STRATEGIC NETWORK INTERDICTION

by

Sunghoon Hong and Myrna Wooders

Working Paper No. 10-W10

June 2010

DEPARTMENT OF ECONOMICS
VANDERBILT UNIVERSITY
NASHVILLE, TN 37235

www.vanderbilt.edu/econ

Strategic Network Interdiction*

Sunghoon Hong Myrna Wooders
Department of Economics,
Vanderbilt University
Nashville, TN 37235 USA
sunghoon.hong@vanderbilt.edu; myrna.wooders@vanderbilt.edu

June 20, 2010

Abstract

We develop a strategic model of network interdiction in a non-cooperative game of flow. A security agency operates a network with arc capacities. An adversary, endowed with a bounded quantity of bads, chooses a flow that specifies a plan for carrying bads through the network from a base to a target. Simultaneously, the agency chooses a blockage, which specifies a plan for blocking the transport of bads through arcs in the network. However, the blockage of arcs disrupts the operation of the network. The adversary gains and the agency loses from the target damage and the network disruption. The adversary incurs the expense of carrying bads. We characterize the Nash equilibria in terms of the primitives of our model. Our model contributes to the literature of game theory by introducing non-cooperative behavior into a Kalai-Zemel type mode of a (cooperative) game of flow. Our research also advances models and results on network interdiction.

JEL classification: C72; D85; H56

Keywords: Network interdiction; Noncooperative game of flow; Blockage; Nash equilibrium; Kalai-Zemel game of flow

*This paper has been presented at the Stony Brook International Conference on Game Theory, July 2009 and the Newport Beach Behavioral and Quantitative Game Theory Conference, May 2010. We gratefully acknowledge comments of the participants and also comments of Andrew Daughety, Jennifer Reinganum, and other colleagues at Vanderbilt University.

1 Introduction

National security is a public good whose reliable provision reduces economic uncertainty and promotes economic activity. Although the government of a country provides its citizens with national security, the country's security level is not determined only by the government's decision. Non-state actors, as well as other governments, may also affect the country's security level. For example, as a violent non-state actor, a terrorist group can critically affect a country's security level.

In this paper we develop a strategic model of network interdiction. A security agency operates a network with arc capacities; airlines, railroads, pipelines, public transportation, and computer networks are examples of such networks. An adversary is endowed with a bounded quantity of bads; explosives, biochemicals, nuclear weapons, and computer viruses are examples of bads. The adversary chooses a flow that specifies a plan for carrying bads through the network from a base to a target. By carrying bads through the network, the adversary can damage the target. Simultaneously, the agency chooses a blockage that specifies a plan for blocking arcs in the network. By blocking arcs, the agency can decrease the amount of bads carried to the target. However, the blockage of arcs disrupts the operation of the network. The adversary gains and the agency loses from the target damage and the network disruption. The adversary incurs the expense of carrying bads.

The properties of the Nash equilibria depend on the primitives of the model. When the marginal loss of the target is no greater than the marginal expense of carrying bads, there are pure strategy Nash equilibria in which the adversary carries no bads from the base to the target and the agency blocks no arcs, regardless of the bound on the quantity of bads. When the marginal target loss is greater than the marginal expense, there are various types of Nash equilibria, depending on the marginal target loss, the network and the bound on the quantity of bads. First, if the bound on the quantity of bads is small, there are pure strategy Nash equilibria in which the adversary carries bads from the base to the target up to the bound and the agency blocks no arcs. Second, if the bound on the quantity of bads is large, there are mixed strategy Nash equilibria in which the adversary carries the maximum possible amount of bads through the network with positive probability and the agency blocks no arcs with positive probability. Third, if the bound on the quantity of bads is intermediate, there are mixed strategy Nash equilibria in which the adversary carries bads up to the bound with positive probability and the agency blocks no arcs with positive probability.

In a Nash equilibrium the adversary damages the target if and only if the adversary successfully carries bads from the base to the target and the agency blocks none of the arcs used. By computing the probability of this joint event, we calculate the equilibrium probability of the adversary damaging the target. When the marginal target loss is greater than the marginal expense and the bound on the quantity of bads is either intermediate or large, the equilibrium probabilities can be expressed as negative power functions of the marginal target loss. Assuming constant marginal target loss, we can conclude that the equilibrium probabilities are negative power functions of the target damage. This theoretical finding is consistent with empirical evidence.[1]

This paper contributes to the game theory literature by introducing noncooperative behavior into a Kalai-Zemel network flow model. Kalai and Zemel [14] define a (transferable utility) cooperative game, called a flow game, in which the worth of a coalition is defined as the value of a maximum flow in the network restricted to the members of the coalition.[2] Their main result is that a cooperative flow game is totally balanced and thus has a nonempty core (that is, there are distributions of the total payoff of the game that are stable against the formation of coalitions). Our framework differs in that players interact strategically. Moreover, one might think of the nodes in our game as points in a transportation route or as servers in the Internet rather than as players themselves or locations of players.

This paper also contributes to the literature on network interdiction. Washburn and Wood [18] introduce a zero-sum game between an evader and an interdictor, in which the evader chooses a path to move through a network and the interdictor chooses an arc at which to set up an inspection site. If the evader traverses a path that includes the inspected arc, the evader is detected with some exogenously given positive probability; otherwise, the evader is not detected. The detection probability of each arc in the network is exogenously given. Both players are allowed to choose mixed strategies. Given a mixed strategy profile, the interdiction probability is defined to be the average probability of the evader being detected. The evader aims to minimize the interdiction probability by choosing a path-selection mixed strategy while the interdictor aims to maximize the interdiction probability by choosing an arc-inspection mixed strategy. By using linear programming and network flow techniques, Washburn and Wood [18] find Nash equilibria of this game. Kodialam and Lakshman [15] also introduce a related game of network interdiction in the context of network security.[3]

Our model differs from the existing models on network interdiction in four aspects:

(i) Networks are capacitated. That is, a capacity is assigned to each arc in a network.

(ii) The adversary is endowed with a bounded quantity of bads, which may, in equilibrium, be binding.

(iii) The adversary chooses a flow rather than a path. If there are multiple paths in a network, the adversary can choose them at once.

(iv) Our network interdiction game is not a zero-sum game nor even a strictly competitive game.

Because of (i), we do not need to take as given detection probabilities; in our model these probabilities are determined by player behaviors. By virtue of (ii) and (iii), our model has a larger set of strategies for the

[1] In empirical research Bohorquez et al. [5] and Clauset et al. [6] show that the fatality distribution of terrorist events follows a power law.

[2] For other studies on cooperative flow games, see Kalai and Zemel [13], Granot and Granot [10], Potters et al. [16], and Reijnierse et al. [17].

[3] Other than these papers, most of the literature on network interdiction deals with an interdictor's optimization problem subject to some budget constraints. See Cormican et al. [7], Israeli and Wood [11], and Wood [19].

adversary than Washburn and Wood [18]. Our consideration of flows rather than paths as in [18], however, creates a more tractable environment and enables us to obtain sharper characterizations of equilibrium strategies. Because of (iv), we need to use a different solution technique to find equilibria. We exploit the idea that in any Nash equilibrium each player makes rival players indifferent between the pure strategies played with positive probability.

We remark that security in network games has attracted significant interest. For example, Ballester et al. [4] study the interaction between players whose payoffs depend on a network. They obtain a proportional relationship between how much effort a player exerts and how central position the player has in the network. Baccara and Bar-Isaac [2] study the formation of networks between criminals and terrorists and find optimal policies for law enforcement agencies. Baccara and Bar-Isaac [3] further study how the choice of interrogation methods affects the formation of terrorist networks. Goyal and Vigier [9] study the design and protection of networks robust to attacks from outside on the networks' nodes.[4]

The remainder of this paper is organized as follows. Section 2 develops a game-theoretic model of network interdiction. Section 3 studies Nash equilibria of the model. Section 4 discusses our theoretical finding, together with empirical evidence, and also discusses future research topics.

2 The Model

2.1 Networks

Let N be a set of nodes with a base node s and a target node t. Let $A \subset N \times N$ be a set of arcs where each arc is an ordered pair of distinct nodes. Let $c := (c_{ij})_{(i,j)\in A}$ be a (row) vector of arc capacities where each entry $c_{ij} \geq 0$ denotes the capacity of arc (i, j). A *network* is defined as a collection $G := (N, A, s, t, c)$.

For each $j \in N$, let $RS(j) := \{(i, j) : (i, j) \in A\}$ be the *reverse star* of node j in network G. For each $j \in N$, let $FS(j) := \{(j, i) : (j, i) \in A\}$ be the *forward star* of node j in network G.

A *bound quantity* $q > 0$ is given. This bound limits the quantity of bads that are available to an adversary to carry through network G from base s to target t. A *flow* $f := (f_{ij})'_{(i,j)\in A}$ with bound quantity q in network G is a (column) vector satisfying the following constraints:

$$0 \leq f_{ij} \leq c_{ij} \qquad \text{for each } (i, j) \in A, \tag{1}$$

$$\sum_{(i,j)\in RS(j)} f_{ij} - \sum_{(j,i)\in FS(j)} f_{ji} = 0 \qquad \text{for each } j \in N \setminus \{s, t\}, \tag{2}$$

$$-q \leq \sum_{(i,s)\in RS(s)} f_{is} - \sum_{(s,i)\in FS(s)} f_{si} \leq 0 \qquad \text{and} \tag{3}$$

$$0 \leq \sum_{(i,t)\in RS(t)} f_{it} - \sum_{(t,i)\in FS(t)} f_{ti} \leq q. \tag{4}$$

[4] For a survey on other literature on networks, see Jackson [12].

Constraint (1) requires that for each arc (i,j), the flow f_{ij} be between zero and the capacity c_{ij}. Constraint (2) requires that for each node j other than base s and target t, the sum of the flows f_{ij} carried to node j be equal to the sum of the flows f_{ji} carried from node j. Constraint (3) requires that the sum of the flows f_{is} carried to base s less the sum of the flows f_{si} carried from base s be between zero and the bound $-q$. Constraint (4) requires that the sum of the flows f_{it} carried to target t less the sum of the flows f_{ti} carried from target t be between zero and the bound q. Let $\mathcal{F}$ be the set of all flows with bound quantity q in network G.

Let $v := (v_{ij})_{(i,j)\in A}$ be a (row) vector with $v_{it} = 1$ for each $(i,t) \in RS(t)$, $v_{ti} = -1$ for each $(t,i) \in FS(t)$, and $v_{ij} = 0$ for each $(i,j) \notin RS(t) \cup FS(t)$. Then the *value* of any flow f is defined as

$$v \cdot f = \sum_{(i,t)\in RS(t)} f_{it} - \sum_{(t,i)\in FS(t)} f_{ti}; \tag{5}$$

that is, the value of a flow $f = (f_{ij})'_{(i,j)\in A}$ is the sum of the flows f_{it} carried to target t less the sum of the flows f_{ti} carried from target t. Constraint (4) implies that for each $f \in \mathcal{F}$,

$$0 \le v \cdot f \le q. \tag{6}$$

A flow $f^o \in \mathcal{F}$ is the *zero flow* if f^o is the vector of zeros. A flow $f^* \in \mathcal{F}$ is a *maximum flow* if for each $f \in \mathcal{F}$, we have $v \cdot f^* \ge v \cdot f$.

A *cut* $(C, \overline{C})$ in network G is a partition of the node set N such that $s \in C$ and $t \in \overline{C}$. For each cut $(C, \overline{C})$, let $A_{(C,\overline{C})} := \{(i,j) \in A : i \in C \text{ and } j \in \overline{C}\}$ be the set of all arcs directed from a node in C to a node in $\overline{C}$.

A *blockage* $b := (b_{ij})'_{(i,j)\in A}$ in network G is a (column) vector with $b_{ij} \in \{0,1\}$ for each $(i,j) \in A$. If $b_{ij} = 1$, arc (i,j) is blocked; if $b_{ij} = 0$, arc (i,j) is not blocked. Let $\mathcal{B}$ be the set of all blockages in network G. The *capacity* of any blockage b is defined as

$$c \cdot b = \sum_{(i,j)\in A} c_{ij} b_{ij}; \tag{7}$$

that is, the capacity of a blockage is the sum of the capacities of the blocked arcs.

A blockage $b^o \in \mathcal{B}$ is the *zero blockage* if b^o is the vector of zeros. For each $b \in \mathcal{B}$, let $A_b := \{(i,j) \in A : b_{ij} = 1\}$ be the set of all arcs blocked under b. A blockage $b \in \mathcal{B}$ is a *cut blockage* if there is a cut $(C, \overline{C})$ such that $A_{(C,\overline{C})} = A_b$. A cut blockage $b^* \in \mathcal{B}$ is a *minimum cut blockage* if for each cut blockage b, we have $c \cdot b^* \le c \cdot b$.

If $q > c \cdot b^*$, we have

$$v \cdot f^* = c \cdot b^*. \tag{8}$$

That is, if the bound quantity is greater than the capacity of a minimum cut blockage, the value of a maximum flow is equal to the capacity of a minimum cut blockage. Equality (8) is called the *max-flow* *min-cut theorem.*[5] If $q \leq c \cdot b^*$, however, we have

$$v \cdot f^* = q. \tag{9}$$

That is, if the bound quantity is less than or equal to the capacity of a minimum cut blockage, the value of a maximum flow is equal to the bound quantity.

