DYNAMIC CLUB FORMATION WITH COORDINATION

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Dynamic Club Formation with Coordination*

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We present a dynamic model of club formation in a society of identical people. Coalitions consisting of members of the same club can form for one period and coalition members can jointly deviate. The dynamic process is described by a Markov chain defined by myopic optimization on the part of coalitions. We define a Nash club equilibrium (NCE) as a strategy profile that is immune to such coalitional deviations. For single–peaked preferences, we show that, if one exists, the process will converge to a NCE profile with probability one. NCE is unique up to a renaming of players and locations. Further, NCE corresponds to strong Nash equilibrium in the club formation game. Finally, we deal with the case where NCE fails to exist due to a nonbalancedness problem. When the population size is not an integer multiple of an optimal club size, there may be 'left over' players who prevent the process from 'settling down'. To treat this case, we define the concept of *k–remainder NCE*, which requires that all but k players are playing a Nash club equilibrium, where k is defined by the minimal number of left over players. We show that the process converges to an ergodic NCE, that is, a set of states consisting only of k–remainder NCE.

Keywords: Club formation, Cooperation, Best–reply dynamics, Nash club equilibrium, Ergodic Nash club equilibrium.

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1 Introduction

We provide a dynamic model of club formation within the framework of a local public good economy. Each period, individuals choose one out of a finite set of locations. In interpretation, those individuals selecting the same location form a club in order to provide a local public good for themselves exclusively or to commonly share a facility. We assume that the public good is financed by a poll tax, or in other words, equal cost sharing on the part of the members of the club. An individual's utility depends on his consumption of a private good, the public good, and on the size of the club; that is, we consider anonymous crowding – people care about the number of members of their club, but not about their identities, and there may be congestion. Examples are the choice of a leisure club, hospital, or restaurant.

In the presence of congestion effects, an increase in the number of members of a club has two opposing effects on the members' utilities: On the one hand, the cost shares are diminished; on the other hand, congestion may be exacerbated. Thus there is a trade-off between cost sharing and crowding effects. Note, however, as in the literature on local public good economies with anonymous crowding, crowding effects are not necessarily negative. For instance, there might be positive externalities in consumption, or fashion effects. Finally, crowding effects might be both positive and negative over different ranges of the club size. In any of these cases, an agent's marginal utility from an increase in club size is increasing up to a point, the 'agent optimal' club size, and then decreasing. Such models now have a long history going back to Buchanan (1965).

We define a non–cooperative game where each player's strategy set is the set of all locations, or clubs (each club is identified with its location), and each player's payoff is a function of the number of players choosing the same strategy, i. e. location, as himself. This model is a simple version of the local public good games analyzed by, among others, Konishi, Le Breton and Weber (1997a, 1998). The existence of pure strategy Nash equilibria of such local public good games has been shown, for various specifications of the game, by several authors. While Konishi et al. (1997a), and, for an even more general model including external effects of group formation on non–members Hollard (2000), prove existence in the general case, Holzman and Law–Yone (1997), Konishi et al. (1997b), and Milchtaich (1996) are concerned with the special case of congestion games, where each player's payoff is non–increasing in the number of players choosing the same strategy as himself. The latter two articles also provide conditions for the existence of strong Nash equilibria. Games with positive externalities are analyzed in Konishi et al. (1997c).

The setup of our model is closely related to the model of Konishi, Le Breton and Weber (1997a). They define a free mobility equilibrium of a local public goods economy as an assignment of players to clubs (locations, or jurisdictions) that partitions the population and has the property that no individual can gain by either moving to any other existing club, or creating his own club. The partition derived from the players' strategy choices is thus stable against unilateral deviations by individuals.

We extend this model in two ways. First, the players' mobility is modelled explicitly: We provide a *dynamic model* where the club formation game is played by myopically optimizing players, who can move between existing clubs (or if an empty location is available, create their own) at each step in time. Second, we take up an issue often addressed in the context of club formation, namely the possibility of *coordinated action* on the part of a group of players. For instance, if a club becomes too crowded, a subset of its members might decide to move to another club or to jointly open a new club, even if every single individual would not want to do that if he were on his own. To ensure stability against such coordinated deviations, we analyze equilibria that are immune to joint deviations by groups of players. Allowing for joint deviations by groups, the appropriate equilibrium concept would be the strong Nash equilibrium. However, in many cases strong equilibria may not exist, e.g. if the optimal club size is not an integer divisor of the population size. In a recent article, Conley and Konishi (2002) resolve the problem of nonexistence due to left over players by analyzing *migration proof* equilibria, which are stable only against *credible* deviations on the part of a coalition. A coalitional deviation to another jurisdiction is credible if no outsiders to the coalition will want to follow the deviators and, within the deviating group, no player can gain by a further deviation. Conley and Konishi show that the migration proof equilibrium exists for the class of games under consideration, is unique, and asymptotically efficient in the sense that payoffs approach their maximum as the number of players goes to infinity.

We pursue a different approach that emphasizes the mobility aspect, and is based on the assumption of myopic optimization on the part of the players. We allow for coalitional deviations only by *groups of players within a club*. That is, a subgroup of the set of members of a club may form a coalition for one period, and deviate jointly to some other strategy, e. g. join another club, move to an unoccupied location, or distribute themselves across different locations or different existing clubs. In the next period, new coalitions will be formed. In each period, each coalition's opportunity to move arises at random.

The model is also related to the literature on evolutionary learning models, e. g. Ellison (1993), Kandori, Mailath, and Rob (1993), Young (1993), and many others. The aim of these models was primarily to justify game theoretic solution concepts on the grounds of bounded rationality. The concept of Nash equilibrium and its refinements have been criticized because they require to high a degree of rationality on the part of the players. The evolutionary learning models disprove this criticism by showing that, in the long run, even boundedly rational players can learn to play a Nash equilibrium. In these models, the players select their strategies in each period by myopic optimization. The opportunity for any one player to revise his strategy occurs at randomly chosen moments in time. Further, the players 'tremble', i. e. they pick a new strategy with a small but positive probability. The dynamics of the system are described by a Markov chain with the state space given by the set of all possible strategy configurations across the population. The solution concept is the *long run equilibrium* introduced by Kandori, Mailath, and Rob (1993). They analyze the stationary distribution of the Markov chain when the probability of the players' making mistakes goes to zero.

