THE NORMATIVE APPROACH TO THE MEASUREMENT OF MULTIDIMENSIONAL INEQUALITY

by

John A. Weymark

Working Paper No. 03-W14R

July 2003
Revised January 2004

DEPARTMENT OF ECONOMICS
VANDERBILT UNIVERSITY
NASHVILLE, TN 37235

www.vanderbilt.edu/econ

The Normative Approach to the Measurement of Multidimensional Inequality

John A. Weymark
Department of Economics, Box 1819, Station B,
Vanderbilt University, Nashville, TN 37235, U.S.A.
(e-mail: john.weymark@vanderbilt.edu)

First Version: June 2003
Current Version: January 2004

Abstract. This article provides an introduction to the normative approach to multidimensional inequality measurement. Multivariate generalizations of the procedures used to construct univariate inequality indices from social evaluation orderings are described. Axiomatizations of multivariate Atkinson, Kolm–Pollak, and generalized Gini indices are discussed. Maasoumi's [*Econometrica* (1986)] two-stage procedure for constructing a multivariate inequality index is critically examined. A dominance criterion proposed by Tsui [*Social Choice and Welfare* (1999)] that takes account of the dependence between the individual distributions of the attributes is also considered.

Keywords and Phrases: multidimensional inequality, inequality indices, normative inequality measurement, multivariate majorization, multivariate dependence.

JEL Classification Number: D63.

1. Introduction

Univariate indices of income inequality provide an inadequate basis on which to compare the inequality of well-being within and between populations. Recognition of this fact has lead to an explosion of research recently on multidimensional economic inequality, beginning with the seminal articles by Kolm (1977) and Atkinson and Bourguignon (1982). These articles are primarily concerned with developing dominance criteria for ranking multivariate distributions. When there are multiple attributes of well-being being compared, one distribution may be more equal than a second if the former exhibits less dispersion than the latter or if it reduces the positive dependence between the individual distributions of the attributes. Kolm focused on the first of these ways in which inequality may manifest itself, whereas Atkinson and Bourguignon focused on the second.

Dominance criteria only provide partial orderings of the possible distributions of attributes. In contrast, an inequality index can be used to completely order all distributions. In the normative approach to inequality measurement, a social evaluation (or its representation, a social evaluation function) is used to construct an inequality index. A social evaluation ranks alternative distributions according to their social desirability. The use of a social evaluation makes explicit the value judgements underlying an inequality index. For univariate distributions, the normative approach was pioneered by Atkinson (1970) and Kolm (1969), who introduced general procedures for constructing an inequality index from a social evaluation. Multi-attribute extensions of their methodologies have been proposed by Kolm (1977) and Tsui (1995).[1]

The purpose of this article is to provide an introduction to the normative approach to the measurement of multidimensional inequality. While my focus is on indices of inequality, rather than on dominance criteria, it is desirable for an inequality index to be consistent with normatively-based dominance criteria. As a consequence, it necessary to devote some attention to this issue. However, it is beyond the scope of this article to provide a systematic survey of the literature on multivariate dominance criteria.[2]

[1] Inequality is not the only social phenonmenon for which multivariate indices have been developed. For example, the United Nations Human Development Index (see United Nations Development Programme (1990)) aggregates indicators of longevity, education, and command over resources into an overall measure of the standard of living.

[2] See Trannoy (2004) for a detailed discussion of multivariate dominance criteria. Some of the issues in the measurement of multidimensional inequality that are not considered

Here, as in much of the literature on the measurement of inequality, it is assumed that the population is homogeneous in the sense that individuals do not differ in welfare-relevant characteristics other than the attributes that are the focus of the analysis. In a heterogeneous society, individuals may differ for a number of reasons—they may belong to households of different size, they may have different preferences, or, even if they have the same preferences, they may have different cardinal utility functions because of differences in their physical characteristics. In the past decade, considerable progress has been made on extending the theory of inequality measurement for homogeneous populations to the heterogeneous case. See, for example, Blackorby, Bossert, and Donaldson (1999), Ebert (1995), Shorrocks (1995), and Weymark (1999).

In Section 2, some of the notation used in this article is introduced. If a social evaluation is to provide a satisfactory basis on which to construct a normative inequality index, it should satisfy a number of basic properties. These properties are considered in Section 3. Among these properties are multi-attribute generalizations of the Pigou–Dalton transfer principle.

The procedure proposed by Atkinson (1970) and Kolm (1969) for constructing relative (i.e., scale invariant) univariate inequality indices and the procedure proposed by Kolm (1969) for constructing absolute (i.e., translation invariant) univariate inequality indices are reviewed in Section 4. Multi-attribute generalizations of these procedures are discussed in 5.

The normative approach has been used by Tsui (1995) to define multi-attribute generalizations of the univariate Atkinson and Kolm–Pollak classes of inequality indices and to axiomatically characterize their underlying social evaluations. These indices are considered in Section 6.

The social evaluation functions used to construct Tsui's indices have a two-stage aggregation property. In the first stage, a utility function is used to determine the distribution of utilities and then these utilities are summed. Maasoumi (1986) has suggested constructing inequality indices directly using a two-stage procedure in which a univariate inequality index is applied to the distribution of utilities obtained in the first stage. Maasoumi's proposal is the subject of Section 7.

Section 8 discusses the multi-attribute generalized Gini indices introduced by Gajdos and Weymark (2003). The social evaluation functions from which these indices are derived also have a two-stage aggregation property, but,

---

here are discussed in the surveys by Maasoumi (1999) and Savaglio (2002).

in contrast to Tsui (1995) and Maasoumi (1986), the order of aggregation is reversed—first the distributions of each attribute are aggregated using univariate generalized Gini social evaluation functions and then the values of these functions are aggregated into an overall evaluation in a second stage.

Section 9 considers a dominance criterion proposed by Tsui (1999) that takes account of the dependence between the individual distributions of the attributes. Some further issues in the measurement of multidimensional inequality are briefly discussed in Section 10.

2. Preliminaries

There is a fixed set of individuals $N = \{1, \ldots, n\}$, with $n \geq 2$. The set of attributes is $Q = \{1, \ldots, q\}$. It is assumed that the quantity of each of these attributes can be continuously varied. Examples of such attributes include income, life expectancy, educational attainment, and health status.[3] Attributes need not differ in kind. For example, the attributes could be incomes in different time periods or in different states of the world. In the latter case, we are measuring inequality under uncertainty.

A *distribution* of attributes among the population is an $n \times q$ real-valued matrix. The ijth entry of a distribution matrix X is x_{ij}, individual i's quantity of the jth attribute. The ith row of X is denoted $x_{i\cdot}$ and the jth column is denoted $x_{\cdot j}$. If there is only one attribute, the distribution is written as x rather than as X or $x_{\cdot 1}$. Three sets of distribution matrices are considered for the *domain* $\mathcal{D}$ of admissible distributions. The first, denoted $\mathcal{M}_1$, is the set of all possible distribution matrices. The second, denoted $\mathcal{M}_2$, is the set of all distribution matrices X for which both (a) $x_{ij} \geq 0$ for all $i \in N$ and all $j \in Q$ and (b) $x_{\cdot j} \in \mathbb{R}^{n*}_{+} := \mathbb{R}^n_+ \setminus \{0_n\}$ for all $j \in Q$.[4] The third, denoted $\mathcal{M}_3$, is the set of all distribution matrices X for which $x_{ij} > 0$ for all $i \in N$ and all $j \in Q$. Note that for distribution matrices in $\mathcal{M}_2$ and $\mathcal{M}_3$, the mean value of any attribute is positive. Except where otherwise specified, $\mathcal{D}$ can be any one of these three domains in the rest of this article.

For any $x \in \mathbb{R}^n$, $\mu(x)$ is the mean of x, $\tilde{x}$ is the permutation of x for which

[3] While, in principle, each of these attributes is subject to continuous variation, in practice, they may only take on a finite number of values. For example, qualitative measures of health status employ discrete categories. For an analysis of inequality measurement for categorical data, see Allison and Foster (2004).

[4] $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ denote the set of real numbers, nonnegative real numbers, and positive real numbers, respectively. 0_n (resp. 1_n) is an n-vector of zeros (resp. ones).

$\tilde{x}_1 \geq \tilde{x}_2 \geq \cdots \geq \tilde{x}_n$, and $\hat{x}$ is the permutation of x for which $\hat{x}_1 \leq \hat{x}_2 \leq \cdots \leq \hat{x}_n$. For $u, v \in \mathbb{R}^n$, u *strictly generalized Lorenz dominates* v if $u \neq v$ and for all $k \in N$, $\sum_{i=1}^k \hat{u}_i \geq \sum_{i=1}^k \hat{v}_i$.[5] If $\sum_{i=1}^n \hat{u}_i = \sum_{i=1}^n \hat{v}_i$ in this definition, then u *strictly Lorenz dominates* v.

3. Basic Properties for a Social Evaluation Relation

The inequality indices considered in this article are derived from explicit social evaluations of the possible distribution matrices. A *social evaluation* is a binary relation $\succeq$ on the set of distribution matrices $\mathcal{D}$. The relation $\succeq$ is interpreted as meaning "weakly socially preferred to". The symmetric and asymmetric factors of $\succeq$ are $\sim$ and $\succ$, respectively. A function $W\colon \mathcal{D} \to \mathbb{R}$ that represents $\succeq$ is called a *social evaluation function*.

By defining $\succeq$ directly on $\mathcal{D}$, the analysis is not limited to welfarist social evaluations. *Welfarism* is the principle that the only feature of a distribution that is socially relevant is the vector of utilities associated with this distribution. Welfarist social objectives can be described using a social welfare function. A *social welfare function* is a real-valued function defined on n-tuples of utilities. If the utility functions are known, a social welfare function can be used to construct a social evaluation function—for each distribution, the value of the social evaluation function is the value assigned by the social welfare function to the utilities obtained with this distribution. For example, suppose that the social welfare function is utilitarian and, in keeping with the assumption that the society is homogeneous, that everybody has the same utility function. The corresponding social evaluation function then has the form

$$W(X) = \sum_{i=1}^{n} U(x_{i\cdot}), \quad \forall X \in \mathcal{D}, \tag{1}$$

where $U\colon \mathbb{D} \to \mathbb{R}$ is the common utility function, where $\mathbb{D} = \mathbb{R}^q$ (resp. $\mathbb{R}^q_+$, $\mathbb{R}^q_{++}$) if $\mathcal{D} = \mathcal{M}_1$ (resp. $\mathcal{M}_2$, $\mathcal{M}_3$).[6] Lack of information about individual utility functions limits the applicability of this approach. In contrast, it

[5] See Shorrocks (1983) for a detailed discussion of generalized Lorenz domination. In the mathematics literature, generalized Lorenz domination is known as weak supermajorization. See Marshall and Olkin (1979, p. 10).