An $s - t$ *path* in network G is a sequence of distinct nodes $i_1, \ldots, i_K$ such that $(i_k, i_{k+1}) \in A$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = s$ and $i_K = t$. In this case, we say that the $s - t$ path *includes* arcs $(i_1, i_2), \ldots, (i_{K-1}, i_K)$. A *cycle* in network G is a sequence of distinct nodes $i_1, \ldots, i_K$ such that $(i_k, i_{k+1}) \in A$ for each $k \in \{1, \ldots, K-1\}$ with $(i_K, i_1) \in A$. In this case, we say that the cycle *includes* arcs $(i_1, i_2), \ldots, (i_{K-1}, i_K)$, and (i_K, i_1). Let H be the set of all $s - t$ paths and cycles in network G.

The *arc-path-cycle incidence matrix*[6] of network G is a matrix $M := (m_{ah})_{a \in A, h \in H}$ with

$$m_{ah} = \begin{cases} 1 & \text{if } h \in H \text{ includes } a \in A \\ 0 & \text{otherwise.} \end{cases}$$

For each flow f, there is a (column) vector $x := (x_h)'_{h \in H}$ such that

$$f = Mx$$

by the flow decomposition algorithm, which is stated in Appendix A. For such vector x and each blockage b, let $x^b := (x^b_h)'_{h \in H}$ be a (column) vector with

$$x^b_h = \begin{cases} x_h & \text{if } h \text{ is an } s - t \text{ path including no blocked arcs} \\ 0 & \text{otherwise.} \end{cases}$$

For each $(f, b) \in \mathcal{F} \times \mathcal{B}$, the *net flow* to target t under f and b is a (column) vector $f^b := (f^b_{ij})'_{(i,j) \in A}$ such that

$$Mx^b = f^b.$$

In any net flow, all cycles have zero flow and only $s - t$ paths have positive flow. For each $(f, b) \in \mathcal{F} \times \mathcal{B}$, we have

$$\begin{aligned} f - f^b &= Mx - Mx^b \\ &= M(x - x^b) \\ &\geq 0 \end{aligned} \tag{10}$$

because $f = Mx$, $f^b = Mx^b$, and $x \geq x^b$. In other words, if $f^b = (f^b_{ij})'_{(i,j) \in A}$ is the net flow under a flow $f = (f_{ij})'_{(i,j) \in A}$ and a blockage b, for each arc (i, j), the net flow f^b_{ij} cannot exceed the flow f_{ij}. The following example illustrates how to find net flows.

[5] For a detailed discussion, see Ahuja et al. [1] and Ford and Fulkerson [8].

[6] This extends the arc-path incidence matrix used in Washburn and Wood [18] to allow cycles.

Example 1 *Let $G = (N, A, s, t, c)$ be a network, where $N = \{s, i_1, i_2, t\}$ is a node set with base s and target t, $A = \{(s, i_1),\ (s, i_2),\ (i_1, i_2),\ (i_2, t),\ (t, i_1)\}$ is an arc set, and $c = (c_{si_1}, c_{si_2}, c_{i_1i_2}, c_{i_2t}, c_{ti_1}) = (4, 1, 2, 5, 2)$ is an arc capacity vector. Suppose that $q = 3$. A flow with bound quantity q in network G is given as $f = (f_{si_1}, f_{si_2}, f_{i_1i_2}, f_{i_2t}, f_{ti_1})' = (1, 1, 2, 3, 1)'$. See Figure 1. The arc-path-cycle incidence matrix of network G is*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where the first two columns describes the $s-t$ paths s, i_1, i_2, t and s, i_2, t respectively and the third column describes the cycle i_1, i_2, t, i_1. By the flow decomposition algorithm, there is a vector $x = (1, 1, 1)'$ such that $f = Mx$. That is,

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Now a blockage in network G is given as $b = (b_{si_1}, b_{si_2}, b_{i_1i_2}, b_{i_2t}, b_{ti_1})' = (0, 1, 0, 0, 0)'$. Then, $x^b = (1, 0, 0)'$. Thus, the net flow to target t under f and b is $f^b = (1, 0, 1, 1, 0)'$. As in inequality (10), we have $f \geq f^b$. $\square$

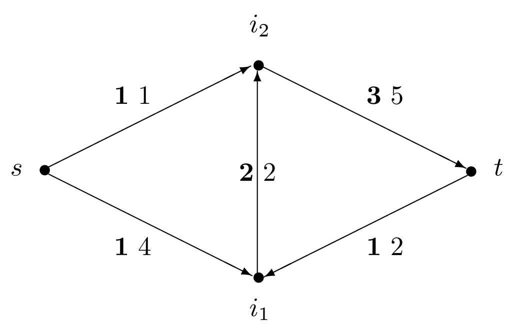

Figure 1 Each solid circle indicates a node; each arrow indicates an arc; in each pair of numbers, the first bold number indicates a flow and the second light number indicates the capacity of an arc.

2.2 Players and Strategies

A network $G = (N, A, s, t, c)$ is given. There are two players, say player 1 and player 2. Let player 1 be an adversary and let player 2 be an agency. Suppose that both players move *simultaneously*.

Player 1 is endowed with a bound quantity $q > 0$ of bads located at base node s. Player 1 plans to damage target node t and disrupt the operation of network G. Player 1 chooses a plan for carrying bads through network G from base s to target t. Formally, player 1 chooses a *flow* (of bads) with bound quantity q in network G. Player 1's set of pure strategies is denoted by $\mathcal{F}$, the set of all flows with bound quantity q in network G. For each $f \in \mathcal{F}$, player 1 carries f_{ij} amount of bads through arc (i, j).

Player 2 aims to protect target t and operate network G effectively. Player 2 chooses a plan for blocking arcs in the network. Formally, player 2 chooses a *blockage* (of arcs) in network G. Player 2's set of pure strategies is denoted by $\mathcal{B}$, the set of all blockages in network G. For each $b \in \mathcal{B}$, if $b_{ij} = 1$, player 2 blocks arc (i, j); otherwise, she does not block the arc.

Players are allowed to choose *mixed strategies*. Player 1's set of mixed strategies is denoted by $\Delta(\mathcal{F})$ and player 2's set of mixed strategies is denoted by $\Delta(\mathcal{B})$.

2.3 Payoffs

The net flow to the target determines how severely the target is damaged. If player 1 chooses a flow f and player 2 chooses a blockage b, then f^b is the net flow to target t under f and b. Let $\ell := (\ell_{ij})_{(i,j)\in A}$ be a (row) vector of target losses where each entry ℓ_{ij} denotes the target damage caused by one unit of bads carried through arc (i, j). We assume that $\ell_{it} = \ell_t > 0$ for each $(i, t) \in RS(t)$ and $\ell_{ij} = 0$ for each $(i, j) \notin RS(t)$. We call ℓ_t the *marginal target loss*. For each $(f, b) \in \mathcal{F} \times \mathcal{B}$, the *target damage* amounts to

$$\ell \cdot f^b = \ell_t \sum_{(i,t)\in RS(t)} f^b_{it}. \tag{11}$$

Then player 1 gains $\ell \cdot f^b$ while player 2 loses the same amount.

By blocking arcs, player 2 decreases the net flow of bads to the target. However, the blockage of arcs disrupts the operation of network G. The capacity of the arc blockage determines how severely the network is disrupted. For each $b \in \mathcal{B}$, the *network disruption* amounts to

$$c \cdot b = \sum_{(i,j)\in A} c_{ij} b_{ij}. \tag{12}$$

Then player 1 gains $c \cdot b$ while player 2 loses the same amount.

Player 1 incurs the expense of carrying bads from the base to the target. Let $e := (e_{ij})_{(i,j)\in A}$ be a (row) vector with $e_{si} = 1$ for each $(s, i) \in FS(s)$ and $e_{ij} = 0$ for each $(i, j) \notin FS(s)$. For each $f \in \mathcal{F}$, the *expense of carrying bads* amounts to

$$e \cdot f = \sum_{(s,i)\in FS(s)} f_{si}. \tag{13}$$

Player 2 obtains a constant worth $w \geq 0$ while operating network G.

For each $(f, b) \in \mathcal{F} \times \mathcal{B}$, the payoff of player 1 is defined as

$$u_1(f, b) = \ell \cdot f^b + c \cdot b - e \cdot f,$$

and the payoff of player 2 is defined as

$$u_2(f, b) = w - \ell \cdot f^b - c \cdot b.$$

For each $\sigma = (\sigma_1, \sigma_2) \in \Delta(\mathcal{F}) \times \Delta(\mathcal{B})$, the *expected payoffs* of the players are

$$u_1(\sigma_1, \sigma_2) = E_\sigma[u_1(f, b)] \text{ and } u_2(\sigma_1, \sigma_2) = E_\sigma[u_2(f, b)].$$

Let $\Gamma := (G, q, \Delta(\mathcal{F}), \Delta(\mathcal{B}), u_1, u_2)$ be a *network interdiction game.*

3 Nash Equilibria of the Model

A network interdiction game is given as $\Gamma = (G, q, \Delta(\mathcal{F}), \Delta(\mathcal{B}), u_1, u_2)$. We use Nash equilibrium as our solution concept. In a Nash equilibrium, no player can be (strictly) better off by changing his or her strategy.

Definition 1 *A strategy profile $(\sigma_1, \sigma_2) \in \Delta(\mathcal{F}) \times \Delta(\mathcal{B})$ constitutes a Nash equilibrium of network interdiction game Γ if for each $\sigma_1' \in \Delta(\mathcal{F})$ and each $\sigma_2' \in \Delta(\mathcal{B})$,*

$$u_1(\sigma_1, \sigma_2) \geq u_1(\sigma_1', \sigma_2) \text{ and } u_2(\sigma_1, \sigma_2) \geq u_2(\sigma_1, \sigma_2').$$

Depending on whether the marginal target loss is greater than one, we consider two cases. Suppose first that the marginal target loss is less than or equal to one.

A flow $f^\tau \in \mathcal{F}$ is a *trivial flow* if $e \cdot f^\tau = 0$. In any trivial flow, all $s - t$ paths have zero flow. Thus, the net flow to the target under any trivial flow f^τ and any blockage b is the zero flow. That is, for each $b \in \mathcal{B}$,

$$(f^\tau)^b = f^o. \tag{14}$$

We call (f^τ, b^o) a *trivial-flow zero-blockage strategy profile.* If the marginal target loss is less than or equal to one, player 1, the adversary, does not have an incentive to carry bads from the base to the target and player 2, the agency, does not have an incentive to block arcs. Thus, we have the following proposition.

Proposition 1 *If $\ell_t \leq 1$, any trivial-flow zero-blockage strategy profile (f^τ, b^o) is a Nash equilibrium of network interdiction game Γ.*

The proof of Proposition 1 is presented in Appendix B. Notice that Proposition 1 holds regardless of the bound quantity. We provide an example of trivial-flow zero-blockage Nash equilibria.

Example 2 *Consider network G in Example 1. Suppose that bound quantity q is any positive real number. A trivial flow is given as $f^\tau = (0, 0, 2, 2, 2)'$. See Figure 2. Suppose that $\ell_t \leq 1$. From Proposition 1, strategy profile (f^τ, b^o) is a Nash equilibrium of the network interdiction game. In this Nash equilibrium, player 1 carries no bads from the base to the target and player 2 blocks no arcs.* □

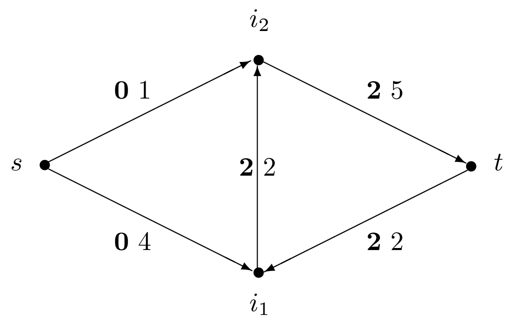

Figure 2 The bold numbers indicate the trivial flow.

Now suppose that the marginal target loss is greater than one. Then player 1 has the incentive to carry bads from the base to the target. Thus, any trivial-flow zero-blockage strategy profile is not a Nash equilibrium. To study Nash equilibria in this case, we divide into three subcases, depending on the marginal target loss, the capacity of a minimum cut blockage, and the bound quantity. The bound quantity q is *small* if $q \leq (1/\ell_t)c \cdot b^*$. The bound quantity q is *intermediate* if $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$. The bound quantity q is *large* if $c \cdot b^* < q$.

Let f^* denote a maximum flow with large bound quantity q in network G. Because q is large, that is, because $q > c \cdot b^*$, from equality (8), we have $v \cdot f^* = c \cdot b^*$. A maximum flow f^* with large bound quantity q in network G is *acyclic* if $f^* = (f^*)^{b^o}$. Let f^α denote an acyclic maximum flow with large bound quantity q in network G. Then equality (8) can be rewritten as

$$v \cdot f^\alpha = c \cdot b^*. \tag{15}$$

We first study the subcase when $\ell_t > 1$ and $q \leq (1/\ell_t)c \cdot b^*$. A flow $f^\beta \in \mathcal{F}$ is a *binding flow* if $f^\beta = (q/(c \cdot b^*))f^\alpha$. Because $v \cdot f^\alpha = c \cdot b^*$ from equality (15),

$$\begin{aligned} v \cdot f^\beta &= (q/(c \cdot b^*))v \cdot f^\alpha \\ &= q; \end{aligned} \tag{16}$$

that is, the value of a binding flow is equal to the bound quantity. We call (f^β, b^o) a *binding-flow zero-blockage strategy profile*. If the marginal target loss is greater than one and the bound quantity is small, player 1, the adversary, has an incentive to carry bads from the base to the target up to the bound quantity but player 2, the agency, does not have the incentive to block arcs. Thus, we have the following proposition.

