int_{\mathscr{F}} \big(\int_{\mathscr{D}} \operatorname{tr}(H_z E_z^* A E_w H_w^* H_w f(w) E_w^* E_z H_z^*) d\mu(w) \big) d\mu(z)
\end{aligned}
$$

As in the proof of Proposition 3.8, we consider the diagonal action of Γ on $\mathscr{D}^2$. The fundamental domain is $\mathscr{F} \times \mathscr{D}$. Since μ^2 is a Γ-invariant measure and the integrand $\operatorname{tr}(H_z E_z^* A E_w H_w^* H_w f(w) E_w^* E_z H_z^*)$ is also Γ-invariant, we can also replace it with another fundamental domain $\mathscr{D} \times \mathscr{F}$ by changing (z,w) to (w,z), which leaves the integration invariant. Hence the integral above equals to:

$$
\begin{aligned}
& \frac{1}{c_\pi \mu(\mathscr{F})} \int_{\mathscr{F}} \big(\int_{\mathscr{D}} \operatorname{tr}(H_z E_z^* A E_w H_w^* H_w f(w) E_w^* E_z H_z^*) d\mu(z) \big) d\mu(w) \\
=& \frac{1}{nc_\pi \mu(\mathscr{F})} \sum_{i=1}^{n} \int_{\mathscr{F}} \big(\int_{\mathscr{D}} \langle H_z E_z^* A E_w H_w^* v_i, H_z E_z^* E_w f(w)^* H_w^* v_i \rangle_\pi d\mu(z) \big) d\mu(w) \\
=& \frac{1}{nc_\pi \mu(\mathscr{F})} \sum_{i=1}^{n} \int_{\mathscr{F}} \langle A E_w H_w^* v_i, E_w f(w)^* H_w^* v_i \rangle_{H_\pi} d\mu(w) \\
=& \frac{1}{c_\pi \mu(\mathscr{F})} \int_{\mathscr{F}} \operatorname{tr}(H_w f(w) E_w^* A E_w H_w^*) d\mu(w) \\
=& \frac{1}{\mu(\mathscr{F})} \int_{\mathscr{F}} \operatorname{tr}(H_w f(w) H_w^{-1} S(A)(w)) d\mu(w) \\
=& \frac{1}{\mu(\mathscr{F})} \int_{\mathscr{F}} \operatorname{tr}(K_A(w) H_w^* H_w f(w)) d\mu(w)
\end{aligned}
$$

□

If $f \in L^\infty(\mathscr{D})$, this formula of the trace can be simplified as follows.

Corollary 3.16 *For $f \in L^\infty(\mathscr{F})$, we have*

$$\tau(AT_f) = \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}} f(z)\,\mathrm{tr}(Q(A)(z))d\mu(z).$$

Proof: It follows the fact $\mathrm{tr}(f(z)Q(A)(z)) = f(z)\,\mathrm{tr}(Q(A)(z))$ if $f(z)$ is a scalar. $\square$

Chapter 4

The Commutant of the Group von Neumann Algebras

4.1 The L^2-space of matrix-valued functions

We keep the notations as in the previous sections. For a measurable function $f\colon \mathscr{F} \to \mathrm{End}(V_\pi)$, we denote by f_H the following function

$$f_H(z) = H_z \cdot f(z) \cdot H_z^{-1},\ z \in \mathscr{F}.$$

Consider the following vector space of $\mathrm{End}(V)$-valued functions on $\mathscr{F}$:

$$L^2_H(\mathscr{F}, \mathrm{End}(V), \mu) = \{f\colon \mathscr{F} \to \mathrm{End}(V) \ \text{ measurable} |\int_{\mathscr{F}} \mathrm{Tr}(f_H(z) f_H(z)^*) d\mu(z) < \infty\}.$$

Here Tr is the trace on $\mathrm{End}(V)$ which is not normalized.

We also denote a sesquilinear form defined on $L^2_H(\mathscr{F}, \mathrm{End}(V), \mu)$ by

$$\langle f, h\rangle = \langle f, h\rangle_{L^2_H} = \int_{\mathscr{F}} \mathrm{Tr}(f_H(z) h_H(z)^*) d\mu(z) = \int_{\mathscr{F}} \mathrm{Tr}(H_z f(z) H_z^{-1} (H_z^{-1})^* h(z)^* H_z^*) d\mu(z),$$

where $f, h \in L^2_H(\mathscr{F}, \mathrm{End}(V_\pi), \mu)$.

Lemma 4.1 *The sesquilinear form $\langle\cdot,\cdot\rangle$ is an inner product on $L^2_H(X, \mathrm{End}(V), \mu)$.*

Proof: We can check $\langle \alpha f_1 + \beta f_2, h\rangle = \alpha\langle f_1, h\rangle + \beta\langle f_2, h\rangle$ for $\alpha, \beta \in \mathbb{C}$. As $\mathrm{Tr}(f_H(z) h_H(z)^*) = \mathrm{Tr}(H_z f(z) H_z^{-1} (H_z^{-1})^* h(z)^* H_z^*) = \overline{\mathrm{Tr}(H_z h(z) H_z^{-1} (H_z^{-1})^* f(z)^* H_z^*)} = \overline{\mathrm{Tr}(h_H(z) f_H(z)^*)}$, we have $\langle f, h\rangle = \overline{\langle h, f\rangle}$.

Now we assume $\langle f, f\rangle = 0$. Then $\|H_z f(z) H_z^{-1}\|_F = 0$ almost everywhere on $\mathscr{F}$. Hence $f \overset{\text{a.e.}}{=} 0$. □

Therefore we obtain a Hilbert space $L^2_H(\mathscr{F}, \mathrm{End}(V), \mu)$ or simply $L^2_H(\mathscr{F}, \mathrm{End}(V))$

Now we consider the following space

$$L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi)) = \{f\colon \mathscr{F} \to \mathrm{End}(V_\pi) \text{ measurable} | \|f_H(z)\|_F \in L^\infty(\mathscr{F})\}.$$

One can show this is a complex algebra by the fact $(fh)_H(z) = f_H(z)h_H(z)$ and the Frobenius norm $\|\cdot\|_F$ is sub-multiplicative, i.e., $\|AB\|_F \leq \|A\|_F \cdot \|B\|_F$. We denote $\|(\|f_H(z)\|_F)\|_\infty$, which is the essential norm of $\|f_H(z)\|_F \in L^\infty(\mathscr{F})$, by $\|f\|_{H,\infty}$. Note that $\|f_H(z)\|_{\text{op}} \leq \|f\|_{H,\infty}$ for all $z \in \mathscr{F}$.

Furthermore, we also define another Hilbert space

$$L^2_H(\mathscr{F}, V_\pi) = \{\phi\colon \mathscr{F} \to V_\pi \text{ measurable} | \int_{\mathscr{F}} \langle H_z\phi(z), H_z\phi(z)\rangle_\pi d\mu(z) < \infty\},$$

where the inner product is given as $\langle \cdot,\cdot\rangle_1 = \int_{\mathscr{F}} \langle H_z\phi(z), H_z\psi(z)\rangle_\pi d\mu(z)$ for $\phi, \psi \in L^2_H(\mathscr{F}, V_\pi)$.

Define an action σ of $f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$ on $L^2_H(\mathscr{F}, V_\pi)$ as

$$f \circ \phi(z) = f(z) \cdot \phi(z),$$

where $f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$ and $\phi \in L^2_H(\mathscr{F}, V_\pi)$.

Proposition 4.2 *The action σ defined above gives a well-defined faithful C^*-representation of the algebra $L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$ such that*

1. *The adjoint of $\sigma(f(z))$ is $\sigma(f(z))^* = \sigma(H_z^{-1}(H_z^*)^{-1} f(z)^* H_z^* H_z)$,*
2. *$\sigma(f(z))$ is a positive operator iff $H_z f(z) H_z^{-1} = g(z)^* g(z)$ for some $g \in L^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$.*

Proof: Assume $\|f\|_{H,\infty} = C$. Then we have

$$\begin{aligned}\langle H_z f(z)\phi(z), H_z f(z)\phi(z)\rangle_\pi &= \langle f_H(z)H_z\phi(z), f_H(z)H_z\phi(z)\rangle_\pi \\ &\leq \|f_H(z)\|_{\text{op}} \cdot \langle H_z\phi(z), H_z\phi(z)\rangle_\pi \leq C \cdot \langle H_z\phi(z), H_z\phi(z)\rangle_\pi.\end{aligned}$$

Hence $\|\sigma(f)\|_{L^2_H(\mathscr{F}, V_\pi)} \leq C$ which is well-defined.

It is straightforward to check $\sigma(f) = 0$ if and only if $f = 0$. Moreover, we have

$$\begin{aligned}\langle \sigma(f)\phi, \psi\rangle_H &= \int_{\mathscr{F}} \langle H_z f(z)\phi(z), H_z\psi(z)\rangle_\pi d\mu(z) \\ &= \int_{\mathscr{F}} \langle H_z f(z) H_z^{-1} H_z\phi(z), H_z\psi(z)\rangle_\pi d\mu(z) \\ &= \int_{\mathscr{F}} \langle H_z\phi(z), H_z H_z^{-1}(H_z^*)^{-1} f(z)^* H_z^* H_z\psi(z)\rangle_\pi d\mu(z).\end{aligned}$$

This proves $\sigma(f(z))^* = \sigma(H_z^{-1}(H_z^*)^{-1}f(z)^*H_z^*H_z)$. By the second line above, $\sigma(f(z))$ is positive iff $H_z f(z)H_z^{-1} = g(z)^*g(z)$ for some $g \in L^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$. □

As σ is faithful, we denote also by $L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$ its image under σ and equipped it with the C^*-structure as above.

Now we regard $L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$ as the Γ-invariant functions in $L_H^\infty(\mathscr{D}, \mathrm{End}(V_\pi))$ (see 3.13). For a given $f \in L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$, we denote by $\tilde{f}$ its Γ-invariant lifting to $L^\infty(\mathscr{D}, \mathrm{End}(V_\pi))$ as follows. For $w \in \mathscr{D}$, there is a unique $\gamma \in \Gamma$ such that $w = \gamma z$. Let $\gamma_\pi(z) = \pi(J(\gamma, z)) = \pi(\kappa(\gamma \exp z))$, the function $\tilde{f}(w)$ with $w \in \mathscr{D}$ is given as following:

$$\tilde{f}(w) = \tilde{f}(\gamma z) = \gamma_\pi(z) f(z) \gamma_\pi(z)^{-1}, \text{ for all } \gamma \in \Gamma, z \in \mathscr{F}.$$

Lemma 4.3 *For $z \in \mathscr{F}$, we have*

$$H_{\gamma z}\tilde{f}(\gamma z)H_{\gamma z}^{-1} = \pi(k(\gamma, z))^* H_z f(z) H_z^{-1} \pi(k(\gamma, z))$$

with some $k(\gamma, z) \in K$. Hence the $\mathrm{End}(V_\pi)$*-valued function $\tilde{f}$ is in $L_H^\infty(\mathscr{D}, \mathrm{End}(V_\pi))$ and gives a well-defined Γ-intertwining Toeplitz operator $T_{\tilde{f}}$.*

Proof: Recall $H_z = \pi(h_z^{-1}) = \pi(\kappa(g_z)^{-1})$ (see Section 3.1). In the proof of Proposition 3.5.(v), we know there is $k(\gamma, z) \in K$ such that $k(\gamma, z) = h_z^{-1}\kappa(\gamma \exp z)^{-1}h_{\gamma z}$. Then $\pi(k(\gamma, z)) = H_z \pi(\kappa(\gamma \exp z)^{-1})H_{\gamma z}^{-1}$. We obtain

$$\begin{aligned} H_{\gamma z}\tilde{f}(\gamma z)H_{\gamma z}^{-1} &= \pi(k(\gamma, z))^{-1} H_z \pi(\gamma \exp z)^{-1})\tilde{f}(\gamma z)\pi(\gamma \exp z))H_z^{-1}\pi(k(\gamma, z)) \\ &= \pi(k(\gamma, z))^{-1} H_z f(z) H_z^{-1} \pi(k(\gamma, z)). \end{aligned}$$

Note $k(\gamma, z) \in K$ and π is a unitary representation of K. We have $\|H_{\gamma z}\tilde{f}(\gamma z)H_{\gamma z}^{-1}\|_{\mathrm{op}} = \|H_z f(z) H_z^{-1}\|_{\mathrm{op}}$.

The Γ-intertwining property follows from the definition of $\tilde{f}$ and Proposition 3.13. □

Now we define a map B on $L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$ by

$$Bf(z) = \tfrac{1}{c_\pi} E_z^* T_{\tilde{f}} E_z H_z^* H_z, \; z \in \mathscr{D},$$

for $f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$. It is related to the Berezin symbols by $Bf(z) = H_z^{-1} S(T_{\tilde{f}})(z) H_z$. We denote the $\pi(k(\gamma,z))^*$ by $k_\pi(\gamma,z)$.

Lemma 4.4 *For $f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$, we have $Bf(z) \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$ which can be given as*

$$\begin{aligned} Bf(z) &= H_z^{-1} \int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma \in \Gamma} R(\gamma w, z)(H_{\gamma w} \gamma_\pi(w) f(w) \gamma_\pi(w)^{-1} H_{\gamma w}^{-1}) R(\gamma w, z)^* d\mu(w) H_z \\ &= H_z^{-1} \int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma \in \Gamma} R(\gamma w, z) k_\pi(\gamma, z)(H_w f(w) H_w^{-1})) k_\pi(\gamma, z)^* R(\gamma w, z)^* d\mu(w) H_z \end{aligned}$$

Furthermore, if we take $f = I_{V_\pi}$, $BI_{V_\pi}(z) = I_{V_\pi}$.