The model presented here is similar in its dynamic setting. The players are boundedly rational, they practise myopic optimization, and the dynamics is described by a Markov chain. The difference is that, in our model, the players do not tremble. Randomness is introduced by assuming that the players are randomly selected to revise their strategies. The model adds to the literature on evolutionary learning by offering a justification for the concept of strong Nash equilibrium in the presence of bounded rationality: In our model, the decision makers learn to play a strong Nash equilibrium, if one exists, or else they will arrive at a strategy configuration that is, in a certain sense, close to equilibrium.

In contrast to other models on the endogenous formation of coalitions, our model is truly dynamic: At each time step, a probability distribution determines the state, i. e. the strategy vector, for the next period, where the transition probabilities are derived from the myopic best–reply rules together with the random opportunities of strategy revision.[1] Also, the restrictions on joint deviations are appealing; typically clubs are assumed to form so that individuals within the club can interact with each other. Coordination of strategies is one form of interaction. We call a vector of strategy choices that is immune to improving deviations by coalitions contained in any club a *Nash club equilibrium (NCE).*

As suggested above, if the optimal coalition size, say s^* – the club size that maximizes per capita utility of the club membership – is not an integer divisor of the population size then, provided that the population is larger than the optimal club size, a Nash club equilibrium may fail to exist because some players will be 'left over'. This problem often arises in the literature on club economies. Here, we adopt an approach similar to that used in cooperative game theoretic approaches and consider equilibrium outcomes that allow for left over players.[2] For this purpose, we introduce the concept of a *k–remainder Nash club equilibrium.* This

[1]This is in strong contrast to models of club formation where, once a group of players has agreed upon forming a coalition, this group will drop out of the population and out of negotiations, as in Ray and Vohra (1997), for example.

[2]See, for example, Kovalenkov and Wooders (2003) for nonemptiness of approximate cores of economies with clubs, where an approximate core notion allows left over players.

is a strategy profile with the following property: k players can be removed from the population in such a way that the remaining $n-k$ players are playing a Nash club equilibrium (on the reduced strategy space), and k is the minimal number of left over players. We demonstrate:

Existence of a k–remainder Nash club equilibrium, and

Convergence to an ergodic set of k–remainder Nash club equilibria.

The remainder of the paper is organized as follows. The next section presents the formal framework of our model, i. e. the stage game. Section 3 describes coalitional deviations and the myopic adaptation process on the part of the players. We define the equilibrium concept, Nash club equilibrium (NCE). Section 4 defines the finite Markov chain, while sections 5 and 6 deal with existence and efficiency of NCE, respectively. Convergence of the Markov chain is analyzed in section 7. Finally, we deal with the problem that NCE might not exist. We introduce the concept of ergodic NCE in section 8 and prove convergence. The last section concludes.

2 The Basic Model

We consider a finite set $N = \{1, \ldots, n\}$ of individuals, or players. Each player can choose a location from a finite set[3] $G = \{a, b, \ldots, m\}$. Individuals choosing the same location form a club in order to provide a public good for themselves exclusively, or to use a common facility and share costs equally. The cost of the facility is exogenously given. Since there is no danger of confusion, we will identify a club with the location at which it is formed, e. g. club a is the name of the club formed at location $a \in G$.

A player's strategy is his choice of a club. A *strategy profile* is thus a vector $g = (g_1, \ldots, g_n) \in G^n$, indicating a club (location) for each player. We consider only pure strategies. Note that a strategy profile induces a partition of players into clubs, a *club structure.*

Each person's utility depends on the size of his club, the number of players choosing the same club. Formally, this crowding effect is captured by a function $h : G \times N \to \mathbb{R}$, where $h(a, s)$ is the (dis)utility to a member of club a when the total number of members (himself included) is s.

[3] This assumption is needed for technical reasons. However, if $|G|$ is large relative to the set of players, it is not at all restrictive. Further, we are able to analyse the interesting case of $|G|$ being small, so our model is in fact richer than one with an unlimited set of locations.

We design a non–cooperative game $\Gamma = \{N, G, (u_i)_{i\in N}\}$ where N is the set of players, G is the common strategy set — each player can choose any one of the possible locations — and $u_i : G^n \to \mathbb{R}$ is player i's payoff function.

For any given strategy profile $g = (g_1, \ldots, g_n)$, let $n_a(g)$ denote the number of players choosing strategy $a \in G$, and let $c(s)$ denote the cost of providing the optimal amount of public good for s club members. The payoff to player i playing strategy $g_i = a$ in strategy profile g is then given by the indirect utility function[4]

$$(1) \quad u_i(g) = v(a) - \frac{c(n_a(g))}{n_a(g)} + h(a, n_a(g)),$$

where $v(a)$ denotes his utility derived from the local public good. A *Nash equilibrium* of Γ is a strategy profile g with the property that $u_i(g) \geq u_i(g_{-i}, b)$ for all $i \in N$ and all $b \in G$, where $(g_{-i}, b) := (g_1, \ldots, g_{i-1}, b, g_{i+1}, \ldots, g_n)$, i. e.

$$-\frac{c(a)}{n_a(g)} + h(a, n_a(g)) \geq -\frac{c(b)}{n_b(g)+1} + h(b, n_b(g)+1)$$

for all $i \in N$ and for all $b \in G$, where a is the strategy adopted by player i in strategy profile g.

Konishi, Le Breton and Weber (1997a) and Hollard (2000) show the existence of a pure strategy Nash equilibrium for the game Γ.[5]

Proposition 1 *(Konishi et al. (1997a)) The game* Γ *admits a Nash equilibrium in pure strategies.*

3 Coordination of Coalitions within Clubs

We now turn to coalitional deviations and Nash club equilibrium. In each period, the strategy choices of that period induce a partition of the set of agents into clubs. Given this partition, we assume that the only admissible coalitions consist of players within the same club. These coalitions last for one period. A *coalition* is thus a subset of the set of members of a club. Note that the set of admissible coalitions may change from period to period. If a coalition is formed, its members will jointly decide which location each of them will choose. That is, coalition

[4]To simplify notation, we drop the dependence of utility on the amount of the private good.