Proposition 2 *If $\ell_t > 1$ and $q \leq (1/\ell_t)c \cdot b^*$, any binding-flow zero-blockage strategy profile (f^β, b^o) is a Nash equilibrium of network interdiction game Γ.*

The proof of Proposition 2 is presented in Appendix B. We provide an example of binding-flow zero-blockage Nash equilibria.

Example 3 *Consider network G in Example 1. Note that the acyclic maximum flow is $f^{\alpha} = (2, 1, 2, 3, 0)'$ and that the minimum cut blockage is $b^* = (0, 1, 1, 0, 0)'$. The capacity of the minimum cut blockage is $c \cdot b^* = 3$. Suppose that $\ell_t = 2$ and $q = 1$. Then the binding flow is $f^{\beta} = (2/3, 1/3, 2/3, 1, 0)'$. See Figure 3. From Proposition 2, strategy profile (f^{β}, b^o) is a Nash equilibrium of the network interdiction game. In this Nash equilibrium, player 1 carries bads from the base to the target up to the bound quantity but player 2 does not block any arc.* $\square$

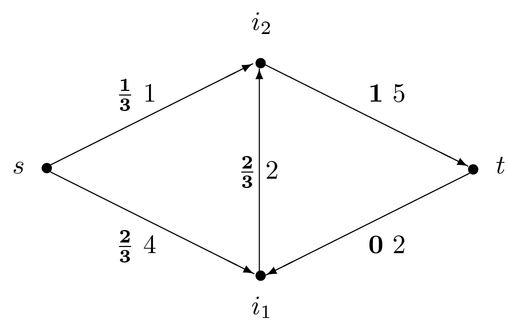

Figure 3 The bold numbers indicate the binding flow.

We next study the subcase when $\ell_t > 1$ and $c \cdot b^* < q$. Recall that f^{α} is an acyclic maximum flow with large bound quantity q in network G. A mixed strategy $\sigma_1^* \in \Delta(\mathcal{F})$ is a *max-flow strategy* for player 1 if $\sigma_1^*(f^{\tau}) = 1 - 1/\ell_t$ and $\sigma_1^*(f^{\alpha}) = 1/\ell_t$. A mixed strategy $\sigma_2^* \in \Delta(\mathcal{B})$ is a *min-cut strategy* for player 2 if $\sigma_2^*(b^o) = 1/\ell_t$ and $\sigma_2^*(b^*) = 1 - 1/\ell_t$. We call (σ_1^*, σ_2^*) a *max-flow min-cut strategy profile.* We provide max-flow min-cut Nash equilibria of the network interdiction game.

Proposition 3 *If $\ell_t > 1$ and $c \cdot b^* < q$, any max-flow min-cut strategy profile (σ_1^*, σ_2^*) is a Nash equilibrium of network interdiction game Γ.*

The proof of Proposition 3 is presented in Appendix B.

Here we show that given a min-cut strategy σ_2^*, player 1 is indifferent between f^{τ} and f^{α}, and that given a max-flow strategy σ_1^*, player 2 is indifferent between b^o and b^*. In any max-flow min-cut Nash equilibrium, strategy profiles (f^{τ}, b^o), (f^{τ}, b^*), (f^{α}, b^o), and (f^{α}, b^*) are assigned positive probability. Figure 4 illustrates the players' payoffs for these strategy profiles. The calculation of these payoffs can be found in the proof of Proposition 3. Because $\sigma_2^*(b^o) = 1/\ell_t$ and $\sigma_2^*(b^*) = 1 - 1/\ell_t$, player 1's expected payoff for (f^{τ}, σ_2^*) is calculated as

$$\begin{aligned} u_1(f^{\tau}, \sigma_2^*) &= \sigma_2^*(b^o) u_1(f^{\tau}, b^o) + \sigma_2^*(b^*) u_1(f^{\tau}, b^*) \\ &= (1/\ell_t)(0) + (1 - 1/\ell_t) c \cdot b^* \\ &= (1 - 1/\ell_t) c \cdot b^*. \qquad (17) \end{aligned}$$

Similarly, player 1's expected payoff for (f^α, σ_2^*) is

$$
\begin{aligned}
u_1(f^\alpha, \sigma_2^*) &= \sigma_2^*(b^o)u_1(f^\alpha, b^o) + \sigma_2^*(b^*)u_1(f^\alpha, b^*) \\
&= (1/\ell_t)(\ell_t - 1)c \cdot b^* + (1 - 1/\ell_t)(0) \\
&= (1 - 1/\ell_t)c \cdot b^*. \qquad (18)
\end{aligned}
$$

Thus, $u_1(f^\tau, \sigma_2^*) = u_1(f^\alpha, \sigma_2^*)$. That is, by choosing a min-cut strategy σ_2^*, player 2 makes player 1 indifferent between f^τ and f^α. Because $\sigma_1^*(f^\tau) = 1 - 1/\ell_t$ and $\sigma_1^*(f^\alpha) = 1/\ell_t$, player 2's expected payoff for (σ_1^*, b^o) is calculated as

$$
\begin{aligned}
u_2(\sigma_1^*, b^o) &= \sigma_1^*(f^\tau)u_2(f^\tau, b^o) + \sigma_1^*(f^\alpha)u_2(f^\alpha, b^o) \\
&= (1 - 1/\ell_t)w + (1/\ell_t)(w - (\ell_t)c \cdot b^*) \\
&= w - c \cdot b^*. \qquad (19)
\end{aligned}
$$

For (σ_1^*, b^*), player 2's expected payoff is

$$u_2(\sigma_1^*, b^*) = w - c \cdot b^*. \qquad (20)$$

Thus, $u_2(\sigma_1^*, b^o) = u_2(\sigma_1^*, b^*)$. That is, by choosing a max-flow strategy σ_1^*, player 1 makes player 2 indifferent between b^o and b^*.

1╲2	b^o	b^*
f^τ	$0, w$	$c \cdot b^*, w - c \cdot b^*$
f^α	$(\ell_t - 1)c \cdot b^*, w - (\ell_t)c \cdot b^*$	$0, w - c \cdot b^*$

Figure 4 Payoffs for strategy profiles (f^τ, b^o), (f^τ, b^*), (f^α, b^o), and (f^α, b^*).

In any max-flow min-cut Nash equilibrium (σ_1^*, σ_2^*) of the game, player 1 gains a payoff of $(1 - 1/\ell_t)c \cdot b^*$ and player 2 gains $w - c \cdot b^*$. That is, $u_1(\sigma_1^*, \sigma_2^*) = (1 - 1/\ell_t)c \cdot b^*$ and $u_2(\sigma_1^*, \sigma_2^*) = w - c \cdot b^*$. Thus, any two max-flow min-cut Nash equilibria are payoff equivalent.

In any max-flow min-cut Nash equilibrium (σ_1^*, σ_2^*) of the network interdiction game, player 1, the adversary, chooses a trivial flow f^τ with probability $\sigma_1^*(f^\tau) = 1 - 1/\ell_t$ and chooses an acyclic maximum flow f^α with probability $\sigma_1^*(f^\alpha) = 1/\ell_t$. That is, player 1 carries no bads from the base to the target with probability $1 - 1/\ell_t$ and carries the maximum possible amount of bads through the network with probability $1/\ell_t$. In this equilibrium, player 2, the agency, chooses the zero blockage b^o with probability $\sigma_2^*(b^o) = 1/\ell_t$ and chooses a minimum cut blockage b^* with probability $\sigma_2^*(b^*) = 1 - 1/\ell_t$. That is, player 2 blocks no arcs with probability $1/\ell_t$ and blocks all the arcs necessary to disconnect the target from the base with probability $1 - 1/\ell_t$.

In any max-flow min-cut Nash equilibrium, we can calculate the probability of the adversary damaging the target. Note that in this equilibrium, player 1 damages the target if and only if player 1 carries the maximum amount of bads and player 2 blocks no arcs. This joint event takes place with probability $(1/\ell_t)(1/\ell_t) = \ell_t^{-2}$. Thus, in any max-flow min-cut Nash equilibrium, the probability of the adversary damaging the target is ℓ_t^{-2}.

We provide an example of max-flow min-cut Nash equilibria.

Example 4 *Consider network G in Example 1. Note that the acyclic maximum flow is $f^\alpha = (2, 1, 2, 3, 0)'$ and that the minimum cut blockage is $b^* = (0, 1, 1, 0, 0)'$. The capacity of the minimum cut blockage is $c \cdot b^* = 3$. Suppose that $\ell_t = 4$ and $q = 5$. In a max-flow min-cut Nash equilibrium (σ_1^*, σ_2^*), player 1 chooses the zero flow f^o with probability $\sigma_1^*(f^o) = 3/4$ and the acyclic maximum flow f^α with probability $\sigma_1^*(f^\alpha) = 1/4$. Player 2 chooses the zero blockage b^o with probability $\sigma_2^*(b^o) = 1/4$ and the minimum cut blockage b^* with probability $\sigma_2^*(b^*) = 3/4$. Thus, the equilibrium probability of player 1 damaging the target is 1/16. See Figure 5.* □

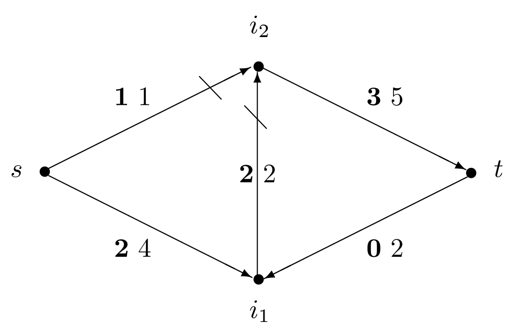

Figure 5 The bold numbers indicate the acyclic maximum flow; the line segments indicate the minimum cut blockage.

A mixed strategy $\sigma_1^\lambda \in \Delta(\mathcal{F})$ is a λ-*scaled max-flow strategy*, or simply a λ-*flow strategy*, for player 1 if for some $\lambda \in [1/\ell_t, 1]$, $\sigma_1^\lambda(f^\tau) = 1 - 1/\lambda\ell_t$ and $\sigma_1^\lambda(\lambda f^\alpha) = 1/\lambda\ell_t$. Note that any λ-flow strategy with $\lambda = 1$ is a max-flow strategy. We call $(\sigma_1^\lambda, \sigma_2^*)$ a λ-*flow min-cut strategy profile*. We present λ-flow min-cut Nash equilibria of the network interdiction game.

Proposition 4 *If $\ell_t > 1$ and $c \cdot b^* < q$, any λ-flow min-cut strategy profile $(\sigma_1^\lambda, \sigma_2^*)$ is a Nash equilibrium of network interdiction game Γ.*

The proof of Proposition 4 is presented in Appendix B.

In any λ-flow min-cut Nash equilibrium $(\sigma_1^\lambda, \sigma_2^*)$ of the network interdiction game, player 1 gains a payoff of $(1-1/\ell_t)c \cdot b^*$ and player 2 gains $w - c \cdot b^*$. That is, $u_1(\sigma_1^\lambda, \sigma_2^*) = (1-1/\ell_t)c \cdot b^*$ and $u_2(\sigma_1^\lambda, \sigma_2^*) = w - c \cdot b^*$. Thus, any two λ-flow min-cut Nash equilibria are payoff equivalent.

In any λ-flow min-cut Nash equilibrium $(\sigma_1^\lambda, \sigma_2^*)$ of the game, player 1, the adversary, chooses a trivial flow f^τ with probability $\sigma_1^\lambda(f^\tau) = 1 - 1/\lambda\ell_t$ and chooses a flow λf^α with probability $\sigma_1^\lambda(\lambda f^\alpha) = 1/\lambda\ell_t$. Player 2, the agency, chooses the zero blockage b^o with probability $\sigma_2^*(b^o) = 1/\ell_t$ and chooses a minimum cut blockage b^* with probability $\sigma_2^*(b^*) = 1 - 1/\ell_t$. Note that in this equilibrium, player 1 damages the target if and only if player 1 chooses λf^α and player 2 chooses b^o. This joint event takes place with probability $(1/\lambda\ell_t)(1/\ell_t) = (1/\lambda)\ell_t^{-2}$. Thus, in any λ-flow min-cut Nash equilibrium, the probability of the adversary damaging the target is $(1/\lambda)\ell_t^{-2}$.