[6] This functional form is used by Atkinson (1970) in the unidimensional case and by Atkinson and Bourguignon (1982) in the multidimensional case. However, Atkinson (1983, p. 5) has said that "there is nothing inherently *utilitarian* in the formulation" given in (1).

is possible to use a social evaluation even if nothing is known about the individuals' utility functions other than that they are increasing in their arguments.

There are a number of basic properties that a social evaluation should satisfy if it is to serve as a satisfactory basis from which to construct an inequality index. These properties are formulated as axioms. There are two types of basic axioms: (a) axioms that are not concerned with the distributional sensitivity of the social evaluation and (b) axioms that are multi-attribute generalizations of the Pigou (1912)–Dalton (1920) transfer principle.

3.1. Non-Distributional Axioms

The first axiom requires $\succeq$ to be a complete preorder.

Ordering (ORD). The binary relation $\succeq$ is reflexive, complete, and transitive on $\mathcal{D}$.

The second axiom requires $\succeq$ to be continuous; i.e., any strict ranking of two distribution matrices is invariant to small perturbations in these matrices. Continuity ensures that the analysis is not overly sensitive to errors in measurement of the distributions.

Continuity (CONT). The sets $\{Y \in \mathcal{D} \mid Y \succ X\}$ and $\{Y \in \mathcal{D} \mid X \succ Y\}$ are open for all $X \in \mathcal{D}$.[7]

It is assumed that all of the attributes are desirable. Then, regardless of the exact form of individual preferences, the Pareto principle requires that increasing the quantity of any attribute for any individual is socially desirable provided that nobody's allocation of any attribute is decreased. The following monotonicity axiom states this principle formally.

Monotonicity (MON). For all $X, Y \in \mathcal{D}$, if $X \neq Y$ and $x_{ij} \geq y_{ij}$ for all $i \in N$ and all $j \in Q$, then $X \succ Y$.

If U is not interpreted as a utility function, then (1) simply amounts to saying that the social evaluation function is symmetric and additively separable.

[7] A matrix in $\mathcal{D}$ can be thought of as a vector in $\mathbb{R}^{nq}$. A subset of $\mathcal{D}$ is open if the corresponding set of vectors is open in $\mathbb{R}^{nq}$.

Equal treatment of individuals is captured by an anonymity axiom. In a homogeneous society, individuals are treated symmetrically if permuting the individual distributions is a matter of social indifference.

Anonymity (ANON). For all $n \times n$ permutation matrices Π and all $X \in \mathcal{D}$, $X \sim \Pi X$.

3.2. Multidimensional Transfer Principles

Distributional sensitivity of the social evaluation is obtained by requiring $\succeq$ to satisfy some form of the Pigou (1912)–Dalton (1920) transfer principle. The single attribute case is considered first. For concreteness, whenever there is a single attribute, it is supposed that this attribute is income.

A *Pigou–Dalton transfer* is a transfer of income from a richer to a poorer person that results in the initially poorer person ending up with less income than the initially richer person starts with.[8] Formally, if the initial incomes are x_{i_1} and x_{i_2} with $x_{i_1} < x_{i_2}$ and the size of the transfer is $\delta > 0$, then i_1's post-transfer income is $y_{i_1} = x_{i_1} + \delta < x_{i_2}$ (and, hence, i_2's post-transfer income is $y_{i_2} = x_{i_2} - \delta > x_{i_1}$). A Pigou–Dalton transfer can be equivalently expressed in terms of a strict T-transform. A *strict T-transform* is a linear transform defined by an $n \times n$ matrix T of the form

$$T = \lambda I_n + (1 - \lambda)\Pi_{i_1 i_2} \tag{2}$$

for some $\lambda \in (0, 1)$ and some $i_1, i_2 \in N$, where I_n is the $n \times n$ identity matrix and $\Pi_{i_1 i_2}$ is the $n \times n$ permutation matrix that interchanges the i_1 and i_2 coordinates. Letting $y = Tx$, it is easy to verify that $y_{i_1} = \lambda x_{i_1} + (1 - \lambda)x_{i_2}$, $y_{i_2} = (1 - \lambda)x_{i_1} + \lambda x_{i_2}$, and $y_k = x_k$ for all $k \notin \{i_1, i_2\}$.

If the distribution y is obtained from the distribution x by a sequence of Pigou–Dalton transfers (possibly involving a number of different pairs of individuals), then *y Pigou–Dalton majorizes x*. As is well-known (see Marshall and Olkin (1979, Chapter 1) or Hardy, Littlewood, and Pólya (1934)), (a) y Pigou–Dalton majorizes x if and only if (b) y strictly Lorenz dominates x if and only (c) $y = Bx$ for some $n \times n$ bistochastic matrix B that is not

[8]Note that in this formulation of a Pigou–Dalton transfer, the relative positions in the income distributions of the two individuals are permitted to differ pre- and post-transfer. It is sometimes assumed that the transfer does not reverse the rank order. In the presence of Anonymity, this restriction is of no consequence.

a permutation matrix.[9] The *Pigou–Dalton transfer principle* requires y to be socially preferred to x if y Pigou–Dalton majorizes x. When $q = 1$, a symmetric social evaluation function W satisfies this principle if and only if W is strictly S-concave.[10]

There have been a number of different ways proposed for generalizing the unidimensional Pigou–Dalton transfer principle so that it can be applied when there are multiple attributes. See Kolm (1977), Marshall and Olkin (1979, Chapter 15), and Savaglio (2002). Two of these multi-attribute Pigou–Dalton transfer principles are considered here.

In the first of these principles, the definition of Pigou–Dalton majorization is generalized by applying the same sequence of T-transforms to all attributes.

Definition. For all $X, Y \in \mathcal{D}$ for which $X \neq Y$, Y *uniformly Pigou–Dalton majorizes* X, denoted $Y \succ_{\text{PD}} X$, if $Y = PX$, where P is the product of a finite number of $n \times n$ strict T-transforms.

The corresponding multidimensional transfer principle is the *Uniform Pigou–Dalton Majorization Principle*.

Uniform Pigou–Dalton Majorization Principle (UPM). For all $X, Y \in \mathcal{D}$ for which $X \neq Y$, if $Y \succ_{\text{PD}} X$, then $Y \succ X$.

As noted above, if there is only one attribute, a sequence of Pigou–Dalton transfers can be equivalently expressed in terms of a bistochastic matrix. This observation suggests using bistochastic matrices to define a multi-attribute version of Pigou–Dalton majorization.

Definition. For all $X, Y \in \mathcal{D}$ for which $X \neq Y$, Y *uniformly majorizes* X, denoted $Y \succ_{\text{U}} X$, if $Y = BX$ for some $n \times n$ bistochastic matrix B that is not a permutation matrix.[11]

---

[9] A nonnegative square matrix is *bistochastic* if all of its row and column sums are equal to 1.

[10] A function $f\colon D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, is *S-concave* if $f(Bx) \geq f(x)$ for all $x \in D$ and all $n \times n$ bistochastic matrices B and it is *strictly S-concave* if the inequality is strict when B is not a permutation matrix. *S-convexity* and *strict S-convexity* are defined analogously by reversing the inequality signs.

[11] The terminology used here is based on Tsui (1999). In the terminology of Marshall and Olkin (1979), $Y \succ_{\text{PD}} X$ is equivalent to saying that X *chain majorizes* Y and $Y \succ_{\text{U}} X$ is equivalent to saying that X *majorizes* Y.

Note that the same bistochastic matrix is being used to smooth the distributions of each attibute. The corresponding multidimensional transfer principle is the *Uniform Majorization Principle*.

Uniform Majorization Principle (UM). For all $X, Y \in \mathcal{D}$ for which $X \neq Y$, if $Y \succ_{\mathrm{U}} X$, then $Y \succ X$.

The product of strict T-transform matrices is a non-permutation bistochastic matrix. If either $q = 1$ or $n = 2$, the converse is also true. However, if $n \geq 3$ and $q \geq 2$, there exist non-permutation bistochastic matrices that are not products of strict T-transforms. See Marshall and Olkin (1979, p. 431). As a consequence, except in the special cases noted above, UM is a more restrictive assumption than UPM.

Kolm (1977) has shown that if a common increasing and strictly concave utility function U is used to evaluate individuals' allocations of attributes, then for all $X, Y \in \mathcal{D}$, the vector of utilities $(U(y_1), \ldots, U(y_n))$ strictly generalized Lorenz dominates the vector of utilities $(U(x_1), \ldots, U(x_n))$ if $Y \succ_{\mathrm{U}} X$. Hence, the ordering defined by the utilitarian social evaluation function (1) satisfies UM if U is increasing and strictly concave.

If $Y \succ_{\mathrm{PD}} X$ or if $Y \succ_{\mathrm{U}} X$, then Y exhibits less dispersion than X. Furthermore, the mean value of each attribute is the same in both X and Y. It is for these reasons that UPM and UM have so much appeal as ways of incorporating inequality aversion into a social evaluation. However, in a society in which individuals have different preferences, Fleurbaey and Trannoy (2003) have shown that multi-attribute versions of the Pigou–Dalton transfer principle can conflict with the Pareto principle. Thus, a welfarist would want to limit the domain of applicability of multidimensional transfer principles in heterogeneous societies.

4. Normative Univariate Inequality Indices

In the normative approach to inequality measurement, an inequality index is constructed from a social evaluation ordering. This approach has its origins in the articles by Atkinson (1970) and Kolm (1969) on univariate inequality measurement. In this section, I review the procedures that were proposed by Atkinson and Kolm for deriving univariate inequality indices from social evaluation orderings.

