PERCOLATION AND ISING MODEL ON GRAPHS WITH TREE-LIKE STRUCTURE

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

This thesis belongs both to the group theory and probability. In the past 20 years there has been a growing interest in the use of probability in finite groups and in infinite groups given by defining relations. The first motivation for study generic properties of infinite groups is the effort to understand how a typical group looks like, what properties typical elements and subgroups of a given group have. One of the first statements is that "most" groups are hyperbolic, by Gromov. Later the result was made precise and the density model of random group was introduced, by Gromov and Ol′shanskiĭ [34]. The model of random group refers to a random presentation, where the lengths of relators are long enough and approximately the same. Using random groups Gromov [19] obtained a probabilistic proof that there exists a finitely presented group that is not uniformly embeddable into a Hilbert space. It also turns out that random group-theoretic objects exhibit various kinds of algebraic rigidity properties. In particular, Kapovich, Schupp and Shpilrain [24] proved that random one-relator groups satisfy a strong Mostow-type rigidity.

Arzhantseva and Ol′shanskiĭ [2] showed that, generically, for most groups given by m generators and n defining relators, any subgroup with fewer than m generators is free. In subsequent work Arzhantseva applied this approach to prove a number of other results about "generic" properties of finitely presented groups. Kapovich and Schupp [23] showed that this approach can be combined with other techniques to yield precise results about subgroups of bounded rank in some well-known classes of groups, namely, Coxeter groups, Artin groups and one-relator groups with torsion.

The notion of genericity in the work cited above concerns the collection of finitely presented groups. Kapovich, Myasnikov, Schupp and Shpilrain [22] considered generic properties of algorithmic problems in individual groups with respect to asymptotic density. They showed that for a very large class of finitely generated groups the classical decision problems of group theory - the word, conjugacy and membership problems - all have linear-time generic-case complexity. This was used to organize successful attacks on group theory based cryptography protocols.

Other connection between group theory and probability includes probability measures and random processes on Cayley graphs. Properties of random walks, percolation, Ising model or other random processes are determined in part by the underlying group structure. First important result was made by Kesten [26] showing the connection between random walks and amenability. Various models of statistical physics was originally studied on cubic lattice $\mathbb{Z}^d$ and later generalized to other graphs, see [30]. Benjamini, Lyons, Peres, and Schramm [5] introduced group-invariant percolation and showed a connection between Cheeger constant, spectral radius and the threshold value. They also proved that critical percolation on any non-amenable Cayley graph has no infinite clusters.

The main part of this thesis treats two random models on a special class of graphs. We define a tree-like structure of a graph and prove basic properties of these graphs in Chapter II. Then we present algorithms for finding values of the critical probability of the percolation and the critical temperature of the Ising model. The background on these models and statements of the results can be found in the following sections of the introduction.

In the last chapter we study random one-relator groups. We show that almost all one-relator groups with more than two generators are residually finite. It is a joint work with Mark Sapir. The background and formulation of the result are in the last section of this introductory chapter.

I.1 Percolation

We will use the notation $\mathcal{G} = (V, E)$ for a graph with the vertex set V and the edge set E. All graphs are assumed to be locally finite (vertices have finite degrees) and transitive (for any two vertices there exists an automorphism of $\mathcal{G}$ mapping one vertex to the other). We fix one vertex of the graph and call it the *origin*.

For every $p \in (0, 1)$, the *Bernoulli bond percolation* on $\mathcal{G}$ is a product probability measure P_p on the space $\Omega = \{0, 1\}^E$, the subsets of the edge set E. The product measure is defined via $\mathrm{P}_p(\omega(e) = 1) = p$ for all $e \in E$. The σ-algebra of P_p-measurable sets does not depend on p. We denote the σ-algebra by Σ and the expected value by E_p.

For any realization $\omega \in \Omega$, open edges form a random subgraph of $\mathcal{G}$. The *percolation function* is defined to be the probability that the origin is contained in an infinite cluster. The behavior of the percolation model depends strongly on the value of probability p. There is a critical value p_c of the probability p such that for $0 \le p < p_c$ all clusters are finite, and if $p_c < p \le 1$ there is an infinite cluster P_p-almost surely (see Grimmett [18]).

Explicit values of p_c have been known only for some special cases. In particular, for lattices in $\mathbb{R}^2$ the value of p_c is obtained using dual graphs (for $\mathbb{R}^d$ with $d \ge 3$ the values of p_c are not known). For the square lattice, Kesten ([25]) proved $p_c = 1/2$, for the triangular lattice $p_c = 2\sin(\pi/18)$, and for the hexagonal lattice $p_c = 1 - 2\sin(\pi/18)$ (see Grimmett [18]). Ziff and Scullard [40] found p_c for a larger class of lattices in $\mathbb{R}^2$ (they considered graphs that can be decomposed onto certain self-dual arrangements). The value of critical probability is also known for trees ($p_c = 1/\text{branching number}$) and Cayley graphs of virtually cyclic groups, where $p_c = 1$. The critical probability can also be found in case of a free product of finite transitive graphs ([27]). As far as we know these are the only known graphs where p_c for the bond percolation has been computed exactly.

If we change the generating set of a group, the graph changes dramatically, and there have been no examples of groups (except virtually cyclic) where p_c was known for all generating sets. It is expected that many properties of percolation (behavior at the critical value) are invariant with respect to changing the generating set, so it is useful to know how

p_c depends on the generating set of a group.

In this article we study the critical probability p_c for a class of graphs that admit the so-called *tree-like structure*. Roughly speaking, such a graph can be decomposed into a rooted tree of edge-disjoint pieces that intersect by the so-called border sets, which are cut-sets of the graph. We always assume that there are finitely many isomorphism classes of pieces. This class of graphs includes, for example, all transitive graphs with more than one end, Cayley graphs of amalgamated products and HNN extensions (for example: $SL(2, \mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$). In the case of an amalgamated product the pieces correspond to the Cayley graphs of factors and the border sets consist of vertices from cosets of the amalgamated subgroup. The precise definition of the tree-like structure is contained in Chapter II and several examples, including the grandparent tree, can be found in Chapter V.

Consider a realization of the percolation on a graph $\mathcal{G}$ with a tree-like structure. For each piece P_i that is not the root, let B_i be the border set that is the intersection of P_i with its parent in the tree.

Since B_i is a cut set, we can define an equivalence relation on it, by saying that two vertices of B_i are equivalent if they are connected by an open path inside the union of all pieces that are not descendants of P_i. The data consisting of the equivalence relation and the distinguished equivalence class connected to the origin is called the *color* of the piece P_i. If no vertex of the border set is connected to the origin we say that this piece has white color. Since the p_c of the whole graph does not exceed the p_c of any subgraph, we can assume that the probability p is smaller than the minimum of all p_c's of pieces. Then it is easy to see that the percolation process dies if and only if the tree of non-white pieces is finite. The fact that the pieces form a tree suggests using a branching process with individuals corresponding to the colored pieces, such that the distribution of children is induced by the percolation process. Unfortunately the color of a piece depends not only on the color of its parent (as required for branching process), but also on the colors of the siblings and their descendants.

Nevertheless, we define a different distribution on the colors of the children, and we obtain a branching process that has finite population size if and only if the expected size of the percolation cluster is finite. The next statement is the main result.

Theorem I.1. *Assume the graph $\mathcal{G}$ has a tree-like structure.*

(i) For a percolation with parameter p there exist a branching process on the tree of pieces, such that the expected size of its population is finite if and only if the expected size of the percolation cluster at the origin is finite.

(ii) If all the border sets are finite, then the branching process has finitely many types, and the first moment matrix is of finite size. In this case p_c is the smallest value of p such that the spectral radius of the first moment matrix is 1.

(ii) If in addition the pieces are finite, then the entries of the first moment matrix are algebraic functions in p. Therefore p_c is an algebraic number.
There exists an algorithm that, given the pieces and their border sets, computes a finite extension K of the field $\mathbb{Q}(p)$ and an algebraic function f in K such that p_c is the smallest positive root of f.

The theorem is proved in Section III.2.

We list here two corollaries of Theorem I.1. The first one is already proved in [27] by the author using different method. The second corollary answers a question of M. Sapir (personal communication) about the special linear group $SL(2, \mathbb{Z})$, which is an simple example of a group that is not free product. Both corollaries are proved in Chapter V.

By the expected sub-critical cluster size $\chi_i(p)$ we mean $E_p(|C|)$, where C is the cluster containing the origin and $p < p_c$.

Corollary I.2. *Let $\mathcal{G}$ be a free product of (transitive) graphs. Denote by $\chi_i(p)$ the expected (sub-critical) cluster size in the i-th factor graph. The critical probability p_c of $\mathcal{G}$ is the infimum of positive solutions of*

$$\sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} \chi_i(p) - (n-1) \prod_{i=1}^{n} \chi_i(p) = 0.$$

Note that in the case of free products (with respect to natural set of generators) the border sets consist of one vertex and so the branching process has just one type of individual (for more details see Section V.1). It is in fact also a special case of Theorem I.4 below.

We explicitly compute the critical probabilities of several Cayley graphs. In particular we prove the following.

Corollary I.3. *The critical probability p_c of the special linear group $SL(2, \mathbb{Z})$ given by presentation $\langle a, b | a^4, b^6, a^2 b^{-3} \rangle$ is an algebraic number that is equal to* $.4291140496\ldots$

The critical probability p_c of the projective special linear group $PSL(2, \mathbb{Z})$ given by presentation $\langle a, b | a^2, b^3 \rangle$ is an algebraic number equal to $.5199\ldots$

The critical probability p_c of the grandparent tree is an algebraic number equal to $.1587\ldots$

The general case of Cayley graphs of amalgamated products and HNN extensions is covered by the following theorem. Consider a group G acting on a simplicial tree T. The standard generating set of the group G is any generating set consisting of elements in the vertex stabilizers and free letters. It follows from the Structure theorems of Bass-Serre theory (see for example [38] or [11]) that the group G is the fundamental group of a graph of groups (see Section II.2.3).

Theorem I.4. *Let G be a group acting on a simplicial tree and let $\mathcal{G}$ be its Cayley graph with respect to a standard generating set. Then $\mathcal{G}$ has a tree-like structure whose pieces* *correspond to the Cayley graphs of the vertex stabilizers and border sets correspond to the edge stabilizers.*

In the Section II.3, we prove that every transitive graph with more than one end has a tree-like structure with finite border sets. We use a strong result of Dunwoody [15] about existence of special cut sets. This can be applied to, say, Cayley graphs of free groups with arbitrary finite generating sets. Moreover, the pieces obtained from the general construction are finite in this case, and are explicitly described in Section V.5.

Theorem I.5. *Let G be a virtually free group, that is, it acts on a simplicial tree with finite vertex stabilizers. Then its Cayley graph with respect to any finite generating set has a tree-like structure with finite border sets and finite pieces. Given a finite generating set, the pieces of the tree-like structure are algorithmically constructed.*

Therefore the p_c is an algebraic number and one can use the algorithm from Theorem I.1 (iii) to compute p_c, given any finite generating set.

This theorem is proved in Section II.2.4.

This gives the first example of a class of Cayley graphs closed under quasi-isometry where we can algorithmically find the value of p_c for every graph in the class (besides the graphs with 0 and 2 ends where $p_c = 1$).

I.2 Ising model

In the Ising model of ferromagnetism, spins (± 1) are assigned randomly to vertices of a (transitive, locally finite) graph $\mathcal{G} = (V, E)$. The strength of interactions along the edges is given by the inverse temperature β in such a way as to favor alignment of the spins. Any probability measure on $\{-1, 1\}^E$ satisfying locally the condition about interactions is called a Gibbs state. For the precise definition see Section IV.1.

The model was introduced by Lenz and Ising as a simple model for magnetization. The inverse temperature stands for K_b/T, where K_b is Boltzmann's constant and T is the temperature. The model was first studied for cubic lattices $\mathbb{Z}^d$, where the existence of phase transition was shown by Dobrushin [13]. It means that there is a critical inverse temperature β_c such that at an inverse temperature lower than β_c, there is only one Gibbs state, while at inverse temperature above β_c, there are at least two - we say that the phase transition occurs.

The value of critical temperature is known for example for $\mathbb{Z}^1$, where it is infinity. Much more difficult result, which started with work by Onsager [36], is that for $\mathbb{Z}^2$ the critical temperature $\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$. A rigorous calculation of the critical value in higher dimensions is beyond current knowledge.

The following result gives an algorithm for finding the critical inverse temperature for Cayley graphs of virtually free groups and similar graphs.

Theorem I.6. *Consider a graph $\mathcal{G}$, which has a tree-like structure with finite pieces. There is a system of polynomial equations that has more than one solution if and only if the phase transition occurs. Assuming there is only one Gibbs state at the critical inverse temperature for the Ising model, the value of* $\exp(\beta_c)$*, the exponent of the critical inverse temperature, is an algebraic number, which can be found as a solution of a system of polynomial equations. These equations depend on the chosen tree-like structure.*

We assumed that the interaction strength is the same along all edges. Nevertheless a similar result can be obtained assuming that the interaction strength varies. Note that the value of the critical inverse temperature is found using Jacobian of the system of equations.

In the Chapter V we compute the critical inverse temperature for the grandparent tree and prove the following corollary. The expected magnetization $M = M(\beta)$ of a graph is the expected sum of spins on the graph provided the origin has a positive spin, at an inverse temperature β.

Corollary I.7. *Let $\mathcal{G}$ be a free product of N (transitive) finite graphs. Denote by M_i the expected magnetization in the i-th factor graph. The critical inverse temperature β_c of $\mathcal{G}$ is the infimum of positive solutions of*

$$(N-1)\prod_i \mathrm{E}(M_i) - \sum_j \prod_{i\neq j} \mathrm{E}(M_i) = 0.$$

In case of the N-regular tree this leads to the critical inverse temperature $\beta_c = \coth^{-1}(N-1)$.

The special case of trees was previously shown by Lyons [29].

A natural generalization of the Ising model is the Potts model, where a larger (finite) set of spins is considered. Similar system of equations can be obtained for the Potts model on graphs with tree-like structure. But the value of the critical inverse temperature for the Potts model can not be obtained by the above method. The underlying reason for this is that the Gibbs state at the critical temperature is unique for Ising model and is not unique for the Potts model, shown for homogeneous trees in [20].

The connection between percolation and Ising model is rather strong. They are linked via the random-cluster model, which is a two-parameter family of processes introduced by Fortuin and Kasteleyn [17]. Study of the random-cluster model has so far been limited to finite graphs, lattices $\mathbb{Z}^d$ and homogeneous trees. The random-cluster model on graphs with tree-like structure is a natural option for future research.

I.3 One-relator groups

Residual finiteness of 1-related groups is one of the main topics in combinatorial group theory since 1960s. The first non-residually finite examples were given in [4] (say, the Baumslag–Solitar groups $BS(p,q) = \langle a, b \mid b^{-1}a^p b = a^q \rangle$ where p and q are different primes).

Possibly the strongest "positive", non-probabilistic result so far is the result by D. Wise [39]: any one-relator group whose relator is a positive word satisfying the condition $C'(1/6)$ is residually finite. The strongest "negative" result appeared recently in the paper by Baumslag, Miller and Traeger [3]: Let $G = \langle a, b, \dots | r = 1\rangle$ be a one-relator group with at least two generators and let $G(r, w) = \langle a, b, \dots | r^{r^w} = r^2\rangle$ where w is an element of a free group with free generators $a, b, \dots$ that does not commute with r. Then [3, Theorem 1] asserts that the group $G(r, w)$ is not residually finite. Note that the length of the relator of $G(r, w)$ is at most a constant multiple of the length of the relator of G.

In [7], Borisov and Sapir reported a result of computations saying that, apparently, more than 94% of 1-related groups with 2 generators and a relator of size $n \gg 1$ are ascending HNN extensions of free groups (that is represented as HNN extensions of a free group where one of the associated subgroups is the free group itself). By the main result of [7], such groups are residually finite. Borisov and Sapir used the Monte-Carlo method for $n \approx 10^6$. Schupp and later Dunfield and Thurston [14] conducted similar experiments on their own and came to the same conclusion. At the same time, Dunfield and Thurston noticed [14] that a 2-generated 1-related group is not almost surely an ascending HNN extension of a free group (that is the probability that a 2-generated 1-related group with a relator of size n is an ascending HNN extension of a free group is bounded away from 1 as $n \to \infty$). Recently, Borisov and Sapir [8] strengthened their result from [7] by proving that every ascending HNN extension of a free group is virtually residually (finite p-)group for every sufficiently large prime p. Thus many 1-related groups with two generators satisfy this property as well.

Note that by the result of Feighn and Handel [16], ascending HNN extensions of free groups are *coherent* that is every finitely generated subgroup of them is finitely presented. Coherence is a very important property of groups also. Coherence of all one-relator groups is a long-standing open problem.

In this paper, we consider two natural models of choosing a random 1-related group.

Model NR. For every $r \geq 0$, consider the set S_r of all group words R of length $r \geq 1$ in a free group $F_k = \langle x_1, ..., x_k\rangle$. On that set, we choose the uniform probability measure. By a *random 1-related group with k generators of complexity* r we mean the group with presentation $\langle x_1, ..., x_k \mid R = 1\rangle$ where R is a random word from S_r.

Model CR. In this model, we consider the set CS_r of cyclically reduced words in F_k of length r and consider the uniform probability measure on that set. Then a *random 1-related group with k generators of complexity* r is a group $\langle x_1, ..., x_k \mid R\rangle$ where R is a random word from CS_r.

Now given any property P of groups, consider the probability p_r that a random k-generator 1-relator group of complexity r has property P. If p_r has a limit p, we say that a random k-generator 1-relator group has this property with probability p.

We prove below (see Lemma VI.9) that if the limit of probabilities p_r exists in the random model CR, then it coincides with the limit in model NR.

In this paper, we mostly study 1-related groups with at least 3 generators. By Brown's result [9, Page 490] such a group is **never** an ascending HNN extension of a free group. Nevertheless by using some deep results of Ol′shanskiĭ [35] from combinatorial group theory together with some deep results from the probability theory (more specifically, the theory of Brownian motions in $\mathbb{R}^k$) we prove the following

Theorem I.8. *A random k-generator 1-relator group, $k \geq 3$, can be embedded into an ascending HNN extension of a finitely generated free group with probability 1. In particular, almost surely, such a group is residually finite, virtually residually (finite p-)group for every sufficiently large prime p, and coherent.*

Note that almost all (with probability tending to 1) 1-related groups of complexity $r \gg 1$ satisfy the small cancellation condition $C'(1/6)$. Hence they are hyperbolic almost surely. It is still a major open question in group theory whether every hyperbolic group is residually finite. The positive answer would of course imply a part of Theorem I.8. On the other hand, Theorem I.8 gives a new large class of residually finite hyperbolic groups.

As an immediate corollary of Theorem I.8, we deduce that one cannot replace a multiplicative constant in the result of Baumslag-Miller-Troeger [3, Theorem 1] mentioned above by an additive constant: if $n \geq 3$, then there exists no maps $\phi\colon F_n \to F_n$ such that $|\phi(R)| - |R|$ is bounded from above by some constant C and such that for every non-trivial $R \in F_n$, the group $\langle F_n \mid \phi(R) = 1\rangle$ is not residually finite. Indeed, it is easy to see that if such a map exists, the probability of a 1-related group with n generators to be residually finite would be bounded away from 1 as $|R|$ tends to ∞.