[5]The class of games considered in Konishi *et al.* (1997a) is not restricted to finite strategy sets. Hollard (2000) allows for external effects on players outside the group.

members may jointly deviate to another location, or the members may distribute themselves across different locations. A coalition will form whenever it is in the best interest of every single coalition member to do so. That is, each member must strictly benefit from forming the coalition and, for each member, there must be no other coalition (within the same club) that yields a higher payoff to that player. Such coalitions will be called *viable*, which will be defined formally below. In the next period, all coalitions dissolve, and new ones can be formed. Formally:

Given any strategy profile g, define the resulting partition of the player set by

$$N(g) = \{N_a(g) \ldots, N_m(g)\},$$

where $N_a(g)$ denotes the set of all players choosing location a under the strategy profile g.

Definition 1 *Given any strategy profile g, a* coalition *C is a nonempty subset of a club induced by g, i. e. $C \subset N_a(g)$ for any $N_a(g) \in N(g)$.*

3.1 Myopic Strategy Choice

For the remainder of the paper, we assume that all locations are identical. Thus the utility of membership in a club depends only on the number of members of the club. We can therefore express utility as a function of club size. Let a be the location of player i in strategy profile g. Define

$$u_i(g) \equiv u(n_a(g)).$$

That is, the payoff to a player i who is a member of club a under any strategy profile g with $n_a(g) = s$ will realize the payoff $u(s)$.

Further we assume, as Conley and Konishi (2002) for example, that preferences are single peaked. This implies that, from the point of view of an individual, there exists an optimal club size, which may be any number between 1 (singleton clubs) and n (a grand club). Denote this number by s^*.[6]

Finally, we use the term 'potential coalition' to refer to any subset of the total player set. Given a club structure the only admissible coalitions are those that are subsets of the membership of some club.

Assumption Preferences for clubs are single peaked over club sizes. That is, we assume that there exists an integer $s^* \in \{1, \ldots, n\}$ such that

[6] Alternatively, following Wooders (1978), we could assume that there is an interval of optimal club sizes or, more generally, that there are at least two optimal club sizes and these are relatively prime numbers. This would reduce the number of cases where NCE does not exist.

1. for any clubs a, b with $n_a < n_b \leq s^*$ we have $u(n_a) < u(n_b)$, and
2. for any clubs a, b with $n_a > n_b \geq s^*$ we have $u(n_a) < u(n_b)$.

We now turn to the dynamic adaptation process.[7] Time is divided into discrete periods $t = 0, 1, 2, \ldots$. In the initial period $t = 0$, we start with an arbitrary strategy profile $g \in G^n$.[8] Each period, each player receives a payoff, determined by the strategy choices of all players in that period. This payoff depends on the club size.

In any period t, given any strategy profile g and the resulting club structure $N(g)$, the adaptation process consists of the following steps.

1. For every club a under the strategy profile g, members of a can form coalitions. Given any strategy profile g, a coalition C is called *viable* if (i) it forms a subset of a given club, i.e. $C \subset N_a(g), C \neq \emptyset$; (ii) there exists a strategy profile $y = (y_C, g_{-C})$ such that $u_i(y) > u_i(g)$ for all $i \in C$, and (iii) for any $i \in C$, there is no other coalition $C' \ni i$ and strategy $x = (x_{C'}, g_{-C'})$ such that i gets a higher payoff in C' than in C, that is, a payoff $u_i(x) > u_i(y)$.

   Note that there may be more than one viable coalition within each club and viable coalitions may overlap. Also, viable coalitions may consist of individual players.

2. In each period, one potential coalition (not necessarily admissible nor viable)[9] is picked at random and then gets the opportunity to revise its strategy. [10]

3. If any coalition gets the opportunity to revise, it will do so if and only if the coalition is viable. A viable coalition, if picked, will revise its strategy

[7] Adaptive models have been applied, in a context formally similar to the one of this paper, to both cooperative and non–cooperative games, see e. g. Dieckmann (1999) for non–cooperative games, and e. g. Arnold and Schwalbe (2002) for cooperative games. Milchtaich and Winter (2002) provide a different dynamic model of group formation where in each period, one player and one group (or club) are randomly selected, and if the selected player prefers the selected group to his present one, he will move. However, they do not consider the possibility of coalitions being formed among the members of a given group.

[8] We will later see that the choice of the initial strategy profile is irrelevant with respect to the results of the model.

[9] That is, each element of the set $2^N \backslash \emptyset$. Although not all of these potential coalitions are possible under any given club structure, we take them all into account in order to correctly define the dynamic process.

[10] This "inertia" can be justified by the assumption that strategy adjustments involve a nominal cost, for example the cost of moving from one location to another.

so that the coalition members will receive their highest possible payoffs in the next period, i. e. they chose a best reply to the current strategy configuration. If there is more than one best reply, the coalition randomizes, placing strictly positive probability on each.

4. Coalitions that do not get the chance to move will stay put.

Note that we do not explicitly model the formation of viable coalitions within clubs, nor procedures for arriving at joint strategy profiles.[11] We assume that a coalition forms if it is in the interest of all its members, i. e. if they can coordinate their strategies in such a way that each member's payoff will be increased.

Also note that in our model, the formation of both coalitions (for one period) and clubs is always reversible: Clubs may form and dissolve again since at each time step decision makers are free to choose their strategies.

We define a *Nash club equilibrium* as a strategy profile that is stable against deviations by coalitions, that is, in a Nash club equilibrium, no viable coalition exists.

Definition 2 *A strategy profile g is a* Nash club equilibrium (NCE) *if there exists no viable coalition. A* NCE club structure *is the partition of the population induced by a NCE.*

That is, a strategy profile g is a NCE if there is no club $a \in G$, no coalition $C \subset N_a(g)$, and no strategy profile $y = (y_C, g_{-C})$ such that $u_i(y) > u_i(g)$ for all $i \in C$. In other words, there is no coalition that would want to deviate if it were given the opportunity.

4 The Dynamics

The myopic best–reply rules together with the stochastic opportunities for strategy revision on the part of coalitions define a Markov chain on the finite state space G^n. A *state* of the system is a strategy profile, i.e. a strategy (choice of location) for each player. Note that a state of the system induces a partition of the set of players into clubs. The transition probabilities between states are determined by the best–reply rules and the fact that each coalition's opportunity to revise its strategy arises at random.

[11] That is, we do not model a negotiation process like e. g. Ray and Vohra (1997). We simply assume that, if it is advantageous, players coordinate their moves. The process by which they arrive at their mutually beneficial strategies is not modelled.