4.1. The Atkinson–Kolm–Sen Inequality Index

Suppose that $q = 1$ and that the social evaluation ordering is $\succeq$. The *equally-distributed-equivalent income* $\Xi_{\succeq}(x)$ associated with a given univariate income distribution x is the per capita income that, if distributed equally, is indifferent to the actual income distribution according to $\succeq$. Formally, $\Xi_{\succeq}(x)$ is defined implicitly by

$$(\Xi_{\succeq}(x) \cdot 1_n) \sim x, \quad \forall x \in \mathcal{D}. \tag{3}$$

ORD, CONT, and MON ensure that $\Xi_{\succeq}(x)$ is well-defined. The *equally-distributed-equivalent income function* is the mapping $\Xi_{\succeq} \colon \mathcal{D} \to \mathbb{R}$ that assigns the equally-distributed-equivalent income to each income distribution in the domain. $\Xi_{\succeq}$ is a particular representation of $\succeq$.

Now suppose that that $\mathcal{D} \in \{\mathcal{M}_2, \mathcal{M}_3\}$ (so that the mean income is always positive). The *Atkinson–Kolm–Sen inequality index* corresponding to $\succeq$ is the function $I^{\text{AKS}}_{\succeq} \colon \mathcal{D} \to \mathbb{R}$ defined by

$$I^{\text{AKS}}_{\succeq}(x) = 1 - \frac{\Xi_{\succeq}(x)}{\mu(x)}, \quad \forall x \in \mathcal{D}. \tag{4}$$

If $\succeq$ satisfies the Pigou–Dalton transfer principle, the value of this index is bounded above by 1 and bounded below by 0 (and the lower bound is only attained if incomes are equally distributed). This procedure for constructing an inequality index was independently proposed by Atkinson (1970) and Kolm (1969), and was later popularized by Sen (1973).[12] This index has a simple interpretation. $I^{\text{AKS}}_{\succeq}(x)$ is the fraction of the total income that could be destroyed if incomes are equalized and the resulting distribution is indifferent to x according to $\succeq$. Thus, the Atkinson–Kolm–Sen inequality index is a measure of the waste due to inequality.

Given any univariate inequality index $I \colon \mathcal{D} \to \mathbb{R}$, (4) can be used to determine the underlying social evaluation that generates this index using the Atkinson–Kolm–Sen methodology. This social evaluation is represented by

$$\Xi_{\succeq}(x) = \mu(x)[1 - I(x)], \quad \forall x \in \mathcal{D}. \tag{5}$$

A univariate or multivariate inequality index $I \colon \mathcal{D} \to \mathbb{R}$ is *normatively significant* if for all $X, Y \in \mathcal{D}$ for which $\mu(x_{\cdot j}) = \mu(y_{\cdot j})$ for all $j \in Q$,

[12] Atkinson (1970) assumed that the social evaluation function has the form given in (1).

$I(X) \geq I(Y)$ if and only if $Y \succeq X$. By construction, an Atkinson–Kolm–Sen inequality index is normatively significant.

An index is a *relative* index if it is invariant to a proportional scaling of all its variables; i.e., if it is homogeneous of degree 0. From (4) it follows that $I_{\succeq}^{\text{AKS}}$ is a relative inequality index if and only if $\Xi_{\succeq}$ is homogeneous of degree 1, which is equivalent to requiring $\succeq$ to be homothetic.

4.2. The Kolm Inequality Index

As in the preceding subsection, suppose that $q = 1$ and that the social evaluation ordering is $\succeq$. Further suppose that $\mathcal{D}$ is any one of the three domains defined in Section 2. Kolm (1969) has proposed an alternative to $I_{\succeq}^{\text{AKS}}$ for the measurement of univariate income inequality. The *Kolm inequality index* corresponding to $\succeq$ is the function $I_{\succeq}^{\text{K}} \colon \mathcal{D} \to \mathbb{R}$ defined by

$$I_{\succeq}^{\text{K}}(x) = \mu(x) - \Xi_{\succeq}(x), \quad \forall x \in \mathcal{D}. \tag{6}$$

If $\succeq$ satisfies the Pigou–Dalton transfer principle, the value of this index is always nonnegative and it is only equal to 0 if everyone has the same income. $I_{\succeq}^{\text{K}}(x)$ is the per capita income that could be destroyed if incomes are equalized and the resulting distribution is indifferent to x according to $\succeq$. As is the case with $I_{\succeq}^{\text{AKS}}$, $I_{\succeq}^{\text{K}}$ is normatively significant.

An index is an *absolute* index if it is invariant to an increase or decrease of all of its variables by a common amount. Kolm intended for $I_{\succeq}^{\text{K}}$ to be used as a measure of absolute inequality, which implicitly places an invariance restriction on $\succeq$ analogous to the homotheticity requirement for $I_{\succeq}^{\text{AKS}}$ to be a relative index. From (6), it can be seen that $I_{\succeq}^{\text{K}}$ is an absolute inequality index if and only if $\Xi_{\succeq}$ is unit-translatable, which is equivalent to requiring $\succeq$ to be translatable.[13]

If one subscribes to Kolm's procedure, the equally-distributed-equivalent income function underlying the univariate inequality index $I \colon \mathcal{D} \to \mathbb{R}$ is given by

$$\Xi_{\succeq}(x) = \mu(x) - I(x), \quad \forall x \in \mathcal{D}. \tag{7}$$

[13] A function $f \colon D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, is *unit-translatable* if $f(x + \lambda 1_n) = f(x) + \lambda$ for all $x \in D$ and all $\lambda \in \mathbb{R}$ for which $x + \lambda 1_n \in D$. A binary relation on a subset of a Euclidean space is *translatable* if it can be represented by a unit-translatable function.

Comparing (5) and (7), we see that the social evaluation that provides the normative foundation for an inequality index is indeterminate unless one has adopted a particular procedure for deriving inequality indices from social evaluations.[14] For this reason, Foster (1994), among others, has criticized the various proposals for constructing normative inequality indices.

Both the $I_{\succeq}^{\mathrm{AKS}}$ and $I_{\succeq}^{\mathrm{K}}$ indices are cardinal. Replacing I by an ordinally-equivalent function, say by squaring I, in either (5) or (7) changes the underlying social evaluation. While there has been some research on ordinal inequality indices for univariate distributions (see Blackorby, Bossert, and Donaldson (1999), Chakravarty (1990), and Dutta (2002)), this is an issue that has not been considered in the multivariate literature.

5. Normative Multivariate Inequality Indices

In this section, I describe how the $I_{\succeq}^{\mathrm{AKS}}$ and $I_{\succeq}^{\mathrm{K}}$ indices have been generalized for multivariate distributions by Kolm (1977) and Tsui (1995). Throughout this discussion, it is assumed that $\succeq$ satisfes ORD, CONT, MON, ANON, and UM.

5.1. The Multi-Attribute Kolm Inequality Index

In this subsection, it is supposed that $\mathcal{D} \in \{\mathcal{M}_2, \mathcal{M}_3\}$. The Kolm (1977) multi-attribute generalization of the Atkinson–Kolm–Sen inequality index measures the inequality of a distribution matrix by the fraction of the aggregate amount of *each* attribute that could be destroyed if every attribute is equalized and the resulting distribution is indifferent to the original distribution according to $\succeq$.[15] To define this index formally, some preliminary definitions are needed.

[14] While the two procedures described above are the most commonly-used for constructing normative inequality indices, they do not exhaust the possibilities. For a discussion of some other approaches, see Blackorby, Bossert, and Donaldson (1999), Chakravarty (1990), and Dutta (2002).

[15] Bourguignon (1999) and List (1999) have proposed alternate multi-attribute generalizations of the $I_{\succeq}^{\mathrm{AKS}}$ index. Bourguignon's index is constructed by first computing the ratio of the utilitarian sum in (1) to the utilitarian sum that would have been obtained if eveyone had the mean value of each attribute and then subtracting this number from 1. Bourguignon assumes that the utility function used to aggregate the individual allocations is a CES function. List's proposal is considered in Section 9.

For all $X \in \mathcal{D}$, let X_μ denote the distribution matrix in which, for all $j \in Q$, the entries in the jth column are all set equal to $\mu(x_{\cdot j})$. Define the function $\Delta_{\succeq}\colon \mathcal{D} \to \mathbb{R}$ by setting, for all $X \in \mathcal{D}$, $\Delta_{\succeq}(X)$ equal to the scalar that solves

$$\Delta_{\succeq}(X)X_\mu \sim X. \tag{8}$$

By ORD, CONT, and MON, this function is well-defined.

The *multi-attribute Kolm inequality index* associated with $\succeq$ is the function $I^{\mathrm{KM}}_{\succeq}\colon \mathcal{D} \to \mathbb{R}$ defined by setting

$$I^{\mathrm{KM}}_{\succeq}(X) = 1 - \Delta_{\succeq}(X), \quad \forall X \in \mathcal{D}. \tag{9}$$

If $q = 1$, $\Delta_{\succeq}(x) = \Xi_{\succeq}(x)/\mu(x)$ for all $x \in \mathcal{D}$. Hence, for univariate distributions, $I^{\mathrm{KM}}_{\succeq} = I^{\mathrm{AKS}}_{\succeq}$.

$\Delta_{\succeq}$ is not a representation of $\succeq$. Nevertheless, for any $X, Y \in \mathcal{D}$ for which $X_\mu = Y_\mu$, MON implies that $\Delta_{\succeq}(X) \geq \Delta_{\succeq}(Y)$ if and only if $X \succeq Y$. It then follows from (9) that $I^{\mathrm{KM}}_{\succeq}$ is normatively significant.

Consider an arbitrary inequality index $I\colon \mathcal{D} \to \mathbb{R}$. While (9) can be used to to solve for the $\Delta_{\succeq}$ function that generates I, $\Delta_{\succeq}$ does not provide sufficient information to determine $\succeq$ when $q > 1$. The problem is that the social ranking of X and Y is not known if $X_\mu \neq Y_\mu$ and neither X_μ weakly dominates Y_μ (attribute by attribute) nor Y_μ weakly dominates X_μ.