Let us present the main ideas of the proof of Theorem I.8. Let $G = \langle a_1, a_2, ..., a_k \mid R = 1\rangle$ be a 1-related group. Consider the Cayley graph Γ_k of $\mathbb{Z}^k$ viewed as the abelianization of the free group $\langle a_1, ..., a_n\rangle$. Then there is a path w in Γ_k starting at $O = (0, 0, ..., 0)$ and reading the word R. Let P be the end point of w. By a result of Brown [9], if $k = 2$ and a support line of w that is parallel to the vector $\vec{OP}$ intersects w at a single vertex or a single edge, then G is an ascending HNN extension of a free group. If $k > 2$ then one needs to consider the convex hull H of w and its projections onto the hyperplane $\mathbb{R}^{k-1}$ that is perpendicular to $\vec{OP}$. The projection is a (convex) polyhedron L in $\mathbb{R}^{k-1}$. We prove, using a result of Ol′shanskiĭ [35] about factor-groups of hyperbolic groups, that if one of the vertices of L is visited exactly once by the projection of w onto $\mathbb{R}^{k-1}$, then G is embeddable into a 2-generated 1-related group that is an ascending HNN extension of a free group, i.e. the conclusions of Theorem I.8 hold. It remains to prove that a random walk w in Γ_k of length r satisfies that condition with probability tending to 1 as $r \to \infty$. In case of two generators the projection L is an interval, and so there are only two vertices in L, hence the projection of w visits each of these vertices several times with probability bounded away zero (that is essentially proved in [14]). But if $k > 2$, then the number of vertices of L grows with the length of the relator. In order to prove that, we use the fact that the boundary of the convex hull of a Brownian motion in $\mathbb{R}^2$ is a smooth curve [12]. Therefore it is "very likely" that one of the vertices of L is visited only once by the projection of the random walk w onto $\mathbb{R}^{k-1}$.

CHAPTER II

GRAPHS WITH TREE-LIKE STRUCTURE

In this chapter we study a class of graphs that admit the so-called *tree-like structure*. We show that this class of graphs includes all transitive graphs with more than one end and Cayley graphs of virtually free groups. Several examples, including the grandparent tree and $SL(2, \mathbb{Z})$, can be found in Chapter V.

Recall that the *Cayley graph* of a group G with respect to the finite set of generators S is the (nonoriented) graph $\mathcal{G}$ with vertex set $V = G$ and edges $\{g, h\} \in E$ if and only if $g^{-1}h \in S$ (with the appropriate multiplicity). We equip the Cayley graph with the usual graph metric.

Definition II.1. *A* tree-like structure *on a (transitive, locally finite, connected, infinite) graph $\mathcal{G}$ is a triple $(\mathbf{P}, J, \gamma)$, where $\mathbf{P}$ consists of pairs of non-empty subgraphs (P_i, B_i) of $\mathcal{G}$, $i \in I$ (P_i are called the* pieces*, B_i are called the* border sets*), J is a finite subset of I, γ is a* model map *from I to J, and the following conditions are satisfied.*

(1) For every $i \in I$, P_i is a subgraph of $\mathcal{G}$ and $B_i \subseteq V(P_i)$.

(2) For every $i \neq j$, $E(P_i) \cap E(P_j) = \emptyset$ and $\bigcup_{i \in I} E(P_i) = E(\mathcal{G})$.

(3) There is a partial order on the pieces with maximal element P_0, such that its graphical representation is a tree with a root P_0. Moreover, if P_i is a child of P_j, then $P_i \cap P_j = B_i$.

Denote by $U(P_i)$ the union of the pieces in the descendant subtree of P_i (P_i including), for all $i \in I$.

(4) For every $i \neq j$, if $P_i \not\subseteq U(P_j)$, then $P_i \cap U(P_j) \subseteq B_j$.

(5) For every $i \in I$, there is an isomorphism between $U(P_i)$ and $U(P_{\gamma(i)})$ taking pieces to pieces and border sets to border sets, respecting the order on the pieces.

We say that there is a finite number of isomorphism classes of pairs (P_i, B_i)'s. And for $j \in J$ the pieces P_j are called the model pieces.

If there is a piece P_i with no edges, then it consists only of vertices and we can remove such a piece (and add the vertices to the parent piece) and change the tree-like structure accordingly. Therefore we will assume that each P_i contains at least one edge. Clearly the tree-like structure can be degenerated in the sense that the whole graph is just one piece or the number of pieces is finite. In what follows we will always assume that it is non-degenerated, that is, the number of pieces is infinite. In all cases considered in this paper, the pieces have at most finitely many components (this simplifies the computations).

For every $i \in I$, denote by $\Lambda_i \subset I$ the set of indices of children of P_i (given by part (3) of Definition II.1). Now we will present several basic properties of the tree-like structure.

Lemma II.2. *For every i, the set $U(P_i)$ is covered by P_i and the collection of $U(P_\lambda)$ for $\lambda \in \Lambda_i$. Moreover, pairwise intersections of $U(P_\lambda)$'s for $\lambda \in \Lambda_i$ consist only of vertices of P_i.*

Proof. By property (2) each edge is included in exactly one piece and therefore if $P_k \subset U(P_{\lambda_1})$, then $P_k \not\subset U(P_{\lambda_2})$, provided $\lambda_2 \neq \lambda_1$. Therefore by property (4) the intersection $U(P_{\lambda_1}) \cap U(P_{\lambda_2})$ is included in $B_{\lambda_1} \cap B_{\lambda_2}$. Since each $B_\lambda \subset P_i$ by property (3), the claim follows. $\square$

Lemma II.3. *Every border set (except possibly B_0, the border set of the root) is a vertex cut set of graph $\mathcal{G}$. In particular, let a subgraph $U(P_i)^c$ of $\mathcal{G}$ be induced by the edges $E(\mathcal{G}) \setminus E(U(P_i))$. Then $U(P_i) \cup U(P_i)^c = \mathcal{G}$ and $U(P_i) \cap U(P_i)^c = B_i$.*

Proof. If the complement $U(P_i)^c$ satisfies $U(P_i) \cap U(P_i)^c = B_i$, then B_i is indeed a cut set. By property (4) for any $P_k \subset U(P_i)^c$ we have $P_k \cap U(P_i) \subset B_i$ so the claim follows. $\square$

II.1 Changing the generating set

The following lemmas apply to the Cayley graphs of groups. In that case the group itself acts on its Cayley graph (by multiplication from the right) and so it is a subgroup of the automorphism group $Aut(\mathcal{G})$. If the isomorphisms of part (5) of Definition II.1 are (almost) in the group G, we can make the generating set smaller or slightly bigger and still obtain a tree-like structure.

Lemma II.4. *Let S be a finite generating set of a group G. Suppose that there is a tree-like structure of the Cayley graph $\mathcal{G}$ of G with respect to S, and that each of the isomorphisms from part (5) of Definition II.1 can be extended to the whole $\mathcal{G}$, and that these extensions form a subgroup H of $Aut(\mathcal{G})$. If $G \cap H$ has finite index in H, then for any generating subset $S' \subset S$ the Cayley graph $\mathcal{G}'$ of G with respect to S' has a tree-like structure with pieces containing the same vertices (and fewer edges).*

Proof. If we restrict the isomorphisms in part (5) of Definition II.1 from H to $G \cap H$, we will increase the number of isomorphism classes by finitely many because $G \cap H$ has finite index in H. Thus we can assume that $H \subset G$.

The graph $\mathcal{G}'$ is obtained from $\mathcal{G}$ by removing edges with labels in $S \setminus S'$. The pieces P_i' of the tree-like structure on $\mathcal{G}'$ are obtained in the same way (just by removing edges with labels in $S \setminus S'$). Let us verify the conditions of Definition II.1 Every edge belongs to exactly one piece, thus property (2) follows. Conditions (1),(3),(4) depend only on vertices of pieces. Since vertices do not change, these conditions are satisfied.

It remains to show property (5). Since the isomorphisms between $U(P_i)$'s in $\mathcal{G}$ are given by elements in G, they are also isomorphisms of $U(P_i')$'s in $\mathcal{G}'$. Thus there are finitely many isomorphism classes between pieces of G'. □

Lemma II.5. *Let S, S' be finite generating sets of a group G. Suppose that there is a tree-like structure with pieces P_i of the Cayley graph $\mathcal{G}$ of G with respect to S, and that the isomorphisms from part (5) of Definition II.1 form a subgroup H of $Aut(\mathcal{G})$. Assume that $G \cap H$ has finite index in H and that the following condition is satisfied:*

($*$) for all $x \in S' \setminus S$ and $g \in G$ there is a piece P_i containing both vertices labeled by g and gx.

Then the Cayley graph $\mathcal{G}'$ of G with respect to S' has a tree-like structure with pieces P_i', where P_i' contains the same vertices as P_i.

Proof. Again we can assume that $H \subset G$. If we prove the claim for $S'' = S \cup S'$, then for S' it follows from Lemma II.4. Therefore we can assume that $S \subset S'$.

The graph $\mathcal{G}'$ is obtained from $\mathcal{G}$ by adding edges labeled by elements in $S' \setminus S$. In the tree-like structure, we add each new edge to the oldest (i.e. the maximal in the partial order) piece P_i containing both its endpoints. Such a piece exists by assumption ($*$). Suppose that there are two such oldest pieces P_i and P_j containing end points of an edge e. Then none of them is a descendant of the other and their intersection is in $B_i \cup B_j$ by condition (4). Therefore e should be added to their predecessor that is older than both of them. Therefore there is a unique such oldest piece for each edge. This implies property (2). Conditions (1),(3),(4) now depend only on vertices of pieces that have not been changed. Therefore these conditions hold as in the tree-like structure on $\mathcal{G}$.

It remains to prove property (5). Consider an isomorphism from property (5) of the graph $\mathcal{G}$. Let $h \in H \subset G$ such that $U(P_i) = hU(P_j)$, and h takes pieces to pieces and border sets to border sets (we can assume $i, j \neq 0$). We will show that the descendant trees of the modified pieces P_i' and P_j' in $\mathcal{G}'$ can be mapped by the same isomorphism h, and it takes the modified pieces to pieces and border sets to border sets as well. The vertices of pieces have not been changed so a difficulty can arise only for edges. By contradiction, assume that $U(P_i') \neq hU(P_j')$ respective that there is a descendant P_k' of P_i' and P_l' of P_j' such that $P_k = hP_l$ but $P_k' \neq hP_l'$. There exists an edge e in P_k' that is missing in hP_l' (or vice versa). If $e \notin hP_l'$, then there is a older (in the ordering) piece than P_l' containing both endpoints of $h^{-1}e$. Therefore these endpoints are in the border set B_l'. Consequently the endpoints of e are in B_k', and so the edge must be included in some ancestor of P_k', a contradiction. □

Lemma II.5 can be applied only if the pieces are large enough to make the condition ($*$) satisfied. The following lemma allows us to enlarge pieces. The new piece P_i' is a union of the children pieces of the piece P_i. If we started with finite pieces, then the constructed pieces are again finite. Moreover, the tree of pieces (partial order) remains unchanged.

Lemma II.6. *Assume graph $\mathcal{G}$ admits a tree-like structure with pieces P_i and border sets B_i. Define new pieces P_i' and border set B_i' as follows*

$$P_0' = P_0 \cup \bigcup_{k \in \Lambda_0} P_k, \qquad B_0' = B_0,$$
$$P_i' = V(P_i) \cup \bigcup_{k \in \Lambda_i} P_k, \qquad B_i' = V(P_i), \qquad \text{for } i \neq 0.$$

Define the partial order on the pieces P_i' to be the same as on the original pieces P_i. Then the modified pieces P_i' and border sets B_i' form a new tree-like structure of $\mathcal{G}$.

Proof. We need to verify conditions of Definition II.1.

(1)-(2) Clearly follow from the definition of P_i' and B_i'.

(3) Let P_j' be a child of P_i'. Then $P_i' \cap P_j'$ contains vertices of P_j, that is B_j'. If $k \in \Lambda_j$, then by property (4) of the original tree-like structure, $U(P_k) \cap P_i' \subset B_k \subset V(P_j) = B_j'$. Therefore $P_i' \cap P_j' = B_j'$.

(4) Assume P_i' is not a descendant of P_j' and $i \neq j$. Then P_i is not a descendant of P_j and no child of P_i is a descendant of P_j (it can be P_j itself). Therefore using property (4) of the original tree-like structure we see that $P_i' \cap U(P_j) \subset P_j$. Since the intersection contains only vertices (by (2)), we can write

$$P_i' \cap U(P_j') = V((P_i \cup \bigcup_{k \in \Lambda_i} P_k) \cap U(P_j)) \subset V(P_j) = B_j'.$$

(5) Assume that there exists an isomorphism from property (5) between $U(P_i)$ and $U(P_j)$ such that the pieces and border sets are respected. The modification of the piece P_i into P_i' uses the children of P_i that are preserved by the isomorphism. Thus the modified pieces and their border sets are also preserved by this isomorphism between $U(P_i')$ and $U(P_j')$. Thus there are finitely many isomorphism classes.

□

II.2 Fundamental groups of graphs of groups

In this section we will generalize the example of $SL(2, \mathbb{Z})$ to arbitrary graphs of groups by showing that the Cayley graphs of fundamental group of graphs of groups have the tree-like structure. To make the transition simpler, we first consider amalgamated products and HNN extensions before proceeding to arbitrary graphs of groups.

II.2.1 Amalgamated products with standard generating sets

Recall that an amalgamated product $G_1 *_H G_2$ is obtained from two groups G_1 and G_2 provided monomorphisms $i_k : H \to G_k$, $k = 1, 2$. The group $G_1 *_H G_2$ is a quotient of the free product $G_1 * G_2/N$, where N is the smallest normal subgroup containing elements $i_1(h)i_2^{-1}(h)$, $h \in H$.

Consider the right Cayley graph $\mathcal{G}$ of an amalgamated product $G_1 *_H G_2$ with respect to the generating set $S_1 \cup S_2$, where $G_1 = \langle S_1 \rangle$ and $G_2 = \langle S_2 \rangle$. The structure of the Cayley graph $\mathcal{G}$ is the following. First consider the copies of the Cayley graph $\mathcal{G}_1$ of G_1 and the Cayley graph $\mathcal{G}_2$ of G_2 containing the origin. These subgraphs intersect by the vertices of H, which we consider as a border set. Each coset of H inside G_1 is another border set connecting $\mathcal{G}_1$ with another copy of $\mathcal{G}_2$; each coset of H inside G_2 is a border set connecting it with another copy of $\mathcal{G}_1$. The copies of $\mathcal{G}_1$ and $\mathcal{G}_2$ form a tree, it is a G-tree (see [11] Section 8.6) which is usually called the Bass-Serre tree of the amalgamated product.

Let us pick one of the pieces, say $\mathcal{G}_1$, and call it a root P_0. It contains $[G_1 : H]$ border sets B_λ, $\lambda \in \Lambda_1$. All of these P_λ, $\lambda \in \Lambda_1$, are in the same isomorphism class, say I_2, because there exists an isomorphism that takes P_λ to $P_{\lambda'}$ with $\lambda, \lambda' \in \Lambda_1$, and carries descendants to descendants. This isomorphism is just a left multiplication by an element of $G_1 *_H G_2$. Children of pieces in the isomorphism class I_2 are in the same isomorphism class again, denote it by I_1. So the isomorphism class of a piece depends only on the parity of the generation. More precisely there are three isomorphism classes: the root, the set of pieces of odd generations and the set of pieces of even generations. The root differs from an even generation piece by the number of children, but it is only one such exceptional piece.

II.2.2 HNN extensions with standard generating sets

Recall that an HNN extension G is constructed from a base group G_1 having a presentation $G_1 = \langle S \mid R \rangle$, and from an isomorphism α between two subgroups H and K of G_1. Let t be a new symbol not in S (free letter), and define $G = G_1 *_\alpha = \langle S, t \mid R, tht^{-1} = \alpha(h), \forall h \in H \rangle$.

Consider the Cayley graph $\mathcal{G}$ of this HNN extension G with respect to the generating set $S \cup \{t\}$. The Cayley graph $\mathcal{G}_1$ of G_1 is a part of the Cayley graph $\mathcal{G}$. Each coset of H (resp. K) is attached to another copy of $\mathcal{G}_1$, the attachment is done by edges labeled by t, these edges correspond to the isomorphism α. Therefore the piece $\mathcal{G}_1$ is connected to $[G_1 : H] + [G_1 : K]$ other pieces. This way we obtain the Bass-Serre tree of the HNN extension.

Let us denote by P_0 a subgraph of $\mathcal{G}$ containing the graph $\mathcal{G}_1$ and all t-edges incident to at least one vertex in $\mathcal{G}_1$ (and we also add its other endpoint). In $\mathcal{G}$, the that contains the copy of $\mathcal{G}_1$ at v and all t-edges incident to at least one vertex in this copy that are not in P_0. The border set B_1 consists of vertices in H in the copy of $\mathcal{G}_1$. Repeating this for all t-edges emerging from P_0 we obtain $[G_1 : K]$ pieces in the first generation. Similarly by following edges labeled by t^{-1} emerging from P_0 we obtain $[G_1 : H]$ other pieces in the first generation. Then we do the same for next generations. This procedure, gives the tree-like structure of the graph. Each piece P_i contains a copy of $\mathcal{G}_1$ and some t-edges connected to it (in fact it contains all such t-edges except the $|K|$ edges that connect the border set B_i and the parent). There are three isomorphism classes of pieces: the root, pieces whose border sets are copies of K and pieces whose border sets are copies of H.

II.2.3 Graphs of groups

Let us recall the definition of the fundamental group of a graph of groups [11].

Definition II.7. *A* graph of groups $\mathfrak{G}$ *consists of*

(i) a connected graph X with vertex set $V(X)$ and edge set $E(X)$,

(ii) for each vertex v of X a group G_v, and for each edge e of X a group G_e, and

(iii) for each edge $e = (v_1, v_2)$ monomorphisms $\tau : G_e \to G_{v_1}$ and $\sigma : G_e \to G_{v_2}$.

Denote by E the free group with basis $\{t_e; e \in E(X)\}$. Let $F(\mathfrak{G})$ be the quotient group $(E * \ast_{v\in V(X)} G_v)/N$, where N is the normal closure of the subset $\{t_e^{-1}\tau(a)t_e\sigma(a)^{-1} : e \in E(X), a \in G_e\}$.
Let T be a maximal tree in X. We define the fundamental group $\pi(\mathfrak{G}, X, T)$ *to be $F(\mathfrak{G})/M$, where M is the normal closure of $\{t_e, e \in E(T)\}$.*

Note that the groups $\pi(\mathfrak{G}, X, T)$ are independent of T up to isomorphism.

We will consider the Cayley graph $\mathcal{G}$ of the fundamental group $\pi(\mathfrak{G}, X, T)$ with the following generators: $(\bigcup_{v\in V(X)} S_v) \cup \{t_e : e \in E(X) \setminus E(T)\}$, where $G_v = \langle S_v \rangle$ for $v \in V(X)$. This set of generators depends on T. We call this set of generators *standard*. In order to obtain locally finite graph we will consider only finite sets of generators, in particular this restricts us to finite graphs X of finitely generated groups.

Observe that if X consists of one edge between two distinct vertices, then its fundamental group is an amalgamated product of the vertex groups. If the two vertices coincide, then the fundamental group is a HNN extension. The tree-like structure we described for these specific cases will be now generalized to the Cayley graph of any (finite) graph of groups.

Theorem I.4. *The Cayley graph of the fundamental group of a graph of groups has a tree-like structure.*

Proof. We will define the pieces and verify the conditions (1)-(5) in Definition II.1.