Observe that, once a NCE profile is reached, no player or coalition will switch clubs. An NCE profile is thus an *absorbing state* of the process, i.e. a state that cannot be left again once it has been entered. That is, once the process has reached a NCE, it will 'settle down' in that state forever. Conversely, any strategy profile that is not NCE cannot be an absorbing state, since at least one viable coalition will exist and will gain by deviating when it gets the chance to do so, which will happen with positive probability.

Observation A strategy profile is NCE if and only if it is an absorbing state of the Markov process.

The above observation, however, does not ensure convergence of the process to an absorbing state. Instead, the process may get trapped in a set of states, and perpetually oscillate between these states. We will show that this is not the case. First, however, we will deal with existence of NCE.

5 Existence of a Nash Club Equilibrium

The existence of NCE depends on the relationship between several parameters of the model. These are the size of the population n, the optimal club size s^*, and the number of locations $|G|$. First note that if $s^* \geq n$, there is a unique NCE club structure, namely the grand coalition. This is unique up to a relabelling of locations. In what follows, we focus on the case of $s^* < n$. Three cases have to be considered:

1. If $|G| \geq n/s^*$ and n/s^* is an integer, NCE exists. A NCE club structure consists of n/s^* clubs of size s^*.

2. If $|G| > n/s^*$ and n/s^* is not an integer, a NCE might not exist.

   Example Let $n = 10$, $|G| = 5$, $s^* = 3$, and assume $u(1) < u(4)$. In this example, no NCE exists: If the players form three clubs of size 3, there will be one left over player who can gain by joining any of the three clubs. If there are two clubs of size 3 and one club of size 4, a coalition of any three members of the latter can gain by jointly deviating to an unoccupied location. (Note that if $u(1) \geq u(4)$, then a NCE does exist.)

3. If $|G| < n/s^*$, a NCE exists. In a NCE club structure, all players are distributed as evenly as possible across all locations, and location size is given by integers $n_a(g^*)$ satisfying

$$n_a(g^*) \in \left\{ \left\lfloor \frac{n}{|G|} \right\rfloor, \left\lceil \frac{n}{|G|} \right\rceil \right\},$$

where $\lfloor n/|G| \rfloor$ is the largest integer weakly smaller than $n/|G|$ and $\lceil n/|G| \rceil$ is the smallest integer weakly larger than $n/|G|$. Note that if $n/|G|$ is an integer then $\lfloor n/|G| \rfloor = \lceil n/|G| \rceil$.

Example: Existence of NCE with 'few locations'. Let $n = 100$, $s^* = 10$, $|G| = 7$. A NCE club structure consists of five clubs of size 14 and two clubs of size 15. The optimal club size cannot be reached in NCE in this example. The reason is that, even though any coalition of ten players would prefer to jointly deviate to a new club, this is impossible because there are no unoccupied locations. Clearly a member of a 15 person club cannot benefit from moving to a 14 person club.

In case 2. above, NCE exists under the condition stated in the following proposition.

Proposition 2 Existence of NCE for the case $|G| > \frac{n}{s^*}$. *Let $n = rs^* + \ell$ where r and ℓ are positive integers and $\ell < s^*$. Then a NCE exists if and only if $u(\ell) > u(s^*+1)$. Moreover, if $u(\ell) > u(s^*+1)$ and g is a NCE, then the induced club structure will have r clubs of size s^* and 1 club of size ℓ.*

Proof. See appendix.

Note that, if a NCE exists, there are multiple NCE that differ only with respect to the names of players and locations. That is, the club structure induced by NCE is uniquely characterized by the number of clubs and the sizes of their memberships as follows:

- If $|G| \geq \frac{n}{s^*}$ and s^* divides n, then all nonempty clubs are of size s^*.
- If $|G| > \frac{n}{s^*}$ but s^* does not divide n, i.e. $n = rs^* + \ell$, $0 < \ell < s^*$, and $u(\ell) > u(s^* + 1)$, there are r clubs of size s^* and one club of size ℓ.
- If $|G| < \frac{n}{s^*}$, players are distributed across clubs as evenly as possible, i.e. each club is either of size $\lfloor n/|G| \rfloor$ or of size $\lceil n/|G| \rceil$.

This gives rise to the following observation.

Observation If a NCE exists, it will be unique up to a relabelling of players and locations.

6 Efficiency of Nash Club Equilibrium

We will next show that, in our model, a NCE is a strong Nash equilibrium profile, i.e. no group of players (not even from different clubs) could gain by jointly deviating.

Formally, a strategy profile g is a *strong Nash equilibrium* if for every subset $S \subset N$ and all strategy profiles $y_S = \{y_i : i \in S\}$ for the members S, there exists at least one player $i \in S$ such that

$$u_i(y_S, g_{-S}) < u_i(g).$$

It is obvious that every strong Nash equilibrium is a NCE. The converse is also true, as stated in the following proposition.

Proposition 3 *A Nash club equilibrium is a strong Nash equilibrium of the game* Γ.

Proof. See appendix.

We will now analyze the convergence properties of the Markov process.

7 Convergence to a Nash Club Equilibrium

We will now show that, if a NCE exists, the adaptation process will converge to a NCE profile with probability one. We provide an algorithm describing a path of moves of viable coalitions that terminates in a NCE.

There is one situation that is slightly more delicate; this is the case where the NCE club size is s^* (or s^* and ℓ for some $\ell < s^*$) and where $u(\ell') < u(s^* + 1)$ for some positive ℓ' greater than one. For simplicity, to discuss this case, suppose that the NCE club size is unique and equals $s^* = 4$. Now suppose that $|G| = 5$ and the size of the population n is eight ($n = 8$).(Convergence would be quicker if there were fewer locations, subject still to the condition that $|G| \geq |N| / s^*$). Let us also suppose that $u(2) < u(3) < u(s^* + 1)$. It is clear that a person in a club consisting of himself alone will prefer to move to a club with $s^* + 1$ members than to another club containing only one member. To illustrate the treatment of this situation, we describe a path by a series of lists where the k^{th} number in a list represents the number of people in the k^{th} club, $k = a, ..., d$.