The value of $I^{\mathrm{KM}}_{\succeq}$ is clearly bounded above by 1. If $X = X_\mu$, then $\Delta_{\succeq}(X) = 1$ and $I^{\mathrm{KM}}_{\succeq}(X) = 0$. Because $X_\mu = BX$ for the bistochastic matrix B in which all entries are equal to $1/n$, UM and MON imply that $I^{\mathrm{KM}}_{\succeq}(X) > 0$ when $X \neq X_\mu$.[16] ANON implies that $I^{\mathrm{KM}}_{\succeq}$ treats individuals symmetrically; i.e., $I^{\mathrm{KM}}_{\succeq}(X)$ is invariant to a permutation of the rows of X.

$I^{\mathrm{KM}}_{\succeq}$ is a relative index if and only if $\succeq$ is homothetic. For future reference, this property of $\succeq$ is stated as a formal axiom.

[16]The same conclusion holds if UPM is substituted for UM because, as the following argument demonstrates, when $X \neq X_\mu$, X_μ can be obtained from X by a finite sequence of strict T-transforms. If attribute 1 is not equally distributed in X, then by Lemma 2.B.1 in Marshall and Olkin (1979), there exists a finite sequence of strict T-transforms that, when applied to X, results in a distribution matrix Y in which attribute 1 is equally distributed. Note that if a T-transform is applied to Y, the distribution of attribute 1 is unchanged. Hence, reasoning as above, if attribute 2 is not equally distributed in Y, by applying a finite sequence of strict T-transforms to Y, it is possible to equalize the distributions of both attributes 1 and 2. By applying the same argument to each attribute sequentially, it follows that X_μ can be obtained from X by a finite sequence of strict T-transforms.

Homotheticity (HOM). For all $X, Y \in \mathcal{D}$ and all $\lambda > 0$, $X \succeq Y$ if and only if $\lambda X \succeq \lambda Y$.

For univariate distributions of income, one justification that has been offered for this axiom is that the social evaluation should be invariant to the units in which income is measured (dollars, euros, yen, etc.). In the multi-attribute case, HOM implies that $\succeq$ is invariant to a common proportional change in the units in which all goods are measured. If the various attributes correspond to different kinds of goods, this line of reasoning suggests that *independent* changes in the units in which different attributes are measured should not affect the social evaluation ordering.

Strong Homotheticity (SHOM). For all $X, Y \in \mathcal{D}$ and all $q \times q$ diagonal matrices Λ for which $\lambda_{jj} > 0$ for all $j \in Q$, $X \succeq Y$ if and only if $X\Lambda \succeq Y\Lambda$.

SHOM was proposed by Tsui (1995). The appeal of this kind of scale invariance assumption has been questioned by Bourguignon (1999, p. 479) when applied to inequality measurement. He has argued that if, say, incomes are doubled, then the contribution of other attributes to overall inequality may well be affected. Even if one accepts SHOM when the atrributes are different kinds of goods, as noted by Gajdos and Weymark (2003), this axiom is not appropriate if some of the attributes are naturally measured in the same units. For example, if the attributes are incomes in different states of the world, their units of measurement cannot be varied independently. For some goods, even stronger invariance properties may be appropriate. This would be the case if an attribute is measured on an ordinal scale.[17]

5.2. The Multi-Attribute Tsui Inequality Index

In this subsection, it is supposed that $\mathcal{D} \in \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\}$. Tsui (1995) has provided a multi-attribute generalization of Kolm's univariate inequality index. Tsui's index measures inequality by the amount of each attribute that must be taken away from every individual in order to obtain an allocation that is indifferent to the original allocation according to $\succeq$ if the distribution of each attribute is equalized.

[17]HOM and SHOM (and TRA and STRA defined below) are formally equivalent to invariance assumptions used in the literature on social choice with interpersonal utility comparisons. See Bossert and Weymark (2004) and Gajdos and Weymark (2003).

The formal definition of this index uses the function $\Gamma_{\succeq}\colon \mathcal{D} \to \mathbb{R}$ defined by setting, for all $X \in \mathcal{D}$, $\Gamma_{\succeq}(X)$ equal to the scalar that solves

$$(X_\mu - \Gamma_{\succeq}(X)\mathbf{1}) \sim X, \tag{10}$$

where $\mathbf{1}$ is a distribution matrix whose entries are all equal to 1. By ORD, CONT, and MON, this function is well-defined.

The *multi-attribute Tsui inequality index* associated with $\succeq$ is the function $I^{\mathrm{TM}}_{\succeq}\colon \mathcal{D} \to \mathbb{R}$ defined by setting

$$I^{\mathrm{TM}}_{\succeq}(X) = \Gamma_{\succeq}(X), \quad \forall X \in \mathcal{D}. \tag{11}$$

If $q = 1$, $I^{\mathrm{TM}}_{\succeq} = I^{\mathrm{K}}_{\succeq}$ because $\Gamma_{\succeq}(x) = \mu(x) - \Xi_{\succeq}(x)$ for all $x \in \mathcal{D}$. Using reasoning similar to that used in the discussion of $I^{\mathrm{KM}}_{\succeq}$, it is straightforward to show that $I^{\mathrm{TM}}_{\succeq}$ is normatively significant, $I^{\mathrm{TM}}_{\succeq}(X) = 0$ if $X = X_\mu$, $I^{\mathrm{TM}}_{\succeq}(X) > 0$ if $X \neq X_\mu$, and $I^{\mathrm{TM}}_{\succeq}(X) = I^{\mathrm{TM}}_{\succeq}(\Pi X)$ for any $n \times n$ permutation matrix Π. Furthermore, given an arbitrary inequality index $I\colon \mathcal{D} \to \mathbb{R}$, it is not possible to determine the social evaluation $\succeq$ that generates I using Tsui's procedure.

$I^{\mathrm{TM}}_{\succeq}$ is an absolute index if and only if $\succeq$ is translatable.

Translatability (TRA). For all $X, Y \in \mathcal{D}$ and all $\lambda \in \mathbb{R}$ for which $X + \lambda\mathbf{1} \in \mathcal{D}$ and $Y + \lambda\mathbf{1} \in \mathcal{D}$, $X \succeq Y$ if and only if $X + \lambda\mathbf{1} \succeq Y + \lambda\mathbf{1}$.

If TRA is satisfied, then $\succeq$ is invariant to a common change in the origins from which the quantities of the various attributes are measured. Tsui (1995) has also considered a stronger version of this translatability axiom that requires $\succeq$ be invariant to independent changes in these origins.

Strong Translatability (STRA). For all $X, Y \in \mathcal{D}$ and all $q \times q$ diagonal matrices Λ for which $X + \mathbf{1}\Lambda \in \mathcal{D}$ and $Y + \mathbf{1}\Lambda \in \mathcal{D}$, $X \succeq Y$ if and only if $X + \mathbf{1}\Lambda \succeq Y + \mathbf{1}\Lambda$.

As with SHOM, STRA would not be appropriate if there are attributes that should be measured in the same units.

6. Multidimensional Atkinson and Kolm–Pollak Indices

Starting with the work of Atkinson (1970) and Kolm (1969), the procedures described in Section 4 have been used to derive new functional forms for univariate inequality indices. Prominent among them are the Atkinson (1970) class of relative inequality indices and the Kolm (1969, 1976)–Pollak (1971) class of absolute inequality indices. In this section, I consider the multi-attribute generalizations of these indices proposed by Tsui (1995).

6.1. Multidimensional Atkinson Indices

In this subsection, it is supposed that $\mathcal{D} = \mathcal{M}_3$. With minor modifications, the analysis also applies to the domain $\mathcal{M}_2$. Atkinson (1970) considered univariate distributions and assumed that the social evaluation $\succeq$ is represented by a symmetric and additively separable social evaluation function, as in (1). Suppose that U is increasing and strictly concave. With these assumptions, $\succeq$ is homothetic if and only if there is a scalar $r < 1$ such that for all $x_i > 0$,

$$U(x_i) = \begin{cases} a + b\frac{x_i^r}{r}, & \text{if } r < 1 \text{ and } r \neq 0; \\ a + b\ln(x_i), & \text{if } r = 0 \ , \end{cases} \tag{12}$$

for some $a \in \mathbb{R}$ and some $b > 0$. For the social evaluation function obtained by substituting (12) into (1), the equally-distributed-equivalent income function is given by

$$\Xi^{\mathrm{A}}_{\succeq}(x) = \begin{cases} \left[\frac{1}{n}\sum_{i=1}^n x_i^r\right]^{1/r}, & \text{if } r < 1 \text{ and } r \neq 0; \\ \prod_{i=1}^n x_i^{1/n}, & \text{if } r = 0 \ , \end{cases} \tag{13}$$

for all $x \in \mathcal{D}$. $\Xi^{\mathrm{A}}_{\succeq}$ is a mean of order r function for $r < 1$. The Atkinson–Kolm–Sen relative inequality index corresponding to (13) is

$$I^{\mathrm{A}}_{\succeq}(x) = \begin{cases} 1 - \left[\frac{1}{n}\sum_{i=1}^n \left(\frac{x_i}{\mu(x)}\right)^r\right]^{1/r}, & \text{if } r < 1 \text{ and } r \neq 0; \\ 1 - \prod_{i=1}^n \left(\frac{x_i}{\mu(x)}\right)^{1/n}, & \text{if } r = 0 \ , \end{cases} \tag{14}$$

for all $x \in \mathcal{D}$. An index of the form given in (14) is an *Atkinson index of inequality.*[18]

[18] The same class of indices was also characterized by Kolm (1969, Theorem 13).