In any λ-flow min-cut Nash equilibrium, if the adversary damages the target, the target damage amounts to $\ell \cdot (\lambda f^\alpha)^{b^o} = (\lambda\ell_t)c \cdot b^*$. Let $TD_\lambda := (\lambda\ell_t)c \cdot b^*$ denote the target damage in a λ-flow min-cut Nash equilibrium. Let $p_\lambda := (1/\lambda)\ell_t^{-2}$ denote the probability of the adversary damaging the target in a λ-flow min-cut Nash equilibrium. Because $p_\lambda = (1/\lambda)\ell_t^{-2}$ and $\ell_t = (1/\lambda)(1/(c \cdot b^*))TD_\lambda$, we have

$$p_\lambda = \lambda(c \cdot b^*)^2(TD_\lambda)^{-2} \tag{21}$$

where $\lambda \in (1/\ell_t, 1]$. Thus, in any λ-flow min-cut Nash equilibrium with $\lambda \in (1/\ell_t, 1]$, the probability of the adversary damaging the target is a negative power function of the target damage. In addition, if $\lambda = (\ell_t)^{-\theta}$ for some $\theta \in [0, 1)$, equality (21) can be rewritten as

$$p_\lambda = (c \cdot b^*)^{\frac{\theta-2}{\theta-1}}(TD_\lambda)^{-\frac{\theta-2}{\theta-1}} \tag{22}$$

because $p_\lambda = (\ell_t)^{\theta-2}$ and $\ell_t = (c \cdot b^*)^{\frac{1}{\theta-1}}(TD_\lambda)^{-\frac{1}{\theta-1}}$. However, if $\lambda = 1/\ell_t$, the equilibrium probability is independent of the target damage, because $p_\lambda = \ell_t^{-1}$ and $TD_\lambda = c \cdot b^*$.

We finally study the subcase when $\ell_t > 1$ and $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$. Recall that f^β is a binding flow. A mixed strategy $\sigma_1^\beta \in \Delta(\mathcal{F})$ is a *β-flow strategy* for player 1 if $\sigma_1^\beta(f^\tau) = 1 - (1/\ell_t)((c \cdot b^*)/q)$ and $\sigma_1^\beta(f^\beta) = (1/\ell_t)((c \cdot b^*)/q)$. We call $(\sigma_1^\beta, \sigma_2^*)$ a *β-flow min-cut strategy profile.* We provide β-flow min-cut Nash equilibria of the network interdiction game.

Proposition 5 *If $\ell_t > 1$ and $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$, any β-flow min-cut strategy profile $(\sigma_1^\beta, \sigma_2^*)$ is a Nash equilibrium of network interdiction game Γ.*

The proof of Proposition 5 is presented in Appendix B.

Here we show that given a min-cut strategy σ_2^*, player 1 is indifferent between f^τ and f^β, and that given a β-flow strategy σ_1^β, player 2 is indifferent between b^o and b^*. In any β-flow min-cut Nash equilibrium, strategy profiles (f^τ, b^o), (f^τ, b^*), (f^β, b^o), and (f^β, b^*) are assigned positive probability. Figure 6 illustrates the players' payoffs for these strategy profiles. The calculation of these payoffs can be found in the proof of Proposition 5. Because $\sigma_2^*(b^o) = 1/\ell_t$ and $\sigma_2^*(b^*) = 1 - 1/\ell_t$, player 1's expected payoff for (f^τ, σ_2^*) is calculated as

$$\begin{aligned} u_1(f^\tau, \sigma_2^*) &= \sigma_2^*(b^o)u_1(f^\tau, b^o) + \sigma_2^*(b^*)u_1(f^\tau, b^*) \\ &= (1/\ell_t)(0) + (1 - 1/\ell_t)c \cdot b^* \\ &= (1 - 1/\ell_t)c \cdot b^*. \end{aligned} \tag{23}$$

Similarly, player 1's expected payoff for (f^β, σ_2^*) is

$$\begin{aligned} u_1(f^\beta, \sigma_2^*) &= \sigma_2^*(b^o)u_1(f^\beta, b^o) + \sigma_2^*(b^*)u_1(f^\beta, b^*) \\ &= (1/\ell_t)(\ell_t - 1)q + (1 - 1/\ell_t)(c \cdot b^* - q) \\ &= (1 - 1/\ell_t)c \cdot b^*. \end{aligned} \tag{24}$$

Thus, $u_1(f^\tau, \sigma_2^*) = u_1(f^\beta, \sigma_2^*)$. That is, by choosing a min-cut strategy σ_2^*, player 2 makes player 1 indifferent between f^τ and f^β. Because $\sigma_1^\beta(f^\tau) = 1-(1/\ell_t)((c\cdot b^*)/q)$ and $\sigma_1^\beta(f^\beta) = (1/\ell_t)((c\cdot b^*)/q)$, player 2's expected payoff for (σ_1^β, b^o) is calculated as

$$\begin{aligned} u_2(\sigma_1^\beta, b^o) &= \sigma_1^\beta(f^\tau)u_2(f^\tau, b^o) + \sigma_1^\beta(f^\beta)u_2(f^\beta, b^o) \\ &= (1 - (1/\ell_t)((c \cdot b^*)/q))w + (1/\ell_t)((c \cdot b^*)/q)(w - (\ell_t)q) \\ &= w - c \cdot b^*. \end{aligned} \tag{25}$$

For (σ_1^β, b^*), player 2's expected payoff is

$$u_2(\sigma_1^\beta, b^*) = w - c \cdot b^*. \tag{26}$$

Thus, $u_2(\sigma_1^\beta, b^o) = u_2(\sigma_1^\beta, b^*)$. That is, by choosing a β-flow strategy σ_1^β, player 1 makes player 2 indifferent between b^o and b^*.

$1\diagdown 2$	b^o	b^*
f^τ	$0, w$	$c \cdot b^*, w - c \cdot b^*$
f^β	$(\ell_t - 1)q, w - (\ell_t)q$	$c \cdot b^* - q, w - c \cdot b^*$

Figure 6 Payoffs for strategy profiles (f^τ, b^o), (f^τ, b^*), (f^β, b^o), and (f^β, b^*).

In any β-flow min-cut Nash equilibrium $(\sigma_1^\beta, \sigma_2^*)$ of the game, player 1 gains a payoff of $(1 - 1/\ell_t)c \cdot b^*$ and player 2 gains $w - c \cdot b^*$. That is, $u_1(\sigma_1^\beta, \sigma_2^*) = (1 - 1/\ell_t)c \cdot b^*$ and $u_2(\sigma_1^\beta, \sigma_2^*) = w - c \cdot b^*$. Thus, any two β-flow min-cut Nash equilibria are payoff equivalent.

In any β-flow min-cut Nash equilibrium $(\sigma_1^\beta, \sigma_2^*)$ of the network interdiction game, player 1, the adversary, chooses a trivial flow f^τ with probability $\sigma_1^\beta(f^\tau) = 1 - (1/\ell_t)((c \cdot b^*)/q)$ and chooses a binding flow f^β with probability $\sigma_1^\beta(f^\beta) = (1/\ell_t)((c \cdot b^*)/q)$. That is, player 1 carries no bads from the base to the target with probability $1 - (1/\ell_t)((c \cdot b^*)/q)$ and carries the bound quantity of bads with probability $(1/\ell_t)((c \cdot b^*)/q)$. In this equilibrium, player 2, the agency, chooses the zero blockage b^o with probability $\sigma_2^*(b^o) = 1/\ell_t$ and chooses a minimum cut blockage b^* with probability $\sigma_2^*(b^*) = 1 - 1/\ell_t$. That is, player 2 blocks no arcs with probability $1/\ell_t$ and blocks all the arcs necessary to disconnect the target from the base with probability $1 - 1/\ell_t$.

In any β-flow min-cut Nash equilibrium, we can calculate the probability of the adversary damaging the target. Note that in this equilibrium, player 1 damages the target if and only if player 1 carries the bound quantity of bads and player 2 blocks no arcs. This joint event takes place with probability $((c \cdot b^*)/q)\ell_t^{-2}$. Thus, in any β-flow min-cut Nash equilibrium, the probability of the adversary damaging the target is $((c \cdot b^*)/q)\ell_t^{-2}$.

We provide an example of β-flow min-cut Nash equilibria.

Example 5 *Consider network G in Example 1. Note that the acyclic maximum flow is $f^\alpha = (2, 1, 2, 3, 0)'$ and that the minimum cut blockage is $b^* = (0, 1, 1, 0, 0)'$. The capacity of the minimum cut blockage is $c \cdot b^* = 3$. Suppose that $\ell_t = 4$ and $q = 3/2$. Then the binding flow is $f^\beta = (1, 1/2, 1, 3/2, 0)'$. In a β-flow min-cut Nash equilibrium $(\sigma_1^\beta, \sigma_2^*)$, player 1 chooses the zero flow f^o with probability $\sigma_1^\beta(f^o) = 1/2$ and the binding flow f^β with probability $\sigma_1^\beta(f^\beta) = 1/2$. Player 2 chooses the zero blockage b^o with probability $\sigma_2^*(b^o) = 1/4$ and the minimum cut blockage b^* with probability $\sigma_2^*(b^*) = 3/4$. Thus, the equilibrium probability of player 1 damaging the target is $1/8$. See Figure 7.* $\square$

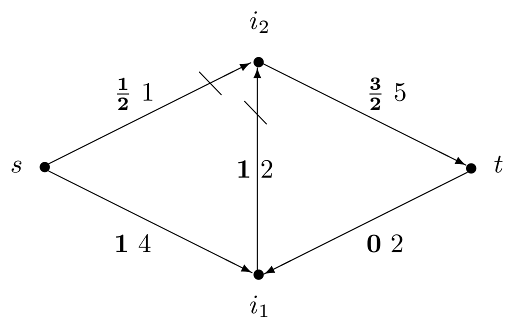

Figure 7 The bold numbers indicate the binding flow; the line segments indicate the minimum cut blockage.

A mixed strategy $\sigma_1^\mu \in \Delta(\mathcal{F})$ is a *μ-flow strategy* for player 1 if for some $\mu \in [(1/\ell_t)((c \cdot b^*)/q), 1]$, $\sigma_1^\mu(f^\tau) = 1 - (1/\mu\ell_t)((c \cdot b^*)/q)$ and $\sigma_1^\mu(\mu f^\beta) = (1/\mu\ell_t)((c \cdot b^*)/q)$. Note that any μ-flow strategy with $\mu = 1$ is a β-flow strategy. We call $(\sigma_1^\mu, \sigma_2^*)$ a *μ-flow min-cut strategy profile.* We present μ-flow min-cut Nash equilibria of the network interdiction game.

Proposition 6 *If $\ell_t > 1$ and $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$, any μ-flow min-cut strategy profile $(\sigma_1^\mu, \sigma_2^*)$ is a Nash equilibrium of network interdiction game Γ.*

The proof of Proposition 6 is presented in Appendix B.

In any μ-flow min-cut Nash equilibrium $(\sigma_1^\mu, \sigma_2^*)$ of the network interdiction game, player 1 gains a payoff of $(1 - 1/\ell_t)c \cdot b^*$ and player 2 gains $w - c \cdot b^*$. That is, $u_1(\sigma_1^\mu, \sigma_2^*) = (1 - 1/\ell_t)c \cdot b^*$ and $u_2(\sigma_1^\mu, \sigma_2^*) = w - c \cdot b^*$. Thus, any two μ-flow min-cut Nash equilibria are payoff equivalent.

In any μ-flow min-cut Nash equilibrium $(\sigma_1^\mu, \sigma_2^*)$ of the game, player 1, the adversary, chooses a trivial flow f^τ with probability $\sigma_1^\mu(f^\tau) = 1 - (1/\mu\ell_t)((c \cdot b^*)/q)$ and chooses a flow μf^β with probability $\sigma_1^\mu(\mu f^\beta) =$

$(1/\mu\ell_t)((c \cdot b^*)/q)$. Player 2, the agency, chooses the zero blockage b^o with probability $\sigma_2^*(b^o) = 1/\ell_t$ and chooses a minimum cut blockage b^* with probability $\sigma_2^*(b^*) = 1 - 1/\ell_t$. Note that in this equilibrium, player 1 damages the target if and only if player 1 chooses μf^β and player 2 chooses b^o. This joint event takes place with probability $(1/\mu)((c \cdot b^*)/q)\ell_t^{-2}$. Thus, in any μ-flow min-cut Nash equilibrium, the probability of the adversary damaging the target is $(1/\mu)((c \cdot b^*)/q)\ell_t^{-2}$.

In any μ-flow min-cut Nash equilibrium, if the adversary damages the target, the target damage amounts to $\ell \cdot (\mu f^\beta)^{b^o} = (\mu\ell_t)q$. Let $TD_\mu := (\mu\ell_t)q$ denote the target damage in a μ-flow min-cut Nash equilibrium. Let $p_\mu := (1/\mu)((c \cdot b^*)/q)\ell_t^{-2}$ denote the probability of the adversary damaging the target in a μ-flow min-cut Nash equilibrium. Because $p_\mu = (1/\mu)((c \cdot b^*)/q)\ell_t^{-2}$ and $\ell_t = (1/\mu)(1/q)TD_\mu$, we have

$$p_\mu = \mu q (c \cdot b^*)(TD_\mu)^{-2} \tag{27}$$

where $\mu \in ((1/\ell_t)((c \cdot b^*)/q), 1]$. Thus, in any μ-flow min-cut Nash equilibrium with $\mu \in ((1/\ell_t)((c \cdot b^*)/q), 1]$, the probability of the adversary damaging the target is a negative power function of the target damage. In addition, if $\mu = (q/(c \cdot b^*))^{-\theta}(\ell_t)^{-\theta}$ for some $\theta \in [0, 1)$, equality (27) can be rewritten as

$$p_\mu = (q/(c \cdot b^*))^{\frac{1}{\theta-1}}(q)^{\frac{\theta-2}{\theta-1}}(TD_\mu)^{-\frac{\theta-2}{\theta-1}} \tag{28}$$

because $p_\mu = (q/(c \cdot b^*))^{\theta-1}(\ell_t)^{\theta-2}$ and $\ell_t = (q/(c \cdot b^*))^{-\frac{\theta}{\theta-1}}(q)^{\frac{1}{\theta-1}}(TD_\mu)^{-\frac{1}{\theta-1}}$. However, if $\mu = (1/\ell_t)((c \cdot b^*)/q)$, the equilibrium probability is independent of the target damage, because $p_\mu = \ell_t^{-1}$ and $TD_\mu = c \cdot b^*$.