First observe that the Cayley graph $\mathcal{G}$ is covered by translates of the Cayley graphs $\mathcal{G}_v$ of the vertex groups G_v, $v \in V(X)$ by left multiplications by elements of G, and edges labeled by t_e, $e \in E(X) \setminus E(T)$. For each $e = (v_1, v_2) \in E(X) \setminus E(T)$ denote by R_e the set of edges labeled by t_e such that they start at a vertex of $\mathcal{G}_{v_1}$ and end at a vertex of the translate of $\mathcal{G}_{v_2}$ by t_e. Then we denote by $g\mathcal{G}_v$ the translate of the subgraph $\mathcal{G}_v$ by g, the representative of a coset in $\pi(\mathfrak{G}, X, T)/G_v$. Denote by ghR_e the translate of the set R_e by gh, where g is a representative of a coset in $\pi(\mathfrak{G}, X, T)/G_{v_1}$ and h is a representative of a coset in $G_{v_1}/\tau(G_e)$. These sets will be used in the construction of pieces of the tree-like structure.

Consider the quotient map $\pi(\mathfrak{G}, X, T) \to \pi_1(X)$ taking all G_v to identity. The fundamental group $\pi_1(X)$ of a finite graph X is free of rank $|E(X) \setminus E(T)|$. Every translate of R_e is a preimage of one edge in the Cayley graph of $\pi_1(X)$, thus it is an edge cut set of $\mathcal{G}$. Moreover, if $e = (v_1, v_2) \in E(T)$, then any translate $g\tau(G_e)$ by $g \in \pi(\mathfrak{G}, X, T) \to \pi_1(X)$ is a vertex cut set.

We say that $g_1\mathcal{G}_{v_1}$ and $g_2\mathcal{G}_{v_2}$ are neighbors if there is an edge $e = (v_1, v_2) \in E(T)$ (or $e = (v_2, v_1)$) and $g_1\mathcal{G}_{v_1} \cap g_2\mathcal{G}_{v_2} \neq \emptyset$. We say that $g_1\mathcal{G}_{v_1}$ and $g_2\mathcal{G}_{v_2}$ are delayed neighbors if there is an edge $e = (v_1, v_2) \in E(X) \setminus E(T)$ ($e = (v_2, v_1)$ resp.) and there is translate hR_e of a set of edges labeled by t_e connecting a vertex of $g_1\mathcal{G}_{v_1}$ to a vertex of $g_2\mathcal{G}_{v_2}$, we say the neighbors are delayed by hR_e.

Consider a graph with vertices $g\mathcal{G}_v$ and edges between neighbors and delayed neighbors. This graph is a tree. Assume not and let there be a cycle. If some edge of the cycle arises from delayed neighbors, then this R_e is not a cut set. If all edges in the cycle are neighbors, they induce a loop in T, which is also a contradiction. We choose a root of this tree to be $\mathcal{G}_{v_o}$ for some $v_0 \in V(X)$.

Now we are ready to define pieces and border sets. We take the tree from above and define the pieces: For the root $\mathcal{G}_{v_0}$ we define a piece $P_{v_0,o}$ to be a union of $\mathcal{G}_{v_0}$ and sets hR_e for all its delayed neighbors. For any other $g\mathcal{G}_v$ we define a piece $P_{v,g}$ to be a union of $\mathcal{G}_{v_0}$ and sets hR_e for all its delayed neighbors that are not parent of $g\mathcal{G}_v$ in the above tree order. Define the border sets of a piece $P_{v,g}$ to be its intersection with the parent piece. For the root define the border set to contain only the origin.

Let us now verify the conditions in Definition II.1:

(1) The Cayley graphs $\mathcal{G}_v$ are connected and the pieces remain connected after adding the incident edges t_e. The intersection of P_{v_1,g_1} with its parent P_{v_2,g_2} is a translate of $\tau(G_e)$ ($\sigma(G_e)$ resp.) inside P_{v_1,g_1} provided $e = (v_1, v_2)$ ($e = (v_2, v_1)$ resp.). Thus the border set is indeed subset of the vertex set of the piece.

(2) We need to show that every edge $e \in E(\mathcal{G})$ is in exactly one P_i. Clearly this holds for edges labeled by $s \in S_v$. Every edge labeled by t_e has endpoints in two different pieces, which by the construction become parent and child, and thus it is included only in the parent piece.

(3) The tree was defined above, such that the border sets satisfy required condition (3).

(4) For all $j \neq 0$ the border set B_j is a cut set. Using the notation $U(P_i)^c$ for the subgraph

induced on edges in $E(G) \setminus E(U(P_i))$, we obtain $U(P_i) \cap U(P_i)^c = B_i$. So claim (4) follows.

(5) Now we want to show that there are only finitely many isomorphism classes of pieces. The root is a unique piece of its class and we exclude it from the following consideration. Assume P_i was obtained from translate of $\mathcal{G}_{v_1}$ and its parent from $\mathcal{G}_{v_2}$ that are neighbors due to an edge $e = (v_1, v_2) \in E(X)$. This edge together with its direction characterizes the isomorphism class of P_i. Indeed if P_j is characterized by the same edge $e = (v_1, v_2) \in E(X)$, then clearly there is an element $g \in \pi(\mathfrak{G}, X, T)$ acting on the Cayley graph as an isomorphism f such that $f(P_i) = P_j$ and $f(B_i) = B_j$. Therefore $f(U(P_i)) = U(P_j)$ as well. Moreover, since the pieces (and border sets) are identified in each subtree by the same procedure, this isomorphism respects the structure of pieces in the subtree.

□

Corollary II.8. *For the Cayley graph of a fundamental group of a finite graph of finite groups, with the natural generating sets, the p_c is an algebraic number.*

Proof. It follows from Theorem I.1 (iii) and Theorem I.4. □

Remark II.9. *A fundamental group of a graph of groups with trivial edge groups is a free product of its vertex groups and copies of $\mathbb{Z}$, so the first moment matrix M can be expressed as in the example V.1.*

II.2.4 Groups acting on trees with finite vertex stabilizers

The goal of this section is to prove Theorem I.5. To this end, we combine the result about standard generating sets (Theorem I.4) with the general results from Chapter II.

Theorem I.5. *Let G be a virtually free group, that is, it acts on a simplicial tree T with finite vertex stabilizers. Then its Cayley graph with respect to any finite generating set has a tree-like structure with finite pieces. Given a finite generating set of G, the pieces of the tree-like structure are algorithmically constructed.*

Proof. The group G in this case is a fundamental group of a finite graph of finite groups. Recall that the standard generating set, as defined in Section II.2.3, consists of generators of the vertex groups and free letters corresponding to the edges outside the spanning tree of the factor graph T/G. Assume S_1 is maximal generating set such that it contains all elements of the vertex groups and free letters. In Section II.2.3 we constructed a tree-like structure for the Cayley graph, with pieces corresponding to vertex groups. This tree of pieces P_i is a starting point of our generalization to an arbitrary generating set.

Denote by m the number of vertex groups in T/G. Elements of an arbitrary (finite) generating set S can be represented by reduced words in S_1 of bounded length (depending on S). Denote the maximal length by N.

The next step is to apply Lemma II.6 several times, so that the piece P'_i contains all its descendants up to n-th generation. Every application of that lemma enlarge the pieces by one generation. We will find n large enough to guarantee that there exists a piece containing both endpoints of each edge labeled by an element in S.

Let P'_i be a piece in the n-times "enlarged" tree-like structure. Then the vertex set $V(P'_i) = \bigcup_{k\in\Lambda_i(n)} V(P_k)$, where $\Lambda_i(0) = \{i\}$ and $\Lambda_i(n+1) = \{i\} \cup \bigcup_{s\in\Lambda_i(n)} \Lambda_s$. Since the original pieces were finite, Λ_j's are finite and so are the modified pieces P'_i. An intuitive picture of how such enlarged pieces look like, comes from the free groups, see Corollary V.3.

Recall that the pieces P_i correspond to transitions of vertex groups gG_v, and they are connected in the tree of pieces if they are so-called neighbors (corresponding to edge in spanning tree) or delayed neighbors (corresponding to other edges in the graph of groups), see Section II.2.3. Now we can use a graph distance between pieces. Let $s \in S_1$ and x be a vertex in some piece $g_1G_{v_1}$. How far is a piece containing the vertex xs?

Assume $s \in G_{v_2}$, so it is in some vertex stabilizer. There is a path between v_1 and v_2 in the spanning tree of the graph T/G, visiting vertices $v_1 = u_1, u_2, \dots, u_k = v_2$. The size of the spanning tree is equal to the the number of vertex groups minus 1, that is $m-1$. For every i there is $g_i \in G$ such that $x \in g_iG_{u_i}$. Therefore there is a path in the tree of pieces through vertices $g_1G_{u_1}, g_2G_{u_2}, \dots, g_kG_{u_k}$. Moreover, $xs \in g_kG_{v_2}$. Thus the distance between pieces containing x and xs is at most $m-1$.

Assume s is a free letter. Then it corresponds to an edge e in the graph of groups outside the spanning tree, starting at vertex v_2 and terminating at vertex v_3. As before there is a path from $g_1G_{v_1}$ to a piece $g_kG_{v_2}$, and $xs \in g_{k+1}G_{v_3}$ a delayed neighbor of $g_kG_{v_2}$. Thus the distance between pieces containing x and xs is at most m, in this case. Similarly for s^{-1}.

Let $x \in G$ with word length (in S_1) at most N. The distance between pieces containing g and gx is at most mN, for any $g \in G$. If the distance between two pieces is at most mN, then their closest common ancestor differs from them by less than mN generations. In particular, if $\operatorname{dist}(P_i, P_j) \leq mN$, then there is P_k, such that P'_k contains vertices of P_i and P_j provided $n > mN$.

Now we can apply Lemma II.5 to conclude that the Cayley graph with respect to $S \cup S_1$ has a tree-like structure with finite pieces. Consequently by Lemma II.4 the Cayley graph with respect to S has also a tree-like structure with finite pieces. □

II.3 Transitive graphs with more than one end

In what follows we will generalize the above decomposition into the tree-like structure, which was natural for amalgamated products and HNN extensions, to the case of transitive graphs with more than one end.

Recall that the number of ends of a graph is the supremum of the number of connected components of any of its subgraphs that was obtained by removing a finite set of vertices. An infinite transitive graph can have one, two or infinitely many ends.

First let us recall the notation and a result obtained by Dunwoody in [15].

Let $\mathcal{G}$ be an infinite connected graph with more than one end. Let c be a subset of vertices of $\mathcal{G}$, denote by $c^* = V(\mathcal{G}) \setminus c$ and by $\partial_E c$ the set of edges having one endpoint in c and the other in c^*. Denote by $\partial_V c$ the set of vertices in c having a neighbor in c^*. A set of vertices c such that $\partial_E c$ is finite is called a cut. A cut is said to be non-trivial if both c and c^* are infinite.

Lemma II.10 (Dunwoody). *Let $\mathcal{G}$ be a graph with more than one end and let $H \subset Aut(\mathcal{G})$. There exists a non-trivial cut d such that for any $g \in H$ one of the inclusions $d \subset gd$, $d \subset gd^*$, $d^* \subset gd$, $d^* \subset gd^*$ holds.*

Let d be a cut satisfying the above lemma and for any other cut b we denote

$$\begin{aligned}
T_b &= \{gd | g \in Aut(\mathcal{G}), gd \subsetneq b, \text{ and there is no } h \in Aut(\mathcal{G}) \\
&\quad \text{such that } gd \subsetneq hd \subsetneq b \text{ or } gd \subsetneq hd^* \subsetneq b\}, \\
T_b^* &= \{gd^* | g \in Aut(\mathcal{G}), gd^* \subsetneq b, \text{ and there is no } h \in Aut(\mathcal{G}) \\
&\quad \text{such that } gd^* \subsetneq hd \subsetneq b \text{ or } gd^* \subsetneq hd^* \subsetneq b\}, \\
Q_1 &= d \cap \bigcap_{c \in T_d \cup T_d^*} (c^* \cup \partial_V c), \\
Q_2 &= d^* \cap \bigcap_{c \in T_{d^*} \cup T_{d^*}^*} (c^* \cup \partial_V c). \qquad \text{(II.1)}
\end{aligned}$$

If there is $g \in Aut(\mathcal{G})$ such that $gd = d^*$ the above definition gives us $T_d = T_d^*$. In order not to consider each cut twice, we set $T_d^* = \emptyset$ in this case.

Now we will decompose the graph into pieces isomorphic to Q_1 and Q_2.

Lemma II.11. *Let $c_1, c_2 \in T_d \cup T_d^*$ and $c_1 \neq c_2$. Then the distance from $c_1 \setminus Q_1$ to $c_2 \setminus Q_1$ in $\mathcal{G}$ is at least 2. Moreover, $d = Q_1 \dot\cup \dot{\bigcup}_{c \in T_d \cup T_d^*} (c \setminus \partial_V c)$. Similarly for d^*.*

Proof. From the property that there is no $h \in Aut(\mathcal{G})$ such that $c_i \subsetneq hd \subsetneq d$ or $c_i \subsetneq hd^* \subsetneq d$ for $i = 1, 2$ it follows that $c_1 \not\subset c_2$ and $c_1^* \not\subset c_2^*$. Since $d^* \subset c_i^*$, we have $c_1^* \not\subset c_2$. Therefore by Dunwoody's result in Lemma II.10 $c_1 \subset c_2^*$ and $c_2 \subset c_1^*$. If $\partial_V c_i \subset Q_1$ the claim would follow.

Assume $c \in T_d \cup T_d^*$ such that $\partial_V c \not\subset Q_1$. It means that there is $c' \in T_d \cup T_d^*$ such that $(c'^* \cup \partial_V C') \cap d$ does not contain $\partial_V c$, and therefore $(c' \setminus \partial_V c') \cap \partial_V c \neq \emptyset$. But from above we have that $c' \subset c^*$, what is a contradiction. □

Theorem II.12. *Every transitive graph with more than one end admits a tree-like structure with finite border sets.*

Proof. The pieces we consider are Q_1, Q_2 and their translates. Assume d is a cut satisfying the above Dunwoody's Lemma II.10 and that the origin is in $\partial_V d^*$. Let the root piece P_0 be a subgraph of $\mathcal{G}$ induced on the vertices $Q_2 \cup \partial_V d^*$.

Now with each translate of d (d^* resp.) we have a translate of Q_1 (Q_2 resp.). Consider the cuts in $T_{d^*} \cup T^*_{d^*}$ and the translates of Q_1 and Q_2 corresponding to them. The pieces of the first generation are subgraphs induced by the vertices of these translates of Q_1 and Q_2 (we include all the edges that are not already in P_0). Let the first generation also contain a piece induced by edges in the original Q_1.

The whole tree of pieces is constructed inductively. We always consider translates of Q_1 and Q_2 corresponding to the cuts in $T_b \cup T^*_b$, where b are the cuts used in the previous generation. The piece is a subgraph induced on the vertices of the appropriate translate of Q_1 and Q_2, and we exclude all edges which are already in some piece of previous generation. The border set B_i is defined as the intersection of P_i with its parent ($B_0 := \{o\}$).

Now we can start verifying the properties of Definition II.1.

(1) The pieces and border sets such that B_i is a subset of vertex set of P_i.

(2) Every edge is in exactly one piece - in the first appearing in the construction and containing both endpoints of the edge.

(3) The partial order of the elements follows from the construction, as well as the property that border set is the intersection of the piece with its parent.

(4) If we remove a border set B_i the graph falls apart. In particular $U(P_i)$ is the subgraph induced by vertices of c' and by Lemma II.11 it intersects the $U(P_i)^c$ only by vertices in $\partial_V c$, which are in B_i.

(5) There are two isomorphism classes of pieces (excluding the root), based on whether P_i arises from transition Q_1 or Q_2.

□

CHAPTER III

PERCOLATION ON TREE-LIKE GRAPHS

III.1 Partitions of the border sets

Consider a graph $\mathcal{G}$ with a tree-like structure as above and take a realization $\omega \in \Omega$ of a percolation process on $\mathcal{G}$. Some pairs of vertices of the border set B_0 can be connected by open paths in $\mathcal{G}$. In this way the realization determines a partition of the set B_0: two vertices are in the same class of the partition if they are connected by an open path in $\mathcal{G}$. The percolation process induces a probability measure on the set of all partitions of B_0. Similarly we obtain a partition of each B_j by looking at the open paths in the subgraph $U(P_j)$. We call this a *descendant partition* of the border set B_j. We say a partition of a set B is *induced by percolation* on H, if H is a subgraph of $\mathcal{G}$, B is a subset of vertices of H, and two vertices of B are in the same class of the partition if and only if they are connected by an open path in H.

In this section, we will use the decomposition into pieces to find the measure on the set of descendant partitions of border sets using recurrent relations.

Let $Z^{(i)}$ be the set of all partitions of the border set B_i that can be induced by the percolation on $U(P-i)$. Let $q_i : \Omega \to Z^{(i)}$ be a map assigning to each realization $\omega \in \Omega$ the partition on the border set B_i. We consider only partitions induced by percolation on $U(P-i)$ and thus the map q_i is surjective. The measure on $Z^{(i)}$ is a pullback of the percolation measure P_p by q_i. Thus $A \subset Z^{(i)}$ is measurable if and only if $q_i^{-1}(A) \in \Sigma$. For simplicity we use the same notation P_p for the probability measure on partitions and denote the σ-algebra of measurable sets of partitions by $\mathcal{Z}^{(i)}$.

For every finite subset F of B_i and every partition $z^{(i)}$ of B_i, denote by $z^{(i)}(F)$ the set of all partitions that coincide with $z^{(i)}$ on F. These sets are obviously in $\mathcal{Z}^{(i)}$.

If the subgraphs $U(P_i)$ and $U(P_j)$ are isometric (by an isomorphism from part (5) of the definition), then the σ-algebras of partitions $Z^{(i)}$ and $Z^{(j)}$ are isomorphic and the measures induced by the same percolation process are preserved.

Lemma III.1. *A descendant partition of B_i is determined by the state of the edges in P_i and by the descendant partitions of border sets $B_\lambda, \lambda \in \Lambda_i$ (which are children of P_i).*

Proof. Indeed using property (2) of Definition II.1 and Lemma II.2 we see that every edge of an open path connecting two vertices of B_i in $U(P_i)$ is in exactly one of the following graphs: P_i or $U(P_\lambda), \lambda \in \Lambda_i$. We can split the path into several segments, each of them being in some $U(P_\lambda)$ or P_i. A segment in $U(P_\lambda)$ starts at some point of B_λ and ends at some other point of B_λ. Therefore the existence of the open segment is determined by the partition on B_λ. Thus in order to decide whether there is an open path connecting certain vertices in B_i it is sufficient to know the states of edges in P_i and the partitions of $B_\lambda, \lambda \in \Lambda_i$. $\square$

For every measurable set $A \in \mathcal{Z}^{(i)}$ and every vector $\zeta \in \prod_{\lambda\in\Lambda_i} Z^{(\lambda)}$ of partitions on B_λ's, $\lambda \in \Lambda_i$, we denote the conditional probability of A provided ζ by $\mathrm{P}_p(A|\zeta)$. Denote by $\mu_p^{(i)}$ the product measure on $\prod_{\lambda\in\Lambda_i} Z^{(\lambda)}$ such that the measure on each $Z^{(\lambda)}$ is given by P_p.

Then we can write:

$$\mathrm{P}_p(A) = \int_{\zeta\in\prod_{\lambda\in\Lambda_i} Z^{(\lambda)}} \mathrm{P}_p(A|\zeta) d\mu_p^{(i)}(\zeta). \tag{III.1}$$

Note that since for pieces of the same isomorphism class we have isomorphic σ-algebras of the set of partitions and measures on them, we can consider only one such equation for each model border set (of index in the finite set J from the definition).

This defines an operator on the space of all measures on the direct product of $Z^{(i)}$'s. More precisely, let X_j be the space of the probability measures on $Z^{(j)}$ with the σ-algebra $\mathcal{Z}^{(j)}$. In fact $X_j \subset [0,1]^{\mathcal{Z}^{(j)}}$. Denote $\prod_{j\in J} X_j$ by $\mathbf{X}$.