The multi-attribute generalization of this class of indices proposed by Tsui (1995) was identified axiomatically. This axiomatization requires that $n \geq 3$. Tsui's axioms are multi-attribute analogues of the axioms used by Blackorby, Donaldson, and Auersperg (1981) to axiomatize the social evaluation functions for the Atkinson class of indices. They include ORD, CONT, MON, ANON, UM, and SHOM.[19]

Tsui's final axiom is a separability assumption. For all $X \in \mathcal{D}$ and all nonempty $S \subset N$, let X^S denote the submatrix of X containing the distributions of attributes for individuals in S and let $X^{S^{\mathrm{C}}}$ be the submatrix containing the remaining rows of X. The individuals in S are *individual separable* in $\succeq$ from their complement S^{C} if the conditional ordering of the subdistribution matrices for the individuals in S obtained by fixing the subdistribution matrix of the individuals in S^{C} does not depend on the choice of the latter submatrix.

Minimal Individual Separability (MIS). There exists a non-singleton set of individuals $S \subset N$ that is individual separable in $\succeq$ from S^{C}.

ANON implies that if S is individual separable in $\succeq$ from S^{C}, then any set of individuals with the same cardinality is individual separable from its complement. When $n \geq 3$, combining ANON and MIS with ORD, CONT, MON, and UM implies that there exists an increasing, strictly concave function $U \colon \mathcal{D} \to \mathbb{R}$ such that $\succeq$ can be represented by an additive social evaluation function of the form given in (1). The function U is a utility function that the social evaluator uses to aggregate any individual's allocation of the q attributes into a summary statistic. The function U need not coincide with any individual's actual utility function. Adding SHOM to the six axioms used to obtain the additive representation of the social evaluation order places considerable structure on the aggregator function U, as the following theorem due to Tsui (1995) demonstrates.

Theorem 1. *Suppose that $n \geq 3$. A social evaluation $\succeq$ on $\mathcal{D} = \mathcal{M}_3$ satisfies* ORD, CONT, MON, ANON, UM, MIS, *and* SHOM *if and only if there exists an increasing, strictly concave function $U \colon \mathbb{R}^q_{++} \to \mathbb{R}$ such that $\succeq$ can*

[19]Tsui formulated his axioms in terms of a social evaluation function, rather than in terms of the underlying binary relation. Tsui assumed that the social welfare function is strictly quasiconcave. However, this assumption can be replaced by the weaker assumption UM in his theorems.

be represented by an additive social evaluation function of the form given in (1) *where, for all* $x_i > 0$,

$$U(x_i) = a + b \prod_{j=1}^{q} x_j^{r_j} \tag{15}$$

or

$$U(x_i) = a + \sum_{j=1}^{q} s_j \ln(x_j), \tag{16}$$

where $a \in \mathbb{R}$, $s_j > 0$ *for all* $j \in Q$, *and the parameters* b *and* r_j, $j \in Q$, *are chosen so that the function* U *in* (15) *is increasing and strictly concave.*[20]

A function $I_{\succeq}^{\mathrm{AM}} \colon \mathcal{D} \to \mathbb{R}$ is a member of the corresponding class of multi-attribute Kolm relative inequality indices if for all $X \in \mathcal{D}$,

$$I_{\succeq}^{\mathrm{AM}}(X) = 1 - \left[\frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{q} \left(\frac{x_{ij}}{\mu(x_{\cdot j})} \right)^{r_j} \right]^{1/\sum_j r_j} \tag{17}$$

or

$$I_{\succeq}^{\mathrm{AM}}(X) = 1 - \prod_{i=1}^{n} \left[\prod_{j=1}^{q} \left(\frac{x_{ij}}{\mu(x_{\cdot j})} \right)^{s_j/\sum_j s_j} \right]^{1/n}, \tag{18}$$

where the parameters satisfy the restrictions in Theorem 1. $I_{\succeq}^{\mathrm{AM}}$ is a *multi-attribute Atkinson inequality index*. If $q = 1$, (17) and (18) are equivalent to the formulae for the univariate Atkinson index given in (14).

6.2. Multidimensional Kolm–Pollak Indices

In this subsection, it is supposed that $\mathcal{D} = \mathcal{M}_1$. With minor modifications, the analysis also applies to the other two domains.

For the univariate case, assume as in Atkinson (1970) that the social evaluation $\succeq$ is symmetric and additively separable. If the function U in (1)

[20] The parameter restrictions on b and the r_j are quite complicated. See Tsui (1995) for details.

is increasing and strictly concave, then $\succeq$ is translatable if and only if there is a scalar $r > 0$ such that for all $x_i \in \mathbb{R}$,

$$U(x_i) = a + b\left(\frac{\exp(-rx_i)}{-r}\right), \tag{19}$$

for some $a \in \mathbb{R}$ and some $b > 0$. The equally-distributed-equivalent income function for $\succeq$ is given by

$$\Xi^{\mathrm{KP}}_{\succeq}(x) = -\frac{1}{r}\ln\left[\frac{1}{n}\sum_{i=1}^{n}\exp(-rx_i)\right], \quad r > 0, \tag{20}$$

for all $x \in \mathcal{D}$. The Kolm absolute inequality index corresponding to (20) can be written as

$$I^{\mathrm{KP}}_{\succeq}(x) = \frac{1}{r}\ln\left[\frac{1}{n}\sum_{i=1}^{n}\exp(r(\mu(x) - x_i))\right], \quad r > 0, \tag{21}$$

for all $x \in \mathcal{D}$. An index of the form given in (21) is a *Kolm–Pollak index of inequality*. This class of inequality indices was introduced by Kolm (1969) (see also Kolm (1976)). In consumer theory, the functional form in (20) was shown by Pollak (1971) to characterize the additive utility functions that have linear Engel curves.

Given an ordering R on the positive othant of $\mathbb{R}^n$, an ordering R^* on all of $\mathbb{R}^n$ can be defined by setting, for all $u, v \in \mathbb{R}^n$, uR^*v if and only if $(\exp(u_1), \ldots, \exp(u_n))R(\exp(v_1), \ldots, \exp(v_n))$. The ordering R is homothetic if and only if the ordering R^* is translatable. This observation accounts for why the equally-distributed income functions for the Kolm–Pollak indices can be obtained from the equally-distributed income functions for the Atkinson indices by an exponential change of variables.[21]

By substituting STRA for SHOM in the axioms of Theorem 1, Tsui (1995) has characterized a class of multi-attribute Kolm–Pollak social evaluation orderings.

Theorem 2. *Suppose that $n \geq 3$. A social evaluation $\succeq$ on $\mathcal{D} = \mathcal{M}_1$ satisfies* ORD, CONT, MON, ANON, UM, MIS, *and* STRA *if and only if there*

[21] In the Kolm–Pollak counterpart to the $r = 0$ case of the Atkinson index, U is concave, but not strictly concave. Furthermore, the equally-distributed-equivalent income function is always equal to average income and inequality as measured by the Kolm index is identically 0.

exists an increasing, strictly concave function $U\colon \mathbb{R}^q \to \mathbb{R}$ *such that* $\succeq$ *can be represented by an additive social evaluation function of the form given in* (1) *where, for all* $x_i > 0$,

$$U(x_i) = a + b\prod_{j=1}^{q} \exp(r_j x_j), \tag{22}$$

where $a \in \mathbb{R}$ *and the parameters* b *and* r_j, $j \in Q$, *are chosen so that the function* U *in* (22) *is increasing and strictly concave.*

The functional form in (22) can be obtained from (15) by an exponential change of variables. A function $I_{\succeq}^{\mathrm{KPM}}\colon \mathcal{D} \to \mathbb{R}$ is a member of the corresponding class of multi-attribute Tsui absolute inequality indices if for all $X \in \mathcal{D}$,

$$I_{\succeq}^{\mathrm{KPM}}(X) = \frac{1}{\sum_j r_j} \ln\left[\frac{1}{n}\sum_{i=1}^{n}\exp\left(\sum_{j=1}^{q} r_j(\mu(x_{\cdot j}) - x_{ij})\right)\right], \tag{23}$$

where the parameters satisfy the restrictions in Theorem 2. $I_{\succeq}^{\mathrm{KPM}}$ is a *multi-attribute Kolm–Pollak inequality index*. This class of indices coincides with the univariate Kolm–Pollak class when $q = 1$.

Note that Theorems 1 and 2 use the stronger forms, SHOM and STRA, of the invariance axioms. The implications of using HOM and TRA instead have not been determined.

7. Maasoumi's Two-Stage Aggregation Procedure

The multi-attribute social evaluation functions considered in the preceding section are defined using a two-stage aggregation procedure. In the first stage, for each individual, a utility function is used to aggregate the individual's allocation of the q attributes into a summary measure of well-being. This initial aggregation results in a unidimensional distribution of utilities. In the second stage, the individual utilities are summed to provide the overall social evaluation of a distribution matrix.

Maasoumi (1986) (see also Maasoumi (1999)) has suggested that a multi-attribute *inequality index* should be constructed directly using a two-stage procedure. Specifically, he proposed using a utility function in the first stage to aggregate the individual allocations of the attributes into a vector of utilities and then, in the second stage, using a univariate inequality index applied to this distribution to obtain a measure of the inequality in the distribution matrix. He also proposed functional forms for these aggregators. For the second-stage aggregator, he suggested using a member of the class of generalized entropy inequality indices. This class of indices contains the Atkinson class and all of the indices that are ordinally equivalent to some member of the Atkinson class. Using information-theoretic considerations, Maasoumi argued that the utility function should be a weighted mean of order r. The multi-attribute Atkinson and Kolm–Pollak inequality indices are not two-stage aggregators in the sense of Maasoumi, as is apparent from inspection of their functional forms.

Dardanoni (1995) has shown that an inequality index constructed according to Maasoumi's proposal need not satisfy the inequality counterpart of UM. His argument is based on the following example.

Let

$$X = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 90 & 10 \\ 90 & 10 & 10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}.$$

We then have

$$Y = BX = \begin{bmatrix} 10 & 10 & 10 \\ 50 & 50 & 10 \\ 50 & 50 & 10 \end{bmatrix}.$$

Because the distribution matrix Y is obtained by multiplying X by a non-permutation bistochastic matrix, it seems reasonable to say that X exhibits more inequality than Y. Indeed, this is the case for any normatively-significant inequality index if the underlying social evaluation satisfies UM.