Proof: As $Bf(z) = H_z^{-1} S(T_f)(z) H_z$, it follows then by Proposition 3.13 and Proposition 4.3. Then the case $f = I_{V_\pi}$ is straightforward by Proposition 3.5. □

Proposition 4.5 *The map B defined in Lemma 4.4 can be extended to a bounded operator on $L^2_H(\mathscr{F}, \mathrm{End}(V_\pi))$.*

Proof: As $\mu(\mathscr{F}) < \infty$, $L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$ is a dense subspace of $L^2_H(\mathscr{F}, \mathrm{End}(V_\pi))$. It suffices to show

$$\|Bf\|^2_{L^2_H(\mathscr{F}, \mathrm{End}(V_\pi))} \leq C \cdot \|f\|^2_{L^2_H(\mathscr{F}, \mathrm{End}(V_\pi))}$$

for any $f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$.

Take any $z \in \mathscr{F}$ and consider the following map

$$\begin{aligned} \phi_z : \quad L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi)) &\rightarrow \mathrm{End}(V_\pi) \\ f &\mapsto \phi_z f = H_z \cdot Bf(z) \cdot H_z^{-1} \end{aligned}$$

We first show ϕ_z is a unital positive map. By Proposition 4.2.2, we assume $\sigma(f)$ (or simply f) is positive, i.e., $H_z f(z) h_z^{-1} = g(z)^* g(z)$ for some $g \in L^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$. Hence ϕ_z is positive as $\phi_z f = \int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma \in \Gamma} R(\gamma w, z) k_\pi(\gamma, z)^* (g(w)^* g(w)) R(\gamma w, z)^* d\mu(w)$ is positive in $\mathrm{End}(V_\pi)$. Furthermore, ϕ_z is unital by Lemma 4.4.

Witout loss of generality, we may assume f is normal (or self-adjoint). Then, by Kadison's inequality [24], we have

$$
\begin{aligned}
&\phi_z(f)\phi_z(f^*) \leq \phi_z(ff^*) \\
=&\int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma\in\Gamma} R(\gamma w,z)k_\pi(\gamma,z)^*(H_w f(w)H_w^{-1}(H_w^*)^{-1} f(w)^* H_w^* H_w H_w^{-1}) R(\gamma w,z)^* d\mu(w) \\
=&\int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma\in\Gamma} R(\gamma w,z)k_\pi(\gamma,z)^*(H_w f(w)H_w^{-1}(H_w^*)^{-1} f(w)^* H_w^*) R(\gamma w,z)^* d\mu(w).
\end{aligned}
$$

Consider the L^2_H-norm of Bf. Note that $R(w,z)^* = R(z,w)$ and $\|R(w,z)\|_F^2$ is Γ-invariant by Lemma 3.7. Hence we have

$$
\begin{aligned}
&\|Bf\|^2_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))} \\
=&\int_{\mathscr{F}} \mathrm{Tr}(H_z Bf(z) H_z^{-1}(H_z^{-1})^* Bf(z)^* H_z^*) d\mu(z) \\
=&\int_{\mathscr{F}} \mathrm{Tr}(\phi_z(f)\phi_z(f^*)) d\mu(z) \leq \int_{\mathscr{F}} \mathrm{Tr}(\phi_z(ff^*)) d\mu(z) \\
=&\int_{\mathscr{F}} \mathrm{Tr}(\int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma\in\Gamma} R(\gamma w,z)k_\pi(\gamma,z)^*(H_w f(w)H_w^{-1}(H_w^*)^{-1} f(w)^* H_w^*) k_\pi(\gamma,z) R(\gamma w,z)^* d\mu(w)) d\mu(z) \\
=&\frac{1}{c_\pi} \int_{\mathscr{F}} \int_{\mathscr{F}} \sum_{\gamma\in\Gamma} \mathrm{Tr}(R(\gamma w,z)k_\pi(\gamma,z)^*(H_w f(w)H_w^{-1}(H_w^*)^{-1} f(w)^* H_w^*) k_\pi(\gamma,z) R(\gamma w,z)^*) d\mu(w) d\mu(z) \\
=&\frac{1}{c_\pi} \int_{\mathscr{F}} \int_{\mathscr{F}} \sum_{\gamma\in\Gamma} \|R(\gamma w,z)k_\pi(\gamma,z)^* H_w f(w) H_w^{-1}\|_F^2 d\mu(w) d\mu(z) \\
\leq&\frac{1}{c_\pi} \int_{\mathscr{F}} \int_{\mathscr{F}} \sum_{\gamma\in\Gamma} \|R(\gamma w,z)\|_F^2 \|H_w f(w) H_w^{-1}\|_F^2 d\mu(w) d\mu(z) \\
=&\frac{1}{c_\pi} \int_{\mathscr{F}} \big(\|H_w f(w)H_w^{-1}\|_F^2 \cdot \int_{\mathscr{F}} \sum_{\gamma\in\Gamma} \|R(\gamma w,z)\|_F^2 d\mu(z)\big) d\mu(w) \\
=&\frac{1}{c_\pi} \int_{\mathscr{F}} \big(\|H_w f(w)H_w^{-1}\|_F^2 \cdot \int_{\mathscr{F}} \sum_{\gamma\in\Gamma} \|R(w,\gamma^{-1}z)\|_F^2 d\mu(z)\big) d\mu(w) \\
=&\frac{1}{c_\pi} \int_{\mathscr{F}} \big(\|H_w f(w)H_w^{-1}\|_F^2 \cdot \int_{\mathscr{F}} \sum_{\gamma\in\Gamma} \mathrm{Tr}(R(w,\gamma^{-1}z)R(w,\gamma^{-1}z)^*) d\mu(z)\big) d\mu(w) \\
=&\frac{1}{c_\pi} \int_{\mathscr{F}} \big(\|H_w f(w)H_w^{-1}\|_F^2 \cdot \int_{\mathscr{F}} \sum_{\gamma\in\Gamma} \mathrm{Tr}(R(\gamma^{-1}z,w)^* R(\gamma^{-1}z,w)^*) d\mu(z)\big) d\mu(w) \\
=&\frac{1}{c_\pi} \int_{\mathscr{F}} \big(\|H_w f(w)H_w^{-1}\|_F^2 \cdot \mathrm{Tr}(\int_{\mathscr{F}} \sum_{\gamma\in\Gamma} R(\gamma^{-1}w,z)^* R(\gamma^{-1}w,z)^* d\mu(z))\big) d\mu(w)
\end{aligned}
$$

$$=\int_{\mathscr{F}} n \cdot \|H_w f(w) H_w^{-1}\|_F^2 \mu(w)$$
$$=n \cdot \|f\|^2_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))},$$

where we also apply $BI_{V_\pi}(z) = I_{V_\pi}$ (see Lemma 4.4). Hence B is a bounded. $\square$

Corollary 4.6 *The operator B is injective on $L^\infty_H(\mathscr{F},\mathrm{End}(V_\pi))$.*

Proof: Take $f \in L^\infty_H(\mathscr{F},\mathrm{End}(V_\pi))$, we know $H_z f(z) H_z^{-1} \in L^\infty(\mathscr{F},\mathrm{End}(V_\pi)$. Rewrite $H_z f(z) H_z^{-1} = g(z) + i \cdot h(z)$ for some $f, g \in L^\infty(\mathscr{F},\mathrm{End}(V_\pi))$ such that $g(z)^* = g(z), h(z)^* = h(z)$ for all $z \in \mathscr{F}$.

We assume $Bf = 0$. From the proof of 4.5, we know

$$\|Bf\|^2_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))} = \int_{\mathscr{F}} \mathrm{Tr}(\phi_z(f)\phi_z(f^*))d\mu(z) = 0.$$

So $\phi_z(f) = 0$ for all z, which is to say

$$\begin{aligned}\mathrm{Tr}(\phi_z) &= \int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma\in\Gamma} R(\gamma w, z) k_\pi(\gamma, z)(g(w) + i \cdot h(w)) k_\pi(\gamma, z)^* R(\gamma w, z)^* d\mu(w) \\ &= \mathrm{Tr}\big(\int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma\in\Gamma} R(\gamma w, z) k_\pi(\gamma, z) g(w) k_\pi(\gamma, z)^* R(\gamma w, z)^* d\mu(w)\big) \\ &\quad + i \cdot \mathrm{Tr}\big(\int_{\mathscr{F}} \frac{1}{c_\pi} \sum_{\gamma\in\Gamma} R(\gamma w, z) k_\pi(\gamma, z) h(w) k^*_\pi(\gamma, z) R(\gamma w, z)^* d\mu(w)\big).\end{aligned}$$

Hence it imlplies $g(w) = h(w) = 0$ for all $w \in \mathscr{F}$ and $f = 0$. $\square$

We denote the extended map also by B and write $T_{\tilde{f}}$ simply as T_f.

Proposition 4.7 *The map $T : L^\infty_H(\mathscr{F},\mathrm{End}(V_\pi)) \to A_\pi$ given by $f \mapsto T_f$ can be extended to a bounded linear operator $L^2_H(\mathscr{F},\mathrm{End}(V_\pi)) \to L^2(A_\pi, \tau)$.*

Proof: Note $L^\infty_H(\mathscr{F},\mathrm{End}(V_\pi))$ is dense in $L^2_H(\mathscr{F},\mathrm{End}(V_\pi))$ since $\mu(\mathscr{F})$ is finite. Take

$f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$ and consider the trace $\tau(T_f^* T_f)$.

$$\begin{aligned}
\|T_f\|^2_{L^2(A_\pi,\tau)} = \tau(T_f^* T_f) &= \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}} \mathrm{tr}(f(z)Q(T_f^*)(z))d\mu(z) \\
&= \frac{1}{\mu(\mathscr{F})}\langle f, Bf\rangle_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))} \leq \frac{\|B\|_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))}}{\mu(\mathscr{F})}\|f\|^2_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))}.
\end{aligned}$$

Hence T can be extended to a bounded operator on $L^2_H(\mathscr{F}, \mathrm{End}(V_\pi))$:

$$T\colon L^2_H(\mathscr{F}, \mathrm{End}(V_\pi)) \to L^2(A_\pi, \tau)$$

with $\|T\| \leq \frac{\|B\|_{L^2}}{\mu(\mathscr{F})}$, which is bounded by Proposition 4.5. □

Corollary 4.8 $T^*(A) = \frac{1}{n\cdot\mu(\mathscr{F})}Q(A)$

Proof: Let us consider $\langle T^*(A), f\rangle_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))}$ for an arbitrary $f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))$. By Proposition 3.15 and 5.8.(ii), we have

$$\begin{aligned}
\langle T^*(A), f\rangle_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))} &= \langle A, T_f\rangle_{L^2(A_\pi,\tau)} = \tau(AT_f^*) = \tau(AT_{H_z^{-1}(H_z^{-1})^* f(z)^* H_z^* H_z}) \\
&= \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}} \mathrm{tr}(K_A(z)H_z^* H_z H_z^{-1}(H_z^{-1})^* f(z)^* H_z^* H_z)d\mu(z) \\
&= \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}} \mathrm{tr}(H_z(K_A(z)H_z^* H_z)H_z^{-1}\cdot(H_z^{-1})^* f(z)^* H_z^*)d\mu(z) \\
&= \frac{1}{\mu(\mathscr{F})}\int_{\mathscr{F}} \mathrm{tr}(H_z Q(A)(z)H_z^{-1}\cdot(H_z^{-1})^* f(z)^* H_z^*)d\mu(z) \\
&= \frac{1}{n\cdot\mu(\mathscr{F})}\int_{\mathscr{F}} \mathrm{Tr}(H_z Q(A)(z)H_z^{-1}\cdot(H_z^{-1})^* f(z)^* H_z^*)d\mu(z) \\
&= \langle \frac{1}{n\cdot\mu(\mathscr{F})}Q(A)(z), f(z)\rangle_{L^2_H(\mathscr{F},\mathrm{End}(V_\pi))}
\end{aligned}$$

As the L^∞-space is dense in $L^2(\mathscr{F}, \mathrm{End}(V_\pi))$, this implies $T^*(A) = \frac{1}{n\cdot\mu(\mathscr{F})}Q(A)$. □

Proposition 4.9 *The range of T is dense in $L^2(A_\pi, \tau)$.*

Proof: It suffices show T^* is injective on $L^2(A_\pi, \tau)$. Let ν be the measure $\frac{c_\pi}{\mu(\mathscr{F})}(\mu\times\mu)$ on $\mathscr{F}\times\mathscr{D}$. Consider the following Hilbert space

$$K = L^2(\mathscr{F}\times\mathscr{D}, \mathrm{End}(V_\pi), \nu) = \{f\colon \mathscr{F}\times\mathscr{D} \to \mathrm{End}(V_\pi) | \langle f, f\rangle_K < \infty\}.$$

Here the inner product is given by

$$\langle f, h\rangle_K = \int_{\mathscr{F}\times\mathscr{D}} \mathrm{tr}(f(z,w)h(z,w)^*)d\nu(z,w).$$

We can check this gives an inner product which gives the Hilbert space.

For any $A, B \in A_\pi$, we have

$$\begin{aligned}
\tau(AB^*) &= \frac{1}{c_\pi \mu(\mathscr{F})} \sum_{1\le i\le n} \int_{\mathscr{F}} \langle H_z E_z^* B^* A E_z H_z^* v_i, v_i\rangle_\pi d\mu(z) \\
&= \frac{1}{c_\pi \mu(\mathscr{F})} \sum_{1\le i\le n} \int_{\mathscr{F}} \langle A E_z H_z^* v_i, B E_z H_z^* v_i\rangle_{H_\pi} d\mu(z) \\
&= \frac{1}{c_\pi \mu(\mathscr{F})} \sum_{1\le i\le n} \int_{\mathscr{F}} \left(\int_{\mathscr{D}} \langle H_w E_w^* A E_z H_z^* v_i, H_w E_w^* B E_z H_z^* v_i\rangle_\pi d\mu(w)\right) d\mu(z) \\
&= \int_{\mathscr{F}\times\mathscr{D}} \mathrm{tr}(R(A)(z,w)(R(B)(z,w))^* d\nu(z,w).
\end{aligned}$$

Hence $\langle A, B\rangle_\tau = \langle R(A), R(B)\rangle_K$ and R is an isometry from $L^2(A_\pi, \tau)$ to K, i.e. $R^*R = \mathrm{id}$.