Initial state:	$4,1,1,1,1$
	$5,1,1,1,0$
	$2,4,1,1,0$
	$2,5,1,0,0$
	$4,3,1,0,0$
final NCE state:	$4,4,0,0,0$

The subtlety is to first allow a viable coalition contained in one of the smallest clubs to move to a location containing s^* members and then to allow a viable coalition contained in the club with s^*+1 members to move to the largest club with fewer than s^* members. To take account of situations such as that illustrated by the penultimate state $4,3,1,0,0$, during each 'loop' in the procedure, there is a positive probability that some viable coalition moves to a location with fewer than s^* members.

The situation above is slightly different if we have $u(2) < u(s^*+1)$ but $u(3) > u(s^*+1)$. In this case, the following list illustrates a path terminating in a NCE.

Initial state:	$4,1,1,1,1$
	$5,1,1,1,0$
	$2,4,1,1,0$
	$3,4,1,0,0$
final NCE state:	$4,4,0,0,0$

Proposition 4 *If the set of Nash club equilibria of the game Γ is nonempty, the adaptation process will converge to a Nash club equilibrium profile with probability one as time tends towards infinity, no matter where the process starts.*

Proof. Suppose a NCE exists and let s^{**} denote a club size induced by NCE. Observe that if $|G| \geq \frac{n}{s^*}$ then, in the case that s^* divides n, we have $s^{**} = s^*$ and otherwise $s^{**} \in \{s^*, \ell\}$ where $n = rs^* + \ell$, r and ℓ are positive integers, $\ell < s^*$, and if $|G| < n/s^*$ then $s^{**} \in \left\{ \left\lfloor \frac{n}{|G|} \right\rfloor, \left\lceil \frac{n}{|G|} \right\rceil \right\}$. We split the proof into three cases:

Case (A) $G| \geq \frac{n}{s^*}$ and s^* divides n.

Case (B) $n = rs^* + \ell$, r and ℓ are positive integers, $\ell < s^*$.

Case (C) $|G| < n/s^*$ and $s^{**} \in \left\{ \left\lfloor \frac{n}{|G|} \right\rfloor, \left\lceil \frac{n}{|G|} \right\rceil \right\}$.

Starting from a state that is not NCE, we construct a path, i.e. a sequence of states with positive transition probabilities, that terminates in a NCE. This shows that any state that is not NCE must be transient, which implies that it cannot be absorbing.

Case (A). Consider a state g that is not NCE. Then there must be clubs of nonoptimal size. There are two mutually exclusive possibilities: either (a) there is a club a with $n_a(g) > s^*$ or (b) all clubs are of size s^* or less, with some clubs being strictly smaller than s^*. Our approach is to first give coalitions contained in clubs with more than s^* members an opportunity to move until there are no clubs of size greater than s^*. We then consider viable coalitions smaller than s^* and give all of them (one after the other) the opportunity to move, where their movement will not result in coalitions of size greater than s^*. Finally, we deal with the situation where the only viable coalitions are those whose optimal move is to join a coalition with s^* members. Here, as illustrated in the examples above, we mix the movement of singletons to clubs of size s^* with movements of viable coalitions contained in clubs of size greater than s^* to join smaller clubs until we arrive at an equilibrium.

Step 1. Suppose there is a club a with $n_a(g) > s^*$. This implies that there exists a location b with $n_b(g) < s^*$ (including the case of $n_b = 0$). Thus, any coalition $C \subset N_a(g)$ with $s^* - n_b(g)$ players is viable since $n_b(g) + |C| = s^*$ and s^* maximizes per capita utility. Suppose one such coalition gets the chance to move, which happens with positive probability. Let g' denote the state after the move of C. Note that

$$|\{d \in G : n_{g'}(d) = s^*\}| > |\{d \in G : n_g(d) = s^*\}|,$$

i.e. the number of clubs of optimal size is increased. Repeat this argument until all club sizes are equal to or smaller than s^*. If all clubs are now of size s^* we are done. Otherwise, we continue by next treating cases where no clubs are larger than s^* and some are smaller.

Let $S^*(g^1)$ denote the set of clubs of size s^* in state g^1 at the conclusion of Step 1.

Step 2. We first need to consider the possibility that $S^*(g^1) = \varnothing$. For this case let $\widehat{S}(g^1)$ denote the set of clubs of maximal size, say $\widehat{s}$. Observe that $\widehat{S}(g) \neq \varnothing$ and, at this stage in our proof, $\widehat{s} < s^*$. (Equality will hold only if $S^*(g) \neq \varnothing$.) In this case, at least one viable coalition's best move is to join a club of size $\widehat{s}$ so as to form the largest possible club with no more than s^* members. Suppose one such viable coalition C gets the chance to move. The movement of C to a club in $\widehat{S}(g^1)$ will induce a new strategy profile g^2 with a (weakly) increased number of empty locations and with an increase in the size of the membership of at least one nonempty location. Repeating this argument as many times as possible will lead to a situation where eventually there are clubs of size s^* and it is not possible for a new club of size s^* to be formed by the movement of a viable coalition (with fewer than s^* members) to a new location. Let g^3 denote the state at the end of Step 2.

Step 3. Suppose that there remain clubs containing fewer than s^* members. There are two possible cases:

(3.1) A viable coalition will move to a location with fewer than s^* members to create a club with $h \leq s^*$ members. In this case it must hold that $u(h) \geq u(s^* + 1)$.

(3.2) A viable coalition is a singleton and, if given the chance, will move to a club with s^* members.

Step 3a. We next sequentially give all viable coalitions satisfying the conditions of (3.1) the opportunity to move. Let g^4 denote the resulting state. (If there are no viable coalitions satisfying the conditions of (3.1) then $g^4 = g^3$). Note that in the state g^4, if there are any viable coalitions, they satisfy the conditions of (3.2).

Step 3b. Suppose in state g^4 there are viable coalitions satisfying the conditions of (3.2). (Note that for this to occur it must be the case that $s^* \geq 3$. Otherwise this procedure would terminate after Step 3a.)

Let s^1 denote the size of one of the smallest clubs in the state g^4 and suppose without loss of generality that $n_a = s^1$. XXNote that in this case $s^1 < s^* - 1$.

(*) Let one member of N_a move to a club with s^* members, say to N_b. Now N_b has $s^* + 1$ members. Let g^5 denote the resulting state.

Next, let N_c be the largest club with fewer than s^* members in state g^4. Then there is a viable coalition C contained in N_b with $|C| + n_c = s^*$. Give such a coalition the opportunity to move. Note that this leaves the same number of clubs of size s^* as in g^4 and with an increase in the size of at least one club containing fewer then s^* members.