Now suppose that the inequality index is constructed using Maasoumi's two-stage procedure. Assume that the univariate inequality index I used in the second stage is a strictly S-convex relative index. Also assume that the utility function U used in the first stage is a symmetric, increasing, concave function of its arguments. We then have $U(x_{1\cdot}) = U(y_{1\cdot}) < U(x_{2\cdot}) = U(x_{3\cdot}) \leq U(y_{2\cdot}) = U(y_{3\cdot})$. If the weak inequality is in fact an equality, there has been no change in the distribution of the utilities, so I registers no change in inequality. If the weak inequality is strict, then the situation of the two individuals with the largest pre-transfer utilities is improved after the transfers and, as can be seen by considering the pre- and post-transfer

Lorenz curves for the utility distributions, I regards Y as exhibiting more inequality than X. In either case, the ordering of X and Y differs from the one described in the preceding paragraph.

The conclusion that Dardanoni (1995, p. 202) draws from this example is that uniform majorization is "uninformative for evaluating the amount of inequality in society." This seems unwarranted. A more natural conclusion would be to question the appropriateness of Maasoumi's two-stage aggregation procedure for constructing multidimensional inequality indices.

8. Multidimensional Generalized Gini Indices

The most widely-used univariate index of relative inequality is the Gini index. The Gini social evaluation ordering is a member of the class of generalized Gini social evaluations introduced by Weymark (1981). These social evaluations are both homothetic and translatable. As a consequence, the procedures described in Section 4 can be used to define a class of generalized Gini relative inequality indices and a class of generalized Gini absolute inequality indices. In this section, the multi-attribute generalizations of these indices recently introduced by Gajdos and Weymark (2003) are considered.

8.1. Multidimensional Generalized Gini Social Evaluations

Before turning to the multidimensional generalized Ginis, it is first necessary to formally define the univariate generalized Ginis. A *generalized Gini social evaluation* is a binary relation $\succeq$ on $\mathcal{D}$ whose equally-distributed-equivalent income function $\Xi^{\mathrm{G}}_{\succeq}\colon \mathcal{D} \to \mathbb{R}$ has the form

$$\Xi^{\mathrm{G}}_{\succeq}(x) = \sum_{i=1}^{n} a_i \tilde{x}_i, \quad \forall x \in \mathcal{D}, \tag{24}$$

where $0 < a_1 < a_2 < \cdots < a_n$ and $\sum_{i=1}^{n} a_i = 1$.[22] The *Gini equally-distributed-equivalent income function* is the special case of (24) in which $a_i = (2i-1)/n^2$ for all $i \in N$. As noted above, a generalized Gini social evaluation is both homothetic and translatable. The corresponding classes of generalized Gini relative and absolute inequality indices are obtained by substituting (24) into (4) and (6), respectively.

[22]Weymark (1981) merely requires that $a_1 \leq a_2 \leq \cdots \leq a_n$. If any of these weak inequalities hold with equality, $\Xi^{\mathrm{G}}_{\succeq}$ is S-concave, but not strictly S-concave.

The multi-attribute versions of the generalized Gini social evaluations introduced by Gajdos and Weymark (2003) were obtained by supplementing multidimensional generalizations of the axioms Weymark (1981) used to characterize the univariate generalized Ginis with a assumption that requires the social evaluation to be separable with respect to attributes. Aside from the separability assumption, the only one of these axioms that has not already been introduced is a comonotonic additivity axiom. In order to define this axiom, some additional notation is required.

A distribution matrix X is *nonincreasing comonotonic* if $x_{1j} \geq x_{2j} \geq \cdots \geq x_{nj}$ for all $j \in Q$. In other words, person 1 has at least as much of every attribute as person 2, person 2 has at least as much of every attribute as person 3, etc. Let $\mathcal{D}^{\mathrm{D}}$ denote the set of nonincreasing comonotonic matrices in $\mathcal{D}$.

Weak Comonotonic Additivity (WCA). For all $X, Y \in \mathcal{D}^{\mathrm{D}}$ and all $Z \in \mathcal{M}_1^{\mathrm{D}}$ for which there exists a $j_0 \in N$ such that (i) $x_{\cdot j} = y_{\cdot j}$ for all $j \neq j_0$, (ii) $z_{ij} = 0$ for all $i \in N$ and all $j \neq j_0$, and (iii) $X + Z \in \mathcal{D}^{\mathrm{D}}$ and $Y + Z \in \mathcal{D}^{\mathrm{D}}$, $X \succeq Y$ if and only if $X + Z \succeq Y + Z$.[23]

The distribution matrices X, Y, $X + Z$, and $Y + Z$ in the definition of WCA are all nonincreasing comonotonic and have identical distributions of all the attributes except j_0. Hence, the social ranking of these matrices coincides with the ranking obtained with the conditional ordering of attribute j_0. $X+Z$ and $Y + Z$ are obtained from X and Y by adding a common distribution of attribute j_0 to both $x_{\cdot j_0}$ and $y_{\cdot j_0}$. WCA requires the social ranking of two comonotonic distribution matrices to be invariant to this kind of change. In other words, in any conditional ordering of two distributions of an attribute, the ordering only depends on the amounts by which the distributions differ.[24]

For all $X \in \mathcal{D}$ and all nonempty $M \subset Q$, let X^M denote the submatrix of X containing the distributions of the attributes in M and let $X^{M^{\mathrm{C}}}$ be the submatrix containing the other columns of X. The attributes in M are *attribute separable* in $\succeq$ from their complement M^{C} if the conditional ordering of the subdistribution matrices for the attributes in M obtained by fixing the subdistribution matrix of the attributes in M^{C} is independent of the choice of the latter submatrix. The separability axiom used by Gajdos and Weymark (2003) is given by the following axiom.

[23] The definition of WCA given here differs slightly from the one given in Gajdos and Weymark (2003). Theorems 3 and 4 below are valid with either definition of WCA.

[24] Gajdos and Weymark (2003) also considered a stronger version of this axiom.

Strong Attribute Separability (SAS). For all nonempty $M \subset Q$, M is attribute separable in $\succeq$ from M^{C}.

Gajdos and Weymark (2003) have shown that if a social evaluation $\succeq$ on $\mathcal{D}$ satisfies SAS and ANON, then the two multi-attribute transfer principles, UPM and UM, place equivalent structure on $\succeq$. They have also shown that ORD, CONT, MON, and SAS imply that $\succeq$ has a two-stage aggregation representation, but the order of aggregation differs from that used in the multi-attribute Atkinson and Kolm–Pollak inequality indices. In the first stage, the distributions of each attribute are aggregated, resulting in a q-dimensional vector of scalars. In the second stage, the components of this vector are aggregated to provide an overall evaluation of the distribution matrix. If there are at least three attributes, this second-stage aggregator is additively separable. If it is additionally assumed that $\succeq$ satisfies WCA, then the first-stage aggregators must be generalized Gini equally-distributed-equivalent income functions.

8.2. Multidimensional Generalized Gini Relative Inequality Indices

Now assume that $\mathcal{D} = \mathcal{M}_2$. The following theorem, due to Gajdos and Weymark (2003), characterizes the set of social evaluations that satisfy HOM in addition to the axioms considered in the preceding paragraph.[25]

Theorem 3. *If* $q \geq 3$*, then the binary relation* $\succeq$ *on* $\mathcal{D} = \mathcal{M}_2$ *satisfies* ORD, CONT, MON, ANON, UPM, SAS, WCA, *and* HOM *if and only if there exists an* $n \times q$ *matrix* A *of positive coefficients with* $a_{\cdot j}$ *increasing and* $\sum_{i=1}^{n} a_{ij} = 1$ *for all* $j \in Q$*, a positive vector* $\gamma \in \mathbb{R}^q$ *with* $\sum_{j=1}^{q} \gamma_j = 1$*, and a scalar* r *such that* $\succeq$ *can be represented by a social evaluation function* $W_{\succeq}^{\mathrm{GMR}}\colon \mathcal{M}_2 \to \mathbb{R}$ *for which*

$$W_{\succeq}^{\mathrm{GMR}}(X) = \left[\sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \right)^{r} \right]^{\frac{1}{r}}, \quad \forall X \in \mathcal{M}_2, \tag{25}$$

[25] Strictly speaking, in this and the following theorem, Gajdos and Weymark used a variant of UM in which the conclusion that $X \succ Y$ is weakened to $X \succeq Y$. With this weaker assumption, the weights $a_{\cdot j}$ only need to be nondecreasing, rather than increasing.

if $r \neq 0$ *and*

$$W_{\succeq}^{\mathrm{GMR}}(X) = \prod_{j=1}^{q} \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \right)^{\gamma_j}, \quad \forall X \in \mathcal{M}_2, \tag{26}$$

if $r = 0$.

Thus, the second-stage aggregator is a mean of order r function, where r is unrestricted, if $\succeq$ is assumed to satisfy the axioms of Theorem 3. If HOM is strengthened to SHOM, then only the $r = 0$ case is possible; i.e., the second-stage aggregator must be a Cobb–Douglas function.

A function $I_{\succeq}^{\mathrm{GMR}}\colon \mathcal{D} \to \mathbb{R}$ is a member of the corresponding class of multi-attribute Kolm relative inequality indices if

$$I_{\succeq}^{\mathrm{GMR}}(X) = 1 - \frac{\left[\sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij}\right)^r\right]^{\frac{1}{r}}}{\left[\sum_{j=1}^{q} \gamma_j \mu(x_{\cdot j})^r\right]^{\frac{1}{r}}}, \quad \forall X \in \mathcal{M}_2, \tag{27}$$

when $r \neq 0$ and

$$I_{\succeq}^{\mathrm{GMR}}(X) = 1 - \frac{\prod_{j=1}^{q} \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij}\right)^{\gamma_j}}{\prod_{j=1}^{q} \mu(x_{\cdot j})^{\gamma_j}}, \quad \forall X \in \mathcal{M}_2, \tag{28}$$

when $r = 0$. $I_{\succeq}^{\mathrm{GMR}}$ is a *multi-attribute generalized Gini relative inequality index*.

8.3. Multidimensional Generalized Gini Absolute Inequality Indices

Now assume that $\mathcal{D} = \mathcal{M}_1$. Using a simple exponential change of variables, Gajdos and Weymark (2003) have established the counterpart of Theorem 3 for translatable social evaluations.