For $x = (x_1, \dots, x_{|J|}) \in \mathbf{X}$ denote by $\mu_x^{(j)}$ the product measure on the space $\prod_{\lambda\in\Lambda_j} Z^{(\lambda)}$ such that the measure on each $Z^{(\lambda)}$ is given by $x_{\gamma(\lambda)} \in X_{\gamma(\lambda)}$.
Define an operator $\Psi_p : \mathbf{X} \to \mathbf{X}$ (where $\Psi_p(x) = (\Psi_p(x)_1, \dots, \Psi_p(x)_{|J|})$) by

$$\Psi_p(x)_j(A) = \int_{\zeta\in\prod_{\lambda\in\Lambda_j} Z^{(\lambda)}} \mathrm{P}_p(A|\zeta) d\mu_x^{(j)}(\zeta), \text{ for all } A \in \mathcal{Z}^{(j)}. \tag{III.2}$$

For any $p \in (0, 1)$ the measure on $Z^{(j)}, j \in J$ induced by the percolation on $U(P_j)$ is a fixed point of this operator Ψ_p, indeed compare (III.2) with (III.1).

If the pieces are finite, then the equations (III.1) form a finite system of polynomial equations in the unknown variables $P_p(z^{(j)})$. The probabilities of partitions are important for evaluating the first moment matrix of the branching process defined in Section III.2.5.

We can endow each $Z^{(j)}$ with a topology generated by the cylindrical sets $z^{(j)}(F)$. This topological space is second countable and Hausdorff. Moreover, the space is compact since each sequence has an accumulation point. (Indeed let $F_i, i = 1, \dots$ be an increasing sequence of finite subsets of B_j, and $\bigcup F_i = B_j$. In order to find an accumulation point of a sequence x_k, we can consider a set of partitions that agree on F_i with infinitely many x_k's, and let i go to infinity.)

The space $\mathbf{X}$ of the probability measures on a compact metric space is a convex compact metric space in the weak* topology induced by continuous functions (follows from the Riesz representation theorem). Then a continuous operator on $\mathbf{X}$ has a fixed point. The set of its fixed points is closed and since the operator Ψ_p is linear, the set is also convex. The operator Ψ_p acts linearly on the space of measures and has norm at most one because it preserves the subset of probability measures. Therefore the set of its fixed points is convex, compact.

III.2 Branching processes

III.2.1 Preliminaries

Recall that a multi-type branching process is a Markov process that models a population in which each individual in generation n produces some random number of offspring of the various types in generation $n+1$, according to a fixed probability distribution that depends only on the type of the individual.

Assume S is a set of types of individuals and that a type s individual produces children of different types according to a probability distribution p_s on $\mathbb{N}_0^S$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Assume that all individuals produce offspring independently of each other and of the history of the process. Let vector $X_n \in \mathbb{N}_0^S$ represent the n-th generation, where each coordinate $X_n[s]$ represents the number of individuals of type s in the n-th generation. It is given by the recurrent relation

$$X_{n+1} = \sum_{s \in S} \sum_{i=1}^{X_n[s]} \xi_{ns}^i, \tag{III.3}$$

where ξ_{ns}^i are independent random variables with the distribution p_s. The sequence $\{X_n\}_0^\infty$ is called a *multi-type Galton-Watson branching process* with initial population size $X_0 \in \mathbb{N}_0^S$ and offspring distribution $p_s, s \in S$.

Let $e_s \in \mathbb{N}_0^S$ be a vector with 1 at the position $s \in S$ and zeros at other positions. Let $M_n(e_s, B)$ be the expected number of individuals of the n-th generation of types in $B \subset S$. For any fixed initial e_s, $M_n(e_s, .)$ is a measure on S given by

$$M_n(e_s, B) = E\left(\sum_{s \in B} X_n[s] | X_0 = e_s\right). \tag{III.4}$$

The initial population can be given by any vector a in $\mathbb{N}_0^S$ (usually it is some e_s, an atomic measure). We will give some properties of the M_n's using a notation from the book of Nummelin [33]. Assume the expected size of the first generation is bounded, we can condition on the states in the first generation and obtain:

$$M_n(a, B) = \int_{s \in S} M_{n-1}(s, B) dM_1(a, s). \tag{III.5}$$

Each $M_n : (S, P(S)) \to \mathbb{R}_+$ is a kernel acting as an operator $\tilde{M}_n$ on the space of measures on S by

$$\tilde{M}_n(\mu)(B) = \int_{s \in S} M_n(s, B) d\mu(s).$$

The product of two kernels is defined as in expression (III.5), thus $\tilde{M}_n$ is the n-th iterate kernel of $\tilde{M}_1$.

A branching process is called *singular* if each individual has exactly one offspring almost surely. M is *irreducible* if and only if there exist a σ-finite measure ϕ on S such that for all ϕ-positive sets $B \subset S$ and $s \in S$, $M_n(e_s, B) > 0$ for some n.

If a non-singular branching process is irreducible, then the population becomes extinct or explodes exponentially. The irreducibility condition is very important here. The branching process we will construct can be reducible in general. This prevents us from making claims about the extinction of the process in general. If we restrict ourself to the finite set of types, we can make further claims, see Section III.2.5.

If S is finite, then the operators M_n are matrices. The expected size of the first generation (the first moment matrix M) is given by

$$m_{rs} = \mathrm{E}(X_1[s] | X_0 = e_r). \tag{III.6}$$

Two types r and s are said to be in the same class if an individual of type r is in the offspring of an individual of type s with positive probability and vice versa (i.e. for some n, the (r, s) entry of M^n, $m_{rs}^{(n)}$, is positive). Now the multi-type branching process is irreducible if all types are in the same class.

The process is *positively regular* if there exists n such that all elements of M^n are strictly positive. If the process is irreducible but non-positively regular, then it is *periodic*. The period of a branching process is a number d such that the matrix M may be represented, after reordering the types of individuals, in the form

$$M = \begin{bmatrix} 0 & M(1,2) & 0 & \dots & 0 \\ 0 & 0 & M(2,3) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M(d-1,d) \\ M(d,1) & 0 & 0 & \dots & 0 \end{bmatrix},$$

where $M(i, i + 1)$ denotes a non-zero matrix.

We will need the following result from the theory of branching processes.

Lemma III.2. *If an irreducible multi-type Galton-Watson process with finite number of types is non-singular, then the population becomes extinct with probability one, whenever the maximal eigenvalue of the first moment matrix M is at most one.*

For the proof see Mode [32], Theorem 7.1 on page 16 and Theorem 2.1 on page 54. Note that by The Perron-Frobenius theorem the maximal (in absolute value) eigenvalue of M will always be a non-negative real number.

III.2.2 Coloring of the tree of pieces according to the percolation

Next we will relate the percolation cluster size to the population of a multi-type branching process. The number of types will be as small as possible in order to simplify the computation at the cost that the population size will not match, only approximate the cluster size, in particular it will be finite if and only if the percolation cluster was also finite.

Assume that there is a distinct vertex o called the origin in the border set B_0. Each realization of a percolation gives rise to a coloring of the tree of pieces in the following way.

Consider a piece P_i that is not the root. Consider a subgraph $U(P_i)^c$ of $\mathcal{G}$ induced by the edges $E(\mathcal{G}) \setminus E(U(P_i))$. Note that the intersection of $U(P_i)^c$ and $U(P_i)$ is the border set B_i, and it is a cut set by Lemma II.3. The subgraph $U(P_i)^c$ contains the origin. Vertices of the border set B_i may be connected by open paths in $U(P_i)^c$. This gives a new partition on B_i and one class of the partition may be connected to the origin. We assigned a color to a piece based on the data consisting of the partition and the distinct origin-connected class (which may be empty). Say that a piece is white if the class connected to the origin is empty. All other possible partitions and origin-connected classes give rise to a new color. Denote all possible colors $y^{(i)}$ by $Y^{(i)}$, that is all colors that appear for some realization of percolation $\omega \in \Omega$. Observe that in this model the colors of pieces depend not only on the parent but on the colors of its siblings and the whole subtree of their descendants. This does not give us a branching process directly, but this last difficulty is to be overcome.

Consider the maps $q_i' : \Omega \to Y^{(i)}$ assigning to each realization a color of the piece P_i. We can pullback the percolation probability measure P_p to $Y^{(i)}$. This determines a σ-algebra $\mathcal{Y}^{(i)}$ on $Y^{(i)}$. We can identify colors of the pieces P_i and $P_{\gamma(i)}$ (γ is the model map from part (5) of Definition II.1) because the spaces $Y^{(i)}$ and $Y^{(\gamma(i))}$ are isomorphic.

Lemma III.3. *For every $i \in I$ and $\upsilon \in \Lambda_i$, $U(P_\upsilon)^c$ is covered by P_i, $U(P_i)^c$ and the collection of $U(P_\lambda)$ for $\upsilon \neq \lambda \in \Lambda_i$. So the color of P_υ is determined by the state of the edges in P_i, the color of P_i and descendant partitions of border sets B_λ for $\upsilon \neq \lambda \in \Lambda_i$.*

Proof. The argument is identical to the one used in Lemma II.2, and Lemma III.1. □

Note that only a finite number of children will be non-white if $p < \min\{p_c(P_i)\}$, and all children of a white piece will be white as well.

III.2.3 The complete branching process

In the previous sections we assigned to each piece P_i (or its border set B_i) a descending partition and a color based on the realization of the percolation process, that is a pair $(z^{(i)}, y^{(i)}) \in \bigcup_{j \in J}(Z^{(j)} \times Y^{(j)})$. Denote $\bigcup_{j \in J}(Z^{(j)} \times Y^{(j)})$ by Θ.

Assume a piece P_j with border set B_j has a descendant partition $z^{(j)}$ and a color $y^{(j)}$. The descendant partitions and the colors of the pieces $P_\lambda, \lambda \in \Lambda_j$ (the offspring pieces of P_j) are random variables taking values in $Z^{(\lambda)} \times Y^{(\lambda)}$. Denote by (ζ, η) the vector of these random variables, with values in $\prod_{\lambda\in\Lambda_j} Z^{(\lambda)} \times \prod_{\lambda\in\Lambda_j} Y^{(\lambda)}$. Let Q be a random variable representing all the colors and descending partitions of pieces in $U(P_j)^c$.

Lemma III.4. *We claim that the distribution of the descendant partitions and colors of the offspring pieces depends only on the parent, in particular*

$$\mathrm{P}_p((\zeta,\eta) \in A|(z^{(j)}, y^{(j)})\&Q) = \mathrm{P}_p((\zeta,\eta) \in A|(z^{(j)}, y^{(j)})), \tag{III.7}$$

for any measurable set $A \subset \prod_{\lambda\in\Lambda_j} Z^{(\lambda)} \times \prod_{\lambda\in\Lambda_j} Y^{(\lambda)}$.

Proof. From Lemma III.3 and Lemma III.1 we have that (ζ, η) is independent of Q for a given $(z^{(j)}, y^{(j)})$. Therefore the claim follows.

In other words, the descendant partition and the color of the parent encodes every connectedness relation coming from $U(P_j)^c$. So given the descendant partition and the color of the piece P_i, the descendant partitions and the colors of pieces in $U(P_j)^c$ and of pieces in $U(P_j)$ are independent.

□

The purpose of this claim becomes clear after the following definition.

Let us define a measure $\mathcal{D}(z^{(i)}, y^{(i)})$ on $\prod_{\lambda\in\Lambda_i} Z^{(\lambda)} \times \prod_{\lambda\in\Lambda_i} Y^{(\lambda)}$ according to the probabilities in Lemma III.4. We identify the space $Z^i \times Y_i$ with $Z^{\gamma(i)} \times Y^{\gamma(i)}$ and we count the number of repetitions of each $(z^{(j)}, y^{(j)}), j \in J$ among the offspring. In this way we define a map $\prod_{\lambda\in\Lambda_j} Z^{(\lambda)} \times \prod_{\lambda\in\Lambda_i} Y^{(\lambda)} \to \mathbb{N}_0^{\Theta}$. For any $i \in J$, we can pull back the distribution $\mathcal{D}(z^{(i)}, y^{(i)})$ to a distribution $\mathcal{D}'(z^{(i)}, y^{(i)})$ on $\mathbb{N}_0^{\Theta}$.

Definition III.5. *The* complete *multi-type Galton–Watson branching process induced by the percolation with parameter p on a graph with a tree-like structure is given by the following conditions.*

The type of an individual is given by the color and the descendant partition. The set of types is $\Theta = \bigcup_{j\in J}(Z^{(j)} \times Y^{(j)})$.

There is one initial individual of type (z_0, y_0), *where* z_0 *is the diagonal descendant partition of* B_0 *(i.e. all classes are of size* 1*) and* y_0 *is the color with the diagonal partition of* B_0 *with only the origin in the distinct origin-connected class.*

Every individual of some type $(z^{(j)}, y^{(j)})$ *gives birth to* $|\Lambda_j|$ *individuals, with the distribution of types* $\mathcal{D}'(z^{(j)}, y^{(j)})$.

We can represent this branching process by a tree. Clearly it will match the tree of pieces of the graph.

Lemma III.6. *The coloring of the tree of pieces according to the percolation has the same distribution as the above defined complete branching process.*

Proof. Let us represent the coloring of the tree of pieces by a random process (X_n), where $X_n \in \mathbb{N}_0^{\Theta}$. The colors and the descendant partitions are represented by elements of Θ and each coordinate of X_n gives a number of pieces with a specific color and a specific descending partition in the n-th generation of the tree of pieces .

By Lemma III.4 the distribution of the offspring of an individual in the n-th generation is independent of the other individuals (in generation at most n) and depends only on its type $(z^{(j)}, y^{(j)})$. The distribution is $\mathcal{D}'(z^{(j)}, y^{(j)})$.

Therefore the process X_n coincides with the complete branching process above. □

III.2.4 The branching process with a reduced number of types

Next we will reduce the number of types and obtain a different branching process. We will also impose an independency condition on the offspring of every individual, that is, the joint distribution of the offspring $\mathcal{D}(y^{(j)})$ will be a product measure. Nevertheless we will show that the expected population size of this reduced branching process contains enough information about p_c.

Using Lemma III.3 we can make the following observation about the coloring of the tree of pieces. We look at the piece P_i and evaluate the conditional probability of a specific piece P_v having color $y^{(v)}$ (resp. $y^{(\gamma(v))}$) assuming the descendant partitions of the other border sets are given by the vector $\zeta \in \prod_{\lambda \in \Lambda_i} Z^{(\lambda)}$ (note that the descendant partition on P_v has no influence). Taking an expected value of these conditional probabilities over all possible descendant partitions gives us the probability $\mathrm{P}_p(y^{(v)}|y^{(i)})$ of having a child piece of a given color $y^{(v)}$ from a piece of color $y^{(i)}$. These probabilities are the same for the piece P_i as for its model piece $P_{\gamma(i)}$, so we can write for $j \in J$

$$\mathrm{P}_p(y^{(\gamma(v))} \in B|y^{(j)}) = \int_{\zeta \in \prod_{\Lambda_j} Z^{(\lambda)}} \mathrm{P}_p(y^{(\gamma(v))} \in B|y^{(j)} \& \zeta) d\mu^{(j)}(\zeta), \qquad \text{(III.8)}$$

where $v \in \Lambda_j$ and $B \in \mathcal{Y}^{(\gamma(v))}$, a measurable set of colors. These conditional probabilities may differ for different children v_1 and $v_2 \in \Lambda_j$ even if $\gamma(v_1) = \gamma(v_2)$ (that is if the set of possible colors coincide).

Denote by $Y'^{(j)}$ the subset of $Y^{(j)}$ such that $y^{(j)} \in Y'^{(j)}$ if and only if it is not white. Let $\mathbf{Y} = \bigcup_{j \in J} Y'^{(j)}$. Then $\mathbf{Y}$ is the set of all non-white colors. Let the σ-algebra $\mathcal{Y}$ on $\mathbf{Y}$ be generated by the intersections of sets in $\mathcal{Y}^{(j)}$ with $Y'^{(j)}$.

Definition III.7. *The* reduced *multi-type Galton-Watson branching process induced by the percolation with parameter p on a graph with a tree-like structure is given by the following conditions.*

The types of individuals are colors in $\mathbf{Y}$.

There is one initial individual with color $y_0^{(0)}$ that is the diagonal partition of B_0 with only the origin in the distinct origin-connected class.

Every individual of some color $y^{(j)}$ gives birth to possibly $|\Lambda_j|$ individuals, each of which is born and has its color assigned independently of the others. The distribution of the color

$y^{(\gamma(\lambda))}$ of a child indexed by $\lambda \in \Lambda_j$ follows the law in formula (III.8).

Note that white is no longer a legitimate color. A child indexed by $\lambda \in \Lambda_j$ is not born in the reduced branching process with the same probability as the probability, that the corresponding piece is white in the percolation, that is $\mathrm{P}_p(y^{(\gamma(v))}\, is\, white | y^{(j)})$. This is not the only difference between the reduced branching process and the coloring of the tree of pieces according to the percolation. The joint distribution of the offspring of an individual is different (because of the independence), and only the first moment is the same. But the first moment is all we need.

Remark III.8. *In the definition of the tree-like structure we allowed some isomorphism classes of pieces to be finite. In particular very often the root is not isometric to any other piece. In such case, it is useful to start the branching process not at the root (with one element), but with a generation that already consists only of pieces that are in infinite isomorphism classes. This gives us the initial measure on the colors. Therefore we consider only a subset J' of J such that $j \in J'$ if and only if $|\gamma^{-1}(j)| = \infty$. Then $\mathbf{Y} = \bigcup_{j \in J'} Y'^{(j)}$.*

Theorem I.1 (i). *For a percolation with parameter p the reduced branching process on the tree of pieces has the property that the expected size of its population is finite if and only if the expected size of the percolation cluster at the origin is finite.*

Proof. Note that, by the result of Aizenman and Barsky [1], the sub-critical phase on transitive graphs is equivalently characterized by the finiteness of the expected cluster size (they proved this for $\mathbb{Z}^d$, the generalization to all transitive graphs was pointed out by Lyons and Peres [31]).

Clearly $p_c(\mathcal{G}) \leq p_c(P_i)$. If there is an infinite cluster at the origin for p, such that $p < p_c(P_i)$, then the cluster has to intersect infinitely many border sets almost surely. Therefore the number of border sets connected to the origin is finite if and only if the original cluster was also finite. In what follows we will always assume $p \leq p_c(P_i)$ (if P_i is finite, then we set $p_c(P_i) = 1$).

Given a realization of percolation we have introduced a coloring of the tree of pieces. The percolation cluster at the origin is infinite if and only if the non-white colored component of the tree of pieces is infinite (by the assumption that p is smaller than p_c of the pieces). The expected size of the colored component is a sum of the probabilities that a piece is non-white over all pieces of the tree.

The branching process from Definition III.7 can be naturally illustrated by a tree isomorphic to the tree of pieces. The distribution of colors of a specific individual in this branching process equals to the distribution of non-white colors of the related piece in the percolation. Therefore the expected population size of this branching process equals the expected number of non-white colored pieces in the percolation.

Therefore $p < p_c$ if and only if the related branching process has finite expected population size. □

The branching process is not singular because for $p < 1$ (and some non-white color y) there is no color, which appears as an only child of y almost surely (i.e. with positive probability there are less or more children with different colors). Assume the initial measure is $\nu : \mathcal{Y} \to \mathbb{R}_+$ (it can be the atomic measure from Definition III.7 or distribution of the first generation of pieces with infinite isomorphism classes from Remark III.8). Then the expected population size of the branching process is $\sum_{n=0}^{\infty} \tilde{M}_n \nu(\mathbf{Y})$. The critical probability p_c is then a supremum of all p such that $\sum_{n=0}^{\infty} \tilde{M}_n \nu(\mathbf{Y}) < \infty$. In general, the convergence of $\sum_{n=0}^{\infty} \tilde{M}_n \nu(\mathbf{Y})$ may depend strongly on the initial measure ν. But we will show that if the border sets are finite, then it actually depends only on the spectral radius of M_1.