In summary, the Nash equilibria of the network interdiction game depend on the marginal target loss, the capacity of a minimum cut blockage, and the bound quantity. When the marginal target loss is less than or equal to one, we find trivial-flow zero-blockage Nash equilibria. When the marginal target loss is greater than one and the bound quantity is small, we find binding-flow zero-blockage Nash equilibria. When the marginal target loss is greater than one and the bound quantity is large, we find λ-flow min-cut Nash equilibria. Thus, we can view max-flow min-cut Nash equilibria as a special case of λ-flow min-cut Nash equilibria. When the marginal target loss is greater than one and the bound quantity is intermediate, we find μ-flow min-cut Nash equilibria. We can then view β-flow min-cut Nash equilibria as a special case of μ-flow min-cut Nash equilibria.

When the marginal target loss is greater than one and the bound quantity is either intermediate or large, the equilibrium probabilities of the adversary damaging the target can be expressed as negative power functions of the target damage. In the following section we discuss our theoretical finding together with empirical evidence.

4 Discussion

We first relate our results to some empirical studies of terrorist events and then discuss related research in progress and further directions.

4.1 Fatality Distribution of Terrorist Events

Let z denote the number of fatalities in a terrorist event and let $p(z)$ denote the frequency of a terrorist event in which the number of fatalities is z. The fatality distribution of terrorist events follows a *power law* if for each $z \geq z_{\min}$,

$$p(z) \propto z^{-\gamma}$$

where $z_{\min}$ and γ are the parameters of the distribution. The estimates of the parameters are derived from data and denoted by $\hat{z}_{\min}$ and $\hat{\gamma}$.

Recent empirical studies show that the fatality distribution of terrorist events follows a power law. Clauset et al. [6] use the database of National Memorial Institute for the Prevention of Terrorism (MIPT) and conclude that the fatality distribution follows a power law. The estimate of the scaling parameter is $\hat{\gamma} = 2.38$. Bohorquez et al. [5] construct a data set on insurgent wars and conclude that for each insurgent war the fatality distribution follows a power law. The estimates of the scaling parameter are clustered around 2.5.

Recall that in any λ-flow min-cut Nash equilibrium with $\lambda = (\ell_t)^{-\theta}$ for some $\theta \in [0, 1)$, the probability of the adversary damaging the target, p_λ, is a negative power function of the target damage TD_λ. Precisely, from equality (22), we have

$$p_\lambda = (c \cdot b^*)^{\frac{\theta-2}{\theta-1}} (TD_\lambda)^{-\frac{\theta-2}{\theta-1}},$$

which can be rewritten as

$$p_\lambda(TD_\lambda) \propto (TD_\lambda)^{-\frac{\theta-2}{\theta-1}}.$$

To link this theoretical finding and empirical evidence we make two additional assumptions. Suppose that the target damage is measured by the number of fatalities and that the probability of the adversary damaging the target is proportional to the frequency of a terrorist event.

Now suppose that the estimate of the scaling parameter, $\hat{\gamma} \geq 2$, is derived from data. By setting $\hat{\gamma} = \frac{\hat{\theta}-2}{\hat{\theta}-1}$ and solving for $\hat{\theta}$, we have $\hat{\theta} = \frac{\hat{\gamma}-2}{\hat{\gamma}-1}$. Notice that $\hat{\theta} \in [0, 1)$. Therefore, in the λ-flow min-cut Nash equilibrium with $\lambda = (\ell_t)^{-\hat{\theta}}$, the fatality distribution is predicted to be

$$p_\lambda(TD_\lambda) \propto (TD_\lambda)^{-\hat{\gamma}}$$

and is consistent with data. Similarly, in the μ-flow min-cut Nash equilibrium with $\mu = (q/(c \cdot b^*))^{-\hat{\theta}} (\ell_t)^{-\hat{\theta}}$, the predicted fatality distribution, $p_\mu(TD_\mu) \propto (TD_\mu)^{-\hat{\gamma}}$, is consistent with data.

4.2 Further Research

This paper presents a strategic model of network interdiction in which two players, an adversary and an agency, have complete information and simultaneously choose their strategies. Building on this research a further approach is to assume that the players have incomplete information about each other's strategies and payoffs and about the sources and targets; the agency may not know the adversary's type – its feasible strategies, its payoffs, the sources of the bads and also their targets. The extension to incomplete information is, in our view, of clear importance. (It may also be interesting to have more than two players.) An alternative approach is to assume that the players sequentially choose their strategies. The agency may observe the adversary's choice of strategy and choose her own strategy conditional on this observation or, alternatively, the agency may move first in setting up a security system. Both these approaches are subjects of our current and future planned research.

Appendix A

In this appendix we provide the flow decomposition algorithm.[7] A network is given as $G = (N, A, s, t, c)$. For each $f \in \mathcal{F}$, we can find a vector $x = (x_h)'_{h \in H}$ such that $f = Mx$ by using the flow decomposition algorithm. Initially we are given a flow f and the zero vector x. At each step of the algorithm, we construct a sequence of distinct nodes, and obtain either an $s - t$ path or a cycle. We then update vector x and flow f. This algorithm terminates when the updated flow is the zero flow.

Algorithm 1 *Flow Decomposition*

Let $f = (f_{ij})'_{(i,j) \in A} \in \mathcal{F}$ be given. We set $x = (x_h)'_{h \in H} = \mathbf{0}$, the vector of zeros.

At Step $k = 1, 2, \ldots,$ if f is the zero flow, this algorithm terminates and yields vector x. If f is not the zero flow, there is an arc $(i, j) \in A$ with $f_{ij} > 0$.

(i) We start from base s. If there is $(i_1, i_2) \in A$ with $i_1 = s$ and $f_{i_1 i_2} > 0$, beginning with the two nodes i_1, i_2 we begin the construction of a sequence of distinct nodes with the two nodes i_1 and i_2. If there is $(i_2, i_3) \in A$ with $f_{i_2 i_3} > 0$, we add node i_3 to the sequence. Repeat this until we add target t or a previously added node to the sequence. In the former case, an $s - t$ path is obtained and, in the latter case, a cycle is obtained. We denote the outcome by $h \in H$. We replace $x_h = 0$ with the minimum flow of the arcs included in h. We then replace f_{ij} with $f_{ij} - x_h$ if h includes (i, j). We proceed to the next step.

(ii) If there is no $(i_1, i_2) \in A$ with $i_1 = s$ and $f_{i_1 i_2} > 0$, we find another arc (i, j) with $f_{ij} > 0$. We start from node i. By applying the argument in (i), we obtain a cycle and update vector x and flow f. We proceed to the next step. □

Appendix B

A network interdiction game is given as $\Gamma = (G, q, \Delta(\mathcal{F}), \Delta(\mathcal{B}), u_1, u_2)$. We start from the following lemmas.

Lemma 1 *For each $(f, b) \in \mathcal{F} \times \mathcal{B}$, $e \cdot f^b = (1/\ell_t)\ell \cdot f^b$.*

[7] See for reference Ahuja et al. [1].

Proof. Since f^b is the net flow to the target under flow f and blockage b, only $s-t$ paths have positive flow. Thus, $\sum_{(s,i)\in FS(s)} f^b_{si} = \sum_{(i,t)\in RS(t)} f^b_{it}$. Because $e \cdot f^b = \sum_{(s,i)\in FS(s)} f^b_{si}$ and $\ell \cdot f^b = \ell_t \sum_{(i,t)\in RS(t)} f^b_{it}$, we have $e \cdot f^b = (1/\ell_t)\ell \cdot f^b$. $\square$

Lemma 2 *For each $(f,b) \in \mathcal{F} \times \mathcal{B}$, $e \cdot f^b = v \cdot f^b$.*

Proof. Since f^b is the net flow to the target under flow f and blockage b, only $s-t$ paths have positive flow. This implies that $\sum_{(s,i)\in FS(s)} f^b_{si} = \sum_{(i,t)\in RS(t)} f^b_{it}$ and that $v \cdot f^b = \sum_{(i,t)\in RS(t)} f^b_{it}$. Because $e \cdot f^b = \sum_{(s,i)\in FS(s)} f^b_{si}$, we have $e \cdot f^b = v \cdot f^b$. $\square$

Lemma 3 *For each $(f,b) \in \mathcal{F} \times \mathcal{B}$, $(1/\ell_t)\ell \cdot f^{b^o} \leq c \cdot b + (1/\ell_t)\ell \cdot f^b$.*

Proof. Because f^{b^o} is the net flow to the target under flow f and the zero blockage b^o, we have $f^{b^o}_{ij} \leq c_{ij}$ for each $(i,j) \in A$. Thus, blocking arc (i,j) decreases the value of the net flow by at most c_{ij}. Hence, for each $b \in \mathcal{B}$, $v \cdot f^{b^o} - v \cdot f^b \leq \sum_{(i,j)\in A_b} c_{ij}$. Because $\sum_{(i,j)\in A_b} c_{ij} = c \cdot b$, we have $v \cdot f^{b^o} \leq c \cdot b + v \cdot f^b$. Therefore, from Lemmas 1 and 2, $(1/\ell_t)\ell \cdot f^{b^o} \leq c \cdot b + (1/\ell_t)\ell \cdot f^b$. $\square$

From Lemmas 1, 2, and 3, we deduce the following.

Lemma 4 *If f^α is an acyclic maximum flow with large bound quantity q in network G, then $e \cdot f^\alpha = (1/\ell_t)\ell \cdot f^\alpha$. Furthermore, if f^β is a binding flow, it holds that $e \cdot f^\beta = (1/\ell_t)\ell \cdot f^\beta$.*

Proof. Note that $e \cdot (f^\alpha)^{b^o} = (1/\ell_t)\ell \cdot (f^\alpha)^{b^o}$ from Lemma 1. Because f^α is acyclic, $(f^\alpha)^{b^o} = f^\alpha$. Thus, $e \cdot f^\alpha = (1/\ell_t)\ell \cdot f^\alpha$. Now multiplying both sides by $q/(c \cdot b^*)$, we have $(q/(c \cdot b^*))e \cdot f^\alpha = (1/\ell_t)(q/(c \cdot b^*))\ell \cdot f^\alpha$. Because f^β is a binding flow, $f^\beta = (q/(c \cdot b^*))f^\alpha$. Because $e \cdot f^\beta = (q/(c \cdot b^*))e \cdot f^\alpha$ and $\ell \cdot f^\beta = (q/(c \cdot b^*))\ell \cdot f^\alpha$, we have $e \cdot f^\beta = (1/\ell_t)\ell \cdot f^\beta$. $\square$

Lemma 5 *If f^α is an acyclic maximum flow with large bound quantity q in network G, then $e \cdot f^\alpha = v \cdot f^\alpha$. Furthermore, if f^β is a binding flow, it holds that $e \cdot f^\beta = v \cdot f^\beta$.*

Proof. Note that $e \cdot (f^\alpha)^{b^o} = v \cdot (f^\alpha)^{b^o}$ from Lemma 2. Because f^α is acyclic, $(f^\alpha)^{b^o} = f^\alpha$. Thus, $e \cdot f^\alpha = v \cdot f^\alpha$. Now multiplying both sides by $q/(c \cdot b^*)$, we have $(q/(c \cdot b^*))e \cdot f^\alpha = (q/(c \cdot b^*))v \cdot f^\alpha$. Because f^β is a binding flow, $f^\beta = (q/(c \cdot b^*))f^\alpha$. Because $e \cdot f^\beta = (q/(c \cdot b^*))e \cdot f^\alpha$ and $v \cdot f^\beta = (q/(c \cdot b^*))v \cdot f^\alpha$, we have $e \cdot f^\beta = v \cdot f^\beta$. $\square$

Lemma 6 *If f^α is an acyclic maximum flow with large bound quantity q in network G, for each $b \in \mathcal{B}$, $(1/\ell_t)\ell \cdot f^\alpha \leq c \cdot b + (1/\ell_t)\ell \cdot (f^\alpha)^b$. Furthermore, if $q \leq (1/\ell_t)c \cdot b^*$ and f^β is a binding flow, for each $b \in \mathcal{B}$, $\ell \cdot f^\beta \leq c \cdot b + \ell \cdot (f^\beta)^b$.*

Proof. Note that for each $b \in \mathcal{B}$, $(1/\ell_t)\ell \cdot (f^\alpha)^{b^o} \leq c \cdot b + (1/\ell_t)\ell \cdot (f^\alpha)^b$ from Lemma 3. Since f^α is acyclic, $(f^\alpha)^{b^o} = f^\alpha$. Thus, for each $b \in \mathcal{B}$, we have $(1/\ell_t)\ell \cdot f^\alpha \leq c \cdot b + (1/\ell_t)\ell \cdot (f^\alpha)^b$. Now multiplying both sides by $(\ell_t)(q/(c \cdot b^*))$, we have $(q/(c \cdot b^*))\ell \cdot f^\alpha \leq (\ell_t)(q/(c \cdot b^*))c \cdot b + (q/(c \cdot b^*))\ell \cdot (f^\alpha)^b$. This implies that $(q/(c \cdot b^*))\ell \cdot f^\alpha \leq c \cdot b + (q/(c \cdot b^*))\ell \cdot (f^\alpha)^b$ because $q \leq (1/\ell_t)c \cdot b^*$. Since f^β is a binding flow, $f^\beta = (q/(c \cdot b^*))f^\alpha$. Because $\ell \cdot f^\beta = (q/(c \cdot b^*))\ell \cdot f^\alpha$ and $\ell \cdot (f^\beta)^b = (q/(c \cdot b^*))\ell \cdot (f^\alpha)^b$, for each $b \in \mathcal{B}$, we have $\ell \cdot f^\beta \leq c \cdot b + \ell \cdot (f^\beta)^b$. □

We now present the proofs of the propositions.