Note we have $T^*(A) = \frac{1}{n\cdot\mu(\mathscr{F})} Q(A)$ by Corollary 4.8. Hence the map T^*R^* on $R(A_\pi)$ is exactly the map given by

$$T^*R^* \colon R(A)(z,w) \mapsto \tfrac{1}{n\cdot\mu(\mathscr{F})} Q(A)(z), A \in A_\pi.$$

Note each element in $R(A_\pi)$ can be written as $\frac{1}{c_\pi} H_w K_A(w,z) H_z^*$ for some $A \in A_\pi$. So this map is exactly the map given by

$$\tfrac{1}{c_\pi} H_w K_A(w,z) H_z^* \mapsto \tfrac{1}{n\cdot c_\pi\cdot\mu(\mathscr{F})} K_A(z,z) H_z^* H_z$$

It can be further extended to a well-defined bounded map on $R(L^2(A_\pi, \tau))$, the range of R. Note by Proposition 3.5, the map $R(A) \mapsto Q(A)$ is injective.

For any $A \in A_\pi$, by Proposition 3.4, we know $c_\pi H_w^{-1} R(A)(z,w)(H_z^*)^{-1} = K_A(w,z)$ is holomorphic in $w, \bar{z}$. Hence that the range $R(L^2(A_\pi))$ are also in the following set

$$\{h(z,w) \colon \mathscr{F}\times\mathscr{D} \to \mathrm{End}(V_\pi) | H_w^{-1} h(z,w)(H_z^*)^{-1} \text{is holomorphic in } w, \bar{z}\}$$

As the map T^*R^* on $R(A_\pi)$ above, we obtain the explicit formula for T^*R^* on $R(L^2(A_\pi))$ given by

$$T^*R^*:\, h(z,w) \mapsto q(z) = \tfrac{1}{n\cdot\mu(\mathscr{F})} H_z^{-1} h(z,z) H_z, \forall h(z,w) \in R(L^2(A_\pi)).$$

Note $H_w^{-1}h(z,w)(H_z^*)^{-1}$ is sesqui-holomorphic. By Theorem 3.3, we know the map

$$H_w^{-1}h(z,w)(H_z^*)^{-1} \mapsto H_z^{-1}h(z,z)(H_z^*)^{-1}$$

is injective. Hence the map $h(z,w) \mapsto q(z)H_z^{-1}(H_z^*)^{-1}$ is injective and so is $T^*R^*:\, h(z,w) \mapsto q(z)$.

□

4.2 The commutant and its subalgebras

We will show the operators T_f above with $f \in L_H^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$ generate the commutant A_π of the group von Neumann algebra $L_\pi(\Gamma)''$. Some results on subalgebras and subfactors are also included. We recall a well-known fact:

Lemma 4.10 *Let $M \subset B(H)$ be a von Neumann algebra with a positive, faithful, normal, normalized trace* tr. *Then the topology induced by $\|x\|_2 = \mathrm{tr}(xx^*)^{1/2}$ coincides with the strong operator topology on any bounded subset of M.*

Proof: It suffices to consider the unit ball $M_1 = \{x \in M | \|x\|_{B(H)} \leq 1\}$ in M. Let us consider the GNS construction for tr and we get a normal faithful representation

$$\pi_{\mathrm{tr}}:\, M \to B(L^2(M, \mathrm{tr})).$$

Note π_{tr} is injective, strong operator topology to strong operator topology continuous and $\|\pi_{\mathrm{tr}}(x)\| \leq \|x\|$.

Take a sequence $\{x_i\}_{i\geq 1}$ in M_1 and suppose we $\|x_i\|_2 \to 0$. Let $\Omega = \hat{1}$ be the cyclic vector in $L^2(M, \mathrm{tr})$. We take an arbitrary $y \in M$ and $\hat{y} = y\Omega \in L^2(M, \mathrm{tr})$.

$$\begin{aligned}\|\pi_{\mathrm{tr}}(x_i)\hat{y}\|_2^2 &= \langle \pi_{\mathrm{tr}}(x_i)y\Omega, \pi_{\mathrm{tr}}(x_i)y\Omega\rangle_{L^2(M,\mathrm{tr})} = \mathrm{tr}(x_i y y^* x_i^*) \\ &\leq \|y\|^2\, \mathrm{tr}(x_i x_i^*) \to 0.\end{aligned}$$

Note M is $\|\cdot\|_2$-dense in $L^2(M,\mathrm{tr})$ by the GNS construction. For any $v \in L^2(M,\mathrm{tr})$ and positive integer N, there exists a $y \in M$ such that $\|v-\hat{y}\| < \frac{1}{N}$. We have $\|\pi_{\mathrm{tr}}(x_i)v\|_2 \leq \|x_i\|\|v-\hat{y}\|_2 + \|\pi_{\mathrm{tr}}(x_i)\hat{y}\|_2 \leq \frac{1}{N} + \|\pi_{\mathrm{tr}}(x_i)\hat{y}\|_2$. Hence $\pi_{\mathrm{tr}}(x_i)v \to 0$ and we can apply π_{tr}^{-1} (which is also strong operator continuous, see [25] 7.1.16) so that $x_i \to 0$ in the strong operator topology on $B(H)$.

Conversely, if $x_i \to 0$ in the strong operator topology on $B(H)$, we have $\pi_{\mathrm{tr}}(x_i)$ also converges to 0 in the strong operator topology on $L^2(M)$. Then

$$\|x_i\|_2^2 = \mathrm{tr}(x_i x_i^*) = \langle \pi_{\mathrm{tr}}(x_i)\Omega, \pi_{\mathrm{tr}}(x_i)\Omega\rangle_{L^2(M)} \to 0.$$

□

As shown above, $T_f \in A_\pi$ for $f \in L^\infty(\mathscr{F}, \mathrm{End}(V_\pi))$. A natural question is how large is the subalgebra of A_π generated by these operators.

Proposition 4.11 *Let $M \subset B(H)$ be a von Neumann algebra with a positive, faithful, normal trace* tr *and $A \subset M$ be a $*$-subalgebra of M. Then A is L^2-dense in $L^2(M,\mathrm{tr})$ if and only if it is weak operator dense in M, i.e.,*

$$\overline{A}^{\|\cdot\|_2} = L^2(M) \textit{ if and only if } \overline{A}^{\mathrm{w.o}} = M.$$

Proof: As the norm topology is finer than the weak operator topology, we assume A is norm closed, i.e., A is a C^*-algebra. Since A is convex, $\overline{A}^{\mathrm{w.o}} = \overline{A}^{\mathrm{s.o}}$.

Take any self-adjoint $x \in M$ with $\|x\| \leq 1$. There exists a net $\{a_n\}_{n\geq 1}$ in A such that $\|a_n - x\|_2 \to 0$. Also we have $\|a_n^* - x^*\|_2 \to 0$ as $\|a\|_2 = \mathrm{tr}(aa^*) = \mathrm{tr}(a^*a) = \|a^*\|_2$ for all $a \in M$. So $\|\frac{a_n + a_n^*}{2} - x\|_2 \to 0$ and hence we can further assume $\{a_n\}_{n\geq 1}$ are self-adjoint, i.e. $a_n \in A_{\mathrm{s.a}}$.

Consider $f(t) = \frac{2t}{t^2+1}$ which is a bijection on $[-1,1]$. Let $g = f^{-1}$ and $y = g(x)$ then $y \in M_{\mathrm{s.a}}$. By the argument above, there are $\{b_n\}_{n\geq 1}$ in $A_{\mathrm{s.a}}$ such that $\|b_n - y\|_2 \to 0$.

We want to show $\|f(b_n) - x\|_2 \to 0$. Note that $f(b_n) \in A$ and $\|f(b_n)\| \leq 1$, hence by continuous functional calculus, we have

$$f(b_n)-f(y)=\frac{2(b_n(1+y^2)-y(1+b_n^2))}{(1+b_n^2)(1+y^2)}=\frac{2(b_n-y)}{(1+b_n^2)(1+y^2)}+\frac{2b_ny(y-b_n)}{(1+b_n^2)(1+y^2)}.$$

Note that $\|ab\|_2\le\|a\|\|b\|_2$ and $\|ab\|_2\le\|b\|\|a\|_2$. Moreover, $\|(1+b_n^2)^{-1}\|\le 1$ and $\|(1+y^2)^{-1}\|\le 1$. Hence

$$\begin{aligned}\|f(b_n)-f(y)\|_2 &\le 2\|(1+y^2)^{-1}\|\cdot\|b_n-y\|_2+2\|f(b_n)\|\|y(1+y^2)^{-1}\|\cdot\|b_n-y\|_2\\ &\le 4\|b_n-y\|_2\to 0.\end{aligned}$$

Hence $\|f(b_n)-x\|_2=\|f(b_n)-f(y)\|_2\to 0$.

Note $\|f(b_n)\|\le 1$. That is to say the closure of unit ball of $A_{\mathrm{s.a}}$ (inside M) in $\|\cdot\|_2$ is just the unit ball of $M_{\mathrm{s.a}}$. By Lemma 4.10, we obtain $\overline{(A_{\mathrm{s.a}})_1}^{\mathrm{s.o}}=(M_{\mathrm{s.a}})_1$ and hence $\overline{A}^{\mathrm{w.o}}=\overline{A}^{\mathrm{s.o}}=M$.

For the converse, it suffices to prove $\overline{A_1}^{\|\cdot\|_2}=\overline{M_1}^{\|\cdot\|_2}$ or equivalently for any $x\in M_1$, there exists a sequence $\{x_k\}_{k\ge1}$ in A_1 such that $x_k\xrightarrow{\|\cdot\|_2}x$. This is guaranteed by the assumption $\overline{A}^{\mathrm{s.o}}=M$, Lemma 4.10 and also the Kaplansky density thoerem [22]. □

Finally we can determine the von Neumann algebra generated by these T_f's with $f\in L_H^\infty(\mathscr{F},\mathrm{End}(V_\pi))$.

Theorem 4.12 *We have*

$$\overline{\langle T_f|f\in L_H^\infty(\mathscr{F},\mathrm{End}(V_\pi))\rangle}^{s.o.}=A_\pi.$$

Proof: As $L_H^\infty(\mathscr{F},\mathrm{End}(V_\pi))$ is dense in $L_H^2(\mathscr{F},\mathrm{End}(V_\pi))$, by Proposition 4.9, we know $\{T_f|f\in L_H^\infty(\mathscr{F},\mathrm{End}(V_\pi))\}$ is a dense subspace of $L^2(A_\pi,\tau)$. Then, by Proposition 4.11, these T_f's generated A_π in the strong operator (hence also in the weak operator topology).

□

Corollary 4.13 *We have* $\overline{\langle\{T_f|f\in L^\infty(\mathscr{F})\}\otimes\mathrm{End}(V_\pi)\rangle}^{s.o.}=A_\pi$.

Proof: Take $f\in L_H^\infty(\mathscr{F},\mathrm{End}(V_\pi))$ and assume $H_zf(z)H_z^{-1}=g(z)=[g_{i,j}(z)]_{1\le i,j\le n}$ with each $g_{i,j}\in L^\infty(\mathscr{F})$. Then we have

$$T_f = T_{H_z^{-1} g(z) H_z} = T_{\sum_{i,j} H_z^{-1} g_{i,j}(z) e_{i,j} H_z} = \sum_{i,j} T_{g_{i,j}(z) H_z^{-1} e_{i,j} H_z}..$$

Now we define a map between two complex vector spaces:

$$\Phi\colon \{T_f | f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi))\} \to \{T_f | f \in L^\infty(\mathscr{F})\} \otimes \mathrm{Mat}_n(\mathbb{C}),$$

which is given by

$$\Phi\colon T_f = \sum_{i,j} T_{g_{i,j}(z) H_z^{-1} e_{i,j} H_z} \mapsto [T_{g_{i,j}}]_{1 \le i,j \le n}.$$

It is straightforward to check Φ is linear and surjective.

For the injectivity, we suppose there are two $g_{i,j}, g'_{i,j} \in L^\infty(\mathscr{F})$ such that $T_{g_{i,j}(z) H_z^{-1} e_{i,j} H_z} = T_{g'_{i,j}(z) H_z^{-1} e_{i,j} H_z}$ as Toeplitz operators. Then we have $S(T_{g_{i,j}(z) H_z^{-1} e_{i,j} H_z}) = S(T_{g'_{i,j}(z) H_z^{-1} e_{i,j} H_z})$. Hence $B(g_{i,j}(z) H_z^{-1} e_{i,j} H_z) = B(g'_{i,j}(z) H_z^{-1} e_{i,j} H_z)$. Then, by Corollary 4.6, we know $g_{i,j} = g'_{i,j}$ as they are scalar functions and Φ is injective. □

Remark 4.14 *For $f \in L^\infty(\mathscr{F})$, these T_f's will certainly generate a von Neumann subalgebra of A_π. For $n \neq 1$, it is still unknown that how large this subalgebra is. This is related to the Toeplitz C^*-algebras with continuous symbols on the bounded symmetric domains, see [39].*

Indeed $\{p_i = M_{H_z^{-1} e_{i,i} H_z}\}_{1 \le i \le n}$ gives a family of orthogonal projections in $B(L^2(\mathscr{D}, V_\pi))$ satisfying $\sum_{1 \le i \le n} p_i = 1$. One can show

$$\langle M_f | f \in L^\infty_H(\mathscr{D}, \mathrm{End}(V_\pi)) \rangle \cong \langle M_f | f \in L^\infty(\mathscr{D}) \rangle \otimes \mathrm{End}(V_\pi)$$

as an isomorphism of von Neumann algebras acting on $L^2(\mathscr{D}, V_\pi), p_i L^2(\mathscr{D}, V_\pi), V_\pi$ respectively ($e_{i,j} \in \mathrm{End}(V_\pi)$ acts as $M_{H_z^{-1} e_{i,j} H_z}$). Note these p_i's commute with the action of G. We can also consider the Γ-invariant case:

$$\langle M_f | f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi)) \rangle \cong \langle M_f | f \in L^\infty(\mathscr{F}) \rangle \otimes \mathrm{End}(V_\pi).$$

We have $A_\pi = P_\pi \cdot \langle M_f | f \in L^\infty_H(\mathscr{F}, \mathrm{End}(V_\pi)) \rangle \cdot P_\pi$.