III.2.5 The case of finite border sets

If the border sets are finite, then the space of partitions is finite and so is the space of colors $\mathbf{Y}$. Denote by $M = [m_{ab}]_{a,b \in \mathbf{Y}}$ the first moment matrix of the branching process, that is the matrix of expected number of offspring of each color $m_{ab} = \mathrm{E}_p(\#b|a)$, where

$$\begin{aligned} \mathrm{P}_p(y^{(\gamma(v))}|y^{(j)}) &= \int_{\zeta \in \prod_{\Lambda_j} Z^{(\lambda)}} \mathrm{P}_p(y^{(\gamma(v))}|y^{(j)} \& \zeta) d\mu^{(j)}(\zeta) \\ \mathrm{E}_p(\# y^{(k)}|y^{(j)}) &= \sum_{\lambda \in \Lambda_j, \gamma(\lambda) = k} \mathrm{P}_p(y^{(\gamma(\lambda))}|y^{(j)}). \end{aligned} \tag{III.9}$$

The expected number of individuals of the n-th generation is then given by the n-th power of M.

Theorem I.1 (ii). *If all the border sets are finite, then the branching process has finitely many types, and the first moment matrix is of finite size. In this case p_c is the smallest value of p such that the spectral radius of the first moment matrix is* 1*.*

Proof. Assume the graph $\mathcal{G}$ has a tree-like structure such that the border sets B_i's are finite. Using the already proved part (i) of Theorem I.1 we need to decide for which p the expected population size of the constructed branching process is finite.

This branching process is non-singular for $p < 1$ because if a piece has more than one child, then the offspring size is bigger than one with positive probability, and if every piece has exactly one child, then it has no offspring (only white) with positive probability. The expected population size is $\sum M^n$ applied to the initial measure. If the spectral radius is less than 1, then the sum $\sum M^n$ is always finite. If the spectral radius is at least 1, then there exists a possible initial measure, for which the expected population size is infinite (follows from The Perron-Frobenius theorem). If the process is irreducible, then it is independent of the choice of initial measure.

In case M is irreducible the result follows directly from Lemma III.2. In the other case there are several classes of types and we denote by $M_1, \ldots, M_k$ the first moment matrices of each class. We can reorder the types in $\mathbf{Y}$ such that M has diagonal blocks equal to M_i's and all entries above these diagonal blocks are zero. Thus the spectral radius of M is the spectral radius of some M_s. There is a type (color from $\mathbf{Y}$) in the s-th class such that with positive probability there is a piece P_i with this color (because we considered only those colors that are realized by percolation). The process starting at this P_i will have infinite cluster size whenever the spectral radius of M_s is at least one. Therefore the expected size of the percolation cluster is finite whenever the maximal eigenvalue of M is less than one.

Note that $det(M-1)=0$ if 1 is an eigenvalue of M and for $p=0$ all eigenvalues of M are zero. The eigenvalues depend continuously on the matrix entries, which are continuous functions of p. Therefore p_c is the first positive value of p such that $det(M-1)=0$. □

Theorem I.1 (iii). *If all the pieces are finite, then the entries of the first moment matrix of the reduced branching process are algebraic functions in p. Therefore p_c is algebraic.*

There exists an algorithm that, given the model pieces and their border sets, computes a finite extension K of the field $\mathbb{Q}(p)$ and a function f in K such that p_c is the smallest positive root of f.

Proof. If the pieces are finite, then the probabilities of the descending partitions, $P_p(z^{(j)})$, can be found as solutions of the system of equations (III.1) introduced in the previous section. Let us index the partitions on B_j by natural numbers $1,\ldots,|Z^{(j)}|$. Denote by $x_{j,i}$ the probability of the i-th possible partition on the border set of model piece P_j, that is $x_{j,i}=P_p(z_i^{(j)})$. Then we have the following system of equations in the unknown variables $x_{j,i}$.

$$x_{j,i}=\sum_k\left(\sum_{\Gamma\in L}\alpha(\Gamma)\right)\prod_{\lambda\in\Lambda_j}x_{\gamma(\lambda),k(\lambda)}, \tag{III.10}$$

where

- k is a map that assigns partitions to the border sets B_λ's. That is $k:\Lambda_j\to\mathbb{N}$, and for all $\lambda\in\Lambda_j$, $k(\lambda)\in\{1,2,\ldots,|Z^{(\lambda)}|\}$. We sum over all such possible functions k.

- For a given k, L is the set of subgraphs of P_j such that $\Gamma\in L$ if and only if the following holds: provided the partitions on the children pieces are $z_{k(\lambda)}^{(\lambda)}$ and the open edges of P_j are given by Γ, the partition on B_j (induced by $U(P_j)$) equals $z_i^{(j)}$. Note that by Lemma III.1 the partition on B_j is uniquely determined by the provided informations.

- $\alpha(\Gamma)$ is the probability of the subgraph Γ in the percolation, that is $\alpha(\Gamma)=p^{E(\Gamma)}(1-p)^{E(P_i)-E(\Gamma)}$.

The number of equations is equal to the number of variables. The degree in p of each equation (III.10) is equal to the number of edges in a piece and the degree in the unknown variables $x_{j,i}$'s is $|\Lambda_j|$, the number of offspring of the piece P_j. Denote by K the algebraic extension of $\mathbb{Q}(p)$ containing roots of this system of equations.

Similarly as above we can rewrite the formulas (III.9) as follows

$$P(y^{(\gamma(v))}|y^{(j)}) = \sum_k \left(\sum_{\Gamma \in L'} \alpha(\Gamma) \right) \prod_{\lambda \in \Lambda_j} x_{\gamma(\lambda),k(\lambda)}, \tag{III.11}$$

where

- k and $\alpha(\Gamma)$ are defined as above, and
- L' depends on k and it is the set of subgraphs of P_j such that $\Gamma \in L'$ if and only if the color on B_v is $y^{(\gamma(v))}$ provided the color of the parent is $y^{(j)}$, the descendant partitions on the children pieces are $z_{k(\lambda)}^{(\lambda)}$ and the open edges of P_j are given by Γ (by Lemma III.3 these uniquely determine the color on B_v).

Consequently the entries of the first moment matrix M, given by formula (III.9), are polynomial functions in p and $x_{j,i}$'s. The critical probability p_c is the first positive value of p such that $det(M-1) = 0$. The function $det(M-1)$ is an element of the field K, and p_c is an algebraic number in this case. □

CHAPTER IV

ISING MODEL ON TREE-LIKE GRAPHS

IV.1 Preliminaries

Consider a finite subgraph W of $\mathcal{G}$ and a boundary condition $\delta \in \{-1,1\}^{\partial W}$, where ∂W is the outside vertex boundary of W. The *finite volume Gibbs state on W at inverse temperature β and with boundary condition δ* is a probability measure $\nu^{\delta}_{W,\beta}$ on $\{-1,1\}^{W}$ given by

$$\nu^{\delta}_{W,\beta}(\omega) = \frac{1}{Z^{\delta}_{W,\beta}} \exp\left(\beta \sum_{u,v\in W: u\sim v} \omega(u)\omega(v)\right), \tag{IV.1}$$

where $\omega \in \{-1,1\}^{W}$ and $Z^{\delta}_{W,\beta}$ is a normalizing constant.

Let W_n be an increasing sequence of finite subgraphs of the graph $\mathcal{G}$ such that $\bigcup W_n = G$. Denote by δ_n a boundary condition on the outside vertex boundary of W_n. Any weak subsequential limit of $\nu^{\delta_n}_{W_n,\beta}$ as $n \to \infty$ is a Gibbs state.

The following discussion is based on Liggett [28] and Jonasson and Steif [21]. Consider a sequence of boundary conditions $\delta_n \equiv +1$ and $\delta_n \equiv -1$. By the monotonicity of the Ising model, for $J, \beta > 0$ and for any other boundary condition δ_n, $\nu^{-}_{W_n,\beta} \le \nu^{\delta_n}_{W_n,\beta} \le \nu^{+}_{W_n,\beta}$. The limits $\nu^{+}_{\beta} = \lim_{n\to\infty} \nu^{+}_{W_n,\beta}$ and $\nu^{-}_{\beta} = \lim_{n\to\infty} \nu^{-}_{W_n,\beta}$ exist and are independent of the choice of W_n. Therefore the phase transition occurs (i.e. the Gibbs state is not unique) if and only if $\nu^{+}_{\beta} \neq \nu^{-}_{\beta}$. Note that ν^{+}_{J} and ν^{-}_{J} are invariant under all graph automorphisms of $\mathcal{G}$.

IV.2 Recurrent relation

In this section we will prove Theorem I.6.

It might be useful to keep in mind a simple example. Consider a rooted binary tree and its subgraph W_n containing vertices up to level n. Denote by x_n the probability that there is a positive spin at the origin in the measure $\nu^{\delta_n}_{W_n,\beta}$ for some boundary condition δ_n. Assume we know x_{n-1}, we can write:

$$x_n = (e^{2\beta} x_{n-1}^2 + 2x_{n-1}(1-x_{n-1}) + e^{-2\beta}(1-x_{n-1})^2)/d,$$

where d is a normalizing constant, in this case:

$$d = (e^{\beta} x_{n-1} + e^{-\beta}(1-x_{n-1}))^2 + (e^{-\beta} x_{n-1} + e^{\beta}(1-x_{n-1}))^2.$$

Any limiting invariant Gibbs state has to satisfy this recurrent equation (with $x_n = x_{n-1}$). It remains to decide for which values of β it has more than one solution. Figure IV.1 shows the extreme solutions for different values β. Using the Jacobian of the equation we can show that the critical temperature for binary tree is $\frac{1}{2}ln(3) = .5493\ldots$.

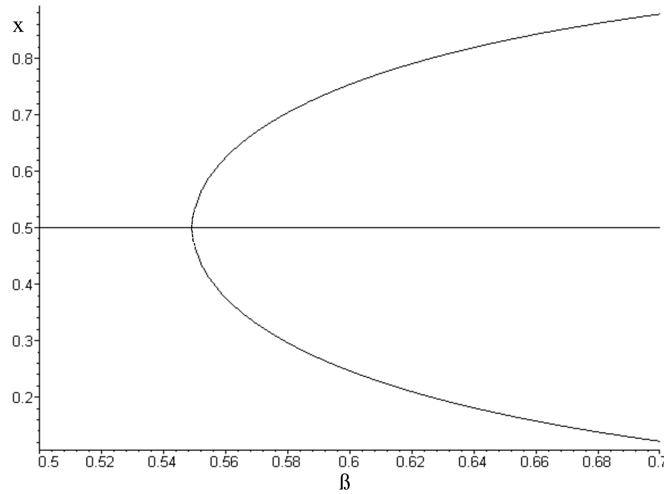

Figure IV.1: Values of the solution x in dependency on β.

Let us present the argument in greater generality. Recall the statement of the result which covers virtually free groups:

Theorem I.6. *Consider a graph $\mathcal{G}$, which has a tree-like structure with finite pieces. There is a system of polynomial equations that has more than one solution if and only if the phase transition occurs. Assuming there is only one Gibbs state at the critical inverse temperature for the Ising model, the value of* $\exp(\beta_c)$*, the exponent of the critical inverse temperature, is an algebraic number, which can be found as a solution of a system of polynomial equations. These equations depend on the chosen tree-like structure.*

Proof. Given a piece P with a border set B and a subtree $U(P)$, denote by $\nu^{\delta}_{B,\beta}$ a restriction of the measure $\lim_{n\to\infty} \nu^{\delta_n}_{U(P)\cap W_n,\beta}$ to $\{-1,1\}^B$ for $\delta = \pm 1$. Given two pieces P_i, P_j in the same isomorphism class we have that $\nu^{\delta}_{B_i,\beta} = \nu^{\delta}_{B_j,\beta}$.

The measure $\nu^{\delta}_{B_i,\beta}$ is determined by the structure of the piece P_i and measures $\nu^{\delta}_{B_\lambda,\beta}$'s for $\lambda \in \Lambda_i$, the children of the piece P_i. This leads to a finite system of equations, which are polynomial (in $\exp(\beta)$) for finite pieces (see Chapter V for examples). The phase transition occurs, i.e, there is more than one Gibbs state, if and only if $\nu^{+}_{B_o,\beta} \neq \nu^{-}_{B_o,\beta}$. Since every positive solution of the system of equations leads to a Gibbs state, the phase transition occurs if and only if the system of equation has more than one solution.

For an $\omega \in \{-1,1\}^B$ denote by $-\omega$ an element with all spins inverted. If $\nu^{\delta}_{B_\lambda,\beta}$ is a solution of the above equations then $\nu^{\delta}_{B_\lambda,\beta}(-\omega)$ is also a solution (corresponding to an inverted boundary). In particular for any β there is a symmetric solution, such that $\nu^{o}_{B_\lambda,\beta}(\omega) = \nu^{o}_{B_\lambda,\beta}(-\omega)$.

The following discussion is an application of the Implicit function theorem and properties of the Jacobian. If there is only one Gibbs state at the critical inverse temperature for the Ising model, then for the inverse temperature lower or equal to β_c the equations has only one solution. Therefore these equations implicitly define a function $[0, \beta_c] \to \nu^{\delta}_{*,\beta}$. This solution is symmetric. Since the equations are polynomial, all the solutions are continuous in β and as soon as there is more than one solution, by the symmetry, there are at least three of them. As β approaches β_c from above, the solutions approach the symmetric one. Therefore at β_c the Jacobian of the system of equations is 0. Conversely, if the Jacobian is zero at β, then there is no neighborhood of the point β, such that the equation would implicitly define a function. In other words, in any neighborhood of β, there is a value of the inverse temperature for which there is more than one solution.

Thus the critical inverse temperature β_c is the minimal value of β such that the Jacobian is equal to 0 at the point of symmetric solution of the system of equations. $\square$

CHAPTER V

EXAMPLES

All examples illustrates the tree-like structure. Most detailed analysis of the percolation process can be found in the section about $SL(2,\mathbb{Z})$. The computation for Ising model is carried out in details for the grandparent tree. The free products are considered for both, percolation and Ising model.

V.1 Free products of (transitive) graphs

The free product $\mathcal{G}$ of transitive graphs $G_1,\dots,G_n$ is an infinite connected graph constructed as a union of copies of G_i's such that each vertex of $\mathcal{G}$ belongs to exactly one copy of each G_i, and every simple closed path in $\mathcal{G}$ is included in one copy of some G_i. In particular, if $G_1=\langle S_1\rangle$ and $G_2=\langle S_2\rangle$, a Cayley graph of $G_1 * G_2$ with respect to $S_1\cup S_2$ is a free product of Cayley graphs of G_1 and G_2 with respect to S_1 and S_2 respectively.

First we show the result from percolation.

Corollary I.2. *Let $\mathcal{G}$ be a free product of transitive graphs $\mathcal{G}_1,\dots,\mathcal{G}_n$. Denote by $\chi_i(p)$ the expected sub-critical cluster size in the i-th factor graph G_i. The critical probability p_c of $\mathcal{G}$ is the infimum of positive solutions of*

$$\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}\chi_i(p)-(n-1)\prod_{i=1}^{n}\chi_i(p)=0.$$

Proof. The tree-like structure is very natural in this case. The copies of factor graphs $\mathcal{G}_1,\dots,\mathcal{G}_n$ are pieces (thus $J=\{1,\dots,n\}$) and each vertex is in fact a border set.

The partitions of border sets are trivial (the spaces of partitions have size 1) in this case. For the coloring we need only two colors $Y^{(i)}=\{y_1^{(i)},y_2^{(i)}\}$. Assuming that $y_2^{(i)}$ is white, then $\mathbf{Y}=\{y_1^{(1)},\dots,y_1^{(n)}\}$.

Denote by b_i the border vertex of piece P_i and by $\tau_p^{(i)}(a\leftrightarrow b)$ the probability that a and b are connected in $\mathcal{G}_i$. Denote by $\chi_i(p)$ the expected cluster size in $\mathcal{G}_i$. Now we are ready to compute the matrix $M=(m_{ij})$.

$$\begin{aligned}
\mathrm{P}_p(y_1^{(\gamma(\lambda))}|y_1^{(j)})&=\tau_p^{(j)}(b_j\leftrightarrow b_\lambda)\\
\mathrm{E}_p(\#y_1^{(k)}|y_1^{(j)})&=\chi_j(p)-1\\
m_{ij}&=\chi_i(p)-1\text{ , for } i\neq j\\
m_{ii}&=0
\end{aligned}\tag{V.1}$$

In order to find for which p the spectral radius of M equals 1 we solve the equation $\det(M-1)=0$ by Theorem I.1 (iii). The determinant is computed as follows (we subtract the first column from all the others and then expand the determinant along the first row):

$$\det(M-1) = \det\begin{bmatrix} -1 & \chi_1(p)-1 & \dots & \chi_1(p)-1 \\ \chi_2(p)-1 & -1 & \dots & \chi_2(p)-1 \\ \vdots & \vdots & \ddots & \vdots \\ \chi_n(p)-1 & \chi_n(p)-1 & \dots & -1 \end{bmatrix}$$

$$\begin{aligned} \det(M-1) &= \det\begin{bmatrix} -1 & \chi_1(p) & \chi_1(p) & \dots & \chi_1(p) \\ \chi_2(p)-1 & -\chi_2(p) & 0 & \dots & 0 \\ \chi_3(p)-1 & 0 & -\chi_3(p) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_n(p)-1 & 0 & 0 & \dots & -\chi_n(p) \end{bmatrix} \\ &= -\prod_{i=2}^{n}(-\chi_i(p)) + \sum_{j=2}^{n}(\chi_j(p)-1)\prod_{i=1,i\neq j}^{n}(-\chi_i(p)) \\ &= (-1)^n\left(\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}\chi_i(p) - (n-1)\prod_{i=1}^{n}\chi_i(p)\right) \qquad \text{(V.2)} \end{aligned}$$

Therefore p_c is the infimum of positive solutions of

$$\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}\chi_i(p) - (n-1)\prod_{i=1}^{n}\chi_i(p) = 0.$$

□

The projective special linear group $PSL(2,\mathbb{Z})$ is an example of a free product, namely it is $\mathbb{Z}_2 * \mathbb{Z}_3$. To find its p_c we need to evaluate the expected cluster size in a cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$. As obtained by the author in [27] the cluster size in cyclic group of size m, denoted by $|C|_{\mathbb{Z}_m}$ satisfies

$$\mathrm{E}_p(|C|_{\mathbb{Z}_m}) = \frac{1+p}{1-p} - \frac{p^m(m+1)-p^{m+1}(m-1)}{1-p}. \qquad \text{(V.3)}$$

Therefore the critical probability on $PSL(2,\mathbb{Z})$ is equal to $.5199\dots$.

Now we shift our attention to the Ising model. Recall that the expected magnetization $M = M(\beta)$ of a graph is the expected sum of spins on the graph provided the origin has a positive spin, at an inverse temperature β.

Corollary I.7. *Let $\mathcal{G}$ be a free product of (transitive) finite graphs. Denote by $M_i(\beta)$ the expected magnetization in the i-th factor graph. The critical inverse temperature β_c of $\mathcal{G}$ is* *the infimum of positive solutions of*

$$(N-1)\prod_i \mathrm{E}(M_i) - \sum_j \prod_{i\neq j} \mathrm{E}(M_i) = 0.$$

In case of the N-regular tree this leads to the critical inverse temperature $\beta_c = \coth^{-1}(N-1)$.

Proof. Assume the graph $\mathcal{G}$ is a free product of N transitive graphs and consider the tree-like structure as above. Denote by x_i (resp. y_i) the probability that border set (root vertex) in the i-th piece has a positive (resp. negative) spin. Denote by M_i the magnetization of the i-th piece assuming the root has positive spin.