Proof of Proposition 1. Suppose that $\ell_t \leq 1$. We show that no player can be (strictly) better off by unilaterally deviating from (f^τ, b^o). Note that $u_1(f^\tau, b^o) = 0$ and $u_2(f^\tau, b^o) = w$. Since $c \cdot b^o = 0$, $f \geq f^{b^o}$, $e \cdot f^{b^o} = (1/\ell_t)\ell \cdot f^{b^o}$ from Lemma 1, and $\ell_t \leq 1$, if player 1 chooses any flow f, his payoff is calculated as

$$\begin{aligned} u_1(f, b^o) &= \ell \cdot f^{b^o} + c \cdot b^o - e \cdot f \\ &\leq \ell \cdot f^{b^o} - e \cdot f^{b^o} \\ &= \ell \cdot f^{b^o} - (1/\ell_t)\ell \cdot f^{b^o} \\ &\leq 0. \end{aligned}$$

Thus, player 1 cannot be better off. If player 2 chooses any blockage b, her payoff is calculated as

$$\begin{aligned} u_2(f^\tau, b) &= w - \ell \cdot (f^\tau)^b - c \cdot b \\ &= w - c \cdot b \\ &\leq w, \end{aligned}$$

since $(f^\tau)^b = f^o$ and $c \cdot b \geq 0$. Thus, player 2 cannot be better off. Therefore, (f^τ, b^o) is a Nash equilibrium of Γ. □

Proof of Proposition 2. Suppose that $\ell_t > 1$ and $q \leq (1/\ell_t)c \cdot b^*$. We show that no player can be (strictly) better off by unilaterally deviating from (f^β, b^o). Calculate the players' payoffs for strategy profile (f^β, b^o). Since $(f^\beta)^{b^o} = f^\beta$, $\ell \cdot f^\beta = (\ell_t)e \cdot f^\beta$ from Lemma 4, $e \cdot f^\beta = v \cdot f^\beta$ from Lemma 5,

$$\begin{aligned} u_1(f^\beta, b^o) &= \ell \cdot (f^\beta)^{b^o} + c \cdot b^o - e \cdot f^\beta \\ &= \ell \cdot f^\beta - e \cdot f^\beta \\ &= (\ell_t - 1)e \cdot f^\beta \\ &= (\ell_t - 1)v \cdot f^\beta \\ &= (\ell_t - 1)q. \end{aligned}$$

The last equality comes from equality (16). Similarly,

$$\begin{aligned} u_2(f^\beta, b^o) &= w - \ell \cdot (f^\beta)^{b^o} - c \cdot b^o \\ &= w - (\ell_t)q. \end{aligned}$$

If player 1 chooses any flow f, his payoff is calculated as

$$\begin{aligned} u_1(f, b^o) &= \ell \cdot f^{b^o} + c \cdot b^o - e \cdot f \\ &\leq \ell \cdot f^{b^o} - e \cdot f^{b^o} \\ &= (\ell_t - 1)e \cdot f^{b^o} \\ &= (\ell_t - 1)v \cdot f^{b^o} \\ &\leq (\ell_t - 1)q, \end{aligned}$$

because $f \geq f^{b^o}$, $\ell \cdot f^{b^o} = (\ell_t)e \cdot f^{b^o}$ from Lemma 1, $e \cdot f^{b^o} = v \cdot f^{b^o}$ from Lemma 2, and $v \cdot f^{b^o} \leq q$ from inequality (6). Thus, player 1 cannot be better off. If player 2 chooses any blockage b, her payoff is calculated as

$$\begin{aligned} u_2(f^\beta, b) &= w - \ell \cdot (f^\beta)^b - c \cdot b \\ &\leq w - \ell \cdot f^\beta \\ &= w - (\ell_t)e \cdot f^\beta \\ &= w - (\ell_t)v \cdot f^\beta \\ &= w - (\ell_t)q, \end{aligned}$$

because $\ell \cdot f^\beta \leq c \cdot b + \ell \cdot (f^\beta)^b$ from Lemma 6, $\ell \cdot f^\beta = (\ell_t)e \cdot f^\beta$ from Lemma 4, $e \cdot f^\beta = v \cdot f^\beta$ from Lemma 5. Note that the last equality comes from equality (16). Thus, player 2 cannot be better off. Therefore, (f^β, b^o) is a Nash equilibrium of Γ. $\square$

Proof of Proposition 3. Suppose that $\ell_t > 1$ and $c \cdot b^* < q$. To show that (σ_1^*, σ_2^*) is a Nash equilibrium of Γ, it suffices to show that (i) $u_1(f^\tau, \sigma_2^*) = u_1(f^\alpha, \sigma_2^*)$, (ii) $u_2(\sigma_1^*, b^o) = u_2(\sigma_1^*, b^*)$, (iii) for each $f \in \mathcal{F}$, $u_1(f^\tau, \sigma_2^*) \geq u_1(f, \sigma_2^*)$ and (iv) for each $b \in \mathcal{B}$, $u_2(\sigma_1^*, b^o) \geq u_2(\sigma_1^*, b)$.

(i) We show that $u_1(f^\tau, \sigma_2^*) = u_1(f^\alpha, \sigma_2^*)$. First, calculate player 1's payoffs for strategy profiles (f^τ, b^o), (f^τ, b^*), (f^α, b^o), and (f^α, b^*). Because $(f^\tau)^{b^o} = f^o$ and $e \cdot f^\tau = 0$,

$$\begin{aligned} u_1(f^\tau, b^o) &= \ell \cdot (f^\tau)^{b^o} + c \cdot b^o - e \cdot f^\tau \\ &= 0. \end{aligned}$$

Because $(f^\tau)^{b^*} = f^o$ and $e \cdot f^\tau = 0$,

$$\begin{aligned} u_1(f^\tau, b^*) &= \ell \cdot (f^\tau)^{b^*} + c \cdot b^* - e \cdot f^\tau \\ &= c \cdot b^*. \end{aligned}$$

Because $(f^\alpha)^{b^o} = f^\alpha$ and $\ell \cdot f^\alpha = (\ell_t)e \cdot f^\alpha$ from Lemma 4,

$$\begin{aligned} u_1(f^\alpha, b^o) &= \ell \cdot (f^\alpha)^{b^o} + c \cdot b^o - e \cdot f^\alpha \\ &= (\ell_t - 1)e \cdot f^\alpha \\ &= (\ell_t - 1)c \cdot b^*. \end{aligned}$$

The last equality comes from Lemma 5 and equality (15). Because $(f^\alpha)^{b^*} = f^o$ and $e \cdot f^\alpha = v \cdot f^\alpha$ from Lemma 5,

$$\begin{aligned} u_1(f^\alpha, b^*) &= \ell \cdot (f^\alpha)^{b^*} + c \cdot b^* - e \cdot f^\alpha \\ &= c \cdot b^* - v \cdot f^\alpha \\ &= 0. \end{aligned}$$

The last equality comes from equality (15). Then, as in (17), player 1's expected payoff for (f^τ, σ_2^*) is $u_1(f^\tau, \sigma_2^*) = (1 - 1/\ell_t)c \cdot b^*$. As in (18), player 1's expected payoff for (f^α, σ_2^*) is $u_1(f^\alpha, \sigma_2^*) = (1 - 1/\ell_t)c \cdot b^*$. Thus, $u_1(f^\tau, \sigma_2^*) = u_1(f^\alpha, \sigma_2^*)$.

(ii) We show that $u_2(\sigma_1^*, b^o) = u_2(\sigma_1^*, b^*)$. First, calculate player 2's payoffs for strategy profiles (f^τ, b^o), (f^α, b^o), (f^τ, b^*), and (f^α, b^*). Because $(f^\tau)^{b^o} = f^o$,

$$\begin{aligned} u_2(f^\tau, b^o) &= w - \ell \cdot (f^\tau)^{b^o} - c \cdot b^o \\ &= w. \end{aligned}$$

Because $(f^\alpha)^{b^o} = f^\alpha$ and $\ell \cdot f^\alpha = (\ell_t)e \cdot f^\alpha$ from Lemma 4,

$$\begin{aligned} u_2(f^\alpha, b^o) &= w - \ell \cdot (f^\alpha)^{b^o} - c \cdot b^o \\ &= w - (\ell_t)e \cdot f^\alpha \\ &= w - (\ell_t)c \cdot b^*. \end{aligned}$$

The last equality comes from Lemma 5 and equality (15). Because $(f^\tau)^{b^*} = f^o$,

$$\begin{aligned} u_2(f^\tau, b^*) &= w - \ell \cdot (f^\tau)^{b^*} - c \cdot b^* \\ &= w - c \cdot b^*. \end{aligned}$$

Because $(f^\alpha)^{b^*} = f^o$,

$$\begin{aligned} u_2(f^\alpha, b^*) &= w - \ell \cdot (f^\alpha)^{b^*} - c \cdot b^* \\ &= w - c \cdot b^*. \end{aligned}$$

Then, as in (19), player 2's expected payoff for (σ_1^*, b^o) is $u_2(\sigma_1^*, b^o) = w - c \cdot b^*$. As in (20), player 2's expected payoff for (σ_1^*, b^*) is $u_2(\sigma_1^*, b^*) = w - c \cdot b^*$. Thus, $u_2(\sigma_1^*, b^o) = u_2(\sigma_1^*, b^*)$.

(iii) We show that for each $f \in \mathcal{F}$, $u_1(f^\tau, \sigma_2^*) \geq u_1(f, \sigma_2^*)$. For each $f \in \mathcal{F}$, we calculate player 1's payoffs. Because $f \geq f^{b^o}$,

$$\begin{aligned} u_1(f, b^o) &= \ell \cdot f^{b^o} + c \cdot b^o - e \cdot f \\ &\leq \ell \cdot f^{b^o} - e \cdot f^{b^o}. \end{aligned}$$

Because $f^{b^*} = f^o$ and $f \geq f^{b^o}$,

$$\begin{aligned} u_1(f, b^*) &= \ell \cdot f^{b^*} + c \cdot b^* - e \cdot f \\ &\leq c \cdot b^* - e \cdot f^{b^o}. \end{aligned}$$

Then, for each $f \in \mathcal{F}$,

$$\begin{aligned} u_1(f, \sigma_2^*) &= \sigma_2^*(b^o) u_1(f, b^o) + \sigma_2^*(b^*) u_1(f, b^*) \\ &\leq (1/\ell_t)(\ell \cdot f^{b^o} - e \cdot f^{b^o}) + (1 - 1/\ell_t)(c \cdot b^* - e \cdot f^{b^o}) \\ &= (1/\ell_t)\ell \cdot f^{b^o} - e \cdot f^{b^o} + (1 - 1/\ell_t) c \cdot b^* \\ &= (1 - 1/\ell_t) c \cdot b^*, \end{aligned}$$

because $\sigma_2^*(b^o) = 1/\ell_t$ and $\sigma_2^*(b^*) = 1 - 1/\ell_t$. Note that the last equality comes from Lemma 1. From (i), we know that $u_1(f^\tau, \sigma_2^*) = (1 - 1/\ell_t) c \cdot b^*$. Thus, for each $f \in \mathcal{F}$, $u_1(f^\tau, \sigma_2^*) \geq u_1(f, \sigma_2^*)$.

(iv) We show that for each $b \in \mathcal{B}$, $u_2(\sigma_1^*, b^o) \geq u_2(\sigma_1^*, b)$. For each $b \in \mathcal{B}$, we calculate player 2's payoffs. Because $(f^\tau)^b = f^o$,

$$\begin{aligned} u_2(f^\tau, b) &= w - \ell \cdot (f^\tau)^b - c \cdot b \\ &= w - c \cdot b. \end{aligned}$$

Also, we have $u_2(f^\alpha, b) = w - \ell \cdot (f^\alpha)^b - c \cdot b$. Then, for each $b \in \mathcal{B}$,

$$\begin{aligned} u_2(\sigma_1^*, b) &= \sigma_1^*(f^\tau) u_2(f^\tau, b) + \sigma_1^*(f^\alpha) u_2(f^\alpha, b) \\ &= (1 - 1/\ell_t)(w - c \cdot b) + (1/\ell_t)(w - \ell \cdot (f^\alpha)^b - c \cdot b) \\ &= w - c \cdot b - (1/\ell_t)\ell \cdot (f^\alpha)^b \\ &\leq w - (1/\ell_t)\ell \cdot f^\alpha \end{aligned}$$

because $\sigma_1^*(f^\tau) = 1 - 1/\ell_t$ and $\sigma_1^*(f^\alpha) = 1/\ell_t$. Note that the last inequality comes from Lemma 6. Because $(1/\ell_t)\ell \cdot f^\alpha = e \cdot f^\alpha$ and $e \cdot f^\alpha = c \cdot b^*$, we have $u_2(\sigma_1^*, b) \leq w - c \cdot b^*$. From (ii), we know that $u_2(\sigma_1^*, b^o) = w - c \cdot b^*$. Thus, for each $b \in \mathcal{B}$, $u_2(\sigma_1^*, b^o) \geq u_2(\sigma_1^*, b)$. This proves that (σ_1^*, σ_2^*) is a Nash equilibrium of Γ. □

Proof of Proposition 4. Suppose that $\ell_t > 1$ and $c \cdot b^* < q$. To show that $(\sigma_1^\lambda, \sigma_2^*)$ is a Nash equilibrium of Γ, it suffices to show that (i) $u_1(f^\tau, \sigma_2^*) = u_1(\lambda f^\alpha, \sigma_2^*)$, (ii) $u_2(\sigma_1^\lambda, b^o) = u_2(\sigma_1^\lambda, b^*)$, (iii) for each $f \in \mathcal{F}$, $u_1(f^\tau, \sigma_2^*) \geq u_1(f, \sigma_2^*)$ and (iv) for each $b \in \mathcal{B}$, $u_2(\sigma_1^\lambda, b^o) \geq u_2(\sigma_1^\lambda, b)$.