Theorem 4. *If* $q \geq 3$*, then the binary relation* $\succeq$ *on* $\mathcal{D} = \mathcal{M}_1$ *satisfies* ORD, CONT, MON, ANON, UPM, SAS, WCA, *and* TRA *if and only if there exists an* $n \times q$ *matrix* A *of positive coefficients with* $a_{\cdot j}$ *increasing and* $\sum_{i=1}^{n} a_{ij} = 1$ *for all* $j \in Q$*, a positive vector* $\gamma \in \mathbb{R}^q$*, and a scalar* r *such that* $\succeq$ *can be represented by a social evaluation function* $W_{\succeq}^{\mathrm{GMA}}\colon \mathcal{M}_1 \to \mathbb{R}$ *for which*

$$W_{\succeq}^{\mathrm{GMA}}(X) = \frac{1}{r} \ln \left[\sum_{j=1}^{q} \gamma_j \exp\left(r \sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \right) \right], \quad \forall X \in \mathcal{M}_1, \tag{29}$$

if $r \neq 0$ *and*

$$W_{\succeq}^{\text{GMA}}(X) = \sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \right), \quad \forall X \in \mathcal{M}_1, \tag{30}$$

if $r = 0$.

The second-stage aggregators in Theorem 4 are Kolm–Pollak functions. Note that, in contrast to (20), the parameter r can take on any value in $\mathbb{R}$. If TRA is strengthened to STRA, then $r = 0$, in which case the second-stage aggregator is linear.

A function $I_{\succeq}^{\text{GMA}} \colon \mathcal{D} \to \mathbb{R}$ is a member of the corresponding class of multi-attribute Tsui absolute inequality indices if

$$I_{\succeq}^{\text{GMA}}(X) = \frac{1}{r} \ln \left[\frac{\sum_{j=1}^{q} \gamma_j \exp\left(r \mu(x_{\cdot j})\right)}{\sum_{j=1}^{q} \gamma_j \exp\left(r \sum_{i=1}^{n} a_{ij} \tilde{x}_{ij}\right)} \right], \quad \forall X \in \mathcal{M}_1, \tag{31}$$

when $r \neq 0$ and

$$I_{\succeq}^{\text{GMA}} = \sum_{j=1}^{q} \gamma_j \left[\mu(x_{\cdot j}) - \sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \right], \quad \forall X \in \mathcal{M}_1, \tag{32}$$

when $r = 0$. $I_{\succeq}^{\text{GMA}}$ is a *multi-attribute generalized Gini absolute inequality index*.

9. Correlation Increasing Majorization

The two multi-attribute generalizations of the Pigou–Dalton transfer principle, UPM and UM, ensure that the social evaluation is inequality averse in the sense that uniform mean-preserving decreases in the spreads of the distributions of the attributes are socially desirable. Atkinson and Bourguignon (1982) have argued that a multi-attribute inequality index should also take account of the statistical dependence between the attribute distributions. Tsui (1999) has investigated one way in which an inequality index can be be sensitive to the dependence properties of distribution matrices.[26] Tsui's axiom requires the value of a multi-attribute inequality index to increase if two individuals' allocations are rearranged so that one of these individuals

[26] For an overview of different dependence concepts, see Joe (1997).

receives at least as much of every attribute as the other and strictly more of at least one attribute (and this was not the case before the rearrangement).[27] In this section, the corresponding principle for social evaluations is considered.

For $x, y \in \mathbb{R}^q$, $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_q, y_q\})$ and $x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_q, y_q\})$.

Definition. For all $X, Y \in \mathcal{D}$, Y is obtained from X by a *correlation-increasing transfer* if $X \neq Y$, X is not a permutation of Y, and there exist $i_1, i_2 \in N$ such that (i) $y_{i_1\cdot} = x_{i_1\cdot} \wedge x_{i_2\cdot}$, (ii) $y_{i_2\cdot} = x_{i_1\cdot} \vee x_{i_2\cdot}$, and (iii) $y_{i\cdot} = x_{i\cdot}$ for all $i \notin \{i_1, i_2\}$.

Note that a correlation-increasing transfer preserves the mean of each attribute.

Definition. For all $X, Y \in \mathcal{D}$, Y is *more correlated* than X, denoted $Y \succ_{\mathrm{C}} X$, if Y can be obtained from X by a finite sequence of correlation-increasing transfers.[28]

The social evaluation version of Tsui's dependence-sensitivity axiom requires that if Y is more correlated than X, then X should be socially preferred to Y.

Correlation Increasing Majorization (CIM). For all $X, Y \in \mathcal{D}$, if $Y \succ_{\mathrm{C}} X$, then $X \succ Y$.

Tsui (1999) has shown that CIM and UM (resp. UPM) are independent principles. In other words, any pair of distribution matrices that can be ordered by $\succ_{\mathrm{C}}$ cannot be ordered by $\succ_{\mathrm{U}}$ (resp. $\succ_{\mathrm{PD}}$) and vice versa.

A function $f\colon D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^q$, is *L-superadditive* if for all $x, y \in D$, $f(x \wedge y) + f(x \vee y) \geq f(x) + f(y)$. If the inequality is strict when $x \wedge y \neq x$ and $x \wedge y \neq y$, then f is *strictly L-superadditive.*[29] *L-subadditive* and *strictly*

[27] Dardanoni (1995) has investigated a variant of this principle that requires Y to exhibit at least as much inequality as X if Y is a comonotonic rearrangement of X.

[28] Tsui's definition of $\succ_{\mathrm{C}}$ permits the sequence of correlation-increasing transfers to be supplemented with permutations of the individual allocations.

[29] See Marshall and Olkin (1979, Chapter 6) for a detailed discussion of L-superadditive functions. Marshall and Olkin define f to be L-superadditive if for all distinct $j_1, j_2 \in Q$ and all $x, y \in D$ for which $x_j = y_j$ for all $j \notin \{j_1, j_2\}$, $f(x \wedge y) + f(x \vee y) \geq f(x) + f(y)$. This definition is equivalent to the one in the text.

L-*subadditive* functions are defined analogously by reversing the inequality signs. If f is twice differentiable, then f is L-superadditive (resp. strictly L-superadditive, L-subadditive, strictly L-subadditive) if and only if for all distinct $j_1, j_2 \in Q$ and all $x \in D$, $\partial^2 f(x)/\partial x_{j_1} \partial x_{j_2} \geq 0$ (resp. > 0, ≤ 0, < 0).

It follows from the definition of strict L-subadditivity that the utilitarian social evaluation (1) satisfies CIM if U is strictly L-subadditive. Furthermore, a straightforward extension of Proposition 4 in Tsui (1999) shows that, for all $X, Y \in \mathcal{D}$, the vector of utilities $u = (U(x_{1\cdot}), \ldots, U(x_{n\cdot}))$ strictly generalized Lorenz dominates the vector of utilities $v = (U(y_{1\cdot}), \ldots, U(y_{n\cdot}))$ if $Y \succ_{\mathrm{C}} X$ and U is increasing and strictly L-subadditive.[30]

For the two-attribute case, the implications of correlation-increasing transfers for the utilitarian social evaluation function have also been considered by Atkinson and Bourguignon (1982).[31] The two attributes are *substitutes* if the utility function U is strictly L-subadditive and they are *complements* if it is strictly L-superadditive. In contradiction to CIM, the value of a utilitarian social evaluation function increases in response to a correlation-increasing transfer when the two goods are complements. For this reason, Bourguignon and Chakravarty (2003) criticized Tsui's use of CIM, arguing that he has implicitly assumed that all goods are substitutes.

The multi-attribute generalized Gini social evaluations do not satisfy CIM. The reason is quite simple—the separability across attributes implied by SAS is inconsistent with $\succeq$ being sensitive to the statistical dependence of the individual attribute distributions. As Gajdos and Weymark (2003) have shown, the inconsistency with CIM holds even if there is a single attribute that is attribute separable in $\succeq$ from the other attributes if $\succeq$ satisfies ANON.

There are, however, multi-attribute relative inequality indices that satisfy the inequality versions of both UM and CIM. For the domain $\mathcal{M}_3$, Tsui (1999) has axiomatized a class of such indices. They are multi-attribute generalized entropy relative inequality indices. List (1999) has constructed multidimensional generalizations of the Gini and Atkinson indices that also satisfy both of these properties.[32]

List has also proposed a procedure for constructing multidimensional rel-

[30] If, in addition, U is symmetric and additive, then u strictly Lorenz dominates v.

[31] Bourguignon (1999) has considered the implications of correlation-increasing transfers for the multi-attribute inequality indices proposed by Maasoumi (1986).

[32] These indices are not members of the families of multi-attribute generalized Gini and Atkinson indices discussed Sections 6 and 8.

ative inequality indices on $\mathcal{M}_2$ that are consistent with both of these inequality dominance principles. List's construction bears some relationship to the two-stage aggregation procedure proposed by Maasoumi (1986) that is described in Section 7. For any $X \in \mathcal{M}_2$, X is first replaced by the distribution matrix $X_{\mathrm{C}} = X\Lambda_{\mathrm{C}}$, where Λ_{C} is the $q \times q$ diagonal matrix in which $\lambda_{jj} = 1/\mu(x_{\cdot j})$ for all $j \in Q$. The mean value of each attribute in X_{C} is 1. This step ensures that the resulting index is invariant to independent changes in the units in which attributes are measured. Next, a common utility function is used to aggregate each person's allocation of the attributes in X_{C}. In order for the resulting index to satisfy both inequality dominance principles, this utility function must satisfy a number of properties, such as strict concavity. Finally, the resulting vector of utilities is aggregated using a generalized-Lorenz-consistent univariant inequality index.