Then the equations takes following form:

$$x_i = \sum_{j=1}^{n_i} \mathrm{P}(M_i = 2j - n_i) \prod_{k\neq i} x_k^{j-1} y_k^{n_i-j}/d,$$
$$y_i = \sum_{j=1}^{n_i} \mathrm{P}(M_i = 2j - n_i) \prod_{k\neq i} y_k^{j-1} x_k^{n_i-j}/d,$$

where d is a sum of the right sites. The number of solution does not change if we set $d = 1$. Then $x_i = y_i = 1$ is always a solution. The Jacobian of the system of equations is determinant of the following matrix

$$\begin{bmatrix} -1 & 0 & b_1-1 & n_1-b_1 & \dots & b_1-1 & n_1-b_1 \\ 0 & -1 & n_1-b_1 & b_1-1 & \dots & n_1-b_1 & b_1-1 \\ b_2-1 & n_2-b_2 & -1 & 0 & \dots & b_2-1 & n_2-b_2 \\ n_2-b_2 & b_2-1 & 0 & -1 & \dots & n_2-b_2 & b_2-1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_N-1 & n_N-b_N & b_N-1 & n_N-b_N & \dots & -1 & 0 \\ n_N-b_N & b_N-1 & n_N-b_N & b_N-1 & \dots & 0 & -1 \end{bmatrix},$$

where $b_i = (\mathrm{E}(M_i) + n_i)/2$, the expected number of positive spins. Using few row and column transformations we get that the Jacobian equals

$$\det \begin{bmatrix} -1 & 0 & n_1-1 & 0 & \dots & n_1-1 & \\ 0 & -1 & 0 & E(M_1)-1 & \dots & 0 & E(M_1)-1 \\ n_2-1 & 0 & -1 & 0 & \dots & n_2-1 & 0 \\ 0 & E(M_2)-1 & 0 & -1 & \dots & 0 & E(M_2)-1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n_N-1 & 0 & n_N-1 & 0 & \dots & -1 & 0 \\ 0 & E(M_N)-1 & 0 & E(M_N)-1 & \dots & 0 & -1, \end{bmatrix}.$$

Now we can move odd rows up and odd columns left to obtain a block diagonal matrix. So the Jacobian is equal to

$$\left((N-1)\prod_i n_i - \sum_j \prod_{i\neq j} n_i\right)\left((N-1)\prod_i E(M_i) - \sum_j \prod_{i\neq j} E(M_i)\right). \tag{V.4}$$

Therefore the critical inverse temperature is a solution of the following equation involving the expected magnetization of the pieces:

$$(N-1)\prod_i \mathrm{E}(M_i) - \sum_j \prod_{i\neq j} \mathrm{E}(M_i) = 0.$$

Note that in case $G * G$ it simplifies to $\mathrm{E}(M) = 2$.

Consider a regular tree of degree N, we have

$$\begin{aligned}
\mathrm{P}(M = 2j - N) &= \binom{N-1}{j-1}\gamma^{j-1}\gamma^{-(N-j)}/d, \\
d &= \sum_{j=1}^{N}\binom{N-1}{j-1}\gamma^{j-1}\gamma^{-(N-j)} = (\gamma+\gamma^{-1})^{N-1}, \\
\mathrm{E}\left(\frac{M+N}{2}\right) &= \frac{\sum_{j=1}^{N}\binom{N-1}{j-1} j\gamma^{j-1}\gamma^{-(N-j)}}{(\gamma+\gamma^{-1})^{N-1}} = \frac{n\gamma+\gamma^{-1}}{\gamma+\gamma^{-1}},
\end{aligned} \tag{V.5}$$

where $\gamma = \exp(\beta)$. The critical inverse temperature β_c is then obtained as a solution of the following equation:

$$\begin{aligned}
\frac{n\gamma+\gamma^{-1}}{\gamma+\gamma^{-1}} &= \frac{N}{2} + 1, \\
\gamma^2 &= \frac{N}{N-2}.
\end{aligned}$$

Therefore $\beta_c = \frac{1}{2}\ln\left(\frac{N}{N-2}\right) = \coth^{-1}(N-1)$ as proved before by Lyons [29] (in fact his result covers more general trees). □

Consider again the projective special linear group $PSL(2,\mathbb{Z})$. The expected magnetization $\mathrm{E}(M_n)$ of a cyclic group of size n is

$$\mathrm{E}(M_n) = \frac{ne^{n\beta} + \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor}\sum_{k=s}^{n-s}(2k-1)\binom{k}{s}\binom{n-k-1}{s-1}e^{(n-4s)\beta}}{\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2s}e^{(n-4s)\beta}}. \tag{V.6}$$

Thus the critical inverse temperature $\beta_c = \frac{1}{4}\ln(4 + sqrt(17)) = 1.688\ldots$.

The easiest example of a Cayley graph with a tree-like structure that is not a free product, is $SL(2,\mathbb{Z})$. It is the amalgamated product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ with standard generating set $\{a, b\}$ so that $\mathbb{Z}_4 = \langle a \rangle$ and $\mathbb{Z}_6 = \langle b \rangle$. We can illustrate the general method here because the required computations (for obtaining p_c) are relatively simple, or at least doable.

Corollary I.3. *The critical probability p_c of the special linear group $SL(2,\mathbb{Z})$ given by presentation $\langle a, b | a^4, b^6, a^2 b^{-3} \rangle$ is an algebraic number equal to* $.4291140496\ldots$

Proof. The Cayley graph has the tree-like structure with pieces of two isomorphism classes (squares P_1 and hexagons P_2) corresponding to the factor groups, see Figure V.1. The hexagonal pieces consist of six-tuples of vertices connected by bold lines in Figure V.1 and squares contain dashed edges. Each vertex is contained in exactly one square and one hexagon and each neighboring square and hexagon share two vertices - the border sets.

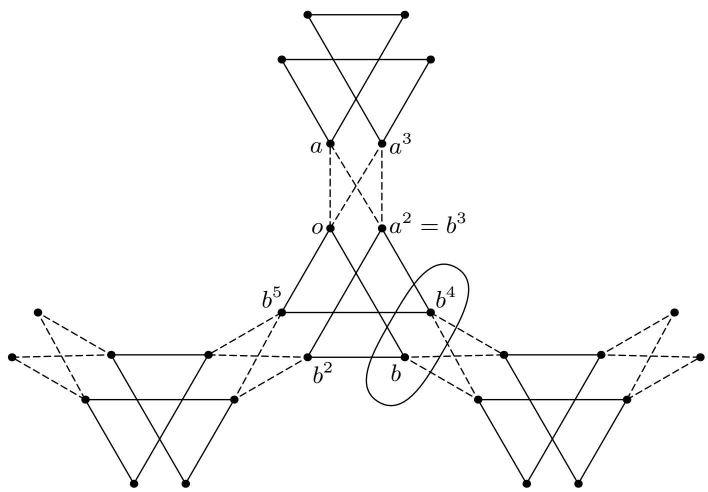

Figure V.1: The Cayley graph of $SL(2,\mathbb{Z})$ with one border set circled

First we need to find the distribution of the descendant partitions of the border sets. There are two possible descendant partitions on each border set, B_1 and B_2. Denote them by $Z^{(1)} = \{z_1^{(1)}, z_2^{(1)}\}$ and $Z^{(2)} = \{z_1^{(2)}, z_2^{(2)}\}$, where $z_1^{(i)}$ means that the two vertices of the border set B_i are connected and $z_2^{(i)}$ means that they are not connected. Let us use a simplified notation: $S = \mathrm{P}_p(z_1^{(1)})$ and $H = \mathrm{P}_p(z_1^{(2)})$ (note that then $\mathrm{P}_p(z_2^{(1)}) = 1 - S$ and $\mathrm{P}_p(z_2^{(2)}) = 1 - H$). Now we will find the system of equations for $x_{1,1} = S$ and $x_{2,1} = H$ given by formula (III.10). In particular, we can write

$$S = \left(\sum_{\Gamma \in L_1} \alpha(\Gamma) \right) H + \left(\sum_{\Gamma \in L_2} \alpha(\Gamma) \right) (1 - H),$$

where L_i is the set of subgraphs of the square, such that $\Gamma \in L_i$ if and only if the vertices of the border set of the square are connected by an open path under the condition that the partition on the children is $z_i^{(2)}$ and the open edges of the square are given by Γ. In fact, $\sum_{\Gamma\in L_1}\alpha(\Gamma)$ is the probability that the vertices of the border set of the square are connected under the condition that the vertices of the border set of its child (hexagon) are connected. It means that at least one edge emerging from each vertex of the border set has to be open, which happens with probability $(2p-p^2)^2$. If the children border set has descendant partition $z_2^{(2)}$ (not connected), then the probability of connected parent is equal to $2p^2-p^4$. Therefore we get the following expression of S.

$$
\begin{aligned}
S &= (2p-p^2)^2 H + (2p^2-p^4)(1-H) && \text{(V.7)}\\
H &= ((2p-p^2)^3 + (1-p)^2(2p^2-p^4))S^2 + 2(p+p^2-p^3)^2 S(1-S) && \text{(V.8)}\\
&\quad + (2p^3-p^6)(1-S)^2
\end{aligned}
$$

The second equation for H is obtained in a similar way. These formulas lead to a quadratic equation and if we choose a root in [0,1] we obtain

$$
S = \frac{4p^4(2p^2+2p+1)(1-p)^4 - 1 + \sqrt{D}}{4p^5(5p-2)(1-p)^4}, \tag{V.9}
$$

where

$$
\begin{aligned}
D = 1 - 8p^4(p-1)^4(2p^{12} - 8p^{11} + 18p^{10} - 32p^9 + 38p^8\\
- 28p^7 + 17p^6 - 2p^5 - 12p^4 + 4p^3 + 2p^2 + 2p + 1).
\end{aligned}
$$

This give us the measure on the set of partitions of the border sets.

In Section III.2 we introduced the coloring of a tree. For a border set of size two there are three non-white colors: either both vertices are connected to the origin or exactly one of them is connected to the origin. The two different situations when exactly one vertex of the border set is connected to the origin are symmetric, i.e. there is an obvious isomorphism of $U(P_i)$ to itself such that each vertex of B_i is mapped onto the other one. Therefore these two colors have the same distribution of offspring. This simplification is special for the Cayley graph of $SL(2,\mathbb{Z})$, because this new symmetry is not one of the isomorphisms in condition (5) of the definition of the tree-like structure. Since the distribution of offspring does not depend on which of the two vertices is connected to the origin, we identify these two colors as one.

The set of colors becomes $Y^{(i)} = \{y_r^{(i)}, y_b^{(i)}, y_w^{(i)}\}$ where r (red) means that both vertices of the border set are connected to the origin, b (blue) means that exactly one vertex is connected to the origin (thus the border set is disconnected) and w stands for white - no vertex connected to the origin. We evaluate the probabilities of getting a child of a given color, according to formula (III.11). Again thanks to the symmetry of the hexagon, the probabilities for both children (P_{λ_0}, P_{λ_1}) are the same.

$$
\begin{aligned}
\mathrm{P}_p(y_r^{(\lambda)}|y_r^{(1)}) &= (1-(1-p)^2)^2 \\
\mathrm{P}_p(y_b^{(\lambda)}|y_r^{(1)}) &= 2(1-(1-p)^2)(1-p)^2 \\
\mathrm{P}_p(y_w^{(\lambda)}|y_r^{(1)}) &= (1-p)^4 \\
\mathrm{P}_p(y_b^{(\lambda)}|y_b^{(1)}) &= 2p(1-p)(1-p^2) \\
\mathrm{P}_p(y_r^{(\lambda)}|y_b^{(1)}) &= p^4 + 4p^3(1-p) + p^2(1-p)^2 \\
\mathrm{P}_p(y_w^{(\lambda)}|y_b^{(1)}) &= (1-p)^2 \\
\mathrm{P}_p(y_r^{(\lambda)}|y_r^{(2)}) &= (1-(1-p^2)(1-p))^2(1-S) + ((2p-p^2)^3 \\
&\quad + (1-p)^2(p(1-(1-p)(1-p^2)) + (1-p)p^3))S \\
\mathrm{P}_p(y_b^{(\lambda)}|y_r^{(2)}) &= 2(1-(1-p)(1-p^2))(1-p)(1-p^2)(1-S) \\
&\quad + 2((1-p)^2p(1-p)(1-p^2) + (2p-p^2)^2(1-p)^2)S \\
\mathrm{P}_p(y_w^{(\lambda)}|y_r^{(2)}) &= (1-p)^2(1-p^2)^2(1-S) \\
&\quad + (1-p)^4(1+2p-p^2)S \\
\mathrm{P}_p(y_r^{(\lambda)}|y_b^{(2)}) &= (p^3 + p^4(1-p^2) + p^5(1-p))(1-S) \\
&\quad + (p^3 + 3p^2(1-p))(1-(1-p)(1-p^2))S \\
\mathrm{P}_p(y_b^{(\lambda)}|y_b^{(2)}) &= (1-p^3)(p(1-p^2) + p^2(1-p))(1-S) \\
&\quad + p(3p+1)(1+p-p^2)(1-p)^2S \\
\mathrm{P}_p(y_w^{(\lambda)}|y_b^{(2)}) &= (1-p)(1-p^2)(1-S) \\
&\quad + (1-p)(1-p(1-(1-p)^2(1-p^2)))S
\end{aligned}
$$

Let us illustrate the derivation of the above expressions on $\mathrm{P}_p(y_r^{(\lambda)}|y_r^{(2)})$. We will work with the hexagon. Its boundary consists of three border sets. One is the parent and one is the new child in question B_λ. Now $\mathrm{P}_p(y_r^{(\lambda)}|y_r^{(2)})$ is the probability that this child is red if the parent is red. The third border set of this hexagon has two possible descendant partitions, the vertices of that border set are either connected outside the piece with probability S or not connected with probability $1-S$.

Let us first assume they are not connected. Each vertex in the child border set can be connected to the parent either by a single edge or by a pair of edges and these two events are independent. Therefore the probability that a given vertex of the child border set is not connected to the parents is $(1-p)(1-p^2)$. Thus the probability that both vertices of the child border set are connected to the parent is $(1-(1-p^2)(1-p))^2$.

Assume now that the vertices in the third border set are connected in the descendant partition. This happens with the probability S. Then the hexagon can be contracted as in Figure V.2, i.e. we contract the border sets with connected vertices into one vertex. There

are several ways how to have both vertices in the border set B_λ connected to the parent:
1. If the dashed edges are open, then the probability that both vertices of the child border set are connected to the parent in this situation is $(2p-p^2)^3$.
2. If the dashed edges are closed and the dash-dotted edge is open, then the probability of this situation is $(1-p)^2p(1-(1-p)(1-p^2))$.
3. If all dashed and dash-dotted edges are closed, then all the remaining edges must be open in order to connect both vertices of the child border set to the parent. The probability of this event is $(1-p)^3p^3$.

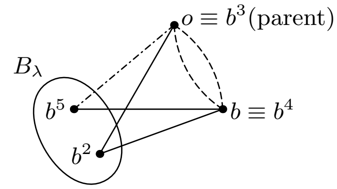

Figure V.2: Contracted hexagon

Now $\mathbf{Y} = \{y_r^{(1)}, y_b^{(1)}, y_r^{(2)}, y_b^{(2)}\}$ and the first moment matrix M takes the following form.

$$M = \begin{bmatrix} 0 & 0 & \mathrm{P}_p(y_r^{(\lambda)}|y_r^{(1)}) & \mathrm{P}_p(y_b^{(\lambda)}|y_r^{(1)}) \\ 0 & 0 & \mathrm{P}_p(y_r^{(\lambda)}|y_b^{(1)}) & \mathrm{P}_p(y_b^{(\lambda)}|y_b^{(1)}) \\ 2\mathrm{P}_p(y_r^{(\lambda)}|y_r^{(2)}) & 2\mathrm{P}_p(y_b^{(\lambda)}|y_r^{(2)}) & 0 & 0 \\ 2\mathrm{P}_p(y_r^{(\lambda)}|y_b^{(2)}) & 2\mathrm{P}_p(y_b^{(\lambda)}|y_b^{(2)}) & 0 & 0 \end{bmatrix}. \tag{V.10}$$

If we solve the equation $\det(M-1)=0$ we obtain $p_c = .4291140496\ldots$ □

V.3 Grandparent tree

An interesting example of a transitive graph with infinitely many ends that is not a graph is the grandparent tree. It is obtained from the three-regular tree. We pick one end and add an extra edges connecting every vertex with the vertex at graph distance 2 in the direction of the chosen end. The obtained graph is called grandparent tree. This graph has infinitely many ends and it is not unimodular. Nevertheless it has a tree-like structure.

In the suggested decomposition bellow, we obtain pieces of size four with border sets of size two. Note that in the Figure V.3 the graph is oriented so that the distinguished end is to the top. In the right picture we see that the border set of a piece consist of the two upper vertices and each of its two children intersects it by a pair of vertices - the middle one and the left (resp. right) one.

Note that the pieces that lie on the path from the root to the distinguished end have different orientation - the border set is a different pair of vertices inside the piece (exercise to the reader).

A careful analysis of the branching process can be carried out as in the case of $SL(2,\mathbb{Z})$.

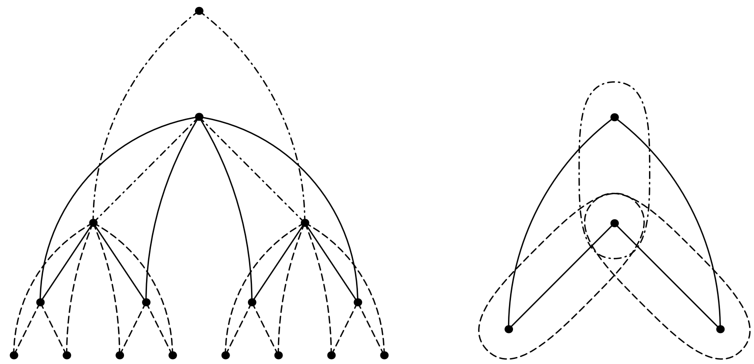

Figure V.3: Part of the grandparent tree and its piece with cycled border sets.

The p_c obtained is $.158656326\ldots$

Let us treat the Ising model in more details.

Denote the probability of each element in $\{-1,1\}^B$ by a new variable (the number of variables needed is $2^2 = 4$.) Let x (resp. u) be the probability that both vertices of the border set have positive (resp. negative) spins. Let y (resp. t) be the probability that the upper (resp. lower) vertex has a positive spin while the other has a negative one. Then we can write

$$\begin{aligned} x &= (\gamma^2 x + \gamma^{-2} y)^2/d, \\ y &= (t+u)^2/d, \\ t &= (x+y)^2/d, \\ u &= (\gamma^2 u + \gamma^{-2} t)^2/d, \\ d &= (\gamma^2 x + \gamma^{-2} y)^2 + (t+u)^2 + (x+y)^2 + (\gamma^2 u + \gamma^{-2} t)^2, \end{aligned}$$

where $\gamma = \exp(\beta)$.

The measures $\nu^{\pm}_{B_\lambda,\beta}$ have to satisfy the above system of equations. We can "rescale" the equations by setting $d = 1$. This does not change the number of solutions. For any value of γ this system of equations has exactly one solution such that $x = u$ and $y = t$.