(i) We show that $u_1(f^\tau, \sigma_2^*) = u_1(\lambda f^\alpha, \sigma_2^*)$. Calculate player 1's expected payoff for $(\lambda f^\alpha, \sigma_2^*)$. Because $(\lambda f^\alpha)^{b^o} = \lambda f^\alpha$ and $(\lambda)\ell \cdot f^\alpha = (\lambda \ell_t) e \cdot f^\alpha$ from Lemma 4,

$$\begin{aligned} u_1(\lambda f^\alpha, b^o) &= \ell \cdot (\lambda f^\alpha)^{b^o} + c \cdot b^o - e \cdot (\lambda f^\alpha) \\ &= (\lambda \ell_t) e \cdot f^\alpha - (\lambda) e \cdot f^\alpha \\ &= \lambda(\ell_t - 1) e \cdot f^\alpha. \end{aligned}$$

Because $(\lambda f^\alpha)^{b^*} = f^o$,

$$\begin{aligned} u_1(\lambda f^\alpha, b^*) &= \ell \cdot (\lambda f^\alpha)^{b^*} + c \cdot b^* - e \cdot (\lambda f^\alpha) \\ &= c \cdot b^* - (\lambda) e \cdot f^\alpha. \end{aligned}$$

Because $\sigma_2^*(b^o) = 1/\ell_t$ and $\sigma_2^*(b^*) = 1 - 1/\ell_t$, player 1's expected payoff for $(\lambda f^\alpha, \sigma_2^*)$ is

$$\begin{aligned} u_1(\lambda f^\alpha, \sigma_2^*) &= \sigma_2^*(b^o) u_1(\lambda f^\alpha, b^o) + \sigma_2^*(b^*) u_1(\lambda f^\alpha, b^*) \\ &= (1/\ell_t)\lambda(\ell_t - 1) e \cdot f^\alpha + (1 - 1/\ell_t)(c \cdot b^* - (\lambda) e \cdot f^\alpha) \\ &= \lambda(1 - 1/\ell_t) e \cdot f^\alpha + (1 - 1/\ell_t) c \cdot b^* - \lambda(1 - 1/\ell_t) e \cdot f^\alpha \\ &= (1 - 1/\ell_t) c \cdot b^*. \end{aligned}$$

From part (i) of the proof of Proposition 3, we have $u_1(f^\tau, \sigma_2^*) = (1 - 1/\ell_t) c \cdot b^*$. Thus, $u_1(f^\tau, \sigma_2^*) = u_1(\lambda f^\alpha, \sigma_2^*)$.

(ii) We show that $u_2(\sigma_1^\lambda, b^o) = u_2(\sigma_1^\lambda, b^*)$. From part (ii) of the proof of Proposition 3, we have $u_2(f^\tau, b^o) = w$ and $u_2(f^\tau, b^*) = w - c \cdot b^*$. Now calculate player 2's payoffs for strategy profiles $(\lambda f^\alpha, b^o)$ and $(\lambda f^\alpha, b^*)$. Because $(\lambda f^\alpha)^{b^o} = \lambda f^\alpha$ and $(\lambda)\ell \cdot f^\alpha = (\lambda \ell_t) e \cdot f^\alpha$ from Lemma 4,

$$\begin{aligned} u_2(\lambda f^\alpha, b^o) &= w - \ell \cdot (\lambda f^\alpha)^{b^o} - c \cdot b^o \\ &= w - (\lambda \ell_t) e \cdot f^\alpha \\ &= w - (\lambda \ell_t) c \cdot b^*. \end{aligned}$$

The last equality comes from Lemma 5 and equality (15). Because $(\lambda f^\alpha)^{b^*} = f^o$,

$$\begin{aligned} u_2(\lambda f^\alpha, b^*) &= w - \ell \cdot (\lambda f^\alpha)^{b^*} - c \cdot b^* \\ &= w - c \cdot b^*. \end{aligned}$$

Because $\sigma_1^\lambda(f^\tau) = 1 - 1/\lambda \ell_t$ and $\sigma_1^\lambda(\lambda f^\alpha) = 1/\lambda \ell_t$, player 2's expected payoff for (σ_1^λ, b^o) is

$$\begin{aligned} u_2(\sigma_1^\lambda, b^o) &= \sigma_1^\lambda(f^\tau) u_2(f^\tau, b^o) + \sigma_1^\lambda(\lambda f^\alpha) u_2(\lambda f^\alpha, b^o) \\ &= (1 - 1/\lambda \ell_t) w + (1/\lambda \ell_t)(w - (\lambda \ell_t) c \cdot b^*) \\ &= w - c \cdot b^*. \end{aligned}$$

Also, player 2's expected payoff for (σ_1^λ, b^*) is $u_2(\sigma_1^\lambda, b^*) = w - c \cdot b^*$. Thus, $u_2(\sigma_1^\lambda, b^o) = u_2(\sigma_1^\lambda, b^*)$.

(iii) This is already shown in part (iii) of the proof of Proposition 3.

(iv) We show that for each $b \in \mathcal{B}$, $u_2(\sigma_1^\lambda, b^o) \geq u_2(\sigma_1^\lambda, b)$. For each $b \in \mathcal{B}$, we calculate player 2's payoffs. From part (iv) of the proof of Proposition 3, we have $u_2(f^\tau, b) = w - c \cdot b$. Because $(\lambda f^\alpha)^b = \lambda(f^\alpha)^b$,

$$\begin{aligned} u_2(\lambda f^\alpha, b) &= w - \ell \cdot (\lambda f^\alpha)^b - c \cdot b \\ &= w - (\lambda)\ell \cdot (f^\alpha)^b - c \cdot b. \end{aligned}$$

Because $\sigma_1^\lambda(f^\tau) = 1 - 1/\lambda\ell_t$ and $\sigma_1^\lambda(\lambda f^\alpha) = 1/\lambda\ell_t$, player 2's expected payoff for (σ_1^λ, b) is

$$\begin{aligned} u_2(\sigma_1^\lambda, b) &= \sigma_1^\lambda(f^\tau)u_2(f^\tau, b) + \sigma_1^\lambda(\lambda f^\alpha)u_2(\lambda f^\alpha, b) \\ &= (1 - 1/\lambda\ell_t)(w - c \cdot b) + (1/\lambda\ell_t)(w - (\lambda)\ell \cdot (f^\alpha)^b - c \cdot b) \\ &= w - c \cdot b - (1/\lambda\ell_t)(\lambda)\ell \cdot (f^\alpha)^b \\ &= w - c \cdot b - (1/\ell_t)\ell \cdot (f^\alpha)^b \\ &\leq w - (1/\ell_t)\ell \cdot f^\alpha. \end{aligned}$$

Note that the last inequality comes from Lemma 6. Because $(1/\ell_t)\ell \cdot f^\alpha = e \cdot f^\alpha$ and $e \cdot f^\alpha = c \cdot b^*$, we have $u_2(\sigma_1^\lambda, b) \leq w - c \cdot b^*$. From (ii), we know that $u_2(\sigma_1^\lambda, b^o) = w - c \cdot b^*$. Thus, for each $b \in \mathcal{B}$, $u_2(\sigma_1^\lambda, b^o) \geq u_2(\sigma_1^\lambda, b)$.

Therefore, $(\sigma_1^\lambda, \sigma_2^*)$ is a Nash equilibrium of Γ. □

Proof of Proposition 5. Suppose that $\ell_t > 1$ and $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$. To show that $(\sigma_1^\beta, \sigma_2^*)$ is a Nash equilibrium of Γ, it suffices to show that (i) $u_1(f^\tau, \sigma_2^*) = u_1(f^\beta, \sigma_2^*)$, (ii) $u_2(\sigma_1^\beta, b^o) = u_2(\sigma_1^\beta, b^*)$, (iii) for each $f \in \mathcal{F}$, $u_1(f^\tau, \sigma_2^*) \geq u_1(f, \sigma_2^*)$ and (iv) for each $b \in \mathcal{B}$, $u_2(\sigma_1^\beta, b^o) \geq u_2(\sigma_1^\beta, b)$.

(i) We show that $u_1(f^\tau, \sigma_2^*) = u_1(f^\beta, \sigma_2^*)$. First, calculate player 1's payoffs for strategy profiles (f^τ, b^o), (f^τ, b^*), (f^β, b^o), and (f^β, b^*). Because $(f^\tau)^{b^o} = f^o$ and $e \cdot f^\tau = 0$,

$$\begin{aligned} u_1(f^\tau, b^o) &= \ell \cdot (f^\tau)^{b^o} + c \cdot b^o - e \cdot f^\tau \\ &= 0. \end{aligned}$$

Because $(f^\tau)^{b^*} = f^o$ and $e \cdot f^\tau = 0$,

$$\begin{aligned} u_1(f^\tau, b^*) &= \ell \cdot (f^\tau)^{b^*} + c \cdot b^* - e \cdot f^\tau \\ &= c \cdot b^*. \end{aligned}$$

Because $(f^\beta)^{b^o} = f^\beta$ and $\ell \cdot f^\beta = (\ell_t)e \cdot f^\beta$ from Lemma 4,

$$\begin{aligned} u_1(f^\beta, b^o) &= \ell \cdot (f^\beta)^{b^o} + c \cdot b^o - e \cdot f^\beta \\ &= (\ell_t - 1)e \cdot f^\beta \\ &= (\ell_t - 1)q. \end{aligned}$$

The last equality comes from Lemma 5 and equality (16). Because $(f^\beta)^{b^*} = f^o$ and $e \cdot f^\beta = v \cdot f^\beta$ from Lemma 5,

$$
\begin{aligned}
u_1(f^\beta, b^*) &= \ell \cdot (f^\beta)^{b^*} + c \cdot b^* - e \cdot f^\beta \\
&= c \cdot b^* - v \cdot f^\beta \\
&= c \cdot b^* - q.
\end{aligned}
$$

The last equality comes from equality (16). Then, as in (23), player 1's expected payoff for (f^τ, σ_2^*) is $u_1(f^\tau, \sigma_2^*) = (1 - 1/\ell_t)c \cdot b^*$. As in (24), player 1's expected payoff for (f^β, σ_2^*) is $u_1(f^\beta, \sigma_2^*) = (1 - 1/\ell_t)c \cdot b^*$. Thus, $u_1(f^\tau, \sigma_2^*) = u_1(f^\beta, \sigma_2^*)$.

(ii) We show that $u_2(\sigma_1^\beta, b^o) = u_2(\sigma_1^\beta, b^*)$. First, calculate player 2's payoffs for strategy profiles (f^τ, b^o), (f^β, b^o), (f^τ, b^*), and (f^β, b^*). Because $(f^\tau)^{b^o} = f^o$,

$$
\begin{aligned}
u_2(f^\tau, b^o) &= w - \ell \cdot (f^\tau)^{b^o} - c \cdot b^o \\
&= w.
\end{aligned}
$$

Because $(f^\beta)^{b^o} = f^\beta$ and $\ell \cdot f^\beta = (\ell_t)e \cdot f^\beta$ from Lemma 4,

$$
\begin{aligned}
u_2(f^\beta, b^o) &= w - \ell \cdot (f^\beta)^{b^o} - c \cdot b^o \\
&= w - (\ell_t)e \cdot f^\beta \\
&= w - (\ell_t)q.
\end{aligned}
$$

The last equality comes from Lemma 5 and equality (16). Because $(f^\tau)^{b^*} = f^o$,

$$
\begin{aligned}
u_2(f^\tau, b^*) &= w - \ell \cdot (f^\tau)^{b^*} - c \cdot b^* \\
&= w - c \cdot b^*.
\end{aligned}
$$

Because $(f^\beta)^{b^*} = f^o$,

$$
\begin{aligned}
u_2(f^\beta, b^*) &= w - \ell \cdot (f^\beta)^{b^*} - c \cdot b^* \\
&= w - c \cdot b^*.
\end{aligned}
$$

Then, as in (25), player 2's expected payoff for (σ_1^β, b^o) is $u_2(\sigma_1^\beta, b^o) = w - c \cdot b^*$. As in (26), player 2's expected payoff for (σ_1^β, b^*) is $u_2(\sigma_1^\beta, b^*) = w - c \cdot b^*$. Thus, $u_2(\sigma_1^\beta, b^o) = u_2(\sigma_1^\beta, b^*)$.