Chapter 5

Actions of Cuspidal Automorphic Forms

We construct a family of Γ-invariant bounded linear operators from the cuspidal automorphic forms, with actions between these square-integrable representations. We mainly consider functions that are right-equivariant for the action of the maximal compact subgroup K and left-invariant for the discrete subgroup Γ.

5.1 Automorphic forms on a real reductive group

We review automorphic forms on real reductive Lie groups. A comprehensive treatment of this theory can be found in many resources, for example, see [7, 8, 18].

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$. Let Γ be an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Furthermore, as in the previous sections, we let K be a maximal compact subgroup of $G = G(\mathbb{R})$, $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K respectively and $Z(\mathbb{R})$ be the center of G. Let $U(\mathfrak{g})$ be the universal enveloping algebra and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Let $\mathscr{H} = \mathscr{H}(G(\mathbb{R}), K)$ be the Hecke algebra which is the convolution algebra of all K-finite distributions on G with support in K [27].

Let (π, V) be a representation of K, and we do not assume it is finite dimensional. Let $\widehat{K}$ be the set of equivalence classes of irreducible representations of K and take $\rho \in \widehat{K}$. We denote

$$V(\rho) = \{v \in V \,|\, \mathrm{span}_{k \in K}\, \pi(k)v \cong \rho\}.$$

By a $(\mathfrak{g}, K)$-*module*, we mean a complex vector space V with a representation π of $\mathfrak{g}$ and K which satisfies the following:

(i) The space V is a countable algebraic direct sum $V = \oplus_i V_i$ where each V_i is a finite dimensional K-invariant vector space.

(ii) For $X \in \mathfrak{p}, v \in V$, we have

$$\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v|_{t=0} = \lim_{h\to 0}\frac{\pi(exp(hX))v-v}{h},$$

where the limit exists.

(iii) For $k \in K, X \in \mathfrak{g}$, we have $\pi(k)\pi(X)\pi(k^{-1})v = \pi(\mathrm{Ad}(k)X)v$.

Furthermore, we call it an *admissible* (g,K)*-modules* if $V(\rho)$ is finite dimensional for each $\rho \in \widehat{K}$. Note for a smooth function $\phi\colon G(\mathbb{R}) \to \mathbb{C}$, there is a natural action of K and $\mathfrak{g}$ (hence $U(\mathfrak{g})$) on ϕ. The complex span of ϕ over the action of $\mathfrak{g}$ and K gives us a $(\mathfrak{g},K)$-*module* [41]. Moreover, we say ϕ is *slowly increasing* if there are constants $c, r \in \mathbb{R}_{>0}$ such that

$$|\phi(g)| \leq C \cdot \|g\|^r,$$

where $\|g\| = (\mathrm{tr}(\sigma(g)^*\sigma(g)))^{1/2}$ and σ is a finite dimensional complex representation with finite kernel and closed image. This condition does not depend on the choice of the representation σ [8] but the constant C does.

A smooth complex valued function ϕ on $G(\mathbb{R})$ is an *automorphic form* for (Γ, K) if it satisfies the following conditions:

(i) It is Γ-left invriant: $\phi(\gamma \cdot g) = \phi(g)$, $g \in G(\mathbb{R}), \gamma \in \Gamma$.

(ii) The right translates of ϕ by elements of K span a finite dimensional vector space.

(iii) There is an ideal $I \subset Z(\mathfrak{g})$ of finite codimension such that $x \circ \phi = 0$ for all $x \in I$.

(iv) It is slow increasing.

Note the condition (ii) is equivalent to the existence of an idempotent $\zeta \in \mathscr{H}$ such that the convolution $\phi * \zeta = \phi$. We let $\mathscr{A}(\Gamma, \zeta, I, K)$ be the space of all the automorphic forms of this type.

Assume N is the unipotent radical of any proper maximal parabolic $\mathbb{Q}$-subgroup of $G(\mathbb{Q})$. Given $\phi \in \mathscr{A}(\Gamma,\zeta,I,K)$, we call it a *cuspidal automorphic form*, or simply a *cusp form* if

$$\int_{(\Gamma\cap N(\mathbb{R}))\backslash N(\mathbb{R})} \phi(n\cdot g)dn = 0, \forall g \in G_{\mathbb{R}},$$

where dn is the measure on the quotient space $(\Gamma\cap N(\mathbb{R}))\backslash N(\mathbb{R})$ obtained from the measure dg on G. We let $\mathscr{A}^0(\Gamma,\zeta,I,K)$ be the subspace of cusp forms in $\mathscr{A}(\Gamma,\zeta,I,K)$. By [8, 18], a cusp form is bounded and square-integrable modulo $Z(\mathbb{R})\cdot\Gamma$.

Let $\rho\colon K \to GL(V_\rho)$ be a finite dimensional unitary representation of K and V_ρ is equipped with a Hermitian product $\langle,\rangle_\rho$ and hence a norm $\|\cdot\|_\rho$. We also have the following definition of Harish-Chandra [18, 7] of the *vector-valued automorphic forms* as follows.

Definition 5.1 *A smooth function $F\colon G(\mathbb{R}) \to V_\rho$ is called a V_ρ-valued automorphic form if it satisfies*

(i) $F(\gamma\cdot g) = F(g)$*,* $g \in G(\mathbb{R}), \gamma \in \Gamma$.

(ii) $F(g\cdot k) = \rho(k^{-1})\cdot F(g)$*,* $g \in G(\mathbb{R}), k \in K$.

(iii) There is an ideal $I \subset Z(\mathfrak{g})$ of finite codimension such that $x\circ F = 0$ for all $x \in I$.

(iv) F is slow increasing, i.e., $\|F(g)\|_\rho \le C\cdot\|g\|^n$ for some $C > 0$, for all $g \in G$.

We denote the space of all such V_ρ-value functions by $\mathscr{A}(\Gamma,I,\rho)$ or simply $\mathscr{A}(\Gamma,\rho)$. Furthermore, if we also have $\int_{(\Gamma\cap N(\mathbb{R}))\backslash N(\mathbb{R})} F(n\cdot g)dn = 0$, for all $g \in G_{\mathbb{R}}$, it is called a V_ρ-valued cuspidal form or cusp form. This vector space is denoted by $\mathscr{A}^0(\Gamma,I,\rho)$ or simply $\mathscr{A}^0(\Gamma,\rho)$.

As in the scalar-valued case, a vector-valued cusp form is also bounded (in each coordinate of V_ρ) and square-integrable modulo $Z(\mathbb{R})\cdot\Gamma$.

Now we introduce automorphic forms on the domain $\mathscr{D} = G/K$. Although the definition given by A. Borel [7] involves a general cocycle μ defined on $\Gamma \times \mathscr{D}$, we only focus on the special case of the canonical automorphy factor $J(g,x)\colon G \times \mathscr{D} \to K_{\mathbb{C}}$ (see Section 2.2). We also fix the finite dimensional unitary representation $\rho\colon K \to GL(V_\rho)$ and also denote by ρ its extension to $K_{\mathbb{C}}$.

Definition 5.2 *Let J be the canonical automorphy factor. A vector-valued automorphic form of type ρ is a smooth function $f\colon \mathscr{D} = G/K \to V_\rho$ satisfying*

$$f(\gamma x) = \rho(J(\gamma,x)) \cdot f(x),\ \forall x \in \mathscr{D}, \forall \gamma \in \Gamma.$$

We denote the space of such functions by $\mathscr{A}_{\mathscr{D}}(\Gamma,J)$.

Example 5.3 *Let $G = SL_2(\mathbb{R})$, $K = SO(2)$ and $\Gamma \subset G$ is the modular group. Let $J(g,z) = cz+d$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. We also take the irreducible representation $\rho = \rho_m\colon K \to S^1$ given by $k \mapsto k^m$. Then a function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying*

$$f\left(\tfrac{az+b}{cz+d}\right) = (cz+d)^m f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ gives us a classical modular form for Γ (with some holomorphy conditions, see [37]).

Now we can relate $\mathscr{A}_{\mathscr{D}}(\Gamma,J)$ with the classical automorphic forms $\mathscr{A}(\Gamma,\rho)$ above.

Lemma 5.4 *Given $F \in \mathscr{A}(\Gamma,\rho)$, the map given by $\Phi(F)(\dot{g}) = f(\dot{g}) = \rho(J(g,0))F(g)$ is well-defined and $\Phi(\mathscr{A}(\Gamma,\rho)) \subset \mathscr{A}_{\mathscr{D}}(\Gamma,J)$.*

Proof: Take $g_1, g_2 \in G$ such that $\dot{g_1} = \dot{g_2}$ and we assume $g_1 = g_2 k$ for some $k \in K$. Then we have

$$\begin{aligned}
\rho(J(g_1,0))F(g_1) &= \rho(J(g_2k,0))F(g_2k) = \rho(J(g_2k,0))\rho(k)^{-1}F(g_2) \\
&= \rho(J(g_2,0))\rho(J(k,0))\rho(k)^{-1}F(g_2) \\
&= \rho(J(g_2,0))\rho(k)\rho(k)^{-1}F(g_2) = \rho(J(g_2,0))F(g_2),
\end{aligned}$$

where we apply the cocycle condition of J and the fact $J(k,0) = k$ for $k \in K$. We also have

$$f(\dot{\gamma g}) = \rho(J(\gamma g, 0))F(\gamma g) = \rho(J(\gamma, \dot{g})J(g, 0))F(g) = \rho(J(\gamma, \dot{g}))f(\dot{g}).$$

So $f \in \mathscr{A}_{\mathscr{D}}(\Gamma, J)$. $\square$

For a general cocyle μ, we may not be able to define an inverse map from $\mathscr{A}_{\mathscr{D}}(\Gamma, \mu)$ to $\mathscr{A}(\Gamma, \rho)$ since the cocyle μ on Γ cannot always be extended to G. An example on $\Gamma_0(4)$ with half-integral weight was given by G. Shimura [38]. Recall there is a smooth embedding $i\colon \mathscr{D} = G/K \hookrightarrow NA \subset G$ and we denote this map from $\mathscr{D} \to G$ by $i(z) = g_z$ (see Section 2.2). Please note we have $z = \dot{g_z} = g_z \cdot K$ as a coset in G/K.

Corollary 5.5 *Let $F \in \mathscr{A}(\Gamma, \rho)$ and $f = \Psi(F)$. Then $F(g_z) = \rho(J(g_z, 0)^{-1})f(z)$.*

Proof: It follows from the fact $F(g) = \rho(J(g, 0)^{-1})f(\dot{g})$. $\square$

5.2 The intertwining operators

Take a cusp form $F \in \mathscr{A}^0(\Gamma, \rho)$ and let $f = \Phi(F)$ which is a vector-valued automorphic form on the domain $\mathscr{D}$. We also call a function of such type a *cusp form* and denote them by $\mathscr{A}^0_{\mathscr{D}}(\Gamma, \rho)$.

Lemma 5.6 *Let $\phi \in L^2(\mathscr{D}, V_\pi)$, then the map*

$$M_f\colon f(z) \mapsto M_f(\phi)(z) = f(z) \otimes \phi(z)$$

is a well-defined bounded map with image in $L^2(\mathscr{D}, V_{\rho\otimes\pi})$

Proof: Let us consider the norm $\|M_f(\phi)\|_{L^2(\mathscr{D}, V_{\rho\otimes\pi})}$. We have

$$\begin{aligned}
&\|M_f(\phi)\|^2 = \langle f \otimes \phi, f \otimes \phi \rangle_{L^2} \\
&= \int_{\mathscr{D}} \langle (\rho \otimes \pi)(\kappa(w, w)^{-1}) f(w) \otimes \phi(w), f(w) \otimes \phi(w) \rangle_{V_{\rho\otimes\pi}} d\mu(w) \\
&= \int_{\mathscr{D}} \langle \rho(\kappa(w, w)^{-1}) f(w), f(w) \rangle_{V_\rho} \cdot \langle \pi(\kappa(w, w)^{-1})\phi(w), \phi(w) \rangle_\pi d\mu(w).
\end{aligned}$$

Suppose $w \in \mathscr{D} = G/K$ has a representative $\dot{g}$ with $g \in G$. Note $f(\dot{g}) = \rho(J(g, 0))F(g)$ and $\kappa(\dot{g}, \dot{g}) = J(g, 0)\overline{J(g, 0)^{-1}}$. Following Remark 2.7, one has $\rho(\kappa(\dot{g}, \dot{g})) = \rho(J(g, 0))\rho(J(g, 0))^*$.