The Jacobian of the rescaled system of equations is determinant of the following matrix

$$\begin{bmatrix} 2\gamma^2(\gamma^2 x + \gamma^{-2} y) - 1 & 2\gamma^{-2}(\gamma^2 x + \gamma^{-2} y) & 0 & 0 \\ 0 & -1 & 2(t+u) & 2(t+u) \\ 2(x+y) & 2(x+y) & -1 & 0 \\ 0 & 0 & 2\gamma^{-2}(\gamma^2 u + \gamma^{-2} t) & 2\gamma^2(\gamma^2 u + \gamma^{-2} t) - 1 \end{bmatrix}.$$

Setting the determinant equal zero we obtain that the critical inverse temperature β_c is equal to $\ln(\frac{1}{3}(72\sqrt{6} - 27)^{1/4})$.

V.4 An example with infinite pieces

Consider an amalgamated product $G_1 *_{\mathbb{Z}_2} G_2$ with $G_1 = \mathbb{Z}_2 \times \mathbb{Z}$ and $G_2 = \mathbb{Z}_4$. Its Cayley graph with respect to natural generators can be decomposed similarly as above, with border sets of size two. Therefore the set of partition $Z^{(i)}$ has two elements, and the set of colors $\mathbf{Y}$ has four elements as for the Cayley graph of $SL(2, Z)$. The first moment matrix M has again two anti-diagonal blocks.

Let A (B resp.) be the probability that the two vertices of the border set in G_1 (G_2 resp.) are connected (in the descendant subtree). Then A and B satisfy:

$$\begin{aligned}
A &= 2p - p^2 + (1-p)^2(2C - C^2),\\
B &= (2p - p^2)^2 A + (2p^2 - p^4)(1 - A),\\
C &= p^2(1 - (1-p)^2(1-B) + (1-p)^2(1-B)C),\\
C &= \frac{p^2(1 - (1-p)^2(1-B))}{1 - p^2(1-p)^2(1-B)}.
\end{aligned}$$

In order to express A, we split the piece $\mathbb{Z}_2 \times \mathbb{Z}$ into three parts, the first one containing only two edges between vertices of B_i. Then we split the rest at the B_i and obtain two other (identical) parts (corresponding to $\mathbb{Z}_2 \times \mathbb{N}$, see for example the last block in Figure V.4). The probability that vertices of B_i are connected in one of these two latter parts is equal to C .

These formulas lead to a cubic equation, and exactly one of the solutions is real and in $[0, 1]$. Now we are ready to express the entries of first moment matrix M. There are two anti-diagonal blocks M_{12} and M_{21}, again the one corresponding to $\mathbb{Z}_4$ is already known from previous example. For the other factor group $\mathbb{Z}_2 \times \mathbb{Z}$ we denote by T_n the transition matrix from the parent to the n-th child (order them by distance from the origin). In order to obtain the transition matrix, we split the graph of $\mathbb{Z}_2 \times \mathbb{Z}$ into several blocks, see Figure V.4. For each block we express the transition probabilities and thus T_n is composition of these probabilities, in particular it is a product of matrices corresponding to each block.

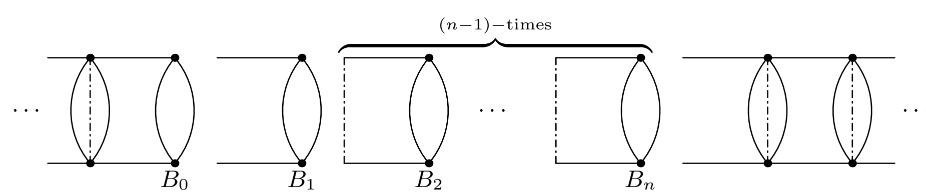

Figure V.4: Blocks of the Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}$

The first block consists of vertices to the left from the parent and including the parent. The probability of the connection in the left part is C and the probability that the two vertices in B_0 are connected is $2p - p^2$. This gives the first matrix in formula (V.11). The second block is used to express matrix T i.e. the transition to the first child B_1 from the parent B_0. The third one is repeated $(n-1)$-times in order to reach the n-th child. It differs from T by the factor of B coming from the descendant partition on the sibling border set (this connection is drawn by the dash-dotted line). The last factor in the product corresponds to the rightmost part of the picture and equals the probability of connection on the right side of the child in question (it is again equal to C).

$$\begin{aligned} T_n &= \begin{bmatrix} \mathrm{P}_p(r_n|r) & \mathrm{P}_p(b_n|r) \\ \mathrm{P}_p(r_n|b) & \mathrm{P}_p(b_n|b) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1-(1-C)(1-p)^2 & (1-C)(1-p)^2 \end{bmatrix} T \\ &\quad \cdot \left(\begin{bmatrix} 1 & 0 \\ B & 1-B \end{bmatrix} T \right)^{n-1} \begin{bmatrix} 1 & 0 \\ C & (1-C) \end{bmatrix}, \end{aligned} \tag{V.11}$$

$$\begin{aligned} T &= \begin{bmatrix} \mathrm{P}_p(r|r) & \mathrm{P}_p(b|r) \\ \mathrm{P}_p(r|b) & \mathrm{P}_p(b|b) \end{bmatrix} = \begin{bmatrix} p^2+2p(1-p)(2p-p^2) & 2p(1-p)^3 \\ p(2p-p^2) & p(1-p)^2 \end{bmatrix}, \\ M_{21} &= \begin{bmatrix} \mathrm{E}_p(r|r) & \mathrm{E}_p(b|r) \\ \mathrm{E}_p(r|b) & \mathrm{E}_p(b|b) \end{bmatrix} = 2\sum_{n=1}^{\infty} T_n \\ &= 2 \begin{bmatrix} 1 & 0 \\ C & (1-C) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2p-p^2 & (1-p)^2 \end{bmatrix} T \\ &\quad \cdot \left(1 - \begin{bmatrix} 1 & 0 \\ B & 1-B \end{bmatrix} T \right)^{-1} \begin{bmatrix} 1 & 0 \\ C & (1-C) \end{bmatrix} \end{aligned} \tag{V.12}$$

Again we can solve the equation $\det(M-1)=0$ and we obtain $p_c = .2951\ldots$

V.5 Free groups with non-standard generators

The simplest example of a transitive graph with infinitely many ends is a regular tree. The Cayley graph of free group with respect to free generators is a regular tree and the percolation on it is well understood. The result of this section gives us a simple way to find p_c for any finite generating set.

Assume $F_n = \langle x_1, \ldots, x_n \rangle$. Let $\mathcal{G}$ be its Cayley graph with respect to the standard (free) generators. Denote by H the subgroup of $Aut(\mathcal{G})$ generated by left translations by elements in F_n and isomorphisms arising from permutations of the generators.

The following lemma shows that the the cut d satisfying the condition in Dunwoody's Lemma II.10 can be found explicitly.

Lemma V.1. *Consider the free group F_n with any finite set of generators. Denote by d the set of vertices labeled by words starting with letter x_1 (one element of any free generating set). Then d is such that for any $g \in H$ one of the inclusions $d \subset gd$, $d \subset gd^*$, $d^* \subset gd$,*

$d^* \subset gd^*$ holds.

Proof. For any finite generating set, d and d^* are infinite with finite boundaries. Indeed for any m there are only finitely many pairs $(a, b); a \in d, b \in d^*$ with the word distance less than m. Without loss of generality we assume that the generating set is symmetric on the permutation of generators, so the whole group H acts on the Cayley graph by isomorphisms.

Now for any $h \in H$, if the origin 1 is contained in hd, then there is a word x such that all vertices in hd^* are labeled by reduced words starting with x and $x \in d^*$. If the origin is in hd^*, then hd has a similar property. Assume that for a given $h \in H$ such a word is w and the origin is in hd (or in hd^* respectively). Then if w starts with letter x_1, then $d^* \subset hd^*$ (or $d^* \subset hd$ respectively), otherwise $d \subset hd$ (or $d \subset hd^*$ respectively). □

The following theorem is a special case of Theorem I.5. We include here a different, more hands-on proof. We present here the pieces of the tree-like structure of free groups explicitly. They are constructed using a cut set satisfying Dunwoody's Lemma II.10. We could as well use the Lemma II.6 (starting with pieces that are single edges of Cayley graph of free group with respect to free generators), the pieces obtained would be exactly the same.

Theorem V.2. *The Cayley graph of free group F_n (with n free generators) with respect to any finite generating set has a tree-like structure with finite pieces. Therefore its p_c is an algebraic number and one can use the algorithm from Theorem I.1 (iii) to compute p_c, given any finite generating set.*

Proof. Recall that we want to show that the Cayley graph of F_n with respect to any finite set of generators has a tree-like structure with finite pieces.

Any finite set of generators is contained in some ball S_k, i.e. the set of all words of length at most k (in the standard word metric). Therefore, using Lemma II.4, it is enough to find the tree-like structure with finite pieces for the generating set S_k. The isomorphisms will be from the group H, which admits the group F_n as a finite index subgroup.

Consider the cut d from Lemma V.1. We observed that always either hd or hd^* is a set of all vertices labeled by a reduced word starting with some x. We denote such a cut D_x and its complement (containing the origin) by D_x^*. For example for d we write D_{x_1}.

Applying the construction in the proof of Lemma II.12 to the cut d we obtain the following tree-like structure. The pieces are isomorphic to Q_1 and Q_2. Since d and d^* are isomorphic, it is enough to describe just one of them, say Q_2 (general formula (II.1)).

$$T_{d^*} \cup T_{d^*}^* = \{D_x : |x| = 1, x \neq x_1\}$$
$$Q_2 = D_{x_1}^* \cap \bigcap_{|x|=1, x\neq x_1} (D_x^* \cup \partial_V D_x)$$

If a vertex is labeled by a reduced word longer than k, then it is in $D_x \setminus \partial_V D_x$ for some $|x| = 1$, thus not in Q_2. Therefore $Q_2 \subset S_k$, hence it is finite.

Note that since $|T_{d^*} \cup T^*_{d^*}| = 2n - 1$ the branching number of the tree of pieces is $2n - 1$. □

The tree-like structure is explicitly given by the procedure of Lemma II.12. For the Cayley graph of F_2 with respect to S_2 it results in pieces as in Figure V.5. In the picture we see that the whole Q_2 splits into three parts. For the tree-like structure we can take pieces corresponding to these thirds of Q_2. In particular the middle piece P_i (with labeled vertices in the picture) has border set $B_i = \{o, a\}$, and its children pieces (three of them) share with P_i the middle vertex a and one of the remaining three vertices (ab, a^2, ab^{-1}). The value of the critical probability is $.139\ldots$ in this case.

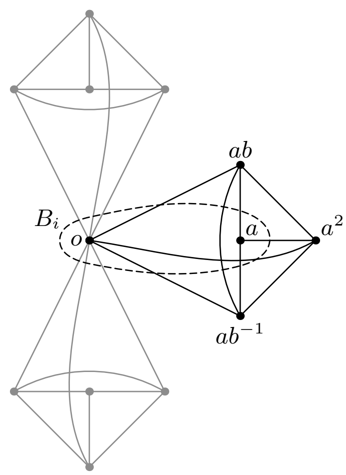

Figure V.5: Piece in Cayley graph of F_2 with respect to S_2

If the generating set is a general ball S_k we can identify the following tree-like structure.

Corollary V.3. *Consider the free group $F_2 = \langle a, b \rangle$ and its Cayley graph with respect to the generating set S containing all words of length at most k (in the standard word metric). There is a tree-like structure with border sets of size $\frac{3^{k-1}+1}{2}$, pieces of size $\frac{3^k+1}{2}$, and the tree of pieces has branching number* 3.

Proof. The root piece P_0 is a subgraph induced by vertices of S_{k-1} with border set containing the origin. The vertices of each of the four pieces of first generation form a subset of S_k that contains the origin and vertices labeled by words starting with a specific letter. The children of the piece corresponding to the letter x are induced by subsets of xS_k containing x and vertices labeled by words starting with xy, where $x \neq y \in \{a, a^{-1}, b, b^{-1}\}$. Clearly each such piece has exactly three children. Since $|S_k| = 4 * (3^k - 1)/(3 - 1) + 1$ we can see $|P_i| = (|S_k| - 1)/4 + 1 = (3^k + 1)/2$ for all $i \neq 0$. Again we included every edge in exactly one piece, that is in the oldest one containing both its endpoints. The border set is defined as the intersection of a piece with its parent. It is not difficult to see that B_i coincides with vertices of a piece in Cayley graph with respect to S_{k-1}, and thus $|B_i| = (3^{k-1} + 1)/2$. $\square$

CHAPTER VI

ONE RELATOR GROUPS WITH MORE THAN TWO GENERATORS

VI.1 The theory of 1-related groups

VI.1.1 The case of two generators

Let $G = \langle a, b \mid R = 1\rangle$ be a 1-related group, R is a cyclically reduced word in $F_2 = \langle a, b\rangle$. Consider a square lattice Γ in $\mathbb{R}^2$, the Cayley graph of $\mathbb{Z}^2$. We assume that horizontal edges are labeled by a and the vertical edges are labeled by b. Let $\psi : F_2 \to \mathbb{Z}^2$ be the abelianization map. Let w be the path in Γ starting at the origin $(0, 0)$ and reading the word R. This w is called the *trace* of the relator R. Note that w can visit every vertex (edge) many times. Vertices (edges) visited only once are called *simple*. A line L in $\mathbb{R}^2$ is said to be a *supporting* line of w if the path w lies on one side of L and has a common vertex with L.

Theorem VI.1. *(Brown [9, Theorem 4.4]) Let $G = \langle a, b|R = 1\rangle$, where R is a nontrivial cyclically reduced word in the free group on $\{a, b\}$ and $R \notin [F_2, F_2]$. Let w be the trace of R, ending at a point (m, n).*

G is an ascending HNN extension of a free group if and only if one of the two supporting lines of w parallel to the vector (m, n) intersects w in one simple vertex or one simple edge.

VI.1.2 Embedding into 2-generated groups

Let $G = \langle x_1, ..., x_k \mid R = 1\rangle$. If the sum of exponents of x_i in R is 0, then we can apply the Magnus rewriting to R. It consists of

- removing all occurrences of x_i in R;
- replacing every occurrence of a letter x_j in R by the letter $x_{j,p}$ where p is the sum of exponents of x_i in the prefix of R before that occurrence of x_j.

Let R' be the resulting word. The second indices p of letters in R' will be called the *Magnus x_i-indexes*. We say that certain Magnus index is *unique* if it occurs only once in R'.

We are going to use the following statement, which can be deduced from, say, a general result in [35] about hyperbolic groups. Recall that words $w_1, ..., w_k$ satisfy small cancellation condition $C'(1/k)$ if and only if whenever u is a subword of word w_i and at the same time a subword of word W_j (if $j = i$ then the subword should appear at two different places of w_i), then $|u| < 1/k|w_i|$.

Lemma VI.2. *Let $w_1, ..., w_k$ be words in the free group F_n satisfying $C'(\frac{1}{12})$. Then the subgroup $H = \langle w_1, ..., w_k\rangle$ of F_n satisfies the* congruence extension property*, that is for* *every normal subgroup N of H, the intersection of the normal closure N^G of N in F_2 with H is N. In particular, the natural homomorphism $H/N \to G/N^G$ is injective.*

Let ϕ be the map $F_k \to F_n$ (where $F_n = \langle x_1, ..., x_n \rangle$) given by $x_i \mapsto w_i, i = 1, ..., k$ where $w_1, ..., w_k$ satisfy $C'(\frac{1}{12})$. Lemma VI.2 immediately implies

Lemma VI.3. *The map ϕ induces an injective homomorphism from $G = \langle x_1, ..., x_k \mid R = 1\rangle$ to the 1-related n-generated group $\langle x_1, ..., x_n \mid \phi(R) = 1\rangle$.*

Theorem VI.4. *Consider a group $G = \langle x_1, x_2, \ldots, x_k | R = 1\rangle$, where R is a word in the free group on $\{x_1, x_2, \ldots, x_k\}$, $k \geq 2$. Assume the sum of exponents of x_k in R is zero and that the maximal Magnus x_k-index of x_1 is unique. Then G can be embedded into an ascending HNN extension of a finitely generated free group.*

Proof. We may assume that the maximal Magnus x_k-index of x_1 is bigger than the one of x_i, for $1 < i < k$, otherwise apply automorphism $x_i \to x_k^{-m} x_i x_k^m, x_j \to x_j (j \neq i)$ for m large enough.

Let $n \gg 1$. Consider the following words $w_1, ..., w_k \in F_2$.

$$\begin{aligned} w_1 &= aba^2b...a^nba^{n+1}ba^{-n-1}ba^{-n}b...a^{-2}ba^{-1}b \\ w_i &= ab^ia^2b^i...a^nb^ia^{-n}b^i...a^{-2}b^ia^{-1}b^i, \quad \text{for} \quad 1 < i < k \\ w_k &= ab^ka^2b^k...a^nb^ka^{-n}b^k...a^{-2}b^k \end{aligned}$$

These words satisfy the following conditions

(1) For a large enough n, these words and their cyclic shifts satisfy the small cancellation condition $C'(\frac{1}{12})$. Indeed, the maximal length of a subword repeating twice as a prefix of cyclic shifts of w_i does not exceed $2n + 3 + k$, and the length of each w_i is at least n^2. For a large enough n, we have $\frac{2n+3+k}{n^2} < \frac{1}{12}$.

(2) The sum of exponents of a in w_i, $i < k$, is equal to 0, the sum of exponents of a in w_k is 1.

(3) The maximal Magnus a-index of b in w_1 is $\frac{(n+1)(n+2)}{2}$, and this index is unique. The maximal Magnus a-indices of b in all other words are strictly smaller than the one in w_1.

By Lemma VI.3, the group G embeds into the two-generated one-relator group with presentation $\langle a, b \mid R(w_1, ..., w_k) = 1\rangle$.

It remains to prove that word $R(w_1, ..., w_k)$ satisfies the conditions of Lemma VI.1. Let $R' = R(w_1, ..., w_k)$. Clearly the sum of exponents of a in R' is zero. Every letter b with maximal Magnus a-index in R' comes from some occurrence of a word w_i substituted for letter x_i. The sum of exponents of a is nonzero only in w_k. Therefore the Magnus a-index of a letter b is a sum of the Magnus x_k-index of the letter x_i in R, for which it was substituted, and the Magnus a-index of b in w_i. The Magnus x_k-index in R is maximal for the letter x_1 and the maximum is unique in R. The maximal Magnus a-index of b in w_1 is also unique (and bigger than in all other w_i's). This gives a uniqueness of the maximal Magnus a-index in R'. Therefore there is a supporting line parallel to the b-axes that intersects the trace of R' in one simple edge corresponding to the letter b with the maximal Magnus a-index. Therefore by Lemma VI.1 the group $\langle a, b | R' = 1\rangle$ is an ascending HNN extension of a finitely generated free group. □

VI.1.3 More than 2 generators and walks in $\mathbb{Z}^k$

In the case of more than two generators we generalize the notion of supporting line in the following way. Given a relator R, a nontrivial word in the free group on $\{x_1, x_2, \dots, x_k\}$, let w be its trace in the lattice $\mathbb{Z}^k$. For a letter $t \in \{x_1, x_2, \dots, x_k\}$, let w_t be a set of edges labeled by t in w. A vertex on w_t is called *simple* if it does not belong to two edges of w_t. In particular, if w contains two consecutive edges with labels t, t^{-1}, then the endpoints of these edges are not simple vertices.

Definition VI.5. *A hyperplane P is a* supporting *hyperplane of w_t if the trace w_t lies on one side of P and has a common vertex with P. A hyperplane P is said to be* touching *w if*

- *P is parallel to the line containing the origin and the endpoint of w,*
- *there is $t \in \{x_1, x_2, \dots, x_k\}$, such that P is a supporting hyperplane of w_t,*
- *the intersection of P and w_t consists of one simple vertex or one simple edge.*

Lemma VI.6. *Let $G = \langle x_1, x_2, \dots, x_k | R = 1\rangle$, where R is a word in the free group on $\{x_1, x_2, \dots, x_k\}$, $k \geq 2$. Let w be a trace of R in the lattice $\mathbb{Z}^k$. If there is a hyperplane P touching w, then G can be embedded into an ascending HNN extension of a free group.*

Proof. We will embed G into a one-relator group on $k + 1$ generators that satisfies the condition of Theorem VI.4.