Return to Step 3a. and repeat the procedure until $n_a = 0$. (Note that this is possible since there is at least one club with s^* members.) This brings us to a state, say g^5, with strictly more empty locations, at least one club with s^* members, and an increase of at least 1 in the size of the smallest club. (**)

Return to Step 3a and repeat the this procedure until it is no longer possible. We must then have reached a NCE.

Case B. Our procedure in this case is basically the same as for Case (A) except that at some point the largest club smaller than s^* will contain ℓ members. Once such a club exists do not give possible viable coalitions contained in one such club an opportunity to move (except at the end of the procedure described for Case A – but then there will no longer be any viable coalitions).

Case C. Let S^* denote the set of NCE club sizes and let $\overline{s}$ denote the maximal club size in S^* and let $\underline{s}$ denote the minimal club size in S^* (that is, $\overline{s} = \max\{s : s \in S^*\}$ and similarly, $\underline{s} = \min\{s : s \in S^*\}$). Either $\overline{s} = \underline{s}$ or $\underline{s} = \overline{s} - 1$. Note that in Case C in any state g it must hold that the average number of members of each club, say s^{Av}, must lie between (or be equal to) $\overline{s}$ and $\underline{s}$.

Suppose that g is not a NCE. Then, there must exist at least one club, say N_a, with $n_a > \overline{s}$. Moreover, there must exist another club, say N_b, with $n_b < \overline{s}$. Suppose, without any loss, that $n_b \leq n_c$ for all other clubs n_c. Let $C \subset N_a$ be a viable coalition which, if given the opportunity, will move to a location with size n_b. We give C the opportunity to move. It is then a best reply for C to move to location a. Repeat this procedure until it is no longer possible to find a viable coalition. This will happen only when all clubs are of size $\overline{s}$ or $\underline{s}$; thus the outcome of the procedure is a NCE. □

8 Ergodic Nash Club Equilibrium

Note that, if $|G| \leq n/s^*$, under our assumptions a NCE always exists, where all players are distributed evenly across all clubs. In this section, we focus on the case $|G| > n/s^*$.

When the optimal club size is such that $n/s^* \notin I$, and $|G| > n/s^*$, a NCE might fail to exist, as we demonstrated by an example in Section 5. Here is another example more convenient for the current purposes.

Example.[12] $G = \{a, b, \ldots, f\}$, $N = \{1, 2, \ldots, 5\}$, and $u_i(g) = 1 + \phi(s)$ where

$$\phi(s) = \begin{cases} 0 & \text{for} \quad s = 1 \\ 2/s & \text{for} \quad s \geq 2. \end{cases}$$

The table shows each club member's payoff for each possible club size s:

s	$u_i(\cdot)$
1	1
2	2
3	1.66
4	1.5
5	1.4

[12] This example is taken from Arnold and Schwalbe (2002).

In this game the optimal club size is $s^* = 2$. But at most two clubs of size 2 can be formed. The left over player can then gain by joining any of these two clubs, since this increases his payoff from 1 to 1.66. However, in a club of size 3, any two players can gain by forming a coalition and deviating to an unoccupied location. Thus, no NCE exists.

The nonexistence of a NCE is due to an indivisibility of optimally sized clubs or, in other words, a 'non-balancedness' problem.[13] We now define a notion of NCE that takes this problem into account.

Definition 3 *We define a strategy profile g as a* k–remainder NCE *if there exist k players, $k \geq 0$, such that, if these players are removed from the population, the strategies of the remaining $n-k$ players will form a NCE (on the reduced strategy set G^{n-k}) where $k = n - s^* \lfloor n/s^* \rfloor$.*

In the example above, the strategy profiles $g' = (a, a, b, b, c)$ and $g'' = (a, a, a, b, b)$ both form 1–remainder NCE: removing player 5 from g' and player 3 from g'' yields a NCE in both cases. In contrast, the profiles (a, a, a, a, b) and (a, b, c, d, d) are not 1–remainder Nash club equilibria. Proposition 5 characterizes k–remainder NCE for those cases where $k > 0$.

Proposition 5 *Assume that $|G| > n/s^*$ and $n = rs^* + \ell$ for some positive integers r and $\ell < s^*$. Also assume that $u(\ell) < u(s^* + 1)$. Then:*
(i) Any strategy profile g with an induced partition of the set of players into r clubs with no fewer than s^* members and no more than $s^* + l$ members and with the remaining clubs of size less than or equal to ℓ is a k-remainder NCE for $k = \ell$.
(ii) Any strategy profile g with an induced partition of the set of players into fewer than r clubs of size greater than or equal to s^* is not a k-remainder equilibrium.

Proof. Let g be a strategy profile satisfying the conditions of part (i) of the proposition. Since there are r clubs each containing at least s^* members we can remove players from these clubs so that there are only s^* players remaining in each of these clubs. Also, remove all players from the clubs of size less than s^*. The number of players removed is equal to k and the strategy choices of the remaining players constitute a NCE, since each nonempty club now contains s^* players.

[13] The indivisibility problem is that the optimal club is indivisible. This would be solved if there were constant per capita benefits to club formation – in which case clubs containing more than one member would be redundant. An alternative approach, following Wooders (1978), would be to allow a range of optimal club sizes containing two relatively prime integers, for example, s^* and $s^* + 1$. Then, since any sufficiently large population size n can be written as the sum of nonnegative integer multiples of s^* and $s^* + 1$, for all sufficiently large populations, an NCE would exist.

The proof of part (ii) of the Proposition follows from the observation that it is impossible to remove only ℓ agents and have all nonempty clubs of size s^* (so that the outcome, restricted to the remaining agents, would be a NCE). □

Obviously, if preferences are single peaked, k–remainder NCE always exist. The special case of $k = 0$ corresponds to the definition of NCE.

For $k > 0$, a k–remainder NCE is not an absorbing state since it is not a NCE, i.e. there are coalitions that will switch locations when they get the opportunity to adjust their strategies. In the example, for instance, in the strategy profile $g^* = (a, a, b, b, c)$, player 5 would switch to either a or b, and in state $g'' = (a, a, a, b, b)$, a coalition of players 1 and 2 (or 1 and 3, or 2 and 3) would switch to an unoccupied location. Our main result is to show that, in the long run, only k–remainder NCE will be observed. To this end, we need the definition of an ergodic set.