(Note that $F(g)$ is independent of the choice of the representative $\dot{g}$ in the coset $w \in \mathscr{D} = G/K$.) We have

$$
\begin{aligned}
&\langle \rho(\kappa(w,w)^{-1}) f(w), f(w) \rangle_{V_\rho} \\
=&\langle (\rho(J(g,0))^{-1})^* \rho(J(g,0)^{-1}) \rho(J(g,0)) F(g), \rho(J(g,0)) F(g) \rangle_{V_\rho} \\
=&\langle F(g), F(g) \rangle_{V_\rho}.
\end{aligned}
$$

Note F is bounded on G since it is a cusp form, i.e., there is a positive constant C_F such that $\langle F(g), F(g) \rangle_{V_\rho} \le C_F$ for all $g \in G$. So we get $\|M_f(\phi)\|^2_{L^2(\mathscr{D}, V_{\rho\otimes\pi})} \le C_F \cdot \|\phi\|^2_{L^2(\mathscr{D}, V_\pi)}$. $\square$

Now we define a Toeplitz-type operator on the holomorphic discrete series. Recall that P_π is the orthogonal projection from $L^2(\mathscr{D}, V_\pi)$ to $H_\pi = L^2_{\text{hol}}(\mathscr{D}, V_\pi)$. Let $T_f \colon H_\pi \to H_{\rho\otimes\pi}$ be the operator defined by

$$T_f(\phi) = P_{\rho\otimes\pi} M_f P_\pi(\phi) = P_{\rho\otimes\pi} M_f(\phi) = P_{\rho\otimes\pi}(f(z) \otimes \phi(z)).$$

where $\phi \in H_\pi$. Moreover, when ρ is the trivial representation of K, f is an essentially bounded function on $\mathscr{D}$, i.e., $f \in L^\infty(\mathscr{D}, d\mu)$. Then M_f is bounded $L^2(\mathscr{D}, V_\pi)$ and the definition above coincides with the classical Toeplitz operator $T_f = P_\pi M_f \in B(H_\pi)$ for any finite dimensional representation π (see Section 3.2).

Remark 5.7 *The tensor product above is pointwise defined. Indeed, the vector-valued function $\rho(J(g,0)^{-1}) f(\dot{g})$ is essentially bounded so that $f(z) \otimes \phi(z)$ is still in the Hilbert space $L^2(\mathscr{D}, V_{\rho\otimes\pi})$.*

In general, the tensor Hilbert space $H_\rho \otimes H_\pi$ is an infinite direct sum of discrete series representations, which is much larger than $H_{\rho\otimes\pi}$. J. Repka gave a clear description of the decomposition of arbitrary tensor products of these holomorphic discrete series representations [31].

Proposition 5.8 *$T_f \in B(H_\pi, H_{\rho\otimes\pi})$ and its adjoint operator $T_f^* \in B(H_{\rho\otimes\pi}, H_\pi)$ is given by*

$$T_f^*(\psi \otimes \eta) = T_{\overline{(f,\psi)}}\eta(w), \textit{for } \psi(z) \in H_\rho, \eta(z) \in H_\pi$$

where $(f,\psi) = \langle \rho(\kappa(w,w)^{-1})f(w), \psi(w)\rangle_{V_\rho}$ *is a function on* $\mathscr{D}$.

Proof: As T_f is a composition of M_f and a projection, the boundedness follows from Lemma 5.6. Note $H_{\rho\otimes\pi}$ can be densely generated by the vector-valued functions of form $\psi(z)\otimes\eta(z)\colon \mathscr{D} \to V_\rho \otimes V_\pi$ with $\psi(z) \in H_\rho$ and $\eta(z) \in H_\pi$. Let $\phi \in H_\pi$. Assume $\dot{g_w} = w$ and consider the inner product $\langle \phi, T_f^*(\psi\otimes\eta)\rangle_{H_\pi} = \langle T_f(\phi), \psi\otimes\eta\rangle_{H_{\rho\otimes\pi}}$. We have

$$\begin{aligned}
&\langle T_f(\phi), \psi\otimes\eta\rangle_{H_{\rho\otimes\pi}} = \langle P_{\rho\otimes\pi}(f\otimes\phi), \psi\otimes\eta\rangle_{L^2} = \langle (f\otimes\phi), \psi\otimes\eta\rangle_{L^2}\\
=&\int_{\mathscr{D}} \langle (\rho\otimes\pi)(\kappa(w,w)^{-1})(f\otimes\phi)(w), (\psi\otimes\eta)(w)\rangle_{V_{\rho\otimes\pi}} d\mu(w)\\
=&\int_{\mathscr{D}} \langle \rho(\kappa(w,w)^{-1})f(w), \psi(w)\rangle_{V_\rho}\cdot\langle \pi(\kappa(w,w)^{-1})\phi(w), \eta(w)\rangle_\pi d\mu(w)\\
=&\int_{\mathscr{D}} \langle \pi(\kappa(w,w)^{-1})\phi(w), \overline{\langle \rho(\kappa(w,w)^{-1})f(w), \psi(w)\rangle_{V_\rho}}\eta(w)\rangle_\pi d\mu(w)\\
=&\langle \phi, M_{\overline{(f,\psi)}}\eta\rangle_{L^2} = \langle \phi, P_\pi M_{\overline{(f,\psi)}}\eta\rangle_{H_\pi} = \langle \phi, T_{\overline{(f,\psi)}}\eta\rangle_{H_\pi}.
\end{aligned}$$

This is to say

$$T_f^*(\psi\otimes\eta)(w) = T_{\overline{(f,\psi)}}\eta(w).$$

□

Note $\overline{(f,\psi)}$ may not be essentially bounded, but $M_{\overline{(f,\psi)}}\eta$ is still in the space $L^2(\mathscr{D}, V_\pi)$ by Cauchy-Schwarz inequality. As F is left-invariant for Γ, a natural question is how T_f intertwines the action of Γ.

Proposition 5.9 *With the assumption above,* T_f *commutes with the action of* Γ*, i.e.,* $T_f L_\pi(\gamma) = L_{\rho\otimes\pi}(\gamma) T_f$*, for any* $\gamma \in \Gamma$.

Proof: Take $\phi, \eta \in H_\pi$, $\psi(z) \in H_\rho$ and $z \in \mathscr{D}$. We have

$$\begin{aligned}
&\langle L_{\rho\otimes\pi}(\gamma)T_f(\phi(z)), (\psi\otimes\eta)(z)\rangle_{H_{\rho\otimes\pi}}\\
=&\langle L_{\rho\otimes\pi}(\gamma)P_{\rho\otimes\pi}M_f(\phi(z)), (\psi\otimes\eta)(z)\rangle_{H_{\rho\otimes\pi}} = \langle P_{\rho\otimes\pi}L_{\rho\otimes\pi}(\gamma)(f\otimes\phi)(z), (\psi\otimes\eta)(z)\rangle_{L^2}\\
=&\langle (L_\rho(\gamma)(f)\otimes L_\pi(\gamma)(\phi))(z), (\psi\otimes\eta)(z)\rangle_{L^2} = \langle T_f L_\pi(\gamma)(\phi)(z), (\psi\otimes\eta)(z)\rangle_{H_{\rho\otimes\pi}}
\end{aligned}$$

where we apply Proposition 2.9 and use the fact $L_\rho(\gamma)f = f$. $\square$

Take another cusp form $H \in \mathscr{A}^0(\Gamma, \rho)$ and let $h(z) = \Phi(H)$.

Corollary 5.10 *The composite operator $T_h^* T_f \in B(H_\pi)$ commutes with the action of Γ, i.e., $T_h^* T_f L_\pi(\gamma) = L_\pi(\gamma) T_h^* T_f$ for $\gamma \in \Gamma$. Moreover, if either one of f or h is holomorphic, we have*

$$T_h^* T_f \phi(z) = T_{(f,h)} \phi(z) = T_{\langle F(g_z), H(g_z)\rangle_{V_\rho}} \phi(z)$$

for any $\phi(z) \in H_\pi$.

Proof: The first claim is straightforward by Proposition 5.8. Assume f is holomorphic and take an arbitrary $\eta \in H_\pi$. Note $f(z) \otimes \phi(z) \in H_{\rho\otimes\phi}$ so $\langle T_h^* T_f \phi, \eta\rangle_{H_\pi} = \langle T_h^* f \otimes \phi, \eta\rangle_{H_\pi}$. Then the first equality follows from Proposition 5.8.

Moreover, by Remark 2.7, we have

$$\rho(\kappa(z,z)) = \rho(J(g_z,0))\rho(J(g_z,0)^*) = \rho(J(g_z,0))\rho(J(g_z,0))^*.$$

Then

$$\begin{aligned}
(f,\psi)(z) &= \langle \rho(\kappa(z,z)^{-1}) f(z), \psi(z)\rangle_{V_\rho} \\
&= \langle \rho(J(g_z,0)^{-1}) f(z), \rho(J(g_z,0)^{-1}) h(z)\rangle_{V_\rho} \\
&= \langle F(g_z), H(g_z)\rangle_{V_\rho},
\end{aligned}$$

which is bounded since F, H are bounded in each coordinate. Hence $\langle F(g_z), H(g_z)\rangle_{V_\rho} \in L^\infty(\mathscr{D})$ that makes the associated Toeplitz operator well-defined. $\square$

Chapter 6

An Example on $SL(2,\mathbb{R})$ and Fuchsian Subgroups

In this section, we consider the Lie group $SL(2,\mathbb{R})$ and apply the previous results to its discrete series representations and its Fuchsian subgroups. We remind the reader that by a Fuchsian subgroup, we mean a Fuchsian group of the first kind, i.e., a discrete subgroup Γ of $SL(2,\mathbb{R})$ (or $PSL(2,\mathbb{R})$) such that $\Gamma\backslash\mathbb{H}^*$ is compact [37]. Let $\{(L_m,H_m)|m\geq 2\}$ be the family of holomorphic discrete series representations of $SL(2,\mathbb{R})$ (see below) and let $A_m = L_m(\Gamma)'\cap B(H_m)$'s be the commutants.

This section is devoted to the proof of the following result:

Theorem 6.1 *Let $\Gamma\subset SL(2,\mathbb{R})$ be a Fuchsian subgroup and T_f be the Toeplitz operator associated with a cusp form f of Γ. Then*

$$\overline{\{span_{f,g}(T_g)^*T_f\}}^{w.o.} = A_m$$

as f,g run through all cusp forms of Γ of same weights.

Note for $G = SL(2,\mathbb{R})$, $K = SO(2)$ is a maximal subgroup. Hence by Theorem 2.6, the symmetric domain $\mathscr{D} = G/K$ is just the open unit disk. For the convenience to discuss automorphic forms, we identify it with the Poincaré upper-half plane

$$\mathbb{H} = \{z = x+iy\in\mathbb{C}|y>0\}.$$

with the invariant measure $d\mu = y^{-2}dxdy$. Moreover, since $K = SO(2)$ is abelian, all its irreducible representation are one-dimensional and can be characterized as (π_m,V_m) such that $\pi_m(g) = g^m\in S^1$ for an integer m.

Note $K_{\mathbb{C}} = \mathbb{C}^\times$ is the complexified group of $K = SO(2)$. By definition, the canonical automorphy factor $J\colon SL(2,\mathbb{R})\times\mathbb{H}\to\mathbb{C}^\times$ is given by

$$J(g,z) = cz+d \text{ for } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R}),\ z \in \mathbb{H}.$$

Following Section 2.2, we can describe the holomorphic discrete series representations of $SL(2,\mathbb{R})$. One can show that $\pi_m(\kappa(z,z)^{-1}) = y^m$ where $z = x+iy$. So we move the term y^m to the measure and denote $y^m d\mu = y^{m-2}dxdy$ by $d\mu_m$.

Let $L^2(\mathbb{H},\mu_m)$ be the square-integrable functions on $\mathbb{H}$ with respect to the measure $\mu_m = y^{m-2}dxdy$ (and μ_0 is just μ). Let H_m be the subspace of all holomorphic functions in $L^2(\mathbb{H},\mu_m)$, i.e.,

$$H_m = L^2_{\text{hol}}(\mathbb{H},\mu_m).$$

As in Section 2.2, for a given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \in SL_2(\mathbb{R})$ and $f \in H_m$, the action on H_m is given by

$$(L_m(g)f)(z) = f(g^{-1}z)(cz+d)^{-m}$$

where $g^{-1}z = \frac{az+b}{cz+d}$.

Proposition 6.2 ([16]) *For any integer $m \geq 2$, (L_m,H_m) is an irreducible unitary representation of $SL(2,\mathbb{R})$. Moreover, it is square-integrable with formal dimensions $d_m = \frac{m-1}{4\pi}$.*

To have a positive formal dimension, we will focus on the case $m \geq 2$ from now on.

6.1 Berezin transform and the trace

Let (L_m,H_m) be the holomorphic discrete series of $SL(2,\mathbb{R})$ associated with the one dimensional representation (π_m,V_m) of $K = SO(2)$ (and also of $K_{\mathbb{C}} = \mathbb{C}^{\times}$). Note all the matrix-valued ($\text{End}(V_\pi)$-valued) functions defined in Section 3.1 reduce to the scalar-valued functions since $V_m = \mathbb{C}$. In this section, we use the following simplified notation for Berezin symbols:

(i) $K(z,w) = E_z^* E_w$ and $K(z,z) = E_z^* E_z$.

(ii) $\widehat{A}(z) = S(A)(z)$ for $A \in B(H_m)$

(iii) $\widehat{A}(z,w) = R(A)(z,w)$ for $A \in B(H_m)$ (defined in the proof of Proposition 4.9).

Let $v_1 \in V_m$ be a unit vector, one can further show $\widehat{A}(z,w) = \frac{\langle AE_w(v_1), E_z(v_1)\rangle_{H_m}}{\langle E_w(v_1), E_z(v_1)\rangle_{H_m}}$.

Corollary 6.3 *Given $A, B \in \mathbb{B}(H_m)$, then*

(i) $\widehat{A}(z,w)$ is sesqui-holomorphic (i.e., holomorphic in z and anti-holomorphic in w),

(ii) the map $A \mapsto \widehat{A}(z,w)$ or $\widehat{A}(z)$ is one-to-one,

(iii) $\sup_{z\in\mathbb{H}} |\widehat{(A)}(z)| \leq ||A||$,

(iv) $\widehat{A^*}(z,w) = \overline{\widehat{A}(w,z)}$,

(v) $\widehat{AB}(z,w) = \int_{\mathbb{H}} \frac{K(z,\eta)K(\eta,w)}{K(z,w)} \widehat{A}(z,\eta)\widehat{B}(\eta,w)d\mu_m(\eta)$.