(iii) This is already shown in part (iii) of the proof of Proposition 3.

(iv) We show that for each $b \in \mathcal{B}$, $u_2(\sigma_1^\beta, b^o) \geq u_2(\sigma_1^\beta, b)$. For each $b \in \mathcal{B}$, we calculate player 2's payoffs. Because $(f^\tau)^b = f^o$,

$$
\begin{aligned}
u_2(f^\tau, b) &= w - \ell \cdot (f^\tau)^b - c \cdot b \\
&= w - c \cdot b.
\end{aligned}
$$

Also, we have $u_2(f^\beta, b) = w - \ell \cdot (f^\beta)^b - c \cdot b$. Then, for each $b \in \mathcal{B}$,

$$\begin{aligned}
u_2(\sigma_1^\beta, b) &= \sigma_1^\beta(f^\tau)u_2(f^\tau, b) + \sigma_1^\beta(f^\beta)u_2(f^\beta, b) \\
&= (1 - (1/\ell_t)((c \cdot b^*)/q))(w - c \cdot b) + (1/\ell_t)((c \cdot b^*)/q)(w - \ell \cdot (f^\beta)^b - c \cdot b) \\
&= w - c \cdot b - (1/\ell_t)((c \cdot b^*)/q)\ell \cdot (f^\beta)^b \\
&= w - c \cdot b - (1/\ell_t)((c \cdot b^*)/q)(q/(c \cdot b^*))\ell \cdot (f^\alpha)^b \\
&= w - c \cdot b - (1/\ell_t)\ell \cdot (f^\alpha)^b \\
&\leq w - (1/\ell_t)\ell \cdot f^\alpha
\end{aligned}$$

because $\sigma_1^\beta(f^\tau) = 1 - (1/\ell_t)((c \cdot b^*)/q)$, $\sigma_1^\beta(f^\beta) = (1/\ell_t)((c \cdot b^*)/q)$, and $f^\beta = (q/(c \cdot b^*))f^\alpha$. Note that the last inequality comes from Lemma 6. Because $(1/\ell_t)\ell \cdot f^\alpha = e \cdot f^\alpha$ and $e \cdot f^\alpha = c \cdot b^*$, we have $u_2(\sigma_1^\beta, b) \leq w - c \cdot b^*$. From (ii), we know that $u_2(\sigma_1^\beta, b^o) = w - c \cdot b^*$. Thus, for each $b \in \mathcal{B}$, $u_2(\sigma_1^\beta, b^o) \geq u_2(\sigma_1^\beta, b)$.

Therefore, $(\sigma_1^\beta, \sigma_2^*)$ is a Nash equilibrium of Γ. □

Proof of Proposition 6. Suppose that $\ell_t > 1$ and $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$. To show that $(\sigma_1^\mu, \sigma_2^*)$ is a Nash equilibrium of Γ, it suffices to show that (i) $u_1(f^\tau, \sigma_2^*) = u_1(\mu f^\beta, \sigma_2^*)$, (ii) $u_2(\sigma_1^\mu, b^o) = u_2(\sigma_1^\mu, b^*)$, (iii) for each $f \in \mathcal{F}$, $u_1(f^\tau, \sigma_2^*) \geq u_1(f, \sigma_2^*)$ and (iv) for each $b \in \mathcal{B}$, $u_2(\sigma_1^\mu, b^o) \geq u_2(\sigma_1^\mu, b)$.

(i) We show that $u_1(f^\tau, \sigma_2^*) = u_1(\mu f^\beta, \sigma_2^*)$. Calculate player 1's expected payoff for $(\mu f^\beta, \sigma_2^*)$. Because $(\mu f^\beta)^{b^o} = \mu f^\beta$ and $(\mu)\ell \cdot f^\beta = (\mu \ell_t)e \cdot f^\beta$ from Lemma 4,

$$\begin{aligned}
u_1(\mu f^\beta, b^o) &= \ell \cdot (\mu f^\beta)^{b^o} + c \cdot b^o - e \cdot (\mu f^\beta) \\
&= (\mu \ell_t)e \cdot f^\beta - (\mu)e \cdot f^\beta \\
&= \mu(\ell_t - 1)e \cdot f^\beta.
\end{aligned}$$

Because $(\mu f^\beta)^{b^*} = f^o$,

$$\begin{aligned}
u_1(\mu f^\beta, b^*) &= \ell \cdot (\mu f^\beta)^{b^*} + c \cdot b^* - e \cdot (\mu f^\beta) \\
&= c \cdot b^* - (\mu)e \cdot f^\beta.
\end{aligned}$$

Because $\sigma_2^*(b^o) = 1/\ell_t$ and $\sigma_2^*(b^*) = 1 - 1/\ell_t$, player 1's expected payoff for $(\mu f^\beta, \sigma_2^*)$ is

$$\begin{aligned}
u_1(\mu f^\beta, \sigma_2^*) &= \sigma_2^*(b^o)u_1(\mu f^\beta, b^o) + \sigma_2^*(b^*)u_1(\mu f^\beta, b^*) \\
&= (1/\ell_t)\mu(\ell_t - 1)e \cdot f^\beta + (1 - 1/\ell_t)(c \cdot b^* - (\mu)e \cdot f^\beta) \\
&= \mu(1 - 1/\ell_t)e \cdot f^\beta + (1 - 1/\ell_t)c \cdot b^* - \mu(1 - 1/\ell_t)e \cdot f^\beta \\
&= (1 - 1/\ell_t)c \cdot b^*.
\end{aligned}$$

From part (i) of the proof of Proposition 5, we have $u_1(f^\tau, \sigma_2^*) = (1 - 1/\ell_t)c \cdot b^*$. Thus, $u_1(f^\tau, \sigma_2^*) = u_1(\mu f^\beta, \sigma_2^*)$.

(ii) We show that $u_2(\sigma_1^\mu, b^o) = u_2(\sigma_1^\mu, b^*)$. From part (ii) of the proof of Proposition 5, we have $u_2(f^\tau, b^o) = w$ and $u_2(f^\tau, b^*) = w - c \cdot b^*$. Now calculate player 2's payoffs for strategy profiles $(\mu f^\beta, b^o)$ and $(\mu f^\beta, b^*)$. Because $(\mu f^\beta)^{b^o} = \mu f^\beta$ and $(\mu)\ell \cdot f^\beta = (\mu \ell_t) e \cdot f^\beta$ from Lemma 4,

$$\begin{aligned} u_2(\mu f^\beta, b^o) &= w - \ell \cdot (\mu f^\beta)^{b^o} - c \cdot b^o \\ &= w - (\mu \ell_t) e \cdot f^\beta \\ &= w - (\mu \ell_t) q. \end{aligned}$$

The last equality comes from Lemma 5 and equality (16). Because $(\mu f^\beta)^{b^*} = f^o$,

$$\begin{aligned} u_2(\mu f^\beta, b^*) &= w - \ell \cdot (\mu f^\beta)^{b^*} - c \cdot b^* \\ &= w - c \cdot b^*. \end{aligned}$$

Because $\sigma_1^\mu(f^\tau) = 1 - (1/\mu\ell_t)((c \cdot b^*)/q)$ and $\sigma_1^\mu(\mu f^\beta) = (1/\mu\ell_t)((c \cdot b^*)/q)$, player 2's expected payoff for (σ_1^μ, b^o) is

$$\begin{aligned} u_2(\sigma_1^\mu, b^o) &= \sigma_1^\mu(f^\tau) u_2(f^\tau, b^o) + \sigma_1^\mu(\mu f^\beta) u_2(\mu f^\beta, b^o) \\ &= (1 - (1/\mu\ell_t)((c \cdot b^*)/q))w + (1/\mu\ell_t)((c \cdot b^*)/q)(w - (\mu\ell_t)q) \\ &= w - c \cdot b^*. \end{aligned}$$

Also, player 2's expected payoff for (σ_1^μ, b^*) is $u_2(\sigma_1^\mu, b^*) = w - c \cdot b^*$. Thus, $u_2(\sigma_1^\mu, b^o) = u_2(\sigma_1^\mu, b^*)$.

(iii) This is already shown in part (iii) of the proof of Proposition 3.

(iv) We show that for each $b \in \mathcal{B}$, $u_2(\sigma_1^\mu, b^o) \geq u_2(\sigma_1^\mu, b)$. For each $b \in \mathcal{B}$, we calculate player 2's payoffs. From part (iv) of the proof of Proposition 5, we have $u_2(f^\tau, b) = w - c \cdot b$. Because $(\mu f^\beta)^b = \mu (f^\beta)^b$,

$$\begin{aligned} u_2(\mu f^\beta, b) &= w - \ell \cdot (\mu f^\beta)^b - c \cdot b \\ &= w - (\mu)\ell \cdot (f^\beta)^b - c \cdot b. \end{aligned}$$

Because $\sigma_1^\mu(f^\tau) = 1 - (1/\mu\ell_t)((c \cdot b^*)/q)$, $\sigma_1^\mu(\mu f^\beta) = (1/\mu\ell_t)((c \cdot b^*)/q)$, and $f^\beta = (q/(c \cdot b^*)) f^\alpha$, player 2's expected payoff for (σ_1^μ, b) is

$$\begin{aligned} u_2(\sigma_1^\mu, b) &= \sigma_1^\mu(f^\tau) u_2(f^\tau, b) + \sigma_1^\mu(\mu f^\beta) u_2(\mu f^\beta, b) \\ &= (1 - (1/\mu\ell_t)((c \cdot b^*)/q))(w - c \cdot b) + (1/\mu\ell_t)((c \cdot b^*)/q)(w - (\mu)\ell \cdot (f^\beta)^b - c \cdot b) \\ &= w - c \cdot b - (1/\mu\ell_t)((c \cdot b^*)/q)(\mu)\ell \cdot (f^\beta)^b \\ &= w - c \cdot b - (1/\mu\ell_t)((c \cdot b^*)/q)(q/(c \cdot b^*))(\mu)\ell \cdot (f^\alpha)^b \\ &= w - c \cdot b - (1/\ell_t)\ell \cdot (f^\alpha)^b \\ &\leq w - (1/\ell_t)\ell \cdot f^\alpha. \end{aligned}$$

Note that the last inequality comes from Lemma 6. Because $(1/\ell_t)\ell \cdot f^\alpha = e \cdot f^\alpha$ and $e \cdot f^\alpha = c \cdot b^*$, we have $u_2(\sigma_1^\mu, b) \leq w - c \cdot b^*$. From (ii), we know that $u_2(\sigma_1^\mu, b^o) = w - c \cdot b^*$. Thus, for each $b \in \mathcal{B}$, $u_2(\sigma_1^\mu, b^o) \geq u_2(\sigma_1^\mu, b)$.

Therefore, $(\sigma_1^{\mu}, \sigma_2^{*})$ is a Nash equilibrium of Γ. $\square$

References

[1] Ahuja, R.K., Magnanti, T.L., Orlin, J.B., 1993. Network Flows. Prentice-Hall, New Jersey.

[2] Baccara, M., Bar-Isaac, H., 2008. How to organize crime. Review of Economic Studies 75, 1039–1067.

[3] Baccara, M., Bar-Isaac, H., 2008. Interrogation methods and terror networks. Working paper.

[4] Ballester, C., Calvó-Armengol, A., Zenou, Y., 2006. Who's who in networks. Wanted: the key player. Econometrica 74, 1403–1417.

[5] Bohorquez, J.C., Gourley, S., Dixon, A.R., Spagat, M., Johnson, N.F., 2009. Common ecology quantifies human insurgency. Nature 462, 911–914.

[6] Clauset, A., Young, M., Gleditsch, K.S., 2007. On the frequency of severe terrorist events. Journal of Conflict Resolution 51, 58–87.

[7] Cormican, K.J., Morton, D.P., Wood, R.K., 1998. Stochastic network interdiction. Operations Research 46, 184–197.

[8] Ford, L.R., Fulkerson, D.R., 1962. Flows in Networks. Princeton University Press, New Jersey.

[9] Goyal, S., Vigier, A., 2008. Robust networks. Working paper.

[10] Granot, D., Granot, F., 1992. On some network flow games. Mathematics of Operations Research 17, 792–841.

[11] Israeli, E., Wood, R.K., 2002. Shortest-path network interdiction. Networks 40, 97–111.

[12] Jackson, M.O., 2008. Social and Economic Networks. Princeton University Press, New Jersey.

[13] Kalai, E., Zemel, E., 1982. Generalized network problems yielding totally balanced games. Operations Research 30, 998–1008.

[14] Kalai, E., Zemel, E., 1982. Totally balanced games and games of flow. Mathematics of Operations Research 7, 476–478.

[15] Kodialam, M., Lakshman, T.V., 2003. Detecting network intrusion via sampling: A game theoretic approach. Proceedings of Infocom 2003, 1880–1889.

[16] Potters, J., Reijnierse, H., Biswas, A., 2006. The nucleolus of balanced simple flow networks. Games and Economic Behavior 54, 205–225.

[17] Reijnierse, H., Maschler, M., Potters, J., Tijs, S., 1996. Simple flow games. Games and Economic Behavior 16, 238–260.

[18] Washburn, A., Wood, K., 1995. Two-person zero-sum games for network interdiction. Operations Research 43, 243–251.

[19] Wood, R.K., 1993. Deterministic network interdiction. Mathematical and Computer Modelling 17, 1–18.