Definition 4 *An ergodic set $E \subset G^N$ is a set of states such that, first, each state in E can be reached from every other state in E in a finite number of steps, and second, once the set E is reached, it cannot be left again, i. e. the probability of the system's going from some state $g \in E$ to some other state $g' \notin E$ is equal to zero. Further, ergodic sets are minimal in the sense that there is no proper subset of E satisfying the above conditions.*

Note that an absorbing state is the same as a singleton ergodic set.

An ergodic NCE is then defined as follows.

Definition 5 *Given any stage game Γ with optimal club size s^*, an* ergodic Nash club equilibrium *is a set of states $M \subset G^N$ with the following properties:*

1. *For $k = n - s^* \lfloor n/s^* \rfloor$, every state $g \in M$ is a k–remainder NCE, and*
2. *M is an ergodic set.*

Obviously, an ergodic NCE is a subset of the set of all k–remainder NCE. Also, for the case of $k = 0$, each NCE is an ergodic NCE.

The following proposition establishes our convergence result for $k > 0$.

Proposition 6 *The adaptation process will converge to an ergodic Nash club equilibrium with probability one as time goes to infinity, no matter where the process starts.*

This implies that, once the process has reached a k–remainder NCE, only k–remainder NCE will be observed forever after.

Proof The first part of the proof (step 1) is analogous to the one of convergence to NCE for the case $n = rs^* + l$ and $l > 0$.

The theory of finite Markov chains states that the process will reach an ergodic set with probability one as time goes to infinity.[14] Given this, the proof proceeds in two steps. First, we show that, from every state that is not a k–remainder NCE, there is a path terminating in a k–remainder NCE. Second, we show that once a k–remainder NCE is reached, any state that can be reached from there will also be a k–remainder NCE. The two steps together imply that any state that is not a k–remainder NCE cannot be part of an ergodic set. This in turn implies that any ergodic set contains only states that are k–remainder NCE, and the theory of finite Markov chains ensures that this set will be reached with probability one as time goes to infinity.

Step 1. The first step of the proof simply follows the procedure in our prior convergence result. Eventually, a state will be reached where there are r clubs of size s^*. Any such state is a k-remainer NCE with $k = \ell$.

Step 2. Suppose the process has reached a state g that is a k–remainder NCE. Note that, in this state, any viable coalition is of size s^* or less. Let $n = rs^* + k$. Then there are r clubs of size $s \geq s^*$ and at most k clubs of sizes between one and k.

(i) Consider a club a with $n_a(g) > s^*$. If there is a viable coalition $C \subset N_a(g)$, since $|G| > n/s^*$ there must be another location b with $n_b(g) < s^*$ and $n_b(g) + |C| = s^*$ (possibly $n_b(g) = 0$). Suppose C moves to b. Call the resulting state g'. Notice that g' contains one club of size s^* more than the original state g. Now let us compare the number of leftover players before and after the move of C from a to b.

Before the move, there were $n_a(g) - s^*$ players left over at a and, in total, $n_b(g)$ players at b (since $n_b(g) < s^*$). In addition, there may be other left over players at other locations. Denote the number of those by m, $m \geq 0$. Thus, the number of left over players in state g is

$$n_a(g) - s^* + n_b(g) + m.$$

Let g' denote the state after the move. In this state, club b will have s^* members so there will not be any left over players at location b. At club a, the number

[14] Kemeny/Snell (1976), Theorem 3.1.1 on page 43.

of players left over will be $n_a - |C|$, where $|C| = s^* - n_b$. As a result, the total number of left over players in state g' is

$$n_a(g) - |C| + m = n_a(g) - (s^* - n_b(g)) + m,$$

which is exactly the same as in state g before the move. Since g is a k–remainder NCE, it follows that g' must also be a k–remainder NCE.

(ii) Now consider a club b with $n_b(g) < s^*$. There are two possible cases. Either a viable coalition can optimally move into a club containing no fewer than s^* members or a viable coalition can optimally move into a club with fewer than s^* members.In the first case, suppose some player j from a club b with $n_b(g) < s^*$ forms a viable (singleton) coalition and gets the chance to move. The "best reply" for j is to join the smallest of the clubs of size s^* or more (this is not necessarily unique). But this move does not affect the number of left over players: While the number of left over players at b is reduced by one (player j), at the same time it is increased by one at the club j has moved to. This means that, since the state before the move is a k–remainder NCE, the state after the move will also be a k–remainder NCE. In the second case, the both the club containing the viable coalition and the club into which the viable coalition moved consists of left overs, so the total number of left overs is unchanged. □

An ergodic NCE is a subset of the set of all k–remainder NCE. All states in an ergodic NCE are characterized as follows.

Proposition 7 *Let $h \in G^n$ be element of an ergodic NCE. Then there is no club a with $n_a(h) > s^* + \lceil k/r \rceil$.*

Proof We show that any state that contains a club of size larger than $s^* + \lceil k/r \rceil$ will never be visited again once it has been left.

Start from a state g with $n_a(g) > s^* + \lceil k/r \rceil$ for some club a. According to step 1 above, there is a positive probability of the process reaching a state with r clubs of size s^* and one of size k. Call this state g'.

Now, if g were in the ergodic set, there would be a path from g' to g. However, this is not the case. In state g', without loss of generality suppose the clubs $a_1, a_2, \ldots, a_r$ are of size s^* and state a_{r+1} is of size k. Then, there are exactly k viable coalitions, namely the singleton subsets of club a_{r+1}. If all of these viable coalitions get the chance to move one after the other, we will end up with the k players evenly distributed across clubs $a_1, a_2, \ldots, a_r$ (because for $s > s^*$ utility is decreasing in club size). But then each club will have no more than $s^* + \lceil k/r \rceil$ members. At this stage, since $|G| \leq r + 1$, there will be at least one unoccupied location such that the only viable coalitions are of size s^*. But any move by any such coalition will induce club sizes of s^* or less. Thus, state g can never be reached again once it has been left. This contradicts g being part of an ergodic set. □

Further characterizations of the ergodic NCE depend on the exact relationships between s^*, k, and $|G|$.