Proof: It follows from Proposition 3.4 and Proposition 3.5. □

Now we define the commutant by

$$A_m = L_m(\Gamma)' \cap B(H_\pi) = \{A \in B(H_m) | AL_m(\gamma) = L_m(\gamma)A, \ \forall \gamma \in \Gamma\},$$

which will be shown to be a tracial von Neumann algebra. It is a II_1 factor if $L_m(\Gamma)''$ is a II_1 factor and the coupling constant of $\dim_{L_m(\Gamma)''} H_m$ is finite, which holds when Γ is an ICC lattice.

Corollary 6.4 *If $A, B \in \mathbb{B}(H_m)$ and $g \in SL_2(\mathbb{R})$, the Berezin transform of $L_m(g)^{-1}AL_m(g)$ is $\widehat{A}(gz, gw)$. $A \in \mathscr{A}_m$ if and only if $\widehat{A}$ is Γ-invariant.*

Proof: It follows from Proposition 3.5 (iv). □

Corollary 6.5 *For $m \geq 2$, the following linear functional defines a faithful normal tracial state on A_m: Let $A \in A_m$, then*

$$\tau(A) = \frac{1}{\mu(\mathscr{F})} \int_{\mathscr{F}} \widehat{A}(z) d\mu(z).$$

τ is the unique tracial state if A_m is a type II_1 factor.

Proof: It follows from Proposition 3.8. $\square$

In the special case $\Gamma = SL(2,\mathbb{Z})$, most of the results above are known, see [34, 30].

Remark 6.6 *Let $Z(\Gamma)$ be the center of Γ. The representation (L_m, H_m) is indeed a projective unitary representation of Γ/Z, which may also give a factor. For example, in the case $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm I\}$, as $H^2(PSL(2,\mathbb{Z}), S^1) = 0$, the representation is ordinary and gives the factor $L_m(PSL(2,\mathbb{Z}))''$ since each Fuchsian subgroup of $PSL(2,\mathbb{Z})$ is ICC [1].*

One may also consider the lattices in $SL(2,\mathbb{R})$. Indeed, they are all essentially ICC: there are only finitely many conjugacy classes that are finite. Following [30], in this case the A_m's are also factors of type II_1.

6.2 The action of cusp forms

Let Γ be a Fuchsian subgroup of the first kind. Recall that a cusp form of weight p of Γ is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying

(i) $f(z) = (cz+d)^{-p} f(\frac{az+b}{cz+d})$, $z \in \mathbb{H}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

(ii) f vanishes at each cusp of Γ.

By definition, cusps are some elements in $\mathbb{R} \cup \{\infty\}$. One can refer [28, 37] for the precise description. Let $S_p(\Gamma)$ be the vector space generated by all cusp forms of weight p of Γ, which is finite dimensional. It is well-known that for any $f \in S_k(\Gamma)$, there is a constant $B_f \geq 0$ such that $|f(x+iy)| \leq B_f \cdot y^{-p/2}$ [28].

Recall that $\mathscr{A}^0(\Gamma, \pi_p)$ is the space of cusp forms defined on the real Lie group $G = SL(2,\mathbb{R})$ (see Section 4.1).

Lemma 6.7 *Each $f \in S_p(\Gamma)$ is the image of some $F \in \mathscr{A}^0(\Gamma, \pi_p)$ under the map $\Phi(F)(\dot{g}) = \pi_p(J(g,0))F(g)$.*

Proof: Following [10], it is straightforward to check $\pi_p(J(g,0))^{-1} f(\dot{g})$ is a well-defined function on $SL(2,\mathbb{R})$ which is also a cusp form. $\square$

Indeed, let Δ be the Casimir element of $\mathfrak{gl}_2^{\mathbb{C}}$ and Z be the 2-by-2 identity matrix in $\mathfrak{gl}_2$, there is an isomorphism

$$S_p(\Gamma) \to \mathscr{A}^0(\Gamma, \langle \Delta - \frac{p^2-1}{4}, Z\rangle, \pi_p)$$
$$f \mapsto \phi_f(g) = J(g,i)^{-p} f(gi)$$

which also illustrates the correspondence between two types of cusp forms. We refer to [10] for more details about the relation between the automorphic forms on $SL(2,\mathbb{R})$ and the classical automorphic forms defined on the upper-half plane $\mathbb{H}$.

As representations of $SO(2)$, we have

$$V_p \otimes V_m \cong V_{p+m}.$$

Hence the Toeplitz-type operator $P_{\pi_p\otimes\pi_m}(f\otimes\cdot)$ defined in Section 5.2 reduced to the classical Toeplitz operator $P_{\pi_p\otimes\pi_m}M_f$, or simply $P_{m+p}M_f$. Now, given an arbitrary $f \in S_p(\Gamma)$ and any $m \geq 2$, let $T_f = P_{m+p}M_f P_m \in B(H_m, H_{m+p})$ be the Toeplitz operator associated with f. The following two results are the special cases of the ones in Section 5.3. For the reader's convenience, we also give separate proofs that emphasize more scalar-valued cusp forms instead of the vector-valued ones.

Proposition 6.8 *The Toeplitz operator T_f satisfies the following conditions.*

(i) $T_f \in B(H_m, H_{m+p})$,

(ii) T_f *intertwines the action of* Γ, *i.e*

$$T_f\pi_m(g) = \pi_{m+p}(g)T_f,\ \forall g \in \Gamma.$$

(iii) $(T_g)^* = P_m M_{\overline{g}\cdot y^p} P_{m+p} \in B(H_{m+p}, H_m)$, *which also intertwines the action of* Γ.

Proof: (i) Let $\phi \in H_m$ and $\psi \in H_{m+p}$. Then

$$\begin{aligned}\|T_f\phi\|^2_{m+p} &= \int_{\mathbb{H}} |f(z)\phi(z)|^2 y^{m+p-2} dxdy \\ &\leq \int_{\mathbb{H}} B^2 |\phi(z)|^2 y^{m-2} dxdy,\end{aligned}$$

where we used $|f(z)| \leq By^{-p/2}$. So $T_f \in B(H_m, H_{m+p})$.

(ii) Take $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \Gamma$. We have

$$\begin{aligned} L_{m+p}(g)T_f\phi(z) &= f\left(\frac{az+b}{cz+d}\right)\phi\left(\frac{az+b}{cz+d}\right)(cz+d)^{-(m+p)} \\ &= f(z)\phi\left(\frac{az+b}{cz+d}\right)(cz+d)^{-m} \\ &= T_f L_m(g)\phi(z). \end{aligned}$$

(iii) Consider the inner product on H_{m+p}. We have

$$\begin{aligned} \langle T_g^m\phi, \psi\rangle_{m+p} &= \int_{\mathbb{H}} g(z)\phi(z)\overline{\psi(z)}y^{m+p-2}dxdy \\ &= \int_{\mathbb{H}} \phi(z)(\overline{P_m\overline{g}y^p\psi(z)})y^m dxdy, \end{aligned}$$

which is $\langle \phi, (T_g)^*\psi\rangle_m$. Hence $T_g{}^* = P_m M_{\overline{g}\cdot y^p} P_{m+p}$. □

Corollary 6.9 *Given $f, g \in S_p(\Gamma)$, we have*

$$(T_g)^*T_f = P_m M_{f\overline{g}y^p} P_m = T_{f\overline{g}y^p} \in A_m,$$

where $A_m = L_m(\Gamma)'$ is the factor.

Proof: Recall $(f(z), g(z)) = \langle \kappa(z,z)^{-p} f(z), g(z)\rangle_{V_m} = f(z)\overline{g(z)}y^p$. Note both f, g are holomorphic, then it follows from Corollary 5.10. □

6.3 Two existence results

This part is devoted to the proof of Theorem 6.1. Before this, we need two theorems for the existence of some meromorphic functions and holomorphic functions on a compact Riemann surface. We refer to [14] for the theory of Riemann surfaces.

Let $\Gamma \subset SL(2,\mathbb{R})$ be an arbitrary Fuchsian group of the first kind and P_Γ be the set of all cusps of Γ. Let $\mathbb{H}^* = \mathbb{H} \cup P_\Gamma$. Denote $\mathscr{F} = \Gamma\backslash\mathbb{H}$ and $\mathscr{F}^* = \Gamma\backslash\mathbb{H}^*$ be the quotient spaces by the action of Γ. It is well-known that $\mathscr{F}^*$ is a compact Hausdorff space and also a compact

Riemann surface [28]. We denote a Riemann surface by $\mathscr{M}$ and the field of meromorphic functions on $\mathscr{M}$ by $A(\mathscr{M})$.

Theorem 6.10 *If $\mathscr{M}$ is a compact Riemann surface and $P_1,\ldots,P_n \in \mathscr{M}$ are distinct points and $z_1,\ldots,z_n \in \mathbb{C}$, there exists $\phi \in A(\mathscr{M})$ such that $\phi(P_i) = z_i$ for all $1 \le i \le n$.*

Proof: Take integers i, j such that $1 \le i \ne j \le n$. Let us consider the divisor $D = kP_i - P_j$ where $k = k_{i,j} \in \mathbb{Z}$. Apply the Riemann-Roch Theorem for the divisor D, we get

$$l(D) = \deg(D) - g + 1 + l(\mathrm{div}(\omega) - D) = m - g + l(\mathrm{div}(\omega) - D),$$

where $l(D) = \dim_{\mathbb{C}} L(D)$ with $L(D) = \{f \in A(\mathscr{M}) | f = 0 \text{ or } \mathrm{div}(f) + D \ge 0\}$ and $\mathrm{div}(\omega)$ is a canonical divisor. Take k sufficiently large, there would be a desired $k_{i,j}$ such that $\deg(\mathrm{div}(\omega) - kP_i + P_i) < 0$ and hence $l(\mathrm{div}(\omega) - kP_i + P_i) = 0$. Then, as $l(kP_i - P_j) > l((k-1)P_i - P_j)$, there must be some $\psi_{i,j} \in L(kP_i - P_j) - L((k-1)P_i - P_j)$. So we get a meromorphic function $\psi_{i,j}$ with

$$v_{P_i}(\psi_{i,j}) = -k_{i,j} < 0 \text{ and } v_{P_j}(\psi_{i,j}) \ge 1.$$

Let $\phi_{i,j} = \frac{\psi_{i,j}}{\psi_{i,j}+1}$ then $\phi_{i,j}(P_i) = 1$ and $\phi_{i,j}(P_j) = 0$. Now we define $\phi_i = \prod_{1 \le j \le n, j \ne i} \phi_{i,j}$ which satisfies

$$\phi_i(P_i) = 1, \phi_i(P_j) = 0 \text{ for } j \ne i.$$

Then the function $\phi = \sum_{1 \le i \le n} c_i \phi_i$ is the one we want.

□

We further assume $\mathscr{F}^*$ to be the compact Riemann surface given by $\Gamma \backslash \mathbb{H}^*$.

Theorem 6.11 *Let $P_1,\ldots,P_n \in \mathscr{F}^*$ be distinct points, then there exists a holomorphic automorphic form f such that $f(P_i) \ne 0$ for all $1 \le i \le n$. Moreover, if $\{P_i\}_{1 \le i \le n}$ are not cusps, we can further require f to be a cusp form.*

Proof: We first focus on a single point P_1. Suppose all holomorphic automorphic forms have P_1 as zeros. We take a holomorphic f_1 such that $v_{P_1}(f_1) = m \geq 1$ is minimal.

Consider the divisor kP. By the Riemann-Roch Theorem, we have

$$l(kP) = \deg(kP) - g + 1 + l(\mathrm{div}(\omega) - kP) = k - g + 1 + l(\mathrm{div}(\omega) - kP).$$

For sufficiently large k, we have $\deg(\mathrm{div}(\omega) - kP) < 0$ and $l(kP) = k - g + 1$. Then there exists $\phi_1 \in A(\mathscr{F}^*)$ with a single pole of order k. Then $g_1 = f_1^k \phi_1^m$ is holomorphic and does not vanish at P_1.

Now suppose we have such holomorphic automorphic forms $\{g_i\}_{1\leq i\leq n}$ such that $g_i(P_i) \neq 0$ and the weight of g_i is k_i for $1 \leq i \leq n$. Now let N be a common multiple of all these k_i's. Then a linear combinations $f = \sum_{1\leq i\leq n} \lambda_i g_i^{N/k_i}$ (with some suitable λ_i's) will give us a desired automorphic form of weight N.

If $\{P_i\}_{1\leq i\leq n}$ are not cusps, we can further assume from the beginning that all g_i's are cusp forms. Then we get a desired f in the same way. □

6.4 II_1 factors from cusp forms on $SL(2,\mathbb{R})$

Now we let $A_k(\Gamma)$ be the space of automorphic forms of weight k so $S_k(\Gamma)$ is a subspace of $A_k(\Gamma)$ spanned by the cusp forms. We know the Petersson inner-product on $S_k(\Gamma)$ is given by

$$\langle f, g\rangle = \tfrac{1}{\mu(\mathscr{F})} \int_{\mathscr{F}} f(z)\overline{g(z)} y^k d\mu(z),$$

which is Hermitian. We denote the term in the integral by $(f,g)_k = f(z)\overline{g(z)}y^k$ with a emphasis on the weight k.

Now let

$$\mathscr{F}_1 = \mathscr{F}^*/(\Gamma\backslash P_\Gamma) = \Gamma\backslash(\mathbb{H}^*/P_\Gamma) = \mathscr{F} \cup \{\mathrm{pt}\}.$$

by identifying all cusps in the fundamental domain with one point. Hence $\mathscr{F}_1$ is compact.

Proposition 6.12 *$\{(f,g)_k | k \in \mathbb{N}, f,g \in S_k(\Gamma)\}$ are well-defined functions on $\mathscr{F}_1$ and separate points of $\mathscr{F}_1$.*

Proof: As all $(f,g)_k$'s vanish on all cusps, it is well-defined on the quotient space of $\mathscr{F}^*$ by identifying all cusps.