With the exception of List's generalized Atkinson indices, none of the inequality indices proposed by List (1999) and Tsui (1999) are provided with a social-evaluation foundation. List's generalized Atkinson indices are defined using the function $\mu^{\mathrm{e}}_{\succeq} \colon \mathcal{M}_2 \to \mathbb{R}$, where, for all $X \in \mathcal{M}_2$, $\mu^{\mathrm{e}}_{\succeq}(X)$ is the scalar that solves

$$\mu^{\mathrm{e}}_{\succeq}(X)\mathbf{1} \sim X_{\mathrm{C}}. \tag{33}$$

Using this function, a *List multi-attribute relative inequality index* $I^{\mathrm{LM}}_{\succeq} \colon \mathcal{M}_2 \to \mathbb{R}$ is defined by setting

$$I^{\mathrm{LM}}_{\succeq}(X) = 1 - \mu^{\mathrm{e}}_{\succeq}(X), \quad \forall X \in \mathcal{M}_2. \tag{34}$$

List's multidimensional Atkinson indices are defined using a specific functional form for the function $\mu^{\mathrm{e}}_{\succeq}$. $I^{\mathrm{LM}}_{\succeq}$ provides an alternative to Kolm's procedure for constructing multi-attribute relative inequality indices. At present, nothing is known about the properties of $I^{\mathrm{LM}}_{\succeq}$ except when $\mu^{\mathrm{e}}_{\succeq}$ has the particular functional form assumed by List.

10. Concluding Remarks

It is also natural to investigate the implications of requiring social evaluations to satisfy multivariate versions of univariate transfer sensitivity, decomposability, and population replication invariance axioms. The latter two properties require social evaluations to be defined for different population sizes. Tsui (1999) has employed decomposability and replication invariance axioms in his characterization of a class of multi-attribute generalized entropy inequality indices. However, these indices were axiomatized directly, rather than indirectly using a social evaluation. Moyes (1999) has formulated multivariate transfer sensitivity axioms, but, to the best of my knowledge, multivariate generalizations of univariate transfer sensitivity axioms have yet to be used to help construct normative inequality indices.[33]

A number of functional forms for multivariate inequality indices have been proposed that do not have explicit normative foundations. Examples of such indices are List's multivariate Gini indices (see List (1999)), Tsui's multivariate generalized entropy indices (see Tsui (1999)), and Koshevoy and Mosler's multivariate generalizations of the Gini index (see Koshevoy and Mosler (1997)).

The framework employed here has also been used to analyze inequality under uncertainty. See Ben-Porath, Gilboa, and Schmeidler (1997) and Gajdos and Maurin (2004). In this application, attributes are incomes in different states of the world. By working in this more structured environment, it is possible to formulate axioms that are appropriate for this specific problem that may not be appropriate in other interpretations of the model. Of particular note in this regard is that incomes in different states are measured in the same units, a fact that is exploited in some of the axioms used by Ben-Porath, Gilboa, and Schmeidler (1997) and Gajdos and Maurin (2004).

A related area of research is the measurement of multidimensional poverty. See, for example, Bourguignon and Chakravarty (2003) and Tsui (2002). Multidimensional poverty raises many of the same issues that have been explored in the multidimensional inequality literature. In addition, there are issues that relate specifically to the concern with poverty. For example, there is the basic issue of who should be counted as being poor—someone who is poor in all dimensions or someone who falls below the poverty threshold in any dimension?

Although much has already been learned about multidimensional normative inequality indices, much more remains to be discovered. Compared to the theory of univariate inequality measurement, the analysis of multidimensional inequality is in its infancy.

[33]For a discussion of transfer sensitivity for univariate distributions, see Shorrocks and Foster (1987).

Acknowlegements

This article was prepared for the 2003 Summer School on Inequality and Economic Integration organized by the International School of Economic Research at the University of Siena. I am grateful to Andrea Brandolini, Satya Chakravarty, Marc Fleurbaey, Thibault Gajdos, Christian List, Ernesto Savaglio, and the participants in the Summer School for their comments. I am especially grateful to Claudio Zoli for the proof that appears in footnote 16.

References

Allison, R. A., Foster, J. E., 2004. Measuring health inequality using qualitative data. Journal of Health Economics, forthcoming.

Atkinson, A. B., 1970. On the measurement of inequality. Journal of Economic Theory 2, 244–263.

Atkinson, A. B., 1983. Social Justice and Public Policy. MIT Press, Cambridge, MA.

Atkinson, A. B., Bourguignon, F., 1982. The comparison of multi-dimensioned distributions of economic status. Review of Economic Studies 49, 183–201.

Ben-Porath, E., Gilboa, I., Schmeidler, D., 1997. On the measurement of inequality under uncertainty. Journal of Economic Theory 75, 194–204.

Blackorby, C., Bossert, W., Donaldson, D., 1999. Income inequality measurement: The normative approach. In: Silber, J. (Ed.), Handbook of Income Inequality Measurement. Kluwer Academic Publishers, Boston, pp. 133–157.

Blackorby, C., Donaldson, D., Auersperg, M., 1981. A new procedure for the measurement of inequality within and among population subgroups. Canadian Journal of Economics 14, 665–685.

Bossert, W., Weymark, J. A., 2004. Utility in social choice. In: Barberà, S., Hammond, P. J., Seidl, C. (Eds.), Handbook of Utility Theory, Volume 2: Extensions. Kluwer Academic Publishers, Boston, pp. 1099–1177.

Bourguignon, F., 1999. Comment on Maasoumi (1999). In: Silber, J. (Ed.), Handbook of Income Inequality Measurement. Kluwer Academic Publishers, Boston, pp. 477–484.

Bourguignon, F., Chakravarty, S. R., 2003. The measurement of multidimensional poverty. Journal of Economic Inequality 1, 25–49.

Chakravarty, S. R., 1990. Ethical Social Index Numbers. Springer-Verlag, Heidelberg.

Dalton, H., 1920. The measurement of the inequality of incomes. Economic Journal 30, 348–361.

Dardanoni, V., 1995. On multidimensional inequality measurement. In: Dagum, C., Lemmi, A. (Eds.), Income Distribution, Social Welfare, Inequality, and Poverty. Vol. 6 of Research on Economic Inequality. JAI Press, Stamford, CT, pp. 201–207.

Dutta, B., 2002. Inequality, poverty, and welfare. In: Arrow, K. J., Sen, A. K., Suzumura, K. (Eds.), Handbook of Social Choice and Welfare. Vol. 1. North-Holland, Amsterdam, pp. 597–633.

Ebert, U., 1995. Income inequality and differences in household size. Mathematical Social Sciences 30, 37–55.

Fleurbaey, M., Trannoy, A., 2003. The impossibility of a Paretian egalitarian. Social Choice and Welfare 21, 243–263.

Foster, J. E., 1994. Normative measurement: Is theory relevant? American Economic Review (Papers and Proceedings) 84, 265–370.

Gajdos, T., Maurin, E., 2004. Unequal uncertainties and uncertain inequalities: An axiomatic approach. Journal of Economic Theory, forthcoming.

Gajdos, T., Weymark, J. A., 2003. Multidimensional generalized Gini indices. Working Paper No. 03-W11, Department of Economics, Vanderbilt University.

Hardy, G. H., Littlewood, J. E., Pólya, G., 1934. Inequalities. Cambridge University Press, Cambridge.

Joe, H., 1997. Multivariate Models and Dependence Concepts. Chapman and Hall, London.

Kolm, S.-C., 1969. The optimal production of social justice. In: Margolis, J., Guitton, H. (Eds.), Public Economics. Macmillan, London, pp. 145–200.

Kolm, S.-C., 1976. Unequal inequalities. I. Journal of Economic Theory 12, 416–442.

Kolm, S.-C., 1977. Multidimensional egalitarianisms. Quarterly Journal of Economics 91, 1–13.

Koshevoy, G. A., Mosler, K., 1997. Multivariate Gini indices. Journal of Multivariate Analysis 60, 252–276.

List, C., 1999. Multidimensional inequality measurement: A proposal. Working Paper in Economics No. 1999-W27, Nuffield College, Oxford.

Maasoumi, E., 1986. The measurement and decomposition of multidimensional inequality. Econometrica 54, 991–997.

Maasoumi, E., 1999. Multidimensioned approaches to welfare analysis. In: Silber, J. (Ed.), Handbook of Income Inequality Measurement. Kluwer Academic Publishers, Boston, pp. 437–477.

Marshall, A. W., Olkin, I., 1979. Inequalities: Theories of Majorization and Its Applications. Academic Press, New York.

Moyes, P., 1999. Comparisons de distributions hétérogènes et critères de dominance. Économie et Prévision 138–139, 125–146.

Pigou, A. C., 1912. Wealth and Welfare. Macmillan, London.

Pollak, R. A., 1971. Additive utility functions and linear Engel curves. Review of Economic Studies 38, 401–414.

Savaglio, E., 2002. Multidimensional inequality: A survey. Working Paper No. 362, Dipartimento di Economia Politica, Università degli Studi di Siena.

Sen, A. K., 1973. On Economic Inequality. Clarendon Press, Oxford.

Shorrocks, A. F., 1983. Ranking income distributions. Economica 50, 3–17.

Shorrocks, A. F., 1995. Inequality and welfare evaluation of heterogeneous income distributions. Discussion Paper No. 447, Department of Economics, University of Essex.

Shorrocks, A. F., Foster, J. E., 1987. Transfer sensitive inequality measures. Review of Economic Studies 54, 485–497.

Trannoy, A., 2004. Multidimensional egalitarianism: The dominance approach. In: Farina, F., Savaglio, E. (Eds.), Inequality and Economic Integration. Routledge, London.

Tsui, K.-Y., 1995. Multidimensional generalizations of the relative and absolute inequality indices: The Atkinson–Kolm–Sen approach. Journal of Economic Theory 67, 251–265.

Tsui, K.-Y., 1999. Multidimensional inequality and multidimensional generalized entropy measures: An axiomatic derivation. Social Choice and Welfare 16, 145–157.

Tsui, K.-Y., 2002. Multidimensional poverty indices. Social Choice and Welfare 19, 69–93.

United Nations Development Programme, 1990. Human Development Report 1990. Oxford University Press, New York.

Weymark, J. A., 1981. Generalized Gini inequality indices. Mathematical Social Sciences 1, 409–430.

Weymark, J. A., 1999. Comment on Blackorby, Bossert, and Donaldson (1999). In: Silber, J. (Ed.), Handbook of Income Inequality Measurement. Kluwer Academic Publishers, Boston, pp. 157–161.