If the normal vector of P has irrational entries, then there is a hyperplane P' whose normal vector has rational entries that is also touching w. Thus we can assume the normal vector of P pointing toward the half-space not containing w_t is $(n(1), n(2), ..., n(k))$ with integer entries.

Consider the following substitution ϕ:

$$x_i \mapsto x_i z^{n(i)}, \quad i = 1, \dots, k.$$

Let $H = \langle x_1, x_2, \dots, x_k, z | \phi(R)\rangle$. Then G is embedded into H by ϕ. Since the normal vector of P is orthogonal to the line connecting the origin and the endpoint of w, the sum of exponents of z in $\phi(R)$ is zero.

It remains to show that the maximal Magnus z-index of x_t in $\phi(R)$ is unique.

We can assume that the edge in w_t intersecting P corresponds to the first letter of R. Assume that there is another letter x_t (at position j) in $\phi(R)$ with at least the same Magnus z-index as the first letter x_t in the word. Let $m(i)$ be the total sum of exponents of letter x_i between these two occurrences of x_t (note that it is the same in R as in $\phi(R)$). If the exponent of the first letter x_t is 1, then add 1 to $m(t)$. If the exponent of the other letter x_t (at position g) is -1, then subtract 1 to $m(t)$. The Magnus z-index of the latter letter x_t differs from the Magnus z-index of the first letter by precisely $m(1)n(1) + m(2)n(2) + \cdots + m(k)n(k)$.

Consider the edge corresponding to the first letter x_t and the edge of letter x_t at position j. Connect their initial points in $\mathbb{Z}^k$ by a vector (the vector connecting their terminal points is the same). It is easy to see that the coordinates of this vector are $(m(1), m(2), \ldots, m(k))$. If the scalar product of this vector with the normal vector of P is non-negative, then one of the endpoints of the edge of letter x_t at position g lies at P or on the other side than w_t. This is impossible, because P is a hyperplane touching w (with respect to x_t). □

Remark VI.7. *Let R be a non-reduced word in $\{x_1, ..., x_n\}$, and let R' be the cyclically reduced form of R. Let w, w' be the walks corresponding to R and R' respectively. If there exists a touching plane for w, then there exists a touching plane for w'. The proof easily proceeds by induction on the number of reductions.*

Let w be the walk in $\mathbb{Z}^k$ corresponding to R. Let ξ be the vector connecting the initial and the terminal points of w. Let $t \in \{1, ..., k\}$. For every supporting plane P of w_t let P^+ be the closed half-space of $\mathbb{R}^k$ bounded by P and containing w_t. The intersection of all P^+ is a convex polyhedron in $\mathbb{R}^k$. We shall call $\Delta_0(t)$ the projection of the boundary $\Delta(t)$ of that polyhedron onto the hyperplane orthogonal to ξ. Then $\Delta(t)$ is the right cylinder with base $\Delta_0(t)$, i.e. the direct product $\Delta_0(t) \times \mathbb{R}$. A vertex of the random walk projected to a 0-cell of $\Delta_0(t)$ is called a *corner*. For every vertex x that is a 0-cell of a $\Delta_0(t)$, the line $x + \mathbb{R}\xi \subseteq \Delta(t)$ will be called the *support line* of w_t.

Lemma VI.6 immediately implies

Lemma VI.8. *If one of the support lines of w_t intersects w_t in a simple vertex or a simple edge, then G is embeddable into an ascending HNN extension of a free group.*

VI.2 Random walks in $\mathbb{Z}^k$

VI.2.1 Preliminaries

Denote by P_n^{NR} the (uniform) measure on simple random walks of length n (not necessary reduced) and by P_n^{NB} the uniform measure on non-backtracking simple random walks of length n. To model cyclically reduced words, we denote by P_n^{CR} the uniform measure on non-backtracking simple random walks with last edge that is not inverse of the first edge of the walk (note that asymptotically this happens with probability $(2k-1)/2k$). In all cases we can consider the sample space Ω containing all walks of any finite length.

We say that an event A depends only on the cyclically reduced path of the random walk if $w \in A$ if and only if $w' \in A$, where w' is the cyclically reduced path of w. An example of such event is an event that a support line of the cyclically reduced path w' of a random walk w intersects w' in a simple vertex or a simple edge.

Lemma VI.9. *Let A be an event depending only on the cyclically reduced path of the random walk. Assume $\lim_{n\to\infty} P_n^{CR}(A)$ exists, then*

$$\lim_{n\to\infty} P_n^{NR}(A) = \lim_{n\to\infty} P_n^{CR}(A).$$

Proof. Let $\lim_{n\to\infty} P_n^{CR}(A) = a$ and assume n_0 is such that for all $n > n_0$

$$|P_n^{CR}(A) - a| < \epsilon.$$

If an event A depends only on the cyclically reduced path w' of a random walk w, then conditioning on the length of the cyclically reduced path $|w'|$ we see that $P_n^{NR}(A||w'| = k) = P_k^{CR}(A)$, provided $P_n^{NR}(|w'| = k) > 0$. Let n_1 be such that for all $n > n_1$, $P_n^{NR}(|w'| < n_0) \leq \epsilon$.

Then

$$P_n^{NR}(A) = \sum_{k=0}^{n} P_n^{NR}(|w'| = k) P_k^{CR}(A),$$

and we can split the sum in two parts ($k \leq n_0$ and $k > n_0$) and obtain for $n > n_1$

$$(1-\epsilon)(a-\epsilon) < P_n^{NR}(A) < \epsilon + (a+\epsilon).$$

Therefore $\lim_{n\to\infty} P_n^{NR}(A) = a$. $\square$

Next we will need a modified version of The Donsker's invariance principle. Denote by C the space of all continuous function $f : [0,1] \to \mathbb{R}^k$ such that $f(0) = 0$, equipped with the sup norm.

Theorem VI.10Donsker's Theorem modified. *Consider a piecewise linear function $Y_n(t) : [0,1] \to \mathbb{R}^k$, where the line segments are connecting points $Y_n(t) = S_{nt}/\sqrt{n}$ for $t = 0, 1/n, 2/n, \ldots, n/n = 1$, where (S_n) has a distribution according to P_n^{CR}. Then $Y_n(t)$ converges in distribution to a Brownian motion, as $n \to \infty$.*

Proof. First we prove that conditioning on the first step of non-backtracking random walk has asymptotically no influence on $Y_n(t)$, which allows us to switch between P^{NB} and P^{CR}.

Next, we basically repeat the proof of the Donsker's Theorem in [6, Theorem 10.1]. The Central Limit Theorem for non-backtracking walks that we will use was proved in [37].

Let (R_n) be a non-backtracking random walk. We cut the walk at time $\ln(n)$, splitting the walk into two (dependent) parts $\left(R^{(1)}_{\ln(n)}\right)$ and $\left(R^{(2)}_{n-\ln(n)}\right)$. Define piecewise linear functions $X(t)$ and $Z(t)$ connecting points $X(t) = R_{nt}/\sqrt{n}$ and $Z(t) = R^{(2)}_{nt}/\sqrt{n-\ln(n)}$ respectively. Clearly, the distance (in the sup norm) between $X(t)$ and $Z(t)$ goes to 0, as $n \to \infty$. Moreover the latter part of the walk $\left(R^{(2)}_{n-\ln(n)}\right)$ tends to be independent of the first step of (R_n), as $n \to \infty$. Therefore the piecewise linear functions obtained from walks with measures P^{NB} and P^{CR} have the same limit in distribution.

Next we show that the finite-dimensional distribution of $Y_n(t)$ converges to the one of Brownian motion. By the result of Rivin [37, Theorem 5.1] the probability distribution of $S_n/\sqrt{n}$ converges to a normal distribution on $\mathbb{R}^k$, whose mean is 0 and covariance matrix is diagonal, with entries

$$\sigma^2 = \frac{1}{\sqrt{2k-1}}\left[1 + \left(\frac{c+1}{c-1}\right)^{1/2}\right],$$

where $c = k/\sqrt{2k-1}$. By the previous paragraph this holds for $R_n/\sqrt{n}$ as well.

Consider now the two-dimensional distribution, that is the position at two time points, $s < t$. It is enough to show that $S_{ns}/\sqrt{n}$ and $(S_{nt} - S_{ns})/\sqrt{n})$ are asymptotically independent (the normal distribution of each of them was already established). The first step of $(S_{nt} - S_{ns})$ is not independent of (S_{ns}), but asymptotically the distribution of $(S_{nt} - S_{ns})/\sqrt{n}$ is independent of the first step. The convergence of finite-dimensional distribution for more time points can be proved in the same way.

It remains to show the tightness of the process. We refer ourselves to the proof in Billingsley [6, Page 69], and here we prove only the lemma needed. The claim is:

$$\mathrm{P}^{NB}_n\left(\max_{i<n}|S_i| \geq \lambda\sigma\sqrt{n}\right) \leq \mathrm{P}^{NB}_n\left(|S_n| \geq (\lambda - \sqrt{2})\sigma\sqrt{n}\right).$$

In order to prove this, we define events $E_i = \{\max_{j<i}|S_j| < \lambda\sigma\sqrt{n} \leq |S_i|\}$. Now we have:

$$\begin{aligned}
\mathrm{P}^{NB}_n\left(\max_{i<n}|S_i| \geq \lambda\sigma\sqrt{n}\right) &\leq \mathrm{P}^{NB}_n\left(|S_n| \geq (\lambda - \sqrt{2})\sigma\sqrt{n}\right) \\
&\quad + \sum_{i=1}^{n-1}\mathrm{P}^{NB}_n\left(E_i \cap \left\{|S_n| < (\lambda - \sqrt{2})\sigma\sqrt{n}\right\}\right), \\
\mathrm{P}^{NB}_n\left(E_i \cap \left\{|S_n| < (\lambda - \sqrt{2})\sigma\sqrt{n}\right\}\right) &\leq \mathrm{P}^{NB}_n\left(E_i \cap \left\{|S_n - S_i| \geq \sigma\sqrt{2n}\right\}\right) \\
&= \mathrm{P}^{NB}_n(E_i)\mathrm{P}^{NB}_n\left(|S_n - S_i| \geq \sigma\sqrt{2n}\right).
\end{aligned}$$

The last equality follows from the fact that the length of $S_n - S_i$ is independent of the walk up to the time i. Now by Chebyshev's inequality $\mathrm{P}_n^{NB}\left(|S_n - S_i| \geq \sigma\sqrt{2n}\right) \leq 1/2$. The claim follows from

$$\sum_{i=1}^{n-1} \mathrm{P}_n^{NB}\left(E_i \cap \left\{|S_n| < (\lambda - \sqrt{2})\sigma\sqrt{n}\right\}\right) \leq \frac{1}{2}\sum_{i=1}^{n-1} \mathrm{P}_n^{NB}(E_i)$$
$$\leq \frac{1}{2} P_n^{NB}\left(\max_{i<n} |S_i| \geq \lambda\sigma\sqrt{n}\right).$$

□

VI.2.2 Corners of random walk

Let (S_n) be a non-backtracking random walk in $\mathbb{Z}^k$ of length n with last edge that is not inverse of the first edge (according to the measure P_n^{CR}). Recall that Δ_0 is the projection of the boundary of the convex hull of the random walk (S_n) onto the hyperplane orthogonal to ξ, the vector connecting the initial and the terminal points of the random walk. Denote by H_n, the set of corners, which are the vertices of the random walk that project to the 0-cells of Δ_0. We count the corners with their multiplicities.

Lemma VI.11. *Let (S_n) be a non-backtracking random walk in $\mathbb{Z}^k$ of length n with last edge that is not inverse of the first edge (according to the measure P_n^{CR}). Let H_n be the set of its corners as defined above. Then for any integer m*

$$\mathrm{P}_n^{CR}(|H_n| < m) \to 0 \text{ as } n \to \infty.$$

Proof. Consider a piecewise linear function $X_n(t) : [0,1] \to \mathbb{R}^k$, where the line segments are connecting points $X_n(t) = S_{nt}/\sqrt{n}$ for $t = 0, 1/n, 2/n, \ldots, n/n = 1$. Recall that C is the space of all continuous function $f : [0,1] \to \mathbb{R}^k$ such that $f(0) = 0$, equipped with the sup norm. By Theorem VI.10, $X_n(t)$ converges in distribution to a Brownian motion, as $n \to \infty$. Denote by A_m a subset of C such that $f \in A_m$ if the convex hull of the projection of f to a hyperplane orthogonal to $f(1)$ is a $k-1$-dimensional (convex) polytope with at most m 0-cells. We will show that the set A_m is a closed subset of C in the sup norm and that the Wiener measure of A_m is zero. It follows from the weak convergence that $\mathrm{P}(X_n(t) \in A_m) \to 0$ as $n \to \infty$.

First show that A_m is closed. Let $f \notin A_m$ be a limit (in the sup norm) of $f_n \in A_m$. Let p_n (resp. p) denotes a projection on a hyperplane at the origin orthogonal to $f_n(1)$ (resp. $f(1)$). For any $\epsilon > 0$ and for all but finitely many n, we have $|p(f(t)) - p(p_n(f_n(t)))| < \epsilon$ for all t. If the convex hull of $p_n(f_n)$ is a polytope with at most m 0-cells, then the same holds for $p(p_n(f_n))$. Denote by B_n (resp. B) the convex hull of $p(p_n(f_n))$ (resp. $p(f)$). Then for any $\epsilon > 0$ the boundary of B is in Hausdorff ϵ-neighborhood of the boundary of B_n for all but finitely many n. We need to prove that if a convex body in $\mathbb{R}^k$ is arbitrarily close to some polytope with at most m 0-cells, then the body itself is such a polytope. To prove that, enumerate the 0-cells of B_n somehow $\{v_{n,1}, ..., v_{n,m}\}$ (the last few $0-$cells may coincide if the total number of $0-$cells is smaller than m). Choose one convergent subsequence $\{v_{n_j(i),i}\}_{j=1,2,...}$ of $\{v_{n,i}\}$ for each i in such a way that the set $\{n_j(i+1), j = 1, 2, ...\}$ is a subset of $\{n_j(i), j = 1, 2, ...\}$. Let $N^{(0)}$ be the sequence $\{n_j(m), j = 1, 2, ...\}$. There exists a subset $N^{(1)}$ of $N^{(0)}$ such that for every $i, j \in \{1, 2, ..., m\}$ either $v_{t,i}$ and $v_{t,j}$ span a 1-cell in all B_t, $t \in N^{(1)}$ or they don't span a 1-cell in all B_t, $t \in N^{(1)}$. Proceeding by induction on the dimension of a cell, we can find an infinite subset N of natural numbers such that for every subset $M \subset \{1, 2, ..., m\}$ either vertices $v_{t,i}$, $i \in M$, span a cell in B_t for all $t \in N$ or they span a cell in none of these B_t. For every $M \subseteq \{1, 2, ..., m\}$ such that $v_{t,i}$, $i \in M$, span a cell $F_M(t)$ of dimension j in all $B_t, t \in N$, the limit $\lim_{t \in N} F_M(t)$ exists and is an Euclidean convex polytope of dimension j spanned by the 0-cells $v_i, i \in M$. Hence the convex hull B of $p(f)$ is a convex polytope with at most m 0-cells. To see that A_m has measure 0 we introduce the following set D of continuous functions $[0, 1] \to \mathbb{R}^k$. A function $f \in C$ is in D if a convex hull of its projection to some 2-dimensional plane orthogonal to $f(1)$ has a smooth boundary, i.e. it is a C^1 curve in the plane. Clearly $D \cap A_m$ is empty for all m. Let X_t be a standard Brownian motion in $\mathbb{R}^k$. Then $X(t) - tX(1)$ is a Brownian bridge in $\mathbb{R}^k$. All projections of this Brownian bridge to $\mathbb{R}^2$ are equivalent in distribution and give Brownian bridges in $\mathbb{R}^2$. To conclude that D has Wiener measure 1, it is enough to show that the convex hull of a planar Brownian bridge has a smooth boundary almost surely. For Brownian motions, that is proved in [12]. We are going to use almost the same argument.

Consider a Brownian bridge and pick any of its extreme points. Move the beginning of the time from 0 to this extreme point and rotate the plane so that the path is in the upper half plane. The obtained process Y_t is a Brownian excursion, i.e. it stays in the upper half plane and return to the starting point. The same is true for Y_{1-t}. Then the transformation $V_t := (1 + t)Y_{t/(1+t)}$ is a Brownian meander (see for example [6], p.68, exercise 3). Let $V_t = (V_t(1), V_t(2))$, by [10], for any $c > 0$, we have

$$\mathrm{P}(\inf\{t : t > 0, |V_t(2)| \leq c|V_t(1)|\} = 0) = 1.$$

By reversing the transformation we obtain the same property for Y_t and Y_{1-t}. Now the claim follows using the argument from Theorem 1 in [12]. □

We say that a random walk is *bad* if there is no 0-cell of Δ_0 such that only a single vertex is projected to it.

Lemma VI.12. *The probability that a k-dimensional non-backtracking simple random walk (with last edge that is not inverse of the first edge of the walk) is bad in the above sense tends to* 0, *for* $k > 2$.

Proof. Let (S_n) be a k-dimensional non-backtracking simple random walk (with last edge that is not inverse of the first edge of the walk), $k > 2$. The number of all cyclically reduced walks, $|T_n|$, equals asymptotically $(2d-1)^n$. Let B_n be a set of all "bad" walks, i.e. for all 0-cells of Δ_0 we have at least two vertices projected to it.

Define a map $\tau_i : B_n \to T_{n+4}$ that inserts a commutator at an i-th corner of the random walk in such a way that it produces a new corner. For example, if the corner is between letters $x_i x_j$, we can insert $x_i x_j^{-1} x_i^{-1} x_j$ in between them, so that the second vertex of these three new vertices projects outside of Δ_0 of the original walk. Note that the new walk is not bad anymore.

This map τ_i is injective. Moreover images of the same walk under τ_i for different i are disjoint. The set of bad walks with more than K corners, $U_K := \{w | w \in B_n, H_n(w) > K\}$, is mapped by $\tau_1, \tau_2, \dots \tau_K$ into T_{n+4}. The union of their images $\bigcup_{i=1}^{K} \tau_i(U_K)$ is of size $K|U_K|$.

For any integer K we can write:

$$\begin{aligned}
|B_n| &= |B_n \setminus U| + |U| \\
&\leq \mathrm{P}(H_n < K)|T_n| + \frac{|T_{n+4}|}{K} \\
\frac{|B_n|}{|T_n|} &\leq \mathrm{P}(H_n < K) + \frac{1}{K}\frac{|T_{n+4}|}{|T_n|} \\
\frac{|B_n|}{|T_n|} &\leq \mathrm{P}(H_n < K) + \frac{(2d-1)^4}{K} \to \frac{(2d-1)^4}{K}
\end{aligned}$$

This holds for K arbitrarily large. The first summand tends to 0, as $n \to \infty$, by Lemma VI.12. It implies that the probability of a bad walk is less than any positive number. □

Remark VI.13. *Using Lemma VI.9, the same result as in Lemma VI.12 follows for a simple random walk (we consider corners of its reduced form).*

Proof of Theorem I.8. Let $G = \langle x_1, x_2, \dots, x_k | R = 1 \rangle$ be a random k-generator 1-relator group, $k > 2$. If the trace of R is not bad in the above sense, then there is a hyperplane touching its cyclic reduction. Thus, by Lemma VI.12, there is a hyperplane touching R with probability tending to 1. By Lemma VI.6, this implies that the group can be embedded into an ascending HNN extension of a free group almost surely. □

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