Now we take a pair of distinct points $P, Q \in \mathscr{F}_1$. By Theorem 6.10, there is a meromorphic function $\phi \in A_0(\Gamma)$ such that $\phi(P), \phi(Q)$ are distinct.

Case 1: $P, Q \in \mathscr{F}$.

We take $f(z), g(z) \in S_k(\Gamma)$ with all $f(P), f(Q), g(P), g(Q)$ are nonzero. The existence follows from Theorem 6.11.

For sufficiently large k, we may assume the multiplication by f eliminates all the poles of ϕ. So we can further assume $f(z)$ satisfies $f(z)\phi(z) \in S_k(\Gamma)$. We assume $(f,g)_k$ cannot separate P, Q, i.e.,

$$(f,g)_k(P) = f(P)\overline{g(P)}y_P^k = f(Q)\overline{g(Q)}y_Q^k = (f\phi, g)_k(Q).$$

Then we have

$$(f\phi, g)_k(P) = f(P)\phi(P)\overline{g(P)}y_P^k \neq f(Q)\phi(Q)\overline{g(Q)}y_Q^k = (f\phi, g)_k(Q).$$

Case 2: $P \in \mathscr{F}$, $Q = \{\text{pt}\}$.

As Q stands for cusps, it suffices to show some $(f,g)_k(P) \neq 0$. But this follows from Theorem 6.11. $\square$

As $\mathscr{F}_1$ is compact, we apply Stone-Weierstrass to get the following corollary.

Corollary 6.13 *The functions of the form $(f,g)_k$ generate the function space of continuous functions on $\mathscr{F}_1$ that vanish at the the point pt, or equivalently, $(f,g)_k$'s generate the space of continuous functions on $\mathscr{F}$ that vanish on all cusps, i.e.,*

$$\overline{\{(f,g)_k | k \in \mathbb{N}, f,g \in S_k(\Gamma)\}}^{||\cdot||_\infty} = \{\psi \in C(\mathscr{F}^*) \mid \psi|_{\text{cusps}} = 0\}.$$

As there are only finitely many cusps in $\mathscr{F}$ and $\mu(\mathscr{F}) < \infty$, we obtain:

Proposition 6.14 *For $m \geq 2$, we have*

$$\overline{\{\mathrm{span}_{f,g}(T_g)^* T_f\}}^{\text{w.o.}} = A_m,$$

where f, g run through all cusp forms of same weights of Γ.

Proof: By Corollary 6.13, we know the these $(f,g)_k$'s generate the space of continuous functions on $\mathscr{F}^*$ that vanish at cusps. Hence the restriction of $(f,g)_k$'s on $\mathscr{F}$ also span a dense subspace of $L^2(\mathscr{F})$.

Note that $T((f,g)_k) = T(f\overline{g}y^k) = (T_g)^* T_f$. Then, by Theorem 4.9, we know these $(T_g)^* T_f$'s give a $*$-closed subalgebra which is dense in $L^2(A_m)$. Then the claim follows from Proposition 4.11 or Proposition 4.12. $\square$

Chapter 7

The von Neumann Algebras from Cusp Forms

Let $H_\pi = L^2_{\text{holo}}(\mathscr{D}, V_\pi)$ be the holomorphic discrete series representation of G associated with an irreducible representation (π, V_π) of its maximal compact subgroup K. Given a cusp form $f : \mathscr{D} \to V_\rho$ for Γ, we denote by T_f the Toeplitz-type operator associated with f as in Section 5.

This section will be largely devoted to proving the following result.

Theorem 7.1 *Assume* $\dim_{\mathbb{C}} V_\pi = 1$*, the von Neumann algebra* $L_\pi(\Gamma)'$ *of* Γ*-intertwining operators on the holomorphic discrete series* H_π *can be generated as:*

$$L_\pi(\Gamma)' = \overline{\{span_{f,g} T_g^* T_f\}}^{w.o.},$$

where f, g *run through the cusp forms for* Γ *of same types.*

The proof is based on the Baily-Borel compactification and the cusp forms given by Poincaré series.

7.1 Baily-Borel compactification

We review some basic facts of the Baily-Borel compactification of the quotient space $\Gamma\backslash\mathscr{D}$. It is a generalization of the compactification of the fundamental domain $SL(2,\mathbb{Z})\backslash\mathbb{H}$, or equivalently,

$$SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2) \cong \{z \in \mathbb{H} |\ |z| > 1, |\mathrm{Re}(z)| < \tfrac{1}{2}\}$$

to a general Lie group G and an arithmetic subgroup Γ. From now on, we assume G has no normal $\mathbb{Q}$-subgroup of dimension 3. More details can be found in [2, 3, 35].

Let $\mathbf{G}$ be a semi-simple linear algebraic group defined over $\mathbb{Q}$ such that $G = G_{\mathbb{R}}$ is the real Lie group we discussed in the previous sections. Let $\mathscr{D} = G/K$ as before.

By the Harish-Chandra realization, we regard $\mathscr{D} \subset \mathfrak{p}^+ = \mathbb{C}^N$ as a smooth manifold with the natural action of G. The action also extends to the closure $\overline{\mathscr{D}} \subset \mathfrak{p}^+$. We say a real affine hyperplane $H \subset \mathbb{C}^N$ is a *supporting hyperplane* if $H \cap \overline{\mathscr{D}} \neq \emptyset$ and $H \cap \mathscr{D} = \emptyset$. Let H be such a supporting plane and denote its intersection with the closure $\overline{\mathscr{D}}$ by $\overline{F}$, i.e., $\overline{F} = H \cap \overline{\mathscr{D}}$. Furthermore, there is a minimal complex affine subspace $L \subset \mathbb{C}^m$ contains $\overline{F}$. The *boundary component* F is a nonempty open subset of L whose closure is $\overline{F}$, which is also a bounded symmetric domain in L. The detailed construction of boundary components is based on the $\mathbb{R}$-roots and $\mathbb{Q}$-roots of $\mathbf{G}$ [2, 3].

Meanwhile, there is a maximal set $(\alpha_1, \ldots, \alpha_t) \subset \Delta_n^+$ such that they are strongly orthogonal, i.e., neither $\alpha_i + \alpha_j$ nor $\alpha_i - \alpha_j$ is a root. The *standard boundary components* F_b are indexed by $1 \leq b \leq t$, which can generate all the boundary components by the transfroms of G (see [3] 1.5). For such a boundary component F_b, there is also an unbounded realization $S_b \subset \mathfrak{p}^+$ and also a cocyle $J_b(g,x)\colon G \times S_b \to \mathbb{C}$ which is defined as the determinant of the Jacobian of $g \in G$ at $x \in S_b$.

One should keep in mind that $\mathscr{D}$ itself is an improper rational boundary component. For each rational boundary component F, there is a canonical projection $\sigma_F\colon \mathscr{D} \to F$.

For a boundary component F, its normalizer is defined by

$$N_G(F) = \{g \in G | g \cdot F \subset F\},$$

where the action is induced from that on $\overline{\mathscr{D}}$. It is well-known $N_G(F)$ is a parabolic subgroup of G. We call F a *rational boundary component* if $N_G(F)$ is defined over $\mathbb{Q}$ as a subgroup of the linear algebraic group (see $\mathbf{G}$ [3].3 and [2].III). There are countably many rational boundary components.

Now we let $\mathscr{D}^*$ be the union of $\mathscr{D}$ and its rational boundary components, equipped with the *Satake topology*, which is the unique topology with some properties related to the arithmetic group Γ (see [2] III. or [3] Theorem 4.9). Let $\mathscr{F} = \Gamma \backslash \mathscr{D}$ be the fundamental domain.

Definition 7.2 *The Baily-Borel compactification $\mathscr{F}^*$ of $\mathscr{F}$ is defined to be the quotient*

$$\mathscr{F}^* = \Gamma \backslash \mathscr{D}^*,$$

equipped with the quotient topology.

Baily and Borel proved the following result.

Theorem 7.3 ([2] III.6) *The Baily-Borel compactification $\mathscr{F}^*$ is a compact Hausdorff space containing $\mathscr{F}$ as an open dense subset. $\mathscr{F}^*$ is a finite union of subspaces of the form*

$$\Gamma_F \backslash F,$$

where F is a rational boundary component and $\Gamma_F = \Gamma \cap N_G(F)$.

Moreover, the closure of $\Gamma_F \backslash F$ is the union of $\Gamma_F \backslash F$ and subspaces $\Gamma_{F'} \backslash F'$ of strictly smaller dimension.

They actually proved $\mathscr{F}^*$ is isomorphic to a projective subvariety of some complex projective space $\mathbf{P}^N_{\mathbb{C}}$. From now on, we will denote the compactafication as

$$\mathscr{F}^* = V_0 \cup V_1 \cup \cdots \cup V_t,$$

where $V_0 = \mathscr{F}$ and $V_i = \Gamma_{F_i} \backslash F_i$ for some rational boundary component F_i with $0 \leq i \leq t$. It can also be proved that $\dim(\mathscr{F}^* - \mathscr{F}) \leq \dim(\mathscr{F}) - 2$ if G has no normal $\mathbb{Q}$-subgroup of dimension 3.

7.2 Poincaré-Eisenstein series

The Poincaré series were first constructed for automorphic forms on a Lie group by Poincaré. Intuitively, for $SL(2,\mathbb{R})$, Poincaré series apply group averages of an infinite sum, which is a natural way to construct functions invariant under the automorphy action of the modular group. We have the following result proved by Harish-Chandra (see [3] Theorem 5.4) and R. Godement [15]. Note we focus on the functions with the left action of Γ and usually the right action of K instead of the ones with actions on different sides in these references.

Theorem 7.4 (Harish-Chandra, Godement) *Let Γ be a discrete subgroup of G and V be a finite dimensional complex vector space. Let $\tilde{f}\colon G \to V$ be a function in $L^1(G) \otimes V$, which is $Z(\mathfrak{g})$-finite and is K-finite on the left. Then the series*

$$P_{\tilde{f}}(g) = \sum_{\gamma\in\Gamma} \tilde{f}(\gamma \cdot g),\ P_{\|\tilde{f}\|}(g) = \sum_{\gamma\in\Gamma} \|\tilde{f}(\gamma \cdot g)\|,$$

are absolutely and uniformly convergent on compact subsets and are bounded on G.

Furthermore, if $\tilde{f}$ is of finite type on the right instead, $P_{\tilde{f}}$ is absolutely and uniformly convergent on compact sets, but not necessarily bounded.

The series $P_{\tilde{f}}$ are called a *Poincaré series on the group G* [15]. We are more interested in the following Poincaré series defined on the bounded symmetric domain $\mathscr{D}$. For the canonical autormorphy factor $J\colon G \times \mathscr{D} \in K_{\mathbb{C}}$ and a representation (ρ, V_ρ) of K, we associate a function $f\colon \mathscr{D} \to V_\rho$ to $\tilde{f}\colon G \to V\rho$ by

$$f(\dot{g}) = \rho(J(g,0))\tilde{f}(g).$$

They satisfies the property that f is holomorphic on $\mathscr{D}$ if and only if $Y \circ \tilde{f} = 0$ for all $Y \in \mathfrak{p}^-$.

Now we consider the action of G on $G/K = \mathscr{D} \subset \mathfrak{p}^+$. For any $z \in \mathscr{D}$, we let $J_{\mathscr{D}}(g,z) \in \mathbb{C}$ be the determinant of the Jacobian of $z \mapsto g(z)$.

Lemma 7.5 ([35] II.5.3.) *The Jacobian of $z \mapsto g(z)$ is the adjoint representation* ad *of the canonical automorphy factor J, i.e.,*

$$\mathrm{Jac}(z \mapsto g(z)) = \mathrm{ad}_{\mathfrak{p}^+}(J(g,z)).$$

where $\mathrm{ad}_{\mathfrak{p}^+}$ is the restriction of ad *on $\mathfrak{p}^+$.*

Hence we have $J_{\mathscr{D}}(g,z) = \det(\mathrm{ad}_{\mathfrak{p}^+}(J(g,z)))$.

Theorem 7.6 ([7, 12]) *Let f be a polynomial function on $\mathscr{D}$, $m \geq 4$ be an integer. Then the series*

$$P_{m,f}(z) = \sum_{\gamma\in\Gamma} J_{\mathscr{D}}(\gamma, z)^m f(\gamma \cdot z)$$

converges absolutely and uniformly on compact sets. It defines a holomorphic automorphic form of weight m, i.e.,

$$P_{m,f}(z) = J_{\mathscr{D}}(\gamma,z)^m P_{m,f}(\gamma z).$$

The function $g \mapsto f(g\cdot 0)J_{\mathscr{D}}(g,0)^{-m}$ *is bounded on* G*.*

The functions $P_{m,f}$'s above are called *Poincaré series on the domain* $\mathscr{D}$ [3]. Indeed, these series are sufficient to separate the points.

Proposition 7.7 *Let* $z_1,\dots,z_N$ *be* Γ*-inequivalent points in* $\mathscr{D}$*. Take a set of points* $c_1,\dots,c_N \in \mathbb{C}$*. For* m *sufficiently large, we can find a polynomial* f *such that* $P_{f,m}(z_i) = c_i$ *for all* $1 \le i \le N$*.*

Proof: Let us consider the linear map given by

$$E_{\vec{z}}\colon f \mapsto (P_{m,f}(z_1),\dots,P_{m,f}(z_N)).$$

Its image is a linear subspace of $\mathbb{C}^N$. We will show this map is surjective for sufficiently large m.