9 Concluding Discussion

This paper provides a game theoretic model of club formation where player mobility is explicitly modelled by a dynamic process. In each period, the members of any given club may form coalitions, and then choose locations by a myopic best reply rule. We define a Nash club equilibrium (NCE) by a strategy configuration no coalition wants to deviate from. If a NCE exists, the club structure induced by it is unique, and the state is efficient in the sense of strong Nash equilibrium. Further, we show that our dynamic process defined by the myopic best–reply rules on the part of the coalitions converges to a NCE when time tends towards infinity.

To broaden our existence results in an intuitive way, we define an ergodic Nash club equilibrium as a set of club profiles each of which constitutes a k–remainder NCE, where all but k players are playing NCE, and k is the minimal number of left over players. We show that an ergodic Nash club equilibrium exists and that, as time tends towards infinity, the process will converge to an ergodic Nash club equilibrium with probability one.

Our results are novel and interesting in several ways. First, there is an interesting relationship between different equilibrium concepts. On the one hand, even though strong Nash equilibrium takes into account deviations by all kinds of coalitions, whereas we allow only for deviations by subsets of the set of members of a club, our concept of NCE is shown to correspond to strong Nash equilibrium, in the context of our model. On the other hand, a NCE (and therefore a strong Nash equilibrium) may fail to exist. But, since non–existence is merely due to problems of 'numbers mismatching', this problem can be solved by our notion of a k–remainder Nash club equilibrium, which always exists.

Further, our equilibrium notions can be related to concepts of cooperative game theory, as employed in the literature on coalition formation in hedonic games, e.g. Bogomolnaia and Jackson (2002). Their concept of *Nash stability* corresponds to a Nash equilibrium of our club formation game, i.e. a strategy profile (player partition) that is immune to deviations by single players. Bogomolnaia and Jackson prove existence of a Nash stable partition of the population for the case of hedonic preferences, where a player's utility depends only on the members of a club but not on the location itself. The result on the existence of Nash equilibrium in the case of location dependent preferences can be seen as a corollary: If individuals are identical, and their preferences depend only on the club size, then a Nash equilibrium, and therefore a Nash stable partition, exists.

Moreover, the cooperative notion of a *core stable partition* corresponds to a strong Nash equilibrium of our game: No group of players can gain by deviating. Since our concept of NCE is weaker than that of a strong Nash equilibrium, the set of strong Nash equilibria (and thus the set of core stable partitions) is a subset of the set of NCE. Thus, we have the following relationship between cooperative and non–cooperative concepts:

Strong Nash equilibrium (core stability) $\Rightarrow$ NCE $\Rightarrow$ Nash equilibrium (Nash stability).

Second, our model is dynamic. In each period, new coalitions may form, and new locations can be chosen. This reflects player mobility, or the agents' "voting with their feet". Finally, even though players are myopic, equilibrium club profiles will be reached in the long run. Thus, being myopic may "help" a population to reach a desirable outcome.

It is easy to see that, if the number of locations is sufficiently large, then replicating both the player set and the set of locations will lead to a situation where the percentage of left-over players becomes small. Were transfer payments allowed, each agent in an optimal club could be charged a small fee (in the nature of unemployment insurance) and the totality of these fees could compensate the left over players, which would yield outcomes satisfying another notion of approximate stability. Similar ideas have appeared in game theory since Shubik (1971) and been used in club theory since Pauly (1970) and Wooders (1980). We do not pursue this further here.

We plan to continue the investigation in this paper in several directions. First, we propose to introduce crowding types (that is, external effects of players on each other, independent of their preferences), as in Conley and Wooders (2001) and earlier papers. A particularly interesting extension may be to situations where players choose their crowding types, their skills, or educational levels, as in Conley and Wooders (2001).

Since our process of coalition formation is myopic, an interesting direction of research would be to allow long term coalition formation, so that, for example, two players who meet in a club may decide to commit themselves to act jointly for several periods.

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Appendix

Proof of proposition 2 Suppose $u(\ell) > u(s^* + 1)$. Let g be any strategy profile with the property that the induced club structure has r clubs of size s^* and one of size ℓ. It is immediate that g is an NCE. For suppose not. No club of size s^* contains a viable coalition. So suppose C is a viable coalition contained in the club with ℓ members. The only interesting alternatives available to the members of C are to move to an empty club, which cannot increase the payoffs of the members of C, or to move to a club containing s^* members. Then $u(s^* + |C|)$ is decreasing in $|C|$. It follows from the definition of a viable coalition that C must contain only one member. Since $u(\ell) > u(s^* + 1)$ the movement of the individual player constituting C to a club of size s^* would decrease his utility.

Now suppose that $u(\ell) < u(s^*+1)$. Observe that $|G| \geq n/s^*$ and n/s^* is not an integer implies that $|G| > n/s^*$ (and also that $|G| > r$) since $|G|$ is an integer. Let g be any strategy profile with an induced club structure where at least one location contains fewer than s^* members. Suppose a is such a location. Now if all the other nonempty clubs contain s^* members, from our assumption that $u(\ell) < u(s^* + 1)$, $N_a(g)$ contains a viable coalition and therefore g is not a NCE. If some other nonempty club contains fewer than s^* members, again $N_a(g)$ contains a viable coalition. Thus, if g is a NCE, the only remaining possibility is that all clubs contain at least s^* members. But then at least one club contains more than s^* members. Moreover, there exists at least one location with no members so any club containing more than s^* members contains a viable coalition and g cannot be a NCE. □

Proof of proposition 3. We show that, if a strategy profile g is not a strong Nash equilibrium, then it is not NCE. Suppose g is not a strong Nash equilibrium. Then, there exists a coalition $S \subset N$ and a strategy profile $y = (y_S, g_{-S})$ such that $u_i(y) > u_i(g)$ for all $i \in S$. Now let S_a denote the nonempty intersection of S with the membership of some club a (clearly there exists such a club a). Since g is not a strong Nash equilibrium we can select a so that $|S_a| \neq s^{**}$ where s^{**} denotes a club size induced by NCE; if $|S_a|$ were equal to s^{**} for all $a \in G$, then we would have a contradiction to the supposition that there exists a strategy profile $y = (y_S, g_{-S})$ such that $u_i(y) > u_i(g)$ for all $i \in S$. Now observe that from our arguments in the preceding proof, it follows that there is some improving strategy for some (possibly different) coalition S', which implies that g is not a NCE. □