Take some $0 < u < 1$. By [3] Lemma 5.8, we know the function $g \mapsto |J_{\mathscr{D}}(g,0)|^a$ is in $L^1(G)$ if $a \ge 2$. Then, by Theorem 7.4, we conclude $\sum_{\gamma\in\Gamma}|J_{\mathscr{D}}(\gamma,z)|^2$ converges absolutely and uniformly on compact sets. Hence the set $\Gamma_{u,i} = \{\gamma\in\Gamma \,|\, |J_{\mathscr{D}}(\gamma,z_i)| > u\}$ is a finite. Now let f be a polynomial on $\mathfrak{p}^+$ such that

$$f(z_i) = c_i \text{ and } f(\gamma z_i) = 0 \text{ for } \gamma\in\Gamma_{u,i},\ 1 \le i \le N.$$

Note $J_{\mathscr{D}}(e,z_i)^l f(z_i) = f(z_i) = c_i$, we have $|P_{m,f}(z_i) - c_i| \le \sum_{\gamma\notin\Gamma_{u,i}} |u^l f(\gamma z_i)|$ which converges to 0 as $l \to \infty$. So for any $\varepsilon > 0$, there exists an integer $m(\varepsilon)$ such that $\|E(f) - \vec{c}\|_{\mathbb{C}^N} < \varepsilon$ if $m > m(\varepsilon)$. Now let $\vec{c}$ runs through the standard basis $\{e_1,\dots,e_N\}$of $\mathbb{C}^N$ and let ε be small enough, the argument above implies the map E contains a basis for m sufficiently large. Hence E is surjective. □

Now we introduce the Poincaré-Eisenstein series. We first introduce some notations.

(i) P : a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$,

(ii) $U : R_u(P)$, the unipotent radical of P,

(iii) Γ_∞ : a subgroup of finite index of $\Gamma \cap P$,

(iv) B : a normal connected $\mathbb{Q}$-subgroup of P which contains the split radical of P,

(v) $\Gamma_0 = \Gamma_\infty \cap B$.

Let $\tilde{f}$ be a function on G. Following [3], The *Poincaré-Eisenstein series* are defined to be:

$$E_{\tilde{f}}(g) = \sum_{\gamma \in \Gamma/\Gamma_0} \tilde{f}(\gamma \cdot g),\ g \in G.$$

When $\tilde{f}$ satisfies certain conditions (see [3] Theorem 6.4), we can show $E_{\tilde{f}}$ converges absolutely and uniformly on compacts sets.

Let us fix $F = F_b$, a standard rational boundary component. We further assume

- $P = N_G(F)$,
- B : a normal connected $\mathbb{Q}$-subgroup containing the split radical of P such that $B^0_{\mathbb{R}} \subset Z(F)$ and $Z(F)/B^0_{\mathbb{R}}$ is compact,
- $\Gamma_\infty = P \cap \Gamma$,
- $\Gamma_0 = B^0_{\mathbb{R}} \cap \Gamma$.

Let ϕ be a polynomial on F in the coordinates of the canonical bounded realization of F and let l be a positive integer. Consider the series

$$E(z) = E_{\phi,l,\Gamma}(z) = \sum_{\gamma \in \Gamma/\Gamma_0} \phi(\sigma_b(\gamma \cdot z)) \cdot J_F(\gamma, z)^l,$$

which are called *Poincaré-Eisenstein series adapted to F* in [3]. We also consider the transform of E by g:

$$(E \circ g)(z) = J_F(g^{-1}, z)^l E(g^{-1} \cdot z).$$

For the convergence of these series, Baily and Borel also gave the following result.

Theorem 7.8 ([3] Section 7.2) *There exists a positive integer l_0 such that if l is a positive multiple of l_0, then the series $E_{\phi,l,\Gamma} \circ g$ converges absolutely and uniformly on compact sets.*

Now let f be an integral automorphic form (see [3] Section 8.5) on $\mathscr{D}$ of weight l, i.e., $f(z) = J_{\mathscr{D}}(\gamma, z)^l f(\gamma \cdot z)$ for all $\gamma \in \Gamma$. Then for each rational boundary component F, it has an extension to an automorphic form for Γ_F on F, which we denote by $\Phi_F(f)$.

Theorem 7.9 ([3] Theorem 8.6) *Let E be a Poincaré-Eisenstein series adapted to the rational boundary component F, for the arithmetic group Γ of weight l. Then E is an integral automorphic form. Let F' be a rational boundary component. Then $\Phi_{F'}E = 0$ if $\dim F' \leq \dim F$ and $F' \subsetneq \Gamma \cdot F$. The operator Φ_F maps the module of Poincaré-Eisenstein series adapted to F of weight l, onto the module of Poincaré series for $\Gamma(F)$ of weight l.*

Remark 7.10 *By this theorem, the Poincaré series $P_{m,\phi}$ defined on the improper component $\mathscr{D}$ satisfies $\Phi_F(P_{m,\phi}) = 0$ for any proper boundary components F. This also implies $P_{m,\phi}$ is a cusp form (see [3] Section 8.10).*

Since the function $g \mapsto f(g \cdot 0) J_{\mathscr{D}}(g, 0)^{-m}$ is bounded (Theorem 7.6), as in the proof of Lemma 5.6, we are able to associate to each $P_{m,f}$ a well-defined Toeplitz operator $T_{P_{m,f}}$. We will focus on the Toeplitz operators of this type later for the proof of Theorem 7.1.

7.3 Proof of the main theorem

Two cusp forms f, g on the bounded symmetric domain $\mathscr{D}$ are called of a same type ρ if both f, g take values in a representation V_ρ of K and $f(\gamma x) = \rho(J(\gamma, x)) \cdot f(x), g(\gamma x) = \rho(J(\gamma, x)) \cdot g(x)$ as in Definition 5.2. For the proof of Theorem 7.1, we will focus on the cusp forms of type $\det^{-m}(\mathrm{ad}_{\mathfrak{p}^+})$, which are nothing but the cusp forms of weight m by Lemma 7.5.

Proof: [Proof of Theorem 7.1] By Corollary 5.10, if f, g are cusp forms of type (ρ, V_ρ) and at least one of them is holomorphic, we know the composite operator $T_g^* T_f$ is just the Toeplitz operator $T_{(f,g)}$ associated with the essentially bounded function $(f,g)(w) = \langle \rho(\kappa(w,w)^{-1})f(w), g(w)) \rangle_{V_\rho}$. Hence, by Corollary 4.2, it suffices to show these (f,g)'s span a dense subspace of $L^2(\mathscr{F})$ (or equivalently $L^\infty(\mathscr{F})$, $C(\mathscr{F})$.)

By Theorem 7.3, we know $\mathscr{F}^*$ is a compact Hausdorff space which contains $\mathscr{F}$ as a dense open subset. Let us consider the quotient space $\mathscr{F}_1$ of $\mathscr{F}^*$ by identifying all the elements do not belong to $\mathscr{F}$ (which form a closed subset of $\mathscr{F}^*$), i.e.,

$$\mathscr{F}_1 = \mathscr{F}^*/(\mathscr{F}^* \backslash \mathscr{F}).$$

This is also the disjoint union of $\mathscr{F}$ and a single point, denoted as $\{\mathrm{pt}\}$ (which represents all the proper boundary components), i.e., $\mathscr{F}_1 = \mathscr{F} \sqcup \{\mathrm{pt}\}$.

Let us consider the Poincaré series $P_{m,\phi}$ for a polynomial ϕ on $\mathscr{D}$. By Theorem 7.9, $\Phi_F(P_{m,\phi}) = 0$ for any proper boundary components F. Hence every $P_{m,\phi}$ gives a well-defined function on $\mathscr{F}_1$ which vanishes at pt. Then, by Remark 7.10, it suffices to consider the functions of the type $(P_{m,\phi}, P_{m,\psi})$.

Take any two distinct points $z_1, z_2 \in \mathscr{F}_1$ and consider the following two cases: (i) $z_1 = \mathrm{pt}, z_2 \in \mathscr{F}$, or (ii) $z_1, z_2 \in \mathscr{F}$. In either of the two cases, by Proposition 7.7, there is a polynomial ϕ such that $P_{m,\phi}(z_1) \neq P_{m,\phi}(z_2)$ for some m. So $(P_{m,\phi}, P_{m,\psi})(z_1) \neq (P_{m,\phi}, P_{m,\psi})(z_2)$ for a suitable ψ such that $P_{m,\psi}(z_2) \neq 0$. Note $\mathscr{F}_1$ is compact and Hausdorff, by Stone-Weierstrass Theorem, the forms $(P_{m,\phi}, P_{m,\psi})$ generate the space

$$\{h\colon \mathscr{F}_1 \to \mathbb{C} | h \text{ is continuous, } h(\mathrm{pt}) = 0\}.$$

Hence their restriction on $\mathscr{F}$ is dense in $L^\infty(\mathscr{F})$, which completes the proof. □

Remark 7.11 *Assume G is a connected semi-simple linear algebraic group over $\mathbb{R}$ and a lattice Γ is Zariski-dense in G. We can show Γ is an ICC group (see [16] 3.3.b). This gives a large family of the cases that A_π is a II_1 factor.*

BIBLIOGRAPHY

[1] C. A. Akemann. Operator algebras associated with Fuchsian groups. *Houston J. Math.*, 7(3):295–301, 1981.

[2] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai. *Smooth compactifications of locally symmetric varieties*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. With the collaboration of Peter Scholze.

[3] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[4] F. A. Berezin. Quantization. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:1116–1175, 1974.

[5] F. A. Berezin. General concept of quantization. *Comm. Math. Phys.*, 40:153–174, 1975.

[6] S. Bochner and W. T. Martin. *Several Complex Variables*. Princeton Mathematical Series, vol. 10. Princeton University Press, Princeton, N. J., 1948.

[7] A. Borel. Introduction to automorphic forms. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 199–210. Amer. Math. Soc., Providence, R.I., 1966.

[8] A. Borel and H. Jacquet. Automorphic forms and automorphic representations. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 189–207. Amer. Math. Soc., Providence, R.I., 1979. With a supplement "On the notion of an automorphic representation" by R. P. Langlands.

[9] A. Böttcher and B. Silbermann. *Analysis of Toeplitz operators*. Springer Monographs

in Mathematics. Springer-Verlag, Berlin, second edition, 2006. Prepared jointly with Alexei Karlovich.

[10] D. Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[11] B. Cahen. Berezin quantization for holomorphic discrete series representations: the non-scalar case. *Beitr. Algebra Geom.*, 53(2):461–471, 2012.

[12] H. Cartan. Fonctions automorphes et séries de Poincaré. *J. Analyse Math.*, 6:169–175, 1958.

[13] M. P. De Oliveira. Some formulas for the canonical kernel function. *Geom. Dedicata*, 86(1-3):227–247, 2001.

[14] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[15] R. Godement. Domaines fondamentaux des groupes arithmétiques. In *Séminaire Bourbaki, 1962/63. Fasc. 3, No. 257*, page 25. Secrétariat mathématique, Paris, 1964.

[16] F. M. Goodman, P. de la Harpe, and V. F. R. Jones. *Coxeter graphs and towers of algebras*, volume 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1989.

[17] Harish-Chandra. Discrete series for semisimple Lie groups. I. Construction of invariant eigendistributions. *Acta Math.*, 113:241–318, 1965.

[18] Harish-Chandra. *Automorphic forms on semisimple Lie groups*. Notes by J. G. M. Mars. Lecture Notes in Mathematics, No. 62. Springer-Verlag, Berlin-New York, 1968.

[19] H. Hedenmalm, B. Korenblum, and K. Zhu. *Theory of Bergman spaces*, volume 199 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[20] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[21] V. Jones and V. S. Sunder. *Introduction to subfactors*, volume 234 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1997.

[22] V. F. R. Jones. *Von Neumann algebras*. Course notes, Vanderbilt University, 2015.

[23] V. F. R. Jones. Bergman space zero sets, modular forms, von Neumann algebras and ordered groups, 2020. arXiv preprint arXiv:2006.16419.

[24] R. V. Kadison. A generalized Schwarz inequality and algebraic invariants for operator algebras. *Ann. of Math. (2)*, 56:494–503, 1952.

[25] R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. II*, volume 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original.

[26] A. W. Knapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[27] A. W. Knapp and D. A. Vogan, Jr. *Cohomological induction and unitary representations*, volume 45 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1995.

[28] T. Miyake. *Modular forms*. Springer-Verlag, Berlin, 1989. Translated from the Japanese by Yoshitaka Maeda.

[29] K.-H. Neeb. *Holomorphy and convexity in Lie theory*, volume 28 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 2000.

[30] F. Radulescu. Free group factors and Hecke operators. In *The varied landscape of operator theory*, volume 17 of *Theta Ser. Adv. Math.*, pages 241–257. Theta, Bucharest, 2014.

[31] J. Repka. Tensor products of holomorphic discrete series representations. *Canadian J. Math.*, 31(4):836–844, 1979.

[32] A. Robert. *Introduction to the representation theory of compact and locally compact groups*, volume 80 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1983.

[33] F. Rădulescu. On the von Neumann algebra of Toeplitz operators with automorphic symbol. In *Subfactors (Kyuzeso, 1993)*, pages 268–273. World Sci. Publ., River Edge, NJ, 1994.

[34] F. Rădulescu. The Γ-equivariant form of the Berezin quantization of the upper half plane. *Mem. Amer. Math. Soc.*, 133(630):viii+70, 1998.

[35] I. Satake. *Algebraic structures of symmetric domains*, volume 4 of *Kanô Memorial Lectures*. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N.J., 1980.

[36] W. Schmid. L^2-cohomology and the discrete series. *Ann. of Math. (2)*, 103(2):375–394, 1976.

[37] G. Shimura. *Introduction to the arithmetic theory of automorphic functions*. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. Kanô Memorial Lectures, No. 1.

[38] G. Shimura. On Hilbert modular forms of half-integral weight. *Duke Math. J.*, 55(4):765–838, 1987.

[39] H. Upmeier. Toeplitz C^*-algebras on bounded symmetric domains. *Ann. of Math. (2)*, 119(3):549–576, 1984.

[40] N. L. Vasilevski. *Commutative algebras of Toeplitz operators on the Bergman space*, volume 185 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2008.

[41] N. R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.